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**Topological Phenomena in
Quantum Walks owing to
Peculiarity of Floquet and/or
Open Systems**

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Chapter 1

Introduction

Looking around our surroundings, we find many objects with various shapes. Since long ago, the shape of objects has been an object of our primitive interest. Mathematicians have established languages and tools to describe and classify "shapes" of objects based on integers referred to as topological numbers [1]. Topological numbers are determined from global natures of objects, independently of details which can be changed by continuous deformations. Over the past few decades, such mathematically sophisticated methods have been found beneficial for understanding various physical phenomena [2, 3]. Physicists have applied analyses based on topological numbers to not only objects in real space such as spin configurations but also objects in momentum space such as energy band structures. Phases in which topological numbers of energy band structures are zero and non-zero are respectively referred to as trivial and non-trivial topological phases. Since topological numbers of energy bands are determined from global structures of energy eigenstates in momentum space, it depends only on symmetries and spatial dimensions of systems whether non-trivial topological phases can exist or not. Then, features which stem from such symmetry protected topological phases persist in a variety of physical settings which share same symmetries, irrespective of details such as the kind of particles and/or various local perturbations without breaking the symmetries. Based on such universality, explorations on topological phases have made significant progress in the research field of condensed matter physics [2, 3]. The topologies of energy band structures are manifested by topologically protected edge states localizing at boundaries of systems, which is referred to as the bulk-edge correspondence. From the viewpoint of non-trivial topological phases and the bulk-edge correspondence, interpretations on already known phenomena have been constructed and novel phenomena have been predicted theoretically. For instances, the integer quantum Hall effect is caused by chiral edge states which result from non-trivial topological phases characterized by the Chern number [4, 5, 6, 7, 8, 9, 10], theoretical works on topological phases characterized by $\mathbb{Z}_2$ number led to the discovery of topological insulators which accommodate helical edge states [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22], and some of Andreev bound states can be regarded as edge states in topological superconductors [23, 24, 25, 26, 27, 28, 29]. While the physics of symmetry protected topological phases was firstly explored in electronic systems historically, its universality ensures that the concept of topology can also be applied to photonic systems [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47].

A quantum walk is one of such photonic systems in which non-trivial topological phases can exist [48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76]. Quantum walks were mathematically introduced as quantum versions of classical random walks [77, 78, 79] and dynamics in various physical settings are regarded as quantum walks [80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100]. Quantum walks have advantages for exploring topological phases in comparison to other electronic and/or photonic systems, that symmetries and topological numbers are easily manipulated and real space observations of topologically protected edge states are possible [51, 62, 64, 70, 72, 73, 76]. Owing

to these features, various topological phases and resulting edge states of quantum walks have been explored. As we explain in Chap. 3, quantum walks are often referred to as stroboscopic simulators of Hamiltonian dynamics, since the dynamics of quantum walks are similar to the dynamics of Schrödinger or Dirac equation governed by time-independent Hamiltonians. Actually, many topological features of gapped energy bands in photonic systems including quantum walks can be understood from the analogy to electronic systems described by time-independent Hamiltonians. However, unfortunately, the simulation abilities of quantum walks are much inferior to those of computers since it is difficult to prepare large systems and errors are accumulated as time steps are increased in experiments. In addition, while simulating or mimicing topological phases of electronic systems in photonic systems may lead to useful optical devices, it is difficult to yield theoretically novel achievements from such studies.

Then, what is the meaning of studying topological phases in quantum walks? Our answer is that explorations of topological phenomena peculiar to quantum walks have significance, without analogy to conventional systems. Studying unique topological phenomena qualitatively different from conventional systems may lead to extensions of scopes to which theories based on topologies are effectively applicable. To this end, we focus on three distinct features of quantum walks from conventional electronic systems and/or other photonic systems. First is that gain-loss effects of light can be manipulated in quantum walks [86, 93, 64, 72, 76]. Since the probability of photons and the intensity of light are not conserved, novel topological phenomena without analogy to closed systems can occur in such open systems. Using loss effects of light which make systems open to environments, nonlinear effects may also be manipulated in the dynamics based on feed-forward controls [94, 95, 96]. Also, nonlinearity is induced in quantum walks where light pulses with high intensity propagate in optical fiber loops [97, 98, 99, 100]. Then, secondary, nonlinear effects may cause peculiar topological phenomena in quantum walks. Third is that time evolves in a discrete manner, from which quantum walks belong to Floquet systems (periodically driven systems) [48, 49, 50, 51, 53, 54, 55, 56, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 72, 73, 74, 75, 76]. The periodicity in time should be the origin of unique topological phenomena which cannot be understood from the analogy to static systems described by time-independent Hamiltonians. As a result, we clarify novel topological phenomena stemming from these peculiar features which are gain-loss effects of light, nonlinear effects, and discrete periodicity in time direction.

The rest of this doctoral thesis is organized as follows. Chapters 2 and 3 are devoted to the introduction of basic knowledge and backgrounds about topological phases and quantum walks. In Chap. 2, we explain topological phases and the bulk-edge correspondence in static systems, focusing on the winding number which is a kind of topological numbers. In Chap. 3, we give the explanation on how to describe the dynamics and topological phases of quantum walks. Chapters 4, 5, and 6 include our results. In Chap. 4, we derive topological numbers in chiral symmetric open Floquet systems and confirm the bulk-edge correspondence based on the obtained topological numbers. We clarify that discreteness of time can lead to novel bifurcations of edge states in Chap. 5, addressing two types of nonlinear quantum walks. We discuss in Chap. 6 topological phases unique to quantum walks, based on chiral symmetry and/or time-glide symmetry. Chapter 7 gives summary and outlook.

Chapter 2

Topological Phases

In this chapter, we briefly explain topological phases, the bulk-edge correspondence, and resulting edge states. Throughout this doctoral thesis, we focus on topological phases characterized by the winding number, while there are other topological numbers such as the Chern number and the $\mathbb{Z}_2$ topological number. By narrowing the subject rather than comprehensively explaining various topological numbers, we provide an intuitive way of understanding the winding number in Sec. 2.1. We show that topologically protected edge states emerge originating from the winding number in Sec. 2.2, by using a chiral symmetric model. In Sec. 2.3, we explain the classification of topological phases based on symmetries and spatial dimensions of systems.

2.1 Intuitive introduction of the winding number

Topological numbers are integers unchanged under continuous deformations which correspond to transformations without cutting and glueing of objects. The simplest example of topological numbers is the number of holes. Given an object such as a ball, a cup, or fake glasses without lenses, we can count the numbers of holes as in Fig. 2.1 (a), which topologically characterizes the objects. We can easily understand that the number of holes does not change unless the object is cut or glued. In other words, the number is a global invariant under continuous transformations. The number of holes is analogous to the vorticity which characterizes spin configurations in two-dimensional classical spin systems like XY model. Figure 2.1 (b) shows various configurations in such spin systems. Ignoring the magnitude of spins, the whole information of a two-dimensional classical spin system is included in the angle of spins $\varphi(\mathbf{r})$,

$$\mathbf{S}(\mathbf{r}) \propto \begin{pmatrix} \cos[\varphi(\mathbf{r})] \\ \sin[\varphi(\mathbf{r})] \end{pmatrix}, \quad (2.1)$$

where $\mathbf{S}(\mathbf{r})$ represents the direction of spin at each position on the two-dimensional plane. We can characterize various configurations of classical spins by a vorticity, which is defined as

$$\nu = \frac{1}{2\pi} \oint_p d\mathbf{r} \cdot \nabla \varphi(\mathbf{r}), \quad (2.2)$$

where the integral path p is a closed loop. If the vorticity obtained from a loop takes nonzero values as in the center and right configurations of Fig. 2.1 (b), $\varphi(\mathbf{r})$ is not well defined at a point inside the loop. Then, the point is referred to as the vortex core which is schematically described as a red point in Fig. 2.1 (b). A vortex core is analogous to a hole in Fig. 2.1 (a), while the corresponding vorticity is not restricted to 0, 1 and can take arbitrary integer values $\cdots -2, -1, 0, 1, 2, \cdots$. The vorticity is unchanged by deforming the integral path p , unless p goes across the "hole" at which $\varphi(\mathbf{r})$ is ill defined. Roughly speaking, the vorticity in Eq. (2.2) not only counts the number of

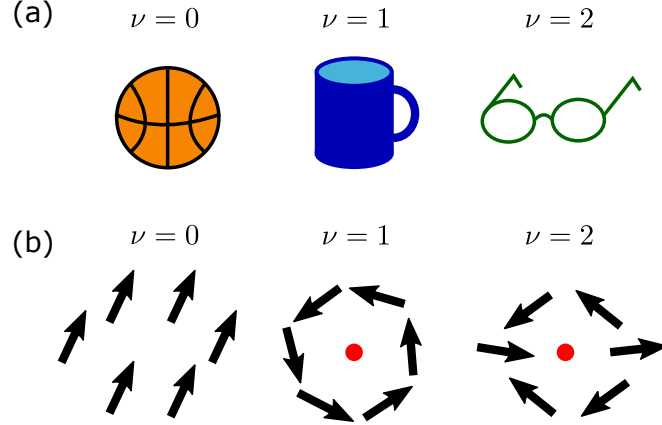


Figure 2.1: (a) Various objects and (b) spin configurations which are topologically different. The integer ν represents the number of holes in (a) and the vorticity in (b). The red points in (b) are analogous to the holes in (a) since the angle of spin cannot be defined at the vortex cores.

”holes” but also clarifies what type of the ”hole” each vortex core is.

We can further proceed the introduction of the winding number based on the analogy to the classical spin systems or counting the number of holes. While the vorticity in Eq. (2.2) is defined on spin configurations in real space, the same quantity can be defined on energy band structures in momentum space. We explain the above statement by considering a simple Hamiltonian in two dimensions

$$\mathcal{H}_e(\mathbf{k}) = \begin{pmatrix} 0 & k_x - ik_y \\ k_x + ik_y & 0 \end{pmatrix} = k \begin{pmatrix} 0 & e^{i\varphi(\mathbf{k})} \\ e^{-i\varphi(\mathbf{k})} & 0 \end{pmatrix}, \quad (2.3)$$

where $\mathbf{k} = (k_x, k_y)$, $k = \sqrt{k_x^2 + k_y^2}$, and $\tan[\varphi(\mathbf{k})] = -k_y/k_x$. This Hamiltonian with only off diagonal elements satisfies chiral symmetry, which is defined as

$$\Gamma \mathcal{H}_e(\mathbf{k}) \Gamma^{-1} = -\mathcal{H}_e(\mathbf{k}), \quad \Gamma = \sigma_3 \quad (2.4)$$

where σ_3 is one of Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.5)$$

Originating from chiral symmetry, there is a topological number as we show below. Note that, when we consider lattice systems with periodic boundary conditions, Hamiltonians can take such a form only approximately near gap closing points since $\mathcal{H}_e(\mathbf{k})$ in Eq. (2.3) violates the periodicity of Brillouin zone. Eigenstates of $\mathcal{H}_e(\mathbf{k})$ become

$$|\phi_+(\mathbf{k})\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{-i\varphi(\mathbf{k})} \end{pmatrix}, \quad |\phi_-(\mathbf{k})\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -e^{-i\varphi(\mathbf{k})} \end{pmatrix} \quad (2.6)$$

where corresponding eigenenergies are $\varepsilon_{\mathbf{k}} = +k$ and $\varepsilon_{\mathbf{k}} = -k$, respectively. Energy bands are objects where eigenenergies and corresponding eigenstates are attached to every coordinate in momentum space. In the present case, eigenstates of $\mathcal{H}_e(\mathbf{k})$ are characterized by $\varphi(\mathbf{k})$. At the gap closing point $\mathbf{k} = 0$ which is referred to as Dirac point, $\varphi(\mathbf{k})$ is ill defined and we can regard the Dirac point as a ”hole”. Then, we define the vorticity or the topological charge of the Dirac point

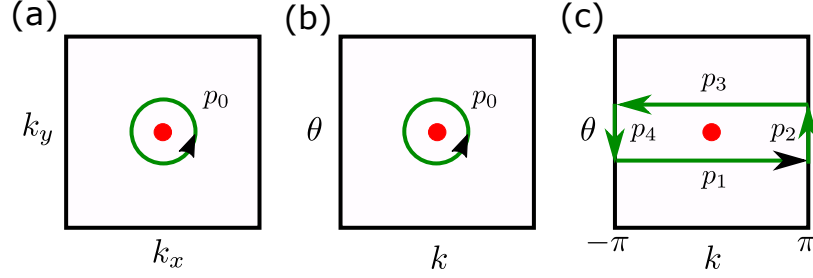


Figure 2.2: Integral paths for calculating the vorticities or topological charges of Dirac points (red circles) in (a) two-dimensional systems and (b,c) one-dimensional systems with a parameter θ . In (c), integrals along paths p_2 and p_4 cancel out due to 2π periodicity of k .

as

$$\nu_D = \frac{1}{2\pi} \oint_p d\mathbf{k} \cdot \nabla \varphi(\mathbf{k}), \quad (2.7)$$

in the same way as Eq. (2.2). The integral path p in Eq. (2.7) is taken such that p encloses the Dirac point as shown in Fig. 2.2 (a). When the integral path p does not enclose a Dirac point, the right hand side of Eq. (2.7) becomes zero, since p can be continuously deformed into a point without passing across a "hole" at which $\varphi(\mathbf{k})$ is ill defined. In other words, for changing the value of vorticity by deforming the integral path, p needs to go across a Dirac point.

We can extend the above discussion to one-dimensional systems. Consider a case that a chiral symmetric Hamiltonian $\mathcal{H}(k)$ with $\Gamma = \sigma_3$ which describes a one-dimensional system is characterized by a parameter θ . Then, we can draw an energy band structure in (k, θ) space where k is momentum. In the case that a band touching occurs at a point in (k, θ) space, we can characterize the Dirac point by the topological charge in Eq. (2.7) where $\varphi(k)$ represents the phase of $\mathcal{H}^{(1,2)}(k)$ and the integral path encloses the Dirac point as shown in Fig. 2.2 (b). Also, we can transform the integral path p_0 in Fig. 2.2 (b) into $p_1 + p_2 + p_3 + p_4$ in Fig. 2.2 (c), where k runs through the whole Brillouin zone from $k = -\pi$ (π) to $k = \pi$ ($-\pi$) along p_1 (p_3) on which θ is fixed at a value. We again note that $\mathcal{H}_e(\mathbf{k})$ in Eq. (2.3) is derived under the approximation that only the linear order of k_x and k_y is taken into account, and an exact Hamiltonian without approximations should be a periodic function of k_x and k_y . This is because we consider lattice systems with periodic boundary conditions. When k_x and k_y are respectively replaced by k and θ as in Fig. 2.2, Hamiltonians which describe one-dimensional systems must satisfy $\mathcal{H}(k) = \mathcal{H}(k + 2\pi)$ under periodic boundary conditions. Since the value of the topological charge does not depend on p as long as the integral path does not pass across a Dirac point, ν_D^0 becomes $\nu_1 + \nu_2 + \nu_3 + \nu_4$ where the superscript 0 and subscripts 1, 2, 3, 4 indicate corresponding integral paths. Since Brillouin zone has 2π periodicity, p_1 and p_3 form closed loops and then ν_1 and ν_3 take integer values, while ν_2 and ν_4 are not necessarily integers. Also, from the periodicity of Brillouin zone, ν_2 is equal to $-\nu_4$ and they cancel out. Then, the topological charge becomes

$$\nu_D^0 = \nu_1 + \nu_3. \quad (2.8)$$

Although ν_D^0 and ν_1, ν_3 are derived from the same concept and both count how many times phases wind along different paths, for explicitly mentioning them in separate names, we refer to the former and latter as the topological charge and the winding number respectively. With varying a parameter θ , the value of the corresponding winding number changes only when the integral path goes across a gap closing point. While winding numbers ν_1 and ν_3 deviate from the analogy to "counting the number of holes", remarkable outcomes emerge originating from these winding

numbers, as we discuss in the next section.

While we have given intuitive explanations of the topological charge and the winding number, mathematically generalized forms of them are as follows. Regarding chiral symmetric Hamiltonians, topological charges can be calculated from

$$\nu_D = \frac{i}{4\pi} \oint_p \text{tr}[\Gamma \mathcal{H}^{-1}(\mathbf{k}) d\mathcal{H}(\mathbf{k})]. \quad (2.9)$$

We can easily confirm that Eq. (2.9) corresponds to Eq. (2.7) in the case of $\mathcal{H}(\mathbf{k}) = \mathcal{H}_e(\mathbf{k})$ with $\Gamma = \sigma_3$ in Eq. (2.3). Note that, in general, $\mathcal{H}(\mathbf{k})$ need not be 2×2 matrices. Also, winding numbers are defined for matrices which are elements of general linear group. In one-dimensional systems, given a matrix $\alpha(k) \in GL(n, \mathbb{C})$, the corresponding winding number is

$$\nu[\alpha] = \frac{1}{2\pi i} \int_{-\pi}^{\pi} dk \frac{d}{dk} \ln \det[\alpha(k)]. \quad (2.10)$$

In the case of a chiral symmetric Hamiltonian with $\Gamma = \sigma_3$, $(1, 2)$ component of the Hamiltonian should be substituted into $\alpha(k)$ for calculating ν_1 and ν_3 in Eq. (2.8). We note that the direction of the integral path p_3 in Fig. 2.2 (c) is opposite to that in Eq. (2.10), while the directions of p_1 in Fig. 2.2 (c) and the integral path in Eq. (2.10) are the same. In general, gap closings can occur at several points in (k, θ) space. Then, we emphasize that the energy gap must be open along the both integral paths for calculating topological charges and winding numbers. This is because phases like $\varphi(\mathbf{k})$ in Eq. (2.3) are ill defined at gap closing points or equivalently $\alpha(k)$ with zero eigenvalues is not the element of $GL(n, \mathbb{C})$.

2.2 Bulk-edge correspondence

When a system is characterized by a non-zero winding number, topologically protected states with zero eigenenergy emerge at boundaries of the system. This is referred to as the bulk-edge correspondence, since the bulk topological number calculated in the system with periodic boundary conditions corresponds to the number of edge states in the same system, except replacing periodic boundary conditions with open boundary conditions. More precisely, when two regions labelled L and R with different winding numbers which are respectively ν_L and ν_R are jointed, topologically protected edge states at zero eigenenergy $\varepsilon = 0$ emerge at the jointed interface, of which the number corresponds to $|\nu_L - \nu_R|$. The bulk-edge correspondence has been proved mathematically or shown to be satisfied actually in various systems, and thus it is widely accepted at present. Since proving the bulk-edge correspondence is a difficult problem to be addressed here, we only comment that the bulk-edge correspondence is related to index theorem, gauge invariance, and so on. In order to make the bulk-edge correspondence acceptable for readers, we show that it is actually satisfied in a concrete example. To this end, we introduce Su-Schrieffer-Heeger model which is shortened as SSH model [101, 102]. The SSH model is described by the Hamiltonian

$$H_{\text{SSH}} = \sum_x t_1 |x, B\rangle \langle x, A| + t_2 |x+1, A\rangle \langle x, B| + \text{h.c.} \quad (2.11)$$

as shown in Fig. 2.3 (a), where A and B are sublattice indices, t_1 and t_2 are intra and inter sublattice hopping terms, and h.c. represents hermitian conjugated terms. When periodic boundary conditions are imposed, the Hamiltonian becomes

$$\mathcal{H}_{\text{SSH}}(k) = \begin{pmatrix} 0 & t_1 + t_2 e^{-ik} \\ t_1 + t_2 e^{ik} & 0 \end{pmatrix} = |t(k)| \begin{pmatrix} 0 & e^{i\varphi(k)} \\ e^{-i\varphi(k)} & 0 \end{pmatrix} \quad (2.12)$$

in momentum space, where $|t(k)| = \sqrt{[t_1 + t_2 \cos(k)]^2 + [t_2 \sin(k)]^2}$, $-\tan[\varphi(k)] = [t_2 \sin(k)]/[t_1 + t_2 \cos(k)]$, and the basis is $|A\rangle = (1, 0)^T$, $|B\rangle = (0, 1)^T$. The SSH model has chiral symmetry,

$$\Gamma \mathcal{H}_{\text{SSH}}(k) \Gamma^{-1} = -\mathcal{H}_{\text{SSH}}(k), \quad \Gamma = \sigma_3, \quad (2.13)$$

which induces non-trivial topological phases characterized by the winding number. In addition to chiral symmetry, $\mathcal{H}_{\text{SSH}}(k)$ has time-reversal symmetry and particle-hole symmetry:

$$\mathcal{T}\mathcal{H}_{\text{SSH}}(k)\mathcal{T}^{-1} = \mathcal{H}_{\text{SSH}}(-k), \quad \mathcal{T} = \mathcal{K}, \quad \Xi\mathcal{H}_{\text{SSH}}(k)\Xi^{-1} = -\mathcal{H}_{\text{SSH}}(-k), \quad \Xi = \sigma_3\mathcal{K}, \quad (2.14)$$

where $\mathcal{K}$ is the complex conjugation. With these three symmetries, the SSH model is classified into a symmetry class referred to as class BDI where non-trivial topological phases can exist, which will be shown in Sec. 2.3. Without loss of generality, we fix $t_2 = 1$ as the energy unit. Energy gaps close at $(t_1, k) = (-1, 0)$ and $(1, \pi)$, which we can understand from the eigenenergy $\varepsilon_k = \pm|t(k)|$. Eigenstates with eigenvalues $\pm|t(k)|$ are the same as Eq. (2.6) while the definition of $\varphi(k)$ is different. Figure 2.3 (b) shows $\varphi(k)$ in (k, t_1) space. From Fig. 2.3 (b), we can see that the phase $\varphi(k)$ winds around the gap closing points, which means that the Dirac points have non-zero topological charges and they are topologically protected. Near these Dirac points, the Hamiltonian can be written as

$$\mathcal{H}_{\text{SSH}}(\delta k) \simeq \begin{pmatrix} 0 & \delta t - i\delta k \\ \delta t + i\delta k & 0 \end{pmatrix} = \delta t\sigma_1 + \delta k\sigma_2 \quad \text{around } (t_1, k) = (-1, 0), \quad (2.15)$$

$$\mathcal{H}_{\text{SSH}}(\delta k) \simeq \begin{pmatrix} 0 & \delta t + i\delta k \\ \delta t - i\delta k & 0 \end{pmatrix} = \delta t\sigma_1 - \delta k\sigma_2 \quad \text{around } (t_1, k) = (1, \pi) \quad (2.16)$$

up to the first order of δt and δk , where $(\delta t, \delta k) = (t_1 + 1, k - 0)$ and $(t_1 - 1, k - \pi)$ in Eqs. (2.15) and (2.16), respectively. We note that any Hamiltonians can be written by linear terms of wave numbers generally near gap closing points, similar to those in Eqs. (2.15) or (2.16), which we will use for the classification of topological phases in Sec. 2.3. Topological charges of Dirac points in Eq. (2.9) are respectively +1 and -1 at $(t_1, k) = (-1, 0)$ and $(t_1, k) = (1, \pi)$. Substituting the off-diagonal term $t_1 + t_2 e^{-ik} = |t(k)|e^{i\varphi(k)}$ in Eq. (2.12) into $\alpha(k)$ in Eq. (2.10), we can obtain the winding number of the SSH model for each t_1 ,

$$\nu_{\text{SSH}} = \begin{cases} -1 & (|t_1| < 1) \\ 0 & (|t_1| > 1) \end{cases} \quad (2.17)$$

With varying t_1 , the value of the winding number ν_{SSH} changes only when the integral path goes across a gap closing point $(t_1, k) = (-1, 0)$ or $(1, \pi)$ at which $\varphi(k)$ is ill defined. Eigenenergies as functions of t_1 under open boundary conditions are shown in Fig. 2.3 (c). Zero energy states which are absent under periodic boundary conditions exist in the parameter range where $\nu_{\text{SSH}} = -1$, while there is no zero energy state when $\nu_{\text{SSH}} = 0$. This indicates that the bulk-edge correspondence is satisfied, since the winding number of vacuum outside the one-dimensional chain is zero. As shown in Fig. 2.3 (d), the topologically protected zero energy states are localized at boundaries, which is the reason why they are referred to as edge states.

2.3 Classification of topological phases

It is determined by symmetries and dimensions of systems whether non-trivial topological phases can exist or not. Actually, an example in Sec. 2.2 is a model satisfying chiral symmetry

$$\Gamma H \Gamma^{-1} = -H, \quad \Gamma \mathcal{H}(k) \Gamma^{-1} = -\mathcal{H}(k), \quad (2.18)$$

as in Eq. (2.13), which is the reason why the SSH model can be characterized by the winding number in Eq. (2.10) where $\alpha(k) = t_1 + t_2 e^{-ik}$. Chiral symmetry operators are unitary and Hermitian operators which satisfy $\Gamma^\dagger = \Gamma^{-1}$, $\Gamma^2 = 1$ where 1 is the $m \times m$ identity matrix ($m = 1, 2, \dots$). In addition to chiral symmetry, time-reversal symmetry and particle-hole symmetry which are respectively defined as

$$\mathcal{T} H \mathcal{T}^{-1} = H, \quad \Xi H \Xi^{-1} = -H \quad (2.19)$$

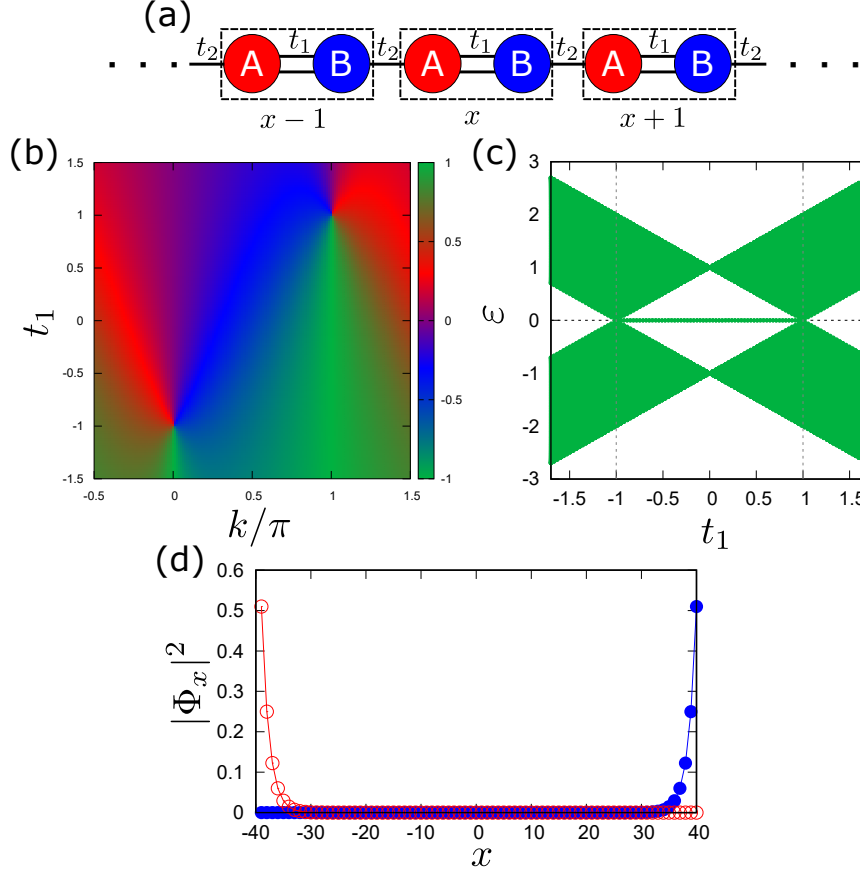


Figure 2.3: (a) A schematic picture of the SSH model, (b) the phase profile in (k, t_1) space, (c) eigenenergies of the SSH model with open boundary conditions, and (d) topologically protected edge states with $\varepsilon = 0$ when $t_1 = 0.7, t_2 = 1$. In (b), we can understand that $\varphi(k)$ cannot be defined at gap closing points $(k, t_1) = (0, -1)$ and $(\pi, 1)$, which are analogous to holes in Fig. 2.1. In (c), topologically protected edge states appear at $\varepsilon = 0$, in the parameter range $-1 < t_1 < 1$ where $\varphi(k)$ winds one time through Brillouin zone as shown in (b).

play important roles. These symmetry operators include complex conjugation $\mathcal{K}$ and such operators are called anti-unitary operators. Thereby $\mathcal{T}$ and Ξ flip the sign of k when these symmetry relations are written in momentum space:

$$\mathcal{T}\mathcal{H}(k)\mathcal{T}^{-1} = \mathcal{H}(-k), \quad \Xi\mathcal{H}(k)\Xi^{-1} = -\mathcal{H}(-k). \quad (2.20)$$

Squares of $\mathcal{T}$ and Ξ become $+1$ or -1 . While notations in the whole Hilbert space and momentum space are different regarding Hamiltonians $H = \sum_k |k\rangle \langle k| \otimes \mathcal{H}(k)$ and time-evolution operators $U = \sum_k |k\rangle \langle k| \otimes \mathcal{U}(k)$ which are considered in the following, we represent a symmetry operator by the same notation in both cases, because Γ , $\mathcal{T}$, and Ξ are independent of x and k . The restrictions on symmetry operators are imposed from the idea that symmetries of physical systems should be described by simple operators independent of details which each system has. In other words, physicists should explore universal features which various physical systems commonly have by considering ensembles of systems classified by the simple symmetry operators with the same basic features. When a system has both time-reversal and particle-hole symmetries, the system inevitably has chiral symmetry with $\Gamma = \mathcal{T}\Xi$, which can be understood from Eqs. (2.18)-(2.20).

Table 2.1: Classification of topological phases based on symmetries in each dimension [103, 104, 105, 106, 107]. The left column shows the names of symmetry classes. TRS, PHS, and CS represent Time-Reversal Symmetry, Particle-Hole Symmetry, and Chiral Symmetry, respectively. The squares of symmetry operators are denoted by ± 1 , where $\mathcal{T}^2$ and Ξ^2 can be -1 since $\mathcal{T}$ and Ξ are anti-unitary operators while $\Gamma^2 = 1$ is always satisfied since Γ is unitary. The number 0 means that the corresponding symmetry does not exist. Symbols $\mathbb{Z}$ and $\mathbb{Z}_2$ mean that the system is characterized by a topological number which can take integers and binary numbers, respectively, while $-$ denotes that there is no topological number which characterizes the system.

class	TRS	PHS	CS	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$	$d = 8$
A	0	0	0	-	$\mathbb{Z}$	-	$\mathbb{Z}$	-	$\mathbb{Z}$	-	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$	-	$\mathbb{Z}$	-	$\mathbb{Z}$	-
AI	+1	0	0	-	-	-	$\mathbb{Z}$	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	+1	+1	1	$\mathbb{Z}$	-	-	-	$\mathbb{Z}$	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-	-	-	$\mathbb{Z}$	-	$\mathbb{Z}_2$
DIII	-1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	-	-	-	$\mathbb{Z}$	-
AII	-1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	-	-	-	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	-	-	-
C	0	-1	0	-	$\mathbb{Z}$	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	-	-
CI	+1	-1	1	-	-	$\mathbb{Z}$	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	-

Therefore, when two of three symmetries are satisfied, the other symmetry is always satisfied, while the existence of chiral symmetry is not the sufficient condition for time-reversal symmetry and particle-hole symmetry.

Table 2.1 shows topological phases which can exist in each symmetry class and dimension. In the following, we explain how the table is obtained. There are several procedures to obtain Tab. 2.1. Historically, topological phases of matter were firstly discovered by constructing topological numbers elaborately, such as the Chern number in class A with $d = 2$ [7] and the $\mathbb{Z}_2$ number in class AII with $d = 2$ [15], to name a few. After that, people have made the classification table using mathematically sophisticated theories and procedures, such as Clifford algebra, K theory, and so on. Identifying concrete topological numbers is beneficial since we can intuitively understand topological properties of systems as explained in Sec. 2.1 regarding the winding number. Actually, in one-dimensional systems, AIII, BDI, and CII classes are topologically characterized by the winding number. The SSH model in Sec. 2.2 is classified into BDI class with $d = 1$, which is the reason why we can characterize the system by the winding number as in Tab. 2.1. However, identifying concrete topological numbers not only in all symmetry classes but also in all dimensions is considerably laborious. Thereby, we adopt a more elegant and efficient procedure, utilizing the extension problem of Clifford algebras and the classification based on homotopy group [103, 104, 105, 106, 107].

In order to classify topological phases, we consider d -dimensional massive Dirac Hamiltonians

$$\mathcal{H}(\mathbf{k}) \simeq \sum_{j=1}^d k_j \gamma_j + \gamma_0, \quad (2.21)$$

where γ_j ($j = 0, \dots, d$) are Gamma matrices which satisfy

$$\gamma_l \gamma_m + \gamma_m \gamma_l = 2\delta_{lm}. \quad (2.22)$$

Since d -dimensional Hamiltonians can be continuously deformed to the form in Eq. (2.21) near topological transition points or equivalently Dirac points, it is sufficient to consider Dirac Hamil-

tonians for the classification of topological phases. We have already seen an example of Eq. (2.21) in Sec. 2.2; equations (2.15) and (2.16) with appropriately rescaled parameters correspond to Eq. (2.21) with $d = 1$. By taking the mass term γ_0 as the energy unit, the condition $\gamma_0^2 = 1$ can be imposed near Dirac points. The mass term γ_0 opens a gap in the spectrum of $\mathcal{H}(\mathbf{k})$ in the same way as $\delta t\sigma_1$ in Eqs. (2.15) and (2.16). The classification of the Hamiltonian in Eq. (2.21) corresponds to the classification of the mass term γ_0 . This can be understood from the example in Sec. 2.2, where topologically distinct phases correspond to different signs of δt near each Dirac point. Actually, we have no choice but to explore structures of the mass term in Eq. (2.21), since topological features cannot be changed by other terms related to momentum. In order to classify γ_0 , we use Clifford algebras and extension problems of them. Generators of a complex Clifford algebra anti-commute with each other and become +1 for the square of each,

$$Cl_q : \{e_1, \dots, e_q\}, \quad e_l e_m + e_m e_l = 2\delta_{lm}, \quad (2.23)$$

where q is the number of generators. The basis which spans 2^q -dimensional complex vector space is constructed from the generators as in $e_1^{b_1} e_2^{b_2} \dots e_q^{b_q}$ where $b_j = 0, 1$ ($j = 1, \dots, q$); vectors in the space are represented as $\sum_{\{b_j\}} c_{b_1 \dots b_q} e_1^{b_1} \dots e_q^{b_q}$ where $\{c_{b_1 \dots b_q}\}$ are complex numbers. Since gamma matrices in Eqs. (2.21) and (2.22) satisfy the conditions to be generators of complex Clifford algebras in Eq. (2.23), it is expected that we can use complex Clifford algebras to classify Dirac Hamiltonians. This expectation is actually true for classes A and AIII in which Hamiltonians do not satisfy anti-unitary symmetries, as shown in the following. When time-reversal symmetry and/or particle-hole symmetry exist, imaginary unit i which anti-commutes with $\mathcal{T}$ and Ξ should be included in generators, and thus we use real Clifford algebras in which generators anti-commute with each other and their squares become +1 or -1,

$$Cl_{p,q} : \{e_1, \dots, e_p | e_{p+1}, \dots, e_{p+q}\}, \quad e_l e_m + e_m e_l = \begin{cases} -2\delta_{lm} & (l = 1, \dots, p) \\ +2\delta_{lm} & (l = p+1, \dots, p+q) \end{cases} \quad (2.24)$$

The subscripts p and q denote the number of generators whose square is -1 and +1, respectively. The basis which spans $2^p 2^q$ -dimensional real vector space is constructed from the generators as in $e_1^{b_1} \dots e_p^{b_p} e_{p+1}^{b_{p+1}} \dots e_{p+q}^{b_{p+q}}$ where $b_j = 0, 1$ ($j = 1, \dots, p+q$); vectors in the space are represented as $\sum_{\{b_j\}} c_{b_1 \dots b_{p+q}} e_1^{b_1} \dots e_{p+q}^{b_{p+q}}$ where $\{c_{b_1 \dots b_{p+q}}\}$ are real numbers. It is known that complex and real Clifford algebras satisfy

$$Cl_q \simeq Cl_{q+2}, \quad Cl_{p,q} \simeq Cl_{p+1,q+1}, \quad Cl_{p,q} \simeq Cl_{p+8,q} \simeq Cl_{p,q+8}, \quad Cl_{p,q} \simeq Cl_{q,p+2}, \quad (2.25)$$

where the first and third relations are referred to as Bott periodicity. Originating from Bott periodicity, the largest value of d is 8 in Tab. 2.1, as is explained below.

We can construct complex or real Clifford algebras from Gamma matrices and symmetry operators of $\mathcal{H}(\mathbf{k})$. For instance, in three dimensional systems classified into class AIII, γ_j and Γ become the generators of the complex Clifford algebra

$$Cl_5 : \{\gamma_1, \gamma_2, \gamma_3, \Gamma, \gamma_0\} \quad \text{for class AIII with } d = 3. \quad (2.26)$$

This is because chiral symmetry operator Γ anti-commutes with gamma matrices γ_j ($j = 0, 1, 2, 3$) owing to chiral symmetry relation $\Gamma \mathcal{H}(\mathbf{k}) + \mathcal{H}(\mathbf{k}) \Gamma = 0$ in Eq. (2.18). In class A, generators of a complex Clifford algebra can be obtained by removing Γ from those in class AIII. In the case that the system has time-reversal symmetry and/or particle-hole symmetry, we must construct real Clifford algebras in which the imaginary unit i is included in generators, since $\mathcal{T}$ and Ξ anti-commute with i . We take class D with $d = 2$ as an example. From the symmetry relation $\Xi \mathcal{H}(\mathbf{k}) + \mathcal{H}(-\mathbf{k}) \Xi = 0$ in Eq. (2.20), Ξ commutes with γ_j for $j = 1, 2$ and anti-commutes with γ_0 . Then, by obtaining anti-commuting generators from combinations of i , Ξ , γ_j , and γ_0 , we can construct the real Clifford algebra as in

$$Cl_{2,3} : \{i\gamma_1, i\gamma_2 | \Xi, i\Xi, \gamma_0\} \quad \text{for class D with } d = 2. \quad (2.27)$$

Table 2.2: Zero-dimensional homotopy groups of C_q and R_q .

q	0	1	q	0	1	2	3	4	5	6	7
$\pi_0(C_q)$	$\mathbb{Z}$	0	$\pi_0(R_q)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

Regarding time-reversal symmetry, $\mathcal{T}$ anti-commutes with γ_j for $j = 1, \dots, d$ and commutes with γ_0 from the corresponding symmetry relation $\mathcal{T}\mathcal{H}(\mathbf{k}) - \mathcal{H}(-\mathbf{k})\mathcal{T} = 0$ in Eq. (2.20). When both time-reversal symmetry and particle-hole symmetry exist, we also need to consider whether their symmetry operators commute or anti-commute with each other for constructing real Clifford algebras. When $\mathcal{T}^2$ and Ξ^2 have the same sign, $\mathcal{T}$ commutes with Ξ since $\Gamma^2 = (\mathcal{T}\Xi)^2 = 1$ is satisfied. In the same way, when $\mathcal{T}^2$ and Ξ^2 have the opposite signs, $\mathcal{T}$ and Ξ anti-commute with each other. Considering class BDI with $[\mathcal{T}, \Xi] = 0$ and $d = 1$ for instance, we can carry out the construction of the real Clifford algebra by combining i , $\mathcal{T}$, Ξ , γ_1 , and γ_0 so that the composites which are generators anti-commute with each other,

$$Cl_{2,3} : \{i, i\Xi\gamma_0 | \Xi\gamma_1, \mathcal{T}, i\Xi\} \text{ for class BDI with } d = 1. \quad (2.28)$$

From the relation between Dirac Hamiltonians and Clifford algebras, we can understand that adding mass terms γ_0 to Dirac Hamiltonians which have no mass term corresponds to extending Clifford algebras where the number of generators is increased by 1. Actually, in the case of Eqs. (2.26), (2.27), and (2.28), their correspondence respectively can be written as

$$\{\gamma_1, \gamma_2, \gamma_3, \Gamma\} \rightarrow \{\gamma_1, \gamma_2, \gamma_3, \Gamma, \gamma_0\} : Cl_4 \rightarrow Cl_5 \text{ for class AIII with } d = 3, \quad (2.29)$$

$$\{i\gamma_1, i\gamma_2 | \Xi, i\Xi\} \rightarrow \{i\gamma_1, i\gamma_2 | \Xi, i\Xi, \gamma_0\} : Cl_{2,2} \rightarrow Cl_{2,3} \text{ for class D with } d = 2, \quad (2.30)$$

$$\{i | \Xi\gamma_1, \mathcal{T}, i\Xi\} \rightarrow \{i, i\Xi\gamma_0 | \Xi\gamma_1, \mathcal{T}, i\Xi\} : Cl_{1,3} \rightarrow Cl_{2,3} \text{ for class BDI with } d = 1. \quad (2.31)$$

For all symmetry classes and spatial dimensions, generators of Clifford algebras which include γ_0 and corresponding extensions become

$$\{\gamma_1, \dots, \gamma_d, \gamma_0\} \text{ for class A: } Cl_d \rightarrow Cl_{d+1} \quad (2.32)$$

$$\{\gamma_1, \dots, \gamma_d, \Gamma, \gamma_0\} \text{ for class AIII: } Cl_{d+1} \rightarrow Cl_{d+2} \quad (2.33)$$

$$\{i\gamma_0 | \gamma_1, \dots, \gamma_d, \mathcal{T}, i\mathcal{T}\} \text{ for class AI: } Cl_{0,d+2} \rightarrow Cl_{1,d+2} \quad (2.34)$$

$$\{i, i\Xi\gamma_0 | \Xi\gamma_1, \dots, \Xi\gamma_d, \mathcal{T}, i\Xi\} \text{ for class BDI: } Cl_{1,d+2} \rightarrow Cl_{2,d+2} \quad (2.35)$$

$$\{i\gamma_1, \dots, i\gamma_d | \Xi, i\Xi, \gamma_0\} \text{ for class D: } Cl_{d,2} \rightarrow Cl_{d,3} \quad (2.36)$$

$$\{i, \mathcal{T}, i\mathcal{T}\gamma_0 | \mathcal{T}\gamma_1, \dots, \mathcal{T}\gamma_d, \Xi\} \text{ for class DIII: } Cl_{2,d+1} \rightarrow Cl_{3,d+1} \quad (2.37)$$

$$\{\mathcal{T}, i\mathcal{T}, i\gamma_0 | \gamma_1, \dots, \gamma_d\} \text{ for class AII: } Cl_{2,d} \rightarrow Cl_{3,d} \quad (2.38)$$

$$\{i, \Xi\gamma_1, \dots, \Xi\gamma_d, \mathcal{T}, i\Xi | i\Xi\gamma_0\} \text{ for class CII: } Cl_{d+3,0} \rightarrow Cl_{d+3,1} \quad (2.39)$$

$$\{i\gamma_1, \dots, i\gamma_d, \Xi, i\Xi | \gamma_0\} \text{ for class C: } Cl_{d+2,0} \rightarrow Cl_{d+2,1} \quad (2.40)$$

$$\{i, \mathcal{T}\gamma_1, \dots, \mathcal{T}\gamma_d, \Xi | \mathcal{T}, i\mathcal{T}\gamma_0\} \text{ for class CI: } Cl_{d+2,1} \rightarrow Cl_{d+2,2} \quad (2.41)$$

In Eqs. (2.32)-(2.41), generators of Clifford algebras which include mass terms and corresponding extensions from those without γ_0 are written in the left and right, respectively. Note that, while ways of constructing Clifford algebras are not solely determined, Clifford algebras in each symmetry class obtained from different generators are the same in the light of Eq. (2.25). We write the classifying spaces of complex and real Clifford algebras as

$$Cl_q \rightarrow Cl_{q+1} : C_q, \quad Cl_{0,q} \rightarrow Cl_{0,q+1} : R_q, \quad (2.42)$$

respectively. The classification of mass terms γ_0 corresponds to the classification of C_q or R_q . It is known that homotopy groups of C_q and R_q satisfy

$$\pi_d(C_q) = \pi_0(C_{q-d}), \quad \pi_d(R_q) = \pi_0(R_{q-d}). \quad (2.43)$$

Therefore, in order to know topological classifications of d -dimensional systems, it is sufficient to explore zero-dimensional homotopy groups. Homotopy groups of R_q and C_q with $d = 0$ are exhibited in Tab. 2.2. Using Eqs. (2.25), (2.43), and Tab. 2.2, we can obtain the classification table in Tab. 2.1. Originating from Bott periodicity for complex Clifford algebras in Eq. (2.25), columns of A and AIII for d -dimensional systems in Tab. 2.1 are the same as that for $d + 2$ -dimensional systems. Also, regarding symmetry classes whose topological phases are determined by real Clifford algebras, Bott periodicity in Eq. (2.25) makes columns for d -dimensional systems be the same as that for $d + 8$ -dimensional systems, which is the reason why Tab. 2.1 is ended at $d = 8$.

Chapter 3

Quantum Walks and Their Topological Phases

In Chap. 3, we explain quantum walks and the associated topological phases. Since many aspects of quantum walks can be understood from the analogy to systems described by time-independent Hamiltonians, quantum walks are often referred to as "simulators" of such systems. On the other hand, we explore distinct features of quantum walks from systems governed by static Hamiltonians. In Secs. 3.1 and 3.2, we explain how to describe the dynamics and study symmetries of quantum walks. In Sec. 3.3, we derive topological numbers of unitary quantum walks from chiral symmetry based on the discussion about the bulk-edge correspondence. In Sec. 3.4, we introduce two kinds of experiments based on optical fiber loops and arrays of optical elements, in which topologically protected edge states were observed. We mention novel physical effects in quantum walks and state objectives of this doctoral thesis in Sec. 3.5.

3.1 Description of dynamics

We consider quantum walks in which walkers move in the one-dimensional position space $|x\rangle$ and have two internal states $|-\rangle = (1, 0)^T$ and $|+\rangle = (0, 1)^T$, where the superscript T denotes the transpose. Using this basis, a state in a time step t ($= 0, 1, 2, \dots$) is written as

$$|\psi(t)\rangle = \sum_{x,\sigma} \psi_{x,\sigma}(t) |x\rangle \otimes |\sigma\rangle, \quad (3.1)$$

where $\psi_{x,\sigma}(t)$ denotes the wave function amplitude of a quantum particle such as a photon at a position x ($= \dots, -1, 0, 1, \dots$) with an internal state σ ($= \pm$). While we explain one-dimensional quantum walks with two internal states in this Chapter, extensions to higher dimensional systems with larger number of internal states are feasible. Actually, d -dimensional quantum walks with $2d$ internal states have been theoretically studied [77, 78]. However, quantum walks with two internal states have higher experimental realizability than quantum walks in which particles have $2m$ ($m = 2, 3, \dots$) internal states. This is because experiments of quantum walks are mostly carried out by using photons with two internal states which are vertical polarization and horizontal polarization. We thereby study quantum walks with two internal states in this doctoral thesis. The dynamics of quantum walks are described by time-evolution operators U as in

$$|\psi(t+1)\rangle = U |\psi(t)\rangle. \quad (3.2)$$

Equation (3.2) means that time evolves in a discrete manner, which is the same feature as periodically driven systems or equivalently Floquet systems described by microscopic time-dependent

Hamiltonians $H(t) = H(t + T)$ where T is the period of driving external fields [108]. Actually, time-evolution operators for such Floquet systems are described by $U(0 \rightarrow T) = \mathbb{T}e^{-i \int_0^T dt H(t)}$ where $\mathbb{T}$ is the time-ordering operator. Therefore, quantum walks share many features with such Floquet systems, while microscopic Hamiltonians cannot exist in some quantum walks as we explain in Sec. 6.3.2. If one carries out the measurement on positions, a quantum particle is found at a lattice position x and a time step t with a probability

$$P(x, t) = |\psi_{x,+}(t)|^2 + |\psi_{x,-}(t)|^2. \quad (3.3)$$

Here, we mention a confusing matter about the dynamics of quantum walks and the interpretation of $P(x, t)$. While quantum walks include "quantum" in its name, dynamics of classical light can also be described by quantum walks. This is because the name "quantum walk" originates from the comparison between classical dynamics and quantum dynamics of a particle. While it is now common knowledge that quantum particles have both "particle like" features and "wave like" features, historically "particle" and "wave" had been considered as separate conceptions. Therefore, for electrons and light, which were considered as "particles" and "waves", the behaviour which represents quantum nature is respectively "wave like" behaviour and "particle like" behaviour. The representative examples are interference of electron wave functions and Compton scatterings of photons. While quantum walks can describe the dynamics of quantum wave functions of single photons and classical electromagnetic wave (or coherent state), its name stems from the former in comparison to the dynamics of particles in classical mechanics. When we consider the dynamics of classical coherent light, $P(x, t)$ in Eq. (3.3) is regarded as the intensity of light.

In standard quantum walks, time-evolution operators are composed of coin operators and shift operators, of which the former and latter respectively play the role of changing internal states and positions of wave functions, as we explain below. The general form of coin operators is written as

$$\sum_x |x\rangle \langle x| \otimes e^{i\alpha(x)} \begin{pmatrix} e^{i\beta(x)} & 0 \\ 0 & e^{-i\beta(x)} \end{pmatrix} \begin{pmatrix} \cos[\gamma(x)] & -\sin[\gamma(x)] \\ \sin[\gamma(x)] & \cos[\gamma(x)] \end{pmatrix} \begin{pmatrix} e^{i\delta(x)} & 0 \\ 0 & e^{-i\delta(x)} \end{pmatrix}, \quad (3.4)$$

where $\alpha(x), \beta(x), \gamma(x)$, and $\delta(x)$ are position dependent real numbers. The 2×2 matrix which acts on the internal states is the general form of $U(2)$. Among them, we use three types of coin operators

$$C_j(\theta) = \sum_x |x\rangle \langle x| \otimes \mathcal{C}_j[\theta(x)], \quad (3.5)$$

$$\mathcal{C}_1[\theta(x)] = \begin{pmatrix} \cos[\theta(x)] & i \sin[\theta(x)] \\ i \sin[\theta(x)] & \cos[\theta(x)] \end{pmatrix} = e^{i\theta(x)\sigma_1}, \quad (3.6)$$

$$\mathcal{C}_2[\theta(x)] = \begin{pmatrix} \cos[\theta(x)] & -\sin[\theta(x)] \\ \sin[\theta(x)] & \cos[\theta(x)] \end{pmatrix} = e^{-i\theta(x)\sigma_2}, \quad (3.7)$$

$$\mathcal{C}_3[\theta(x)] = \begin{pmatrix} e^{i\theta(x)} & 0 \\ 0 & e^{-i\theta(x)} \end{pmatrix} = e^{i\theta(x)\sigma_3} \quad (3.8)$$

in this doctoral thesis, where σ_i ($i = 1, 2, 3$) are Pauli matrices in Eq. (2.5). As we mention in Secs. 3.4 and 4.4.1, $C_1(\theta)$ describes the dynamics of light pulses in optical fiber loops connected by couplers. $C_2(\theta)$ is the well-known rotation matrix, and we use it in Chaps. 5 and 6. The coin operators in Eqs. (3.6) and (3.7) change the internal states of walkers with a position-dependent parameter $\theta(x)$ which determines how internal states are mixed. The third one $C_3(\theta)$ used in Chap. 4, which we often refer to as the phase operator, does not mix internal states but attaches phases to wave function amplitudes depending on internal states. We comment that there is a coin operator referred to as an Hadamard coin operator which is frequently used in experiments of quantum walks, while we do not address it in this thesis. Since the Hadamard coin operator can

be written as

$$C_H(\theta) = \sum_x |x\rangle \langle x| \otimes C_H[\theta(x)], \quad C_H[\theta(x)] = \begin{pmatrix} \cos[\theta(x)] & \sin[\theta(x)] \\ \sin[\theta(x)] & -\cos[\theta(x)] \end{pmatrix} = C_2[\theta(x)]\sigma_3, \quad (3.9)$$

quantum walks which are described by $C_H(\theta)$ share fundamental features with those by $C_2(\theta)$. Shift operators which describe split-step quantum walks are

$$\begin{aligned} S_{\pm} &= \sum_x |x \pm 1\rangle \langle x| \otimes |\pm\rangle \langle \pm| + |x\rangle \langle x| \otimes |\mp\rangle \langle \mp| \\ &= \sum_k |k\rangle \langle k| \otimes e^{\mp i \frac{k}{2}} \mathcal{S}(k/2), \quad \mathcal{S}(k/2) = \begin{pmatrix} e^{i \frac{k}{2}} & 0 \\ 0 & e^{-i \frac{k}{2}} \end{pmatrix} = e^{i \frac{k}{2} \sigma_3}, \end{aligned} \quad (3.10)$$

respectively, where $k = 2\pi n/N$ ($n = -N/2, \dots, N/2 - 1$) is momentum in systems of which the total number of lattices is N with periodic boundary conditions. According to Eq. (3.10), the internal states $|-\rangle$ and $|+\rangle$ correspond to left and right moving components, respectively. The representation in momentum space $e^{\mp i \frac{k}{2}} \mathcal{S}(k/2)$ is obtained from Fourier transformation

$$|x\rangle = \frac{1}{\sqrt{N}} \sum_k e^{-ikx} |k\rangle. \quad (3.11)$$

The shift operators in Eq. (3.10) change the positions of wave function amplitudes depending on the internal states. We also use a shift operator constructed by combining S_+ and S_- ,

$$S = S_+ S_- = \sum_x \begin{pmatrix} |x-1\rangle \langle x| & 0 \\ 0 & |x+1\rangle \langle x| \end{pmatrix} = \sum_k |k\rangle \langle k| \otimes \mathcal{S}(k), \quad (3.12)$$

which changes the positions of wave function amplitudes with both internal states. We show several time-evolution operators as examples:

$$U_{e1} = S C_2(\theta), \quad U_{e2} = S_+ C_2(\theta_2) S_- C_1(\theta_1). \quad (3.13)$$

Using U_{e1} , we explain that quantum walks can be seen as quantum versions of classical random walks. Figure 3.1 compares classical random walks and quantum walks. In one-dimensional classical random walks, a particle goes left or right on a probability p or $1-p$ respectively, at each time step. In quantum walks, coin operators change wave function amplitudes with an internal state $+$ or $-$ into superpositions of those with $+$ and $-$, and shift operators carry out the position change of walkers. Quantum walks and classical random walks are analogous in that the operation of $C_j(\theta)$ in quantum walks is similar to the coin toss in classical random walks determining which direction a particle moves to. Actually, quantum walks were made mathematically based on classical random walks [77, 78, 79]. Meanwhile, interference of wave functions occurs in quantum walks, which is different from classical random walks. Owing to quantum inference, probability distributions in quantum walks spread rapidly in comparison to classical random walks; standard deviations after t time steps are proportional to t and $\sqrt{t}$ respectively in quantum and classical walks.

