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Normal trace for vector fields of bounded mean oscillation

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Abstract

We introduce various spaces of vector fields of bounded mean oscillation (*BMO*) defined in a domain so that normal trace on the boundary is bounded when its divergence is well controlled. The behavior of “normal” component and “tangential” component may be different for our *BMO* vector fields. As a result zero extension of the normal component stays in *BMO* although such property may not hold for tangential components.

Keywords: *BMO*, normal trace, duality, Jones’ extension, Triebel-Lizorkin space.

1 Introduction

One of basic questions on vector fields defined on a domain Ω in $\mathbf{R}^n$ is whether the normal trace is well controlled without estimating all partial derivatives when the divergence is well controlled. Such a type of estimates is well known when a vector field is L^p ($1 < p < \infty$) or L^∞ . Here are examples. Let Ω be a bounded domain with smooth boundary Γ . Let $\mathbf{n}$ denotes its exterior unit normal vector field on Γ . For simplicity we assume that a vector field v satisfies $\operatorname{div} v = 0$. Then there is a constant C independent of v such that

$$\|v \cdot \mathbf{n}\|_{W^{-1/p,p}(\Gamma)} \leq C \|v\|_{L^p(\Omega)} \quad (1) \quad \{\mathbf{N1}\}$$

$$\|v \cdot \mathbf{n}\|_{L^\infty(\Gamma)} \leq C \|v\|_{L^\infty(\Omega)}. \quad (2) \quad \{\mathbf{N2}\}$$

Here $W^{s,p}$ denotes the Sobolev space which is actually a Besov space $B_{p,p}^s$ for non-integer s . The first estimate is a key to establish the Helmholtz decomposition of an L^p vector field; see e.g. [FM]. The second estimate is important to study for example total variation flow; see e.g. [ACM, Appendix C1]. These estimates (1), (2) hold for various domains including the case that Ω is a half space $\mathbf{R}_+^n$ i.e.,

$$\mathbf{R}_+^n = \{(x_1, \dots, x_n) \mid x_n > 0\}.$$

Our goal in this paper is to extend (2) by replacing $\|v\|_{L^\infty(\Omega)}$ by some *BMO* type norm.

However, it turns out that the normal trace of divergence free BMO vector fields may not be bounded. Indeed, consider

$$v = (v^1, v^2), \quad v^1(x) = v^2(x) = \log |x_1 - x_2|. \quad x = (x_1, x_2) \in \mathbf{R}^2.$$

This vector field is in $BMO(\mathbf{R}^2)$ and it is divergence free in distribution sence. Indeed,

$$\int_{\mathbf{R}^2} v \cdot \nabla \varphi \, dx = \frac{1}{2} \int_{\mathbf{R}^2} \log |\zeta| ((\partial_\zeta - \partial_\eta) \tilde{\varphi} + (\partial_\eta + \partial_\zeta) \tilde{\varphi}) \, d\zeta d\eta = 0,$$

$$\zeta = x_1 - x_2, \quad \eta = x_1 + x_2$$

for all compactly supported smooth function φ , i.e., $\varphi \in C_c^\infty(\mathbf{R}^2)$. Here, $\tilde{\varphi}(\zeta, \eta) = \varphi((\zeta + \eta)/2, (\eta - \zeta)/2)$. However, if we consider $\Omega = \mathbf{R}_+^2$ and $\Gamma = \{x_2 = 0\}$, then $v \cdot \mathbf{n} = -v_2$ on Γ is clearly unbounded. This example indicates that we need some control near the boundary. Such a control is introduced in [BG], [BGS], [BGMST], [BGST]. More precisely, for $f \in L_{\text{loc}}^1(\Omega)$, $\nu \in (0, \infty]$ they introduced a seminorm

$$[f]_{b^\nu} := \sup \left\{ r^{-n} \int_{\Omega \cap B_r(x)} |f(y)| \, dy \mid x \in \Gamma, \, 0 < r < \nu \right\},$$

where $B_r(x)$ denotes the closed ball of radius r centered at x . For $\mu \in (0, \infty]$ they define

$$[f]_{BMO^\mu} := \sup \left\{ \frac{1}{|B_r(x)|} \int_{B_r(x)} |f - f_{B_r(x)}| \, dy \mid B_r(x) \subset \Omega, \, r < \mu \right\},$$

where $f_B = \frac{1}{|B|} \int_B f(y) \, dy$, the average over B ; here $|B|$ denotes the Lebesgue measure of B . The BMO type space $BMO_b^{\mu, \nu}$ introduced in these papers is the space of $f \in L_{\text{loc}}^1(\Omega)$ having finite

$$\|f\|_{BMO_b^{\mu, \nu}} := [f]_{BMO^\mu} + [f]_{b^\nu}.$$

This space is very convenient to study the Stokes semigroup in [BG], [BGS], [BGMST], [BGST] as well as the heat semigroup [BGST]. One of our main results (Theorem 25) yields

$$\|v \cdot \mathbf{n}\|_{L^\infty(\Gamma)} \leq C \|v\|_{BMO_b^{\mu, \nu}} \tag{3} \quad \{\mathbf{N3}\}$$

for any $\mu, \nu \in (0, \infty]$ for any uniformly $C^{1+\beta}$ domain.

However, for applications, especially to establish the Helmholtz decomposition $BMO_b^{\mu, \nu}$ norm for all components is too strong so we would like to estimate by a weaker norm. We only use b^ν seminorm for normal component of a vector field v . To decompose the vector field let $d_\Omega(x)$

be the distance of $x \in \Omega$ from the boundary Γ , i.e.,

$$d_\Omega(x) := \inf\{|x - y| \mid y \in \Gamma\}.$$

If Ω is uniformly C^2 , then d_Ω is C^2 in a δ -tubular neighborhood Γ_δ of Γ for some $\delta < R_*$, where R_* is the reach of Γ [GT, Chapter 14, Appendix], [KP, §4.4]; here

$$\Gamma_\delta := \{x \in \Omega \mid d_\Omega(x) < \delta\}.$$

Instead of (3), our main results (Theorem 21, 22) together with Theorem 9 read:

$$\|v \cdot \mathbf{n}\|_{L^\infty(\Gamma)} \leq C([v]_{BMO^\mu} + [\nabla d_\Omega \cdot v]_{b^\nu}) \quad (4) \quad \{\text{MTB}\}$$

for $\nu \leq \delta$, $\mu \in (0, \infty]$ provided that Ω is a bounded $C^{2+\beta}$ domain with $\beta \in (0, 1)$. The quantity $\nabla d_\Omega \cdot v$ is a kind of normal component.

Our main strategy is to use the formula

$$\int_\Gamma (v \cdot \mathbf{n}) \psi \, d\mathcal{H}^{n-1} = \int_\Omega (\operatorname{div} v) \rho \, dx - \int_\Omega v \cdot \nabla \varphi \, dx$$

for any $\varphi \in C_c^\infty(\overline{\Omega})$ with $\varphi|_\Gamma = \psi$, where $d\mathcal{H}^{n-1}$ denotes the surface element. This formula is obtained by integration by parts. If $\operatorname{div} v = 0$, then it reads

$$\int_\Gamma (v \cdot \mathbf{n}) \psi \, d\mathcal{H}^{n-1} = - \int_\Omega v \cdot \nabla \varphi \, dx. \quad (5) \quad \{\text{SINT}\}$$

Our estimate (4) follows from localization, flattening the boundary and duality argument. To get the flavor we explain the case when Ω is the half space $\mathbf{R}_+^n$. For $\psi \in L^1(\Gamma)$ it is known that there is $\varphi \in F_{1,2}^1(\mathbf{R}^n)$ such that its trace to the boundary equals to ψ ; see e.g. [Tr92, Section 4.4.3]. Here $F_{1,2}^1$ denotes the Triebel-Lizorkin space which means that $\nabla \varphi \in h^1$, a localized Hardy space. We may assume that φ is even in x_n . We extend $v = (v', v_n)$ even in x_n for tangential part v' and odd in x_n for the normal part $v_n = \nabla d_\Omega \cdot v$. Although extended v' is still in $BMO^\infty(\mathbf{R}^n)$, the extended v_n may not be in $BMO^\infty(\mathbf{R}^n)$ unless we assume $[v_n]_{b^\nu} < \infty$. Here we invoke $[\nabla d_\Omega \cdot v]_{b^\nu} < \infty$. By these extensions our (5) yields

$$\int_\Gamma (v \cdot \mathbf{n}) \psi \, d\mathcal{H}^{n-1} = - \frac{1}{2} \int_{\mathbf{R}^n} v \cdot \nabla \varphi \, dx, \quad (6) \quad \{\text{KID}\}$$

where v denotes the extended vector field. We apply h^1 - bmo duality [Sa, Theorem 3.22] for (6) to get

$$\left| \int_\Gamma (v \cdot \mathbf{n}) \psi \, d\mathcal{H}^{n-1} \right| \leq C \|v\|_{bmo} \|\varphi\|_{F_{1,2}^1},$$

where $bmo = BMO \cap L_{\text{ul}}^1$ a localized BMO space. Here L_{ul}^1 denotes a uniformly local L^1 space; see Section 2 for details. Since $\|\varphi\|_{F_{1,2}^1} \leq C\|\psi\|_{L^1}$, this implies

$$\|v \cdot \mathbf{n}\|_{L^\infty(\Gamma)} \leq C\|v\|_{bmo(\mathbf{R}^n)} \quad (7) \quad \{\text{MTH}\}$$

$$\leq C \left([v]_{BMO^\infty(\Omega)} + [v]_{L_{\text{ul}}^1(\Omega)} + [\nabla d_\Omega \cdot v]_{b^\infty} \right). \quad (8)$$

Here and hereafter C denotes a constant depends only on space dimension and its numerical value may be different line by line.

In the case of a curved domain we need localization and flattening procedure by using normal (principal) coordinates. The localized space $bmo_\delta^\mu = BMO^\mu \cap L_{\text{ul}}^1(\Gamma_\delta)$ is convenient for this purpose. Again we have to handle normal component $\nabla d_\Omega \cdot v$ separately. If the domain has a compact boundary, we are able to remove L_{ul}^1 term in (7) and we deduce the estimate (4). Note that in this trace estimate only the behavior of v near Γ is important so one may use finite exponents in BMO^μ and b^ν .

As a byproduct we notice the extension problem of BMO functions. In general, zero extension of $v \in BMO^\mu(\Omega)$ may not belong to $BMO^\mu(\mathbf{R}^n)$ but if v is in $BMO_b^{\mu,\nu}$, as noticed in [BGST], its zero extension belongs to $BMO^\mu(\mathbf{R}^n)$ for $\nu \geq 2\mu$. We also note that it is possible to extend general $bmo_\delta^\mu(\Omega)$ to BMO^μ whose support is near $\bar{\Omega}$. We develop such a theory to explain the role of b^ν .

This paper is organized as follows. In Section 2 we introduce several localized BMO spaces and compared these spaces. Some of them are discussed in [BGST]. We introduce a new space $vbmo_\delta^{\mu,\nu}$ which requires that the b^ν seminorm of the normal component is bounded in $(bmo_\delta^\mu)^n$. A key observation is that if the boundary of the domain is compact, i.e., either a bounded or an exterior domain, the requirement in $L_{\text{ul}}^1(\Gamma_\delta)$ is redundant in the definition of $vbmo_\delta^{\mu,\nu}$. In Section 3 we discuss extension problem as well as localization problem. In Section 4 we shall prove our main results. In Appendix we discuss coordinate change of vector fields by normal coordinates for the reader's convenience.

2 Spaces

\{\text{PS}\}

In this section we fix notation of important function spaces. Let $L_{\text{ul}}^1(\mathbf{R}^n)$ be a uniformly L^1 space, i.e., for a fixed $r_0 > 0$

$$L_{\text{ul}}^1(\mathbf{R}^n) := \left\{ f \in L_{\text{loc}}^1(\mathbf{R}^n) \mid \|f\|_{L_{\text{ul}}^1} := \sup_{x \in \mathbf{R}^n} \int_{B_{r_0}(x)} |f(y)| dy < \infty \right\}.$$

The space is independent of the choice of r_0 . For a domain Ω the space L_{ul}^1 is the space of all L_{loc}^1 functions f in Ω whose zero extension belongs to $L_{\text{ul}}^1(\mathbf{R}^n)$. In other words

$$L_{\text{ul}}^1(\Omega) := \left\{ f \in L_{\text{loc}}^1(\Omega) \mid \|f\|_{L_{\text{ul}}^1(\Omega)} := \sup_{x \in \mathbf{R}^n} \int_{B_{r_0}(x) \cap \Omega} |f(y)| dy < \infty \right\}.$$

As in [BG] we set

$$BMO^\mu(\Omega) := \left\{ f \in L_{\text{loc}}^1(\Omega) \mid [f]_{BMO^\mu} < \infty \right\}.$$

For $\delta \in (0, \infty]$ we set

$$bmo_\delta^\mu(\Omega) := BMO^\mu(\Omega) \cap L_{\text{ul}}^1(\Gamma_\delta) = \{f \in BMO^\mu(\Omega) \mid \text{restriction of } f \text{ on } \Gamma_\delta \text{ is in } L_{\text{ul}}^1(\Gamma_\delta)\}.$$

This is a Banach space equipped with the norm

$$\|f\|_{bmo_\delta^\mu} := [f]_{BMO^\mu(\Omega)} + [f]_{\Gamma_\delta}, \quad [f]_{\Gamma_\delta} := \|f\|_{L_{\text{ul}}^1(\Gamma_\delta)},$$

where the restriction of f on Γ_δ is still denoted by f . If there is no boundary we set

$$bmo(\mathbf{R}^n) := BMO^\infty(\mathbf{R}^n) \cap L_{\text{ul}}^1(\mathbf{R}^n)$$

which is a local BMO space and it agrees with the Triebel-Lizorkin space $F_{\infty,2}^0$; see e.g. [Tr92, Section 1.7.1], [Sa, Theorem 3.26].

