Simple matrix representations of the orthogonal polynomials for a rational spectral density on the unit circle

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Abstract

In this note, by using a discrete analog of a projection formula introduced by A. Seghier in 1978, we calculate the orthogonal polynomials on the unit circle for a rational spectral density having no zeros there, and derive simple matrix representations of themselves, their squared norms, and the Verblunsky coefficients.

Keywords: Orthogonal polynomials on the unit circle, Verblunsky coefficients, rational spectral density, Seghier's formula
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1. Introduction

Let μ be a probability measure on the unit circle $\mathbb{T}$ that is supported on an infinite set. By applying the Gram–Schmidt method to the sequence of monomials $1, z, z^2, \ldots$ in the Hilbert space $L^2(\mu)$ with

$$(\varphi, \psi)_\mu = \int_0^{2\pi} \varphi(e^{i\theta})\overline{\psi(e^{i\theta})}d\mu(\theta), \qquad \|\varphi\|_\mu = \left(\int_0^{2\pi} |\varphi(e^{i\theta})|^2 d\mu(\theta) \right)^{1/2},$$

one obtains a sequence of the *monic orthogonal polynomials* $\Phi_0, \Phi_1, \Phi_2, \ldots$ with respect to μ, i.e.,

$$\Phi_n(z) = z^n + \text{lower order}, \qquad (\Phi_n, \Phi_{n'})_\mu = 0 \qquad (n \neq n').$$

Their constant terms $\alpha_n = \Phi_n(0)$ lie in the open unit disc $\mathbb{D}$ for all $n = 1, 2, \ldots$. Therefore, μ yields a sequence $\{\alpha_1, \alpha_2, \ldots\}$ in $\mathbb{D}$. Verblunsky's theorem states that every sequence in $\mathbb{D}$ arises in this way from a unique probability measure on $\mathbb{T}$ that is supported on an infinite set, whence α_n are called the *Verblunsky coefficients*. (There are several other names such as the Schur parameters.) As an application, the monic orthogonal polynomials are used for studying the prediction problem for a centered, complex-valued stationary time series $\{X_k\}$ with spectral measure μ, i.e.,

$$\mathrm{Cov}(X_j, X_k) = \int_0^{2\pi} e^{i(j-k)\theta} d\mu(\theta) \qquad (j, k = 0, \pm 1, \pm 2, \ldots).$$

More precisely, Φ_n is a spectral counterpart of the one-step prediction error in the sense that

$$X_n - \hat{X}_n = X_n - \sum_{k=1}^{n} \phi_{n,k} X_{n-k} \quad \leftrightarrow \quad \Phi_n(z) = z^n - \sum_{k=1}^{n} \phi_{n,k} z^{n-k}, \qquad \mathrm{Var}(X_n - \hat{X}_n) = \|\Phi_n\|_\mu^2,$$

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where $\hat{X}_n$ is the linear predictor of X_n based on $\{X_0, X_1, \ldots, X_{n-1}\}$. Since $\phi_{n,n} = -\alpha_n$, the Verblunsky coefficients are essentially the same as the *partial autocorrelation function* $\{\phi_{n,n}\}$. See Simon [11] for the theory of orthogonal polynomials on the unit circle, and Brockwell–Davis [2] for the prediction problem of stationary time series.

Inoue [3, 4, 5], Inoue–Kasahara [6, 7] and Bingham *et al.* [1] have developed explicit representations of the predictor coefficients and prediction error variances, and studied their applications. Taking a different approach, this note presents simple matrix representations of the monic orthogonal polynomials and related quantities for a probability measure of the form $d\mu(\theta) = w(e^{i\theta})\frac{d\theta}{2\pi}$, where w is a rational function of $e^{i\theta}$ satisfying $w(e^{i\theta}) \neq 0$ on $\mathbb{T}$. As is well known, the latter can be expressed as

$$w(e^{i\theta}) = \frac{|P(e^{i\theta})|^2}{|Q(e^{i\theta})|^2} \tag{1}$$

with some polynomials P and Q which have no common zeros and satisfy

$$P(0) > 0, \qquad Q(0) > 0, \qquad P(z) \neq 0, \qquad Q(z) \neq 0 \qquad (z \in \mathbb{D} \cup \mathbb{T}). \tag{2}$$

