Proceedings of 45th Sapporo Symposium on Partial Differential Equations

Edited by
S.-I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda,
T. Ozawa, T. Sakajo, and K. Tsutaya

Series #179, 2020

HOKKAIDO UNIVERSITY
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Edited by
S.-I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda,
T. Ozawa, T. Sakajo, and K. Tsutaya

Sapporo, 2020

Partially supported by Grant-in-Aid for Scientific Research, the Japan Society for the Promotion of Science.

日本学術振興会科学研究費補助金（基盤研究 B 課題番号 19H01795）

Preface

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 17 through August 19 in 2020 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school. Late Professor Taira Shirota started the symposium more than 40 years ago. Late Professor Kôji Kubota and late Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

Finally, it should be mentioned that some contributions in this volume are written in Japanese, since the symposium is held as a domestic one for preventing from the spread of COVID-19.

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The 45th Sapporo Symposium on Partial Differential Equations

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Program Committee	Shin-Ichiro Ei, Yoshikazu Giga, Nao Hamamuki, Shuichi Jimbo, Hideo Kubo, Hirotoshi Kuroda, Tohru Ozawa, Takashi Sakajo, Kimitoshi Tsutaya
URL	http://www.math.sci.hokudai.ac.jp/sympo/sapporo/program200817.html

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Stability of steady states for geometric evolution equations

高坂良史[1]
(神戸大学大学院海事科学研究科)

1 序論

$\Gamma_t \subset \mathbb{R}^3$ は時間発展する動曲面とし，その挙動が

$$V = -\Delta_{\Gamma_t} H \ \text{ on } \ \Gamma_t \tag{1}$$

によって支配されているとする．ここで，V は Γ_t の法速度，H は Γ_t の平均曲率，Δ_{Γ_t} は Γ_t 上の Laplace-Beltrami 作用素である．(1) は表面拡散方程式と呼ばれる．表面拡散方程式 (1) が最初に紹介されたのは Mullins[8] の論文であり，空間変数が 1 次元の場合の表面拡散方程式が，空間変数が 1 次元の場合の平均曲率流 $V = H$ と共に導出されている．[8] では対称性をもつ粒界溝の挙動の支配法則が考察されており，界面を時間発展する曲線 $\Gamma_t = \{(x, y(x,t)) \,|\, x > 0\}$ と考え，まず粒界溝の移動が蒸発と凝集による原子の再配置によって起こる場合に，粒界溝の挙動を記述する運動方程式として

$$y_t = \frac{y_{xx}}{1 + y_x^2}$$

が導出されている (ここでは，簡単のため物理定数は 1 とする)．曲線 Γ_t の法速度 V と曲率 κ がそれぞれ

$$V = \frac{y_t}{\sqrt{1 + y_x^2}}, \quad \kappa = \frac{y_{xx}}{(1 + y_x^2)^{3/2}}$$

と表されることから，この微分方程式は曲率流方程式 $V = \kappa$ (空間変数 1 次元の場合の平均曲率流方程式) を意味する．一方，粒界溝の移動が表面拡散による原始の再配置によって起こる場合は，粒界溝の挙動を記述する運動方程式として

$$y_t = -\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{1 + y_x^2}}\frac{\partial}{\partial x}\left\{\frac{y_{xx}}{(1 + y_x^2)^{3/2}}\right\}\right]$$

が導出されている (上記と同様に物理定数は 1)．曲線の弧長パラメータに関する微分が

$$\frac{\partial}{\partial s} = \frac{1}{\sqrt{1 + y_x^2}}\frac{\partial}{\partial x}$$

と表されることから，この微分方程式は空間変数 1 次元の場合の表面拡散方程式を意味する．また，表面拡散方程式 (1) は，Cahn, Taylor[11] によって界面 Γ_t の表面エネルギー ($\approx$ 表面積) の H^{-1}-勾配流として得られることが示されている．その概略については第 2 章で紹介する．さらに，Cahn, Elliott, Novick-Cohen[2] によって，易動度が秩序パラメータに依存する退化型で，ポテンシャルが対数関数である Cahn-Hilliard 方程式の特異極限として表面拡散方程式 (1) が得られることが形式的に示されている．

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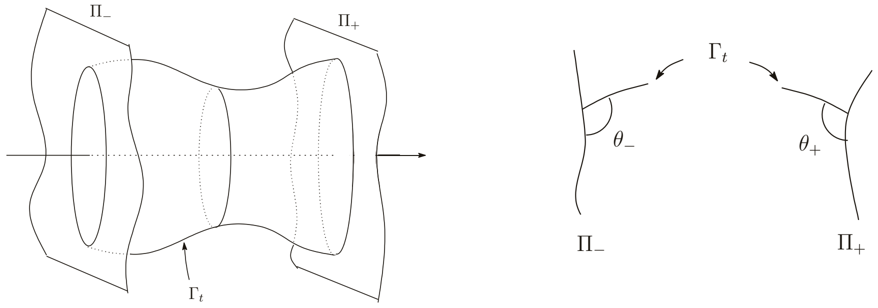


図 1: (2) の設定

本講演では次の問題を考える．今，$\varphi_\pm : \mathbb{R}_+ \to \mathbb{R}$ に対して，

$$\Pi_\pm = \{(\varphi_\pm(|\boldsymbol{\eta}|), \boldsymbol{\eta})^T \mid \boldsymbol{\eta} \in \mathbb{R}^2\}$$

とおく．このとき，動曲面 Γ_t は Π_- と Π_+ の間に存在し，Γ_t の挙動は以下の初期値・境界値問題によって支配されているとする．

$$\begin{cases} V = -\Delta_{\Gamma_t} H \ \text{ on } \ \Gamma_t, \\ (N, N_{\Pi_\pm})_{\mathbb{R}^3} = \cos\theta_\pm \ \text{ on } \ \Gamma_t \cap \Pi_\pm, \\ (\nabla_{\Gamma_t} H, \nu_\pm)_{\mathbb{R}^3} = 0 \ \text{ on } \ \Gamma_t \cap \Pi_\pm, \\ \Gamma_t|_{t=0} = \Gamma_0. \end{cases} \tag{2}$$

ここで，N は Γ_t の外向き単位法線ベクトル，$N_{\Pi_\pm}$ は $\Pi_\pm$ の外向き単位法線ベクトル，$\nu_\pm$ は $\Gamma_t \cap \Pi_\pm$ での Γ_t の外向き単位余法線ベクトルを表す．(2) に対応する定常曲面 (つまり，$V = 0$ を満たす曲面) を Γ_* とすると，Γ_* の平均曲率 H_* は以下を満たす．

$$\begin{cases} \Delta_{\Gamma_*} H_* = 0 \ \text{ on } \ \Gamma_*, \\ (\nabla_{\Gamma_*} H_*, \nu_\pm)_{\mathbb{R}^3} = 0 \ \text{ on } \ \Gamma_* \cap \Pi_\pm. \end{cases}$$

このとき

$$\|\nabla_{\Gamma_*} H_*\|^2_{L^2(\Gamma_*)} = 0$$

を得るので，(2) に対応する定常解は平均曲率一定曲面となる．本講演では軸対称な平均曲率一定曲面である Delaunay 曲面を (2) に対応する定常曲面 Γ_* としてとらえ，Γ_* からの軸対称な摂動のもとで (2) を考える．その線形化問題に対応する固有値問題を解析することで，(2) の定常解としての Delaunay 曲面の安定性について得られた結果を紹介する．

2　勾配流

表面拡散方程式の勾配流の構造についてその概略を述べる．Γ は閉曲面とし，Γ の表面積を $\mathrm{Area}\,[\Gamma]$ と表すことにする．このとき，$\mathrm{Area}\,[\Gamma]$ の第 1 変分 $\delta\mathrm{Area}\,[\Gamma]$ は

$$\delta\mathrm{Area}\,[\Gamma](\varphi) = -\int_\Gamma H\varphi\, dS \quad (\varphi \in E)$$

で与えられる．ただし，E は Hilbert 空間であり，この後に設定する．ここで，$\delta\mathrm{Area}\,[\Gamma](\varphi)$ は線形汎関数であるから，Riez の表現定理より

$$\delta\mathrm{Area}\,[\Gamma](\varphi) = (\nabla_E \mathrm{Area}\,[\Gamma], \varphi)_E$$

となる $\nabla_E \mathrm{Area}\,[\Gamma]$ が存在する．一方，動曲面 Γ_t の表面積 $\mathrm{Area}\,[\Gamma_t]$ の t に関する微分は，

$$\frac{d}{dt}\mathrm{Area}\,[\Gamma_t] = -\int_{\Gamma_t} HV\,dS_t$$

で与えられる．したがって，$\Gamma = \Gamma_t$ とし，$\varphi = V\,(\in E)$ とすると，

$$\frac{d}{dt}\mathrm{Area}\,[\Gamma_t] = (\nabla_E \mathrm{Area}\,[\Gamma_t], V)_E$$

を得る．よって，Γ_t の挙動が $V = -\nabla_E \mathrm{Area}\,[\Gamma_t]$ によって支配されるとき，

$$\frac{d}{dt}\mathrm{Area}\,[\Gamma_t] = -\|\nabla_E \mathrm{Area}\,[\Gamma_t]\|_E^2 \le 0$$

が成り立つ．方程式 $V = -\nabla_E \mathrm{Area}\,[\Gamma_t]$ を E に関する $\mathrm{Area}\,[\Gamma_t]$ の勾配流方程式という．

- **$\boldsymbol{E}$ が積分平均 0 の $\boldsymbol{L^2(\Gamma_t)}$ 関数全体の場合**

$\varphi \in E$ に対して，

$$\int_\Gamma H\varphi\,dS = \int_\Gamma (H - H_{av})\varphi\,dS, \quad H_{av} = \frac{1}{\mathrm{Area}\,[\Gamma]}\int_\Gamma H\,dS$$

と変形できる．$H - H_{av}$ の積分平均は 0 であるから $H - H_{av} \in E$．よって，

$$\nabla_E \mathrm{Area}\,[\Gamma_t] = -(H - H_{av})$$

を得るので，この場合の勾配流方程式は，

$$V = H - H_{av} \ \ (\text{体積保存型平均曲率流方程式}).$$

- **$\boldsymbol{E}$ が積分平均 0 の $\boldsymbol{(H^1(\Gamma_t))^*(= H^1(\Gamma_t)}$ の共役空間) 関数全体の場合**

この場合，E の内積は形式的に次で与えられる．

$$(\varphi_1, \varphi_2)_E = \int_\Gamma \left\{(-\Delta_\Gamma)^{-1}\varphi_1\right\}\varphi_2\,dS.$$

よって，

$$(\nabla_E \mathrm{Area}\,[\Gamma_t], \varphi)_E = \int_{\Gamma_t} \left\{(-\Delta_\Gamma)^{-1}\nabla_E \mathrm{Area}\,[\Gamma_t]\right\}\varphi\,dS_t$$

となるので，

$$(-\Delta_\Gamma)^{-1}\nabla_E \mathrm{Area}\,[\Gamma_t] = -(H - H_{av}).$$

この結果，

$$\nabla_E \mathrm{Area}\,[\Gamma_t] = \Delta_\Gamma H$$

を得るので，この場合の勾配流方程式は，

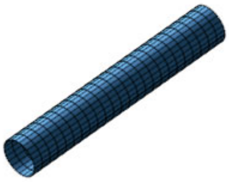
$$V = -\Delta_{\Gamma_t} H \ \ (\text{表面拡散方程式})$$

このとき，表面拡散方程式は H^{-1}-勾配流であるという．

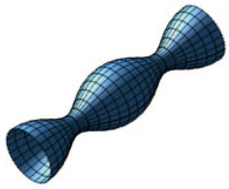
(注) E の元である関数の条件として積分平均 0 が課されると，$V \in E$ のとき V の積分平均は 0 である．Γ_t に囲まれた部分の体積を $\mathrm{Vol}\,[\Gamma_t]$ と表すと

$$\frac{d}{dt}\mathrm{Vol}\,[\Gamma_t] = \int_{\Gamma_t} V\,dS_t = 0 \ \ (V \in E)$$

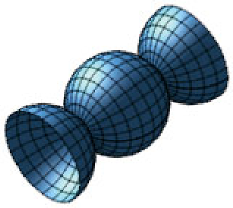
となり，Γ_t の体積は保存される (t に依らず一定である)．



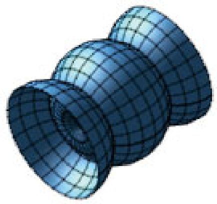
円柱 $(B=0)$



アンデュロイド $(0<B<1)$



(連鎖) 球 $(B=1)$



ノドイド $(B>1)$

図 2: Delaunay 曲面 $(H_* \neq 0)$

3　固有値問題

Γ_* を (2) に対応する軸対称な定常曲面とする．このとき Γ_* は

$$\Gamma_* = \{(x_*(s), y_*(s)\cos\zeta, y_*(s)\sin\zeta)^T \mid s \in [0,d], \zeta \in [0,2\pi]\}$$

で表される．ただし，s は軸対称な曲面 Γ_* の生成曲線の弧長パラメータであり，$(x_*(s), y_*(s))^T$ は生成曲線を表す．平均曲率が 0 でない Delaunay 曲面の表現公式は次のようになる．

命題 3.1 ([5, 7])　H_* は $H_*<0$ を満たす定数とする．平均曲率が H_* である Delaunay 曲面の生成曲線は次で与えられる．

$$\begin{cases} x_*(s) = \displaystyle\int_0^s \frac{1-B\sin(2H_*(\sigma-\tau))}{\sqrt{1+B^2-2B\sin(2H_*(\sigma-\tau))}}\, d\sigma, \\ y_*(s) = -\dfrac{1}{2H_*}\sqrt{1+B^2-2B\sin(2H_*(s-\tau))}. \end{cases} \tag{3}$$

ただし，$B \geq 0$，$\tau \in \mathbb{R}$ は定数である．

Delaunay 曲面は $B=0$ のとき円柱，$0<B<1$ のときアンデュロイド，$B=1$ のとき (連鎖) 球，$B>1$ のときノドイドになる (図 2 参照)．

Delaunay 曲面 Γ_* からの軸対称な摂動を考え，それによって得られる (2) に対応する非線形問題を線形化すると次のようになる．

$$\begin{cases} v_t = -\dfrac{1}{2}\Delta_{\Gamma_*} L[v] \ \text{ for } \ (s,t) \in [0,d]\times[0,T], \\ \partial_s v \pm (\kappa_{\Pi_\pm}\csc\theta_\pm - \kappa_{\Gamma_*}\cot\theta_\pm)v = 0 \ \text{ for } \ s=0,d,\ t\in[0,T], \\ \partial_s L[v] = 0 \ \text{ for } \ s=0,d,\ t\in[0,T]. \end{cases} \tag{4}$$

ただし，

$$
\begin{aligned}
&L[v]=\Delta_{\Gamma_*}v+|A_*|^2v,\\
&\Delta_{\Gamma_*}=\frac{1}{y_*}\left\{\partial_s(y_*\partial_s)+\frac{1}{y_*}\partial_\zeta^2\right\},\quad |A_*|^2=(-x_*''y_*'+x_*'y_*'')^2+\left(\frac{x_*'}{y_*}\right)^2,\\
&\kappa_{\Pi_\pm}=\pm\frac{\ddot{\varphi}_\pm(y_*)}{\{1+(\dot{\varphi}_\pm(y_*))^2\}^{3/2}},\quad \kappa_{\Gamma_*}=-x_*''y_*'+x_*'y_*''.
\end{aligned}
$$

ここで，κ_{Π_-} は $y=y_*(0)$ での曲線 $x=-\varphi_-(y)$ の曲率，κ_{Π_+} は $y=y_*(d)$ での曲線 $x=\varphi_+(y)$ の曲率，κ_{Γ_*} は Γ_* の生成曲線 $(x_*(s),y_*(s))^T$ の曲率を表す．軸対称な摂動で v は ζ に依らないので

$$
\Delta_{\Gamma_*}v=\frac{1}{y_*}\{\partial_s(y_*\partial_s v)\}
$$

となる．また，体積保存性から v は

$$
\int_0^d v\,y_*ds=0
$$

を満たし，この条件下で線形化問題も H^{-1}-勾配流の構造をもつ．定常曲面 Γ_* の安定性を調べるため，(4) に対応する次の固有値問題を考える．

$$
\begin{cases}
-\Delta_{\Gamma_*}L[w]=\lambda w \ \text{ for } \ s\in[0,d],\\
\partial_s w\pm(\kappa_{\Pi_\pm}\csc\theta_\pm-\kappa_{\Gamma_*}\cot\theta_\pm)w=0 \ \text{ at } \ s=0,d,\\
\partial_s L[w]=0 \ \text{ at } \ s=0,d.
\end{cases}
\tag{5}
$$

固有値問題 (5) を解析するため，関数空間等を設定する．まず，

$$
\mathcal{E}=\left\{w\in H^1(\Gamma_*)\ \middle|\ \int_0^d w\,y_*ds=0\right\},\quad \mathcal{X}=\left\{w\in(H^1(\Gamma_*))^*\ \middle|\ \langle w,1\rangle=0\right\}
$$

とする．ただし，$(H^1(\Gamma_*))^*$ は $H^1(\Gamma_*)$ の共役空間である．また，

$$
\mathcal{D}(\mathcal{A})=\left\{w\in H^3(\Gamma_*)\ \middle|\ \partial_s w\pm(\kappa_{\Pi_\pm}\csc\theta_\pm-\kappa_{\Gamma_*}\cot\theta_\pm)w=0 \ \text{ at } \ s=0,d,\ \int_0^d w\,y_*ds=0\right\}
$$

とし，$\mathcal{D}(\mathcal{A})$ 上の線形作用素 $\mathcal{A}:\mathcal{D}(\mathcal{A})\to\mathcal{X}$ を

$$
\langle\mathcal{A}w,\psi\rangle=\int_0^d\partial_s L[w]\,\partial_s\psi\,y_*ds\quad(w\in\mathcal{D}(\mathcal{A}),\,\psi\in\mathcal{E})
$$

で定める．このとき，双線形形式

$$
\begin{aligned}
I[w_1,w_2]=&\int_0^d\left\{\partial_s w_1\partial_s w_2-|A_*|^2w_1w_2\right\}y_*\,ds\\
&+y_*(\kappa_{\Pi_+}\csc\theta_+-\kappa_{\Gamma_*}\cot\theta_+)w_1w_2\big|_{s=d}\\
&+y_*(\kappa_{\Pi_-}\csc\theta_--\kappa_{\Gamma_*}\cot\theta_-)w_1w_2\big|_{s=0}
\end{aligned}
$$

を定めると，線形作用素 $\mathcal{A}$ について

$$(\mathcal{A}w, \psi)_{-1} = -I[w, \psi] \quad (\psi \in \mathcal{E})$$

が成り立つ．ここで $(\cdot, \cdot)_{-1}$ は H^{-1}-内積を表し，以下で定義される．

$$(w_1, w_2)_{-1} = \int_0^d \partial_s u_{w_1} \partial_s u_{w_2}\, y_* ds.$$

ただし，$w_i \in \mathcal{X}$ に対して u_{w_i} は

$$\begin{cases} -\Delta_{\Gamma_*} u_{w_i} = w_i \ \text{ for } \ s \in (0, d), \\ \partial_s u_{w_i} = 0 \ \text{ at } \ s = 0, d \end{cases}$$

の弱解である．このとき，線形作用素 $\mathcal{A}$ とその固有値に関して次が成り立つ．

(P1) $\mathcal{A}$ は H^{-1}-内積に関して自己共役である．

(P2) $\{\lambda_n\}_{n\in\mathbb{N}}$ を $\mathcal{A}$ の固有値とし，$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$ とすると，$\{\lambda_n\}_{n\in\mathbb{N}}$ は次のように表される．

$$\lambda_1 = -\inf_{w\in\mathcal{E}\setminus\{0\}} \frac{I[w,w]}{(w,w)_{-1}}, \quad \lambda_n = -\sup_{\mathcal{W}\in\Sigma_{n-1}} \inf_{w\in\mathcal{W}^\perp\setminus\{0\}} \frac{I[w,w]}{(w,w)_{-1}}.$$

ここで，Σ_n は $\mathcal{E}$ の n 次元部分空間の族であり，$\mathcal{W}^\perp$ は H^{-1}-内積に関する $\mathcal{W}$ の直交補空間である．

(P3) $\mathcal{A}$ の固有値は $\kappa_{\Pi_\pm}$, κ_{Γ_*}, d, $\theta_\pm$ に連続的に依存し，$\kappa_{\Pi_\pm}$ に関して単調に減少する．

4 安定性の解析

(5) のすべての固有値が負である (つまり，最大固有値 λ_1 が負である) とき，定常曲面 Γ_* は軸対称な摂動のもとで線形安定であるという．

補題 4.1 $\kappa_{\Pi_-}, \kappa_{\Pi_+} > m$ かつ $d < \delta$ であれば

$$I[w, w] > 0 \quad (w \in \mathcal{E} \setminus \{0\})$$

となる $m > 0$, $\delta > 0$ が存在する．

補題4.1 と (P2) から，$\kappa_{\Pi_-}, \kappa_{\Pi_+} > m$ かつ $d < \delta$ であれば最大固有値 λ_1 が 0 以下になるような $m > 0$, $\delta > 0$ が存在する．固有値は $\kappa_{\Pi_\pm}$ に連続的に依存し，かつ単調に減少するので，0 が固有値となる条件を求めれば安定性の判定条件を得ることができる．そこで，0-固有値問題

$$\Delta_{\Gamma_*} L[w] = 0 \ \text{ for } \ s \in [0, d], \tag{6}$$
$$\partial_s w \pm (\kappa_{\Pi_\pm} \csc\theta_\pm - \kappa_{\Gamma_*} \cot\theta_\pm) w = 0 \ \text{ at } \ s = 0, d, \tag{7}$$
$$\partial_s L[w] = 0 \ \text{ at } \ s = 0, d \tag{8}$$

を考える．(6) の両辺に $L[w]$ をかけて部分積分し，境界条件 (8) を適用すると

$$\|\partial_s L[w]\|^2_{L^2(\Gamma_*)} = 0.$$

よって，$L[w] =$ (定数) となる．この結果，境界条件 (8) を満たす方程式 (6) の解を得るには

$$L[w] = 0, \quad L[w] = \gamma\,(\neq 0) \tag{9}$$

を解けばよい．ただし，γ は任意定数である．このとき，w_1, w_2 を $L[w] = 0$ の基本解，w_3 を $L[w] = 1$ の特殊解とすると，境界条件 (8) を満たす方程式 (6) の解は

$$w(s) = c_1 w_1(s) + c_2 w_2(s) + \gamma w_3(s) \tag{10}$$

と表される．ただし，c_1, c_2 は任意定数である．ここで

$$\Lambda_\pm := \kappa_{\Pi_\pm} \csc\theta_\pm - \kappa_{\Gamma_*} \cot\theta_\pm$$

とする．(10) で表される解が残りの境界条件 (7) と

$$\int_0^d w\, y_* ds = 0$$

を満たし，かつ非自明解となる条件を求めれば，その条件が 0 が固有値であるための必要十分条件となる．つまり，0 が固有値であるための必要十分条件は，

$$\begin{vmatrix} w_1'(0) - \Lambda_- w_1(0) & w_2'(0) - \Lambda_- w_2(0) & w_3'(0) - \Lambda_- w_3(0) \\ w_1'(d) + \Lambda_+ w_1(d) & w_2'(d) + \Lambda_+ w_2(d) & w_3'(d) + \Lambda_+ w_3(d) \\ \displaystyle\int_0^d w_1\, y_* ds & \displaystyle\int_0^d w_2\, y_* ds & \displaystyle\int_0^d w_3\, y_* ds \end{vmatrix} = 0 \tag{11}$$

が成り立つことである．

$$\boldsymbol{w}(s) = (w_1(s), w_2(s), w_3(s))^T, \quad \boldsymbol{I}(d) = \left(\int_0^d w_1\, y_* ds, \int_0^d w_2\, y_* ds, \int_0^d w_3\, y_* ds \right)^T$$

とすると，(11) は

$$A^w \kappa_{\Pi_-} \kappa_{\Pi_+} + B_-^w \kappa_{\Pi_-} + B_+^w \kappa_{\Pi_+} + C^w = 0 \tag{12}$$

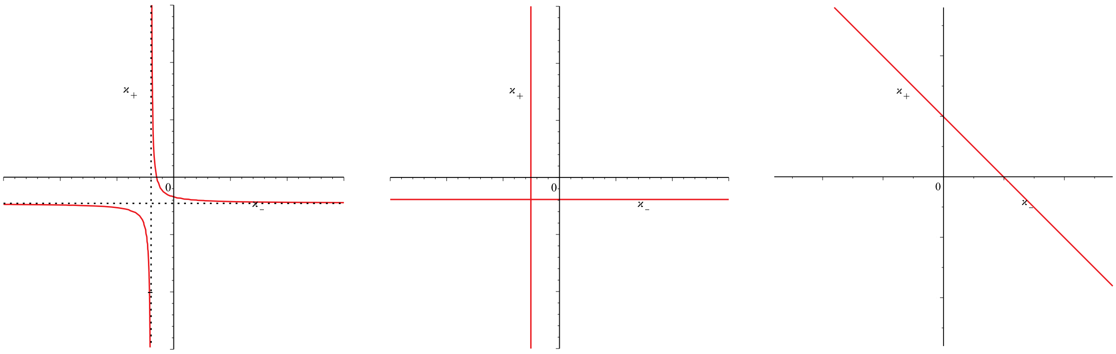
と表記できる．ただし，

$$\begin{aligned} A^w =& -(\boldsymbol{w}(0) \times \boldsymbol{w}(d), \boldsymbol{I}(d))_{\mathbb{R}^3}, \\ B_-^w =& \left\{ -(\boldsymbol{w}(0) \times \boldsymbol{w}'(d), \boldsymbol{I}(d))_{\mathbb{R}^3} + (\boldsymbol{w}(0) \times \boldsymbol{w}(d), \boldsymbol{I}(d))_{\mathbb{R}^3} \kappa_{\Gamma_*}(d) \cot\theta_+ \right\} \sin\theta_+, \\ B_+^w =& \left\{ (\boldsymbol{w}'(0) \times \boldsymbol{w}(d), \boldsymbol{I}(d))_{\mathbb{R}^3} + (\boldsymbol{w}(0) \times \boldsymbol{w}(d), \boldsymbol{I}(d))_{\mathbb{R}^3} \kappa_{\Gamma_*}(0) \cot\theta_- \right\} \sin\theta_-, \\ C^w =& \left\{ (\boldsymbol{w}'(0) \times \boldsymbol{w}'(d), \boldsymbol{I}(d))_{\mathbb{R}^3} \right. \\ & \left. - (\boldsymbol{w}(0) \times \boldsymbol{w}(d), \boldsymbol{I}(d))_{\mathbb{R}^3} \kappa_{\Gamma_*}(d) \kappa_{\Gamma_*}(0) \cot\theta_+ \cot\theta_- \right\} \sin\theta_+ \sin\theta_-. \end{aligned}$$

条件式 (12) を解析する．簡単な考察により次を得る．

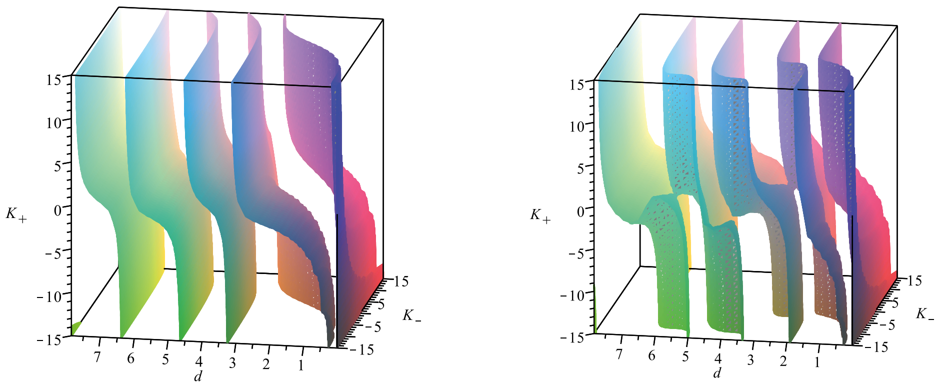
(I) $A^w \neq 0$ かつ $B_-^w B_+^w - A^w C^w \neq 0$ の場合

$$(12) \quad \Leftrightarrow \quad \kappa_{\Pi_+} = -\frac{B_-^w}{A^w} + \frac{\dfrac{B_-^w B_+^w - A^w C^w}{(A^w)^2}}{\kappa_{\Pi_-} - \left(-\dfrac{B_+^w}{A^w} \right)}.$$



(I) (II) (III)

図 3: $(\kappa_{\Pi_-}, \kappa_{\Pi_+})$ 平面上で (12) が描く曲線．ただし，(I) は $B_-^w B_+^w - A^w C^w > 0$ の場合の曲線．



(I)&(II) (I)&(II)&(III)

図 4: $(\kappa_{\Pi_-}, \kappa_{\Pi_+}, d)$ 空間上で (12) が描く曲面．

(II) $A^w \neq 0$ かつ $B_-^w B_+^w - A^w C^w = 0$ の場合

$$(12) \quad \Leftrightarrow \quad \left\{ \kappa_{\Pi_-} - \left(-\frac{B_+^w}{A^w} \right) \right\} \left\{ \kappa_{\Pi_+} - \left(-\frac{B_-^w}{A^w} \right) \right\} = 0.$$

(III) $A^w = 0$ の場合

$$(12) \quad \Leftrightarrow \quad B_-^w \kappa_{\Pi_-} + B_+^w \kappa_{\Pi_+} + C^w = 0.$$

上記 (I)，(II)，(III) を踏まえ，$(\kappa_{\Pi_-}, \kappa_{\Pi_+}, d)$ 空間で (12) を図示すると，図 4 のような曲面になる．補題 4.1 と (P3) から，パラメータ $(\kappa_{\Pi_-}, \kappa_{\Pi_+}, d)$ が一番右側の曲面の右側の領域に含まれていれば，Γ_* は軸対称な摂動に関して線形安定になる．

係数 A^w, $B_\pm^w$, C^w は定常曲面 Γ_* の形状に依存して決まる．その詳細を解析するため，Γ_* を平均曲率 H_* (定数, < 0) をもつ Delaunay 曲面として w_i $(i = 1, 2, 3)$ を求める．Γ_* の

生成曲線は (3) で与えられるので，$|A_*|^2$ と κ_{Γ_*} は次のようになる．

$$|A_*|^2 = \frac{4H_*^2\{B^2(B-\sin(2H_*(s-\tau))^2+(1-B\sin(2H_*(s-\tau))^2\}}{(1+B^2-2B\sin(2H_*(s-\tau)))^2},$$
$$\kappa_{\Gamma_*} = \frac{2BH_*(B-\sin(2H_*(s-\tau))}{1+B^2-2B\sin(2H_*(s-\tau))}.$$

このとき，$L[w]=0$ の基本解 w_1, w_2 と $L[w]=1$ の特殊解 w_3 を求めると

$$\begin{cases} w_1(s) = \dfrac{\cos(2H_*(s-\tau))}{\sqrt{1+B^2-2B\sin(2H_*(s-\tau))}}, \\ w_2(s) = \sin(2H_*(s-\tau)) + 2H_*\left\{\dfrac{1+B^2}{2}I_1(s) - \dfrac{1}{2}I_2(s)\right\}, \\ w_3(s) = \dfrac{1}{4H_*^2} + \dfrac{B}{2H_*}I_1(s)\,w_1(s). \end{cases} \tag{13}$$

ただし，

$$I_1(s) = I_1(s;H_*,B,\tau) := \int_0^s \frac{1}{\sqrt{1+B^2-2B\sin(2H_*(\sigma-\tau))}}\,d\sigma,$$
$$I_2(s) = I_2(s;H_*,B,\tau) := \int_0^s \sqrt{1+B^2-2B\sin(2H_*(\sigma-\tau))}\,d\sigma.$$

ここで，

$$H_*^+ = -H_*\,(>0), \quad \alpha = H_*^+\tau + \frac{\pi}{4}, \quad \beta = H_*^+\tau - \frac{\pi}{4}$$

とおき，$\alpha \in [-\pi/2, \pi/2)$, $\beta < 0$,

$$\begin{cases} -\dfrac{\pi}{2} + m\pi < H_*^+ s - \alpha < \dfrac{\pi}{2} + m\pi \quad (m \in \mathbb{N}\cup\{0\}) \text{ for } B \neq 1, \\ 0 < H_*^+ s - \beta < \pi \text{ for } B = 1. \end{cases}$$

とする．$k = 2\sqrt{B}/(1+B)$ とするとき，$I_1(s;-H_*^+,B,\tau)$, $I_2(s;-H_*^+,B,\tau)$ は

$$I_1(s;-H_*^+,B,\tau) = \begin{cases} \dfrac{1}{H_*^+(1+B)}\big\{2mK(k) + (-1)^m F(\sin(H_*^+ s-\alpha);k) - F(\sin(-\alpha);k)\big\} \quad (B\neq 1), \\ \dfrac{1}{2H_*^+}\left\{\log\left(\tan\left(\dfrac{H_*^+ s-\beta}{2}\right)\right) - \log\left(\tan\left(-\dfrac{\beta}{2}\right)\right)\right\} \quad (B=1), \end{cases}$$

$$I_2(s;-H_*^+,B,\tau) = \begin{cases} \dfrac{1+B}{H_*^+}\big\{2mE(k) + (-1)^m E(\sin(H_*^+ s-\alpha);k) - E(\sin(-\alpha);k)\big\} \quad (B\neq 1), \\ \dfrac{2}{H_*^+}\big\{\cos\beta - \cos(H_*^+ s-\beta)\big\} \quad (B=1), \end{cases}$$

と表される．$K(k)$ は第１種完全楕円積分，$E(k)$ は第２種完全楕円積分，$F(\eta;k)$ は第１種

不完全楕円積分，$E(\eta;k)$ は第 2 種不完全楕円積分であり，それぞれ以下で定義される．

$$K(k)=\int_0^1 \frac{1}{\sqrt{(1-k^2\xi^2)(1-\xi^2)}}\,d\xi,\quad E(k)=\int_0^1 \sqrt{\frac{1-k^2\xi^2}{1-\xi^2}}\,d\xi,$$
$$F(\eta;k)=\int_0^\eta \frac{1}{\sqrt{(1-k^2\xi^2)(1-\xi^2)}}\,d\xi,\quad E(\eta;k)=\int_0^\eta \sqrt{\frac{1-k^2\xi^2}{1-\xi^2}}\,d\xi.$$

(13) を (12) に代入すると，

$$\begin{aligned}&A^D(H_*^+,B,d,\tau)\kappa_{\Pi_-}\kappa_{\Pi_+}+B_-^D(H_*^+,B,d,\tau,\theta_+)\kappa_{\Pi_-}+B_+^D(H_*^+,B,d,\tau,\theta_-)\kappa_{\Pi_+}\\&+C^D(H_*^+,B,d,\tau,\theta_+,\theta_-)=0.\end{aligned}$$

A^D, $B_\pm^D$, C^D は Maple 2015 を援用して表記できるが繁雑であるため，ここでは A^D のみ表記する．

$$\begin{aligned}&A^D(H_*^+,B,d,\tau)\\
&=\frac{1}{8(H_*^+)^3PQ}\Big[(H_*^+)^2(1-B^2)^2I_1^2\cos(2H_*^+\tau)\cos(2H_*^+(d-\tau))\\
&\quad-4(H_*^+)^2(1+B^2)I_1I_2\cos(2H_*^+\tau)\cos(2H_*^+(d-\tau))\\
&\quad+3(H_*^+)^2I_2^2\cos(2H_*^+\tau)\cos(2H_*^+(d-\tau))\\
&\quad+2H_*^+(1+B^2)I_1\big\{P\sin(2H_*^+\tau)\cos(2H_*^+(d-\tau))+Q\cos(2H_*^+\tau)\sin(2H_*^+(d-\tau))\big\}\\
&\quad-4H_*^+BI_1\big\{P\cos(2H_*^+(d-\tau))-Q\cos(2H_*^+\tau)\big\}\\
&\quad-4H_*^+I_2\big\{P\sin(2H_*^+\tau)\cos(2H_*^+(d-\tau))+Q\cos(2H_*^+\tau)\sin(2H_*^+(d-\tau))\big\}\\
&\quad+2PQ\big\{1+\sin(2H_*^+\tau)\sin(2H_*^+(d-\tau))\big\}\\
&\quad-(P^2+Q^2)\cos(2H_*^+\tau)\cos(2H_*^+(d-\tau))\Big].\end{aligned}$$

ただし，$P(H_*^+,B,\tau)$, $Q(H_*^+,B,d,\tau)$ はそれぞれ

$$P(H_*^+,B,\tau)=\sqrt{1+B^2-2B\sin(2H_*^+\tau)},$$
$$Q(H_*^+,B,d,\tau)=\sqrt{1+B^2+2B\sin(2H_*^+(d-\tau))}$$

で与えられる．さらに Maple 2015 を援用すると

$$\begin{aligned}&B_-^D(H_*^+,B,d,\tau,\theta_+)B_+^D(H_*^+,B,d,\tau,\theta_-)-A^D(H_*^+,B,d,\tau)C^D(H_*^+,B,d,\tau,\theta_+,\theta_-)\\
&=\frac{1}{16(H_*^+)^4PQ}\Big[H_*^+\big\{(1+B^2)(1+\sin(2H_*^+\tau)\sin(2H_*^+(d-\tau)))-(P^2+Q^2)\big\}I_1\\
&\quad+H_*^+\big(3-\sin(2H_*^+(d-\tau))\sin(2H_*^+\tau)\big)I_2\\
&\quad-P\cos(2H_*^+\tau)\sin(2H_*^+(d-\tau))-Q\sin(2H_*^+\tau)\cos(2H_*^+(d-\tau))\Big]^2\geq 0\end{aligned}$$

が成り立つことがわかる．このとき以下の定理を得る．

定理 4.1 q_1 を A^D の $H_*^+ d$ に関する最初の零点とする．また

$$D(\kappa_{\Pi_\pm}, H_*^+, B, d, \tau, \theta_\pm) := A^D(H_*^+, B, d, \tau)\kappa_{\Pi_-}\kappa_{\Pi_+} + B_-^D(H_*^+, B, d, \tau, \theta_+)\kappa_{\Pi_-} + B_+^D(H_*^+, B, d, \tau, \theta_-)\kappa_{\Pi_+} + C^D(H_*^+, B, d, \tau, \theta_+, \theta_-)$$

とする．このとき，$\kappa_{\Pi_\pm}, H_*^+, B, d, \tau, \theta_\pm$ が

$$\hat{D}(\kappa_{\Pi_\pm}, H_*^+, B, d, \tau, \theta_\pm) > 0, \;\; \kappa_{\Pi_-} > -\frac{B_+^D(H_*^+, B, d, \tau, \theta_-)}{A^D(H_*^+, B, d, \tau)}, \;\; H_*^+ d < q_1,$$

を満たすならば，Delaunay 曲面は軸対称な摂動に関して線形安定である．

定理 4.2 $H_*^+ d \geq q_1$ ならば，Delaunay 曲面が安定になるような $(\kappa_{\Pi_-}, \kappa_{\Pi_+})$ の組は存在しない．

一般的な解析から得られる結果はここまでである．実際にどのような設定でどのような形状の Delaunay 曲面が安定になるかを調べるには，さらに詳細な解析を必要とする．講演ではいくつかの状況を設定し，具体的な形状の Delaunay 曲面に関して安定性を議論する．

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STABILITY OF LAMB DIPOLES

KEN ABE

1. Introduction

1.1. **Lamb dipoles.** This talk is based on a joint work [1] with Kyudong Choi (UNIST). We consider the two-dimensional vorticity equations:

$$\begin{aligned} \partial_t \zeta + v \cdot \nabla \zeta = 0, \quad v = k * \zeta \quad & \text{in } \mathbb{R}^2 \times (0,\infty), \\ \zeta = \zeta_0 \quad & \text{on } \mathbb{R}^2 \times \{t=0\}, \end{aligned} \tag{1.1}$$

with the kernel $k(x) = (2\pi)^{-1} x^{\perp} |x|^{-2}$, $x^{\perp} = {}^t(-x_2, x_1)$. The equations (1.1) admit *a vortex pair*, i.e., a solution of the form

$$\begin{aligned} v(x,t) &= u(x + u_\infty t) - u_\infty, \\ \zeta(x,t) &= \omega(x + u_\infty t), \end{aligned}$$

vanishing at space infinity with a constant velocity $u_\infty \in \mathbb{R}^2$. Vortex pairs are symmetric dipoles with compactly supported two vorticities having opposite signs translating to one direction. They are theoretical models of coherent vortex structures in large-scale geophysical flows. See, e.g., [14], [10] for experimental works. By rotational invariance of (1.1), we take $u_\infty = {}^t(-W, 0)$, $W > 0$, without loss of generality. Substituting (v, ζ) into (1.1) implies the steady Euler equations for (u, ω) in a half plane:

$$\begin{aligned} u \cdot \nabla \omega = 0 \quad & \text{in } \mathbb{R}^2_+, \\ u \to u_\infty \quad & \text{as } |x| \to \infty. \end{aligned} \tag{1.2}$$

In 1906, H. Lamb [16, p.231] noted an explicit solution to (1.2), generally referred to as *the Lamb dipole* (Chaplygin-Lamb dipole), a solution $\omega_L = \lambda \max\{\Psi_L, 0\}$, $u_L = {}^t(\partial_{x_2}\Psi_L, -\partial_{x_1}\Psi_L)$, $0 < \lambda < \infty$, of the form

$$\Psi_L(x) = \begin{cases} C_L J_1(\lambda^{1/2} r) \sin\theta, & r \le a, \\ -W\left(r - \dfrac{a^2}{r}\right)\sin\theta, & r > a, \end{cases} \tag{1.3}$$

with the constants

Date: June 26, 2020.

$$C_L = -\frac{2W}{\lambda^{1/2} J_0(c_0)}, \quad a = c_0 \lambda^{-1/2},$$

where (r, θ) is the polar coordinate and $J_m(r)$ is the m-th order Bessel function of the first kind. The constant c_0 is the first zero point of J_1, i.e., $J_1(c_0) = 0$, $c_0 = 3.8317\cdots$, $J_0(c_0) < 0$. The parameter $\lambda > 0$ denotes the strength of the vortex and is related with its impulse by

$$\int_{\mathbb{R}^2_+} x_2 \omega_L \mathrm{d}x = \frac{c_0^2 \pi W}{\lambda}.$$

The Lamb dipole (1.3) is the simplest explicit solution to (1.2), symmetric for the x_2-variable, which is a special case of non-symmetric Chaplygin dipoles, independently founded by S. A. Chaplygin in 1903 [8], [9]. See also [22].