We analyse the dynamics of quantum walks based on eigenvalues and eigenstates of time-evolution operators,

$$U |\phi_j\rangle = \lambda_j |\phi_j\rangle, \quad \lambda_j = e^{-i\varepsilon_j}, \quad \langle \phi_l | \phi_m \rangle = \delta_{lm}, \quad \sum_j |\phi_j\rangle \langle \phi_j| = 1. \quad (3.14)$$

Since U is a unitary operator, we can always construct orthonormal basis as in Eq. (3.14). The phases ε_j of eigenvalues λ_j are referred to as quasi-energies from the reason which we mention

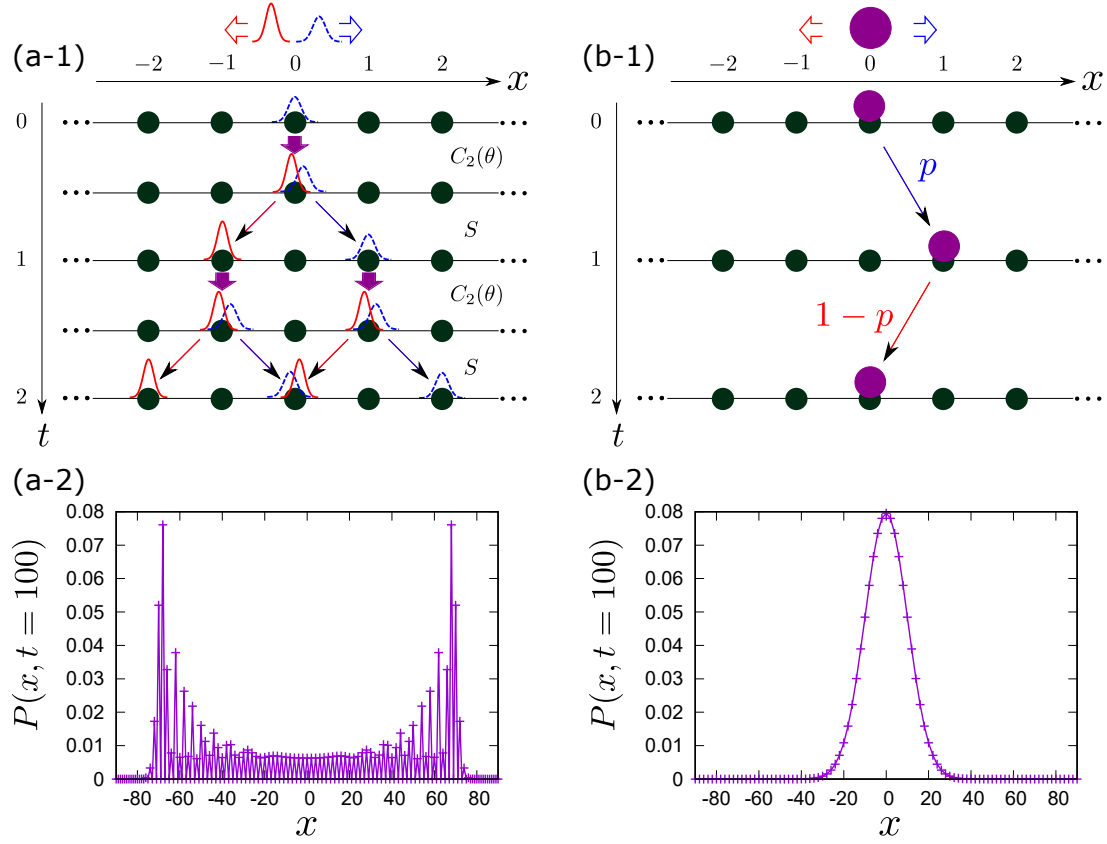


Figure 3.1: Comparison of quantum walks and classical random walks. (a-1) Dynamics of a quantum walk described by U_{e1} . Internal states of a walker, corresponding to $-$ (waves depicted by red solid lines) and $+$ (waves depicted by blue dashed lines) components, are mixed by $C_2(\theta)$, and wave function amplitudes shift to left or right depending on internal states. (b-1) Dynamics of a classical random walk. A particle (purple ball) moves to right or left with a probability p or $1 - p$, respectively. (a-2),(b-2) Probability distributions after 100 time steps of the quantum walk with U_{e1} and the classical random walk. The parameters and initial conditions are (a-2) $\theta(x) = \pi/4$, $|\psi(0)\rangle = |x=0\rangle \otimes (|-\rangle + i|+\rangle)/\sqrt{2}$ and (b-2) $p = 1/2$, $x = 0$.

below. The time evolutions of quantum walks are attributed to $|\phi_j\rangle$ which earns the phase ε_j at each time step

$$|\psi(t)\rangle = \sum_j c_j e^{-i\varepsilon_j t} |\phi_j\rangle, \quad c_j = \langle \phi_j | \psi(0) \rangle. \quad (3.15)$$

When parameters included in U are independent of position x , we represent the time-evolution operator in momentum space

$$U = \sum_k |k\rangle \langle k| \otimes \mathcal{U}(k), \quad (3.16)$$

which is obtained by Fourier transformation in Eq. (3.11). Then, we can obtain eigenvalues and quasi-energies as functions of momentum k :

$$\mathcal{U}(k) |\phi(k)\rangle = \lambda_k |\phi(k)\rangle, \quad \lambda_k = e^{-i\varepsilon_k}. \quad (3.17)$$

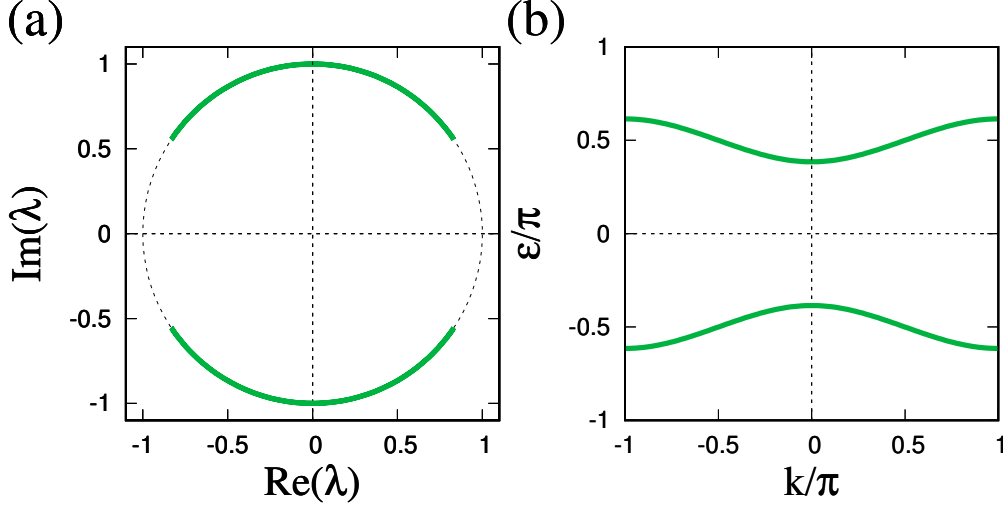


Figure 3.2: (a) eigenvalues and (b) the dispersion of $\mathcal{U}_{e2}(k)$ with $\theta_1 = \pi/7$ and $\theta_2 = \pi/8$, which are given by $\lambda_k = e^{-i\varepsilon_k}$ with $\cos(\varepsilon_k) = \cos(\theta_1)\cos(\theta_2)\cos(k)$. The dashed curve in (a) represents the unit circle on which eigenvalues λ_k distribute.

For instance, U_{e1} and U_{e2} are written as

$$\mathcal{U}_{e1}(k) = \mathcal{S}(k)\mathcal{C}_2(\theta), \quad \mathcal{U}_{e2}(k) = \mathcal{S}(k/2)\mathcal{C}_2(\theta_2)\mathcal{S}(k/2)\mathcal{C}_1(\theta_1) \quad (3.18)$$

in momentum space when θ , θ_1 , and θ_2 are independent of x . Eigenvalues and quasi-energies of $\mathcal{U}_{e2}$ are shown in Fig. 3.2. When a quasi-energy spectrum has a gap at a quasi-energy ε_b as in Fig. 3.2, we can define an effective Hamiltonian or equivalently Floquet Hamiltonian

$$U = e^{-iH}, \quad \mathcal{U}(k) = e^{-i\mathcal{H}(k)}, \quad (3.19)$$

by taking the branch cut at ε_b . In many cases, it is convenient to take the branch cut at $\varepsilon_b = 0$ or π . The phases ε_j and ε_k are eigenvalues of effective Hamiltonians H and $\mathcal{H}(k)$, which is the reason why the phases are referred to as quasi-energies.

3.2 Naive symmetries based on effective Hamiltonians

As shown in Sec. 2.3, symmetries of systems determine whether non-trivial topological phases can exist or not. Symmetries of time-evolution operators are naively defined from effective Hamiltonians. In order to derive symmetry relations of time-evolution operators, we rewrite relations of time-reversal symmetry, particle-hole symmetry, and chiral symmetry of static Hamiltonians

$$\mathcal{T}H\mathcal{T}^{-1} = H, \quad \Xi H\Xi^{-1} = -H, \quad \Gamma H\Gamma^{-1} = -H. \quad (3.20)$$

Here, $\mathcal{T}$ and Ξ are anti-unitary operators which include complex conjugation $\mathcal{K}$, while Γ is unitary. Symmetry operators satisfy $\mathcal{T}^2 = \pm 1$, $\Xi^2 = \pm 1$, and $\Gamma^2 = 1$. From Eqs. (3.19) and (3.20), symmetry relations on time-evolution operators become

$$\mathcal{T}U\mathcal{T}^{-1} = U^{-1}, \quad \Xi U\Xi^{-1} = U, \quad \Gamma U\Gamma^{-1} = U^{-1}. \quad (3.21)$$

In momentum space, Eqs. (3.20) and (3.21) are written as

$$\mathcal{T}\mathcal{H}(k)\mathcal{T}^{-1} = \mathcal{H}(-k), \quad \Xi\mathcal{H}(k)\Xi^{-1} = -\mathcal{H}(-k), \quad \Gamma\mathcal{H}(k)\Gamma^{-1} = -\mathcal{H}(k), \quad (3.22)$$

$$\mathcal{T}\mathcal{U}(k)\mathcal{T}^{-1} = \mathcal{U}^{-1}(-k), \quad \Xi\mathcal{U}(k)\Xi^{-1} = \mathcal{U}(-k), \quad \Gamma\mathcal{U}(k)\Gamma^{-1} = \mathcal{U}^{-1}(k), \quad (3.23)$$

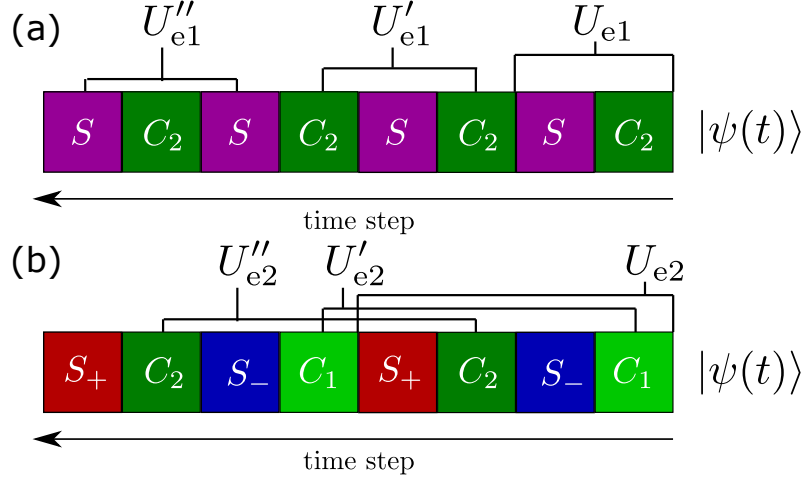


Figure 3.3: Schematic pictures of symmetric time frames for (a) U_{e1} and (b) U_{e2} .

respectively. While $\mathcal{T}$ and Ξ flip the sign of k since they are anti-unitary, Γ does not since it is unitary. Among three symmetries particle-hole symmetry is the simplest one to be checked, since the right hand side of the symmetry relation is the time-evolution operator itself. For example, $\mathcal{U}_{e1}(k)$ in Eq. (3.18) has particle-hole symmetry with $\Xi = \mathcal{K}$, which we can check from relations of each operator,

$$\Xi C_2(\theta) \Xi^{-1} = C_2(\theta), \quad \Xi S(k) \Xi^{-1} = S(-k). \quad (3.24)$$

However, checking time-reversal symmetry and chiral symmetry is slightly more complicated than checking particle-hole symmetry, since the right hand sides of symmetry relations are inverse of time-evolution operators. Then, we use symmetric time frames [53], in which elemental operators are arranged symmetrically and we can check symmetry relations easily. In a symmetric time frame, $\mathcal{U}_{e1}(k)$ and $\mathcal{U}_{e2}(k)$ are redefined as

$$\mathcal{U}'_{e1}(k) = C_2(\theta/2) S(k) C_2(\theta/2), \quad \mathcal{U}'_{e2}(k) = C_1(\theta_1/2) S(k/2) C_2(\theta_2) S(k/2) C_1(\theta_1/2), \quad (3.25)$$

respectively. As shown in Fig. 3.3, taking symmetric time frames corresponds to changing the origin of time steps. Since these time-evolution operators are obtained by unitary transformations, $\mathcal{U}'_{e1}(k) = C_2(\theta/2) \mathcal{U}_{e1}(k) C_2(-\theta/2)$, $\mathcal{U}'_{e2}(k) = C_1(\theta_1/2) \mathcal{U}_{e2}(k) C_1(-\theta_1/2)$, they respectively share the same spectrum with $\mathcal{U}_{e1}(k)$ and $\mathcal{U}_{e2}(k)$. We can also define time-evolution operators in different time frames, such as

$$\mathcal{U}''_{e1}(k) = S(k/2) C_2(\theta) S(k/2), \quad \mathcal{U}''_{e2}(k) = C_2(\theta_2/2) S(k/2) C_1(\theta_1) S(k/2) C_2(\theta_2/2), \quad (3.26)$$

by shifting the origin of time steps. In general, there are infinite number of time frames which we can take and we choose appropriate one for exploring symmetries. In Secs. 3.3.2 and 4.3, we use two types of time frames for deriving topological numbers of chiral symmetric quantum walks. Taking symmetric time frames, we can check symmetry relations from each elemental operator. For example, from Eqs. (3.23) and (3.25), we can understand that $\mathcal{U}'_{e1}(k)$ has time-reversal symmetry and chiral symmetry, since

$$\mathcal{T} C_2(\theta/2) \mathcal{T}^{-1} = C_2^{-1}(\theta/2), \quad \mathcal{T} S(k) \mathcal{T}^{-1} = S^{-1}(-k), \quad (3.27)$$

$$\Gamma C_2(\theta/2) \Gamma^{-1} = C_2^{-1}(\theta/2), \quad \Gamma S(k) \Gamma^{-1} = S^{-1}(k) \quad (3.28)$$

are satisfied with $\mathcal{T} = \sigma_1 \mathcal{K}$ and $\Gamma = \sigma_1$. In the same way, we can check that $\mathcal{U}'_{e2}(k)$ has time-reversal symmetry with $\mathcal{T} = \sigma_1 \mathcal{K}$,

$$\mathcal{T} \mathcal{C}_1(\theta_1/2) \mathcal{T}^{-1} = \mathcal{C}_1^{-1}(\theta_1/2), \quad \mathcal{T} \mathcal{C}_2(\theta_2) \mathcal{T}^{-1} = \mathcal{C}_2^{-1}(\theta_2), \quad \mathcal{T} \mathcal{S}(k/2) \mathcal{T}^{-1} = \mathcal{S}^{-1}(-k/2). \quad (3.29)$$

In the following, we define time-evolution operators in time frames where symmetries can be seen easily when we discuss symmetries, and remove the superscripts ' and '' if these are not necessary.

3.3 Topological numbers originating from chiral symmetry

Quantum walks can possess non-trivial topological phases not analogous to static systems, which accomodate edge states whose quasi-energies are π . For the emergence of such anomalous topological phases characterized by winding numbers, chiral symmetry plays a key role. Discussions in this section are partially based on Ref. [53].

3.3.1 chiral symmetry in quantum walks

While we defined naive chiral symmetry of U based on effective Hamiltonians in Sec. 3.2, we impose a more strict condition for U to have chiral symmetry. Given a time-evolution operator U' in a time frame, we decompose it into two parts,

$$U' = U_2 U_1. \quad (3.30)$$

Since time-evolution operators of quantum walks are composed of elemental operators such as $C_1(\theta)$, S_- , and so on, various decompositions are possible. We define chiral symmetry of quantum walks as a relation on U_1 and U_2 ,

$$\Gamma U_1 \Gamma^{-1} = U_2^{-1}. \quad (3.31)$$

In other words, if there exists a decomposition such that U_1 and U_2 satisfy Eq. (3.31), U' has chiral symmetry. In the case of U'_{e1} in Eq. (3.25), $U_1 = S_+ C_2(\theta/2)$ and $U_2 = C_2(\theta/2) S_-$ with $\Gamma = \sigma_1$ since S can be decomposed into $S_- S_+$ as in Eq. (3.12). When U' satisfies chiral symmetry, we can define another time-evolution operator with chiral symmetry,

$$U'' = U_1 U_2, \quad (3.32)$$

in a different time frame. Equation (3.31) is a sufficient condition for U' and U'' to satisfy naive chiral symmetry in Eq. (3.21),

$$\Gamma U' \Gamma^{-1} = U'^{-1}, \quad \Gamma U'' \Gamma^{-1} = U''^{-1}. \quad (3.33)$$

We remark that chiral symmetry in Eq. (3.31) is essential for deriving topological numbers, while naive chiral symmetry in Eq. (3.33) is insufficient. While the reason why we need to define chiral symmetry as in Eq. (3.31) is related to boundary conditions and anomalous Floquet topological phases, we do not discuss it in detail here since we are working on this problem at the moment when this doctoral thesis is written. Chiral symmetry ensures that quasi-energies of U' appear in pairs of $\pm \varepsilon_n$,

$$U' |\phi'_n\rangle = e^{-i\varepsilon_n} |\phi'_n\rangle, \quad U' \Gamma |\phi'_n\rangle = e^{i\varepsilon_n} \Gamma |\phi'_n\rangle \quad (3.34)$$

These two eigen equations are related by naive chiral symmetry of U' in Eq. (3.33). Then the quasi-energy spectrum becomes symmetric to $\varepsilon = 0$ and π in chiral symmetric quantum walks. Topologically protected edge states appear at such particular quasi-energies when boundaries exist. It is convenient to take the basis such that chiral symmetry operator becomes

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.35)$$

In this basis, time-evolution operators in momentum space have the following form

$$\mathcal{U}'(k) = \begin{pmatrix} A'(k) & B'(k) \\ -B'^{\dagger}(k) & D'(k) \end{pmatrix}, \quad A'(k) = A'^{\dagger}(k), \quad D'(k) = D'^{\dagger}(k), \quad (3.36)$$

because of naive chiral symmetry in Eq. (3.33). Then, $\sin[\mathcal{H}'(k)] = i[\mathcal{U}'(k) - \mathcal{U}'^{\dagger}(k)]/2$ and $\cos[\mathcal{H}'(k)] = [\mathcal{U}'(k) + \mathcal{U}'^{\dagger}(k)]/2$ become

$$\sin[\mathcal{H}'(k)] = i \begin{pmatrix} 0 & B'(k) \\ -B'^{\dagger}(k) & 0 \end{pmatrix}, \quad \cos[\mathcal{H}'(k)] = \begin{pmatrix} A'(k) & 0 \\ 0 & D'(k) \end{pmatrix}. \quad (3.37)$$

From the same discussion, $\mathcal{U}''(k)$ and corresponding effective Hamiltonian $\mathcal{H}''(k)$ have the same structure as Eq. (3.37),

$$\sin[\mathcal{H}''(k)] = i \begin{pmatrix} 0 & B''(k) \\ -B''^{\dagger}(k) & 0 \end{pmatrix}, \quad \cos[\mathcal{H}''(k)] = \begin{pmatrix} A''(k) & 0 \\ 0 & D''(k) \end{pmatrix}, \quad (3.38)$$

where $A''(k) = A''^{\dagger}(k)$ and $D''(k) = D''^{\dagger}(k)$ are satisfied. From naive chiral symmetry of time-evolution operators in Eq. (3.33), $\sin[\mathcal{H}'(k)]$ and $\sin[\mathcal{H}''(k)]$ satisfy

$$\Gamma \sin[\mathcal{H}'(k)] \Gamma^{-1} = -\sin[\mathcal{H}'(k)], \quad \Gamma \sin[\mathcal{H}''(k)] \Gamma^{-1} = -\sin[\mathcal{H}''(k)]. \quad (3.39)$$

Thereby, when both gaps at $\varepsilon = 0$ and $\varepsilon = \pi$ are open, winding numbers of $B'(k)$ and $B''(k)$

$$\nu' = \frac{1}{2\pi i} \int_{-\pi}^{\pi} dk \frac{d}{dk} \ln \det[B'(k)], \quad \nu'' = \frac{1}{2\pi i} \int_{-\pi}^{\pi} dk \frac{d}{dk} \ln \det[B''(k)] \quad (3.40)$$

topologically characterize the systems described by U' and U'' , respectively. The winding number ν' or ν'' in Eq. (3.40) does not solely characterize the system topologically. As we discuss in the following, we need to consider both ν' and ν'' to precisely predict the numbers of edge states with $\varepsilon = 0$ and $\varepsilon = \pi$.

3.3.2 derivation from the bulk-edge correspondence

We can show that linear combinations of ν' and ν'' predict the numbers of edge states when the bulk spectrum is gapped at both $\varepsilon = 0$ and π [53]. To this end, we consider a system in which two regions named L and R with different parameter sets are combined. Since chiral symmetry makes spectra symmetric with respect to $\varepsilon = 0$ and $\varepsilon = \pi$, topologically protected edge states should appear at these special quasi-energies when there are boundaries. The latter edge states at $\varepsilon = \pi$ are intrinsic to Floquet systems since their origin is the 2π periodicity of ε or equivalently the discrete periodicity in time.

In the Hilbert space spanned by eigenstates with $\varepsilon = 0, \pi$, we can simultaneously diagonalize time-evolution operators and chiral symmetry operator, since U' and U'' commute with Γ . Also, U' and U'' have the same eigenvalues because they are related by the unitary transformation $U'' = U_1 U' U_1^{-1}$. Then, we denote eigenvalue equations of U' and U'' with $\varepsilon = 0, \pi$ as

$$U' |\Phi'_{\varepsilon, \eta'}\rangle = e^{-i\varepsilon} |\Phi'_{\varepsilon, \eta'}\rangle, \quad U'' |\Phi''_{\varepsilon, \eta''}\rangle = e^{-i\varepsilon} |\Phi''_{\varepsilon, \eta''}\rangle, \quad \varepsilon = 0, \pi, \quad (3.41)$$

where the subscripts η' and η'' denote ± 1 corresponding to eigenvalues of Γ ,

$$\Gamma |\Phi'_{\varepsilon, \eta'}\rangle = \eta' |\Phi'_{\varepsilon, \eta'}\rangle, \quad \Gamma |\Phi''_{\varepsilon, \eta''}\rangle = \eta'' |\Phi''_{\varepsilon, \eta''}\rangle, \quad (3.42)$$

which we refer to as chirality. We can thereby label these edge states by the quasi-energy and chirality. Then, winding numbers in Eq. (3.40) should satisfy the bulk-edge correspondence

$$\nu'_L - \nu'_R = n'_{0+} + n'_{\pi+} - n'_{0-} - n'_{\pi-}, \quad (3.43)$$

$$\nu''_L - \nu''_R = n''_{0+} + n''_{\pi+} - n''_{0-} - n''_{\pi-}, \quad (3.44)$$

where ν'_L and ν''_L (ν'_R and ν''_R) are respectively winding numbers obtained from U' and U'' in the region L (R). We want to derive topological numbers (ν_0, ν_π) which satisfy the bulk-edge correspondence for zero-energy states and π -energy states,

$$\nu_0^L - \nu_0^R = n'_{0+} - n'_{0-}, \quad \nu_\pi^L - \nu_\pi^R = n'_{\pi+} - n'_{\pi-}. \quad (3.45)$$

The reason why Eqs. (3.43)-(3.45) indicate the bulk-edge correspondence is that eigenstates whose quasi-energies are neither zero nor π inevitably have amplitudes in both internal states $|+\rangle$ and $|-\rangle$. This can be understood from Eq. (3.34),

$$\langle \phi_n | \Gamma | \phi_n \rangle = 0 \rightarrow \langle x, + | \phi_n \rangle = -\langle x, - | \phi_n \rangle \text{ with } \varepsilon_n \neq 0, \pi, \quad (3.46)$$

since $|\phi_n\rangle$ and $\Gamma |\phi_n\rangle$ have different eigenvalues $\lambda_n = e^{-i\varepsilon_n}$ and $\lambda_n^{-1} = e^{i\varepsilon_n}$. Equation (3.46) means that a pair of eigenstates with $\varepsilon = 0$ or π which have opposite chiralities is needed to make an eigenstate whose quasi-energy is not equal to zero and π . Then, the right hand sides of Eq. (3.45) become topological numbers of edgestates with $\varepsilon = 0, \pi$ since these quantities do not change as long as chiral symmetry is satisfied and the gaps are open.

We can derive ν_0 and ν_π by revealing a relation between η' and η'' . Acting U_1 from left on the eigenvalue equation of U' in Eq. (3.41) and taking Eqs. (3.30) and (3.32) into account, we obtain

$$U'' U_1 |\Phi'_{\varepsilon, \eta'}\rangle = e^{-i\varepsilon} U_1 |\Phi'_{\varepsilon, \eta'}\rangle. \quad (3.47)$$

From Eq. (3.47), we can understand that edge states with quasi-energy ε in the system described by U'' are obtained from $|\Phi'_{\varepsilon, \eta'}\rangle$,

$$|\Phi''_{\varepsilon, \eta''}\rangle = U_1 |\Phi'_{\varepsilon, \eta'}\rangle. \quad (3.48)$$

Chirality of $|\Phi''_{\varepsilon, \eta''}\rangle$ is determined by ε and chirality of $|\Phi'_{\varepsilon, \eta'}\rangle$. Acting chiral symmetry operator Γ from left on both sides of Eq. (3.48), $\Gamma |\Phi''_{\varepsilon, \eta''}\rangle$ becomes

$$\Gamma |\Phi''_{\varepsilon, \eta''}\rangle = U_2^{-1} \eta' |\Phi'_{\varepsilon, \eta'}\rangle, \quad (3.49)$$

owing to Eqs. (3.31) and (3.42). Using Eqs. (3.30) and (3.41), the right hand side of Eq. (3.49) can be written as

$$U_2^{-1} \eta' |\Phi'_{\varepsilon, \eta'}\rangle = U_2^{-1} \eta' e^{i\varepsilon} U_2 U_1 |\Phi'_{\varepsilon, \eta'}\rangle. \quad (3.50)$$

From Eqs. (3.48), (3.49), and (3.50), $|\Phi''_{\varepsilon, \eta''}\rangle$ satisfies

$$\Gamma |\Phi''_{\varepsilon, \eta''}\rangle = \eta'' |\Phi''_{\varepsilon, \eta''}\rangle, \quad \eta'' = \eta' e^{i\varepsilon}. \quad (3.51)$$

Therefore, chirality of $|\Phi''_{0, \eta''}\rangle$ is the same as that of $|\Phi'_{0, \eta'}\rangle$ which is η' . On the other hand, $|\Phi''_{\pi, \eta''}\rangle$ has chirality opposite to that of $|\Phi'_{\pi, \eta'}\rangle$ which becomes $-\eta'$. Then, the numbers of edge states satisfy

$$n'_{0+} = n''_{0+}, \quad n'_{\pi+} = n''_{\pi-}, \quad n'_{0-} = n''_{0-}, \quad n'_{\pi-} = n''_{\pi+}. \quad (3.52)$$

Substituting (3.52) into Eqs. (3.43) and (3.44), we can derive

$$(\nu'_L + \nu''_L) - (\nu'_R + \nu''_R) = 2(n'_{0+} - n'_{0-}), \quad (\nu'_L - \nu''_L) - (\nu'_R - \nu''_R) = 2(n'_{\pi+} - n'_{\pi-}). \quad (3.53)$$

From Eqs. (3.45) and (3.53), topological numbers (ν_0, ν_π) become

$$\nu_0 = \frac{\nu' + \nu''}{2}, \quad \nu_\pi = \frac{\nu' - \nu''}{2}. \quad (3.54)$$

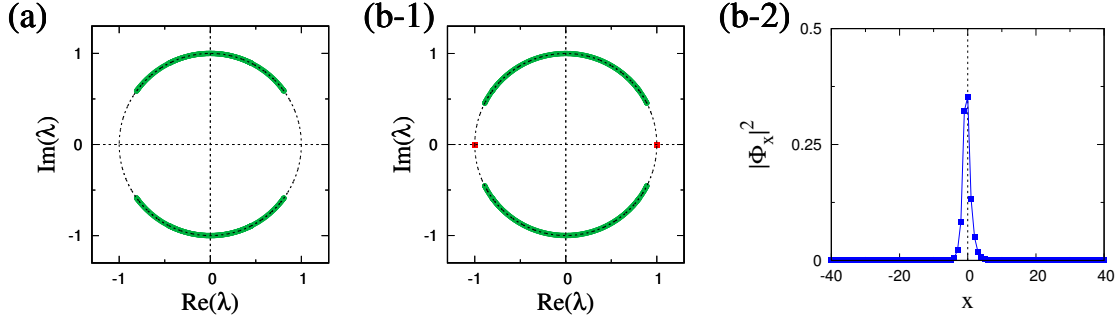


Figure 3.4: Eigenvalues and an edgestate of U'_{e1} when (a) $(\theta_L, \theta_R) = (2\pi/5, 7\pi/10)$ and (b) $(\theta_L, \theta_R) = (2\pi/5, -3\pi/10)$. In (a) and (b-1), green circles and dashed curves represent eigenvalues and the unit circle, respectively. The red squares in (b-1) at $\lambda = +1$ and $\lambda = -1$ correspond to the edge states with $\varepsilon = 0$ and $\varepsilon = \pi$, respectively. (b-2) shows a probability distribution of a topologically protected edgestate, whose eigenvalue corresponds to one of red squares in (b-1).

These winding numbers ν_0 and ν_π predict the numbers of topologically protected edge states at $\varepsilon = 0$ and π , respectively. It should be noted that both gaps at $\varepsilon = 0$ and $\varepsilon = \pi$ must be open for calculating ν' and ν'' , which are respectively winding numbers of $B'(k)$ and $B''(k)$ in Eqs. (3.37) and (3.38), so that $\ln \det[B'(k)]$ and $\ln \det[B''(k)]$ are well defined on the integral path.

We show that the derived topological numbers are valid, by using U_{e1} in Eq. (3.13) as an example. Topological numbers are calculated from $\mathcal{U}'_{e1}(k)$ in Eq. (3.25) and $\mathcal{U}''_{e1}(k)$ in Eq. (3.26) which are defined in different time frames and satisfy chiral symmetry with $\Gamma = \sigma_1$. Carrying out a unitary transformation which results in $\Gamma = \sigma_3$, we can obtain ν' and ν'' from (1, 2) components of $e^{i\frac{\pi}{4}\sigma_2}\mathcal{U}'_{e1}(k)e^{-i\frac{\pi}{4}\sigma_2}$ and $e^{i\frac{\pi}{4}\sigma_2}\mathcal{U}''_{e1}(k)e^{-i\frac{\pi}{4}\sigma_2}$,

$$\nu' = \text{sign}[\sin(\theta)], \nu'' = 0 \rightarrow \nu_0 = \nu_\pi = \frac{1}{2}\text{sign}[\sin(\theta)]. \quad (3.55)$$

Figure 3.4 shows eigenvalues and eigenstates of U_{e1} when the position dependence of $\theta(x)$ is

$$\theta(x) = \begin{cases} \theta_L & (x \leq 0) \\ \theta_R & (x > 0) \end{cases} \quad (3.56)$$

When $\theta_L = 2\pi/5$ and $\theta_R = 7\pi/10$, topological numbers in both regions are the same and edge states do not exist at $\varepsilon = 0, \pi$, as shown in Fig. 3.4 (a). On the other hand, if $\theta_L = 2\pi/5$ and $\theta_R = -3\pi/10$, different topological numbers in the left and right regions result in edge states at $\varepsilon = 0, \pi$ which are described by red squares in Fig. 3.4 (b-1). The corresponding eigenstates are actually localized near the boundary, which is shown in Fig. 3.4 (b-2).

3.4 Experiments

The dynamics of quantum walks have been explored in various experimental settings. Here, we explain two types of photonic experiments.

First one is schematically depicted in Fig. 3.5 (a) where optical fiber loops are used [85, 86, 93, 97, 98, 99, 100]. The system is composed of two loops whose lengths are different. The 50/50 coupler which is the purple rectangle in Fig. 3.5 (a) separates a wave packet which propagates in the long or short fiber into two wave packets which respectively propagate in short and long fibers. This process is regarded as that a coin operator separates $|+\rangle$ state or $|-\rangle$ state into a linear combination of $|+\rangle$ and $|-\rangle$. Actually, the operation of the 50/50 coupler is described by the

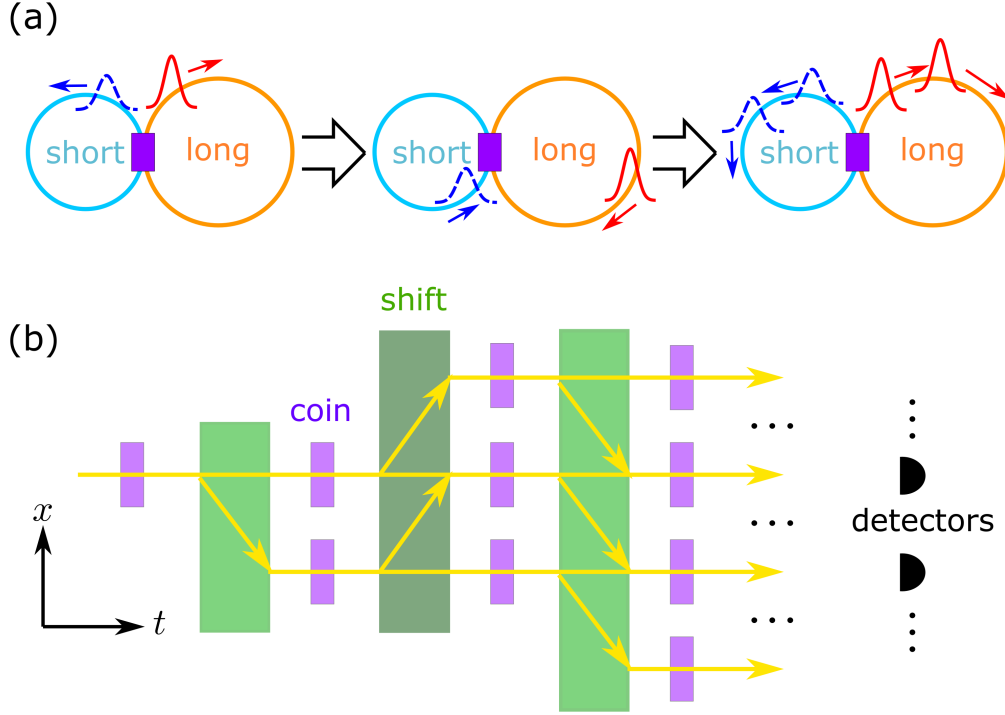


Figure 3.5: Physical settings of quantum walks, using (a) optical fibre loops and (b) arrays of optical elements. In (a), advance and delay in time caused by the difference in the lengths of fibers are respectively regarded as position shift to right and left in real space. In (b), half-wave plates (purple small rectangles) and beam displacers (green large rectangles) play the role of coin operators and shift operators, respectively.

coin operator $\mathcal{C}_1(\pi/4)$ in Eq. (3.5). Since lengths of two fibers are different, the timing at which wave packets in the long (short) fiber loop enter the 50/50 coupler again is delayed (advanced) in comparison to wave packets in the short (long) loop. This process is regarded as that wave packets go left (right); the time delay of wave packets due to the propagation in the long fiber is regarded as the position shift to left direction and vice versa is the shift of wave packets to right direction. In other words, the advance and delay of wave packets in time domain are respectively identified with the position shift to right and left directions in real space. While Fig. 3.5 (a) is for one-dimensional cases, experiments of two-dimensional systems also have been done in similar settings to Fig. 3.5 (a) [99, 100]. In addition, based on the mapping from time delay and advance to position shifts in real space which is similar to Fig. 3.5 (a), topologically protected edge states were observed in a two-dimensional quantum walk [70], while optical fibers were not utilized.

Second one is schematically depicted in Fig. 3.5 (b) where single photons or beams of coherent light pass through various optical elements. In such systems, the direction in which light propagates is identified as "time", and thereby time-evolution operators are composed of several optical elements which are arrayed along the direction of time. We consider that polarizations $|H\rangle$ and $|V\rangle$ of light correspond to internal states $|-\rangle$ and $|+\rangle$ in quantum walks, respectively. Then, combinations of half-wave plates and phase shifters which are described as small purple rectangles in Fig. 3.5 (b) play the role of coin operators. Operations of phase shifters are described by $\mathcal{C}_3(\theta)$ in Eq. (3.8). Half-wave plates are made of birefringent substances in which refractive indices depend

on the directions of polarization. Here, we write polarization directions which propagate rapidly and slowly as $|H'\rangle$ and $|V'\rangle$, respectively. Since half-wave plates are designed so that the difference in propagation distances of $|H'\rangle$ and $|V'\rangle$ becomes the half of a wave length which is focused on, $|V'\rangle$ acquires the phase π relative to $|H'\rangle$ after passing through a half-wave plate. When a photon or coherent light with a polarization $\cos(\theta)|H'\rangle - \sin(\theta)|V'\rangle$ passes through a half-wave plate for the corresponding wave length, the polarization is changed into $\cos(\theta)|H'\rangle + \sin(\theta)|V'\rangle$. In the same way, $\sin(\theta)|H'\rangle + \cos(\theta)|V'\rangle$ is changed into $\sin(\theta)|H'\rangle - \cos(\theta)|V'\rangle$ by the half-wave plate. The angle θ can take arbitrary values because the relative angle between $|H\rangle$ and $|H'\rangle$ or equivalently $|V\rangle$ and $|V'\rangle$ can be changed by rotating half-wave plates. Then, if we set θ so that $|H\rangle = \cos(\theta)|H'\rangle - \sin(\theta)|V'\rangle$ and $|V\rangle = \sin(\theta)|H'\rangle + \cos(\theta)|V'\rangle$ are satisfied, $C_H(2\theta) = C_2(2\theta)\sigma_3$ is acted on polarizations of light which passes through the half-wave plate. Therefore, combining half-wave plates and phase shifters, various types coin operators are realized. Beam displacers which are described as large green rectangles in Fig. 3.5 (b) take the role of shift operators. Beam displacers are made of birefringent substances in which refractive angles depend on polarizations. Then, when light passes through beam displacers, light moves in the direction of x depending on polarizations. For example, passing through a beam displacer, $|H\rangle$ components go left and $|V\rangle$ components go straight. Observations of topologically protected edge states based on experimental settings in Fig. 3.5 (b) were carried out in 2012 for the first time [51]. They constructed a system described by $U = S_- C_2(\theta_2/2)\sigma_3 S_+ C_2(\theta_1/2)\sigma_3$ in which topological numbers depend on θ_1 and θ_2 . In Ref. [51], edge states were directly observed through probability distributions of single photons by varying topological numbers spatially and preparing initial states which localize near the boundary at which topological numbers change. Such controls of topological numbers are possible since rotation angles θ_1 and θ_2 can be manipulated as mentioned above. It is an advantage of photonic systems including quantum walks that such real-space observations of edge states are possible, in comparison to electronic systems in which existence of edge states are indirectly confirmed by transport properties.

3.5 Novel physical effects and objectives

So far, we have explained quantum walks in which walkers experience linear and unitary time-evolutions. However, novel physical effects can be induced artificially or can emerge spontaneously. These effects give rise to interesting topological phenomena which cannot be understood from the analogy to conventional systems. In order to mention our objectives, we briefly explain these effects, while details are written in Secs. 4.4.1 and 5.2.

Gain-loss effects of light can be manipulated in both settings (a) and (b) of Fig. 3.5, while gain effects cannot (can) exist in (only in) quantum walks of single photons (classical light). Gain-loss effects on classical light in optical fiber loops are introduced by combinations of erbium doped fiber amplifiers and couplers which separate light in a fiber into two fibers [86, 93, 98]. Loss effects on single photons in arrays of optical elements originate from partially polarizing beam splitters which leak photons to environments in a probability which depends on polarizations [64, 72, 76]. Under these gain and/or loss effects, total intensity of light and probability of single photons are not conserved. In such nonunitary time evolutions, unique topological phenomena can occur, without analogy to closed systems. Then, our first objective is to explore peculiar topological phenomena in open quantum walks, which is addressed in Chap. 4.

Nonlinear effects can also emerge in quantum walks of classical light. When light pulses propagate in optical fibers with high intensities, Kerr nonlinearity cannot be ignored [97, 98, 99, 100]. In such a case, phases attached to wave function amplitudes at each position as in Eq. (3.8) depend on wave function amplitudes itself, which results in nonlinear dynamics. In addition, it has been proposed to artificially induce nonlinear effects [94, 95, 96]. In the proposed schemes, light intensities at each position is measured during time evolutions, and the rotation angles of

coin operators are manipulated depending on the results of the measurements. In other words, $\theta(x)$ in Eq. (3.5) depends on intensities of light, which makes the dynamics nonlinear. Our second objective is to study peculiar topological phenomena which should be caused by nonlinear effects in quantum walks, which is addressed in Chap. 5.

While gain-loss effects and nonlinear effects explained above are additional physical effects to ordinary quantum walks, quantum walks intrinsically has a unique feature. The unique feature is the discrete periodicity in time, as we have stated throughout this chapter. In such Floquet systems, there are topological phases which cannot be understood in the same way as static systems described by time-independent Hamiltonians. For example, a unique winding number which is referred to as the Rudner winding number and resulting edge states can exist in Floquet systems [109]. Also, edge states at $\varepsilon = \pi$ which we have discussed in Sec. 3.3 do not exist in static systems. While we have focused on chiral symmetry and resulting winding numbers in Chaps. 2 and 3, symmetries which have no counterpart in static systems can exist in Floquet systems [110]. Then, our third objective is to explore topological phases which stem from unique symmetries in Floquet systems, which is addressed in Chap. 6.

Chapter 4

Bulk-Edge Correspondence in Chiral Symmetric Open Floquet Systems

4.1 Introduction

While topological phases have been explored mainly in closed systems described by Hermitian Hamiltonians, non-trivial topological phases can also exist in open systems described by non-Hermitian "Hamiltonians" [111, 112], of which the former and latter correspond to unitary and nonunitary time-evolution operators in quantum walks. Historically, theory of such open systems has been developed independently of research fields on topological phases. First, in 1996, Hatano and Nelson found that localization-delocalization transitions can occur in a one-dimensional non-Hermitian system with asymmetric hopping amplitudes and on-site disorder, while the delocalized states are absent in one-dimensional Hermitian systems [113]. Non-Hermitian Hamiltonians which satisfy $\mathcal{PT}$ symmetry were secondary paid attention to, where $\mathcal{PT}$ symmetry is a combined symmetry of parity and time-reversal symmetries. This is because in 1998 Bender and Boettcher discovered that $\mathcal{PT}$ symmetric non-Hermitian Hamiltonians can have entirely real spectra [114]. Inspired by their discoveries, various researches have been carried out for further understanding quantum systems [115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136] and exploring novel phenomena [137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 86, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163] which cannot be realized in closed Hermitian systems. Third, topological phases in open systems started to attract attention in 2011 [164, 165]. As a result, unique features have been revealed [164, 165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 43, 44, 176, 111, 177, 178, 179, 180, 72, 181, 182, 183, 184, 185, 186, 187, 112, 188, 189, 190, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 76, 204], of which some recast the Hatano-Nelson model from the viewpoint of the bulk-edge correspondence [111, 185, 202, 203].

Non-Hermitian "Hamiltonians" can describe not only open quantum systems [205, 206, 156] but also classical systems, which is now widely known although Refs. [113, 114] did not mention the fact. Quantum walks are ideal platforms to study both types of open systems which are classical systems with gain-loss effects and post selected quantum systems. This is because rate of gain-loss and post selection can be manipulated in quantum walks, which we explain in Sec. 4.4.1. As we have mentioned, quantum walks are also versatile testbeds of topological phenomena, since real-space observation of topological edge states and manipulation of symmetries and/or topological numbers are possible [51, 62, 70]. Although topological phases in open systems described by non-Hermitian Hamiltonians [164, 165, 169, 172, 173, 174, 177, 168, 170, 175, 112] and closed

quantum walks described by unitary time-evolution operators [48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 73, 74, 75, 207] have been extensively studied, those of open quantum walks have not been studied so much. Since quantum walks described by nonunitary time-evolution operators are not only open systems without the conservation law on probability or intensity but also Floquet systems in which time evolves in a discrete manner, it should be interesting to explore topological phases of open quantum walks.

In this work, we study topological phases associated with chiral symmetry and the corresponding edge states in one-dimensional open Floquet systems with gain and loss. We derive a procedure to calculate topological numbers for Floquet topological phases driven by chiral symmetric nonunitary time-evolutions. While a method to calculate topological numbers in chiral symmetric closed Floquet systems was obtained in Ref. [53], we generalize the procedure to open Floquet systems. Our derivation gives a microscopic foundation of topological numbers which have been used in open Floquet systems without definitive ground. We construct a model of open quantum walks based on the experiments in Refs. [86, 72] and show that the derived topological numbers correctly predict the numbers of edge states. We also show that the constructed model satisfies $\mathcal{PT}$ symmetry in some situations and emergent topological edge states break $\mathcal{PT}$ symmetry, which results in exponential amplification of their wave function amplitudes with increasing time steps.

The rest of this chapter is organized as follows. In Secs. 4.2 and 4.3, we derive symmetry relations and chiral topological numbers of open Floquet systems, respectively. The discussions in Secs. 4.2 and 4.3 are not restricted to quantum walks but can be applied to general open Floquet systems. In Sec. 4.4, we introduce experimental settings and the way of describing the dynamics of open quantum walks. In Sec. 4.5, constructing a chiral symmetric model of open quantum walks, we confirm the bulk-edge correspondence using the derived topological numbers and discuss the dynamics based on $\mathcal{PT}$ symmetry which the model has in some situations. Section 4.6 is conclusion.

4.2 Symmetries of open Floquet systems

We clarify symmetries related to topological phases and the dynamics in general open Floquet systems, of which the former and latter are respectively discussed in Secs. 4.2.1 and 4.2.2.

4.2.1 global symmetries related to topological phases

We extend global symmetries in closed systems to open systems, which are relevant to topological phases. To this end, we rewrite relations of time-reversal symmetry, particle-hole symmetry, and chiral symmetry of Hermitian Hamiltonians $\mathcal{H}(k) = \mathcal{H}^\dagger(k)$ in Eqs. (2.18) and (2.20),

$$\mathcal{T}\mathcal{H}(k)\mathcal{T}^{-1} = \mathcal{H}(-k), \quad \Xi\mathcal{H}(k)\Xi^{-1} = -\mathcal{H}(-k), \quad \Gamma\mathcal{H}(k)\Gamma^{-1} = -\mathcal{H}(k). \quad (4.1)$$

The squares of anti-unitary symmetry operators $\mathcal{T}$ and Ξ are $+1$ or -1 , while the unitary symmetry operator Γ satisfies $\Gamma^2 = 1$. Naive symmetry relations of unitary time-evolution operators $\mathcal{U}^{-1}(k) = \mathcal{U}^\dagger(k)$ are obtained from Eq. (4.1),

$$\mathcal{T}\mathcal{U}(k)\mathcal{T}^{-1} = \mathcal{U}^{-1}(-k), \quad \Xi\mathcal{U}(k)\Xi^{-1} = \mathcal{U}(-k), \quad \Gamma\mathcal{U}(k)\Gamma^{-1} = \mathcal{U}^{-1}(k), \quad (4.2)$$

with $\mathcal{U}(k) = e^{-i\mathcal{H}(k)}$. Closed systems described by Hermitian Hamiltonians or unitary time-evolution operators are classified from symmetries in Eq. (4.1) or Eq. (4.2), which is referred to as AZ classification shown in Tab. 2.1. On the other hand, there are open systems described by non-Hermitian "Hamiltonians" $\mathcal{M}(k)$ or nonunitary time-evolution operators $\mathcal{V}(k)$,

$$\mathcal{M}(k) \neq \mathcal{M}^\dagger(k), \quad \mathcal{V}^{-1}(k) \neq \mathcal{V}^\dagger(k). \quad (4.3)$$

For instance, non-Hermitian terms stemming from interactions with environment are added to "Hamiltonians" in post-selected quantum systems, and vectorized Lindblad equation which describes the dynamics of density matrix has the same structure as Schrödinger equation with non-Hermitian "Hamiltonians" [205]. Also, deviating from quantum regime, non-Hermitian operators ubiquitously emerge in various systems since there is no constraint of Hermiticity on operators describing dynamics. In such open quantum systems and classical systems, removing the constraint of Hermiticity on "Hamiltonians" results in the possibility of two symmetries from each symmetry in closed systems [208, 112]. While the above examples are systems in which continuous time evolutions are described by "Hamiltonians", Floquet systems which obey discrete time evolutions with post selection or gain-loss effects are described by nonunitary time-evolution operators, as we show in Sec. 4.4.1. Then, through the ramification of symmetries on Hamiltonians, symmetries on time-evolution operators also ramify based on the definition of effective Hamiltonians $\mathcal{V}(k) = e^{-i\mathcal{M}(k)}$,

$$\text{TRS} : \begin{cases} \mathcal{T}\mathcal{M}(k)\mathcal{T}^{-1} = \mathcal{M}(-k) \rightarrow \mathcal{T}\mathcal{V}(k)\mathcal{T}^{-1} = \mathcal{V}^{-1}(-k) \\ \mathcal{T}\mathcal{M}(k)\mathcal{T}^{-1} = \mathcal{M}^\dagger(-k) \rightarrow \mathcal{T}\mathcal{V}(k)\mathcal{T}^{-1} = \mathcal{V}^\dagger(-k) \end{cases} \quad (4.4)$$

$$\text{PHS} : \begin{cases} \Xi\mathcal{M}(k)\Xi^{-1} = -\mathcal{M}^\dagger(-k) \rightarrow \Xi\mathcal{V}^{-1}(k)\Xi^{-1} = \mathcal{V}^\dagger(-k) \\ \Xi\mathcal{M}(k)\Xi^{-1} = -\mathcal{M}(-k) \rightarrow \Xi\mathcal{V}(k)\Xi^{-1} = \mathcal{V}^\dagger(-k) \end{cases} \quad (4.5)$$

$$\text{CS} : \begin{cases} \Gamma\mathcal{M}(k)\Gamma^{-1} = -\mathcal{M}(k) \rightarrow \Gamma\mathcal{V}(k)\Gamma^{-1} = \mathcal{V}^{-1}(k) \\ \Gamma\mathcal{M}(k)\Gamma^{-1} = -\mathcal{M}^\dagger(k) \rightarrow \Gamma\mathcal{V}(k)\Gamma^{-1} = \mathcal{V}^\dagger(k) \end{cases} \quad (4.6)$$

Note that, while topological phases in open systems have been classified based on symmetries of $\mathcal{M}(k)$ and novel phenomena in such systems have been explored [165, 164, 209, 210, 180, 211, 175, 178, 212, 213, 214, 181, 184, 112, 199], topological phenomena in systems obeying discrete nonunitary time evolutions [64, 72, 76] are less understood in comparison to continuous time systems described by non-Hermitian Hamiltonians.

