博士学位論文

Mean-field behavior for percolation models
（パーコレーション模型に対する平均場臨界現象）

Yoshinori Kamijima （上島芳倫）

北海道大学大学院理学院
数学専攻

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Superviser (Principal referee) Prof. Akira Sakai (Hokkaido University).

Other referees

- Prof. Takao Namiki (Hokkaido University).
- Prof. Takahiro Hasebe (Hokkaido University).
- Prof. Tadahiro Miyao (Hokkaido University).

Preface

Various statistical-mechanical models exhibit critical phenomena, and it is believed that the critical exponents take on their mean-field values in high dimensions. The lace expansion is one of the few mathematically rigorous methods to prove mean-field behavior for such models. It can show that the two-point function for the concerned model, up to the critical point, is bounded by the Green function for the underlying random walk in high dimensions.

First, we explain some background, historical facts and the purposes of this thesis.

Background

Cooperation of infinitely many particles results in various intriguing and challenging problems. One of those is to understand phase transitions and critical behavior of statistical-mechanical models, such as ordinary/oriented percolation and the ferromagnetic Ising model. Percolation, for example, exhibits a phase transition when the bond-occupation parameter p crosses its critical value p_{c}. If p is far below p_{c}, each cluster of occupied vertices is so small that we may use standard probabilistic techniques for i.i.d. random variables to predict what happens in the subcritical phase. If p is far above p_{c}, on the other hand, vacant vertices can only form tiny islands and most of the other vertices are connected to form a single gigantic cluster. However, when p is close to p_{c}, the cluster of connected vertices from the origin may be extremely large but porous in a nontrivial way, and therefore naive perturbation methods fail.

To mention ordinary percolation more precisely, consider a locally finite, amenable and transitive graph as space. A standard example is the d-dimensional integer lattice $\mathbb{Z}^d$, whose edge set is denoted by $\mathbb{B}$. Let D be the 1-step distribution of nearest-neighbor simple random walk (RW): $D(x) = (2d)^{-1}\delta_{|x|,1}$ on $\mathbb{Z}^d$. $(\{0,1\}^{\mathbb{B}}, \mathcal{F}, \mathbb{P}_p)$ is regarded as a probability space, where $\mathcal{F}$ is the σ-algebra generated by cylinder sets, and $\mathbb{P}_p = \bigotimes_{\{u,v\}\in\mathbb{B}} \mathrm{Ber}(pD(v-u))$ is a product Bernoulli measure. We write $\mathbb{E}_p$ in the expectation against $\mathbb{P}_p$. We say that a bond $b \in \mathbb{B}$ is occupied in $\xi \in \{0,1\}^{\mathbb{B}}$ if $\xi(b) = 1$. The main observable to be investigated is the percolation two-point function

$$G_p(x) = \mathbb{P}_p(o \longleftrightarrow x),$$

where $\{\, o \longleftrightarrow x \,\}$ denotes the event that either $o = x$ or there exists a (self-avoiding) path from o to x consisting of occupied bonds. The set $\mathcal{C}(o) = \{\, x \in \mathbb{Z}^d \mid o \longleftrightarrow x \,\}$ is called the cluster of o. There is a critical point $p_{\mathrm{H}} < \|D\|_\infty^{-1}$ such that the percolation probability $\mathbb{P}_p(|\mathcal{C}(o)| = \infty) = \lim_{n\to\infty} \mathbb{P}_p(|\mathcal{C}(o)| \geq n)$ is positive if $p > p_{\mathrm{H}}$ [3]. Also, there is another critical point p_{T} such that the susceptibility (the expected cluster size) $\chi_p \equiv \sum_x G_p(x) = \mathbb{E}_p[|\mathcal{C}(o)|]$ is finite if $p < p_{\mathrm{T}}$ (as $p \uparrow p_{\mathrm{T}}$, χ_p diverges faster than $(p_{\mathrm{c}} - p)^{-1}$ by the second inequality in (0.0.1) below). It is known that $p_{\mathrm{T}} = p_{\mathrm{H}}$ for any translation invariant percolation model on $\mathbb{Z}^d$ [5, 32]. Thus, we denote the critical point $p_{\mathrm{T}} = p_{\mathrm{H}}$ by p_{c}.

Similar phenomena occur in oriented percolation. In this model, we consider (directed) bond variables on space-time $\mathbb{Z}^d \times \mathbb{Z}_+$, where $\mathbb{Z}_+ = \mathbb{N} \sqcup \{0\}$, and the two-point function is defined by

$$\varphi_p(x,t) = \mathbb{P}_p((o,0) \longrightarrow (x,t)),$$

where $\{ (o,0) \longrightarrow (x,t) \}$ denotes the event that either $(o,0) = (x,t)$ or there exists a time-oriented path from $(o,0)$ to (x,t) consisting of occupied bonds. As in ordinary percolation, $\chi_p \equiv \sum_{(x,t)} \varphi_p(x,t)$ is finite if and only if $p < p_{\mathrm{T}} = p_{\mathrm{H}}$ (although χ_p and p_{c} are distinct from ordinary ones, we write them in the same symbols in Preface for convenience).

The way χ_p diverges for ordinary/oriented percolation is intriguing, as it shows power-law behavior as $(p_{\mathrm{c}} - p)^{-\gamma}$ with the critical exponent γ. It is considered to be universal in the sense that the value of γ depends only on d and is insensitive to the details of the lattice structure. For example, the value of γ for bond ordinary percolation on $\mathbb{Z}^2$ is believed to be $\frac{43}{18}$ and equal to that for site ordinary percolation on the 2-dimensional triangular lattice. This is not the case for the critical point p_{c}, as its value may vary depending on the detail lattice structure. Other statistical-mechanical models that exhibit divergence of the susceptibility are also characterized by the critical exponent γ, and many physicists as well as mathematicians have been trying hard to identify the value of γ and classify the models into different universality classes since the last century.

The mean-field theory for ordinary percolation

It is still open to prove, in general, the existence of γ for ordinary/oriented percolation because there are many different paths joining two vertices, and pairs of such paths generally have edges in common. However, the critical exponents are obtained for ordinary percolation on the Bethe lattice with coordination number r (the r-regular tree). The following calculation yields $\gamma = 1$ for ordinary percolation (see, e.g., [26, Chapter 2] for the more details). Let $h(x)$ be the height of x. On the Bethe lattice, note that, if o is connected to x, such a path is unique. As in one-dimensional percolation, we have

$$G_p(x) = (p\|D\|_\infty)^{h(x)} = \Big(\frac{p}{r}\Big)^{h(x)}.$$

By the fact that there are $r(r-1)^{n-1}$ vertices in the Bethe lattice at height $n \geq 1$, we obtain

$$\chi_p = 1 + \sum_{n=1}^{\infty} r(r-1)^{n-1} \Big(\frac{p}{r}\Big)^n = \frac{1 + p/r}{1 - p(r-1)/r},$$

which implies $\gamma = 1$ and $p_{\mathrm{c}} = r/(r-1)$.

Heuristically, when the dimension is high, any pairs of percolation connections are close to being independent. Since each percolation connection is defined as a self-avoiding path, by analogy with high dimensional self-avoiding walk, it is expected that such a path is regarded as a sample path of RW. Also, any different paths starting from o do not intersect each other on the Bethe lattice. These facts suggest that $G_p(x)$ may be approximated by the RW Green function $S_\mu(x) \equiv \sum_{n=0}^{\infty} \mu^n D^{*n}(x)$ with a mean-field fugacity $\mu = \mu(\mathbb{Z}^d, p)$, where D^{*n} is the n-fold convolution of D, and therefore χ_p may be approximated by $\sum_x S_\mu(x) = (1-\mu)^{-1}$. Thus, $\mu(p_{\mathrm{c}}) = 1$, which is the critical point (=the radius of convergence) for RW. If μ is left-differentiable at p_{c}, then this implies $\chi_p \asymp (p_{\mathrm{c}} - p)^{-1}$ (i.e., χ_p is bounded above and below by positive multiples of $(p_{\mathrm{c}} - p)^{-1}$) as $p \uparrow p_{\mathrm{c}}$. In this respect, the mean-field value for the critical exponent γ is 1.

The infrared bound

For ordinary/oriented percolation, instead of deriving the exact solution for χ_p, one may seek for bounds on χ_p or its derivative. Indeed, it is shown in [4, 5] that

$$\epsilon_R\big(1 - \triangle_p(R)\big)\chi_p^2 \leq \frac{\mathrm{d}\chi_p}{\mathrm{d}p} \leq \chi_p^2, \tag{0.0.1}$$

where $\epsilon_R \to 0$ as $R \to \infty$, if the condition

$$\triangle_p(R) \equiv \sup\{\, G_p^{*3}(x) \mid \|x\|_2 \geq R \,\} \xrightarrow[R\to\infty]{} 0 \tag{0.0.2}$$

is satisfied for ordinary percolation. It can be shown that the second inequality in (0.0.1) implies that χ_p is always bounded below by $(p_c - p)^{-1}$. Moreover, the first inequality in (0.0.1) implies that χ_p is also bounded above by a multiple of $(p_c - p)^{-1}$, hence $\gamma = 1$, if

$$G_{p_c}^{*3}(o) = \lim_{p\uparrow p_c} \int_{\mathbb{T}^d} \hat{G}_p(k)^3 \frac{\mathrm{d}^d k}{(2\pi)^d} < \infty$$

(which implies $\lim_{R\to\infty} \triangle_{p_c}(R) = 0$ by [5, Lemma 2.1]), where $\hat{G}_p(k)$ is the Fourier transform of the percolation two-point function and $\mathbb{T}^d \equiv [-\pi, \pi]^d$ is the d-dimensional torus of side length 2π in the Fourier space. It is a sufficient condition for the mean-field behavior for χ_p and is called the triangle condition, named after the shape of the diagram consisting of three line segments. Whether or not the triangle condition holds depends on the behavior of $\hat{G}_p(k)$ in the infrared regime (i.e., around $k = 0$).

Usually, there is no a priori bound on $\hat{G}_p(k)$. However, for some spin models with a strong symmetry condition called reflection positivity (e.g., the ferromagnetic Ising model with symmetric nearest-neighbor couplings satisfies this condition), the two-point function enjoys the following infrared bound [16]: for any $d > 2$, there is a constant $K < \infty$ such that

$$\|(1-\hat{D})\hat{G}_p\|_\infty \equiv \sup_{k\in\mathbb{T}^d} \big(1-\hat{D}(k)\big)|\hat{G}_p(k)| \leq K \qquad \text{uniformly in } p \text{ close to } p_c. \tag{0.0.3}$$

If D is a symmetric, non-degenerate and finite-range distribution with variance σ^2, then $1 - \hat{D}(k) \sim \frac{\sigma^2}{2d}|k|^2$ as $|k| \to 0$. Suppose that the infrared bound holds for ordinary percolation. Then

$$G_{p_c}^{*3}(o) \leq \int_{\mathbb{T}^d} \left(\frac{K}{1-\hat{D}(k)}\right)^3 \frac{\mathrm{d}^d k}{(2\pi)^d} \asymp \int_{\mathbb{T}^d} \frac{\mathrm{d}^d k}{|k|^6},$$

which implies that the triangle condition holds in all dimensions $d > 6$.

On the other hand, there is some evidence (from hyperscaling inequalities, numerical simulations, conformal field theory and so on) to suggest that the critical exponents (if they exist) cannot take on their mean-field values simultaneously if $d < 6$ for ordinary percolation. In this respect, the critical dimension d_c is 6 for ordinary percolation.

By replacing $\triangle_p(R)$ with

$$\sup\left\{ \sum_{(y,s),(w,r)} \varphi_p(w,r)\varphi_p(y-w,s-r)\varphi_p(y-x,s-t) \,\middle|\, \|(x,t)\|_2 \geq R \right\},$$

the convergence (0.0.2) is also sufficient condition for the mean-field behavior for χ_p for oriented percolation. When the models depend on time $\mathbb{Z}_+$, the infrared bound is a little different from (0.0.3) and given by

$$\|(1-\mathrm{e}^{\mathrm{i}\cdot}\hat{D})\hat{\varphi}_p\|_\infty \equiv \sup_{(k,\theta)\in\mathbb{T}^{d+1}} \big|1-\mathrm{e}^{\mathrm{i}\theta}\hat{D}(k)\big|\big|\hat{\varphi}_p(k,\mathrm{e}^{\mathrm{i}\theta})\big| \leq K \qquad \text{uniformly in } p \text{ close to } p_c, \tag{0.0.4}$$

where $\hat{\varphi}_p(k, \mathrm{e}^{\mathrm{i}\theta})$ is the Fourier-Laplace transform of $\varphi_p(x,t)$. It implies that the triangle condition for oriented percolation holds on $\mathbb{Z}^d \times \mathbb{Z}_+$ in all dimensions $d > 4$ [33]. The critical dimension $d_c = 4$ for oriented percolation is different from that for ordinary percolation due to

$$\begin{aligned}
&\sum_{(y,s),(w,r)} \varphi_p(w,r)\varphi_p(y-w,s-r)\varphi_p(y-x,s-t)\\
&=\int_{\mathbb{T}^d}\frac{\mathrm{d}^dk}{(2\pi)^d}\int_{\mathbb{T}}\bigg(\frac{K}{|1-\mathrm{e}^{\mathrm{i}\theta}\hat{D}(k)|}\bigg)^2\frac{K}{|1-\mathrm{e}^{-\mathrm{i}\theta}\hat{D}(k)|}\frac{\mathrm{d}\theta}{2\pi}\\
&\lesssim\int_{\mathbb{T}^d}\frac{\mathrm{d}^dk}{(2\pi)^d}\int_{\mathbb{T}}\frac{1}{\big(\|k\|_2^2+|\theta|\big)^3}\frac{\mathrm{d}\theta}{2\pi},
\end{aligned}$$

where $f(x)\lesssim g(x)$ means that there exists a constant $C<\infty$ such that $f(x)\le Cg(x)$.

To complete the mean-field picture in high dimensions, it thus remains to show that the infrared bounds (0.0.3) and (0.0.4) hold for all dimensions $d>d_{\mathrm{c}}$. Here, the lace expansion comes into play.

The lace expansion

In 1985, Brydges and Spencer [10] came up with a fascinating idea for weakly self-avoiding walk. First, they looked at the naive expansion for the self-avoiding walk two-point function G_p^{SAW}:

$$\begin{aligned}
G_p^{\mathrm{SAW}}(x)&=\sum_{\omega\colon o\to x}p^{|\omega|}\prod_{j=1}^{|\omega|}D(\omega_j-\omega_{j-1})\prod_{0\le s<t\le|\omega|}(1-\lambda\delta_{\omega_s,\omega_t})\\
&=\sum_{\omega\colon o\to x}p^{|\omega|}\prod_{j=1}^{|\omega|}D(\omega_j-\omega_{j-1})\sum_{\Gamma\in\mathcal{G}[0,|\omega|]}(-\lambda)^{|\Gamma|}\prod_{\{s,t\}\in\Gamma}\delta_{\omega_s,\omega_t},
\end{aligned}$$

where Γ, which is called a graph, is a set of pairs of indices on $[0,|\omega|]\equiv\{0,1,\dots,|\omega|\}$, $\mathcal{G}[0,|\omega|]$ is the set of all graphs, and $|\Gamma|$ is the cardinality of Γ. Next, from each $\Gamma\in\mathcal{G}[0,|\omega|]$, they isolated a connected graph $\Gamma_0\subset\Gamma$ of the origin. Then, they extracted a minimally connected graph $\mathcal{L}\subset\Gamma_0$ called a lace, and resummed all the other edges in $\Gamma\setminus\mathcal{L}$ to partially restore the self-avoidance constraint. This is what we nowadays call the algebraic lace expansion, named after the shape of the aforesaid minimally connected graph. Since then, the algebraic lace expansion has been successfully applied to other models, such as oriented percolation [33], lattice trees and lattice animals [20].

Later in 1990s, Hara and Slade (e.g., [22]) came up with a more intuitive derivation of the lace expansion. To distinguish it from the algebraic lace expansion, we sometimes call it the inclusion-exclusion lace expansion. This opened up the possibility of applying the lace expansion to a wider class of models, including (ordinary) percolation [21], the contact process [37], the Ising model [38] and the (one-component) φ^4 model [40].

From now on, we simply call the latter the lace expansion. We will show its derivation for ordinary percolation in Section 2.3 and for oriented percolation in Section 3.3.

The result of the lace expansion is formally explained by the following recursion equation similar to that for the RW Green function: for any $p<p_{\mathrm{c}}$, there are functions I_p and J_p such that

$$G_p(x)=I_p(x)+(J_p*G_p)(x).$$

If I_p and J_p satisfy certain regularity conditions, then it is natural to believe that the global behavior of G_p is also similar to that of the RW Green function and therefore the infrared bound (0.0.3) holds.

However, since I_p and J_p are described by an alternating series of the lace-expansion coefficients $\{\pi_p^{(n)}\}_{n=0}^\infty$, each of which involves complicated local interaction (n represents the degree of complexity), it is certainly not true that the aforesaid regularity conditions always hold. In fact, the regularity conditions require the critical triangle $(D^{*2}*G_{p_{\mathrm{c}}}^{*3})(o)$ for ordinary percolation to be small, not to be merely finite. This seemingly tautological statement (i.e., the critical triangle have to be small in order to prove them to be finite) is taken care of by the so-called bootstrapping argument, which will be explained later in this thesis.

During the course of the bootstrapping argument, we often assume that the number of neighbors per vertex is sufficiently large. Since each vertex has $2d$ neighbors on $\mathbb{Z}^d$, it means that d is assumed to be large. For strictly self-avoiding walk, for example, Hara and Slade [22, 23] succeeded in showing that $d \geq 5$ is large enough to prove mean-field results. For ordinary percolation, however, the situation is not as good as for self-avoiding walk. The best results so far were obtained by Fitzner and van der Hofstad [15], in which they proved mean-field results on $\mathbb{Z}^d$ for $d \geq 11$ by using NoBLE, a perturbation method from non-backtracking random walk (= memory-2 self-avoiding walk). For oriented percolation, the situation is much worse than ordinary percolation. Nguyen and Yang [33, 34] proved mean-field results on $\mathbb{Z}^d \times \mathbb{Z}_+$ for $d \geq d_0 \gg 4$, but they did not identify the value of d_0.

There is another way to increase the number of neighbors per vertex. Instead of taking d large, we may enlarge the range L of neighbors. One such example is the spread-out lattice $\bar{\mathbb{Z}}^d_L$, in which two distinct vertices $x, y \in \mathbb{Z}^d$ satisfying $\|x - y\|_\infty \leq L$ are defined to be neighbors, hence $(2L+1)^d - 1$ neighbors per vertex. By taking L sufficiently large, all the models for which the lace expansion was obtained are proven to exhibit mean-field behavior for all d above the predicted upper-critical dimensions [20, 21, 31, 33, 37, 38, 40].

The purpose of this thesis

Since we believe in universality, the mean-field results on the spread-out lattice $\bar{\mathbb{Z}}^d_L$, as long as $L < \infty$, are believed to hold on $\mathbb{Z}^d$ as well. This is proven to be true for self-avoiding walk, but not yet for ordinary/oriented percolation. We want to get rid of the artificial parameter L and come up to a decent nearest-neighbor lattice, on which 7-dimensional ordinary and $(5+1)$-dimensional oriented percolation are proven to exhibit the mean-field behavior. In this thesis, we analyze the lace expansion for percolation on a d-dimensional version of the body-centered cubic (BCC) lattice, denoted by $\mathbb{L}^d$, which has better features than the standard $\mathbb{Z}^d$, as explained in the next section. Thanks to those features, enumeration of RW quantities relevant to the lace-expansion analysis becomes extremely simple. Also, since those RW quantities are much smaller than the $\mathbb{Z}^d$-counterparts, it is easy to get closer to the predicted upper-critical dimension without introducing too much technical complexity. The purpose of this thesis is to explain the current status of the BCC work and reveal the potential problems to overcome for completion of the mean-field picture in high dimensions. As a result, using the lace expansion, we prove that nearest-neighbor ordinary percolation on $\mathbb{L}^{d\geq 9}$ and oriented percolation on $\mathbb{L}^{d\geq 9} \times \mathbb{Z}_+$ (the constraint $d \geq 9$ is just a coincidence) exhibit the mean-field behavior.

In Chapter 1, the models are defined more precisely, and the triangle conditions and the main results (the infrared bound) for each ordinary/oriented percolation are stated. At the end of Chapter 1, we quote some lemmas and their proofs to use in the later chapters.

In Chapter 2, we prove the main result for ordinary percolation. Also, in Chapter 3, we prove the main result for oriented percolation. The proofs are almost parallelly described in which we use the lace expansion. In Section 2.1 and Section 3.1, we define the bootstrapping functions $\{g_i\}_{i=1}^3$ and state the strategy of proofs. It suffices to show the continuity, the initial condition and the bootstrapping argument for $\{g_i\}_{i=1}^3$. The continuity is proved in Section 2.2 and Section 3.2. By combining the bounds appeared in Section 2.4–2.7 and Section 3.4–3.7, both the initial condition and the bootstrapping argument are proved in Section 2.8 and Section 3.8.

In each Section, briefly speaking, we bound $\{g_i\}_{i=1}^3$ in terms of the lace expansion coefficients $\{\pi_p^{(n)}\}_{n=1}^\infty$, bound $\{\pi_p^{(n)}\}_{n=1}^\infty$ in terms of so-called diagrams, bound their diagrams in terms of RW quantities, and we estimate the numerical values of the RW quantities. The upper bounds on $\{\pi_p^{(n)}\}_{n=1}^\infty$ are obtained by splitting the events of $\{\pi_p^{(n)}\}_{n=1}^\infty$ carefully. Paying attentions for them lead to better bounds than the previous researches. Moreover, thanks to the structure of the BCC lattice $\mathbb{L}^d$, it is easier to compute the RW quantities and to obtain smaller values of them than $\mathbb{Z}^d$, cf., Section 3.7 (or Section 2.7) and Appendix A.

For the oriented percolation, Section 3.5 and Section 3.6 contain two novel tasks (see Section 3.9 for the details). Nguyen and Yang [33] used the inequality $\hat{D}_L(k) + 1 > 0$ for all k, where D_L is the RW transition probability for the spread-out model, to bound g_2 in our notation. They used a similar inequality despite the nearest-neighbor model, but the inequality does not hold because $\hat{D}(k)$ can take -1 for some $k \in \mathbb{T}^d$. The structure of the BCC lattice helps us to solve this problem (see also Proposition 3.5.1). Moreover, Nguyen and Yang showed the finiteness of the weighted open triangle diagram (the derivative of the open triangle diagram), which corresponds to $\hat{V}_{p,1}^{(0,1)}(k)$ in this thesis, for only $d > 8$. We focus that the period of $\hat{D}(k)^2$ equals that of $\hat{D}(2k)$. This equality yields the finiteness of $\hat{V}_{p,m}^{(\lambda,\rho)}(k)$ in Lemma 3.6.1 for $d > 4$. It has a potential that this finiteness is helpful to improve our result and to prove the mean-field behavior near the upper critical dimension.

The result for ordinary percolation is based on the joint work with Handa and Sakai [19], and that for oriented percolation is based on the joint work with Chen and Handa [11].

Contents

Chapter 1

Introduction

1.1 Models

1.1.1 The body-centered cubic (BCC) lattice

The d-dimensional BCC lattice $\mathbb{L}^d$ is a graph that contains the origin $o = (0, \dots, 0)$ and is generated by the set of neighbors $\mathscr{N}^d = \{x = (x_1, \dots, x_d) \in \mathbb{Z}^d \mid \prod_{i=1}^d |x_i| = 1\}$. Although we should generally distinguish a graph from its vertex set, for convenience, we denote by "$x \in \mathbb{L}^d$" that x belongs to the vertex set of the BCC lattice. It is equivalent to $\mathbb{Z}^d$ when $d = 1$ and 2 (modulo rotation by $\pi/4$) but is more crowded in higher dimensions in the sense that the degree of each vertex is 2^d on $\mathbb{L}^d$, while it is $2d$ on $\mathbb{Z}^d$. We write $x \sim y$ if $x, y \in \mathbb{L}^d$ are neighbors, i.e., $\prod_{j=1}^d |x_j - y_j| = 1$. It is a natural extension of the standard 3-dimensional BCC structure (see Figure 1.1). The d-dimensional Brownian motion with the identity covariance matrix can be constructed as the scaling limit of random walk (RW) on $\mathbb{L}^d$ generated by the 1-step distribution

$$D(x) \equiv \frac{1}{|\mathscr{N}^d|} \mathbb{1}_{\{x \in \mathscr{N}^d\}} = \prod_{j=1}^d \frac{1}{2} \delta_{|x_j|,1}, \tag{1.1.1}$$

where $\mathbb{1}_{\{\cdot\}}$ is the indicator function, $\delta_{\cdot,\cdot}$ is the Kronecker delta, and $|A|$ denotes the cardinality of a set A. Due to this factorization and Stirling's formula[1], we can obtain a rather sharp bound on the $2n$-step return probability for all $n \in \mathbb{N}$, as

$$0 \le (\pi n)^{-d/2} - D^{*2n}(o) \le \Big(1 - e^{d(\frac{1}{24n+1} - \frac{1}{6n})}\Big)(\pi n)^{-d/2} \le \frac{2d}{15n}(\pi n)^{-d/2}, \tag{1.1.2}$$

where $f^{*n}(x) = (f^{*(n-1)} * f)(x) \equiv \sum_{y \in \mathbb{L}^d} f^{*(n-1)}(y) f(x - y)$ denotes the n-fold convolution on $\mathbb{L}^d$ for some function $f \colon \mathbb{L}^d \to \mathbb{R}$. This helps us to numerically compute some random-walk quantities as we see in Section 2.7 and Section 3.7. Let $\hat{D}(k)$ be the Fourier transform of $D \colon \mathbb{L}^d \to \mathbb{R}$ defined as

$$\hat{D}(k) \equiv \sum_{x \in \mathbb{L}^d} D(x) \mathrm{e}^{\mathrm{i}k \cdot x} = \prod_{j=1}^d \cos k_j \tag{1.1.3}$$

for $k = (k_1, \dots, k_d) \in \mathbb{T}^d$, where $k \cdot x$ is the Euclidian inner product, and $\mathbb{T}^d \equiv [-\pi, \pi]^d$ is the d-dimensional torus of side length 2π in the Fourier space. In the equality in the above, we used (1.1.1).

[1] The two-sided bound $\dfrac{1}{12n+1} \le \log \dfrac{n!}{\sqrt{2\pi n}\,(\frac{n}{e})^n} \le \dfrac{1}{12n}$ holds for all $n \in \mathbb{N}$ [13, Section II.9].

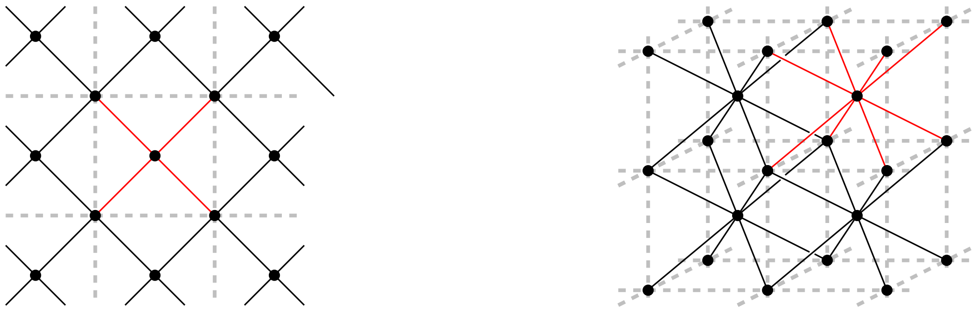


Figure 1.1: The basic structure (in red) of the BCC lattice $\mathbb{L}^d$ for $d = 2, 3$.

1.1.2 Ordinary percolation

Here, we introduce bond percolation on $\mathbb{L}^d$. In this thesis, we simply call ordinary percolation "percolation". Each bond $\{u, v\} \subset \mathbb{L}^d$ randomly takes either one of the two states, occupied or vacant, independently of the other bonds. We define the bond-occupation probability of a bond $\{u, v\}$ as $pD(v - u)$, where $p \in [0, \|D\|_\infty^{-1}]$ is the percolation parameter, and $\|f\|_\infty = \sup_{x\in\mathbb{L}^d} |f(x)|$. p is equal to the expected number of occupied bonds per vertex. Let $\mathbb{P}_p$ be the associated probability measure, and denote its expectation by $\mathbb{E}_p$.

Next, we define the percolation two-point function. In order to do so, we first introduce the notion of connectivity. For $n \in \mathbb{N}$, a path of length n on $\mathbb{L}^d$ is defined to be a sequence $(\omega_0, \omega_1, \dots, \omega_n)$ of vertices in $\mathbb{L}^d$. Let $\Omega_n(x, y)$ be the set of self-avoiding paths of length n from $\omega_0 = x$ to $\omega_n = y$, and let $\Omega(x, y) = \bigsqcup_{n\in\mathbb{Z}_+} \Omega_n(x, y)$, where $\mathbb{Z}_+ \equiv \mathbb{N} \sqcup \{0\}$, and $\sqcup$ is a disjoint union. By convention, $\Omega(x, x) = \Omega_0(x, x) \equiv \{\, (x) \,\}$. We write the number of steps of $\omega \in \Omega(x, y)$ as $|\omega|$. We say that a self-avoiding path $\omega = (\omega_0, \dots, \omega_{|\omega|}) \in \Omega(x, y)$ is occupied if either $x = y$ or every $b_j(\omega) \equiv \{\omega_{j-1}, \omega_j\}$ for $j = 1, \dots, |\omega|$ is occupied. We say that x is connected to y, denoted by $x \longleftrightarrow y$, if there is an occupied self-avoiding path $\omega \in \Omega(x, y)$. Then, we define the percolation two-point function as

$$G_p(x) = \mathbb{P}_p(o \longleftrightarrow x) = \mathbb{P}_p\bigg(\bigcup_{\omega\in\Omega(o,x)} \{\omega \text{ is occupied}\}\bigg). \tag{1.1.4}$$

The susceptibility χ_p and its critical point p_c are defined as

$$\chi_p = \sum_{x\in\mathbb{L}^d} G_p(x) \quad \text{and} \quad p_\mathrm{c} = \sup\Big\{\, p \in [0, \|D\|_\infty^{-1}] \;\Big|\; \chi_p < \infty \,\Big\},$$

respectively.

1.1.3 Oriented percolation

Here, we introduce bond oriented percolation on $\mathbb{L}^d \times \mathbb{Z}_+$. We use the same symbols as in the previous subsection to express some notions. If we emphasize the difference, we add superscripts "pe" for ordinary percolation and "op" for oriented percolation, such as p_c^pe and p_c^op. We call $\mathbb{L}^d \times \mathbb{Z}_+$ a space-time in which we write a vertex in the bold font, i.e., $\boldsymbol{x} = (x, t) \in \mathbb{L}^d \times \mathbb{Z}_+$, while we write an vertex in space $\mathbb{L}^d$ in the normal font, i.e., $x \in \mathbb{L}^d$. For convenience, x and $\tau(\boldsymbol{x})$ denote the spatial and temporal components of $\boldsymbol{x}$, respectively. A bond on $\mathbb{L}^d \times \mathbb{Z}_+$ is an ordered pair $((x, t), (y, s))$ of two vertices in $\mathbb{L}^d \times \mathbb{Z}_+$. Each bond $(\boldsymbol{x}, \boldsymbol{y})$ is occupied with probability $q_p(\boldsymbol{y} - \boldsymbol{x}) \equiv pD(y - x)\delta_{\tau(\boldsymbol{y})-\tau(\boldsymbol{x}),1}$ and vacant with probability $1 - q_p(\boldsymbol{y} - \boldsymbol{x})$ independently of the others, where $p \in [0, \|D\|_\infty^{-1}]$ is the percolation parameter. Let $\mathbb{P}_p$ be the associated probability measure with such bond variables, and denote its expectation by $\mathbb{E}_p$.

For $t \in \mathbb{N}$, a (time-oriented) path of length t on $\mathbb{L}^d \times \mathbb{Z}_+$ is defined to be a sequence $(\boldsymbol{\omega}_0, \boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_t)$ of vertices in $\mathbb{L}^d \times \mathbb{Z}_+$ such that $\tau(\boldsymbol{\omega}_s) - \tau(\boldsymbol{\omega}_{s-1}) = 1$ for $s = 1, \dots, t$. Let $\mathscr{W}(\boldsymbol{x}, \boldsymbol{y})$ be the set of all paths of length $t = \tau(\boldsymbol{y}) - \tau(\boldsymbol{x})$ from $\boldsymbol{x} = \boldsymbol{\omega}_0$ to $\boldsymbol{y} = \boldsymbol{\omega}_t$. By convention, $\mathscr{W}(\boldsymbol{x}, \boldsymbol{x}) \equiv \{ (\boldsymbol{x}) \}$. We say that a path $\vec{\boldsymbol{\omega}} = (\boldsymbol{\omega}_0, \dots, \boldsymbol{\omega}_t) \in \mathscr{W}(\boldsymbol{x}, \boldsymbol{y})$ of length $t = \tau(\boldsymbol{y}) - \tau(\boldsymbol{x})$ is occupied if either $\boldsymbol{x} = \boldsymbol{y}$ or every bond $(\boldsymbol{\omega}_{s-1}, \boldsymbol{\omega}_s)$ for $s = 1, \dots, t$ is occupied. We say that $\boldsymbol{x}$ is connected to $\boldsymbol{y}$, denoted by $\boldsymbol{x} \to \boldsymbol{y}$, if there is an occupied path $\vec{\boldsymbol{\omega}} \in \mathscr{W}(\boldsymbol{x}, \boldsymbol{y})$. Then, we define the two-point function as

$$\varphi_p(\boldsymbol{x}) = \mathbb{P}_p(\boldsymbol{o} \longrightarrow \boldsymbol{x}) = \mathbb{P}_p\Bigg(\bigcup_{\vec{\boldsymbol{\omega}} \in \mathscr{W}(\boldsymbol{o}, \boldsymbol{x})} \{ \vec{\boldsymbol{\omega}} \text{ is occupied} \} \Bigg)$$

for $\boldsymbol{x} \in \mathbb{L}^d \times \mathbb{Z}_+$, where $\boldsymbol{o} \equiv (o, 0)$. The susceptibility and its critical point are defined as

$$\chi_p = \sum_{\boldsymbol{x} \in \mathbb{L}^d \times \mathbb{Z}_+} \varphi_p(\boldsymbol{x}) \quad \text{and} \quad p_{\mathrm{c}} = \sup \left\{ p \in [0, \|D\|_\infty^{-1}] \;\middle|\; \chi_p < \infty \right\}, \tag{1.1.5}$$

respectively.

1.1.4 Addendum: the uniqueness of critical points

In the literature, the percolation threshold p_{H} characterized by the emergence of an infinite cluster of the origin,

$$p_{\mathrm{H}} = \begin{cases} \inf \{ p \in [0, \|D\|_\infty^{-1}] \mid \mathbb{P}_p^{\mathrm{pe}}(o \longleftrightarrow \infty) > 0 \} & \text{[percolation]}, \\ \inf \{ p \in [0, \|D\|_\infty^{-1}] \mid \mathbb{P}_p^{\mathrm{op}}(\boldsymbol{o} \longrightarrow \infty) > 0 \} & \text{[oriented percolation]}, \end{cases}$$

is usually called the "critical point", respectively, while our critical point p_{c} is denoted by p_{T}. However, Menshikov [32] and Aizenman and Barsky [1] independently proved that $p_{\mathrm{H}} = p_{\mathrm{T}}$ for any translation invariant percolation model on $\mathbb{Z}^d$. Recently, Duminil-Copin and Tassion [12] found a particularly simple proof of the uniqueness on arbitrary locally finite vertex-transitive infinite graphs. They also extended the idea to the Ising model and dramatically simplified the proof of the uniqueness of the critical temperature, first proven by Aizenman, Barsky and Fernández [2]. This uniqueness also holds on $\mathbb{L}^d$, hence we do not distinguish from each other.

1.1.5 Fourier-Laplace transform

We mention the Fourier-Laplace transform of $\varphi_p(\boldsymbol{x})$ as a preliminary for the main results. Let $\Phi_p(k; t) = \sum_{x \in \mathbb{L}^d} \varphi_p(x, t) \mathrm{e}^{\mathrm{i}k \cdot x}$ be the Fourier transform of $\varphi_p(x, t)$ with respect to x. The Markov property of the oriented percolation implies that $\{\log \Phi_p(0; t)/t\}_{t=1}^\infty$ is a subadditive sequence, hence there exists a constant $m_p^{-1} \equiv \lim_{t \uparrow \infty} \Phi_p(0; t)^{1/t}$ depending on p by [17, Appendix. II] with $m_{p_{\mathrm{c}}} = 1$ [34]. For every $p < p_{\mathrm{c}}$, the inequality $m_p > 1$ holds. By the definition, m_p is the radius of convergence of the Laplace transform (the generating function) of $\Phi_p(0; t)$, defined by

$$\hat{\varphi}_p(k, z) \equiv \sum_{t \in \mathbb{Z}_+} \Phi_p(k; t) z^t = \sum_{(x,t) \in \mathbb{L}^d \times \mathbb{Z}_+} \varphi_p(x, t) \mathrm{e}^{\mathrm{i}k \cdot t} z^t$$

for $k \in \mathbb{T}^d$ and $z \in \mathbb{C}$ satisfying $|z| \in [0, m_p)$. Also, let $Q_p(x, t) = p^t D^{*t}(x) \mathbb{1}_{\{t \in \mathbb{Z}_+\}}$ be the random-walk two-point function. By convention, 0-fold convolution denotes the Kronecker delta: $f^{*0}(x) = \delta_{o,x}$ for any function $f \colon \mathbb{L}^d \to \mathbb{R}$. The Laplace transform of $Q_1(x, t)$ gives the well-known random-walk Green function as

$$S_p(x) \equiv \sum_{t \in \mathbb{Z}_+} Q_1(x, t) p^t = \sum_{t \in \mathbb{Z}_+} p^t D^{*t}(x)$$

for $x \in \mathbb{L}^d$ and $p \in [0,1]$. Notice that the radius of convergence of $S_p(x)$ is 1, and in particular $S_1(x)$ is well-defined in a proper limit when $d > 2$. By the Boolean inequality, one can easily see that $\varphi_1(x,t) \leq Q_1(x,t)$ (and also $G_1(x) \leq S_1(x)$). Taking the sum of both sides leads to $p_{\mathrm{c}}^{\mathrm{op}} \geq 1$ (and $p_{\mathrm{c}}^{\mathrm{pe}} \geq 1$).