The Lamb dipole is considered as a stable vortex structure in a two-dimensional flow. Its stability has been studied by an experimental work [10] and also by a numerical work [13]. On the other hand, despite the explicit form of this classical solution, its mathematical stability had been an open question since the solution was introduced by S. A. Chaplygin and H. Lamb at the early 20th century. For solutions with a single-signed vortex such as a circular vortex [28], [25] or a rectangular vortex [4], stability results have been developed, while no stability result was known for the Lamb dipole which has a multi-signed vortex and forms a traveling wave.

There is an interesting relation with *solitons* in the theory of nonlinear wave equations. One of classical models that describes propagation of a wave may be the KdV equation [15]. More generally for the gKdV equation,

$$\partial_t w + \partial_x^3 w + \partial_x (w^p) = 0, \quad x \in \mathbb{R}, \ t > 0,$$

for an integer $p \geq 2$, there exists a soliton solution of the form $w(x, t) = Q_c(x - ct)$ for $c > 0$ and $Q_c(x) = c^{1/(p-1)} Q(c^{1/2} x)$, where

$$Q(x) = \left(\frac{p+1}{2\cosh^2((p-1)x/2)} \right)^{1/(p-1)},$$

is called soliton, which is a unique positive solution of the elliptic problem $\partial_x^2 Q + Q^p = Q$, up to translation. Stability of this soliton is well known when the problem is globally well-posed. Indeed for $2 \leq p < 5$, the gKdV equation is globally well-posed, and if initial data is close to the soliton, the solution remains nearby the soliton for all time by admitting translation of Q [5], [29]. Such stability is termed *orbital stability*. For $p = 5$, this soliton is unstable [20] and a finite time blow-up occurs [23], [21]. The Euler equations may have some aspects of the wave equation. Even for the three-dimensional case, vortex rings form traveling waves. We shall establish the orbital stability theorem for the Lamb dipole which is the most typical traveling wave.

In the sequel, we identify a function ζ_0 in $\mathbb{R}^2_+$ with an odd extension to $\mathbb{R}^2$ for the x_2-variable, i.e., $\zeta_0(x_1, x_2) = -\zeta_0(x_1, -x_2)$. Since a classical solution to (1.1) exists and is symmetric for the x_2-variable for sufficiently smooth initial data [18], a standard approximation argument implies the existence of a symmetric global weak solution $\zeta \in BC([0,\infty); L^2 \cap L^1(\mathbb{R}^2))$ for symmetric initial data $\zeta_0 \in L^2 \cap L^1(\mathbb{R}^2)$ [19]. Here, $BC([0,\infty); X)$ denotes the space of all bounded continuous functions from $[0,\infty)$ into a Banach space X. Among other results, our simplest result is the following:

Theorem 1.1. *Let $0 < \lambda,\ W < \infty$. The Lamb dipole ω_L is orbitally stable in the sense that for $\nu > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for $\zeta_0 \in L^2 \cap L^1(\mathbb{R}^2_+)$ satisfying $x_2\zeta_0 \in L^1(\mathbb{R}^2_+)$, $\zeta_0 \geq 0$, $\|\zeta_0\|_1 \leq \nu$ and*

$$\inf_{y \in \partial\mathbb{R}^2_+} \left\{ \|\zeta_0 - \omega_L(\cdot + y)\|_2 + \|x_2(\zeta_0 - \omega_L(\cdot + y))\|_1 \right\} \leq \delta,$$

there exists a global weak solution $\zeta(t)$ of (1.1) satisfying

$$\inf_{y \in \partial\mathbb{R}^2_+} \left\{ \|\zeta(t) - \omega_L(\cdot + y)\|_2 + \|x_2(\zeta(t) - \omega_L(\cdot + y))\|_1 \right\} \leq \varepsilon, \quad \textit{for all } t \geq 0.$$

1.2. **Vorticity method.** Theorem 1.1 is a particular case of our general stability theorem. Let us consider the existence problem (1.2). The equation $(1.2)_1$ can be written by the Jacobian of ${}^t(\Psi, \omega)$ for $u = {}^t(\partial_{x_2}\Psi, -\partial_{x_1}\Psi)$. Therefore ω is represented by $\omega = \lambda f(\Psi)$ with some function $f(t)$ and $\lambda > 0$ and existence of such (u, ω) is reduced to the free-boundary problem for $\gamma \geq 0$:

$$\begin{aligned} -\Delta\Psi &= \lambda f(\Psi) && \text{in } \mathbb{R}^2_+, \\ \Psi &= -\gamma && \text{on } \partial\mathbb{R}^2_+, \\ \partial_{x_1}\Psi \to 0, &\quad \partial_{x_2}\Psi \to -W && \text{as } |x| \to \infty. \end{aligned} \tag{1.4}$$

The function f is called a vorticity function which is prescribed by a non-negative and non-decreasing function. We shall take

$$f(t) = t_+, \quad t_+ = \max\{t, 0\},$$

for which the Lamb dipole Ψ_L is a solution to (1.4) for $\gamma = 0$ and spt $\omega_L = \overline{B(0,a) \cap \mathbb{R}^2_+}$, i.e., $\omega_L = \lambda f(\Psi_L)$. Here $B(0,a)$ is an open disk centered at the origin with the radius $a > 0$. The three parameters $W, \gamma \geq 0$ and $\lambda > 0$ are referred to as propagation speed, flux constant and strength parameter. We chose the flux constant γ so that $\Psi = 0$ on the boundary of the vortex core spt $\omega = \overline{\Omega}$. The problem (1.4) is a free-boundary problem since the vortex core Ω is a priori unknown. Once the core is found, one can find Ψ by solving the two problems:

$$
\begin{aligned}
&-\Delta\Psi = \lambda\Psi \quad \text{in } \Omega, \quad \Psi = 0 \quad \text{on } \partial\Omega, \\
&-\Delta\Psi = 0 \quad \text{in } \mathbb{R}^2_+\backslash\overline{\Omega},\ \Psi = -\gamma \quad \text{on } \partial\mathbb{R}^2_+,\ \partial_{x_1}\Psi \to 0,\ \partial_{x_2}\Psi \to -W \quad \text{as } |x| \to \infty.
\end{aligned}
$$

On the other hand, the core is characterized as $\Omega = \{x \in \mathbb{R}^2_+ \mid \Psi(x) > 0\}$ by a maximum principle. The function $\Psi = \psi - Wx_2 - \gamma$ is represented by the Green function of the Laplace operator subject to the Dirichlet boundary condition in a half plane

$$
(1.5) \qquad \psi(x) = \int_{\mathbb{R}^2_+} G(x,y)\omega(y)\mathrm{d}y, \quad G(x,y) = \frac{1}{4\pi}\log\left(1 + \frac{4x_2y_2}{|x-y|^2}\right).
$$

To study existence and stability of solutions to (1.4), we consider a variational principle based on vorticity, called a vorticity method, originating from the idea of Kelvin [26], initiated by Arnold [2], [3]. See also Benjamin [6] for vortex rings. For vortex pairs, vorticity methods were developed by Turkington [27] and Burton [7]. See also Norbury [24] and Yang [30] for a stream function method.

Our approach is based on the vorticity method of Friedman-Turkington [12], [11] developed for vortex rings. For $0 < \mu, \nu, \lambda < \infty$, we set a space of admissible functions

$$
K_{\mu,\nu} = \left\{ \omega \in L^2(\mathbb{R}^2_+) \ \middle|\ \omega \geq 0,\ \int_{\mathbb{R}^2_+} x_2\omega \mathrm{d}x = \mu,\ \int_{\mathbb{R}^2_+} \omega \mathrm{d}x \leq \nu \right\}.
$$

We construct solutions of (1.4) by maximizing a penalized energy

$$
E_{2,\lambda}[\omega] = E[\omega] - \frac{1}{2\lambda}\int_{\mathbb{R}^2_+} \omega^2 \mathrm{d}x, \quad E[\omega] = \frac{1}{2}\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+} G(x,y)\omega(x)\omega(y)\mathrm{d}x\mathrm{d}y.
$$

For a notational convenience, we formulate the maximization problem as a minimization of $-E_{2,\lambda}$ and denote by

$$
(1.6) \qquad I_{\mu,\nu\,\lambda} = \inf_{\omega \in K_{\mu,\nu}} \{-E_{2,\lambda}[\omega]\}.
$$

The constants $W,\ \gamma \geq 0$ are Lagrange multipliers. This formulation is slightly different from that of [12], [11], where admissible functions are restricted to a space of symmetric functions for $x_1 \in \mathbb{R}$. More precisely, the method in [12], [11] applies to prove compactness of a minimizing sequence satisfying

$$
(1.7) \qquad
\begin{aligned}
&\omega(x_1, x_2) = \omega(-x_1, x_2), \\
&\omega(x_1, x_2) \text{ is non-increasing for } x_1 > 0.
\end{aligned}
$$

The condition (1.7) is essential for the method in [12], [11]. In fact, since the energy $-E_{2,\lambda}$ is invariant by translation for the x_1-variable, translation of any minimizer is a minimizing sequence. We shall show without assuming (1.7) that any minimizing sequence is relatively compact by translation for the x_1-variable by using the concentration compactness principle of Lions [17]. The following Theorem 1.2 is an improvement of [12], [11] in terms of vortex pairs.

Theorem 1.2. *Let $0 < \mu, \nu, \lambda < \infty$. For any minimizing sequence $\{\omega_n\}$ satisfying $\omega_n \in K_{\mu_n,\nu}$, $\mu_n \to \mu$ and $-E_{2,\lambda}[\omega_n] \to I_{\mu,\nu,\lambda}$, there exists a sequence $\{y_n\} \subset \partial\mathbb{R}^2_+$ such that $\{\omega_n(\cdot + y_n)\}$ and $\{x_2\omega_n(\cdot + y_n)\}$ are relatively compact in $L^2(\mathbb{R}^2_+)$ and $L^1(\mathbb{R}^2_+)$, respectively. In particular, the problem (1.6) has a minimizer in $K_{\mu,\nu}$.*

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Regularity for the stationary Navier-Stokes equations over bumpy boundaries and a local wall law

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1 Introduction

In this talk we are concerned with the local regularity of viscous incompressible fluid flows above rough bumpy boundaries $x_3 > \varepsilon\gamma(x'/\varepsilon)$ with γ Lipschitz and the no-slip boundary condition. Although bumpy boundaries have a complicated geometry and low regularity, the flow may paradoxically be better behaved than for smooth or flat boundaries. It is well documented in the physical [23, 35] and the mathematical [31, 12, 19] literature that roughness favors slip of the fluid on the boundary in certain regimes. In the striking paper [8] it is even showed experimentally that roughness may delay the transition to turbulence. This also supports the idea that the vanishing viscosity limit from Navier-Stokes to Euler may be less singular above highly oscillating boundaries than above flat ones [21, 11, 33].

Our goal is to investigate these effects, such as the enhanced slip, or the delay of the transition to turbulence, from the point of view of the regularity theory. Due in particular to vorticity creation at the boundary, the boundary regularity of fluid flows with the no-slip boundary conditions is delicate. In the nonstationary case, it is for instance not known whether there is an analogue of Constantin and Fefferman's [7] celebrated geometric regularity criteria for supercritical blow-up scenarios. For perfect slip or Navier-slip boundary conditions on the contrary, the situation is brighter. In particular an extension of the criteria of [7] is known in this case; see the work [6] by Beirão da Veiga and Berselli and [29] by Li. We expect that fluids over bumpy boundaries have an intermediate behavior between these two extreme no-slip and (full-)slip situations, especially as far as the mesoscopic regularity properties are concerned.

Our approach grounds on the use of asymptotic analysis to prove regularity estimates. The success of such methods to prove the regularity to certain Partial Differential Equations is spectacular. One of the striking examples is that of homogenization. The basic idea is that the large-scale regularity is determined by the macroscopic properties of the systems, i.e. in the homogenization limit, while the small-scale regularity is determined by the regularity of the data (coefficients, boundary). Two approaches were developed: (a) blow-up and compactness arguments in periodic homogenization in the wake of the pioneering works [4, 5], (b) quantitative arguments based on suboptimal local error estimates as developed for periodic homogenization [36, 9, 34], almost periodic homogenization [3], and stochastic homogenization [15, 1, 18, 38].

In this work, we focus on the regularity for stationary problems. We consider the three-

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dimensional stationary Navier-Stokes equations

$$\begin{cases} -\Delta u^\varepsilon + \nabla p^\varepsilon = -u^\varepsilon \cdot \nabla u^\varepsilon & \text{in } B^\varepsilon_{1,+}(0) \\ \nabla \cdot u^\varepsilon = 0 & \text{in } B^\varepsilon_{1,+}(0) \\ u^\varepsilon = 0 & \text{on } \Gamma^\varepsilon_1(0)\,. \end{cases} \tag{NS$^\varepsilon$}$$

The functions $u^\varepsilon = u^\varepsilon(x)$, $u^\varepsilon = (u^\varepsilon_1, u^\varepsilon_2, u^\varepsilon_3)^\top \in \mathbb{R}^3$, and $p^\varepsilon = p^\varepsilon(x) \in \mathbb{R}$ denote respectively the velocity field and the pressure field of the fluid. We have set for $\varepsilon \in (0,1]$ and $r \in (0,1]$,

$$B^\varepsilon_{r,+}(0) = \{x \in \mathbb{R}^3 \mid x' \in (-r,r)^2\,,\ \ \varepsilon\gamma(\frac{x'}{\varepsilon}) < x_3 < \varepsilon\gamma(\frac{x'}{\varepsilon}) + r\}\,,$$

$$\Gamma^\varepsilon_r(0) = \{x \in \mathbb{R}^3 \mid x' \in (-r,r)^2\,,\ \ x_3 = \varepsilon\gamma(\frac{x'}{\varepsilon})\}\,.$$

The boundary function $\gamma \in W^{1,\infty}(\mathbb{R}^2)$ is assumed to satisfy $\gamma(x') \in (-1,0)$ for all $x' \in \mathbb{R}^2$. Note that, since our interest is in the local boundary regularity of (NS$^\varepsilon$), the boundary condition is prescribed only on the lower part of $\partial B^\varepsilon_{r,+}(0)$. We work in the framework of weak solutions of (NS$^\varepsilon$). A vector function $u^\varepsilon \in H^1(B^\varepsilon_{1,+}(0))^3$ is said to be a weak solution to (NS$^\varepsilon$) if u^ε satisfies $\nabla \cdot u^\varepsilon = 0$ in the sense of distributions, $u^\varepsilon|_{\Gamma^\varepsilon_1(0)} = 0$ in the trace sense, and

$$\int_{B^\varepsilon_{1,+}(0)} \nabla u^\varepsilon \cdot \nabla \varphi = -\int_{B^\varepsilon_{1,+}(0)} (u^\varepsilon \cdot \nabla u^\varepsilon) \cdot \varphi \tag{1}$$

for any $\varphi \in C^\infty_{0,\sigma}(B^\varepsilon_{1,+}(0))$. Here $C^\infty_{0,\sigma}(\Omega)$ denotes the space of test functions $\{f \in C^\infty_0(\Omega)^3 \mid \nabla\cdot f = 0\}$ when Ω is an open set in $\mathbb{R}^3$. For the pressure p^ε, we emphasize that the unique existence in $L^2(B^\varepsilon_{1,+}(0))$ up to an additive constant can be proved in a functional analytic way using the weak formulation (1).

Our use of compactness arguments to tackle the regularity for solutions of (NS$^\varepsilon$) is reminiscent of the pioneering work of Avellaneda and Lin [4, 5] in homogenization, and of the works by Gérard-Varet [10], Gu and Shen [16], and Kenig and Prange [24, 25]. We separate the small-scale regularity, i.e. at scales $\lesssim \varepsilon$, from the mescopic- or large-scale regularity, i.e. at scales $\varepsilon \lesssim r \leq 1$. Concerning the small scales, the classical Schauder regularity theory for the Stokes and the Navier-Stokes equations was started by Ladyženskaja [28] using potential theory and by Giaquinta and Modica [14] using Campanato spaces. These classical estimates require some smoothness of the boundary and typically depend on the modulus of continuity of $\nabla\gamma$ when the boundary is given by $x_3 = \gamma(x')$. Therefore, these estimates degenerate for highly oscillating boundaries $x_3 = \varepsilon\gamma(x'/\varepsilon)$ with sufficiently small $\varepsilon \in (0,1)$. As for the large scales, on the contrary, the regularity is inherited from the limit system when $\varepsilon \to 0$ posed in a domain with a flat boundary. Here no regularity is needed for the original boundary, beyond the boundedness of γ and of its gradient. The mechanism for the regularity at small scales and at large scales is hence completely different. Moreover, it is possible to prove, at the large scales, improved estimates that are known to be false at the small scales. An example of this is our large-scale Lipschitz estimate of Theorem 2.1 below that is known to be false over a Lipschitz graph at the small scales even in the case of a linear elliptic operator [26, 27, 36].

Beyond improved regularity estimates, our objective is to develop local error estimates for the homogenization of viscous incompressible fluids over bumpy boundaries and derive local wall laws. The wall law catches an averaged effect from the $O(\varepsilon)$-scale on large scale flows of order $O(1)$ through homogenization. In the wall law, a rough boundary is modeled as a smooth one and an appropriate condition is imposed on it reflecting the roughness of the original boundary. In typical situations, this process gives a Navier-type condition with slip length of $O(\varepsilon)$, the so-called Navier wall law. This effective boundary condition reads for instance in two dimensions

$$u_1 = \varepsilon\alpha\partial_2 u_1\,, \quad u_2 = 0 \quad \text{on} \quad \partial\mathbb{R}^2_+ \tag{2}$$

with a constant α depending only on the boundary function γ. We now briefly review the literature concerned with the derivation of wall laws such as (2) and the proof of error estimates in the global setting. The literature is vast and it is impossible to be exhaustive here. The wall law for simple stationary shear flows is analyzed in the pioneering work Jäger and Mikelić [22] when the boundary is periodic. This result is extended to a random setting by Gérard-Varet [10] and to the almost periodic setting by Gérard-Varet and Masmoudi [12]. Nonstationary cases are studied in Mikelić, Nečasová, and Neuss-Radu [32] under the assumption that the limit flows are space-time C^2 functions. The strong regularity condition in [32] implies that a careful analysis is needed when we study Initial Boundary Value Problems (IBVPs). Indeed, for these cases, no matter how regular the initial data are, there is the loss of regularity of solutions due to the boundary compatibility condition. Higaki [19] considers an IBVP in a bumpy half-space and verifies the Navier wall law for C^1 initial data under natural compatibility conditions. A key ingredient is to make use of the L^∞-regularity theory of the Navier-Stokes equations in the half-spaces. Theorem 2.3 below provides a local counterpart of these global error estimates in the case of the stationary Navier-Stokes equations.

2 Outline and novelty of our results

Our main results are given in the two theorems below. In Theorem 2.1 we state a uniform Lipschitz estimate. In Theorem 2.3 we give a local error estimate and identify the building blocks of the regularity theory. Both results hold for weak solutions of the nonlinear equations (NS^ε) and hold without any smallness assumption on the size of the solutions. For an open set $\Omega \subset \mathbb{R}^3$ and a Lebesgue measurable function f on Ω, we set

$$\fint_\Omega |f| = \frac{1}{|\Omega|} \int_\Omega |f| \, , \qquad (\overline{f})_\Omega = \frac{1}{|\Omega|} \int_\Omega f \, ,$$

where $|\Omega|$ denotes the Lebesgue measure of Ω.

Theorem 2.1 ([20], mesoscopic Lipschitz estimate)**.** *For all $M \in (0,\infty)$, there exists a constant $\varepsilon^{(1)} \in (0,1)$ depending on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$ and M such that the following statement holds. For all $\varepsilon \in (0,\varepsilon^{(1)}]$ and $r \in [\varepsilon/\varepsilon^{(1)},1]$, any weak solution $u^\varepsilon \in H^1(B^\varepsilon_{1,+}(0))^3$ to (NS^ε) with*

$$\Bigg(\fint_{B^\varepsilon_{1,+}(0)} |u^\varepsilon|^2 \Bigg)^{\frac12} \le M \tag{3}$$

satisfies

$$\Bigg(\fint_{B^\varepsilon_{r,+}(0)} |u^\varepsilon|^2 \Bigg)^{\frac12} \le C^{(1)}_M r \, , \tag{4}$$

where the constant $C^{(1)}_M$ is independent of ε and r, and depends on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$ and M. Moreover, $C^{(1)}_M$ is a monotone increasing function of M and converges to zero as M goes to zero.

Remark 2.2. (i) By using the Caccioppoli inequality, one can easily prove

$$\Bigg(\fint_{B^\varepsilon_{r,+}(0)} |\nabla u^\varepsilon|^2 \Bigg)^{\frac12} \le \widetilde{C^{(1)}_M}$$

for $r \in [\varepsilon/\varepsilon^{(1)}, \frac{1}{2}]$. Here the constant $\widetilde{C_M^{(1)}}$ satisfies the same property as $C_M^{(1)}$.

(ii) In the paper [10], Gérard-Varet obtains a uniform Hölder estimate for weak solutions of the Stokes equations when $\gamma \in C^{1,\omega}(\mathbb{R}^2)$ for a fixed modulus of continuity ω. Let us emphasize that there is a gap in difficulty between the uniform Hölder estimate (right-hand side of (4) replaced by Cr^μ with $\mu \in (0,1)$) and the uniform Lipschitz estimate (4). Indeed, at least in the compactness argument, the Lipschitz estimate requires the analysis of the boundary layer corrector. Moreover, let us emphasize that the Lipschitz estimate is the best that can be proved for u^ε uniformly in ε. This comment does not contradict the uniform $C^{1,\mu}$ estimate below. Indeed the estimate in Theorem 2.3 is a measure of the oscillation between u^ε and affine functions, and is not an estimate for u^ε directly. The reader is referred to the recent works Zhuge [38] and Gu and Zhuge [18] obtaining large-scale Lipschitz estimates by stochastic homogenization.

(iii) As in the works [4, 10, 24] one can combine the mesoscopic regularity estimate with the classical regularity, provided the boundary is regular enough, i.e. when $\nabla\gamma$ is Hölder continuous. In that case, we can prove the full Lipschitz estimate $\|\nabla u^\varepsilon\|_{L^\infty(B^\varepsilon_{1,+}(0))}$ for (NS$^\varepsilon$). However, one cannot expect such an estimate to hold in Lipschitz domains even for the Laplace equation with the Dirichlet boundary condition.

(iv) There is a version of Theorem 2.1 for the linear Stokes equations; see [20, Theorem 3]. An important application of such uniform Lipschitz estimates is for estimating the Green and Poisson kernels associated to the Stokes equations in the Lipschitz half-space $\{y_3 > \gamma(y')\}$. Following [4], such estimates were proved for elliptic systems in bumpy domains in [24], or the Stokes equations with periodic coefficients [17]. Such estimates play a crucial role for the homogenization of boundary layer correctors, in particular in the works [13, 2, 37]

Next let us state the result which gives a local justification of the Navier wall law. The following theorem is concerned with the polynomial approximation of weak solutions to (NS$^\varepsilon$) at mesoscopic scales. Remark 2.4 below states consequences of the theorem and Remark 2.5 establishes the connection between our theorem and the Navier wall law.

Theorem 2.3 ([20], polynomial approximation)**.** *Fix $M \in (0,\infty)$ and $\mu \in (0,1)$. Then there exists a constant $\varepsilon^{(2)} \in (0,1)$ depending on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$, M, and μ such that for all weak solutions $u^\varepsilon \in H^1(B^\varepsilon_{1,+}(0))^3$ to* (NS$^\varepsilon$) *satisfying the bound* (3), *the following statements hold.*

(i) *For all $\varepsilon \in (0,\varepsilon^{(2)}]$ and $r \in [\varepsilon/\varepsilon^{(2)}, 1]$, we have*

$$\left(\fint_{B^\varepsilon_{r,+}(0)} \big|u^\varepsilon(x) - \sum_{j=1}^{2} c^\varepsilon_{r,j} x_3 \mathbf{e}_j\big|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \le C_M^{(2)} (r^{1+\mu} + \varepsilon^{\frac{1}{2}} r^{\frac{1}{2}})\,, \tag{5}$$

where the coefficient $c^\varepsilon_{r,j}$, $j \in \{1,2\}$, is a functional of u^ε depending on ε, r, $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$, M, and μ, while the constant $C_M^{(2)}$ is independent of ε and r, and depends on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$, M, and μ.

(ii) *We assume in addition that $\gamma \in W^{1,\infty}(\mathbb{R}^2)$ is 2π-periodic in each variable. Then there exists a constant vector field $\alpha^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)}, 0)^\top \in \mathbb{R}^3$, $j \in \{1,2\}$, depending only on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$ such that for all $\varepsilon \in (0,\varepsilon^{(2)}]$ and $r \in [\varepsilon/\varepsilon^{(2)}, 1]$, we have*

$$\left(\fint_{B^\varepsilon_{r,+}(0)} \big|u^\varepsilon(x) - \sum_{j=1}^{2} c^\varepsilon_{r,j} (x_3 \mathbf{e}_j + \varepsilon\alpha^{(j)})\big|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \le \widetilde{C_M^{(2)}} (r^{1+\mu} + \varepsilon^{\frac{3}{2}} r^{-\frac{1}{2}})\,, \tag{6}$$

where the coefficient $c^\varepsilon_{r,j}$, $j \in \{1,2\}$, is same as in the estimate (5), *while the constant $\widetilde{C_M^{(2)}}$ is independent of ε and r, and depends on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$, M, and μ.*

Remark 2.4. (i) Each of the constants $C_M^{(2)}$ and $\widetilde{C_M^{(2)}}$ satisfies the same property as $C_M^{(1)}$ in Theorem 2.1 as functions of M.

(ii) Note that at the small scale, namely when $r = O(\varepsilon)$, the right-hand side in the estimate (5) is no better than the right-hand side of (4) in Theorem 2.1. Hence there is no improvement at this scale. On the other hand, if we consider the case $r \in [(\varepsilon/\varepsilon^{(2)})^\delta, 1]$ with $\delta \in (0,1)$, then we see that

$$r^{1+\mu} + \varepsilon^{\frac{1}{2}} r^{\frac{1}{2}} \leq (1 + (\varepsilon^{(2)})^{\frac{1}{2}} r^{\frac{1-\delta}{2\delta} - \mu}) r^{1+\mu} .$$

Therefore, we call the estimate (5) a mesoscopic $C^{1,\mu}$ estimate at the scales $r \in [(\varepsilon/\varepsilon^{(2)})^\delta, 1]$ with $\delta \in (0, (2\mu+1)^{-1}]$.

(iii) A comparison between the estimates (5) and (6) highlights the regularity improvement coming from the boundary periodicity. Indeed, the estimate (6) is sharper than (5) at mesoscopic scales because $\varepsilon^{\frac{3}{2}} r^{-\frac{1}{2}} \leq \varepsilon^{\frac{1}{2}} r^{\frac{1}{2}}$ holds whenever $r \in [\varepsilon, 1]$.

Remark 2.5. Let us denote the polynomial in (6) by $P^\varepsilon_{N,j}$, $j \in \{1,2\}$:

$$P^\varepsilon_{N,j}(x) = x_3 \mathbf{e}_j + \varepsilon \alpha^{(j)} .$$

Then each $P^\varepsilon_{N,j}$ is a shear flow in the half-space $\mathbb{R}^3_+$ and is an explicit solution to the Navier-Stokes equations with a Navier-slip boundary condition

$$\begin{cases} -\Delta u^\varepsilon_N + \nabla p^\varepsilon_N = -u^\varepsilon_N \cdot \nabla u^\varepsilon_N & \text{in } \mathbb{R}^3_+ \\ \nabla \cdot u^\varepsilon_N = 0 & \text{in } \mathbb{R}^3_+ \\ u^\varepsilon_{N,3} = 0 & \text{on } \partial\mathbb{R}^3_+ \\ (u^\varepsilon_{N,1}, u^\varepsilon_{N,2})^\top = \varepsilon \overline{M} (\partial_3 u^\varepsilon_{N,1}, \partial_3 u^\varepsilon_{N,2})^\top & \text{on } \partial\mathbb{R}^3_+ \end{cases} \tag{NS$^\varepsilon_N$}$$

with a trivial pressure $p^\varepsilon_N = 0$. Here the 2×2 matrix $\overline{M} = (\alpha_i^{(j)})_{1 \leq i,j \leq 2}$ can be proved to be positive definite; see Proposition 4.2 (ii) below. Thus the estimate (6) in Theorem 2.3 reads as follows: any weak solution u^ε to (NS$^\varepsilon$) can be approximated at any mesoscopic scale by a linear combination of the Navier polynomials $P^\varepsilon_{N,1}$ and $P^\varepsilon_{N,2}$ multiplied by constants depending on u^ε. This is a local version of the Navier wall law at the $O(\varepsilon^\delta)$-scales, which has been widely studied in the global framework.

The novelty of our results can be summarized as follows:

(I) Singular boundary: it is just Lipschitz and has no structure (except in Theorem 2.3 (ii)).

(II) No smallness assumption on the size of solutions.

(III) Derivation of a local wall law and local error estimates.

As is stated in (I), one of the originalities of Theorem 2.1 is that it does not rely on the smoothness of the boundary such as, the Hölder continuity of $\nabla\gamma$. Moreover, one cannot use any Fourier methods due to the lack of structure of the boundary. In fact, when working with Lipschitz boundaries, the classical Schauder theory is not applicable directly since there is no improvement of flatness coming from zooming on the boundary as is explained in [25]. The smoothing happens at scales larger than that of the boundary layer thickness.

Concerning point (II), we are able to remove any smallness assumption on the size of the solutions in Theorem 2.1 and Theorem 2.3. This is in stark contrast with previous works concerned with the regularity of elliptic or Stokes systems [4, 10, 16, 24, 25]. Moreover, as far as we know the error estimates in the stationary global setting are all in the perturbative regime; see for instance [12].

Point (III) is concerned with Theorem 2.3. It is important physically as well as mathematically since we are interested in the effects of rough boundaries on viscous fluids. Our result is a first-step toward understanding roughness effects on the Navier-Stokes flows in view of regularity improvement. As far as we know, estimate (6) is the first justification of a local wall law.

These three aspects are further discussed in connection with our strategy below.

3 Difficulties and strategy

The proof of Theorem 2.1 and Theorem 2.3 is based on a compactness argument as in [24, 25] originating from the works [4, 5] on uniform estimates in homogenization. In principle, we follow the strategy of [25] concerned with the regularity theory of elliptic systems in bumpy domains. The main points in [25] are: (1) construction of a boundary layer corrector in the Lipschitz half-space, (2) proof of the mesoscopic regularity by compactness and iteration. This strategy entails difficulties related to the lack of structure of the boundary which implies a lack of compactness of the solution to the boundary layer problem, and to the unavailability of Fourier methods up to the boundary. In addition to these difficulties, our proof is more involved due to: (i) the vectoriality of the equations (NS^ε) and the divergence-free condition, (ii) the nonlocal pressure, (iii) the nonlinearity of the Navier-Stokes equations and the lack of smallness of the solutions.

Concerning the first point, the (vectorial) divergence-free condition $\nabla \cdot u^\varepsilon = 0$ causes a difficulty in the compactness argument even for the Stokes equations; see [20, Section 4] especially Lemma 13 and its proof. A key idea is that no boundary layer is needed on the vertical component of the velocity. Therefore the boundary layer corrector is naturally constructed as a divergence-free function.

Concerning the second point, let us stress a key difference between the stationary Navier-Stokes equations and the nonstationary ones. For the stationary Stokes equations imposed in a ball $B_1(0)$, one can estimate the pressure directly in terms of the velocity as

$$\|p - (\overline{p})_{B_1(0)}\|_{L^2(B_1(0))} \leq C\|\nabla p\|_{H^{-1}(B_1(0))} \leq C\|\nabla u\|_{L^2(B_1(0))}. \tag{7}$$

Similar estimates in balls intersecting the boundary and for the Navier-Stokes equations are intensively used in our analysis. This is in strong contrast with the nonstationary Navier-Stokes equations where the pressure interacts with the time derivative of the velocity.

The third aspect is partly related to (ii). In typical statements of the partial regularity theory for the nonstationary Navier-Stokes equations, one assumes smallness of certain scale-critical quantities in ε and hence one obtains linear equations in the limit $\varepsilon \to 0$. Then the regularity theory for the linear equations yields a space-time Hölder regularity improvement for the original solution; see Lin [30] for example. However, for the stationary Navier-Stokes equations discussed in our talk, we do not need such a smallness condition; see Theorem 2.1. The limit equations when $\varepsilon \to 0$ are not linear, but we can prove the smoothness of weak solutions because H^1 bounds are enough to control both the nonlinear term and pressure term in L^2 space. Then bootstrapping using the standard elliptic regularity in a smooth domain leads to the (spatial) C^∞-regularity for the limit equations. Estimate (7) is the reason why one can bootstrap the regularity. Once the regularity is inherited at a fixed scale $\theta \in (0,1)$, a serious difficulty arises in the iteration of such an estimate. At each step in the induction, we need to use the Caccioppoli inequality to control the norm $\|u^\varepsilon\|_{L^2}$. A naive approach yields an estimate that depends algebraically on the size M of u^ε as in (3). Hence the naive estimate becomes unbounded in M as the iteration proceeds. This prevents one from closing the induction due to the lack of uniformity. We overcome this difficulty by choosing the free parameter θ in the compactness lemma in terms of the data γ and M. This is done in the spirit of the Newton shooting method. It should finally be emphasized that the boundary layer corrector, entering the scheme for the nonlinear Navier-Stokes equations (NS^ε), solves the linear Stokes equations. This is expected from the following formal heuristics. Indeed, in the boundary layer $u^\varepsilon \simeq \varepsilon v(x/\varepsilon)$, so that v solves $-\frac{1}{\varepsilon}\Delta v + \varepsilon v \cdot \nabla v + \nabla q = 0$, $\nabla \cdot v = 0$.