4.2.2 $\mathcal{PT}$ symmetry in open systems

In addition to global symmetries which are important to topological phases, $\mathcal{PT}$ symmetry is also important in open systems, which is a combined symmetry of Parity symmetry and Time-reversal symmetry. This is because $\mathcal{PT}$ symmetry has crucial effects on the dynamics through real-complex transitions of eigenvalues [114, 115, 116, 117, 118, 119, 120, 121, 122, 124, 130, 141, 144, 145, 146, 147, 86, 150, 151, 152, 155, 156, 158, 159]. $\mathcal{PT}$ symmetry of a non-Hermitian Hamiltonian M is defined as

$$(\mathcal{PT})M(\mathcal{PT})^{-1} = M, \quad (4.7)$$

where $\mathcal{PT}$ includes the complex conjugation $\mathcal{K}$. Since $\mathcal{PT}$ symmetry operator also contains the parity operation, in the momentum space representation of Eq. (4.7), the sign of k is not inverted

$$(\mathcal{PT})\mathcal{M}(k)(\mathcal{PT})^{-1} = \mathcal{M}(k). \quad (4.8)$$

From Eqs. (4.7) and (4.8), $\mathcal{PT}$ symmetry relation of nonunitary time-evolution operators becomes

$$(\mathcal{PT})V(\mathcal{PT})^{-1} = V^{-1}, \quad (\mathcal{PT})\mathcal{V}(k)(\mathcal{PT})^{-1} = \mathcal{V}^{-1}(k). \quad (4.9)$$

When a nonunitary time-evolution operator has $\mathcal{PT}$ symmetry, the dynamics of the system drastically change depending on whether eigenstates of the time-evolution operator satisfy $\mathcal{PT}$ symmetry or not. Considering non-degenerate eigenvalues

$$V|\phi_\lambda\rangle = \lambda|\phi_\lambda\rangle, \quad \lambda = e^{-i\varepsilon} \quad (4.10)$$

and taking Eq. (4.9) into account, $\mathcal{PT}|\phi_\lambda\rangle$ is an eigenstate of V with the eigenvalue $(\lambda^*)^{-1}$ or equivalently the quasi-energy ε^* ,

$$V\mathcal{PT}|\phi_\lambda\rangle = (\lambda^*)^{-1}\mathcal{PT}|\phi_\lambda\rangle, \quad (\lambda^*)^{-1} = e^{-i\varepsilon^*}. \quad (4.11)$$

Then, $|\lambda| = 1$ or equivalently ε is real when $|\phi_\lambda\rangle$ is an eigenstate of $\mathcal{PT}$, even though V is nonunitary. On the other hand, if $|\phi_\lambda\rangle$ does not retain $\mathcal{PT}$ symmetry, $|\lambda| \neq 1$ or equivalently ε is complex. Therefore, there are two phases in open systems described by $\mathcal{PT}$ symmetric time-evolution operators,

1. $\mathcal{PT}$ symmetric phase: all of $|\phi_\lambda\rangle$ are eigenstates of $\mathcal{PT}$ and all of ε take real values.
2. $\mathcal{PT}$ broken phase: some of $|\phi_\lambda\rangle$ are not eigenstates of $\mathcal{PT}$ and complex quasi-energies exist.

In $\mathcal{PT}$ symmetric phase, wave function amplitudes do not diverge and decay to zero with time steps. When a complex quasi-energy with positive (negative) imaginary part exists, wave function amplitudes of the corresponding eigenstate grow (decay) exponentially with increasing time steps. Since quasi-energies appear in complex conjugate pairs in $\mathcal{PT}$ broken phase, there are both growing states and decaying states, which results in divergence of wave function amplitudes with time steps unless the overlap of the initial state and the growing eigenstates is exactly zero.

4.3 Topological numbers protected by chiral symmetry

We derive topological numbers in one-dimensional open Floquet systems, stemming from chiral symmetry. To this end, we take a procedure similar to that in Sec. 3.3.2.

4.3.1 Definitions of chiral symmetry and chirality

We define chiral symmetry of open Floquet systems as

$$\Gamma V_1 \Gamma^{-1} = V_2^\dagger, \quad (4.12)$$

where Γ is a unitary operator which satisfies $\Gamma^2 = 1$ and nonunitary operators V_1 and V_2 are components of time-evolution operators in different time frames

$$V' = V_2 V_1, \quad V'' = V_1 V_2. \quad (4.13)$$

Equation (4.12) is a sufficient condition that V' and V'' have naive chiral symmetry,

$$\Gamma V' \Gamma^{-1} = V'^\dagger, \quad \Gamma V'' \Gamma^{-1} = V''^\dagger, \quad (4.14)$$

which is the bottom one in Eq. (4.6). Chiral symmetry in Eq. (4.12) is essential for deriving topological numbers based on the bulk-edge correspondence, while naive chiral symmetry in Eq. (4.14) is insufficient for predicting the numbers of edge states. Since V' and V'' are related by a similarity transformation $V_2^{-1} V' V_2 = V''$, they share the same eigenvalues. Then, right eigenequations of V' and V'' are written as

$$V' |\phi'_\varepsilon\rangle = e^{-i\varepsilon} |\phi'_\varepsilon\rangle, \quad V'' |\phi''_\varepsilon\rangle = e^{-i\varepsilon} |\phi''_\varepsilon\rangle, \quad (4.15)$$

where ε is the quasi-energy. In order to derive topological numbers, we also need right eigenequations of $V'^\dagger$ and $V''^\dagger$

$$V'^\dagger |\tilde{\phi}'_\varepsilon\rangle = e^{i\varepsilon^*} |\tilde{\phi}'_\varepsilon\rangle, \quad V''^\dagger |\tilde{\phi}''_\varepsilon\rangle = e^{i\varepsilon^*} |\tilde{\phi}''_\varepsilon\rangle, \quad (4.16)$$

which are equivalent to Hermitian conjugations of left eigenequations of V' and V'' , respectively. Since V_1 and V_2 are not unitary, their quasi-energies ε need not be real. We assume that eigenstates satisfy the biorthogonal normalization conditions

$$\langle \phi'_\varepsilon | \tilde{\phi}'_\varepsilon \rangle = \langle \phi''_\varepsilon | \tilde{\phi}''_\varepsilon \rangle = \delta_{\varepsilon\bar{\varepsilon}}. \quad (4.17)$$

While $|\phi'_\varepsilon\rangle = |\tilde{\phi}'_\varepsilon\rangle$ and $|\phi''_\varepsilon\rangle = |\tilde{\phi}''_\varepsilon\rangle$ when time-evolution operators are unitary, right eigenstates and left eigenstates do not coincide when time-evolution operators are not unitary.

From Eqs. (4.14)-(4.16), we can understand that $\Gamma |\phi'_\varepsilon\rangle$ and $\Gamma |\phi''_\varepsilon\rangle$ are respectively proportional to $|\tilde{\phi}'_{-\varepsilon^*}\rangle$ and $|\tilde{\phi}''_{-\varepsilon^*}\rangle$,

$$\Gamma |\phi'_\varepsilon\rangle \propto |\tilde{\phi}'_{-\varepsilon^*}\rangle, \quad \Gamma |\phi''_\varepsilon\rangle \propto |\tilde{\phi}''_{-\varepsilon^*}\rangle. \quad (4.18)$$

Thereby chiral symmetry makes quasi-energies appear in $(\varepsilon, -\varepsilon^*)$ pairs. Then, eigenvalues of topologically protected edge states reside on the symmetric axes $\text{Re}(\varepsilon) = 0$ and $\text{Re}(\varepsilon) = \pi$ in one-dimensional systems. Therefore, left and right eigenstates corresponding to topologically protected edge states satisfy

$$\Gamma |\Phi'_{\varepsilon, \eta'}\rangle = \eta' |\tilde{\Phi}'_{\varepsilon, \eta'}\rangle, \quad \Gamma |\Phi''_{\varepsilon, \eta''}\rangle = \eta'' |\tilde{\Phi}''_{\varepsilon, \eta''}\rangle, \quad \text{Re}(\varepsilon) = 0, \pi. \quad (4.19)$$

The proportionality constant η' takes real values, since the biorthogonal normalization condition in Eq. (4.17) results in

$$\langle \tilde{\Phi}'_{\varepsilon, \eta'} | \Phi'_{\varepsilon, \eta'} \rangle = \langle \tilde{\Phi}'_{\varepsilon, \eta'} | \Gamma^2 | \Phi'_{\varepsilon, \eta'} \rangle = \frac{\eta'}{(\eta')^*} \langle \Phi'_{\varepsilon, \eta'} | \tilde{\Phi}'_{\varepsilon, \eta'} \rangle = 1. \quad (4.20)$$

In the same way, η'' is real. In addition, we can choose normalization coefficients so that

$$|\eta'|^2 = \frac{\langle \Phi'_{\varepsilon, \eta'} | \Phi'_{\varepsilon, \eta'} \rangle}{\langle \tilde{\Phi}'_{\varepsilon, \eta'} | \tilde{\Phi}'_{\varepsilon, \eta'} \rangle} = 1, \quad |\eta''|^2 = \frac{\langle \Phi''_{\varepsilon, \eta''} | \Phi''_{\varepsilon, \eta''} \rangle}{\langle \tilde{\Phi}''_{\varepsilon, \eta''} | \tilde{\Phi}''_{\varepsilon, \eta''} \rangle} = 1 \quad (4.21)$$

are satisfied. Then, the values of η' and η'' are $+1$ or -1 and we can use them as the labels of edge states. We refer to the label as chirality. Note that, if we consider the other symmetry $\Gamma V_1 \Gamma^{-1} = V_2^{-1}$ different from Eq. (4.12), which is a sufficient condition for the top one in Eq. (4.6), edge states satisfy $\varepsilon = 0, \pi$ and become the eigenstates of Γ . In this case, amplifications and attenuations of edge states are prohibited since their quasi-energies do not have finite imaginary part, while the discussions on the bulk-edge correspondence are satisfied in the similar way to those in Sec. 3.3.2. On the other hand, chiral symmetry in Eq. (4.12) considered here does not forbid amplifications and attenuations of edge states, which leads to peculiar phenomena in open systems as we clarify in Sec. 4.5.

4.3.2 how to calculate winding numbers

A procedure to calculate a winding number which satisfies the bulk-edge correspondence in chiral symmetric non-Hermitian systems was obtained in Ref. [164]. We explain the procedure in which the basis is taken so that chiral symmetry operator becomes σ_3 ,

$$\sigma_3 \mathcal{M}(k) \sigma_3 = -\mathcal{M}^\dagger(k). \quad (4.22)$$

From the right and left eigenstates of the Hamiltonian

$$\mathcal{M}(k) |\phi_\pm(k)\rangle = \varepsilon_\pm(k) |\phi_\pm(k)\rangle, \quad \langle \tilde{\phi}_\pm(k) | \mathcal{M}(k) = \langle \tilde{\phi}_\pm(k) | \varepsilon_\pm(k), \quad \sigma_3 |\phi_\pm(k)\rangle = |\tilde{\phi}_\mp(k)\rangle \quad (4.23)$$

where $\varepsilon_-(k) = -\varepsilon_+^*(k)$, we define a Hermitian matrix $\mathcal{Q}(k)$ as

$$\mathcal{Q}(k) = \frac{1}{2} [|\tilde{\phi}_+(k)\rangle \langle \phi_+(k)| + |\phi_+(k)\rangle \langle \tilde{\phi}_+(k)| - |\tilde{\phi}_-(k)\rangle \langle \phi_-(k)| - |\phi_-(k)\rangle \langle \tilde{\phi}_-(k)|]. \quad (4.24)$$

Equations (4.23) and (4.24) ensure that $\{\mathcal{Q}(k), \sigma_3\} = 0$ is satisfied, which results in

$$\mathcal{Q}(k) = \begin{pmatrix} 0 & q(k) \\ q^*(k) & 0 \end{pmatrix}. \quad (4.25)$$

Then, the systems is topologically characterized by the winding number of the off-diagonal term $\nu[q]$, which satisfies the bulk-edge correspondence [164].

4.3.3 discussions on the bulk-edge correspondence

We now discuss the bulk-edge correspondence, considering a system in which two regions labelled by L and R are jointed. Winding numbers ν' and ν'' , which are respectively obtained from $\mathcal{V}'(k)$ and $\mathcal{V}''(k)$ obeying the method in Sec. 4.3.2, are assumed to satisfy

$$\nu'_L - \nu'_R = n'_{0+} + n'_{\pi+} - n'_{0-} - n'_{\pi-}, \quad (4.26)$$

$$\nu''_L - \nu''_R = n''_{0+} + n''_{\pi+} - n''_{0-} - n''_{\pi-}, \quad (4.27)$$

where $n'_{\text{Re}(\varepsilon)\pm}$ and $n''_{\text{Re}(\varepsilon)\pm}$ represent the number of edge states obtained from V' and V'' , whose subscripts denote the real part of ε and chirality $\pm$. In open systems described by time-independent chiral symmetric non-Hermitian Hamiltonians, the winding number $\nu[q]$ in Sec. 4.3.2 satisfies Eqs. (4.26) and (4.27) which are collapsed into a single equation with $n_{\pi\pm} = 0$ without superscripts ' and '' [164]. The reason is as follows why Eqs. (4.26) and (4.27) represent relations of the bulk-edge correspondence in systems described by V' and V'' . When an eigenvalue satisfies neither $\text{Re}(\varepsilon) = 0$ nor $\text{Re}(\varepsilon) = \pi$, $|\tilde{\phi}'_\varepsilon\rangle$ and $\Gamma|\phi'_\varepsilon\rangle \propto |\tilde{\phi}'_{-\varepsilon^*}\rangle$ ($|\tilde{\phi}''_\varepsilon\rangle$ and $\Gamma|\phi''_\varepsilon\rangle \propto |\tilde{\phi}''_{-\varepsilon^*}\rangle$) are independent since their eigenvalues are different $\varepsilon \neq -\varepsilon^*$. Then, from right and left eigenstates in Eqs. (4.15) and (4.16), we can construct biorthonormal states [164]

$$|\phi'_\pm\rangle = \frac{1}{\sqrt{2}}(|\tilde{\phi}'_\varepsilon\rangle \pm \Gamma|\phi'_\varepsilon\rangle), \quad |\tilde{\phi}'_\pm\rangle = \frac{1}{\sqrt{2}}(|\phi'_\varepsilon\rangle \pm \Gamma|\tilde{\phi}'_\varepsilon\rangle), \quad \langle\tilde{\phi}'_\pm|\phi'_\mp\rangle = 0, \quad \langle\tilde{\phi}'_\pm|\phi'_\pm\rangle = 1 \quad (4.28)$$

$$|\phi''_\pm\rangle = \frac{1}{\sqrt{2}}(|\tilde{\phi}''_\varepsilon\rangle \pm \Gamma|\phi''_\varepsilon\rangle), \quad |\tilde{\phi}''_\pm\rangle = \frac{1}{\sqrt{2}}(|\phi''_\varepsilon\rangle \pm \Gamma|\tilde{\phi}''_\varepsilon\rangle), \quad \langle\tilde{\phi}''_\pm|\phi''_\mp\rangle = 0, \quad \langle\tilde{\phi}''_\pm|\phi''_\pm\rangle = 1 \quad (4.29)$$

which satisfy

$$\Gamma|\phi'_\pm\rangle = \pm|\tilde{\phi}'_\pm\rangle, \quad \Gamma|\phi''_\pm\rangle = \pm|\tilde{\phi}''_\pm\rangle. \quad (4.30)$$

This means that eigenstates with $\text{Re}(\varepsilon) \neq 0, \pi$ always bear biorthonormal basis in which the number of bases with chirality $+$ is the same as that with chirality $-$. In other words, edgestates with opposite chirality must be paired when perturbations make edge states acquire quasi-energies with $\text{Re}(\varepsilon) \neq 0, \pi$ as long as chiral symmetry is preserved [164]. Therefore, the number of edge states in the right hand side of Eqs. (4.26) and (4.27) is topologically protected. Topological numbers in open Floquet systems which we want to derive are ν_0 and ν_π satisfying

$$\nu_0^L - \nu_0^R = n'_{0+} - n'_{0-}, \quad \nu_\pi^L - \nu_\pi^R = n'_{\pi+} - n'_{\pi-}, \quad (4.31)$$

which predict the numbers of topologically protected edge states with $\text{Re}(\varepsilon) = 0$ and $\text{Re}(\varepsilon) = \pi$, respectively. We derive ν_0 and ν_π by clarifying a relation between η' and η'' . To this end, operating V_1 and $V_2^\dagger$ to the right eigenequations of V' and $V''^\dagger$ respectively, we obtain

$$V_1 V_2 V_1 |\Phi'_{\varepsilon, \eta'}\rangle = e^{-i\varepsilon} V_1 |\Phi'_{\varepsilon, \eta'}\rangle, \quad (4.32)$$

$$V_2^\dagger V_1^\dagger V_2^\dagger |\tilde{\Phi}'_{\varepsilon, \eta'}\rangle = e^{i\varepsilon^*} V_2^\dagger |\tilde{\Phi}'_{\varepsilon, \eta'}\rangle. \quad (4.33)$$

Then, from Eqs. (4.13)-(4.16), (4.32) and (4.33), we can see that

$$V_1 |\Phi'_{\varepsilon, \eta'}\rangle = v_1 |\Phi''_{\varepsilon, \eta''}\rangle, \quad V_2^\dagger |\tilde{\Phi}'_{\varepsilon, \eta'}\rangle = v_2 |\tilde{\Phi}''_{\varepsilon, \eta''}\rangle \quad (4.34)$$

are satisfied. The complex numbers v_1 and v_2 in Eq. (4.34) are related by

$$v_2 = \frac{e^{i\varepsilon^*}}{v_1^*}, \quad (4.35)$$

which is obtained by using Eq. (4.34) and the normalization conditions in Eq. (4.17),

$$\langle \Phi''_{\varepsilon, \eta''} | \tilde{\Phi}''_{\varepsilon, \eta''} \rangle = \langle \Phi'_{\varepsilon, \eta'} | \frac{1}{v_1^*} V_1^\dagger V_2^\dagger \frac{1}{v_2} | \tilde{\Phi}'_{\varepsilon, \eta'} \rangle = \frac{e^{i\varepsilon^*}}{v_2 v_1^*} \langle \Phi'_{\varepsilon, \eta'} | \tilde{\Phi}'_{\varepsilon, \eta'} \rangle = 1. \quad (4.36)$$

From Eqs. (4.12), (4.19), and (4.34), $\Gamma |\Phi''_{\varepsilon, \eta''}\rangle$ becomes

$$\Gamma |\Phi''_{\varepsilon, \eta''}\rangle = \frac{1}{v_1} \Gamma V_1 \Gamma^2 |\Phi'_{\varepsilon, \eta'}\rangle = \frac{\eta'}{v_1} V_2^\dagger | \tilde{\Phi}'_{\varepsilon, \eta'} \rangle = \frac{v_2}{v_1} \eta' | \tilde{\Phi}''_{\varepsilon, \eta''} \rangle \quad (4.37)$$

Equations (4.35) and (4.37) mean that η' and η'' satisfy

$$\eta'' = \frac{e^{i\varepsilon^*}}{|v_1|^2} \eta'. \quad (4.38)$$

When Eq. (4.21) is satisfied, $|v_1|^2$ should be $|e^{i\varepsilon^*}|$. From Eq. (4.38), we can see that edge states with $\text{Re}(\varepsilon) = 0$ (π) have the same (opposite) chirality in systems described by V' and V'' . This means that the number of edge states satisfy

$$n'_{0+} = n''_{0+}, \quad n'_{\pi+} = n''_{\pi-}, \quad n'_{0-} = n''_{0-}, \quad n'_{\pi-} = n''_{\pi+}. \quad (4.39)$$

From Eqs. (4.26), (4.27), (4.31), and (4.39), we can obtain

$$\nu_0 = \frac{\nu' + \nu''}{2}, \quad \nu_\pi = \frac{\nu' - \nu''}{2}. \quad (4.40)$$

Equation (4.40) has been employed in open Floquet systems described by nonunitary time-evolution operators [72, 69], based on the analogy to closed Floquet systems described by unitary time-evolution operators in which the same formula is proven to be satisfied [53]. Our derivation gives the microscopic foundation of Eq. (4.40) that has been used empirically so far for open Floquet systems.

Before closing this section, we comment on the skin effect. In open systems, eigenvalues of non-Hermitian Hamiltonians or nonunitary time-evolution operators can drastically change depending on boundary conditions, which is referred to as the skin effect [178, 181, 184, 185, 186, 191, 192, 194, 195, 196, 199, 200, 201, 202, 203, 76, 204]. It is known that the skin effect exists when the spectrum under periodic boundary conditions forms closed loops with a non-zero winding number in the complex plane [185, 192, 195, 200, 201, 202, 204]. When there is the skin effect, the bulk-edge correspondence breaks down in the conventional meaning, that is, topological numbers calculated from non-Hermitian Hamiltonians or nonunitary time-evolution operators with periodic boundary conditions do not correctly predict the number of edge states when open boundary conditions are imposed. When systems in which the skin effect exists are analyzed, topological numbers calculated from M or V with open boundary conditions are often used [181, 196, 200]. In such cases, eigenstates under open boundary conditions are obtained by utilizing the concept of "generalized Brillouin zone" in which eigenstates are characterized by complex momentum. Satisfying Eqs. (4.26) and (4.27), ν' and ν'' are either ones calculated in homogeneous systems with periodic boundary conditions without the skin effect or ones calculated using generalized Brillouin zone when there is the skin effect. In this work, we use the former winding number in Sec. 4.5. This is because the spectrum does not form any closed loop and the skin effect is absent in our model addressed in Sec. 4.5.

4.4 Open quantum walks

Open quantum walks in which gain and/or loss effects of light exist can be described by nonunitary time-evolution operators which include gain-loss operators.

4.4.1 experiments

Open quantum walks with gain and/or loss which are described by nonunitary time-evolution operators have been realized in experiments by implementing optical fiber loops [86, 93, 98, 99] and using single photons [64, 72, 76]. Here, we explain these two types of experiments.

The experimental setting based on optical fiber loops is schematically described in Fig. 3.5 (a). As explained in Sec. 3.4, delay and advance of wave packets in time are respectively regarded as the position shift to left and right directions in real space, which are caused by the difference of lengths of two fibers. The red circle in the long loop represents the erbium doped fiber amplifier (EDFA). Erbium ions in EDFA emit light with wave length around 1550 nm when they absorb light with different wave length such as 980 nm or 1480 nm. Then, amplifications of light in optical fiber loops are realized by injecting light with different wave length into EDFA, which is depicted as the red arrows in Fig. 4.1 (a). The loss of light is introduced by 90/10 coupler depicted as the blue circle in Fig. 4.1 (a), in addition to spontaneously induced loss. In the setting of Fig. 4.1 (a), intensities of wave packets in long and short loops respectively increase and decrease. It is also possible to include EDFA and 90/10 coupler in both fiber loops. In such a case, gain-loss effects can be manipulated in both loops through the manipulations on the intensities of light injected into EDFA.

The experimental setting based on arrayed optical elements is schematically described in Fig. 4.1 (b-1). As explained in Sec. 3.4, combinations of half-wave plates and phase shifters correspond to coin operators, and beam displacers play the role of shift operators, of which the former and latter are respectively described as purple small rectangles and green large rectangles. In addition to these elements, partially polarizing beam splitters (PPBSs) are put in the paths of light, which are described by light blue rectangles with diagonal blue lines. A probability or ratio that light passes through a PPBS depends on the polarization. For example, a PPBS makes light with vertical polarization vanish from the system (remain in the system) in a probability $1 - e^{-2l}$ (e^{-2l}), which is described by yellow dashed arrows (solid arrows), while light with horizontal polarization does not experience any loss passing through the PPBS. Then, partially polarizing beam splitters cause polarization dependent loss effects of light. Such processes are referred to as the post selected dynamics, in which we ignore light reflected by PPBSs which goes to environment (yellow dashed arrows) and focus only on light passing through PPBSs or equivalently measured by detectors (yellow solid arrows).

4.4.2 description

In order to describe the dynamics of open quantum walks, we introduce the gain-loss operator

$$D(\gamma) = \sum_x |x\rangle \langle x| \otimes \mathcal{D}[\gamma(x)], \quad \mathcal{D}[\gamma(x)] = \begin{pmatrix} e^{\gamma(x)} & 0 \\ 0 & e^{-\gamma(x)} \end{pmatrix} = e^{\gamma(x)\sigma_3}. \quad (4.41)$$

The gain-loss operator $\mathcal{D}[\gamma(x)]$ with positive $\gamma(x)$ amplifies (dumps) the wave function amplitude at a position x with the internal state $|-\rangle$ ($|+\rangle$) by the factor $e^{\gamma(x)}$ ($e^{-\gamma(x)}$), and $\mathcal{D}[-\gamma(x)]$ vice versa. The gain-loss operator in Eq. (4.41) effectively describes amplifications and attenuations of wave packets in long and short optical fibers, which respectively correspond to $|-\rangle$ and $|+\rangle$, as described in Fig. 4.1 (a). In the case of such open quantum walks, $P(x, t)$ in Eq. (3.3) represent the intensity of classical light at a site x and time step t . In addition to the dynamics with both gain and loss, $D(\gamma)$ describes the dynamics with only loss in Fig. 4.1 (b), which we explain in the

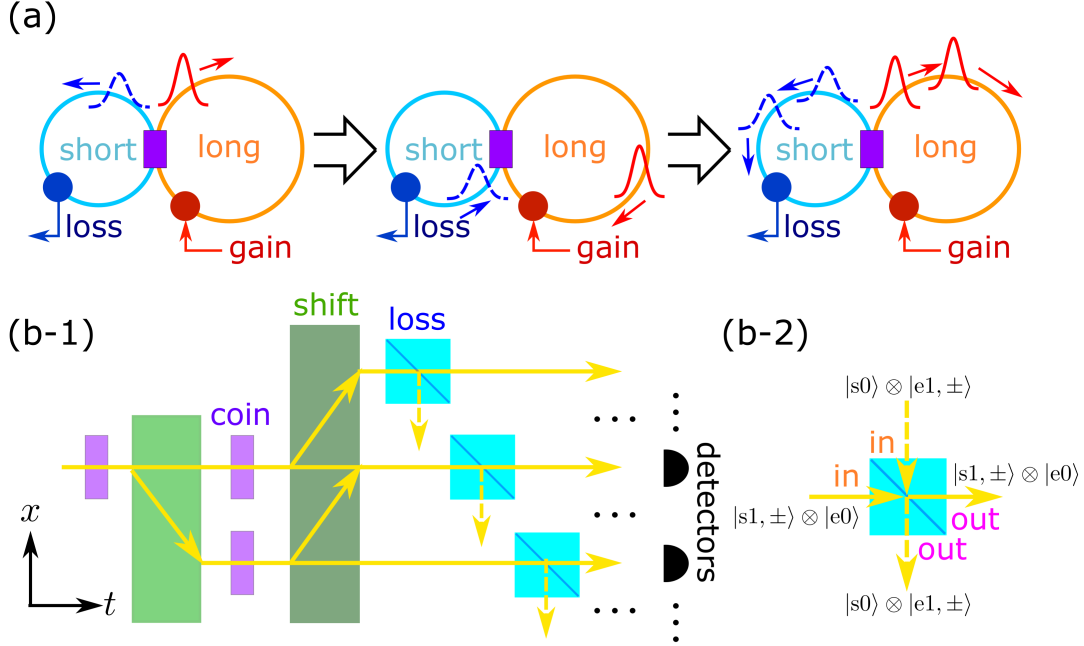


Figure 4.1: Experimental settings of open quantum walks, based on (a) optical fibre loops and (b) arrays of optical elements. In (a), the purple rectangle, red circle, and blue circle respectively represent the 50/50 coupler which mixes wave packets in short and long fibers, erbium doped fiber amplifier (EDFA) which causes the gain effects of light, and 90/10 coupler which induces the loss effects of light. In (b-1), purple, green, and light blue rectangles are respectively combinations of half-wave plates and phase shifters which correspond to coin operators, beam displacers which correspond to shift operators, and partially polarizing beam splitters (PPBSs) which correspond to loss operators in Eq. (4.48). (b-2) schematically depicts the basis of PPBSs which is used in Eqs. (4.42) and (4.43).

following. An operation of a partially polarizing beam splitter can be described by

$$\begin{aligned}
 \mathcal{P}_+(l) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-l} & 0 & \sqrt{1-e^{-2l}} \\ 0 & 0 & 1 & 0 \\ 0 & \sqrt{1-e^{-2l}} & 0 & e^{-l} \end{pmatrix} \\
 &= \tau_0 \otimes \begin{pmatrix} 1 & 0 \\ 0 & e^{-l} \end{pmatrix} + \tau_1 \otimes \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{1-e^{-2l}} \end{pmatrix}
 \end{aligned} \tag{4.42}$$

which reflects light with vertical polarization $|V\rangle$ or equivalently $|+\rangle$ in a probability e^{-2l} . In Eq. (4.42), the basis is taken as $|s1, -\rangle \otimes |e0\rangle = (1, 0, 0, 0)^T$, $|s1, +\rangle \otimes |e0\rangle = (0, 1, 0, 0)^T$, $|s0\rangle \otimes |e1, -\rangle = (0, 0, 1, 0)^T$, and $|s0\rangle \otimes |e1, +\rangle = (0, 0, 0, 1)^T$ where $s1$ and $s0$ ($e1$ and $e0$) respectively represent that a photon is present and absent in the system (environment), which is schematically depicted in Fig. 4.1 (b-2). While we consider only the cases in which the input is $|s1, -\rangle \otimes |e0\rangle$ or $|s1, +\rangle \otimes |e0\rangle$, Eq. (4.42) also describes the cases in which a photon is injected from environment. In the same

way,

$$\begin{aligned}\mathcal{P}_-(l) &= \begin{pmatrix} e^{-l} & 0 & \sqrt{1-e^{-2l}} & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{1-e^{-2l}} & 0 & e^{-l} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \tau_0 \otimes \begin{pmatrix} e^{-l} & 0 \\ 0 & 1 \end{pmatrix} + \tau_1 \otimes \begin{pmatrix} \sqrt{1-e^{-2l}} & 0 \\ 0 & 0 \end{pmatrix}\end{aligned}\quad (4.43)$$

describes an operation of a partially polarizing beam splitter different from the above one, which reflects light with horizontal polarization $|H\rangle$ or equivalently $|-\rangle$ in a probability e^{-2l} . In open quantum systems, interactions with environment are described by Klaus operators $\mathcal{W}_k = \langle e_k | U_{se} | e_i \rangle$ which satisfy $\sum_k \mathcal{W}_k^\dagger \mathcal{W}_k = 1$, where $|e_k\rangle$, U_{se} , and $|e_i\rangle$ are the basis of environment, a unitary operator which describes interactions between the system and environment, and an initial state of environment, respectively [205]. In the present case that the initial state of environment is $|e0\rangle$, Klaus operators become

$$\mathcal{W}_1^+(l) = \langle e1 | \mathcal{P}_+(l) | e0 \rangle = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{1-e^{-2l}} \end{pmatrix}, \quad \mathcal{W}_0^+(l) = \langle e0 | \mathcal{P}_+(l) | e0 \rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{-l} \end{pmatrix}, \quad (4.44)$$

$$\mathcal{W}_1^-(l) = \langle e1 | \mathcal{P}_-(l) | e0 \rangle = \begin{pmatrix} \sqrt{1-e^{-2l}} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{W}_0^-(l) = \langle e0 | \mathcal{P}_-(l) | e0 \rangle = \begin{pmatrix} e^{-l} & 0 \\ 0 & 1 \end{pmatrix}. \quad (4.45)$$

Klaus operators are nonunitary operators, in general. When a density operator of an initial state is written as $\rho_i = |\psi_i\rangle \langle \psi_i|$, which describes a quantum state of a photon in the system, the final state ρ_f after the photon passes through the PPBS described by $\mathcal{P}_\pm(l)$ becomes

$$\rho_f = \text{tr}_e[\mathcal{P}_\pm(l)\rho_i \otimes |e0\rangle \langle e0|] = [\mathcal{W}_0^\pm(l)]\rho_i[\mathcal{W}_0^\pm(l)]^\dagger + [\mathcal{W}_1^\pm(l)]\rho_i[\mathcal{W}_1^\pm(l)]^\dagger, \quad (4.46)$$

where tr_e represents the partial tracing of environment and $\rho_i \otimes |e0\rangle \langle e0|$ is the initial state of the whole system. Considering the dynamics of open quantum walks, we ignore photons which have gone to environment and focus only on photons which remain in the system, which is referred to as the post selection. This corresponds to ignoring the second term in the right hand side of Eq. (4.46). Then, the final quantum state can be obtained by operating Kraus operators to the initial state,

$$|\psi_f\rangle = \mathcal{W}_0^\pm(l) |\psi_i\rangle, \quad (4.47)$$

This means that open quantum walks are described by nonunitary time-evolution operators which include Klaus operators. The Klaus operators introduced above can be written as

$$\mathcal{W}_0^+(l) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-l} \end{pmatrix} = e^{-\frac{l}{2}} \mathcal{D}(-l/2), \quad \mathcal{W}_0^-(l) = \begin{pmatrix} e^{-l} & 0 \\ 0 & 1 \end{pmatrix} = e^{-\frac{l}{2}} \mathcal{D}(l/2). \quad (4.48)$$

Then, ignoring the constant trivial factor $e^{-\frac{l}{2}}$, we can describe post-selected dynamics of single photons based on $\mathcal{D}(\gamma = l/2)$. Regarding dynamics of single photons under post selections, $P(x, t)$ in Eq. (3.3) corresponds to the corrected probability which can be larger than one due to the ignorance of $e^{-\frac{l}{2}}$. Note that, when the factor is taken into account, raw probabilities are inevitably smaller than one.

4.5 Results in a chiral symmetric model

Including gain-loss operators in addition to coin, phase, and shift operators, we define a time-evolution operator of a one-dimensional open quantum walk as

$$V' = C_1(\theta_1/2)D(\gamma)SC_1(\theta_2/2)C_3(\theta_3)C_1(\theta_2/2)SD(-\gamma)C_1(\theta_1/2). \quad (4.49)$$

The time evolution is described by V' as in Eq. (3.2) where a unitary time-evolution operator U is replaced with the nonunitary time-evolution operator V' . Time-evolution operators which include $D(\gamma)$ can describe both classical dynamics of coherent light with gain-loss effects and post-selected quantum dynamics of single photons, as mentioned in Sec. 4.4.2.

4.5.1 symmetries

The nonunitary time-evolution operator V' in Eq. (4.49) has various symmetries which are important to argue topological phases and dynamics. We decompose the time-evolution operator into $V' = V_2 V_1$, where V_1 and V_2 are defines as

$$V_1 = C_3(\theta_3/2)C_1(\theta_2/2)SD(-\gamma)C_1(\theta_1/2), \quad V_2 = C_1(\theta_1/2)D(\gamma)SC_1(\theta_2/2)C_3(\theta_3/2). \quad (4.50)$$

Then, defining chiral symmetry operator as

$$\Gamma = \sum_x |x\rangle \langle x| \otimes \sigma_2, \quad (4.51)$$

V_1 and V_2 satisfy Eq. (4.12). Then, V' is classified into class AIII which can possess non-trivial topological phases [112] but has not been studied in open quantum walks. When $\theta_3(x) = 0$ and the phase operator $C_3(\theta_3)$ has no effect on dynamics, V' also has particle-hole symmetry and time-reversal symmetry,

$$\Xi V' \Xi^{-1} = V', \quad \mathcal{T} V' \mathcal{T}^{-1} = V'^{\dagger}, \quad (4.52)$$

respectively, where symmetry operators are

$$\Xi = \sum_x |x\rangle \langle x| \otimes \sigma_3 \mathcal{K}, \quad \mathcal{T} = \sum_x |x\rangle \langle x| \otimes \sigma_1 \mathcal{K}. \quad (4.53)$$

In this case, the system is classified into class BDI[†] in the AZ[†] classification and possibly possesses non-trivial topological phases [112]. When $\gamma = 0$, the time-evolution operator is unitary $V'^{\dagger} = V'^{-1}$ and classified into class BDI in the AZ classification [103, 104, 105, 106, 107]. Equation (4.9) is also satisfied by defining $\mathcal{PT}$ symmetry operator as

$$\mathcal{PT} = \sum_x |-x\rangle \langle x| \otimes \sigma_0 \mathcal{K}, \quad (4.54)$$

where $\mathcal{K}$ is the complex conjugation, if constraints on position dependences of $\theta_{j=1,2,3}(x)$ and $\gamma(x)$

$$\theta_j(x) = \theta_j(-x), \quad \gamma(x) = \gamma(-x), \quad (4.55)$$

are fulfilled. As we show in Sec. 4.5.3, $\mathcal{PT}$ symmetry plays a crucial role in the dynamics of the open quantum walk.

We clarify a condition for the presence of finite band gaps of the quasi-energy spectrum around $\text{Re}(\varepsilon) = 0$ and π , so that topological numbers are well defined. By assuming a homogeneous system in which $\theta_j(x)$ and $\gamma(x)$ are constant, we apply Fourier transformation to V' and $\mathcal{V}'(k)$ in momentum space becomes

$$\mathcal{V}'(k) = d_0(k)\sigma_0 + id_1(k)\sigma_1 + d_2(k)\sigma_2 + id_3(k)\sigma_3, \quad (4.56)$$

$$d_0(k) = \cos(\theta_1)[\cos(\theta_2)\cos(\theta_3)\cos(2k) - \sin(\theta_3)\sin(2k)] - \sin(\theta_1)\sin(\theta_2)\cos(\theta_3)\cosh(2\gamma), \quad (4.57)$$

$$d_1(k) = \sin(\theta_1)[\cos(\theta_2)\cos(\theta_3)\cos(2k) - \sin(\theta_3)\sin(2k)] + \cos(\theta_1)\sin(\theta_2)\cos(\theta_3)\cosh(2\gamma), \quad (4.58)$$

$$d_2(k) = d_2 = -\sin(\theta_2)\cos(\theta_3)\sinh(2\gamma), \quad (4.59)$$

$$d_3(k) = \cos(\theta_2)\cos(\theta_3)\sin(2k) + \sin(\theta_3)\cos(2k), \quad (4.60)$$

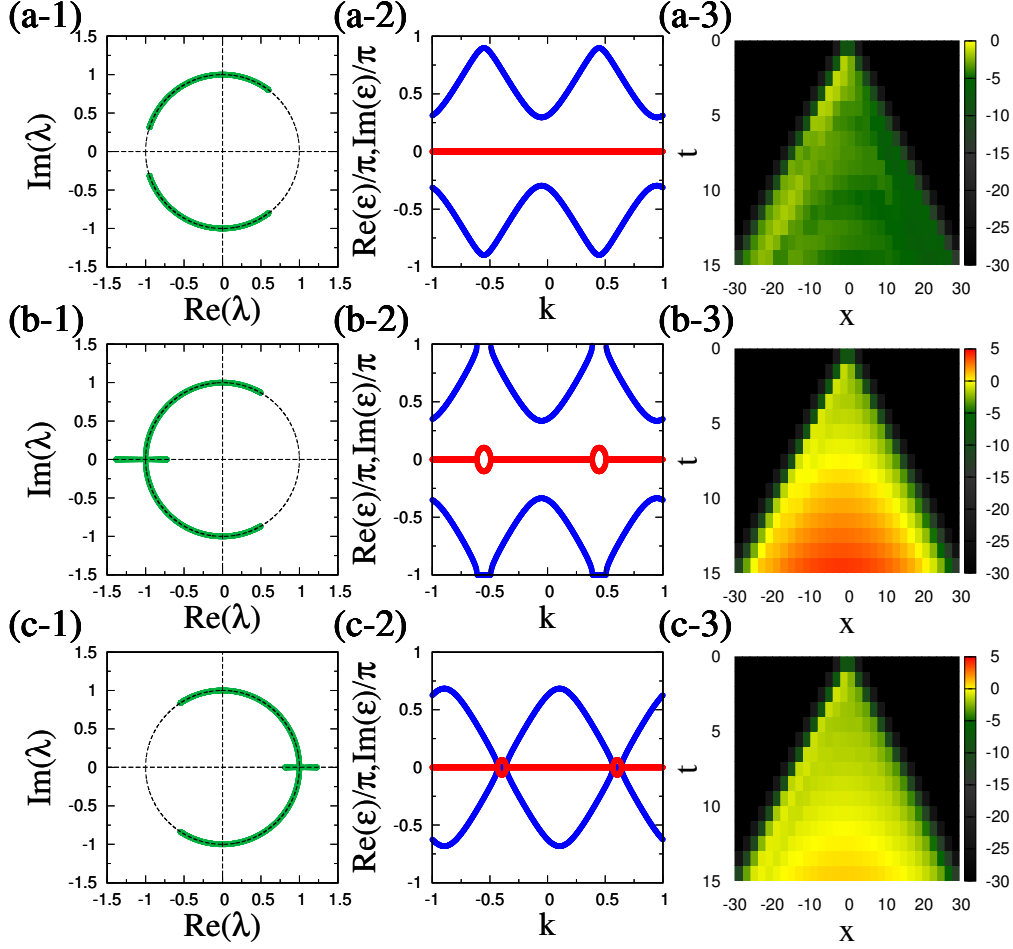


Figure 4.2: (1,2) The bulk spectrum of $\mathcal{V}'(k)$ and (3) time-evolutions of $\ln P(x, t)$ when parameters are set to be $\theta_1 = \pi/5$, $\theta_2 = \pi/10$, with (a) $e^\gamma = 1.1, \theta_3 = \pi/10$, (b) $e^\gamma = 1.7, \theta_3 = \pi/10$, and (c) $e^\gamma = 1.7, \theta_3 = 4\pi/5$. In the left column (1), the dashed curves represent the unit circle. In the center column (2), blue and red lines respectively represent the real and imaginary part of quasi-energies. In the right column (3), the initial state is $|\psi(0)\rangle = |x=0\rangle \otimes |\sigma=-\rangle$.

where $d_0^2(k) + d_1^2(k) - d_2^2 + d_3^2(k) = 1$ is satisfied. Then, the eigenvalues and eigenstates of $\mathcal{V}'(k)$ are derived as

$$\lambda_{\pm}(k) = d_0(k) \pm i\sqrt{1 - d_0^2(k)}, \quad |\phi_{\pm}(k)\rangle = \tilde{N}_{\pm}(k) \begin{pmatrix} d_1(k) - d_2 \\ -d_3(k) \pm \sqrt{1 - d_0^2(k)} \end{pmatrix} \quad (4.61)$$

where $\tilde{N}_{\pm}(k)$ is a normalization constant which satisfies $\langle \tilde{\phi}_{\pm}(k) | \phi_{\pm}(k) \rangle = 1$. On one hand, when $|d_0(k)| \leq 1$ for any $k \in [0, 2\pi)$, all of the quasi-energies are kept to be real since $|\lambda_{\pm}(k)| = 1$, although V' is nonunitary. This is because eigenstates preserve $\mathcal{PT}$ symmetry $\mathcal{PT}|\phi_{\pm}(k)\rangle \propto |\phi_{\pm}(k)\rangle$, which we can understand from Eqs. (4.54) and (4.61) since $d_j(k)$ ($j = 0, 1, 2, 3$) are real numbers. On the other hand, when $|d_0(k)| > 1$ in a certain range of k , $\mathcal{PT}$ symmetry of eigenstates is broken in the range and then the corresponding quasi-energies become complex or equivalently $|\lambda_{\pm}(k)| = |d_0(k) \pm \sqrt{d_0^2(k) - 1}| \neq 1$. The former and latter situations are $\mathcal{PT}$ symmetric phase and broken $\mathcal{PT}$ symmetry phase explained in Sec. 4.2.2, respectively. As shown in Fig. 4.2 (a) and (b), when the value of γ is increased, eigenstates emerge which break $\mathcal{PT}$ symmetry. We

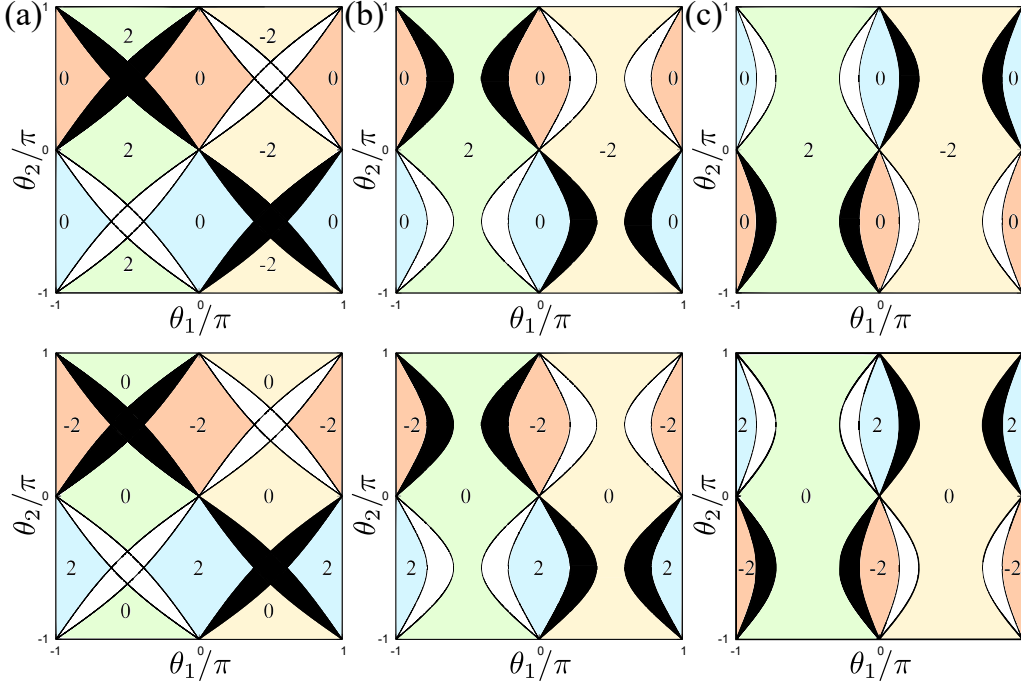


Figure 4.3: θ_1 and θ_2 dependences of ν' (top row) and ν'' (bottom row), with $e^\gamma = 1.2$ and (a) $\theta_3 = 0$, (b) $\theta_3 = \pi/5$, (c) $\theta_3 = 7\pi/10$. In the black (white) region, the quasi-energy gap around $\text{Re}(\varepsilon) = 0$ (π) is closed and topological numbers are not well defined.

can understand from Fig. 4.2 (b) and (c) that it depends on parameters whether the gap closing occurs on $\text{Re}(\varepsilon) = 0$ or π . Figure 4.2 (b-3) and (c-3) show that the intensity or corrected probability $P(x, t)$ exponentially grows with increasing time steps when complex quasi-energies exist, which never occurs as in Fig. 4.2 (a-3) when gaps on $\text{Re}(\varepsilon) = 0, \pi$ are open. The condition for the presence of the quasi-energy band gaps around $\text{Re}(\varepsilon) = 0$ and π , or equivalently $|\lambda_\pm(k)| = 1$ for all k , is also discerned by the above unbroken/broken $\mathcal{PT}$ symmetry phases. From $\max[d_0(k)] < 1$ and $\min[d_0(k)] > -1$, the condition that the gaps on $\text{Re}(\varepsilon) = 0$ and $\text{Re}(\varepsilon) = \pi$ are open becomes

$$|\cos(\theta_1)|\sqrt{\cos^2(\theta_2)\cos^2(\theta_3) + \sin^2(\theta_3)} < 1 \pm \sin(\theta_1)\sin(\theta_2)\cos(\theta_3)\cosh(2\gamma). \quad (4.62)$$

Equation (4.62) gives phase boundaries of topological numbers as shown in Figs. 4.3 and 4.4. Hereafter, we focus on unbroken $\mathcal{PT}$ symmetry phases where both quasi-energy band gaps around $\text{Re}(\varepsilon) = 0$ and π are open. We note that, for the nonunitary quantum walk, the parameter space where topological numbers are not well defined extends in finite regions, while it features lines for the unitary quantum walk ($\gamma = 0$).

4.5.2 topological numbers

Since V_1 and V_2 in Eq. (4.50) satisfy Eq. (4.12) with Γ in Eq. (4.51), we can use Eq. (4.40) to derive ν_0 and ν_π . Since the bulk spectra of our model do not form any closed loop in the complex plane as shown in Fig. 4.2 (a-1) and (b-1), which results in absence of the skin effect [202] which breaks the bulk-edge correspondence. Then, topological numbers obtained when periodic boundary conditions are imposed should expect the number of edge states. Therefore, we calculate a winding number ν' from eigenstates of $\mathcal{V}'(k)$, using a method explained in Sec. 4.3.