Such a rational function is important in time series analysis since it is the spectral density of a causal and invertible ARMA (autoregressive-moving average) process, characterized by the difference equation

$$Q(B)X_k = P(B)Z_k \qquad (k = 0, \pm 1, \pm 2, \ldots),$$

where B is the backward shift operator ($BX_k = X_{k-1}$) and $\{Z_k\}$ is a white noise ($\mathrm{Cov}(Z_j, Z_k) = \delta_{jk}$). The matrix representations are derived by calculating Φ_n via an analog of a projection formula introduced by Seghier [10], which is explained in Section 2. The representations themselves, including those of the squared norms $\|\Phi_n\|_\mu^2$ and the Verblunsky coefficients α_n, are given in Section 3. Among other results, the representation of α_n is particularly simple.

We dedicate this note to Yusuke Nakamura (1992–2016) who discovered a similar matrix representation of the finite predictor for some continuous-time process, in joint work with the first author. It was this discovery that made the latter realize, after his death, the existence of explicit formulas in a simple case of this paper, and thus led us to the present work.

2. Seghier's formula

In his work on the prediction problem for a continuous-time stationary process, Seghier [10] introduced a projection formula which is useful when the process has a rational spectral density. This section is devoted to a brief discussion on its discrete analog and related matters. For details, see Kasahara–Bingham [8].

Let $\mathbb{D}$, $\mathbb{T}$ and $\mathbb{E}$ denote the open unit disc, the unit circle, and the region outside the circle in the extended complex plane, respectively, so that $\mathbb{C} \cup \{\infty\} = \mathbb{D} \cup \mathbb{T} \cup \mathbb{E}$ (disjoint union). Recall the Hilbert space $L^2(\mu)$ from Section 1. Let $H^2(\mu)$ and $K^2(\mu)$ be the closed subspaces of $L^2(\mu)$ spanned by $\{1, e^{i\theta}, e^{2i\theta}, \ldots\}$ and $\{e^{-i\theta}, e^{-2i\theta}, \ldots\}$, respectively. Write L^2, H^2 and K^2 for these spaces with the normalized Lebesgue measure $d\mu(\theta) = \frac{d\theta}{2\pi}$, which gives $L^2 = H^2 \oplus K^2$ (orthogonal decomposition). The functions in H^2 and K^2 have natural analytic extensions into $\mathbb{D}$ and $\mathbb{E}$, respectively. Denote by $\mathbb{C}[e^{i\theta}]_{<n}$ the space of polynomials of $e^{i\theta}$ with degree less than n. The monic orthogonal polynomial Φ_n is the projection of $e^{in\theta}$ onto the orthogonal complement of $\mathbb{C}[e^{i\theta}]_{<n}$ in $L^2(\mu)$.

Let $d\mu(\theta) = w(e^{i\theta})\frac{d\theta}{2\pi} + d\mu_s(\theta)$, where μ_s is the singular part. If $\log w$ is integrable, there exists a unique outer function h, called the *Szegő function*, in H^2 such that $w = |h|^2$ and $h(0) > 0$. By [8, Theorem 3.1], the equation

$$H^2(\mu) \cap e^{in\theta} K^2(\mu) = \mathbb{C}[e^{i\theta}]_{<n} \tag{3}$$

holds for every $n = 0, 1, 2, \ldots$ if and only if $\mu_s = 0$, $\log w$ is integrable, and h^2 is *rigid*, which means that, except for its positive constant multiples, no functions in the classical Hardy space H^1 share with it the same argument. Set

$$\phi^\dagger(z) = \overline{\phi(1/\overline{z})}.$$

For $n = 0, 1, 2, \ldots$, let H_n be the Hankel operator with symbol $e^{in\theta}(h/h^\dagger)$ and H_n^* its adjoint operator, i.e.,

$$H_n(\varphi) = \mathscr{P}_{K^2}\{e^{in\theta}(h/h^\dagger)\varphi\} \qquad (\varphi \in H^2), \qquad H_n^*(\psi) = \mathscr{P}_{H^2}\{e^{-in\theta}(h^\dagger/h)\psi\} \qquad (\psi \in K^2),$$

where $\mathscr{P}_S$ is the orthogonal projection operator of L^2 onto S. By [8, Remark 3.4 and Proposition 3.5], if (3) holds and also if $I - H_n^* H_n$ is invertible, the monic orthogonal polynomial Φ_n can be expressed as

$$\Phi_n = h(0)\{e^{in\theta}(h^\dagger)^{-1}(I - H_n^* H_n)^{-1}1 - h^{-1}H_n(I - H_n^* H_n)^{-1}1\}, \tag{4}$$

which is a discrete analog of Seghier's projection formula in [10, Proposition 3].