4 The boundary layer corrector

We summarize the results concerning the boundary layer problem in Lipschitz half-spaces without proof for easy reference. These results are needed in the proof of Theorems 2.1 and 2.3.

General case. The boundary layer equations for $j \in \{1,2\}$ are written as

$$\begin{cases} -\Delta v + \nabla q = 0\,, & y \in \Omega^{\mathrm{bl}} \\ \nabla \cdot v = 0\,, & y \in \Omega^{\mathrm{bl}} \\ v(y', \gamma(y')) = -\gamma(y')\mathbf{e}_j\,, & \end{cases} \tag{$\mathrm{BL}^{(j)}$}$$

where $\gamma \in W^{1,\infty}(\mathbb{R}^2)$ and Ω^{bl} denotes the Lipschitz half-space $\Omega^{\mathrm{bl}} = \{y \in \mathbb{R}^3 \mid \gamma(y') < y_3 < \infty\}$. The unique existence of weak solutions to ($\mathrm{BL}^{(j)}$) is stated as follows.

Proposition 4.1. *Fix $j \in \{1,2\}$ and let $\gamma \in W^{1,\infty}(\mathbb{R}^2)$. Then there exists a unique weak solution $(v,q) = (v^{(j)}, q^{(j)}) \in H^1_{loc}(\overline{\Omega^{\mathrm{bl}}})^3 \times L^2_{loc}(\overline{\Omega^{\mathrm{bl}}})$ to ($\mathrm{BL}^{(j)}$) satisfying*

$$\sup_{\eta \in \mathbb{Z}^2} \int_{\eta + (0,1)^2} \int_{\gamma(y')}^{\infty} |\nabla v^{(j)}(y', y_3)|^2 \,\mathrm{d}y_3 \,\mathrm{d}y' \le C\,,$$

where the constant C depends only on $\|\gamma\|_{W^{1,\infty}(\mathbb{R}^2)}$.

The basic idea of the proof is to decompose the domain Ω^{bl} into $\overline{\mathbb{R}^3_+} \cup (\Omega^{\mathrm{bl}} \setminus \overline{\mathbb{R}^3_+})$ and to derive an equivalent equations to ($\mathrm{BL}^{(j)}$) on the infinite channel $\Omega^{\mathrm{bl}} \setminus \overline{\mathbb{R}^3_+}$ but involving the Dirichlet-to-Neumann operator. This formulation allows us to apply the Poincaré inequality in the vertical direction when estimating the local energy. The reader is referred to [20, Section 3] for the details.

Periodic case. We note the asymptotic behavior of the boundary layer corrector at spatial infinity when the boundary is periodic; see [20, Proposition 11].

Proposition 4.2. *Fix $j \in \{1,2\}$ and let $\gamma \in W^{1,\infty}(\mathbb{R}^2)$ be 2π-periodic in each variable. Then the weak solution $(v^{(j)}, q^{(j)})$ to ($\mathrm{BL}^{(j)}$) provided by Proposition 4.1 satisfies the following properties.*

(i) *There exists a constant vector field $\alpha^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)}, 0)^\top \in \mathbb{R}^3$ such that*

$$|v^{(j)}(y) - \alpha^{(j)}| + y_3 |q^{(j)}(y)| \le C \|v^{(j)}(\cdot, 0)\|_{L^2((0,2\pi)^2)}\, e^{-\frac{y_3}{2}}\,, \quad y_3 > 1\,,$$

where C is a numerical constant.

(ii) *The 2×2 matrix $\overline{M} \in \mathbb{R}^{2\times 2}$ defined by $\overline{M} = (\alpha_i^{(j)})_{1 \le i,j \le 2}$ is symmetric and positive definite.*

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Numerical computations of split Bregman method for fourth order total variation flow

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1 Introduction

We are interested in gradient flows

$$u_t \in -\partial_H E(u) \text{ for } t > 0, \tag{1}$$

with convex and lower semi-continuous energy functional $E : H \to \mathbb{R} \cup \{\infty\}$ but may be very singular, where H is a Hilbert space and $\partial_H E(u)$ is the subdifferential defined as

$$\partial_H E(u) = \{p \in H : E(v) - E(u) \geq (p, v-u)_H \text{ for all } v \in H\}. \tag{2}$$

We give a few examples. Spohn [13] has proposed a mathematical model for the relaxation of a crystalline surface below the roughening temperature;

$$u_t = -\Delta \left(\operatorname{div} \left(\beta \frac{\nabla u}{|\nabla u|} + |\nabla u|^{p-2} \nabla u \right) \right), \tag{3}$$

where $\beta > 0$ and $p > 1$. Kashima [8] has presented this model as a fourth order H^{-1}-gradient flow for energy functional

$$E(u) = \beta \int_\Omega |Du| + \frac{1}{p} \int_\Omega |Du|^p. \tag{4}$$

Rudin, Osher and Fatemi [12] have proposed to minimize the following functional which is known as Rudin-Osher-Fatemi (ROF) model:

$$\int_\Omega |Du| + \frac{\lambda}{2} \|u - f\|^2_{L^2(\Omega)}. \tag{5}$$

This is closely related to the second order total variation flow:

$$u_t = \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right). \tag{6}$$

Indeed, equation (6) introduces the subdifferential formulation $u_t \in -\partial_{L^2(\Omega)} E(u)$, where $E(u) = \int_\Omega |Du|$. We apply the backward Euler method $u_t \approx (u^{k+1} - u^k)/\tau$ to the subdifferential formulation, then we obtain

$$u^{k+1} = \underset{u \in L^2(\Omega)}{\operatorname{argmin}} \left\{ \int_\Omega |Du| + \frac{1}{2\tau} \|u - u^k\|^2_{L^2(\Omega)} \right\}, \tag{7}$$

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where $\tau > 0$ is time step size. This is essentially the same problem as the ROF model. On the other hand, Osher, Solé and Vese [11] have proposed to minimize

$$\int_\Omega |Du| + \frac{\lambda}{2}\|u - f\|^2_{H^{-1}(\Omega)}. \tag{8}$$

Osher-Solé-Vese (OSV) model (8) is closely related to the fourth order total variation flow, which is formally described as

$$u_t = -\Delta\left(\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right). \tag{9}$$

Performing numerical computations for the Spohn's model, ROF model, the OSV model and total variation flow have difficulties because of its singularity. Several studies have suggested numerical schemes for the ROF model and second order total variation flow. Especially, the split Bregman framework, which is based on the Bregman iterative scheme [1] is well-known as an efficient solver for the ROF model (for example, see [10, 5]).

The aim of this work is to provide a new numerical scheme, which is based on the backward Euler method and split Bregman framework, for fourth order total variation flow, OSV model and Spohn's fourth order model under periodic boundary condition. We introduce spatial discretization by piecewise constant functions, then we compute $\nabla(-\Delta_{\mathrm{av}})^{-1}v_h$ for piecewise constant function v_h approximately or exactly. We also propose a new shrinkage operator for Spohn's model. Numerical experiments are demonstrated for fourth order problems under periodic boundary condition. We can check that our scheme perform quite well. Especially, it has been proved that the solution of fourth order total variation flow becomes discontinuous instantaneously for a class of initial data (see [4]). Our numerical result shows the discontinuity approximately.

2 Preliminary

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the torus. In this paper, we regard $\mathbb{T}$ as an interval $[0,1]$ with periodic boundary condition. The Fourier coefficient $\widehat{f}_T(\xi) \in \mathbb{C}$ for $f \in \mathcal{D}'(\mathbb{T})$ is defined by the generalized Fourier transform (for example, see [7, Chapter 5]):

$$\widehat{f}_T(\xi) = \langle f, e^{-2\pi i\xi x}\rangle_{\mathcal{D}'(\mathbb{T}),\mathcal{D}(\mathbb{T})}, \qquad f(x) = \sum_{\xi\in\mathbb{Z}} \widehat{f}_T(\xi)e^{2\pi i\xi x}. \tag{10}$$

Now we define the spaces of functions on $\mathbb{T}$ whose average are equal to zero:

$$L^2_{\mathrm{av}}(\mathbb{T}) = \left\{ f \in \mathcal{D}'(\mathbb{T}) : \sum_{\xi\neq 0} |\widehat{f}_T(\xi)|^2 < \infty \text{ and } \widehat{f}_T(0) = 0 \right\}, \tag{11}$$

$$H^1_{\mathrm{av}}(\mathbb{T}) = \left\{ f \in \mathcal{D}'(\mathbb{T}) : \sum_{\xi\neq 0} \xi^2|\widehat{f}_T(\xi)|^2 < \infty \text{ and } \widehat{f}_T(0) = 0 \right\}, \tag{12}$$

$$H^{-1}_{\mathrm{av}}(\mathbb{T}) = \left\{ f \in \mathcal{D}'(\mathbb{T}) : \sum_{\xi\neq 0} \xi^{-2}|\widehat{f}_T(\xi)|^2 < \infty \text{ and } \widehat{f}_T(0) = 0 \right\}. \tag{13}$$

We can check that each of these spaces are Hilbert space and the norms satisfy

$$\|f\|_{L^2_{\mathrm{av}}(\mathbb{T})} = \|f\|_{L^2(\mathbb{T})} \qquad \text{for all } f \in L^2_{\mathrm{av}}(\mathbb{T}), \tag{14}$$

$$\|f\|_{H^1_{\mathrm{av}}(\mathbb{T})} = \|\nabla f\|_{L^2(\mathbb{T})} \qquad \text{for all } f \in H^1_{\mathrm{av}}(\mathbb{T}), \tag{15}$$

$$\|f\|_{H^{-1}_{\mathrm{av}}(\mathbb{T})} = \|(-\Delta_{\mathrm{av}})^{-1}f\|_{H^1_{\mathrm{av}}(\mathbb{T})} = \|\nabla(-\Delta_{\mathrm{av}})^{-1}f\|_{L^2(\mathbb{T})} \qquad \text{for all } f \in H^{-1}_{\mathrm{av}}(\mathbb{T}). \tag{16}$$

We recall the spaces of functions of *bounded variation* in one-dimensional torus.

Definition 1 (Definition 3.3.13 of [6]). For a measurable function f on $\mathbb{T}$ which is defined everywhere, we define the *total variation* as

$$\int_{\mathbb{T}} |Df| = \operatorname{ess\,sup} \left\{ \sum_{j=1}^{M} |f(x_j) - f(x_{j-1})| : 0 = x_0 < x_1 < \cdots < x_M = 1 \right\}, \tag{17}$$

where the supremum is taken over all partition of the interval $[0, 1]$. Moreover, we define

$$BV(\mathbb{T}) = \left\{ v \in \mathcal{D}'(\mathbb{T}) : \int_{\mathbb{T}} |Df| < \infty \right\}. \tag{18}$$

We define the functional $E : H^{-1}_{\mathrm{av}}(\mathbb{T}) \to \mathbb{R} \cup \{\infty\}$ as

$$E(v) = \begin{cases} \displaystyle\int_{\mathbb{T}} |Dv| & \text{if } v \in BV(\mathbb{T}) \cap H^{-1}_{\mathrm{av}}(\mathbb{T}), \\ \infty & \text{otherwise.} \end{cases} \tag{19}$$

Note that $E : H^{-1}_{\mathrm{av}}(\mathbb{T}) \to \mathbb{R} \cup \{\infty\}$ is nonnegative, proper, lower semi-continuous and convex. In this paper, we consider the gradient flow equation of the form

$$\text{(gradient flow)} \begin{cases} u_t(t) & \in -\partial_{H^{-1}_{\mathrm{av}}(\mathbb{T})} E(u(t)) \quad \text{for a.e. } t > 0, \\ u(\cdot, 0) & = u_0 \in H^{-1}_{\mathrm{av}}(\mathbb{T}). \end{cases} \tag{20}$$

It is well-known that the theory of maximal monotone operators shows the existence and uniqueness of solution $u \in C([0, \infty), H^{-1}_{\mathrm{av}}(\mathbb{T}))$ to equation (20) (for example, see [9]).

Let $\tau > 0$ be the temporal step size. We consider the backward Euler method for gradient flow equation (20): for given $u^k \in H^{-1}_{\mathrm{av}}(\mathbb{T})$, find $u^{k+1} \in H^{-1}_{\mathrm{av}}(\mathbb{T})$ such that

$$\frac{u^{k+1} - u^k}{\tau} \in -\partial_{H^{-1}_{\mathrm{av}}(\mathbb{T})} E(u^{k+1}). \tag{21}$$

This can be reduced to solving the following minimization problem:

$$u^{k+1} = \operatorname*{argmin}_{u \in H^{-1}_{\mathrm{av}}(\mathbb{T})} \left\{ E(u) + \frac{1}{2\tau} \|u - u^k\|^2_{H^{-1}_{\mathrm{av}}(\mathbb{T})} \right\}. \tag{22}$$

Since E is convex, such u^{k+1} is uniquely determined. The convergence of backward Euler method has been proved in [9]. Note that equation (22) is similar to the the minimization problem which is given by OSV model [11]:

$$\text{(OSV)} \begin{cases} \text{Find } u \in H^{-1}_{\mathrm{av}}(\mathbb{T}) \text{ such that} \\ u = \operatorname{argmin} \left\{ E(u) + \dfrac{\lambda}{2} \|u - f\|^2_{H^{-1}_{\mathrm{av}}(\mathbb{T})} \right\}, \end{cases} \tag{23}$$

where $f \in H^{-1}_{\mathrm{av}}(\mathbb{T})$ is given data and $\lambda > 0$ is an artificial parameter.

Hereafter, we consider the following minimization problem:

$$\text{(P0)} \quad \operatorname*{argmin}_{u \in H^{-1}_{\mathrm{av}}(\mathbb{T})} \left\{ E(u) + \frac{\lambda}{2} \|u - f\|^2_{H^{-1}_{\mathrm{av}}(\mathbb{T})} \right\}, \tag{24}$$

where $f \in H^{-1}_{\mathrm{av}}(\mathbb{T})$ is a given data or $f = u^k$, and λ is a given parameter or $\lambda = 1/\tau$. This involves both of (OSV) and the backward Euler method for (gradient flow). Furthermore, (P0) can be written equivalently as the following constrained minimization problem:

$$\text{(P1)} \quad \operatorname*{argmin}_{u \in H^{-1}_{\mathrm{av}}(\mathbb{T})} \left\{ \int_{\mathbb{T}} |d| + \frac{\lambda}{2} \|u - f\|^2_{H^{-1}_{\mathrm{av}}(\mathbb{T})} : d = Du \right\}. \tag{25}$$

Remark. We can apply the above argument to Spohn's fourth order model, which formally is described as

$$u_t = -\Delta \left(\operatorname{div} \left(\beta \frac{\nabla u}{|\nabla u|} + |\nabla u|^{p-2} \nabla u \right) \right). \tag{26}$$

The subdifferential formulation is given as $u_t \in -\partial_{H^{-1}_{\mathrm{av}}(\mathbb{T})} \widetilde{E}(u)$, where

$$\widetilde{E}(u) = \beta \int_{\mathbb{T}} |Du| + \frac{1}{p} \int_{\mathbb{T}} |Du|^p. \tag{27}$$

Therefore the backward Euler method for the H^{-1}-gradient flow of $\widetilde{E}$ gives

$$\operatorname*{argmin}_{u \in H^{-1}_{\mathrm{av}}(\mathbb{T})} \left\{ \beta \int_{\mathbb{T}} |d| + \frac{1}{p} \int_{\mathbb{T}} |d|^p + \frac{\lambda}{2} \|u - f\|^2_{H^{-1}_{\mathrm{av}}(\mathbb{T})} : d = Du \right\}. \tag{28}$$

3 Discretization for fourth order problems

3.1 Discretization for minimization problem

We introduce the (spatial) discretization. Let $N \in \mathbb{N}$ be the partition number, $h = 1/N$ and $x_n = nh$. We regard $x_0 = x_N$, then $\{x_n\}_{n=0}^N$ gives an uniform partition of $\mathbb{T}$. Furthermore, we let $x_{n+1/2} = (x_n + x_{n+1})/2 = (n + 1/2)h$ for $n = -1, 0, \ldots, N$, where $x_{-1/2}$ and $x_{N+1/2}$ are identified with $x_{N-1/2}$ and $x_{1/2}$, respectively. Then we define the following spaces of piecewise constant functions:

$$V_h = \{ v_h : \mathbb{T} \to \mathbb{R} : v_h|_{I_n} \in \mathbb{P}_0(I_n) \text{ for all } n = 0, \ldots, N \}, \tag{29a}$$

$$V_{h0} = \left\{ v_h = \sum_{n=1}^N v_n \mathbf{1}_{I_n} \in V_h : \sum_{n=1}^N v_n = 0 \right\}, \tag{29b}$$

$$\widehat{V}_h = \{ d_h : I \to \mathbb{R} : d_h|_{I_{n-1/2}} \in \mathbb{P}_0(I_{n-1/2}) \text{ for all } n = 1, \ldots, N \}, \tag{29c}$$

where $I = [0, 1)$, $I_n = [x_{n-1/2}, x_{n+1/2})$, $I_{n-1/2} = [x_{n-1}, x_n)$, $\mathbb{P}_0(J)$ is a space of constant functions on interval J and $\mathbf{1}_{I_n}$ is its characteristic function.

Henceforth, we always denote

$$\mathbf{d} = (d_1, \ldots, d_N)^{\mathrm{T}} \in \mathbb{R}^N, \qquad \widetilde{\mathbf{v}} = (v_1, \ldots, v_N)^{\mathrm{T}} \in \mathbb{R}^N, \qquad \mathbf{v} = (v_1, \ldots, v_{N-1})^{\mathrm{T}} \in \mathbb{R}^{N-1} \tag{30}$$

for given $d_h = \sum_{n=1}^N d_n \mathbf{1}_{I_{n-1/2}} \in \widehat{V}$ and $v_h = \sum_{n=1}^N v_n \mathbf{1}_{I_n} \in V_{h0}$. Here we introduce two matrices

$$R_N = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ -1 & -1 & \ldots & -1 \end{pmatrix} \in \mathbb{R}^{N \times (N-1)}, \ \nabla_h = h^{-1} \begin{pmatrix} 1 & 0 & \ldots & 0 & -1 \\ -1 & 1 & \ldots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & -1 & 1 \end{pmatrix} \in \mathbb{R}^{N \times N}.$$

Note that ∇_h is called the discrete gradient. Since $v_n = -\sum_{n=1}^{N-1} v_n$ for any $v_h \in V_{h0}$, we obtain $\widetilde{\mathbf{v}} = R_N \mathbf{v}$. We define $D_h : V_{h0} \to \widehat{V}_h \cap L^2_{\mathrm{av}}(I)$ as

$$D_h v_h = \sum_{n=1}^N (v_n - v_{n-1}) \mathbf{1}_{[x_{n-1}, x_n)}, \tag{31}$$

where v_0 is identified with v_N. Let $v_h \in V_{h0}$ and $d_h = D_h v_h \in \widehat{V}_h$, then we have

$$\mathbf{d} = S_N \widetilde{\mathbf{v}} = S_N R_N \mathbf{v}, \tag{32}$$

where $S_N = h\nabla_h$. Moreover, we can check that

$$E(v_h) = \sum_{n=1}^{N} |v_n - v_{n-1}| = \|S_N \widetilde{\mathbf{v}}\|_1 = \|\mathbf{d}\|_1, \tag{33}$$

Note that $D_h v_h \in \widehat{V}_h \cap L^2_{\mathrm{av}}(I) \subset L^2_{\mathrm{av}}(I)$ for all $v_h \in V_{h0}$; however, $D_h v_h \notin H^{-1}_{\mathrm{av}}(\mathbb{T})$ because it does not satisfy the periodic boundary condition.

Here we introduce the discretized problem for (P0):

$$(\mathrm{P0})_h \quad \operatorname*{argmin}_{u_h \in V_{h0}} \left\{ E(u_h) + \frac{\lambda}{2} \|u_h - f_h\|^2_{H^{-1}_{\mathrm{av}}(\mathbb{T})} \right\}, \tag{34}$$

where $f_h \in V_{h0}$ is given data or $f_h = u_h^k$. This can be written equivalently as the discretized problem for (P1):

$$(\mathrm{P1})_h \quad \operatorname*{argmin}_{u_h \in V_{h0}} \left\{ \|\mathbf{d}\|_1 + \frac{\lambda}{2} \|u_h - f_h\|^2_{H^{-1}_{\mathrm{av}}(\mathbb{T})} : d_h = D_h u_h \in \widehat{V}_h \right\}. \tag{35}$$

Furthermore, we introduce the following unconstrained problem:

$$(\mathrm{P2})_h \quad \operatorname*{argmin}_{u_h \in V_{h0}, d_h \in \widehat{V}_h} \left\{ \|\mathbf{d}\|_1 + \frac{\lambda}{2} \|u_h - f_h\|^2_{H^{-1}_{\mathrm{av}}(\mathbb{T})} + \frac{\mu}{2} \|d_h - D_h u_h\|^2_{L^2(I)} \right\}, \tag{36}$$

where $\mu > 0$ is another parameter.

Remark. Using $\|d_h - D_h u_h\|^2_{L^2(I)}$ in the unconstrained problem enables to apply the shrinking method to minimization problem in the split Bregman framework.

We define $\nabla_{\mathrm{av},h} = \nabla_h R_N = h^{-1} S_N R_N$ be the discrete gradient for functions whose average are zero, then we obtain

$$\frac{\mu}{2} \|d_h - D_h u_h\|^2_{L^2(I)} = \frac{\mu h}{2} \|\mathbf{d} - h\nabla_{\mathrm{av},h} \mathbf{u}\|_2^2. \tag{37}$$

Henceforth we denote $\widetilde{\mu} = \mu h$.

3.2 Corresponding matrix form for H^{-1} norm: first scheme

Next, we consider two expressions of $\|u_h - f_h\|^2_{H^{-1}_{\mathrm{av}}(\mathbb{T})}$ for $u_h - f_h \in V_{h0}$. Recall that

$$\|v_h\|^2_{H^{-1}_{\mathrm{av}}(\mathbb{T})} = \|\nabla(-\Delta_{\mathrm{av}})^{-1} v_h\|^2_{L^2(\mathbb{T})} \quad \text{for all } v_h \in V_{h0}. \tag{38}$$

We propose two schemes for considering $\nabla(-\Delta_{\mathrm{av}})^{-1}$. The first scheme is proposed to approximate $\nabla(-\Delta_{\mathrm{av}})^{-1}$ by using the discrete gradient $\nabla_h \in \mathbb{R}^{N\times N}$ and the discrete Laplacian $-\Delta_h = \nabla_h^{\mathrm{T}} \nabla_h \in \mathbb{R}^{N \times N}$. Let $w_h \in V_{h0}$ be such that $\widetilde{\mathbf{v}} = -\Delta_h \widetilde{\mathbf{w}}$, then we have

$$R_N \mathbf{v} = -\Delta_h R_N \mathbf{w}. \tag{39}$$

Multiplying the (unique) pseudo-inverse matrix

$$L_N = \frac{1}{N}\begin{pmatrix} N-1 & -1 & \dots & -1 & -1 \\ -1 & N-1 & \dots & -1 & -1 \\ \vdots & & \ddots & & \vdots \\ -1 & -1 & \dots & N-1 & -1 \end{pmatrix} \in \mathbb{R}^{(N-1)\times N} \tag{40}$$

yields $\mathbf{v} = L_N(-\Delta_h)R_N\mathbf{w}$. For simplicity of notation, we let

$$A_N = L_N S_N^{\mathrm{T}} S_N R_N, \tag{41a}$$

$$(-\Delta_{\mathrm{av}})_h = h^{-2}A_N = L_N(-\Delta_h)R_N. \tag{41b}$$

It is easy to check that $\det(-\Delta_{\mathrm{av}})_h \neq 0$. This implies

$$\begin{cases} \widetilde{\mathbf{v}} & = -\Delta_h\widetilde{\mathbf{w}}, \\ \widetilde{\mathbf{w}} & = R_N(-\Delta_{\mathrm{av}})_h^{-1}\mathbf{v}. \end{cases} \tag{42}$$

Our first scheme is proposed to approximate $(-\Delta_{\mathrm{av}})^{-1}$ by $R_N(-\Delta_{\mathrm{av}})_h^{-1}$, instead of $(-\Delta_h)^{-1}$ which does not exist. This yields

$$\begin{aligned} \|\nabla(-\Delta_{\mathrm{av}})^{-1}v_h\|_{L^2(I)}^2 &\approx \left\|\left(\nabla_h R_N(-\Delta_{\mathrm{av}})_h^{-1}\mathbf{v}\right)\cdot(\mathbf{1}_{I_{1/2}},\dots,\mathbf{1}_{I_{N-1/2}})\right\|_{L^2(I)}^2 \\ &= h\|\nabla_{\mathrm{av},h}(-\Delta_{\mathrm{av}})_h^{-1}\mathbf{v}\|_2^2 \\ &= h^3\|S_N R_N A_N^{-1}\mathbf{v}\|_2^2. \end{aligned}$$

As a result, our first scheme is described as

$$\frac{\lambda}{2}\|u_h - f_h\|_{H_{\mathrm{av}}^{-1}(\mathbb{T})}^2 \approx \frac{\lambda h}{2}\|\nabla_{\mathrm{av},h}(-\Delta_{\mathrm{av}})_h^{-1}(\mathbf{u}-\mathbf{f})\|_2^2 = \frac{\lambda h^3}{2}\|S_N R_N A_N^{-1}(\mathbf{u}-\mathbf{f})\|_2^2. \tag{43}$$

3.3 Corresponding matrix form for H^{-1} norm: second scheme

Our second scheme is to compute $\nabla(-\Delta_{\mathrm{av}})^{-1}$ directly. It requires the second degree piecewise polynomial which has continuous derivative. We define the second degree periodic B-spline basis functions

$$B_n(x) = \begin{cases} \dfrac{(x - x_{n-\frac{3}{2}})^2}{2h^2} & \text{if } x \in I_{n-1}, \\ \dfrac{(x - x_{n-\frac{1}{2}})(x_{n+\frac{1}{2}} - x)}{h^2} + \dfrac{1}{2} & \text{if } x \in I_n, \\ \dfrac{(x_{n+\frac{3}{2}} - x)^2}{2h^2} & \text{if } x \in I_{n+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{44}$$

We identify $B_{-1} \equiv B_{N-1}$, $B_0 \equiv B_N$ and $B_1 \equiv B_{N+1}$. Fix $v_h \in V_{h0}$ arbitrarily, then there exits $w = \sum_{n=1}^N w_n B_n$ with $\sum_{n=1}^N w_n = 0$ such that $w = (-\Delta_{\mathrm{av}}^{-1})v_h \in H_{\mathrm{av}}^1(\mathbb{T})$. We let

$$\widetilde{\mathbf{w}} = (w_1,\dots,w_N)^{\mathrm{T}} \in \mathbb{R}^N \text{ and } \mathbf{w} = R_N\widetilde{\mathbf{w}} \in \mathbb{R}^{N-1} \tag{45}$$

for given $w = \sum_{n=1}^N w_n B_n \in H_{\mathrm{av}}^1(\mathbb{T})$. Then we have

$$v_h = -\Delta w = \sum_{n=1}^N w_n(-\Delta B_n) = h^{-2}\sum_{n=1}^N(-w_{n-1} + 2w_n - w_{n+1})\mathbf{1}_{I_n} \in V_{h0}. \tag{46}$$

This implies $w \in H^1_{\mathrm{av}}(\mathbb{T})$ satisfies $w = (-\Delta_{\mathrm{av}})^{-1} v_h$ and

$$R_N \mathbf{v} = \widetilde{\mathbf{v}} = -\Delta_h \widetilde{\mathbf{w}} = -\Delta_h R_N \mathbf{w}. \tag{47}$$

Multiplying the pseudo-inverse matrix L_N yields

$$\mathbf{w} = (-\Delta_{\mathrm{av}})_h^{-1} \mathbf{v}. \tag{48}$$

On the other hand, we can obtain

$$\begin{aligned}
\|v_h\|^2_{H^{-1}_{\mathrm{av}}(\mathbb{T})} &= \|\nabla w\|^2_{L^2(\mathbb{T})} \\
&= \left\| \sum_{n=1}^{N} w_n \nabla B_n \right\|^2_{L^2(\mathbb{T})} \\
&= h(\nabla_h \widetilde{\mathbf{w}})^{\mathrm{T}} M_N \nabla_h \widetilde{\mathbf{w}} \\
&= h(\nabla_h R_N (-\Delta_{\mathrm{av}})_h^{-1} \mathbf{v})^{\mathrm{T}} M_N \nabla_h R_N (-\Delta_{\mathrm{av}})_h^{-1} \mathbf{v},
\end{aligned}$$

where

$$M_N = \begin{pmatrix} 2/3 & 1/6 & 0 & \dots & 0 & 1/6 \\ 1/6 & 2/3 & 1/6 & \dots & 0 & 0 \\ \vdots & & \ddots & & & \vdots \\ 1/6 & 0 & 0 & \dots & 1/6 & 2/3 \end{pmatrix} \in \mathbb{R}^{N \times N}. \tag{49}$$

Let $\sqrt{M_N} \in \mathbb{R}^{N \times N}$ be such that $\sqrt{M_N}^{\mathrm{T}} \sqrt{M_N} = M_N$, then it can be determined uniquely:

$$\sqrt{M_N} = \begin{pmatrix} a & 0 & \dots & 0 & b \\ b & a & \dots & 0 & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & \dots & b & a \end{pmatrix} \in \mathbb{R}^{N \times N}, \tag{50}$$

where $a = \dfrac{\sqrt{3}+1}{2\sqrt{3}}$ and $b = \dfrac{\sqrt{3}-1}{2\sqrt{3}}$. Summarizing the above argument, our second scheme can be described as

$$\frac{\lambda}{2}\|u_h - f_h\|^2_{H^{-1}_{\mathrm{av}}(\mathbb{T})} = \frac{\lambda h}{2}\|\sqrt{M_N}\nabla_{\mathrm{av},h}(-\Delta_{\mathrm{av}})_h^{-1}(\mathbf{u}-\mathbf{f})\|_2^2 = \frac{\lambda h^3}{2}\|\sqrt{M_N} S_N R_N A_N^{-1}(\mathbf{u}-\mathbf{f})\|_2^2. \tag{51}$$

3.4 Split Bregman framework for fourth order problems

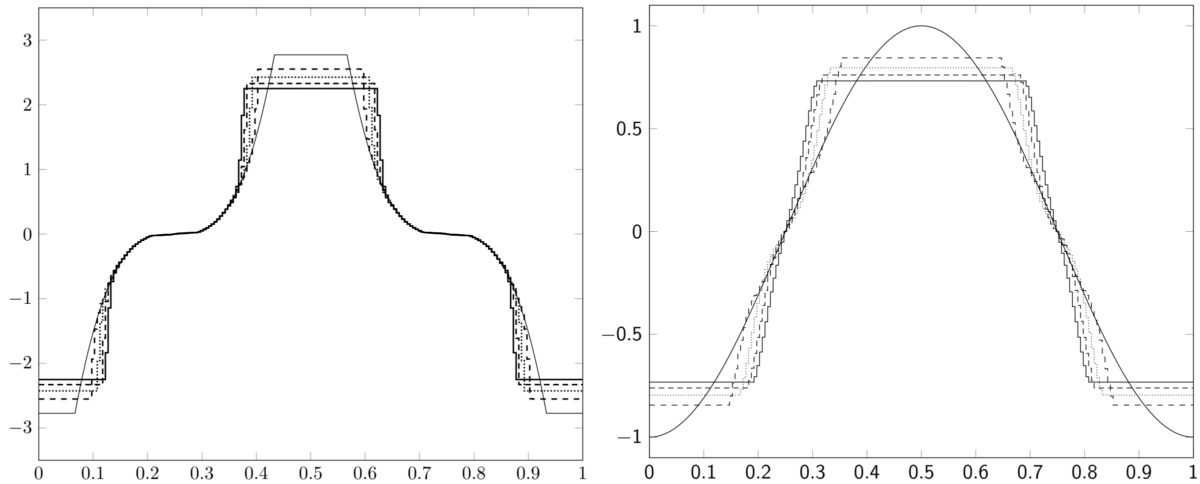
Applying equation (37), (43) and (51) to $(\mathrm{P2})_h$ gives the following discretized problem;

$$(\mathrm{P2})'_h \quad \underset{\mathbf{u}\in\mathbb{R}^{N-1}, \mathbf{d}\in\mathbb{R}^N}{\operatorname{argmin}} \left\{ \|\mathbf{d}\|_1 + \frac{\lambda h}{2}\|K(\mathbf{u}-\mathbf{f})\|_2^2 + \frac{\widetilde{\mu}}{2}\|\mathbf{d} - h\nabla_{\mathrm{av},h}\mathbf{u}\|_2^2 \right\}, \tag{52}$$

where $K = \nabla_{\mathrm{av},h}(-\Delta_{\mathrm{av}})_h^{-1}$ or $\sqrt{M_N}\nabla_{\mathrm{av},h}(-\Delta_{\mathrm{av}})_h^{-1}$. By considering equation (43) and (51), we suggest to let $\lambda = O(h^{-3})$ and $\widetilde{\mu} = O(1)$.

Since this is similar to the matrix formulation for ROF model, we can apply the split Bregman framework to fourth order problems:

$$(\mathrm{P3})_h \quad \begin{cases} \mathbf{u}^{j+1} = \underset{\mathbf{u}\in\mathbb{R}^{N-1}}{\operatorname{argmin}} \left\{ \dfrac{\lambda h}{2}\|K(\mathbf{u}-\mathbf{f})\|_2^2 + \dfrac{\widetilde{\mu}}{2}\|\mathbf{d}^j - h\nabla_{\mathrm{av},h}\mathbf{u} - \boldsymbol{\alpha}^j\|_2^2 \right\}, & \text{(53a)} \\ \mathbf{d}^{j+1} = \underset{\mathbf{d}\in\mathbb{R}^{N}}{\operatorname{argmin}} \left\{ \|\mathbf{d}\|_1 + \dfrac{\widetilde{\mu}}{2}\|\mathbf{d} - h\nabla_{\mathrm{av},h}\mathbf{u}^{j+1} - \boldsymbol{\alpha}^j\|_2^2 \right\}, & \text{(53b)} \\ \boldsymbol{\alpha}^{j+1} = \boldsymbol{\alpha}^j - \mathbf{d}^j + h\nabla_{\mathrm{av},h}\mathbf{u}^{j+1}. & \text{(53c)} \end{cases}$$



(a) Numerical result for fourth order TV flow

(b) Numerical result for Spohn's model

Figure 1: Numerical examples of one dimensional problems.

The convergence result of split Bregman framework [3, Theorem 3.2] is satisfied also for fourth order problems.

Remark. When we consider the backward Euler method, then $(\mathrm{P3})_h$ gives $\{\mathbf{u}^{k,j+1}\}$ for given $u^k \in V_{h0}$. The convergence result yields that the split Bregman iteration provides $u^{k+1} \in V_{h0}$ as the limit $j \to \infty$. In practice we regard $\mathbf{u}^{k+1} \approx \mathbf{u}^{k,m}$ for some $m \in \mathbb{N}$ and obtain the evolution as a result of finite number of split Bregman iteration.

Remark. The split Bregman iteration requires to determine $\mathbf{d}^0$ and $\boldsymbol{\alpha}^0 \in \mathbb{R}^N$. When we consider the backward Euler method, we suggest letting $\mathbf{d}^{k+1,0} = \mathbf{d}^{k,m}$ and $\boldsymbol{\alpha}^{k+1,0} = \boldsymbol{\alpha}^{k,m}$.