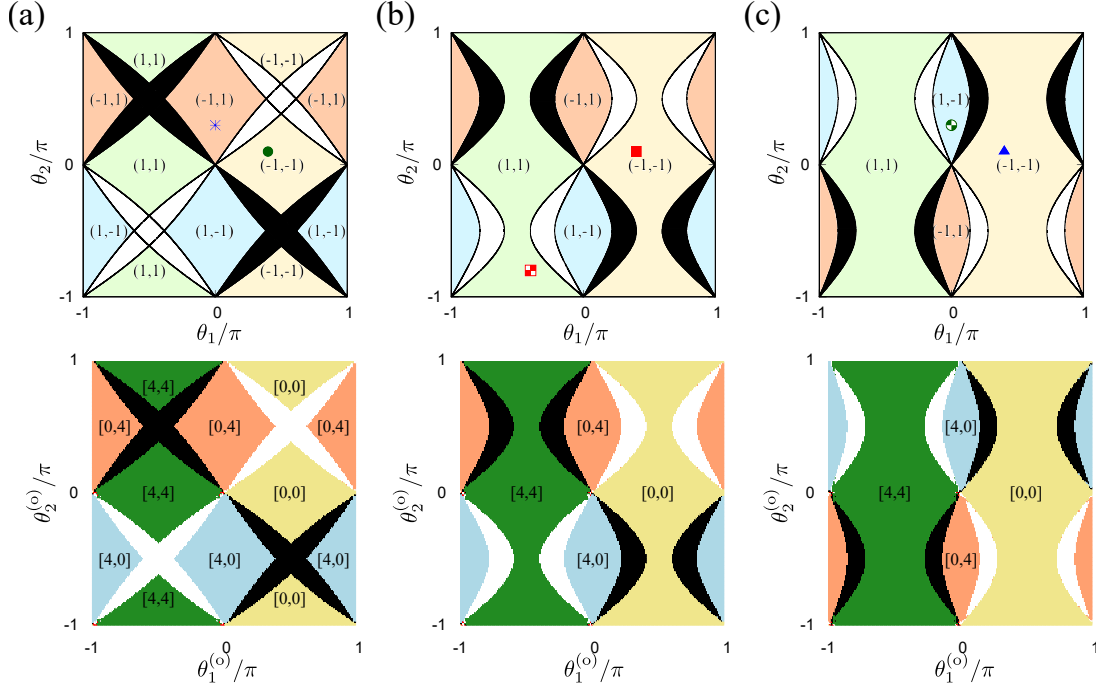


Figure 4.4: Top row: The phase diagrams of topological numbers (ν_0, ν_π) as functions of θ_1 and θ_2 of the nonunitary quantum walk described by V_1 and V_2 in Eq. (4.50) with $e^\gamma = 1.2$ and various θ_3 which are (a) $\theta_3 = 0$, (b) $\theta_3 = \pi/5$, (c) $\theta_3 = 7\pi/10$. The regions with topological numbers (ν_0, ν_π) correspond to unbroken $\mathcal{PT}$ symmetry phases, while black and white regions represent broken $\mathcal{PT}$ symmetry phases with complex quasi-energies whose real parts are $\text{Re}(\varepsilon) = 0$ and π , respectively. The green filled circle, red filled rectangle, and blue triangle represent $(\theta_1^{(i)}, \theta_2^{(i)}) = (2\pi/5, \pi/10)$. The blue asterisk and green half-filled circle are $(\theta_1^{(o)}, \theta_2^{(o)}) = (0, 3\pi/10)$, while the red half-filled rectangle is $(\theta_1^{(o)}, \theta_2^{(o)}) = (-2\pi/5, -4\pi/5)$. The marked parameters are used in numerical calculations in Figs. 4.6, 4.7, and 4.8. Bottom row: The numbers of edge states $[n_0, n_\pi]$ appearing on $\text{Re}(\varepsilon) = 0, \pi$ in the system described in Fig. 4.5 and Eq. (4.75) with parameters in Eq. (4.76), for various values of $(\theta_1^{(o)}, \theta_2^{(o)})$. The randomness is absent $w_j = w_\gamma = 0$ and the values of γ and $\theta_3^{(i)} = \theta_3^{(o)} = \theta_3$ are the same as those in the top row.

First, we obtain ν' from eigenstates of $\mathcal{V}'(k)$. The calculation of ν'' from $\mathcal{V}''(k) = \mathcal{V}_1(k)\mathcal{V}_2(k) = \mathcal{C}_3(\theta_3/2)\mathcal{C}_1(\theta_2/2)\mathcal{S}(k)\mathcal{D}(-\gamma)\mathcal{C}_1(\theta_1)\mathcal{D}(\gamma)\mathcal{S}(k)\mathcal{C}_1(\theta_2/2)\mathcal{C}_3(\theta_3/2)$ is easily carried out in the same way as the procedure for ν' . In order to apply the method explained in Sec. 4.3.2, we change the basis by a unitary transformation so that chiral symmetry operator becomes σ_3 . Then, the time-evolution operator becomes

$$\tilde{\mathcal{V}}'(k) = \exp[i\frac{\pi}{4}\sigma_1]\mathcal{V}'(k)\exp[-i\frac{\pi}{4}\sigma_1] = d_0(k)\sigma_0 + id_1(k)\sigma_1 + id_3(k)\sigma_2 - d_2\sigma_3. \quad (4.63)$$

The eigenstates of $\tilde{\mathcal{V}}'(k)$ with eigenvalues $\lambda_\pm(k)$ in Eq. (4.61) are

$$|\phi_\pm(k)\rangle = \frac{1}{\sqrt{2\cos[2\Omega(k)]}}(e^{\pm i\Omega(k)}, \pm ie^{\mp i\Omega(k)}e^{-i\varphi(k)})^T, \quad (4.64)$$

$$\langle\tilde{\phi}_\pm(k)| = \frac{1}{\sqrt{2\cos[2\Omega(k)]}}(e^{\pm i\Omega(k)}, \mp ie^{\mp i\Omega(k)}e^{i\varphi(k)}), \quad (4.65)$$

respectively, where $\varphi(k)$ and $\Omega(k)$ are defined as

$$|d(k)|e^{i\varphi(k)} = d_3(k) + id_1(k), \quad \cos[2\Omega(k)] = \sqrt{1 - \left(\frac{d_2}{|d(k)|}\right)^2}, \quad \sin[2\Omega(k)] = \frac{d_2}{|d(k)|}. \quad (4.66)$$

Substituting eigenstates in Eqs. (4.64) and (4.65) into Eq. (4.24), we can obtain $q(k)$ in Eq. (4.25),

$$q(k) = -ie^{i\varphi(k)} / \cos[2\Omega(k)]. \quad (4.67)$$

Then, the winding number becomes

$$\nu' = \frac{1}{2\pi i} \oint dk \frac{d}{dk} \ln[q(k)] = \frac{1}{2\pi} \oint d\varphi(k). \quad (4.68)$$

The values of ν' which depend on θ_1 and θ_2 is described in the top row of Fig. 4.3 with $e^\gamma = 1.2$ and various θ_3 .

Second, the other winding number ν'' is obtained from the eigenstates of

$$\mathcal{V}''(k) = \mathcal{V}_1(k)\mathcal{V}_2(k) = e_0(k)\sigma_0 + ie_1(k)\sigma_1 + e_2(k)\sigma_2 + ie_3(k)\sigma_3, \quad (4.69)$$

$$e_0(k) = \cos(\theta_1)[\cos(\theta_2)\cos(\theta_3)\cos(2k) - \sin(\theta_3)\sin(2k)] - \sin(\theta_1)\sin(\theta_2)\cos(\theta_3)\cosh(2\gamma), \quad (4.70)$$

$$e_1(k) = \cos(\theta_1)\sin(\theta_2)\cos(2k) + \sin(\theta_1)\cos(\theta_2)\cosh(2\gamma), \quad (4.71)$$

$$e_2(k) = e_2 = \sin(\theta_1)\sinh(2\gamma), \quad (4.72)$$

$$e_3(k) = \cos(\theta_1)[\cos(\theta_2)\sin(\theta_3)\cos(2k) + \cos(\theta_3)\sin(2k)] - \sin(\theta_1)\sin(\theta_2)\sin(\theta_3)\cosh(2\gamma). \quad (4.73)$$

Obeying the same procedure for calculating ν' explained above, ν'' becomes

$$\nu'' = \frac{1}{2\pi i} \oint dk \frac{d}{dk} \ln[e_3(k) + ie_1(k)]. \quad (4.74)$$

The bottom row of Fig. 4.3 shows θ_j dependence of ν'' . While ν' is independent of θ_3 as long as the gaps around $\text{Re}(\varepsilon) = 0, \pi$ are open, the sign of ν'' depends on the sign of $\cos(\theta_3)$. Finally, by substituting ν' and ν'' in each parameter region into Eq. (4.40), we can obtain the phase diagram of ν_0 and ν_π , in the top row of Fig. 4.4. We note that the topological numbers for the open quantum walk defined in Eq. (4.40) are the same as those for the isolated quantum walk whose time-evolution operators are unitary ($\gamma = 0$), as long as both two band gaps around $\text{Re}(\varepsilon) = 0$ and π are open. Therefore, the topological numbers are robust against gain and loss of the $\mathcal{PT}$ symmetric nonunitary driving.

4.5.3 edge states and dynamics

Having established the phase diagram in the top row of Fig. 4.4, we confirm that the topological numbers for Floquet topological phases give the correct numbers of edge states even in open systems described by nonunitary time-evolution operators. In order to induce edge states, we spatially change topological numbers through the position dependences of $\theta_j(x)$. In addition, we

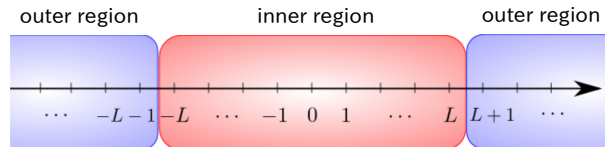


Figure 4.5: The schematic view of the inner and outer regions defined in Eq. (4.75).

introduce randomness to show that edge states are robust against disorder. Taking the conditions in Eq. (4.55) into consideration, we separate the system into three, an inner and two outer regions as shown in Fig. 4.5, which are discerned by the values of parameters,

$$\gamma(x) \in [\gamma - w_\gamma, \gamma + w_\gamma], \quad \theta_j(x) \in \begin{cases} [\theta_j^{(o)} - w_j, \theta_j^{(o)} + w_j] & (x \leq -L - 1) \\ [\theta_j^{(i)} - w_j, \theta_j^{(i)} + w_j] & (-L \leq x \leq L) \\ [\theta_j^{(o)} - w_j, \theta_j^{(o)} + w_j] & (x \geq L + 1) \end{cases}, \quad (4.75)$$

where $j = 1, 2, 3$. Both $\theta_j(x)$ and $\gamma(x)$ are randomly distributed obeying box distributions whose widths are respectively w_j and w_γ . The position dependence of $\gamma(x)$ is different from that of $\theta_j(x)$, because the gain-loss parameter has no effect on ν_0 and ν_π as long as the gaps around $\text{Re}(\varepsilon) = 0, \pi$ are open. There are two interfaces near $x = \pm L$ where the topological numbers vary, so that $\mathcal{PT}$ symmetry of V' is preserved when $w_j = w_\gamma = 0$. Note that, in normal situations to induce edge states of closed quantum walks, $\mathcal{PT}$ symmetry is not taken into account and there is only one boundary at which $\theta_j(x)$ change [51, 62, 70]. In the following numerical simulations of open quantum walks, we fix parameters as

$$e^\gamma = 1.2, \quad (\theta_1^{(i)}, \theta_2^{(i)}) = \left(\frac{2\pi}{5}, \frac{\pi}{10} \right), \quad (4.76)$$

where $\theta_1^{(i)}$ and $\theta_2^{(i)}$ correspond to the green filled circle, red filled rectangle, and blue triangle respectively in the top panels of Fig. 4.4 (a), (b), and (c). When we calculate eigenvalues of the time-evolution operator V' by numerical diagonalizations, periodic boundary conditions are imposed on two ends of the finite system with $-N \leq x \leq N - 1$, $N = 2L = 200$. The results are shown in Fig. 4.6 for the system in Eq. (4.75) with fixed $(\theta_1^{(i)}, \theta_2^{(i)}, e^\gamma)$ and three kinds of $(\theta_1^{(o)}, \theta_2^{(o)}, \theta_3^{(i)}, \theta_3^{(o)})$, which are indicated by symbols in and above the top rows of Figs. 4.4 and 4.6, respectively. According to the phase diagrams of topological numbers in the top row of Fig. 4.4 we expect the existence of four edge states (a) on $\text{Re}(\varepsilon) = \pi$, (b) both on $\text{Re}(\varepsilon) = 0$ and $\text{Re}(\varepsilon) = \pi$, and (c) on $\text{Re}(\varepsilon) = \pi$ from the bulk-edge correspondence, since there are two boundaries as in Eq. (4.75). Figure 4.6 clearly shows that the expectations are actually true and the bulk-edge correspondence is satisfied. From Fig. 4.6 (2), we can understand that edge states are robust against randomness, since they are on the real axis in the presence of disorder $w_j/\pi = w_\gamma = 1/5$. Topologically protected edge states have complex quasi-energies even when V' has $\mathcal{PT}$ symmetry because they cannot preserve $\mathcal{PT}$ symmetry due to the localization at boundaries as shown in Fig. 4.6 (3).

Further we systematically check the above specific results for the whole parameter region of $(\theta_1^{(o)}, \theta_2^{(o)})$ in the following way. We count the numbers of eigenstates with $\text{Re}(\varepsilon) = 0, \pi$ for various $(\theta_1^{(o)}, \theta_2^{(o)})$ by treating two different system sizes $N = 100$ and $N = 200$. Then, we distinguish edge states from other states, which are on the real axis due to band gap closings on $\text{Re}(\varepsilon) = 0$ or π , by the system size dependences of the numbers. If the number of eigenstates with $\text{Re}(\varepsilon) = 0$ (π) are unchanged with changing the system size N , the number is recognized as the number of topologically protected edge states n_0 (n_π). The results are summarized in the bottom row of Fig. 4.4 showing the set $[n_0, n_\pi]$, where w_j and w_γ are fixed to zero in order to explicitly compare analytical results for topological numbers and numerical results for the numbers of edge states. Comparing the top and bottom rows of Fig. 4.4 in the light of Eq. (4.76), in unbroken $\mathcal{PT}$ symmetry phases, the numbers of edge states completely agree with the predictions by the bulk-edge correspondence. In broken $\mathcal{PT}$ symmetry phases, we observe that the number of eigenstates with $\text{Re}(\varepsilon) = 0$ or π increases as the systems size N is increased excepting a few points. However, these exceptions would be due to finite size effects and could be improved by using larger N . Thereby, we conclude that the bulk-edge correspondence works even for open Floquet systems with chiral symmetry.

Finally, we study dynamics of wave functions driven by the time-evolution operator V' when

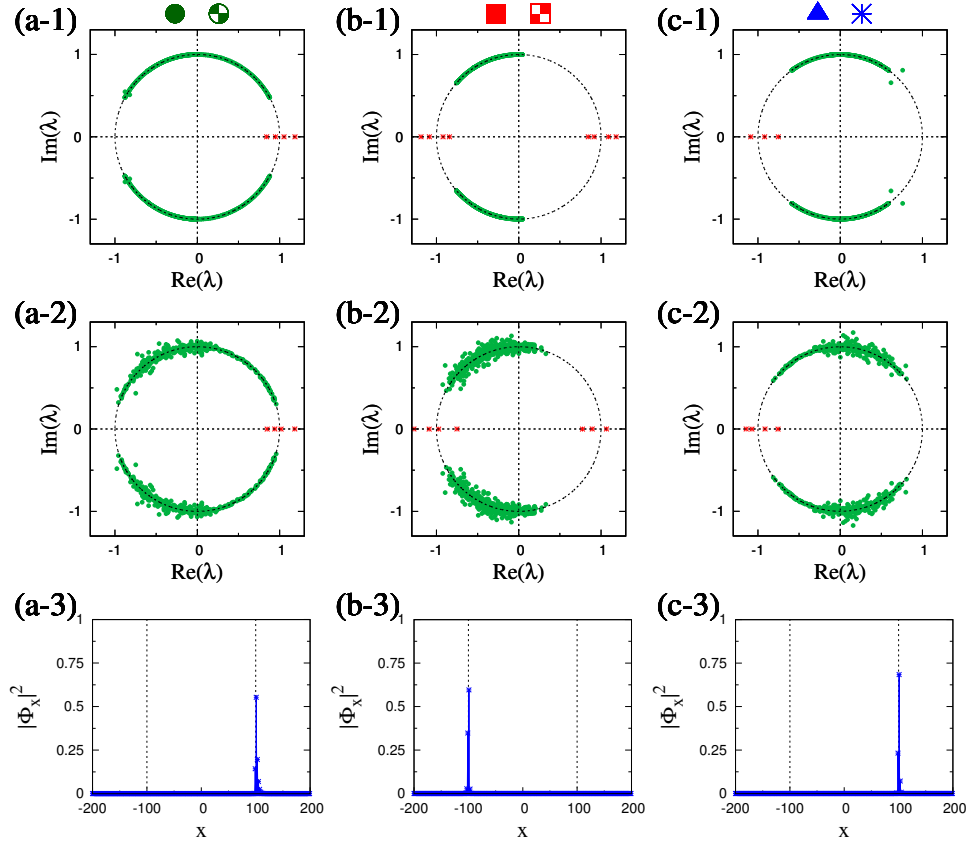


Figure 4.6: The eigenvalues λ of the time-evolution operator on the complex plane (the top and middle rows) and intensities $|\Phi_x|^2 = |\Phi_{x,+}|^2 + |\Phi_{x,-}|^2$ of topologically protected edge states (the bottom row), with (a) $(\theta_1^{(o)}, \theta_2^{(o)}, \theta_3^{(i)}, \theta_3^{(o)}) = (0, 3\pi/10, 0, 7\pi/10)$, (b) $(\theta_1^{(o)}, \theta_2^{(o)}, \theta_3^{(i)}, \theta_3^{(o)}) = (-2\pi/5, -4\pi/5, \pi/5, \pi/5)$, and (c) $(\theta_1^{(o)}, \theta_2^{(o)}, \theta_3^{(i)}, \theta_3^{(o)}) = (0, 3\pi/10, 7\pi/10, 0)$. The red asterisks in (1) and (2) represent eigenvalues of edge states. In (1) and (3) there is no randomness, while $w_j/\pi = w_\gamma = 1/5$ in (2). The symbols above the top row represent adopted parameters plotted in Fig. 4.4, of which the left and right respectively correspond to parameters in the inner and outer regions.

boundaries are around $x = \pm L$ with $L = 20$. Figure 4.7 shows the results of numerical simulations without randomness, where the parameters are the same as those in Fig. 4.6 (b-1) and (b-3) in which only edge states have complex quasi-energies. First, we discuss the case in which the initial state is far from boundaries, $|\psi(0)\rangle = |4\rangle \otimes |-\rangle$. On one hand, topologically protected edge states have no effect on the dynamics of closed quantum walks which are described by unitary time-evolution operators with $e^\gamma = 1$, as shown in Fig. 4.7 (a-1). Then, the unitary time evolution in (a-1) is similar to the nonunitary time evolution in (c-1) where edge states do not exist and all eigenstates preserve $\mathcal{PT}$ symmetry. Note that, even when $\mathcal{PT}$ symmetry is satisfied and all of quasi-energies are real, the total intensity or corrected probability $P(t) = \sum_x P(x, t)$ is not conserved as shown in Fig. 4.7 (c-2). This is because eigenstates of nonunitary time-evolution operators do not satisfy the orthonormal condition in general, $\langle \phi_\lambda | \phi_\mu \rangle \neq \delta_{\lambda\mu}$, $\sum_\lambda |\phi_\lambda\rangle \langle \phi_\lambda| \neq 1$, while they satisfy the bi-orthonormal condition $\langle \tilde{\phi}_\lambda | \phi_\mu \rangle = \delta_{\lambda\mu}$, $\sum_\lambda |\tilde{\phi}_\lambda\rangle \langle \phi_\lambda| = 1$. On the other hand, when there are topologically protected edge states which break $\mathcal{PT}$ symmetry, $P(x, t)$ grows exponentially with time steps in open quantum walks described by nonunitary time-evolution operators. Figure 4.7

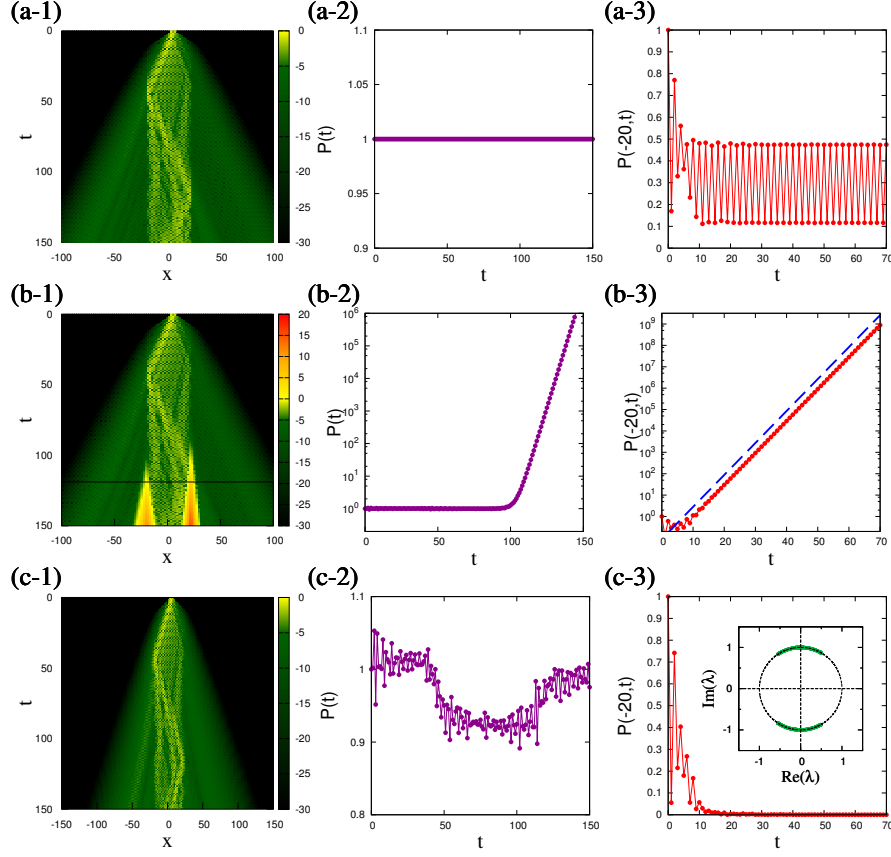


Figure 4.7: Time evolutions of (1) $\ln P(x,t)$, (2) $P(t) = \sum_x P(x,t)$, and (3) $P(x,t)$ at a specific position when $\theta_1^{(i)} = 2\pi/5$, $\theta_2^{(i)} = \pi/10$, $w_j = w_\gamma = 0$, $L = 20$, and other parameters are (a) $(\theta_1^{(o)}, \theta_2^{(o)}, \theta_3^{(i)}, \theta_3^{(o)}, e^\gamma) = (-2\pi/5, -4\pi/5, \pi/5, \pi/5, 1)$, (b) $(\theta_1^{(o)}, \theta_2^{(o)}, \theta_3^{(i)}, \theta_3^{(o)}, e^\gamma) = (-2\pi/5, -4\pi/5, \pi/5, \pi/5, 1.2)$, and (c) $(\theta_1^{(o)}, \theta_2^{(o)}, \theta_3^{(i)}, \theta_3^{(o)}, e^\gamma) = (2\pi/5, \pi/10, \pi/5, 7\pi/10, 1.2)$. In (1) and (2) the initial state is $|\psi(0)\rangle = |x=4\rangle \otimes |\sigma=-\rangle$, while $|\psi(0)\rangle = |-20\rangle \otimes |-\rangle$ in (3). The dashed line in (b-3) and the inset in (c-3) represent Eq. (4.77) with $\beta \approx \ln(1.19)$ and eigenvalues of which all satisfy $|\lambda| = 1$, respectively.

(b-1) and (b-2) exhibit that $P(x,t)$ at two interfaces increase with increasing time steps, although the initial state is not near the interfaces. These enhancements occur as long as the overlaps of edge states and the initial state are not exactly zero. Secondary, in Fig. 4.7 (a-3), (b-3), and (c-3), we show the intensity or corrected probability $P(-20,t)$ near an interface, when the initial state is put near the interface $|\psi(0)\rangle = |-20\rangle \otimes |-\rangle$. In the case of closed quantum walks described by unitary time-evolution operators, as shown in Fig. 4.7 (a-3), finite values of $P(x,t)$ at the interface remain after time evolutions owing to the existence of edge states. The oscillation of $P(x,t)$ with time steps in Fig. 4.7 (a-3) originates from the interference of edge states at $\varepsilon = 0$ and $\varepsilon = \pi$, which is one of unique topological phenomena in Floquet systems. As shown in Fig. 4.7 (c-3), when there is no edge state and quasi-energies are entirely real, $P(x,t)$ at the interface rapidly decays. On the contrary, from Fig. 4.7 (b-2), we find that $P(-20,t)$ increases exponentially with time steps. Taking Eqs. (3.2) and (4.10) into account, this enhancement of the intensity or corrected probability is the manifestation of edge states with largest $\text{Im}(\varepsilon)$. The dashed line in Fig. 4.7 (b-2)

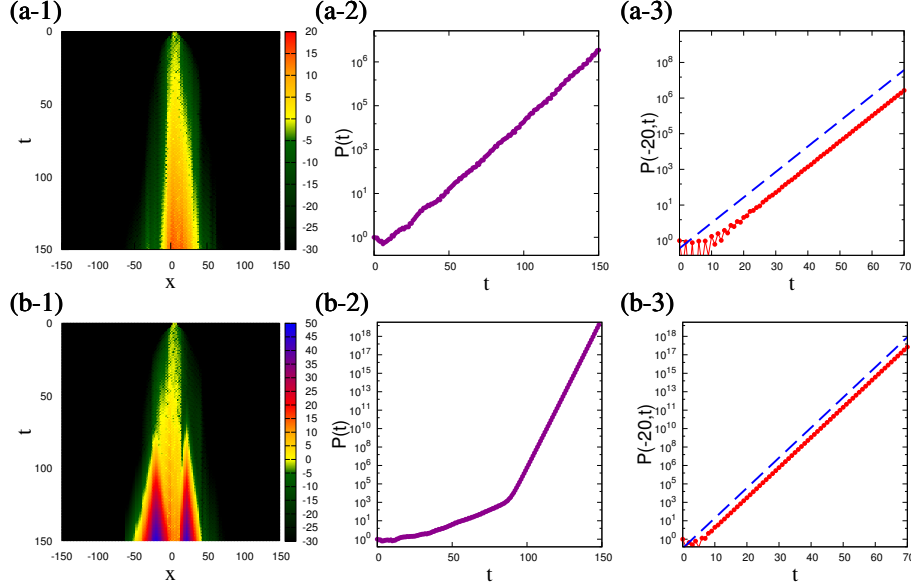


Figure 4.8: Time evolutions of (1) $\ln P(x, t)$, (2) $P(t) = \sum_x P(x, t)$, and (3) $P(x, t)$ at a boundary with $w_j/\pi = w_\gamma = 1/5$ when other parameters are the same as those in Fig. 4.7 (b). In (a) and (b), the realizations of $\{\theta_j(x)\}$ and $\{\gamma(x)\}$ are different, while the strength of randomness is the same. The initial states are (1),(2) $|\psi(0)\rangle = |x=4\rangle \otimes |\sigma=-\rangle$ and (3) $|\psi(0)\rangle = |-20\rangle \otimes |-\rangle$. In (a-3) and (b-3), the blue dashed lines represent Eq. (4.77) respectively with $\beta \approx \ln(1.15)$ and $\beta \approx \ln(1.38)$.

showing

$$P_e(t) \propto \exp(2\beta t), \quad \beta = \max[\text{Im}(\varepsilon)] \quad (4.77)$$

confirms that the manifestation originates from $\mathcal{PT}$ symmetry breaking of the topologically protected edge states. From the dynamics of open quantum walks in Fig. 4.7 (b), we can understand that manifestations of edge states can be observed not only when the initial state is close to a boundary but also in the case that the initial state is not near interfaces. This is contrasted to closed quantum walks in which the initial state must be put very close to the interface for observing edge states, which we can understand from Fig. 4.7 (a). Therefore, the time evolution in Fig. 4.7 (b-1) and (b-2) digests peculiar phenomena of $\mathcal{PT}$ and chiral symmetric open quantum walks. While only eigenvalues of edge states deviate from the unit circle in the above example of time evolutions, other eigenstates can have complex quasi-energies in general. For example, in Fig. 4.6 (a-1) and (c-1), several eigenstates other than edge states satisfy $|\lambda| \neq 1$, while V' has $\mathcal{PT}$ symmetry. In addition, when there is randomness, all of eigenvalues deviate from the unit circle. Then, imaginary parts of quasi-energies depend on the realizations of $\{\theta_j(x)\}$ and $\{\gamma(x)\}$, not only for bulk states but also for edge states. This is because there is no constraint on $\text{Im}(\varepsilon)$ of edge states, while their eigenvalues are on the real axis $\text{Re}(\varepsilon) = 0, \pi$. Figure 4.8 actually shows that it depends on ensembles of $\{\theta_j(x)\}$ and $\{\gamma(x)\}$ whether edge states mainly contribute to the dynamics or not when the initial state is put away from the interfaces $|\psi(0)\rangle = |4\rangle \otimes |-\rangle$. When the initial state is put near an interface $|\psi(0)\rangle = |-20\rangle \otimes |-\rangle$, an edge state is amplified but the rates of growths are different depending on ensembles, as shown in Fig. 4.8 (a-3) and (b-3).

4.6 Conclusion and discussion

We have studied topological phases in one-dimensional open Floquet systems which have chiral symmetry. We have established a procedure to calculate topological numbers in chiral symmetric open Floquet systems in Eq. (4.40). The derivation has been carried out through generalizing the definition of chirality which characterizes topological edge states and discussing the bulk-edge correspondence. While the method has been applied to open Floquet systems [215, 72, 69] from the analogy to closed Floquet systems [53], our results give the microscopic foundation for the validity of the procedure in open Floquet systems. To our knowledge, this is the first study which systematically clarifies features resulting from chiral symmetry defined as Eq. (4.12) in open Floquet systems. We have constructed a model of open quantum walks which has chiral symmetry and confirmed that the derived topological numbers precisely predict the numbers of edge states. The model is classified into class BDI[†] or AIII [112] depending on parameters, of which the latter is explored for the first time in open quantum walks. While open quantum walks similar to our model in Eq. (4.49) have been studied in Ref. [69] without mentioning chiral symmetry in Eq. (4.12), we can confirm that the quantum walk is classified into class BDI[†].

We have also found that the constructed model satisfies $\mathcal{PT}$ symmetry, which has crucial effects on the dynamics when there is no randomness. In this case, topologically protected edge states break $\mathcal{PT}$ symmetry and have complex quasi-energies. Then, edge states whose quasi-energies have positive imaginary part attribute to exponential amplifications of intensities or corrected probabilities. This gives the explanation of edge states with longer lifetimes compared to bulk states, observed in an open quantum walk of single photons based on the setting in Fig.4.1 (b) [215]. The experiment has significant meanings in both research fields of Floquet topological phases and open quantum systems, since the observed behavior is not only a topological phenomenon peculiar to $\mathcal{PT}$ and chiral symmetric open Floquet systems but also the first experimental confirmation of $\mathcal{PT}$ symmetry breaking in an open quantum system under post selections. When parameters are randomized, the time-evolution operator does not retain $\mathcal{PT}$ symmetry and all of eigenstates have complex quasi-energies. In this case, it depends on parameters, ensembles, and initial states whether edge states mainly contribute to the dynamics or not. The phenomena which we have shown can also be investigated from the time-step dependence of light intensity in the experimental settings of classical coherent light [86, 93, 98, 99]. While realizations of position dependent coin operators may be difficult, classical systems have several advantages in comparison to quantum systems [215, 64, 72, 76] that time-steps can be larger and gain effects can be introduced, and so on [86, 93, 98, 99].

Chapter 5

Bifurcation of Edge States due to Discrete Periodicity in Time

5.1 Introduction

As explained in Chaps. 2 and 3, topological phases and resulting edge states have been explored in systems described by linear equations. Actually, both Schrödinger equation which describes usual quantum systems and the time-evolution equation of quantum walks are linear equations. On the other hand, it has been extensively studied what is caused by nonlinear effects in topologically non-trivial systems [216, 217, 218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228]. For example, emergence of solitons [216, 217, 219, 222] and enhancements of harmonic generations [224, 226, 227] have been reported, to name a few. While there are no fundamental theory and formulation to describe nonlinear effects on edge states and/or topological phases in nonlinear systems, there exist much possibility that interesting topological phenomena occur in nonlinear systems which are not analogous to linear systems.

One of such phenomena was reported in a nonlinear quantum walk that wave function amplitudes spontaneously accumulate to a topological edge state in time evolutions which include a nonlinear coin operator, even when initial states are far from a boundary in which the edge state resides [96]. This phenomenon is contrast to the behaviour in usual linear quantum walks where wave function amplitudes are localized due to topological phases only when an initial state is put near a boundary which accommodates topologically protected edge states [51, 62, 70]. The accumulation of wave function amplitudes into an edge state indicates that the edge state is a stable attractor in the nonlinear dynamics. From a linear stability analysis carried out in Ref. [96], it is concluded that there is always a stable edge state to which wave function amplitudes accumulate in the whole parameter region of the nonlinear quantum walk. However, a peculiar feature of quantum walks is not taken into consideration, that is, the discreteness of time is ignored in Ref. [96]. This is caused by considering the dynamics in the continuum limit and not taking edge states with $\varepsilon = \pi$ into account. As has been stated repeatedly, exploring topological phenomena in quantum walks, uniqueness of quantum walks should be paid attention to. Actually, as we show in the following, all of edge states become unstable repellers in a wide range of the parameter region, which suggests that the peculiarity of quantum walks is related to the instability of edge states. Thereby, in this work, we study the relation between the stability of edge states and the discrete periodicity in time. As a result, through the specific problem whether topological edge states are stable or unstable in nonlinear quantum walks, we show that the discreteness of time generally has significant effects on stationary states in nonlinear systems.

The rest of this chapter is organized as follows. In Sec. 5.2, we introduce nonlinear quantum walks based on feed-forward controls. We explain in Sec. 5.3 two models which we address and

linear stability analysis which we use to explore the stability of topological edge states. We clarify when edge states are stable attractors or unstable repellers and how the discreteness of time has effects on the conditions for the stability in Secs. 5.4 and 5.5, respectively addressing single and two-step nonlinear quantum walks. Section 5.6 is conclusion.

5.2 Nonlinear quantum walks

It has been proposed to introduce nonlinear effects in quantum walks, using feed-forward controls in optical systems [94, 95]. In Fig. 5.1, the scheme of such feed-forward controls is schematically described, where the originally proposed procedure is similar to (b). In the proposed feed-forward scheme [94, 95], light is separated into two paths during the dynamics of quantum walks, of which one remains in the system and the other is injected into detectors. When the rate of the separation is fixed, the intensity of light remaining the system at a lattice position x is known from the measured intensity of light injected into a detector. Then, the conditions of optical elements through which remaining light passes after the separation are changed depending on the results of the measurement. The manipulations of optical elements are done before light in the system reaches the next optical elements and remaining light is affected by its intensity. The manipulated optical elements are for example half-wave plates and/or phase shifters, of which the former was considered originally in Refs. [94, 95]. Carrying out such feed-forward controls results in adding nonlinear terms in time-evolution equations, since parameters such as rotation angles in coin operators and polarization dependent phases in phase operators include wave function amplitudes of light.

We comment on expected possibilities and difficulties, while experiments of nonlinear quantum walks have not been carried out using feed-forward controls. In the physical setting described in Fig. 5.1 (a), manipulations of parameters in coin operators such as Eqs. (3.6) and (3.7) are considered to be difficult, while manipulations of $\theta(x)$ in Eq. (3.8) may be possible. This is because it is difficult to dynamically change the mixing ratios of couplers, which play the role of coin operators (purple rectangles) in experiments based on optical fiber loops. In the physical setting described in Fig. 5.1 (b), the manipulations of parameters in both coin operators and phase operators may be possible, while it would be difficult to control optical elements before light reaches those. The difficulty originates from that the manipulations will be carried out by electronic devices whose speed cannot be faster than that of light. The same difficulty regarding the speed of manipulations may be avoided in the setting in Fig. 5.1 (a) by preparing long fibers and then making light propagate long distance before manipulations are done. While both physical settings have both advantages and disadvantages, we expect that the difficulties should be overcome in the near future.

5.3 Models and methods

In order to explore relationships between the stability of stationary states and the peculiarity to Floquet systems, we address two models of nonlinear quantum walks which are introduced in Sec. 5.3.1. Using a procedure which we explain in Sec. 5.3.2, we analyse the stability of edge states taking the discreteness of time into account.

5.3.1 two models of quantum walks

In the physical settings shown in Fig. 5.1, a time-evolution operator at a time step is composed of linear parts and nonlinear parts. Before introducing nonlinear effects, we consider linear components of two models which we address in the present work. The first one is a single-step quantum walk, whose time-evolution operator is given by

$$U_s = U_{s2}U_{s1}, \quad U_{s1} = S_+C_2(\theta_0/2), \quad U_{s2} = C_2(\theta_0/2)S_- . \quad (5.1)$$

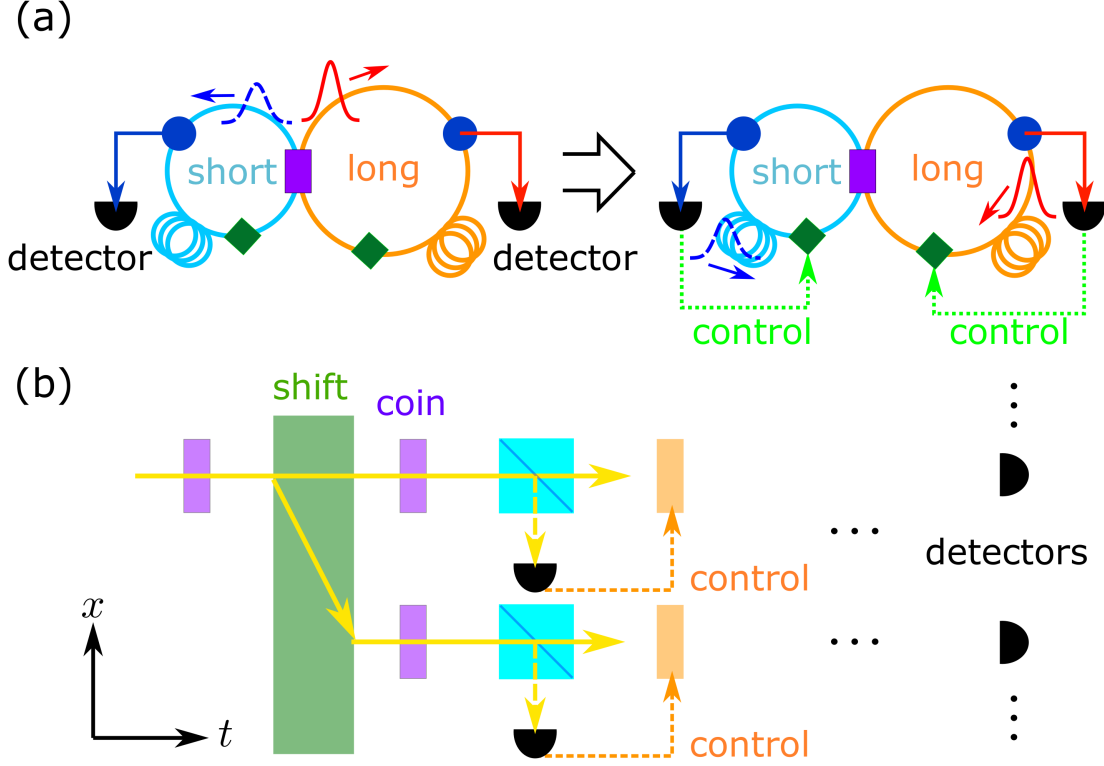


Figure 5.1: Schematic pictures of nonlinear quantum walks, based on (a) optical fibre loops and (b) arrayed optical elements. In both cases, the intensity of light is measured during the dynamics and conditions of optical elements through which light passes after the measurement are controlled according to the results of the measurement. The controlled optical elements are described by (a) green squares and (b) orange rectangles.

U_{s1} corresponds to U'_{e1} in Sec. 3.2. Note that, we remove superscripts ' and ' ' attached to time-evolution operators in Chaps. 3 and 4, since we do not consider several time frames in this chapter. The time-evolution operator in Eq. (5.1) has chiral symmetry with $\Gamma = \sigma_1$. Since U_s also has time-reversal symmetry and particle-hole symmetry with $\mathcal{T} = \sigma_1 \mathcal{K}$ and $\Xi = \mathcal{K}$, the system described by U_s is classified into class BDI in Tab. 2.1. Then, non-trivial topological phases can exist since this model is a one-dimensional model. Actually, following the procedure explained in Sec. 3.3, we can calculate topological numbers from $e^{i\frac{\pi}{4}\sigma_2} \mathcal{U}_{s2}(k) \mathcal{U}_{s1}(k) e^{-i\frac{\pi}{4}\sigma_2}$ and $e^{i\frac{\pi}{4}\sigma_2} \mathcal{U}_{s1}(k) \mathcal{U}_{s2}(k) e^{-i\frac{\pi}{4}\sigma_2}$ with homogeneous $\theta_0(x) = \theta_0$,

$$\nu_0 = \nu_\pi = \frac{1}{2} \text{sign}[\sin(\theta_0)]. \quad (5.2)$$

The second model is a two-step quantum walk described by

$$U_t = U_{t2} U_{t1}, \quad U_{t1} = C_2(\theta_2/2) S C_2(\theta_1/2), \quad U_{t2} = C_2(\theta_1/2) S C_2(\theta_2/2) \quad (5.3)$$

in the linear regime. Note that, since U_t shifts wave function amplitudes on even (odd) sites to even (odd) sites as it includes two shift operators, U_t is decomposed to a block matrix structure for even sites and odd sites. The time-evolution operator in Eq. (5.3) satisfies time-reversal symmetry, particle-hole symmetry, and chiral symmetry respectively with $\mathcal{T} = \sigma_1 \mathcal{K}$, $\Xi = \mathcal{K}$, and $\Gamma = \sigma_1$ which are the same as symmetry operators of U_s . Thus, U_t is also classified into class BDI and topological numbers become

$$\nu_0 = \text{sign} \left[\cos \left(\frac{\theta_1 - \theta_2}{2} \right) \sin \left(\frac{\theta_1 + \theta_2}{2} \right) \right], \quad \nu_\pi = \text{sign} \left[\cos \left(\frac{\theta_1 + \theta_2}{2} \right) \sin \left(\frac{\theta_1 - \theta_2}{2} \right) \right], \quad (5.4)$$

which are obtained from (1, 2) components of $e^{i\frac{\pi}{4}\sigma_2} \mathcal{U}_{t2}(k) \mathcal{U}_{t1}(k) e^{-i\frac{\pi}{4}\sigma_2}$ and $e^{i\frac{\pi}{4}\sigma_2} \mathcal{U}_{t1}(k) \mathcal{U}_{t2}(k) e^{-i\frac{\pi}{4}\sigma_2}$ with position-independent parameters $\theta_1(x), \theta_2(x) = \theta_1, \theta_2$. From Eqs. (5.2) and (5.4), we can understand that the sign flips of θ_j ($j = 0, 1, 2$) change the values of topological numbers. Therefore, if we consider the system depicted in Fig. 5.2,

$$\theta_j(x) = \begin{cases} +\theta_j & (|x| \leq m), \\ -\theta_j & (|x| > m), \end{cases} \quad (5.5)$$

there exist topologically protected edge states with $\varepsilon = 0$ and $\varepsilon = \pi$ near $x = \pm m$ where the signs of $\theta_j(x)$ are changed. We write edge states in single and two-step quantum walks as

$$|\Phi_{\varepsilon, \eta}\rangle = \sum_{x, \sigma} \Phi_{x, \sigma}^{(\varepsilon, \eta)} |x\rangle \otimes |\sigma\rangle, \quad (5.6)$$

where the labels $\varepsilon = 0, \pi$ and $\eta = \pm$ represent the quasi-energy and chirality of edge states, respectively. For simplicity, we use the same notations $|\Phi_{\varepsilon, \eta}\rangle$ for topological edge states in both models. Thereby, eigenequations of the edge states are

$$U_{s/t} |\Phi_{\varepsilon, \eta}\rangle = e^{-i\varepsilon} |\Phi_{\varepsilon, \eta}\rangle, \quad \Gamma |\Phi_{\varepsilon, \eta}\rangle = \eta |\Phi_{\varepsilon, \eta}\rangle. \quad (5.7)$$

Since chiral symmetry operator is $\Gamma = \sigma_1$ in the present cases, wave function amplitudes of topologically protected edge states satisfy

$$\Phi_{x, -}^{(\varepsilon, \eta)} = \eta \Phi_{x, +}^{(\varepsilon, \eta)}. \quad (5.8)$$

By taking Eq. (5.8) into account, we use a simplified notation

$$|\Phi_x^{(\varepsilon)}| \equiv |\Phi_{x, -}^{(\varepsilon, \eta)}| = |\Phi_{x, +}^{(\varepsilon, \eta)}|, \quad (5.9)$$

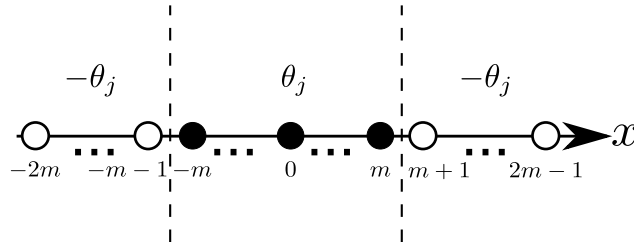


Figure 5.2: Position dependence of $\theta_j(x)$ with $j = 0, 1, 2$. There are two boundaries between $x = \pm m$ and $x = \pm(m+1)$. The number of sites is $4m$ where m is an even number. We refer to the region in which $|x| \leq m$ ($|x| > m$) is satisfied as the inner (outer) region. Periodic boundary conditions are imposed on both edges at $x = -2m$ and $x = 2m-1$.

when we focus on absolute values of wave function amplitudes.

Since the explanations about the linear regime of two models are finished, we introduce nonlinear effects. In Ref. [96], a nonlinear coin operator

$$C_2(\kappa\Theta) = \sum_x |x\rangle \langle x| \otimes \mathcal{C}_2[\kappa\Theta(x, t)] \quad (5.10)$$

was introduced, where $\Theta(x, t)$ is defined as

$$\Theta(x, t) = |\psi_{x,-}(t)|^2 - |\psi_{x,+}(t)|^2, \quad (5.11)$$

corresponding to the rotation angle of the coin operator. Since $\Theta(x, t)$ depends on wave function amplitudes at each time step t , the rotation by $C_2(\kappa\Theta)$ introduces nonlinear effects in the dynamics. The purpose of introducing nonlinear coin operators in Eqs. (5.10) and (5.11) was to accumulate wave function amplitudes into topological edge states [96], as we explain in the following. The parameter κ denotes the strength of nonlinearity. For convenience, we fix the sign of κ as $\kappa \geq 0$ in the following. The time evolutions of a single-step nonlinear quantum walk are described by U_s and $C_2(\kappa\Theta)$,

$$|\psi(t+1)\rangle = U_s C_2(\kappa\Theta) |\psi(t)\rangle. \quad (5.12)$$

We also consider a two-step nonlinear quantum walk whose time evolutions are described as

$$|\psi(t+1)\rangle = U_{t2} C_2(-\kappa\tilde{\Theta}) |\tilde{\psi}(t)\rangle, \quad |\tilde{\psi}(t)\rangle = U_{t1} C_2(\kappa\Theta) |\psi(t)\rangle. \quad (5.13)$$

Here, $\tilde{\Theta}(x, t)$ is also defined from $|\tilde{\psi}(t)\rangle$ in a similar way to $\Theta(x, t)$ in Eq. (5.11),

$$\tilde{\Theta}(x, t) = |\tilde{\psi}_{x,-}(t)|^2 - |\tilde{\psi}_{x,+}(t)|^2. \quad (5.14)$$

As internal states are rotated by angles which include wave function amplitudes themselves, $|\psi(t)\rangle$ governed by Eq. (5.12) or (5.13) experiences nonlinear time-evolutions. Since $|\tilde{\Phi}_{\varepsilon, \tilde{\eta}}\rangle = U_{t1} |\Phi_{\varepsilon, \eta}\rangle$ becomes an eigenstate of $U_{t1} U_{t2}$ with the eigenvalue $e^{-i\varepsilon}$, replacing $\Phi_{x, \sigma}^{(\varepsilon, \eta)}$ and η with $\tilde{\Phi}_{x, \sigma}^{(\varepsilon, \tilde{\eta})}$ and $\tilde{\eta}$, $\tilde{\Phi}_{x, \sigma}^{(\varepsilon, \tilde{\eta})}$ also satisfies Eqs. (5.8) and (5.9), as mentioned in Sec. 3.3.2. Then, from Eqs. (5.8)-(5.14), we can understand that the nonlinear coin operator has no effect on edge states because of $|\Phi_{x,-}^{(\varepsilon, \eta)}|^2 - |\Phi_{x,+}^{(\varepsilon, \eta)}|^2 = |\tilde{\Phi}_{x,-}^{(\varepsilon, \tilde{\eta})}|^2 - |\tilde{\Phi}_{x,+}^{(\varepsilon, \tilde{\eta})}|^2 = 0$, or equivalently topologically protected edge states always satisfy $C_2(\kappa\Theta) |\Phi_{\varepsilon, \eta}\rangle = C_2(0) |\Phi_{\varepsilon, \eta}\rangle = |\Phi_{\varepsilon, \eta}\rangle$ and $C_2(\kappa\tilde{\Theta}) |\tilde{\Phi}_{\varepsilon, \tilde{\eta}}\rangle = |\tilde{\Phi}_{\varepsilon, \tilde{\eta}}\rangle$. Therefore, topological edge states are stationary states in both single and two-step nonlinear quantum walks.

5.3.2 methods for exploring the stability of edge states

As clarified in Sec. 5.3.1, topologically protected edge states are stationary states in the nonlinear quantum walks which we consider in this work. In general, there are two types of stationary states in nonlinear systems, which are stable stationary states (attractors) and unstable stationary states (repellers). As these names indicate, states of systems spontaneously accumulate to attractors during nonlinear dynamics, while repellers are not the case.

The stability of a stationary state, or equivalently whether the stationary state becomes an attractor or a repeller, is revealed from linear stability analysis which we apply to topologically protected edge states. Suppose that $|\psi(t)\rangle$ consists of the stationary edge state with quasi-energy ε and chirality η , which is written as $|\Phi_{\varepsilon, \eta}\rangle$, and an infinitesimally weak fluctuating state $|\delta\psi(t)\rangle$ around the edge state,

$$|\psi(t)\rangle = |\Phi_{\varepsilon, \eta}\rangle + |\delta\psi(t)\rangle, \quad |\delta\psi(t)\rangle = \sum_{x, \sigma} \delta\psi_{x, \sigma}(t) |x\rangle \otimes |\sigma\rangle, \quad (5.15)$$

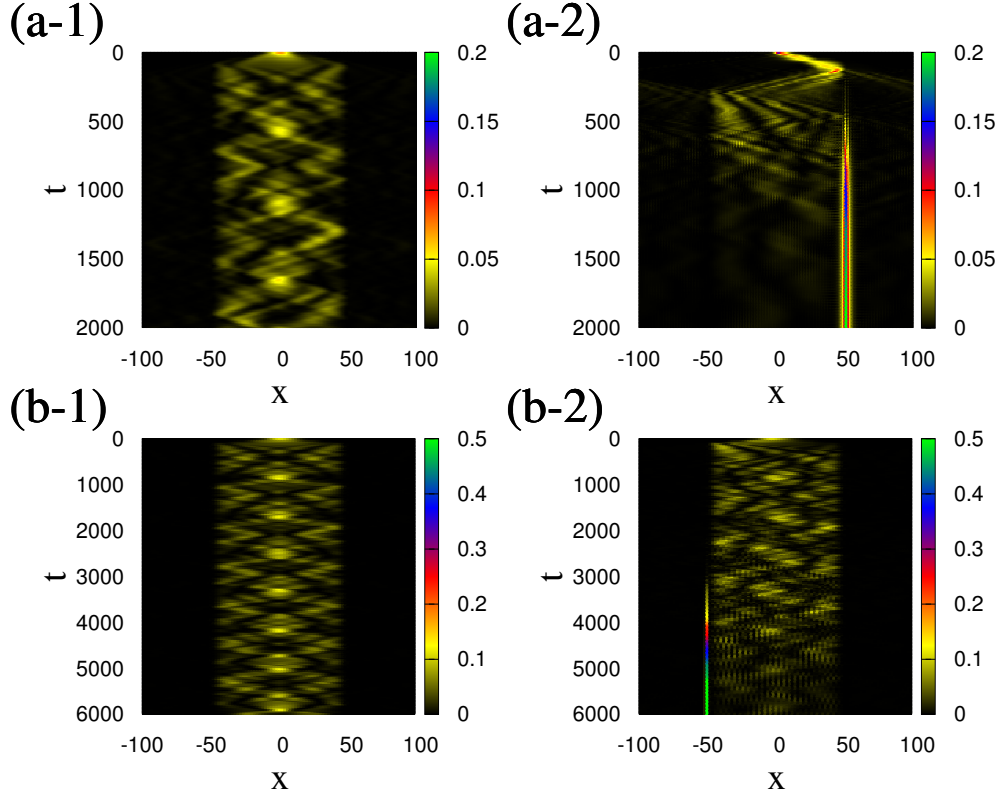


Figure 5.3: Time evolutions of $P(x, t) = |\psi_{x,+}(t)|^2 + |\psi_{x,-}(t)|^2$ of the (a) single-step nonlinear quantum walk with $\theta_0 = \pi/10$ and (b) two-step nonlinear quantum walk with $\theta_1 = -\pi/4, \theta_2 = 2\pi/5$. The nonlinear parameter in each figure is (a-1),(b-1) $\kappa = 0$, (a-2) $\kappa = 4.2$, and (b-2) $\kappa = 0.9$, while other parameters are $m = 50$ and $\Delta^2 = 50$ in all panels. When the sign of κ is flipped in (a-2) and (b-2), wave function amplitudes accumulate to the other boundary.

where $|\delta\psi_{x,\sigma}(t)/\Phi_{x,\sigma}^{(\varepsilon,\eta)}| \ll 1$ is assumed. Taking only the first order of $\delta\psi_{x,\sigma}(t)$ into account, the state in Eq. (5.15) satisfies the normalization condition, since $|\Phi_{\varepsilon,\eta}\rangle$ and $|\delta\psi(t)\rangle$ are orthogonal. Substituting Eq. (5.15) into Eq. (5.12) or (5.13) and taking only the first order of $\delta\psi_{x,\sigma}(t)$ into account, we can obtain a linearized time-evolution equation of $|\delta\psi(t)\rangle$

$$|\delta\psi(t+1)\rangle = V_{s/t}^{(\varepsilon,\eta)} |\delta\psi(t)\rangle. \quad (5.16)$$

The zeroth order terms of $\delta\psi_{x,\sigma}(t)$ cancel out since $\Phi_{x,\sigma}^{(\varepsilon,\eta)}$ is a stationary state in the nonlinear dynamics. The time-evolution operators $V_s^{(\varepsilon,\eta)}$ and $V_t^{(\varepsilon,\eta)}$ lose unitarity, and include wave function amplitudes of the stationary state $\Phi_{x,\sigma}^{(\varepsilon,\eta)}$. The stability of $|\Phi_{\varepsilon,\eta}\rangle$ is determined by the nonunitary dynamics of $|\delta\psi(t)\rangle$ in Eq. (5.16). If $\delta\psi_{x,\sigma}(t)$ diverge with time steps, $|\delta\psi(t)\rangle$ dominates in the dynamics and then $|\Phi_{\varepsilon,\eta}\rangle$ becomes a repeller (unstable). If $\delta\psi_{x,\sigma}(t)$ attenuate with time steps, $|\Phi_{\varepsilon,\eta}\rangle$ becomes an attractor (stable) and wave function amplitudes accumulate to $\Phi_{x,\sigma}^{(\varepsilon,\eta)}$ during nonlinear time-evolutions. The behaviour of $|\delta\psi(t)\rangle$, which is either divergence or attenuation, is determined from eigenvalues of the nonunitary operator including $\Phi_{x,\sigma}^{(\varepsilon,\eta)}$. If $V_{s/t}^{(\varepsilon,\eta)}$ has an eigenvalue whose absolute value is larger than one, $|\delta\psi(t)\rangle$ exponentially diverges with time steps, which makes $|\Phi_{\varepsilon,\eta}\rangle$ unstable.