3. Results

We calculate the monic orthogonal polynomials for a rational spectral density having no zeros on the unit circle, and derive matrix representations of themselves and the related quantities.

Throughout this section, assume that $\mu_s = 0$ and w is a rational function of $e^{i\theta}$ having no zeros on $\mathbb{T}$. In this case, h^2 is rigid since its reciprocal lies in H^1; see Yabuta [12]. Also, H_n is strictly contractive for all $n = 0, 1, 2, \ldots$ since the Helson–Szegő theorem implies $\|H_0\| < 1$, and also $\|H_n\|$ is non-increasing in n; see Peller [9, Chapter 9]. (The value of $\|H_n\|$ is equal to the cosine of the angle between "past" $K^2(\nu)$ and "future" $e^{in\theta}H^2(\nu)$, where $d\nu = w^{-1}\frac{d\theta}{2\pi}$.) Consequently, (3) holds true, $I - H_n^* H_n$ is invertible, and thus, Seghier's formula (4) is available for all $n = 0, 1, 2, \ldots$.

Let P and Q be the polynomials satisfying (1) and (2). Without loss of generality, one may assume that $P(0) = 1$, and also that P is non-constant. (For, if $P(z) = 1$, then $\Phi_n(z) = z^n Q^\dagger(z)$ for $n \geq \deg Q$, and $\alpha_n = 0$ for $n > \deg Q$, hence there is nothing to do; see [2, Example 3.4.3] and [11, Corollary 1.7.6].) This assumption permits us to write

$$P(z) = \prod_{i=1}^{d}(1 - \overline{p}_i z)^{m_i},$$

where $p_1, p_2, \ldots, p_d$ are distinct points in $\mathbb{D} \setminus \{0\}$ and $m_1, m_2, \ldots, m_d$ are positive integers. Define

$$m = \deg P \, (= m_1 + m_2 + \cdots + m_d), \qquad n_0 = \max\{\deg Q - \deg P, 0\}.$$

By the conditions (1) and (2), the Szegő function h is given by

$$h(z) = \frac{P(z)}{Q(z)}.$$

Since

$$\frac{1}{h^\dagger(z)} = \frac{z^m Q^\dagger(z)}{\displaystyle\prod_{i=1}^{d}(z - p_i)^{m_i}},$$

one may write

$$\frac{h(z)}{h^\dagger(z)} = \sum_{i=1}^{d}\sum_{j=1}^{m_i}\frac{a_{ij}}{(z - p_i)^j} + R(z), \tag{5}$$

where a_{ij} is the j-th coefficient in the principal part of the Laurent series expansion at a pole $z = p_i$, i.e.,

$$a_{ij} = \lim_{z \to p_i}\frac{1}{(m_i - j)!}\frac{d^{m_i - j}}{dz^{m_i - j}}\frac{(z - p_i)^{m_i}\, h(z)}{h^\dagger(z)},$$

and R is a rational function which is holomorphic in $\mathbb{D} \cup \mathbb{T}$ except for a pole at $z = 0$ of multiplicity n_0 in the case $n_0 > 0$. Set

$$\phi^\sharp(z) = (1/z)\overline{\phi(1/\overline{z})},$$

which provides us with

$$(\phi^\sharp)^\sharp = \phi, \qquad H^2 \ni \varphi \mapsto \varphi^\sharp \in K^2, \qquad K^2 \ni \psi \mapsto \psi^\sharp \in H^2, \qquad H_n^* \psi = (H_n \psi^\sharp)^\sharp.$$

To check the last equation, use $(\mathscr{P}_{H^2}\phi)^\sharp = \mathscr{P}_{K^2}(\phi^\sharp)$ and $(\chi\psi)^\sharp = \chi^\dagger \psi^\sharp$ in the following way:

$$[\mathscr{P}_{H^2}\{e^{-in\theta}(h^\dagger/h)\psi\}]^\sharp = \mathscr{P}_{K^2}[\{e^{-in\theta}(h^\dagger/h)\psi\}^\sharp] = \mathscr{P}_{K^2}[\{e^{-in\theta}(h^\dagger/h)\}^\dagger \psi^\sharp] = \mathscr{P}_{K^2}\{e^{in\theta}(h/h^\dagger)\psi^\sharp\}.$$