1.2 Main results

As we mentioned in Preface, it is believed that the susceptibility χ_p obeys the power law $(p_{\mathrm{c}} - p)^{-\gamma}$ in the vicinity of p_{c} with a critical exponent γ. In particular, it is believed that $\gamma = 1$ for ordinary percolation on $\mathbb{Z}^{d>6}$ and for oriented percolation on $\mathbb{Z}^{d>4} \times \mathbb{Z}_+$. Aizenman and Newman [4] showed that the triangle condition is a sufficient condition for percolation models to exhibit the mean-field behavior (or $\gamma = 1$). Barsky and Aizenman [5] extended/reformulated the Aizenman-Newman triangle condition in order to deal with oriented percolation unifiedly (moreover, they also showed that other critical exponents δ and β take the mean-field values 2 and 1, respectively, if the triangle condition is satisfied). An alternative argument involving operator theory is given by Kozma [30]. Let

$$\triangle_p(R) = \begin{cases} \sup \left\{ G_p^{*3}(x) \;\middle|\; \|x\|_2 \geq R \right\} & \text{[percolation]}, \\ \sup \left\{ \displaystyle\sum_{\boldsymbol{y} \in \mathbb{L}^d \times \mathbb{Z}_+} \varphi_p^{\star 2}(\boldsymbol{y}) \varphi_p(\boldsymbol{y} - \boldsymbol{x}) \;\middle|\; \|\boldsymbol{x}\|_2 \geq R \right\} & \text{[oriented percolation]}, \end{cases}$$

where $\|x\|_2 = (\sum_{i=1}^d |x_i|^2)^{1/2}$, $\|(x,t)\|_2 = (\|x\|_2^2 + t^2)^{1/2}$, and $f^\star(x,t) = (f^{\star(n-1)} \star f)(x,t) \equiv \sum_{(y,s) \in \mathbb{L}^d \times \mathbb{Z}_+} f^{\star(n-1)}(y,s) f(x-y, t-s)$ denotes the n-fold convolution on $\mathbb{L}^d \times \mathbb{Z}_+$ for some function $f \colon \mathbb{L}^d \times \mathbb{Z}_+ \to \mathbb{R}$. The triangle condition is expressed that

$$\lim_{R \to \infty} \triangle_{p_{\mathrm{c}}}(R) = 0. \tag{1.2.1}$$

If (1.2.1) holds, then solving the differential inequality

$$\epsilon_R \big(1 - \triangle_p(R)\big) \chi_p^2 \leq \frac{\mathrm{d}\chi_p}{\mathrm{d}p} \leq \chi_p^2,$$

where $\epsilon_R \to 0$ as $R \to \infty$, implies $\gamma = 1$.

To verify (1.2.1), we use the infrared bounds which are our main theorem.

Theorem 1.2.1 (Infrared bound)**.** For the nearest-neighbor model of ordinary percolation on $\mathbb{L}^{d \geq 9}$ and of oriented percolation on $\mathbb{L}^{d \geq 9} \times \mathbb{Z}_+$, there exist model-dependent constants $K^{\mathrm{pe}}, K^{\mathrm{op}} \in (0, \infty)$ such that

$$\left| \hat{G}_p(k) \right| \leq \frac{K^{\mathrm{pe}}}{1 - \hat{D}(k)} = K^{\mathrm{pe}} \hat{S}_1(k) \tag{1.2.2}$$

for any $p \in [1, p_{\mathrm{c}}^{\mathrm{pe}})$ and $k \in \mathbb{T}^d$, and

$$|\hat{\varphi}_p(k,z)| \leq \frac{K^{\mathrm{op}}}{|1 - \hat{q}_1(k, \mathrm{e}^{\mathrm{i} \arg z})|} = K^{\mathrm{op}} \left| \hat{Q}_1(k, \mathrm{e}^{\mathrm{i} \arg z}) \right| \tag{1.2.3}$$

for any $p \in [0, p_{\mathrm{c}}^{\mathrm{op}})$, $k \in \mathbb{T}^d$ and $z \in \mathbb{C}$ with $|z| \in [1, m_p)$.

It is not so difficult to see that the infrared bounds (1.2.2) and (1.2.3) yield the triangle conditions (1.2.1) for ordinary and oriented percolation, respectively. We follow [4, 5, 33].

Lemma 1.2.2. For $n \in \mathbb{N}$, $S_1^{*n}(o) < \infty$ on $\mathbb{L}^d$ if $d > 2n$.

Proof. By the square summability of S_μ for $\mu < 1$ and the monotone convergence theorem, the Fourier transform of S_1 is expressed as

$$S_1^{*n}(o) = \lim_{\mu \uparrow 1} S_\mu^{*n}(o) = \lim_{\mu \uparrow 1} \int_{\mathbb{T}^d} \hat{S}_\mu(k)^n \frac{\mathrm{d}^d k}{(2\pi)^d} = \int_{\mathbb{T}^d} \frac{1}{\big(1 - \hat{D}(k)\big)^n} \frac{\mathrm{d}^d k}{(2\pi)^d}.$$

To show the integrability of $(1 - \hat{D}(k))^{-n}$, it suffices to verify the finiteness around the singular points. For example, by (1.1.3) and its expansion $\hat{D}(k) = 1 - \|k\|_2^2/2 + o(\|k\|_2^2)$,

$$\begin{aligned}\int_{\{\, k \in \mathbb{T}^d \mid \|k\|_2 \le \delta \,\}} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\big(1 - \hat{D}(k)\big)^n} &\lesssim \int_{\{\, k \in \mathbb{T}^d \mid \|k\|_2 \le \delta \,\}} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\|k\|_2^{2n}} \\ &\lesssim \int_0^\delta \frac{1}{r^{2n}} \cdot r^{d-1} \, \mathrm{d}r = \int_0^\delta r^{d-1-2n} \, \mathrm{d}r\end{aligned}$$

for $\delta < 1$, where $f(x) \lesssim g(x)$ means that there exists a constant $C < \infty$ such that $f(x) \le C g(x)$. In the last line, we used the change of variables to the polar coordinates. The right-most integral is finite when $d > 2n$. Similarly, the integrability around the other singular points is shown. □

Lemma 1.2.3 ([4, Lemma 3.3])**.** $\hat{G}_p(k) \ge 0$ for every $p \in [0, p_\mathrm{c})$ and $k \in \mathbb{T}^d$.

Proof. First, by translation-invariance, we can use any vertex y to rewrite $\hat{G}_p(k)$ as

$$\hat{G}_p(k) = \sum_x \mathrm{e}^{\mathrm{i}k \cdot x} \mathbb{P}_p(o \longleftrightarrow x) = \sum_x \mathrm{e}^{\mathrm{i}k \cdot x} \mathbb{P}_p(y \longleftrightarrow x + y) = \mathbb{E}_p\Bigg[\sum_z \mathrm{e}^{\mathrm{i}k \cdot (z-y)} \mathbb{1}_{\{y \longleftrightarrow z\}}\Bigg].$$

Then, by using the identity $1 = \sum_y \mathbb{1}_{\{y \in \mathcal{C}(o)\}}/|\mathcal{C}(o)|$, where $\mathcal{C}(o)$ is the set of vertices connected from o, we can rewrite the rightmost expression as

$$\begin{aligned}\mathbb{E}_p\Bigg[\frac{1}{|\mathcal{C}(o)|} \sum_{y \in \mathcal{C}(o)} \sum_z \mathrm{e}^{\mathrm{i}k \cdot (z-y)} \mathbb{1}_{\{y \longleftrightarrow z\}}\Bigg] &= \mathbb{E}_p\Bigg[\frac{1}{|\mathcal{C}(o)|} \sum_{y, z \in \mathcal{C}(o)} \mathrm{e}^{\mathrm{i}k \cdot (z-y)}\Bigg] \\ &= \mathbb{E}_p\Bigg[\bigg|\frac{1}{\sqrt{|\mathcal{C}(o)|}} \sum_{z \in \mathcal{C}(o)} \mathrm{e}^{\mathrm{i}k \cdot z}\bigg|^2\Bigg] \ge 0.\end{aligned}$$

□

Lemma 1.2.4 ([5, Lemma 2.1])**.** If the infrared bound (1.2.2) holds, then $G_{p_\mathrm{c}}^{*3}(o) < \infty$, which implies the triangle condition (1.2.1) for ordinary percolation.

Proof. By (1.2.2), Lemma 1.2.2, Lebesgue's dominated convergence theorem and Lemma 1.2.3, we obtain

$$\begin{aligned}G_{p_\mathrm{c}}^{*3}(o) = \lim_{p \uparrow p_\mathrm{c}} G_p^{*3}(o) &= \lim_{p \uparrow p_\mathrm{c}} \int_{\mathbb{T}^d} \hat{G}_p(k)^3 \frac{\mathrm{d}^d k}{(2\pi)^d} \\ &= \int_{\mathbb{T}^d} \hat{G}_{p_\mathrm{c}}(k)^3 \frac{\mathrm{d}^d k}{(2\pi)^d} \\ &= \int_{\mathbb{T}^d} \Big|\hat{G}_{p_\mathrm{c}}(k)\Big|^3 \frac{\mathrm{d}^d k}{(2\pi)^d} < \infty,\end{aligned}$$

which shows the first half. Simultaneously, the above means that $\hat{G}_{p_\mathrm{c}}(k)^3$ is a L^1-function. Therefore, by the Riemann-Lebesgure lemma,

$$G_{p_\mathrm{c}}^{*3}(x) = \int_{\mathbb{T}^d} \hat{G}_{p_\mathrm{c}}(k)^3 \mathrm{e}^{-\mathrm{i}k \cdot x} \frac{\mathrm{d}^d k}{(2\pi)^d} \xrightarrow[\|x\|_2 \uparrow \infty]{} 0.$$

This completes the proof of the second half. □

Lemma 1.2.5 ([33]). If the infrared bound (1.2.3) holds, then the triangle condition (1.2.1) for oriented percolation is satisfied.

Proof. Fix $\epsilon \in (0, (d-4)/6)$. By the Hausdorff-Young inequality and (1.2.3), there is a constant C_ϵ depending on ϵ such that

$$\left(\sum_{\boldsymbol{x}} \left| \sum_{\boldsymbol{y}} \varphi_p^{\star 2}(\boldsymbol{y}) \varphi_p(\boldsymbol{y} - \boldsymbol{x}) \right|^{1+1/\epsilon} \right)^{\epsilon/(1+\epsilon)} \tag{1.2.4}$$

$$\le C_\epsilon \left(\int_{\mathbb{T}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \left| \hat{\varphi}_p(k, \mathrm{e}^{\mathrm{i}\theta})^2 \hat{\varphi}_p(k, \mathrm{e}^{-\mathrm{i}\theta}) \right|^{1+\epsilon} \right)^{1/(1+\epsilon)}$$

$$= C_\epsilon \left(\int_{\mathbb{T}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \left| \hat{\varphi}_p(k, \mathrm{e}^{\mathrm{i}\theta}) \right|^{3(1+\epsilon)} \right)^{1/(1+\epsilon)}$$

$$\le C_\epsilon \left(\int_{\mathbb{T}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \frac{K^{3(1+\epsilon)}}{\left|1 - \mathrm{e}^{\mathrm{i}\theta} \hat{D}(k)\right|^{3(1+\epsilon)}} \right)^{1/(1+\epsilon)}. \tag{1.2.5}$$

In the equality, we used $\hat{\varphi}_p(k, z^*) = \hat{\varphi}_p(k, z)^*$ for every $k \in \mathbb{T}^d$ and $z \in \mathbb{C}$, where z^* is the complex conjugate of z. As in the proof of Lemma 1.2.2, the integrability of the right-most in the above follows from

$$\iint_{\{ (k,\theta) \in \mathbb{T}^{d+1} \mid \|k\|_2 \le \delta_1,\ |\theta| \le \delta_2 \}} \frac{1}{\left(\|k\|_2^2 + |\theta|\right)^{3(1+\epsilon)}} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}\theta}{2\pi}$$

$$= 2 \int_{\{ k \in \mathbb{T}^d \mid \|k\|_2 \le \delta_1 \}} \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^{\delta_2} \frac{1}{\left(\|k\|_2^2 + \theta\right)^{3(1+\epsilon)}} \frac{\mathrm{d}\theta}{2\pi}$$

$$= \frac{2}{2 + 3\epsilon} \int_{\{ k \in \mathbb{T}^d \mid \|k\|_2 \le \delta_1 \}} \frac{1}{\left(\|k\|_2^2 + \theta\right)^{2+3\epsilon}} \Bigg|_{\theta=\delta_2}^{0} \frac{\mathrm{d}^d k}{(2\pi)^d}$$

$$\lesssim \int_{\{ k \in \mathbb{T}^d \mid \|k\|_2 \le \delta_1 \}} \frac{1}{\|k\|_2^{2(2+3\epsilon)}} \frac{\mathrm{d}^d k}{(2\pi)^d} \lesssim \int_0^{\delta_1} r^{d-5-6\epsilon} \, \mathrm{d}r < \infty$$

for $\delta_1, \delta_2 < 1$. Since the upper bound on (1.2.5) does not depend on p, $\sum_{\boldsymbol{y}} \varphi_{p_\mathrm{c}}^{\star 2}(\boldsymbol{y}) \varphi_{p_\mathrm{c}}(\boldsymbol{y} - \boldsymbol{x})$ is a $(1 + 1/\epsilon)$-summable function over $\mathbb{L}^d \times \mathbb{Z}_+$. This immediately implies

$$\sum_{\boldsymbol{y}} \varphi_{p_\mathrm{c}}^{\star 2}(\boldsymbol{y}) \varphi_{p_\mathrm{c}}(\boldsymbol{y} - \boldsymbol{x}) \xrightarrow[\|\boldsymbol{x}\|_2 \uparrow \infty]{} 0,$$

and completes the proof. □

1.3 Preliminary

In this section, we enumerate four lemmas used in the later chapters. Their proof are almost quotations from the references. Let

$$\hat{\Delta}_k \hat{f}(l) = \frac{\hat{f}(l+k) + \hat{f}(l-k)}{2} - \hat{f}(l)$$

for a function $\hat{f}$ on $\mathbb{T}^d$ and $k, l \in \mathbb{T}^d$, which is the notation for a sort of the second derivative in the Fourier space in a particular direction. Also, let $a \vee b$ and $a \wedge b$ denote $\max\{a, b\}$ and $\min\{a, b\}$, respectively. These symbols are often used in the field of probability theory.

The next lemma is crucial for our proofs of Theorem 1.2.1 because they are based on this lemma. See Section 2.1 and Section 3.1 for more details

Lemma 1.3.1 ([41, Lemma 5.9])**.** Fix an interval $[p_1, p_2) \subset \mathbb{R}$ and a function $f\colon [p_1, p_2) \to \mathbb{R}$. Suppose there exist $0 < a < b < \infty$ such that f satisfies the following conditions:

(i) f is continuous on $[p_1, p_2)$;

(ii) The initial condition $f(p_1) \leq a$ holds;

(iii) $\forall p \in (p_1, p_2)$, $[f(p) \leq b \implies f(p) \leq a]$.

Then, $f(p) \leq a$ for all $p \in [p_1, p_2)$.

Proof. We use the method of reductio ad absurdum. Assume that there exists $p \in [p_1, p_2)$ such that $f(p) > a$ when f satisfies the hypotheses. By the condition (ii), p must be greater than p_1. If $a < f(p) \leq b$, then (iii) yields a contradiction. Thus, we assume $f(p) > b$. By the continuity, there exists $q \in (p_1, p)$ such that $a < f(q) \leq b$. Since (iii) holds, $f(q)$ must be less than or equal to a. Therefore, we arrive at a contradiction. $\square$

The next lemma is used to bound function g_3 defined in Section 2.1 for ordinary percolation and Section 3.1 for oriented percolation (also, g_1 and g_2 are defined). In particular, we apply a modified version of this lemma, which is proved in Section 3.5, to the latter. Using the inequality $\sin^2\alpha = 1 - \cos^2\alpha \leq 2(1-\cos\alpha)$ in (3.5.9) instead of (3.5.10), one can immediately see the proof of this lemma, hence we omit it.

Lemma 1.3.2 ([41, Lemma 5.7])**.** Suppose that $a(-x) = a(x)$ for all $x \in \mathbb{L}^d$, and let

$$\hat{A}(k) = \frac{1}{1 - \hat{a}(k)}.$$

Then, for all $k, l \in \mathbb{T}^d$,

$$\left|\hat{\Delta}_k \hat{A}(l)\right| \leq \left(\widehat{|a|}(0) - \widehat{|a|}(k)\right)\left(\frac{\hat{A}(l+k) + \hat{A}(l-k)}{2}\hat{A}(l) + 4\hat{A}(l)\hat{A}(l+k)\hat{A}(l-k)\left(\widehat{|a|}(0) - \widehat{|a|}(l)\right)\right).$$

The following two lemmas are used to prove the continuities of $\{g_i\}_{i=1}^3$ in Section 2.2 for ordinary percolation and Section 3.2 for oriented percolation. In particular, the latter is also used for decomposition of diagrams, which are sums over products of two-point functions, in Section 2.4 for ordinary percolation and in Section 3.4 for oriented percolation.

Lemma 1.3.3 ([41, Lemma 5.13])**.** Fix $0 \leq a \leq p_0 < p_{\mathrm{c}}$. Let $\{f_k(p)\}_{k\in\mathbb{T}^d}$ be an equicontinuous family of functions in $p \in [a, p_0]$. Suppose that $\sup_{k\in\mathbb{T}^d} f_k(p) < \infty$ for every $p \in [a, p_0]$. Then, $\sup_{k\in\mathbb{T}^d} f_k(p) < \infty$ is continuous in $p \in [a, p_0]$.

Proof. Let $\bar{f} = \sup_k f_k$, and let $\epsilon > 0$ be given. Since $\{f_k(p)\}_{k\in\mathbb{T}^d}$ is an equicontinuous family, $\exists\delta > 0$ such that, $\forall k \in \mathbb{T}^d$ and $\forall p_1, p_2 \in [a, p_0]$ with $|p_1 - p_2| < \delta$, $|f_k(p_1) - f_k(p_2)| < \epsilon/2$. By the definition of the supremum, for $p \in [a, p_0]$, we can take $l \in \mathbb{T}^d$ such that $0 \leq \bar{f}(p) - f_l(p) < \epsilon/2$. Then, for every $p_1, p_2 \in [a, p_0]$ with $|p_1 - p_2| < \delta$ and $\bar{f}(p_1) \geq \bar{f}(p_2)$,

$$\begin{aligned}
0 &\leq \bar{f}(p_1) - \bar{f}(p_2) \\
&\leq \bar{f}(p_1) - f_l(p_2) \\
&\leq \left(\bar{f}(p_1) - f_l(p_1)\right) + \left(f_l(p_1) - f_l(p_2)\right) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Therefore, $\bar{f}$ is continuous. $\square$

Lemma 1.3.4 ([15, Lemma 2.13] or [26, Lemma 7.3])**.** Let $J \geq 1$ and $t_j \in \mathbb{R}$ for $j = 1, \ldots, J$. Then,

$$0 \leq 1 - \cos \sum_{j=1}^{J} t_j \leq J \sum_{j=1}^{J} (1 - \cos t_j). \tag{1.3.1}$$

Proof. First, take the real part of the telescopic identity $1 - \exp(\mathrm{i} \sum_{j=1}^{J} t_j) = \sum_{j=1}^{J} (1 - \exp(\mathrm{i} t_j)) \exp(\mathrm{i} \sum_{h=1}^{j-1} t_h)$, where the empty sum for $j = 1$ is regarded as zero. Then, use the inequalities $|\sin \sum_{h=1}^{j-1} t_h| \leq \sum_{h=1}^{j-1} |\sin t_h|$, $|\sin t_j||\sin t_h| \leq (\sin^2 t_j + \sin^2 t_h)/2$ and $\sin^2 t_j \leq 2(1 - \cos t_j)$ to obtain

$$\begin{aligned} 1 - \cos\left(\sum_{j=1}^{J} t_j\right) - \sum_{j=1}^{J}(1 - \cos t_j) &= -\sum_{j=1}^{J} \underbrace{(1 - \cos t_j)\left(1 - \cos \sum_{h=1}^{j-1} t_h\right)}_{\geq 0} + \sum_{j=1}^{J} (\sin t_j) \sin \sum_{h=1}^{j-1} t_h \\ &\leq \sum_{j=1}^{J} \sum_{h=1}^{j-1} \frac{\sin^2 t_j + \sin^2 t_h}{2} \leq (J-1) \sum_{j=1}^{J} (1 - \cos t_j), \end{aligned} \tag{1.3.2}$$

which implies (1.3.1). □

Chapter 2

Lace-expansion analysis for percolation

2.1 Bootstrapping functions

The proof of Theorem 1.2.1 is straightforward, assuming the following three propositions. To state those propositions, we first define

$$g_1(p) = p, \qquad\qquad g_2(p) = \|(1-\hat{D})\hat{G}_p\|_\infty. \tag{2.1.1}$$

Obviously, what we want to do is to show that $g_2(p)$ is bounded uniformly in $p \in [1, p_\mathrm{c})$. By Lemma 1.3.2, the Fourier transform of the RW Green function $\hat{S}_1(k) \equiv (1-\hat{D}(k))^{-1}$ obeys the inequality

$$\begin{aligned}&|\hat{\Delta}_k \hat{S}_1(l)| \\ &\le \hat{U}(k,l) \equiv \big(1-\hat{D}(k)\big)\bigg(\frac{\hat{S}_1(l+k)+\hat{S}_1(l-k)}{2}\hat{S}_1(l) + 4\hat{S}_1(l+k)\hat{S}_1(l-k)\bigg).\end{aligned} \tag{2.1.2}$$

Finally, we define

$$g_3(p) = \sup_{k,l} \frac{|\hat{\Delta}_k(\hat{G}_p(l)p\hat{D}(l))|}{\hat{U}(k,l)} \tag{2.1.3}$$

where the supremum near $k=0$ should be interpreted as the supremum over the limit as $\|k\|_2 \to 0$.

Now, we state the aforementioned three propositions and show that they indeed imply Theorem 1.2.1 for percolation by Lemma 1.3.1.

Proposition 2.1.1 (Continuity)**.** The functions $\{g_i(p)\}_{i=1}^3$ are continuous in $p \in [1, p_\mathrm{c})$.

Proposition 2.1.2 (Initial conditions)**.** For percolation on $\mathbb{L}^{d\ge 8}$, there are model-dependent finite constants $\{K_i\}_{i=1}^3$ such that $g_i(1) < K_i$ for $i = 1,2,3$.

Proposition 2.1.3 (Bootstrapping argument)**.** For percolation on $\mathbb{L}^{d\ge 9}$, we fix $p \in (1, p_\mathrm{c})$ and assume $g_i(p) \le K_i$, $i = 1,2,3$, where $\{K_i\}_{i=1}^3$ are the same constants as in Proposition 2.1.2. Then, the stronger inequalities $g_i(p) < K_i$, $i = 1,2,3$, hold.

In the rest of this chapter, we prove Proposition 2.1.1 in Section 2.2 and Proposition 2.1.2–2.1.3 in Section 2.8. In particular, Proposition 2.1.3 is shown in the following steps:

1. Bound the bootstrapping functions $\{g_i(p,m)\}_{i=1}^3$ above by the lace expansion.

2. Bound the lace expansion coefficients ($\{\pi_p^{(N)}\}_{N=0}^\infty$) above by basic diagrams (B_p, T_p and $\hat{V}_p^j(k)$, $i = 0,1,2,3$).

3. Bound the basic diagrams above by the bootstrapping hypotheses $g_i(p) \le K_i$ for $i = 1, 2, 3$ and the random-walk quantities (2.6.1).

4. Verify the stronger inequalities $g_i(p) < K_i$ holds for $i = 1, 2, 3$ by combining the bounds in Step 1–3 and substituting specifically numerical values into $\{K_i\}_{i=1}^3$.

These steps are explained in Section 2.5, 2.4, 2.6 and 2.8, respectively.

2.2 Continuity of the bootstrapping functions

In this section, we prove Proposition 2.1.1. First, we recall (2.1.1) and (2.1.3) for the bootstrapping functions $\{g_i(p)\}_{i=1}^3$. Obviously, $g_1(p) \equiv p$ is continuous. To prove continuity of the other two, we introduce

$$\tilde{g}_{2,k}(p) = \big(1 - \hat{D}(k)\big)\hat{G}_p(k), \tag{2.2.1}$$

$$\tilde{g}_{3,k,l}(p) = \frac{\hat{\Delta}_k(\hat{G}_p(l)\hat{D}(l))}{\hat{U}(k,l)} \tag{2.2.2}$$

and show that they are continuous in $p \in [1, p_{\mathrm{c}})$ for every $k, l \in \mathbb{T}^d$. However, since

$$g_2(p) = \sup_{k \in \mathbb{T}^d} |\tilde{g}_{2,k}(p)|, \qquad\qquad g_3(p) = \sup_{k,l \in \mathbb{T}^d} |\tilde{g}_{3,k,l}(p)|, \tag{2.2.3}$$

and the supremum of continuous functions is not necessarily continuous, we must be a bit more cautious here. Lemma 1.3.3 provides a sufficient condition for the supremum to be continuous.

Therefore, in order to prove continuity of $\{g_i(p)\}_{i=2,3}$ in $p \in [1, p_{\mathrm{c}})$, we want to show that $\{\tilde{g}_{2,k}(p)\}_{k \in \mathbb{T}^d}$ and $\{\tilde{g}_{3,k,l}(p)\}_{k,l \in \mathbb{T}^d}$ are equicontinuous families of functions in $p \in [1, p_0]$ for each $p_0 \in [1, p_{\mathrm{c}})$. To prove this, it then suffices to show that the following (i) and (ii) hold.

(i) $\tilde{g}_{2,k}(p)$ and $\partial_p \tilde{g}_{2,k}(p)$ are finite uniformly in $k \in \mathbb{T}^d$ and $p \in [1, p_0]$.

(ii) $\tilde{g}_{3,k,l}(p)$ and $\partial_p \tilde{g}_{3,k,l}(p)$ are finite uniformly in $k, l \in \mathbb{T}^d$ and $p \in [1, p_0]$.

To prove (i) is not so hard. By $0 \le 1 - \hat{D}(k) \le 2$, $|\hat{G}_p(k)| \le \chi_p$ and the monotonicity of χ_p in p, we obtain $|\tilde{g}_{2,k}(p)| \le 2\chi_{p_0} < \infty$ uniformly in $k \in \mathbb{T}^d$ and $p \in [1, p_0]$. Moreover, by Russo's formula and the BK inequality (see, e.g., [17]), and then using translation-invariance, we obtain

$$0 \le \partial_p G_p(x) \le (D * G_p^{*2})(x), \tag{2.2.4}$$

hence

$$|\partial_p \tilde{g}_{2,k}(p)| \le 2 \sum_x (D * G_p^{*2})(x) \le 2\chi_{p_0}^2 < \infty, \tag{2.2.5}$$

uniformly in $k \in \mathbb{T}^d$ and $p \in [1, p_0]$, as required.

To prove (ii) needs extra care, especially near $k = 0$, because of the factor $1 - \hat{D}(k)$ in $\hat{U}(k,l)$. First, by the telescopic inequality (1.3.1), we obtain

$$\begin{aligned} \big|\hat{\Delta}_k\big(\hat{G}_p(l)\hat{D}(l)\big)\big| \quad &\le \quad \sum_x (1 - \cos k \cdot x)(G_p * D)(x) \\ &\overset{(1.3.1)}{\le} \quad 2\Big(|\hat{\Delta}_k \hat{G}_p(0)| + \chi_p\big(1 - \hat{D}(k)\big)\Big), \end{aligned} \tag{2.2.6}$$

and that

$$\begin{aligned}
\left|\hat{\Delta}_k\Big(\partial_p\hat{G}_p(l)\hat{D}(l)\Big)\right| &\leq \sum_x(1-\cos k\cdot x)(\partial_p G_p * D)(x) \\
&\overset{(2.2.4)\,\&\,(1.3.1)}{\leq} 2\Big(\Big|\hat{\Delta}_k\partial_p\hat{G}_p(0)\Big| + \chi_p^2\big(1-\hat{D}(k)\big)\Big) \\
&\overset{(2.2.4)}{\leq} 2\Big(\sum_x(1-\cos k\cdot x)(D*G_p^{*2})(x) + \chi_p^2\big(1-\hat{D}(k)\big)\Big) \\
&\overset{(1.3.1)}{\leq} 2\Big(6\chi_p|\hat{\Delta}_k\hat{G}_p(0)| + 4\chi_p^2\big(1-\hat{D}(k)\big)\Big). \qquad (2.2.7)
\end{aligned}$$

Therefore, to evaluate $\tilde{g}_{3,k,l}(p)$ and $\partial_p\tilde{g}_{3,k,l}(p)$, it suffices to evaluate $|\hat{\Delta}_k\hat{G}_p(0)|$.

To evaluate $|\hat{\Delta}_k\hat{G}_p(0)|$ by using (1.3.1), we first rewrite the expression (1.1.4) for $G_p(x)$. To do so, we introduce ordering among self-avoiding paths from o to x as follows. For each vertex x, let $B(x)$ be the set of bonds incident on x. Order the elements in $B(x)$ in an arbitrary but fixed manner. For a pair of bonds $b, b' \in B(x)$, we write $b \prec b'$ if b is lower than b' in that ordering. For a pair of self-avoiding paths $\omega, \omega' \in \Omega(x,y)$, we write $\omega \prec \omega'$ if at the first time τ when ω becomes incompatible with ω' (therefore $b_j(\omega) = b_j(\omega')$ for all $j < \tau$) we have $b_\tau(\omega) \prec b_\tau(\omega')$. We say that ω is occupied if all $b_1(\omega), \dots, b_{|\omega|}(\omega)$ are occupied. Let $E_{x,y}(\omega)$ be the event that $\omega \in \Omega(x,y)$ is the lowest occupied path from x to y:

$$E_{x,y}(\omega) = \{\omega \text{ is occupied}\} \setminus \bigcup_{\substack{\omega'\in\Omega(x,y)\\(\omega'\prec\omega)}} \{\omega' \text{ is occupied}\}. \qquad (2.2.8)$$

Then, we can rewrite the expression (1.1.4) for $G_p(x)$ as

$$G_p(x) = \sum_{\omega\in\Omega(o,x)} \mathbb{P}_p\big(E_{o,x}(\omega)\big). \qquad (2.2.9)$$

We can bound $|\hat{\Delta}_k\hat{G}_p(0)|$ as

$$\begin{aligned}
\Big|\hat{\Delta}_k\hat{G}_p(0)\Big| &= \sum_x (1-\cos k\cdot x)G_p(x) \\
&= \sum_x \sum_{\omega\in\Omega(o,x)} \left(1-\cos\sum_{i=1}^{|\omega|} k\cdot(\omega_i-\omega_{i-1})\right)\mathbb{P}_p\big(E_{o,x}(\omega)\big) \\
&\leq \sum_x \sum_{\omega\in\Omega(o,x)} |\omega|\sum_{i=1}^{|\omega|}\big(1-\cos k\cdot(\omega_i-\omega_{i-1})\big)\,\mathbb{P}_p\big(E_{o,x}(\omega)\big) \\
&= \sum_{u,v,x}\big(1-\cos k\cdot(v-u)\big)\sum_{\omega\in\Omega(o,x)}|\omega|\sum_{i=1}^{|\omega|}\mathbb{1}_{\{b_i(\omega)=(u,v)\}}\,\mathbb{P}_p\big(E_{o,x}(\omega)\big).
\end{aligned}$$

Let $\eta = (\omega_0, \dots, \omega_i)$ and $\xi = (\omega_i, \dots, \omega_{|\omega|})$ and denote their concatenation in that order by $\eta\circ\xi$. Then, the above inequality is equivalent to

$$\begin{aligned}
|\hat{\Delta}_k\hat{G}_p(0)| \leq \sum_{u,v,x}\big(1-\cos k\cdot(v-u)\big)&\sum_{\eta\in\Omega(o,v)}\mathbb{1}_{\{b_{|\eta|}(\eta)=(u,v)\}} \\
&\times \sum_{\substack{\xi\in\Omega(v,x)\\(\eta\circ\xi\in\Omega(o,x))}}(|\eta|+|\xi|)\,\mathbb{P}_p\big(E_{o,x}(\eta\circ\xi)\big). \qquad (2.2.10)
\end{aligned}$$

Next, we rewrite $\mathbb{P}_p(E_{o,x}(\eta \circ \xi))$. To do so, we introduce a peculiar cluster of η as follows. Given a vertex y and a bond $b \in B(y)$, we define $\mathcal{C}_{\prec b}(y)$ to be the set of vertices that are connected from y via an occupied bond $b' \in B(y)$ with $b' \prec b$; if there are no such occupied bonds, then we define $\mathcal{C}_{\prec b}(y) = \{y\}$. Given a self-avoiding path η, we let

$$\mathcal{C}_{\prec \eta} = \bigcup_{j=1}^{|\eta|} \mathcal{C}_{\prec b_j(\eta)}(\eta_{j-1}). \tag{2.2.11}$$

Notice that the terminal point $\eta_{|\eta|}$ is not in $\mathcal{C}_{\prec \eta}$. Using this notation and recalling (2.2.8), we can rewrite the event $E_{o,x}(\eta \circ \xi)$ for $\eta \in \Omega(o,v)$ and $\xi \in \Omega(v,x)$ with $\eta \circ \xi \in \Omega(o,x)$ as

$$E_{o,x}(\eta \circ \xi) = E_{o,v}(\eta) \cap \big\{E_{v,x}(\xi) \text{ occurs on } \mathbb{L}^d \setminus \mathcal{C}_{\prec \eta}\big\}. \tag{2.2.12}$$

For a $V \subset \mathbb{L}^d$, we let $\mathbb{P}_p^V$ be the percolation measure defined by making all bonds b with $b \cap (\mathbb{L}^d \setminus V) \neq \varnothing$ vacant. Then, we obtain the rewrite

$$\mathbb{P}_p(E_{o,x}(\eta \circ \xi)) = \mathbb{E}_p\Big[\mathbb{1}_{E_{o,v}(\eta)}\, \mathbb{P}_p^{\mathbb{L}^d \setminus \mathcal{C}_{\prec \eta}}\big(E_{v,x}(\xi)\big)\Big], \tag{2.2.13}$$

hence

$$\begin{aligned}
|\hat{\Delta}_k \hat{G}_p(0)| \leq \sum_{u,v,x} \big(1 - \cos k \cdot (v-u)\big) &\sum_{\eta \in \Omega(o,v)} \mathbb{1}_{\{b_{|\eta|}(\eta) = (u,v)\}} \\
\times \Bigg(&|\eta|\, \mathbb{E}_p\Bigg[\mathbb{1}_{E_{o,v}(\eta)} \sum_{\substack{\xi \in \Omega(v,x) \\ (\eta \circ \xi \in \Omega(o,x))}} \mathbb{P}_p^{\mathbb{L}^d \setminus \mathcal{C}_{\prec \eta}}\big(E_{v,x}(\xi)\big)\Bigg] \\
&+ \mathbb{E}_p\Bigg[\mathbb{1}_{E_{o,v}(\eta)} \sum_{\substack{\xi \in \Omega(v,x) \\ (\eta \circ \xi \in \Omega(o,x))}} |\xi|\, \mathbb{P}_p^{\mathbb{L}^d \setminus \mathcal{C}_{\prec \eta}}\big(E_{v,x}(\xi)\big)\Bigg]\Bigg).
\end{aligned} \tag{2.2.14}$$

The contribution from the first expectation is evaluated as follows. First, we note that the sum over ξ can be replaced by the sum over $\xi \in \Omega(v,x)$ that are restricted in $\mathbb{L}^d \setminus \mathcal{C}_{\prec \eta}$, or $\mathbb{P}_p^{\mathbb{L}^d \setminus \mathcal{C}_{\prec \eta}}(E_{v,x}(\xi)) = 0$ otherwise. Then, the resulting sum equals the restricted two-point function on $\mathbb{L}^d \setminus \mathcal{C}_{\prec \eta}$ and is bounded by the full two-point function $G_p(x-v)$. Therefore,

$$\begin{aligned}
&\sum_{\eta \in \Omega(o,v)} \mathbb{1}_{\{b_{|\eta|}(\eta) = (u,v)\}} |\eta|\, \mathbb{E}_p\Bigg[\mathbb{1}_{E_{o,v}(\eta)} \sum_{\substack{\xi \in \Omega(v,x) \\ (\eta \circ \xi \in \Omega(o,x))}} \mathbb{P}_p^{\mathbb{L}^d \setminus \mathcal{C}_{\prec \eta}}\big(E_{v,x}(\xi)\big)\Bigg] \\
&\leq \sum_{\eta \in \Omega(o,v)} \mathbb{1}_{\{b_{|\eta|}(\eta) = (u,v)\}} |\eta|\, \mathbb{P}_p\big(E_{o,v}(\eta)\big)\, G_p(x-v).
\end{aligned} \tag{2.2.15}$$

We apply the same analysis to $\eta = \zeta \circ (u,v)$, where $\zeta = (\eta_0, \dots, \eta_{|\eta|-1})$, and obtain

$$\begin{aligned}
(2.2.15) &\leq pD(v-u)\, G_p(x-v) \sum_{\zeta \in \Omega(o,u)} (|\zeta| + 1)\, \mathbb{P}_p\big(E_{o,u}(\zeta)\big) \\
&= pD(v-u)\, G_p(x-v) \sum_y \sum_{\substack{\zeta' \in \Omega(o,y) \\ \zeta'' \in \Omega(y,u) \\ (\zeta' \circ \zeta'' \in \Omega(o,u))}} \mathbb{P}_p\big(E_{o,u}(\zeta' \circ \zeta'')\big),
\end{aligned} \tag{2.2.16}$$

where the equality is due to the identity $|\zeta|+1=\sum_y \mathbb{1}_{\{y\in\zeta\}}$. Again, by the same analysis as discussed above, we finally obtain

$$(2.2.16) \le G_p^{*2}(u)\, pD(v-u)\, G_p(x-v). \tag{2.2.17}$$

The contribution from the second expectation in (2.2.14) can be evaluated in a similar way, and the result is

$$\begin{aligned}&\sum_{\eta\in\Omega(o,v)} \mathbb{1}_{\{b_{|\eta|}(\eta)=(u,v)\}}\mathbb{E}_p\Bigg[\mathbb{1}_{E_{o,v}(\eta)} \sum_{\substack{\xi\in\Omega(v,x)\\(\eta\circ\xi\in\Omega(o,x))}} |\xi|\, \mathbb{P}_p^{\mathbb{L}^d\setminus\mathcal{C}_{\prec\eta}}\big(E_{v,x}(\xi)\big)\Bigg]\\ &\le G_p(u)\, pD(v-u)\, G_p^{*2}(x-v).\end{aligned} \tag{2.2.18}$$

Substituting (2.2.17)–(2.2.18) back into (2.2.14), we obtain

$$|\hat{\Delta}_k \hat{G}_p(0)| \le 2p_0\big(1-\hat{D}(k)\big)\chi_{p_0}^3, \tag{2.2.19}$$

which implies finiteness of $\tilde{g}_{3,k,l}(p)$ and $\partial_p\tilde{g}_{3,k,l}(p)$ uniformly in $k,l\in\mathbb{T}^d$ and $p\in[1,p_0]$, as required. This completes the proof of Proposition 2.1.1.