The functional in (53a) is differentiable with respect to $\mathbf{u}$, and the minimization (53b) can be reduced to the shrinkage method:

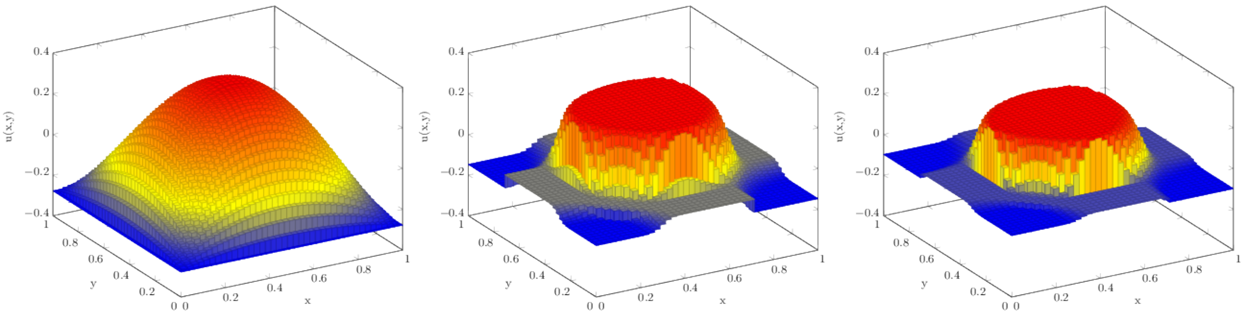
$$\left\{\begin{aligned} \mathbf{u}^{j+1} &= \left(\lambda h K^{\mathrm{T}} K + \widetilde{\mu} h^2 (\nabla_{\mathrm{av},h})^{\mathrm{T}} \nabla_{\mathrm{av},h}\right)^{-1} \left(\lambda h K^{\mathrm{T}} K \mathbf{f} + \widetilde{\mu} h (\nabla_{\mathrm{av},h})^{\mathrm{T}} (\mathbf{d}^j - \boldsymbol{\alpha}^j)\right), \\ (\mathbf{d}^{j+1})_n &= \operatorname{shrink}\left((h\nabla_{\mathrm{av},h}\mathbf{u}^{j+1} + \boldsymbol{\alpha}^j)_n, \widetilde{\mu}^{-1}\right) \text{ for all } n = 1, \dots, N, \\ \boldsymbol{\alpha}^{j+1} &= \boldsymbol{\alpha}^j - \mathbf{d}^{j+1} + h \nabla_{\mathrm{av},h} \mathbf{u}^{j+1}. \end{aligned}\right.$$

Remark. The backward Euler method and the split Bregman framework for Spohn's model give

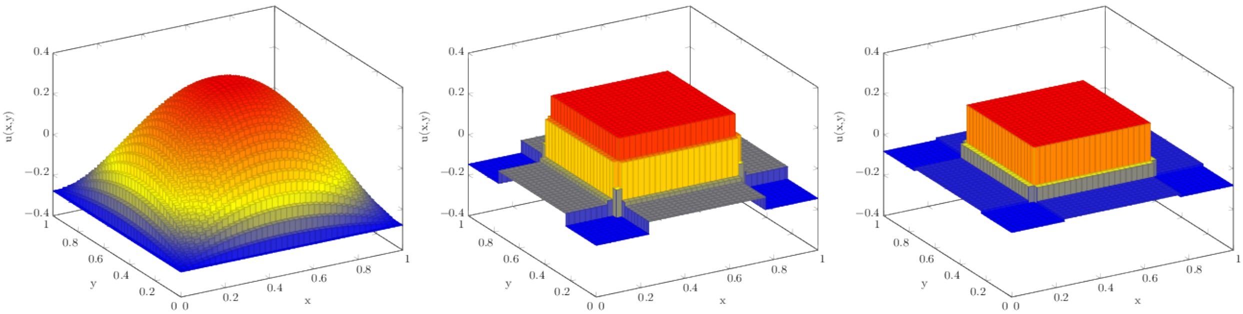
$$\left\{\begin{aligned} \mathbf{u}^{k,j+1} &= \operatorname*{argmin}_{\mathbf{u} \in \mathbb{R}^{N-1}} \left\{ \frac{\lambda h}{2} \|K(\mathbf{u} - \mathbf{u}^k)\|_2^2 + \frac{\widetilde{\mu}}{2} \|\mathbf{d}^{k,j} - h\nabla_{\mathrm{av},h}\mathbf{u} - \boldsymbol{\alpha}^{k,j}\|_2^2 \right\}, && \text{(54a)} \\ \mathbf{d}^{k,j+1} &= \operatorname*{argmin}_{\mathbf{d} \in \mathbb{R}^N} \left\{ \beta \|\mathbf{d}\|_1 + \frac{1}{p} \|\mathbf{d}\|_p^p + \frac{\widetilde{\mu}}{2} \|\mathbf{d} - h\nabla_{\mathrm{av},h}\mathbf{u}^{k,j+1} - \boldsymbol{\alpha}^{k,j}\|_2^2 \right\}, && \text{(54b)} \\ \boldsymbol{\alpha}^{k,j+1} &= \boldsymbol{\alpha}^{k,j} - \mathbf{d}^{k,j+1} + h\nabla_{\mathrm{av},h}\mathbf{u}^{k,j+1}. && \text{(54c)} \end{aligned}\right.$$

The shrinkage formula for Spohn's model is provided by solving the quadratic equations given by the Euler-Lagrange equation for (54b):

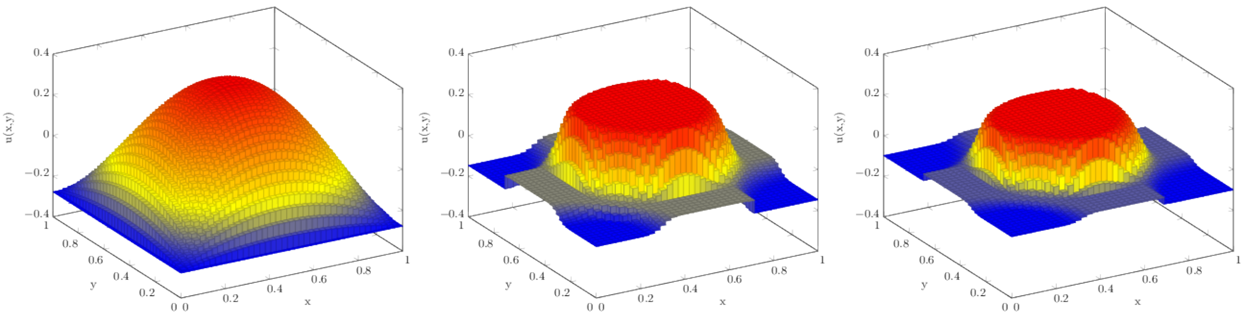
$$\left\{\begin{aligned} \mathbf{u}^{k,j+1} &= \left(\lambda h K^{\mathrm{T}} K + \widetilde{\mu} h^2 (\nabla_{\mathrm{av},h})^{\mathrm{T}} \nabla_{\mathrm{av},h}\right)^{-1} \left(\lambda h K^{\mathrm{T}} K \mathbf{f} + \widetilde{\mu} h (\nabla_{\mathrm{av},h})^{\mathrm{T}} (\mathbf{d}^{k,j} - \boldsymbol{\alpha}^{k,j})\right), \\ (\mathbf{d}^{k,j+1})_n &= \frac{\widetilde{\mu}\rho_n^{k,j+1}}{2|\rho_n^{k,j+1}|} \left(-1 + \sqrt{1 + \frac{4}{\widetilde{\mu}} \max\left\{ |\rho_n^{k,j+1}| - \frac{\beta}{\widetilde{\mu}}, 0 \right\}} \right) \text{ for all } n = 1, \dots, N, \\ \boldsymbol{\alpha}^{k,j+1} &= \boldsymbol{\alpha}^{k,j} - \mathbf{d}^{k,j+1} + h\nabla_{\mathrm{av},h}\mathbf{u}^{k,j+1}. \end{aligned}\right.$$



(a) Fourth order isotropic total variation flow



(b) Fourth order anisotropic total variation flow



(c) Spohn's fourth order model on $\mathbb{T}^2$

Figure 2: Numerical examples of two-dimensional problems.

where $\rho_n^{k,j+1} = (h\nabla_{\mathrm{av},h}\mathbf{u}^{k,j+1} + \boldsymbol{\alpha}^{k,j})_n$.

Remark. The split Bregman framework for two-dimensional cases can be provided by the similar way to one-dimensional cases.

Acknowledgements

A part of the work of the second author was done when he was a postdoc fellow at the University of Tokyo. Its hospitality is gratefully acknowledged. The work of the first author was partly supported by the Japan Society for the Promotion of Science through the grant No. 26220702 (Kiban S), No. 19H00639 (Kiban A), No. 18H05323 (Kaitaku), No. 17H01091 (Kiban A) and No. 16H03948 (Kiban B).

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Heat or wave?
A classification of semilinear damped wave equations with time-dependent coefficients *

In memory of Professor Kôji Kubota

Hiroyuki Takamura †

In this talk, I would like to discuss about a characterization of semilinear damped wave equations with time-dependent coefficients. They have been regarded as almost like semilinear heat equations according to the critical exponent, decay estimates and asymptotic behaviors of the solution. But in recent a few years, we find that it is not true in the case that the coefficient has a strong time-decay, so that we have a threshold which may have a mixed situation between heat and wave according to the size of the coefficient. Here we propose a new classification by lifespan estimates. As a result, we find that such a threshold must belong to the situation like wave equations. The main result in this talk is based on our recent work, Imai, Kato, Takamura and Wakasa [8].

We are concerned with the following initial value problem for semilinear wave equations with the scale-invariant damping:

$$\left\{\begin{array}{ll} v_{tt}-\Delta v+\dfrac{\mu}{1+t}v_t=|v|^p & \text{in} \quad \mathbf{R}^n\times[0,\infty), \\ v(x,0)=\varepsilon f(x),\ v_t(x,0)=\varepsilon g(x), & x\in\mathbf{R}^n, \end{array}\right. \tag{1}$$

where $v=v(x,t)$ is a real valued unknown function, $\mu>0$, $p>1$, $n\in\mathbf{N}$, the initial data $(f,g)\in H^1(\mathbf{R}^n)\times L^2(\mathbf{R}^n)$ has compact support, and $\varepsilon>0$ is a "small" parameter.

It is interesting to look for the critical exponent $p_c(n)$ such that

$$\left\{\begin{array}{ll} p>p_c(n) \text{ (and may have an upper bound)} & \Longrightarrow\ T(\varepsilon)=\infty, \\ 1<p\le p_c(n) & \Longrightarrow\ T(\varepsilon)<\infty, \end{array}\right.$$

*Main results in this talk are based on the joint works with Takuto Imai (Accenture Japan Ltd), Masakazu Kato (Muroran Institute of Technology), and Kyouhei Wakasa (National Institute of Technology, Kushiro College).

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where $T(\varepsilon)$ is the lifespan, the maximal existence time, of the energy solution of (1) with an arbitrary fixed non-zero data. Then, we have the following conjecture:

$$\begin{cases} \mu > \mu_0(n) & \Longrightarrow \ p_c(n) = p_F(n) & \text{(heat-like)}, \\ \mu = \mu_0(n) & \Longrightarrow \ p_c(n) = p_F(n) = p_S(n+\mu) & \text{(intermediate)}, \\ 0 < \mu < \mu_0(n) & \Longrightarrow \ p_c(n) = p_S(n+\mu) & \text{(wave-like)}, \end{cases} \tag{2}$$

where

$$\mu_0(n) := \frac{n^2+n+2}{n+2}. \tag{3}$$

Here

$$p_F(n) := 1 + \frac{2}{n} \tag{4}$$

is the so-called Fujita exponent which is the critical exponent of the associated semilinear heat equations $v_t - \Delta v = v^p$, and

$$p_S(n) := \begin{cases} \infty & (n=1), \\ \dfrac{n+1+\sqrt{n^2+10n-7}}{2(n-1)} & (n \geq 2) \end{cases} \tag{5}$$

is the so-called Strauss exponent which is the critical exponent of the associated semilinear wave equations $v_{tt} - \Delta v = |v|^p$. We note that $p_S(n)$ $(n \geq 2)$ is the positive root of

$$\gamma(p,n) := 2 + (n+1)p - (n-1)p^2 = 0 \tag{6}$$

and $0 < \mu < \mu_0(n)$ is equivalent to $p_F(n) < p_S(n+\mu)$.

The conjecture (2) shows the critical situation of our problem in the following sense. If one replaces the damping term $\mu v_t/(1+t)$ in (1) by $\mu v_t/(1+t)^\beta$, then one can see that there is no such a $p_c(n)$, namely $T(\varepsilon) = \infty$ for any $p > 1$ when $\beta < -1$, the so-called over damping case. Moreover one has $p_c(n) = p_F(n)$ for any $\mu > 0$ when $-1 \leq \beta < 1$, the so-called effective damping case, and $p_c(n) = p_S(n)$ for any $\mu > 0$ (it can be any $\mu \in \mathbf{R}$) when $\beta > 1$, the so-called scattering damping case. Therefore one may say that the so-called scale-invariant case, $\beta = 1$, is an intermediate situation between wave-like, in which the critical exponent is related to $p_S(n)$, and heat-like, in which the critical exponent is $p_F(n)$. To see all the references above results, for example, see introductions of related papers to the scattering damping case, Lai and Takamura [13] (the sub-critical case), Wakasa and Yordanov [19] (the critical case), Liu and Wang [15] (partial result of the super-critical case).

For the conjecture (2), D'Abbicco [2] has obtained the heat-like existence partially with

$$\mu \geq \begin{cases} 5/3 & \text{for } n=1 \ (\text{cf. } \mu_0(1) = 4/3), \\ 3 & \text{for } n=2 \ (\text{cf. } \mu_0(2) = 2), \\ n+2 & \text{for } n \geq 3, \end{cases}$$

while Wakasugi [22] has obtained the blow-up parts in $1 < p < p_F(n)$ for $\mu \geq 1$ and $1 < p < p_F(n+\mu-1)$ for $0 < \mu < 1$. In the damped case $\mu > 0$, his second result is the first blow-up result for super-Fujita exponents which are larger than $p_F(n)$.

In this paper, we consider a special case of $\mu = 2$. The speciality of this value is clarified by setting

$$u(x,t) := (1+t)^{\mu/2} v(x,t),$$

where v is the solution to (1). Then, u satisfies

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu(2-\mu)}{4(1+t)^2} u = \dfrac{|u|^p}{(1+t)^{\mu(p-1)/2}} & \text{in} \quad \mathbf{R}^n \times [0,\infty), \\ u(x,0) = \varepsilon f(x), \ u_t(x,0) = \varepsilon\{\mu f(x)/2 + g(x)\}, & x \in \mathbf{R}^n, \end{cases} \tag{7}$$

so that all the technics in the analysis on semilinear wave equations can be employed and we may discussed about not only the energy solution but also the classical solution. In fact, via this reduced problem (7), D'Abbicco, Lucente and Reissig [4] have proved the intermediate part of the conjecture (2) for $n = 2$ and the wave-like part for $n = 3$ when $\mu = 2$. We note that the assumption of the radial symmetry is considered in [4] for the existence part in $n = 3$. Moreover, D'Abbicco and Lucente [3] have obtained the wave-like existence part of (2) for odd $n \geq 5$ when $\mu = 2$ also with radial symmetry.

In the case of $\mu \neq 2$, Lai, Takamura and Wakasa [14] have first studied the wave-like blow-up of the conjecture (2) with a loss replacing μ by $\mu/2$ in the sub-critical case. Initiating this result, Ikeda and Sobajima [5] have obtained the blow-up part of (2).

For the lifespan estimate, one may expect that

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-(p-1)/\{2-n(p-1)\}} & \text{for } 1 < p < p_F(n), \\ \exp\left(C\varepsilon^{-(p-1)}\right) & \text{for } p = p_F(n) \end{cases} \tag{8}$$

for the heat-like domain $\mu > \mu_0(n)$ and

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-2p(p-1)/\gamma(p,n+\mu)} & \text{for } 1 < p < p_S(n+\mu), \\ \exp\left(C\varepsilon^{-p(p-1)}\right) & \text{for } p = p_S(n+\mu) \end{cases} \tag{9}$$

for the wave-like domain $0 < \mu < \mu_0(n)$. Recall the definitions of $\mu_0(n)$, $p_F(n)$, $p_S(n)$ and $\gamma(p,n)$ in (3), (4), (5) and (6). Here $T(\varepsilon) \sim A(\varepsilon, C)$ stands for the fact that there are positive constants, C_1 and C_2, independent of ε satisfying $A(\varepsilon, C_1) \leq T(\varepsilon) \leq A(\varepsilon, C_2)$. Actually, (8) for $n = 1$ and $\mu = 2 > \mu_0(1) = 4/3$ is obtained by Wakasa [21], and (9) is obtained by Kato and Sakuraba [10] for $n = 3$ and $\mu = 2 < \mu_0(3) = 14/5$. One may refer Lai [12] for the existence part of weaker solution. Moreover, the upper bound of (8) in the sub-critical case is obtained by Wakasugi [22]. Also the upper bound of (9) is obtained by Ikeda and Sobajima [5] in the critical case, later it is reproved by Tu and Li [18], and Tu and Li [17] in the sub-critical case.

In the non-damped case of $\mu = 0$, it is known that (9) is true for $n \geq 3$, or $p > 2$ and $n = 2$, The open part around this fact is $p = p_S(n)$ for $n \geq 9$. In other cases, (9) is still true if $\int_{\mathbf{R}^n} g(x)dx = 0$. On the other hand, we have

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-(p-1)/2} & \text{for } n=1, \\ C\varepsilon^{-(p-1)/(3-p)} & \text{for } n=2 \text{ and } 1<p<2, \\ Ca(\varepsilon) & \text{for } n=2 \text{ and } p=2 \end{cases} \tag{10}$$

if $\int_{\mathbf{R}^n} g(x)dx \neq 0$, where $a = a(\varepsilon)$ is a positive number satisfying $\varepsilon^2 a^2 \log(1+a) = 1$. We note that the bounds in (10) are smaller than the one of the first line in (9) with $\mu = 0$ in each case. For all the references in the case of $\mu = 0$, see Introduction of Imai, Kato, Takamura and Wakasa [7].

The remarkable fact is that even if μ is in the heat-like domain, the lifespan estimate for (1) is similar to the one for non-damped case. Indeed, for $n = 1$ and $\mu = 2 > \mu_0(1) = 4/3$, Kato, Takamura and Wakasa [11] show that the result on (8) by Wakasa [21] mentioned above is true only if $\int_{\mathbf{R}}\{f(x)+g(x)\}dx \neq 0$. More precisely, they have obtained that

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-2p(p-1)/\gamma(p,3)} & \text{for } 1<p<2, \\ Cb(\varepsilon) & \text{for } p=2, \\ C\varepsilon^{-p(p-1)/(3-p)} & \text{for } 2<p<3, \\ \exp(C\varepsilon^{-p(p-1)}) & \text{for } p=p_F(1)=3, \end{cases} \tag{11}$$

if $\int_{\mathbf{R}}\{f(x)+g(x)\}dx = 0$, where $b = b(\varepsilon)$ is a positive number satisfying $\varepsilon^2 b \log(1+b) = 1$. We note that the bounds in (11) are larger than those in (8) with $n = 1$ and $\mu = 2$ in each case.

The main aim in this talk is to show the lifespan estimates for (1) in two dimensional case, $n = 2$, with $\mu = 2$ which is similar to one dimensional case as above. We note $p_c(2) = p_F(2) = p_S(2+2) = 2$ and $\mu_0(2) = 2$. More precisely, we shall show that

$$T(\varepsilon) \sim \begin{cases} c\varepsilon^{-(p-1)/(4-2p)} & \text{for} \quad 1<p<2, \\ \exp(c\varepsilon^{-1/2}) & \text{for} \quad p=2 \end{cases} \tag{12}$$

if $\int_{\mathbf{R}^2}\{f(x)+g(x)\}dx \neq 0$, and

$$T(\varepsilon) \sim \begin{cases} c\varepsilon^{-2p(p-1)/\gamma(p,4)} & \text{for} \quad 1<p<2, \\ \exp(c\varepsilon^{-2/3}) & \text{for} \quad p=2 \end{cases} \tag{13}$$

if $\int_{\mathbf{R}^2}\{f(x)+g(x)\}\,dx = 0$. We note that the critical cases in (12) and (13) are new in the sense that they are different from (8) and (9). (12) and (13) are announced in Introduction of Lai and Takamura [13], but there are typos in the exponents of ε in the critical case.

The strategy of proofs is based on point-wise estimates of the solution. In the existence part, we employ the classical iteration argument for semilinear wave equations without damping term, which is first introduced by John [9] in three space dimensions, and its variant, which is developed by Imai, Kato, Takamura and Wakasa [7] in two space dimensions. In the blow-up part, we also employ an improved version of Kato's lemma on ordinary differential inequality by Takamura [16] for the sub-critical cases. We note that, till now, the so-called test function method such as in Ikeda, Sobajima and Wakasa [6] cannot be applicable to delicate analysis to catch the logarithmic growth of the solution in the case of $p = 2$ in (10), (11), (12) and (13). Therefore we employ the so-called slicing method of the blow-up domain for the critical case, which is introduced by Agemi, Kurokawa and Takamura [1] to handle weakly coupled systems of non-damped semilinear wave equations with critical exponents.

Acknowledgement

This series of works started when the speaker was working in Future University Hakodate. It has been partially supported by Special Research Expenses in FY2017, General Topics (No.B21), Future University Hakodate, also by the Grant-in-Aid for Scientific Research (B) (No.18H01132) and (C) (No.15K04964), Japan Society for the Promotion of Science.

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EQUIVALENCE OF SOLUTIONS OF EIKONAL EQUATION IN METRIC SPACES

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1. Introduction

We discuss the equivalence between some known notions of solutions to the eikonal equation and more general analogs of the Hamilton-Jacobi equations in complete and rectifiably connected metric spaces. This talk is based on joint work with N. Shanmugalingam (University of Cincinnati) and X. Zhou (Okinawa Institute of Science and Technology Graduate University).

The Hamilton-Jacobi equations in the Euclidean spaces are widely applied in various fields such as optimal control, geometric optics, computer vision, image processing. It is well known that the notion of viscosity solutions provides a nice framework for the well-posedness of first order fully nonlinear equations; we refer to [10, 4] for comprehensive introduction.

In seeking to further develop various fields such as optimal transport [3, 26], mean field games [9], topological networks [25, 17, 1, 15, 16] etc., the Hamilton-Jacobi equations in a general metric space $(\mathbf{X}, d)$ have recently attracted great attention, see for example [12, 13]. Typical forms of the equations include

$$H(x, u, |\nabla u|) = 0 \quad \text{in } \Omega, \tag{1.1}$$

and its time-dependent version

$$\partial_t u + H(x, t, u, |\nabla u|) = 0 \quad \text{in } (0, \infty) \times \mathbf{X} \tag{1.2}$$

with necessary boundary or initial value conditions. Here $\Omega \subsetneq \mathbf{X}$ is an open set and $H : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function called the Hamiltonian of the Hamilton-Jacobi equation. While ∂_t denotes the time differentiation, $|\nabla u|$ stands for a generalized notion of the gradient norm of u in metric spaces.

We pay particular attention to the so-called eikonal equation

$$|\nabla u|(x) = f(x) \quad \text{in } \Omega. \tag{1.3}$$

Here $f : \Omega \to [0, \infty)$ is a given continuous function satisfying

$$\inf_{\Omega} f > 0.$$

The eikonal equation in the Euclidean space has important applications in various fields such as geometric optics, electromagnetic theory and image processing [18, 21].

Several new notions of viscosity solutions have recently been proposed in the general metric setting. We refer the reader to Section 2 for a review with precise definitions of solutions and to the relevant papers we mention below.

Using rectifiable curves in the space $\mathbf{X}$, Giga, Hamamuki and Nakayasu [13] discussed a notion of metric viscosity solutions to (1.3) and established well-posedness under the Dirichlet condition

$$u = \zeta \quad \text{on } \partial\Omega, \tag{1.4}$$

where ζ is a given bounded continuous function on $\partial\Omega$ satisfying an appropriate regularity assumption to be discussed later. In the sequel, this type of solutions will be called curve-based solutions (or c-solutions for short). The definition of c-solutions (Definition 2.1) essentially relies on optimal control interpretations along rectifiable curves and requires very little structure of the space. The same approach is used in [22] to study the evolution problem (1.2) with the Hamiltonian $H(x, p)$ convex in p. Moreover, unique viscosity solutions of the eikonal equation in the sense of [13] are also constructed on fractals like the Sierpinski gasket [7].

On the other hand, when $(\mathbf{X}, d)$ is a complete geodesic space, by interpreting $|\nabla u|$ as the local slope of a locally Lipschitz function u, Ambrosio and Feng [2] provide a different viscosity approach to (1.2) for a class of convex Hamiltonians. This was extended to the class of potentially nonconvex Hamiltonians H by Gangbo and Święch [11, 12], who proposed a generalized notion of viscosity solutions via appropriate test classes and proved uniqueness and existence of the solutions to more general Hamilton-Jacobi equations in length spaces. Stability and convexity of such solutions are studied respectively in [23] and in [19]. Since this definition of solutions is based on the local slope, we shall call them slope-based solutions (or s-solutions for short) below. See the precise definition in Definition 2.3.

Since either approach above provides a generalized viscosity solution theory for first order nonlinear equations, especially the uniqueness and existence results, it is natural to expect that they actually agree with each other in the wider setting of metric spaces. This motivates us to explore the relations between both types of viscosity solutions and to understand further connections to other possible approaches.

2. Metric viscosity solutions

Let us briefly review the two notions of viscosity solutions to Hamilton-Jacobi equations in metric spaces, mentioned in the first section. We focus on the eikonal equation (1.3).

2.1. **Curve-based solutions.**

For any interval $I \subset \mathbb{R}$ with $0 \in I$, we say an absolutely continuous curve $\xi : I \to \mathbf{X}$ is admissible if

$$|\xi'| \leq 1 \quad \text{a.e. in } I.$$

Note that these are 1-Lipschitz curves.

For any $x \in \mathbf{X}$, we write $\xi \in \mathcal{A}_x(I, \mathbf{X})$ if $\xi : I \to \mathbf{X}$ is an admissible curve satisfying $\xi(0) = x$. We also need to define the exit time and entrance time of a curve ξ:

$$\begin{aligned} T_\Omega^+[\xi] &:= \inf\{t \in I : t \geq 0, \xi(t) \notin \Omega\}; \\ T_\Omega^-[\xi] &:= \sup\{t \in I : t \leq 0, \xi(t) \notin \Omega\}. \end{aligned} \tag{2.1}$$

Definition 2.1 (Definition 2.1 in [13])**.** An upper semicontinuous (USC) function u in Ω is called a curve-based viscosity subsolution or c-subsolution of (1.3) if for any $x \in \Omega$ and $\xi \in \mathcal{A}_x(\mathbb{R}, \Omega)$, we have

$$\left|\phi'(0)\right| \leq f(x) \tag{2.2}$$

whenever $\phi \in C^1(\mathbb{R})$ such that $t \mapsto u(\xi(t)) - \phi(t)$, $t \in \xi^{-1}(\Omega)$, attains a local maximum at $t = 0$.

A lower semicontinuous (LSC) function u in Ω is called a curve-based viscosity supersolution or c-supersolution of (1.3) if for any $\varepsilon > 0$ and $x \in \Omega$, there exists $\xi \in \mathcal{A}_x(\mathbb{R}, \mathbf{X})$ and $w \in LSC(T^-, T^+)$ with $-\infty < T^{\pm} = T^{\pm}_{\Omega}[\xi] < \infty$ such that

$$w(0) = u(x), \quad w \geq u \circ \xi - \varepsilon, \tag{2.3}$$

and

$$\left|\phi'(t_0)\right| \geq f(\xi(t_0)) - \varepsilon \tag{2.4}$$

whenever $\phi \in C^1(\mathbb{R})$ such that $w(t) - \phi(t)$ attains a minimum at $t = t_0 \in (T^-, T^+)$. A function $u \in C(\Omega)$ is said to be a curve-based viscosity solution or c-solution if it is both a c-subsolution and a c-supersolution of (1.3).

The regularity of u above can be relaxed, since we only need its semicontinuity along each curve ξ. In fact, one can require a c-subsolution (resp., c-supersolution) to be merely arcwise upper (resp., lower) semicontinuous; consult [13] for details. However, in order to obtain our main results, we need to impose the conventional continuity of u rather than the arcwise continuity.

Uniqueness of c-solutions of (1.3) with boundary data (1.4) is shown by proving a comparison principle [13, Theorem 3.1]. The existence of solutions in $C(\overline{\Omega})$, on the other hand, is based on an optimal control interpretation; in particular, it is shown in [13, Theorem 4.2 and Theorem 4.5] that

$$u(x) = \inf_{\xi \in C_x} \left\{ \int_0^{T^+_{\Omega}[\xi]} f(\xi(s))\, ds + \zeta\left(\xi(T^+_{\Omega}[\xi])\right) \right\} \tag{2.5}$$

is a c-solution of (1.3) and (1.4) provided that for each $x \in \overline{\Omega}$ we have

$$C_x := \left\{\xi \in \mathcal{A}_x([0,\infty), \mathbf{X}) \,:\, T^+_{\Omega}[\xi] \in (0, \infty)\right\} \neq \emptyset$$

and ζ satisfies a boundary regularity.

Note in the definition of c-supersolution, for each (x, ε) the conditions (2.3) and (2.4) for (ξ, w) are satisfied for all $t \in (T^-, T^+)$. We localize this definition as follows.

Definition 2.2 (Local curve-based solutions)**.** A function $u \in LSC(\Omega)$ is said to be a local c-supersolution if for each $x \in \Omega$ there exists $r > 0$ with an open ball in metric d $B^d_r(x) \subset \Omega$, and for each $\varepsilon > 0$ we can find a curve $\xi_\varepsilon \in \mathcal{A}_x(\mathbb{R}, \mathbf{X})$ with $\xi_\varepsilon(0) = x$ and a function $w \in LSC(t^-_r, t^+_r)$ with $t^-_r := T^-_{B^d_r(x)}[\xi_\varepsilon]$ and $t^+_r := T^+_{B^d_r(x)}[\xi_\varepsilon]$ such that

$$w(0) = u(x), \quad w(t) \geq u \circ \xi_\varepsilon(t) - \varepsilon \ \text{ for all } t \in (t^-_r, t^+_r),$$

and

$$\left|\phi'(t_0)\right| \geq f(\xi_\varepsilon(t_0)) - \varepsilon$$

whenever $\phi \in C^1(\mathbb{R})$ such that $w(t) - \phi(t)$ attains a minimum at $t = t_0 \in (t^-_r, t^+_r)$. A function $u \in C(\Omega)$ is said to be a local c-solution if it is both a c-subsolution and a local c-supersolution of (1.3).

2.2. **Slope-based solutions.** We next discuss the definition proposed in [11], which relies more on the property of geodesic or length metric. We denote by $\mathrm{Lip}_{loc}(\Omega)$ the set of locally Lipschitz continuous functions on an open subset Ω of a complete length space $(\mathbf{X}, d)$. For $u \in \mathrm{Lip}_{loc}(\Omega)$ and for $x \in \Omega$, we define the local slope of u to be

$$|\nabla u|(x) := \limsup_{y \to x} \frac{|u(y) - u(x)|}{d(x, y)}. \tag{2.6}$$

Let

$$\begin{aligned} \overline{\mathcal{C}}(\Omega) &:= \{u \in \mathrm{Lip}_{loc}(\Omega) \; : \; |\nabla^+ u| = |\nabla u| \text{ and } |\nabla u| \text{ is continuous in } \Omega\}, \\ \underline{\mathcal{C}}(\Omega) &:= \{u \in \mathrm{Lip}_{loc}(\Omega) \; : \; |\nabla^- u| = |\nabla u| \text{ and } |\nabla u| \text{ is continuous in } \Omega\}, \end{aligned} \tag{2.7}$$

where, for each $x \in \mathbf{X}$,

$$|\nabla^{\pm} u|(x) := \limsup_{y \to x} \frac{[u(y) - u(x)]_{\pm}}{d(x, y)} \tag{2.8}$$

with $[a]_+ := \max\{a, 0\}$ and $[a]_- := -\min\{a, 0\}$ for any $a \in \mathbb{R}$. We call $|\nabla^+ u|$ and $|\nabla^- u|$ the (local) super- and sub-slopes of u respectively; they are also named super- and sub-gradient norms in the literature (cf. [20]).

Definition 2.3 (Definition 2.6 in [12])**.** A function $u \in USC(\Omega)$ (resp., $u \in LSC(\Omega)$) is called a slope-based viscosity subsolution (resp., slope-based viscosity supersolution) or s-subsolution (resp., s-supersolution) of (1.3) if

$$|\nabla \psi_1|(x) \leq f(x) + |\nabla \psi_2|^*(x) \tag{2.9}$$

$$(\text{resp.,} \quad |\nabla \psi_1|(x) \geq f(x) - |\nabla \psi_2|^*(x)) \, . \tag{2.10}$$

holds for any $\psi_1 \in \underline{\mathcal{C}}(\Omega)$ (resp., $\psi_1 \in \overline{\mathcal{C}}(\Omega)$) and $\psi_2 \in \mathrm{Lip}_{loc}(\Omega)$ such that $u - \psi_1 - \psi_2$ attains a local maximum (resp., minimum) at a point $x \in \Omega$, We say that $u \in C(\Omega)$ is an s-solution of (1.3) if it is both an s-subsolution and an s-supersolution of (1.3).

Comparison principles for s-solutions are given in [12, Theorem 5.3] for the eikonal equation and in [12, Theorem 5.1, Theorem 5.3] for more general Hamilton-Jacobi equations.

3. Equivalence of solutions to the Dirichlet problem

We first compare c- and s-solutions of the eikonal equation and show their equivalence when continuous solutions to the Dirichlet problem exist. To this end, we assume that $(\mathbf{X}, d)$ is complete and rectifably connected; namely, for any $x, y \in \mathbf{X}$, there exists a rectifiable curve in $\mathbf{X}$ joining x and y.

Our key idea is to use the induced intrinsic metric of the space $\mathbf{X}$, which is given by

$$\tilde{d}(x, y) = \inf\{\ell(\xi) : \xi \text{ is a rectifiable curve connecting } x \text{ and } y\} \tag{3.1}$$

for $x, y \in \mathbf{X}$, where $\ell(\xi)$ denotes the length of the curve ξ with respect to the original metric d. It is then easily seen that $(\mathbf{X}, \tilde{d})$ is a length space. Since this change of metric preserves the property of c-solutions, we can directly compare the notion of c-solutions in $(\mathbf{X}, d)$ with s-solutions in $(\mathbf{X}, \tilde{d})$. In order to preserve the completeness of the metric space, we assume throughout this work that

$$\tilde{d} \to 0 \text{ as } d \to 0. \tag{3.2}$$

Before introducing our main results, we emphasize that in the sequel, by "local" we mean the property in question holds in sufficiently small open balls with respect to the intrinsic metric of the space.

Our first main result is as follows.

Theorem 3.1 (Equivalence between c- and s-solutions)**.** *Let $(\mathbf{X}, d)$ be a complete rectifiably connected metric space and $\tilde{d}$ be the intrinsic metric given by* (3.1). *Assume that $(\mathbf{X}, d)$ also satisfies* (3.2). *Suppose that $\Omega \subsetneq \mathbf{X}$ is an open set bounded with respect to the metric $\tilde{d}$. Assume that $f \in C(\Omega)$ satisfies $\inf_\Omega f > 0$. If u is a c-solution of* (1.3) *with respect to d, then u is a locally Lipschitz s-solution of* (1.3) *with respect to $\tilde{d}$. In addition, if f is uniformly continuous in Ω, ζ is uniformly continuous on $\partial\Omega$, and there exists a modulus of continuity σ such that*

$$|\zeta(y) - u(x)| \leq \sigma(\tilde{d}(x, y)) \quad \text{for all } x \in \Omega \text{ and } y \in \partial\Omega, \tag{3.3}$$

then u is the unique c-solution and s-solution of (1.3) *satisfying* (3.3).

In the second result of the theorem above, we can replace the assumptions of uniform continuity of u and f by their continuity if $(\mathbf{X}, d)$ is additionally assumed to be proper, that is, any bounded closed subset of $\mathbf{X}$ is compact.

As the metric d was replaced with the metric $\tilde{d}$, the proof of the first statement in Theorem 3.1 is more or less analogous to the classical arguments to show the relation between a viscosity solution and its optimal control interpretation (cf. [4]).

The c-solutions and f considered in [13] are in general only continuous along curves, and therefore might not be continuous with respect to either the metric d or the metric $\tilde{d}$; see [13, Example 4.9]. Here we assume the stronger requirement of continuity with respect to the metric d. Then (3.3) in the second result of Theorem 3.1 guarantees that the c-solution u is uniformly continuous up to $\partial\Omega$. Recall that the comparison principle (and thus the uniqueness) for s-solutions [12] needs such an assumption. More precisely, it was shown in [12, Theorem 5.3] that any s-subsolution u and any s-supersolution v satisfy $u \leq v$ in $\overline{\Omega}$ whenever there exists a modulus σ such that

$$u(x) \leq \zeta(y) + \sigma(\tilde{d}(x, y)), \quad v(x) \geq \zeta(y) - \sigma(\tilde{d}(x, y)) \tag{3.4}$$

for all $x \in \Omega$ and $y \in \partial\Omega$. One thus needs to use (3.3) to validate the comparison principle and conclude the uniqueness and equivalence.

4. Local equivalence of solutions

Our result in Theorem 3.1 states that c- and s-solutions of (1.3) coincide. However, we show the equivalence in presence of the Dirichlet boundary condition so as to use the comparison principle. One may wonder about a more direct proof of the equivalence without using the boundary condition.

The second part is devoted to answering this question. Our method is based on localization of our arguments in the first part. To this end, we introduce a local version of the notion of c-solutions, which we call local c-solutions, by restricting the definition in a small metric ball centered at each point in Ω. We also include a third notion, called Monge solutions, in our discussion. We compare locally the notions of s-, local c-, and Monge s olutions of (1.3) in a complete length space.

The Monge solution is known to be an alternative notion of solutions to Hamilton-Jacobi equations in Euclidean spaces [24, 5]. In a complete length space $(\mathbf{X}, d)$, our generalized definition of Monge solutions to (1.3) is quite simple; it only requires a locally Lipschitz function u to satisfy

$$|\nabla^- u|(x) = f(x) \quad \text{for } every\ x \in \Omega,$$

where $|\nabla^- u|(x)$ is given in (2.8). One advantage of this notion is that it does not involve any viscosity tests and the comparison principle can be easily established. We show that locally uniformly continuous c-, s- and Monge solutions of the eikonal equation are actually equivalent under a weaker positivity assumption on f. A more precise statement is given below.

Theorem 4.1 (Local equivalence between solutions of eikonal equation)**.** *Let $(\mathbf{X}, d)$ be a complete length space and $\Omega \subset \mathbf{X}$ be an open set. Assume that f is locally uniformly continuous and $f > 0$ in Ω. Let $u \in C(\Omega)$. Then the following statements are equivalent:*

(a) *u is a local c-solution of* (1.3)*;*
(b) *u is a locally uniformly continuous s-solution of* (1.3)*;*
(c) *u is a Monge solution of* (1.3)*.*

In addition, if any of (a)–(c) holds, then u is locally Lipschitz with

$$|\nabla u|(x) = |\nabla^- u|(x) = f(x) \quad \textit{for all } x \in \Omega. \tag{4.1}$$

We actually prove more:

- The notions of all locally uniformly continuous subsolutions are equivalent and locally Lipschitz;
- The notions of locally Lipschitz s-supersolutions and Monge supersolutions are equivalent;
- Any locally Lipschitz local c-supersolution is a Monge supersolution;
- Any Monge solution is a local c-solution.

We however do not know whether an s- or Monge supersolution needs to be a local c-supersolution.