In order to confirm expectations from the linear stability analysis, we simulate the nonlinear

dynamics in Eqs. (5.12) and (5.13) numerically. We employ the Gaussian initial state

$$\psi_{x,-}(0) = \psi_{x,+}(0) = N \exp(-x^2/2\Delta^2), \quad (5.17)$$

where the standard deviation Δ is a parameter and N is a real normalization constant. As wave function amplitudes of the initial state are all real, $\psi_{x,\sigma}(t)$ are always real during the nonlinear dynamics. Figure 5.3 shows time evolutions of linear and nonlinear quantum walks with the initial state in Eq. (5.17) and boundaries in Eq. (5.5) or equivalently Fig. 5.2. When nonlinear effects are absent ($\kappa = 0$) and the initial state is far from boundaries, topological edge states have no effect on the dynamics as shown in Fig. 5.3 (a-1) and (b-1). When there are nonlinear effects, it seems that edge states are stable attractors in both quantum walks, since wave function amplitudes accumulate to boundaries as shown in Fig. 5.3 (a-2) and (b-2). However, such judges depending only on appearances of distributions of intensities are insufficient, since it is possible that edge states are unstable repellers even when wave function amplitudes are localized at a boundary. Actually, we show that this is the case in the wide range of parameter space in the following. Then, we quantitatively evaluate the stability of edge states based on the fidelity which is defined as

$$F_{\varepsilon,\eta}(t) = |\langle \Phi_{\varepsilon,\eta} | \psi(t) \rangle|. \quad (5.18)$$

If an edge state $|\Phi_{\varepsilon,\eta}\rangle$ is stable (unstable), the corresponding fidelity $F_{\varepsilon,\eta}(t)$ is close to one (much smaller than one) after many time steps. Note that, in both single and two-step nonlinear quantum walks, there are cases in which soliton-like states appear when an initial state $|\psi(0)\rangle$ is localized at a single site. We empirically know that soliton-like states do not appear if $\{\psi_{x,\sigma}(0)\}$ forms a Gaussian wave packet in Eq. (5.17), and we do not focus on soliton-like states in the present work.

5.4 Single-step nonlinear quantum walks

Substituting Eq. (5.15) into Eq. (5.12) we obtain the time-evolution equation for $|\delta\psi(t)\rangle$,

$$|\delta\psi(t+1)\rangle = V_s^{(\varepsilon,\eta)} |\delta\psi(t)\rangle, \quad V_s^{(\varepsilon,\eta)} = U_s D_{\kappa,\eta}(|\Phi_x^{(\varepsilon)}|), \quad (5.19)$$

up to the first order of $\delta\psi_{x,\sigma}(t)$, where $D_{\kappa,\eta}(|\Phi_x^{(\varepsilon)}|)$ is

$$D_{\kappa,\eta}(|\Phi_x^{(\varepsilon)}|) = \sum_x |x\rangle \langle x| \otimes \mathcal{D}_{\kappa,\eta}(|\Phi_x^{(\varepsilon)}|), \quad (5.20)$$

$$\mathcal{D}_{\kappa,\eta}(|\Phi_x^{(\varepsilon)}|) = \sigma_0 - 2\kappa |\Phi_x^{(\varepsilon)}|^2 \begin{pmatrix} \eta & -1 \\ -1 & \eta \end{pmatrix} = (1 - 2\kappa\eta |\Phi_x^{(\varepsilon)}|^2) \sigma_0 + 2\kappa |\Phi_x^{(\varepsilon)}|^2 \sigma_1. \quad (5.21)$$

If $\kappa = 0$, $D_{\kappa,\eta}(|\Phi_x^{(\varepsilon)}|)$ is the identity operator and $V_s^{(\varepsilon,\eta)} = U_s$ is a unitary operator. Then, the eigenvalues of $V_s^{(\varepsilon,\eta)}$ written as $\lambda_s^{(\varepsilon,\eta)}$ are complex numbers on the unit circle. For $\kappa > 0$, since $D_{\kappa,\eta}(|\Phi_x^{(\varepsilon)}|)$ and then $V_s^{(\varepsilon,\eta)}$ are nonunitary operators, $\lambda_s^{(\varepsilon,\eta)}$ are generally not on the unit circle in the complex plane. On one hand, if $\max(|\lambda_s^{(\varepsilon,\eta)}|) > 1$, $\delta\psi_{x,\sigma}(t)$ diverge with time steps and then $|\Phi_{\varepsilon,\eta}\rangle$ is unstable. On the other hand, if $\max(|\lambda_s^{(\varepsilon,\eta)}|) \leq 1$, $\delta\psi_{x,\sigma}(t)$ attenuate or remain small and then $|\Phi_{\varepsilon,\eta}\rangle$ is stable. While the stability of edge states in single-step nonlinear quantum walks was explored in Ref. [96], our procedure explained above differs from the analysis of the previous work in that we take the discrete nature of quantum walks into account which was ignored in Ref. [96]. We show below that the discrete periodicity in time is essential to bifurcations where topological edge states become unstable repellers. To this end, before showing results obtained from our analysis in Secs. 5.4.2 and 5.4.3, we devote Sec. 5.4.1 to explaining the stability analysis in the continuum limit which was applied in Ref. [96], for comparison with our scheme.

5.4.1 stability analysis in the continuum limit

Taking the continuum limit in x and t , Eq. (5.12) is transformed into a nonlinear Dirac equation

$$i \frac{\partial}{\partial t} |\psi_x(t)\rangle = -i\sigma_3 \frac{\partial}{\partial x} |\psi_x(t)\rangle + [\theta_0(x) + \kappa\Theta(x, t)]\sigma_2 |\psi_x(t)\rangle, \quad (5.22)$$

where $|\psi_x(t)\rangle = [\psi_{x,-}(t), \psi_{x,+}(t)]^T$ and only the first order of $\theta_0(x)$ and $\kappa\Theta(x, t)$ is taken into account. Using Eq. (5.22), the stability of edge states was studied in Ref. [96].

Substituting Eq. (5.15) into Eq. (5.22) and expanding the equation up to the first order of $\delta\psi_{x,\sigma}(t)$, the time-evolution equation for $|\delta\psi(t)\rangle$

$$i \frac{\partial}{\partial t} |\delta\psi_x(t)\rangle = \Omega |\delta\psi_x(t)\rangle, \quad \Omega = -i\sigma_3 \frac{\partial}{\partial x} + \theta_0\sigma_2 - 2i\kappa|\Phi|^2[\eta\sigma_0 - \sigma_1] \quad (5.23)$$

is obtained, where $|\delta\psi_x(t)\rangle = [\delta\psi_{x,-}(t), \delta\psi_{x,+}(t)]^T$. In Eq. (5.23), for simplicity, x dependence of $\theta_0(x)$ is ignored. In addition, x and ε dependences of $|\Phi_x^{(\varepsilon)}|$ are ignored, and we write these as $|\Phi|$. These approximations may be justified when $\theta_0(x)$ and $|\Phi_x^{(\varepsilon)}|$ are not strongly dependent on x . Assuming a plane wave state which is written as $|\delta\psi_x(t)\rangle = e^{i(qx - \omega t)} |\delta\psi_0(t)\rangle$ with a wave number q , Eq. (5.23) becomes

$$i \frac{\partial}{\partial t} |\delta\psi_0(t)\rangle = \Omega(q) |\delta\psi_0(t)\rangle, \quad (5.24)$$

where $\Omega(q)$ corresponds to a non-Hermitian operator

$$\Omega(q) = -2i\kappa\eta|\Phi|^2\sigma_0 + \tilde{\Omega}(q), \quad \tilde{\Omega}(q) = \begin{pmatrix} q & -i\theta_0 + 2i\kappa|\Phi|^2 \\ i\theta_0 + 2i\kappa|\Phi|^2 & -q \end{pmatrix}. \quad (5.25)$$

We argue the non-Hermitian operator in Eq. (5.25) from the viewpoint of $\mathcal{PT}$ symmetry which is a combined symmetry of parity and time-reversal symmetries. As discussed in Sec. 4.2.2, when a non-Hermitian operator has $\mathcal{PT}$ symmetry, there are two phases which are the $\mathcal{PT}$ symmetric phase and $\mathcal{PT}$ broken phase [114, 159]. While all eigenvalues are real in the $\mathcal{PT}$ symmetric phase, eigenvalues are partially or fully complex in the $\mathcal{PT}$ broken phase. The transition between the two phases is referred to as $\mathcal{PT}$ symmetry breaking. Here, we focus on the second term $\tilde{\Omega}(q)$ in Eq. (5.25) for the moment, since the first term $-2i\kappa\eta|\Phi|^2$ is the constant and trivial term which describes a uniform attenuation common to all eigenstates. Then, we identify that $\tilde{\Omega}(q)$ has $\mathcal{PT}$ symmetry

$$(\mathcal{PT})\tilde{\Omega}(q)(\mathcal{PT})^{-1} = \tilde{\Omega}(q), \quad (5.26)$$

where $\mathcal{PT}$ symmetry operator is

$$\mathcal{PT} = \sigma_3\mathcal{K}. \quad (5.27)$$

Since the eigenvalues of $\tilde{\Omega}(q)$ are

$$\tilde{\omega} = \pm \sqrt{q^2 + \theta_0^2 - 4\kappa^2|\Phi|^4}, \quad (5.28)$$

the eigenvalues of $\Omega(q)$ are given by

$$\omega = -2i\kappa\eta|\Phi|^2 \pm \sqrt{q^2 + \theta_0^2 - 4\kappa^2|\Phi|^4}. \quad (5.29)$$

If the imaginary part of ω is entirely negative (partially positive), $|\delta\psi_x(t)\rangle$ attenuates (diverges) with time and the edge state $|\Phi_{\varepsilon,\eta}\rangle$ is stable (unstable). The corresponding eigenstate of $\Omega(q)$ and $\tilde{\Omega}(q)$ is

$$|\phi_q\rangle = \begin{pmatrix} i\theta_0 - 2i\kappa|\Phi|^2 \\ \tilde{\omega} - q \end{pmatrix}. \quad (5.30)$$

When $\kappa < |\theta_0|/4|\Phi|^2$, $\tilde{\omega}$ is entirely real because $\mathcal{PT}$ symmetry is preserved $\mathcal{PT}|\phi_q\rangle = -|\phi_q\rangle$. In this case, the stability of the edge states is determined by the first pure imaginary term in Eq. (5.29). Increasing κ , $\tilde{\omega}$ can become pure imaginary with positive and negative signs since $\mathcal{PT}$ symmetry is broken when $\theta_0 < 2\kappa|\Phi|^2$. However, $\mathcal{PT}$ symmetry breaking does not affect the stability of edge states since the first imaginary term in Eq. (5.29) always dominates the sign of $\text{Im}(\omega)$ due to

$$2\kappa|\Phi|^2 \geq |\text{Im}(\tilde{\omega})|. \quad (5.31)$$

Then, based on the analysis in terms of the non-Hermitian operator which describes continuous time evolutions of fluctuations, $\mathcal{PT}$ symmetry breaking is irrelevant to the stability of $|\Phi_{\varepsilon,\eta}\rangle$, which indicates that $|\Phi_{\varepsilon,+}\rangle$ and $|\Phi_{\varepsilon,-}\rangle$ are always attractors and repellers respectively with positive κ . In other words, from the analysis done in Ref. [96], it is concluded that the strength of nonlinearity $\kappa (> 0)$ is irrelevant to determining whether an edge state becomes an attractor or a repeller. On the other hand, from our stability analysis in which the discreteness of time is taken into account, it is concluded that the strength of κ has effects on the stability of topological edge states, as we show in the following.

5.4.2 derivation of edge states

In order to obtain eigenvalues of $V_s^{(\varepsilon,\eta)}$ which determine the stability of edge states, analytical forms of $\Phi_{x,\sigma}^{(\varepsilon,\eta)}$ are needed since $V_s^{(\varepsilon,\eta)}$ includes $|\Phi_x^{(\varepsilon)}|$ in its argument. Therefore, we derive wave function amplitudes of edge states analytically. For simplicity, we restrict the range of θ_0 as $0 < \theta_0 < \pi/2$. Since edge states satisfy Eq. (5.8), in the inner region where $-m \leq x \leq m$ and $\theta_0(x) = \theta_0$, the time-evolution equation of $|\Phi_{\varepsilon,\eta}\rangle$ becomes

$$e^{-i\varepsilon}\Phi_{x,-}^{(\varepsilon,\eta)} = \cos(\theta_0/2)\alpha_\eta\Phi_{x+1,-}^{(\varepsilon,\eta)} - \sin(\theta_0/2)\beta_\eta\Phi_{x-1,-}^{(\varepsilon,\eta)}, \quad (5.32)$$

$$\eta e^{-i\varepsilon}\Phi_{x,+}^{(\varepsilon,\eta)} = \sin(\theta_0/2)\alpha_\eta\Phi_{x+1,+}^{(\varepsilon,\eta)} + \cos(\theta_0/2)\beta_\eta\Phi_{x-1,+}^{(\varepsilon,\eta)}, \quad (5.33)$$

where α_η and β_η are

$$\alpha_\eta = \cos(\theta_0/2) - \eta \sin(\theta_0/2), \quad \beta_\eta = \sin(\theta_0/2) + \eta \cos(\theta_0/2). \quad (5.34)$$

Considering edge states localized near the right boundary, we assume

$$\Phi_{x,\pm}^{(\varepsilon,\eta)} = \eta\Phi_{x,\pm}^{(\varepsilon,\eta)} = N_s(-1)^{\frac{\varepsilon}{\pi}x}e^{-\gamma(m-x)}, \quad (5.35)$$

where N_s is the normalization constant. While $\Phi_{x,\sigma}^{(0,\eta)}$ have the same sign in any positions, the sign of $\Phi_{x,\sigma}^{(\pi,\eta)}$ is opposite to that of $\Phi_{x\pm 1,\sigma}^{(\pi,\eta)}$. Substituting Eq. (5.35) into Eqs. (5.32) and (5.33), we obtain

$$1 = \cos(\theta_0/2)\alpha_\eta e^\gamma - \sin(\theta_0/2)\beta_\eta e^{-\gamma}, \quad (5.36)$$

$$\eta = \sin(\theta_0/2)\alpha_\eta e^\gamma + \cos(\theta_0/2)\beta_\eta e^{-\gamma}, \quad (5.37)$$

for both $\varepsilon = 0$ and $\varepsilon = \pi$. Since Eqs. (5.36) and (5.37) are quadratic equations of e^γ , we obtain two solutions of e^γ from each equation. We employ one solution which is obtained from both Eqs. (5.36) and (5.37). Then, e^γ becomes

$$e^\gamma = \beta_\eta/\alpha_\eta. \quad (5.38)$$

From Eqs. (5.34) and (5.38), η must be + for the solutions of localized states which are described by positive γ . In the outer region where $x \geq m+1$ and $\theta_0(x) = -\theta_0$, we obtain the same solution of γ in the same way, while Eq. (5.35) is changed to $\Phi_{x,-}^{(\varepsilon,\eta)} = \eta\Phi_{x,+}^{(\varepsilon,\eta)} = N_s(-1)^{\frac{\varepsilon}{\pi}x}e^{-\gamma[x-(m+1)]}$. In

the case that $|\Phi_{\varepsilon,\eta}\rangle$ is localized near the left boundary, η must be $-$ while the localization length γ^{-1} is the same as that with $\eta = +$. The boundary conditions of $|\Phi_{\varepsilon,+}\rangle$ are

$$e^{-i\varepsilon}\Phi_{m,-}^{(\varepsilon,+)} = \alpha_-[\cos(\theta_0/2)\Phi_{m+1,-}^{(\varepsilon,+)} - \sin(\theta_0/2)e^{-\gamma}(-1)^{\frac{\varepsilon}{\pi}}\Phi_{m,-}^{(\varepsilon,+)}], \quad (5.39)$$

$$e^{-i\varepsilon}\Phi_{m,+}^{(\varepsilon,+)} = \beta_+[\sin(\theta_0/2)\Phi_{m+1,+}^{(\varepsilon,+)} + \cos(\theta_0/2)e^{-\gamma}(-1)^{\frac{\varepsilon}{\pi}}\Phi_{m,+}^{(\varepsilon,+)}], \quad (5.40)$$

near the right boundary. Assuming

$$\Phi_{m,\sigma}^{(\varepsilon,+)} = (-1)^{\frac{\varepsilon}{\pi}}\Phi_{m+1,\sigma}^{(\varepsilon,+)}, \quad (5.41)$$

Eqs. (5.39) and (5.40) are satisfied. For edge states with $\eta = -$ which are localized near the left boundary, the boundary conditions are satisfied by the same assumption as Eq. (5.41), changing $m, m+1$, and $+$ into $-m, -m-1$, and $-$, respectively. In the case of $0 < \theta_0 < \pi/2$, the explicit form of $\Phi_{x,\sigma}^{(\varepsilon,\eta)}$ becomes

$$\Phi_{x,-}^{(\varepsilon,\eta)} = \eta\Phi_{x,+}^{(\varepsilon,\eta)} = N_s(-1)^{\frac{\varepsilon}{\pi}x}e^{-\gamma|x-\eta m'|}, \quad \gamma = \log\left[\frac{1+\sin(\theta_0)}{\cos(\theta_0)}\right], \quad N_s = \sqrt{\frac{1-e^{-2\gamma}}{4}}, \quad (5.42)$$

with $m' = m$ for $|x| \leq m$ and $m' = m+1$ for $|x| \geq m+1$. We use the analytical solutions of edge states in Eq. (5.42) for deriving bifurcation thresholds at which edge states become unstable in Secs. 5.4.3 and 5.4.4.

5.4.3 derivation of the thresholds

In order to clarify whether edge states are stable or unstable analytically, we use a property of the spectral radius and the spectral norm, since the spectral radius of $V_s^{(\varepsilon,\eta)}$ which is defined as $\max(|\lambda_s^{(\varepsilon,\eta)}|)$ determines the stability of $|\Phi_{\varepsilon,\eta}\rangle$. Given a square matrix A which is diagonalizable, its spectral radius $\max(|\lambda_A|)$ and spectral norm $\sigma(A)$ always satisfy

$$\max(|\lambda_A|) \leq \sigma(A), \quad \sigma(A) = \sqrt{\max(|\lambda_{A^\dagger A}|)}, \quad (5.43)$$

where λ_A and $\lambda_{A^\dagger A}$ respectively represent the eigenvalues of A and $A^\dagger A$. Here, $A^\dagger$ denotes the Hermitian conjugation of A . Using Eq. (5.43), we can derive the bifurcation thresholds in the single-step nonlinear quantum walks where edge states become unstable, and also in the two-step nonlinear quantum walks as discussed in Sec. 5.5.2). The stability of the edge state $|\Phi_{\varepsilon,\eta}\rangle$ is determined by the maximum value of $|\lambda_s^{(\varepsilon,\eta)}|$, which is equivalent to the spectral radius in Eq. (5.43) with $A = V_s^{(\varepsilon,\eta)} = U_s D_{\kappa,\eta}(|\Phi_x^{(\varepsilon)}|)$. Since U_s is unitary, $V_s^{(\varepsilon,\eta)}$ satisfies

$$[V_s^{(\varepsilon,\eta)}]^\dagger [V_s^{(\varepsilon,\eta)}] = D_{\kappa,\eta}^\dagger(|\Phi_x^{(\varepsilon)}|) D_{\kappa,\eta}(|\Phi_x^{(\varepsilon)}|). \quad (5.44)$$

Therefore, the upper bound of $\max(|\lambda_s^{(\varepsilon,\eta)}|)$ is obtained by diagonalizing the positive-definite matrix $D_{\kappa,\eta}^\dagger(|\Phi_x^{(\varepsilon)}|) D_{\kappa,\eta}(|\Phi_x^{(\varepsilon)}|)$, in the light of Eq. (5.43). We can easily calculate eigenvalues of $D_{\kappa,\eta}^\dagger(|\Phi_x^{(\varepsilon)}|) D_{\kappa,\eta}(|\Phi_x^{(\varepsilon)}|)$ because it is a block diagonal matrix for x , which we can understand from Eq. (5.20). Since $\mathcal{D}_{\kappa,\eta}(|\Phi_x^{(\varepsilon)}|)$ defined in Eq. (5.21) is Hermitian, the eigenvalues of $\mathcal{D}_{\kappa,\eta}^\dagger(|\Phi_x^{(\varepsilon)}|) \mathcal{D}_{\kappa,\eta}(|\Phi_x^{(\varepsilon)}|)$ correspond to δ_1^2 and $\delta_2^2(x)$, where δ_1 and $\delta_2(x)$ are the eigenvalues of $\mathcal{D}_{\kappa,\eta}(|\Phi_x^{(\varepsilon)}|)$ given by

$$\delta_1 = 1, \quad \delta_2(x) = 1 - 4\kappa\eta|\Phi_x^{(\varepsilon)}|^2. \quad (5.45)$$

From Eqs. (5.43)-(5.45), the spectral norm $\sigma[V_s^{(\varepsilon,\eta)}]$ is determined by

$$\sigma[V_s^{(\varepsilon,\eta)}] = \max_x[\delta_1, |\delta_2(x)|]. \quad (5.46)$$

On one hand, in the case of $\eta = -$, $\delta_2(x) = 1 + 4\kappa|\Phi_x^{(\varepsilon)}|^2$ becomes larger than 1 and

$$\max(|\lambda_s^{(\varepsilon,-)}|) \leq 1 + 4\kappa \max_x(|\Phi_x^{(\varepsilon)}|^2) (> 1) \quad (5.47)$$

is satisfied for positive κ . Thereby, $|\Phi_{\varepsilon,-}\rangle$ can always be unstable. On the other hand, in the case of $\eta = +$, it depends on κ whether the value of $\sigma[V_s^{(\varepsilon,\eta)}]$ becomes 1 or $\delta_2(x) = |1 - 4\kappa \max_x(|\Phi_x^{(\varepsilon)}|^2)|$. When $\max_x(|1 - 4\kappa|\Phi_x^{(\varepsilon)}|^2|) \leq 1$ is satisfied for small κ , $|\Phi_{\varepsilon,+}\rangle$ is stable because of

$$\max(|\lambda_s^{(\varepsilon,+)}|) \leq 1. \quad (5.48)$$

When $1 - 4\kappa \max_x(|\Phi_x^{(\varepsilon)}|^2) < -1$ is satisfied, $|\Phi_{\varepsilon,+}\rangle$ can be unstable due to

$$\max(|\lambda_s^{(\varepsilon,+)}|) \leq |1 - 4\kappa \max_x(|\Phi_x^{(\varepsilon)}|^2)| (> 1). \quad (5.49)$$

The threshold between the stable region in which $|\Phi_{\varepsilon,+}\rangle$ are inevitably attractors and the unstable region in which $|\Phi_{\varepsilon,+}\rangle$ can be repellers becomes

$$\kappa_c = \frac{1}{2 \max_x(|\Phi_x^{(\varepsilon)}|^2)}, \quad (5.50)$$

from the condition that $1 - 4\kappa_c \max_x(|\Phi_x^{(\varepsilon)}|^2) = -1$. When the value of $\max_x(|\Phi_x^{(\varepsilon)}|^2)$ is small, large κ is needed to make $|\Phi_{\varepsilon,+}\rangle$ unstable. Therefore, $|\Phi_{\varepsilon,+}\rangle$ tends to be stable (unstable) if the localization length $[\log\{1/\cos(\theta_0) + \tan(\theta_0)\}]^{-1}$ is large (small). While Eq. (5.49) gives only the upper bound of $\max(|\lambda_s^{(\varepsilon,\eta)}|)$, as we numerically demonstrate later, this inequality can correctly estimate the criteria of the linear stability analysis, whether $\max(|\lambda_s^{(\varepsilon,\eta)}|) \leq 1$ or $\max(|\lambda_s^{(\varepsilon,\eta)}|) > 1$, at least for the present model. Thereby, here we summarize the results derived from the above discussion under the facts that $\kappa > 0$ and $|\Phi_x^{(\varepsilon)}| > 0$:

1. In the case of $\eta = -$, since $\max(|\lambda_s^{(\varepsilon,-)}|) \leq 1 + 4\kappa \max_x(|\Phi_x^{(\varepsilon)}|^2) > 1$ is satisfied for any $\kappa > 0$, $|\Phi_{\varepsilon,-}\rangle$ are always unstable repellers.
2. In the case of $\eta = +$, $|\Phi_{\varepsilon,+}\rangle$ are inevitably stable attractors when κ is smaller than a threshold value $\kappa_c = 1/2 \max_x(|\Phi_x^{(\varepsilon)}|^2)$, which guarantees $\max(|\lambda_s^{(\varepsilon,+)}|) \leq 1$. For $\kappa > \kappa_c$, however, an additional bifurcation occurs since $\max(|\lambda_s^{(\varepsilon,+)}|)$ is actually larger than one.

The first result is consistent with the conclusion in Ref. [96]. However, the second result claims that $|\Phi_{\varepsilon,+}\rangle$ are unstable for $\kappa > \kappa_c$ and an additional bifurcation occurs with increasing the strength of nonlinearity κ , which is different from the claim in Ref. [96]. The bifurcations where edge states change from stable attractors to unstable repellers are unique to Floquet systems since these cannot be revealed by the stability analysis in the continuum limit but by our procedure taking the discrete nature of time evolutions into consideration.

5.4.4 numerical confirmation

In order to confirm the validity of Eq. (5.50), we numerically calculate eigenvalues of $V_s^{(\varepsilon,\eta)}$. For simplicity, we consider the case of $0 < \theta_0 < \pi/2$ and use Eq. (5.42) where $\Phi_{x,+}^{(\varepsilon,+)}$ and $\Phi_{x,+}^{(\varepsilon,-)}$ are respectively localized at the right and left boundaries. From Eqs. (5.42) and (5.50), we obtain the expected threshold value κ_c . Alternating sign changes $(-1)^{\frac{\varepsilon}{\pi}x}$ do not influence $D_{\kappa,\eta}(|\Phi_x^{(\varepsilon)}|)$ and $V_s^{(\varepsilon,\eta)}$, which can be understood from Eq. (5.21). Thus, in the single-step nonlinear quantum walk, $V_s^{(0,\eta)} = V_s^{(\pi,\eta)}$ and $\lambda_s^{(0,\eta)} = \lambda_s^{(\pi,\eta)}$ are satisfied, since $|\Phi_{0,\eta}\rangle$ and $|\Phi_{\pi,\eta}\rangle$ have the same localization length γ^{-1} . Therefore, we focus only on $V_s^{(0,\eta)}$ and $\lambda_s^{(0,\eta)}$ hereafter. Because of this equivalence, however, there are always two stable states $|\Phi_{0,\eta}\rangle$ and $|\Phi_{\pi,\eta}\rangle$ when $\kappa < \kappa_c$ or

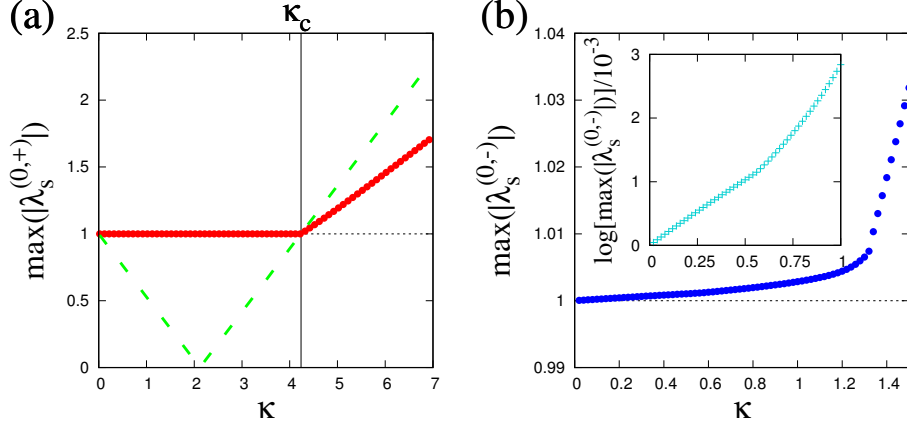


Figure 5.4: κ dependences of $\max(|\lambda_s^{(0,\eta)}|)$ with $\theta_0 = \pi/10$, $\kappa_c \simeq 4.24$, and $m = 150$. (a) In the case of $\eta = +$, the green dashed line represents the absolute value of $1 - 4\kappa \max_x(|\Phi_x^{(\varepsilon)}|^2)$ which is one eigenvalue of $\mathcal{D}_{\kappa,+}(|\Phi_x^{(\varepsilon)}|)$ in Eq. (5.45). (b) In the case of $\eta = -$, the inset shows the logarithm of $\max(|\lambda_s^{(0,-)}|)$.

equivalently $\max(|\lambda_s^{(0,\eta)}|) = \max(|\lambda_s^{(\pi,\eta)}|) \leq 1$, and we cannot predict which state is realized after long time evolutions from the linear stability analysis. Substituting Eq. (5.42) into Eqs. (5.19)-(5.21), we calculate $\lambda_s^{(\varepsilon,\eta)}$ by numerical diagonalizations. Figure 5.4 shows κ dependences of $\max(|\lambda_s^{(0,\eta)}|)$. In the case of $\eta = -$, $\max(|\lambda_s^{(0,-)}|) > 1$ is satisfied for all $\kappa > 0$ as shown in Fig. 5.4 (b). Therefore, $|\Phi_{\varepsilon,-}\rangle$ are always unstable repellers. In the case of $\eta = +$, $\max(|\lambda_s^{(0,+)}|)$ remains to be 1 when κ is smaller than a threshold value κ_c , as shown in Fig. 5.4 (a). In this case, $|\Phi_{\varepsilon,+}\rangle$ are stable attractors. However, for $\kappa > \kappa_c$, $\max(|\lambda_s^{(0,+)}|)$ becomes larger than one, which means that $|\Phi_{\varepsilon,+}\rangle$ also become unstable. These numerical results completely agree with the theoretical predictions from Eq. (5.50). In Fig. 5.4 (a), the absolute value of $1 - 4\kappa \max_x(|\Phi_x^{(\varepsilon)}|^2)$ which is one of the eigenvalues of $\mathcal{D}_{\kappa,+}(|\Phi_x^{(\varepsilon)}|)$ is also plotted to support this conclusion.

Here, we verify the additional bifurcation predicted from the stability analysis by numerically simulating time evolutions of the nonlinear quantum walk in Eq. (5.12) with the initial state in Eq. (5.17). To begin with, we employ the same parameters as shown in Fig. 5.4 where the stability analysis predicts the additional bifurcation at a threshold value $\kappa_c \approx 4.24$, from Eq. (5.50). Below the threshold, the normalized intensities are accumulated around a boundary near $x = 150$, forming a stationary state as shown in Fig. 5.5 (a-1) and (a-2). Figure 5.5 (a-3) compares the distribution of intensities for the stationary state at $T = 5 \times 10^5$ and that for an edge state $|\Phi_{0,+}\rangle$ of the quantum walk without nonlinearity. Since both states are almost identical, we can see that the edge state $|\Phi_{0,+}\rangle$ is a stable attractor for $\kappa < \kappa_c$. Above the threshold, the time evolution is similar to that with $\kappa < \kappa_c$ up to a certain time step where the distribution of intensities is accumulated around the boundary as shown in Fig. 5.5 (b-1). Figure 5.5 (b-2) clarifies that the distribution of intensities near the boundary abruptly shrinks around $t = 2.3 \times 10^5$. The states after the abrupt shrink are not stationary as the intensities slightly fluctuate. From Fig. 5.5 (b-3), we can see that the edge state $|\Phi_{0,+}\rangle$ is unstable for $\kappa > \kappa_c$, since the localization length of the final state is smaller than that of the edge state $|\Phi_{0,+}\rangle$. Since the distribution of intensities for the other edge state $|\Phi_{\pi,+}\rangle$ localized near $x = 150$ is the same as that of $|\Phi_{0,+}\rangle$, we can also see that $|\Phi_{\pi,+}\rangle$ is unstable with $\kappa > \kappa_c$ from Fig. 5.5 (b-3).

Figure 5.6 shows fidelities at each time step. In the case of $\kappa = 4.2 < \kappa_c$, as shown in Fig. 5.6 (a), the fidelity for the edge state $|\Phi_{0,+}\rangle$ becomes almost one after long time steps. In the case of $\kappa = 4.5 > \kappa_c$ as shown in Fig. 5.6 (b), $F_{0,+}(t)$ and also $F_{\pi,+}(t)$ do not reach one and

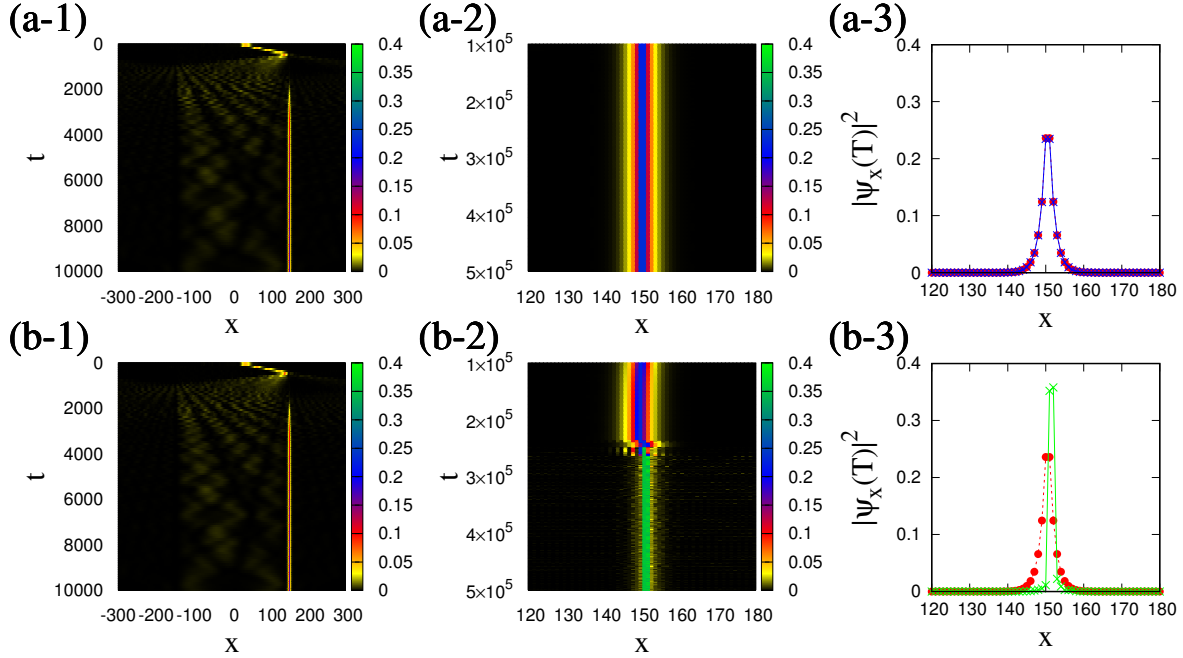


Figure 5.5: Time evolutions of the normalized intensity $P(x,t) = |\psi_{x,+}(t)|^2 + |\psi_{x,-}(t)|^2$ of the single-step nonlinear quantum walk with $\theta_0 = \pi/10$ and $\Delta^2 = 60$. Two boundaries locate at $x = \pm 150$. The nonlinear parameter κ is (a) below the threshold value $\kappa = 4.2 < \kappa_c$ and (b) above the threshold value $\kappa = 4.5 > \kappa_c$. The left and center columns show the time evolutions in shorter and longer time scales, respectively. The right column shows distributions of intensities at the final time step $T = 5 \times 10^5$, which correspond to crosses and solid lines whose colors are (a-3) blue and (b-3) green. In (a-3) and (b-3), the intensities of an edge state $|\Phi_{0,+}\rangle$ is also shown, which corresponds to red circles and dashed lines.

the former abruptly decreases around $t = 2.3 \times 10^5$, which is consistent with the observation in Fig. 5.5 (b-2). Fidelities for other edge states with $\eta = -$ are almost zero during both dynamics. These observations clearly validate the prediction from the linear stability analysis. We note that the abrupt shrink of the intensity distribution in Fig. 5.5 (b-2) or the sharp drop of $F_{0,+}(t)$ in Fig. 5.6 (b) cannot be predicted from the linear stability analysis. This is because the linear stability analysis cannot predict behaviours in the unstable region where fluctuations become dominant, since the analysis assumes weak fluctuations around the edge states, while it can predict stable-unstable transitions. While the additional bifurcation cannot be predicted from the stability analysis explained in Sec. 5.4.1, our procedure explained in Secs. 5.3.2 and 5.4.3 can predict that edge states can be unstable repellers. Then, the additional bifurcation is a unique phenomenon in Floquet systems with nonlinearity since it is revealed by taking the discrete nature of time evolutions into account.

We can also see the uniqueness to discrete time systems from motions of eigenvalues with varying the value of κ . Figure 5.7 shows eigenvalues of $V_s^{(0,+)}$ for various values of κ in the complex plane. The eigenvalues $\lambda_s^{(\varepsilon,\eta)}$ appear as quartets because of symmetries of the nonunitary time-evolution operator $V_s^{(\varepsilon,\eta)}$. Due to the bipartite structure of $V_s^{(\varepsilon,\eta)}$ that wave function amplitudes on even (odd) sites always move to odd (even) sites, sublattice symmetry $\tau_3 V_s^{(\varepsilon,\eta)} \tau_3 = -V_s^{(\varepsilon,\eta)}$ is satisfied where τ_3 is $\sum_x |x\rangle \langle x| (-1)^x$. Also, $V_s^{(\varepsilon,\eta)}$ satisfies $[V_s^{(\varepsilon,\eta)}]^* = V_s^{(\varepsilon,\eta)}$. These two symmetries

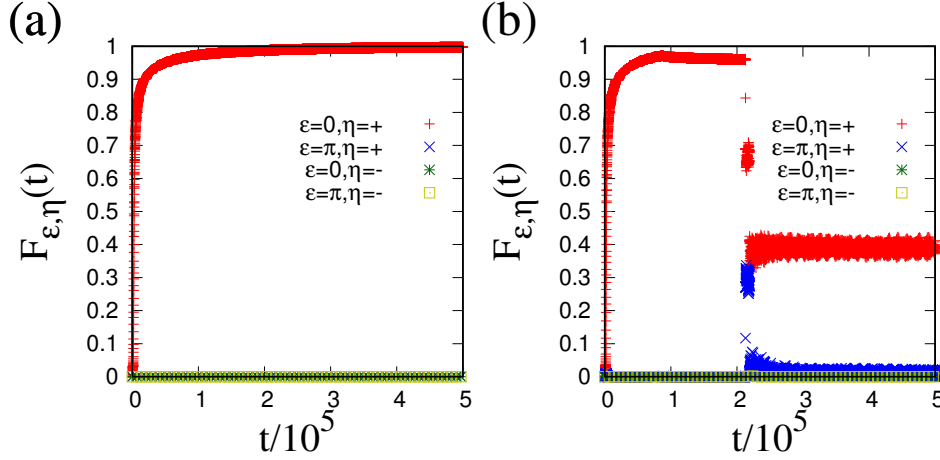


Figure 5.6: Time-step dependence of the fidelity $F_{\epsilon,\eta}(t)$ for edge states $|\Phi_{0,+}\rangle$, $|\Phi_{\pi,+}\rangle$, $|\Phi_{0,-}\rangle$, and $|\Phi_{\pi,0}\rangle$ when parameters are $\theta_0 = \pi/10$, $\Delta^2 = 50$, $m = 150$, and (a) $\kappa = 4.2 < \kappa_c$, (b) $\kappa = 4.5 > \kappa_c$.

guarantee that eigenvalues appear as quartets $\pm\lambda_s^{(\epsilon,\eta)}$ and $\pm(\lambda_s^{(\epsilon,\eta)})^*$. Here, the inside (outside) of the unit circle in the complex plane corresponds to a stable (unstable) region of the edge states. First of all, we mention that absolute values of eigenvalues which are depicted by gray circles in Fig. 5.7 (a-1)-(a-4) are equal to one or slightly less than one. This is because these eigenvalues are exactly same as or very closed to eigenvalues of U_s which are on the unit circle due to the unitarity of U_s by following reasons. The first reason is that the topological edge states of U_s are also eigenstates of $V_s^{(\epsilon,\eta)}$ with the same eigenvalue $e^{-i\epsilon}$, because $D_{\kappa,\eta}(|\Phi_x^{(\epsilon)}\rangle)|\Phi_{\epsilon,\eta}\rangle = |\Phi_{\epsilon,\eta}\rangle$ is satisfied, which can be understood from Eqs. (5.8), (5.20), and (5.21). The second reason is that $\mathcal{D}_{\kappa,\eta}(|\Phi_x^{(\epsilon)}\rangle) \simeq \sigma_0$ unless x is near the boundaries at $x = \pm m$, since $|\Phi_x^{(\epsilon)}|^2$ takes exponentially small values far away from the boundaries. Then, extended bulk states of U_s which we write as $|\phi_b\rangle$ whose eigenvalues are λ_b satisfy $D_{\kappa,\eta}(|\Phi_x^{(\epsilon)}\rangle)|\phi_b\rangle \simeq |\phi_b\rangle$ and $V_s^{(\epsilon,\eta)}|\phi_b\rangle \simeq \lambda_b|\phi_b\rangle$, as the bulk states have small wave function amplitudes around $x = \pm m$. The only exceptions are topologically trivial localized states or equivalently impurity states, which are localized near boundaries but whose quasi-energies are neither 0 nor π . These states take important roles for us to understand the additional bifurcation. We find that the absolute values of eigenvalues which correspond to trivial localized states strongly depend on κ which is the strength of nonlinearity. Increasing κ , the four eigenvalues which are plotted as purple squares flow toward inside of the unit circle and reach on the real axis, as shown in Fig. 5.7 (b-1). Further increasing κ , two eigenvalues (blue squares) go toward the origin, while other two eigenvalues (red triangles) proceed to the opposite direction, which corresponds to Fig. 5.7 (b-2). After two eigenvalues (blue squares) collide and pass through each other at the origin [Fig. 5.7 (b-2)], two pairs of two eigenvalues (red triangles and blue squares) symmetrically flow toward right and left being on the real axis, while additional four eigenvalues (green tilted squares) proceed to the origin [Fig. 5.7 (b-3)]. Again increasing κ , as shown in Fig. 5.7 (b-4), the eigenvalues described as red triangles get out of the unit circle, which makes $|\Phi_{\epsilon,+}\rangle$ unstable. As we have explained, the motions of $\lambda_s^{(0,+)}$ in the complex plane change $|\Phi_{\epsilon,+}\rangle$ from stable attractors to unstable repellers as κ is increased. This additional bifurcation is related to a fact that the stable region of $\lambda_s^{(\epsilon,\eta)}$ is bounded, which is one of peculiar features in Floquet systems (see also Sec. 5.6). Furthermore, as mentioned in Sec. 5.4.1, the linear stability analysis in the continuum limit does not predict the additional bifurcation. These facts suggest that the bifurcation which we have clarified is unique to Floquet systems or equivalently discrete time systems.

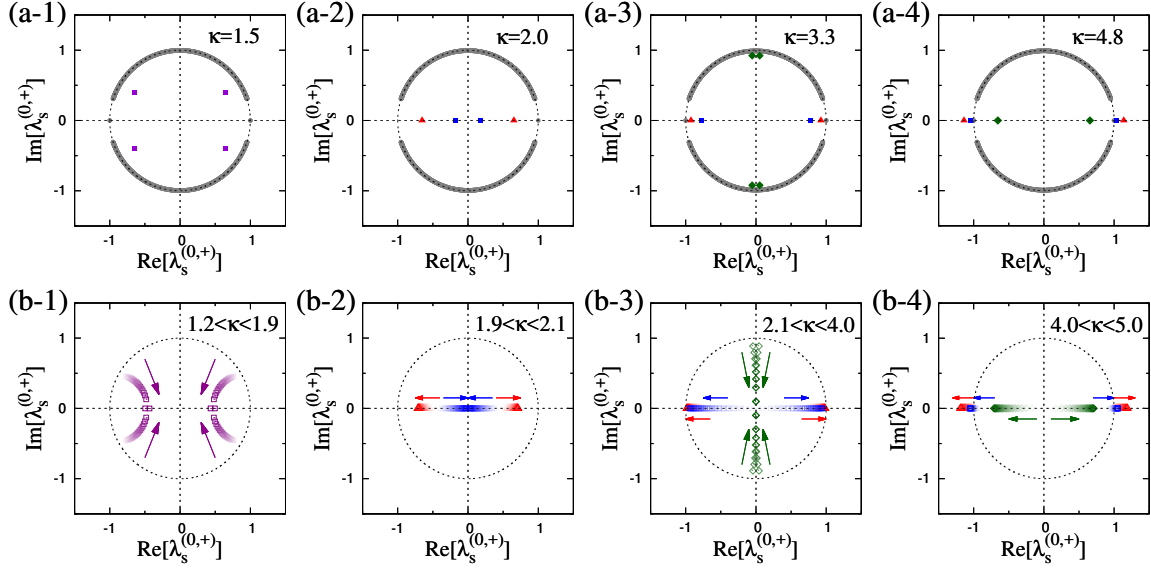


Figure 5.7: Eigenvalues of $V_s^{(0,+)}$ which are written as $\lambda_s^{(0,+)}$ in the complex plane, with $\theta_0 = \pi/10$ and $m = 150$. Circles depicted in dashed lines are the unit circles. In (a-1) $\kappa = 1.5$, (a-2) $\kappa = 2.0$, (a-3) $\kappa = 3.3$, (a-4) $\kappa = 4.8$, all eigenvalues are plotted. (b-1) $1.2 \leq \kappa \leq 1.9$, (b-2) $1.9 \leq \kappa \leq 2.1$, (b-3) $2.1 \leq \kappa \leq 4.0$, and (b-4) $4.0 \leq \kappa \leq 5.0$ show trajectories of particular eigenvalues, which strongly depend on κ and/or contribute to the bifurcation of $|\Phi_{\varepsilon,+}\rangle$. The eigenvalues flow toward the direction of the arrows and the colors are deepened as κ is increased in each figure.

Before closing this subsection, we extensively check the validity of the additional bifurcations predicted from our stability analysis. Figure 5.8 (a) and (b) respectively show $F_{0,+}(T)$ and $F_{\pi,+}(T)$ at $T = 10^7$ for various values of θ_0 and κ . Below the green line which is obtained from Eq. (5.50), the blue (yellow) regions in Fig. 5.8 (a) correspond to the yellow (blue) regions in Fig. 5.8 (b), which means that wave function amplitudes accumulate to $\Phi_{x,\sigma}^{(0,+)}$ or $\Phi_{x,\sigma}^{(\pi,+)}$. This is because both edge states $|\Phi_{0,+}\rangle$ and $|\Phi_{\pi,+}\rangle$ are stable attractors when κ is smaller than κ_c . In other words, since $V_s^{(0,+)} = V_s^{(\pi,+)}$ is satisfied, the linear stability analysis which we have done cannot predict which state is realized after nonlinear time evolutions. It depends on initial states whether wave function amplitudes converge to $\Phi_{x,\sigma}^{(0,+)}$ or $\Phi_{x,\sigma}^{(\pi,+)}$. If we choose larger one among $F_{0,+}(T)$ and $F_{\pi,+}(T)$, we can obtain Fig. 5.9 (b) in which the stable blue region and unstable red region are clearly separated by the green line. From Fig. 5.9 (b), we can understand that the analytical result in Eq. (5.50) precisely predicts the critical value of κ , below (above) which both $|\Phi_{0,+}\rangle$ and $|\Phi_{\pi,+}\rangle$ become stable attractors (unstable repellers) and wave function amplitudes accumulate to either $\Phi_{x,\sigma}^{(0,+)}$ or $\Phi_{x,\sigma}^{(\pi,+)}$ (neither $\Phi_{x,\sigma}^{(0,+)}$ nor $\Phi_{x,\sigma}^{(\pi,+)}$ is realized). Numerically obtained $\max(|\lambda_s^{(0,+)}|)$ is also shown in Fig. 5.9 (a). The conspicuous similarity of Fig. 5.9 (a) and (b) indicates that our stability analysis agrees well with the simulation results of time evolutions. We note that, as shown in Fig. 5.9, κ_c is a decreasing function of θ_0 in $0 < \theta_0 < \pi/2$. This is because $\max_x(|\Phi_x^{(\varepsilon)}|^2)$ is an increasing function of θ_0 as can be understood from Eq. (5.42), due to which κ_c decreases following Eq. (5.50) as θ_0 is increased.

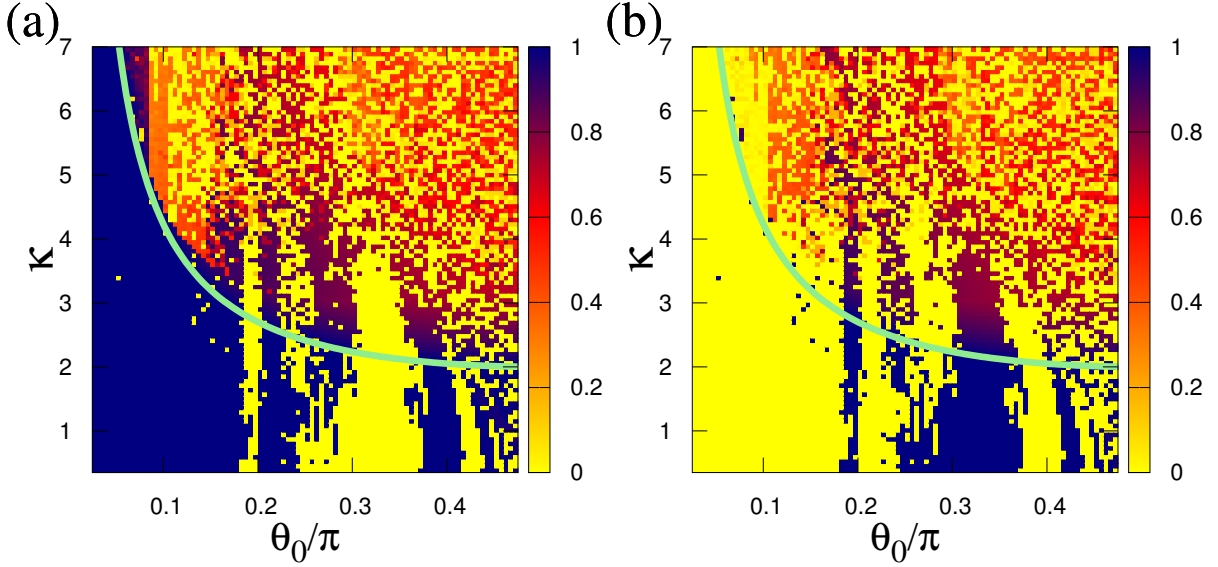


Figure 5.8: (a) $F_{0,+}(T)$ and (b) $F_{\pi,+}(T)$ at $T = 10^7$ with various parameters θ_0 and κ with $m = 100$. The threshold predicted from Eq. (5.50) is shown by the green solid curves. The initial state is given by Eq. (5.17) with $\Delta^2 = 40$.