As will be seen presently, the Hankel operators H_n are closely related with the rational functions

$$f_{ij}(z) = \frac{z^{j-1}}{(1-\overline{p}_i z)^j}, \qquad f_{ij}^\sharp(z) = \frac{1}{(z-p_i)^j}.$$

Define a vector-valued function $\boldsymbol{f}$ by the $(m_1, m_2, \ldots, m_d)$-block representation

$$\boldsymbol{f}(z) = {}^T(f_{11}(z), \ldots, f_{1m_1}(z) \,|\, f_{21}(z), \ldots, f_{2m_2}(z) \,|\, \cdots \,|\, f_{d1}(z), \ldots, f_{dm_d}(z)).$$

Also, for a holomorphic function φ in $\mathbb{D}$, define an m-dimensional column vector $\boldsymbol{v}[\varphi]$ by

$$\boldsymbol{v}[\varphi] = {}^T([\varphi]_{10}, \ldots, [\varphi]_{1,m_1-1} \,|\, [\varphi]_{20}, \ldots, [\varphi]_{2,m_2-1} \,|\, \cdots \,|\, [\varphi]_{d0}, \ldots, [\varphi]_{d,m_d-1}),$$

where $[\varphi]_{ij}$ is the j-th coefficient in the Taylor series expansion at $z = p_i$, i.e.,

$$[\varphi]_{ij} = \frac{1}{j!}\frac{d^j\varphi(z)}{dz^j}\Big|_{z=p_i}.$$

In addition, let $\mathbf{1}$ denote a constant vector $\boldsymbol{f}(0) = \boldsymbol{v}[1] = {}^T(1, 0, \ldots, 0 \,|\, 1, 0, \ldots, 0 \,|\, \cdots \,|\, 1, 0, \ldots, 0)$.

To obtain the main results of this note, let us prepare three lemmas.

Lemma 1. *Let $n \geq n_0$. If φ is a holomorphic function in $\mathbb{D} \cup \mathbb{T}$, then $(H_n\varphi)(z)$ is a linear combination of $f_{ij}^\sharp(z)$ expressed in the form*

$$(H_n\varphi)(z) = {}^T\boldsymbol{f}^\sharp(z) A B_n \boldsymbol{v}[\varphi],$$

where A and B_n respectively are $d \times d$ block diagonal matrices with $m_i \times m_i$ submatrices

$$A_i = \begin{pmatrix} a_{i,j+k-1} \end{pmatrix} \qquad \text{(upper anti-triangular Hankel matrices)}$$

and

$$B_{n,i} = \begin{pmatrix} \binom{n}{j-k} p_i^{n-(j-k)} \end{pmatrix} \qquad \text{(lower triangular Toeplitz matrices)},$$

the displayed terms being the (j,k)-entries.

Proof. The value of the function $H_n\varphi$ can be evaluated by the formula

$$(H_n\varphi)(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\zeta^n h(\zeta)\varphi(\zeta)}{h^\dagger(\zeta)(z-\zeta)}\, d\zeta \qquad (z \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T})).$$

In view of the partial fraction expansion (5), it is seen that the integrand is holomorphic in $\mathbb{D} \cup \mathbb{T}$ except for poles at $z = p_1, p_2, \ldots, p_d$ since the singularity at $z = 0$ is removable for $n \geq n_0$. By Leibniz's rule,

$$\begin{aligned} \frac{1}{(r-1)!}\frac{d^{r-1}}{d\zeta^{r-1}}\frac{\zeta^n\varphi(\zeta)}{z-\zeta} &= \sum_{j=1}^{r} \frac{1}{(z-\zeta)^j} \cdot \frac{1}{(r-j)!}\frac{d^{r-j}}{d\zeta^{r-j}}\{\zeta^n\varphi(\zeta)\} \\ &= \sum_{j=1}^{r} \frac{1}{(z-\zeta)^j} \sum_{k=1}^{r-j+1} \binom{n}{r-j+1-k} \zeta^{n-(r-j+1-k)} \frac{1}{(k-1)!}\frac{d^{k-1}}{d\zeta^{k-1}}\varphi(\zeta). \end{aligned}$$