In proving Theorem 4.1, it turns out that the local Lipschitz continuity of solutions is an important ingredient. Note that the local Lipschitz continuity holds for Monge solutions by definition, and it can be easily deduced for c-subsolutions as well because the space $(\mathbf{X}, \tilde{d})$ is a length space. In contrast, the Lipschitz regularity of s-solutions of (1.3) is less straightforward. Our proof requires the assumption on the local uniform continuity of s-solutions and f due to possible lack of compactness for general length spaces. We can remove such an assumption if $(\mathbf{X}, d)$ is proper (and therefore the length space space X is a geodesic space because of the generalized Hopf-Rinow theorem in metric spaces [14, 6], which can be viewed as an immediate consequence of the Arzela-Ascoli theorem). The notions of continuity and uniform continuity in a compact set are clearly equivalent.

We will assume that $(\mathbf{X}, d)$ satisfies the hypotheses of Theorem 3.1. It is worth stressing that here $(\mathbf{X}, d)$ is assumed to be a length space only for simplicity from the point of view of c-solutions. The s-solutions and Monge solutions are not defined for the case that $\mathbf{X}$ is not a length space. For metric spaces that are not length spaces, the slopes in the definition of s-solutions and Monge solutions require us to replace the metric d with $\tilde{d}$.

It is not difficult to extend our discussion on the equivalence to the general equation (1.1) under a monotonicity assumption on $p \to H(x, r, p)$. When $p \to H(x, r, p)$ is increasing, we can simply apply an implicit-function-type argument to locally reduce the problem to the eikonal equation.

Besides, as in (4.1) in Theorem 4.1, in this general case for any solution u we can obtain the continuity of $|\nabla u|$ as well as the following property:

$$|\nabla u|(x) = |\nabla^- u|(x) \quad \text{for all } x \in \Omega; \tag{4.2}$$

in other words, the solution itself actually lie in the test class for s-subsolutions proposed in [11, 12]. This type of properties also appears in the study of time-dependent Hamilton-Jacobi equations on metric spaces (cf. [20]). Our analysis reveals that (4.2) resembles the semi-concavity in the Euclidean space. In fact, in the Euclidean space, (4.2) implies the existence of a C^1 test function everywhere from above, which is a typical property of semi-concave functions [8]. We expect more applications of the regularity property (4.2), since in general metric spaces defining convex functions is not trivial at all. It would be interesting to see further properties on such regular solutions in relation to the structure of PDEs and the geometric property.

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Nonlinear diffusion equation with dynamic boundary conditions and related topics

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Abstract

境界条件に時間微分を含む「動的境界条件」を課した初期値境界値問題は, 境界上での 1 つの系を記述するとともに, 領域内部の系との接合問題となる. 本講演では, 動的境界条件下での非線形拡散方程式の初期値境界値問題の適切性を Cahn–Hilliard 型の方程式と動的境界条件によって近似し, その極限として特徴付ける方法を紹介する. また, fast diffusion 型の方程式と動的境界条件の問題に対して, 既存の結果を応用して有限時刻消滅の証明を試みる.

1. 導入

時間と空間変数を独立変数に持つ, スカラー値関数を求める放物型偏微分方程式において, 空間領域が有界もしくは境界をもつ場合, 初期条件と境界条件を補助的に課した初期値境界値問題を考察する場合がある. 例えば Dirichlet(ディリクレ), Neumann(ノイマン), Robin(ロビン, ロバン) 境界条件が代表的であり, それぞれを第 1 種, 第 2 種, 第 3 種境界条件と呼ぶことからもこれらの条件が非常に一般的で様々な問題において古くからの研究対象となってきた補助条件であることが分かる. 近年, dynamic(動的あるいは力学的) 境界条件と呼ばれる条件を含む問題の研究が盛んに行われ始めた. これは境界条件に未知関数の時間微分を含む様な境界条件であり, 領域内部の主たる方程式と比較して境界上でも同様に時間発展する方程式を考察することになる. もはや動的境界条件は領域内部の方程式を解くための補助条件では無く, この場合, 境界値問題は領域内部の方程式と境界上の方程式の連立系あるいは接合問題と見なすこともできる. よって, 内部と境界で反応項の強さが異なる系や, そもそも全く異なる系の連立系を表現することも可能となり, 動的境界条件を含む問題は興味深い.

2. 動的境界条件下での非線形拡散方程式

$0 < T < \infty$, $\Omega \subset \mathbb{R}^d (d = 2, 3)$ は十分滑らかな境界 $\Gamma := \partial\Omega$ をもつ有界領域とする. 未知関数を $u : (0, T) \times \Omega \to \mathbb{R}$ とし, 次の非線形拡散方程式を考察する:

$$\partial_t u - \Delta\beta(u) = g \quad \text{in } Q := (0, T) \times \Omega, \tag{1}$$
$$\partial_t u + \partial_{\boldsymbol{\nu}}\beta(u) - \Delta_\Gamma\beta(u) = g_\Gamma \quad \text{on } \Sigma := (0, T) \times \Gamma. \tag{2}$$

これに適切な初期条件を課して初期値境界値問題を解いていくが, しばらく初期条件については割愛する. ここで, ∂_t は時間微分, Δ は Laplace 作用素, $\partial_{\boldsymbol{\nu}}$ は外向き法線 $\boldsymbol{\nu}$ 方向微分, そして, Δ_Γ は Laplace–Beltrami 作用素 (例えば, Grigor'yan[53] を参照) を意味する. $\beta : \mathbb{R} \to \mathbb{R}$ の選び方によって (1)–(2) は動的境界条件下での様々な非線形拡散方程式を意味する. 例えば

$$\beta(r) := \begin{cases} k_{\mathrm{S}} r & r < 0, \\ 0 & 0 \le r \le \ell, \\ k_{\mathrm{L}}(r - \ell) & r > \ell, \end{cases} \quad \beta(r) := |r|^{m-1} r \quad (m > 0), \quad \beta(r) := \partial I_{[0,1]}(r)$$

のように選べば, (1)–(2) はそれぞれ Stefan 問題のエンタルピー形式 ($k_{\mathrm{S}}, k_{\mathrm{L}}, \ell$ は正定数), 多孔質媒体方程式 ($m > 1$), fast diffusion 方程式 ($0 < m < 1$), Hele-Shaw 問題の弱形式となる. 最後の例では $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ は, 支持関数 $I_{[0,1]}$ の劣微分 $\partial I_{[0,1]}$ であり, 多価写像である. いずれにせよ, β は単調増加なグラフという仮定のみで可解性の議論をしていく. 動的境界条件下での Stefan

本研究は科研費 (課題番号:17K05321) および, 公益財団法人 住友財団 基礎科学研究助成 (No.190367) の助成を受けたものである.

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問題については愛木 [1, 2, 3] によって適切性の研究がされており, そこでは Laplace–Beltrami 作用素は登場せず, Laplace–Beltrami 作用素から得られる解の滑らかさの情報に頼ること無く解の存在を議論している点に注意したい. また, この結果は Igbida らによって, より一般的な問題に拡張されている [8, 55]. Hele-Shaw 問題の弱形式に対しては [54], その他の非線形拡散 [82] なども参照せよ. 一方, 関連する動的境界条件下での非線形拡散方程式については Gal の一連の結果 [48, 49] が詳しい.

本講演の前半では上記 (1)–(2) のような動的境界条件下での非線形拡散方程式に対して, Cahn–Hilliard 方程式系による近似から接近を試みる. Cahn–Hilliard 方程式系 [12] はスピノーダル分解と呼ばれる 2 種の合金の相分離現象を記述する偏微分方程式としてよく知られている (例えば, その歴史的背景については Cahn が「京都賞」を受賞した際の講演記録 [11], 数学的取り扱いについては Miranville[72, 73] のサーベイを参照):

$$\partial_t u - \Delta\mu = 0 \quad \text{in } Q, \tag{3}$$
$$\mu = -\varepsilon\Delta u + F'(u) - f \quad \text{in } Q. \tag{4}$$

方程式を特徴付ける非線形項 F' について, F はそのグラフの形状から二重井戸型ポテンシャルと呼ばれており, 例えば $F(r) = (1/4)(r^2 - 1)$, つまり $F'(r) = r^3 - r$ が原型のポテンシャルとしてよく取り扱われる. 未知関数の 1 つ $u := u(t, x)$ は, 秩序変数もしくはオーダーパラメータと呼ばれる相の状況を記述する関数であり, 例えば $u = 1$ ならば全て金属 A, $u = -1$ ならば全て金属 B, $u = 0$ ならば両金属が五分五分といった解釈ができる. $F'(r)$ を $F'(r) := \beta(r) + \varepsilon\pi(r)$ のように F の凸部分と非凸部分に分け, $\varepsilon \to 0$ の極限操作を行えば (3)–(4) は形式的に (1) へと接近することが見て取れる. また, (2) は対応する境界上での Cahn–Hilliard 型の動的境界条件を用意すればよい (例えば, [17, 47, 52]). ただし, 動的境界条件下の Cahn–Hilliard 方程式といっても様々な種類が研究されており, この「対応する境界上での Cahn–Hilliard 型の動的境界条件」とは, Goldstein–Miraville–Schimperna[52] のモデルを指す. この極限操作の流れを紹介するのが本講演の前半部分である. 時間微分を弱い意味で捉え, 適切な弱解を定義し, 劣微分作用素に支配される抽象発展方程式の枠組みで可解性を論じる ([39, 40]).

本講演の後半では, 次のような動的境界条件下での非線形拡散方程式に対する長時間挙動を考察する.

$$\partial_t u - \Delta u^m + a u^m = \lambda u^p \quad \text{in } \Omega, \quad t > 0, \tag{5}$$
$$\partial_t u + \partial_{\boldsymbol{\nu}} u^m - \Delta_\Gamma u^m + b u^m = \mu u^q \quad \text{on } \Gamma, \quad t > 0, \tag{6}$$

ここで, $0 < m < 1$, $p, q \geq 1$, $(a, b), (\lambda, \mu) \in \{(1, 0), (0, 1)\}$ とする. $a = \lambda = 0$ であれば (5) は fast diffusion 方程式としてよく知られ (例えば, Vázquez[84] を参照), 有限時刻で解は恒等的に 0 となる性質「有限時刻消滅 (*finite time extinction*)」を持つ (Sabinina[79]). この種の問題は非常に多くの先行研究があり, 例えば以下の設定

$$\begin{aligned} \partial_t u - \Delta u^m &= u^p \quad \text{in } \Omega, \quad t > 0, \\ u &= 0 \quad \text{on } \Gamma, \quad t > 0 \end{aligned}$$

であれば, Ball[9], 藤田 [38] の $m = 1$ の結果を元に, $0 < m < 1$ の場合にも非常に数多くの様々な研究 (例えば [5, 7, 24, 25, 26, 35, 36, 67, 85]) が続いている. なお $\Omega = \mathbb{R}^3$ の場合に戻れば, Leoni[65] や望月–向井 [74] の $0 < m < 1$ の場合の結果もある. さらに, (5) を非線形境界条件下で考察した結果も, 例えば [16, 23, 32, 37, 66, 80, 87] のように数多くある. 一方, 動的境界条件 (6) に注目すると, $m = 1$, $a = 0$, $\lambda = 1$ の場合に関連して Fila–石毛–川上 [27, 28, 29, 30, 31] の一連の研究や, その他 [33, 48, 49, 86] などの結果がある. なお, Fila–Quittner[34] の動的境界条件下での問題設定 ($m = 1$ の場合) が本講演で対象としている問題に最も近い.

本講演の後半で対象とする問題をはっきりさせるため, 以下に係数と方程式の対応をあげて

おく.

$$(a,b,\lambda,\mu)=(1,0,1,0),\qquad \begin{cases}\partial_t u-\Delta u^m+u^m=u^p & \text{in } \Omega,\\ \partial_t u+\partial_{\boldsymbol{\nu}} u^m-\Delta_\Gamma u^m=0 & \text{on } \Gamma;\end{cases} \tag{7}$$

$$(a,b,\lambda,\mu)=(0,1,1,0),\qquad \begin{cases}\partial_t u-\Delta u^m=u^p & \text{in } \Omega,\\ \partial_t u+\partial_{\boldsymbol{\nu}} u^m-\Delta_\Gamma u^m+u^m=0 & \text{on } \Gamma;\end{cases} \tag{8}$$

$$(a,b,\lambda,\mu)=(1,0,0,1),\qquad \begin{cases}\partial_t u-\Delta u^m+u^m=0 & \text{in } \Omega,\\ \partial_t u+\partial_{\boldsymbol{\nu}} u^m-\Delta_\Gamma u^m=u^q & \text{on } \Gamma;\end{cases} \tag{9}$$

$$(a,b,\lambda,\mu)=(0,1,0,1),\qquad \begin{cases}\partial_t u-\Delta u^m=0 & \text{in } \Omega,\\ \partial_t u+\partial_{\boldsymbol{\nu}} u^m-\Delta_\Gamma u^m+u^m=u^q & \text{on } \Gamma.\end{cases} \tag{10}$$

ここでも, 初期条件は省略している. ここでは Filo[35], Fila–Filo[25, 26] の, Dirichlet 境界条件もしくは非線形境界条件下での先行研究に従い, ある安定集合から初期値を選んだ場合, 解は大域的に一意存在し有限時刻消滅が起こることを示す. 解の存在については $0<m<1$ であることと動的境界条件が Laplace–Beltrami 作用素を含むことから, 境界での解の滑らかさが獲得でき適切な弱解が得られる点に注意したい.

3. Cahn–Hilliard 方程式系による接近

ここでは, (1)–(2) の初期値境界値問題に対し Cahn–Hilliard 方程式系による近似によって接近を試みるにあたり, 以下の問題 (GMS) を設定する. $\varepsilon\in(0,1]$ を近似変数, 未知関数を $u,\mu:Q\to\mathbb{R}$ と $u_\Gamma,\mu_\Gamma:\Sigma\to\mathbb{R}$ とし, 次の Cahn–Hilliard 方程式系を考察する:

$$\partial_t u-\Delta\mu=0\quad \text{a.e. in } Q, \tag{11}$$

$$\mu=-\varepsilon\Delta u+\xi+\varepsilon\pi(u)-f,\quad \xi\in\beta(u)\quad \text{a.e. in } Q, \tag{12}$$

$$u_{|\Gamma}=u_\Gamma,\quad \mu_{|\Gamma}=\mu_\Gamma,\quad \partial_t u_\Gamma+\partial_{\boldsymbol{\nu}}\mu-\Delta_\Gamma\mu_\Gamma=0\quad \text{a.e. on } \Sigma, \tag{13}$$

$$\mu_\Gamma=\partial_{\boldsymbol{\nu}} u-\varepsilon\Delta_\Gamma u_\Gamma+\xi_\Gamma+\varepsilon\pi_\Gamma(u_\Gamma)-f_\Gamma,\quad \xi_\Gamma\in\beta_\Gamma(u_\Gamma)\quad \text{a.e. on } \Sigma, \tag{14}$$

$$u(0)=u_0\quad \text{a.e. in } \Omega,\quad u_\Gamma(0)=u_{\Gamma,0}\quad \text{a.e. on } \Gamma, \tag{15}$$

ただし, $u_{|\Gamma}$ は u のトレースを意味する. つまり, (13)–(14) は u と u_Γ, μ と μ_Γ の接合条件になっている. 前節までのように u のトレース $u_{|\Gamma}$ を改めて u と略記すれば u_Γ や μ_Γ を持ち出す必要はないが, 内部と境界の関数 u,μ と u_Γ,μ_Γ をそれぞれ別に考えた方が後に扱いやすいため, ここでは未知関数を分けて考える. また, β の原始関数 $\widehat{\beta}$ を F の凸部分, π の原始関数 $\widehat{\pi}$ を F の非凸部分と表現し, より広い二重井戸型ポテンシャル $F=\widehat{\beta}+\varepsilon\widehat{\pi}$ まで取り扱う. さらに, (12) と (14) とでポテンシャルが異なってもかまわない. 以下の 3 つの例は Cahn–Hilliard 方程式としてよく取り上げられるものであるが, β を $\mathbb{R}\to 2^{\mathbb{R}}$ と, 一般の多価写像まで広げて適切性が論じられることで, $\varepsilon\to 0$ の極限操作を行った際に, より多くの非線形拡散方程式がこの枠組みに乗る.

- $\beta(r)=r^3$, $\pi(r)=-r$, $D(\beta)=\mathbb{R}$ の場合, F は次の原型の二重井戸型ポテンシャル

  $$F(r)=\frac{1}{4}(r^2-1)^2;$$

- $\beta(r)=\ln((1+r)/(1-r))$, $\pi(r)=-2cr$, $D(\beta)=(-1,1)$ の場合, F は次の log 型ポテンシャル

  $$F(r)=\big((1+r)\ln(1+r)+(1-r)\ln(1-r)\big)-cr^2,$$

  ただし, $c>0$ は凸性を崩す (二重井戸構造を作る) 十分大きな定数;

- $\beta(r)=\partial I_{[-1,1]}(r)$, $\pi(r)=-r$, $D(\beta)=[-1,1]$ の場合, F は次の特異型ポテンシャル

  $$F(r)=I_{[-1,1]}(r)-\frac{1}{2}r^2,$$

ここで, $I_{[-1,1]}$ は閉区間 $[-1,1]$ 上の指示関数, すなわち

$$I_{[-1,1]}(r)=\begin{cases}0 & \text{if } r\in[-1,1],\\ \infty & \text{otherwise.}\end{cases}$$

また, $\partial I_{[-1,1]}$ は $I_{[-1,1]}$ の劣微分を意味する (例えば, Barbu[10] を参照).

前述問題 (GMS) は Gal[47] や Goldstein–Miranville–Schimperna[52] で提唱され, 様々な研究 (例えば [14, 15, 17, 44, 45, 57]) が続いているが, 特徴的な保存則を持つことに注意したい. 実際, (11), (13), (15) より, 次の総質量保存則を得る:

$$\int_\Omega u(t)dx+\int_\Gamma u_\Gamma(t)d\Gamma=\int_\Omega u_0dx+\int_\Gamma u_{\Gamma,0}d\Gamma \quad \text{for all } t\in[0,T].$$

以後, 次の表記を用いる: $H:=L^2(\Omega)$, $V:=H^1(\Omega)$, $W:=H^2(\Omega)$ $H_\Gamma:=L^2(\Gamma)$, $V_\Gamma:=H^1(\Gamma)$, $W_\Gamma:=H^2(\Gamma)$. また, $\boldsymbol{H}:=H\times H_\Gamma$ や $\boldsymbol{V}:=\bigl\{\boldsymbol{z}:=(z,z_\Gamma)\in V\times V_\Gamma : z_\Gamma=z_{|_\Gamma} \text{ a.e. on } \Gamma\bigr\}$, $\boldsymbol{W}:=(W\times W_\Gamma)\cap\boldsymbol{V}$, $\boldsymbol{L}^\infty:=L^\infty(\Omega)\times L^\infty(\Gamma)$ としておく. さらに, 総質量保存則の効果的な利用から次の関数空間を用意する: $\boldsymbol{H}_0:=\{\boldsymbol{z}\in\boldsymbol{H}:m(\boldsymbol{z})=0\}$, ここで, $m:\boldsymbol{H}\to\mathbb{R}$ は

$$m(\boldsymbol{z}):=\frac{1}{|\Omega|+|\Gamma|}\left\{\int_\Omega zdx+\int_\Gamma z_\Gamma d\Gamma\right\} \quad \text{for all } \boldsymbol{z}\in\boldsymbol{H}$$

と定義し, さらに簡単のため $m_0:=m(\boldsymbol{u}_0)$ とおく. ただし, $\boldsymbol{u}_0:=(u_0,u_{\Gamma,0})$. また, $\boldsymbol{V}_0:=\boldsymbol{V}\cap\boldsymbol{H}_0$ とおく. それぞれの関数空間は通常のノルムを備えているが, $\boldsymbol{V}_0$ のノルムについては, 総質量保存則に対応した Poincaré–Wirtinger の不等式を得ることで $|\boldsymbol{z}|_{\boldsymbol{V}_0}:=a(\boldsymbol{z},\boldsymbol{z})^{1/2}$ $(\boldsymbol{z}\in\boldsymbol{V}_0)$ が採用できる. ここで, 双線形形式 $a(\cdot,\cdot):\boldsymbol{V}\times\boldsymbol{V}\to\mathbb{R}$ を

$$a(\boldsymbol{z},\tilde{\boldsymbol{z}}):=\int_\Omega\nabla z\cdot\nabla\tilde{z}dx+\int_\Gamma\nabla_\Gamma z_\Gamma\cdot\nabla_\Gamma\tilde{z}_\Gamma d\Gamma \quad \text{for all } \boldsymbol{z},\tilde{\boldsymbol{z}}\in\boldsymbol{V}$$

で定義しておく. 実際, $\boldsymbol{V}_0$ からその共役空間 $\boldsymbol{V}_0^*$ への写像 $\boldsymbol{F}$ を $\langle\boldsymbol{F}\boldsymbol{z},\tilde{\boldsymbol{z}}\rangle_{\boldsymbol{V}_0^*,\boldsymbol{V}_0}:=a(\boldsymbol{z},\tilde{\boldsymbol{z}})$ と定義すれば $\boldsymbol{F}:\boldsymbol{V}_0\to\boldsymbol{V}_0^*$ は双対写像となり, さらにある定数 $C_{\mathrm{P}}>0$ が存在して

$$C_{\mathrm{P}}|\boldsymbol{z}|_{\boldsymbol{V}}^2\le\langle\boldsymbol{F}\boldsymbol{z},\boldsymbol{z}\rangle_{\boldsymbol{V}_0^*,\boldsymbol{V}_0}=|\boldsymbol{z}|_{\boldsymbol{V}_0}^2 \quad \text{for all } \boldsymbol{z}\in\boldsymbol{V}_0$$

を満たす. また, $\boldsymbol{V}_0\hookrightarrow\hookrightarrow\boldsymbol{H}_0\hookrightarrow\hookrightarrow\boldsymbol{V}_0^*$ なる稠密かつコンパクトな埋め込みが成立する (Colli–深尾 [17, Appendix] を参照).

以上の設定の下, 剣持–Niezgódka–Pawłow[60] (より一般的な形で, 久保 [63] も参照) による発展方程式の抽象理論による接近に倣い (GMS) の適切性を論じることができる. そこで, 次の変数変換を用意する: $\boldsymbol{w}:=(v,v_\Gamma):=(u-m_0,u_\Gamma-m_0)$, $\boldsymbol{w}_0:=(w_0,w_{\Gamma,0}):=(u_0-m_0,u_{\Gamma,0}-m_0)$. また, 次の表記を用いる: $\boldsymbol{\beta}(\boldsymbol{z}):=(\beta(z),\beta_\Gamma(z_\Gamma))$, $\boldsymbol{\pi}(\boldsymbol{z}):=(\pi(z),\pi_\Gamma(z_\Gamma))$.

以後, 以下を仮定する:

(A1) β, β_Γ: $\mathbb{R}\to 2^{\mathbb{R}}$ は極大単調で, ある適正下半連続凸関数 $\widehat{\beta},\widehat{\beta}_\Gamma:\mathbb{R}\to[0,\infty]$ で $\widehat{\beta}(0)=\widehat{\beta}_\Gamma(0)=0$ を満たすものによって $\beta=\partial\widehat{\beta}$, $\beta_\Gamma=\partial\widehat{\beta}_\Gamma$ と劣微分で表現されている;

(A2) π, $\pi_\Gamma:\mathbb{R}\to\mathbb{R}$ は Lipshitz 連続;

(A3) $D(\beta_\Gamma)\subset D(\beta)$ で, さらにある定数 $\varrho,c_0>0$ が存在して

$$\bigl|\beta^\circ(r)\bigr|\le\varrho\bigl|\beta_\Gamma^\circ(r)\bigr|+c_0 \quad \text{for all } r\in D(\beta_\Gamma),$$

ここで, $\beta^\circ(r):=\{r^*\in\beta(r):|r^*|=\min_{s\in\beta(r)}|s|\}$;

(A4) $m_0 \in \mathrm{int}D(\beta_\Gamma)$ でさらに $\widehat{\beta}(u_0) \in L^1(\Omega)$, $\widehat{\beta}_\Gamma(u_{\Gamma,0}) \in L^1(\Gamma)$ が成立する;

(A5) $\boldsymbol{f} := (f, f_\Gamma) \in W^{1,1}(0,T;\boldsymbol{H})$ または $\boldsymbol{f} \in L^2(0,T;\boldsymbol{V})$, $\boldsymbol{u}_0 \in \boldsymbol{V}$.

β が一価写像であればもちろん $\beta^\circ = \beta$ である．また $\beta = \beta_\Gamma$ の場合には (A3) の仮定は自動的に成立する．すなわち, (A3) は内部と境界で異なるポテンシャルを取るための条件で, β は (A3) の意味で β_Γ に支配されている (Calatroni–Colli[13]). Allen–Cahn 型の動的境界条件の場合であれば, 支配関係が逆の仮定での取り扱いも Gilardi–Miranville–Schimperna[51] にある.

このとき, (GMS) に対して以下の弱解の存在定理が成立する:

定理 3.1.[17, Theorem 2.2]

(A1)–(A5) の下, ある組み $(\boldsymbol{w}, \boldsymbol{\mu}, \boldsymbol{\xi})$ が次のクラス

$$\boldsymbol{w} \in H^1(0,T;\boldsymbol{V}_0^*) \cap L^\infty(0,T;\boldsymbol{V}_0) \cap L^2(0,T;\boldsymbol{W}),$$
$$\boldsymbol{\mu} \in L^2(0,T;\boldsymbol{V}), \quad \boldsymbol{\xi} \in L^2(0,T;\boldsymbol{H}),$$
$$\xi \in \beta(w + m_0) \ \textit{a.e. in } Q, \quad \xi_\Gamma \in \beta_\Gamma(w_\Gamma + m_0) \ \textit{a.e. on } \Sigma$$

に存在して

$$\big\langle \boldsymbol{w}'(t), \boldsymbol{z} \big\rangle_{\boldsymbol{V}_0^*, \boldsymbol{V}_0} + a\big(\boldsymbol{\mu}(t), \boldsymbol{z}\big) = 0 \quad \textit{for all } \boldsymbol{z} \in \boldsymbol{V}_0,$$
$$\big(\boldsymbol{\mu}(t), \boldsymbol{z}\big)_{\boldsymbol{H}} = \varepsilon a\big(\boldsymbol{w}(t), \boldsymbol{z}\big) + \big(\boldsymbol{\xi}(t) + \varepsilon \boldsymbol{\pi}(\boldsymbol{w}(t) + m_0 \mathbf{1}) - \boldsymbol{f}(t), \boldsymbol{z}\big)_{\boldsymbol{H}} \quad \textit{for all } \boldsymbol{z} \in \boldsymbol{V},$$

for a.a. $t \in (0,T)$ で, さらに $\boldsymbol{w}(0) = \boldsymbol{w}_0$ *in* $\boldsymbol{H}_0$ を満たす.

次の仮定を追加すれば, β が多価写像という強い非線形性の仮定 (A1) の下でも (GMS) に対する強解の存在が証明できる:

(A5)’ $\boldsymbol{f} \in H^1(0,T;\boldsymbol{H})$, $\boldsymbol{f}(0) \in \boldsymbol{V}$, $\boldsymbol{u}_0 \in \boldsymbol{V} \cap (H^3(\Omega) \times H^3(\Gamma))$, $\boldsymbol{\beta}^\circ(\boldsymbol{u}_0) \in \boldsymbol{V}$.

定理 3.2.[17, Theorem 4.2]

(A1)–(A4), (A5)’ の下, ある組み $(\boldsymbol{u}, \boldsymbol{\mu}, \boldsymbol{\xi})$ が次のクラス

$$\boldsymbol{u} \in W^{1,\infty}(0,T;\boldsymbol{V}^*) \cap H^1(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{W}),$$
$$\boldsymbol{\mu} \in L^\infty(0,T;\boldsymbol{V}) \cap L^2(0,T;\boldsymbol{W}), \quad \boldsymbol{\xi} \in L^\infty(0,T;\boldsymbol{H}),$$
$$\xi \in \beta(u) \ \textit{a.e. in } Q, \quad \xi_\Gamma \in \beta_\Gamma(u_\Gamma) \ \textit{a.e. on } \Sigma$$

に存在して (11)–(15) を満たす.

弱解の存在定理の証明の本質的な着想は, 次の抽象発展方程式の初期値問題とその近似解に対する一様評価の工夫にある:

$$\boldsymbol{F}^{-1}\boldsymbol{w}'(t) + \varepsilon \partial\varphi\big(\boldsymbol{w}(t)\big) = \boldsymbol{P}\big(-\boldsymbol{\xi}(t) - \varepsilon\boldsymbol{\pi}\big(\boldsymbol{w}(t) + m_0\mathbf{1}\big) + \boldsymbol{f}(t)\big) \quad \text{in } \boldsymbol{H}_0,$$
$$\boldsymbol{\xi}(t) \in \boldsymbol{\beta}\big(\boldsymbol{w}(t) + m_0\mathbf{1}\big) \quad \text{in } \boldsymbol{H}, \quad \text{for a.a. } t \in (0,T),$$
$$\boldsymbol{w}(0) = \boldsymbol{w}_0 \quad \text{in } \boldsymbol{H}_0,$$

ここで, $\varphi : \boldsymbol{H}_0 \to [0,\infty]$ は次の適正下半連続凸関数

$$\varphi(\boldsymbol{z}) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx + \dfrac{1}{2}\displaystyle\int_\Gamma |\nabla_\Gamma z_\Gamma|^2 d\Gamma & \text{if } \boldsymbol{z} \in \boldsymbol{V}_0, \\ \infty \quad \text{if } \boldsymbol{z} \in \boldsymbol{H}_0 \setminus \boldsymbol{V}_0. \end{cases}$$

また, $\boldsymbol{P}:\boldsymbol{H}\to\boldsymbol{H}_0$ は $\boldsymbol{P}\boldsymbol{z}:=\boldsymbol{z}-m(\boldsymbol{z})\boldsymbol{1}$, ただし $\boldsymbol{1}:=(1,1)$. このとき, 楕円型正則性と Δz の評価を有効に使えば $D(\partial\varphi)=\boldsymbol{W}\cap\boldsymbol{V}_0$ でさらに, $\partial\varphi(\boldsymbol{z})=(-\Delta z,\partial_{\boldsymbol{\nu}}z-\Delta_\Gamma z_\Gamma)$ を得る (Colli–深尾 [17, Appendix] を参照). この場合, $\boldsymbol{\mu}=\varepsilon\partial\varphi(\boldsymbol{w})+\boldsymbol{\xi}+\varepsilon\boldsymbol{\pi}(\boldsymbol{w}+m_0\boldsymbol{1})-\boldsymbol{f}$ であるが, その $\boldsymbol{V}$ の一様評価を得るのに $\nabla\mu$, $\nabla_\Gamma\mu_\Gamma$ の評価に加え $m(\boldsymbol{\mu})$ の評価を工夫することで Poincaré–Wirtinger の不等式が応用できる点が本質的な着想の 1 つである.

問題 (GMS) について, 解の連続依存性により弱解や強解の一意性も得られる [17, Theorem 2.1]. また, 時間周期問題は元田 [75], 境界上の熱源 f_Γ を制御項とする際の最適制御問題は深尾–山崎 [44], 純粋層からの分離定理および長時間挙動について深尾–Wu[43], 最大正則性については梶原 [57], 境界を主領域とし内部の方程式を補助条件とする準静的問題への応用は Colli–深尾 [19] で扱われている. その他, 関連する結果として有限要素法による数値計算に関連して Cherfils–Petcu[14], Cherfils–Petcu–Pierre[15] や構造保存数値計算に強い降旗–松尾 [46] の離散変分導関数法に関連して深尾–吉川–和田 [45] の結果もある. また, 内部と境界でそれぞれ個別に保存則が成立する異なる動的境界条件下の問題として Liu–Wu[69] の新しいモデル (LW) も提唱され, Colli–深尾–Wu[20], Garcke–Knopf[50] へと研究は続いている. 最新の話題として, (GMS) と (LW) との間の漸近解析 Knopf–Lam–Liu–Metzger[61] が非常に興味深い.

4. (GMS) から非線形拡散方程式への接近

本講演の主眼としている非線形拡散方程式 (ND) を確認する: 未知関数を $u,\xi:Q\to\mathbb{R}$, $u_\Gamma,\xi_\Gamma:\Sigma\to\mathbb{R}$ とし

$$\partial_t u-\Delta\xi=g,\quad \xi\in\beta(u)\quad \text{a.e. in } Q, \tag{16}$$

$$\xi_{|\Gamma}=\xi_\Gamma,\quad \partial_t u_\Gamma+\partial_{\boldsymbol{\nu}}\xi-\Delta_\Gamma\xi_\Gamma=g_\Gamma,\quad \xi_\Gamma\in\beta(u_\Gamma)\quad \text{a.e. on } \Sigma, \tag{17}$$

$$u(0)=u_0\quad \text{a.e. in } \Omega,\quad u_\Gamma(0)=u_{\Gamma,0}\quad \text{a.e. on } \Gamma. \tag{18}$$

本節では $\boldsymbol{g}=(g,g_\Gamma)$ と $\boldsymbol{u}_0$ について, 次を仮定する:

(A6) $\boldsymbol{g}\in L^2(0,T;\boldsymbol{H}_0)$, $\boldsymbol{u}_0\in\boldsymbol{H}$;

このとき, $\boldsymbol{f}\in L^2(0,T;\boldsymbol{V}_0)$ を $\partial\varphi(\boldsymbol{f})=\boldsymbol{g}$ in $\boldsymbol{H}_0$ a.e. in $(0,T)$, $\boldsymbol{w}_{0,\varepsilon}\in\boldsymbol{V}_0$ を $\boldsymbol{w}_{0,\varepsilon}+\varepsilon\partial\varphi(\boldsymbol{w}_{0,\varepsilon})=\boldsymbol{w}_0:=\boldsymbol{u}_0-m_0\boldsymbol{1}$ in $\boldsymbol{H}_0$ を満たすように取れば, ある定数 $C_1>0$ が存在して

$$|\boldsymbol{w}_{0,\varepsilon}|^2_{\boldsymbol{H}_0}\le C_1,\quad \varepsilon|\boldsymbol{w}_{0,\varepsilon}|^2_{\boldsymbol{V}_0}\le C_1,$$
$$\int_\Omega\widehat{\beta}(w_{0,\varepsilon}+m_0)dx\le C_1,\quad \int_\Gamma\widehat{\beta}(w_{\Gamma,0,\varepsilon}+m_0)d\Gamma\le C_1\quad \text{for all } \varepsilon\in(0,1],$$
$$\boldsymbol{w}_{0,\varepsilon}\to\boldsymbol{w}_0\quad \text{in } \boldsymbol{H}_0,\quad \boldsymbol{u}_{0,\varepsilon}:=\boldsymbol{w}_{0,\varepsilon}+m_0\boldsymbol{1}\to\boldsymbol{u}_0\quad \text{in } \boldsymbol{H}\quad \text{as } \varepsilon\to 0.$$

動的境界条件下での非線形拡散方程式 (16)–(18) は, $\beta=\beta_\Gamma$ とした (11)–(15) からの $\varepsilon\to 0$ なる極限操作によって, (A1) の仮定の下で (ND) の可解性を論じることができる:

定理 4.1.[39, Theorem 2.1] [40, Theorem 2.1]

(A1), (A4), (A6) の下, ある組み $(\boldsymbol{u},\boldsymbol{\xi})$ が次のクラス

$$\boldsymbol{u}\in H^1(0,T;\boldsymbol{V}^*)\cap L^\infty(0,T;\boldsymbol{H}),$$
$$\boldsymbol{\xi}\in L^2(0,T;\boldsymbol{V}),\quad \xi\in\beta(u)\ \textit{a.e. in } Q,\quad \xi_\Gamma\in\beta(u_\Gamma)\ \textit{a.e. on } \Sigma$$

に存在して

$$\left\langle u'(t),z\right\rangle_{V^*,V}+\left\langle u'_\Gamma(t),z_\Gamma\right\rangle_{V_\Gamma^*,V_\Gamma}+\int_\Omega\nabla\xi(t)\cdot\nabla z dx+\int_\Gamma\nabla_\Gamma\xi_\Gamma(t)\cdot\nabla_\Gamma z_\Gamma d\Gamma$$
$$=\int_\Omega g(t)zdx+\int_\Gamma g_\Gamma(t)z_\Gamma d\Gamma\quad \textit{for all } \boldsymbol{z}\in\boldsymbol{V},$$

for a.a. $t\in(0,T)$ で, さらに $\boldsymbol{u}(0)=\boldsymbol{u}_0$ *in* $\boldsymbol{H}$ を満たす.

連続依存性については [39, Theorem 2.2] を参照. 注として Cahn–Hilliard 方程式系から非線形拡散方程式への接近では境界条件は本質ではなく, 例えば, Neumann 境界条件下では Colli–深尾 [18] において誤差評価

$$|u_\varepsilon - u|^2_{C([0,T];V^*)} + \int_0^T \bigl(\xi_\varepsilon(s) - \xi(s), u_\varepsilon(s) - u(s)\bigr)_H ds \le \mathrm{Const.}\varepsilon^{1/2}$$

も含めて同様の結果が, Robin 境界条件下では深尾–元田 [42] において誤差評価や Neumann 境界条件下での問題への収束の結果が得られている.

Cahn–Hilliard 方程式系から非線形拡散方程式への接近では非線形性を記述する β に関する増大条件の緩和について利点がある. 実際, 従来の $H^1(\Omega)$ の共役空間における発展方程式の抽象理論からの接近では $\widehat{\beta}$ の積分をポテンシャルとし, その $H^1(\Omega)$ の共役空間での劣微分による特徴付けを行うが, その際に下半連続性を得るために β の増大条件は欠かせない仮定であった. Neumann 境界条件下では, 剣持 [59] の他, 多孔質媒体方程式の場合に赤木 [4], log 型の場合に久保–Lu[64] など個別の問題ごとに可解性が考察されていた. その後, Robin 境界条件下を例に Damlamian–剣持 [22] で "下半連続拡張" の着想が示され, 増大条件の緩和が可能となった. Cahn–Hilliard 方程式系から退化放物型方程式への接近では, 非線形な拡散項は Cahn–Hilliard 方程式系における摂動項でしかないため "下半連続拡張" を用いる必要が無い点に利点がある. Damlamian[21] から始まった $H^1(\Omega)$ の共役空間における発展方程式の抽象理論による接近で長年課題であった増大条件の緩和について [22] に継ぐ 1 つの解決策を与えたことになる (L^1 の枠組みは Barbu[10] を参照せよ). 領域が非有界な場合にこの利点を利用した深尾–来間–横田 [41] の結果もある.