5.5 Two-step nonlinear quantum walks

In order to further explore bifurcations stemming from the discreteness of time, in addition to the single-step nonlinear quantum walk, we apply our method to the two-step nonlinear quantum walk in Eq. (5.13). We cannot apply the analysis in terms of the continuous Hamiltonian in Ref. [96] to the two-step nonlinear quantum walk. This is because two nonlinear coin operators $C_2(\kappa\Theta)$ and $C_2(-\kappa\tilde{\Theta})$ in Eq. (5.13) cancel out each other up to the first order of $\kappa\Theta(x, t)$ by taking the continuum limit in x and t . Then, not only the stability of edge states but also the dynamics cannot be analysed in the continuum limit. On the contrary, we can both describe the dynamics and predict the stability of edge states precisely, by taking the discreteness of time into account. In the same way as the single-step nonlinear quantum walk, we assume the infinitesimally weak fluctuating state in Eq. (5.15) and substitute it into Eqs. (5.13). Ignoring higher order terms in $\delta\psi_{x,\sigma}(t)$, we can obtain the time-evolution equation of $|\delta\psi(t)\rangle$,

$$|\delta\psi(t+1)\rangle = V_t^{(\varepsilon,\eta)} |\delta\psi(t)\rangle, \quad V_t^{(\varepsilon,\eta)} = U_{t2} D_{-\kappa,\tilde{\eta}}(|\tilde{\Phi}_x^{(\varepsilon)}|) U_{t1} D_{\kappa,\eta}(|\Phi_x^{(\varepsilon)}|), \quad (5.51)$$

where $|\tilde{\Phi}_x^{(\varepsilon)}|$ denotes absolute values of wave function amplitudes of

$$|\tilde{\Phi}_{\varepsilon,\tilde{\eta}}\rangle = U_{t1} |\Phi_{\varepsilon,\eta}\rangle, \quad (5.52)$$

in the same way as Eq. (5.9). The largest value of $|\lambda_t^{(\varepsilon,\eta)}|$, where $\lambda_t^{(\varepsilon,\eta)}$ are eigenvalues of $V_t^{(\varepsilon,\eta)}$, dominates the time evolution in Eq. (5.51) and determines the stability of $|\Phi_{\varepsilon,\eta}\rangle$. While it may be expected that the stability of edge states depend on η , $\tilde{\eta}$, and ε , chirality of $|\tilde{\Phi}_{\varepsilon,\tilde{\eta}}\rangle$ is determined from η and ε . Actually, from the discussion in Sec. 3.3.2, we can understand that $\tilde{\eta}$ becomes

$$\Gamma |\tilde{\Phi}_{\varepsilon,\tilde{\eta}}\rangle = \tilde{\eta} |\tilde{\Phi}_{\varepsilon,\tilde{\eta}}\rangle, \quad \tilde{\eta} = \eta e^{i\varepsilon}, \quad (5.53)$$

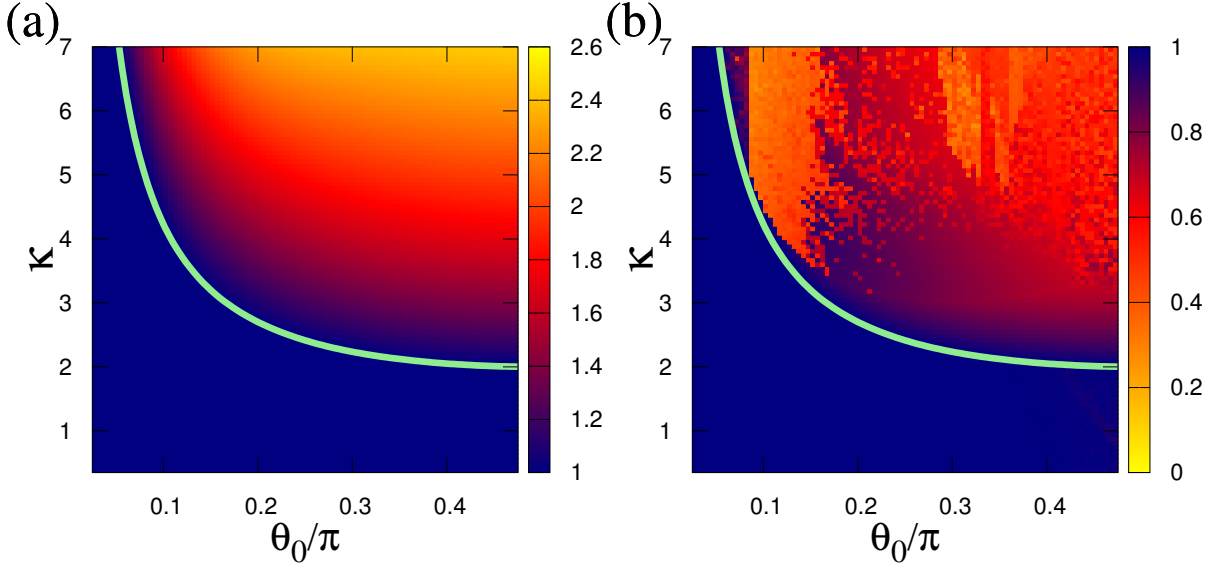


Figure 5.9: θ_0 and κ dependences of (a) $\max(|\lambda_s^{(0,+)}|)$ and (b) $\max[F_{0,+}(T), F_{\pi,+}(T)]$ with $m = 100$ and $T = 10^7$. The threshold predicted from Eq. (5.50) is shown by the green solid curves. The initial state is given by Eq. (5.17) with $\Delta^2 = 40$.

where U_{t1} , U_{t2} , η , $\tilde{\eta}$, $|\Phi_{\varepsilon,\eta}\rangle$, and $|\tilde{\Phi}_{\varepsilon,\tilde{\eta}}\rangle$ in Chap. 5 respectively correspond to U_1 , U_2 , η' , η'' , $|\Phi'_{\varepsilon,\eta}\rangle$, and $|\Phi''_{\varepsilon,\eta}\rangle$ in Sec. 3.3.2. Then, the stability of edge states depends on η and ε , which is different from the single-step nonlinear quantum walk where the stability of edge states depends only on η .

5.5.1 derivation of edge states

In order to clarify the stability of edge states through eigenvalues of $V_t^{(\varepsilon,\eta)}$, we derive analytical forms of topologically protected edge states in two-step quantum walks without nonlinear effects. We assume that two boundaries in Fig. 5.2 are separated enough and edge states localized at different boundaries have no overlap each other. Note that, in the two-step quantum walk, edge states have finite wave function amplitudes only at even sites or odd sites, since wave functions at even (odd) sites always shift to even (odd) sites after one time step in the two-step quantum walk. While we use $|\Phi_{\varepsilon,\eta}\rangle$ at even sites for the linear stability analysis, edge states in odd sites are obtained in the same way and our conclusion does not depend on the even-odd parity. In the inner region where $-m \leq x \leq m$ and $\theta_j(x) = \theta_j$ ($j = 1, 2$), Eq. (5.7) can be written as

$$e^{-i\varepsilon}\Phi_{x,-}^{(\varepsilon,\eta)} = \begin{aligned} &\cos(\theta_1/2)\cos(\theta_2)\alpha_{1,\eta}\Phi_{x+2,-}^{(\varepsilon,\eta)} - \cos(\theta_1/2)\sin(\theta_2)\beta_{1,\eta}\Phi_{x,-}^{(\varepsilon,\eta)} \\ &- \sin(\theta_1/2)\sin(\theta_2)\alpha_{1,\eta}\Phi_{x,-}^{(\varepsilon,\eta)} - \sin(\theta_1/2)\cos(\theta_2)\beta_{1,\eta}\Phi_{x-2,-}^{(\varepsilon,\eta)} \end{aligned} \quad (5.54)$$

$$\eta e^{-i\varepsilon}\Phi_{x,+}^{(\varepsilon,\eta)} = \begin{aligned} &\sin(\theta_1/2)\cos(\theta_2)\alpha_{1,\eta}\Phi_{x+2,+}^{(\varepsilon,\eta)} - \sin(\theta_1/2)\sin(\theta_2)\beta_{1,\eta}\Phi_{x,+}^{(\varepsilon,\eta)} \\ &+ \cos(\theta_1/2)\sin(\theta_2)\alpha_{1,\eta}\Phi_{x,+}^{(\varepsilon,\eta)} + \cos(\theta_1/2)\cos(\theta_2)\beta_{1,\eta}\Phi_{x-2,+}^{(\varepsilon,\eta)} \end{aligned} \quad (5.55)$$

where $\alpha_{j,\eta}$ and $\beta_{j,\eta}$ are defined as

$$\alpha_{j,\eta} = \cos(\theta_j/2) - \eta \sin(\theta_j/2), \quad \beta_{j,\eta} = \sin(\theta_j/2) + \eta \cos(\theta_j/2). \quad (5.56)$$

Considering the edge states localized near the left boundary, we assume

$$\Phi_{x,-}^{(\varepsilon,\eta)} = \eta \Phi_{x,+}^{(\varepsilon,\eta)} = N_t (-1)^{\frac{\varepsilon x}{2\pi}} e^{-\gamma_\varepsilon(x+m)}, \quad (5.57)$$

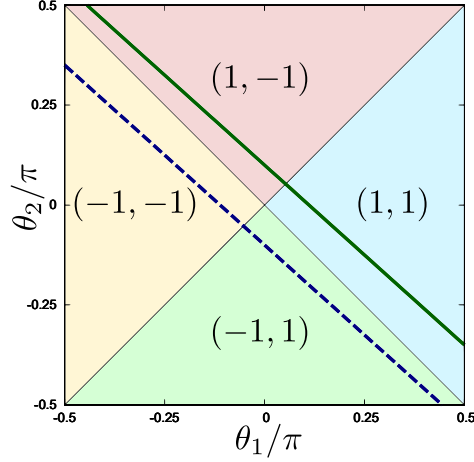


Figure 5.10: θ_1 and θ_2 dependences of topological numbers (ν_0, ν_π) . The green solid line $\theta_2 = -9\theta_1/10 + \pi/10$ and blue dashed line $\theta_2 = -9\theta_1/10 - \pi/10$ are used when the validity of the linear stability analysis is confirmed in Fig. 5.13.

where N_t is the normalization constant. Substituting Eq. (5.57) into Eqs. (5.54) and (5.55), we obtain

$$1 = \frac{\cos(\theta_1/2)[\cos(\theta_2)\alpha_{1,\eta}e^{-2\gamma_\varepsilon} - (-1)^{\frac{\varepsilon}{\pi}}\sin(\theta_2)\beta_{1,\eta}]}{-\sin(\theta_1/2)[(-1)^{\frac{\varepsilon}{\pi}}\sin(\theta_2)\alpha_{1,\eta} + \cos(\theta_2)\beta_{1,\eta}e^{2\gamma_\varepsilon}]} \quad (5.58)$$

$$\eta = \frac{\sin(\theta_1/2)[\cos(\theta_2)\alpha_{1,\eta}e^{-2\gamma_\varepsilon} - (-1)^{\frac{\varepsilon}{\pi}}\sin(\theta_2)\beta_{1,\eta}]}{+\cos(\theta_1/2)[(-1)^{\frac{\varepsilon}{\pi}}\sin(\theta_2)\alpha_{1,\eta} + \cos(\theta_2)\beta_{1,\eta}e^{2\gamma_\varepsilon}]} \quad (5.59)$$

From Eqs. (5.58) and (5.59), we obtain two solutions of γ_ε from each quadratic equation regarding $e^{2\gamma_\varepsilon}$. Among them, we choose one which satisfies both Eqs. (5.58) and (5.59). Then, γ_0 and γ_π become

$$e^{2\gamma_0} = (\alpha_{1,\eta}/\beta_{1,\eta})(\alpha_{2,\eta}/\beta_{2,\eta}), \quad e^{2\gamma_\pi} = (\alpha_{1,\eta}/\beta_{1,\eta})(\beta_{2,\eta}/\alpha_{2,\eta}), \quad (5.60)$$

respectively. In the outer region where $x \leq -m-1$ and $\theta_i(x) = -\theta_i$, γ_0 and γ_π become the same as Eq. (5.60), while Eq. (5.57) is changed to $\Phi_{x,-}^{(\varepsilon,\eta)} = \eta\Phi_{x,+}^{(\varepsilon,\eta)} = N_t(-1)^{\frac{\varepsilon x}{2\pi}}e^{-\gamma_\varepsilon(-m-2-x)}$. For simplicity, we focus only on parameter regions A and B in Fig. 5.10. From Eqs. (5.56) and (5.60), it is known that chirality must be minus for $\varepsilon = 0$, since $\gamma_0 > 0$ should be satisfied. For $\varepsilon = \pi$, $\gamma_\pi > 0$ is satisfied when $\eta = +$, in the parameter region A where $0 < \theta_2 < \pi/2$ and $-\theta_2 < \theta_1 < \theta_2$. In the parameter region B where $0 < \theta_1 < \pi/2$ and $-\theta_1 < \theta_2 < \theta_1$, $\eta = -$ is satisfied for γ_π to be positive. Edge states localized near the right boundary have the opposite chirality, while γ_0 and γ_π have the same values as Eq. (5.60). Near the left boundary, $\Phi_{-m,\sigma}^{(\varepsilon,\eta)}$ satisfies

$$-\sin(\theta_1/2)\cos(\theta_2)\eta\alpha_{1,\eta}\Phi_{-m-2,\sigma}^{(\varepsilon,\eta)} = \frac{[1 + \eta(-1)^{\frac{\varepsilon}{\pi}}\sin(\theta_2)]\Phi_{-m,\sigma}^{(\varepsilon,\eta)}}{-\cos(\theta_1/2)\cos(\theta_2)\alpha_{1,\eta}\Phi_{-m+2,\sigma}^{(\varepsilon,\eta)}} \quad (5.61)$$

Substituting Eqs. (5.57) and (5.60) into Eq. (5.61), the boundary condition of $\Phi_{x,\sigma}^{(0,-)}$ is

$$\cos(\theta_2)\Phi_{-m-2,\sigma}^{(0,-)} = [1 - \sin(\theta_2)]\Phi_{-m,\sigma}^{(0,-)}, \quad (5.62)$$

in both parameter regions. For $|\Phi_{\pi,\eta}\rangle$, taking it into account that η and γ_π are dependent on parameter regions, boundary conditions become

$$\cos(\theta_2)\Phi_{-m-2,\sigma}^{(\pi,+)} = [1 - \sin(\theta_2)]\Phi_{-m,\sigma}^{(\pi,+)} \quad (5.63)$$

in the parameter region A, and

$$[1 - \sin(\theta_2)]\Phi_{-m-2,\sigma}^{(\pi,-)} = \cos(\theta_2)\Phi_{-m,\sigma}^{(\pi,-)} \quad (5.64)$$

in the parameter region B, from Eqs. (5.57), (5.60), and (5.61). The boundary conditions near the right boundary are obtained in the same way.

Explicit forms of edge states are as follows. In both regions A and B, ν_0 has the same value and the analytical form of $|\Phi_{0,\eta}\rangle$ is

$$\Phi_{x,-}^{(0,+)} = \begin{cases} N_t e^{-\gamma_0|x-m|} & (x \leq m) \\ N_t p e^{-\gamma_0|x-(m+2)|} & (x > m) \end{cases} \quad (5.65)$$

$$\Phi_{x,-}^{(0,-)} = \begin{cases} N_t p e^{-\gamma_0|x-(-m-2)|} & (x < -m) \\ N_t e^{-\gamma_0|x-(-m)|} & (x \geq -m) \end{cases} \quad (5.66)$$

where γ_0 , p , and N_t are

$$\gamma_0 = \frac{1}{2} \log \left[\frac{\{1 + \sin(\theta_1)\}\{1 + \sin(\theta_2)\}}{\cos(\theta_1)\cos(\theta_2)} \right], \quad p = \frac{\cos \theta_2}{1 + \sin \theta_2}, \quad N_t = \sqrt{\frac{1 - e^{-4\gamma_0}}{2(1 + p^2)}}. \quad (5.67)$$

Irrespective of the parameter regions A and B, $|\Phi_{0,+}\rangle$ and $|\Phi_{0,-}\rangle$ are localized at the right and left boundaries, respectively. On the other hand, for $\varepsilon = \pi$, the value of ν_π and localization centers of $|\Phi_{\pi,\eta}\rangle$ depend on which parameter region θ_1 and θ_2 exist in. In the region A, $|\Phi_{\pi,\eta}\rangle$ is

$$\Phi_{x,-}^{(\pi,+)} = \begin{cases} N_t p (-1)^{\frac{x}{2}} e^{-\gamma_\pi^A|x-(-m-2)|} & (x < -m) \\ N_t (-1)^{\frac{x}{2}} e^{-\gamma_\pi^A|x-(-m)|} & (x \geq -m) \end{cases} \quad (5.68)$$

$$\Phi_{x,-}^{(\pi,-)} = \begin{cases} N_t (-1)^{\frac{x}{2}} e^{-\gamma_\pi^A|x-m|} & (x \leq m) \\ N_t p (-1)^{\frac{x}{2}} e^{-\gamma_\pi^A|x-(m+2)|} & (x > m) \end{cases} \quad (5.69)$$

where

$$\gamma_\pi^A = \frac{1}{2} \log \left[\frac{\{1 - \sin(\theta_1)\}\{1 + \sin(\theta_2)\}}{\cos(\theta_1)\cos(\theta_2)} \right]. \quad (5.70)$$

Equations (5.68) and (5.69) mean that $|\Phi_{\pi,+}\rangle$ is localized at the left boundary and $|\Phi_{\pi,-}\rangle$ is vice versa, in the region A. In the region B,

$$\Phi_{x,-}^{(\pi,+)} = \begin{cases} N_t p (-1)^{\frac{x}{2}} e^{-\gamma_\pi^B|x-m|} & (x \leq m) \\ N_t (-1)^{\frac{x}{2}} e^{-\gamma_\pi^B|x-(m+2)|} & (x > m) \end{cases} \quad (5.71)$$

$$\Phi_{x,-}^{(\pi,-)} = \begin{cases} N_t (-1)^{\frac{x}{2}} e^{-\gamma_\pi^B|x-(-m-2)|} & (x < -m) \\ N_t p (-1)^{\frac{x}{2}} e^{-\gamma_\pi^B|x-(-m)|} & (x \geq -m) \end{cases} \quad (5.72)$$

where

$$\gamma_\pi^B = \frac{1}{2} \log \left[\frac{\{1 + \sin(\theta_1)\}\{1 - \sin(\theta_2)\}}{\cos(\theta_1)\cos(\theta_2)} \right]. \quad (5.73)$$

Localization centers of $|\Phi_{\pi,+}\rangle$ and $|\Phi_{\pi,-}\rangle$ in the region B are the opposite to those in the region A. In regions A and B, the normalization constants N_t for $|\Phi_{\pi,\pm}\rangle$ are obtained by substituting γ_π^A and γ_π^B into γ_0 in Eq. (5.67), respectively, while p is the same as Eq. (5.67). Contrary to the single-step nonlinear quantum walk in Sec. 5.4, $\max(|\lambda_t^{(\varepsilon,\eta)}|)$ depends on quasi-energies ε , since the localization lengths of $|\Phi_{0,\eta}\rangle$ and $|\Phi_{\pi,\eta}\rangle$ are different. Note that we can obtain $|\Phi_{\varepsilon,\eta}\rangle$ in odd sites by changing m , $m+2$, and p to $m-1$, $m+1$, and p^{-1} respectively, in Eqs. (5.65)-(5.72).

5.5.2 derivation of the thresholds

We derive the thresholds of bifurcations in the two-step nonlinear quantum walk where the edge states $|\Phi_{\varepsilon,\eta}\rangle$ become unstable, by using properties of the spectral radius and the spectral norm. In addition to Eq. (5.43), we also use an inequality for square matrices B and C ,

$$\sigma(BC) \leq \sigma(B)\sigma(C). \quad (5.74)$$

Substituting $V_t^{(\varepsilon,\eta)} = U_{t2}D_{-\kappa,\tilde{\eta}}(|\tilde{\Phi}_x^{(\varepsilon)}|)U_{t1}D_{\kappa,\eta}(|\Phi_x^{(\varepsilon)}|)$ into A in Eq. (5.43) and using Eq. (5.74), we obtain

$$\max(|\lambda_t^{(\varepsilon,\eta)}|) \leq \sigma[D_{\kappa,\eta}(|\Phi_x^{(\varepsilon)}|)]\sigma[D_{-\kappa,\tilde{\eta}}(|\tilde{\Phi}_x^{(\varepsilon)}|)], \quad (5.75)$$

since the spectral norms of the unitary operators U_{t1} and U_{t2} are always equal to 1. The eigenvalues of $\mathcal{D}_{\kappa,\eta}(|\Phi_x^{(\varepsilon)}|)$ and $\mathcal{D}_{-\kappa,\tilde{\eta}}(|\tilde{\Phi}_x^{(\varepsilon)}|)$ are

$$\delta_1 = 1, \delta_2(x) = 1 - 4\kappa\eta|\Phi_x^{(\varepsilon)}|^2, \quad \tilde{\delta}_1 = 1, \tilde{\delta}_2(x) = 1 + 4\kappa\tilde{\eta}|\tilde{\Phi}_x^{(\varepsilon)}|^2, \quad (5.76)$$

respectively. From Eqs. (5.75) and (5.76), the upper bound of $\max(|\lambda_t^{(\varepsilon,\eta)}|)$ is given by

$$\max(|\lambda_t^{(\varepsilon,\eta)}|) \leq \max_x(\delta_1, |\delta_2(x)|) \max_x(\tilde{\delta}_1, |\tilde{\delta}_2(x)|). \quad (5.77)$$

From Eqs. (5.53), (5.76), and (5.77), regarding $(\varepsilon, \eta) = (0, \pm)$ and $(\pi, -)$, we can understand that $\max_x(\delta_1, |\delta_2(x)|)$ and/or $\max_x(\delta_1, |\tilde{\delta}_2(x)|)$ are larger than one for positive κ . Thereby, the edge states $|\Phi_x^{(0,\pm)}\rangle$ and $|\Phi_x^{(\pi,-)}\rangle$ can be unstable for any positive κ . Regarding the edge state with $(\varepsilon, \eta) = (\pi, +)$, it is inevitably stable as long as $\max_x(|\delta_2(x)|) \leq 1$ and $\max_x(|\tilde{\delta}_2(x)|) \leq 1$ or equivalently $\max_x(\delta_1, |\delta_2(x)|) = \max_x(\delta_1, |\tilde{\delta}_2(x)|) = 1$ are satisfied, which means

$$\max(|\lambda_t^{(\pi,+)}|) \leq 1. \quad (5.78)$$

When one of $\max_x(|\delta_2(x)|)$ and $\max_x(|\tilde{\delta}_2(x)|)$ is larger than one, the upper bound of $\max(|\lambda_t^{(\pi,+)}|)$ becomes

$$\max(|\lambda_t^{(\pi,+)}|) \leq |1 - 4\kappa \max_x(|\Phi_x^{(\pi)}|^2, |\tilde{\Phi}_x^{(\pi)}|^2)| (> 1) \quad (5.79)$$

and $|\Phi_{\pi,+}\rangle$ can be unstable. Therefore, from the condition that $\max_x(|\delta_2(x)|)$ or $\max_x(|\tilde{\delta}_2(x)|)$ becomes one, the threshold of the stable-unstable transition is given by

$$\kappa_c = \frac{1}{2 \max_x(|\Phi_x^{(\pi)}|^2, |\tilde{\Phi}_x^{(\pi)}|^2)}. \quad (5.80)$$

Although Eq. (5.77) gives only the upper bound of $\max(|\lambda_t^{(\varepsilon,\eta)}|)$, numerical calculations that we show later guarantee that the inequality properly predicts the thresholds for the present model. Thereby, we again summarize the results obtained from Eqs. (5.77)-(5.80) here:

1. The edge states $|\Phi_{0,\eta=\pm}\rangle$ and $|\Phi_{\pi,-}\rangle$ are unstable for any $\kappa > 0$, since the right hand side of Eq. (5.77) is always larger than one and $\max(|\lambda_t^{(\varepsilon,\eta)}|) > 1$ is satisfied.
2. In the case of $|\Phi_{\pi,\eta=+}\rangle$, the edge state is inevitably stable as long as both $\max_x(|\delta_2(x)|)$ and $\max_x(|\tilde{\delta}_2(x)|)$ are smaller than or equal to one which is satisfied when κ is smaller than a critical value κ_c in Eq. (5.80). For $\kappa > \kappa_c$, $\max(|\lambda_t^{(\pi,+)}|) > 1$ is satisfied since $\max_x(|\delta_2(x)|)$ and/or $\max_x(|\tilde{\delta}_2(x)|)$ is larger than one and a transition from a stable attractor to an unstable repeller occurs.

As we show in the following, the bifurcation of $|\Phi_{\pi,+}\rangle$ predicted above is also unique to Floquet systems, while there is no analysis ignoring the discreteness of time for comparison with our stability analysis.

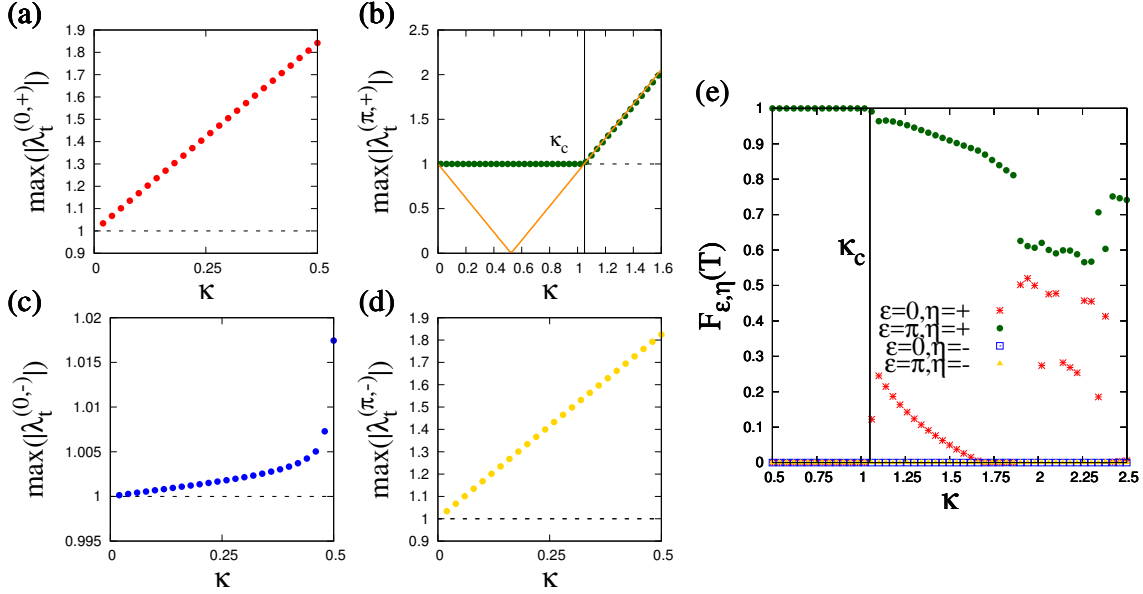


Figure 5.11: (a)-(d) $\max(|\lambda_t^{(\varepsilon,\eta)}|)$ and (e) the fidelity $F_{\varepsilon,\eta}(T)$ at $T = 10^8$, with $\theta_1 = 5\pi/13$, $\theta_2 = -\pi/7$, $\Delta^2 = 10$, and $m = 150$. The black dashed lines in (a)-(d) represent $\max(|\lambda_t^{(\varepsilon,\eta)}|) = 1$, and the orange solid line in (d) represents $|1 - 4\kappa\eta \max_x(|\Phi_x^{(\varepsilon)}|^2, |\tilde{\Phi}_x^{(\varepsilon)}|^2)|$. Note that, $|\psi(0)\rangle$ and $|\Phi_{\varepsilon,\eta}\rangle$ have amplitudes only at even sites.

5.5.3 numerical confirmation

Now, we numerically confirm the validity of the above analytical results. To this end, we calculate eigenvalues of $V_t^{(\varepsilon,\eta)}$ by substituting the analytical solutions of edge states in Eqs. (5.65)-(5.73) into Eq. (5.51) and carrying out numerical diagonalizations of $V_t^{(\varepsilon,\eta)}$. Figure 5.11 (a)-(d) show κ dependence of the maximum value of $|\lambda_t^{(\varepsilon,\eta)}|$ for four kinds of edge states under specific conditions. We clearly observe that $\max(|\lambda_t^{(\varepsilon,\eta)}|)$ for edge states $|\Phi_x^{0,+}\rangle$, $|\Phi_x^{0,-}\rangle$, and $|\Phi_x^{\pi,-}\rangle$ become larger than one for any small $\kappa > 0$. However, $\max(|\lambda_t^{(\pi,+)}|)$ for the edge state $|\Phi_x^{\pi,+}\rangle$ remains to be one as long as $\kappa < \kappa_c$, and starts to increase with increasing κ further. When $\theta_1 = 5\pi/13$ and $\theta_2 = -\pi/7$, the value of κ_c is estimated as $\kappa_c \approx 1.05$ from Eq. (5.80), which correctly predicts the threshold as shown in Fig. 5.11 (b). These observations are consistent with the analytically derived predictions from Eq. (5.77). We verify the above result of the stability analysis by numerically simulating the time evolution of the nonlinear quantum walk in Eq. (5.13). The fidelity $F_{\varepsilon,\eta}(T)$ at a time step $T = 10^8$ is shown in Fig. 5.11 (e). As expected from $\max(|\lambda_t^{(\varepsilon,\eta)}|)$ in Fig. 5.11 (a)-(d), $F_{\pi,+}(T)$ is almost one (much smaller than one) for $\kappa \leq \kappa_c$ ($\kappa > \kappa_c$), while $F_{0,+}(T)$, $F_{0,-}(T)$, and $F_{\pi,-}(T)$ are always much smaller than one.

In order to show that the bifurcation discussed above is peculiar to discrete time systems, we explore details for motions of $\lambda_t^{(\pi,+)}$ which are eigenvalues of $V_t^{(\pi,+)}$. Figure 5.12 exhibits the κ dependences of $\lambda_t^{(\pi,+)}$ in the complex plane. Eigenvalues depicted by gray circles are slightly inside (complex values) or just on (real values) the unit circle, of which the former and latter respectively; corresponds to extended bulk states and topological edge states in the linear regime, in the same way as the case of the single-step quantum walk. Increasing κ , one real eigenvalue described as the green tilted square flows toward the origin, while four complex eigenvalues corresponding to purple squares and pink pentagons proceed to the real axis, as shown in Fig. 5.12 (b-1). Further

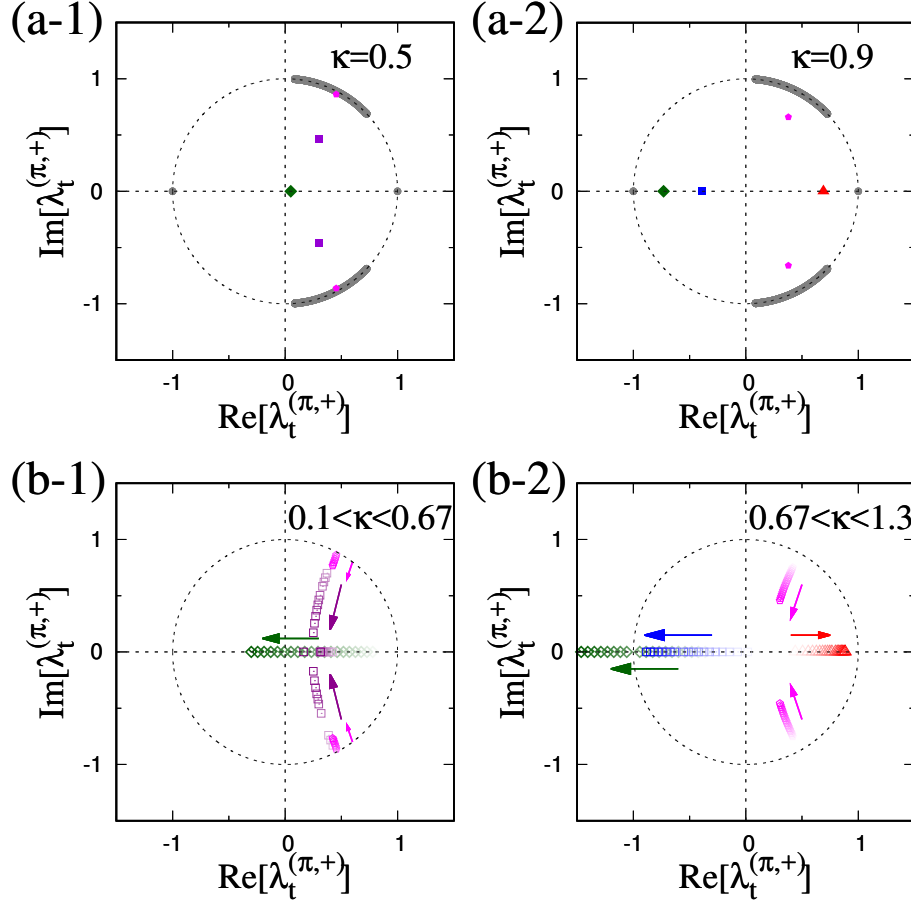


Figure 5.12: Eigenvalues of $V_t^{(\pi,+)}$ which are written as $\lambda_t^{(\pi,+)}$ in the complex plane, with $\theta_1 = 5\pi/13$, $\theta_2 = -\pi/7$, and $m = 150$. The circles depicted in dashed lines are the unit circles. All eigenvalues are shown with (a-1) $\kappa = 0.5$ and (a-2) $\kappa = 0.9$. In (b-1) $0.1 \leq \kappa \leq 0.67$ and (b-2) $0.67 \leq \kappa \leq 1.3$, trajectories of eigenvalues which strongly depend on κ and/or mainly contribute to the bifurcation of $|\Phi_{\pi,+}\rangle$ are plotted. The motions of eigenvalues with increasing κ are represented by the directions of the arrows and the depth of colors.

increasing κ , two eigenvalues which are the green tilted square and blue square flow toward the negative side and go beyond the unit circle, as shown in Fig. 5.12 (b-2). This motion of eigenvalues establishes κ dependences of $\max(|\lambda_t^{(\pi,+)}|)$ in Fig. 5.11. The edge state $|\Phi_{\pi,+}\rangle$ can become unstable with an eigenvalue flowing along a monotonous trajectory which is depicted as green tilted squares, because the stable region of $\lambda_t^{(\pi,+)}$ is bounded inside the unit circle. Since the boundedness of the stable region is due to the discreteness of time, the bifurcation of $|\Phi_{\pi,+}\rangle$ in the nonlinear two-step quantum walk is one of the peculiar phenomena in Floquet systems with nonlinearity, which is detailed in Sec. 5.6.

Finally, we present a comprehensive result to verify the additional bifurcation in the two-step nonlinear quantum walk. We consider that the parameters in the inner (outer) region shown in Fig. 5.2 sweep the green solid (blue dashed) line as shown in Fig. 5.10. Figure 5.13 (a) and (b) show $\max(|\lambda_t^{(\pi,+)}|)$ and $F_{\pi,+}(T)$ at $T = 7 \times 10^6$, respectively, for various θ_1 , accordingly $\theta_2 = -9\theta_1/10 + \pi/10$, and κ . Comparing Fig. 5.13 (a) with (b), it is obvious that the stability analysis based on $\max(|\lambda_t^{(\pi,+)}|)$ gives the correct prediction, excepting near $\theta_1 = \theta_2 = \pi/19$

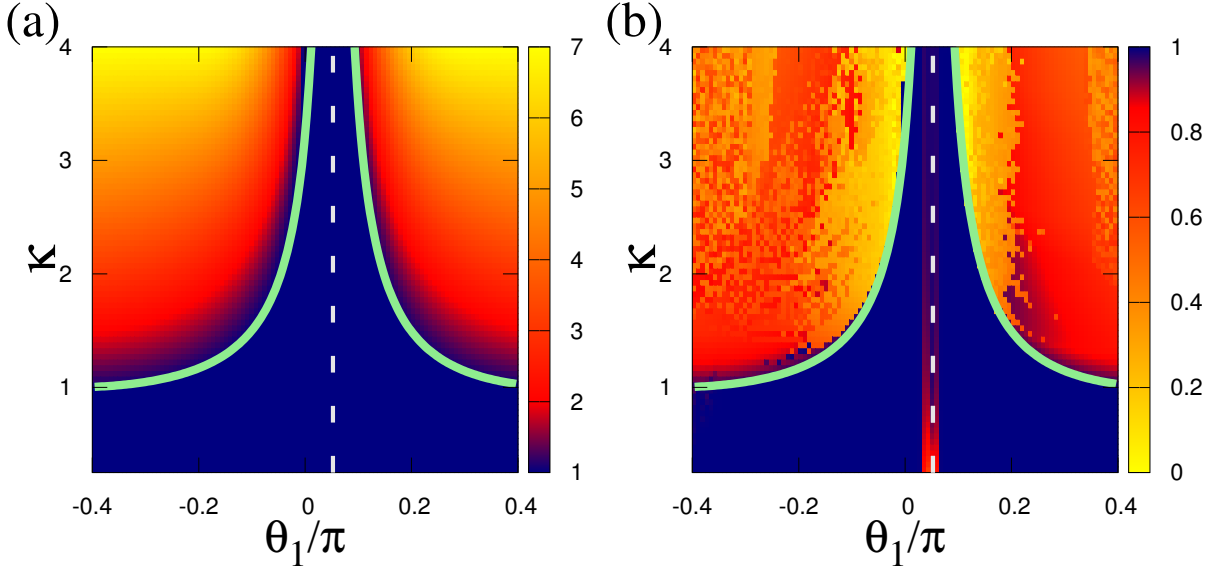


Figure 5.13: θ_1 and κ dependences of (a) $\max(|\lambda_t^{(\pi,+)}|)$ and (b) $F_{\pi,+}(T)$ with $m = 100$, $T = 7 \times 10^6$, and $\Delta^2 = 50$. Note that, $|\psi(0)\rangle$ and $|\Phi_{\pi,+}\rangle$ have amplitudes only at odd sites. In both pictures (a) and (b), rotation angles θ_1 and θ_2 in the inner region (outer region) are scanned along the green solid line $\theta_2 = -9\theta_1/10 + \pi/10$ (blue dashed line $\theta_2 = -9\theta_1/10 - \pi/10$) in Fig. 5.10, avoiding points at which $\theta_1 = 0$, $\theta_2 = 0$, or $\theta_1 = \theta_2$. White dashed lines represent $\theta_1 = \theta_2 = \pi/19$ at which $|\Phi_{\pi,+}\rangle$ does not exist. The green lines represent κ_c as functions of θ_1 , which are obtained from Eq. (5.80).

at which the topological number ν_π is not defined and the edge states $|\Phi_{\pi,\eta}\rangle$ do not exist. In addition, analytically derived thresholds κ_c which are represented as the green lines agree well with the numerically obtained thresholds which are the boundaries of the red and blue regions. The reason why numerical results of $\max(|\lambda_t^{(\pi,+)}|)$ and $F_{\pi,+}(T)$ do not agree when $\theta_1 = \theta_2$ is as follows. By putting $\theta_1 = \theta_2$ into Eqs. (5.70) and (5.73), we obtain $\gamma_\pi^{A/B} = 0$, which means that the localization length of the edge states diverges. Even near this exceptional parameter $\theta_1 = \theta_2 = \pi/19$, $\Phi_{x,\sigma}^{(\pi,\eta)}$ have a huge localization length. Then, edge states localizing at left and right boundaries could largely overlap each other, and the assumption that $|\Phi_{\pi,+}\rangle$ and $|\Phi_{\pi,-}\rangle$ are independent is not suitable. Therefore, we can shrink this exceptional region by making the system size larger. Note that, when $|\Phi_{\pi,+}\rangle$ is stable and the initial state $|\psi(0)\rangle$ in Eq. (5.17) has amplitudes at both even and odd sites with the equal weight, $F_{\pi,+}(T)$ for large T is almost 0.5, since $|\Phi_{\pi,+}\rangle$ has amplitudes only at even or odd sites. Then, we need to replace $|\Phi_{\pi,+}\rangle$ with $|\Phi_{\pi,+}\rangle/\sqrt{2}$ in order to calculate $\lambda_t^{(\pi,+)}$, which makes κ_c become $2\kappa_c$.

5.6 Conclusion and discussion

In this work, we have shown that the stability of topologically protected edge states in the single and two-step nonlinear quantum walks can be predicted by the linear stability analysis, precisely taking the dynamical properties of Floquet systems into consideration. Contrary to the previous work [96], we have analytically found the additional bifurcations depending on the strength of nonlinearity for both nonlinear quantum walks by directly applying time-evolution operators of

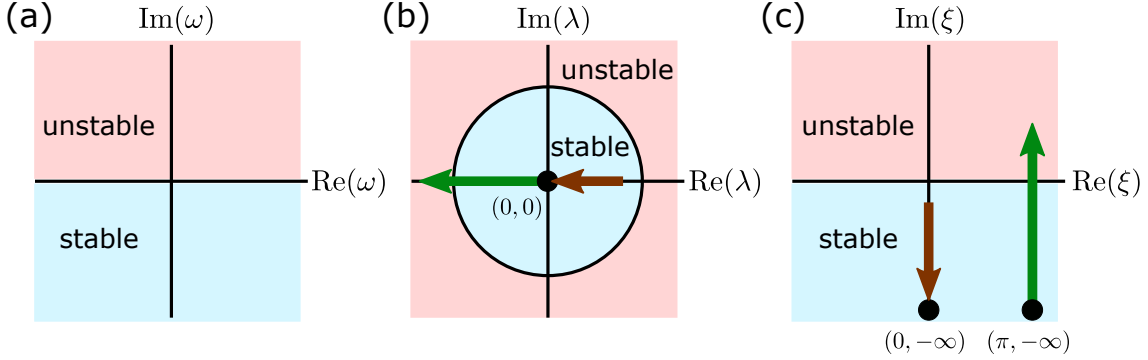


Figure 5.14: Schematic pictures of stable and unstable regions in the complex plane, in the case of (a) static systems which obey continuous time evolutions and (b),(c) discrete time systems. In (b) and (c), brown and green arrows represent the motion of an eigenvalue of a time-evolution operator and the corresponding quasi-energy $\xi = i \log(\lambda)$, respectively.

the quantum walks to the linear stability analysis. Then, we have confirmed the validity of the analysis by numerical calculations.

The bifurcations which we have clarified are unique to discrete time systems, which we can understand from several viewpoints. First is the comparison to the analysis in the continuum limit. In the single-step nonlinear quantum walk addressed in Sec. 5.4, the additional bifurcations cannot be predicted from a Dirac equation where x and t are approximated to take continuous values. In the two-step nonlinear quantum walk addressed in Sec. 5.5, nonlinear dynamics cannot be described in the continuous limit since nonlinear effects vanish in the obtained continuum time equation. On the other hand, our analysis on the nonlinear dynamics and the stability of edge states can predict the bifurcations where edge states become unstable repellers. Thereby it explicitly shows the peculiarity to Floquet systems that the bifurcations are revealed from taking the discrete nature of time evolutions into consideration. Second is the motion of eigenvalues. From the viewpoint of eigenvalues, there is a fundamental difference for the stability analysis between static systems and Floquet systems, in which linearized equations of fluctuations are described by static "Hamiltonians" and discrete time-evolution operators, respectively. On one hand, in static systems, eigenvalues ω of non-Hermitian operators such as Ω in Eq. (5.23) determine the stability of stationary states. On the other hand, in discrete time systems, eigenvalues of nonunitary time-evolution operators such as $V_s^{(\varepsilon, \eta)}$ and $V_t^{(\varepsilon, \eta)}$ respectively in Eqs. (5.19) and (5.51) determine the stability of stationary states. In the former cases, a spectral region which makes a stationary state stable is unbounded as shown in Fig. 5.14 (a), while the stable region is bounded in the latter cases, which is surrounded by the unstable region as shown in 5.14 (b). This difference makes stationary states of Floquet systems more fragile. To clearly see this, we consider a simplified situation of the two-step nonlinear quantum walk in Sec. 5.5 by focusing on one eigenvalue, say λ , described by the green tilted square in Fig. 5.12. When the eigenvalue flows along the brown arrow as parameters are varied, the speed in which fluctuations attenuate becomes fast and the stationary state is stabilized. However, after the eigenvalue goes across the origin which corresponds to the black circle in Fig. 5.14 (b), the eigenvalue flows along the green arrow and the speed of attenuation becomes slow, which results in the collapse of the stationary state when the eigenvalue enters the unstable region. This peculiarity is more highlighted by introducing the quasi-energy defined from λ as $\xi = i \ln(\lambda)$. The quasi-energy ξ plays a similar role to ω , since the stability of stationary states in static and Floquet systems depends on the sign of $\text{Im}(\omega)$ and $\text{Im}(\xi)$ as shown in Fig. 5.14 (a) and (c), respectively. Figure 5.14 (c) shows the trajectory of ξ corresponding to

λ in Fig. 5.14 (b). When λ goes across the origin, ξ jumps from $(0, -\infty)$ to $(\pi, -\infty)$ and the direction of the flow is reversed, which makes ξ enter the unstable region. The discontinuity of ξ originates from the phase singularity of λ at the origin, and such a discontinuity never occurs in systems where t takes continuous values. While we need to consider more complicated situations for the single-step nonlinear quantum walk, the boundedness of the stable region is of importance in both nonlinear quantum walks. Since the boundedness stems from the discreteness of time, the additional bifurcations of edge states that we have shown in the present work are unique to Floquet systems.

In the single-step nonlinear quantum walk, the "Hamiltonian" $\tilde{Q}(q)$ in the continuum limit is non-Hermitian as in Eq. (5.25), while our analysis is based on time-evolution operators. As mentioned in Chap. 4, systems described by non-Hermitian Hamiltonians with $\mathcal{PT}$ symmetry have been explored enthusiastically [114, 115, 116, 117, 118, 119, 120, 121, 122, 124, 130, 141, 144, 145, 146, 147, 86, 150, 151, 152, 155, 229, 230, 157, 156, 215, 158, 159], where $\mathcal{P}$ and $\mathcal{T}$ represent parity and time-reversal, respectively. As shown in Sec. 5.4.1, the "Hamiltonian" in the continuum limit has $\mathcal{PT}$ symmetry. Although $\mathcal{PT}$ symmetry does not influence the stability of edge states in the single-step nonlinear quantum walk, $\mathcal{PT}$ symmetry generally can have large effects on the stability of stationary states in nonlinear systems. Then, it may be interesting to study relations between the stability of stationary states and $\mathcal{PT}$ symmetry breaking in other nonlinear systems.

While we have studied bifurcations of topological edge states in specific models of nonlinear quantum walks, the similar bifurcations in which the direction of stabilization is flipped to that of destabilization with monotonous changes of parameters can generally occur in other discrete time nonlinear systems. Such nonlinear systems have been explored in which time-evolution equations have periodicity and/or limit cycles emerge [231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241]. Then, it should be interesting to discuss the same type of bifurcations and apply our method based on inequalities of the spectral radius and the spectral norm to other discrete time systems where stationary states can be unstable due to the boundedness of the stable region. We note that quantum walks are defined by simple time-evolution operators, which makes it possible to derive the thresholds analytically from the method. It depends on how nonlinear terms are included in time-evolution equations, whether the spectral inequalities such as Eqs. (5.43) and (5.74) can be utilized in effective manners.

Chapter 6

Topological Phases from Chiral and/or Time-Glide Symmetries

6.1 Introduction

As mentioned in Chap. 3, many aspects of symmetry protected topological phases in quantum walks have been understood from the analogy to static systems. Actually, the topological classification of Floquet gapped phases with non-spatial AZ symmetry is essentially the same as the static counterpart [242, 243]. Based on the analogy, a variety of Floquet engineering has been proposed theoretically and carried out experimentally, in which features of systems are manipulated by periodic external driving fields [244, 245, 246, 247, 248, 249, 250, 251, 252, 109, 253, 242, 254, 255, 243, 256, 257, 258, 108, 259, 260]. On the other hand, Floquet systems including quantum walks can have distinct features from static systems in both gapped and gapless phases [109, 246, 54, 110, 261]. For instance, anomalous gapless states with quasi-energy windings are possible in both edge and bulk due to the periodicity in quasi-energy spectra [109, 261]. Furthermore, recently, it was pointed out that there exist non-trivial topological phases protected by space-time symmetries unique to time-periodic systems such as time-glide symmetry and time-screw symmetry [110, 262], although the experimental realization of such systems has not been done yet.

Quantum walks provide versatile platforms for exploring topological phenomena [48, 49, 50, 51, 53, 54, 55, 56, 58, 59, 60, 61, 229, 62, 63, 215, 64, 65, 66, 67, 68, 69, 70, 72, 73, 74, 263, 75, 264], which has been mentioned in the previous chapters. Quantum walks are realizable in various physical settings, where the topological numbers can be easily tuned and the exact descriptions of time evolutions are easily obtained. Owing to these features, various topological edge states have been observed in quantum walks [51, 215, 70, 72], which include ones unique to Floquet systems, such as edge states whose quasi-energies are π in one-dimensional quantum walks [62, 72] and anomalous edge states with quasi-energy windings in two-dimensional quantum walks [70]. While quantum walks have experimental feasibility as mentioned above, their space-time symmetries and the corresponding symmetry protected topological phases have not been discussed yet.

In this work, we study quantum walks with chiral symmetry and/or time-glide symmetry, of which the latter is one of space-time symmetries unique to Floquet systems [110]. While topological numbers which stem from chiral symmetry were obtained from the discussions about the bulk-edge correspondence [53] as explained in Sec. 3.3.2, we derive chiral winding numbers following a different procedure. Then, based on the derived winding numbers, we clarify that quantum walks with chiral symmetry are distinct from other systems which are static systems described by time-independent Hamiltonians and/or Floquet systems described by time-periodic Hamiltonians. In addition, following the definition in Eq. (3.31), we identify chiral symmetry as time-reflection symmetry, which enables us to treat chiral symmetry and time-glide symmetry in a unified man-

ner. By explicitly identifying topological numbers, we present topological classification of quantum walks in the presence of both chiral symmetry and time-glide symmetry. We reveal that there can be topological phases protected by a combined symmetry of time-glide and chiral symmetries which we refer to as time-glide chiral symmetry. We also study a model of two-dimensional quantum walks which has both symmetries. While the constructed model is not fully gapped, the system hosts a variety of edge states protected by these symmetries. We clarify relevant topological numbers for dispersive edge states protected by time-glide chiral symmetry and flat bands protected by chiral symmetry. Through simulations of the time evolutions, we also find that anomalous edge states protected by time-glide chiral symmetry can be detected from probability distributions of walkers since edge states are well-localized on the boundary even in the presence of bulk gapless modes. In addition, we reveal that chiral winding numbers which result in flat bands can be understood from topological charges of Dirac points in the same way as Dirac semimetals, which is discussed in quantum walks for the first time.