Thus, by the formula (5) and the residue theorem,

$$\begin{aligned}(H_n\varphi)(z) &= \sum_{i=1}^{d}\sum_{r=1}^{m_i} a_{ir}\cdot\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\zeta^n\varphi(\zeta)}{(\zeta-p_i)^r(z-\zeta)}\,d\zeta\\ &= \sum_{i=1}^{d}\sum_{r=1}^{m_i} a_{ir}\sum_{j=1}^{r}\frac{1}{(z-p_i)^j}\sum_{k=1}^{r-j+1}\binom{n}{r-j+1-k}p_i^{n-(r-j+1-k)}\,[\varphi]_{i,k-1}\\ &= \sum_{i=1}^{d}\sum_{j=1}^{m_i}\frac{1}{(z-p_i)^j}\sum_{k=1}^{m_i-j+1}\left\{\sum_{l=k}^{m_i-j+1} a_{i,j+l-1}\binom{n}{l-k}p_i^{n-(l-k)}\right\}[\varphi]_{i,k-1},\end{aligned}$$

in which $l = r-j+1$, and this may be rewritten into the matrix form of the assertion. □

Lemma 2. *Let $\boldsymbol{u}$ be an m-dimensional column vector. If $\varphi(z) = {}^T\boldsymbol{f}(z)\boldsymbol{u}$, then*

$$\boldsymbol{v}[\varphi] = C\boldsymbol{u},$$

where C is a $d\times d$ block matrix with $m_i \times m_j$ submatrices

$$C_{ij} = \left(\ [f_{jl}]_{i,k-1}\ \right),$$

the displayed term being the (k,l)-entry. The value of $[f_{jl}]_{ik}$ can be evaluated as

$$[f_{jl}]_{ik} = \sum_{r=0}^{k}\binom{k}{r}\binom{l+k-r-1}{k}\frac{p_i^{l-1-r}\,\overline{p}_j^{k-r}}{(1-p_i\overline{p}_j)^{l+k-r}}.$$

Proof. This assertion is merely a matrix representation of the linear equations

$$[\varphi]_{ik} = \sum_{j=1}^{d}\sum_{l=1}^{m_j}[f_{jl}]_{ik}u_{jl},$$

subject to $\boldsymbol{u} = {}^T(u_{11},\dots,u_{1m_1}\,|\,u_{21},\dots,u_{2m_2}\,|\,\cdots\,|\,u_{d1},\dots,u_{dm_d})$. □

Lemma 3. *If $n \ge n_0$, then*

$$\{(I-H_n^*H_n)^{-1}1\}(z) = 1 + {}^T\boldsymbol{f}(z)\overline{AB}_nG_n(I-\overline{G}_nG_n)^{-1}\mathbf{1},$$

where

$$G_n = \overline{C}AB_n.$$

In particular, this gives

$$\boldsymbol{v}[(I-H_n^*H_n)^{-1}1] = (I-\overline{G}_nG_n)^{-1}\mathbf{1}.$$

Proof. Assume $n \ge n_0$. Let $\boldsymbol{u}$ be an m-dimensional column vector. It follows from Lemmas 1 and 2 that, for $\varphi(z) = {}^T\boldsymbol{f}(z)\boldsymbol{u}$,

$$(H_n\varphi)(z) = {}^T\boldsymbol{f}^\sharp(z)AB_nC\,\boldsymbol{u}.$$

Since $(H_n^*\psi)(z) = (H_n\psi^\sharp)^\sharp(z)$, this implies that, for $\psi(z) = {}^T\boldsymbol{f}^\sharp(z)\boldsymbol{u}$,

$$(H_n^*\psi)(z) = {}^T\boldsymbol{f}(z)\,\overline{AB_nC}\,\boldsymbol{u}.$$

Clearly, $I-\overline{G}_nG_n$ is invertible if and only if $I-G_n\overline{G}_n$ is invertible. To check the latter, let $(I-G_n\overline{G}_n)\,\boldsymbol{u} = \mathbf{0}$. By applying the above results to $\chi(z) = {}^T\boldsymbol{f}(z)\overline{AB}_n\boldsymbol{u}$,

$$\{(I-H_n^*H_n)\,\chi\}(z) = {}^T\boldsymbol{f}(z)\,\overline{AB}_n(I-G_n\overline{G}_n)\,\boldsymbol{u} = 0.$$