2.3 Derivation of the lace expansion

To prove Proposition 2.1.2 and 2.1.3, we use the lace expansion. It gives a similar recursion equation for the two-point function G_p to that for the RW Green function S_p, which is

$$S_p(x) = \delta_{o,x} + (pD * S_p)(x).$$

Proposition 2.3.1 ([21])**.** For any $p<p_c$ and $N\in\mathbb{Z}_+$, there are nonnegative functions $\{\pi_p^{(n)}\}_{n=0}^N$ on $\mathbb{L}^d$ such that, if we define $\Pi_p^{(N)}$ as

$$\Pi_p^{(N)}(x) = \sum_{n=0}^N (-1)^n \pi_p^{(n)}(x), \tag{2.3.1}$$

then we obtain the recursion equation

$$G_p(x) = \delta_{o,x} + \Pi_p^{(N)}(x) + \big((\delta_{o,\cdot} + \Pi_p^{(N)}) * pD * G_p\big)(x) + (-1)^{N+1} R_p^{(N+1)}(x), \tag{2.3.2}$$

where the remainder $R_p^{(N+1)}(x)$ obeys the bound

$$0 \le R_p^{(N+1)}(x) \le (\pi_p^{(N)} * G_p)(x). \tag{2.3.3}$$

Assume that $\lim_{N\to\infty}\pi_p^{(N)}(x)=0$ for every $x\in\mathbb{L}^d$, in fact, which holds by the absolute convergence of the alternative series of $\{\pi_p^{(N)}\}_{N=1}^\infty$ from Lemma 2.4.1 below. Then, $R_p^{(N)}(x)\to 0$ as $N\to\infty$ for every $x\in\mathbb{L}^d$, so that the limit of the series

$$\Pi_p(x) \equiv \lim_{N\to\infty}\Pi_p^{(N)}(x) = \sum_{N=0}^\infty (-1)^N \pi_p^{(N)}(x) \tag{2.3.4}$$

is well-defined.

To prove the above proposition, we first introduce some notions and notation.

Definition 2.3.2. Fix a bond configuration and let $x,y,u,v\in\mathbb{L}^d$.

(i) Given a bond b, we define $\tilde{\mathcal{C}}^b(x)$ to be the set of vertices connected to x in the new configuration obtained by setting b to be vacant.

(ii) We say that a directed bond (u, v) is pivotal for the connection from x to y if $x \longleftrightarrow u$ occurs in $\tilde{\mathcal{C}}^{\{u,v\}}(x)$ (i.e., x is connected to u without using $\{u, v\}$) and if $v \longleftrightarrow y$ occurs in the complement of $\tilde{\mathcal{C}}^{\{u,v\}}(x)$, denoted by $\tilde{\mathcal{C}}^{\{u,v\}}(x)^{\mathrm{c}}$. Let $\texttt{piv}(x, y)$ be the set of directed pivotal bonds for the connection from x to y.

(iii) We say that x is doubly connected to y, denoted by $x \Longleftrightarrow y$, if either $x = y$ or $x \longleftrightarrow y$ and $\texttt{piv}(x, y) = \varnothing$.

(iv) Given a set of vertices $A \subset \mathbb{L}^d$, we say that x and y are connected in A if either $x = y \in A$ or there is an occupied self-avoiding path from x to y consisting of vertices in A. We write this event as $\{x \longleftrightarrow y \text{ in } A\}$.

(v) Given a set of vertices $A \subset \mathbb{L}^d$, we say that x and y are connected through A if either $x = y \in A$ or every occupied self-avoiding path from x to y contains vertices in A. We write this event as $\{x \overset{A}{\longleftrightarrow} y\}$.

Sketch proof of Proposition 2.3.1. First, we derive the first expansion, i.e., (2.3.2) for $N = 0$. By splitting the event $\{o \longleftrightarrow x\}$ into two depending on whether or not there is a pivotal bond for the connection from o to x, we first obtain

$$G_p(x) = \mathbb{P}_p\big(\underbrace{o \longleftrightarrow x,\ \texttt{piv}(o, x) = \varnothing}_{=\{o \Longleftrightarrow x\}}\big) + \mathbb{P}_p\big(o \longleftrightarrow x,\ \texttt{piv}(o, x) \neq \varnothing\big). \tag{2.3.5}$$

Let

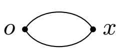
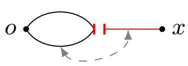
$$\pi_p^{(0)}(x) = \mathbb{P}_p(o \Longleftrightarrow x) - \delta_{o,x}. \tag{2.3.6}$$

Then, by definition, the first term in (2.3.5) is $\delta_{o,x} + \pi_p^{(0)}(x)$. To expand the second term in (2.3.5), we use the first pivotal bond $b \equiv (\underline{b}, \overline{b})$ for the connection from o to x, so that $o \Longleftrightarrow \underline{b}$ in $\tilde{\mathcal{C}}^b(o)$ and $\overline{b} \longleftrightarrow x$ in $\tilde{\mathcal{C}}^b(o)^{\mathrm{c}}$. Since those two events are independent of the occupation status of b, we obtain

$$\begin{aligned}
&\mathbb{P}_p\big(o \longleftrightarrow x,\ \texttt{piv}(o, x) \neq \varnothing\big) \\
&= \sum_b \mathbb{P}_p\Big(o \Longleftrightarrow \underline{b} \text{ in } \tilde{\mathcal{C}}^b(o),\ b \text{ occupied},\ \overline{b} \longleftrightarrow x \text{ in } \tilde{\mathcal{C}}^b(o)^{\mathrm{c}}\Big) \\
&= \sum_b pD(b)\, \mathbb{E}_p^0\Big[\mathbb{1}_{\{o \Longleftrightarrow \underline{b}\}}\ \mathbb{P}_p^1\Big(\overline{b} \longleftrightarrow x \text{ in } \tilde{\mathcal{C}}_0^b(o)^{\mathrm{c}}\Big)\Big],
\end{aligned} \tag{2.3.7}$$

where $D(b)$ is the abbreviation for $D(\overline{b} - \underline{b})$, and the extra indices[1] represent that $\tilde{\mathcal{C}}_0^b(o)$ is random against $\mathbb{E}_p^0$ but deterministic against $\mathbb{P}_p^1$. In the last line, we have dropped "in $\tilde{\mathcal{C}}_0^b(o)$" by using the fact that $\{\overline{b} \longleftrightarrow x \text{ in } \tilde{\mathcal{C}}_0^b(o)^{\mathrm{c}}\} = \varnothing$ when $\mathbb{1}_{\{o \Longleftrightarrow \overline{b} \text{ in } \tilde{\mathcal{C}}_0^b(o)\}} \neq \mathbb{1}_{\{o \Longleftrightarrow \underline{b}\}}$.

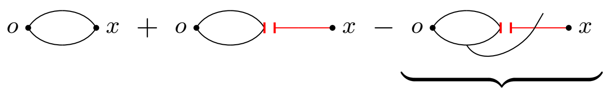
Now we introduce schematic drawings, such as

$\delta_{o,x} + \pi_p^{(0)}(x) =$  , (2.3.7) $=$  . (2.3.8)

In the second drawing, the parallel short line segments in the middle represents $pD(b)$, which is summed over all bonds b and unlabeled. The dashed two-sided arrow represents mutual avoidance

[1] This rewrite is due to the tower property $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid \mathcal{G}]]$, where $\mathbb{E}[X \mid \mathcal{G}]$ is the conditional expectation of a random variable X with respect to a sub-σ-algebra $\mathcal{G}$.

between $\tilde{\mathcal{C}}_0^b(o)$ (in black) and $\tilde{\mathcal{C}}_1^b(x)$ (in red). By the inclusion-exclusion relation $\{\bar{b} \longleftrightarrow x \text{ in } \tilde{\mathcal{C}}_0^b(o)^{\mathrm{c}}\} = \{\bar{b} \longleftrightarrow x\} \setminus \{\bar{b} \xleftrightarrow{\tilde{\mathcal{C}}_0^b(o)} x\}$, we complete the first expansion as

$G_p(x) =$ 

$\equiv R_p^{(1)}(x)$

$$= \delta_{o,x} + \pi_p^{(0)}(x) + \Big(\big(\delta_{o,\cdot} + \pi_p^{(0)}\big) * pD * G_p\Big)(x) - R_p^{(1)}(x). \tag{2.3.9}$$

The precise definition of the remainder $R_p^{(1)}(x)$ is

$$R_p^{(1)}(x) = \sum_b pD(b)\mathbb{E}_p^0\Big[\mathbb{1}_{\{o \Longleftrightarrow \underline{b}\}}\, \mathbb{P}_p^1\Big(\bar{b} \xleftrightarrow{\tilde{\mathcal{C}}_0^b(o)^{\mathrm{c}}} x\Big)\Big]. \tag{2.3.10}$$

Next, we expand the remainder $R_p^{(1)}(x)$ to derive the second expansion, i.e., (2.3.2) for $N = 1$. To do so, and to derive the higher-order expansion later, we have to deal with the event $\{v \xleftrightarrow{A} x\}$ for some vertex v and a vertex set A. Let

$$E(v, x; A) = \big\{v \xleftrightarrow{A} x\big\} \setminus \bigcup_{b \in \mathtt{piv}(v,x)} \big\{v \xleftrightarrow{A} \underline{b}\big\}. \tag{2.3.11}$$

Intuitively, if we regard a percolation cluster of v containing x as a string of sausages from v to x, then $E(v, x; A)$ is considered to be the event that the last sausage is the first one that goes through A. Then, we can split the event $\{v \xleftrightarrow{A} x\}$ into two disjoint events as

$$\Big\{v \xleftrightarrow{A} x\Big\} = E\,(v, x; A) \cup \Big\{\exists b \in \mathtt{piv}(v, x) \text{ occupied } \& \ v \xleftrightarrow{A} \underline{b}\Big\}. \tag{2.3.12}$$

Let

$$\pi_p^{(1)}(x) = \sum_b pD(b)\, \mathbb{E}_p^0\,\Big[\mathbb{1}_{\{o \Longleftrightarrow \underline{b}\}}\, \mathbb{P}_p^1\Big(E\big(\bar{b}, x; \tilde{\mathcal{C}}_0^b(o)\big)\Big)\Big], \tag{2.3.13}$$

so that we have

$$\begin{aligned} R_p^{(1)}(x) = \pi_p^{(1)}(x) + \sum_{b_1} pD(b_1)\, \mathbb{E}_p^0\,\Big[\mathbb{1}_{\{o \Longleftrightarrow \underline{b_1}\}} \\ \times \mathbb{P}_p^1\Big(\exists b_2 \in \mathtt{piv}(\overline{b_1}, x) \text{ occupied } \& \ \overline{b_1} \xleftrightarrow{\tilde{\mathcal{C}}_0^{b_1}(o)} \underline{b_2}\Big)\Big]. \end{aligned} \tag{2.3.14}$$

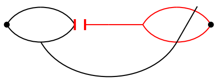
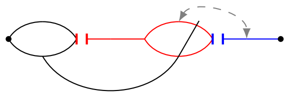
Notice that the event $\{\exists b \in \mathtt{piv}(v, x) \text{ occupied } \& \ v \xleftrightarrow{A} \underline{b}\}$ in (2.3.12) can be rewritten by identifying the first element in $\{b \in \mathtt{piv}(v, x) : v \xleftrightarrow{A} \underline{b}\}$ as

$$\begin{aligned} &\Big\{\exists b \in \mathtt{piv}(v, x) \text{ occupied } \& \ v \xleftrightarrow{A} \underline{b}\Big\} \\ &= \bigcup_b \Big\{E(v, \underline{b}; A) \text{ occurs in } \tilde{\mathcal{C}}^b(v)\Big\} \cap \{b \text{ occupied}\} \cap \Big\{\bar{b} \longleftrightarrow x \text{ in } \tilde{\mathcal{C}}^b(v)^{\mathrm{c}}\Big\}. \end{aligned} \tag{2.3.15}$$

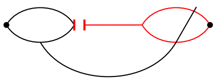
By this rewrite and using the fact that the first and third events on the right-hand side are independent of the occupation status of b, we obtain (cf., (2.3.7))

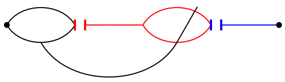
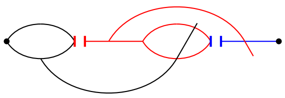
$$\begin{aligned} R_p^{(1)}(x) = \pi_p^{(1)}(x) + \sum_{b_1, b_2} pD(b_1)pD(b_2)\mathbb{E}_p^0\Big[\mathbb{1}_{\{o \Longleftrightarrow \underline{b_1}\}}\, \mathbb{E}_p^1\Big[\mathbb{1}_{E(\overline{b_1}, \underline{b_2}; \tilde{\mathcal{C}}_0^{b_1}(o))} \\ \times \mathbb{P}_p^2\Big(\overline{b_2} \longleftrightarrow x \text{ in } \tilde{\mathcal{C}}_1^{b_2}(\overline{b_1})^{\mathrm{c}}\Big)\Big]\Big], \end{aligned} \tag{2.3.16}$$

where we have dropped “occurs in $\tilde{\mathcal{C}}_1^{b_2}(\overline{b_1})$” by using the fact that $\{\overline{b_2} \longleftrightarrow x \text{ in } \tilde{\mathcal{C}}_1^{b_2}(\overline{b_1})^{\mathrm{c}}\} = \varnothing$ when $\mathbb{1}_{\{E(\overline{b_1},\underline{b_2};\tilde{\mathcal{C}}_0^{b_1}(o)) \text{ occurs in } \tilde{\mathcal{C}}_1^{b_2}(\overline{b_1})\}} \neq \mathbb{1}_{E(\overline{b_1},\underline{b_2};\tilde{\mathcal{C}}_0^{b_1}(o))}$. By using similar schematic drawings to (2.3.8), the above identity for $R_p^{(1)}$ is rewritten as

$R_p^{(1)}(x) = o$  x $+$ o  x , (2.3.17)

where the dashed two-sided arrow represents mutual avoidance between $\tilde{\mathcal{C}}_1^{b_2}(\overline{b_1})$ (in red) and $\tilde{\mathcal{C}}_2^{b_2}(x)$ (in blue). By the inclusion-exclusion relation $\{\overline{b_2} \longleftrightarrow x \text{ in } \tilde{\mathcal{C}}_1^{b_2}(\overline{b_1})^{\mathrm{c}}\} = \{\overline{b_2} \longleftrightarrow x\} \setminus \{\overline{b_2} \xleftrightarrow{\tilde{\mathcal{C}}_1^{b_2}(\overline{b_1})} x\}$, we arrive at the second expansion

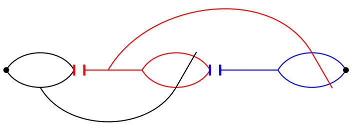
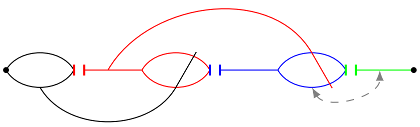
$R_p^{(1)}(x) = o$  x

$+$ o  x $-$ o  x

$\equiv R_p^{(2)}(x)$

$$= \pi_p^{(1)}(x) + \left(\pi_p^{(1)} * pD * G_p\right)(x) - R_p^{(2)}(x).$$

To show how to derive the higher-order expansion coefficients, we further demonstrate the expansion of the remainder $R_p^{(2)}(x)$ by using schematic drawings. Using (2.3.12) and (2.3.15), we can rewrite $R_p^{(2)}(x)$ as

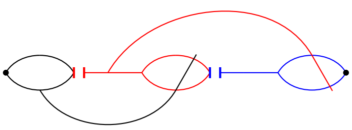
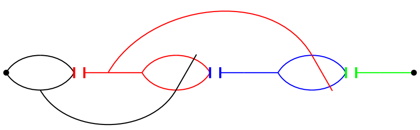
$R_p^{(2)}(x) = o$  x $+$ o  x . (2.3.18)

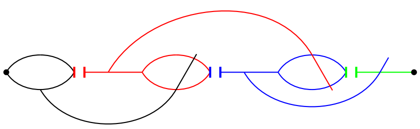
$\equiv \pi_p^{(2)}(x)$

The precise definition of $\pi_p^{(2)}(x)$ is

$$\pi_p^{(2)}(x) = \sum_{b_1,b_2} pD(b_1)pD(b_2)\,\mathbb{E}_p^0\Big[\mathbb{1}_{\{o \Longleftrightarrow \underline{b_1}\}}\,\mathbb{E}_p^1\Big[\mathbb{1}_{E(\overline{b_1},\underline{b_2};\tilde{\mathcal{C}}_0^{b_1}(o))}\,\mathbb{P}_p^2\Big(E(\overline{b_2},x;\tilde{\mathcal{C}}_1^{b_2}(\overline{b_1}))\Big)\Big]\Big]. \tag{2.3.19}$$

As in the previous stages of the expansion, the dashed two-sided arrow in (2.3.18) represents mutual avoidance between $\tilde{\mathcal{C}}_2^{b_3}(\overline{b_2})$ (in blue) and $\tilde{\mathcal{C}}_3^{b_3}(x)$ (in green). Then, by the inclusion-exclusion relation $\{\overline{b_3} \longleftrightarrow x \text{ in } \tilde{\mathcal{C}}_2^{b_3}(\overline{b_2})^{\mathrm{c}}\} = \{\overline{b_3} \longleftrightarrow x\} \setminus \{\overline{b_3} \xleftrightarrow{\tilde{\mathcal{C}}_2^{b_3}(\overline{b_2})} x\}$, we obtain

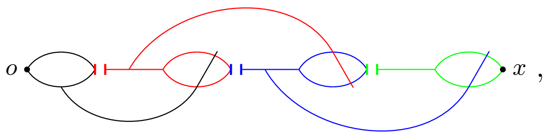
$R_p^{(2)}(x) = o$  x $+$ o  x (2.3.20)

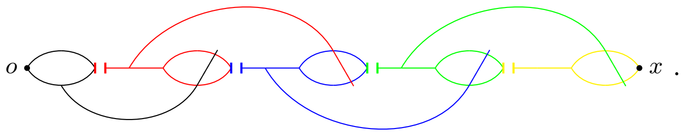
$-$ o  x (2.3.21)

$\equiv R_p^{(3)}(x)$

$$= \pi_p^{(2)}(x) + \left(\pi_p^{(2)} * pD * G_p\right)(x) - R_p^{(3)}(x). \tag{2.3.22}$$

By repeated applications of inclusion-exclusion to the remainders, we can derive the higher-order expansion coefficients, such as

$\pi_p^{(3)}(x) =$  $,$ (2.3.23)

$\pi_p^{(4)}(x) =$  $.$ (2.3.24)

We complete the sketch proof of Proposition 2.3.1. □

2.4 Diagrammatic bounds on the expansion coefficients

The bootstrapping functions $\{g_i(p)\}_{i=1}^3$ are bounded in terms of sums of $\hat{\pi}_p^{(n)}(0)$ and $|\hat{\Delta}_k \hat{\pi}_p^{(n)}(0)|$. In this section, we prove bounds on those quantities in terms of basic diagrams L_p, B_p, T_p, $\hat{V}_p^0$, $\hat{V}_p^1$, $\hat{V}_p^2$ and $\hat{V}_p^3$ defined as

$$L_p = \left\|(pD)^{*2} * G_p\right\|_\infty, \tag{2.4.1}$$
$$B_p = \left\|(pD)^{*2} * G_p^{*2}\right\|_\infty, \tag{2.4.2}$$
$$T_p = \left\|(pD)^{*2} * G_p^{*3}\right\|_\infty, \tag{2.4.3}$$
$$\hat{V}_p^0(k) = \sum_x (pD * G_p)(x)^2 (1 - \cos k \cdot x), \tag{2.4.4}$$
$$\hat{V}_p^1(k) = \sup_x \sum_y (pD * G_p)(y)(1 - \cos k \cdot y)\, G_p(x-y), \tag{2.4.5}$$
$$\hat{V}_p^2(k) = \sup_x \sum_y (pD * G_p)(y)(1 - \cos k \cdot y)(pD * G_p)(x-y), \tag{2.4.6}$$
$$\begin{aligned}\hat{V}_p^3(k) = \sup_{x,y} \sum_{\{v_j\}_{j=1}^5} & G_p(v_1)(pD * G_p)(v_2 - v_1)\big(1 - \cos k \cdot (v_2 - v_1)\big) \\ & \times (pD * G_p)(v_3 - v_2)(pD * G_p)(v_4 - v_1)(pD * G_p)(v_4 - v_2) \\ & \times (pD * G_p)(v_5 - v_4)(pD * G_p)(x - v_5) G_p(y + v_3 - v_5).\end{aligned} \tag{2.4.7}$$

We also define

$$r = p\|D\|_\infty + L_p + B_p,$$
$$\rho = T_p(2r + T_p) + (r + T_p)\left(1 + \frac{B_p}{2} + T_p\right).$$

Lemma 2.4.1 (Diagrammatic bounds on the expansion coefficients)**.** The expansion coefficients $\hat{\pi}_p^{(n)}(0) \equiv \sum_x \pi_p^{(n)}(x)$ and $\left|\hat{\Delta}_k \hat{\pi}_p^{(n)}(0)\right| \equiv \sum_x (1 - \cos k \cdot x)\, \pi_p^{(n)}(x)$, both nonnegative, obey the following bounds:

$$\hat{\pi}_p^{(n)}(0) \le \begin{cases} B_p/2 & [n=0], \\ (1+B_p/2+T_p)^2 r\rho^{n-1} & [n \ge 1], \end{cases} \tag{2.4.8}$$

$$\left|\hat{\Delta}_k\, \hat{\pi}_p^{(0)}(0)\right| \le \frac{1}{2}\hat{V}_p^0(k), \tag{2.4.9}$$

$$\begin{aligned}\left|\hat{\Delta}_k\, \hat{\pi}_p^{(1)}(0)\right| \le{}& \Big(1+2(B_p+r)+\frac{3}{4}B_p(B_p+2r)+3rT_p\Big)\hat{V}_p^0(k) \\ &+(8+6B_p+9T_p)T_p\hat{V}_p^2(k).\end{aligned} \tag{2.4.10}$$

For $m \ge 1$,

$$\begin{aligned}&\left|\hat{\Delta}_k\, \hat{\pi}_p^{(2m)}(0)\right| \\ &\le (4m+1)\Bigg(\rho^{2m-1}\Big(2r\hat{V}_p^0(k)+T_p\hat{V}_p^2(k)+T_p\hat{V}_p^1(k)\Big)\Big(1+\frac{B_p}{2}+T_p\Big) \\ &\qquad + m\rho^{2m-1}\big(\hat{V}_p^2(k)+\hat{V}_p^1(k)\big)\Big(1+\frac{B_p}{2}+T_p\Big)^2 \\ &\qquad + \rho^{2m-2}\Bigg((2m-1)\Big(r^2\hat{V}_p^0(k)+rT_p\hat{V}_p^2(k)+rT_p\hat{V}_p^1(k)+T_p^2\hat{V}_p^1(k)\Big) \\ &\qquad\qquad + (m-1)\big(rT_p\hat{V}_p^0(k)+\hat{V}_p^3(k)\big)\Bigg)\Big(1+\frac{B_p}{2}+T_p\Big)^2\Bigg),\end{aligned} \tag{2.4.11}$$

and

$$\begin{aligned}&\left|\hat{\Delta}_k\, \hat{\pi}_p^{(2m+1)}(0)\right| \\ &\le (4m+3)\Bigg(2\rho^{2m}\Big(r\hat{V}_p^0(k)+T_p\hat{V}_p^2(k)\Big)\Big(1+\frac{B_p}{2}+T_p\Big) \\ &\qquad + \rho^{2m}\big((m+1)\hat{V}_p^2(k)+m\hat{V}_p^1(k)\big)\Big(1+\frac{B_p}{2}+T_p\Big)^2 \\ &\qquad + m\rho^{2m-1}\Bigg(2\Big(r^2\hat{V}_p^0(k)+rT_p\hat{V}_p^2(k)+rT_p\hat{V}_p^1(k)+T_p^2\hat{V}_p^1(k)\Big) \\ &\qquad\qquad + rT_p\hat{V}_p^0(k)+\hat{V}_p^3(k)\Bigg)\Big(1+\frac{B_p}{2}+T_p\Big)^2\Bigg).\end{aligned} \tag{2.4.12}$$

The rest of this subsection is devoted to showing the above bounds on $\hat{\pi}_p^{(n)}(0)$ and $|\hat{\Delta}_k\hat{\pi}_p^{(n)}(0)|$ for $n=0$ in Section 2.4.1, for $n=1$ in Section 2.4.2, for $n=2$ in Section 2.4.3, and for $n \ge 3$ in Section 2.4.4.

2.4.1 Bounds on $\hat{\pi}_p^{(0)}(0)$ and $\left|\hat{\Delta}_k\, \hat{\pi}_p^{(0)}(0)\right|$

By the Boolean and BK inequalities, we obtain

$$\begin{aligned}\pi_p^{(0)}(x) &= \mathbb{P}_p(o \Longleftrightarrow x) - \delta_{o,x} \\ &= \mathbb{P}_p\Bigg(\bigcup_{\substack{b_1,\, b_2 \in B(o) \\ (b_1 \prec b_2)}} \Big\{\big\{b_1 \text{ occupied},\ \overline{b_1} \longleftrightarrow x\big\} \circ \big\{b_2 \text{ occupied},\ \overline{b_2} \longleftrightarrow x\big\}\Big\}\Bigg)\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{\substack{b_1,b_2\in B(o)\\(b_1\prec b_2)}} pD(b_1)G_p(x-\overline{b_1})pD(b_2)G_p(x-\overline{b_2})\\
&\le \frac{1}{2}\left(pD*G_p\right)(x)^2, \qquad (2.4.13)
\end{aligned}
$$

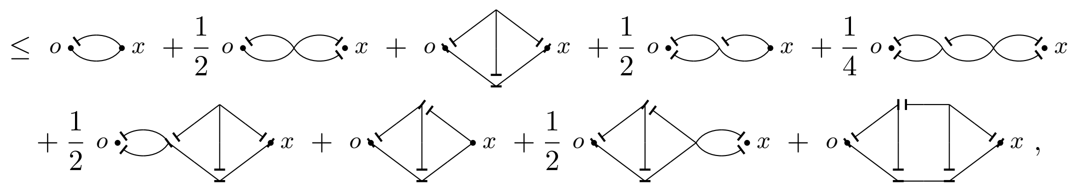
where we have used the ordering $\prec$ introduced above (2.2.8). The factor 1/2 in the last line is due to ignoring the ordering. Then, summing over x yields (2.4.8) for $n=0$.

The bound (2.4.9) on $|\hat{\Delta}_k\hat{\pi}_p^{(0)}(0)|$ is also achieved by multiplying both sides of (2.4.13) by $1-\cos k\cdot x$ and summing the resulting inequality over x.

2.4.2 Bounds on $\hat{\pi}_p^{(1)}(0)$ and $\left|\hat{\Delta}_k\,\hat{\pi}_p^{(1)}(0)\right|$

First, we prove (2.4.8) for $n=1$ and (2.4.10) by assuming the following diagrammatic bound on $\pi_p^{(1)}(x)$:

$\pi_p^{(1)}(x)$



(2.4.14)

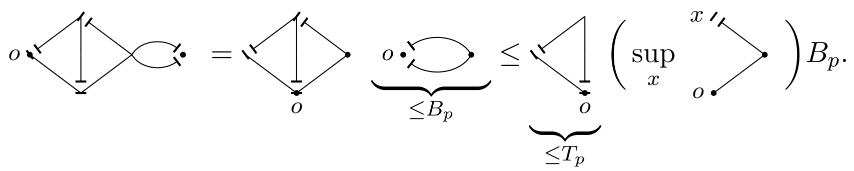
where we have used the following two types of line segments:

$$o\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\, x \;=G_p(x), \qquad\qquad o\,\shortmid\shortmid\!\!-\!\!\!-\!\!\!-\!\!\bullet\, x\;=(pD*G_p)(x). \qquad (2.4.15)$$

The unlabeled vertices are summed over $\mathbb{L}^d$. The proof of (2.4.14) is given at the end of Section 2.4.2. In the following, we repeatedly use the trivial inequality

$$o\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\, x\;\mathbb{1}_{\{x\ne o\}}=G_p(x)\mathbb{1}_{\{x\ne o\}}\le\left(pD*G_p\right)(x)=\;o\,\shortmid\shortmid\!\!-\!\!\!-\!\!\!-\!\!\bullet\, x\;. \qquad (2.4.16)$$

Proof of (2.4.8) *for* $n=1$ *assuming* (2.4.14). The bound on $\hat{\pi}_p^{(1)}(0)$ is obtained by summing both sides of (2.4.14) over x and repeatedly using translation-invariance. For example,

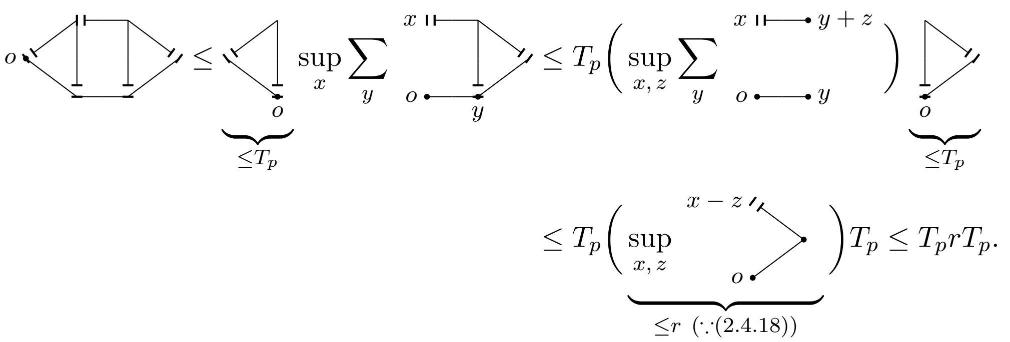


(2.4.17)

The rest of the diagram is bounded above as

$$
\begin{aligned}
\sup_x\sum_y G_p(y)(pD*G_p)(x-y) &= \sup_x\left((pD*G_p)(x)+\sum_y G_p(y)\mathbb{1}_{\{y\ne o\}}(pD*G_p)(x-y)\right)\\
&\overset{(2.4.16)}{\le} \sup_x\Big((pD*G_p)(x)+\big((pD)^{*2}*G_p^{*2}\big)(x)\Big)\\
&= \sup_x\left(pD(x)+\sum_y pD(y)G_p(x-y)\mathbb{1}_{\{x\ne y\}}\right)+B_p\\
&\overset{(2.4.16)}{\le} p\|D\|_\infty+L_p+B_p=r, \qquad (2.4.18)
\end{aligned}
$$

which leads to an upper bounds $T_p r B_p$ on (2.4.17). Moreover,



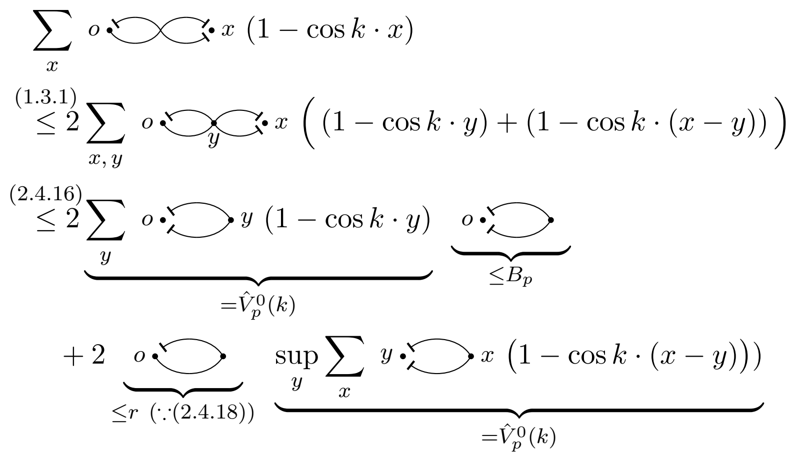
(2.4.19)

Applying the same analysis to the other diagrams, we obtain

$$
\begin{aligned}
\hat{\pi}_p^{(1)}(0) &\leq \left(pD * G_p^{*2}\right)(o) + \frac{1}{2}\left(pD * G_p^{*2}\right)(o) B_p + rT_p + \frac{1}{2} B_p \left(pD * G_p^{*2}\right)(o) \\
&\quad + \frac{1}{4} B_p \left(pD * G_p^{*2}\right)(o) B_p + \frac{1}{2} B_p r T_p + T_p r + \frac{1}{2} T_p r B_p + T_p r T_p \\
&= \left(pD * G_p^{*2}\right)(o) \left(1 + \frac{B_p}{2}\right)^2 + r \left(2\left(1 + \frac{B_p}{2}\right) T_p + T_p^2\right) \\
&\overset{(2.4.18)}{\leq} \left(1 + \frac{B_p}{2} + T_p\right)^2 r,
\end{aligned}
$$

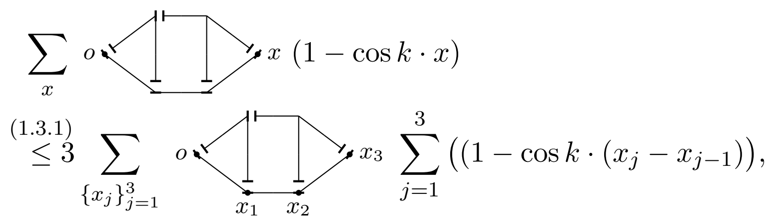
as required. $\square$

Sketch proof of (2.4.10). The bound on $\left|\hat{\Delta}_k \, \hat{\pi}_p^{(1)}(0)\right|$ is obtained by multiplying $1 - \cos k \cdot x$ to both sides of (2.4.14) and summing the resulting expression over x. To decompose the diagrams into the basic diagrams, we also use the telescopic inequality (1.3.1), translation-invariance and (2.4.16). For example,



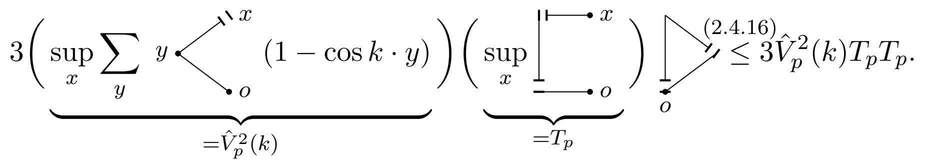
$$
\leq 2\hat{V}_p^0(k) B_p + 2r\hat{V}_p^0(k). \tag{2.4.20}
$$

Another example is the following:



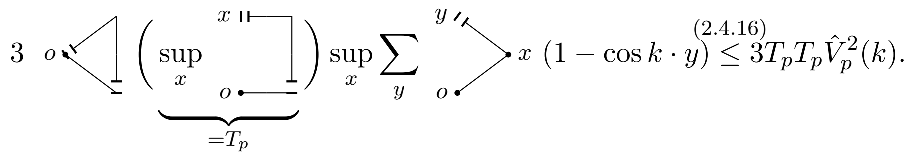
(2.4.21)

where $x_0 = o$. The contribution from $1 - \cos k \cdot x_1$ is bounded by



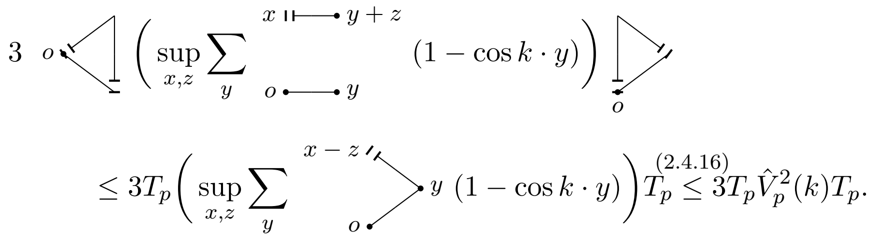
(2.4.22)

The contribution from $1 - \cos k \cdot (x_3 - x_2)$ obeys the same bound, because



(2.4.23)

The contribution from $1 - \cos k \cdot (x_2 - x_1)$ is bounded by



(2.4.24)

As a result, (2.4.21) is bounded by $9T_p^2\hat{V}_p^2(k)$. The other terms can be estimated similarly. We complete the proof of (2.4.10). $\square$

Proof of (2.4.14). First, we recall the definition of $\pi_p^{(1)}(x)$:

$$\pi_p^{(1)}(x) = \sum_b pD(b)\,\mathbb{E}_p^0\Big[\mathbb{1}_{\{o \Longleftrightarrow \underline{b}\}}\,\mathbb{P}_p^1\Big(E\big(\overline{b}, x; \tilde{\mathcal{C}}_0^b(o)\big)\Big)\Big]. \tag{2.4.25}$$

Let $x \overset{A}{\Longleftrightarrow} y$ be the event that x is doubly connected to y through A, i.e., there are at least two occupied paths from x to y and every occupied path from x to y has vertices of A. Then, by definition, we have

$$E(v, x; A) \subset \bigcup_y \Big\{ \{v \longleftrightarrow y\} \circ \{y \overset{A}{\Longleftrightarrow} x\}\Big\}. \tag{2.4.26}$$

Splitting $\{y \overset{A}{\Longleftrightarrow} x\}$ into three events depending on where the double connection traverses A, we obtain

$$\begin{aligned}\big\{y \overset{A}{\Longleftrightarrow} x\big\} &\subset \{y = x \in A\} \cup \{y \Longleftrightarrow x \neq y \in A\} \\ &\quad \cup \bigcup_{z(\neq y)} \Big\{\{y \longleftrightarrow z \in A\} \circ \{z \longleftrightarrow x\} \circ \{y \longleftrightarrow x \neq y\}\Big\},\end{aligned} \tag{2.4.27}$$

hence (see Figure 2.1(a)–(c))

$$\begin{aligned}E(v, x; A) \subset \underbrace{\{v \longleftrightarrow x \in A\}}_{\text{(a)}} \cup \underbrace{\bigcup_y \big\{\{v \longleftrightarrow y \in A\} \circ \{y \Longleftrightarrow x \neq y\}\big\}}_{\text{(b)}} \\ \underbrace{\cup \bigcup_{\substack{y,z \\ (y \neq z)}} \Big\{\{v \longleftrightarrow y\} \circ \{y \longleftrightarrow z \in A\} \circ \{z \longleftrightarrow x\} \circ \{y \longleftrightarrow x \neq y\}\Big\}}_{\text{(c)}}.\end{aligned} \tag{2.4.28}$$

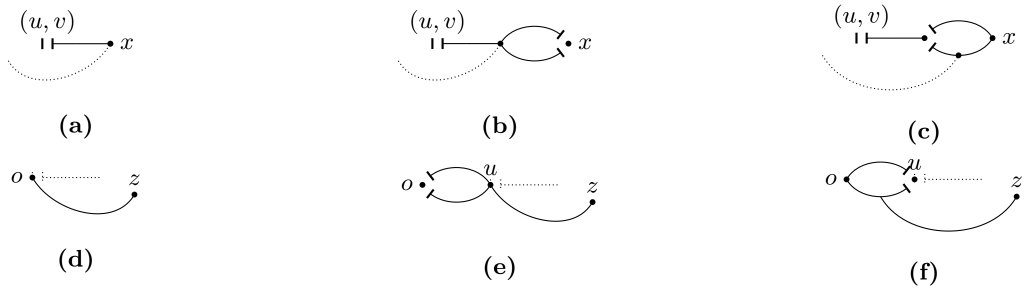


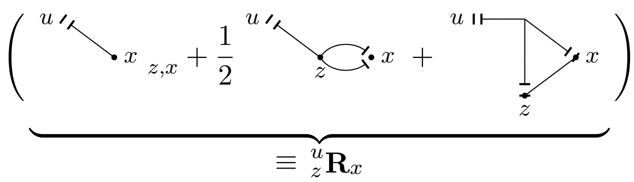
Figure 2.1: Schematic drawings for the events in (2.4.28) and (2.4.31). The real and dotted line segments are on different probability spaces. The arcs having short line segments at one of the two end vertices represent nonzero connections.