For vector-valued function spaces we still write BMO^μ instead of $(BMO^\mu)^n$. For example for vector field v by $v \in bmo_\delta^\mu(\Omega)$ we mean that

$$v = (v_1, \dots, v_n), \quad v_i \in bmo_\delta^\mu(\Omega), \quad 1 \leq i \leq n.$$

We next introduce the space of vector fields whose normal component has finite b^ν of the form

$$vbmo_\delta^{\mu,\nu}(\Omega) := \{v \in bmo_\delta^\mu(\Omega) \mid [\nabla d_\Omega \cdot v]_{b^\nu} < \infty\}$$

for $\nu \in (0, \infty]$. This space is a Banach space equipped with the norm

$$\|v\|_{vbmo_\delta^{\mu,\nu}} := \|v\|_{bmo_\delta^\mu} + [\nabla d_\Omega \cdot v]_{b^\nu}.$$

Similarly, we introduce another space

$$vBMO^{\mu,\nu}(\Omega) := \{v \in BMO^\mu(\Omega) \mid [\nabla d_\Omega \cdot v]_{b^\nu} < \infty\}$$

equipped with a seminorm

$$[v]_{vBMO^{\mu,\nu}} := [v]_{BMO^\mu} + [\nabla d_\Omega \cdot v]_{b^\nu}.$$

Of course, this is strictly larger than the Banach space

$$BMO_b^{\mu,\nu}(\Omega) := \{v \in BMO^\mu(\Omega) \mid [v]_{b^\nu} < \infty\}$$

equipped with the norm

$$\|v\|_{BMO_b^{\mu,\nu}} := [v]_{BMO^\mu} + [v]_{b^\nu}$$

introduced essentially in [BG]. Indeed, in the case when Ω is the half space $\mathbf{R}_+^n$

$$vBMO^{\mu,\nu}(\mathbf{R}_+^n) = (BMO^\mu(\mathbf{R}_+^n))^{n-1} \times BMO_b^{\mu,\nu}(\mathbf{R}_+^n), \quad (9) \quad \{\text{DC}\}$$

where in the right-hand side the each space denotes the space of scalar functions not of vector fields. This shows that $vBMO^{\mu,\nu}(\mathbf{R}_+^n)$ is strictly larger than $BMO_b^{\mu,\nu}(\mathbf{R}_+^n)$ for $n \geq 2$.

Although there are many exponents, the spaces may be the same for different exponents. By definition for $0 < \mu_1 \leq \mu_2 \leq \infty$, $0 < \nu_1 \leq \nu_2 \leq \infty$, $0 < \delta_1 \leq \delta_2 \leq \infty$,

$$[f]_{BMO^{\mu_1}} \leq [f]_{BMO^{\mu_2}}, \quad [f]_{b^{\nu_1}} \leq [f]_{b^{\nu_2}}, \quad [f]_{\Gamma_{\delta_1}} \leq [f]_{\Gamma_{\delta_2}}.$$

\{\text{E1}\}

Proposition 1. *Let Ω be an arbitrary domain in $\mathbf{R}^n$.*

- (i) *Let $0 < \mu_1 < \mu_2 < \infty$. Then seminorms $[\cdot]_{BMO^{\mu_1}}$ and $[\cdot]_{BMO^{\mu_2}}$ are equivalent. If Ω is bounded, one may take $\mu_2 = \infty$.*
- (ii) *Let $0 < \delta_1 < \delta_2 < \infty$ and $\mu \in (0, \infty]$. Then there exists a constant $C > 0$ depending only on $n, \mu, \delta_1, \delta_2$ and Ω such that*

$$[f]_{\Gamma_{\delta_2}} \leq C \left([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}} \right).$$

In particular, the norms $\|\cdot\|_{bmo_{\delta_1}^\mu}$ and $\|\cdot\|_{bmo_{\delta_2}^\mu}$ are equivalent. If Ω is bounded, one may take $\delta_2 = \infty$.

Proof. (i) This is [BGST, Theorem 4] which follows from [BGST, Theorem 3].

(ii) Since the space $L_{\text{ul}}^1(\Gamma_\delta)$ is independent of the radius r_0 in its definition, without loss of generality, we may assume that $r_0 > \delta_1$. Let us firstly consider the case where the dimension

$n > 1$. Let k be the smallest integer such that $2^{-k} < \frac{\delta_1}{\sqrt{n}}$ and $x \in \mathbf{R}^n$. Notice that

$$\int_{B_{r_0}(x) \cap \Gamma_{\delta_2}} |f| dy = \int_{B_{r_0}(x) \cap \Gamma_{\delta_1}} |f| dy + \int_{B_{r_0}(x) \cap (\Gamma_{\delta_2} \setminus \Gamma_{\delta_1})} |f| dy,$$

we can estimate $\|f\|_{L^1(B_{r_0}(x) \cap \Gamma_{\delta_1})}$ directly by $[f]_{\Gamma_{\delta_1}}$. Assume that $\Gamma_{\delta_2} \setminus \Gamma_{\delta_1} \neq \emptyset$. Let $D_k(x)$ be the set of dyadic cubes of side length 2^{-k} that intersect with $B_{r_0}(x) \cap (\Gamma_{\delta_2} \setminus \Gamma_{\delta_1})$. For a dyadic cube $Q_j \in D_k(x)$, we define B_j to be the ball which has radius $\frac{\sqrt{n}}{2} \cdot 2^{-k}$ and shares the same center with Q_j . Let $C_k(x) := \{B_j \mid Q_j \in D_k(x)\}$ and $\Sigma := \{x \in \Omega \mid d_\Omega(x) = \delta_1\}$.

For $Q_j \in D_k(x)$ that intersects Σ , we seek to estimate $\|f\|_{L^1(B_j)}$. Let c_j be a point on $\Sigma \cap Q_j$, we have that $B_{\delta_1}(c_j) \subset \Omega$. Indeed as otherwise, there exists $z \in B_{\delta_1}(c_j) \cap \Omega^c$. Then the line segment joining c_j and z must intersect Γ at some point, say z^* . Then $|z^* - c_j| \leq |z - c_j| < \delta_1$. This contradicts the fact that $d_\Omega(c_j) = \delta_1$. For $y \in B_j$, $|y - c_j| < \sqrt{n} \cdot \ell(Q_j) = \sqrt{n} \cdot 2^{-k} < \delta_1$. So $B_j \subset B_{\delta_1}(c_j)$. Let $d_j \in \Gamma$ be a point such that $|c_j - d_j| = \delta_1$, then on the line segment joining c_j and d_j , we can find a point o_j such that $|o_j - d_j| = \frac{\sqrt{n}}{2} \cdot 2^{-k}$. For $y \in B_{\frac{\sqrt{n}}{2} \cdot 2^{-k}}(o_j)$, we have that $|d_\Omega(y) - d_\Omega(o_j)| \leq |y - o_j|$. Hence $d_\Omega(y) \leq d_\Omega(o_j) + |y - o_j| < \sqrt{n} \cdot 2^{-k} < \delta_1$. This means that $B_{\frac{\sqrt{n}}{2} \cdot 2^{-k}}(o_j) \subset \Gamma_{\delta_1}$. Moreover,

$$|c_j - y| \leq |c_j - o_j| + |o_j - y| \leq \delta_1 - \frac{\sqrt{n}}{2} \cdot 2^{-k} + \frac{\sqrt{n}}{2} \cdot 2^{-k} = \delta_1.$$

Thus $B_{\frac{\sqrt{n}}{2} \cdot 2^{-k}}(o_j) \subset B_{\delta_1}(c_j)$. Denote $B_{\frac{\sqrt{n}}{2} \cdot 2^{-k}}(o_j)$ by B_j^* . We have that

$$\int_{B_j} |f| dy \leq \int_{B_{\delta_1}(c_j)} |f - f_{B_{\delta_1}(c_j)}| dy + \int_{B_{\delta_1}(c_j)} |f_{B_{\delta_1}(c_j)} - f_{B_j^*}| dy + \int_{B_{\delta_1}(c_j)} |f_{B_j^*}| dy.$$

Notice that

$$\begin{aligned} \int_{B_{\delta_1}(c_j)} |f - f_{B_{\delta_1}(c_j)}| dy &\leq C_n \cdot \delta_1^n \cdot [f]_{BMO^\mu}, \\ \int_{B_{\delta_1}(c_j)} |f_{B_{\delta_1}(c_j)} - f_{B_j^*}| dy &\leq \frac{|B_{\delta_1}(c_j)|^2}{|B_j^*|} \cdot [f]_{BMO^\mu}, \\ \int_{B_{\delta_1}(c_j)} |f_{B_j^*}| dy &\leq \frac{|B_{\delta_1}(c_j)|}{|B_j^*|} \cdot [f]_{\delta_1}. \end{aligned}$$

Since $|B_{\delta_1}(c_j)| = C_n \cdot \delta_1^n$ and $\frac{|B_{\delta_1}(c_j)|}{|B_j^*|} = \frac{C_n \cdot \delta_1^n}{(\frac{\sqrt{n}}{2} \cdot 2^{-k})^n} \leq \frac{C_n \cdot \delta_1^n}{(\frac{\delta_1}{4})^n} = C_n$, $\|f\|_{L^1(B_j)}$ is therefore controlled by $C_{\delta_1, n} \cdot ([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}})$.

Next we consider $Q'_j \in D_k(x)$ that does not intersect Σ . Suppose that $Q_j \in D_k(x)$ has a touching edge with Q'_j . There exists a ball B_j^i of radius $\frac{\sqrt{n}-1}{2} \cdot 2^{-k}$ which is contained in $B_j \cap B'_j$ where B_j, B'_j are the smallest balls that contain Q_j, Q'_j respectively. Similar to above,

as $B_i^j \subset B_j$,

$$\begin{aligned} \int_{B_j'} |f| dy &\leq \int_{B_j'} |f - f_{B_j'}| dy + \int_{B_j'} |f_{B_j'} - f_{B_i^j}| dy + \int_{B_j'} |f_{B_i^j}| dy \\ &\leq |B_j'| \cdot [f]_{BMO^\mu} + \frac{|B_j'|^2}{|B_i^j|} \cdot [f]_{BMO^\mu} + \frac{|B_j'|}{|B_i^j|} \cdot \int_{B_j} |f| dy. \end{aligned}$$

Therefore if $\|f\|_{L^1(B_j)}$ is controlled by $C_{\delta_1, n} \cdot ([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}})$, $\|f\|_{L^1(B_j')}$ is also controlled by $C_{\delta_1, n} \cdot ([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}})$.

Since $B_{r_0}(x) \cap (\Gamma_{\delta_2} \setminus \Gamma_{\delta_1})$ is connected, we can estimate $\|f\|_{L^1(B_j)}$ for every $Q_j \in D_k(x)$ where B_j is the smallest ball that contains Q_j . For each $Q_j \in D_k(x)$, there exists $y \in Q_j \cap B_{r_0}(x)$, so for any $z \in B_j$, $|z - x| \leq |z - y| + |y - x| < \sqrt{n} \cdot 2^{-k} + r_0 < r_0 + \delta_1$. Thus $\bigcup_{Q_j \in D_k(x)} B_j \subset B_{r_0 + \delta_1}(x)$.

Let $N(D_k(x))$ be the number of cubes in $D_k(x)$, we have that

$$N(D_k(x)) \leq \frac{|B_{r_0 + \delta_1}(x)|}{2^{-kn}} \leq C_n \cdot \left(\frac{r_0 + \delta_1}{\delta_1} \right)^n.$$

Therefore,

$$\begin{aligned} \int_{B_{r_0}(x) \cap (\Gamma_{\delta_2} \setminus \Gamma_{\delta_1})} |f| dy &\leq \sum_{B_j \in C_{r_0}(x)} \int_{B_j} |f| dy \\ &\leq N(D_k(x)) \cdot C_{\delta_1, n} \cdot ([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}}) \\ &\leq C_{n, \delta_1, r_0} \cdot ([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}}). \end{aligned}$$

For the case where the dimension $n = 1$, we let k to be the smallest integer such that $2^{-k} < \frac{\delta_1}{2}$ and D_k to be the set of dyadic cubes of side length 2^{-k} that intersects $\Gamma_{\delta_2} \setminus \Gamma_{\delta_1}$. Notice that the region $\Gamma_{\delta_2} \setminus \Gamma_{\delta_1}$ is indeed a union of intervals. Without loss of generality, we can assume Ω to be $(0, \infty)$ and take $\mu = \infty$ by part (i) of this proposition. Thus in this case $\Gamma_{\delta_2} \setminus \Gamma_{\delta_1} = (\delta_1, \delta_2)$. For $Q_0 \in D_k$ such that $\delta_1 \in Q_0$,

$$\begin{aligned} \int_{Q_0} |f| dy &\leq \int_{2Q_0} |f| dy \leq \int_{2Q_0} |f - f_{2Q_0}| dy + \int_{2Q_0} |f_{2Q_0} - f_{Q_0^*}| dy + \int_{2Q_0} |f_{Q_0^*}| dy \\ &\leq C \cdot ([f]_{BMO^\infty} + [f]_{\Gamma_{\delta_1}}), \end{aligned}$$

where $Q_0^* = 2Q_0 \setminus (Q_0 \cup [\delta_1, \infty))$ and $\ell(Q_0^*) = \frac{1}{2}\ell(Q_0) = 2^{-(k+1)}$.

We then put an ordering on the elements of D_k in the following way. For $j \in \mathbf{N}$, suppose that we have ordered intervals $Q_0, Q_1, \dots, Q_{j-1}$, we pick $Q_j \in D_k \setminus \{Q_0, Q_1, \dots, Q_{j-1}\}$ such that

Q_j has a touching edge with Q_{j-1} . For $Q_j \in D_k$, similarly we have that

$$\begin{aligned} \int_{Q_j} |f| dy &\leq \int_{2Q_j} |f| dy \leq \int_{2Q_j} |f - f_{2Q_j}| dy + \int_{2Q_j} |f_{2Q_j} - f_{Q_j^*}| dy + \int_{2Q_j} |f_{Q_j^*}| dy \\ &\leq C \cdot ([f]_{BMO^\infty} + [f]_{\Gamma_{\delta_1}}), \end{aligned}$$

where $Q_j^* = 2Q_{j-1} \cap 2Q_j$ and $\ell(Q_j^*) = \ell(Q_j) = 2^{-k}$.