The rest of this chapter is organized as follows. In Sec. 6.2, we derive winding numbers protected by chiral symmetry, based on the unitarity of time-evolution operators. We compare quantum walks with other systems which are static systems and Floquet systems respectively described by time-independent and time-periodic Hamiltonians in Sec. 6.3, clarifying features of quantum walks both common to and distinct from such systems. In Sec. 6.4, we give the definition of time-glide symmetry, how to construct models which have time-glide symmetry and chiral symmetry, and obtain winding numbers stemming from time-glide chiral symmetry. In Sec. 6.5, constructing a concrete model from the method established in Sec. 6.4, we confirm the bulk-edge correspondence, show that edge states protected by time-glide chiral symmetry can be detected through probability distributions, and find relations between chiral winding numbers and charges of Dirac points. Section 6.6 is conclusion.

6.2 Chiral winding numbers from the constraint of unitarity

The procedure to calculate topological numbers in Sec. 3.3.2 can be used only when both bulk gaps at $\varepsilon = 0$ and $\varepsilon = \pi$ are open. Actually, when the bulk spectrum is gapless at $\varepsilon = 0$ or $\varepsilon = \pi$, the winding numbers of $B'(k)$ and $B''(k)$ in Eq. (3.40) cannot be defined since $B'(k)$ and $B''(k)$ become zero at a momentum on the integral path. On the other hand, even when $\mathcal{H}'(k)$ and $\mathcal{H}''(k)$ do not have a gap at $\varepsilon = 0$ (π), there could be midgap edge states at $\varepsilon = \pi$ (0) [54]. The latter edge states intrinsic to Floquet systems are referred to as anomalous Floquet edge states, since their existence can not be explained by topological structures of effective Hamiltonians. In the following, using the definition of chiral symmetry in Eq. (3.31), we introduce topological numbers relevant to the anomalous midgap edge states.

We first derive general properties of $U' = U_2 U_1$ with position-independent parameters, which satisfies chiral symmetry in Eq. (3.31). In momentum space, we write the first half period of the time-evolution operator as

$$U_1(k) = \begin{pmatrix} \alpha(k) & \beta(k) \\ \gamma(k) & \delta(k) \end{pmatrix}, \quad (6.1)$$

in the basis where Γ is σ_3 as in Eq. (3.35). Then, chiral symmetry yields

$$U_2(k) = \begin{pmatrix} \alpha^\dagger(k) & -\gamma^\dagger(k) \\ -\beta^\dagger(k) & \delta^\dagger(k) \end{pmatrix}, \quad (6.2)$$

and thus $U(k)$ is given by

$$U'(k) = \begin{pmatrix} \alpha^\dagger(k)\alpha(k) - \gamma^\dagger(k)\gamma(k) & \alpha^\dagger(k)\beta(k) - \gamma^\dagger(k)\delta(k) \\ -\beta^\dagger(k)\alpha(k) + \delta^\dagger(k)\gamma(k) & -\beta^\dagger(k)\beta(k) + \delta^\dagger(k)\delta(k) \end{pmatrix}. \quad (6.3)$$

Since $\mathcal{U}_1(k)$ satisfies the unitarity $\mathcal{U}_1^\dagger(k)\mathcal{U}_1(k) = \mathcal{U}_1(k)\mathcal{U}_1^\dagger(k) = 1$, we also have

$$\alpha^\dagger(k)\alpha(k) + \gamma^\dagger(k)\gamma(k) = \beta^\dagger(k)\beta(k) + \delta^\dagger(k)\delta(k) = 1, \quad (6.4)$$

$$\alpha^\dagger(k)\beta(k) + \gamma^\dagger(k)\delta(k) = \alpha(k)\gamma^\dagger(k) + \beta(k)\delta^\dagger(k) = 0, \quad (6.5)$$

which lead to

$$\mathcal{U}'(k) - 1 = \begin{pmatrix} -2\gamma^\dagger(k)\gamma(k) & 2\alpha^\dagger(k)\beta(k) \\ -2\beta^\dagger(k)\alpha(k) & -2\beta^\dagger(k)\beta(k) \end{pmatrix}, \quad (6.6)$$

$$\mathcal{U}'(k) + 1 = \begin{pmatrix} 2\alpha^\dagger(k)\alpha(k) & 2\alpha^\dagger(k)\beta(k) \\ -2\beta^\dagger(k)\alpha(k) & 2\delta^\dagger(k)\delta(k) \end{pmatrix}. \quad (6.7)$$

From Eqs. (6.6) and (6.7) we can prove that $\mathcal{U}'(k)$ has quasi-energy spectrum gaps at $\varepsilon = 0$ and $\varepsilon = \pi$ respectively if and only if $\det[\gamma(k)]\det[\beta(k)] \neq 0$ and $\det[\alpha(k)]\det[\delta(k)] \neq 0$. In other words, $\mathcal{U}'(k)$ has an eigenstate with $\varepsilon_k = 0$ ($\varepsilon_k = \pi$) if and only if $\det[\gamma(k)]\det[\beta(k)] = 0$ ($\det[\alpha(k)]\det[\delta(k)] = 0$). Consider an eigenstate $|\phi_{0\pm}(k)\rangle$ of $\mathcal{U}'(k)$ at a momentum k with the quasi-energy $\varepsilon_k = 0$ and chirality $\eta' = \pm$. From Eq. (6.6), the eigen equation of $|\phi_{0\pm}(k)\rangle$ can be written as

$$|\phi_{0+}(k)\rangle = \begin{pmatrix} \xi_{0+}(k) \\ 0 \end{pmatrix}, \quad |\phi_{0-}(k)\rangle = \begin{pmatrix} 0 \\ \xi_{0-}(k) \end{pmatrix}, \quad [\mathcal{U}'(k) - 1] |\phi_{0\pm}(k)\rangle = 0. \quad (6.8)$$

Therefore, the gap at $\varepsilon = 0$ is closed if and only if

$$\gamma^\dagger(k)\gamma(k)\xi_{0+}(k) = \beta^\dagger(k)\alpha(k)\xi_{0+}(k) = 0 \text{ or } \beta^\dagger(k)\beta(k)\xi_{0-}(k) = \alpha^\dagger(k)\beta(k)\xi_{0-}(k) = 0 \quad (6.9)$$

is satisfied with non-trivial $\xi_{0\pm}(k)$. In both cases of Eq. (6.9), the latter conditions $\det[\beta^\dagger(k)\alpha(k)] = 0$ and $\det[\alpha^\dagger(k)\beta(k)] = 0$ respectively follow from the former ones $\det[\gamma(k)] = 0$ and $\det[\beta(k)] = 0$ with Eq. (6.5). Thus, we respectively have $|\phi_{0+}(k)\rangle$ and $|\phi_{0-}(k)\rangle$ if and only if $\det[\gamma(k)] = 0$ and $\det[\beta(k)] = 0$. Combining these results, we find that the gap at $\varepsilon = 0$ is closed if and only if $\det[\gamma(k)]\det[\beta(k)] = 0$. The procedure to derive conditions for closing the other gap at $\varepsilon = \pi$ is the same as that of $\varepsilon = 0$. Considering eigen equations of $|\phi_{\pi\pm}(k)\rangle$ from Eq. (6.7),

$$|\phi_{\pi+}(k)\rangle = \begin{pmatrix} \xi_{\pi+}(k) \\ 0 \end{pmatrix}, \quad |\phi_{\pi-}(k)\rangle = \begin{pmatrix} 0 \\ \xi_{\pi-}(k) \end{pmatrix}, \quad [\mathcal{U}'(k) + 1] |\phi_{\pi\pm}(k)\rangle = 0, \quad (6.10)$$

we have the conditions

$$\alpha^\dagger(k)\alpha(k)\xi_{\pi+}(k) = \beta^\dagger(k)\alpha(k)\xi_{\pi+}(k) = 0 \text{ or } \delta^\dagger(k)\delta(k)\xi_{\pi-}(k) = \alpha^\dagger(k)\beta(k)\xi_{\pi-}(k) = 0 \quad (6.11)$$

for closing the gap at $\varepsilon = \pi$. In both cases of $\eta' = \pm$, the former conditions $\det[\alpha(k)] = 0$ and $\det[\delta(k)] = 0$ respectively lead to the latter conditions $\det[\beta^\dagger(k)\alpha(k)] = 0$ and $\det[\alpha^\dagger(k)\beta(k)] = 0$ from Eq. (6.5). Then, an eigenstate of $\mathcal{U}'(k)$ with $\varepsilon_k = \pi$ exists if and only if $\det[\alpha(k)]\det[\delta(k)] = 0$ is satisfied.

Summarizing the above results, the conditions to close gaps become

$$\det[\beta(k)]\det[\gamma(k)] = 0 \text{ for } \varepsilon_k = 0, \quad \det[\alpha(k)]\det[\delta(k)] = 0 \text{ for } \varepsilon_k = \pi. \quad (6.12)$$

Equation (6.12) implies that if the system hosts a gap at $\varepsilon = 0$ ($\varepsilon = \pi$), $\beta(k)$ and $\gamma(k)$ [$\alpha(k)$ and $\delta(k)$] are elements of $GL(n, \mathbb{C})$. Then, we can topologically characterize systems by two winding numbers for each gap,

$$\nu[\beta] \text{ and } \nu[\gamma] \text{ for the gap at } \varepsilon = 0, \quad \nu[\alpha] \text{ and } \nu[\delta] \text{ for the gap at } \varepsilon = \pi, \quad (6.13)$$

where $\nu[\alpha]$, $\nu[\beta]$, $\nu[\gamma]$, and $\nu[\delta]$ are defined by Eq. (2.10). When there are boundaries, midgap edge states are expected to exist in the gap at $\varepsilon = 0$ ($\varepsilon = \pi$) if $\nu[\gamma]$ and/or $\nu[\beta]$ ($\nu[\alpha]$ and/or

$\nu[\delta]$) is non-zero. When both bulk gaps at $\varepsilon = 0$ and π are open, we can obtain relations between ν_0, ν_π in Eq. (3.54) and $\nu[\alpha], \nu[\beta], \nu[\gamma], \nu[\delta]$ in Eq. (6.13). From Eqs. (6.1) and (6.2), $B'(k)$ and $B''(k)$ in Eqs. (3.37) and (3.38) which are respectively (1, 2) components of $\mathcal{U}'(k) = \mathcal{U}_2(k)\mathcal{U}_1(k)$ and $\mathcal{U}''(k) = \mathcal{U}_1(k)\mathcal{U}_2(k)$ can be written as

$$B'(k) = \alpha^\dagger(k)\beta(k) - \gamma^\dagger(k)\delta(k), \quad B''(k) = -\alpha(k)\gamma^\dagger(k) + \beta(k)\delta^\dagger(k). \quad (6.14)$$

In addition, arbitrary matrices $A(k)$ and $B(k)$ in $GL(n, \mathbb{C})$ generally satisfy

$$\nu[A^\dagger] = -\nu[A], \quad \nu[AB] = \nu[A] + \nu[B]. \quad (6.15)$$

Combining Eqs. (6.5), (6.14), and (6.15), winding numbers of $B'(k)$ and $B''(k)$ become

$$\nu' = \nu[B'] = -\nu[\alpha] + \nu[\beta], \quad \nu'' = \nu[B''] = \nu[\beta] - \nu[\delta]. \quad (6.16)$$

A condition that $\mathcal{U}_1(k)$ is unitary in Eq. (6.5) also puts a constraint on winding numbers,

$$\nu[\alpha] - \nu[\beta] - \nu[\gamma] + \nu[\delta] = 0. \quad (6.17)$$

Equations (3.54), (6.16), and (6.17) lead to

$$\nu_0 = \frac{\nu[\beta] - \nu[\gamma]}{2}, \quad \nu_\pi = \frac{\nu[\delta] - \nu[\alpha]}{2}. \quad (6.18)$$

From Eq. (6.18), we can understand that a linear combination of $\nu[\beta]$ and $\nu[\gamma]$ ($\nu[\alpha]$ and $\nu[\delta]$) leads to the existence of topologically protected edge states at $\varepsilon = 0$ (π). Roles of other linear combinations such as $(\nu[\beta] + \nu[\gamma])/2$ and $(\nu[\alpha] + \nu[\delta])/2$ have not been revealed.

6.3 Comparison to other systems

Chiral symmetric quantum walks have distinct features not only from static systems described by time-independent Hamiltonians but also from Floquet systems described by time-periodic Hamiltonians.

6.3.1 static systems

As mentioned in Sec. 3.1, when a quasi-energy spectrum has a gap at $\varepsilon = \varepsilon_b$, we can define an effective Hamiltonian by taking the branch cut at $\varepsilon = \varepsilon_b$. The definition of effective Hamiltonians in Eq. (3.19) originates from the analogy to static systems described by time-independent Hamiltonians, since states at t in each system are

$$\text{quantum walks: } |\psi(t)\rangle = U^t |\psi(0)\rangle, \quad \text{static systems: } |\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle \quad (6.19)$$

respectively, where t takes integer (continuous) values in the former (latter) case. From Eq. (6.19), quantum walks are often referred to as stroboscopic simulators of quantum systems described by time-independent Hamiltonians. On the contrary, there are phenomena beyond the analogy to such systems. Quantum walks have two distinct features from static systems, as we show in the following.

First is the number of gaps. We mention the reason why a quasi-energy gap is needed for defining an effective Hamiltonian as in Eq. (3.19). This is because quasi-energies have 2π periodicity. When $\varepsilon_k \bmod 2\pi$ takes all values in $(-\pi, \pi]$, the spectrum can violate the continuity imposed from the periodicity of Brillouin zone in static systems. In other words, when a gap of ε_k exists at $\varepsilon = \varepsilon_b$, the spectrum of the effective Hamiltonian $\mathcal{H}(k)$ with the branch cut at ε_b inevitably becomes a smooth and periodic function of k in the same way as static systems. Therefore, when the quasi-energy spectrum is gapless, structures or phenomena which are not analogous to static systems can exist

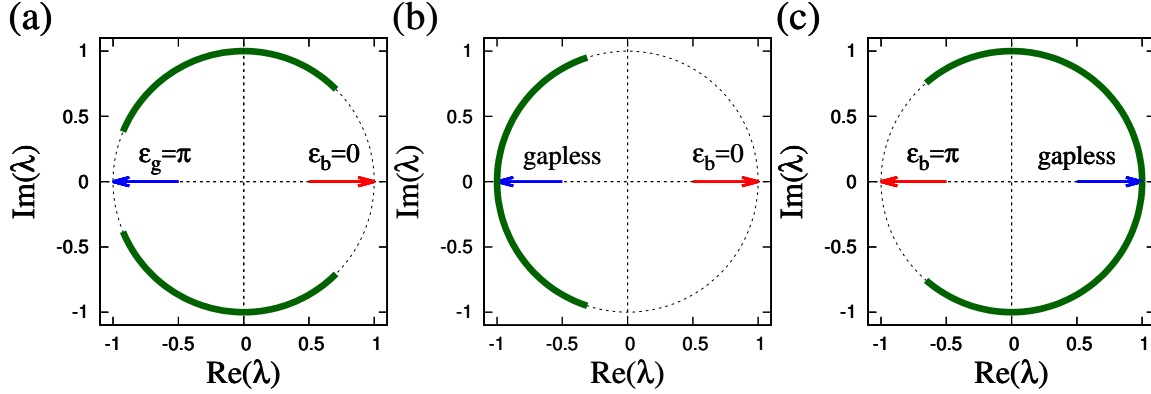


Figure 6.1: Schematics of eigenvalues of time-evolution operators, $\lambda_k = e^{-i\varepsilon_k}$. In (a), the corresponding effective Hamiltonian is gapped at $\varepsilon_g = \pi$ with the branch cut at $\varepsilon_b = 0$, or vice versa. In the cases of (b) and (c), the effective Hamiltonians become gapless respectively at $\varepsilon_g = \pi$ and $\varepsilon_g = 0$, even though the quasi-energy spectra are gapped at branch cuts $\varepsilon_b = 0$ and $\varepsilon_b = \pi$.

in quantum walks. In addition, even when a gap at $\varepsilon = \varepsilon_b$ exists, there are topological numbers which have no counterpart in static systems. Winding numbers $\nu[\alpha]$, $\nu[\beta]$, $\nu[\gamma]$, and $\nu[\delta]$ are such topological numbers. This is because only one quasi-energy gap is needed for the definition of the winding numbers. When an effective Hamiltonian is defined with a branch cut at ε_b , there are two possibilities regarding the spectrum of $\mathcal{H}(k)$. One is that $\mathcal{H}(k)$ has a gap at $\varepsilon = \varepsilon_g (\neq \varepsilon_b)$ and the other is that $\mathcal{H}(k)$ is gapless in terms of $\varepsilon_g (\neq \varepsilon_b)$ as shown in Fig. 6.1. In the former case, we can calculate winding numbers from gapped Hamiltonians $\mathcal{H}(k)$. Actually, the procedure to obtain ν' and ν'' in Eq. (3.40) is based on gapped effective Hamiltonians in the same way as static topological matters. While winding numbers obtained from gapped effective Hamiltonians are not directly related to the number of edge states with $\varepsilon = 0$ or π , the method is similar to that in static systems described by time-independent gapped Hamiltonians. On the other hand, when the spectrum of $\mathcal{H}(k)$ is gapless in terms of ε_g , we cannot rely on winding numbers from gapped Hamiltonians based on the analogy to static systems. If the gap at $\varepsilon_b = 0$ is open, $\nu[\beta]$ and $\nu[\gamma]$ are intrinsic to Floquet systems since these winding numbers are defined even when the gap at $\varepsilon_g = \pi$ is closed. In the same way, $\nu[\alpha]$ and $\nu[\delta]$ are peculiar to Floquet systems, since only one quasi-energy gap at $\varepsilon_b = \pi$ is needed and $\mathcal{H}(k)$ can be gapless at $\varepsilon_g = 0$.

Second distinct feature is the number of topological invariants. When there are two gaps at $\varepsilon = 0$ and $\varepsilon = \pi$, we have four winding numbers $\nu[\alpha]$, $\nu[\beta]$, $\nu[\gamma]$, and $\nu[\delta]$, as shown in Eq. (6.13). Then, we can construct four topological invariants from linear combinations of them, of which two predict the numbers of topological edge states as in Eq. (6.18). While the constraint in Eq. (6.17) from the unitarity of $\mathcal{U}_1(k)$ reduces the kinds of topological invariants which are winding numbers and linear combinations of them, there is a topological invariant different from ν_0 and ν_π . The other topological invariant independent of ν_0 and ν_π is intrinsic to quantum walks, since it has no counterpart in static systems. What is caused by the other topological invariant is still unknown.

6.3.2 Floquet systems with microscopic Hamiltonians

In addition to the difference from static systems explained in Sec. 6.3.1, quantum walks also have uniqueness in comparison to conventional Floquet systems where time evolves in a continuous

manner. Such Floquet systems are described by time-periodic Hamiltonians with period T ,

$$\mathcal{H}(k, t) = \mathcal{H}(k, t + T). \quad (6.20)$$

From Hamiltonians which satisfy Eq. (6.20), time-evolution operators are constructed as

$$\mathcal{U}(k, \tau_1 \rightarrow \tau_2) = \mathbb{T} \exp[-i \int_{\tau_1}^{\tau_2} dt \mathcal{H}(k, t)], \quad (6.21)$$

where $\mathbb{T}$ is the time-ordering operator. Time-evolution operators for one period or equivalently Floquet operators $\mathcal{U}(k, 0 \rightarrow T)$ describe stroboscopic time evolutions at $t = nT$ ($n = 0, 1, 2, \dots$) in the same way as discrete time-evolution operators in quantum walks. From eigenvalues of Floquet operators, quasi-energies ε_k^F are defined as

$$\mathcal{U}(k, 0 \rightarrow T) |\phi_k^F\rangle = \lambda_k |\phi_k^F\rangle, \quad \lambda_k = e^{-i\varepsilon_k^F T}. \quad (6.22)$$

The definition of quasi-energies in conventional Floquet systems as in Eq. (6.22) is the same as the definition of quasi-energies in quantum walks as in Eq. (3.14) with $T = 1$. Also, if a Floquet operator has a gap at a quasi-energy, we can define a Floquet Hamiltonian in the same way as an effective Hamiltonian from a time-evolution operator in quantum walks. Thereby quantum walks are often considered as being equivalent to Floquet systems with microscopic Hamiltonians. However, quantum walks do not always have microscopic Hamiltonians which generate corresponding time-evolution operators, while Floquet operators are constructed from microscopic Hamiltonians which describe continuous time evolutions in $t \in [nT, (n+1)T]$. Among elemental operators included in time-evolution operators of quantum walks, there are two types which are operators constructed from time-dependent Hamiltonians and operators with no microscopic Hamiltonians. The former operators are $\mathcal{C}_2(\theta) = e^{-i\theta\sigma_2}$ and $\mathcal{S}(k) = e^{ik\sigma_3}$, for example. Actually, they are constructed from the Hamiltonian similar to SSH model,

$$H(t) = \sum_x i h_1(t) |x, +\rangle \langle x, -| + i h_2(t) |x+1, +\rangle \langle x, -| + \text{h.c.} \quad (6.23)$$

$$= \sum_k |k\rangle \langle k| \otimes \mathcal{H}(k, t), \quad \mathcal{H}(k, t) = h_2(t) \sin(k) \sigma_1 + [h_1(t) + h_2(t) \cos(k)] \sigma_2, \quad (6.24)$$

with real $h_1(t)$ and $h_2(t)$. The coin operator $\mathcal{C}_2(\theta)$ is made from the Hamiltonian in Eq. (6.24),

$$\mathcal{C}_2(\theta) = e^{-i\theta\sigma_2} = \mathcal{U}(k, \tau_0 \rightarrow \tau_1), \quad (6.25)$$

by varying $h_1(t)$ and $h_2(t)$ as

$$h_1(t) = \frac{\theta}{\tau_1 - \tau_0}, \quad h_2(t) = 0. \quad (6.26)$$

The shift operator $\mathcal{S}(k)$ can also be made from $\mathcal{H}(k, t)$,

$$\mathcal{S}(k) = e^{ik\sigma_3} = e^{-i\frac{\pi}{2}[\sin(k)\sigma_1 + \cos(k)\sigma_2]} e^{i\frac{\pi}{2}\sigma_2} = \mathcal{U}(k, \tau_1 \rightarrow \tau_2) \mathcal{U}(k, \tau_0 \rightarrow \tau_1), \quad (6.27)$$

where the time dependences of $h_1(t)$ and $h_2(t)$ become

$$h_1(t) = \begin{cases} \frac{-\pi}{2(\tau_1 - \tau_0)} & \text{for } \tau_0 \leq t < \tau_1 \\ 0 & \text{for } \tau_1 \leq t \leq \tau_2 \end{cases} \quad h_2(t) = \begin{cases} 0 & \text{for } \tau_0 \leq t \leq \tau_1 \\ \frac{\pi}{2(\tau_2 - \tau_1)} & \text{for } \tau_1 < t \leq \tau_2 \end{cases} \quad (6.28)$$

On the other hand, shift operators $\mathcal{S}_{\pm}(k)$

$$\mathcal{S}_+(k) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-ik} \end{pmatrix}, \quad \mathcal{S}_-(k) = \begin{pmatrix} e^{ik} & 0 \\ 0 & 1 \end{pmatrix} \quad (6.29)$$

cannot be constructed from time-dependent Hamiltonians. This is because $\mathcal{S}_\pm(k)$ have non-zero winding numbers,

$$\nu[\mathcal{S}_\pm] = \frac{1}{2\pi i} \int_{-\pi}^{\pi} dk \frac{d}{dk} \ln \det[\mathcal{S}_\pm(k)] = \mp 1. \quad (6.30)$$

Floquet operators made from microscopic Hamiltonians cannot have non-zero winding numbers. This is because such Floquet operators can always be deformed to trivial one,

$$\mathcal{U}(k, \tau_1 \rightarrow \tau_2) \simeq \mathcal{U}(k, \tau_1 \rightarrow \tau_1) = 1, \quad (6.31)$$

without passing across points at which $\mathcal{U}(k, \tau_1 \rightarrow \tau_2)$ are ill defined. Therefore, operators with non-zero winding numbers cannot be constructed from time-dependent Hamiltonians, in general. The reason why $\mathcal{C}_2(\theta)$ and $\mathcal{S}(k)$ are obtained from $\mathcal{H}(k, t)$ in Eq. (6.24) is that their winding numbers $\nu[\mathcal{C}_2]$ and $\nu[\mathcal{S}]$ are zero.

It makes quantum walks distinct from Floquet systems described by time-dependent Hamiltonians that time-evolution operators of quantum walks can have non-zero winding numbers owing to the absence of microscopic Hamiltonians. In order to clearly see the effect from Eq. (6.31) onto winding numbers, we consider 2×2 time-evolution operators. We can define chiral symmetry of conventional Floquet systems in the same way as Eq. (3.31),

$$\Gamma \mathcal{U}(k, 0 \rightarrow \tau) \Gamma^{-1} = \mathcal{U}^\dagger(k, \tau \rightarrow T). \quad (6.32)$$

If Eq. (6.32) is satisfied, we can obtain winding numbers based on the discussion in Sec. 6.2. However, in such cases, only one winding number is obtained for each gap, while two winding numbers exist for each gap in quantum walks as shown in Eq. (6.13). This is because Eq. (6.31) results in

$$\nu[\mathcal{U}(0 \rightarrow \tau)] = \frac{1}{2\pi i} \oint dk \frac{d}{dk} \ln[\alpha(k)\delta(k) - \beta(k)\gamma(k)] = \nu[\alpha] + \nu[\delta] = \nu[\beta] + \nu[\gamma] = 0 \quad (6.33)$$

with $\det[\mathcal{U}(k, 0 \rightarrow \tau)] = \delta(k)/\alpha^*(k) = -\beta(k)/\gamma^*(k)$ which stems from the unitarity of $\mathcal{U}(k, 0 \rightarrow \tau)$, in the basis of Eqs. (3.35) and (6.1) where $\mathcal{U}_1(k)$ is replaced with $\mathcal{U}(k, 0 \rightarrow \tau)$. From Eqs. (6.17), (6.18), and (6.33), we can obtain

$$\nu[\alpha] = -\nu[\delta], \quad \nu[\beta] = -\nu[\gamma] \quad \rightarrow \quad \nu_0 = \nu[\beta], \quad \nu_\pi = \nu[\delta]. \quad (6.34)$$

Then, two winding numbers for each gap in Eq. (6.13) reduce to one winding number and Floquet systems with microscopic Hamiltonians can have only two kinds of topological invariants. On the other hand, replacing $\mathcal{U}(0 \rightarrow \tau)$ with $\mathcal{U}_1$, Eq. (6.33) can be broken in quantum walks, while Eq. (6.17) is satisfied as long as $\mathcal{U}_1(k)$ possesses unitarity. As a result, quantum walks can have three kinds of topological invariants as linear combinations of $\nu[\alpha]$, $\nu[\beta]$, $\nu[\gamma]$, and $\nu[\delta]$ with Eq. (6.17). Thereby one of these topological invariants cannot exist not only in static systems but also in Floquet systems described by microscopic Hamiltonians. While the above discussion is about 2×2 time-evolution operators in momentum space, Eq. (6.31) generally results in

$$\nu[\mathcal{U}(0 \rightarrow \tau)] = \tilde{\nu}[\alpha] + \tilde{\nu}[\beta] + \tilde{\nu}[\gamma] + \tilde{\nu}[\delta] = 0 \quad (6.35)$$

with $\ln \det[\mathcal{U}(k, 0 \rightarrow \tau)] = \text{tr} \ln[\mathcal{U}(k, 0 \rightarrow \tau)]$, where $\alpha(k)$, $\beta(k)$, $\gamma(k)$, and $\delta(k)$ need not be scalars and $\tilde{\nu}[\alpha]$ is defined as

$$\tilde{\nu}[\alpha] = \frac{1}{2\pi i} \oint dk \text{tr}[\alpha^\dagger(k) \frac{d}{dk} \alpha(k)]. \quad (6.36)$$

Although $\tilde{\nu}[\alpha]$ does not correspond to the winding number of $\alpha(k)$ in general, Eq. (6.35) is expected to put a constraint on $\alpha(k)$, $\beta(k)$, $\gamma(k)$, and $\delta(k)$. When these winding numbers are

unitary matrices, Eq. (6.35) becomes identical to Eq. (6.33) with Eq. (6.17), since $\tilde{\nu}[\alpha]$, $\tilde{\nu}[\beta]$, $\tilde{\nu}[\gamma]$, and $\tilde{\nu}[\delta]$ respectively correspond to $\nu[\alpha]$, $\nu[\beta]$, $\nu[\gamma]$, and $\nu[\delta]$.

The above discussions infer that the topological number unique to quantum walks has a certain relation with the winding number of time-evolution operators, which is actually the case if $\mathcal{U}_1(k)$ is a 2×2 matrix. While we do not address phenomena which stem from the peculiar topological number in this doctoral thesis, we comment that boundary conditions are important for exploring what is caused by the unique topological number. This is because non-trivial winding numbers of time-evolution operators are due to those of $S_{\pm}$, to the best of our knowledge. Constructing time-evolution operators which include S_+ and/or S_- , their winding numbers are usually independent of parameters of coin operators, which we write as θ . Topological phenomena in quantum walks have been observed by spatially varying θ on which ν_0 and ν_{π} depend in many cases. Then, when the value of θ has no effect on the unique topological number different from ν_0 , ν_{π} and resulting phenomena, such conventional methods in which θ is manipulated do not lead to the detection of the peculiar topological phenomena. One of other methods for the detection is to impose boundary conditions on shift operators, not on coin operators. While it is difficult to impose open or reflecting boundary conditions on $S_{\pm}$ preserving their unitarity, it should be interesting to explore various boundary conditions and resulting phenomena. Such studies may reveal roles of the unique topological number with no counterpart in Floquet systems described by microscopic Hamiltonians.

6.4 Time-glide symmetry and topological phases

In 2017, time-glide symmetry was introduced for Floquet systems described by microscopic Hamiltonians [110]. The definition of time-glide symmetry on time-periodic Hamiltonians is

$$\mathcal{G}_T \mathcal{H}(\mathbf{k}, t) \mathcal{G}_T^{-1} = \mathcal{H}(g\mathbf{k}, t + T/2) \quad (6.37)$$

where $\mathcal{G}_T$ is a unitary and Hermitian operator, T is a period of time $\mathcal{H}(\mathbf{k}, t) = \mathcal{H}(\mathbf{k}, t + T)$, and $g\mathbf{k}$ is defined as

$$g\mathbf{k} = (k_1, \dots, k_{d-1}, -k_d) \quad (6.38)$$

for d -dimensional systems. From Eq. (6.37), we can derive the relation of time-glide symmetry on time-evolution operators $\mathcal{U}(\mathbf{k}, \tau_1 \rightarrow \tau_2) = \mathbb{T} \exp[-i \int_{\tau_1}^{\tau_2} dt \mathcal{H}(\mathbf{k}, t)]$,

$$\mathcal{G}_T \mathcal{U}(\mathbf{k}, 0 \rightarrow T/2) \mathcal{G}_T^{-1} = \mathcal{U}(g\mathbf{k}, T/2 \rightarrow T). \quad (6.39)$$

While microscopic Hamiltonians do not always exist in quantum walks as explained in Sec. 6.3.2, we define time-glide symmetry of quantum walks in a similar way to Eq. (6.39). Time-glide symmetry is defined as

$$\mathcal{G}_T \mathcal{U}_1(\mathbf{k}) \mathcal{G}_T^{-1} = \mathcal{U}_2(g\mathbf{k}), \quad \mathcal{U}(\mathbf{k}) = \mathcal{U}_2(\mathbf{k}) \mathcal{U}_1(\mathbf{k}), \quad (6.40)$$

where $\mathcal{U}(\mathbf{k})$ is the momentum-space representation of the full time-evolution operator. Time-glide symmetry is a peculiar symmetry in Floquet systems since it cannot be defined from effective Hamiltonians in Eq. (3.19). This is because time-glide symmetry is defined on $\mathcal{U}_1(\mathbf{k})$ and $\mathcal{U}_2(\mathbf{k})$ as in Eq. (6.40), not on the full time-evolution operator $\mathcal{U}(\mathbf{k}) = e^{-i\mathcal{H}(\mathbf{k})}$.

6.4.1 construction of models

In Chaps. 4 and 5, we have checked symmetries of given models. On the contrary, we can construct models based on relations of chiral symmetry and time-glide symmetry. To this end, we introduce the concept of time units to quantum walks, while time-dependent Hamiltonians are not defined.

We consider that one time step of quantum walks is carried out in the time unit given by $t \in [0, 1]$. Then, chiral symmetry in Eq. (3.31) implies that Γ flips the direction of time evolutions as in

$$t \rightarrow -t. \quad (6.41)$$

Therefore, chiral symmetry for quantum walks can be regarded as time-mirror symmetry, which is an analogue of mirror reflection symmetry in time direction. Note that time-mirror symmetry has nothing to do with ordinary time-reversal symmetry since Γ is unitary. Taking the role of chiral symmetry into account, we separate the one time unit $t \in [0, 1] \simeq [-1/2, 1/2]$ into two units which are $t \in [0, 1/2]$ and $t \in [1/2, 1] \simeq [-1/2, 0]$. Regarding time-glide symmetry, the original definition in Eq. (6.37) with $T = 1$ indicates that the operation of time-glide symmetry is given by

$$\mathbf{x} \rightarrow g\mathbf{x}, \quad t \rightarrow t + 1/2 \quad (6.42)$$

on space-time, where $g\mathbf{x} = (x_1, \dots, x_{d-1}, -x_d)$. Then, we can separate the time unit $t \in [0, 1]$ into $t \in [0, 1/2]$ and $t \in [1/2, 1]$ for time-glide symmetry. Actually, U_1 and U_2 are respectively responsible for time units $t \in [0, 1/2]$ and $t \in [1/2, 1]$, which we can understand from $U = U_2 U_1$.

According to the concept of time units, we can construct models of quantum walks that have both chiral symmetry in Eq. (3.31) and time-glide symmetry in Eq. (6.40). Time-glide symmetry operator $\mathcal{G}_T$ and chiral symmetry operator Γ satisfy commutation or anti-commutation relation,

$$\Gamma \mathcal{G}_T = s \mathcal{G}_T \Gamma, \quad s = \pm 1. \quad (6.43)$$

From operations on space-time by chiral symmetry and time-glide symmetry, which are Eqs. (6.41) and (6.42) respectively, we can again separate the time unit $t \in [0, 1]$ into four units which are $t \in [0, 1/4]$, $t \in [1/4, 1/2]$, $t \in [1/2, 3/4]$, and $t \in [3/4, 1]$. Corresponding to these time units, we introduce four time-evolution operators $\tilde{\mathcal{U}}_1(\mathbf{k})$, $\tilde{\mathcal{U}}_2(\mathbf{k})$, $\tilde{\mathcal{U}}_3(\mathbf{k})$, and $\tilde{\mathcal{U}}_4(\mathbf{k})$ which construct the first and second periods of one time step,

$$\mathcal{U}_1(\mathbf{k}) = \tilde{\mathcal{U}}_2(\mathbf{k})\tilde{\mathcal{U}}_1(\mathbf{k}), \quad \mathcal{U}_2(\mathbf{k}) = \tilde{\mathcal{U}}_4(\mathbf{k})\tilde{\mathcal{U}}_3(\mathbf{k}), \quad (6.44)$$

where the full time-evolution operator is

$$\mathcal{U}(\mathbf{k}) = \mathcal{U}_2(\mathbf{k})\mathcal{U}_1(\mathbf{k}). \quad (6.45)$$

We can construct models which have both chiral symmetry and time-glide symmetry, by imposing conditions written as

$$\Gamma \tilde{\mathcal{U}}_1(\mathbf{k}) \Gamma^{-1} = \tilde{\mathcal{U}}_4^{-1}(\mathbf{k}), \quad \Gamma \tilde{\mathcal{U}}_2(\mathbf{k}) \Gamma^{-1} = \tilde{\mathcal{U}}_3^{-1}(\mathbf{k}), \quad (6.46)$$

$$\mathcal{G}_T \tilde{\mathcal{U}}_1(\mathbf{k}) \mathcal{G}_T^{-1} = \tilde{\mathcal{U}}_3(g\mathbf{k}), \quad \mathcal{G}_T \tilde{\mathcal{U}}_2(\mathbf{k}) \mathcal{G}_T^{-1} = \tilde{\mathcal{U}}_4(g\mathbf{k}), \quad (6.47)$$

on the introduced time-evolution operators which are responsible for the separated time units. Equations (6.45)-(6.47) ensure that chiral symmetry and time-glide symmetry respectively in Eqs. (3.31) and (6.40) are satisfied. From Eqs. (6.46) and (6.47), $\tilde{\mathcal{U}}_1(\mathbf{k})$ which corresponds to $t \in [0, 1/4]$ can generate the time-evolution operators for other time units which are $\tilde{\mathcal{U}}_2(\mathbf{k})$, $\tilde{\mathcal{U}}_3(\mathbf{k})$, and $\tilde{\mathcal{U}}_4(\mathbf{k})$. Actually, $\tilde{\mathcal{U}}_3(\mathbf{k})$ and $\tilde{\mathcal{U}}_4(\mathbf{k})$ are directly obtained from Eqs. (6.47) and (6.46), respectively. In addition, applying Γ and $\mathcal{G}_T$ to $\tilde{\mathcal{U}}_1(\mathbf{k})$, we can obtain $\tilde{\mathcal{U}}_2(\mathbf{k})$ by

$$\tilde{\mathcal{U}}_2(\mathbf{k}) = (\Gamma \mathcal{G}_T) \tilde{\mathcal{U}}_1^{-1}(g\mathbf{k}) (\Gamma \mathcal{G}_T)^{-1}. \quad (6.48)$$

Thereby what we need to do is only to define Γ , $\mathcal{G}_T$, and $\tilde{\mathcal{U}}_1(\mathbf{k})$ for the construction of a model which has time-glide symmetry and chiral symmetry.

6.4.2 winding numbers

In addition to the time-evolution operator in Eq. (6.45), we can also consider another time-evolution operator in a different time frame,

$$\mathcal{U}'(\mathbf{k}) = \mathcal{U}'_2(\mathbf{k})\mathcal{U}'_1(\mathbf{k}), \quad (6.49)$$

where $\mathcal{U}'_1(\mathbf{k})$ and $\mathcal{U}'_2(\mathbf{k})$ are

$$\mathcal{U}'_1(\mathbf{k}) = \tilde{\mathcal{U}}_1(\mathbf{k})\tilde{\mathcal{U}}_4(\mathbf{k}), \quad \mathcal{U}'_2(\mathbf{k}) = \tilde{\mathcal{U}}_3(\mathbf{k})\tilde{\mathcal{U}}_2(\mathbf{k}). \quad (6.50)$$

Since $\mathcal{U}'(\mathbf{k})$ and $\mathcal{U}(\mathbf{k})$ are related by a unitary transformation $\mathcal{U}'(\mathbf{k}) = \tilde{\mathcal{U}}_4^{-1}(\mathbf{k})\mathcal{U}(\mathbf{k})\tilde{\mathcal{U}}_4(\mathbf{k})$, $\mathcal{U}'(\mathbf{k})$ has the same quasi-energy spectrum as $\mathcal{U}(\mathbf{k})$. As a result, $\mathcal{U}'(\mathbf{k})$ also provides topological numbers for anomalous edge states. From Eqs. (6.46) and (6.47), the first and the second half period parts obey

$$\Gamma\mathcal{U}'_1(\mathbf{k})\Gamma^{-1} = \mathcal{U}'_1{}^{-1}(\mathbf{k}), \quad \Gamma\mathcal{U}'_2(\mathbf{k})\Gamma^{-1} = \mathcal{U}'_2{}^{-1}(\mathbf{k}), \quad (6.51)$$

$$\mathcal{G}_T\mathcal{U}'_2(\mathbf{k})\mathcal{G}_T^\dagger = \mathcal{U}'_1(g\mathbf{k}). \quad (6.52)$$

From Eqs. (6.51) and (6.52), we also obtain

$$(\Gamma\mathcal{G}_T)\mathcal{U}'_1(\mathbf{k})(\Gamma\mathcal{G}_T)^{-1} = \mathcal{U}'_2{}^{-1}(g\mathbf{k}), \quad (6.53)$$

which implies a variation of chiral symmetry. We refer to Eq. (6.53) as time-glide chiral symmetry. On the glide symmetric planes $\mathbf{k} = g\mathbf{k} = \mathbf{k}_G$, time-glide chiral symmetry becomes

$$(\Gamma\mathcal{G}_T)\mathcal{U}'_1(\mathbf{k}_G)(\Gamma\mathcal{G}_T)^{-1} = \mathcal{U}'_2{}^{-1}(\mathbf{k}_G). \quad (6.54)$$

Since Eq. (6.54) is equivalent to chiral symmetry on glide symmetric planes $k_d = 0, \pi$, we can apply the discussion in Sec. 6.2. Therefore, when the spatial dimension is $d = 2$, we can define one-dimensional winding numbers on the glide symmetric planes $\mathbf{k} = \mathbf{k}_G$ from $\mathcal{U}'_1(\mathbf{k}_G)$, based on time-glide chiral symmetry in Eq. (6.54). We refer to these winding numbers as time-glide chiral winding numbers. Note that, while time-glide chiral winding numbers can be defined if d is even ($d - 1$ is odd) in general, we consider the simplest case $d = 2$ in which $k_1 = k_x$ and $k_d = k_2 = k_y$.

Time-glide chiral winding numbers are subject to constraints from remaining symmetries depending on whether Γ and $\mathcal{G}_T$ commute or anti-commute. To see this, we take the basis in which $\Gamma\mathcal{G}_T$ is diagonal,

$$\Gamma\mathcal{G}_T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (6.55)$$

In this basis, we denote $\mathcal{U}'_1(\mathbf{k}_G)$ as

$$\mathcal{U}'_1(\mathbf{k}_G) = \begin{pmatrix} \alpha'(\mathbf{k}_G) & \beta'(\mathbf{k}_G) \\ \gamma'(\mathbf{k}_G) & \delta'(\mathbf{k}_G) \end{pmatrix}. \quad (6.56)$$

Then, from time-glide chiral symmetry in Eq. (6.54), $\mathcal{U}'_2(\mathbf{k}_G)$ is written as

$$\mathcal{U}'_2(\mathbf{k}_G) = \begin{pmatrix} \alpha'^\dagger(\mathbf{k}_G) & -\gamma'^\dagger(\mathbf{k}_G) \\ -\beta'^\dagger(\mathbf{k}_G) & \delta'^\dagger(\mathbf{k}_G) \end{pmatrix}. \quad (6.57)$$

Replacing Γ and $\mathcal{U}_1(k)$ with $\Gamma\mathcal{G}_T$ and $\mathcal{U}'_1(\mathbf{k}_G)$, we can apply Eq. (6.13) in Sec. 6.2 to the present case. Thus, if $\mathcal{U}'(\mathbf{k}_G)$ has a gap at $\varepsilon = 0$ ($\varepsilon = \pi$) and $d - 1 = 1$, we can define the one-dimensional winding numbers $\nu[\beta']$ and $\nu[\gamma']$ ($\nu[\alpha']$ and $\nu[\delta']$) in the one-dimensional glide symmetric subspaces.

Time-glide symmetry in Eq. (6.52) puts constraints on time-glide chiral winding numbers. When $\mathcal{G}_T$ commutes with $\Gamma\mathcal{G}_T$, $\mathcal{G}_T$ can be written as

$$\mathcal{G}_T = \begin{pmatrix} g_- & 0 \\ 0 & g_+ \end{pmatrix}, \quad (6.58)$$

and thus Eq. (6.52) leads to the constraints

$$g_- \alpha'(\mathbf{k}_G) g_-^\dagger = \alpha'^\dagger(\mathbf{k}_G), \quad g_+ \delta'(\mathbf{k}_G) g_+^\dagger = \delta'^\dagger(\mathbf{k}_G), \quad (6.59)$$

$$g_- \beta'(\mathbf{k}_G) g_+^\dagger = -\gamma'^\dagger(\mathbf{k}_G), \quad g_+ \gamma'(\mathbf{k}_G) g_-^\dagger = -\beta'^\dagger(\mathbf{k}_G). \quad (6.60)$$

From Eqs. (6.59) and (6.60), $\nu[\alpha'] = \nu[\delta'] = 0$ and $\nu[\beta'] = -\nu[\gamma']$ are satisfied. Consequently, we have a unique winding number $\nu[\beta'] = -\nu[\gamma']$ if $\mathcal{U}'(\mathbf{k}_G)$ has a gap at $\varepsilon = 0$ and $[\Gamma, \mathcal{G}_T] = 0$. When $\{\Gamma, \mathcal{G}_T\} = 0$, there are different constraints. Since $\Gamma\mathcal{G}_T$ and $\mathcal{G}_T$ anti-commute with each other, we can take $\mathcal{G}_T$ as

$$\mathcal{G}_T = \begin{pmatrix} 0 & g_+ \\ g_- & 0 \end{pmatrix}. \quad (6.61)$$

Then, we have constraints

$$g_+ \delta'(\mathbf{k}_G) g_+^\dagger = \alpha'^\dagger(\mathbf{k}_G), \quad g_- \alpha'(\mathbf{k}_G) g_-^\dagger = \delta'^\dagger(\mathbf{k}_G), \quad (6.62)$$

$$g_+ \gamma'(\mathbf{k}_G) g_-^\dagger = -\gamma'^\dagger(\mathbf{k}_G), \quad g_- \beta'(\mathbf{k}_G) g_+^\dagger = -\beta'^\dagger(\mathbf{k}_G). \quad (6.63)$$

From Eqs. (6.62) and (6.63), we can understand that $\nu[\beta'] = \nu[\gamma'] = 0$ and $\nu[\delta'] = -\nu[\alpha']$ are satisfied. Therefore, in the case of $\{\Gamma, \mathcal{G}_T\} = 0$, the presence of time-glide chiral winding number $\nu[\delta'] = -\nu[\alpha']$ is ensured by the gap at $\varepsilon = \pi$. In short, time-glide chiral winding numbers in each case become

$$\nu[\beta'] \text{ for } \varepsilon = 0 \text{ if } [\Gamma, \mathcal{G}_T] = 0, \quad \nu[\delta'] \text{ for } \varepsilon = \pi \text{ if } \{\Gamma, \mathcal{G}_T\} = 0. \quad (6.64)$$

Originating from time-glide winding numbers in Eq. (6.64), topologically protected edge states are expected to emerge. We show that the expectation is actually the case in the following.

6.5 Time-glide and chiral symmetric 2D model

By constructing a concrete model of two-dimensional quantum walks with time-glide symmetry and chiral symmetry, we explore winding numbers and resulting edge states. Shift operators $S_q^\pm$ ($q = x, y$) on the two-dimensional square lattice $\mathbf{r} = (x, y)$ are written as

$$S_x^\pm = \sum_{\mathbf{r}} |\mathbf{r} \pm \hat{\mathbf{x}}\rangle \langle \mathbf{r}| \otimes |\pm\rangle \langle \pm| + |\mathbf{r}\rangle \langle \mathbf{r}| \otimes |\mp\rangle \langle \mp| = \sum_{\mathbf{k}} |\mathbf{k}\rangle \langle \mathbf{k}| \otimes e^{\mp i \frac{k_x}{2}} \mathcal{S}(k_x/2), \quad (6.65)$$

$$S_y^\pm = \sum_{\mathbf{r}} |\mathbf{r} \pm \hat{\mathbf{y}}\rangle \langle \mathbf{r}| \otimes |\pm\rangle \langle \pm| + |\mathbf{r}\rangle \langle \mathbf{r}| \otimes |\mp\rangle \langle \mp| = \sum_{\mathbf{k}} |\mathbf{k}\rangle \langle \mathbf{k}| \otimes e^{\mp i \frac{k_y}{2}} \mathcal{S}(k_y/2), \quad (6.66)$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the unit vectors in the x and y directions, respectively. Also, from $S_x^\pm$ in Eq. (6.65) we introduce a unitary transformed shift operator

$$\tilde{S}_x^\pm = C_1(\pi/4) S_x^\pm C_1(-\pi/4) = \sum_{\mathbf{k}} |\mathbf{k}\rangle \langle \mathbf{k}| \otimes e^{\mp i \frac{k_x}{2}} \tilde{\mathcal{S}}(k_x/2), \quad \tilde{\mathcal{S}}(k_x/2) = e^{i \frac{k_x}{2} \sigma_2}. \quad (6.67)$$

Following the method to construct time-glide and chiral symmetric models explained in Sec. 6.4.1, we systematically obtain such a model from a unitary operator $\tilde{U}_1$

$$\tilde{U}_1 = C_2(\theta_2/2) S_y^- C_1(-\theta_1/2) \tilde{S}_x^+ \quad (6.68)$$

and symmetry operators given by

$$\mathcal{G}_T = \sigma_2, \quad \Gamma = \sigma_1. \quad (6.69)$$

The introduction of two types of coin operators and a strange shift operator in Eq. (6.67) is aimed at breaking time-reversal symmetry and particle-hole symmetry. This is because we explore topological phases protected by time-glide chiral symmetry in the case that other symmetries are absent. In momentum space, $\tilde{U}_1$ is represented as

$$\tilde{\mathcal{U}}_1(\mathbf{k}) = e^{-i\frac{k_x - k_y}{2}} \mathcal{C}_2(\theta_2/2) \mathcal{S}(k_y/2) \mathcal{C}_1(-\theta_1/2) \tilde{\mathcal{S}}(k_x/2), \quad (6.70)$$

when parameters are position independent $\theta_1(\mathbf{r}) = \theta_1$, $\theta_2(\mathbf{r}) = \theta_2$. From Eqs. (6.46), (6.47), (6.48), and (6.70), we obtain

$$\tilde{\mathcal{U}}_2(\mathbf{k}) = e^{i\frac{k_x + k_y}{2}} \tilde{\mathcal{S}}(k_x/2) \mathcal{C}_1(-\theta_1/2) \mathcal{S}(k_y/2) \mathcal{C}_2(\theta_2/2), \quad (6.71)$$

$$\tilde{\mathcal{U}}_3(\mathbf{k}) = e^{-i\frac{k_x + k_y}{2}} \mathcal{C}_2(\theta_2/2) \mathcal{S}(k_y/2) \mathcal{C}_1(\theta_1/2) \tilde{\mathcal{S}}(k_x/2), \quad (6.72)$$

$$\tilde{\mathcal{U}}_4(\mathbf{k}) = e^{i\frac{k_x - k_y}{2}} \tilde{\mathcal{S}}(k_x/2) \mathcal{C}_1(\theta_1/2) \mathcal{S}(k_y/2) \mathcal{C}_2(\theta_2/2). \quad (6.73)$$

The full time-evolution operator U and its unitary equivalent partner U' are given by $U = \tilde{U}_4 \tilde{U}_3 \tilde{U}_2 \tilde{U}_1$ and $U' = \tilde{U}_3 \tilde{U}_2 \tilde{U}_1 \tilde{U}_4$, respectively. In the present case, their first and second half period parts are

$$\mathcal{U}_1(\mathbf{k}) = e^{ik_y} \tilde{\mathcal{S}}(k_x/2) \mathcal{C}_1(-\theta_1/2) \mathcal{S}(k_y/2) \mathcal{C}_2(\theta_2) \mathcal{S}(k_y/2) \mathcal{C}_1(-\theta_1/2) \tilde{\mathcal{S}}(k_x/2), \quad (6.74)$$

$$\mathcal{U}_2(\mathbf{k}) = e^{-ik_y} \tilde{\mathcal{S}}(k_x/2) \mathcal{C}_1(\theta_1/2) \mathcal{S}(k_y/2) \mathcal{C}_2(\theta_2) \mathcal{S}(k_y/2) \mathcal{C}_1(\theta_1/2) \tilde{\mathcal{S}}(k_x/2), \quad (6.75)$$

$$\mathcal{U}'_1(\mathbf{k}) = \mathcal{C}_2(\theta_2/2) \mathcal{S}(k_y/2) \mathcal{C}_1(-\theta_1/2) \tilde{\mathcal{S}}(k_x) \mathcal{C}_1(\theta_1/2) \mathcal{S}(k_y/2) \mathcal{C}_2(\theta_2/2). \quad (6.76)$$

$$\mathcal{U}'_2(\mathbf{k}) = \mathcal{C}_2(\theta_2/2) \mathcal{S}(k_y/2) \mathcal{C}_1(\theta_1/2) \tilde{\mathcal{S}}(k_x) \mathcal{C}_1(-\theta_1/2) \mathcal{S}(k_y/2) \mathcal{C}_2(\theta_2/2). \quad (6.77)$$

Winding numbers which stem from time-glide chiral symmetry and chiral symmetry are respectively calculated from matrix components of $\mathcal{U}'_1(\mathbf{k})$ in Eq. (6.76) and $\mathcal{U}_1(\mathbf{k})$ in Eq. (6.74).