Then, by the BK inequality and the argument around (2.4.13) to derive the factor 1/2, and applying the trivial inequality (2.4.16) to the nonzero connections (e.g., $\mathbb{P}_p(y \longleftrightarrow x \neq y) \leq (pD * G_p)(x-y)$), we obtain

$$\begin{aligned}
&\sum_v pD(v-u)\mathbb{P}_p\big(E(v,x;A)\big) \\
&\leq \sum_{y,z} \mathbb{1}_{\{z\in A\}}\bigg((pD*G_p)(x-u)\delta_{y,x}\delta_{z,x} + \frac{1}{2}(pD*G_p)(y-u)(pD*G_p)(x-y)^2\delta_{z,y} \\
&\qquad + (pD*G_p)(y-u)(pD*G_p)(z-y)G_p(x-z)(pD*G_p)(x-y)\bigg), \qquad (2.4.29)
\end{aligned}$$

and therefore, by using the diagrammatic representations in (2.4.15),

$$\pi_p^{(1)}(x) \leq \sum_{u,z} \mathbb{P}_p(o \Longleftrightarrow u,\ o \longleftrightarrow z)\Bigg(\cdots \Bigg). \qquad (2.4.30)$$



We emphasize that each of the above line segments is a two-point function G_p, and not a connection event described in Figure 2.1.

It remains to investigate $\mathbb{P}_p(o \Longleftrightarrow u,\ o \longleftrightarrow z)$ in (2.4.30). Splitting the event into three depending on which vertex on the backbone from o to u a connection to z comes out of, we have (see Figure 2.1(d)–(f))

$$\begin{aligned}
&\{o \Longleftrightarrow u,\ o \longleftrightarrow z\} \\
&\subset \underbrace{\{o = u \longleftrightarrow z\}}_{\text{(d)}} \cup \underbrace{\big\{\{o \Longleftrightarrow u \neq o\} \circ \{u \longleftrightarrow z\}\big\}}_{\text{(e)}} \\
&\quad \cup \underbrace{\bigcup_w \Big\{\{o \longleftrightarrow u \neq o\} \circ \{o \longleftrightarrow w\} \circ \{w \longleftrightarrow u \neq w\} \circ \{w \longleftrightarrow z\}\Big\}}_{\text{(f)}}. \qquad (2.4.31)
\end{aligned}$$

Then, again, by the BK inequality and the argument around (2.4.13) to derive the factor 1/2, and

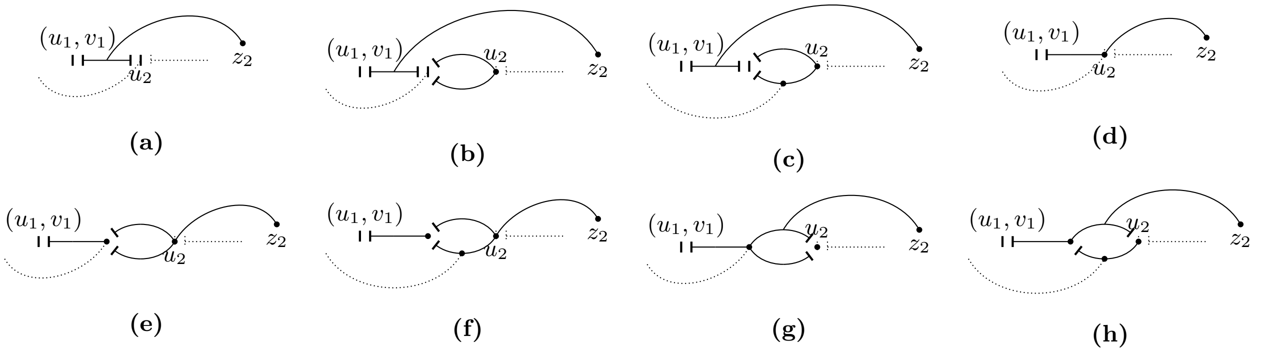
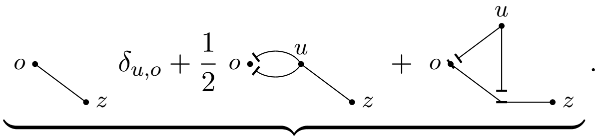


Figure 2.2: Schematic drawings for the events in (2.4.35) (cf., Figure 2.1). A connection to z_2 comes out of the backbone from v_1 to u_2 before the last sausage ((a)–(d)) or from the last sausage ((e)–(h)).

applying the trivial inequality (2.4.16) to the nonzero connections, we obtain

$$\mathbb{P}_p(o \Longleftrightarrow u,\ o \longleftrightarrow z) \le$$



$$\equiv \mathbf{L}_z^u \tag{2.4.32}$$

Combining this with (2.4.30), we obtain the diagrammatic bound (2.4.14), as required. ☐

2.4.3 Bounds on $\hat{\pi}_p^{(2)}(0)$ and $\left|\hat{\Delta}_k\, \hat{\pi}_p^{(2)}(0)\right|$

We organize this section in a different way from the previous section for the case of $n = 1$. We first explain the diagrammatic bound (2.4.38) on $\pi_p^{(2)}(x)$ for a fixed x. Then, by using this, we prove the bounds (2.4.8) for $n = 2$ and (2.4.11) for $m = 1$.

Now we start investigating $\pi_p^{(2)}(x)$ for a fixed x, which is defined as

$$\pi_p^{(2)}(x) = \sum_{b_1, b_2} pD(b_1)pD(b_2)\, \mathbb{E}_p^0 \Big[\mathbb{1}_{\{o \Longleftrightarrow \underline{b_1}\}}\, \mathbb{E}_p^1 \Big[\mathbb{1}_{E(\overline{b_1}, \underline{b_2}; \tilde{\mathcal{C}}_0^{b_1}(o))}\, \mathbb{P}_p^2 \Big(E\big(\overline{b_2}, x; \tilde{\mathcal{C}}_1^{b_2}(\overline{b_1})\big) \Big) \Big] \Big]. \tag{2.4.33}$$

First, by using (2.4.29)–(2.4.30), we obtain

$$\pi_p^{(2)}(x) \le \sum_{b_1} pD(b_1) \sum_{u_2, z_2} \mathbb{E}_p^0 \Big[\mathbb{1}_{\{o \Longleftrightarrow \underline{b_1}\}}\, \mathbb{P}_p^1 \Big(E\big(\overline{b_1}, u_2; \tilde{\mathcal{C}}_0^{b_1}(o)\big) \cap \{\overline{b_1} \longleftrightarrow z_2\} \Big) \Big] \times {}^{u_2}_{z_2}\mathbf{R}_x\, . \tag{2.4.34}$$

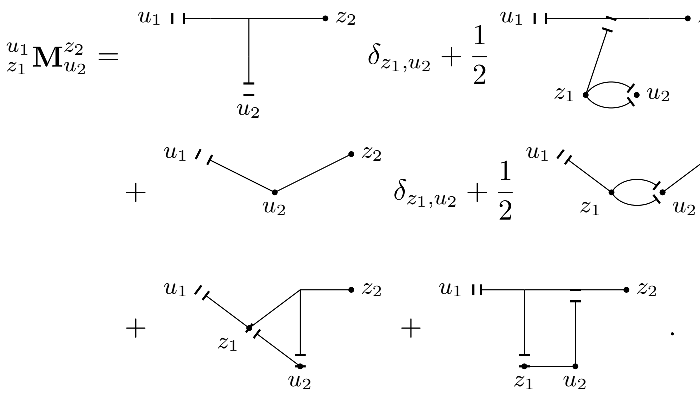
Next, we have to deal with the event $E(v_1, u_2; A) \cap \{v_1 \longleftrightarrow z_2\}$ for a given set A. By (2.4.26), we first obtain the relation

$$E(v_1, u_2; A) \cap \{v_1 \longleftrightarrow z_2\} \subset \bigcup_y \Big\{ \{v_1 \longleftrightarrow y\} \circ \big\{ y \overset{A}{\Longleftrightarrow} u_2 \big\} \Big\} \cap \{v_1 \longleftrightarrow z_2\}. \tag{2.4.35}$$

Next, we split the event into two depending on which vertex on the backbone from v_1 to u_2 a connection to z_2 comes out of: either before or from the last sausage. Then, we split each of the two into four events depending on where the double connection from y to u_2 traverses A. The resulting eight events are depicted in Figure 2.2. Finally, by the BK inequality and the argument around (2.4.13) to derive the factor 1/2, and applying the trivial inequality (2.4.16) to the nonzero connections, we obtain

$$\sum_{b_1} pD(b_1)\,\mathbb{E}_p^0\Big[\mathbb{1}_{\{o \Longleftrightarrow \underline{b_1}\}}\,\mathbb{P}_p^1\Big(E\big(\overline{b_1}, u_2; \tilde{\mathcal{C}}_0^{b_1}(o)\big) \cap \{\overline{b_1} \longleftrightarrow z_2\}\Big)\Big]$$
$$\leq \sum_{u_1, z_1} \mathbb{P}_p(o \Longleftrightarrow u_1,\ o \longleftrightarrow z_1) \times {}^{u_1}_{z_1}\mathbf{M}^{z_2}_{u_2}, \tag{2.4.36}$$

where (in some of the following diagrams, we use the identity $(pD * G_p)(x) = (G_p * pD)(x)$)



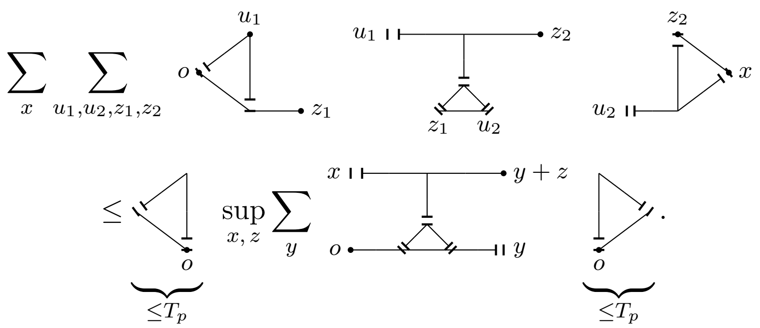
(2.4.37)

Finally, by using (2.4.32), we arrive at

$$\pi_p^{(2)}(x) \leq \sum_{u_1, u_2, z_1, z_2} \mathbf{L}^{u_1}_{z_1} \times {}^{u_1}_{z_1}\mathbf{M}^{z_2}_{u_2} \times {}^{u_2}_{z_2}\mathbf{R}_x, \tag{2.4.38}$$

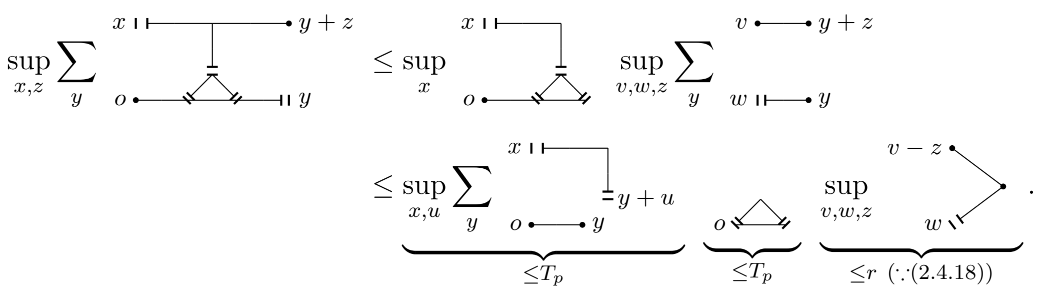
which consists of 72 ($= 3 \times 8 \times 3$) terms. □

Sketch proof of (2.4.8) *for* $n = 2$. The bound on $\hat{\pi}_p^{(2)}(0)$ is obtained by summing both sides of (2.4.38) over x and repeatedly using translation-invariance. For example, the combination of the third diagrams in (2.4.32), (2.4.37) and (2.4.30) is bounded as



(2.4.39)

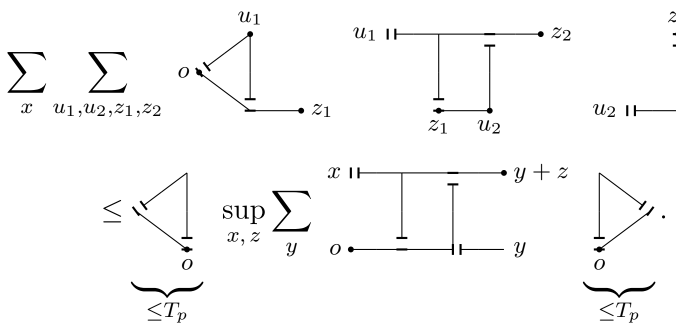
Also, the middle diagram is bounded as



(2.4.40)

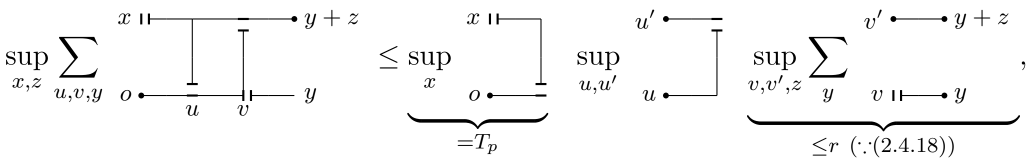
Therefore, (2.4.39) is bounded above by $T_p T_p^2 r T_p$.

Another example is the combination of the third diagram in (2.4.32), the last diagram in (2.4.37) and the third diagram in (2.4.30), which is bounded as



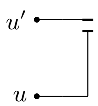
(2.4.41)

Since



(2.4.42)

and



$$= \sum_y (pD * G_p^{*2})(y-u)\, G_p(u'-y)$$

$$\overset{(2.4.16)}{\leq} \underbrace{(pD * G_p^{*2})(u'-u)}_{\leq r\ (\because (2.4.18))} + \underbrace{\big((pD)^{*2} * G_p^{*3}\big)(u'-u)}_{\leq T_p}, \tag{2.4.43}$$

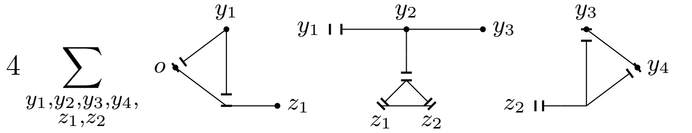
we can bound (2.4.41) above by $T_p T_p (r + T_p) r T_p$.

Applying the same analysis to the other diagrams, we obtain

$$\hat{\pi}_p^{(2)}(0) \leq \left(1 + \frac{B_p}{2} + T_p\right)\Bigg(\underbrace{T_p + \frac{1}{2}T_p B_p + T_p^2 + r + \frac{1}{2} r B_p + T_p r + r T_p + T_p(r + T_p)}_{=\rho}\Bigg)$$
$$\times\, r\left(1 + \frac{B_p}{2} + T_p\right), \tag{2.4.44}$$

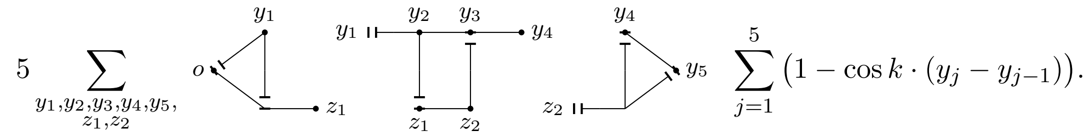
as required. $\square$

Sketch proof of (2.4.11) *for* $m = 1$. The bound on $\left|\hat{\Delta}_k\, \hat{\pi}_p^{(2)}(0)\right|$ is obtained by multiplying $1 - \cos k \cdot x$ to both sides of (2.4.38) and summing the resulting expression over x. During the course, we split $1 - \cos k \cdot x$ by using the telescopic inequality (1.3.1). For example, the combination of the third diagrams in (2.4.32), (2.4.37) and (2.4.30) is bounded by



$$\sum_{j=1}^{4} \big(1 - \cos k \cdot (y_j - y_{j-1})\big), \tag{2.4.45}$$

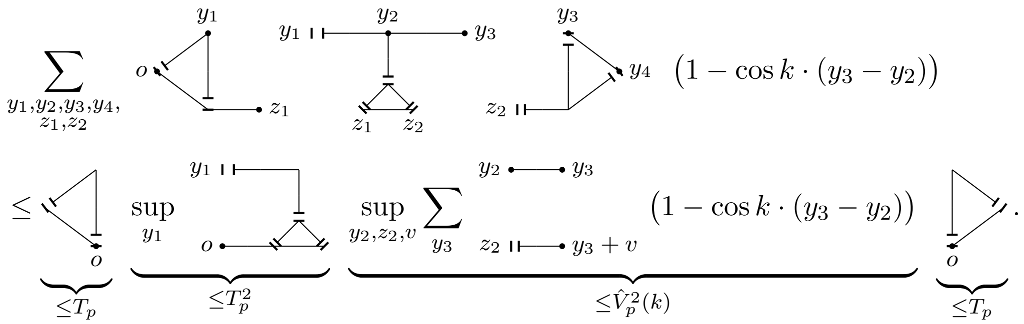
where $y_0 = o$, while the combination of the third diagram in (2.4.32), the last diagram in (2.4.37) and the third diagram in (2.4.30) is bounded by



$$5 \sum_{\substack{y_1, y_2, y_3, y_4, y_5,\\ z_1, z_2}} \cdots \sum_{j=1}^{5} \big(1 - \cos k \cdot (y_j - y_{j-1})\big). \tag{2.4.46}$$

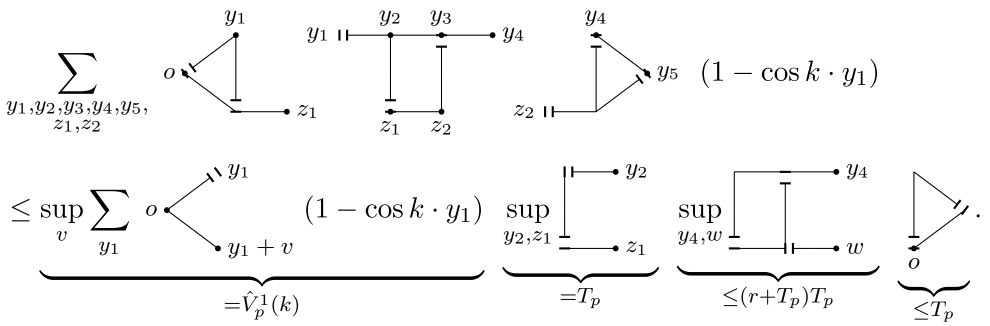
Also, for the other 70 combinations of diagrams in (2.4.32), (2.4.37) and (2.4.30), the number of intervals $y_j - y_{j-1}$ is at most 5. We use this fact to uniformly bound the multiplicative constant in the telescopic inequality (1.3.1) by 5.

Now it remains to bound each combination in terms of basic diagrams. For example, the contribution from $1 - \cos k \cdot (y_3 - y_2)$ in (2.4.45) is bounded as



$$\sum_{\substack{y_1, y_2, y_3, y_4,\\ z_1, z_2}} \cdots \big(1 - \cos k \cdot (y_3 - y_2)\big) \le \underbrace{\cdots}_{\le T_p} \underbrace{\sup_{y_1} \cdots}_{\le T_p^2} \underbrace{\sup_{y_2, z_2, v} \sum_{y_3} \cdots \big(1 - \cos k \cdot (y_3 - y_2)\big)}_{\le \hat{V}_p^2(k)} \underbrace{\cdots}_{\le T_p}. \tag{2.4.47}$$

while the contribution from $1 - \cos k \cdot y_1$ in (2.4.46) is bounded as



$$\sum_{\substack{y_1, y_2, y_3, y_4, y_5,\\ z_1, z_2}} \cdots (1 - \cos k \cdot y_1) \le \underbrace{\sup_{v} \sum_{y_1} \cdots (1 - \cos k \cdot y_1)}_{= \hat{V}_p^1(k)} \underbrace{\sup_{y_2, z_1} \cdots}_{= T_p} \underbrace{\sup_{y_4, w} \cdots}_{\le (r + T_p) T_p} \underbrace{\cdots}_{\le T_p}. \tag{2.4.48}$$

The other combinations can be bounded similarly. We note that each bound uses one of the diagrams $\hat{V}_p^0(k)$, $\hat{V}_p^1(k)$ and $\hat{V}_p^2(k)$ ($\hat{V}_p^3(k)$ is used only in the bounds on $|\hat{\Delta}_k \hat{\pi}_p^{(n)}|$ for $n \ge 3$). Which one is used depends on which pair of two-point functions is multiplied by $1 - \cos k \cdot (y_j - y_{j-1})$: $(pD * G_p)(y_j - y_{j-1})^2$, $(pD * G_p)(y_j - y_{j-1}) G_p(y_j - y_{j-1} - v)$ for some v, or $(pD * G_p)(y_j - y_{j-1})(pD * G_p)(y_j - y_{j-1} - v)$ for some v, respectively. We complete the sketch proof of (2.4.11) for $m = 1$. $\square$

2.4.4 Bounds on $\hat{\pi}_p^{(n \ge 3)}(0)$ and $\left|\hat{\Delta}_k \, \hat{\pi}_p^{(n \ge 3)}(0)\right|$

Recall the definition of $\pi_p^{(n)}(x)$ for $n \ge 3$:

$$\pi_p^{(n)}(x) = \sum_{b_1, \dots, b_n} \prod_{i=1}^{n} pD(b_i) \; \mathbb{E}_p^0 \Bigg[\mathbb{1}_{\{o \Longleftrightarrow \underline{b_1}\}} \; \mathbb{E}_p^1 \Bigg[\mathbb{1}_{E(\overline{b_1}, \underline{b_2}; \tilde{\mathcal{C}}_0^{b_1}(o))} \cdots \\ \times \mathbb{E}_p^{n-1} \Big[\mathbb{1}_{E(\overline{b_{n-1}}, \underline{b_n}; \tilde{\mathcal{C}}_{n-2}^{b_{n-1}}(\overline{b_{n-2}}))} \, \mathbb{P}_p^n \Big(E\big(\overline{b_n}, x; \tilde{\mathcal{C}}_{n-1}^{b_n}(\overline{b_{n-1}})\big) \Big) \Big] \cdots \Bigg] \Bigg]. \tag{2.4.49}$$

Using (2.4.29) first (cf., (2.4.30) and (2.4.34)), then using (2.4.36) for $n-1$ times, and finally using (2.4.32), we obtain the fixed-x bound (cf., (2.4.38))

$$\pi_p^{(n)}(x) \le \sum_{\substack{u_1,\dots,u_n,\\ z_1,\dots,z_n}} \mathbf{L}_{z_1}^{u_1} \times {}_{z_1}^{u_1}\mathbf{M}_{u_2}^{z_2} \times \cdots \times {}_{z_{n-1}}^{u_{n-1}}\mathbf{M}_{u_n}^{z_n} \times {}_{z_n}^{u_n}\mathbf{R}_x \,. \tag{2.4.50}$$

Sketch proof of (2.4.8) *for* $n \ge 3$. We can follow the same line of the proof of (2.4.8) for $n=2$. The only difference is the size of middle section (cf., (2.4.40) and (2.4.42)), and it gives rise to the factor ρ^{n-1}. □

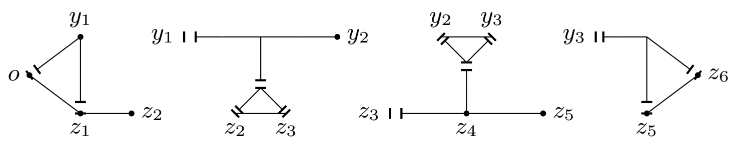
Sketch proof of (2.4.11) *for* $m \ge 2$ *and* (2.4.12). As is done in the proof of (2.4.11) for $m=1$ in Section 2.4.3, we uniformly bound the multiplicative constant in the telescopic inequality (1.3.1) by the maximum number of intervals, which is $2n+1$ for $|\hat{\Delta}_k \hat{\pi}_p^{(n)}(0)|$. The remaining task is almost the same as the previous case, except for the following two:

(i) Use the telescopic inequality (1.3.1) along the upper sequence of line segments for even n (see (2.4.45)–(2.4.46)) or along the lower sequence for odd n (see below).

(ii) Use the basic diagram $\hat{V}_p^3(k)$ to bound certain diagrams to which $1-\cos k\cdot(\cdots)$ is assigned in a peculiar way.

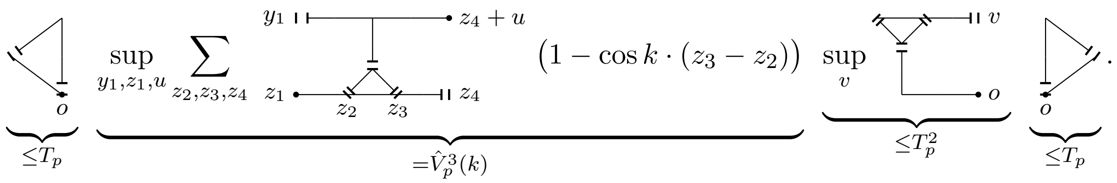
For example, the contribution to $|\hat{\Delta}_k \hat{\pi}_p^{(3)}(0)|$ from the combination of the third diagrams in (2.4.32), (2.4.37) and (2.4.30) is bounded as

$$7 \sum_{\substack{y_1,y_2,y_3,\\ z_1,\dots,z_6}}$$



$$\times \sum_{j=1}^{6} \big(1-\cos k\cdot(z_j - z_{j-1})\big), \tag{2.4.51}$$

where $z_0 = o$. Then, the contribution to the sum from $1-\cos k\cdot(z_3-z_2)$ is bounded by



(2.4.52)

The other combinations can be bounded similarly, and we refrain from showing tedious computations. □

2.5 Diagrammatic bounds on the bootstrapping functions

Let

$$\hat{\Pi}_p^{\text{even}}(k) = \sum_{m=0}^{\infty} \hat{\pi}_p^{(2m)}(k), \qquad \hat{\Pi}_p^{\text{odd}}(k) = \sum_{m=0}^{\infty} \hat{\pi}_p^{(2m+1)}(k). \tag{2.5.1}$$

Suppose that $\rho \equiv T_p(2r+T_p)+(r+T_p)(1+B_p/2+T_p)<1$. Then, by Lemma 2.4.1, we obtain

$$0 \le \hat{\Pi}_p^{\text{even}}(0) \le \frac{B_p}{2} + \frac{(1+B_p/2+T_p)^2 r\rho}{1-\rho^2}, \tag{2.5.2}$$

$$0 \le \hat{\Pi}_p^{\text{even}}(0) \le \frac{(1+B_p/2+T_p)^2 r}{1-\rho^2}, \tag{2.5.3}$$

$$\sup_k \frac{|\hat{\Delta}_k \hat{\Pi}_p^{\text{even}}(0)|}{1-\hat{D}(k)} \le \sum_{j=0}^{3} \varphi_j^{\text{even}} \left\| \frac{\hat{V}_p^j}{1-\hat{D}} \right\|_\infty, \tag{2.5.4}$$

where

$$\varphi_0^{\text{even}} = \frac{1}{2} + \left(1+\frac{B_p}{2}+T_p\right)\frac{10r\rho}{(1-\rho^2)^2} + \left(1+\frac{B_p}{2}+T_p\right)^2 \frac{r^2(5+12\rho^2)+3T_p r\rho^2(3+\rho)}{(1-\rho^2)^3}, \tag{2.5.5}$$

$$\varphi_1^{\text{even}} = \left(1+\frac{B_p}{2}+T_p\right)\frac{5T_p\rho}{(1-\rho^2)^2} + \left(1+\frac{B_p}{2}+T_p\right)^2 \frac{\rho(5+3\rho^2)+T_p(r+T_p)(5+12\rho^2)}{(1-\rho^2)^3}, \tag{2.5.6}$$

$$\varphi_2^{\text{even}} = \left(1+\frac{B_p}{2}+T_p\right)\frac{5T_p\rho}{(1-\rho^2)^2} + \left(1+\frac{B_p}{2}+T_p\right)^2 \frac{\rho(5+3\rho^2)+T_p r(5+12\rho^2)}{(1-\rho^2)^3}, \tag{2.5.7}$$

$$\varphi_3^{\text{even}} = \left(1+\frac{B_p}{2}+T_p\right)^2 \frac{3\rho^2(3+\rho)}{(1-\rho^2)^3}, \tag{2.5.8}$$

and

$$\sup_k \frac{|\hat{\Delta}_k \hat{\Pi}_p^{\text{odd}}(0)|}{1-\hat{D}(k)} \le \sum_{j=0}^{3} \varphi_j^{\text{odd}} \left\| \frac{\hat{V}_p^j}{1-\hat{D}} \right\|_\infty, \tag{2.5.9}$$

where

$$\begin{aligned}
\varphi_0^{\text{odd}} &= 1 + 2(B_p+r) + \frac{3}{4}B_p(B_p+2r) + 3T_p r + \left(1+\frac{B_p}{2}+T_p\right)\frac{14r\rho^2}{(1-\rho^2)^2} \\
&\quad + \left(1+\frac{B_p}{2}+T_p\right)^2 \frac{r\rho(2r+T_p)(7+\rho^2)}{(1-\rho^2)^3}, && (2.5.10) \\
\varphi_1^{\text{odd}} &= \left(1+\frac{B_p}{2}+T_p\right)^2 \frac{\rho(\rho+2T_p r+2T_p^2)(7+\rho^2)}{(1-\rho^2)^3}, && (2.5.11) \\
\varphi_2^{\text{odd}} &= T_p(8+6B_p+9T_p) + \left(1+\frac{B_p}{2}+T_p\right)\frac{14T_p\rho^2}{(1-\rho^2)^2} \\
&\quad + \left(1+\frac{B_p}{2}+T_p\right)^2 \frac{\rho^2(14+3\rho^2)+2T_p r\rho(7+\rho^2)}{(1-\rho^2)^3}, && (2.5.12) \\
\varphi_3^{\text{odd}} &= \left(1+\frac{B_p}{2}+T_p\right)^2 \frac{\rho(7+\rho^2)}{(1-\rho^2)^3}. && (2.5.13)
\end{aligned}$$

These bounds imply the next lemma in which we obtain upper bounds on the bootstrapping functions $\{g_i(p)\}_{i=1}^3$. In the following, we assume that

$$\sum_{n=0}^{\infty} \hat{\pi}_p^{(n)}(0) + \sup_k \sum_{n=0}^{\infty} \frac{-\hat{\Delta}_k \hat{\pi}_p^{(n)}(0)}{1-\hat{D}(k)} < 1. \tag{2.5.14}$$

Then, (2.3.4) is well-defined.