Let $N(D_k)$ be the number of elements of D_k , we have that

$$N(D_k) \leq \frac{\delta_2 - \delta_1}{2^{-k}} + 2 \leq \frac{4(\delta_2 - \delta_1)}{\delta_1} + 2$$

and therefore

$$\int_{\Gamma_{\delta_2} \setminus \Gamma_{\delta_1}} |f| dy \leq C_{\delta_2, \delta_1} \cdot ([f]_{BMO^\mu} + [f]_{\Gamma_{\delta_1}}).$$

The proof is now complete. $\square$

By this observation when we discuss the space bmo_δ^μ there are only four types of spaces

$$bmo_\delta^\mu, \quad bmo_\delta^\infty, \quad bmo_\infty^\mu, \quad bmo_\infty^\infty$$

for finite $\mu, \delta > 0$. If Ω is bounded, it is clear that these four spaces agree with each other. However, if Ω is unbounded, there four spaces may be different because they requires different growth at infinity. Indeed, if $\Omega = (0, \infty)$

$$bmo_\infty^\infty \subsetneq bmo_\delta^\infty$$

since $\log(x+1) \in bmo_\delta^\infty$ while it does not belong to bmo_∞^∞ . Moreover, since $x \in bmo_\delta^\mu$ but it does not belong to neither bmo_∞^μ nor bmo_δ^∞ , we see that

$$bmo_\delta^\infty \subsetneq bmo_\delta^\mu, \quad bmo_\infty^\mu \subsetneq bmo_\delta^\mu.$$

It is possible to prove that $bmo_\infty^\infty = bmo_\infty^\mu$. Indeed as $bmo_\infty^\infty(\Omega) \subset bmo_\infty^\mu(\Omega)$ is simply by the definition of the BMO seminorm. It is sufficient to show the contrary, i.e. $[f]_{BMO^\infty} \leq C \cdot ([f]_{BMO^\mu} + [f]_{\Gamma_\infty})$. Without loss of generality, in defining the seminorm $[\cdot]_{L_{ul}^1(\Gamma_\infty)}$, we set the radius of the ball to be $\frac{\sqrt{n}}{2}$. For $B_r(x) \subset \Omega$ with $r < \mu$,

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |f - f_{B_r(x)}| dy \leq [f]_{BMO^\mu}.$$

For $B_r(x) \subset \Omega$ with $r \geq \mu$, if $r \leq \frac{\sqrt{n}}{2}$, then

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |f - f_{B_r(x)}| dy \leq \frac{2}{|B_r(x)|} \int_{B_{\frac{\sqrt{n}}{2}}(x) \cap \Omega} |f| dy \leq C_{\mu,n} \cdot [f]_{\Gamma_\infty}.$$

If $r > \frac{\sqrt{n}}{2}$, $B_r(x)$ is contained in the cube Q_r with center x and side length $2([r] + 1)$, here $[r]$ is the largest integer less than or equal to r . By dividing each side length of Q_r equally into $2([r] + 1)$ parts, we can divide the cube Q_r into $(2[r] + 2)^n$ subcubes of side length 1. Let S_{Q_r} be the set of these $(2[r] + 2)^n$ subcubes of Q_r . For $Q_r^i \in S_{Q_r}$, let B_r^i be the smallest ball that contains Q_r^i . Let $C_{Q_r} := \{B_r^i \mid Q_r^i \in S_{Q_r}\}$. We have that

$$\int_{B_r(x)} |f| dy \leq \sum_{i=1}^{(2[r]+2)^n} \int_{B_r^i \cap \Omega} |f| dy \leq (2[r] + 2)^n \cdot [f]_{\Gamma_\infty}.$$

Since $r \geq \mu$,

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |f - f_{B_r(x)}| dy \leq \frac{2}{|B_r(x)|} \int_{B_r(x)} |f| dy \leq C_{\mu,n} \cdot [f]_{\Gamma_\infty}.$$

Therefore $bmo_\infty^\infty = bmo_\infty^\mu$ and thus $bmo_\infty^\mu \subsetneq bmo_\mu^\infty$.

We summarize these equivalences.

{B1}

Theorem 2. *Let Ω be an arbitrary domain in $\mathbf{R}^n$. Then*

$$bmo_\infty^\infty(\Omega) = bmo_\infty^\mu(\Omega) \subset bmo_\delta^\infty(\Omega) \subset bmo_\delta^\mu(\Omega)$$

for finite $\delta, \mu > 0$. The inclusions can be strict when Ω is unbounded. If Ω is bounded, all four spaces are the same.

As a simple application of Proposition 1, we conclude that the space $BMO_b^{\mu,\nu}$ is included in bmo_ν^μ since $[f]_\nu \leq c[f]_{b^\nu}$ ($\nu < \infty$) with $c > 0$ depending on ν .

{B2}

Theorem 3. *Let Ω be an arbitrary domain in $\mathbf{R}^n$. For $\mu \in (0, \infty]$ the inclusion*

$$BMO_b^{\mu,\nu}(\Omega) \subset bmo_\nu^\mu(\Omega)$$

holds for $\nu \in (0, \infty)$.

Since b^ν -seminorm controls boundary growth stronger than L^1 sense, this inclusion is in general strict even when Ω is bounded. Here is a simple example when $\Omega = (0, 1)$. The b^ν -seminorm of $f(x) = \log x$ is infinite but $\|f\|_{L^1(\Omega)}$ is finite.

We next discuss the space $vbmo_\delta^{\mu,\nu}$.

{MY}

Remark 4. As proved in [BGST, Theorem 9], if Ω is a bounded Lipschitz domain, the space $BMO_b^{\mu,\nu}$ ($\mu, \nu \in (0, \infty]$) agrees with the Miyachi BMO space [Miy] defined by

$$\begin{aligned} BMO^M(\Omega) &= \{f \in L_{\text{loc}}^1(\Omega) \mid \|f\|_{BMO^M} < \infty\}, \\ \|f\|_{BMO^M} &:= [f]_{BMO^M} + [f]_{b^M}, \\ [f]_{BMO^M} &:= \sup \left\{ \frac{1}{|B_r(x)|} \int_{B_r(x)} |f - f_{B_r(x)}| dy \mid B_{2r}(x) \subset \Omega \right\}, \\ [f]_{b^M} &:= \sup \left\{ \frac{1}{|B_r(x)|} \int_{B_r(x)} |f| dy \mid B_{2r}(x) \subset \Omega \text{ and } B_{5r}(x) \cap \Omega^c \neq \emptyset \right\}. \end{aligned}$$

{E2}

Proposition 5. Let Ω be an arbitrary domain in $\mathbf{R}^n$. Let $0 < \nu_1 \leq \nu_2 \leq \delta \leq \infty$. Then there exists a constant $c > 0$ depending only on n, ν_1, ν_2, δ such that

$$[\nabla d_\Omega \cdot v]_{b^{\nu_2}} \leq [\nabla d_\Omega \cdot v]_{b^{\nu_1}} + c[v]_{\Gamma_\delta}$$

for all $v \in L_{\text{ul}}^1(\Gamma_\delta)$.

Proof. We may assume that $\nu_1 < \infty$. Let $Q_r(x)$ denote a cube centered at x with side length $2r$. Since $|\nabla d| = 1$ and $B_r(x) \subset Q_r(x)$, we see that

$$\begin{aligned} [\nabla d_\Omega \cdot v]_{b^{\nu_2}} - [\nabla d_\Omega \cdot v]_{b^{\nu_1}} &\leq \sup \left\{ \frac{1}{r^n} \int_{B_r(x) \cap \Omega} |\nabla d_\Omega \cdot v| dy \mid x \in \partial\Omega, \nu_1 \leq r < \nu_2 \right\} \\ &\leq \sup \left\{ \frac{1}{r^n} \int_{Q_r(x)} |\tilde{v}| dy \mid x \in \partial\Omega, \nu_1 \leq r \leq \nu_2 \right\} \end{aligned}$$

where $\tilde{v}$ denotes the zero extension of v to $\mathbf{R}^n$. Since $\nu_2 \leq \delta$ so that $Q_r(x) \cap \Omega \subset \Gamma_\delta$, we see that

$$\sup_{x \in \partial\Omega} \int_{Q_r(x)} |\tilde{v}| dy \leq \|v\|_{L_{\text{ul}}^1(\Gamma_\delta)} \quad \text{for } \nu_1 \leq r \leq \nu_2$$

provided that ν_2 is finite by taking an equivalent norm of L_{ul}^1 ; in fact we take $r_0 = \sqrt{n} \nu_2$. This implies that

$$[\nabla d_\Omega \cdot v]_{b^{\nu_2}} - [\nabla d_\Omega \cdot v]_{b^{\nu_1}} \leq \frac{1}{\nu_1^n} [v]_{\Gamma_\delta}.$$

If $\nu_2 = \delta = \infty$, we may assume $r = 2^\ell \nu_1$. We divide $Q_r(x)$ into subcube Q_j , $j = 1, \dots, 2^{\ell n}$ of side length $2\nu_1$. Then

$$\frac{1}{|Q_r(x)|} \int_{Q_r(x)} |\tilde{v}| dy \leq \frac{1}{2^{\ell n} (2\nu_1)^n} \sum_{j=1}^{2^{\ell n}} \int_{Q_j} |\tilde{v}| dy \leq \frac{2^{\ell n}}{2^{\ell n} (2\nu_1)^n} \|\tilde{v}\|_{L_{\text{ul}}^1} \leq \frac{1}{(2\nu_1)^n} \|\tilde{v}\|_{L_{\text{ul}}^1}$$

where r_0 in L^1_{ul} norm is taken as $\sqrt{n} \nu_1$. We thus observe that

$$[\nabla d \cdot v]_{b^{\nu_2}} - [\nabla d \cdot v]_{b^{\nu_1}} \leq c[v]_{\Gamma_\delta}.$$

□

By Propositions 1, 5 we do not need to care about ν . More precisely,

{F3}

Theorem 6. *Let Ω be an arbitrary domain in $\mathbf{R}^n$. Assume that $\mu \in (0, \infty]$ and that $\delta \in (0, \infty]$. Then norms $\|\cdot\|_{vbm\sigma_\delta^{\mu, \nu_1}}$ and $\|\cdot\|_{vbm\sigma_\delta^{\mu, \nu_2}}$ are equivalent provided that $0 < \nu_1 < \nu_2 < \infty$. In the case $\delta = \infty$, we may take $\nu_2 = \infty$.*

In general, different from Theorem 3, the space $vBMO^{\mu, \nu}$ may not be included in bmo_ν^μ even for finite μ by the decomposition (9) and the fact that BMO^μ is not contained in $L^1_{\text{ul}}(\Gamma_\delta)$ for any δ . However, if each connected component of the boundary Γ of Ω has a curved part, we are able to compare these spaces.

{FC}

Definition 7. Let Ω be a uniformly C^1 domain in $\mathbf{R}^n$ and Γ^0 be a connected component of the boundary Γ of Ω . We say that Γ^0 has a *fully curved part* if the set of all normals of Γ^0 spans $\mathbf{R}^n$. In other words, the set $\{\mathbf{n}(x) \in \mathbf{R}^n \mid x \in \Gamma^0\}$ contains n linearly independent vectors, when $\mathbf{n}$ denotes the unit exterior normal of Γ^0 .

We introduce $b^\nu(\Gamma^0)$ -seminorm for convenience. Let us decompose Γ into its connected component Γ^j so that $\Gamma = \bigcup_{j=1}^m \Gamma^j$. We set

$$[f]_{b^\nu(\Gamma^j)} := \sup \left\{ r^{-n} \int_{\Omega \cap B_r(x)} |f(y)| dy \mid x \in \Gamma^j, 0 < r < \nu \right\}.$$

Evidently, $[f]_{b^\nu} = \max_{1 \leq j \leq m} [f]_{b^\nu(\Gamma^j)}$ at least for small $\nu > 0$.

The existence of a fully curved part implies “non-degeneracy” of the seminorm $[\nabla d \cdot f]_{b^\nu}$.

{ND}

Lemma 8. *Let Ω be a uniformly C^2 domain in $\mathbf{R}^n$. Let Γ^j be a connected component of the boundary Γ of Ω . If $c \in \mathbf{R}^n$ satisfies*

$$[\nabla d_\Omega \cdot c]_{b^\nu(\Gamma^j)} = 0,$$

for some $\nu > 0$, then $c = 0$ provided that Γ^j has a fully curved part.

Proof. If Ω is uniformly C^2 , then d_Ω is C^2 in $(\Gamma^j)_\delta$ for sufficiently small $\delta > 0$. Since $-\nabla d_\Omega(x)$ at $x \in \Gamma^j$ equals $\mathbf{n}(x)$, we see that

$$\frac{1}{r^n} \int_{B_r(x) \cap \Omega} \nabla d_\Omega(y) dy \rightarrow c_0 \mathbf{n}(x) \quad \text{as } r \rightarrow 0$$

with scalar constant c_0 . Our assumption implies that $c \cdot \mathbf{n}(x) = 0$ for $x \in \Gamma^j$. If Γ^j has a curved part, then by definition this implies that $c = 0$. $\square$

Here is a few comments on examples of such domains. All connected components of the boundary of a bounded domain, exterior domain has a fully curved part. A perturbed half space

$$\mathbf{R}_\psi^n = \{(x', x_n) \in \mathbf{R}^n \mid x_n > \psi(x'), x' = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}\}$$

with $\psi \in C_c^1(\mathbf{R}^{n-1})$, $\psi \not\equiv 0$ is another example. However, a half-space $\mathbf{R}_+^n$, cylindrical domain $G \times \mathbf{R}^{n-k}$ with $k \geq 1$, $G \subset \mathbf{R}^k$ does not have a boundary having a fully curved part. Our goal is to show that for a domain with boundary components having a fully curved part the space $vBMO^{\mu,\nu}$ is comparable with $vbm\sigma_\delta^{\mu,\nu}$ space if the boundary is compact.