Since $I - H_n^* H_n$ is invertible, one has $\chi(z) = 0$, so that $\overline{AB}_n \boldsymbol{u} = \boldsymbol{0}$. Also, $\overline{AB}_n$ is invertible, as is seen from

$$\det(A_i) = \operatorname{sgn}(\sigma)(a_{i,m_i})^{m_i} \neq 0, \qquad \det(B_{i,n}) = (p_i^n)^{m_i} \neq 0,$$

where σ is a permutation defined by $\sigma(k) = m_i - k + 1$ ($k = 1, 2, \ldots, m_i$). Hence, $\boldsymbol{u} = \boldsymbol{0}$, and $I - G_n \overline{G}_n$ is invertible. Now, by using Lemma 1 and the above results,

$$\begin{aligned}\{H_n^* H_n (1 + {}^T\boldsymbol{f}\, \overline{AB}_n G_n (I - \overline{G}_n G_n)^{-1} \boldsymbol{1})\}(z) &= \{{}^T\boldsymbol{f} \overline{AB}_n G_n (I + \overline{G}_n G_n (I - \overline{G}_n G_n)^{-1}) \boldsymbol{1}\}(z) \\ &= \{{}^T\boldsymbol{f} \overline{AB}_n G_n (I - \overline{G}_n G_n)^{-1} \boldsymbol{1}\}(z),\end{aligned}$$

which gives

$$\{(I - H_n^* H_n)(1 + {}^T\boldsymbol{f}\, \overline{AB}_n G_n (I - \overline{G}_n G_n)^{-1} \boldsymbol{1})\}(z) = 1.$$

Again, since $I - H_n^* H_n$ is invertible, the first half of the assertion holds. The other half can be shown by applying Lemma 2 to $\{(I - H_n^* H_n)^{-1} 1\}(z) = 1 + {}^T\boldsymbol{f}(z)\, \boldsymbol{u}$ with $\boldsymbol{u} = \overline{AB}_n G_n (I - \overline{G}_n G_n)^{-1} \boldsymbol{1}$, and then doing a calculation similar to the above one. $\square$

The following result presents an explicit matrix representation of a discrete analog of Seghier's prediction formula in [10, Corollary 2]. (The original result implicitly gives the "form" of the formula under some general assumption.)

Theorem 4. *For $n \geq n_0$, the monic orthogonal polynomials are expressed as*

$$\Phi_n(z) = z^n \frac{h(0)}{h^\dagger(z)} (1 + {}^T\boldsymbol{f}(z) \overline{AB}_n G_n (I - \overline{G}_n G_n)^{-1} \boldsymbol{1}) - \frac{h(0)}{h(z)}\ {}^T\boldsymbol{f}^\sharp(z) AB_n (I - \overline{G}_n G_n)^{-1} \boldsymbol{1},$$

and their squared norms admit the representation

$$\|\Phi_n\|_\mu^2 = h(0)^2 (1 + {}^T\boldsymbol{g}_n (I - \overline{G}_n G_n)^{-1} \overline{G}_n \boldsymbol{1}),$$

where

$$\boldsymbol{g}_n = AB_n \boldsymbol{1}.$$

Proof. The first formula is readily obtained by applying Lemmas 1 and 3 to Seghier's formula (4). To prove the other one, note that $\|\Phi_n\|_\mu^2 = (\Phi_n, z^n)_\mu$, for which Seghier's formula (4) gives

$$\frac{\Phi_n(z) w(z)}{z^n} = h(0) h(z) \{(I - H_n^* H_n)^{-1} 1\}(z) - \frac{h(0) h^\dagger(z)}{z^n} \{H_n (I - H_n^* H_n)^{-1} 1\}(z).$$

Since the second term of the RHS is holomorphic in $\mathbb{T} \cup \mathbb{E}$ and vanishes at $z = \infty$,

$$\begin{aligned}\|\Phi_n\|_\mu^2 &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\Phi_n(\zeta) w(\zeta)}{\zeta^n} \frac{d\zeta}{\zeta} = h(0) \cdot \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{h(\zeta) \{(I - H_n^* H_n)^{-1} 1\}(\zeta)}{\zeta}\, d\zeta \\ &= h(0)^2 \{(I - H_n^* H_n)^{-1} 1\}(0).\end{aligned}$$