Lemma 2.5.1. Suppose $\rho < 1$ and that $L_p, B_p, T_p, \|\hat{V}_p^j/(1-\hat{D})\|_\infty$, $j = 0,1,2,3$, are so small that the inequality (2.5.14) holds. Then, we have

$$g_1(p) \le \left(1 - \frac{(1+B_p/2+T_p)^2 r}{1-\rho^2}\right)^{-1}, \tag{2.5.15}$$

$$g_2(p) \le \left(1 - \left(1 - \frac{B_p}{2} - \frac{(1+B_p/2+T_p)^2 r}{1-\rho}\right)^{-1} \sum_{j=0}^{3} \varphi_j^{\mathrm{odd}} \left\| \frac{\hat{V}_p^j}{1-\hat{D}} \right\|_\infty \right)^{-1}, \tag{2.5.16}$$

$$\begin{aligned} g_3(p) \le \max\left\{ g_2(p) \left(1 - \frac{B_p}{2} - \frac{(1+B_p/2+T_p)^2 r}{1-\rho}\right)^{-1}, 1\right\}^3 \\ \times p^2 \left(1 + B_p + \frac{2(1+B_p/2+T_p)^2 r}{1-\rho} + 2\sum_{j=0}^{3} (\varphi_j^{\mathrm{even}} + \varphi_j^{\mathrm{odd}}) \left\| \frac{\hat{V}_p^j}{1-\hat{D}} \right\|_\infty \right)^2. \end{aligned} \tag{2.5.17}$$

Proof. The bound on $g_1(p)$ is easy; since $\chi_p = \hat{G}_p(0) = (1+\hat{\Pi}_p(0))/(1-p(1+\hat{\Pi}_p(0)))$ by (2.3.2) and $\hat{\Pi}_p(0) = \hat{\Pi}_p^{\mathrm{even}}(0) - \hat{\Pi}_p^{\mathrm{odd}}(0)$, we obtain

$$g_1(p) = \frac{1}{1+\hat{\Pi}_p(0)} - \frac{1}{\chi_p} \le \frac{1}{1-\hat{\Pi}_p^{\mathrm{odd}}(0)} \overset{(2.5.3)}{\le} \left(1 - \frac{(1+B_p/2+T_p)^2 r}{1-\rho^2}\right)^{-1}. \tag{2.5.18}$$

Next, we bound $g_2(p)$. By (2.3.2),

$$\begin{aligned} \hat{G}_p(k) &= \frac{1+\hat{\Pi}_p(k)}{1-p\hat{D}(k)\big(1+\hat{\Pi}_p(k)\big)} \\ &= \frac{1+\hat{\Pi}_p(k)}{(p+\chi_p^{-1})\big(1+\hat{\Pi}_p(0)\big) - p\hat{D}(k)\big(1+\hat{\Pi}_p(k)\big)} \\ &\le \frac{1+\hat{\Pi}_p(k)}{p + p\hat{\Pi}_p(0) - p\hat{D}(k) - p\hat{D}(k)\hat{\Pi}_p(k)} \\ &\le \frac{1+\hat{\Pi}_p(k)}{p\big(1-\hat{D}(k)\big) + p\big(\hat{\Pi}_p(0)-\hat{\Pi}_p(k)\big) + p\hat{\Pi}_p(k)\big(1-\hat{D}(k)\big)} \\ &= \frac{1}{p\big(1-\hat{D}(k)\big)} \left(1 + \frac{1}{1+\hat{\Pi}_p(k)} \cdot \frac{\hat{\Pi}_p(0)-\hat{\Pi}_p(k)}{1-\hat{D}(k)}\right)^{-1}. \end{aligned}$$

Since $-\hat{\Delta}_k \hat{\Pi}_p(0) = |\hat{\Delta}_k \hat{\Pi}_p^{\mathrm{even}}(0)| - |\hat{\Delta}_k \hat{\Pi}_p^{\mathrm{odd}}(0)|$ and $1 + \hat{\Pi}_p(k) \ge 1 - \hat{\Pi}_p^{\mathrm{even}}(0) - \hat{\Pi}_p^{\mathrm{odd}}(0)$ (> 0 as long as the inequality (2.5.14) holds), we obtain

$$\begin{aligned} g_2(p) &\le \sup_k \left(1 - \frac{1}{1-\hat{\Pi}_p^{\mathrm{even}}(0)-\hat{\Pi}_p^{\mathrm{odd}}(0)} \frac{|\hat{\Delta}_k \hat{\Pi}_p^{\mathrm{odd}}(0)|}{1-\hat{D}(k)}\right)^{-1} \\ &\le \left(1 - \left(1 - \frac{B_p}{2} - \frac{(1+B_p/2+T_p)^2 r}{1-\rho}\right)^{-1} \sum_{j=0}^{3} \varphi_j^{\mathrm{odd}} \left\| \frac{\hat{V}_p^j}{1-\hat{D}} \right\|_\infty \right)^{-1}. \end{aligned} \tag{2.5.19}$$

Finally, we bound $g_3(p)$. Notice that, by (2.3.2), we obtain

$$\hat{G}_p(k) p\hat{D}(k) = \frac{p\hat{D}(k)\big(1+\hat{\Pi}_p(k)\big)}{1-p\hat{D}(k)\big(1+\hat{\Pi}_p(k)\big)} = \frac{1}{1-p\hat{D}(k)\big(1+\hat{\Pi}_p(k)\big)} - 1 \equiv \hat{A}_p(k) - 1, \tag{2.5.20}$$

hence $\hat\Delta_k(\hat G_p(l)p\hat D(l)) = \hat\Delta_k\hat A_p(l)$. Then, by Lemma 1.3.2 with $a_p(x) = (pD*(\delta_{o,\cdot}+\Pi_p))(x)$, since $|\hat G_p(k)| \le g_2(p)\hat S_1(k) \equiv g_2(p)/(1-\hat D(k))$, we obtain

$$
\begin{aligned}
&g_3(p)\\
&\le \sup_{k,l}\frac{1-\hat D(k)}{\hat U(k,l)}\Bigg(\frac{\hat S_1(l+k)+\hat S_1(l-k)}{2}\hat S_1(l)\bigg(\frac{g_2(p)}{1-\hat\Pi_p^{\rm even}(0)-\hat\Pi_p^{\rm even}(0)}\bigg)^2\frac{|\hat\Delta_k\hat a_p(l)|}{1-\hat D(k)}\\
&\qquad+4\hat S_1(l+k)\hat S_1(l-k)\bigg(\frac{g_2(p)}{1-\hat\Pi_p^{\rm even}(0)-\hat\Pi_p^{\rm even}(0)}\bigg)^3\frac{-\hat\Delta_l\widehat{|a_p|}(0)}{1-\hat D(l)}\,\frac{-\hat\Delta_k\widehat{|a_p|}(0)}{1-\hat D(k)}\Bigg)\\
&\overset{(2.1.2)}{\le}\max\bigg\{\frac{g_2(p)}{1-\hat\Pi_p^{\rm even}(0)-\hat\Pi_p^{\rm even}(0)},1\bigg\}^3\max\bigg\{\sup_{k,l}\frac{|\hat\Delta_k\hat a_p(l)|}{1-\hat D(k)},\ \bigg(\sup_k\frac{-\hat\Delta_k\widehat{|a_p|}(0)}{1-\hat D(k)}\bigg)^2\bigg\}. \qquad (2.5.21)
\end{aligned}
$$

Here,

$$
\begin{aligned}
\frac{|\hat\Delta_k\hat a_p(l)|}{1-\hat D(k)} &= \frac{p}{1-\hat D(k)}\bigg|\sum_x(1-\cos k\cdot x)e^{il\cdot x}\big(D(x)+(\Pi_p*D)(x)\big)\bigg|\\
&\le p\bigg(\sum_x\frac{1-\cos k\cdot x}{1-\hat D(k)}D(x)+\sum_{x,y}\frac{1-\cos k\cdot x}{1-\hat D(k)}|\Pi_p(y)|D(x-y)\bigg)\\
&\le p\bigg(1+\sum_{x,y}\frac{1-\cos k\cdot x}{1-\hat D(k)}\big(\Pi_p^{\rm even}(y)+\Pi_p^{\rm odd}(y)\big)D(x-y)\bigg), \qquad (2.5.22)
\end{aligned}
$$

which is larger than 1, since $p\ge 1$. By the telescopic inequality (1.3.1), the sum over x,y is bounded as

$$
\begin{aligned}
&\sum_{x,y}\frac{1-\cos k\cdot x}{1-\hat D(k)}\big(\Pi_p^{\rm even}(y)+\Pi_p^{\rm odd}(y)\big)D(x-y)\\
&\le 2\sum_y\frac{(1-\cos k\cdot y)(\Pi_p^{\rm even}(y)+\Pi_p^{\rm odd}(y))}{1-\hat D(k)}\sum_x D(x-y)\\
&\quad+2\sum_y\big(\Pi_p^{\rm even}(y)+\Pi_p^{\rm odd}(y)\big)\sum_x\frac{(1-\cos k\cdot(x-y))D(x-y)}{1-\hat D(k)}\\
&\le 2\bigg(\frac{|\hat\Delta_k\hat\Pi_p^{\rm even}(0)|}{1-\hat D(k)}+\frac{|\hat\Delta_k\hat\Pi_p^{\rm odd}(0)|}{1-\hat D(k)}\bigg)+2\big(\hat\Pi_p^{\rm even}(0)+\hat\Pi_p^{\rm odd}(0)\big). \qquad (2.5.23)
\end{aligned}
$$

As a result,

$$
\frac{|\hat\Delta_k\hat a_p(l)|}{1-\hat D(k)} \le p\bigg(1+2\big(\hat\Pi_p^{\rm even}(0)+\hat\Pi_p^{\rm odd}(0)\big)+2\bigg(\frac{|\hat\Delta_k\hat\Pi_p^{\rm even}(0)|}{1-\hat D(k)}+\frac{|\hat\Delta_k\hat\Pi_p^{\rm odd}(0)|}{1-\hat D(k)}\bigg)\bigg). \qquad (2.5.24)
$$

It is not difficult to check if $-\hat\Delta_k\widehat{|a_p|}(0)/(1-\hat D(k))$, which is nonnegative, obeys the same bound. Therefore, we obtain

$$
\begin{aligned}
g_3(p) &\le \max\bigg\{g_2(p)\bigg(1-\frac{B_p}{2}-\frac{(1+B_p/2+T_p)^2r}{1-\rho}\bigg)^{-1},1\bigg\}^3\\
&\quad\times p^2\bigg(1+B_p+\frac{2(1+B_p/2+T_p)^2r}{1-\rho}+2\sum_{j=0}^3(\varphi_j^{\rm even}+\varphi_j^{\rm odd})\bigg\|\frac{\hat V_p^j}{1-\hat D}\bigg\|_\infty\bigg)^2, \qquad (2.5.25)
\end{aligned}
$$

as required. □

2.6 Bounds on diagrams in terms of random-walk quantities

In this subsection, we evaluate the diagrams for $p \in [1, p_c)$. First, we evaluate the diagrams for $p \in (1, p_c)$ under the bootstrapping assumptions. We define the RW quantities: the RW loop ε_1, the RW bubble ε_2 and the RW triangle ε_3, as

$$\varepsilon_j = (D^{*2} * S_1^{*j})(o) = \sum_{n=1}^{\infty} D^{*2n}(o) \times \begin{cases} 1 & [j=1], \\ (2n-1) & [j=2], \\ (2n-1)n & [j=3]. \end{cases} \tag{2.6.1}$$

Lemma 2.6.1. Let $d \geq 7$ and $p \in (1, p_c)$ and suppose that $g_i(p) \leq K_i$, $i = 1, 2, 3$, for some constants $\{K_i\}_{i=1}^3$. Then, we have

$$L_p \leq K_1^2 K_2 \varepsilon_1, \tag{2.6.2}$$

$$B_p \leq K_1^2 K_2^2 \varepsilon_2, \tag{2.6.3}$$

$$T_p \leq K_1^2 K_2^3 \varepsilon_3, \tag{2.6.4}$$

$$\left\| \frac{\hat{V}_p^0}{1-\hat{D}} \right\|_\infty \leq K_1^2 K_2 \varepsilon_1 + 5K_1^2 K_2 K_3 \varepsilon_3, \tag{2.6.5}$$

$$\left\| \frac{\hat{V}_p^1}{1-\hat{D}} \right\|_\infty \leq K_1 + \frac{5K_1^2}{2^d} + 5K_1^3 K_2 \varepsilon_1 + 6K_1^3 K_2^2 \varepsilon_2 + 20K_1^2 K_2 K_3 \varepsilon_3, \tag{2.6.6}$$

$$\left\| \frac{\hat{V}_p^2}{1-\hat{D}} \right\|_\infty \leq \frac{K_1^2}{2^d} + K_1^3 K_2 \varepsilon_1 + 2K_1^3 K_2^2 \varepsilon_2 + 10K_1^2 K_2 K_3 \varepsilon_3, \tag{2.6.7}$$

$$\left\| \frac{\hat{V}_p^3}{1-\hat{D}} \right\|_\infty \leq 5K_1^5 K_2^7 K_3 (1 + 3\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3)\varepsilon_3^2. \tag{2.6.8}$$

Proof. By using the assumptions $g_i(p) \leq K_i$ for $i = 1, 2, 3$,

$$L_p \leq p^2 \int_{\mathbb{T}^d} \hat{D}(k)^2 |\hat{G}_p(k)| \frac{\mathrm{d}^d k}{(2\pi)^d} \leq K_1^2 K_2 \underbrace{\int_{\mathbb{T}^d} \frac{\hat{D}(k)^2}{1-\hat{D}(k)} \frac{\mathrm{d}^d k}{(2\pi)^d}}_{=\varepsilon_1}. \tag{2.6.9}$$

Similarly,

$$B_p \leq K_1^2 K_2^2 \varepsilon_2, \qquad T_p \leq K_1^2 K_2^3 \varepsilon_3. \tag{2.6.10}$$

To prove (2.6.5), we first use the trivial inequality $G_p \leq \delta_{o,\cdot} + pD * G_p$ to obtain

$$\begin{aligned} \frac{\hat{V}_p^0(k)}{1-\hat{D}(k)} \leq & \sum_{x \sim o} \frac{1 - \cos k \cdot x}{1-\hat{D}(k)} pD(x) \, (pD * G_p)(x) \\ & + \sum_x \frac{1 - \cos k \cdot x}{1-\hat{D}(k)} \big((pD)^{*2} * G_p\big)(x) \, (pD * G_p)(x). \end{aligned} \tag{2.6.11}$$

By symmetry, the first term is bounded by

$$p \sup_{x \sim o} (pD * G_p)(x) = \underbrace{\frac{p}{2^d} \sum_{x \sim o} (pD * G_p)(x)}_{=((pD)^{*2} * G_p)(o)} = L_p \leq K_1^2 K_2 \varepsilon_1. \tag{2.6.12}$$

For the second term, we use the Fourier representation to obtain

$$\begin{aligned}
&\sum_x \big((pD)^{*2} * G_p\big)(x) \frac{(pD * G_p)(x)(1-\cos k\cdot x)}{1-\hat{D}(k)} \\
&= \int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \big(p\hat{D}(l)\big)^2 \hat{G}_p(l) \frac{-\hat{\Delta}_k(p\hat{D}(l)\hat{G}_p(l))}{1-\hat{D}(k)} \\
&\le K_1^2 K_2 K_3 \int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \hat{D}(l)^2 \hat{S}_1(l) \frac{\hat{U}(k,l)}{1-\hat{D}(k)}. \qquad (2.6.13)
\end{aligned}$$

Recall the definition (2.1.2) of $\hat{U}(k,l)$, in which we have three terms: $\frac{1}{2}\hat{S}_1(l+k)\hat{S}_1(l)$, $\frac{1}{2}\hat{S}_1(l-k)\hat{S}_1(l)$ and $4\hat{S}_1(l+k)\hat{S}_1(l-k)$. By inversion, we have, e.g.,

$$\begin{aligned}
&\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \hat{D}(l)^2 \hat{S}_1(l) \hat{S}_1(l+k) \hat{S}_1(l-k) \\
&= \sum_{x,y} S_1(x) S_1(y) \mathrm{e}^{\mathrm{i}k\cdot(x-y)} \underbrace{\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \hat{D}(l)^2 \hat{S}_1(l) \mathrm{e}^{\mathrm{i}l(x+y)}}_{=(D^{*2}*S_1)(x+y)} \\
&\le (D^{*2} * S_1^{*3})(o) = \varepsilon_3. \qquad (2.6.14)
\end{aligned}$$

It is not difficult to check if the other combinations obey the same bound. This completes the proof of (2.6.5).

Next, we prove (2.6.7) before showing (2.6.6). First, by the trivial inequality $G_p \le \delta_{o,\cdot} + pD * G_p$, we obtain

$$\begin{aligned}
\frac{\hat{V}_p^2(k)}{1-\hat{D}(k)} &\le \sup_x \sum_y \frac{pD(y)(1-\cos k\cdot y)}{1-\hat{D}(k)} (pD * G_p)(x-y) \\
&\quad + \sup_x \sum_y \frac{((pD)^{*2} * G_p)(y)(1-\cos k\cdot y)}{1-\hat{D}(k)} (pD * G_p)(x-y). \qquad (2.6.15)
\end{aligned}$$

Since $\|pD * G_p\|_\infty \le p/2^d + L_p$, the first term is bounded by $K_1(K_1/2^d + K_1^2 K_2 \varepsilon_1)$. For the second term, we use the telescopic inequality (1.3.1) to obtain

$$\begin{aligned}
&2\sum_{y,z} \frac{pD(z)(pD * G_p)(y-z)}{1-\hat{D}(k)} \Big((1-\cos k\cdot z) + \big(1-\cos k\cdot(y-z)\big)\Big)(pD * G_p)(x-y) \\
&\le 2\sum_z \frac{pD(z)(1-\cos k\cdot z)}{1-\hat{D}(k)} \big((pD)^{*2} * G_p^{*2}\big)(x-z) \\
&\quad + 2\sum_{y'} \frac{(pD * G_p)(y')(1-\cos k\cdot y')}{1-\hat{D}(k)} \big((pD)^{*2} * G_p\big)(x-y'), \qquad (2.6.16)
\end{aligned}$$

where we have used the replacement $y' = y - z$. Since $((pD)^{*2} * G_p^{*2})(x-z) \le B_p$, which is due to the Schwarz inequality, the first term is bounded by $2pB_p \le 2K_1^3 K_2^2 \varepsilon_2$. On the other hand, by the Fourier representation, the second term is bounded by (cf., (2.6.13)–(2.6.14))

$$\begin{aligned}
&2\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{|\hat{\Delta}_k(p\hat{D}(l)\hat{G}_p(l))|}{1-\hat{D}(k)} (p\hat{D}(l))^2 \hat{G}_p(l) \\
&\le 2K_1^2 K_2 K_3 \int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{\hat{U}(k,l)}{1-\hat{D}(k)} \hat{D}(l)^2 \hat{S}_1(l) \le 10 K_1^2 K_2 K_3 \varepsilon_3. \qquad (2.6.17)
\end{aligned}$$

This completes the proof of (2.6.7).

Similarly, by using the trivial inequality $G_p \le \delta_{o,\cdot} + pD * G_p$ and the telescopic inequality (1.3.1), we have

$$\begin{aligned}\frac{\hat{V}_p^1(k)}{1-\hat{D}(k)} &\le \sup_x \sum_y \frac{pD(y)(1-\cos k\cdot y)}{1-\hat{D}(k)} \underbrace{G_p(x-y)}_{\le \|G_p\|_\infty} \\ &\quad + 2\sup_x \sum_y \frac{pD(y)(1-\cos k\cdot y)}{1-\hat{D}(k)} \underbrace{(pD*G_p^{*2})(x-y)}_{\le r} \\ &\quad + \underbrace{2\sup_x \sum_y \frac{(pD*G_p)(y)(1-\cos k\cdot y)}{1-\hat{D}(k)} (pD*G_p)(x-y)}_{=\|\hat{V}_p^2/(1-\hat{D})\|_\infty}. \end{aligned} \tag{2.6.18}$$

By $\|G_p\|_\infty \le 1 + p/2^d + L_p \le 1 + K_1/2^d + K_1^2K_2\varepsilon_1$ and using (2.6.7), we obtain (2.6.6).

It remains to show the bound (2.6.8) of order ε_3^2. This is an improvement from a naive bound of order ε_3, and is a result of repeated applications of the Hölder and Schwarz inequalities, as explained now. First, we recall the definition of $\hat{V}_p^3(k)$:

$$\begin{aligned}\hat{V}_p^3(k) = \sup_{x,y} \sum_{\{v_j\}_{j=1}^5} & G_p(v_1)\,(pD*G_p)(v_2-v_1)\big(1-\cos k\cdot(v_2-v_1)\big) \\ &\times (pD*G_p)(v_3-v_2)\,(pD*G_p)(v_4-v_1)\,(pD*G_p)(v_4-v_2) \\ &\times (pD*G_p)(v_5-v_4)\,(pD*G_p)(x-v_5)\,G_p(y+v_3-v_5). \end{aligned} \tag{2.6.19}$$

By using the Fourier representation and then the assumptions $g_j(p) \le K_j$ for $j = 1,2,3$, the above sum over $\{v_j\}_{j=1}^5$ is bounded above as

$$\begin{aligned}&\int \prod_{j=1}^3 \frac{\mathrm{d}^d l_j}{(2\pi)^d} \hat{G}_p(l_1)\Big(-\hat{\Delta}_k\big(\widehat{pD*G_p}(l_2)\big)\Big)\widehat{pD*G_p}(l_3)\,\widehat{pD*G_p}(l_1-l_2) \\ &\qquad \times \widehat{pD*G_p}(l_3-l_2)\,\widehat{pD*G_p}(l_1-l_3)\,\widehat{pD*G_p}(l_1)\,\hat{G}_p(l_3)\,\mathrm{e}^{-\mathrm{i}l_1\cdot x+il_3\cdot y} \\ &\le p^5 \int \prod_{j=1}^3 \frac{\mathrm{d}^d l_j}{(2\pi)^d} |\hat{G}_p(l_1)|\,\big|\hat{\Delta}_k\big(p\hat{D}(l_2)\hat{G}_p(l_2)\big)\big|\,|\hat{D}(l_3)\hat{G}_p(l_3)|\,|\hat{D}(l_1-l_2)\hat{G}_p(l_1-l_2)| \\ &\qquad \times |\hat{D}(l_3-l_2)\hat{G}_p(l_3-l_2)|\,|\hat{D}(l_1-l_3)\hat{G}_p(l_1-l_3)|\,|\hat{D}(l_1)\hat{G}_p(l_1)|\,|\hat{G}_p(l_3)| \\ &\le K_1^5K_2^7K_3 \int \prod_{j=1}^3 \frac{\mathrm{d}^d l_j}{(2\pi)^d} \hat{S}_1(l_1)\,\hat{U}(k,l_2)\,|\hat{D}(l_3)|\hat{S}_1(l_3)\,|\hat{D}(l_1-l_2)|\hat{S}_1(l_1-l_2) \\ &\qquad \times |\hat{D}(l_3-l_2)|\hat{S}_1(l_3-l_2)\,|\hat{D}(l_1-l_3)|\hat{S}_1(l_1-l_3)\,|\hat{D}(l_1)|\hat{S}_1(l_1)\,\hat{S}_1(l_3), \end{aligned} \tag{2.6.20}$$

where we have used the abbreviation $\int = \iiint_{(\mathbb{T}^d)^3}$ and the fact that $\hat{S}_1 \ge 0$.

To investigate the above integral, we introduce the notation, such as $\hat{S}_{1-2} = \hat{S}_1(l_1-l_2)$ and $\hat{S}_{2+} = \hat{S}_1(l_2+k)$ (n.b. the new subscripts are not the values of p). By repeated applications of the Hölder and Schwarz inequalities and periodicity, the contribution from, e.g., $\hat{S}_{2+}\hat{S}_{2-}$ in $\hat{U}(k,l_2)$ is bounded as

$$
\begin{aligned}
&\int \prod_{j=1}^{3} \frac{\mathrm{d}^d l_j}{(2\pi)^d} \hat{S}_1\, \hat{S}_{2+} \hat{S}_{2-}\, |\hat{D}_3| \hat{S}_3\, |\hat{D}_{1-2}| \hat{S}_{1-2}\, |\hat{D}_{3-2}| \hat{S}_{3-2}\, |\hat{D}_{1-3}| \hat{S}_{1-3}\, |\hat{D}_1| \hat{S}_1\, \hat{S}_3 \\
&\le \bigg(\int \prod_{j=1}^{3} \frac{\mathrm{d}^d l_j}{(2\pi)^d} \Big(\hat{S}_{2+}\, |\hat{D}_{3-2}| \hat{S}_{3-2}\, |\hat{D}_{1-3}| \hat{S}_{1-3} \Big)^3 \bigg)^{1/3} \\
&\quad \times \bigg(\int \prod_{j=1}^{3} \frac{\mathrm{d}^d l_j}{(2\pi)^d} \Big(|\hat{D}_1| \hat{S}_1^2\, |\hat{D}_3| \hat{S}_3^2\, \hat{S}_{2-}\, |\hat{D}_{1-2}| \hat{S}_{1-2} \Big)^{3/2} \bigg)^{2/3} \\
&= \bigg(\underbrace{\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l_2}{(2\pi)^d} \hat{S}_{2+}^3}_{\equiv \triangledown} \underbrace{\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l_3}{(2\pi)^d} |\hat{D}_{3-2}|^3 \hat{S}_{3-2}^3}_{\le \varepsilon_3} \underbrace{\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l_1}{(2\pi)^d} |\hat{D}_{1-3}|^3 \hat{S}_{1-3}^3}_{\le \varepsilon_3} \bigg)^{1/3} \\
&\quad \times \bigg(\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l_1}{(2\pi)^d} |\hat{D}_1|^{3/2} \hat{S}_1^3 \int_{\mathbb{T}^d} \frac{\mathrm{d}^d l_3}{(2\pi)^d} |\hat{D}_3|^{3/2} \hat{S}_3^3 \underbrace{\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l_2}{(2\pi)^d} \hat{S}_{2-}^{3/2}\, |\hat{D}_{1-2}|^{3/2} \hat{S}_{1-2}^{3/2}}_{\le \triangledown^{1/2} \varepsilon_3^{1/2}\ (\because \text{Schwarz})} \bigg)^{2/3} \\
&\le \triangledown^{2/3} \varepsilon_3 \bigg(\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l_1}{(2\pi)^d} |\hat{D}_1|^{3/2} \hat{S}_1^3 \bigg)^{4/3} \\
&\le \triangledown^{2/3} \varepsilon_3 \Bigg(\bigg(\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l_1}{(2\pi)^d} \hat{D}_1^2 \hat{S}_1^3 \bigg)^{3/4} \bigg(\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l_1}{(2\pi)^d} \hat{S}_1^3 \bigg)^{1/4} \Bigg)^{4/3} \\
&\le \triangledown\, \varepsilon_3^2.
\end{aligned}
\tag{2.6.21}
$$

By the identity $S_1 = \delta_{o,\cdot} + D * S_1$, we further obtain

$$
\triangledown = \int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \hat{S}_1(l)^3 = 1 + 3\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3. \tag{2.6.22}
$$

The contributions from the other terms in $\hat{U}(k, l_2)$ obey the same bound. This completes the proof of (2.6.8), hence the proof of Lemma 2.6.1. ☐

Next, we evaluate the diagrams at $p = 1$ by using the trivial inequality $G_1(x) \le S_1(x)$. Here, we do not need the bootstrapping assumptions.

Lemma 2.6.2. Let $d \ge 7$ and $p = 1$. Then, we have

$$
\begin{aligned}
L_1 &\le \varepsilon_1, &(2.6.23)\\
B_1 &\le \varepsilon_2, &(2.6.24)\\
T_1 &\le \varepsilon_3, &(2.6.25)\\
\bigg\| \frac{\hat{V}_1^0}{1 - \hat{D}} \bigg\|_\infty &\le \varepsilon_1 + 5\varepsilon_3, &(2.6.26)\\
\bigg\| \frac{\hat{V}_1^1}{1 - \hat{D}} \bigg\|_\infty &\le 1 + \frac{5}{2^d} + 5\varepsilon_1 + 6\varepsilon_2 + 20\varepsilon_3, &(2.6.27)\\
\bigg\| \frac{\hat{V}_1^2}{1 - \hat{D}} \bigg\|_\infty &\le \frac{1}{2^d} + \varepsilon_1 + 2\varepsilon_2 + 10\varepsilon_3, &(2.6.28)\\
\bigg\| \frac{\hat{V}_1^3}{1 - \hat{D}} \bigg\|_\infty &\le 5(1 + 3\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3)\varepsilon_3^2. &(2.6.29)
\end{aligned}
$$

Proof. We only need to use the trivial inequality $G_1(x) \le S_1(x)$, $x \in \mathbb{L}^d$, to obtain that, for $d > 2$ (as mentioned earlier, $\hat{S}_1(k) \equiv (1 - \hat{D}(k))^{-1}$ is well-defined in a proper limit when $d > 2$),

$$L_1 \le \|D^{*2} * S_1\|_\infty = \int_{\mathbb{T}^d} \frac{\hat{D}(k)^2}{1 - \hat{D}(k)} \frac{\mathrm{d}^d k}{(2\pi)^d} = (D^{*2} * S_1)(o) \equiv \varepsilon_1. \tag{2.6.30}$$

Similarly, we can show the other inequalities. For example (cf., (2.6.11)–(2.6.13)),

$$\begin{aligned} \frac{\hat{V}_1^0(k)}{1 - \hat{D}(k)} &\le \sum_{x \sim o} \frac{D(x)(1 - \cos k \cdot x)}{1 - \hat{D}(k)} (D * S_1)(x) \\ &\quad + \sum_x (D^{*2} * S_1)(x) \frac{(D * S_1)(x)(1 - \cos k \cdot x)}{1 - \hat{D}(k)} \\ &\le \underbrace{(D^{*2} * S_1)(o)}_{=\varepsilon_1} + \int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \hat{D}(l)^2 \hat{S}_1(l) \frac{|\hat{\Delta}_k(\hat{D}(l)\hat{S}_1(l))|}{1 - \hat{D}(k)}. \end{aligned} \tag{2.6.31}$$

Since

$$\big|\hat{\Delta}_k\big(\hat{D}(l)\hat{S}_1(l)\big)\big| = \big|\hat{\Delta}_k\big(\hat{S}_1(l) - 1\big)\big| = |\hat{\Delta}_k \hat{S}_1(l)| \le \hat{U}(k, l), \tag{2.6.32}$$

the integral in (2.6.31) is bounded by $5\varepsilon_3$ (cf., (2.6.14)). To avoid redundancy, we refrain from showing the other inequalities. This completes the proof of Lemma 2.6.2. ☐

2.7 Numerically computing the random-walk quantities

As we mentioned in Section 1.1.1, we estimate the RW quantities (2.6.1) by Stirling's formula. For example, if we split the sum into two at $n = N$, then the RW bubble ε_2 in dimensions $d > 4$ can be estimated as

$$0 \le \varepsilon_2 - \sum_{n=1}^{N} (2n - 1) D^{*2n}(o) \overset{(1.1.2)}{\le} 2\pi^{-d/2} \int_N^\infty t^{1-d/2} \,\mathrm{d}t = \frac{4\pi^{-d/2}}{d - 4} N^{(4-d)/2}. \tag{2.7.1}$$

If we choose $d = 5$ and $N = 100$ and use a calculator to evaluate the sum over $n \le N$, then we obtain $\varepsilon_2 \le 0.178465$. Table 2.1 summarizes the bounds on those RW quantities in different dimensions by choosing $N = 500$ (so that, by (1.1.2), we can show that the RW triangle ε_3 for $d = 7$ takes a value around the indicated number within 10^{-6}).

Table 2.1: Upper bounds on the RW loop, bubble and triangle for $3 \le d \le 9$.

	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$
ε_1	0.393216	0.118637	0.046826	0.020461	0.009406	0.004451	0.002144
ε_2	∞	∞	0.178332	0.044004	0.015302	0.006156	0.002678
ε_3	∞	∞	∞	∞	0.052689	0.012354	0.004148

2.8 Completing the bootstrapping arguments

In this subsection, we complete the proof of Propositions 2.1.2–2.1.3.

Proof of Proposition 2.1.2. Since ε_1, ε_2 and ε_3 are finite for $d \geq 7$ (see Table 2.1 in Section 1.1.1) and decreasing in d (because $D^{*2n}(o) \equiv \big(\binom{2n}{n}2^{-2n}\big)^d$ on $\mathbb{L}^d$ is decreasing in d), we have, by Lemma 2.6.2,

$$r = \|D\|_\infty + L_1 + B_1 \leq 2^{-d} + \varepsilon_1 + \varepsilon_2 \leq \begin{cases} 0.0326 & [d = 7], \\ 0.0146 & [d = 8], \\ 0.0068 & [d \geq 9], \end{cases} \tag{2.8.1}$$

and

$$\begin{aligned} \rho &= T_1(2r + T_1) + (r + T_1)\left(1 + \frac{B_1}{2} + T_1\right) \\ &\leq \varepsilon_3(2r + \varepsilon_3) + (r + \varepsilon_3)\left(1 + \frac{\varepsilon_2}{2} + \varepsilon_3\right) \leq \begin{cases} 0.0967 & [d = 7], \\ 0.0279 & [d = 8], \\ 0.0111 & [d \geq 9]. \end{cases} \end{aligned} \tag{2.8.2}$$

In addition, by (2.5.2)–(2.5.13) and Lemma 2.6.2, we have

$$\sum_{n=0}^\infty \hat\pi_1^{(n)}(0) + \sup_k \sum_{n=0}^\infty \frac{-\hat\Delta_k \hat\pi_1^{(n)}(0)}{1 - \hat D(k)} \leq \begin{cases} 2.7700 & [d = 7], \\ 0.3623 & [d = 8], \\ 0.1124 & [d \geq 9], \end{cases} \tag{2.8.3}$$

which implies that the inequality (2.5.14) holds for all $d \geq 8$ (but not for $d = 7$). Then, by Lemma 2.5.1, we obtain

$$g_1(1) \leq \left(1 - \frac{(1 + \varepsilon_2/2 + \varepsilon_3)^2 r}{1 - \rho^2}\right)^{-1} \leq \begin{cases} 1.0154 & [d = 8], \\ 1.0070 & [d \geq 9], \end{cases} \tag{2.8.4}$$

$$\begin{aligned} g_2(1) &\leq \left(1 - \left(1 - \frac{\varepsilon_2}{2} - \frac{(1 + \varepsilon_2/2 + \varepsilon_3)^2 r}{1 - \rho}\right)^{-1} \sum_{j=0}^3 \varphi_j^{\text{odd}} \left\|\frac{\hat V_1^j}{1 - \hat D}\right\|_\infty\right)^{-1} \\ &\leq \begin{cases} 1.1049 & [d = 8], \\ 1.0272 & [d \geq 9], \end{cases} \end{aligned} \tag{2.8.5}$$

$$\begin{aligned} g_3(1) &\leq \Big[\text{the bounds in (2.8.5)}\Big]^3 \times \left(1 - \frac{\varepsilon_2}{2} - \frac{(1 + \varepsilon_2/2 + \varepsilon_3)^2 r}{1 - \rho}\right)^{-3} \\ &\qquad \times \left(1 + \varepsilon_2 + \frac{2(1 + \varepsilon_2/2 + \varepsilon_3)^2 r}{1 - \rho} + 2\sum_{j=0}^3 (\varphi_j^{\text{even}} + \varphi_j^{\text{odd}}) \left\|\frac{\hat V_1^j}{1 - \hat D}\right\|_\infty\right)^2 \\ &\leq \begin{cases} 4.2433 & [d = 8], \\ 1.6673 & [d \geq 9]. \end{cases} \end{aligned} \tag{2.8.6}$$

Proposition 2.1.2 for percolation holds as long as $K_1 > 1.0154$, $K_2 > 1.1049$, $K_3 > 4.2433$ for $d = 8$, and $K_1 > 1.0070$, $K_2 > 1.0272$, $K_3 > 1.6673$ for $d \geq 9$. □

Proof of Proposition 2.1.3. Let

$$K_1 = 1.01, \qquad K_2 = 1.09, \qquad K_3 = 2.70, \tag{2.8.7}$$

so that Proposition 2.1.2 holds for $d \geq 9$. Then, by Table 2.1 and Lemma 2.6.4, we obtain

$$r \leq K_1 2^{-d} + K_1^2 K_2 \varepsilon_1 + K_1^2 K_2^2 \varepsilon_2 \leq 0.0077, \tag{2.8.8}$$

and

$$\rho \leq K_1^2 K_2^3 \varepsilon_3 \left(2r + K_1^2 K_2^3 \varepsilon_3\right) + \left(r + K_1^2 K_2^3 \varepsilon_3\right) \left(1 + \frac{K_1^2 K_2^2 \varepsilon_2}{2} + K_1^2 K_2^3 \varepsilon_3\right)$$
$$\leq 0.0134. \tag{2.8.9}$$

In addition, by (2.5.2)–(2.5.13) and Lemma 2.6.1, we have

$$\sum_{n=0}^{\infty} \hat{\pi}_p^{(n)}(0) + \sup_k \sum_{n=0}^{\infty} \frac{-\hat{\Delta}_k \hat{\pi}_p^{(n)}(0)}{1 - \hat{D}(k)} \leq 0.2151, \tag{2.8.10}$$

which implies that the inequality (2.5.14) holds. Then, similarly to (2.8.4)–(2.8.6), we obtain

$$g_1(p) \leq \left(1 - \frac{(1 + K_1^2 K_2^2 \varepsilon_2/2 + K_1^2 K_2^3 \varepsilon_3)^2 r}{1 - \rho^2}\right)^{-1} \leq 1.0080 < K_1, \tag{2.8.11}$$

$$g_2(p) \leq \left(1 - \left(1 - \frac{K_1^2 K_2^2 \varepsilon_2}{2} - \frac{(1 + K_1^2 K_2^2 \varepsilon_2/2 + K_1^2 K_2^3 \varepsilon_3)^2 r}{1 - \rho}\right)^{-1} \sum_{j=0}^{3} \varphi_j^{\text{odd}} \left\| \frac{\hat{V}_p^j}{1 - \hat{D}} \right\|_\infty \right)^{-1}$$
$$\leq 1.0810 < K_2, \tag{2.8.12}$$

$$g_3(p) \leq (1.081)^3 \left(1 - \frac{K_1^2 K_2^2 \varepsilon_2}{2} - \frac{(1 + K_1^2 K_2^2 \varepsilon_2/2 + K_1^2 K_2^3 \varepsilon_3)^2 r}{1 - \rho}\right)^{-3} K_1^2$$
$$\times \left(1 + K_1^2 K_2^2 \varepsilon_2 + \frac{2(1 + K_1^2 K_2^2 \varepsilon_2/2 + K_1^2 K_2^3 \varepsilon_3)^2 r}{1 - \rho} + 2 \sum_{j=0}^{3} (\varphi_j^{\text{even}} + \varphi_j^{\text{odd}}) \left\| \frac{\hat{V}_p^j}{1 - \hat{D}} \right\|_\infty \right)^2$$
$$\leq 2.6606 < K_3. \tag{2.8.13}$$

This completes the proof of Proposition 2.1.3 for percolation. □

2.9 Further discussion

We have been able to prove convergence of the lace expansion for percolation on $\mathbb{L}^{d \geq 9}$ in full detail. As compared to the NoBLE analysis on $\mathbb{Z}^{d \geq 11}$ [14, 15], the current analysis is much simpler, shorter and more transparent. This is due to the simple structure of the BCC lattice $\mathbb{L}^d$ and the choice of the bootstrapping functions $\{g_i(p)\}_{i=1}^3$.