{V1}

Theorem 9. *Let Ω be a C^2 bounded or exterior domain in $\mathbf{R}^n$ so that each component of the boundary has a fully curved part. For $\mu \in (0, \infty]$ and $\nu \in (0, R_*)$ the identity holds:*

$$vBMO^{\mu,\nu}(\Omega) = vbm\sigma_\nu^{\mu,\nu}.$$

Proof. Let Γ^j be a j -th connected component of the boundary $\Gamma = \partial\Omega$ such that $\Gamma = \bigcup_{j=1}^m \Gamma^j$. Since $(\Gamma^j)_\nu$ is C^2 and compact, there is a number $r_0 \in (0, \nu/2)$ such that

$$(\Gamma^j)_\nu = \bigcup_{x \in \Lambda} \text{int } B_{r_0}(x), \quad \Lambda \subset (\Gamma^j)_\nu,$$

where Γ^j is a connected component of Γ and $(\Gamma^j)_\nu$ denotes its ν -neighborhood. The next lemma shows that

$$vBMO^{\nu,\nu}(\Omega) \subset L_{\text{ul}}^1(\Gamma_\nu)$$

which yields the desired result. Note that we may assume $\nu \leq \mu$ by Proposition 1. $\square$

{V2}

Lemma 10. *Under the same assumption of Theorem 9 with $\mu \leq \nu$ assume that $r_0 < \nu/2 < R_*/2$ is taken so that*

$$(\Gamma^i)_\nu = \bigcup_{x \in \Lambda} \text{int } B_{r_0}(x)$$

with some $\Lambda \subset (\Gamma^j)_\nu$. Then there exists $C > 0$ depending only on r_0, n, Γ^j, ν such that

$$\sup_{x \in \Lambda} \frac{1}{|B_{r_0}(x)|} \int_{B_{r_0}(x)} |f(y)| dy \leq C \left([f]_{BMO^\mu((\Gamma^j)_\nu)} + [\nabla d_\Omega \cdot f]_{b^\nu(\Gamma^j)} \right).$$

Proof. We shall suppress r_0 dependence since it is fixed. We shall prove the average $f_{B(x)} =$

$\frac{1}{|B(x)|} \int_{B(x)} f \, dy$ has an estimate

$$\sup_{x \in \Lambda} |f_{B(x)}| \leq C \left([f]_{BMO^\mu((\Gamma^j)_\nu)} + [\nabla d \cdot f]_{b^\nu(\Gamma^j)} \right). \quad (10) \quad \{\text{KI}\}$$

If this is proved, applying the triangle inequality

$$(|f|)_{B(x)} \leq \frac{1}{|B(x)|} \int_{B(x)} |f - f_{B(x)}| \, dy + |f_{B(x)}|$$

yields the desired result.

We shall prove the key inequality (10) by contradiction argument. Assume the inequality (10) were false. Then, there would exist a sequence $\{f^k\}_{k=1}^\infty$ such that

$$1 = \sup_{x \in \Lambda} |f_{B(x)}^k| \geq k \left([f^k]_{BMO^\mu} + [\nabla d_\Omega \cdot f^k]_{b^\nu} \right).$$

Here we suppress $(\Gamma^i)_\nu$ and Γ^j in the right-hand side. Since

$$\sup_{x \in \Lambda} |c^k(x)| = 1 \quad \text{with} \quad c^k(x) = f_{B(x)}^k \in \mathbf{R}^n,$$

there is a sequence $\{x_k\}_{k=1}^\infty$ in Λ with the property

$$1 \geq |c^k(x_k)| \geq 1/2.$$

By taking a subsequence we may assume that x_k converges to some $\hat{x} \in (\Gamma^j)_\nu$ since Γ^j is compact and $d(x_k, \partial(\Gamma^j)_\nu) \geq r_0$, where $d(x_k, A)$ denotes the distance from a point x_k to a set A . Since Γ^j is connected, there is an increasing sequence $\{K_\ell\}_{\ell=1}^\infty$ of connected compact sets in $(\Gamma^j)_\nu$ such that $\text{int } K_\ell \ni \hat{x}$ for $\ell \geq 1$ and $(\Gamma^j)_\nu = \bigcup_{\ell=1}^\infty K_\ell$. By compactness, there is a finite subset Λ_ℓ of Λ with the property that

$$K_\ell \subset \bigcup_{x \in \Lambda_\ell} \text{int } B(x), \quad \Lambda_\ell \subset \Lambda_{\ell+1}$$

and the right-hand side is connected. By taking a further subsequence we may assume that $c^k(x) \rightarrow c(x)$ for $x \in \Lambda_\ell$. However, since $[f^k]_{BMO^\mu} \rightarrow 0$ so that

$$\int_{B(x)} |f^k - c^k| \, dx \rightarrow 0$$

as $k \rightarrow \infty$, we see that $c(x) = c(y)$ if $\text{int } B(x) \cap \text{int } B(y) \neq \emptyset$. Since

$$\bigcup_{x \in \Lambda_\ell} \text{int } B(x)$$

is connected, $c(x)$ is independent of $x \in \Lambda_\ell$, say $c = c_\ell$. By taking a further subsequence of $\{f^k\}$ we may assume that $c^k(x) \rightarrow c_\ell$ in Λ_ℓ . By a diagonal argument there is a subsequence of $\{f^k\}$ such that

$$c^k(x) \rightarrow c \quad \text{for } x \in \bigcup_{\ell=1}^{\infty} \Lambda_\ell =: \Lambda_\infty \subset \Lambda.$$

We thus observe that

$$\int_{B(x)} |f^k(y) - c| dy \rightarrow 0 \quad \text{for } x \in \Lambda_\infty \quad \text{as } k \rightarrow \infty.$$

If we take $B(x)$ such that $\hat{x} \in \text{int } B(x)$, c should not be equal to zero since $|c^k(x_k)| \geq 1/2$ and $x_k \rightarrow \hat{x}$ as $k \rightarrow \infty$. We now invoke the property that

$$\left[\nabla d_\Omega \cdot f^k \right]_{b^\nu} \rightarrow 0.$$

Since

$$(\Gamma^j)_\nu = \bigcup_{x \in \Lambda_\infty} B(x),$$

we observe that $f^k \rightarrow c$ in $L^1_{\text{loc}}((\Gamma^j)_\nu)$. By taking a subsequence we may assume that $f^k(x) \rightarrow c$ for a.e. $x \in (\Gamma^j)_\nu$ so that $\nabla d_\Omega \cdot f^k \rightarrow \nabla d_\Omega \cdot c$, a.e. By lower semicontinuity of integrals (Fatou's lemma) and supremum operation, the seminorm b^ν is lower semicontinuous under this convergence. We thus conclude that

$$[\nabla d_\Omega \cdot c]_{b^\nu} \leq \liminf_{k \rightarrow \infty} \left[\nabla d_\Omega \cdot f^k \right]_{b^\nu} = 0.$$

By Lemma 8 this c must be zero which leads a contradiction. We thus proved the key estimate (10). This completes the proof of Lemma 10. $\square$

3 A variant of Jones' extension theorem

{JE}

Different from L^∞ function it is in general impossible to extend BMO function by setting zero outside the domain. Indeed, the zero-extension of $\log \min(x, 1) \in bmo_\infty^\infty(\mathbf{R}_+^1)$ does not belong to $BMO^\infty(\mathbf{R})$. The goal in this section is to give a linear, extension operator of BMO type function so that the support of extended function is contained in an ε -neighborhood of the original domain, of a function.

For this purpose we recall an extension given by P. W. Jones [PJ]. Since we modify the way of construction, we will give a sketch of this construction. We first recall a dyadic Whitney decomposition of a set A in $\mathbf{R}^n$. Let $\mathcal{A} = \{Q_j\}_{j \in \mathbf{N}}$ be a set of dyadic closed cubes with side

length $\ell(Q_j)$ contained in A satisfying following four conditions.

- (i) $A = \cup_j Q_j$,
- (ii) $\text{int } Q_j \cap \text{int } Q_k = \emptyset$ if $j \neq k$,
- (iii) $\sqrt{n} \leq d(Q_j, \mathbf{R}^n \setminus A) / \ell(Q_j) \leq 4\sqrt{n}$ for all $j \in \mathbf{N}$,
- (iv) $1/4 \leq \ell(Q_k) / \ell(Q_j) \leq 4$ if $Q_j \cap Q_k \neq \emptyset$.

We say that $\mathcal{A}$ is called a dyadic Whitney decomposition of A . Such a decomposition exists for any open sets; see [Ste, Chapter VI, Theorem 1]. Here $d(B, C)$ for sets B, C in $\mathbf{R}^n$ is defined as

$$d(B, C) = \inf \{ |x - y| \mid x \in B, y \in C \}.$$

If B is a point x , we write $d(x, C)$ instead of $d(\{x\}, C)$.

There are at least two important distance functions on $\mathcal{A}$. For $Q_j, Q_k \in \mathcal{A}$, a family $\{Q(\ell)\}_{\ell=0}^m \subset \mathcal{A}$ is called a Whitney chain of length m if $Q(0) = Q_j$ and $Q(m) = Q_k$ such that $Q(\ell) \cap Q(\ell+1) \neq \emptyset$ for ℓ with $0 \leq \ell \leq m-1$. Then the length of the shortest Whitney chain connecting Q_j and Q_k gives a distance on $\mathcal{A}$, which is denoted by $d_1(Q_j, Q_k)$. The second distance for $Q_j, Q_k \in \mathcal{A}$ is defined as

$$d_2(Q_j, Q_k) := \log \left| \frac{\ell(Q_j)}{\ell(Q_k)} \right| + \log \left| \frac{\ell(Q_j, Q_k)}{\ell(Q_j) + \ell(Q_k)} + 1 \right|.$$

Note that d_1 and d_2 are invariant under dilation as well as translation and rotation. P. W. Jones [PJ] gives a necessary and sufficient condition for a domain such that there exists a linear extension operator. A domain Ω is called a uniform domain if there exists constants $a, b > 0$ such that for all $x, y \in \Omega$ there exists a rectifiable curve $\gamma \subset \Omega$ of length $s(\gamma) \leq a|x - y|$ with $\min \{s(\gamma(x, z)), s(\gamma(y, z))\} \leq b d(z, \partial\Omega)$, where $\gamma(x, z)$ denotes the part of γ between x and z on the curve; see e.g. [GO]. It is equivalent to say that there is a constant $K > 0$ such that

$$d_1(Q_j, Q_k) \leq K d_2(Q_j, Q_k) \tag{11} \quad \{\text{UD}\}$$

for all $Q_j, Q_k \in \mathcal{A}$ and some dyadic Whitney decomposition $\mathcal{A}$ of Ω . \{PJ\}

Theorem 11. *Let $A \subset \mathbf{R}^n$ be a uniform domain. Then there is a constant $C(K)$ depending only on K in (11) such that for each $f \in BMO^\infty(A)$ there is an extension $\bar{f} \in BMO^\infty(\mathbf{R}^n)$ satisfying*

$$[\bar{f}]_{BMO^\infty(\mathbf{R}^n)} \leq C(K)[f]_{BMO^\infty(A)}.$$

The operator $f \mapsto \bar{f}$ is a bounded linear operator. Conversely, if there exists such an extension, then A is a uniform domain.

A bounded Lipschitz domain is a typical example of a uniform domain. The constant K in (11) depends only on the Lipschitz regularity of the domain. A Lipschitz half space $\mathbf{R}_\psi^n$ is another example of a uniform domain; here ψ is a Lipschitz function on $\mathbf{R}^{n-1}$.

We next note that if we modify the construction by P. W. Jones, the support of the extension $\bar{f}$ is contained in an ε -neighborhood of $\bar{\Omega}$ if f is also in L_{ul}^1 type space.

{PJM}

Theorem 12. *Let $\Omega \subset \mathbf{R}^n$ be a uniform domain. For each $\varepsilon > 0$ there is a constant $C = C(K, \varepsilon)$ with K in (11) such that for each $f \in bmo_\infty^\infty(\Omega)$ there is an extension $\bar{f} \in bmo_\infty^\infty(\Omega_{2\varepsilon})$ such that*

$$[\bar{f}]_{bmo_\infty^\infty(\Omega_{2\varepsilon})} \leq C[f]_{bmo_\infty^\infty(\Omega)}$$

and $\text{supp } \bar{f} \subset \bar{\Omega}_\varepsilon$, where

$$\Omega_\varepsilon := \{x \in \mathbf{R}^n \mid d(x, \bar{\Omega}) < \varepsilon\}.$$

The operator $f \mapsto \bar{f}$ is a bounded linear operator.

This can be proved almost along the same way as in [PJ]. We shall give an explicit proof.

Proof. Let k_ε be the smallest integer such that $2^{-k_\varepsilon} < \frac{\varepsilon}{5\sqrt{n}}$. So $2^{-k_\varepsilon} \geq \frac{\varepsilon}{10\sqrt{n}}$. Let $E = \{Q_j\}$ be the Whitney decomposition of Ω and $E' = \{Q'_j\}$ be the Whitney decomposition of Ω^c . Let E_* be the set of Whitney cubes in E whose side length is strictly greater than 2^{-k_ε} . For each $Q_m \in E_*$, we define a function g_m on Ω by

$$g_m(x) := \begin{cases} f_{Q_m}, & \text{if } x \in Q_m \\ 0, & \text{else} \end{cases}$$

and we further define a function g on Ω by

$$g := \sum_{Q_m \in E_*} g_m.$$

Here $f_{Q_m} = \frac{1}{|Q_m|} \int_{Q_m} f(y) dy$ for each $Q_m \in E_*$. Let $\tilde{g}$ be the zero extension of g from Ω to $\mathbf{R}^n$.

Without loss of generality, we assume that the radius r_0 of the ball equals 1 in defining the space $L_{\text{ul}}^1(\Omega)$. Notice that

$$\|g_m\|_{L^\infty(\Omega)} \leq \frac{1}{|Q_m|} \cdot \int_{Q_m} |f| dy.$$

Let k_0 be the smallest integer such that $2^{-k_0} < \frac{2}{\sqrt{n}}$. If $\ell(Q_m) \leq 2^{-k_0}$, then $\|f\|_{L^1(Q_m)} \leq [f]_{\Gamma_\infty}$. In this case, as $\ell(Q_m) > 2^{-k_\varepsilon}$,

$$\|g_m\|_{L^\infty(\Omega)} \leq \frac{1}{|Q_m|} \cdot \int_{Q_m} |f| dy \leq \left(\frac{10\sqrt{n}}{\varepsilon}\right)^n \cdot [f]_{\Gamma_\infty}.$$

If $\ell(Q_m) > 2^{-k_0}$, we divide Q_m into $(\frac{\ell(Q_m)}{2^{-k_0}})^n$ small subcubes of side length 2^{-k_0} . Hence,

$$\int_{Q_m} |f| dy = \sum_{i=1}^{(\ell(Q_m)/2^{-k_0})^n} \int_{Q_m^i} |f| dy \leq \left(\frac{\ell(Q_m)}{2^{-k_0}}\right)^n \cdot [f]_{\Gamma_\infty} \leq |Q_m| \cdot n^{\frac{n}{2}} \cdot [f]_{\Gamma_\infty},$$

in this case $\|g_m\|_{L^\infty(\Omega)} \leq n^{\frac{n}{2}} \cdot [f]_{\Gamma_\infty}$. Therefore,

$$\|g\|_{L^\infty(\Omega)} \leq C_{n,\varepsilon} \cdot [f]_{\Gamma_\infty}$$

and we deduce that $g \in bmo_\infty^\infty(\Omega)$ as $L^\infty(\Omega) \subset bmo_\infty^\infty(\Omega)$.