6.5.1 bulk Dirac points

We first examine the bulk quasi-energy spectrum of the system. From Eqs. (6.74) and (6.75), the matrix components of the time-evolution operator $\mathcal{U}(\mathbf{k}) = \mathcal{U}_2(\mathbf{k}) \mathcal{U}_1(\mathbf{k})$ are

$$\mathcal{U}^{(1,1)}(\mathbf{k}) = [a(\mathbf{k}) + b(\mathbf{k}) + c(\mathbf{k})]^2 - [d(\mathbf{k}) + e(\mathbf{k}) + if(\mathbf{k})]^2, \quad (6.78)$$

$$\mathcal{U}^{(1,2)}(\mathbf{k}) = 2[a(\mathbf{k}) + b(\mathbf{k})][d(\mathbf{k}) + e(\mathbf{k})] + 2c(\mathbf{k})f(\mathbf{k}), \quad (6.79)$$

$$\mathcal{U}^{(2,2)}(\mathbf{k}) = [\mathcal{U}^{(1,1)}(\mathbf{k})]^*, \quad \mathcal{U}^{(2,1)}(\mathbf{k}) = -\mathcal{U}^{(1,2)}(\mathbf{k}), \quad (6.80)$$

where $\mathcal{U}^{(i,j)}(\mathbf{k})$ denotes the (i, j) component of $\mathcal{U}(\mathbf{k})$ and $a(\mathbf{k})$ - $f(\mathbf{k})$ are defined as

$$a(\mathbf{k}) = \cos(\theta_1) \cos(\theta_2) \cos(k_x) \cos(k_y), \quad (6.81)$$

$$b(\mathbf{k}) = \sin(\theta_2) \sin(k_x), \quad (6.82)$$

$$c(\mathbf{k}) = \cos(\theta_2) \sin(k_y), \quad (6.83)$$

$$d(\mathbf{k}) = \cos(\theta_1) \cos(\theta_2) \sin(k_x) \cos(k_y), \quad (6.84)$$

$$e(\mathbf{k}) = -\sin(\theta_2) \cos(k_x), \quad (6.85)$$

$$f(\mathbf{k}) = \sin(\theta_1) \cos(\theta_2) \cos(k_y). \quad (6.86)$$

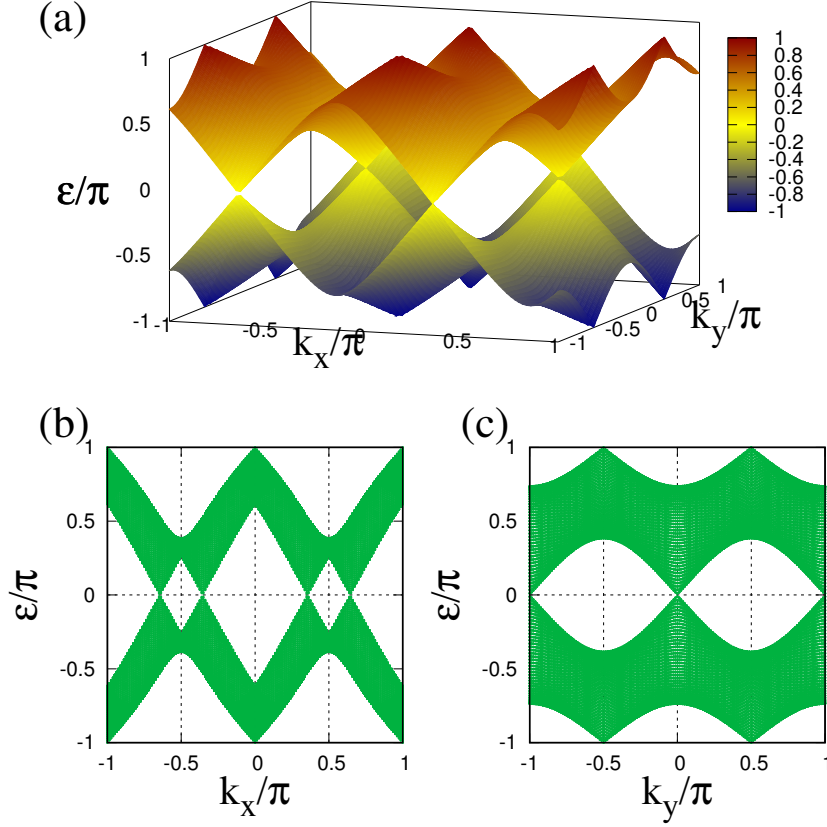


Figure 6.2: (a) A dispersion relation for the homogeneous system with $\theta_1 = -\pi/4$ and $\theta_2 = -9\pi/13$. (b) and (c) show the same dispersion relation respectively mapped on k_x - ε plane and k_y - ε plane.

Due to the structure of $\mathcal{U}(\mathbf{k})$ in Eq. (6.80) and $\det[\mathcal{U}(\mathbf{k})] = 1$, the condition for $\varepsilon_{\mathbf{k}} = 0$ is $\mathcal{U}^{(1,1)}(\mathbf{k}) = \mathcal{U}^{(2,2)}(\mathbf{k}) = 1$. Then, substituting $k_y = 0, \pi$ into Eqs. (6.78)-(6.86), the condition becomes

$$\mathcal{U}^{(1,1)}(k_x, 0/\pi) - 1 = v \sin(k_x \mp w) = 0, \quad (6.87)$$

$$v = \sqrt{\cos^2(\theta_1) \cos^2(\theta_2) + \sin^2(\theta_2)} \quad (6.88)$$

$$\cos(w) = \cos(\theta_1) \cos(\theta_2)/v, \quad \sin(w) = \sin(\theta_2)/v \quad (6.89)$$

where $-$ and $+$ respectively correspond to $k_y = 0$ and $k_y = \pi$. Therefore, gap closing points always exist at $(k_x, k_y, \varepsilon) = (w, 0, 0)$, $(\pi + w, 0, 0)$, $(-w, \pi, 0)$, and $(\pi - w, \pi, 0)$. Gap closings also occur at $(k_x, k_y, \varepsilon) = (0, \pm\pi/2, \pi)$ and $(\pi, \pm\pi/2, \pi)$ for arbitrary θ_1 and θ_2 . Figure 6.2 shows the quasi-energy $\varepsilon_{\mathbf{k}}$ of our model for certain specific values of θ_1 and θ_2 .

These Dirac points are protected by topological numbers associated with chiral symmetry in Eq. (3.33). To see this, we rewrite $\mathcal{U}(\mathbf{k})$ in terms of the effective Hamiltonian $\mathcal{H}(\mathbf{k})$,

$$\mathcal{U}(\mathbf{k}) = e^{-i\mathcal{H}(\mathbf{k})} = \cos[\mathcal{H}(\mathbf{k})] - i \sin[\mathcal{H}(\mathbf{k})]. \quad (6.90)$$

Chiral symmetry of $\mathcal{U}(\mathbf{k})$ implies $\Gamma \mathcal{U}(\mathbf{k}) \Gamma^{-1} = \mathcal{U}^{-1}(\mathbf{k})$, which results in

$$\Gamma \sin[\mathcal{H}(\mathbf{k})] \Gamma^{-1} = -\sin[\mathcal{H}(\mathbf{k})]. \quad (6.91)$$

Then, in the same way as Eq. (2.9) for chiral symmetric Hamiltonians $\Gamma\mathcal{H}(\mathbf{k})\Gamma^{-1} = -\mathcal{H}(\mathbf{k})$, one can define a topological charge of a Dirac point,

$$\nu_D = \frac{i}{4\pi} \oint_p \text{tr} \left(\Gamma \{ \sin[\mathcal{H}(\mathbf{k})] \}^{-1} d \sin[\mathcal{H}(\mathbf{k})] \right), \quad (6.92)$$

where p is a loop enclosing the Dirac point. Note that $\{ \sin \mathcal{H}(\mathbf{k}) \}^{-1}$ is well-defined on p since the quasi-energy spectrum on p has gaps at $\varepsilon = 0$ and π . For the calculation of ν_D , it is convenient to take the basis where Γ is diagonal as in Eq. (3.35). Thus, we have

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad -i \sin[\mathcal{H}(\mathbf{k})] = \begin{pmatrix} 0 & B(\mathbf{k}) \\ -B^\dagger(\mathbf{k}) & 0 \end{pmatrix}, \quad (6.93)$$

from which we obtain

$$\nu_D = \frac{1}{2\pi i} \oint_p d \ln \det \{ B(\mathbf{k}) \}. \quad (6.94)$$

This topological charge is non-zero for each Dirac point. Regarding Dirac points at $(k_x, k_y, \varepsilon) = (w, 0, 0)$ and $(-w, \pi, 0)$, the topological charges take the same value,

$$\nu_D^0 = \text{sign}[\cos(\theta_2)]. \quad (6.95)$$

At $(k_x, k_y, \varepsilon) = (\pi + w, 0, 0)$ and $(\pi - w, \pi, 0)$, the topological charges are $-\nu_D^0$ for both Dirac points. Regarding Dirac points at $(k_x, k_y, \varepsilon) = (0, \pi/2, \pi)$ and $(\pi, -\pi/2, \pi)$, the topological charges become

$$\nu_D^\pi = \text{sign}[\sin(\theta_1) \sin(\theta_2) \cos(\theta_2)] \quad (6.96)$$

in both cases, while the other Dirac points at $(k_x, k_y, \varepsilon) = (0, -\pi/2, \pi)$ and $(\pi, \pi/2, \pi)$ have the opposite topological charge $-\nu_D^\pi$ in common. The topological charges and positions of Dirac points are summarized in Fig.6.3.

In the same way as Sec. 6.2, we can represent ν_D in terms of $\mathcal{U}_1(\mathbf{k}) = \tilde{\mathcal{U}}_2(\mathbf{k})\tilde{\mathcal{U}}_1(\mathbf{k})$. We denote $\mathcal{U}_1(\mathbf{k})$ and $\mathcal{U}_2(\mathbf{k})$ in the basis of Eq. (6.93) as

$$\mathcal{U}_1(\mathbf{k}) = \begin{pmatrix} \alpha(\mathbf{k}) & \beta(\mathbf{k}) \\ \gamma(\mathbf{k}) & \delta(\mathbf{k}) \end{pmatrix}, \quad \mathcal{U}_2(\mathbf{k}) = \begin{pmatrix} \alpha^\dagger(\mathbf{k}) & -\gamma^\dagger(\mathbf{k}) \\ -\beta^\dagger(\mathbf{k}) & \delta^\dagger(\mathbf{k}) \end{pmatrix}, \quad (6.97)$$

Since the unitarity of $\mathcal{U}_1(\mathbf{k})$ ensures $\alpha^\dagger(\mathbf{k})\beta(\mathbf{k}) + \gamma^\dagger(\mathbf{k})\delta(\mathbf{k}) = 0$, we have

$$B(\mathbf{k}) = 2\alpha^\dagger(\mathbf{k})\beta(\mathbf{k}) = -2\gamma^\dagger(\mathbf{k})\delta(\mathbf{k}) \quad (6.98)$$

from the relation $\sin[\mathcal{H}(\mathbf{k})] = i[\mathcal{U}(\mathbf{k}) - \mathcal{U}^\dagger(\mathbf{k})]/2$. Equation (6.98) implies that

$$\nu_D = \nu_D[\beta] - \nu_D[\alpha] = \nu_D[\delta] - \nu_D[\gamma], \quad (6.99)$$

where $\nu_D[\alpha]$, $\nu_D[\beta]$, $\nu_D[\gamma]$, $\nu_D[\delta]$ are the one-dimensional winding numbers of $\alpha(\mathbf{k})$, $\beta(\mathbf{k})$, $\gamma(\mathbf{k})$ and $\delta(\mathbf{k})$ along the contour p which encloses the Dirac point. Equation (6.99) results in

$$\nu_D = \frac{\nu_D[\beta] - \nu_D[\gamma]}{2} + \frac{\nu_D[\delta] - \nu_D[\alpha]}{2}, \quad (6.100)$$

which we use in Sec. 6.5.3. In the right hand side of Eq. (6.100), the first term and second term respectively correspond to ν_0 and ν_π , which we can understand from Eq. (6.18) while integral paths are different.

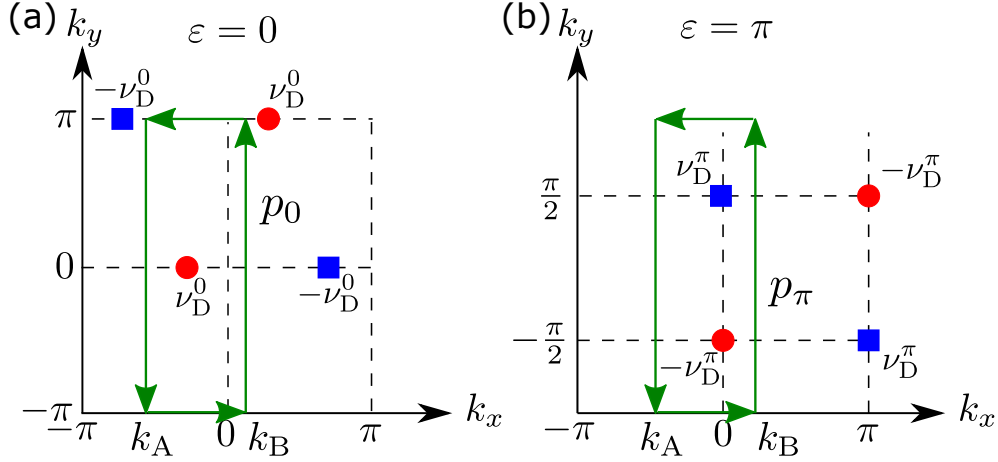


Figure 6.3: The positions and topological charges of Dirac points, (a) at $\varepsilon = 0$ when $-\pi/2 < w < 0$ and (b) at $\varepsilon = \pi$. Dirac points indicated by red circles and blue squares have the opposite topological charges respectively in (a) and (b). Green vertical arrows represent integral paths when we calculate winding numbers fixing k_x at some value, while left ones with $k_x = k_A$ are directed to the opposite of the integral paths for winding numbers which are the same as that of the right ones in which k_x is fixed at k_B . Since horizontal arrows cancel out each other due to the 2π periodicity of k_y , the sum of topological charges enclosed by the paths, which are named as p_0 for $\varepsilon = 0$ and p_π for $\varepsilon = \pi$, corresponds to the difference between winding numbers with $k_x = k_B$ and k_A .

6.5.2 winding numbers and edge states from time-glide chiral symmetry

Since d is even, $\{\mathcal{G}_T, \Gamma\} = 0$, and there exist gaps at $\varepsilon = \pi$ on time-glide symmetric planes $k_y = 0, \pi$, we can characterize the system by time-glide chiral winding number $\nu[\delta']$, as shown in Eq. (6.64). We calculate $\nu[\delta']$ by integrating out k_x and fixing k_y at $0, \pi$. When k_x is fixed at some value, there is no winding number from time-glide chiral symmetry, since preserving time-glide chiral symmetry is impossible when we integrate out k_y . In order to obtain the winding number, we write down the $(2, 2)$ component of $\mathcal{U}'_1(\mathbf{k}) = \tilde{\mathcal{U}}_1(\mathbf{k})\mathcal{U}_4(\mathbf{k})$ with $k_y = 0$,

$$\delta'(k_x, 0) = X \cos(k_x - Y) - iZ \sin(k_x), \quad (6.101)$$

$$X = \sqrt{\cos^2(\theta_2) + \cos^2(\theta_1) \sin^2(\theta_2)}, \quad (6.102)$$

$$\cos(Y) = \cos(\theta_2)/X, \quad \sin(Y) = \cos(\theta_1) \sin(\theta_2)/X, \quad (6.103)$$

$$Z = \sin(\theta_1) \quad (6.104)$$

Integrating out k_x , time-glide chiral winding number becomes

$$\nu_{\text{tgc}} = \nu[\delta'] = -\text{sign}[\sin(\theta_1) \cos(\theta_2)] \text{ at } k_y = 0, \quad (6.105)$$

which predicts the number of edge states with $\varepsilon = \pi$. In the case of $k_y = \pi$, the winding number becomes the same as ν_{tgc} . This relation can be understood by the additional symmetry for this model $\mathcal{U}'_1(k_x, k_y + \pi) = -\mathcal{U}'_2(-k_x, k_y)$, due to which the winding number obeys

$$\nu[\delta'(k_x, 0)] = -\nu[\delta'^\dagger(k_x, 0)] = -\nu[\delta'(-k_x, \pi)] = \nu[\delta'(k_x, \pi)], \quad (6.106)$$

where the additional symmetry is used in the second equality. For $\varepsilon = 0$, there is no winding number from time-glide chiral symmetry as shown in Eq. (6.64). The value of ν_{tgc} depends on

Table 6.1: A Table of various winding numbers studied in Secs. 6.5.2 and 6.5.3. The left column represents the momentum which is fixed as a parameter. The first row in the third and fourth columns shows symmetries which are origins of corresponding winding numbers. Note that, while chiral winding number ν_c depends on k_x , its value only takes $\nu_c = \pm 1$ at $|k_x| = |\pi/2|$ or $\nu_c = 0$ at $k_x = 0, \pi$.

fixing	gap	chiral symmetry	time-glide chiral symmetry
k_x	$\varepsilon = 0$	$\nu_c = \pm 1, 0$	undefined
k_x	$\varepsilon = \pi$	zero	undefined
k_y	$\varepsilon = 0$	zero	undefined
k_y	$\varepsilon = \pi$	zero	$\nu_{\text{tgc}} = \pm 1$

θ_1 and θ_2 , as is described in Fig. 6.4. We also remark that the winding number originating from time-glide chiral symmetry is not related to Dirac points of the effective Hamiltonians in Fig. 6.3 since time-glide symmetry cannot be defined for the whole time-evolution operators or equivalently the effective Hamiltonians.

Edge states originating from time-glide chiral symmetry are expected to appear at $(k_y, \varepsilon) = (0, \pi)$ and (π, π) , when there are boundaries at which ν_{tgc} changes and k_y is well defined. Making θ_1 and θ_2 position dependent, we can vary ν_{tgc} spatially, since the value of ν_{tgc} depends on θ_1 and θ_2 as in Fig. 6.4. In order to explore the edge states protected by time-glide chiral symmetry, we consider the case in which rotation angles depend only on x as described in Fig 6.5 (a),

$$\theta_j(x) = \begin{cases} \theta_j^L & (1 \leq x \leq N_x/2) \\ \theta_j^R & (N_x/2 + 1 \leq x \leq N_x) \end{cases}, \quad j = 1, 2, \quad (6.107)$$

where an even number N_x is the number of lattices in the x direction. Since periodic boundary conditions are imposed on x and y directions, there are two boundaries at $x = 0$ and $N_x/2$ where values of $\theta_j(x)$ change. Dispersion relations as functions of k_y are shown in Fig. 6.6 for several sets of θ_1^R and θ_2^R by fixing the values of θ_1^L and θ_2^L . Comparing dispersion relations for inhomogeneous systems shown in Fig. 6.6 and the dispersion relation for the homogeneous system in Fig. 6.2 (c), we can recognize that edge states appear at $(k_y, \varepsilon) = (0, \pi)$ and (π, π) in Fig. 6.6 (a) and (c), while they do not appear in Fig. 6.6 (b) and (d). Although other edge states originating from an extra symmetry appear at $(k_y, \varepsilon) = (\pm\pi/2, 0)$ in Fig. 6.6 (c) and (d), we do not focus on them in this section. We explain the additional symmetry and resulting edge states in Sec. 6.5.4. In Fig. 6.6 (a) and (c), edge states at $(k_y, \varepsilon) = (0, \pi)$ have four-fold degeneracy. We see from Fig. 6.4 that $|\nu_{\text{tgc}}^L - \nu_{\text{tgc}}^R| = 2$ in the case of Fig. 6.6 (a) and (c), where ν_{tgc}^L and ν_{tgc}^R are time-glide chiral winding numbers in the left and right regions respectively, while $|\nu_{\text{tgc}}^L - \nu_{\text{tgc}}^R| = 0$ in the other cases (b) and (d). We note that the number of edge states predicted by the bulk-edge correspondence becomes twice of $|\nu_{\text{tgc}}^L - \nu_{\text{tgc}}^R|$, because of the presence of two boundaries at $x = 0$ and $N_x/2$. Therefore, ν_{tgc} correctly predicts the number of edge states, or equivalently time-glide chiral winding number satisfies the bulk-edge correspondence. It is a theoretically new result that there are edge states at $\varepsilon = \pi$ protected by time-glide chiral symmetry when gap closing points exist at $\varepsilon = \pi$ with $k_y \neq 0, \pi$, in comparison to the classification in Ref. [110] focusing on cases where the bulk spectrum has the quasi-energy gap around $\varepsilon = \pi$. When $k_y \neq 0, \pi$, the winding number originating from time-glide chiral symmetry in Eq. (6.54) is not defined, and the edge states at $k_y \neq 0, \pi$ deviate from $\varepsilon = \pi$. As a result, the group velocity of edge states is not equal to zero, which is different from that of flat bands whose origin is chiral symmetry discussed in Sec. 6.5.3.

Figure 6.7 (a) shows the probability distributions of a walker at each time step in the case that edge states at $\varepsilon = \pi$ exist. We can clearly see that two peaks of probabilities propagate in the opposite directions along the boundary near $x = N_x/2$, which reflects the existence of edge states

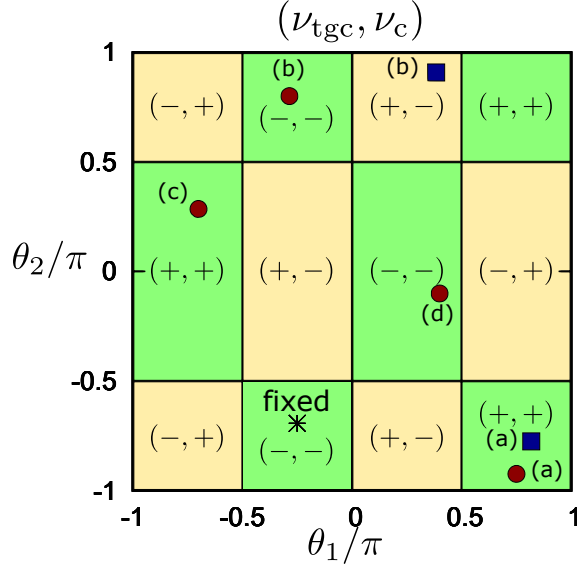


Figure 6.4: θ_1 and θ_2 dependences of winding numbers $(\nu_{\text{tgc}}, \nu_c)$, defined at $k_y = 0$ and $k_x = \pi/2$ respectively, where $+1$ (-1) is represented as $+$ ($-$). The black asterisk represents fixed parameters (θ_1^L, θ_2^L) and (θ_1^B, θ_2^B) in both Figs. 6.6 and 6.8. Red circles marked (a)-(d) represent parameters (θ_1^R, θ_2^R) used in Fig. 6.6. Blue squares marked (a) and (b) are parameters (θ_1^T, θ_2^T) used in Fig. 6.8.

with nonzero group velocity. Figure 6.7 (b) shows the time evolution when there is no edge state. Probabilities diffuse in both x and y directions, and there are no outstanding peaks in Fig. 6.7 (b). Comparing Fig. 6.7 (a) and (b), it is obvious that the propagation of probability peaks along the boundary in Fig. 6.7 (a) is due to the existence of edge states. Note that, in the dynamics of a two-dimensional quantum walk where topological edge states exist and time bins are regarded as positions in real space [70], a peak of probabilities propagates only to one direction, which is different from the behaviour observed in Fig. 6.7 (a).

6.5.3 winding numbers and edge states from chiral symmetry

By fixing k_x or k_y as a parameter, we can regard the two-dimensional system as a one-dimensional chiral symmetric system, which is characterized by winding numbers in Eq. (6.13). The values of winding numbers are listed in Tab. 6.1. First, we consider the case of $\varepsilon = 0$ fixing k_x as a parameter. Based on Eq. (6.18), we define the winding number for $\varepsilon = 0$ as

$$\nu_c = \frac{\nu[\beta] - \nu[\gamma]}{2}. \quad (6.108)$$

While chiral winding number in Eq. (6.108) can be calculated at arbitrary k_x , it is sufficient to calculate the winding number at $k_x = 0, \pm\pi/2$, and π , since Dirac points are located as shown in Fig. 6.3 (a). This is because the value of the winding number changes when the integral path goes across a Dirac point with varying k_x . In order to calculate chiral winding numbers, we apply a unitary transformation with $e^{i\frac{\pi}{4}\sigma_2}$ to $\mathcal{U}_1(\mathbf{k})$ and change the basis to make chiral symmetry operator be σ_3 . Then, from Eq. (6.74), $\beta(\mathbf{k}) = [e^{i\frac{\pi}{4}\sigma_2}\mathcal{U}_1(\pm\pi/2, k_y)e^{-i\frac{\pi}{4}\sigma_2}]^{(1,2)}$ and

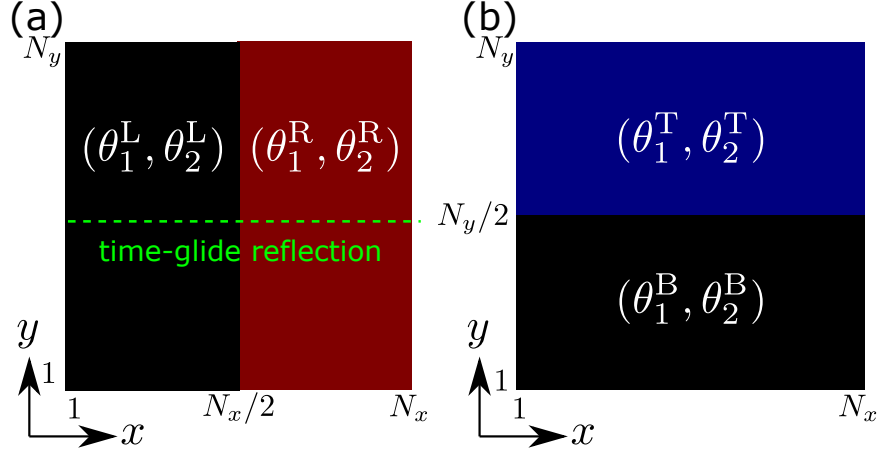


Figure 6.5: Systems with boundaries in the direction of (a) x and (b) y . In both cases (a) and (b), periodic boundary conditions are imposed in both directions. In (a), where $\theta_1(x)$ and $\theta_2(x)$ change at $x = 0, N_x/2$, the green dashed line represents the reflection axis in the y direction by time-glide symmetry. In (b), $\theta_1(y)$ and $\theta_2(y)$ change at $y = 0, N_y/2$.

$\gamma(\mathbf{k})[e^{i\frac{\pi}{4}\sigma_2}\mathcal{U}_1(\pm\pi/2, k_y)e^{-i\frac{\pi}{4}\sigma_2}]^{(2,1)}$ become

$$\beta(k_x = \pm\pi/2, k_y) = e^{ik_y} \cos(\theta_2) [\cos(\theta_1) \cos(k_y) \mp i \sin(k_y)], \quad (6.109)$$

$$\gamma(k_x = \pm\pi/2, k_y) = e^{ik_y} \cos(\theta_2) [\cos(\theta_1) \cos(k_y) \pm i \sin(k_y)], \quad (6.110)$$

at $k_x = \pm\pi/2$. Integrating out k_y , the value of ν_c becomes

$$\nu_c = \mp \text{sign}[\cos(\theta_1)] \text{ at } k_x = \pm\pi/2, \quad (6.111)$$

$$\nu_c = 0 \text{ at } k_x = 0, \pi. \quad (6.112)$$

Chiral winding number ν_c at $k_x = \pi/2$ depends on θ_1 as shown in Fig. 6.4, while $\nu_c = 0$ for arbitrary θ_1 and θ_2 at $k_x = 0, \pi$. For $\varepsilon = \pi$, we consider a winding number $\frac{\nu[\delta] - \nu[\alpha]}{2}$ at $k_x = \pm\pi/2$ in the same way as the case of $\varepsilon = 0$. Then, we find that the winding number becomes zero for arbitrary θ_1, θ_2 , and k_x .

Here, we mention the relation between topological charges of Dirac points in Eq. (6.100) and winding numbers originating from chiral symmetry in Eq. (6.108), by considering integral paths p_0 and p_π enclosing Dirac points in Fig. 6.3. We note that the path p_π always encloses an even number of Dirac points, while the path p_0 is not the case. The vertical line with the upward arrow is identical to the integration path for the winding number at $k_x = k_B$, while the vertical line with the downward arrow is the opposite direction to the path for the winding number at $k_x = k_A$. Further, integrals along the two horizontal lines cancel out each other due to 2π periodicity of k_y . Then, the difference between winding numbers at $k_x = k_A$ and $k_x = k_B$, related to edge states at $\varepsilon = 0$ ($\varepsilon = \pi$), becomes total topological charges of Dirac points enclosed by p_0 (p_π). This is the reason why chiral winding number ν_c is not equal to zero between two Dirac points with opposite topological charges for $\varepsilon = 0$ [Fig. 6.3 (a)]. In the case of $\varepsilon = \pi$, the total topological charges of Dirac points enclosed by integral paths are always zero, which ensures that the winding number $\frac{\nu[\delta] - \nu[\alpha]}{2}$ is always zero [Fig. 6.3 (b)]. When k_y is fixed as a parameter, the winding numbers $\frac{\nu[\delta] - \nu[\gamma]}{2}$ for $\varepsilon = 0$ and $\frac{\nu[\delta] - \nu[\alpha]}{2}$ for $\varepsilon = \pi$ are zero with arbitrary θ_1 and θ_2 . This behavior of winding numbers can also be understood from topological charges and positions of Dirac points in Fig. 6.3, in the same way as the case of fixing k_x .

We study the edge states at interfaces between two regions with different winding numbers.

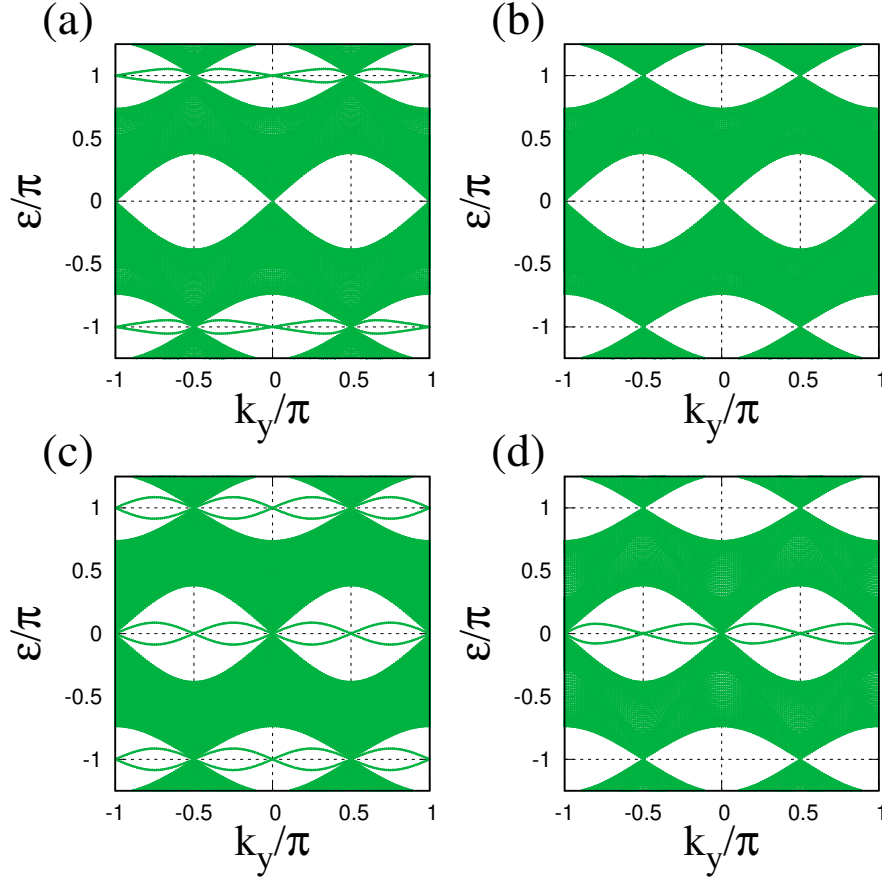


Figure 6.6: Numerically obtained quasi-energies. In (a) and (c), edge states with $\varepsilon = \pi$ are protected by time-glide chiral symmetry. In the left region which is the black region in Fig. 6.5 (a), θ_1^L and θ_2^L are fixed as $(\theta_1^L, \theta_2^L) = (-\pi/4, -9\pi/13)$ and the system size is $N_x = 160$, $N_y = 160$. Periodic boundary conditions are imposed for both x and y directions. Parameters in the right region are (a) $\theta_1^R = 3\pi/4$, $\theta_2^R = -12\pi/13$, (b) $\theta_1^R = -2\pi/7$, $\theta_2^R = 4\pi/5$, (c) $\theta_1^R = -7\pi/10$, $\theta_2^R = 2\pi/7$, and (d) $\theta_1^R = 2\pi/5$, $\theta_2^R = -\pi/10$, which are plotted as red circles in Fig. 6.4. Note that, there are gap closing points at $(k_y, \varepsilon) = (\pm\pi/2, \pi)$.

Edge states which are protected by chiral symmetry in Eq. (3.31) emerge when we make boundaries to y direction,

$$\theta_j(y) = \begin{cases} \theta_j^B & (1 \leq y \leq N_y/2) \\ \theta_j^T & (N_y/2 + 1 \leq y \leq N_y) \end{cases}, \quad j = 1, 2, \quad (6.113)$$

as in Fig. 6.5 (b). We call the regions of $1 \leq y \leq N_y/2$ and $N_y/2 + 1 \leq y \leq N_y$ as the bottom region and top region, respectively. Since the winding number in Eq. (6.108) remains ± 1 for a finite range of k_x , there appear flat bands at $\varepsilon = 0$. Figure 6.8 shows quasi-energies as functions of k_x , in systems shown in Fig. 6.5 (b). In Fig. 6.8 (a) and (b), two Dirac points closest to $k_x = \pi/2$ labeled as B originate from the bulk spectrum in the bottom region. The other two Dirac points labeled as T are those of the top region. The system in Fig. 6.8 (a) has $\nu_c = -1$ in the bottom region and $\nu_c = +1$ in the top region, as shown in Fig. 6.4. As predicted from the bulk-edge correspondence, there appear flat bands at $\varepsilon = 0$ between Dirac points. The eigenstates with $\varepsilon = 0$ between two Dirac points B and B have four-fold degeneracy, which is also consistent

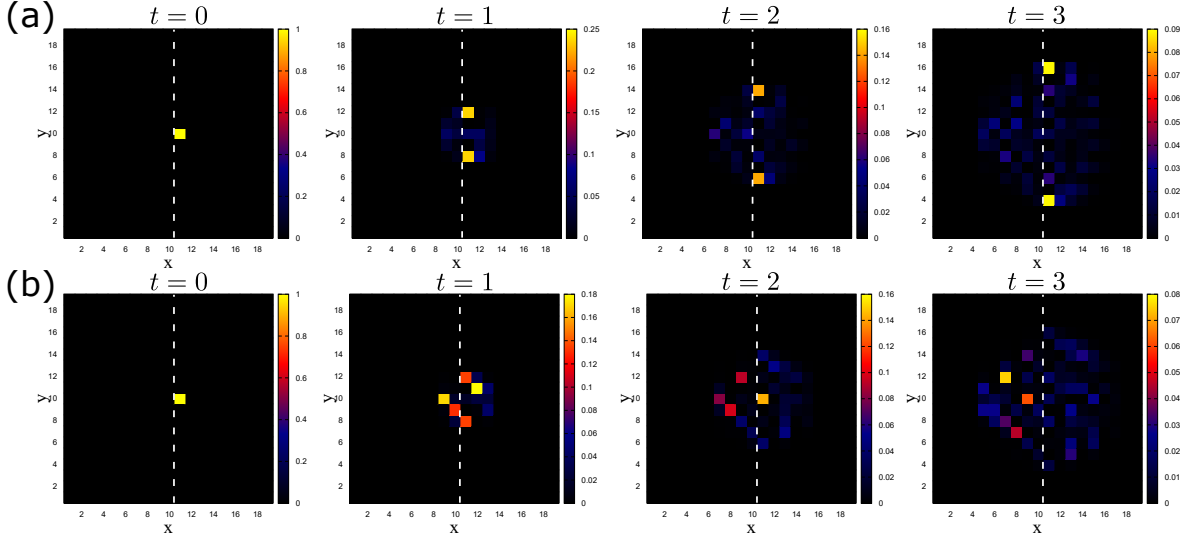


Figure 6.7: The probability distributions $P(\mathbf{r}, t) = |\psi_+(\mathbf{r}, t)|^2 + |\psi_-(\mathbf{r}, t)|^2$ at each time step with $(\theta_1^L, \theta_2^L) = (-\pi/4, -9\pi/13)$ and (a) $\theta_1^R = 3\pi/4$, $\theta_2^R = -12\pi/13$, (b) $\theta_1^R = -2\pi/7$, $\theta_2^R = 4\pi/5$. In (a) and (b), parameters are respectively the same as Fig. 6.6 (a) and (b), which indicates that edge states with $\varepsilon = \pi$ exist in (a) and there is no edge state in (b). The initial state is $|\psi(0)\rangle = |x=11, y=10, \sigma=+\rangle$ and $N_x = N_y = 20$ in both (a) and (b). White dashed lines represent the boundary at which parameters (θ_1, θ_2) change.

with the bulk-edge correspondence since there are two boundaries. As explained in Sec. 6.5.3, the winding numbers at $k_x = 0, \pi$ are zero. Varying k_x from $\pi/2$ to 0 or π and passing through one nearest Dirac point B, the winding number of the bottom region changes from -1 to 0. Then, the degeneracy on $\varepsilon = 0$ at the specific k_x becomes 2, since the winding number in the top region is still -1 . After a second Dirac point T is passed, the flat band vanishes, since chiral winding numbers are zero in both regions. In Fig. 6.8 (b), $\nu_c = -1$ in both regions. Therefore, flat bands vanish around $k_x = \pi/2$, between two Dirac points labeled as B. On the other hand, there are flat bands in the k_x range where the winding number is 0 in the bottom region and -1 in the top region, between two Dirac points B and T, of which one is that of the bottom region and the other is that of the top region. The flat bands in Fig. 6.8 correspond to Fermi arcs in Dirac semimetals, since the behavior of chiral winding numbers can be understood from topological charges of Dirac points. While flat bands have been discussed in quantum walks [63], the correspondence to Fermi arcs is stated for the first time in this work by clarifying the relation between Dirac points in the bulk and flat bands on the edge. In Fig. 6.8 (b), there also appear edge states at $(k_x, \varepsilon) = (0, 0)$ and $(\pi, 0)$. We explain the origin of them in Sec. 6.5.4.

6.5.4 additional symmetries and resulting winding numbers

The system has an additional symmetry at $k_y = \pm\pi/2$. For simplicity, we focus only on $k_y = \pi/2$. Substituting $k_y = \pi/2$ into $\mathcal{U}'(\mathbf{k})$, we obtain

$$E(\theta_1)\mathcal{U}'(k_x, \pi/2)E^\dagger(\theta) = -\mathcal{U}_2(k_x)\mathcal{U}_1(k_x), \quad (6.114)$$

$$\mathcal{U}_1(k_x) = \mathcal{C}_1(\theta_2/2)\mathcal{S}(k_x)\mathcal{C}_1(-\theta_2/2), \quad (6.115)$$

$$\mathcal{U}_2(k_x) = \mathcal{C}_1(-\theta_2/2)\mathcal{S}(-k_x)\mathcal{C}_1(\theta_2/2), \quad (6.116)$$

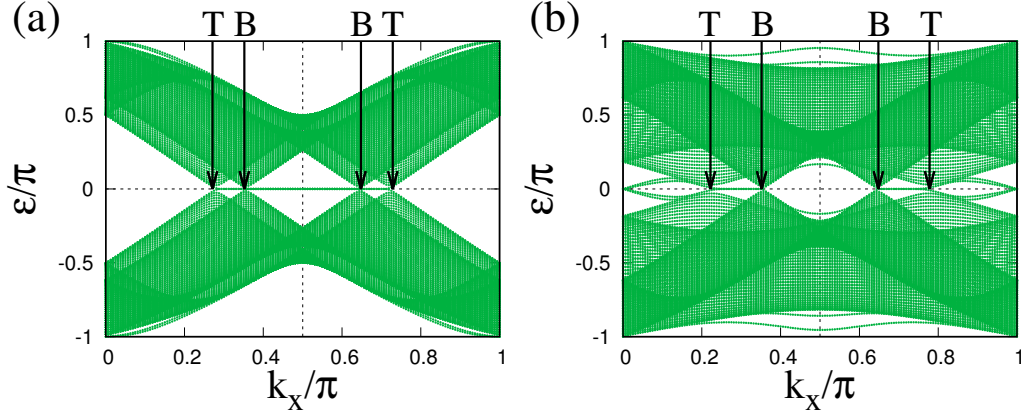


Figure 6.8: Numerically obtained quasi-energies. Flat bands at $\varepsilon = 0$ connect Dirac points, corresponding to Fermi arcs of Dirac semimetals. In the bottom region which is the black region in Fig. 6.5 (b), θ_1^B and θ_2^B are fixed as $(\theta_1^B, \theta_2^B) = (-\pi/4, -9\pi/13)$ and the system size is $N_x = 160$, $N_y = 160$. Periodic boundary conditions are imposed for both x and y directions. Parameters in the top region are (a) $\theta_1^T = 9\pi/11$, $\theta_2^T = -3\pi/4$ and (b) $\theta_1^T = 5\pi/13$, $\theta_2^T = 10\pi/11$, which are plotted as blue squares in Fig. 6.4.

with $E(\theta_1) = e^{i(\frac{\theta_1}{2} - \frac{\pi}{4})\sigma_1} e^{i\frac{\pi}{4}\sigma_3}$. $\dot{\mathcal{U}}_1(k_x)$ and $\dot{\mathcal{U}}_2(k_x)$ satisfy

$$\sigma_3 \dot{\mathcal{U}}_1(k_x) \sigma_3 = \dot{\mathcal{U}}_2^\dagger(k_x). \quad (6.117)$$

Based on the additional symmetry in Eq. (6.117), we can obtain winding numbers of

$$\dot{\alpha}(k_x) = \dot{\mathcal{U}}_1^{(1,1)}(k_x) = \cos(k_x) + i \cos(\theta_2) \sin(k_x), \quad \dot{\delta}(k_x) = \dot{\mathcal{U}}_1^{(2,2)}(k_x) = [\dot{\mathcal{U}}_1^{(1,1)}(k_x)]^* \quad (6.118)$$

which correspond to winding numbers for $\varepsilon = \pi$ in Eq. (6.13). Although discussions on the bulk-edge correspondence or equivalently the derivation of Eq. (6.18) is based on bulk quasi-energy gaps at $\varepsilon = 0$ and $\varepsilon = \pi$, $(\nu[\dot{\delta}] - \nu[\dot{\alpha}])/2$ correctly predicts the number of edge states at $\varepsilon = \pi$ in the system described by $\dot{\mathcal{U}}_2(k_x)\dot{\mathcal{U}}_1(k_x)$ where the gap at $\varepsilon = 0$ is always closed. At $k_y = -\pi/2$, the winding number has the opposite sign to that at $k_y = \pi/2$. The origin of edge states at $(k_y, \varepsilon) = (\pm\pi/2, 0)$ in Fig. 6.6 (c) and (d) is the additional symmetry in Eq. (6.117) or the resulting winding number, since there is the minus sign in the right hand side of Eq. (6.114).

The system also has an additional symmetry at $k_x = 0, \pi$. Substituting $k_x = 0$ and ignoring $e^{\pm ik_y}$ which cancel out, $\mathcal{U}'(0, k_y)$ becomes

$$e^{i\frac{\pi}{4}\sigma_2} \mathcal{U}'(0, k_y) e^{-i\frac{\pi}{4}\sigma_2} = \dot{\mathcal{U}}_2(k_y) \dot{\mathcal{U}}_1(k_y), \quad (6.119)$$

$$\dot{\mathcal{U}}_1(k_y) = \dot{\mathcal{U}}_2(k_y) = \mathcal{C}_2(\theta_2/2) e^{-ik_y\sigma_1} \mathcal{C}_2(\theta_2/2). \quad (6.120)$$

Since $\mathcal{U}'_1(0, k_y)$ satisfies

$$\sigma_3 \dot{\mathcal{U}}_1(k_y) \sigma_3 = [\dot{\mathcal{U}}_1(k_y)]^\dagger, \quad (6.121)$$

we can obtain winding numbers of

$$\dot{\beta}(k_y) = \dot{\mathcal{U}}_1^{(1,2)}(k_y) = -\sin(\theta_2) \cos(k_y) - i \sin(k_y), \quad \dot{\gamma}(k_y) = \dot{\mathcal{U}}_1^{(2,1)}(k_y) = -[\dot{\mathcal{U}}_1^{(1,2)}(k_y)]^*, \quad (6.122)$$

which are winding numbers in Eq. (6.13) for $\varepsilon = 0$. The number of edge states at $(k_x, \varepsilon) = (0, 0)$ in Fig. 6.8 (b) can be predicted from $(\nu[\dot{\beta}] - \nu[\dot{\gamma}])/2$. This is remarkable since the bulk quasi-energy gap at $\varepsilon = \pi$ is closed, while Eq. (6.18) is derived under the condition that both gaps at $\varepsilon = 0$ and $\varepsilon = \pi$ are open. At $k_x = \pi$, there appear edge states in the same way.

6.6 Conclusion and discussion

We have studied Floquet topological phases of quantum walks which satisfy chiral symmetry and/or time-glide symmetry. We have derived chiral winding numbers based on the unitarity of time-evolution operators. Through the derivation which is different from that in Sec. 3.3.2, we have clarified that chiral symmetric quantum walks are distinct from not only static systems described by time-independent Hamiltonians but also Floquet systems described by time-periodic Hamiltonians. In addition, identifying chiral symmetry as a discrete version of time-reflection symmetry, we have shown the way of constructing models which have these symmetries, based on the concept of time units. We have clarified topological numbers which characterize anomalous topological phases protected by time-glide chiral symmetry intrinsic to Floquet systems. While Ref. [110] classifies Floquet topological phases when bulk quasi-energy spectra are fully gapped around $\varepsilon = \pi$, we have clarified that robust edge states protected by time-glide chiral symmetry appear even when bulk gaps are closed at $\varepsilon = \pi$. In addition, we have revealed that topological edge states can exist not only at $\varepsilon = \pi$ but also at $\varepsilon = 0$, while the previous work [110] explored only the former case. Our first study on space-time symmetries and resulting topological phases in quantum walks not only has theoretical novelty but also helps observations of edge states peculiar to Floquet systems whose origin is these symmetries, since quantum walks are feasible experimental platforms to explore Floquet topological phases.

Using the method based on time units, we have made a model of two-dimensional quantum walks, which satisfies time-glide symmetry and chiral symmetry. We have calculated winding numbers and have shown that there are two types of edge states. One is protected by time-glide chiral symmetry, which appears when there are boundaries only in x direction. Since the edge states have nonzero group velocities, two peaks of probabilities propagate in two opposite directions along the boundary, which can be observed in experiments. The other is the flat band which stems from chiral symmetry, appearing when boundaries exist in y direction and time-glide symmetry is broken. Existence or absence of flat bands can be understood from topological charges of Dirac points. The flat bands correspond to Fermi arcs in Dirac semimetals since the way of understanding is the same as those of Fermi arcs as explained in Sec. 6.5.3, which is stated in quantum walks for the first time.

Chapter 7

Summary and Outlook

In this doctoral thesis, we have studied topological phenomena in quantum walks where time evolves in the discrete manner and gain-loss effects can be manipulated. As a result, we have clarified three phenomena unique to Floquet and/or open systems in both linear and nonlinear regimes.

First, we have revealed the emergence of topologically protected edge states which stem from chiral symmetry in open Floquet systems and the enhancement of intensities due to $\mathcal{PT}$ symmetry breaking of edge states. The emergent edge states satisfy $\text{Re}(\varepsilon) = 0$ or $\text{Re}(\varepsilon) = \pi$, of which the latter is unique to Floquet systems. In addition, the enhancement of edge states has no counterpart in closed systems, since total intensity or probability is conserved in closed systems and $\mathcal{PT}$ symmetry breaking is a peculiar phenomenon in open systems. We have derived topological numbers which correctly predict the numbers of edge states, by defining appropriate chiral symmetry of nonunitary time-evolution operators and generalizing the definition of chirality which characterizes edge states. Our derivation based on the bulk-edge correspondence gives the microscopic foundation on topological numbers which have been employed in open Floquet systems [64, 69, 72]. It has also importance that our theoretical expectation was confirmed through the observation of edge states with longer lifetime in comparison to bulk states in a single-photon experiment [215]. The experiment is not only an observation of the peculiar phenomenon in open Floquet systems but also the first detection of $\mathcal{PT}$ symmetry breaking in quantum regime. Therefore, our study has significant meanings in both research fields of Floquet topological phases and open quantum systems. It should be interesting to explore other unique phenomena in open systems employing quantum walks as platforms, since quantum walks are realizable in physical settings where topological features and gain-loss effects can be manipulated.

Second phenomenon is the bifurcation stemming from the discrete nature of time evolutions. While the stability analysis in the continuum limit [96] cannot expect that edge states can be unstable in a nonlinear quantum walk, the stability analysis taking the discrete nature of quantum walks into account correctly predicts the bifurcation at which edge states change from stable attractors to unstable repellers. The complicated motions of eigenvalues which determine the stability of edge states show that the boundedness of the stable region in discrete time nonlinear systems makes stationary states fragile in comparison to nonlinear systems which obey continuous time-evolution equations. In other words, when parameters in a discrete time nonlinear system are changed monotonously to a direction along which stationary states tend to be stable, the tendency may be reversed by the boundedness of the stable region. Such bifurcations generally exist in other discrete time nonlinear systems and have been explored in different research fields [231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241]. In addition, we have analytically derived thresholds at which edge states become unstable, which is considered to be difficult in general nonlinear systems. We have carried out the derivation of thresholds by utilizing inequalities of the spectral radius and the spectral norm. Since this procedure is not restricted to nonlinear quantum

walks and stationary states need not to be topological edge states, our method is generally applicable to discrete time nonlinear systems other than specific models addressed in this work, which indicates that analytical derivations of thresholds may be possible in various systems.

Third, we have pursued the uniqueness of quantum walks in comparison to static systems and Floquet systems respectively described by time-independent and time-periodic Hamiltonians, based on chiral symmetry, time-glide symmetry, and/or the discreteness of time. We have shown that quantum walks with chiral symmetry can possess non-trivial topological phases which are absent in other systems mentioned above, from the viewpoint of the number of gaps and/or topological invariants. In addition, we have clarified that topologically protected edge states can exist originating from time-glide chiral symmetry which is a combined symmetry of time-glide symmetry and chiral symmetry. Time-glide symmetry is a unique symmetry of Floquet systems, since it cannot be defined on effective Hamiltonians based on the analogy to static systems. This is because the operation of time-glide symmetry includes the half translation in time direction. We have defined time-glide symmetry of quantum walks based on the original one in Floquet systems with microscopic Hamiltonians [110], and established the procedure to construct models which have time-glide symmetry and chiral symmetry. Based on time-glide symmetry and chiral symmetry of quantum walks, we have derived time-glide chiral winding number and classified topological phases. In comparison to Ref. [110] which proposed time-glide symmetry for the first time, we have revealed two novel facts. One is that time-glide chiral winding number is valid even when the gap at $\varepsilon = 0$ or π is closed excepting the glide symmetric plane, while Ref. [110] addressed the case in which the gap is fully opened. The other is that time-glide chiral winding number is related not only to the gap at $\varepsilon = \pi$ but also the gap at $\varepsilon = 0$, of which the latter was not addressed in Ref. [110]. Constructing a concrete two-dimensional model, we have clarified that the derived time-glide chiral winding number correctly predicts the number of edge states. We have also clarified that chiral winding numbers which result in flat bands can be understood from topological charges of Dirac points. This is carried out for the first time in quantum walks, while the similar concept for Dirac semimetals is widely known. Based on our study addressing chiral symmetry and time-glide symmetry, of which the former is regarded as one of space-time symmetries for the first time, it should be interesting to explore other space-time symmetries and resulting topological phases in quantum walks.

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List of Publications

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