To go down to the desired 7 dimensions, we must improve our analysis in various aspects. Some improvement ideas are summarized as follows.

1. The largest contribution comes from $|\hat{\Delta}_k \hat{\pi}_p^{(0)}(0)|$ and $|\hat{\Delta}_k \hat{\pi}_p^{(1)}(0)|$, and their common leading term is proportional to the diagram function $\hat{V}_p^0(k)$. To improve its bound, we may introduce extra diagrams, such as $B'_p \equiv \|(pD)^{*4} * G_p^{*2}\|_\infty$ and $T'_p \equiv \|(pD)^{*4} * G_p^{*3}\|_\infty$, or even longer diagrams, such as $T_p^{(n)} \equiv \|(pD)^{*2n} * G_p^{*3}\|_\infty$. Although its RW counterpart $(D^{*2n} * S_1^{*3})(o)$ is decreasing in n, the bound on $T_p^{(n)}$ may attain the minimum at some $n_* \in \mathbb{N}$, due to the exponentially growing factor p^{2n}. So far, we have not investigated a result of using $T_p^{(n_*)}$, since introducing such new diagrams increases the number of terms to deal with, which may cause extra complication.

2. We used the Schwarz inequality to bound $\|\hat{V}_p^j/(1 - \hat{D})\|_\infty$, $j = 0, 1, 2, 3$, in Lemmas 2.6.1–2.6.2. As a result, the relatively large factor 5 appeared (see, e.g., (2.6.17)). It would be of great help if we could do away with the Schwarz inequality to achieve a better bound than Lemma 1.3.2.

3. We ignored the contributions from $\hat{\Pi}_p^{\rm even}(0)$ and $|\hat{\Delta}_k\hat{\Pi}_p^{\rm even}(0)|$ in (2.5.15)–(2.5.16). If we include their effect into computation, then $g_1(p)$ could be much closer to 1 (cf. (2.5.18)) and $g_2(p)$ could be even smaller than 1 (cf. (2.5.19)), and as a result, we could achieve convergence of the lace expansion on $\mathbb{L}^{d\ge7}$. However, to make use of those even terms, we must also control lower bounds on $g_1(p)$ and $g_2(p)$, and to do so, we need nontrivial lower bounds on the lace-expansion coefficients. Achieving this goal without causing too much complication would be a challenging task.

4. Instead of estimating $\hat{G}_p(k)$ by $\hat{S}_1(k) \equiv (1-\hat{D}(k))^{-1}$ uniformly in $k \in \mathbb{T}^d$, which may not be efficient in the ultraviolet regime, we may split the estimate of $G_p(x)$ between for large x and for small x. For small x, estimating $G_p(x)$ by $S_1(x)$ is expected to be a bit too crude. Therefore, estimating the lace-expansion coefficients in the ultraviolet regime by naively using the BK inequality would be too primitive. Here, we may need incorporate the ultraviolet regularization of [4] or large-field analysis in the rigorous renormalization group for spin systems.

Chapter 3

Lace-expansion analysis for oriented percolation

3.1 Bootstrapping functions

We start to prove Theorem 1.2.1 for oriented percolation from this section. By Lemma 1.3.1, it suffices for us to verify the following three propositions, of which in particular we use the lace expansion in the third one. Let

$$\hat{U}_\mu(k,l) = \big(1-\hat{D}(k)\big)\Bigg(\frac{|\hat{S}_\mu(l+k)| + |\hat{S}_\mu(l-k)|}{2}\Big|\hat{S}_\mu(l)\Big| + \big(1-\hat{D}(2l)\big)\Big|\hat{S}_\mu(l)\Big|\Big|\hat{S}_\mu(l+k)\Big|\Big|\hat{S}_\mu(l-k)\Big|\Bigg) \tag{3.1.1}$$

for $k, l \in \mathbb{T}^d$ and $\mu \in \mathbb{C}$ with $|\mu| \in [0,1]$, and let

$$\mu_p(z) = \big(1-\hat{\varphi}_p(0,|z|)^{-1}\big)\mathrm{e}^{\mathrm{i}\arg z}$$

for $p \in [0, p_\mathrm{c})$ and $z \in \mathbb{C}$ with $|z| \in [1, m_p)$. We define the bootstrapping functions $\{g_i(p,m)\}_{i=1}^3$ as

$$g_1(p,m) \equiv p(m \vee 1), \tag{3.1.2}$$

$$g_2(p,m) \equiv \sup_{\substack{k\in\mathbb{T}^d,\\ z\in\mathbb{C}:\ |z|\in\{1,m\}}} \frac{|\hat{\varphi}_p(k,z)|}{|\hat{S}_{\mu_p(z)}(k)|}, \tag{3.1.3}$$

$$g_3(p,m) \equiv \sup_{\substack{k,l\in\mathbb{T}^d,\\ z\in\mathbb{C}:\ |z|\in\{1,m\}}} \frac{\big|\hat{\Delta}_k\big(\hat{q}_p(l,z)\hat{\varphi}_p(l,z)\big)\big|}{\hat{U}_{\mu_p(z)}(k,l)}. \tag{3.1.4}$$

In (3.1.4), the supremum near $k=0$ should be interpreted as the supremum over the limit as $\|k\|_2 \to 0$. The following propositions are sufficient conditions of Lemma 1.3.1. Thus, if Proposition 3.1.1–3.1.3 hold, then it is true that there exist K_i such that $g_i(p,m) < K_i$ for each $i = 1,2,3$. In particular, the bound on $g_2(p,m)$ and Lemma 3.5.3 below immediately imply Theorem 1.2.1 for oriented percolation.

Proposition 3.1.1 (Continuity)**.** The functions $\{g_i(p,m)\}_{i=1}^3$ are continuous in $m \in [1, m_p)$ for every $p \in [0, p_\mathrm{c})$, and the functions $\{g_i(p,1)\}_{i=1}^3$ are continuous in $p \in [0, p_\mathrm{c})$.

Proposition 3.1.2 (Initial conditions)**.** There are model-dependent finite constants $\{K_i\}_{i=1}^3$ such that $g_i(0,1) < K_i$ for $i = 1,2,3$.

Proposition 3.1.3 (Bootstrapping argument)**.** For nearest-neighbor oriented percolation on $\mathbb{L}^{d\geq 9} \times \mathbb{Z}_+$, we fix both $p \in (0, p_\mathrm{c})$ and $z \in \mathbb{C}$ with $|z| \in (1, m_p)$ and assume $g_i(p, m) \leq K_i$, $i = 1, 2, 3$, where $\{K_i\}_{i=1}^3$ are the same constants as in Proposition 3.1.2. Then, the stronger inequalities $g_i(p, m) < K_i$, $i = 1, 2, 3$, hold.

In the rest of this chapter, we prove Proposition 3.1.1 in Section 3.2 and Proposition 3.1.2–3.1.3 in Section 3.8. In particular, Proposition 3.1.3 is shown in the following steps:

1. Bound the bootstrapping functions $\{g_i(p, m)\}_{i=1}^3$ above by the lace expansion.
2. Bound the lace expansion coefficients $(\{\pi_p^{(N)}\}_{N=0}^\infty)$ above by basic diagrams $(B_{p,m}^{(\lambda,\rho)}$, $T_{p,m}^{(\lambda,\rho)}$ and $\hat{V}_{p,m}^{(\lambda,\rho)}(k))$.
3. Bound the basic diagrams above by the bootstrapping hypotheses $g_i(p, m) \leq K_i$ for $i = 1, 2, 3$ and the random-walk quantities (3.6.1).
4. Verify the stronger inequalities $g_i(p, m) < K_i$ holds for $i = 1, 2, 3$ by combining the bounds in Step 1–3 and substituting specifically numerical values into $\{K_i\}_{i=1}^3$.

These steps are explained in Section 3.5, 3.4, 3.6 and 3.8, respectively.

3.2 Continuity of the bootstrapping functions

Let

$$\tilde{g}_{2,k,\theta}(p, m) = \frac{\hat{\varphi}_p(k, m\mathrm{e}^{\mathrm{i}\theta})}{\hat{S}_{\mu_p(m\mathrm{e}^{\mathrm{i}\theta})}(k)} \quad \text{and} \quad \tilde{g}_{3,k,l,\theta}(p, m) = \frac{\hat{\Delta}_k(\hat{q}_p(l, m\mathrm{e}^{\mathrm{i}\theta})\hat{\varphi}_p(l, m\mathrm{e}^{\mathrm{i}\theta}))}{\hat{U}_{\mu_p(m\mathrm{e}^{\mathrm{i}\theta})}(k, l)},$$

and fix $\tilde{p} \in [0, p_\mathrm{c})$ and $\tilde{m} \in [1, m_{\tilde{p}})$ arbitrarily. To show that $\{\, g_i \,\}_{i=1}^3$ are continuous, we use Lemma 1.3.3, whose sufficieint conditions follow from

(i) $\tilde{g}_{2,k,\theta}(p, m)$, $\partial_m \tilde{g}_{2,k,\theta}(p, m)$ and $\partial_p \tilde{g}_{2,k,\theta}(p, 1)$ are bounded uniformly in $k \in \mathbb{T}^d$, $\theta \in \mathbb{T}$, $p \in [0, \tilde{p}]$ and $m \in [1, \tilde{m}]$,

(ii) $\tilde{g}_{3,k,l,\theta}(p, m)$, $\partial_m \tilde{g}_{3,k,l,\theta}(p, m)$ and $\partial_p \tilde{g}_{3,k,l,\theta}(p, 1)$ are bounded uniformly in $k \in \mathbb{T}^d$, $\theta \in \mathbb{T}$, $p \in [0, \tilde{p}]$ and $m \in [1, \tilde{m}]$.

For example, for any $p \in [0, \tilde{p}]$ and $\epsilon > 0$, suppose that there exists a constant $C < \infty$ such that $|\partial_m \tilde{g}_{2,k,\theta}(p, m)| \leq C$ (independent of k, θ, p and m). Then,

$$\big|\tilde{g}_{2,k,\theta}(p, m') - \tilde{g}_{2,k,\theta}(p, m'')\big| = \left| \int_{m'}^{m''} \partial_m \tilde{g}_{2,k,\theta}(p, m) \, \mathrm{d}m \right| \leq C\big|m'' - m'\big| \leq \epsilon$$

for every $m', m'' \in [1, \tilde{m}]$ satisfying $|m' - m''| \leq \epsilon/C$, which means that $\{\, \tilde{g}_{2,k,\theta}(p, m) \,\}_{k,\theta}$ is an equicontinuous family. $\{\, \tilde{g}_{3,k,l,\theta}(p, m) \,\}_{k,l,\theta}$ is also similar.

Lemma 3.2.1. For every $(x, t) \in \mathbb{L}^d \times \mathbb{Z}_+$,

$$\begin{aligned} (t+1)\varphi_p(x, t) &\leq \varphi_p^{\star 2}(x, t), \\ t\varphi_p(x, t) &\leq \big(q_p \star \varphi_p^{\star 2}\big)(x, t). \end{aligned}$$

Proof. By the Boolean inequality and the Markov property,

$$
\begin{aligned}
(t+1)\varphi_p(x,t) &= \sum_{s=0}^{t} \mathbb{P}_p\Bigg(\bigcup_{y\in\mathbb{L}^d} \{\,(o,0)\to(y,s)\to(x,t)\,\}\Bigg)\\
&\le \sum_{(y,s)} \mathbb{P}_p\big((o,0)\to(y,s)\to(x,t)\big)\\
&= \sum_{(y,s)} \mathbb{P}_p\big((o,0)\to(y,s)\big)\mathbb{P}_p\big((y,s)\to(x,t)\big)\\
&= \varphi_p^{\star 2}(x,t).
\end{aligned}
$$

Similarly, noticing $t=\sum_{s=0}^{t}\mathbb{1}_{\{s\ge 1\}}$ and the trivial inequality

$$
\varphi_p(\boldsymbol{x})\mathbb{1}_{\{\boldsymbol{x}\neq\boldsymbol{o}\}} \le (q_p\star\varphi_p)(\boldsymbol{x}), \tag{3.2.1}
$$

we obtain the second inequality. $\square$

Lemma 3.2.2. For every $p\in[0,p_{\mathrm{c}})$, $m\in[1,m_p)$ and $k\in\mathbb{T}^d$,

$$
\frac{1}{2} \le \Big|\hat{S}_{\mu_p(me^{\mathrm{i}\theta})}(k)\Big| \le \hat{\varphi}_p(0,m), \tag{3.2.2}
$$

$$
\frac{1}{4} \le \frac{\hat{U}_{\mu_p(me^{\mathrm{i}\theta})}(k,l)}{1-\hat{D}(k)} \le \hat{\varphi}_p(0,m)^2\Big(1+\big(1-\hat{D}(2l)\big)\hat{\varphi}_p(0,m)\Big), \tag{3.2.3}
$$

$$
\Big|\hat{\Delta}_k\hat{\varphi}_p(0,m)\Big| \le 2pm\big(1-\hat{D}(k)\big)\hat{\varphi}_p(0,m)^3. \tag{3.2.4}
$$

Proof. By the inequality $-1\le\hat{D}(k)\le 1$, the lower bound in (3.2.2) is clear. By the definition of $\mu_p(me^{\mathrm{i}\theta})$,

$$
\begin{aligned}
\Big|\hat{S}_{\mu_p(me^{\mathrm{i}\theta})}(k)\Big| &= \frac{1}{|1-\mu_p(me^{\mathrm{i}\theta})\hat{D}(k)|}\\
&\le \frac{1}{1-(1-\hat{\varphi}_p(0,m)^{-1})|\hat{D}(k)|}\\
&\le \frac{1}{1-(1-\hat{\varphi}_p(0,m)^{-1})}\\
&= \hat{\varphi}_p(0,m).
\end{aligned}
$$

The inequality (3.2.3) immediately follows from (3.1.1) and (3.2.2).

To prove (3.2.4), we introduce an ordering among paths from $\boldsymbol{o}$ to $\boldsymbol{x}\in\mathbb{L}^d\times\mathbb{Z}_+$ as follows. Let $\mathscr{B}((x,t)) = \{((x,t),(y,t+1))\in(\mathbb{L}^d\times\mathbb{Z}_+)^2 \mid x-y\in\mathscr{N}^d\}$ for $(x,t)\in\mathbb{L}^d\times\mathbb{Z}_+$, which is the set of directed bonds whose bottoms are (x,t). We can order the elements in $\mathscr{B}((x,t))$ because it is a finite set. For a pair of bonds $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$, we write $\boldsymbol{b}_1\prec\boldsymbol{b}_2$ if $\boldsymbol{b}_1$ is smaller than $\boldsymbol{b}_2$ in that ordering. Fix $\boldsymbol{x},\boldsymbol{y}\in\mathbb{L}^d\times\mathbb{Z}_+$, we also write $\vec{\boldsymbol{\omega}}^{(1)}\prec\vec{\boldsymbol{\omega}}^{(2)}$ for two paths $\vec{\boldsymbol{\omega}}^{(1)},\vec{\boldsymbol{\omega}}^{(2)}\in\mathscr{W}(\boldsymbol{x},\boldsymbol{y})$ if we have $\boldsymbol{b}_t(\vec{\boldsymbol{\omega}}^{(1)})\prec\boldsymbol{b}_t(\vec{\boldsymbol{\omega}}^{(2)})$ at time t such that $\boldsymbol{b}_s(\vec{\boldsymbol{\omega}}^{(1)})=\boldsymbol{b}_s(\vec{\boldsymbol{\omega}}^{(2)})$ for every $s\le t$ and $\boldsymbol{b}_t(\vec{\boldsymbol{\omega}}^{(1)})\neq\boldsymbol{b}_t(\vec{\boldsymbol{\omega}}^{(2)})$, where $\boldsymbol{b}_j(\vec{\boldsymbol{\omega}})=(\boldsymbol{\omega}_{j-1},\boldsymbol{\omega}_j)$ denotes the jth bond along the path $\vec{\boldsymbol{\omega}}=(\boldsymbol{\omega}_0,\boldsymbol{\omega}_1,\dots,\boldsymbol{\omega}_{|\omega|})$. Let $E_{\boldsymbol{x},\boldsymbol{y}}(\vec{\boldsymbol{\omega}})$ be the event that $\vec{\boldsymbol{\omega}}$ is the lowest occupied path from $\boldsymbol{x}$ to $\boldsymbol{y}$:

$$
E_{\boldsymbol{x},\boldsymbol{y}}(\vec{\boldsymbol{\omega}}) = \{\,\vec{\boldsymbol{\omega}}\text{ is occupied}\,\}\setminus\bigcup_{\substack{\vec{\boldsymbol{\omega}}\in\mathscr{W}(\boldsymbol{x},\boldsymbol{y})\\(\vec{\boldsymbol{\omega}}'\prec\vec{\boldsymbol{\omega}})}}\big\{\,\vec{\boldsymbol{\omega}}'\text{ is occupied}\,\big\}.
$$

Then, we can rewrite the expression for $\varphi_p(\boldsymbol{x})$ as

$$\varphi_p(\boldsymbol{x}) = \mathbb{P}_p\Bigg(\bigcup_{\vec{\boldsymbol{\omega}} \in \mathscr{W}(\boldsymbol{o},\boldsymbol{x})} \{ \, \vec{\boldsymbol{\omega}} \text{ is occupied } \} \Bigg) = \sum_{\vec{\boldsymbol{\omega}} \in \mathscr{W}(\boldsymbol{o},\boldsymbol{x})} \mathbb{P}_p(E_{\boldsymbol{o},\boldsymbol{x}}(\vec{\boldsymbol{\omega}})).$$

By the telescopic inequality (1.3.1), we obtain

$$\begin{aligned}
\Big|\hat{\Delta}_k \hat{\varphi}_p(0,m)\Big| &= \sum_{(x,t)} \varphi_p(x,t)(1-\cos k\cdot x)m^t \\
&\le \sum_{\boldsymbol{x}} \sum_{\vec{\boldsymbol{\omega}} \in \mathscr{W}(\boldsymbol{o},\boldsymbol{x})} \mathbb{P}_p(E_{\boldsymbol{o},\boldsymbol{x}}(\vec{\boldsymbol{\omega}})) m^{\tau(\boldsymbol{x})} \tau(\boldsymbol{x}) \sum_{s=1}^{\tau(\boldsymbol{x})} \big(1-\cos k\cdot(\omega_s-\omega_{s-1})\big) \\
&= \sum_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{x}} m^{\tau(\boldsymbol{x})}\tau(\boldsymbol{x})\big(1-\cos k\cdot(v-u)\big) \sum_{s=1}^{\tau(\boldsymbol{x})} \sum_{\vec{\boldsymbol{\omega}} \in \mathscr{W}(\boldsymbol{o},\boldsymbol{x})} \mathbb{1}_{\{\boldsymbol{b}_s(\vec{\boldsymbol{\omega}})=(\boldsymbol{u},\boldsymbol{v})\}} \mathbb{P}_p(E_{\boldsymbol{o},\boldsymbol{x}}(\vec{\boldsymbol{\omega}}))
\end{aligned}$$

where recall the notation $\boldsymbol{x} = (x, \tau(\boldsymbol{x})) \in \mathbb{L}^d \times \mathbb{Z}_+$, and we write $\vec{\boldsymbol{\omega}}$ as $((\omega_0,0),(\omega_1,1),\dots,(\omega_{\tau(\boldsymbol{x})},\tau(\boldsymbol{x})))$. Next, we decompose $\vec{\boldsymbol{\omega}}$ into the first part $\vec{\boldsymbol{\zeta}}$ from $\boldsymbol{o}$ to $\boldsymbol{v}$, $(\boldsymbol{u},\boldsymbol{v})$ and the last part $\vec{\boldsymbol{\xi}}$ from $\boldsymbol{v}$ to $\boldsymbol{x}$, written as $\vec{\boldsymbol{\omega}} = \vec{\boldsymbol{\zeta}} \circ (\boldsymbol{u},\boldsymbol{v}) \circ \vec{\boldsymbol{\xi}}$. By this decomposition, the above inequality is bounded above as

$$\begin{aligned}
\Big|\hat{\Delta}_k \hat{\varphi}_p(0,m)\Big| &\le \sum_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{x}} m^{\tau(\boldsymbol{x})}\tau(\boldsymbol{x})\big(1-\cos k\cdot(v-u)\big) \\
&\qquad \times \sum_{\vec{\boldsymbol{\zeta}}\circ(\boldsymbol{u},\boldsymbol{v})\circ\vec{\boldsymbol{\xi}} \in \mathscr{W}(\boldsymbol{o},\boldsymbol{x})} \mathbb{P}_p\big(E_{\boldsymbol{o},\boldsymbol{u}}(\vec{\boldsymbol{\zeta}}) \cap \{ \, (\boldsymbol{u},\boldsymbol{v}) \text{ is occupied } \} \cap E_{\boldsymbol{v},\boldsymbol{x}}(\vec{\boldsymbol{\xi}})\big) \\
&\le \sum_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{x}} m^{\tau(\boldsymbol{x})}\tau(\boldsymbol{x})\big(1-\cos k\cdot(v-u)\big) \\
&\qquad \times \sum_{\vec{\boldsymbol{\zeta}} \in \mathscr{W}(\boldsymbol{o},\boldsymbol{u})} \mathbb{P}_p(E_{\boldsymbol{o},\boldsymbol{u}}(\vec{\boldsymbol{\zeta}})) q_p(\boldsymbol{v}-\boldsymbol{u}) \sum_{\vec{\boldsymbol{\xi}} \in \mathscr{W}(\boldsymbol{v},\boldsymbol{x})} \mathbb{P}_p(E_{\boldsymbol{v},\boldsymbol{x}}(\vec{\boldsymbol{\xi}})) \\
&= \sum_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{x}} \big(1-\cos k\cdot(v-u)\big) m^{\tau(\boldsymbol{u})} m m^{\tau(\boldsymbol{x})-\tau(\boldsymbol{v})} \\
&\qquad \times \big(\tau(\boldsymbol{u})+1+\tau(\boldsymbol{x})-\tau(\boldsymbol{v})\big)\varphi_p(\boldsymbol{u}) q_p(\boldsymbol{v}-\boldsymbol{u}) \varphi_p(\boldsymbol{x}-\boldsymbol{v}).
\end{aligned} \tag{3.2.5}$$

By Lemma 3.2.1,

$$\begin{aligned}
\Big|\hat{\Delta}_k \hat{\varphi}_p(0,m)\Big| &\le \sum_{\boldsymbol{u},\boldsymbol{v},\boldsymbol{x}} \big(1-\cos k\cdot(v-u)\big) m^{\tau(\boldsymbol{u})} m m^{\tau(\boldsymbol{x})-\tau(\boldsymbol{v})} \\
&\qquad \times \big(\varphi_p^{\star 2}(\boldsymbol{u}) q_p(\boldsymbol{v}-\boldsymbol{u})\varphi_p(\boldsymbol{x}-\boldsymbol{v}) + \varphi_p(\boldsymbol{u}) q_p(\boldsymbol{v}-\boldsymbol{u}) \varphi_p^{\star 2}(\boldsymbol{x}-\boldsymbol{v})\big) \\
&= 2\big(\hat{q}_p(0,m) - \hat{q}_p(k,m)\big)\hat{\varphi}_p(0,m)^3 \\
&= 2pm\big(1-\hat{D}(k)\big)\hat{\varphi}_p(0,m)^3.
\end{aligned}$$

This completes the proof of (3.2.4). □

Using the above lemmas, we now prove (i) and (ii) in the beginning of this section.

Proof of (i). Clearly, $\tilde{g}_{2,k,\theta}(p,m)$ is bounded uniformly in $k \in \mathbb{T}^d$ and $\theta \in \mathbb{T}$ by (3.2.2) and the inequality $\big|\hat{\varphi}_p(k, m\mathrm{e}^{\mathrm{i}\theta})\big| \le \hat{\varphi}_p(0,m)$. Note that $\hat{\varphi}_p(0,m) < \infty$ for $p \in [0,\tilde{p}]$ and $m \in [1,\tilde{m}]$.

Next, we show that $\partial_m \tilde{g}_{2,k,\theta}(p,m)$ is uniformly bounded. By the chain rule for derivatives,

$$\begin{aligned}
\frac{\partial \tilde{g}_{2,k,\theta}(p,m)}{\partial m} &= \frac{\partial \hat{\varphi}_p(k, m\mathrm{e}^{\mathrm{i}\theta})}{\partial m} \frac{1}{\hat{S}_{\mu_p(m\mathrm{e}^{\mathrm{i}\theta})}(k)} - \frac{\hat{\varphi}_p(k, m\mathrm{e}^{\mathrm{i}\theta})}{\hat{S}_{\mu_p(m\mathrm{e}^{\mathrm{i}\theta})}(k)^2} \frac{\partial \hat{S}_\mu(k)}{\partial \mu}\bigg|_{\mu=\mu_p(m\mathrm{e}^{\mathrm{i}\theta})} \frac{\partial \mu_p(m\mathrm{e}^{\mathrm{i}\theta})}{\partial m} \\
&= \frac{\partial \hat{\varphi}_p(k, m\mathrm{e}^{\mathrm{i}\theta})}{\partial m} \frac{1}{\hat{S}_{\mu_p(m\mathrm{e}^{\mathrm{i}\theta})}(k)} \qquad (3.2.6)\\
&\quad - \frac{\hat{\varphi}_p(k, m\mathrm{e}^{\mathrm{i}\theta})}{\hat{S}_{\mu_p(m\mathrm{e}^{\mathrm{i}\theta})}(k)^2} \cdot \hat{D}(k)\hat{S}_{\mu_p(m\mathrm{e}^{\mathrm{i}\theta})}(k)^2 \cdot \frac{\mathrm{e}^{\mathrm{i}\theta}}{\hat{\varphi}_p(0,m)^2} \frac{\partial \hat{\varphi}_p(0,m)}{\partial m}. \qquad (3.2.7)
\end{aligned}$$

By Lemma 3.2.1,

$$\left| \frac{\partial \hat{\varphi}_p(k, m\mathrm{e}^{\mathrm{i}\theta})}{\partial m} \right| \le \sum_{(x,t)} \varphi_p(x,t) m^{t-1} t \le \sum_{(x,t)} \big(q_p \star \varphi_p^{\star 2}\big)(x,t) m^{t-1} = \hat{q}_p(0,1)\hat{\varphi}_p(0,m)^2 = p\hat{\varphi}_p(0,m)^2 \qquad (3.2.8)$$

for every $k \in \mathbb{T}^d$ and $\theta \in \mathbb{T}$. Applying the triangle inequality, the above inequality and (3.2.2) to (3.2.7), we arrive in

$$\left| \frac{\partial \tilde{g}_{2,k,\theta}(p,m)}{\partial m} \right| \le p\hat{\varphi}_p(0,m)(2\hat{\varphi}_p(0,m)+1),$$

which is finite for $p \in [0, \tilde{p}]$ and $m \in [1, \tilde{m}]$.

Finally, we show that $\partial_p \tilde{g}_{2,k,\theta}(p,m)$ is uniformly bounded. As with the derivative with respect to m,

$$\begin{aligned}
\frac{\partial \tilde{g}_{2,k,\theta}(p,1)}{\partial p} &= \frac{\partial \hat{\varphi}_p(k, \mathrm{e}^{\mathrm{i}\theta})}{\partial p} \frac{1}{\hat{S}_{\mu_p(\mathrm{e}^{\mathrm{i}\theta})}(k)} - \frac{\hat{\varphi}_p(k, \mathrm{e}^{\mathrm{i}\theta})}{\hat{S}_{\mu_p(\mathrm{e}^{\mathrm{i}\theta})}(k)^2} \frac{\partial \hat{S}_\mu(k)}{\partial \mu}\bigg|_{\mu=\mu_p(\mathrm{e}^{\mathrm{i}\theta})} \frac{\partial \mu_p(\mathrm{e}^{\mathrm{i}\theta})}{\partial p} \\
&= \frac{\partial \hat{\varphi}_p(k, \mathrm{e}^{\mathrm{i}\theta})}{\partial p} \frac{1}{\hat{S}_{\mu_p(\mathrm{e}^{\mathrm{i}\theta})}(k)} - \mathrm{e}^{\mathrm{i}\theta}\hat{D}(k) \frac{\hat{\varphi}_p(k, \mathrm{e}^{\mathrm{i}\theta})}{\hat{\varphi}_p(0,1)^2} \frac{\partial \hat{\varphi}_p(0,1)}{\partial p}. \qquad (3.2.9)
\end{aligned}$$

By Russo's formula [36][1] ($\mathtt{piv}(\boldsymbol{x},\boldsymbol{y})$ and $\tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{x})$ are defined in Definition 3.3.2),

$$\begin{aligned}
\left| \frac{\partial \hat{\varphi}_p(k, \mathrm{e}^{\mathrm{i}\theta})}{\partial p} \right| &= \left| \sum_{(x,t)} \sum_{\boldsymbol{b}} q_1(\boldsymbol{b}) \mathbb{P}_p\Big(\boldsymbol{b} \in \mathtt{piv}\big((o,0),(x,t)\big)\Big) \mathrm{e}^{\mathrm{i}k\cdot x}\mathrm{e}^{\mathrm{i}\theta t} \right| \\
&\le \sum_{\boldsymbol{x}} \sum_{\boldsymbol{b}} q_1(\boldsymbol{b}) \mathbb{P}_p\Big(\{\, \boldsymbol{o} \to \underline{\boldsymbol{b}} \,\} \cap \Big\{\, \overline{\boldsymbol{b}} \to \boldsymbol{x} \text{ in } \tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{o})^{\mathrm{c}} \,\Big\}\Big) \\
&\le \sum_{\boldsymbol{x}} \sum_{\boldsymbol{b}} q_1(\boldsymbol{b}) \varphi_p(\underline{\boldsymbol{b}}) \varphi_p(\boldsymbol{x} - \overline{\boldsymbol{b}}) \\
&= \hat{q}_1(0,1)\hat{\varphi}_p(0,1)^2 = \chi_p^2
\end{aligned}$$

for every $k \in \mathbb{T}^d$ and $\theta \in \mathbb{T}$, where $q_p(\boldsymbol{b})$ is the abbreviation for $q_p(\overline{\boldsymbol{b}} - \underline{\boldsymbol{b}})$. Therefore, the absolute value of (3.2.9) is bounded above as

$$\left| \frac{\partial \tilde{g}_{2,k,\theta}(p,1)}{\partial p} \right| \le \chi_p(2\chi_p + 1),$$

which is finite for $p \in [0, \tilde{p}]$ and $m \in [1, \tilde{m}]$. $\square$

[1] It is more useful for readers to see [4, (3.10)] rather than [36] due to our notation of the occupation probability.

Proof of (ii). First, we show the boundedness of $\tilde{g}_{3,k,l,\theta}(p,m)$. By the telescopic inequality (1.3.1),

$$
\begin{aligned}
&\Big|\hat{\Delta}_k\Big(\hat{q}_p(l,m\mathrm{e}^{\mathrm{i}\theta})\hat{\varphi}_p(l,m\mathrm{e}^{\mathrm{i}\theta})\Big)\Big| \\
&\le \sum_{(x,t)} (q_p \star \varphi_p)(x,t)(1-\cos k\cdot x)m^t \\
&\le 2\sum_{\substack{(x,t)\\(y,s)}} q_p(y,s)\varphi_p(x-y,t-s)\Big((1-\cos k\cdot y)+\big(1-\cos k\cdot(x-y)\big)\Big)m^s m^{t-s} \qquad (3.2.10)\\
&\le pm\Big(\big(1-\hat{D}(k)\big)\hat{\varphi}_p(0,m)+\Big|\hat{\Delta}_k\hat{\varphi}_p(0,m)\Big|\Big), \qquad (3.2.11)
\end{aligned}
$$

which is finite for $p\in[0,\tilde{p}]$ and $m\in[1,\tilde{m}]$ due to (3.2.4). By (3.2.3), we conclude that $\tilde{g}_{3,k,l,\theta}(p,m)$ is bounded.

Next, we show that $\partial_m\tilde{g}_{3,k,l,\theta}(p,m)$ is bounded. By the chain rule for derivatives,

$$
\begin{aligned}
\frac{\partial\tilde{g}_{3,k,l,\theta}(p,m)}{\partial m} &= \frac{1}{\hat{U}_{\mu_p(m\mathrm{e}^{\mathrm{i}\theta})}(k,l)}\hat{\Delta}_k\Big(\frac{\partial\hat{q}_p(l,m\mathrm{e}^{\mathrm{i}\theta})}{\partial m}\cdot\hat{\varphi}_p(l,m\mathrm{e}^{\mathrm{i}\theta})+\hat{q}_p(l,m\mathrm{e}^{\mathrm{i}\theta})\cdot\frac{\partial\hat{\varphi}_p(l,m\mathrm{e}^{\mathrm{i}\theta})}{\partial m}\Big)\\
&\quad-\hat{\Delta}_k\Big(\hat{q}_p(l,m\mathrm{e}^{\mathrm{i}\theta})\hat{\varphi}_p(l,m\mathrm{e}^{\mathrm{i}\theta})\Big)\frac{1}{\hat{U}_{\mu_p(m\mathrm{e}^{\mathrm{i}\theta})}(k,l)^2}\,\frac{\partial\hat{U}_\mu(k,l)}{\partial\mu}\bigg|_{\mu=\mu_p(m\mathrm{e}^{\mathrm{i}\theta})}\frac{\partial\mu_p(m\mathrm{e}^{\mathrm{i}\theta})}{\partial m}\\
&=\frac{1}{\hat{U}_{\mu_p(m\mathrm{e}^{\mathrm{i}\theta})}(k,l)}\hat{\Delta}_k\Big(\hat{q}_p(l,\mathrm{e}^{\mathrm{i}\theta})\varphi_p(l,m\mathrm{e}^{\mathrm{i}\theta})\Big)\\
&\quad+\frac{1}{\hat{U}_{\mu_p(m\mathrm{e}^{\mathrm{i}\theta})}(k,l)}\hat{\Delta}_k\Big(\hat{q}_p(l,m\mathrm{e}^{\mathrm{i}\theta})\frac{\partial\varphi_p(l,m\mathrm{e}^{\mathrm{i}\theta})}{\partial m}\Big)\\
&\quad-\frac{\tilde{g}_{3,k,l,\theta}(p,m)}{\hat{U}_{\mu_p(m\mathrm{e}^{\mathrm{i}\theta})}(k,l)}\,\frac{\partial\hat{U}_\mu(k,l)}{\partial\mu}\bigg|_{\mu=\mu_p(m\mathrm{e}^{\mathrm{i}\theta})}\frac{\mathrm{e}^{\mathrm{i}\theta}}{\hat{\varphi}_p(0,m)^2}\frac{\partial\hat{\varphi}_p(0,m)}{\partial m}. \qquad (3.2.12)
\end{aligned}
$$

By (3.2.3) and (3.2.11), the first term in (3.2.12) is trivial. In the second term in (3.2.12), we use an upper bound

$$
\begin{aligned}
&\Big|\hat{\Delta}_k\Big(\hat{q}_p(l,m\mathrm{e}^{\mathrm{i}\theta})\frac{\partial\hat{\varphi}_p(l,m\mathrm{e}^{\mathrm{i}\theta})}{\partial m}\Big)\Big|\\
&\le\sum_{\substack{(x,t)\\(y,s)}} q_p(y,s)(t-s)\varphi_p(x-y,t-s)(1-\cos k\cdot x)m^{t-1}\\
&\le 2\sum_{\substack{(x,t)\\(y,s)}} q_p(y,s)(t-s)\varphi_p(x-y,t-s)\Big((1-\cos k\cdot y)+\big(1-\cos k\cdot(x-y)\big)\Big)m^{t-1}\\
&=2\Big(-\hat{\Delta}_k\hat{q}_p(0,1)\sum_{(x,t)}\underbrace{t\varphi_p(x,t)}_{\le(q_p\star\varphi_p^{\star2})(x,t)}m^t+\hat{q}_p(0,1)\sum_{(x,t)}t\varphi_p(x,t)(1-\cos k\cdot x)m^t\Big).
\end{aligned}
$$

As in (3.2.10),

$$
\begin{aligned}
&\Big|\hat{\Delta}_k\Big(\hat{q}_p(l,m\mathrm{e}^{\mathrm{i}\theta})\frac{\partial\hat{\varphi}_p(l,m\mathrm{e}^{\mathrm{i}\theta})}{\partial m}\Big)\Big|\\
&\le 2\Big(-\hat{\Delta}_k\hat{q}_p(0,1)\hat{q}_p(0,m)\hat{\varphi}_p(0,m)^2
\end{aligned}
$$

$$+ 3\hat{q}_p(0,1)\Big(-\hat{\varphi}_p(0,m)^2\hat{\Delta}_k\hat{q}_p(0,m) - 2\hat{q}_p(0,m)\hat{\varphi}_p(0,m)\hat{\Delta}_k\hat{\varphi}_p(0,m)\Big)\Bigg)$$
$$= 4p^2m\Big(2\big(1-\hat{D}(k)\big)\hat{\varphi}_p(0,m)^2 + 3\hat{\varphi}_p(0,m)\Big|\hat{\Delta}_k\hat{\varphi}_p(0,m)\Big|\Big),$$

which is finite for $p \in [0,\tilde{p}]$ and $m \in [1,\tilde{m}]$ due to (3.2.4). Noticing that $|\mathrm{d}|f(z)|/\mathrm{d}z| \le |\mathrm{d}f(z)/\mathrm{d}z|$ holds[2] for every $z \in \mathbb{C}$ and function $f\colon \mathbb{C} \to \mathbb{C}$, and combining the bounds (3.2.11), (3.2.2), (3.2.3) and (3.2.8), the third term in (3.2.12) is bounded. Therefore, the absolute value of (3.2.12) is finite for $p \in [0,\tilde{p}]$ and $m \in [1,\tilde{m}]$.