5. Fast diffusion 方程式と有限時刻消滅

本節では $0 < m < 1$ とし, $\beta(r) = \beta_\Gamma(r) := |r|^{m-1}r$ の場合を考える. $\gamma := \beta^{-1}$, $v := u^m$, $v_\Gamma := u_\Gamma^m$ なる変数変換によって (5)–(6) は次の形に変形できる:

$$\begin{aligned}
\partial_t\gamma(v) - \Delta v + av = \lambda\gamma(v)^p &\quad \text{a.e. in } \Omega, \quad t>0,\\
v_{|_\Gamma} = v_\Gamma &\quad \text{a.e. on } \Gamma, \quad t>0\\
\partial_t\gamma(v_\Gamma) + \partial_{\boldsymbol{\nu}} v - \Delta_\Gamma v_\Gamma + bv_\Gamma = \mu\gamma(v_\Gamma)^q &\quad \text{a.e. on } \Gamma, \quad t>0\\
v(0) = v_0 \quad \text{a.e. in } \Omega, \quad v_\Gamma(0) = v_{\Gamma,0} &\quad \text{a.e. on } \Gamma,
\end{aligned}$$

ここで, $v_0 : \Omega \to [0,\infty)$ と $v_{\Gamma,0} : \Gamma \to [0,\infty)$ はそれぞれ与えられた初期値で, $v_0 := u_0^m$, $v_{\Gamma,0} := u_{\Gamma,0}^m$ である. また, $p, q \ge 1$, $(a,b), (\lambda,\mu) \in \{(1,0),(0,1)\}$ とする.

有限時刻消滅を議論するにあたり, 数多くの先行研究 (例えば, [25, 26, 58, 68, 71, 77, 81, 83]) に従い, 安定集合 $\mathcal{W}$ とエネルギー E を以下のように定義する: 以後, $\alpha := 1/m$, $p_* := \lambda p + \mu q$ とおく. すなわち, $(\lambda,\mu) = (1,0)$ のときは $p_* = p$, $(\lambda,\mu) = (0,1)$ のとき $p_* = q$ である.

$$\begin{gathered}
\mathcal{W} := \bigl\{\boldsymbol{z} \in \boldsymbol{V} \setminus \{\boldsymbol{0}\} \ : \ z \ge 0, z_\Gamma \ge 0, E(\boldsymbol{z}) < d, 2\varphi_1(\boldsymbol{z}) > (\alpha p_* + 1)\varphi_2(\boldsymbol{z})\bigr\} \cup \{\boldsymbol{0}\},\\
E(\boldsymbol{z}) := \varphi_1(\boldsymbol{z}) - \varphi_2(\boldsymbol{z}),\\
\varphi_1(\boldsymbol{z}) := \frac{1}{2}\int_\Omega |\nabla z|^2 dx + \frac{a}{2}\int_\Omega |z|^2 dx + \frac{1}{2}\int_\Gamma |\nabla_\Gamma z_\Gamma|^2 d\Gamma + \frac{b}{2}\int_\Gamma |z_\Gamma|^2 d\Gamma,\\
\varphi_2(\boldsymbol{z}) := \frac{1}{\alpha p_* + 1}\left(\lambda\int_\Omega |z|^{\alpha p+1} dx + \mu\int_\Gamma |z_\Gamma|^{\alpha q+1} d\Gamma\right).
\end{gathered}$$

安定集合 $\mathcal{W}$ に登場する定数 d は *depth of the potential well*(例えば, Lions[68], 中尾 [71], Sattinger[81], 堤 [83] を参照) と呼ばれ以下で定義できる:

$$d := \inf\bigl\{E(\boldsymbol{z}) \ : \ \boldsymbol{z} \in \boldsymbol{V} \setminus \{\boldsymbol{0}\}, 2\varphi_1(\boldsymbol{z}) = (\alpha p_* + 1)\varphi_2(\boldsymbol{z})\bigr\}.$$

これらは各種具体的な問題に対して, 例えば [25, 26, 68, 71, 81, 83] で取り扱われている. また, 主要-摂動項が $\partial\varphi_1 - \partial\varphi_2$ のように劣微分の差で表現されている抽象発展方程式 (例えば [6, 62, 76]) に対して, その抽象的構造が石井 [56] や大谷 [77] で明らかになった.

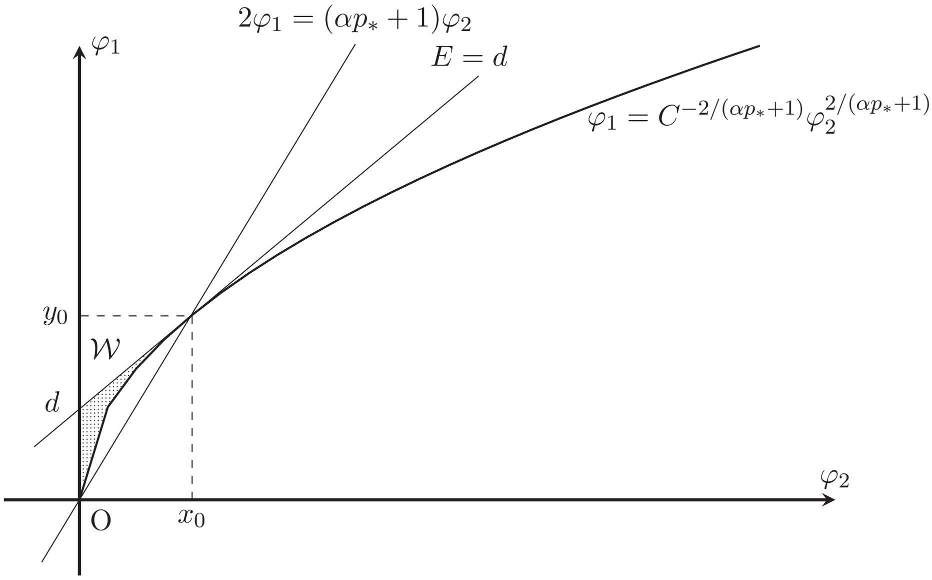

Figure 1: 安定集合 $\mathcal{W}$.

先行研究 Fila–Filo[26] の着想に従い以下を得ることができる: 問題 (7) や (8) に対しては, $q \geq 1$ に対して有限時刻消滅が成立する.

定理 5.1.

$(a, b) \in \{(1,0),(0,1)\}$, $(\lambda, \mu) = (0,1)$, $1/5 < m < 1$ とし, $\boldsymbol{v}_0 := (v_0, v_{\Gamma,0}) \in \mathcal{W} \cap \boldsymbol{L}^\infty$ を仮定する. このとき, $\boldsymbol{v}_0$ に依存する $T_{\mathrm{ext}} \in (0,\infty)$ が存在し, $v(t) \equiv 0$, $v_\Gamma(t) \equiv 0$ が $t \geq T_{\mathrm{ext}}$ に対して成立する. さらに, m に依存する正定数 $C(m) > 0$ が存在して

$$\big|v(t)\big|_{L^{(1+m)/m}(\Omega)} + \big|v_\Gamma(t)\big|_{L^{(1+m)/m}(\Gamma)} \leq C(m)(T_{\mathrm{ext}} - t)^{m/(1-m)} \tag{19}$$

がすべての $t \in [0, T_{\mathrm{ext}}]$ に対して成立する.

系

$v_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ で $v_0 \geq 0$, そして $0 < |v_0|_V^2 < 2d$ を仮定する (つまり, $\boldsymbol{v}_0 := (v_0, 0)$). このときも, 有限時刻消滅が成立する.

問題 (9) や (10) に対しては, $p \geq 1$ に対する付加条件の下で有限時刻消滅が成立する.

定理 5.2.

$(a, b) \in \{(1,0),(0,1)\}$, $(\lambda, \mu) = (1,0)$, $1/5 < m < 1$, $1 \leq p < 5m$ とし, $\boldsymbol{v}_0 := (v_0, v_{\Gamma,0}) \in \mathcal{W} \cap \boldsymbol{L}^\infty$ を仮定する. このとき, $\boldsymbol{v}_0$ に依存する $T_{\mathrm{ext}} \in (0,\infty)$ が存在し, $v(t) \equiv 0$, $v_\Gamma(t) \equiv 0$ が $t \geq T_{\mathrm{ext}}$ に対して成立する. さらに (19) が成立する.

形式的には $m = 1$ の場合の数多くの先行研究の着想に従うが, fast diffusion 方程式 ($0 < m < 1$) では, 時間微分の取り扱いが難しさの 1 つとなる. 実際, 弱解は任意の $T \in [0,\infty)$ に対して $\boldsymbol{v}^{(\alpha+1)/2} = \boldsymbol{v}^{(1+m)/2m} \in H^1(0,T;\boldsymbol{H})$, ならびに

$$\frac{d}{dt} I\big(\boldsymbol{v}(t)\big) + 2\varphi_1\big(\boldsymbol{v}(t)\big) = \lambda \int_\Omega v^{\alpha p+1}(t)dx + \mu \int_\Gamma v_\Gamma^{\alpha q+1}(t)d\Gamma \quad \text{for a.a. } t \in (0,\infty)$$

や

$$\frac{4\alpha}{(\alpha+1)^2}\int_s^t \big|\partial_t \boldsymbol{v}^{(\alpha+1)/2}(\tau)\big|_{\boldsymbol{H}}^2 d\tau + E\big(\boldsymbol{v}(t)\big) \leq E\big(\boldsymbol{v}(s)\big) \quad \text{for all } s, t \in [0,\infty)$$

が得られる一方で, $\boldsymbol{v}'$ には十分な正則性が期待できない. ここで,

$$
\begin{aligned}
I(\boldsymbol{z}) &:= \frac{1}{1+m}\int_\Omega z^{(1+m)/m}dx + \frac{1}{1+m}\int_\Gamma z_\Gamma^{(1+m)/m}d\Gamma \\
&= \frac{\alpha}{\alpha+1}\int_\Omega z^{\alpha+1}dx + \frac{\alpha}{\alpha+1}\int_\Gamma z_\Gamma^{\alpha+1}d\Gamma.
\end{aligned}
$$

ここでは, まず Filo [35] の着想に従い基礎的な代数不等式から $\gamma(\boldsymbol{v}) \in H^1(0,T;\boldsymbol{H})$ なる時間微分の正則性を獲得する. 弱解としての局所可解性は定理 2.1 と同様に Δ_Γ から楕円型正則性の靴ひも理論によって得られる正則性も重要である. また, Fila–Filo[25] にならい $\mathcal{W}$ の不変性を得て, 先行研究の手法を踏襲することが可能となる. L^∞-評価に関しては Alikakos[7]($m=1$), 中尾 [71]($m>1$) の手法を拡張した Fila–Filo[26] の結果 ($0<m<1$) が応用できる.

なお, Dirichret 境界条件下での fast diffusion 方程式に対しては, 時間微分が H^{-1} で得られるという枠組み内で, $u_0 \in H_0^1(\Omega) \cap L^{\alpha+1}(\Omega)$ に対して有限時刻消滅の精密な結果が赤木–梶木屋 [5] で得られている.

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Solvability of doubly nonlinear parabolic equations with p-Laplacian

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1. Introduction

We consider the following initial boundary value problem of doubly nonlinear equation:

$$
\text{(P)} \qquad \begin{cases} \partial_t \beta(u(x,t)) - \Delta_p u(x,t) \ni f(x,t) & (x,t) \in Q := \Omega \times (0,T), \\ u(x,t) = 0 & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}
$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$ and $f = f(x,t)$ is a given external force. Moreover, $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is a (multi-valued) maximal monotone graph on $\mathbb{R}$ satisfying $0 \in \beta(0)$ and the exponent in p-Laplacian $\Delta_p u := \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ is $p \in (1, \infty)$.

The case where $\beta(s) = |s|^{r-2}s$ $(r > 0)$ is one of the well-investigated examples of the above formulation (see, e.g., Raviart [14], Bamberger [2], and Tsutsumi [16]). When $p = 2$ and $\beta(s) = e^s - 1$ in (P), the equation is equivalent to $\partial_t v = \Delta \log v$, considered in, e.g., Berryman–Holland [8] and Esteban–Rodríguez–Vázquez [10]. Recently, Miyoshi–Tsutsumi [12] derived the following equation from the singular limit of a generalized Carleman model (2-speed model of rarefied gas):

$$
\partial_t v - \nabla \cdot (|\nabla \log v|^{p-2} \nabla \log v) = f.
$$

In order to follow them and more general cases, we aim to show the existence of solution to (P) with less assumptions of β.

As for previous studies of solvability, Alt–Luckhaus [1], Otto [13], Benilan-Wittbold [5], and Carrillo–Wittbold [9] deal with the case where β is single-valued based on L^1-contraction principle. Grange–Mignot [11] is concerned with an abstract doubly nonlinear evolution equation in Banach space and assures the solvability under the boundedness condition for the realization of β. Barbu [3] avoids assumptions for growth order or coerciveness by using angular condition and obtains the existence of solution to (P) with $p \geq 2$ in L^2-framework for general β.

This work was supported by JSPS KAKENHI Grant Number #18KK0073.
2010 Mathematics Subject Classification: 35K92, 34G25, 35K55.
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Main purpose of this talk is to construct a solution to (P) for every $p \in (1, \infty)$ without any assumptions of β except $0 \in \beta(0)$. Especially, we do not impose growth condition, coerciveness, single-valuedness, or $D(\beta) = \mathbb{R}$ on β.

2. Preliminary

We here define some notations and review basic properties in nonlinear evolution equation theory (see e.g., [4], [7], and [15]).

Let V be a reflexive Banach space and V^* its dual space. A multi-valued operator $A : V \to 2^{V^*}$ is said to be monotone if $\langle \eta_1 - \eta_2, u_1 - u_2 \rangle_V \geq 0$ for every $u_i \in D(A)$ and $\eta_i \in Au_i$ $(i = 1, 2)$. A monotone operator $A : V \to 2^{V^*}$ is said to be maximal monotone if there is no monotonic extension of A. Moreover, A is said to be bounded if $\bigcup_{u \in B} Au$ is bounded in V^* for any bounded subset $B \subset V$ and coercive if $\lim_{n\to\infty} \frac{\langle \eta_n, u_n \rangle_V}{\|u_n\|_V} = +\infty$ holds for any $u_n \in D(A)$ and $\eta_n \in Au_n$ such that $\|u_n\|_V \to \infty$ as $n \to \infty$. A typical example of maximal monotone operator is subdifferential. Let $\varphi : V \to (-\infty, +\infty]$ be a proper ($\varphi \not\equiv +\infty$) lower semi-continuous convex functional. Then we define

$$\partial_V \varphi(u) := \{\eta \in V^*; \langle \eta, v - u \rangle_V \leq \varphi(v) - \varphi(u) \quad \forall v \in V\}.$$

Obviously $0 \in \partial\varphi(u)$ implies that φ attains its minimum at u.

For a proper lower semi-continuous convex function φ and $\lambda > 0$,

$$\varphi_\lambda(u) := \inf_{v \in V} \left\{ \frac{\|u - v\|_V^2}{2\lambda} + \varphi(v) \right\}$$

is called the Moreau–Yosida regularization of φ. If V and V^* are strictly convex, $\partial_V \varphi_\lambda$ coincides with the Yosida approximation of $\partial_V \varphi$ with parameter $\lambda > 0$. Moreover, the Legendre–Fenchel transformation (conjugate) of φ is defined by

$$\varphi^*(\eta) := \sup_{v \in V} \left\{ \langle \eta, v \rangle_V - \varphi(v) \right\}.$$

Remark that $\varphi^* : V^* \to (-\infty, +\infty]$ is proper lower semi-continuous convex and $\partial_{V^*} \varphi^* = (\partial_V \varphi)^{-1}$.

When $V = V^* = \mathbb{R}$, the maximal monotone operator always possesses a primitive function. That is to say, if $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone graph, then there is a proper lower semi-continuous convex function $j : \mathbb{R} \to (-\infty, +\infty]$ such that $\beta = \partial j$. If $0 \in \beta(0)$ holds, we can assume $j(0) = 0$ and $j \geq 0$ without loss of generality. Let $V = L^p(\Omega)$ and $V^* = L^{p'}(\Omega)$, where $p \in (1, \infty)$ and p' is the Hölder conjugate of p. Then

$$\psi(u) := \begin{cases} \displaystyle\int_\Omega j(u(x))dx & \text{if } u \in L^p(\Omega), \quad j(u) \in L^1(\Omega), \\ +\infty & \text{otherwise}, \end{cases}$$

is proper lower semi-continuous convex on V and $\xi \in \partial_V \psi(u)$ if and only if $\xi \in L^{p'}(\Omega)$ and $\xi(x) \in \beta(u(x))$ for a.e. $x \in \Omega$. In this sense, we write $\tilde{\beta} := \partial_V \psi$ and call it the realization of β in $L^p(\Omega)$. By the canonical restriction, we can regard ψ as a proper lower semi-continuous functional on $X = W_0^{1,p}(\Omega)$ and then we can consider its subdifferential on X, written by $\partial_X \psi$. However, $\xi \in \partial_X \psi(u)$ dose not imply $\xi(x) \in \beta(u(x))$ since ξ belongs to $X^* = W^{-1,p'}(\Omega)$, the dual of $W_0^{1,p}(\Omega)$ (except the case of $D(\beta) = \mathbb{R}$, see Brézis [6]).

Define the resolvent of β by $J_\lambda := (Id + \lambda\beta)^{-1}$ and the Yosida approximation by $\beta_\lambda = (Id - J_\lambda)/\lambda$. It is easy to see that these are Lipschitz continuous function on $\mathbb{R}$. Although the fact immediately yields the Lipschitz continuity of the realization of β_λ in $L^p(\Omega)$ if $p \geq 2$, this dose not hold if $p < 2$. We also remark that the Moreau–Yosida regularization of ψ given above dose not coincide with $u \mapsto \int_\Omega j_\lambda(u(x))dx$ except the case of $p = 2$, where $j_\lambda(s) := \inf_{\sigma\in\mathbb{R}} \left\{ \frac{|s-\sigma|^2}{2\lambda} + j(\sigma) \right\}$. This means the mismatch between the Yosida approximation of $\partial_V \psi$ and $\tilde{\beta_\lambda}$, the realization of β_λ.

3. Main Result

In this talk, we show the existence of the following solution:

Definition 1. Let (u_0, ξ_0) satisfy $\xi_0(x) \in \beta(u_0(x))$ for a.e. $x \in \Omega$. A pair (u, ξ) is said to be a solution to (P) with the initial data (u_0, ξ_0) if $u \in L^\infty(0, T; W_0^{1,p}(\Omega))$ and $\xi \in W^{1,\infty}(0, T; W^{-1,p'}(\Omega)) \cap L^\infty(0, T; L^{p'}(\Omega))$ satisfies $\xi(x, t) \in \beta(u(x, t))$ for a.e. $(x, t) \in Q$ and the equation of (P) in the following sense:

$$\begin{aligned} &\partial_t \xi(t) - \Delta_p u(t) = f(t) \quad \text{in} \quad W^{-1,p'}(\Omega) \quad \text{for a.e. } t \in (0, T), \\ &\xi(\cdot, 0) = \xi_0. \end{aligned}$$

Remark 1. Let $\beta \equiv 0$ and $f \equiv 0$. Then energy equality (1) below implies that for any given $u_0 \in W_0^{1,p}(\Omega)$, (P) has a unique solution

$$u(x,t) = \begin{cases} u_0(x) & t = 0, \\ 0 & t > 0. \end{cases}$$

Hence we deal with cases where u is not continuous and the initial condition for u may be satisfied in just the sense of representative.

Theorem 3.1. Let $p \in (1, \infty)$, $q \in [p', \infty]$, and $0 \in \beta(0)$. Then for any $u_0 \in W_0^{1,p}(\Omega)$, $\xi_0 \in L^{p'}(\Omega) \cap L^q(\Omega)$, and $f \in W^{1,p'}(0, T; L^{p'}(\Omega)) \cap L^\infty(0, T; L^q(\Omega))$, there exist at least one solution to (P) with the initial data (u_0, ξ_0) satisfying

$$\sup_{0\le t\le T} \|\xi(t)\|_{L^{p'}} \le T \sup_{0\le t\le T} \|f(t)\|_{L^{p'}} + \|\xi_0\|_{L^{p'}},$$
$$\sup_{0\le t\le T} \|\xi(t)\|_{L^{q}} \le T \sup_{0\le t\le T} \|f(t)\|_{L^{q}} + \|\xi_0\|_{L^{q}}.$$

Furthermore, (u, ξ) fulfills for every $t_1, t_2 \in [0, T]$

$$(1)\quad \int_\Omega j^*(\xi(x,t_2))dx - \int_\Omega j^*(\xi(x,t_1))dx + \int_{t_1}^{t_2} \|\nabla u(t)\|_{L^p}^p dt = \int_{t_1}^{t_2}\int_\Omega f(x,t)u(x,t)dxdt,$$

where j is the primitive function of β and j^* its Fenchel–Legendre transformation.

With additional conditions of initial data, we can obtain a solution which is Lipschitz continuous with respect to t:

Theorem 3.2. In addition to assumptions in Theorem 3.1, let $\Delta_p u_0 \in L^{p'}(\Omega) \cap L^q(\Omega)$. Then there exist at least one solution to (P) with the initial data (u_0, ξ_0) satisfying

$$\sup_{0\le t\le T} \|\xi(t)\|_{L^{p'}} \le T \sup_{0\le t\le T} \|f(t)\|_{L^{p'}} + \|\xi_0 - \Delta_p u_0\|_{L^{p'}},$$
$$\sup_{0\le t\le T} \|\xi(t)\|_{L^{q}} \le T \sup_{0\le t\le T} \|f(t)\|_{L^{q}} + \|\xi_0 - \Delta_p u_0\|_{L^{q}},$$
$$\|\xi(t_1) - \xi(t_2)\|_{L^1} \le C|t_1 - t_2| \quad \forall t_1, t_2 \in [0, T].$$

Our proof relies on the standard time discretization technique by Raviart [14] and Grange–Mignot [11]. Let $N \in \mathbb{N}$ and define $\tau := T/N$. Then we determine $u_\tau = \{u_\tau^0, u_\tau^1, \dots, u_\tau^N\}$ and $\xi_\tau = \{\xi_\tau^0, \xi_\tau^1, \dots, \xi_\tau^N\}$ satisfying $u_\tau^n \in W_0^{1,p}(\Omega)$, $-\Delta_p u_\tau^n, \xi_\tau^n \in L^{p'}(\Omega) \cap L^q(\Omega)$ for $n = 1, 2, \dots, N$ by solving the following equation inductively:

$$(2)\qquad \begin{cases} \dfrac{\xi_\tau^{n+1}(x) - \xi_\tau^n(x)}{\tau} - \Delta_p u_\tau^{n+1}(x) = f_\tau^n(x) & x \in \Omega, \\ \xi_\tau^{n+1}(x) \in \beta(u_\tau^{n+1}(x)) & x \in \Omega, \\ u_\tau^{n+1}(x) = 0 & x \in \partial\Omega, \end{cases}$$

where $u_\tau^0 := u_0$, $\xi_\tau^0 := \xi_0$, and $f_\tau^n := \frac{1}{\tau}\int_{n\tau}^{(n+1)\tau} f(\cdot, s)ds$. We next connect $u_\tau = \{u_\tau^0, u_\tau^1, \dots, u_\tau^N\}$ and $\xi_\tau = \{\xi_\tau^0, \xi_\tau^1, \dots, \xi_\tau^N\}$ between $[0, T]$ by horizontal lines and a polygonal line, respectively:

$$\Pi_\tau u_\tau(t) := \begin{cases} u_\tau^{n+1} & \text{if } t \in (n\tau, (n+1)\tau], \\ u_\tau^0 & \text{if } t = 0, \end{cases}$$
$$\Lambda_\tau \xi_\tau(t) := \frac{\xi_\tau^{n+1} - \xi_\tau^n}{\tau}(t - n\tau) + \xi_\tau^n \quad \text{if } t \in [n\tau, (n+1)\tau].$$

By establishing a priori estimates independent of N or τ from (2) and discussing the limits as $\tau \to 0$ with some properties of maximal monotone operator and convex analysis, we observe the convergence $\Pi_\tau u_\tau$ and $\Lambda_\tau \xi_\tau$ to a desired solution to (P).

In order to complete this argument, we have to assure the solvability of (2), namely, begin with the following elliptic problem:

$$\begin{cases} \xi(x) - \Delta_p u(x) = h(x) & x \in \Omega, \\ \xi(x) \in \beta(u(x)) & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega. \end{cases} \tag{3}$$

Theorem 3.3. Let $p \in (1, \infty)$, $q \in [p', \infty]$, and $0 \in \beta(0)$. Then for every $h \in L^{p'}(\Omega) \cap L^q(\Omega)$, (3) possesses a unique solution $u \in W_0^{1,p}(\Omega)$ such that $\xi, \Delta_p u \in L^{p'}(\Omega) \cap L^q(\Omega)$ and

$$\|\xi\|_{L^{p'}} \leq \|h\|_{L^{p'}}, \quad \|\xi\|_{L^q} \leq \|h\|_{L^q}. \tag{4}$$

Formally, we can interpret (4) as the result of integration by parts:

$$\int_\Omega |\beta(u)|^{q-2}\beta(u)\Delta_p u dx = -(q-1)\int_\Omega |\beta(u)|^{q-2}\beta'(u)|\nabla u|^p dx \leq 0,$$

where $\beta' \geq 0$ since β is monotone.

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Existence of traveling wave solutions to a nonlocal scalar equation with sign-changing kernel *

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1 はじめに

近年，積分核との合成積で表される非局所項を持つ時間発展方程式が，神経科学 [2] や物質科学 [4]，生物学 [10, 12, 13] などの分野における数理モデルとして提案されている．その一例として，次の単独の時間発展方程式が挙げられる：

$$u_t = du_{xx} + K * u + f(u),\ t > 0,\ x \in \mathbb{R}. \tag{1.1}$$

ここで $u = u(t,x)$ は時刻 t，位置 x における何らかの物質の濃度や密度，状態などを表すものとし，$d \geq 0$，$K \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ は偶関数であり，

$$(K * u)(t,x) := \int_{\mathbb{R}} K(y)u(t,x-y)dy$$

と定め，$f : \mathbb{R} \to \mathbb{R}$ は非線形関数とする．du_{xx} は物質の拡散の効果，非局所項 $K * u$ は物質に生じる非局所効果，$f(u)$ は物質の反応の効果を表している．K の典型例としては，神経科学における脳の発火現象の数理モデルで用いられる，局所活性・非局所抑制型の積分核である [2]（図 1）．

方程式 (1.1) は非局所拡散方程式 [3] や離散モデルを連続化した方程式 [11]，生物の表皮パターンに関する数理モデル [10, 13]，非局所的な内部エネルギーを考慮した相分離現象の数理モデル [4]，生物の分散現象の数理モデル [12] などとして現れる．方程式 (1.1) の数学解析について，積分核が非負のときには非自明の定常解の存在，一定速度で形状を変えずに進行する進行波解の存在，またそれらの安定性などの局在パターンに関する多くの結果が報告されている [5, 7, 8, 16, 17, 18]．非負の積分核を持つ方程式の多くは，比較原理などの反応拡散方程式の数学解析で用いられた道具が適用できることもあり，反応拡散方程式の拡張としての解析結果が得られている．

一方，積分核が図 1 のように符号変化する場合には，方程式 (1.1) などの単独の非局所項を持つ時間発展方程式は複雑な挙動の解を有することが，数値シミュレーションによって調べられている [10, 13, 15]．符号変化する積分核を与えた場合の方程式 (1.1) の数値シミュレーションでは，非単調

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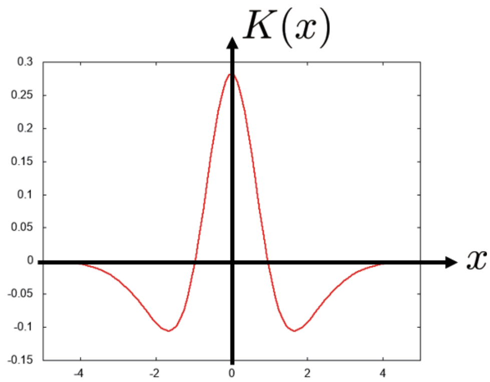

図 1　局所活性・非局所抑制型の積分核のグラフ

な進行波解やチューリング不安定性により空間周期的な解が現れることが報告されている [15]．また非局所項を持つ単独の時間発展方程式に対する 2 次元平面上の数値シミュレーションでは，符号変化する適切な球対称の積分核を与えることで，チューリング不安定性により動物や魚の表皮に現れるストライプやスポット，迷路パターンなどを再現できることが報告されている [13]．これらの結果より，符号変化する積分核の場合には単独の方程式であっても，非単調な進行波解やチューリング不安定性などの多成分の反応拡散方程式と同等な解の性質が内在されていることが予想される．このようなことから符号変化する積分核の場合には，定常解や進行波解の解析は特別な場合のみが扱われているだけであった [6]．

本研究では符号変化する積分核を持つ時間発展方程式における空間パターンの解析を動機として，単安定な非線形項を与えた場合の進行波解の存在について考える．以降は本研究の問題設定と先行研究，および得られた主結果とその証明の方針を説明する．

2　問題設定

本研究では以下の非局所項を持つ半線形スカラー方程式について考える．

$$u_t = du_{xx} + K * u - \alpha u + f(u),\ t > 0,\ x \in \mathbb{R}. \tag{2.1}$$

ここで本稿を通して $d \geq 0$，積分核 K は

$$K(x) = K(-x)\ (x \in \mathbb{R}), \quad C(\mathbb{R}) \cap L^1(\mathbb{R}) \ni K \not\equiv 0, \quad \alpha := \int_{\mathbb{R}} K(x)dx \geq 0 \tag{K1}$$

をみたし，かつ

$$\forall \lambda > 0, \quad \int_{\mathbb{R}} |K(x)|e^{-\lambda x}dx < \infty. \tag{K2}$$

をみたすものとする．また，非線形項 f は

$$\begin{cases} f \in \mathrm{Lip}_{loc}(\mathbb{R}),\ f(0) = f(1) = 0, \\ f > 0 \text{ in } (0,1),\ f < 0 \text{ in } (1,\infty),\ f'(0) > 0 \end{cases} \tag{N1}$$

をみたし，かつある定数 $u^+ \geq 1$ が存在して

$$\forall u \in [0,\ u^+], \quad f(u) \leq f'(0)u \tag{N2}$$

をみたすものとする．

上記の仮定 (K1)，(K2) をみたす積分核 K の具体例は

$$K(x) = \frac{1}{2\sqrt{\pi}}\left(e^{-x^2} - \frac{1}{2}e^{-x^2/4}\right) \tag{2.2}$$

であり，仮定 (N1), (N2) をみたす非線形項 f の具体例は

$$f(u) = \begin{cases} 2-2u, & u \in \left(\frac{1}{2}, +\infty\right), \\ 2u, & u \in \left(-\infty, \frac{1}{2}\right] \end{cases} \tag{2.3}$$

である．(2.3) は任意の $u^+ \geq 1$ に対して (N2) をみたすことに注意する．

本研究では，局在パターンの 1 つである進行波解について考える．方程式 (2.1) の進行波解とは，十分なめらかな関数 ϕ と進行速度 c に対して，$u(t,x) = \phi(x+ct)$ と表される解のことを呼ぶ．また，動座標系 $\xi := x + ct$ を導入すると，進行波解が存在することは，

$$c\phi' = d\phi'' + K * \phi - \alpha\phi + f(\phi),\ \xi \in \mathbb{R} \tag{TW1}$$

をみたす関数 ϕ と進行速度 c を求める問題に帰着される．さらに本研究では，不安定な平衡点 $u=0$ と常微分の意味で安定な平衡点 $u=1$ を繋ぐような進行波解を考える：

$$\phi(-\infty) = 0, \quad \phi(+\infty) = 1. \tag{TW2}$$

2.1 先行研究

これまで方程式 (2.1) は，反応拡散方程式の局所拡散を非局所拡散 [3] に置き換えた方程式として，積分核が非負かつ $d=0$ の場合に，(TW1)，(TW2) をみたす進行波解の存在が調べられてきた [8, 16, 17]．以下の仮定をみたす積分核 K と非線形項 f に対して，次の進行波解の存在に関する結果が知られている．

仮定 2.1　積分核 K は (K1)，(K2) をみたす非負関数とする．非線形項 f は (N1) をみたし，さらに

$$\begin{cases} f \in C^1(\mathbb{R}),\ f'(0) > 0,\ f'(1) < 0, \\ \exists M > 0,\ \exists p \in (0,1]\ s.t.\ -Mu^{1+p} \leq f(u) - f'(0)u \leq 0\ (u \in (0,1)) \end{cases} \tag{2.4}$$

をみたす．

定理 2.2　([16])

$d=0$ として，非線形項 f と積分核 K は仮定 2.1 をみたすとする．このとき，

$$c^* = \inf_{\lambda \in (0,+\infty)} \frac{Q_0(\lambda)}{\lambda}, \quad Q_0(\lambda) = \int_{\mathbb{R}} K(y)e^{-\lambda y}dy - \alpha + f'(0) \tag{2.5}$$

が存在して，$c \geq c^*$ のときには (TW1)，(TW2) をみたす単調増加な関数 ϕ が存在する．$c < c^*$ のときには，そのような関数 ϕ は存在しない．

この他に，$K \geq 0$ かつ $d > 0$ であっても，方程式 (2.1) の進行波解の存在が知られている [8]．しかし積分核が符号変化するときには，方程式 (2.1) の進行波解の存在は示されていなかった．

3　主結果

以降，積分核 K は負の部分を持つものとする．すなわち，

$$K^{\pm}(x) := \max\{\pm K(x), 0\}$$

と定めたとき，$K^{-}(x) \not\equiv 0$ とする．また，$K(x) = K^{+}(x) - K^{-}(x)$，$|K(x)| = K^{+}(x) + K^{-}(x)$ をみたすことに注意する．

主結果を述べる前に，いくつかの量と仮定を用意する．まず次の 2 つの量を定義する．

$$c_Q := \inf_{\lambda \in (0,\hat{\lambda})} \frac{Q(\lambda)}{\lambda}, \quad Q(\lambda) := d\lambda^2 + \int_{\mathbb{R}} K(y) e^{-\lambda y} dy - \alpha + f'(0), \tag{3.1}$$

$$c_R := \inf_{\lambda \in (0,\infty)} \frac{R(\lambda)}{\lambda}, \quad R(\lambda) := d\lambda^2 + \int_{\mathbb{R}} |K(y)| e^{-\lambda y} dy - \alpha + f'(0). \tag{3.2}$$

ここで $\hat{\lambda}$ は，$Q(\lambda) = 0$ となる $\lambda > 0$ が存在すれば，$f'(0) > 0$ なので $\hat{\lambda} := \min\{\lambda > 0 \mid Q(\lambda) = 0\}$ として $c_Q = 0$，もし存在しなければ $\hat{\lambda} = +\infty$ とする．このとき $R(\lambda)$ が $\lambda > 0$ に対して狭義凸関数であることから $c_R > 0$ は定義でき，$c_R > c_Q$ をみたす．

注意 3.1　もし K が十分小さい負の部分を持つ場合でも，$c_Q = 0$ となることがある．例えば，

$$K(x) = (1+\varepsilon)J(x) - \frac{\varepsilon}{2}(J(x-1) + J(x+1))$$

とおく．ただし $\varepsilon > 0$ であり，J は $\left[-\frac{1}{2}, \frac{1}{2}\right]$ 上に台を持つ積分量が 1 の非負の連続な偶関数とする．このとき $\alpha = 1$ であり，簡単のため $f'(0) = 1$, $d = 0$ とおくと，

$$Q(\lambda) = \int_{\mathbb{R}} K(y) e^{-\lambda y} dy = \{1 + \varepsilon - \varepsilon \cosh(\lambda)\} \int_{\mathbb{R}} J(y) e^{-\lambda y} dy$$

である．よって，$\int_{\mathbb{R}} J(y) e^{-\lambda y} dy$ は任意の $\lambda > 0$ に対して正なので，$\varepsilon > 0$ のときには $\hat{\lambda} = \cosh^{-1}\left(\frac{1+\varepsilon}{\varepsilon}\right) \in (0,\infty)$．つまり $c_Q = 0$ となる．

また，非線形項と積分核の積分量に対して，次の仮定を導入する．

仮定 3.2　f は (N1) と (N2) をみたすとする．またある $\eta \in (0,1)$ が存在して，すべての $u \in [0,\eta]$ に対して

$$f(u) = f'(0)u \tag{3.3}$$

をみたし，f は $f'(0) > \alpha$ をみたすとする．さらに $\delta \in (1, u^{+})$ に対して，K^{-} は次をみたす．

$$\int_{\mathbb{R}} K^{-}(y) dy \le \min\left\{\frac{-f(\delta)}{\delta}, \frac{(f'(0) - \alpha)\eta}{\delta}\right\}. \tag{3.4}$$

例 3.3　f を (2.3) として定める．このとき $\eta = \frac{1}{2}$，$f'(0) = 2$ であり，任意の $u^{+} \ge 1$ に対して，f は (N2) をみたす．また $\alpha = 0$ とすると，

$$\frac{-f(\delta)}{\delta} = \frac{2\delta - 2}{\delta}, \quad \frac{(f'(0) - \alpha)\eta}{\delta} = \frac{1}{\delta}$$