Let $f^* := f - g \in bmo_\infty^\infty(\Omega)$. We do Jones extension to f^* . If Ω is unbounded, for each $Q'_j \in E'$, we find a nearset $Q_j \in E$ satisfying $\ell(Q_j) \geq \ell(Q'_j)$. We define that $\tilde{f}^* = f^*$ on Ω and $\tilde{f}^*(x) = f_{Q_j}^*$ for $x \in Q'_j$. If Ω is bounded, we pick $Q_0 \in E$ such that $\ell(Q_0) = \sup_{Q_j \in E} \ell(Q_j)$. We define that $\tilde{f}^* = f^*$ on Ω , $\tilde{f}^*(x) = f_{Q_j}^*$ for $x \in Q'_j$ where $\ell(Q'_j) \leq \ell(Q_0)$ and $\tilde{f}^*(x) = f_{Q_0}^*$ for $x \in Q'_j$ where $\ell(Q_j) > \ell(Q_0)$. By Jones [PJ], $\tilde{f}^* \in BMO$ and $[\tilde{f}^*]_{BMO} \leq C_K \cdot [f^*]_{BMO^\infty(\Omega)}$. By this extension, for $\tilde{f}^*(x) \neq 0$, either $x \in \Omega$ or $x \in Q'_j$ such that $\ell(Q'_j) \leq 2^{-k_\varepsilon}$. Since $d(Q'_j, \Omega) \leq 4\sqrt{n} \cdot \ell(Q'_j)$, pick $x \in \overline{Q'_j}$ and $z \in \Gamma$ such that $|x - z| = d(Q'_j, \Omega)$. For any $y \in Q'_j$, $|y - z| \leq |y - x| + |x - z| \leq 5\sqrt{n} \cdot \ell(Q'_j)$. So $\text{int } Q'_j \subset B_{5\sqrt{n} \cdot \ell(Q'_j)}(z)$ for some $z \in \Gamma$. Since $5\sqrt{n} \cdot \ell(Q'_j) \leq 5\sqrt{n} \cdot 2^{-k_\varepsilon} < \varepsilon$, $\text{int } Q'_j \subset \Omega_\varepsilon$. Let $\tilde{f} := \tilde{f}^* + \tilde{g}$ and $\bar{f} = \tilde{f}|_{\Omega_{2\varepsilon}}$, we have that $\text{supp } \bar{f} \subset \overline{\Omega_\varepsilon}$ and by previous calculation,

$$\begin{aligned} [\bar{f}]_{BMO^\infty(\Omega_{2\varepsilon})} &\leq [\tilde{f}]_{BMO} \leq [\tilde{f}^*]_{BMO} + [\tilde{g}]_{BMO} \leq C_K \cdot [f^*]_{BMO^\infty(\Omega)} + 2\|g\|_\infty \\ &\leq C_{K,n,\varepsilon} \cdot ([f]_{BMO^\infty(\Omega)} + [f]_{\Gamma_\infty}). \end{aligned}$$

Let $B(x)$ denotes the ball of radius 1 centered at x and $\Gamma^\varepsilon := \{x \in \Omega^c \mid d_\Omega(x) < \varepsilon\}$. For $B(x) \cap \Omega_\varepsilon \neq \emptyset$,

$$\int_{B(x) \cap \Omega_\varepsilon} |\bar{f}| dy = \int_{B(x) \cap \Omega} |f| dy + \int_{B(x) \cap \Gamma^\varepsilon} |\bar{f}| dy.$$

The first integral on the right hand side is directly estimated by $[f]_\infty$, so we only need to consider the second integral. Let Q'_* be a largest Whitney cube in E' that intersects $B(x) \cap \Gamma^\varepsilon$. For $Q'_j \in E'$, [PJ, Lemma 2.10] says that if $Q_j \in E$ is a nearest Whitney cube satisfying $\ell(Q_j) \geq \ell(Q'_j)$, then $d(Q_j, Q'_j) \leq 65K^2 \cdot \ell(Q'_j)$. Consider $Q'_j \in E'$ such that $Q'_j \cap B(x) \cap \Gamma^\varepsilon \neq \emptyset$, let $x_j \in Q_j$ where Q_j is a nearest Whitney cube satisfying $\ell(Q_j) \geq \ell(Q'_j)$, let $x'_j \in Q'_j \cap B(x) \cap \Gamma^\varepsilon$ and $x'_* \in Q'_* \cap B(x) \cap \Gamma^\varepsilon$. By choosing K large such that $K^2 \geq 2\sqrt{n}$, we have that

$$|x_j - x'_*| \leq |x'_* - x'_j| + |x'_j - x_j| \leq 2 + 2\sqrt{n} \cdot \ell(Q'_j) + 65K^2 \cdot \ell(Q'_j) \leq 2 + 66K^2 \cdot \ell(Q'_j).$$

Since $\ell(Q_j) \leq 2\ell(Q'_j) \leq 2\ell(Q'_*) \leq 2\ell(Q_*)$ where $Q_* \in E$ is a nearest cube satisfying $\ell(Q_*) \geq$

$$\ell(Q'_*), |x_j - x'_*| \leq 2 + 132K^2 \cdot \ell(Q_*).$$

If $B(x) \cap \Gamma \neq \emptyset$, then $\sqrt{n} \cdot \ell(Q'_*) \leq d(Q'_*, \Omega) \leq 2$. Hence $\ell(Q_*) \leq 2\ell(Q'_*) \leq \frac{4}{\sqrt{n}}$, for any $x_j \in Q_j$, $|x_j - x'_*| < 2 + 132K^2 \cdot \frac{4}{\sqrt{n}}$. Consider the cube $\widetilde{Q}'_*$ with center x'_* and side length $4 + \frac{1064K^2}{\sqrt{n}}$. For each $Q'_j \in E'$ such that $Q'_j \cap B(x) \cap \Gamma^\varepsilon \neq \emptyset$, the corresponding nearest $Q_j \in E$ such that $\ell(Q_j) \geq \ell(Q'_j)$ we choose to define $\widetilde{f}^*$ is contained in $\widetilde{Q}'_*$, i.e. $Q_j \subset \widetilde{Q}'_*$. Hence,

$$\int_{B(x) \cap \Gamma^\varepsilon} |\bar{f}| dy = \sum_{\substack{Q'_j \in E', \\ Q'_j \cap B(x) \cap \Gamma^\varepsilon \neq \emptyset}} \int_{Q'_j \cap B(x) \cap \Gamma^\varepsilon} |f_{Q'_j}^*| dy \leq \int_{\widetilde{Q}'_* \cap \Omega} |f^*| dy.$$

Let p be the largest integer such that $2^{-p} > 4 + \frac{1064K^2}{\sqrt{n}}$, so $2^{-p} \leq 8 + \frac{2128K^2}{\sqrt{n}}$. Let $\widetilde{Q}'_*$ be contained in a larger cube $\widetilde{Q}$ where $\widetilde{Q}$ has center x'_* and side length 2^{-p} . We can divide $\widetilde{Q}$ into $(\frac{2^{-p}}{2^{-k_0}})^n$ subcubes of side length 2^{-k_0} , thus

$$\int_{\widetilde{Q}'_* \cap \Omega} |f^*| dy \leq \sum_{i=1}^{(2^{-p}/2^{-k_0})^n} \int_{\widetilde{Q}_i \cap \Omega} |f^*| dy \leq \left(\frac{2^{-p}}{2^{-k_0}}\right)^n \cdot [f^*]_{\Gamma_\infty} \leq C_{K,n} \cdot [f^*]_{\Gamma_\infty}.$$

If $B(x) \cap \Gamma = \emptyset$, i.e. $B(x) \subset \overline{\Omega}^c$. Let $E'_1 := \{Q'_j \in E' \mid Q'_j \cap B(x) \neq \emptyset\}$. Let $\ell_m := \inf_{Q'_j \in E'_1} \ell(Q'_j)$ and Q'_* be a largest $Q'_j \in E'_1$. If $\ell_m = 0$, then there exists $z \in \Gamma \cap \partial B(x)$. In this case, $\sqrt{n} \cdot \ell(Q'_*) \leq d(Q'_*, \Omega) \leq 2$. Therefore same argument as in the case where $B(x) \cap \Gamma \neq \emptyset$ gives that $\|\bar{f}\|_{L^1(B(x) \cap \Gamma^\varepsilon)} \leq C_{K,n} \cdot [f^*]_\infty$. If $0 < \ell_m \leq 2$, then pick $Q'_m \in E'_1$ such that $\ell(Q'_m) = \ell_m$. Since $\sqrt{n} \cdot \ell(Q'_*) \leq d(Q'_*, \Omega) \leq 2 + \sqrt{n} \cdot \ell(Q'_m) + d(Q'_m, \Omega) \leq 2 + 10\sqrt{n}$, we have that $\ell(Q_*) \leq \frac{4}{\sqrt{n}} + 20$. Hence $|x_j - x'_*| \leq 2 + 132K^2 \cdot (\frac{4}{\sqrt{n}} + 20)$. Following the argument as in the case where $B(x) \cap \Gamma \neq \emptyset$, we can deduce that $\|\bar{f}\|_{L^1(B(x) \cap \Gamma^\varepsilon)} \leq C_{K,n} \cdot [f^*]_{\Gamma_\infty}$. If $\ell_m > 2$, then $B(x)$ intersects at most 2^n Whitney cubes in E' . Without loss of generality, assume that E'_1 has 2^n elements. Then

$$\int_{B(x) \cap \Gamma^\varepsilon} |\bar{f}| dy \leq \sum_{Q'_i \in E'_1} \int_{B(x) \cap Q'_i} |f_{Q'_i}^*| dy \leq \sum_{Q'_i \in E'_1} \frac{|B(x) \cap Q'_i|}{|Q_i|} \cdot \int_{Q_i} |f^*| dy.$$

Divide Q_i into $\left(\frac{\ell(Q_i)}{2^{-k_0}}\right)^n$ subcubes of side length 2^{-k_0} , we have that

$$\int_{Q_i} |f^*| dy \leq \left(\frac{\ell(Q_i)}{2^{-k_0}}\right)^n \cdot [f^*]_{\Gamma_\infty} \leq |Q_i| \cdot n^{\frac{n}{2}} \cdot [f^*]_{\Gamma_\infty}.$$

Therefore,

$$\int_{B(x) \cap \Gamma^\varepsilon} |\bar{f}| dy \leq \left(\sum_{Q'_i \in E'_1} |B(x) \cap Q'_i| \right) \cdot n^{\frac{n}{2}} \cdot [f^*]_{\Gamma_\infty} \leq C_n \cdot [f^*]_{\Gamma_\infty}.$$

Since $[f^*]_{\Gamma_\infty} \leq [f]_{\Gamma_\infty} + [g]_{\Gamma_\infty}$ and $[g]_{\Gamma_\infty}$ is estimated by $C_{n,\varepsilon} \cdot [f]_{\Gamma_\infty}$, we are done. $\square$

As an application we give an estimate for the product of a Hölder function and a function in bmo_∞^∞ . We first recall properties of point multipliers. It is known that for a local hardy space $h^1 = F_{1,2}^0$ [Sa, Theorem 3.18], there is a constant C such that

$$\|\varphi g\|_{F_{1,2}^0} \leq C \|\varphi\|_{C^\gamma} \|g\|_{F_{1,2}^0} \quad g \in F_{1,2}^0 \quad (12) \quad \{\text{MHA}\}$$

for $\varphi \in C^\gamma(\mathbf{R}^n)$, $\gamma \in (0, 1)$, where

$$\|\varphi\|_{C^\gamma} = \sup_{x \in \mathbf{R}^n} |\varphi(x)| + \sup_{\substack{x, y \in \mathbf{R}^n \\ x \neq y}} |\varphi(x) - \varphi(y)| / |x - y|^\gamma;$$

see e.g. [Sa, Remark 4.4]. Since

$$bmo = BMO^\infty(\mathbf{R}^n) \cap L_{\text{ul}}^1(\mathbf{R}^n)$$

equals to $F_{\infty,2}^0$ [Sa, Theorem 3.26], it is a dual space of $h^1 = F_{1,2}^0$ [Sa, Theorem 3.22]. Thus

$$\|\varphi f\|_{bmo} \leq C \|\varphi\|_{C^\gamma} \|f\|_{bmo}. \quad (13) \quad \begin{matrix} \{\text{ME}\} \\ \{\text{PMU}\} \end{matrix}$$

Theorem 13. *Let $\Omega \subset \mathbf{R}^n$ be a uniform domain. Let $\varphi \in C^\gamma(\Omega)$, $\gamma \in (0, 1)$. For each $f \in bmo_\infty^\infty(\Omega)$, the function $\varphi f \in bmo_\infty^\infty(\Omega)$ satisfies*

$$\|\varphi f\|_{bmo_\infty^\infty(\Omega)} \leq C \|\varphi\|_{C^\gamma(\Omega)} \|f\|_{bmo_\infty^\infty(\Omega)}$$

with C independent of φ and f .