By the almost identical calculation for $\partial_m\tilde{g}_{3,k,l,\theta}(p,m)$, we can show that $\partial_p\tilde{g}_{3,k,l,\theta}(p,m)$ is bounded straightforwardly. We omit its proof. □

3.3 Derivation of the lace expansion

In the literature, there are three different ways to derive the lace expansion for oriented percolation. The first is based on an algebraic approach using the Markov property [33], the second is to use inclusion-exclusion relations and nested expectations [21], the third is to use inclusion-exclusion relations and the Markov property [37]. In this thesis, we use the third approach and its representations of $\{\pi_p^{(N)}\}_{N=0}^\infty$.

The lace expansion gives a similar recursion equation for the two-point function φ_p to that for the random-walk two-point function Q_p, which is

$$Q_p(\boldsymbol{x}) = \delta_{\boldsymbol{o},\boldsymbol{x}} + (q_p \star Q_p)(\boldsymbol{x})$$

for $\boldsymbol{x} = (x,t) \in \mathbb{L}^d \times \mathbb{Z}_+$, where $\delta_{\boldsymbol{o},\boldsymbol{x}} = \delta_{o,x}\delta_{o,t}$ (cf., the recursion equation for the random-walk Green function S_p).

Proposition 3.3.1 ([33])**.** For any $p < p_\mathrm{c}$ and $N \in \mathbb{Z}_+$, there exist model-dependent nonnegative functions $\{\pi_p^{(n)}\}_{n=0}^N$ on $\mathbb{L}^d \times \mathbb{Z}_+$ such that, if we define $\Pi_p^{(N)}$ as

$$\Pi_p^{(N)}(\boldsymbol{x}) = \sum_{n=0}^N (-1)^n \pi_p^{(n)}(\boldsymbol{x}),$$

we obtain the recursion equation

$$\varphi_p(\boldsymbol{x}) = \delta_{o,\boldsymbol{x}} + \Pi_p^{(N)}(\boldsymbol{x}) + \Big(\big(\delta_{\boldsymbol{o},\bullet} + \Pi_p^{(N)}\big) \star q_p \star \varphi_p\Big)(\boldsymbol{x}) + (-1)^{N+1}R_p^{(N+1)}(\boldsymbol{x}), \tag{3.3.1}$$

where the remainder $R_p^{(N+1)}(\boldsymbol{x})$ obeys the bound

$$0 \le R_p^{(N+1)}(\boldsymbol{x}) \le \Big(\pi_p^{(N)} \star \varphi_p\Big)(\boldsymbol{x}).$$

Assume that $\lim_{N\to\infty}\pi_p^{(N)}(\boldsymbol{x}) = 0$ for every $\boldsymbol{x} \in \mathbb{L}^d \times \mathbb{Z}_+$, in fact, which holds by the absolute convergence of the alternative series of $\{\pi_p^{(N)}\}_{N=1}^\infty$ from Lemma 3.4.1 below. Then, $R_p^{(N)}(\boldsymbol{x}) \to 0$ as $N \to \infty$ for every $\boldsymbol{x} \in \mathbb{L}^d \times \mathbb{Z}_+$, so that the limit of the series

$$\Pi_p(\boldsymbol{x}) \equiv \lim_{N\to\infty}\Pi_p^{(N)}(\boldsymbol{x}) = \sum_{N=0}^\infty (-1)^N \pi_p^{(N)}(\boldsymbol{x}) \tag{3.3.2}$$

[2]
$$\left|\frac{\mathrm{d}|f(z)|}{\mathrm{d}z}\right| = \left|\frac{\mathrm{d}}{\mathrm{d}z}\sqrt{f(z)^*f(z)}\right| = \left|\frac{1}{2|f(z)|}\left(\frac{\mathrm{d}f(z)^*}{\mathrm{d}z}\cdot f(z) + f(z)^*\cdot\frac{\mathrm{d}f(z)}{\mathrm{d}z}\right)\right| \le \left|\frac{\mathrm{d}f(z)}{\mathrm{d}z}\right|.$$

is well-defined.

To define the lace expansion coefficients $\{\pi_p^{(N)}\}_{N=0}^{\infty}$, we introduce some notions and notation. Here, $\underline{\boldsymbol{b}}$ and $\overline{\boldsymbol{b}}$ denote the bottom and top of a bond $\boldsymbol{b}$, respectively, that is $\boldsymbol{b}=(\underline{\boldsymbol{b}},\overline{\boldsymbol{b}})$.

Definition 3.3.2. Fix a bond configuration and let $\boldsymbol{x},\boldsymbol{y},\boldsymbol{u},\boldsymbol{v}\in\mathbb{L}^d\times\mathbb{Z}_+$.

(i) Given a bond $\boldsymbol{b}$, we define $\tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{x})$ to be the set of vertices connected to $\boldsymbol{x}$ in the new configuration obtained by setting $\boldsymbol{b}$ to be vacant.

(ii) We say that a bond $(\boldsymbol{u},\boldsymbol{v})$ is pivotal for $\boldsymbol{x}\to\boldsymbol{y}$ if $\boldsymbol{x}\to\boldsymbol{u}$ occurs in $\tilde{\mathcal{C}}^{(\boldsymbol{u},\boldsymbol{v})}(\boldsymbol{x})$ (i.e., $\boldsymbol{x}$ is connected to $\boldsymbol{u}$ without using $(\boldsymbol{u},\boldsymbol{v})$) and if $\boldsymbol{v}\to\boldsymbol{y}$ occurs in the complement of $\tilde{\mathcal{C}}^{(\boldsymbol{u},\boldsymbol{v})}(\boldsymbol{x})$, denoted by $\tilde{\mathcal{C}}^{(\boldsymbol{u},\boldsymbol{v})}(\boldsymbol{x})^{\mathrm{c}}$. Let $\mathtt{piv}(\boldsymbol{x},\boldsymbol{y})$ be the set of pivotal bonds for the connection from $\boldsymbol{x}$ to $\boldsymbol{y}$.

(iii) We say that $\boldsymbol{x}$ is doubly connected to $\boldsymbol{y}$, denoted by $\boldsymbol{x}\Rightarrow\boldsymbol{y}$, if either $\boldsymbol{x}=\boldsymbol{y}$ or $\boldsymbol{x}\to\boldsymbol{y}$ and $\mathtt{piv}(\boldsymbol{x},\boldsymbol{y})=\varnothing$.

Sketch proof of Proposition 3.3.1. First, we derive the first expansion, i.e., (3.3.1) for $N=0$. By splitting the event $\{\,\boldsymbol{o}\to\boldsymbol{x}\,\}$ into two depending on whether or not there is a pivotal bond for the connection from $\boldsymbol{o}$ to $\boldsymbol{x}$, we obtain

$$\varphi_p(\boldsymbol{x})=\mathbb{P}_p\big(\underbrace{\boldsymbol{o}\to\boldsymbol{x},\ \mathtt{piv}(\boldsymbol{o},\boldsymbol{x})=\varnothing}_{=\{\,\boldsymbol{o}\Rightarrow\boldsymbol{x}\,\}}\big)+\mathbb{P}_p\big(\boldsymbol{o}\to\boldsymbol{x},\ \mathtt{piv}(\boldsymbol{o},\boldsymbol{x})\neq\varnothing\big).$$

Let

$$\pi_p^{(0)}(x)=\mathbb{P}_p(\boldsymbol{o}\Rightarrow\boldsymbol{x})-\delta_{\boldsymbol{o},\boldsymbol{x}}.$$

Then, by definition, the first term in (3.3) is $\delta_{o,x}+\pi_p^{(0)}(x)$. To expand the second term in (3.3), we use the first pivotal bond $\boldsymbol{b}\equiv(\underline{\boldsymbol{b}},\overline{\boldsymbol{b}})$ for $\boldsymbol{o}\to\boldsymbol{x}$, so that

$$\begin{aligned}&\mathbb{P}_p\big(\boldsymbol{o}\to\boldsymbol{x},\ \mathtt{piv}(\boldsymbol{o},\boldsymbol{x})\neq\varnothing\big)\\&=\sum_{\boldsymbol{b}}\mathbb{P}_p\Big(\boldsymbol{o}\Rightarrow\underline{\boldsymbol{b}},\ \boldsymbol{b}\text{ occupied},\ \overline{\boldsymbol{b}}\to\boldsymbol{x}\notin\tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{o})\Big)\\&\equiv\sum_{\boldsymbol{b}}\mathbb{P}_p\Big(\boldsymbol{o}\Rightarrow\boldsymbol{b}\to\boldsymbol{x}\notin\tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{o})\Big).\end{aligned}$$

In the last line, we introduce the abbreviation for the event $\{\cdots\to\underline{\boldsymbol{b}},\ \boldsymbol{b}\text{ occupied},\ \overline{\boldsymbol{b}}\to\cdots\}$. By the inclusion-exclusion relation $\{\,\overline{\boldsymbol{b}}\to\boldsymbol{x}\notin\tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{o})\,\}=\{\,\overline{\boldsymbol{b}}\to\boldsymbol{x}\,\}\setminus\{\,\overline{\boldsymbol{b}}\to\boldsymbol{x}\in\tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{o})\,\}$ and the Markov property, we complete the first expansion as

$$\begin{aligned}\varphi_p(\boldsymbol{x})&=\delta_{\boldsymbol{o},\boldsymbol{x}}+\pi_p^{(0)}(x)+\sum_{\boldsymbol{b}}\mathbb{P}_p(\{\,\boldsymbol{o}\Rightarrow\boldsymbol{b}\to\boldsymbol{x}\,\})-\underbrace{\sum_{\boldsymbol{b}}\mathbb{P}_p\Big(\Big\{\,\boldsymbol{o}\Rightarrow\boldsymbol{b}\to\boldsymbol{x}\in\tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{o})\,\Big\}\Big)}_{\equiv R_p^{(1)}(\boldsymbol{x})}\\&=\delta_{\boldsymbol{o},\boldsymbol{x}}+\pi_p^{(0)}(x)+\Big(\big(\delta_{\boldsymbol{o},\bullet}+\pi_p^{(0)}\big)\star q_p\star\varphi_p\Big)(\boldsymbol{x})-R_p^{(1)}(\boldsymbol{x}).\end{aligned}\tag{3.3.3}$$

Next, we expand the remainder $R_p^{(1)}(x)$ to derive the second expansion, i.e., (3.3.1) for $N=1$. To do so, and to derive the higher-order expansion later, we have to deal with the event $\{\,\boldsymbol{b}\to\boldsymbol{x}\in A\,\}$ for some bond $\boldsymbol{b}$, some vertex $\boldsymbol{x}$ and a vertex set A. Let

$$E(\boldsymbol{b},\boldsymbol{x};A)=\{\,\boldsymbol{b}\to\boldsymbol{x}\in A\,\}\setminus\bigcup_{\boldsymbol{b}'\in\mathtt{piv}(\overline{\boldsymbol{b}},\boldsymbol{x})}\{\,\underline{\boldsymbol{b}'}\in A\,\}.$$

Intuitively, if we regard a percolation cluster of $\underline{\boldsymbol{b}}$ containing $\boldsymbol{x}$ as a string of sausages from $\underline{\boldsymbol{b}}$ to $\boldsymbol{x}$, then $E(\boldsymbol{b},\boldsymbol{x};A)$ is considered to be the event that the last sausage is the first one that goes through A. Then, we can split the event $\{\, \boldsymbol{b} \to \boldsymbol{x} \in A \,\}$ into two disjoint events as

$$\begin{aligned}\{\, \boldsymbol{b} \to \boldsymbol{x} \in A \,\} &= E(\boldsymbol{b},\boldsymbol{x};A) \sqcup \bigcup_{\boldsymbol{b}'} \{\{\, \boldsymbol{b} \to \boldsymbol{x} \in A \,\} \cap \{\, \underline{\boldsymbol{b}} \in A \,\} \cap \{\, \boldsymbol{b}' \in \mathtt{piv}(\overline{\boldsymbol{b}},\boldsymbol{x}) \,\}\} \\ &= E(\boldsymbol{b},\boldsymbol{x};A) \sqcup \bigcup_{\boldsymbol{b}'} \Big\{ E(\boldsymbol{b},\underline{\boldsymbol{b}'};A) \cap \Big\{\, \boldsymbol{b}' \to \boldsymbol{x} \notin \tilde{\mathcal{C}}^{\boldsymbol{b}'}(\overline{\boldsymbol{b}}) \,\Big\}\Big\}.\end{aligned} \tag{3.3.4}$$

Let

$$\pi_p^{(1)}(\boldsymbol{x}) = \sum_{\boldsymbol{b}} \mathbb{P}_p\Big(\{\, \boldsymbol{o} \Rightarrow \underline{\boldsymbol{b}} \,\} \cap E\Big(\boldsymbol{b},\boldsymbol{x};\tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{o})\Big)\Big),$$

so that we have

$$R_p^{(1)}(\boldsymbol{x}) = \pi_p^{(1)}(\boldsymbol{x}) + \sum_{\boldsymbol{b}_1,\boldsymbol{b}_2} \mathbb{P}_p\Big(\{\, \boldsymbol{o} \Rightarrow \underline{\boldsymbol{b}_1} \,\} \cap E(\boldsymbol{b}_1,\underline{\boldsymbol{b}_2};A) \cap \Big\{\, \boldsymbol{b}_2 \to \boldsymbol{x} \notin \tilde{\mathcal{C}}^{\boldsymbol{b}_2}(\overline{\boldsymbol{b}_1}) \,\Big\}\Big).$$

By the inclusion-exclusion relation $\{\, \overline{\boldsymbol{b}_2} \to \boldsymbol{x} \notin \tilde{\mathcal{C}}^{\boldsymbol{b}_2}(\overline{\boldsymbol{b}_1}) \,\} = \{\, \overline{\boldsymbol{b}_2} \to \boldsymbol{x} \,\} \setminus \{\, \overline{\boldsymbol{b}_2} \to \boldsymbol{x} \in \tilde{\mathcal{C}}^{\boldsymbol{b}_2}(\overline{\boldsymbol{b}_1}) \,\}$ and the Markov property, we arrive at the second expansion

$$\begin{aligned}R_p^{(1)}(\boldsymbol{x}) = \pi_p^{(1)}(\boldsymbol{x}) &+ \Big(\pi_p^{(1)} \star q_p \star \varphi_p\Big)(\boldsymbol{x}) \\ &- \underbrace{\sum_{\boldsymbol{b}_1,\boldsymbol{b}_2} \mathbb{P}_p\Big(\{\, \boldsymbol{o} \Rightarrow \underline{\boldsymbol{b}_1} \,\} \cap E(\boldsymbol{b}_1,\underline{\boldsymbol{b}_2};A) \cap \Big\{\, \boldsymbol{b}_2 \to \boldsymbol{x} \in \tilde{\mathcal{C}}^{\boldsymbol{b}_2}(\overline{\boldsymbol{b}_1}) \,\Big\}\Big)}_{\equiv R_p^{(2)}(\boldsymbol{x})}.\end{aligned}$$

By repeated applications of inclusion-exclusion and the Markov property to the remainders, we can derive the higher-order expansion coefficients. For $N \in \mathbb{Z}_+$, let

$$\tilde{E}_{\vec{\boldsymbol{b}}_N}^{(N)}(\boldsymbol{x}) = \{\, \boldsymbol{o} \Rightarrow \underline{\boldsymbol{b}_1} \,\} \cap \bigcap_{i=1}^{N} E\Big(\boldsymbol{b}_i,\underline{\boldsymbol{b}_{i+1}};\tilde{\mathcal{C}}^{\boldsymbol{b}_i}(\overline{\boldsymbol{b}_{i-1}})\Big),$$

where $\vec{\boldsymbol{b}}_N = (\boldsymbol{b}_1,\dots,\boldsymbol{b}_N)$ is an ordered set of bonds on $\mathbb{L}^d \times \mathbb{Z}_+$, $\underline{\boldsymbol{b}_0} = \boldsymbol{o}$ and $\overline{\boldsymbol{b}_{N+1}} = \boldsymbol{x}$. Then, the higher-order expansion coefficient is defined as

$$\pi_p^{(N)}(\boldsymbol{x}) = \sum_{\vec{\boldsymbol{b}}_N} \mathbb{P}_p\Big(\tilde{E}_{\vec{\boldsymbol{b}}_N}^{(N)}(\boldsymbol{x})\Big),$$

and the higher-order remainder term is defined as

$$R_p^{(N+1)}(\boldsymbol{x}) = \sum_{\vec{\boldsymbol{b}}_{N+1}} \mathbb{P}_p\Big(\tilde{E}_{\vec{\boldsymbol{b}}_N}^{(N)}(\underline{\boldsymbol{b}_{N+1}}) \cap \Big\{\, \boldsymbol{b}_{N+1} \to \boldsymbol{x} \in \tilde{\mathcal{C}}^{\boldsymbol{b}_{N+1}}(\overline{\boldsymbol{b}_N}) \,\Big\}\Big).$$

We complete the sketch proof of Proposition 3.3.1. □

3.4 Diagrammatic bounds on the expansion coefficients

3.4.1 Overview

We state upper bounds on the Fourier-Laplace transform of the sum (3.3.2) of the alternative series of the lace expansion coefficients in terms of basic diagrams. Let $\varphi_p^{(m)}(x,t) = m^t \varphi_p(x,t)$. Recall the notation $\boldsymbol{x} = (x, \tau(\boldsymbol{x})) \in \mathbb{L}^d \times \mathbb{Z}_+$. We define basic diagrams as, for $\lambda, \rho \in \mathbb{N}$, $m > 0$ and $k \in \mathbb{T}^d$,

$$B_{p,m}^{(\lambda,\rho)} \equiv \sup_{\boldsymbol{x}} \sum_{\boldsymbol{y}} \Big(q_p^{\star\lambda} \star \varphi_p\Big)(\boldsymbol{y})\Big(m^\rho q_p^{\star\rho} \star \varphi_p^{(m)}\Big)(\boldsymbol{y}-\boldsymbol{x}),$$

$$T_{p,m}^{(\lambda,\rho)} \equiv \sup_{\boldsymbol{x}} \sum_{\boldsymbol{y}} \Big(q_p^{\star\lambda} \star \varphi_p^{\star 2}\Big)(\boldsymbol{y})\Big(m^\rho q_p^{\star\rho} \star \varphi_p^{(m)}\Big)(\boldsymbol{y}-\boldsymbol{x}),$$

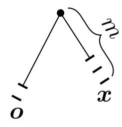
$$\hat{V}_{p,m}^{(\lambda,\rho)}(k) \equiv \sup_{\boldsymbol{x}} \sum_{\boldsymbol{y}} \Big(q_p^{\star\lambda} \star \varphi_p\Big)(\boldsymbol{y})(1-\cos k\cdot y)\Big(m^\rho q_p^{\star\rho} \star \varphi_p^{(m)}\Big)(\boldsymbol{y}-\boldsymbol{x}).$$

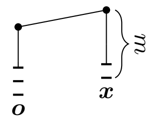
As in Section 2.4, we represent the transition probability q_p by a pair of either parallel lines or a line and a dot, and we represent the two-point function φ_p by a line segment. For example,

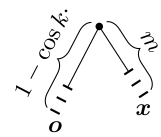
$\varphi_p(\boldsymbol{x}) =$ 

, $\quad (q_p^{\star 2} \star \varphi_p)(\boldsymbol{x}) =$ 

,

$B_{p,m}^{(1,2)} = \sup_{\boldsymbol{x}}$ 

, $\quad T_{p,m}^{(2,1)} = \sup_{\boldsymbol{x}}$ 

, $\quad \hat{V}_{p,m}^{(2,2)}(k) = \sup_{\boldsymbol{x}}$ 

,

where the unlabeled vertices (short lines and dots) are summed over $\mathbb{L}^d \times \mathbb{Z}_+$. The segments emphasized by the braces mean weighted two-point functions or weighted space-time transition probabilities: φ_p or q_p multiplied by m's or $1 - \cos k \cdot \bullet$. Time increases from the beginning point to the ending point of a line. Moreover, we divide $\Pi_p(\boldsymbol{x})$ into two parts: $\Pi_p^{\mathrm{even}}(\boldsymbol{x}) = \sum_{N=1} \pi_p^{(2N)}(\boldsymbol{x})$ and $\Pi_p^{\mathrm{odd}}(\boldsymbol{x}) = \pi_p^{(1)}(\boldsymbol{x}) - \pi_p^{(0)}(\boldsymbol{x}) + \sum_{N=1} \pi_p^{(2N+1)}(\boldsymbol{x})$. Although the latter contains the zeroth lace expansion coefficient, we name it "odd" for convenience.

Lemma 3.4.1. Let $\varepsilon = \varepsilon_1^{(1)}\varepsilon_1^{(2)} \vee \varepsilon_1^{(1)}\varepsilon_2^{(2)} \vee \varepsilon_1^{(2)}\varepsilon_2^{(1)} \vee \varepsilon_2^{(1)}\varepsilon_2^{(2)} \vee (\varepsilon_1^{(1)})^3 \vee (\varepsilon_2^{(1)})^3$. Suppose that $2T_{p,m}^{(1,1)} < 1$. Then, the alternative series (3.3.2) absolutely converges, and we obtain

$$\begin{aligned}
&\hat{\Pi}_p^{\mathrm{even}}(0,m) \le \mathcal{O}(\varepsilon),\\
&\hat{\Pi}_p^{\mathrm{odd}}(0,m) \le \frac{1}{2} B_{p,m}^{(2,2)} + \mathcal{O}(\varepsilon),\\
&\sum_{(x,t)} \Big(\Pi_p^{\mathrm{even}}(x,t) + \Pi_p^{\mathrm{odd}}(x,t)\Big) m^t t \le \frac{1}{2}\Big(B_{p,m}^{(2,2)} + T_{p,m}^{(2,2)}\Big) + \mathcal{O}(\varepsilon),\\
&\sum_{(x,t)} \Big(\Pi_p^{\mathrm{even}}(x,t) + \Pi_p^{\mathrm{odd}}(x,t)\Big) m^t (1-\cos k\cdot x)\\
&\qquad \le \frac{1}{2}\hat{V}_{p,m}^{(2,2)}(k) + 2\hat{V}_{p,1}^{(2,2)}(k) B_{p,m}^{(0,2)} + \mathcal{O}(\varepsilon)\Big(1 - \hat{D}(k)\Big).
\end{aligned}$$

This lemma is a brief version of Lemma 3.4.4 in order to simplify the presentation. The proof is based on the diagrammatic representations of $\{\pi_p^{(N)}\}_{N=0}^\infty$, which is bounded above in terms of sums over products of two-point functions, and the method in [19, 39], which is splitting events of percolation connections into some events whether each line collapses or not. In particular, the coefficient $1/2$ of each leading term results from [27]. The bounds in Lemma 3.4.1 suffices for us to prove $g_i(p,m) < K_i$

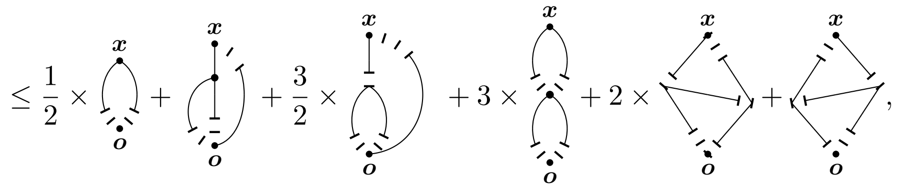
for $i = 1, 2, 3$ because $\mathcal{O}(\varepsilon)$ is less than the leading terms. The definition of ε results from isolating the leading terms, which contain at least four q_p's. Specifically, $\varepsilon = \mathcal{O}\big(10^{-3}\big)$ on $\mathbb{L}^5$, while upper bounds on both $B_{p,m}^{(2,2)}$ and $T_{p,m}^{(2,2)}$ and $V_{p,m}^{(2,2)}(k)$ in Lemma 3.6.1 are $\mathcal{O}\big(10^{-2}\big)$ and $\mathcal{O}\big(10^{-1}\big)$, respectively. In the next subsection, we show the exact expression of the upper bounds in Lemma 3.4.1.

3.4.2 Precise expressions

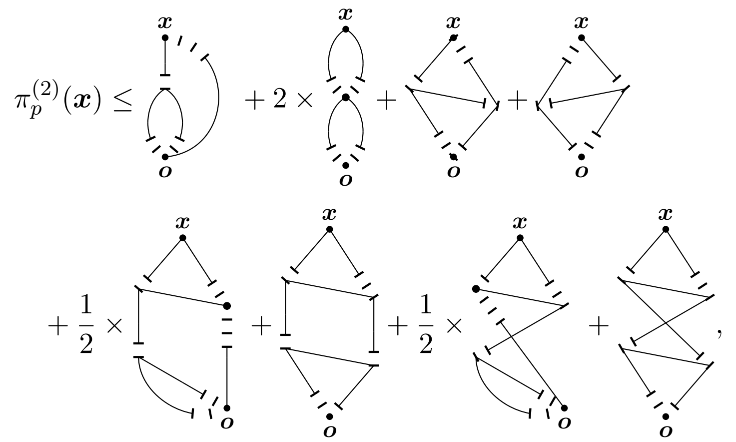
In this subsection, we prove Lemma 3.4.1. First, we bound the lace expansion coefficients $\{\pi_p^{(N)}(\boldsymbol{x})\}_{N=0}^{\infty}$ by diagrams.

Lemma 3.4.2. For $\boldsymbol{x} \in \mathbb{L}^d \times \mathbb{Z}_+$ and $N \geq 2$,

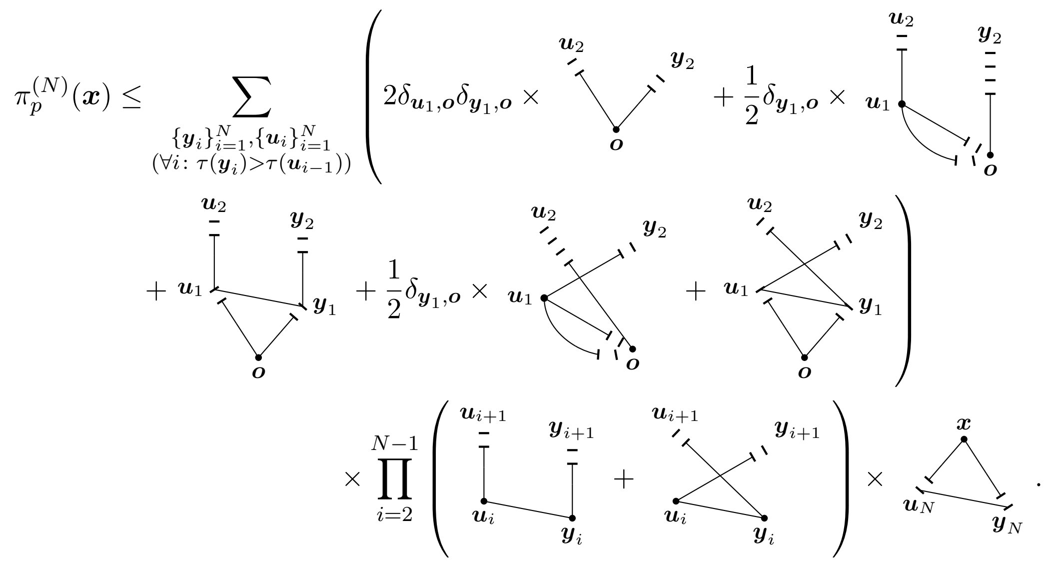
$$\Big|\pi_p^{(0)}(\boldsymbol{x}) - \pi_p^{(1)}(\boldsymbol{x})\Big|$$



(3.4.1)



(3.4.2)



(3.4.3)

Proof of (3.4.1) *in Lemma 3.4.2.* This proof is inspired by [27, Section 3.1]. First, we rewrite the event in $\pi_p^{(0)}(\boldsymbol{x})$. Recall the ordering $\prec$ for $\mathscr{B}(\boldsymbol{o})$ introduced in the proof of (3.2.4). By symmetry,

$$\begin{aligned}
\pi_p^{(0)}(\boldsymbol{x}) &= \mathbb{P}_p(\boldsymbol{o} \Rightarrow \boldsymbol{x}) - \delta_{\boldsymbol{o},\boldsymbol{x}} \\
&= \mathbb{P}_p\Bigg(\bigsqcup_{\boldsymbol{b} \in \mathscr{B}(\boldsymbol{o})} \Big\{ \{ \{ \boldsymbol{o} \to \boldsymbol{x} \} \circ \{ \boldsymbol{b} \to \boldsymbol{x} \} \} \cap \{ \forall \boldsymbol{b}' \prec \boldsymbol{b}, \boldsymbol{b}' \not\to \boldsymbol{x} \} \Big\} \Bigg) \\
&= \mathbb{P}_p\Bigg(\bigsqcup_{\boldsymbol{b} \in \mathscr{B}(\boldsymbol{o})} \Big\{ \{ \{ \boldsymbol{o} \to \boldsymbol{x} \} \circ \{ \boldsymbol{b} \to \boldsymbol{x} \} \} \cap \{ \forall \boldsymbol{b}' \succ \boldsymbol{b}, \boldsymbol{b}' \not\to \boldsymbol{x} \} \Big\} \Bigg) \\
&= \frac{1}{2} \sum_{\boldsymbol{b} \in \mathscr{B}(\boldsymbol{o})} \Big(\mathbb{P}_p\big(\{ \{ \boldsymbol{o} \to \boldsymbol{x} \} \circ \{ \boldsymbol{b} \to \boldsymbol{x} \} \} \cap \{ \forall \boldsymbol{b}' \prec \boldsymbol{b}, \boldsymbol{b}' \not\to \boldsymbol{x} \} \big) \\
&\qquad + \mathbb{P}_p\big(\{ \{ \boldsymbol{o} \to \boldsymbol{x} \} \circ \{ \boldsymbol{b} \to \boldsymbol{x} \} \} \cap \{ \forall \boldsymbol{b}' \succ \boldsymbol{b}, \boldsymbol{b}' \not\to \boldsymbol{x} \} \big) \Big).
\end{aligned} \tag{3.4.4}$$

Next, we rewrite the event in $\pi_p^{(1)}(\boldsymbol{x})$. By the definition, we can easily see that

$$E\Big(\boldsymbol{b}, \boldsymbol{x}; \tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{y})\Big) \subset \{ \boldsymbol{y} \to \boldsymbol{x} \} \circ \{ \boldsymbol{b} \to \boldsymbol{x} \}. \tag{3.4.5}$$

By splitting the event $\{ \boldsymbol{o} \Rightarrow \underline{\boldsymbol{b}} \}$ into two events based on whether $\underline{\boldsymbol{b}}$ equals $\boldsymbol{o}$ or not,

$$\begin{aligned}
\pi_p^{(1)}(\boldsymbol{x}) &= \sum_{\boldsymbol{b}} \mathbb{P}_p\Big(\{ \boldsymbol{o} \Rightarrow \underline{\boldsymbol{b}} \} \cap E(\boldsymbol{b}, \boldsymbol{x}; \tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{o})) \Big) \\
&= \sum_{\boldsymbol{b} \in \mathscr{B}(\boldsymbol{o})} \mathbb{P}_p\Big(E(\boldsymbol{b}, \boldsymbol{x}; \tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{o})) \Big) + \sum_{\boldsymbol{b}} \mathbb{P}_p\Big(\{ \boldsymbol{o} \Rightarrow \underline{\boldsymbol{b}} \neq \boldsymbol{o} \} \cap E(\boldsymbol{b}, \boldsymbol{x}; \tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{o})) \Big) \\
&= \sum_{\boldsymbol{b} \in \mathscr{B}(\boldsymbol{o})} \Big(\mathbb{P}_p\big(\{ \boldsymbol{o} \to \boldsymbol{x} \} \circ \{ \boldsymbol{b} \to \boldsymbol{x} \} \big) - \mathbb{P}_p\Big(\{ \{ \boldsymbol{o} \to \boldsymbol{x} \} \circ \{ \boldsymbol{b} \to \boldsymbol{x} \} \} \setminus E(\boldsymbol{b}, \boldsymbol{x}; \tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{o})) \Big) \Big) \\
&\quad + \sum_{\boldsymbol{b}} \mathbb{P}_p\Big(\{ \boldsymbol{o} \Rightarrow \underline{\boldsymbol{b}} \neq \boldsymbol{o} \} \cap E(\boldsymbol{b}, \boldsymbol{x}; \tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{o})) \Big).
\end{aligned} \tag{3.4.6}$$

Since, clearly, both

$$\{ \forall \boldsymbol{b}' \prec \boldsymbol{b}, \boldsymbol{b}' \not\to \boldsymbol{x} \} \sqcup \{ \exists \boldsymbol{b}'' \prec \boldsymbol{b}, \boldsymbol{b}'' \to \boldsymbol{x} \} \quad \text{and} \quad \{ \forall \boldsymbol{b}' \succ \boldsymbol{b}, \boldsymbol{b}' \not\to \boldsymbol{x} \} \sqcup \{ \exists \boldsymbol{b}'' \succ \boldsymbol{b}, \boldsymbol{b}'' \to \boldsymbol{x} \}$$

are the whole events,

$$\begin{aligned}
&\mathbb{P}_p\big(\{ \boldsymbol{o} \to \boldsymbol{x} \} \circ \{ \boldsymbol{b} \to \boldsymbol{x} \} \big) \\
&\quad = \underbrace{\mathbb{P}_p\big(\{ \{ \boldsymbol{o} \to \boldsymbol{x} \} \circ \{ \boldsymbol{b} \to \boldsymbol{x} \} \} \cap \{ \{ \forall \boldsymbol{b}' \prec \boldsymbol{b}, \boldsymbol{b}' \not\to \boldsymbol{x} \} \cap \{ \forall \boldsymbol{b}'' \succ \boldsymbol{b}, \boldsymbol{b}'' \not\to \boldsymbol{x} \} \} \big)}_{=0} \\
&\qquad + \mathbb{P}_p\big(\{ \{ \boldsymbol{o} \to \boldsymbol{x} \} \circ \{ \boldsymbol{b} \to \boldsymbol{x} \} \} \cap \{ \{ \forall \boldsymbol{b}' \prec \boldsymbol{b}, \boldsymbol{b}' \not\to \boldsymbol{x} \} \cap \{ \exists \boldsymbol{b}'' \succ \boldsymbol{b}, \boldsymbol{b}'' \to \boldsymbol{x} \} \} \big) \\
&\qquad + \mathbb{P}_p\big(\{ \{ \boldsymbol{o} \to \boldsymbol{x} \} \circ \{ \boldsymbol{b} \to \boldsymbol{x} \} \} \cap \{ \{ \forall \boldsymbol{b}' \succ \boldsymbol{b}, \boldsymbol{b}' \not\to \boldsymbol{x} \} \cap \{ \exists \boldsymbol{b}'' \prec \boldsymbol{b}, \boldsymbol{b}'' \to \boldsymbol{x} \} \} \big) \\
&\qquad + \mathbb{P}_p\big(\{ \{ \boldsymbol{o} \to \boldsymbol{x} \} \circ \{ \boldsymbol{b} \to \boldsymbol{x} \} \} \cap \{ \{ \exists \boldsymbol{b}' \prec \boldsymbol{b}, \boldsymbol{b}' \to \boldsymbol{x} \} \cap \{ \exists \boldsymbol{b}'' \succ \boldsymbol{b}, \boldsymbol{b}'' \to \boldsymbol{x} \} \} \big) \\
&\quad = \mathbb{P}_p\big(\{ \{ \boldsymbol{o} \to \boldsymbol{x} \} \circ \{ \boldsymbol{b} \to \boldsymbol{x} \} \} \cap \{ \forall \boldsymbol{b}' \prec \boldsymbol{b}, \boldsymbol{b}' \not\to \boldsymbol{x} \} \big) \\
&\qquad + \mathbb{P}_p\big(\{ \{ \boldsymbol{o} \to \boldsymbol{x} \} \circ \{ \boldsymbol{b} \to \boldsymbol{x} \} \} \cap \{ \forall \boldsymbol{b}' \succ \boldsymbol{b}, \boldsymbol{b}' \not\to \boldsymbol{x} \} \big) \\
&\qquad + \mathbb{P}_p\big(\{ \{ \boldsymbol{o} \to \boldsymbol{x} \} \circ \{ \boldsymbol{b} \to \boldsymbol{x} \} \} \cap \{ \{ \exists \boldsymbol{b}' \prec \boldsymbol{b}, \boldsymbol{b}' \to \boldsymbol{x} \} \cap \{ \exists \boldsymbol{b}'' \succ \boldsymbol{b}, \boldsymbol{b}'' \to \boldsymbol{x} \} \} \big).
\end{aligned} \tag{3.4.7}$$