Observed that ${}^T(AB_n) = AB_n$. By Lemma 3 and $\overline{G}_n (I - G_n \overline{G}_n)^{-1} = (I - \overline{G}_n G_n)^{-1} \overline{G}_n$,

$$\begin{aligned}\{(I - H_n^* H_n)^{-1} 1\}(0) = \overline{\{(I - H_n^* H_n)^{-1} 1\}(0)} &= 1 + {}^T\boldsymbol{1} AB_n \overline{G}_n (I - G_n \overline{G}_n)^{-1} \boldsymbol{1} \\ &= 1 + {}^T\boldsymbol{g}_n (I - \overline{G}_n G_n)^{-1} \overline{G}_n \boldsymbol{1},\end{aligned}$$

which completes the proof. $\square$

The following matrix representation is particularly simple.

Theorem 5. *For $n > n_0$, the Verblunsky coefficients admit the representation*

$$\alpha_n = {}^T\boldsymbol{g}_{n-1} (I - \overline{G}_n G_n)^{-1} \boldsymbol{1}.$$

Proof. Seghier's formula (4) shows that

$$\frac{h(z)\Phi_n(z)}{h(0)} = \frac{z^n h(z)}{h^\dagger(z)}\{(I - H_n^* H_n)^{-1}1\}(z) - \{H_n(I - H_n^* H_n)^{-1}1\}(z),$$

in which the last term, i.e., $\{H_n(I - H_n^* H_n)^{-1}1\}(z)$, is holomorphic in $\mathbb{T} \cup \mathbb{E}$ and vanishes at $z = \infty$. Hence, by definition of the Verblunsky coefficients,

$$\alpha_n = \Phi_n(0) = \frac{h(0)\Phi_n(0)}{h(0)} = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{h(\zeta)\Phi_n(\zeta)}{h(0)}\frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\zeta^{n-1}h(\zeta)}{h^\dagger(\zeta)}\{(I - H_n^* H_n)^{-1}1\}(\zeta)\, d\zeta.$$

Now, let φ be a holomorphic function in $\mathbb{D} \cup \mathbb{T}$. By Leibniz's rule,

$$\frac{1}{(r-1)!}\frac{d^{r-1}}{d\zeta^{r-1}}\{\zeta^{n-1}\varphi(\zeta)\} = \sum_{j=1}^{r}\binom{n-1}{r-j}\zeta^{(n-1)-(r-j)} \cdot \frac{1}{(j-1)!}\frac{d^{j-1}}{d\zeta^{j-1}}\varphi(\zeta).$$

So, as in the proof of Lemma 1, the formula (5) and the residue theorem imply that, for $n > n_0$,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\zeta^{n-1}h(\zeta)\varphi(\zeta)}{h^\dagger(\zeta)}\, d\zeta &= \sum_{i=1}^{d}\sum_{r=1}^{m_i} a_{ir} \cdot \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\zeta^{n-1}\varphi(\zeta)}{(\zeta - p_i)^r}\, d\zeta \\
&= \sum_{i=1}^{d}\sum_{r=1}^{m_i} a_{ir}\sum_{j=1}^{r}\binom{n-1}{r-j}p_i^{(n-1)-(r-j)}[\varphi]_{i,j-1} \\
&= \sum_{i=1}^{d}\sum_{j=1}^{m_i}\left\{\sum_{k=1}^{m_i-j+1} a_{i,j+k-1}\binom{n-1}{k-1}p_i^{(n-1)-(k-1)}\right\}[\varphi]_{i,j-1},
\end{aligned}$$

in which $k = r - j + 1$, and this may be expressed as ${}^T\boldsymbol{g}_{n-1}\boldsymbol{v}[\varphi]$. So, Lemma 3 gives the representation. □

Remark 6. If $m = d$, i.e., if P has only simple zeros, the above results take simple forms without block structure. Indeed, the basic three matrices are written as

$$A = \mathrm{diag}(a_1, \ldots, a_d), \qquad B_n = \mathrm{diag}(p_1^n, \ldots, p_d^n), \qquad C = \begin{pmatrix} \frac{1}{1-p_1\overline{p}_1} & \cdots & \frac{1}{1-p_1\overline{p}_d} \\ \vdots & & \vdots \\ \frac{1}{1-p_d\overline{p}_1} & \cdots & \frac{1}{1-p_d\overline{p}_d} \end{pmatrix},$$

where $a_i = a_{i1}$, the residue of $h/h^\dagger$ at its pole $z = p_i$.

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