Proof. By [McS], there exists $\bar{\varphi} \in C^\gamma(\mathbf{R}^n)$ such that $\bar{\varphi}|_\Omega = \varphi$ and

$$\|\bar{\varphi}\|_{C^\gamma(\mathbf{R}^n)} \leq \|\varphi\|_{C^\gamma(\Omega)}.$$

For our current purpose it suffices to set $\bar{\varphi} = \max\{\min\{\varphi_*, \|\varphi\|_\infty\}, -\|\varphi\|_\infty\}$ with

$$\varphi_*(x) = \inf_{y \in \Omega} \{\varphi(y) + [\varphi]_{C^\gamma} \cdot |x - y|^\gamma\},$$

where $\|\varphi\|_{C^\gamma(\Omega)} = \|\varphi\|_{L^\infty(\Omega)} + [\varphi]_{C^\gamma(\Omega)}$, $\|\varphi\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} |\varphi(x)|$ and $[\varphi]_{C^\gamma(\Omega)} = \sup_{\substack{x, y \in \Omega \\ x \neq y}} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\gamma}$; we often suppress Ω . By definition $\varphi_*(x) \leq \varphi(x)$. Moreover, since $\varphi(x) \leq \varphi(y) + [\varphi]_{C^\gamma} \cdot |x - y|^\gamma$ for $x, y \in \Omega$, we see that $\varphi(x) \leq \varphi_*(x)$ which implies $\varphi = \varphi_*$ on Ω . For any $x \in \mathbf{R}^n$ and $\varepsilon > 0$ there is $y_\varepsilon \in \Omega$ such that

$$\varphi(y_\varepsilon) + [\varphi]_{C^\gamma} \cdot |x - y_\varepsilon|^\gamma \leq \varphi_*(x) + \varepsilon.$$

For $x_1 \in \mathbf{R}^n$ we observe that

$$\varphi_*(x_1) - \varphi_*(x) \leq \varphi(y_\varepsilon) + [\varphi]_{C^\gamma} \cdot |x_1 - y_\varepsilon|^\gamma - \{\varphi(y_\varepsilon) + [\varphi]_{C^\gamma} \cdot |x - y_\varepsilon|^\gamma\} + \varepsilon \leq [\varphi]_{C^\gamma} \cdot |x - x_1|^\gamma + \varepsilon.$$

Since ε is arbitrary, we see that $\varphi_*(x_1) - \varphi_*(x) \leq [\varphi]_{C^\gamma} \cdot |x - x_1|^\gamma$. Interchanging the role of x_1 and x , we conclude that

$$[\varphi_*]_{C^\gamma(\mathbf{R}^n)} \leq [\varphi]_{C^\gamma(\Omega)}.$$

Since $\|\varphi\|_\infty < \infty$, $\bar{\varphi} = \varphi$ on Ω and $\bar{\varphi}$ is still Hölder. More precisely, $[\bar{\varphi}]_{C^\gamma} \leq [\varphi_*]_{C^\gamma}$. By definition $\|\bar{\varphi}\|_\infty \leq \|\varphi\|_\infty$ so we conclude that $\|\bar{\varphi}\|_{C^\gamma} \leq \|\varphi\|_{C^\gamma}$.

Extending $f \in bmo_\infty^\infty(\Omega)$ to $\bar{f} \in bmo$ by Theorem 12, we conclude from multiplication estimate (13) that

$$\begin{aligned} \|\varphi f\|_{bmo_\infty^\infty(\Omega)} &\leq \|\bar{\varphi} \bar{f}\|_{bmo} \\ &\leq C \cdot \|\bar{\varphi}\|_{C^\gamma(\mathbf{R}^n)} \cdot \|\bar{f}\|_{bmo} \\ &\leq C \cdot \|\varphi\|_{C^\gamma(\Omega)} \cdot \|f\|_{bmo_\infty^\infty(\Omega)}. \end{aligned}$$

□ {POZ}

Remark 14. If we prove that the extension $f \mapsto \bar{f}$ constructed in Theorem 11 is bounded from bmo_∞^∞ to $bmo = BMO \cap L_{ul}^1$, then the support condition will follow by taking $\varphi \in C^\gamma(\mathbf{R}^n)$ in Theorem 13 as a cutoff function of Ω , i.e. $\varphi \equiv 1$ on Ω with $\text{supp} \varphi \subset \Omega_\varepsilon$. In other words, we consider $f \mapsto \varphi \bar{f}$. However, the proof that $\bar{f} \in L_{ul}^1$ needs some argument so we give a direct proof of Theorem 12.

For $BMO_b^{\mu, \infty}$ function in Ω it is easy to see that its zero extension is in BMO space; see e.g. [BGST, Lemma 4].

{ZE}

Theorem 15. Let Ω be an arbitrary domain in $\mathbf{R}^n$. Assume that $\mu \in (0, \infty]$. For $f \in BMO_b^{\mu, \nu}(\Omega)$ with $\nu \geq 2\mu$, let f_0 be the zero extension to $\mathbf{R}^n$, i.e. $f_0(x) = 0$ for $x \in \Omega^c$ and $f_0(x) = f(x)$ for $x \in \Omega$. Then $f_0 \in BMO^\mu(\mathbf{R}^n)$ and $[f_0]_{BMO^\mu} \leq C[f]_{BMO_b^{\mu, \nu}}$ with C independent of f .

Proof. If the ball B of radius $\leq \mu$ is in Ω , then

$$\frac{1}{|B|} \int_B |f_0 - f_{0B}| dy \leq [f]_{BMO^\mu}.$$

If B is in Ω^c , then $\int_B |f_0 - f_{0B}| dy = 0$. It remains to estimate the integral if B has nonempty intersection with the boundary $\Gamma = \partial\Omega$. For each $B_r(x) \cap \Gamma \neq \emptyset$, $r < \mu$, we take $x_0 \in B_r(x) \cap \Gamma$.

Then, $B_r(x) \subset B_{2r}(x_0)$ and thus

$$\frac{1}{|B_r(x)|} \int_{B_r(x)} |f_0 - f_{0_{B_r(x)}}| dy \leq \frac{2}{|B_r(x)|} \int_{B_{2r}(x_0)} |f_0| dy \leq \frac{2^{n+1}}{\omega_n} \cdot [f]_{b^{2\mu}},$$

where ω_n is the volume of an n -dimensional ball. □

{BGST}

Remark 16. In [BGST, Lemma 4], it is assumed that $\Omega = \Omega' \times \mathbf{R}^{n-k}$ where Ω' is a bounded Lipschitz domain in $\mathbf{R}^k$. However, from the proof above it is clear that we do not need this requirement. Thus we give a full proof here.

As an application of boundedness of multiplication, we give invariance of function spaces under coordinate changes. We say that Ψ is a global $C^{k+\beta}$ (resp. C^k)-diffeomorphism if $C^{k+\beta}$ (resp. C^k)-norm of Ψ and Ψ^{-1} are bounded in $\mathbf{R}^n$, where $k \in \mathbf{N}$ and $\beta \in (0, 1)$. {CC}

Proposition 17. *The space bmo is invariant under bi-Lipschitz coordinate change and the space h^1 is invariant under global $C^{1+\beta}$ -diffeomorphism.*

Proof. For $f \in bmo$, by a simple change of variables on the equivalent definition of the seminorm $[f]_{BMO}$ where

$$[f]_{BMO} = \sup_{B \subset \mathbf{R}^n} \inf_{c \in \mathbf{R}} \int_B |f(y) - c| dy,$$

see e.g. [Gra, Proposition 3.1.2], we can easily deduce that bmo is invariant under bi-Lipschitz coordinate change.

Let $g \in h^1(\mathbf{R}^n)$ and Ψ be a global $C^{1+\beta}$ -diffeomorphism. We have that

$$\|g \circ \Psi\|_{h^1} = \sup_{\|f\|_{bmo} \leq 1} \left| \int_{\mathbf{R}^n} f \cdot g \circ \Psi dy \right|.$$

By change of variable we have that

$$\left| \int_{\mathbf{R}^n} f(y) \cdot g \circ \Psi(y) dy \right| = \left| \int_{\mathbf{R}^n} f \circ \Psi^{-1}(x) \cdot g(x) \cdot J_{\Psi^{-1}}(x) dx \right|,$$

where $J_{\Psi^{-1}}$ is the Jacobian which is of regularity C^β . Then by the $bmo-h^1$ duality [Sa, Theorem 3.22] and multiplication estimate (12), we deduce that

$$\left| \int_{\mathbf{R}^n} f \circ \Psi^{-1} \cdot g \cdot J_{\Psi^{-1}} dx \right| \leq \|f \circ \Psi^{-1}\|_{bmo} \cdot \|g J_{\Psi^{-1}}\|_{h^1} \leq \|f \circ \Psi^{-1}\|_{bmo} \cdot \|J_{\Psi^{-1}}\|_{C^\beta} \cdot \|g\|_{h^1}.$$

Since bmo is independent of bi-Lipschitz coordinate change, we have that

$$\|g \circ \Psi\|_{h^1} \leq C \cdot \|\nabla \Psi^{-1}\|_{L^\infty} \cdot \|J_{\Psi^{-1}}\|_{C^\beta} \cdot \|g\|_{h^1}$$

for some constant C independent of g and Ψ . □

{CCL}

Proposition 18. *The space $F_{1,2}^1(\mathbf{R}^n)$ is independent of global $C^{1+\beta}$ -diffeomorphism.*

Proof. Let $g \in F_{1,2}^1$ and Ψ be a global $C^{1+\beta}$ -diffeomorphism. By multiplication estimate (12) and Proposition 17, we have that

$$\|\nabla(g \circ \Psi)\|_{F_{1,2}^0} \leq C \cdot \|\nabla \Psi\|_{C^\beta} \cdot \|(\nabla g) \circ \Psi\|_{F_{1,2}^0} \leq C \cdot \|\nabla \Psi\|_{C^\beta} \cdot \|\nabla \Psi^{-1}\|_{L^\infty} \cdot \|J_{\Psi^{-1}}\|_{C^\beta} \cdot \|g\|_{F_{1,2}^1},$$

where $J_{\Psi^{-1}}$ is the Jacobian for Ψ^{-1} and C is a constant independent of g and Ψ . Hence $\nabla(g \circ \Psi) \in F_{1,2}^0$. Since the differentiation mapping is bounded from $F_{p,q}^s$ to $F_{p,q}^{s-1}$ for $p \in (0, \infty)$, $q \in (0, \infty]$ and $s \in \mathbf{R}$, see e.g. [Sa, Theorem 2.12], we have that $\Delta(g \circ \Psi) \in F_{1,2}^{-1}$. Since $F_{1,2}^1 \hookrightarrow F_{1,2}^0$, Proposition 17 tells us that $g \circ \Psi \in F_{1,2}^0$ and thus $g \circ \Psi \in F_{1,2}^{-1}$. Therefore, $(I - \Delta)(g \circ \Psi) \in F_{1,2}^{-1}$. Notice that [Sa, Theorem 2.12] also tells us that for $\sigma \in \mathbf{R}$, $(I - \Delta)^\sigma$ is an isomorphism from $F_{p,q}^s$ to $F_{p,q}^{s-2\sigma}$. Hence by letting $\sigma = -1$, we deduce that

$$\begin{aligned} \|g \circ \Psi\|_{F_{1,2}^1} &= \|(I - \Delta)^{-1}(I - \Delta)(g \circ \Psi)\|_{F_{1,2}^1} \\ &= C \cdot \|(I - \Delta)(g \circ \Psi)\|_{F_{1,2}^{-1}} \\ &\leq C \cdot \left(\|g \circ \Psi\|_{F_{1,2}^0} + \|\nabla(g \circ \Psi)\|_{F_{1,2}^0} \right) \\ &\leq C \cdot (1 + \|\nabla \Psi\|_{C^\beta}) \cdot \|\nabla \Psi^{-1}\|_{L^\infty} \cdot \|J_{\Psi^{-1}}\|_{C^\beta} \cdot \|g\|_{F_{1,2}^1}, \end{aligned}$$

where C is a constant independent of g and Ψ . □

{CCLR}

Remark 19. The proof of Proposition 18 also says that $F_{1,2}^1 = \{f \in F_{1,2}^0 \mid \nabla f \in (F_{1,2}^0)^n\}$.

4 Trace problems

{TR}

In this section we show that the normal trace of a vector field in $vbmo_\delta^{\mu,\nu}$ is in $L^\infty(\Gamma)$ if its divergence is well controlled. We begin with the case that Ω is the half space $\mathbf{R}_+^n$.

We first recall that the trace operator $(Tr f)(x') = f(x', 0)$ for $f \in F_{1,2}^1(\mathbf{R}^n)$ gives a surjective bounded linear operator from $F_{1,2}^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^{n-1})$; see [Tr92, Section 4.4.3].

{HA}

Proposition 20 ([Tr92]). *The operator Tr from $F_{1,2}^1$ to $L^1(\mathbf{R}^{n-1})$ is surjective for $n \geq 2$. Actually, surjectivity holds for a smaller space $B_{1,1}^1$. The inverse operator is called the extension and it is a bounded operator.*

For a C^2 domain Ω a normal trace $v \cdot \mathbf{n}$ on $\Gamma = \partial\Omega$ of v is well-defined as an element of $W_{p,\text{loc}}^{-1/p}(\Gamma)$ if v and $\text{div } v$ is in L_{loc}^p ; see e.g. [FM] or [Gal]. If $v \in vbmo_\delta^{\mu,\nu}(\Omega)$ so that $v \in L_{\text{loc}}^1$,

then by an interpolation inequality (see e.g. [BGST, Theorem 11]) v is in L_{loc}^p for any $p \geq 1$. Thus if $\text{div } v$ is in L_{loc}^p , $v \cdot \mathbf{n}$ is well-defined. We derive L^∞ estimate for $v \cdot \mathbf{n}$ when Ω is the half space.

{NTH}

Theorem 21. *Let μ, ν, δ be in $(0, \infty]$ and $n \geq 2$. Then there is a constant $C = C(\mu, \nu, \delta, n)$ such that*

$$\|v \cdot \mathbf{n}\|_{L^\infty(\mathbf{R}^{n-1})} \leq C \left(\|v\|_{vbm o_\delta^{\mu, \nu}(\mathbf{R}_+^n)} + \|\text{div } v\|_{L_{\text{ul}}^n(\Gamma_\delta)} \right)$$

for all $v \in vbm o_\delta^{\mu, \nu}(\mathbf{R}_+^n)$.