である．このとき $\delta = \dfrac{3}{2}$ として与えると，

$$\int_{\mathbb{R}} K^+(y)dy = \int_{\mathbb{R}} K^-(y)dy \leq \frac{2}{3}$$

をみたす積分核であれば仮定 3.2 をみたす．したがって，積分核 K と非線形項 f として (2.2) と (2.3) を与えれば，仮定 3.2 をみたす．

また，次の関数空間を用意する．

$$C_b^k(\mathbb{R}) := \{g \in C^k(\mathbb{R}) \mid ||g^{(j)}||_\infty < \infty,\ j = 0, 1, \ldots, k\}, \quad \|g\|_\infty := \sup_{\xi \in \mathbb{R}} |g(\xi)|.$$

特に，$C_b(\mathbb{R}) := C_b^0(\mathbb{R})$ とする．このとき，次の結果が得られた．

定理 3.4 ([9])
非線形項 f と積分核 K は仮定 3.2 をみたすとする．このとき，すべての $c > c_R$ に対して (TW1)，$\phi(-\infty) = 0$ かつ

$$\liminf_{\xi \to +\infty} \phi(\xi) > 0 \tag{3.5}$$

をみたす正値関数 $\phi \in C_b(\mathbb{R})$ が存在する．特に $c > \max\{c_R, c_K\}$ のとき，関数 ϕ が $\phi \in C_b^2(\mathbb{R})$ をみたせば，$\phi(+\infty) = 1$ となる．ただし，c_K は次で定まる定数である．

$$c_K := \sqrt{\left(\int_{\mathbb{R}} |K(y)|dy\right)\left(\int_{\mathbb{R}} y^2|K(y)|dy\right)}.$$

注意 3.5 $\phi \in C_b(\mathbb{R})$ は方程式 (TW1) の解であるので，$d > 0$ のときには $\phi \in C_b^2(\mathbb{R})$ であることがわかる．また $d = 0$ のときには，$f \in C^1(\mathbb{R})$ であれば $\phi \in C_b^2(\mathbb{R})$ である．

4 主結果の証明の方針

本節では定理 3.4 の証明の方針を紹介する．定理 3.4 の前半部分である進行波解の存在，後半部分の解の漸近形の解析に分けて説明する．

4.1 進行波解の存在

積分核が符号変化する場合には方程式 (2.1) は一般に比較原理を持たないことから，シャウダーの不動点定理を用いた手法により進行波解の存在を示す．またシャウダーの不動点定理を用いるために，積分核が符号変化する場合の方程式 (TW1) に対する優解・劣解の新たな定義を導入し，その定義をみたす解のペアを構成を行った．

まず，与えられた $c > 0$ に対して (TW1) をみたす関数の存在を証明するための準備として，優解・劣解を導入する．

定義 4.1 $c > 0$ に対して，連続関数の組 $\{\overline{\phi}, \underline{\phi}\}$ が (TW1) の優解・劣解であるとは，

$$c\overline{\phi}'(\xi) \geq d\overline{\phi}''(\xi) + (K^+ * \overline{\phi})(\xi) - (K^- * \underline{\phi})(\xi) - \alpha\overline{\phi}(\xi) + f(\overline{\phi}(\xi)),\ \forall \xi \in \mathbb{R} \setminus A, \tag{4.1}$$

$$c\underline{\phi}'(\xi) \leq d\underline{\phi}''(\xi) + (K^+ * \underline{\phi})(\xi) - (K^- * \overline{\phi})(\xi) - \alpha\underline{\phi}(\xi) + f(\underline{\phi}(\xi)),\ \forall \xi \in \mathbb{R} \setminus A, \tag{4.2}$$

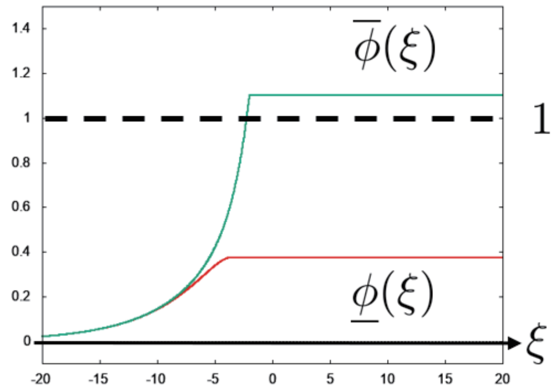

図 2　補題 4.2 における優解・劣解の形状のイメージ図

が成り立つことである．ただし $A \subset \mathbb{R}$ はある有限集合とする．

積分核が負の部分を持つことから，(4.1) には $K^- * \underline{\phi}$，(4.2) には $K^- * \overline{\phi}$ が含まれていることに注意する．仮定 4.2 のもとで定義 4.1 をみたす (TW1) の優解・劣解を構成する．任意の $c > c_R$ に対して，$\lambda_1 \in (0, \hat{\lambda})$ となるような以下の λ_1 を取ることができる．

$$Q(\lambda_1) = c\lambda_1, \quad Q(\lambda) > c\lambda \text{ for all } \lambda \in [0, \lambda_1). \tag{4.3}$$

また c_R の定義より，$R(\lambda) = c\lambda$ をみたす 2 つの根 $\lambda_2 < \lambda_3$ が存在し，

$$R(\lambda) < c\lambda,\ \forall\lambda \in (\lambda_2, \lambda_3); \quad R(\lambda) > c\lambda,\ \forall\lambda \in [0, \lambda_2) \cup (\lambda_3, \infty) \tag{4.4}$$

をみたす．ここで $\lambda_2 > \lambda_1$ であることに注意して，$\nu\lambda_1 \in (\lambda_2, \lambda_3)$ をみたす $\nu > 1$ をとる．また，$h > 1$ に対して，

$$\rho(\xi) := e^{\lambda_1 \xi} - h e^{\nu\lambda_1 \xi},\ \xi_M := \frac{-\ln(h\nu)}{(\nu - 1)\lambda_1} \tag{4.5}$$

を考える．$\xi < \xi_M$ のときには $\rho(\xi) > 0$ であり，

$$\rho(\xi) \le \rho(\xi_M) = C(\nu) h^{-1/(\nu-1)},\ \forall\xi \in \mathbb{R} \tag{4.6}$$

をみたす．ただし $C = C(\nu) := \nu^{-1/(\nu-1)}(1 - 1/\nu) > 0$ である．また，(4.5) の ρ と ξ_M に対して，$\rho(\xi_M) = \eta$ となるように $h > 0$ をとる．このとき，以下の補題が得られる．

補題 4.2　非線形項 f と積分核 K は仮定 3.2 をみたすとする．このとき $c > c_R$ に対して，

$$\overline{\phi}(\xi) = \min\{e^{\lambda_1\xi} + h e^{\nu\lambda_1\xi}, \delta\}, \quad \underline{\phi}(\xi) = \begin{cases} e^{\lambda_1\xi} - h e^{\nu\lambda_1\xi}, & \xi \le \xi_M, \\ \eta, & \xi \ge \xi_M \end{cases}$$

とおく．このとき，$\{\overline{\phi}, \underline{\phi}\}$ は (TW1) の優解・劣解である．

補題 4.2 における優解・劣解の形状は，図 2 のようになっている．また，$\underline{\phi} \in C^1(\mathbb{R})$ となっている．

次に進行波解の存在を示すために，適切な作用素とその不変集合を準備する．$\kappa := \sup\left\{\frac{|f(u)-f(v)|}{|u-v|} \mid u, v \in [0, u^+],\ u \ne v\right\}$ とおく．$c > 0$，$d \ge 0$ に対して，以下の積分作用素

P_d^c を導入する．

$$P_0^c[\psi](\xi) := \frac{1}{c}\int_{-\infty}^{\xi} e^{-\mu(\xi-y)} G[\psi](y)dy \quad (d=0\,\text{のとき})$$

$$P_d^c[\psi](\xi) := \frac{1}{d(\mu^+-\mu^-)}\left[\int_{\xi}^{+\infty} e^{\mu^+(\xi-y)} + \int_{-\infty}^{\xi} e^{\mu^-(\xi-y)}\right] G[\psi](y)dy \quad (d>0\,\text{のとき})$$

ここで，$\mu := (\alpha+\kappa)/c$，$G[\psi](\xi) := (K*\psi)(\xi) + \kappa\psi(\xi) + f(\psi(\xi))$ とおき，$\mu^+ > 0 > \mu^-$ は $d\mu^2 - c\mu - (\alpha+\kappa) = 0$ の実根とする．この P_d^c は，ϕ が P_d^c の不動点であることと ϕ が (TW1) をみたすことが同値であるように構成されている．これより，(TW1) をみたす ϕ の存在を示すために，P_d^c の不動点の存在を示す．

今，$c>0$ に対して以下をみたす (TW1) の優解・劣解 $\{\overline{\phi}, \underline{\phi}\}$ が存在すると仮定する．

(1) $\underline{\phi},\ \overline{\phi}\ :\ \mathbb{R} \to [0, u^+]$,
(2) $\underline{\phi}(\xi) \le \overline{\phi}(\xi)$ $(\forall \xi \in \mathbb{R})$,
(3) $\overline{\phi}'(z-) \ge \overline{\phi}'(z+)$, $\underline{\phi}'(z-) \le \underline{\phi}'(z+)$, $(\forall z \in \mathcal{A})$.

ここで，$\overline{\phi}(z\pm) := \lim_{\xi\to z\pm 0}\overline{\phi}(\xi)$，$\underline{\phi}(z\pm) := \lim_{\xi\to z\pm 0}\underline{\phi}(\xi)$ として与え，$\mathcal{A}$ は定義 4.1 で与えられたものとする．また $\Gamma := \{\psi \in C(\mathbb{R}) \mid \underline{\phi}(\xi) \le \psi(\xi) \le \overline{\phi}(\xi)\}$ とおく．このとき，優解・劣解の定義より集合 $\Gamma \subset C(\mathbb{R})$ は積分作用素 P_d^c の不変集合であることが直ちにわかる．

さらに [14] と同様な議論により，$P_d^c : \Gamma \to \Gamma$ は次のノルムに対してコンパクト作用素であることが示せる：

$$\|\psi\|_\theta := \sup_{\xi\in\mathbb{R}} |\psi(\xi)| e^{-\theta|\xi|},$$

ここで θ は十分小さい定数とする．この部分についての詳細は割愛する．これよりシャウダーの不動点定理から，次の存在定理が得られる．

命題 4.3　$c>0$ に対して，以下をみたす (TW1) の優解・劣解 $\{\overline{\phi}, \underline{\phi}\}$ が存在すると仮定する．

(1) $\underline{\phi},\ \overline{\phi}\ :\ \mathbb{R} \to [0, u^+]$,
(2) $\underline{\phi}(\xi) \le \overline{\phi}(\xi)$ $(\forall \xi \in \mathbb{R})$,
(3) $\overline{\phi}'(z-) \ge \overline{\phi}'(z+)$, $\underline{\phi}'(z-) \le \underline{\phi}'(z+)$, $(\forall z \in \mathcal{A})$.

このとき，(TW1) は解 ϕ を持ち $\mathbb{R}$ 上で $\underline{\phi} \le \phi \le \overline{\phi}$ をみたす．

補題 4.2 で構成した優解・劣解は命題 4.3 の仮定をみたしていることから，$c > c_R$ のときには正値な関数 ϕ が存在するという定理 3.4 の前半部分が得られる．

4.2　関数 ϕ の漸近形

ここでは，(TW1) をみたす関数 ϕ の $\xi \to +\infty$ における極限についての結果を紹介する．積分核が負の部分を持つとき，方程式 (2.1) の平衡点 $u=1$ は偏微分の意味で安定であるとは限らない [15]．そのため，平衡点の安定性によらない関数 ϕ の右方向の極限を得るために，[1] を参考に ϕ' の L^2 評価を用いた解析を行った．まず ϕ' について，以下の補題が得られる．

補題 4.4 $(c,\phi)\in\mathbb{R}\times C_b^2(\mathbb{R})$ を (TW1) をみたす解とする．また，$c>c_K$ と仮定する．このとき，$\phi'\in L^2(\mathbb{R})$ であり，$\phi'(+\infty)=0$ である．

この補題とアスコリ-アルツェラの定理とソボレフの埋め込み定理を用いた議論により，ϕ の無限遠方の集積点 l は $f(l)=0$ をみたすことが示せる．したがって，ϕ の無限遠方における極限の結果として以下の命題が得られる．

命題 4.5 (c,ϕ) を (TW1) をみたし，$\phi\in C_b^2(\mathbb{R})$ である進行波解とする．また，集合 $\mathcal{B}\subset\mathbb{R}$ を

$$\mathcal{B}:=\left\{u\in\left[\inf_{\xi\in\mathbb{R}}\phi(\xi),\ \sup_{\xi\in\mathbb{R}}\phi(\xi)\right]\mid f(u)=0\right\}$$

とおく．$\mathcal{B}$ が有限集合のとき，$c>c_K$ であれば $\phi(+\infty)$ が存在して $\phi(+\infty)\in\mathcal{B}$ である．

これより，定理 3.4 の後半部分が得られる．

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分解されたメビウス・エネルギーの上界・下界と連続度評価

Upper and lower bounds and modulus of continuity of decomposed Möbius energies

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石関 彩 (埼玉大学) との共同研究

概 要

Möbius エネルギーは結び目のエネルギーの一つで、Möbius 変換で不変である事が名前の由来となっている。このエネルギーは定義と異なる幾つかの表現が知られている。一つは共計角の余弦を用いたもので、余弦公式と呼ばれる。他方は、Möbius 不変な複数の項に分解するもので、分解されたものは分解エネルギーと呼ばれる。この事から、余弦公式は分解エネルギーの和と等しく、分解エネルギーによって余弦公式が評価されることが分かる。逆はどうであろうか。すなわち、個々の分解エネルギーは余弦公式を用いて評価されるだろうか。この問題に対して肯定的な解答を与える。すなわち、分解エネルギーの上下限評価や連続度を余弦公式を用いて評価する。更に、2 つの結び目 (閉曲線) の分解エネルギーの差の評価についても考察する。

これは、石関 彩 (埼玉大学) との共同研究 [18] に基づくものである。

1 序

結び目の標準形を決定する目的で O'Hara は結び目のエネルギーを設計した ([23])。これは後に O'Hara エネルギーと呼ばれるようになった。結び目は $\mathbb{R}^3$ 内の自己交叉を持たない閉曲線であるが、エネルギーは $\mathbb{R}^n$ 内のそのような曲線に対しても定義される。本講演では、O'Hara エネルギーの一つである Möbius エネルギーを $\mathbb{R}^n$ 内の閉曲線に対して考える。

$\boldsymbol{f} : \mathbb{R}/\mathcal{L}\mathbb{Z} \to \mathbb{R}^n$ を $\mathbb{R}^n$ に埋め込まれた長さ $\mathcal{L}$ の閉曲線の弧長パラメータ表示とする。閉曲線上の 2 点 $\boldsymbol{f}(s_1)$, $\boldsymbol{f}(s_2)$ の外来的距離を $\|\Delta \boldsymbol{f}\| = \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}$, 内在的距離を $|\Delta s| = \mathrm{dist}_{\mathbb{R}/\mathcal{L}\mathbb{Z}}(s_1, s_2)$ で表す。内在的距離とは、曲線上の距離である。Möbius エネルギーは、

$$\mathcal{E}(\boldsymbol{f}) = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathscr{M}(\boldsymbol{f})\, ds_1 ds_2$$

で定義される。ここで、エネルギー密度は

$$\mathscr{M}(\boldsymbol{f}) = \frac{1}{\|\Delta \boldsymbol{f}\|^2} - \frac{1}{|\Delta s|^2}$$

である。Möbius エネルギーの名称の由来は、Möbius 変換によってエネルギー値が不変である事による。この性質は 1994 年に Freedman-He-Wang [9] によって示された。

Möbius 変換は、合同変換、相似変換、球面に関する反転とそれらの合成である。従って、特に Möbius エネルギーは、相似変換によって不変である。この事は最小化列においてエネルギー密度の集中を許容する事を意味する。そのため、最小化列の収束は一般には期待出来ない。Freedman-He-Wang は、エネルギーの Möbius 不変性を示しただけでなく、結び目型が自明または素である場合は、最小化列に球面に関する反転を組み込むことで、エネルギーを最小化する列である事を保ったままエネルギー密度の集中を回避する事が可能である事を示し、その結果、そのような場合は最小元が存在する事を証明した。結び目型が合成の場合には、エネルギー密度が 2 か所以上で集中してしまう可能性があるため、彼らの方法は機能しない。Kusner-Sullivan [22] は数値計算によって、結び目型が合成の場合にはその結び目型内には Möbius エネルギーの最小元は存在しないと予想した。この予想は結び目エネルギーの積年の未解決問題の一つで、**Kusner-Sullivan 予想**と呼ばれている。

多くの研究者がこの予想に興味を持ち、解決を試みているが、現状では解決に至っていない。解析的に問題を扱う場合、結び目型の差をどのように解析学の手法に取り込むかが鍵となるはずであるが、決定打は生み出されていないように思える。

にも拘わらず、近年、Möbius エネルギーに関する理解が深まりつつある。特に、本講演では触れないが、停留点の正則性については最終決着に至った。すなわち、Blatt-Reiter-Shikorra [6] が停留点は C^∞ 級である事を示し、それを用いて、Blatt-Vorderobermeier [8] によって C^ω 級である事が示された。Blatt-Reiter-Shikorra は [7] において Möbius エネルギーを含む相似変換で不変なエネルギーの停留点の正則性について議論している。その他に Blatt [3] は、Möbius エネルギーを含む幾つかの結び目のエネルギーの未解決問題を解決した。Möbius エネルギーの Möbius 不変性を保った離散化については、Blatt-Ishizeki-Nagasawa [4, 5] で提唱された。Möbius 不変性を考慮しない離散化については、近年のものに限れば Kawakami [19, 20], Okamoto [27] がある。Ishizeki-Nagasawa [14] で Möbius エネルギーの変分公式とその評価を行ったが、O'Hara エネルギーについての同様の結果は、Kawakami-Nagasawa [21], Ishizeki-Nagasawa [17] で得られている。但し、[21] の方法は [14] のそれとは異なる。

2　余弦公式と分解エネルギー

Möbius エネルギーは、エネルギー密度が Möbius 不変でないにも関わらず、エネルギーは Möbius 不変になる。一見すると不思議に思えるこの性質について、エ

ネルギーの別表現によってその不思議さは解消される。それは、

$$\mathcal{E}(\boldsymbol{f}) = \mathcal{E}_0(\boldsymbol{f}) + 4 \tag{1}$$

である。ここで、

$$\mathcal{E}_0(\boldsymbol{f}) = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathscr{M}_0(\boldsymbol{f})\, ds_1 ds_2, \quad \mathscr{M}_0(\boldsymbol{f}) = \frac{1-\cos\varphi}{\|\Delta \boldsymbol{f}\|^2}$$

である。これは、Auckly-Sadun [1] に従い、Doyle-Schramm によって示された ([22])。φ は共計角と呼ばれるもので、(1) にはその余弦が用いられるため、**余弦公式**と呼ばれる。共形角は次で定義されるもので、Möbius 不変量である。

Definiton 2.1 (共計角) C_{12} を $\boldsymbol{f}(s_1)$ を通りその点で $\mathrm{Im}\boldsymbol{f}$ に接し、更に $\boldsymbol{f}(s_2)$ を通る円とする。C_{12} と C_{21} が $\boldsymbol{f}(s_1)$ (または $\boldsymbol{f}(s_2)$) においてなす角 $\varphi(s_1, s_2)$ を **共計角** という。

また、Möbius 不変による交差比 (複比) の不変性より $\dfrac{ds_1 ds_2}{\|\Delta \boldsymbol{f}\|^2}$ が Möbius 不変な 2 次形式である事が分かる。従って、余弦公式の測度 $ds_1 ds_2$ まで込めた被積分関数は Möbius 不変である事が分かり、これにより Möbius エネルギーの Möbius 不変性が容易に見て取れるようになったのである。

$\mathcal{E}$ の他の表現として、分解を用いたものがある (Ishizeki-Nagasawa [13])。Möbius エネルギーは、

$$\mathcal{E}(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{f}) + \mathcal{E}_2(\boldsymbol{f}) + 4$$

と分解される。ここで、

$$\mathcal{E}_i(\boldsymbol{f}) = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathscr{M}_i(\boldsymbol{f})\, ds_1 ds_2 \quad (i = 1,\, 2),$$

$$\mathscr{M}_1(\boldsymbol{f}) = \frac{\|\Delta \boldsymbol{\tau}\|^2}{2\|\Delta \boldsymbol{f}\|^2}, \quad \mathscr{M}_2(\boldsymbol{f}) = \frac{2}{\|\Delta \boldsymbol{f}\|^2} \left\langle \boldsymbol{\tau}(s_1) \wedge \frac{\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|}, \boldsymbol{\tau}(s_2) \wedge \frac{\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|} \right\rangle$$

である。$\boldsymbol{\tau}$ は曲線の単位接ベクトルである。$\wedge$ は $\mathbb{R}^n$ 上の 1-vector 同士のウェッジ積、$\langle \cdot, \cdot \rangle$ は $\mathbb{R}^n$ 上の 2-vector 同士の内積である。興味深い点は $\mathcal{E}_1$ と $\mathcal{E}_2$ が共に Möbius 不変になる事である ([13, 15])。

分解の幾何学的な意味は以下の通りである。$\mathscr{M}_1$ は単位接ベクトルの差分の平方の定数倍であるので、$\mathcal{E}_1$ は曲線の「曲がり具合」を測っている。$|\Delta s| < \dfrac{\mathcal{L}}{2}$ となる s_1, s_2 に対し、$\Delta s = s_1 - s_2$ とおく。ウェッジ積の性質より

$$\begin{aligned}
&\left\langle \boldsymbol{\tau}(s_1) \wedge \frac{\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}, \boldsymbol{\tau}(s_2) \wedge \frac{\Delta \boldsymbol{f}}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right\rangle_{\bigwedge^2 \mathbb{R}^n} \\
&\quad = \frac{(\Delta s)^2}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} \left\langle \left(\boldsymbol{\tau}(s_1) - \frac{\Delta \boldsymbol{f}}{\Delta s} \right) \wedge \frac{\Delta \boldsymbol{f}}{\Delta s}, \left(\boldsymbol{\tau}(s_2) - \frac{\Delta \boldsymbol{f}}{\Delta s} \right) \wedge \frac{\Delta \boldsymbol{f}}{\Delta s} \right\rangle_{\bigwedge^2 \mathbb{R}^n}
\end{aligned}$$

が成り立つ。曲線が C^2 であるとし、$\boldsymbol{\kappa}$ を曲率ベクトルとしよう。$\boldsymbol{\tau}(s_i) - \dfrac{\Delta \boldsymbol{f}}{\Delta s}$ は $|\Delta s|$ が小さいとき凡そ $\boldsymbol{\kappa}(s_i)(\Delta s)$ であるので、$\left(\boldsymbol{\tau}(s_i) - \dfrac{\Delta \boldsymbol{f}}{\Delta s}\right) \wedge \dfrac{\Delta \boldsymbol{f}}{\Delta s}$ の方向は凡そ陪法線方向となる。従って、粗く言って、$\mathscr{M}_2$ は曲線上の 2 点の陪法線の間の角と関連する。ゆえに、$\mathcal{E}_2$ は曲線の「捩じれ具合」を測っていると解釈できる。

分解エネルギーは、エネルギー・クラス ($\mathcal{E}(\boldsymbol{f}) < \infty$ となる $\boldsymbol{f}$ の族) において、絶対可積分な積分になる。このことにより, これらの変分公式と様々な関数空間における評価を可能にした ([14, 16, 12]) と言う点で解析的にも有益である。

余弦公式とエネルギーの分解は、一般化された O'Hara エネルギーについても成り立つ。詳細は、Ishizeki-Nagasawa [17] を見よ。また、余弦公式の多次元化については、O'Hara [25] を見よ。

3 分解エネルギーの上下限評価など

Kusner-Sullivan 予想に取り組むためには、Möbius エネルギーの最小化列が同じ結び目型において収束するか否かを考える必要がある。そのために、分解エネルギー値を追跡したい。最小化列であれ、その連続版である勾配流であれ、エネルギー値は初期のもので押されられる。しかし、分解エネルギーはそうとは限らない。そこで、分解エネルギーを元のエネルギーで評価する事を考える。元のエネルギー $\mathcal{E}$ と余弦公式 (1) で定義される $\mathcal{E}_0$ は定数差であるので、分解エネルギーを元の $\mathcal{E}_0$ で評価すると言ってもよい。分解エネルギーが非負であれば、その和が $\mathcal{E}_0$ であるので評価は自明である。しかし、第 2 エネルギーは非負とは限らないので、評価は自明でない。ここでは、$\mathcal{E}_0$ の他に以下に記す X と $\mathcal{C}$ という量を用いる。前者は相似変換で不変な量で Gromov の distotion (捩じれ) に関連する。後者は Möbius 不変量である。

$\mathcal{E}_0(\boldsymbol{f}) < \infty$ とすると、$\boldsymbol{f}$ は双 Lipschitz である事が知られている ([23])。すなわち、$\displaystyle\sup_{\substack{(s_1,s_2)\in(\mathbb{R}/\mathcal{L}\mathbb{Z})^2 \\ s_1\neq s_2}} \frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|} < \infty$ である。これが *distortion* である。[10, 11, 24, 22] を見よ。[7] ではスケール不変な結び目エネルギーと distortion の関連が議論されている。ここでは、distortion に関連する

$$X(s_1, s_2) = \log \frac{|\Delta s|^2}{\|\Delta \boldsymbol{f}\|^2} = \log \frac{1}{\|\Delta \boldsymbol{f}\|^2} - \log \frac{1}{|\Delta s|^2}$$

を用いる。$\|\Delta \boldsymbol{f}\| \leqq |\Delta s|$ であるので X は非負である。分解エネルギーの上界・下界評価と連続度評価を $\mathcal{E}_0(\boldsymbol{f})$ と X によって与える。

定理 3.1 ([18]) $\mathcal{E}_0(\boldsymbol{f}) < \infty$ のとき、

$$\begin{aligned} 0 \leqq \mathcal{E}_1(\boldsymbol{f}) &\leqq \left(3 + \|X\|_{L^\infty}\right)\mathcal{E}_0(\boldsymbol{f}) + 4\left(4 + \|X\|_{L^\infty}\right), \\ -\left(2 + \|X\|_{L^\infty}\right)\mathcal{E}_0(\boldsymbol{f}) &- 4\left(4 + \|X\|_{L^\infty}\right) \\ \leqq \mathcal{E}_2(\boldsymbol{f}) &\leqq \min\left\{\|X\|_{L^\infty}\left(\mathcal{E}_0(\boldsymbol{f}) + 4\right) - 8, \mathcal{E}_0(\boldsymbol{f})\right\} \end{aligned}$$

が成り立つ。

定理 3.2 ([18]) $\boldsymbol{f}$ と $\widetilde{\boldsymbol{f}}$ は全長が同じものとし、$\mathcal{E}_0(\boldsymbol{f}) < \infty$, $\mathcal{E}_0(\widetilde{\boldsymbol{f}}) < \infty$ とする。$a \in \mathbb{R}/\mathcal{L}\mathbb{Z}$ に対し、$\widetilde{\boldsymbol{f}}_a(\cdot) = \widetilde{\boldsymbol{f}}(\cdot + a)$, $\widetilde{X}_a = \log \dfrac{(\Delta s)^2}{\|\Delta \widetilde{\boldsymbol{f}}_a\|^2}$ とおく。このとき、

$$\begin{aligned} &\left|\mathcal{E}_1(\boldsymbol{f}) - \mathcal{E}_1(\widetilde{\boldsymbol{f}})\right| \\ &\quad \leqq \left|\mathcal{E}_0(\boldsymbol{f}) - \mathcal{E}_0(\widetilde{\boldsymbol{f}})\right| \\ &\qquad + 2 \inf_{a \in \mathbb{R}/\mathcal{L}\mathbb{Z}} \left\{\|X - \widetilde{X}_a\|_{L^\infty}\left(\mathcal{E}_0(\boldsymbol{f}) + \mathcal{E}_0(\widetilde{\boldsymbol{f}}) + 4\right) + \left|\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{X - \widetilde{X}_a}{(\Delta s)^2}\, ds_1 ds_2\right|\right\}, \\ &\left|\mathcal{E}_2(\boldsymbol{f}) - \mathcal{E}_2(\boldsymbol{f}_0)\right| \\ &\quad \leqq 2 \inf_{a \in \mathbb{R}/\mathcal{L}\mathbb{Z}} \left\{\|X - \widetilde{X}_a\|_{L^\infty}\left(\mathcal{E}_0(\boldsymbol{f}) + \mathcal{E}_0(\widetilde{\boldsymbol{f}}) + 4\right) + \left|\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{X - \widetilde{X}_a}{(\Delta s)^2}\, ds_1 ds_2\right|\right\} \end{aligned}$$

が成り立つ。

注意 3.1 $\mathcal{E}_0(\boldsymbol{f}) = \mathcal{E}_0(\widetilde{\boldsymbol{f}})$ であっても $X - \widetilde{X} \equiv 0$ とは限らない。例えば $\mathrm{Im}\boldsymbol{f}$ と $\mathrm{Im}\widetilde{\boldsymbol{f}}$ が Möbius 変換で写り合う場合はエネルギーは等しいが、$X - \widetilde{X} \equiv 0$ とは限らない。よって、$X - \widetilde{X}$ を $\left|\mathcal{E}_0(\boldsymbol{f}) - \mathcal{E}_0(\widetilde{\boldsymbol{f}})\right|$ で評価する事は不可能である。

$\boldsymbol{f}$ と $\widetilde{\boldsymbol{f}}$ を $\mathbb{R}^n$ 内の 2 つの閉曲線とし、$\theta \in \mathbb{R}/\mathbb{Z}$ でパラメトライズされているとする。$\boldsymbol{f}$ が Möbius 変換で $\widetilde{\boldsymbol{f}}$ に写されるときエネルギー差 $\mathcal{E}_i(\boldsymbol{f}) - \mathcal{E}_i(\widetilde{\boldsymbol{f}})$ は 0 になる。これが読み取れるようにエネルギー差を Möbius 不変量の差で表したい。ここでは、

$$\mathscr{C}(\boldsymbol{f})(\theta_1, \theta_2) = \frac{\|\dot{\boldsymbol{f}}(\theta_1)\|\|\dot{\boldsymbol{f}}(\theta_2)\|}{\|\Delta \boldsymbol{f}\|^2}$$

を用いる。複比

$$\frac{\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_3)\|\|\boldsymbol{f}(\theta_2) - \boldsymbol{f}(\theta_4)\|}{\|\boldsymbol{f}(\theta_1) - \boldsymbol{f}(\theta_2)\|\|\boldsymbol{f}(\theta_3) - \boldsymbol{f}(\theta_4)\|}$$

の Möbius 不変性より $\mathscr{C}$ の Möbius 不変性が従う。次が得られる。

定理 3.3 ([18]) $\mathcal{E}(\boldsymbol{f}) < \infty$, $\mathcal{E}(\widetilde{\boldsymbol{f}}) < \infty$, $\|\dot{\boldsymbol{f}}\| \in C^{0,1}$, $\|\dot{\widetilde{\boldsymbol{f}}}\| \in C^{0,1}$ とする。この

とき、

$$\begin{aligned}
&\mathcal{E}_1(\boldsymbol{f})-\mathcal{E}_1(\widetilde{\boldsymbol{f}})\\
&\quad=\iint_{(\mathbb{R}/\mathbb{Z})^2}\left\{\mathscr{C}(\boldsymbol{f})-\mathscr{C}(\widetilde{\boldsymbol{f}})+\frac{1}{2}\frac{\partial^2}{\partial\theta_1\partial\theta_2}\left(\log\mathscr{C}(\boldsymbol{f})-\log\mathscr{C}(\widetilde{\boldsymbol{f}})\right)\right\}d\theta_1 d\theta_2\\
&\quad+\frac{1}{2}\lim_{\varepsilon\to+0}\iint_{|\theta_1-\theta_2|\geqq\varepsilon}\left\{\frac{\partial}{\partial\theta_1}\left(\log\mathscr{C}(\boldsymbol{f})-\log\mathscr{C}(\widetilde{\boldsymbol{f}})\right)\right\}\left\{\frac{\partial}{\partial\theta_2}\left(\log\mathscr{C}(\boldsymbol{f})+\log\mathscr{C}(\widetilde{\boldsymbol{f}})\right)\right\}d\theta_1 d\theta_2\\
&\mathcal{E}_2(\boldsymbol{f})-\mathcal{E}_2(\widetilde{\boldsymbol{f}})\\
&\quad=-\frac{1}{2}\lim_{\varepsilon\to+0}\iint_{|\theta_1-\theta_2|\geqq\varepsilon}\left\{\frac{\partial}{\partial\theta_1}\left(\log\mathscr{C}(\boldsymbol{f})-\log\mathscr{C}(\widetilde{\boldsymbol{f}})\right)\right\}\left\{\frac{\partial}{\partial\theta_2}\left(\log\mathscr{C}(\boldsymbol{f})+\log\mathscr{C}(\widetilde{\boldsymbol{f}})\right)\right\}d\theta_1 d\theta_2
\end{aligned}$$

が成り立つ。

$\mathcal{E}_2$ は O'Hara-Solanes [26] で導入されたエネルギーと本質的に同値である。その Möbius 不変性は [13] と [26] で独立に異なる方法で示された。しかし、$\mathcal{E}$ と $\mathcal{E}_1$ の Möbius 不変性より間接的に導くという点は共通である。ここで用いる方法は、$\mathcal{E}_2$ の Möbius 不変性を直接示す最初の試みである。

いずれの結果も分解の第二エネルギーの密度の新しい表現が出発点となっている。再び、曲線は弧長パラメータで表されているとする。$\mathcal{E}(\boldsymbol{f})<\infty$ より $\boldsymbol{f}\in W^{\frac{1}{2},2}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ である ([2])。従って、$\boldsymbol{f}'=\boldsymbol{\tau}$ は a.e. $s\in\mathbb{R}/\mathcal{L}\mathbb{Z}$ で存在する。これより $\dfrac{\partial^2}{\partial s_1\partial s_2}\|\boldsymbol{f}(s_1)-\boldsymbol{f}(s_2)\|^2$ は a.e. $(s_1,s_2)\in(\mathbb{R}/\mathcal{L}\mathbb{Z})^2$ で存在する。$\dfrac{\partial^2}{\partial s_1\partial s_2}|\Delta s|^2$ は $|s_1-s_2|=\dfrac{\mathcal{L}}{2}$ において Sobolev 空間の元としては定義されない。また、$s_1=s_2$ においては X は定義されない。よって、$0<|s_1-s_2|<\dfrac{\mathcal{L}}{2}$ で考える。

補題 3.1 $0<|s_1-s_2|<\dfrac{\mathcal{L}}{2}$ においては、$\Delta s=s_1-s_2$ としてよい。このとき、

$$\mathscr{M}_2(\boldsymbol{f})=\frac{\partial^2 X}{\partial s_1\partial s_2}-\frac{1}{2}\frac{\partial X}{\partial s_1}\frac{\partial X}{\partial s_2}-\frac{1}{\Delta s}\left(\frac{\partial X}{\partial s_1}-\frac{\partial X}{\partial s_2}\right)$$

が成り立つ。

これにより、分解の第二エネルギーの評価は X の評価より導かれる事が読み取れる。また、第二エネルギーは曲線の捩じれ具合を測っていると考えられた。実際、Gromov の distortion (捩じれ) に関連する量 X でエネルギー密度が表される事で、式の上でもこれが確認できたと言ってよいだろう。

従来の分解の第二エネルギーの結果は、第一エネルギーを解析し、分解前のエネルギーの性質を利用する事で間接的に導いていた。この補題は、第二エネルギーの諸性質を直接導く貴重な手がかりを与える。著者自身もまだこの補題をうまく使いこなせていない感があり、今後、Kusner-Sullivan 予想解決に繋がる諸性質が得られる事を期待している。特に、ある結び目型内の最小化列の収束問題を、X の収束問題に帰着させ、収束の位相によって最小元の存在の有無が判定できないかと考えている。

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On an inverse crack problem in a linearized elasticity by the enclosure method

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1 Introduction

We consider a reconstruction problem for several linear cracks located on a line between two linearized elastic plates from measured data which are a loading surface traction and the resulted displacement field on the boundary of the joined plates. This is a typical inverse crack problem arising from the nondestructive testing of materials. For this problem, we introduce an extraction formula of the cracks from a single set of the data by means of the enclosure method [2].

In the case of a *single* linear crack, the extraction formula of the location and length of an unknown crack has been established by using the enclosure method for isotropic [5] and anisotropic [6] elasticity. However, this result cannot be extended to *several* cracks case directly because the original enclosure method can give information of the convex hull of cracks. As one of ways to overcome the difficulty, we apply the Kelvin transform to the indicator function of the classical enclosure method. The idea of using Kelvin transform has been applied also to the probe method by Ikehata [4]. In [1, 7], by use of this transform in the enclosure method we succeeded to derive the extraction procedure of information about the location of tips of several cracks located on a line between two electric conductive plates from a single set of an electric current density and the corresponding voltage potential on the boundary of the material formed by the plates. Moreover, by a collaborative project with M. Ikehata, A. Hauptmann (University College London, University of Oulu) and S. Siltanen (University of Helsinki), we demonstrate computationally that the Kelvin transformed enclosure method can be used for robust detection of multiple crack locations [1]. In the present paper, we consider an extension of the result [1] to the linearized elastic case. Originally, this problem should be considered in three dimensions, however, our results obtained in 2D are not directly applicable to the actual 3D situation. In order to do that, we need to analyze the asymptotic behavior of the solution of the linearized elasticity equation near the crack edges. Also, the further extension to the multilayered case, that is, cracks are located on the interface of two different elastic bodies (cf. [3] for the Laplace equation), can be expected.