Proof. Let $v \in vbm o_\delta^{\mu, \nu}(\mathbf{R}_+^n)$, by definition the n -th component v_n of $v = (v', v_n)$ belongs to $BMO_b^{\mu, \nu}(\mathbf{R}_+^n)$. For $x'_0 \in \mathbf{R}^{n-1}$, we consider the region $U = B_1(x'_0) \times (-\delta, \delta)$ where $B_1(x'_0)$ denotes the ball in $\mathbf{R}^{n-1}$ centered at x'_0 with radius 1. Let v_{re} denotes the restriction of v on $U \cap \mathbf{R}_+^n$, i.e. $v_{\text{re}} = v|_{U \cap \mathbf{R}_+^n}$. We have that $v_{\text{re}} \in bmo_\infty^\infty(U \cap \mathbf{R}_+^n)$ and

$$\sup_{\substack{x' \in B_1(x'_0) \\ r < \nu}} \frac{1}{|B_r((x', 0))|} \int_{B_r((x', 0))} |(v_{\text{re}})_n| dy < \infty.$$

Let $\overline{(v_{\text{re}})_n}$ be the zero extension of $(v_{\text{re}})_n$ to U . By Theorem 15, $\overline{(v_{\text{re}})_n}$ is in $BMO^\infty(U)$. Let $\overline{v'_{\text{re}}}$ be the even extension of v'_{re} to U of the form

$$\overline{v'_{\text{re}}}(x', x_n) = \begin{cases} v'_{\text{re}}(x', x_n), & x' \in B_1(x'_0) \text{ and } x_n > 0 \\ v'_{\text{re}}(x', -x_n), & x' \in B_1(x'_0) \text{ and } x_n < 0 \end{cases} \quad (14)$$

and set $\tilde{v} = (\overline{v'_{\text{re}}}, \overline{(v_{\text{re}})_n})$. We have that $\tilde{v} \in bmo_\infty^\infty(U)$. By Theorem 12 its Jones' extension v_U belongs to $bmo_\infty^\infty(\mathbf{R}^n)$.

Integration by parts formally yields

$$\int_{\mathbf{R}^{n-1}} v_U \cdot \mathbf{n} \rho dx' = \int_{\mathbf{R}_+^n} (\text{div } v_U) \rho dx - \int_{\mathbf{R}_+^n} v_U \cdot \nabla \rho dx. \quad (15) \quad \{\text{INT}\}$$

By Proposition 20 there is an extension operator $\text{Ext} : L^1(\mathbf{R}^{n-1}) \rightarrow F_{1,2}^1(\mathbf{R}^n)$ such that $\text{Tr} \circ \text{Ext}$ is the identity operator on L^1 . For $\varphi \in C_c^\infty(B_{\frac{1}{2}}(x'_0))$ we set $\sigma = \text{Ext } \varphi$. By multiplying a cut off function $\theta \in C_c^\infty(U)$ such that $\theta \equiv 1$ in $\frac{1}{2}U$ and consider $\rho = \theta \sigma$ we still find $\rho \in F_{1,2}^1(\mathbf{R}^n)$ by a multiplier theorem [Sa, Theorem 3.18], [Tr92, Section 4.2.2]. We estimate (15) to get

$$\begin{aligned} \left| \int_{\mathbf{R}^{n-1}} v_U \cdot \mathbf{n} \rho dx' \right| &\leq \left| \int_U (\text{div } v_U) \rho dx \right| + \left| \int_{\mathbf{R}_+^n} v'_U \cdot \nabla' \rho dx \right| \\ &\quad + \left| \int_{\mathbf{R}^n} v_{U^n} \frac{\partial \rho}{\partial x_n} dx \right| = I + II + III. \end{aligned}$$

We may assume that ρ is even in x_n by taking $(\rho(x', x_n) + \rho(x', -x_n))/2$ so that the second term is estimated by bmo - h^1 duality $(h^1)^* = (F_{1,2}^0)^* = F_{\infty,2}^0 = bmo$ as follows

$$\begin{aligned} II &= \left| \int_{\mathbf{R}_+^n} v'_U \cdot \nabla' \rho \, dx \right| = \frac{1}{2} \left| \int_{\mathbf{R}^n} v'_U \cdot \nabla' \rho \, dx \right| \\ &\leq C \|v'_U\|_{bmo} \|\nabla' \rho\|_{h^1}. \end{aligned}$$

The third term is estimated as

$$III \leq C \|v_{U^n}\|_{bmo} \left\| \frac{\partial \rho}{\partial x_n} \right\|_{h^1}.$$

The first term is estimated by

$$\begin{aligned} I &\leq \|\operatorname{div} v_U\|_{L^n(U)} \|\rho\|_{L^{n/(n-1)}(U)} \\ &\leq C \|\operatorname{div} v\|_{L_{\text{ul}}^n(\Gamma_\delta)} \|\nabla \rho\|_{L^1(U)} \end{aligned}$$

by the Sobolev inequality. Since $\|\nabla \rho\|_{L^1} \leq \|\nabla \rho\|_{h^1}$ and $\|\nabla \rho\|_{h^1} \leq \|\rho\|_{F_{1,2}^1} \leq C \|\varphi\|_{L^1(B_{\frac{1}{2}}(x'_0))}$, collecting these estimates yields

$$\left| \int_{B_{\frac{1}{2}}(x'_0)} v \cdot \mathbf{n} \varphi \, dx' \right| \leq C \|\varphi\|_{L^1(B_{\frac{1}{2}}(x'_0))} \left(\|v\|_{vbm o_\delta^{\mu,\nu}(\mathbf{R}_+^n)} + \|\operatorname{div} v\|_{L_{\text{ul}}^n(\Gamma_\delta)} \right).$$

This yields the desired estimate since $C_c^\infty(B_{\frac{1}{2}}(x'_0))$ is dense in $L^1(B_{\frac{1}{2}}(x'_0))$ and C in the right-hand side is independent of $x'_0 \in \mathbf{R}^{n-1}$. $\square$

We now consider a curved domain. Let Ω be a uniformly C^2 domain in $\mathbf{R}^n$ so that the reach R_* of Γ is positive and $\beta \in (0, 1)$.

{NTG}

Theorem 22. *Let Ω be a uniformly $C^{2+\beta}$ domain in $\mathbf{R}^n$ with $n \geq 2$. Let μ, ν, δ be in $(0, \infty]$. Then there is a constant $C = C(\mu, \nu, \delta, \Omega)$ such that*

$$\|v \cdot \mathbf{n}\|_{L^\infty(\Gamma)} \leq C \left(\|v\|_{vbm o_\delta^{\mu,\nu}(\Omega)} + \|\operatorname{div} v\|_{L_{\text{ul}}^n(\Gamma_\delta)} \right)$$

for all $v \in vbm o_\delta^{\mu,\nu}(\Omega)$.

We shall prove this result by localizing the problems near the boundary and use normal (principal) coordinates. Let Ω be a uniformly $C^{2+\beta}$ domain. In other words, there exist $r_*, \delta_* > 0$ such that for each $z_0 \in \Gamma$, up to translation and rotation, there exists a function $h_{z_0} \in$

$C^{2+\beta}(B_{r_*}(0'))$ with

$$\begin{aligned} |\nabla^k h_{z_0}| &\leq L \text{ in } B_{r_*}(0') \text{ for } k = 0, 1, 2, \\ [\nabla^2 h_{z_0}]_{C^\beta(B_{r_*}(0'))} &< \infty, \nabla' h_{z_0}(0') = 0', h_{z_0}(0') = 0 \end{aligned}$$

such that the neighborhood

$$U_{r_*, \delta_*, h_{z_0}}(z_0) := \{(x', x_n) \in \mathbf{R}^n \mid h_{z_0}(x') - \delta_* < x_n < h_{z_0}(x') + \delta_*, |x'| < r_*\}$$

satisfies

$$\Omega \cap U_{r_*, \delta_*, h_{z_0}}(z_0) = \{(x', x_n) \in \mathbf{R}^n \mid h_{z_0}(x') < x_n < h_{z_0}(x') + \delta_*, |x'| < r_*\}$$

and

$$\partial\Omega \cap U_{r_*, \delta_*, h_{z_0}}(z_0) = \{(x', x_n) \in \mathbf{R}^n \mid x_n = h_{z_0}(x'), |x'| < r_*\}.$$

For $x \in \Omega$, let πx be a point on Γ such that $|x - \pi x| = d_\Omega(x)$. If x is within the reach of Γ , then this πx is unique. There exist $r < r_*$ and $\delta < \delta_*$ such that

$$U(z_0) = \{x \in U_{r_*, \delta_*, h_{z_0}}(z_0) \mid (\pi x)' \in B_r(0'), d_\Gamma(x) < \delta\}$$

is contained in $U_{r_*, \delta_*, h_{z_0}}(z_0)$. Since d_Ω is $C^{2+\beta}$ in $\bar{\Gamma}_\sigma$ for $\sigma < R_*$ [GT, Chap. 14, Appendix] [KP, §4.4], we may take δ smaller (independent of z_0) so that d_Ω is $C^{2+\beta}$ in $\overline{U(z_0)} \cap \Omega$.

We next consider the normal coordinate in $U(z_0)$

$$\begin{cases} x' = y' + y_n \nabla' d_\Omega(y', \psi(y')) \\ x_n = \psi(y') + y_n \partial_{x_n} d_\Omega(y', \psi(y')) \end{cases} \quad (16) \quad \{\text{NC}\}$$

or shortly

$$x = \pi x - d_\Omega(x) \mathbf{n}(\pi x).$$

Let this coordinate change be denoted by $x = \psi(y)$, $\psi \in C^{1+\beta}(B_r(0'))$. Notice that $\nabla \psi(0) = I$. If we consider r and δ small, this coordinate change is indeed a local C^1 -diffeomorphism which maps $U(z_0)$ to V where $V := B_r(0') \times (-\delta, \delta)$. Moreover, by [Lew], we extend ψ to a global C^1 -diffeomorphism $\tilde{\psi}$ such that $\tilde{\psi}|_V = \psi$ and $\|\nabla \tilde{\psi}\|_{L^\infty(\mathbf{R}^n)} < 2$. Let the inverse of ψ in V be denoted by ϕ , i.e. $\phi = \psi^{-1}$.

\{\text{VFG}\}

Lemma 23. *Let W be a vector field with measurable coefficient in Γ_σ , $\sigma < R_*$ of the form*

$$W = \sum_{i=1}^n w_i \frac{\partial}{\partial x_i}.$$

Let y be the normal coordinate such that $y_n = d_\Omega(x)$. Let $\tilde{W}$ be W in y coordinate of the form $\overline{W} = \sum_{j=1}^n \tilde{w}_j(y) \partial/\partial y_j$. Then

$$\tilde{w}_n(y) = \nabla d_\Omega(x(y)) \cdot w(x(y)).$$

We shall prove this lemma in Appendix which follows from a simple linear algebra.

Proof of Theorem 22. We first observe that the restriction v on $U(z_0) \cap \Omega$ is in $bmo_\infty^\infty(U(z_0) \cap \Omega)$. By considering the following equivalent definition of the seminorm $[f]_{BMO^\infty(D)}$ where

$$[f]_{BMO^\infty(D)} = \sup_{B_r(x) \subset D} \inf_{c \in \mathbf{R}} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - c| dy,$$

see [Gra, Proposition 3.1.2], we can deduce that the space bmo_∞^∞ on a bounded domain is independent of bi-Lipschitz coordinate change. We introduce normal coordinate for a vector field $v = \sum_{i=1}^n v_i \partial/\partial x_i$ with $v_i \in bmo_\infty^\infty(U(z_0) \cap \Omega)$. Let w be the transformed vector field of normal coordinate y . By Lemma 23, w_n of $w = \sum_{i=1}^n w_i \partial/\partial y_i$ fulfills $w_n = \nabla d_\Omega(x(y)) \cdot v(x(y))$. Since $v \in vbmo_\delta^{\mu, \nu}(\Omega)$, this implies that $w \in bmo_\infty^\infty(V \cap \mathbf{R}_+^n)$ and moreover,

$$\sup_{\ell < \delta, B_\ell(x) \subset V} \ell^{-n} \int_{B_\ell(x) \cap \mathbf{R}_+^n} |w_n| dy < \infty.$$

Thus, as in the proof of Theorem 21 the zero extension of w_n for $y_n < 0$ is in $bmo_\infty^\infty(V)$, we still denote this extension by w_n . Let $J = J(y)$ denote the Jacobian of the mapping $y \mapsto x$ in V . For tangential part w' of $w = (w', w_n)$, we take an even extension with weight J of the form

$$\hat{w}'(y', y_n) = \begin{cases} w'(y', y_n), & y_n > 0 \\ w'(y', -y_n) J(y', -y_n) / J(y', y_n), & y_n < 0 \end{cases} \quad (17)$$

and set $\tilde{w}(y', y_n) = (\hat{w}', w_n)$. Let $\overline{w}'$ denote the normal even extension of w' to V , thus $w \in bmo_\infty^\infty(V \cap \mathbf{R}_+^n)$ implies that $\overline{w}' \in bmo_\infty^\infty(V)$. Let f be the function defined on V such that $f \equiv 1$ for $y_n \geq 0$ and $f = J(y', -y_n) / J(y', y_n)$ for $y_n < 0$. Since $J(y)^{-1} = |\det D\psi(y)|^{-1} = |\det D\phi(\psi(y))|$ for $y \in V$, we have that $f \in C^\beta(V)$. Notice that $\hat{w}'(y) = \overline{w}'(y) f(y)$, therefore by Theorem 13, we can deduce that $\tilde{w}$ belongs to $bmo_\infty^\infty(V)$. By Theorem 12, the Jones' extension w_U of $\tilde{w}$ belongs to $bmo_\infty^\infty(\mathbf{R}^n)$. Its expression in x coordinate is v_U which is only defined near Γ .

If the support of ρ is in $U(z_0)$, then integration by parts implies that

$$\int_\Gamma v_U \cdot \mathbf{n} \rho d\mathcal{H}^{n-1} = \int_\Omega (\operatorname{div} v_U) \rho dx - \int_\Omega v_U \cdot \nabla \rho dx. \quad (18) \quad \{\text{INT2}\}$$

We shall estimate the left-hand side as in the case of $\mathbf{R}_+^n$. The first integral in the right-hand

side can be estimated similarly as in the proof of Theorem 21. It is sufficient to only consider the second integral. Let $\Psi : B_r(0') \rightarrow \Gamma \cap U(z_0)$ by $(y', 0) \mapsto (y', h_{z_0}(y'))$. Extend $h_{z_0} \in C^2(B_r(0'))$ to $\tilde{h} \in C_c^2(\mathbf{R}^{n-1})$ such that $\tilde{h}|_{B_r(0')} = h_{z_0}$. Define $\tilde{\Psi} : \mathbf{R}^{n-1} \rightarrow \tilde{h}(\mathbf{R}^{n-1})$ by $(y', 0) \mapsto (y', \tilde{h}(y'))$. Hence $\tilde{\Psi}|_{B_r(0')} = \Psi$. Extend further $\tilde{\Psi}$ to $\tilde{\Psi}^* : \mathbf{R}^n \rightarrow \mathbf{R}^n$ by $(y', d) \mapsto \tilde{\Psi}(y', 0) + (0', d)$. Notice that this $\tilde{\Psi}^*$ is a global C^2 -diffeomorphism whose derivatives are bounded in $\mathbf{R}^n$ up to second-order. Let $\zeta > 0$ be a constant, for $\varphi \in C_c^1(\Gamma \cap \zeta U(z_0))$, we have that $\varphi \circ \Psi \in C_c^1(B_\zeta(0'))$. Let $\tilde{\sigma} = \text{Ext}(\varphi \circ \Psi)$ as in the proof of Theorem 21 and let $\sigma = \tilde{\sigma} \circ (\tilde{\Psi}^*)^{-1}$. With this choice of σ , we have that for $(y', h_{z_0}(y')) \in \Gamma \cap \zeta U(z_0)$,

$$\sigma(y', h_{z_0}(y')) = \tilde{\sigma} \circ (\tilde{\Psi}^*)^{-1}(y', h_{z_0}(y')) = \tilde{\sigma}(y', 0) = \varphi \circ \Psi(y', 0) = \varphi(y', h_{z_0}(y')).$$

Thus φ is an extension of σ . Since $(\tilde{\Psi}^*)^{-1}$ is a global C^2 -diffeomorphism and $\tilde{\sigma} \in F_{1,2}^1(\mathbf{R}^n)$, we observe that $\sigma \in F_{1,2}^1(\mathbf{R}^n)$, see e.g. see Proposition 18 or [Tr92, Section 4.3.1].

For each $z_0 \in \Gamma$, there exists $\epsilon_{z_0} > 0$ such that we can find a cutoff function $\theta_{z_0} \in C_c^\infty(U(z_0))$ for which $\theta_{z_0} \equiv 1$ within $\epsilon_{z_0}U(z_0)$ and

$$\sum_{|\alpha| \leq 2} \|D^\alpha \theta_{z_0}\|_{L^\infty(\mathbf{R}^n)} \leq M$$

for some fixed universal constant $M > 1$ independent of z_0 . By multiplying this cutoff function θ_{z_0} , we have that $\rho = \theta_{z_0} \sigma \in F_{1,2}^1(\mathbf{R}^n)$ and $\|\rho\|_{F_{1,2}^1(\mathbf{R}^n)} \leq M \cdot \|\sigma\|_{F_{1,2}^1(\mathbf{R}^n)}$. Hence we take the constant ζ above to be ϵ_{z_0} .

By coordinate change, we observe that

$$\int_{\Omega} v_U \cdot \nabla \rho \, dx = \int_{U(z_0) \cap \Omega} \sum_{i=1}^n v_i \frac{\partial}{\partial x_i} \rho \, dx = \int_{V \cap \mathbf{R}_+^n} \sum_{j=1}^n w_{U_j}(y) J(y) \frac{\partial}{\partial y_j} (\rho \circ \psi(y)) \, dy.$$

The n -th component equals

$$\int_{V \cap \mathbf{R}_+^n} w_{U_n}(y) J(y) \frac{\partial}{\partial y_n} (\rho \circ \psi(y)) \, dy = \int_V w_{U_n}(y) J(y) \frac{\partial}{\partial y_n} (\rho \circ \psi(y)) \, dy$$

since w_{U_n} equals zero for $y_n < 0$. Consider extensions of Hölder functions [McS] and local diffeomorphism [Lew], by the $F_{1,2}^0 - F_{\infty,2}^0$ duality [Sa, Theorem 3.22] and Proposition 17, we conclude that

$$\begin{aligned} \left| \int_V w_{U_n}(y) J(y) \frac{\partial}{\partial y_n} (\rho \circ \psi(y)) \, dy \right| &\leq C \cdot \sum_{i=1}^n \|w_{U_n}\|_{bmo} \cdot \|J\|_{C^\beta(V)} \cdot \|\partial_{y_n} \psi\|_{C^\beta(V)} \cdot \|\nabla \rho \circ \tilde{\psi}\|_{h^1} \\ &\leq C \cdot \|w_{U_n}\|_{bmo} \cdot \|\nabla \rho\|_{h^1}. \end{aligned}$$

For tangential part we may assume that

$$(\rho \circ \psi)(y', y_n) = (\rho \circ \psi)(y', -y_n) \quad \text{for } y_n < 0. \quad (19) \quad \{\text{EEV}\}$$

In fact, for a given ρ we take

$$g(y', y_n) = (\rho \circ \psi(y', y_n) + \rho \circ \psi(y', -y_n))/2$$

which satisfies evenness $g(y', y_n) = g(y', -y_n)$ and

$$g(y', 0) = \theta \circ \psi(y', 0) \cdot \sigma \circ \psi(y', 0) = \theta(y', h_{z_0}(y')) \cdot \varphi(y', h_{z_0}(y')).$$

It suffices to take ρ such that $\rho \circ \psi(y) = g(y)$. Thus, we may assume that $\rho \circ \psi$ is even in y_n so that $\partial_{y_j}(\rho \circ \psi)$ is also even in y_n for $j = 1, 2, \dots, n-1$. Since $w_{U_j}J$ is even in y_n for y in V , we observe that

$$\int_{V \cap \mathbf{R}_+^n} w_{U_j}(y) J(y) \frac{\partial}{\partial y_j} (\rho \circ \psi) dy = \frac{1}{2} \int_V w_{U_j}(y) J(y) \frac{\partial}{\partial y_j} (\rho \circ \psi) dy$$

for $1 \leq j \leq n-1$. Similar to the case for the n -th component, we thus conclude that

$$\left| \int_V w_{U_j}(y) J(y) \frac{\partial}{\partial y_j} (\rho \circ \psi) dy \right| \leq C \cdot \|w_{U_j}\|_{bmo} \cdot \|\nabla \rho\|_{h^1}.$$

Collecting these estimates, we conclude that

$$\begin{aligned} \left| \int_{\Gamma \cap \epsilon_{z_0} U(z_0)} v \cdot \mathbf{n} \varphi dx^{n-1} \right| &\leq C \|w_U\|_{bmo} \|\nabla \rho\|_{h^1} \\ &\leq C \|v\|_{vbmo_\delta^{\mu, \nu}(\Omega)} \|\varphi\|_{L^1(\Gamma \cap \epsilon_{z_0} U(z_0))}. \end{aligned}$$

Thus $\|v \cdot \mathbf{n}\|_{L^\infty} \leq C \|v\|_{vbmo_\delta^{\mu, \nu}(\Omega)}$. □

{\text{NTGB}}

Remark 24. (i) Since $BMO_b^{\mu, \nu} \subset vbmo_\delta^{\mu, \nu}$ for $\delta < \infty$, the estimate in Theorem 22 holds if we replace $vbmo_\delta^{\mu, \nu}$ by $BMO_b^{\mu, \nu}$. Moreover, since we are able to use zero extension in this case. We can follow the proof of Theorem 22 directly without the necessity to invoke normal coordinates. We shall state a version of Theorem 22 for $BMO_b^{\mu, \nu}$ in the end of this section.

(ii) By Theorem 9 we may replace $vbmo_\delta^{\mu, \nu}$ by $vBMO^{\mu, \nu}(\Omega)$ in the estimate in Theorem 22 since we may always take $\delta \leq \nu < R_*$ provided that Ω is a bounded or an exterior domain.

{\text{TRBB}}

Theorem 25. *Let Ω be a uniformly $C^{1+\beta}$ domain in $\mathbf{R}^n$ with $n \geq 2$. Let μ, ν, δ be in $(0, \infty]$.*

Then there is a constant $C = C(\mu, \nu, \delta, \Omega)$ such that

$$\|v \cdot \mathbf{n}\|_{L^\infty(\Gamma)} \leq C \cdot (\|v\|_{BMO_b^{\mu, \nu}(\Omega)} + \|\operatorname{div} v\|_{L_{\text{ul}}^n(\Gamma_\delta)})$$

for all $v \in BMO_b^{\mu, \nu}(\Omega)$.

Proof. For $z_0 \in \Gamma$, let $U(z_0) = U_{r_*, \delta_*, h_{z_0}}(z_0)$ with $\delta_* \leq R_*$. We then follow the proof of Theorem 22 without invoking the normal coordinates. For $v \in BMO_b^{\mu, \nu}(\Omega)$, let v_0 be the zero extension of v . We have that $v_0 \in bmo_\infty^\infty(U(z_0))$. Let v_U be the Jones' extension of $r_{U(z_0)}v_0$ by Theorem 12 where $r_{U(z_0)}v_0$ denotes the restriction of v_0 on $U(z_0)$. For $\varphi \in C_c^1(\Gamma \cap \frac{1}{2}U(z_0))$, we construct the function σ in the same way as in the proof of Theorem 22. Since the boundary Γ is uniformly $C^{1+\beta}$, $\tilde{\Psi}^*$ is a global $C^{1+\beta}$ -diffeomorphism. By Proposition 18, we have that $\sigma = \tilde{\sigma} \circ (\tilde{\Psi}^*)^{-1} \in F_{1,2}^1(\mathbf{R}^n)$. Pick θ in $C_c^\infty(U(z_0))$ such that $\theta \equiv 1$ within $\frac{1}{2}U(z_0)$ and let $\rho = \theta\sigma$, we deduce that $\rho \in F_{1,2}^1(\mathbf{R}^n)$ and

$$\left| \int_{\Omega} v_U \cdot \nabla \rho \, dx \right| \leq C \cdot \|v_U\|_{bmo} \cdot \|\nabla \rho\|_{h^1} \leq C \cdot \|v\|_{BMO_b^{\mu, \nu}(\Omega)} \cdot \|\nabla \rho\|_{h^1}.$$

Therefore,

$$\left| \int_{\Gamma \cap \frac{1}{2}U(z_0)} v \cdot \mathbf{n} \varphi \, dx^{n-1} \right| \leq C \cdot \|v\|_{BMO_b^{\mu, \nu}(\Omega)} \cdot \|\varphi\|_{L^1(\Gamma \cap \frac{1}{2}U(z_0))}.$$

The proof is therefore complete. $\square$

5 Appendix

{APP}

We shall prove Lemma 23. We first recall a simple property of a matrix.

{LA}

Proposition 26. *Let A be an invertible matrix*

$$A = (\vec{a}_1, \dots, \vec{a}_n)$$

when $\vec{a}_j = {}^t(a_{ij})_{1 \leq i \leq n}$ is an column vector. Assume that $\vec{a}_n$ is a unit vector and orthogonal to $\vec{a}_j$ with $1 \leq j \leq n-1$. Then n -row vector of A^{-1} equals ${}^t\vec{a}_n$. In other words, if one writes $A^{-1} = (b_{ij})_{1 \leq i, j \leq n}$, then $b_{nj} = a_{jn}$ for $1 \leq j \leq n$.

Proof. By definition the row vector $\vec{b} = (b_{nj})_{1 \leq j \leq n}$ must satisfies $\vec{b} \cdot \vec{a}_j = 0$ ($j = 1, \dots, n-1$), $\vec{b} \cdot \vec{a}_n = 1$. Since $\{\vec{a}_j\}_{j=1}^{n-1}$ spans $\mathbf{R}^{n-1}$ orthogonal to $\vec{a}_n$, first identities imply that $\vec{b}$ is parallel to $\vec{a}_n$. We thus conclude that $\vec{b} = \vec{a}_n$ since $\vec{b} \cdot \vec{a}_n = 1$ and $|\vec{a}_n| = 1$. $\square$

Proof of Lemma 23. We recall the explicit representation (16) of the normal coordinate. The Jacobi matrix from $y \mapsto x$ is of the form

$$A = (\vec{a}_1, \dots, \vec{a}_n)$$

with $\vec{a}_j = {}^t(\delta_{ij} - y_n \partial_j \mathbf{n}_i(y', \psi(y')), \partial_j \psi(y') - y_n \partial_j \mathbf{n}_n(y', \psi(y')))$ $1 \leq j \leq n-1$,

$$\vec{a}_n = -{}^t \mathbf{n}(y', \psi(y')) \quad \text{where} \quad \mathbf{n} = -\nabla d_\Omega.$$

Note that the vector $(\delta_{ij}, \partial_j \psi(y'))_{1 \leq i \leq n-1}$ is a tangential vector to Γ . Moreover, $(\partial_j \mathbf{n}_1, \dots, \partial_j \mathbf{n}_n)$ is also tangential since $\partial_j \mathbf{n} \cdot \mathbf{n} = \partial_j |\mathbf{n}|^2/2 = 0$. Thus $\vec{a}_j$ is orthogonal to $\vec{a}_n$ for $1 \leq j \leq n-1$. The invertibility of A is guaranteed if $y_n < R_*$.

By a chain rule we have

$$\begin{aligned} \bar{w} &= \sum_{j=1}^n \tilde{w}_j(y) (\partial/\partial y_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n \tilde{w}_j \frac{\partial x_i}{\partial y_j} \frac{\partial}{\partial x_i} \end{aligned}$$

so that

$$w_i(x(y)) = \sum_{j=1}^n \tilde{w}_j(y) \frac{\partial x_i}{\partial y_j} \quad \text{i.e.,} \quad w = A \tilde{w},$$

where $A = (\partial x_i / \partial y_j)_{1 \leq i, j \leq n}$, $\tilde{w} = {}^t(\tilde{w}_1, \dots, \tilde{w}_n)$, $w = {}^t(w_1, \dots, w_n)$. Thus

$$\tilde{w} = A^{-1} w.$$

By Proposition 26, the last row of A^{-1} equals ∇d_Ω .

We thus conclude that $\tilde{w}_n = \nabla d_\Omega \cdot w$. This is what we would like to prove. $\square$

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