2 Formulation

Let $\Omega \subset \mathbb{R}^2$ be a rectangular domain described as $\Omega :=]0, a[\times]0, b[$ with $a > 0$ and $b > 0$. Let $c \in]0, b[$, $m \geq 1$ and $0 = c_0 < c_1 < \cdots < c_{2m+1} = a$. We consider the set

$$\Sigma := ([c_0, c_1] \cup [c_2, c_3] \cup [c_4, c_5] \cup \cdots \cup [c_{2m}, c_{2m+1}]) \times \{c\}.$$

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This is a joint work with Masaru Ikehata (Hiroshima University).

a mathematical model of the *crack* lying on the interface of two plates $\overline{\Omega^+}$ and $\overline{\Omega^-}$ ($\Omega^\pm := \Omega \cap \{\pm(x_2 - c) > 0\}$) which consist of the same isotropic homogeneous elastic materials. Set

$$W := ([0,\, a] \times \{c\}) \setminus \Sigma.$$

which has the decomposition

$$W = (]c_1, c_2[\, \cup \cdots \cup \,]c_{2m-1}, c_{2m}[) \times \{c\}$$

and both c_1 and c_{2m} are in $]0,\, a[$. See Figure 1 for an illustration of the geometry of the situation.

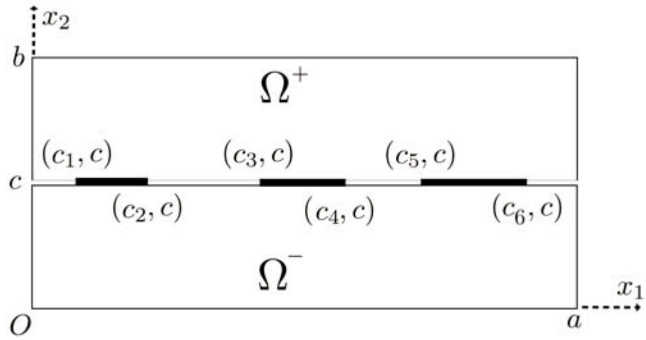

Figure 1: an example of the domain Ω

Let $\boldsymbol{\nu}$ denote the unit outward normal vector field on $\partial\Omega \cup \Sigma$ and $\boldsymbol{\nu}$ is fixed as $(0, 1)$ on Σ. Let $\boldsymbol{g} \in L^2(\partial\Omega)$ be a function defined on $\partial\Omega$ and denote the surface force acting on $\partial\Omega$. By $\boldsymbol{u} = (u_i)_{i=1,2}$, $\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{i,j=1,2}$ and $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1,2}$ we denote the displacement vector, the strain tensor and the stress tensor in the state of a plane strain, respectively. The linear elasticity equations for a homogeneous isotropic material consist of the constitutive law (Hooke's law)

$$\boldsymbol{\sigma} = \lambda \mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} \tag{2.1}$$

and the equilibrium conditions without any body forces

$$\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{0}, \tag{2.2}$$

where $\boldsymbol{I}$ is the identity tensor. λ and μ are *Lamé* constants assumed to be $\mu > 0$ and $\lambda + \mu > 0$. In the context of infinitesimal deformation the linearized strain tensor is given by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}\right) \tag{2.3}$$

where the superscript T denotes matrix transposition.

Then (2.1), (2.2) and (2.3) give the system of equations for $\boldsymbol{u} = (u_1, u_2)$

$$\mu \triangle \boldsymbol{u} + (\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) = \boldsymbol{0}. \tag{2.4}$$

On the crack Σ we assume the free traction condition

$$\boldsymbol{\sigma}^+\boldsymbol{\nu} = \boldsymbol{\sigma}^-\boldsymbol{\nu} = \boldsymbol{0} \quad \text{on} \quad \Sigma^\pm, \tag{2.5}$$

where $\Sigma^\pm$ means both sides of Σ and $\boldsymbol{\sigma}^\pm\boldsymbol{\nu}$ is the stress vector on $\Sigma^\pm$, respectively. We define the space of rigid displacements;

$$\mathcal{R} = \{\boldsymbol{v} \mid \boldsymbol{v}(\boldsymbol{x}) = (k_1 + k_0 x_2, k_2 - k_0 x_1),\ k_0, k_1, k_3 \in \mathbb{R}\}.$$

Definition 1. *We say that for given* $\boldsymbol{g} \in L^2(\partial\Omega)$ *satisfying*

$$\forall \boldsymbol{\rho}(\boldsymbol{x}) \in \mathcal{R}, \quad \int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{\rho} \, \mathrm{d}s_{\boldsymbol{x}} = 0, \tag{2.6}$$

$\boldsymbol{u} \in H^1(\Omega \setminus \overline{\Sigma}) \setminus \mathcal{R}$ *is a weak solution of the problem*

$$(*) \begin{cases} \mu \triangle \boldsymbol{u} + (\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) = \boldsymbol{0} \quad \text{in} \quad \Omega \setminus \overline{\Sigma}, \\ \boldsymbol{\sigma}^{\pm}\boldsymbol{\nu} = \boldsymbol{0} \quad \text{on} \quad \Sigma^{\pm}, \\ \boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{g} \quad \text{on} \quad \partial\Omega \end{cases}$$

if for arbitrary $\boldsymbol{\varphi} \in H^1(\Omega \setminus \overline{\Sigma}) \setminus \mathcal{R}$ *it holds*

$$\int_{\Omega \setminus \overline{\Sigma}} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) \, \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{\varphi} \, \mathrm{d}s_{\boldsymbol{x}}, \tag{2.7}$$

where the double dot in $\boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{\varphi}) = \Sigma_{i,j=1,2}\sigma_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{\varphi})$ *implies the scalar product of matrices.*

By virtue of Lax-Milgram Lemma and Korn's inequality one can prove the problem $(*)$ has a unique weak solution.

In this paper, we consider the following inverse problem.

Inverse Problem . *Fix* $\boldsymbol{g} \neq \boldsymbol{0}$ *satisfying* (2.6). *Extract information about the location of* Σ *from* $\boldsymbol{u}$ *on* $\partial\Omega$.

3 Statement of the main result

Now we state our results for the Inverse Problem defined above. We denote the unit circle by S^1. Let $\boldsymbol{\omega} \in S^1$ and take $\boldsymbol{\omega}^{\perp} \in S^1$ perpendicular to $\boldsymbol{\omega}$ satisfying $\det(\boldsymbol{\omega}^{\perp} \;\; \boldsymbol{\omega}) > 0$. Following [5], for $\tau > 0$ we define a function $\boldsymbol{v}$ satisfying (2.4) in $\mathbb{R}^2$ by

$$\boldsymbol{v} = (\boldsymbol{\omega} + i\boldsymbol{\omega}^{\perp})e^{\tau \boldsymbol{x} \cdot (\boldsymbol{\omega} + i\boldsymbol{\omega}^{\perp})}.$$

Based on the Kelvin transform to $\boldsymbol{v}$, we define a new function

$$\boldsymbol{v}_{\tau} = \boldsymbol{v}_{\tau}(\boldsymbol{y}; \boldsymbol{x}) := (\boldsymbol{e}_1 + i\boldsymbol{e}_2) \exp\left\{ -\tau \frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|^2} \cdot (\boldsymbol{e}_2 + i\boldsymbol{e}_1) \right\}.$$

which satisfies (2.4) in $\mathbb{R}^2 \setminus \{\boldsymbol{x}\}$, also

$$\triangle_{\boldsymbol{y}} \boldsymbol{v}_{\tau} = \boldsymbol{0} \quad \text{and} \quad \nabla_{\boldsymbol{y}} \cdot \boldsymbol{v}_{\tau} = 0.$$

Moreover, let $s > 0$ and write

$$e^{-\frac{\tau}{2s}} \boldsymbol{v}_{\tau} = (\boldsymbol{e}_1 + i\boldsymbol{e}_2) \exp\left\{ -\tau \left(\frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|^2} \cdot \boldsymbol{e}_2 + \frac{1}{2s} \right) \right\} \exp\left\{ -i\tau \frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|^2} \cdot \boldsymbol{e}_1 \right\}.$$

Since

$$\frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|^2} \cdot \boldsymbol{e}_2 + \frac{1}{2s} = \frac{2s(\boldsymbol{y} - \boldsymbol{x}) \cdot \boldsymbol{e}_2 + |\boldsymbol{y} - \boldsymbol{x}|^2}{2s|\boldsymbol{y} - \boldsymbol{x}|^2} = \frac{|\boldsymbol{y} - (\boldsymbol{x} - s\boldsymbol{e}_2)|^2 - s^2}{2s|\boldsymbol{y} - \boldsymbol{x}|^2}, \tag{3.1}$$

we see the following facts;

(i) if $|\boldsymbol{y}-(\boldsymbol{x}-s\boldsymbol{e}_2)|>s$, then $\lim_{\tau\longrightarrow\infty} e^{-\tau/(2s)}|\boldsymbol{v}_\tau|=0$;

(ii) if $|\boldsymbol{y}-(\boldsymbol{x}-s\boldsymbol{e}_2)|<s$, then $\lim_{\tau\longrightarrow\infty} e^{-\tau/(2s)}|\boldsymbol{v}_\tau|=\infty$;

(iii) if $|\boldsymbol{y}-(\boldsymbol{x}-s\boldsymbol{e}_2)|=s$, then each components of $e^{-\tau/(2s)}\boldsymbol{v}_\tau$ is highly oscillating as $\tau\longrightarrow\infty$.

(i), (ii) and (iii) suggest that the circle centred at $\boldsymbol{x}-s\boldsymbol{e}_2$ with radius s plays the role of the line in the previous enclosure method using a single set of Cauchy data [2].

For fixed $\epsilon>0$ set a line segment $\Gamma_\epsilon:=[0,a]\times\{b+\epsilon\}$. Using the function $\boldsymbol{v}_\tau$, we define a mathematical indicator.

Definition 2. *Let $\boldsymbol{u}$ be a weak solution of $(*)$. Given $\boldsymbol{x}\in\Gamma_\epsilon$ and $\tau>0$ define*

$$I(\tau;\boldsymbol{x}):=\int_{\partial\Omega}\boldsymbol{g}(\boldsymbol{y})\cdot\boldsymbol{v}_\tau(\boldsymbol{y};\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{y})\cdot\boldsymbol{\sigma}(\boldsymbol{v}_\tau(\boldsymbol{y};\boldsymbol{x}))\boldsymbol{\nu}\ \mathrm{d}s_{\boldsymbol{y}}.$$

Next, we define the function of $\boldsymbol{x}$ given by

$$s_\Sigma(\boldsymbol{x})=\sup\left\{s>0\mid B_s(\boldsymbol{x}-s\boldsymbol{e}_2)\subset\mathbb{R}^2\setminus\Sigma\right\}.$$

The value $s_\Sigma(\boldsymbol{x})$ at $\boldsymbol{x}\in\Gamma_\epsilon$ is the largest radius of the disc centered at $\boldsymbol{x}-s\boldsymbol{e}_2$ with radius s whose exterior encloses Σ.

Now we state our main result to extract the $s_\Sigma(\boldsymbol{x})$ at $\boldsymbol{x}\in\Gamma_\epsilon$.

Theorem 1. *Let the non-trivial $\boldsymbol{g}\in L^2(\partial\Omega)$ satisfying (2.6) be such that the following condition (†) is fulfilled:*

(†) $\mathrm{supp}(\boldsymbol{g})\subset(\partial\Omega\cap\{|x_2-c|>\gamma\})\setminus(B_\gamma(\boldsymbol{O})\cup B_\gamma(0,b)\cup B_\gamma(a,b)\cup B_\gamma(a,0))$ *for some* $\gamma>0$ *and there exists $\boldsymbol{\rho}_0\in\mathcal{R}$ such that*

$$\int_{\partial\Omega\cap\{x_2>c\}}\boldsymbol{g}\cdot\boldsymbol{\rho}_0\ \mathrm{d}s_{\boldsymbol{x}}\neq 0\quad\textit{or}\quad\int_{\partial\Omega\cap\{x_2<c\}}\boldsymbol{g}\cdot\boldsymbol{\rho}_0\ \mathrm{d}s_{\boldsymbol{x}}\neq 0.$$

Let further $\boldsymbol{x}\in\Gamma_\epsilon$ satisfy that the intersection of $\overline{B_{s_\Sigma(\boldsymbol{x})}(\boldsymbol{x}-s_\Sigma(\boldsymbol{x})\boldsymbol{e}_2)}$ with tips of crack Σ in Ω consists of a single point, namely there exists a $j\in\{1,\cdots,2m\}$ such that

$$\overline{B_{s_\Sigma(\boldsymbol{x})}(\boldsymbol{x}-s_\Sigma(\boldsymbol{x})\boldsymbol{e}_2)}\cap\Sigma=\{(c_j,c)\}.\tag{3.2}$$

Then, the function $e^{-\tau/(2s_\Sigma(\boldsymbol{x}))}I(\tau;\boldsymbol{x})$ is truly algebraically decaying as $\tau\to\infty$ and it holds that

$$\lim_{\tau\to\infty}\frac{\log|I(\tau;\boldsymbol{x})|}{\tau}=\frac{1}{2s_\Sigma(\boldsymbol{x})}.\tag{3.3}$$

4 Asymptotic expression for the indicator function and its property

In order to show the formula (3.3) in Theorem 1 we need to study the asymptotic behavior of the indicator function. For this purpose we employ the following two important propositions.

Proposition 1 (Proposition 2 in [5]).

$$I(\tau;\boldsymbol{x})=-\int_\Sigma\left(\boldsymbol{u}^+(\boldsymbol{y})-\boldsymbol{u}^-(\boldsymbol{y})\right)\cdot\left(\boldsymbol{\sigma}(\boldsymbol{v}_\tau(\boldsymbol{y};\boldsymbol{x}))\boldsymbol{e}_2\right)\ \mathrm{d}s_{\boldsymbol{y}}.\tag{4.1}$$

Here $\boldsymbol{u}^{\pm}$ denotes the trace of $\boldsymbol{u}|_{\Omega^{\pm}}$ onto $]0,a[\times\{c\}$, respectively, and (4.1) is obtained from the Green formula.

Next, we choose a polar coordinates system with respect to the center (c_j, c). It is depending on whether j is even or odd. Since the case when j is even can be treated in the same manner as the case when j is odd. Therefore, in what follow we assume j is an odd number, see Figure 2 for the illustration. Choose a positive number η_0 in such a way that

$$\eta_0 < \min_{j=1,\cdots,2m+1}\{c_j - c_{j-1}\} \quad \text{and} \quad \eta_0 < \min\{b-c, c\}.$$

Set $\boldsymbol{x} = (c_j + r\cos\theta, c + r\sin\theta)$ for $r \in]0,\eta_0[$ and $\theta \in]-\pi,\pi[$ and define

$$\boldsymbol{u}(r,\theta) = \boldsymbol{u}(c_j + r\cos\theta, c + r\sin\theta), \quad r \in]0,\eta_0[, \quad \theta \in]-\pi,\pi[.$$

Hence, we have

$$\boldsymbol{u}^{+}(\boldsymbol{x}) = \boldsymbol{u}(r,\theta), \quad \theta \in]0,\pi[$$

and

$$\boldsymbol{u}^{-}(\boldsymbol{x}) = \boldsymbol{u}(r,\theta), \quad \theta \in]-\pi,0[.$$

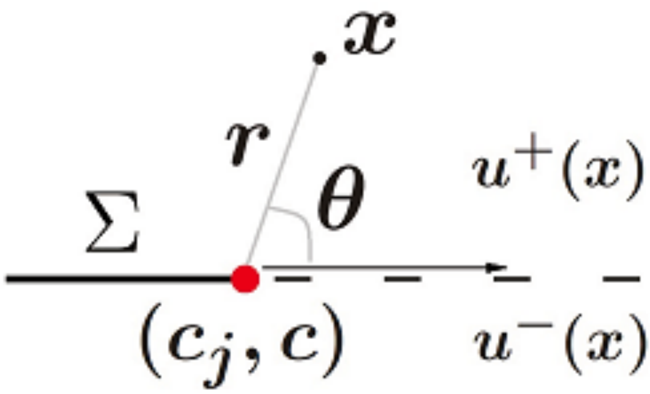

Figure 2: Illustration of the polar coordinates system used

Proposition 2 (Proposition 1 in [5]). *Fix $\eta \in]0,\frac{\eta_0}{2}[$. There exist real numbers $A_k^{(j)}$, $B_k^{(j)}$ $(k = 0,1,2,\cdots)$ such that*

$$\boldsymbol{u}(r,\theta) = \sum_{k=0}^{\infty}\frac{A_k^{(j)}}{2\mu}r^{\frac{k}{2}}\boldsymbol{\varphi}_k(\theta) - \sum_{k=0}^{\infty}\frac{B_k^{(j)}}{2\mu}r^{\frac{k}{2}}\boldsymbol{\psi}_k(\theta) \quad \text{in} \quad B_{2\eta}((c_j,c))\setminus\Sigma, \tag{4.2}$$

where

$$\boldsymbol{\varphi}_k(\theta) = \begin{pmatrix} \kappa\cos\frac{k}{2}\theta - \frac{k}{2}\cos\left(\frac{k}{2}-2\right)\theta + \left\{\frac{k}{2}+(-1)^k\right\}\cos\frac{k}{2}\theta \\ \kappa\sin\frac{k}{2}\theta + \frac{k}{2}\sin\left(\frac{k}{2}-2\right)\theta - \left\{\frac{k}{2}+(-1)^k\right\}\sin\frac{k}{2}\theta \end{pmatrix},$$

$$\boldsymbol{\psi}_k(\theta) = \begin{pmatrix} \kappa\sin\frac{k}{2}\theta - \frac{k}{2}\sin\left(\frac{k}{2}-2\right)\theta + \left\{\frac{k}{2}-(-1)^k\right\}\sin\frac{k}{2}\theta \\ -\kappa\cos\frac{k}{2}\theta - \frac{k}{2}\cos\left(\frac{k}{2}-2\right)\theta + \left\{\frac{k}{2}-(-1)^k\right\}\cos\frac{k}{2}\theta \end{pmatrix}$$

with $\kappa = \frac{\lambda+3\mu}{\lambda+\mu}$*. The series is convergent, absolutely in* $H^1(B_\eta((c_j,c))\cap\Omega^+)$ *and* $H^1(B_\eta((c_j,c))\cap \Omega^-)$*, and uniformly on compact sets in* $B_{2\eta}((c_j,c))$*. Moreover, there exists* $\boldsymbol{\rho}_j \in \mathcal{R}$ *such that for* $n = 1, 2, \cdots$ *the estimate*

$$\left|\boldsymbol{u}(r,\pi) - \boldsymbol{\rho}_j - \left(\sum_{k=1}^{n} \frac{A_k^{(j)}}{2\mu} r^{\frac{k}{2}} \boldsymbol{\varphi}_k(\pi) - \sum_{k=1}^{n} \frac{B_k^{(j)}}{2\mu} r^{\frac{k}{2}} \boldsymbol{\psi}_k(\pi)\right)\right| + \left|\boldsymbol{u}(r,-\pi) - \boldsymbol{\rho}_j - \left(\sum_{k=1}^{n} \frac{A_k^{(j)}}{2\mu} r^{\frac{k}{2}} \boldsymbol{\varphi}_k(-\pi) - \sum_{k=1}^{n} \frac{B_k^{(j)}}{2\mu} r^{\frac{k}{2}} \boldsymbol{\psi}_k(-\pi)\right)\right| \le K_n r^{\frac{n+1}{2}} \tag{4.3}$$

is valid uniformly with respect to $r \in]0, \eta[$*, where* K_n *is a positive constant depending on* n*.*

Remark 1. *For any* $\boldsymbol{\rho} \in \mathcal{R}$*, there exist* $\tilde{A}_0^{(j)}, \tilde{B}_0^{(j)}, \tilde{B}_2^{(j)} \in \mathbb{R}$ *such that*

$$\boldsymbol{\rho} = \tilde{A}_0^{(j)} \boldsymbol{\varphi}_0(\theta) + \tilde{B}_0^{(j)} \boldsymbol{\psi}_0(\theta) + \tilde{B}_2^{(j)} r \boldsymbol{\psi}_2(\theta).$$

Then, it follows from Proposition 2 that for each $n = 1, 2, \cdots$

$$\begin{aligned} &\boldsymbol{u}(c_j - r, c+0) - \boldsymbol{u}(c_j - r, c-0) \\ &\quad = \frac{\kappa+1}{\mu} \sum_{k=1}^{n} (-1)^k r^{\frac{2k-1}{2}} \begin{pmatrix} -B_{2k-1}^{(j)} \\ A_{2k-1}^{(j)} \end{pmatrix} + O\left(r^{\frac{2n+1}{2}}\right) \boldsymbol{k} \end{aligned} \tag{4.4}$$

with a constant vector $\boldsymbol{k}$. In (4.1), by use of (2.1) and (2.3) we calculate

$$\begin{aligned} &\boldsymbol{\sigma}(\boldsymbol{v}_\tau(\boldsymbol{y};\boldsymbol{x}))\boldsymbol{e}_2 \\ &= 2\mu\tau \left(\frac{\boldsymbol{y}-\boldsymbol{x}}{|\boldsymbol{y}-\boldsymbol{x}|^2} \cdot (\boldsymbol{e}_2 + i\boldsymbol{e}_1)\right)^2 \exp\left\{-\tau \frac{\boldsymbol{y}-\boldsymbol{x}}{|\boldsymbol{y}-\boldsymbol{x}|^2} \cdot (\boldsymbol{e}_2 + i\boldsymbol{e}_1)\right\} (\boldsymbol{e}_1 + i\boldsymbol{e}_2). \end{aligned} \tag{4.5}$$

As the result, the following result is obtained.

Proposition 3. *There exist a positive number* A *and an integer* $N \ge 1$ *such that for each fixed* $\boldsymbol{x} \in \Gamma_\epsilon$ *satisfying* (3.2)

$$\lim_{\tau\to\infty} \tau^{\frac{2N-1}{2}} \left| e^{-\frac{\tau}{2s_\Sigma(\boldsymbol{x})}} I(\tau;\boldsymbol{x}) \right| = A, \tag{4.6}$$

which means that the function $e^{-\tau/(2s_\Sigma(\boldsymbol{x}))} I(\tau;\boldsymbol{x})$ *is truly algebraically decaying as* $\tau \to \infty$*.*

This proposition was proved by [1] in the case of Laplace equation. Following [1] the proof is given in the next section.

5 Proof of Proposition 3

The proof is divided into the following steps. In what follows we use the notation for simplicity $s_0 = s_\Sigma(\boldsymbol{x})$.

5.1 Step 1: Decomposition of Σ

Choose $\delta > 0$ in such a way that

$$\overline{B_{s_0+\delta}(\boldsymbol{x} - s_0\boldsymbol{e}_2)} \cap \Sigma \subset [c_{j-1}, c_j] \times \{c\}$$

and

$$\eta_\delta = \sqrt{(s_0 + \delta)^2 - s_0^2} < \eta$$

with η fixed in Proposition 2. Set

$$\eta'_\delta = \sqrt{(s_0 + \delta)^2 - (x_2 - s_0 - c)^2} - |x_1 - c_j|.$$

Then one sees $\eta_\delta > \eta'_\delta > 0$. Now we divide Σ into two parts:

$$\Sigma = (\Sigma \setminus B_{s_0+\delta}(\boldsymbol{x} - s_0\boldsymbol{e}_2)) \cup (\Sigma \cap B_{s_0+\delta}(\boldsymbol{x} - s_0\boldsymbol{e}_2))\,.$$

Firstly, we consider in the case $\boldsymbol{y} \in \Sigma \setminus B_{s_0+\delta}(\boldsymbol{x} - s_0\boldsymbol{e}_2)$. In this case it holds $|\boldsymbol{y} - (\boldsymbol{x} - s_0\boldsymbol{e}_2)| \geq s_0 + \delta$. Consequently, it follows from (3.1) and (4.5) that for all $\boldsymbol{y} \in \Sigma \setminus B_{s_0+\delta}(\boldsymbol{x} - s_0\boldsymbol{e}_2)$

$$e^{-\frac{\tau}{2s_0}}\, |\boldsymbol{\sigma}(\boldsymbol{v}_\tau(\boldsymbol{y};\boldsymbol{x}))\boldsymbol{e}_2| \leq c_1 \tau e^{-c_2 \eta_\delta^2 \tau}$$

with positive constants c_1 and c_2. By using (4.4) one can see that, as $\tau \to \infty$

$$e^{-\frac{\tau}{2s_0}} \int_{\Sigma \setminus B_{s_0+\delta}(\boldsymbol{x} - s_0\boldsymbol{e}_2)} \big(\boldsymbol{u}^+(\boldsymbol{y}) - \boldsymbol{u}^-(\boldsymbol{y})\big) \cdot (\boldsymbol{\sigma}(\boldsymbol{v}_\tau(\boldsymbol{y};\boldsymbol{x}))\boldsymbol{e}_2)\, \mathrm{d}s_{\boldsymbol{y}} = O(\tau e^{-c_3 \tau}), \tag{5.1}$$

where c_3 is a positive constant described as $c_3 = c_2 \eta_\delta^2$.

5.2 Step 2: Estimate of (4.1) for $\boldsymbol{y} \in \Sigma \cap B_{s_0+\delta}(\boldsymbol{x} - s_0\boldsymbol{e}_2)$

In the case $\boldsymbol{y} \in \Sigma \cap B_{s_0+\delta}(\boldsymbol{x} - s_0\boldsymbol{e}_2)$, noting (4.4), (4.5) and Lemma 3.3 in [1], we have

$$\begin{aligned}
&-e^{-\frac{\tau}{2s_0}} \int_{\Sigma \cap B_{s_0+\delta}(\boldsymbol{x} - s_0\boldsymbol{e}_2)} \big(\boldsymbol{u}^+(\boldsymbol{y}) - \boldsymbol{u}^-(\boldsymbol{y})\big) \cdot (\boldsymbol{\sigma}(\boldsymbol{v}_\tau(\boldsymbol{y};\boldsymbol{x}))\boldsymbol{e}_2)\, \mathrm{d}s_{\boldsymbol{y}} \\
&= -e^{-\frac{\tau}{2s_0}} \int_{\Sigma \cap B_{s_0+\delta}(\boldsymbol{x} - s_0\boldsymbol{e}_2)} \left\{ \frac{\kappa+1}{\mu} \sum_{k=1}^{n} (-1)^k r^{\frac{2k-1}{2}} \begin{pmatrix} -B_{2k-1}^{(j)} \\ A_{2k-1}^{(j)} \end{pmatrix} + O\left(r^{\frac{2n+1}{2}}\right) \boldsymbol{k} \right\} \\
&\quad \cdot \left\{ 2\mu\tau \left(\frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|^2} \cdot (\boldsymbol{e}_2 + i\boldsymbol{e}_1) \right)^2 \exp\left\{ -\tau \frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|^2} \cdot (\boldsymbol{e}_2 + i\boldsymbol{e}_1) \right\} (\boldsymbol{e}_1 + i\boldsymbol{e}_2) \right\} \mathrm{d}s_{\boldsymbol{y}} \\
&= 2(\kappa+1)\tau e^{-\frac{\tau}{2s_0}} \sum_{k=1}^{n} (-1)^k \begin{pmatrix} -B_{2k-1}^{(j)} \\ A_{2k-1}^{(j)} \end{pmatrix} \cdot (\boldsymbol{e}_1 + i\boldsymbol{e}_2) I_k(\tau) + O\left(\tau^{-\frac{n+1}{2}} \tau^{\frac{3}{4}}\right).
\end{aligned} \tag{5.2}$$

Here $I_k(\tau)$ is defined as follows:

$$\begin{aligned}
I_k(\tau) &= \int_0^{\eta'_\delta} \frac{r^{\frac{2k-1}{2}}}{(r - s_0\overline{z_\alpha})^2} \exp\left(\frac{i\tau}{r - s_0\overline{z_\alpha}} \right) \mathrm{d}r, \\
z_\alpha &= -\left(e^{-\frac{\pi}{2}i} + i e^{-(\frac{\pi}{2}+\alpha)i} \right) = -\cos\alpha + i(1 + \sin\alpha),
\end{aligned}$$

where $\alpha \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ is uniquely determined as the solution of the equation

$$e^{i\alpha} = \frac{x_1 - c_j}{s_0} + i\frac{x_2 - s_0 - c}{s_0},$$

and $\overline{z_\alpha}$ denotes the complex conjugate of z_α. In this notation for $\boldsymbol{y} = (c_j - r, c)$ one sees

$$\frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|^2} \cdot (\boldsymbol{e}_2 + i\boldsymbol{e}_1) = \frac{-i}{r + (y_1 - c_j) + i(y_2 - c)} = \frac{-i}{r - s_0\overline{z_\alpha}}.$$

Now we provide a lemma for asymptotic behavior of $I_k(\tau)$ as $\tau \to \infty$;

Lemma 1 (Lemma 3.5 in [1]). *Let $n = 1, 2, 3, \cdots$. It holds for all $\alpha \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[$*

$$\lim_{\tau \longrightarrow \infty} \tau^{\frac{2n+1}{2}} e^{-\frac{\tau}{2s_0}} e^{-\frac{i\tau\cos\alpha}{2s_0(1+\sin\alpha)}} I_n(\tau)$$
$$= -is_0^{2n-1} 2^{\frac{2n-1}{2}} (1 + \sin\alpha)^{\frac{2n-1}{2}} e^{i\left(\frac{2n-1}{2}\right)\alpha} \Gamma\left(\frac{2n+1}{2}\right).$$

This lemma implies that as $\tau \to \infty$

$$e^{-\frac{\tau}{2s_0}} I_n(\tau) \sim -is_0^{2n-1} 2^{\frac{2n-1}{2}} (1 + \sin\alpha)^{\frac{2n-1}{2}} e^{i\left(\frac{2n-1}{2}\right)\alpha} e^{\frac{i\tau\cos\alpha}{2s_0(1+\sin\alpha)}} \Gamma\left(\frac{2n+1}{2}\right) \tau^{-\frac{2n+1}{2}}.$$

Combining (5.1) with (5.2), it follows from Lemma 1 that as $\tau \to \infty$

$$e^{-\frac{\tau}{2s_0}} I(\tau; \boldsymbol{x}) = \sum_{n=1}^{n_0} C_n e^{\frac{\cos\alpha}{2s_0(1+\sin\alpha)} i\tau} \tau^{-\frac{2n-1}{2}} + O\left(\tau^{-\frac{n_0+1}{2}} \tau^{\frac{3}{4}}\right), \tag{5.3}$$

where

$$\begin{aligned} C_n &= i(\kappa + 1)(-1)^{n+1} s_0^{2n-1} 2^{\frac{2n+1}{2}} (1 + \sin\alpha)^{\frac{2n-1}{2}} e^{i\left(\frac{2n-1}{2}\right)\alpha} \Gamma\left(\frac{2n+1}{2}\right) \\ &\quad \begin{pmatrix} -B_{2n-1}^{(j)} \\ A_{2n-1}^{(j)} \end{pmatrix} \cdot (\boldsymbol{e}_1 + i\boldsymbol{e}_2). \end{aligned}$$

However, (5.3) is not sufficient to prove (4.6) because it only shows that $e^{-\tau/(2s_0)} I(\tau; \boldsymbol{x})$ is *at most* algebraically decaying as $\tau \to \infty$. Note that $C_n = 0$ if and only if $A_{2n-1}^{(j)} = B_{2n-1}^{(j)} = 0$.

5.3 Step 3: Key lemma and completion of proof of Propostion 3

Next, we provide key lemma for the proof of (4.6).

Lemma 2. *Let $\boldsymbol{g} \neq \boldsymbol{0}$ satisfy the conditions* (2.6) *and* (A). *Then, there exists an integer $n \geq 1$ such that $\left(A_{2n-1}^{(j)}\right)^2 + \left(B_{2n-1}^{(j)}\right)^2 \neq 0$.*

Proof. The proof proceeds along the same line as that of Lemma 2 in [5], see also the proof of Lemma 3.4. in [1] in the case of Laplace equation.

Now we assume that $A_{2n-1}^{(j)} = B_{2n-1}^{(j)} = 0$ for all $n \geq 1$. Then, (4.2) can be reduced to

$$\boldsymbol{u}(r, \theta) = \sum_{k=0}^{\infty} \frac{A_{2k}^{(j)}}{2\mu} r^k \boldsymbol{\varphi}_{2k}(\theta) - \sum_{k=0}^{\infty} \frac{B_{2k}^{(j)}}{2\mu} r^k \boldsymbol{\psi}_{2k}(\theta), \tag{5.4}$$

which means that $\boldsymbol{u}$ is real analytic near the crack tip (c_j, c). Moreover, it is easy to see that $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{e}_2 = \boldsymbol{0}$ on $]c_j,\, c_j+\eta[\,\times\{c\}$. By use of analytic continuation, one can see $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{e}_2 = \boldsymbol{0}$ on W and all the tips of Σ are removable singularities of $\boldsymbol{u}^+$ and $\boldsymbol{u}^-$. Since $\mathrm{supp}(\boldsymbol{g}) \subset (\partial\Omega \cap \{|x_2-c| > \gamma\}) \setminus (B_\gamma(\boldsymbol{O}) \cup B_\gamma(0,b) \cup B_\gamma(a,b) \cup B_\gamma(a,0))$ due to the condition (†), $\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu} = \boldsymbol{0}$ on the edges near corner points $\boldsymbol{O}$, $(a,0)$, $(0,c)$, (a,c), $(0,b)$ and (a,b). For a sufficiently small ϵ' we set $Q^-_{\epsilon'} = \Omega^- \setminus \overline{R_{\epsilon'}}$ with $R_{\epsilon'} = B_{\epsilon'}(\boldsymbol{O}) \cup B_{\epsilon'}(0,c) \cup B_{\epsilon'}(a,0) \cup B_{\epsilon'}(a,c)$. By virtue of interior and boundary regularity results, we see that for a sufficiently small ϵ', $\boldsymbol{u} \in H^2(Q^-_{\epsilon'})$ satisfies

$$\begin{cases} \mu\triangle\boldsymbol{u} + (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) = \boldsymbol{0} & \text{in} \quad Q^-_{\epsilon'}, \\ \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu} = \boldsymbol{0} & \text{on} \quad]\epsilon', a-\epsilon'[\,\times\{c\}, \\ \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu} = \boldsymbol{g} & \text{on} \quad (\partial\Omega \cap \{x_2 < c\}) \setminus R_{\epsilon'}. \end{cases}$$

Then, the divergence theorem yields that for an arbitrary $\boldsymbol{\rho} \in \mathcal{R}$

$$\begin{aligned} 0 &= \int_{Q^-_{\epsilon'}} \boldsymbol{\rho}\cdot(\mu\triangle\boldsymbol{u} + (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}))\,\mathrm{d}\boldsymbol{x} \\ &= \int_{(\partial\Omega\cap\{x_2<c\})\setminus R_{\epsilon'}} \boldsymbol{\rho}\cdot\boldsymbol{g}\,\mathrm{d}s_{\boldsymbol{x}} + \int_{\partial R_{\epsilon'}\cap\Omega^-} \boldsymbol{\rho}\cdot\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu}\,\mathrm{d}s_{\boldsymbol{x}}. \end{aligned}$$

According to [8], as ϵ tends to 0,

$$\left| \int_{\partial R_{\epsilon'}\cap\Omega^-} \boldsymbol{\rho}\cdot\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\nu}\,\mathrm{d}s_{\boldsymbol{x}} \right| \to 0.$$

Thus, we conclude that for an arbitrary $\boldsymbol{\rho} \in \mathcal{R}$

$$\int_{\partial\Omega\cap\{x_2<c\}} \boldsymbol{g}\cdot\boldsymbol{\rho}\,\mathrm{d}s_{\boldsymbol{x}} = \boldsymbol{0}.$$

Since $\boldsymbol{g}$ satisfies the condition (†), it is a contradiction.

Similarly, the above argument is valid for the case of Ω^+. The proof of Lemma 2 is completed. □

In the consequence we can take $N = \min\left\{ n \geq 1 \mid \left(A^{(j)}_{2n-1}\right)^2 + \left(B^{(j)}_{2n-1}\right)^2 \neq 0 \right\}$. Therefore, it follows from (5.3) that for each $\boldsymbol{x} \in \Gamma_\epsilon$ satisfying the condition (3.2), (4.6) is valid with

$$A = (\kappa+1)s_0^{2N-1}2^{\frac{2N+1}{2}}(1+\sin\alpha)^{\frac{2N-1}{2}}\Gamma\left(\frac{2N+1}{2}\right)\left|\begin{pmatrix} -B^{(j)}_{2N-1} \\ A^{(j)}_{2N-1} \end{pmatrix}\cdot(\boldsymbol{e}_1 + i\boldsymbol{e}_2)\right|.$$

6 Examples of $\boldsymbol{g} \in L^2(\partial\Omega)$ satisfying the conditions (2.6) and (†)

We provide two concrete examples of $\boldsymbol{g} \in L^2(\partial\Omega)$ satisfying the conditions (2.6) and (†). For β_1, $\beta_2 \neq 0$,

$$\boldsymbol{g}_1 = \begin{cases} \beta_1\boldsymbol{e}_2 & \text{on} \quad]\gamma,\, a-\gamma[\,\times\{b\}, \\ -\beta_1\boldsymbol{e}_2 & \text{on} \quad]\gamma,\, a-\gamma[\,\times\{0\}, \\ \boldsymbol{0}, & \text{otherwise}. \end{cases}$$

Next, set $\gamma' = \min\{c - 2\gamma, b - c - 2\gamma\}$.

$$\boldsymbol{g}_2 = \begin{cases} \beta_2(a\boldsymbol{e}_1 - (2\gamma+\gamma')\boldsymbol{e}_2) \quad \text{on} \quad \{a\}\times\,]c-\gamma-\gamma',\, c-\gamma[, \\ -\beta_2(a\boldsymbol{e}_1 - (2\gamma+\gamma')\boldsymbol{e}_2) \quad \text{on} \quad \{0\}\times\,]c+\gamma,\, c+\gamma+\gamma'[, \\ \mathbf{0}, \quad \text{otherwise.} \end{cases}$$

Acknowledgments

This work is supported by the Japan Society for the Promotion of Science (JSPS) and the Russian Foundation for Basic Research (RFBR) under the Japan - Russia Research Cooperative Program (project No. J20-720), JSPS KAKENHI (No. 18K03380) and (No. 17H02857).

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