Substituting (3.4.7) into (3.4.6) and subtracting (3.4.4), we obtain

$$\begin{aligned}
\pi_p^{(1)}(\boldsymbol{x}) - \pi_p^{(0)}(\boldsymbol{x}) = \underbrace{\mathbb{P}_p(\boldsymbol{o} \Rightarrow \boldsymbol{x} \neq \boldsymbol{o})}_{\text{(a)}} \\
+ \underbrace{\sum_{\boldsymbol{b} \in \mathscr{B}(\boldsymbol{o})} \mathbb{P}_p\big(\{\{\, \boldsymbol{o} \to \boldsymbol{x} \,\} \circ \{\, \boldsymbol{b} \to \boldsymbol{x} \,\}\} \cap \{\{\, \exists \boldsymbol{b}' \prec \boldsymbol{b}, \boldsymbol{b}' \to \boldsymbol{x} \,\} \cap \{\, \exists \boldsymbol{b}'' \succ \boldsymbol{b}, \boldsymbol{b}'' \to \boldsymbol{x} \,\}\}\big)}_{\text{(b)}} \\
- \underbrace{\sum_{\boldsymbol{b} \in \mathscr{B}(\boldsymbol{o})} \mathbb{P}_p\Big(\{\{\, \boldsymbol{o} \to \boldsymbol{x} \,\} \circ \{\, \boldsymbol{b} \to \boldsymbol{x} \,\}\} \setminus E(\boldsymbol{b}, \boldsymbol{x}; \tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{o}))\Big)}_{\text{(c)}} \\
+ \underbrace{\sum_{\boldsymbol{b}} \mathbb{P}_p\Big(\{\, \boldsymbol{o} \Rightarrow \underline{\boldsymbol{b}} \neq \boldsymbol{o} \,\} \cap E(\boldsymbol{b}, \boldsymbol{x}; \tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{o}))\Big)}_{\text{(d)}}. \qquad (3.4.8)
\end{aligned}$$

Finally, we lead (3.4.8) to the upper bound (3.4.1). In the following, we repeatedly use the trivial inequality (3.2.1) and the fact that, if there are two disjoint connections, then their lengths are at least two for the oriented percolation. By the Boolean and BK [7] inequalities, (a) in (3.4.8) is bounded above as

$$\begin{aligned}
&\mathbb{P}_p(\boldsymbol{o} \Rightarrow \boldsymbol{x} \neq \boldsymbol{o}) \\
&= \mathbb{P}_p\Bigg(\bigcup_{\substack{\boldsymbol{b}_1, \boldsymbol{b}_2 \in \mathscr{B}(\boldsymbol{o}) \\ (\boldsymbol{b}_1 \prec \boldsymbol{b}_2)}} \bigcup_{\substack{\boldsymbol{b}_1' \in \mathscr{B}(\overline{\boldsymbol{b}_1}), \\ \boldsymbol{b}_2' \in \mathscr{B}(\overline{\boldsymbol{b}_2})}} \{\{\, \boldsymbol{b}_1 \text{ is occupied } \& \ \boldsymbol{b}_1' \to \boldsymbol{x} \,\} \circ \{\, \boldsymbol{b}_2 \text{ is occupied } \& \ \boldsymbol{b}_2' \to \boldsymbol{x} \,\}\}\Bigg) \\
&\leq \sum_{\substack{\boldsymbol{b}_1, \boldsymbol{b}_2 \in \mathscr{B}(\boldsymbol{o}) \\ (\boldsymbol{b}_1 \prec \boldsymbol{b}_2)}} \sum_{\substack{\boldsymbol{b}_1' \in \mathscr{B}(\overline{\boldsymbol{b}_1}), \\ \boldsymbol{b}_2' \in \mathscr{B}(\overline{\boldsymbol{b}_2})}} q_p(\boldsymbol{b}_1) q_p(\boldsymbol{b}_1') \varphi_p(\boldsymbol{x} - \overline{\boldsymbol{b}_1'}) \cdot q_p(\boldsymbol{b}_2) q_p(\boldsymbol{b}_2') \varphi_p(\boldsymbol{x} - \overline{\boldsymbol{b}_2'}) \\
&\leq \frac{1}{2} \big(q_p^{\star 2} \star \varphi_p\big)(\boldsymbol{x})^2,
\end{aligned}$$

which corresponds to the 1st term in (3.4.1). The factor 1/2 in the last line is due to ignoring the ordering. To bound (b) in (3.4.8), we note that, for $\boldsymbol{b} \in \mathscr{B}(\boldsymbol{o})$,

$$\begin{aligned}
&\{\{\, \boldsymbol{o} \to \boldsymbol{x} \,\} \circ \{\, \boldsymbol{b} \to \boldsymbol{x} \,\}\} \cap \{\{\, \exists \boldsymbol{b}' \prec \boldsymbol{b}, \boldsymbol{b}' \to \boldsymbol{x} \,\} \cap \{\, \exists \boldsymbol{b}'' \succ \boldsymbol{b}, \boldsymbol{b}'' \to \boldsymbol{x} \,\}\} \\
&\qquad \subset \bigcup_{\boldsymbol{b}', \boldsymbol{b}'' \in \mathscr{B}(\boldsymbol{o})} \bigcup_{\boldsymbol{y}} \{\{\, \boldsymbol{b} \to \boldsymbol{y} \to \boldsymbol{x} \,\} \circ \{\, \boldsymbol{b}' \to \boldsymbol{x} \,\} \circ \{\, \boldsymbol{b}'' \to \boldsymbol{y} \,\}\}.
\end{aligned}$$

By the Boolean and BK inequalities, we have

$$\begin{aligned}
&\sum_{\boldsymbol{b} \in \mathscr{B}(\boldsymbol{o})} \mathbb{P}_p\big(\{\{\, \boldsymbol{o} \to \boldsymbol{x} \,\} \circ \{\, \boldsymbol{b} \to \boldsymbol{x} \,\}\} \cap \{\{\, \exists \boldsymbol{b}' \prec \boldsymbol{b}, \boldsymbol{b}' \to \boldsymbol{x} \,\} \cap \{\, \exists \boldsymbol{b}'' \succ \boldsymbol{b}, \boldsymbol{b}'' \to \boldsymbol{x} \,\}\}\big) \\
&\qquad \leq \sum_{\boldsymbol{y}} \big(q_p^{\star 2} \star \varphi_p\big)(\boldsymbol{y})^2 \big(q_p^{\star 2} \star \varphi_p\big)(\boldsymbol{x}) \varphi_p(\boldsymbol{x} - \boldsymbol{y}),
\end{aligned}$$

which corresponds to 2nd term in (3.4.1). To bound (c) in (3.4.8), we note that, for $\boldsymbol{b} \in \mathscr{B}(\boldsymbol{o})$,

$$
\begin{aligned}
&\big\{\{\,\boldsymbol{o} \to \boldsymbol{x}\,\} \circ \{\,\boldsymbol{b} \to \boldsymbol{x}\,\}\big\} \setminus E(\boldsymbol{b}, \boldsymbol{x}; \tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{o})) \\
&\quad \subset \bigcup_{\boldsymbol{y}} \bigcup_{\boldsymbol{b}'} \Big\{ \big\{\{\,\boldsymbol{o} \to \boldsymbol{y} \to \boldsymbol{x}\,\} \circ \{\,\boldsymbol{b} \to \boldsymbol{b}' \to \boldsymbol{x}\,\} \circ \{\,\boldsymbol{y} \to \underline{\boldsymbol{b}'}\,\}\big\} \\
&\qquad \sqcup \big\{\{\,\boldsymbol{o} \to \boldsymbol{b}' \to \boldsymbol{x}\,\} \circ \{\,\boldsymbol{b} \to \boldsymbol{y} \to \underline{\boldsymbol{b}'}\,\} \circ \{\,\boldsymbol{y} \to \boldsymbol{x}\,\}\big\} \Big\} \\
&\quad = \bigcup_{\boldsymbol{b}'} \big\{\{\,\boldsymbol{o} \to \boldsymbol{x}\,\} \circ \{\,\boldsymbol{o} \to \underline{\boldsymbol{b}'}\,\} \circ \{\,\boldsymbol{b} \to \boldsymbol{b}' \to \boldsymbol{x}\,\}\big\} \\
&\qquad \sqcup \bigcup_{\boldsymbol{b}'} \big\{\{\,\boldsymbol{o} \to \underline{\boldsymbol{b}'} \to \boldsymbol{x}\,\} \circ \{\,\boldsymbol{b} \to \boldsymbol{b}' \to \boldsymbol{x}\,\}\big\} \\
&\qquad \sqcup \bigcup_{\boldsymbol{b}'} \bigcup_{\boldsymbol{y} \neq \boldsymbol{o}, \underline{\boldsymbol{b}'}} \big\{\{\,\boldsymbol{o} \to \boldsymbol{y} \to \boldsymbol{x}\,\} \circ \{\,\boldsymbol{b} \to \boldsymbol{b}' \to \boldsymbol{x}\,\} \circ \{\,\boldsymbol{y} \to \underline{\boldsymbol{b}'}\,\}\big\} \\
&\qquad \sqcup \bigcup_{\boldsymbol{b}'} \big\{\{\,\boldsymbol{o} \to \boldsymbol{b}' \to \boldsymbol{x}\,\} \circ \{\,\boldsymbol{b} \to \underline{\boldsymbol{b}'}\,\} \circ \{\,\underline{\boldsymbol{b}'} \to \boldsymbol{x}\,\}\big\} \\
&\qquad \sqcup \bigcup_{\boldsymbol{b}'} \bigcup_{\boldsymbol{y} \neq \underline{\boldsymbol{b}'}} \big\{\{\,\boldsymbol{o} \to \boldsymbol{b}' \to \boldsymbol{x}\,\} \circ \{\,\boldsymbol{b} \to \boldsymbol{y} \to \underline{\boldsymbol{b}'}\,\} \circ \{\,\boldsymbol{y} \to \boldsymbol{x}\,\}\big\}.
\end{aligned}
\tag{3.4.9}
$$

By the Boolean and BK inequalities, we have

$$
\begin{aligned}
&\sum_{\boldsymbol{b} \in \mathscr{B}(\boldsymbol{o})} \mathbb{P}_p\Big(\big\{\{\,\boldsymbol{o} \to \boldsymbol{x}\,\} \circ \{\,\boldsymbol{b} \to \boldsymbol{x}\,\}\big\} \setminus E(\boldsymbol{b}, \boldsymbol{x}; \tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{o})) \Big) \\
&\le \sum_{\boldsymbol{u}} \big(q_p^{\star 3} \star \varphi_p\big)(\boldsymbol{x}) \big(q_p^{\star 2} \star \varphi_p\big)(\boldsymbol{u})^2 (q_p \star \varphi_p)(\boldsymbol{x} - \boldsymbol{u}) \\
&\quad + \sum_{\boldsymbol{u}} \big(q_p^{\star 2} \star \varphi_p\big)(\boldsymbol{u})^2 \big(q_p^{\star 2} \star \varphi_p\big)(\boldsymbol{x} - \boldsymbol{u})^2 \\
&\quad + \sum_{\boldsymbol{u}, \boldsymbol{y}} \big(q_p^{\star 2} \star \varphi_p\big)(\boldsymbol{u}) (q_p \star \varphi_p)(\boldsymbol{y}) (q_p \star \varphi_p)(\boldsymbol{u} - \boldsymbol{y}) (q_p \star \varphi_p)(\boldsymbol{x} - \boldsymbol{u}) \big(q_p^{\star 2} \star \varphi_p\big)(\boldsymbol{x} - \boldsymbol{y}) \\
&\quad + \sum_{\boldsymbol{u}} \big(q_p^{\star 2} \star \varphi_p\big)(\boldsymbol{u})^2 \big(q_p^{\star 2} \star \varphi_p\big)(\boldsymbol{x} - \boldsymbol{u})^2 \\
&\quad + \sum_{\boldsymbol{u}, \boldsymbol{y}} \big(q_p^{\star 2} \star \varphi_p\big)(\boldsymbol{u}) (q_p \star \varphi_p)(\boldsymbol{y}) (q_p \star \varphi_p)(\boldsymbol{u} - \boldsymbol{y}) \big(q_p^{\star 2} \star \varphi_p\big)(\boldsymbol{x} - \boldsymbol{y}) (q_p \star \varphi_p)(\boldsymbol{x} - \boldsymbol{u}).
\end{aligned}
$$

Each term in the above upper bound corresponds to a contribution of 3rd term, a contribution of 4th term, 5th term, a contribution of 4th term and 6th term in (3.4.1), respectively. To bound (d) in (3.4.8), we apply (3.4.5) and split the below event into two events based on where the branching point is assigned:

$$
\begin{aligned}
\{\,\boldsymbol{o} \Rightarrow \boldsymbol{u} \neq \boldsymbol{o}\,\} \cap \{\,\boldsymbol{o} \to \boldsymbol{x}\,\} &\subset \big\{\{\,\boldsymbol{o} \Rightarrow \boldsymbol{u} \neq \boldsymbol{o}\,\} \circ \{\,\boldsymbol{o} \to \boldsymbol{x}\,\}\big\} \\
&\qquad \cup \bigcup_{\substack{\boldsymbol{y} \neq \boldsymbol{o} \\ (\tau(\boldsymbol{y}) \le \tau(\boldsymbol{u}))}} \big\{\{\,\boldsymbol{o} \to \boldsymbol{u}\,\} \circ \{\,\boldsymbol{o} \to \boldsymbol{y} \to \boldsymbol{u}\,\} \circ \{\,\boldsymbol{y} \to \boldsymbol{x}\,\}\big\}.
\end{aligned}
\tag{3.4.10}
$$

Then, we obtain

$$
\begin{aligned}
&\sum_{\boldsymbol{b}} \mathbb{P}_p\Big(\{\,\boldsymbol{o} \Rightarrow \underline{\boldsymbol{b}} \neq \boldsymbol{o}\,\} \cap E(\boldsymbol{b}, \boldsymbol{x}; \tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{o})) \Big) \\
&\overset{(3.4.5)}{\le} \sum_{\boldsymbol{b}} \mathbb{P}_p\big(\{\,\boldsymbol{o} \Rightarrow \underline{\boldsymbol{b}} \neq \boldsymbol{o}\,\} \cap \big\{\{\,\boldsymbol{o} \to \boldsymbol{x}\,\} \circ \{\,\boldsymbol{b} \to \boldsymbol{x}\,\}\big\} \big)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\boldsymbol{b}} \mathbb{P}_p\big(\{\{\,\boldsymbol{o} \Rightarrow \underline{\boldsymbol{b}} \neq \boldsymbol{o}\,\} \cap \{\,\boldsymbol{o} \to \boldsymbol{x}\,\}\} \circ \{\,\boldsymbol{b} \to \boldsymbol{x}\,\}\big) \\
&\overset{(3.4.10)}{\leq} \sum_{\boldsymbol{b}} \Bigg(\mathbb{P}_p\big(\{\,\boldsymbol{o} \Rightarrow \underline{\boldsymbol{b}} \neq \boldsymbol{o}\,\} \circ \{\,\boldsymbol{o} \to \boldsymbol{x}\,\} \circ \{\,\boldsymbol{b} \to \boldsymbol{x}\,\}\big) \\
&\quad + \sum_{\substack{\boldsymbol{y} \neq \boldsymbol{o} \\ (\tau(\boldsymbol{y}) \leq \tau(\underline{\boldsymbol{b}}))}} \mathbb{P}_p\big(\{\,\boldsymbol{o} \to \underline{\boldsymbol{b}}\,\} \circ \underbrace{\{\,\boldsymbol{o} \to \boldsymbol{y} \to \underline{\boldsymbol{b}}\,\}}_{=\{\,\boldsymbol{o}\to\boldsymbol{y}=\underline{\boldsymbol{b}}\,\} \sqcup \{\,\boldsymbol{o}\to\boldsymbol{y}\to\underline{\boldsymbol{b}}\neq\boldsymbol{y}\,\}} \circ \{\,\boldsymbol{y} \to \boldsymbol{x}\,\} \circ \{\,\boldsymbol{b} \to \boldsymbol{x}\,\}\big) \Bigg). \qquad (3.4.11)
\end{aligned}
$$

Applying the BK [7] inequality and the same method for (a) to the right-most in the above, and paying attention to the disjointness of the connections, we arrive in

$$
\begin{aligned}
&\sum_{\boldsymbol{b}} \mathbb{P}_p\Big(\{\,\boldsymbol{o} \Rightarrow \underline{\boldsymbol{b}} \neq \boldsymbol{o}\,\} \cap E(\boldsymbol{b}, \boldsymbol{x}; \tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{o}))\Big) \\
&\quad \leq \frac{1}{2} \sum_{\boldsymbol{u}} \big(q_p^{\star 2} \star \varphi_p\big)(\boldsymbol{u})^2 (q_p \star \varphi_p)(\boldsymbol{x} - \boldsymbol{u}) \big(q_p^{\star 3} \star \varphi_p\big)(\boldsymbol{x}) \\
&\qquad + \sum_{\boldsymbol{u}} \big(q_p^{\star 2} \star \varphi_p\big)(\boldsymbol{u})^2 \big(q_p^{\star 2} \star \varphi_p\big)(\boldsymbol{x} - \boldsymbol{u})^2 \\
&\qquad + \sum_{\boldsymbol{u}, \boldsymbol{y}} \big(q_p^{\star 2} \star \varphi_p\big)(\boldsymbol{u}) (q_p \star \varphi_p)(\boldsymbol{y}) (q_p \star \varphi_p)(\boldsymbol{u} - \boldsymbol{y}) (q_p \star \varphi_p)(\boldsymbol{x} - \boldsymbol{u}) \big(q_p^{\star 2} \star \varphi_p\big)(\boldsymbol{x} - \boldsymbol{y}).
\end{aligned}
$$

Each term in the above upper bound corresponds to a contribution of 3rd term, of 4th term and of 5th term in (3.4.1), respectively. Combining the upper bounds on (a)–(d) completes the proof of (3.4.1). ☐

Proof of (3.4.2) *in Lemma 3.4.2.* It is not hard to prove the upper bound by using (3.4.5), (3.4.9) and (3.4.12), so that we omit it. ☐

Proof of (3.4.3) *in Lemma 3.4.2.* By the definition,

$$
\begin{aligned}
E(\boldsymbol{b}, \boldsymbol{u}; \tilde{\mathcal{C}}^{\boldsymbol{b}}(\boldsymbol{v})) \cap \{\,\overline{\boldsymbol{b}} \to \boldsymbol{x}\,\} \subset \bigcup_{\substack{\boldsymbol{y} \\ (\tau(\underline{\boldsymbol{b}}) < \tau(\boldsymbol{y}) \leq \tau(\boldsymbol{u}))}} &\Big\{\{\{\,\boldsymbol{b} \to \boldsymbol{u}\,\} \circ \{\,\boldsymbol{v} \to \boldsymbol{y} \to \boldsymbol{u}\,\} \circ \{\,\boldsymbol{y} \to \boldsymbol{x}\,\}\} \\
&\cup \{\{\,\boldsymbol{b} \to \boldsymbol{y} \to \boldsymbol{u}\,\} \circ \{\,\boldsymbol{v} \to \boldsymbol{u}\,\} \circ \{\,\boldsymbol{y} \to \boldsymbol{x}\,\}\}\Big\}. \qquad (3.4.12)
\end{aligned}
$$

Paying attention to the disjointness of connections and the magnitude relationship between times, we obtain

$$
\begin{aligned}
\pi_p^{(N)}(\boldsymbol{x}) &= \sum_{\vec{\boldsymbol{b}}_N} \mathbb{P}_p\Big(\tilde{E}^{(N)}_{\vec{\boldsymbol{b}}_N}(\boldsymbol{x})\Big) = \sum_{\vec{\boldsymbol{b}}_N} \mathbb{P}_p\Big(\tilde{E}^{(N-1)}_{\vec{\boldsymbol{b}}_{N-1}}(\underline{\boldsymbol{b}_N}) \cap E(\boldsymbol{b}_N, \boldsymbol{x}; \tilde{\mathcal{C}}^{\boldsymbol{b}_N}(\overline{\boldsymbol{b}_{N-1}}))\Big) \\
&\overset{(3.4.5)}{\leq} \sum_{\vec{\boldsymbol{b}}_N} \mathbb{P}_p\Big(\tilde{E}^{(N-2)}_{\vec{\boldsymbol{b}}_{N-2}}(\underline{\boldsymbol{b}_{N-1}}) \cap E(\boldsymbol{b}_{N-1}, \boldsymbol{b}_N; \tilde{\mathcal{C}}^{\boldsymbol{b}_{N-1}}(\overline{\boldsymbol{b}_{N-2}})) \cap \{\,\boldsymbol{b}_{N-1} \to \boldsymbol{x}\,\} \circ \{\,\boldsymbol{b}_N \to \boldsymbol{x}\,\}\Big) \\
&\overset{(3.4.12)}{\leq} \sum_{\vec{\boldsymbol{b}}_{N-2}} \sum_{\boldsymbol{v}_{N-1}, \boldsymbol{v}_N} \sum_{\substack{\boldsymbol{y}_N, \boldsymbol{u}_{N-1}, \boldsymbol{u}_N \\ (\tau(\boldsymbol{u}_{N-1}) < \tau(\boldsymbol{y}_N) \leq \tau(\boldsymbol{u}_N))}} \Bigg(\mathbb{P}_p\Big(\tilde{E}^{(N-2)}_{\vec{\boldsymbol{b}}_{N-2}}(\boldsymbol{u}_{N-1}) \cap \{\,\overline{\boldsymbol{b}_{N-2}} \to \boldsymbol{y}_N\,\}\Big) \\
&\qquad \times \mathbb{P}_p\big(\{\,(\boldsymbol{u}_{N-1}, \boldsymbol{v}_{N-1}) \to \boldsymbol{u}_N\,\} \circ \{\,\boldsymbol{y}_N \to \boldsymbol{u}_N\,\} \circ \{\,\boldsymbol{y}_N \to \boldsymbol{x}\,\} \circ \{\,(\boldsymbol{u}_N, \boldsymbol{v}_N) \to \boldsymbol{x}\,\}\big) \\
&\qquad + \mathbb{P}_p\Big(\tilde{E}^{(N-2)}_{\vec{\boldsymbol{b}}_{N-2}}(\boldsymbol{u}_{N-1}) \cap \{\,\overline{\boldsymbol{b}_{N-2}} \to \boldsymbol{u}_N\,\}\Big) \\
&\qquad \times \mathbb{P}_p\big(\{\,(\boldsymbol{u}_{N-1}, \boldsymbol{v}_{N-1}) \to \boldsymbol{y}_N \to \boldsymbol{u}_N\,\} \circ \{\,\boldsymbol{y}_N \to \boldsymbol{x}\,\} \circ \{\,(\boldsymbol{u}_N, \boldsymbol{v}_N) \to \boldsymbol{x}\,\}\big) \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\le \sum_{\vec{\boldsymbol{b}}_{N-2}} \sum_{\substack{\boldsymbol{y}_N,\boldsymbol{u}_{N-1},\boldsymbol{u}_N \\ (\tau(\boldsymbol{u}_{N-1})<\tau(\boldsymbol{y}_N)\le\tau(\boldsymbol{u}_N))}} \Bigg(\mathbb{P}_p\Big(\tilde{E}^{(N-2)}_{\vec{\boldsymbol{b}}_{N-2}}(\boldsymbol{u}_{N-1}) \cap \{\, \overline{\boldsymbol{b}_{N-2}} \to \boldsymbol{y}_N \,\}\Big) \\
&\qquad \times (q_p \star \varphi_p)(\boldsymbol{u}_N - \boldsymbol{u}_{N-1}) \\
&\qquad + \mathbb{P}_p\Big(\tilde{E}^{(N-2)}_{\vec{\boldsymbol{b}}_{N-2}}(\boldsymbol{u}_{N-1}) \cap \{\, \overline{\boldsymbol{b}_{N-2}} \to \boldsymbol{u}_N \,\}\Big)(q_p \star \varphi_p)(\boldsymbol{y}_N - \boldsymbol{u}_{N-1})\Bigg) \\
&\qquad \times \underbrace{\varphi_p(\boldsymbol{u}_N - \boldsymbol{y}_N)(q_p \star \varphi_p)(\boldsymbol{x} - \boldsymbol{y}_N)(q_p \star \varphi_p)(\boldsymbol{x} - \boldsymbol{u}_N)}_{\equiv \Xi(\boldsymbol{y}_N,\boldsymbol{u}_N;\boldsymbol{x},\boldsymbol{x})/2} \\
&\overset{(3.4.12)}{\le} \sum_{\vec{\boldsymbol{b}}_{N-3}} \sum_{\substack{\boldsymbol{y}_{N-1},\boldsymbol{y}_N,\boldsymbol{u}_{N-2},\boldsymbol{u}_{N-1},\boldsymbol{u}_N \\ (\forall i:\ \tau(\boldsymbol{u}_{i-1})<\tau(\boldsymbol{y}_i)\le\tau(\boldsymbol{u}_i))}} \Bigg(\mathbb{P}_p\Big(\tilde{E}^{(N-3)}_{\vec{\boldsymbol{b}}_{N-3}}(\boldsymbol{u}_{N-2}) \cap \{\, \overline{\boldsymbol{b}_{N-3}} \to \boldsymbol{y}_{N-1} \,\}\Big) \\
&\qquad \times (q_p \star \varphi_p)(\boldsymbol{u}_{N-1} - \boldsymbol{u}_{N-2}) \\
&\qquad + \mathbb{P}_p\Big(\tilde{E}^{(N-3)}_{\vec{\boldsymbol{b}}_{N-3}}(\boldsymbol{u}_{N-2}) \cap \{\, \overline{\boldsymbol{b}_{N-3}} \to \boldsymbol{u}_{N-1} \,\}\Big)(q_p \star \varphi_p)(\boldsymbol{y}_{N-1} - \boldsymbol{u}_{N-2})\Bigg) \\
&\qquad \times \underbrace{\begin{aligned}&\big(\varphi_p(\boldsymbol{u}_{N-1} - \boldsymbol{y}_{N-1})(q_p \star \varphi_p)(\boldsymbol{y}_N - \boldsymbol{y}_{N-1})(q_p \star \varphi_p)(\boldsymbol{u}_N - \boldsymbol{u}_{N-1}) \\ &\quad + \varphi_p(\boldsymbol{u}_{N-1} - \boldsymbol{y}_{N-1})(q_p \star \varphi_p)(\boldsymbol{u}_N - \boldsymbol{y}_{N-1})(q_p \star \varphi_p)(\boldsymbol{y}_N - \boldsymbol{u}_{N-1})\big)\end{aligned}}_{\equiv \Xi(\boldsymbol{y}_{N-1},\boldsymbol{u}_{N-1};\boldsymbol{y}_N,\boldsymbol{u}_N)} \\
&\qquad \times \frac{1}{2}\Xi(\boldsymbol{y}_N, \boldsymbol{u}_N; \boldsymbol{x}, \boldsymbol{x}).
\end{aligned}
$$

By applying (3.4.12) to the above repeatedly,

$$
\begin{aligned}
\pi_p^{(N)}(\boldsymbol{x}) \le \sum_{\boldsymbol{v}_1} \sum_{\substack{\{\boldsymbol{y}_i\}_{i=2}^N,\{\boldsymbol{u}_i\}_{i=1}^N \\ (\forall i:\ \tau(\boldsymbol{u}_{i-1})<\tau(\boldsymbol{y}_i)\le\tau(\boldsymbol{u}_i))}} &\Big(\mathbb{P}_p\big(\{\, \boldsymbol{o} \Rightarrow \boldsymbol{u}_1 \,\} \cap \{\, (\boldsymbol{u}_1, \boldsymbol{v}_1) \to \boldsymbol{u}_2 \,\} \circ \{\, \boldsymbol{o} \to \boldsymbol{y}_2 \,\}\big) \\
&+ \mathbb{P}_p\big(\{\, \boldsymbol{o} \Rightarrow \boldsymbol{u}_1 \,\} \cap \{\, (\boldsymbol{u}_1, \boldsymbol{v}_1) \to \boldsymbol{y}_2 \,\} \circ \{\, \boldsymbol{o} \to \boldsymbol{u}_2 \,\}\big)\Big) \\
&\times \left(\prod_{j=2}^{N-1} \Xi(\boldsymbol{y}_j, \boldsymbol{u}_j; \boldsymbol{y}_{j+1}, \boldsymbol{u}_{j+1})\right)\frac{1}{2}\Xi(\boldsymbol{y}_N, \boldsymbol{u}_N; \boldsymbol{x}, \boldsymbol{x}).
\end{aligned}
$$

Finally, using the Boolean and BK [7] inequality, and applying a similar method to (3.4.1), we arrive in (3.4.3). ☐

Multiplying the diagrammatic bounds in Lemma 3.4.2 by the factors m^t, t or $1 - \cos k \cdot x$, and taking the sum of them, we obtain the following bounds.

Lemma 3.4.3. Let N be some integer numbers grater than or equal to 3. When we multiply the upper bounds in Lemma 3.4.2 by m^t,

$$
\begin{aligned}
\Big|\hat{\pi}_p^{(0)}(0, m) - \hat{\pi}_p^{(1)}(0, m)\Big| \le \frac{1}{2}B^{(2,2)}_{p,m} + B^{(2,2)}_{p,1}B^{(0,2)}_{p,m} + \frac{3}{2}B^{(2,2)}_{p,1}B^{(1,3)}_{p,m} \\
+ 3\Big(B^{(2,2)}_{p,m}\Big)^2 + 3B^{(2,1)}_{p,m}T^{(2,2)}_{p,m},
\end{aligned}
\tag{3.4.13}
$$

$$
\begin{aligned}
\hat{\pi}_p^{(2)}(0, m) \le B^{(2,2)}_{p,m}B^{(1,3)}_{p,m} + 2\Big(B^{(2,2)}_{p,m}\Big)^2 + 2B^{(2,1)}_{p,m}T^{(2,2)}_{p,m} \\
+ \frac{1}{2}B^{(2,2)}_{p,1}B^{(1,3)}_{p,m}T^{(1,2)}_{p,m} + B^{(2,1)}_{p,m}T^{(1,1)}_{p,m}T^{(1,2)}_{p,m} \\
+ \frac{1}{2}B^{(2,2)}_{p,1}B^{(1,3)}_{p,m}T^{(2,1)}_{p,m} + B^{(2,1)}_{p,m}T^{(1,1)}_{p,m}T^{(2,1)}_{p,m},
\end{aligned}
\tag{3.4.14}
$$

$$\hat{\pi}_p^{(N)}(0,m) \le \Big(B_{p,m}^{(1,1)} + \frac{1}{2}B_{p,1}^{(2,2)}B_{p,m}^{(1,3)} + T_{p,m}^{(1,1)}B_{p,m}^{(1,1)}\Big)\Big(2T_{p,m}^{(1,1)}\Big)^{N-1}. \tag{3.4.15}$$

When we multiply the upper bounds in Lemma 3.4.2 by m^t and t,

$$\begin{aligned}
\sum_{(x,t)} \Big|\pi_p^{(0)}(x,t) - \pi_p^{(1)}(x,t)\Big| m^t t &\le \frac{1}{2}\Big(B_{p,m}^{(2,2)} + T_{p,m}^{(2,2)}\Big) + B_{p,1}^{(2,2)}\Big(B_{p,m}^{(0,2)} + T_{p,m}^{(0,2)}\Big) \\
&\quad + \frac{3}{2}B_{p,1}^{(2,2)}\Big(2B_{p,m}^{(1,3)} + T_{p,m}^{(1,3)}\Big) \\
&\quad + 6B_{p,m}^{(2,2)}\Big(B_{p,m}^{(2,2)} + T_{p,m}^{(2,2)}\Big) + 3\Big(mT_{p,m}^{(1,2)}T_{p,1}^{(2,2)} + B_{p,m}^{(1,2)}T_{p,m}^{(2,2)} + T_{p,m}^{(2,2)}T_{p,m}^{(2,1)}\Big),
\end{aligned} \tag{3.4.16}$$

$$\begin{aligned}
\sum_{(x,t)} \pi_p^{(2)}(x,t) m^t t &\le B_{p,1}^{(2,2)}\Big(2B_{p,m}^{(1,3)} + T_{p,m}^{(1,3)}\Big) \\
&\quad + 4B_{p,m}^{(2,2)}\Big(B_{p,m}^{(2,2)} + T_{p,m}^{(2,2)}\Big) + 2\Big(mT_{p,m}^{(1,2)}T_{p,1}^{(2,2)} + B_{p,m}^{(1,2)}T_{p,m}^{(2,2)} + T_{p,m}^{(2,2)}T_{p,m}^{(2,1)}\Big) \\
&\quad + \frac{1}{2}B_{p,1}^{(2,2)}\Big(3B_{p,m}^{(1,3)}T_{p,m}^{(1,2)} + 2T_{p,m}^{(1,3)}T_{p,m}^{(1,2)}\Big) + \Big(3T_{p,m}^{(1,2)}T_{p,m}^{(1,1)}T_{p,m}^{(2,1)} + B_{p,m}^{(1,2)}T_{p,m}^{(1,1)}T_{p,m}^{(2,1)}\Big) \\
&\quad + \frac{1}{2}B_{p,1}^{(2,2)}\Big(2B_{p,m}^{(1,3)}T_{p,m}^{(2,1)} + 2T_{p,m}^{(1,3)}T_{p,m}^{(2,1)}\Big) + 3T_{p,m}^{(2,1)}T_{p,m}^{(1,1)}T_{p,m}^{(2,1)},
\end{aligned} \tag{3.4.17}$$

$$\begin{aligned}
&\sum_{(x,t)} \pi_p^{(N)}(x,t) m^t t \\
&\le \Big(T_{p,m}^{(1,1)} + \frac{1}{2}(2B_{p,m}^{(2,2)}B_{p,m}^{(1,3)} + B_{p,m}^{(2,2)}T_{p,m}^{(3,1)}) + 2\Big(T_{p,m}^{(1,1)}\Big)^2\Big)\Big(2T_{p,m}^{(1,1)}\Big)^{N-1} \\
&\quad + (N-2)\Big(T_{p,m}^{(1,1)} + \frac{1}{2}B_{p,m}^{(2,2)}T_{p,m}^{(1,3)} + \Big(T_{p,m}^{(1,1)}\Big)^2\Big)\Big(2T_{p,m}^{(1,1)}\Big)^{N-2} \\
&\quad + \Big(T_{p,m}^{(1,1)} + \frac{1}{2}B_{p,m}^{(2,2)}T_{p,m}^{(1,3)} + \Big(T_{p,m}^{(1,1)}\Big)^2\Big)\Big(2T_{p,m}^{(1,1)}\Big)^{N-1}.
\end{aligned} \tag{3.4.18}$$

When we multiply the upper bounds in Lemma 3.4.2 by m^t and $1 - \cos k \cdot x$,

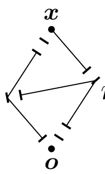
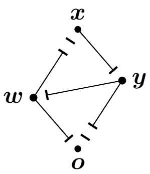
$$\begin{aligned}
&\sum_{(x,t)} \Big|\pi_p^{(0)}(x,t) - \pi_p^{(1)}(x,t)\Big| m^t (1 - \cos k \cdot x) \\
&\quad \le \frac{1}{2}\hat{V}_{p,m}^{(2,2)}(k) + 2\Big(\hat{V}_{p,1}^{(2,2)}(k)B_{p,m}^{(0,2)} + B_{p,1}^{(2,2)}\hat{V}_{p,m}^{(1,2)}(k)\Big) \\
&\qquad + \frac{3}{2}B_{p,m}^{(2,2)}\hat{V}_{p,m}^{(3,1)}(k) \\
&\qquad + 12B_{p,m}^{(2,2)}\hat{V}_{p,m}^{(2,2)}(k) + 6\Big(\hat{V}_{p,m}^{(1,2)}(k)T_{p,m}^{(2,2)} + T_{p,m}^{(2,2)}\hat{V}_{p,m}^{(2,1)}(k)\Big),
\end{aligned} \tag{3.4.19}$$

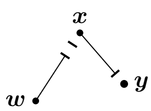
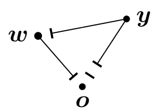
$$\begin{aligned}
\hat{\pi}_p^{(2)}(0,m) - \hat{\pi}_p^{(2)}(k,m) &\le B_{p,m}^{(2,2)}\hat{V}_{p,m}^{(3,1)}(k) \\
&\quad + 8B_{p,m}^{(2,2)}\hat{V}_{p,m}^{(2,2)}(k) + 4\Big(\hat{V}_{p,m}^{(1,2)}(k)T_{p,m}^{(2,2)} + T_{p,m}^{(2,2)}\hat{V}_{p,m}^{(2,1)}(k)\Big) \\
&\quad + B_{p,m}^{(2,2)}\Big(\hat{V}_{p,m}^{(3,1)}(k)T_{p,m}^{(2,1)} + T_{p,m}^{(3,1)}\hat{V}_{p,m}^{(2,1)}(k)\Big) \\
&\quad + 3\Big(\hat{V}_{p,m}^{(1,2)}(k)T_{p,m}^{(1,1)}T_{p,m}^{(2,1)} + T_{p,m}^{(1,2)}\hat{V}_{p,m}^{(1,1)}(k)T_{p,m}^{(2,1)} + T_{p,m}^{(1,2)}T_{p,m}^{(1,1)}\hat{V}_{p,m}^{(2,1)}(k)\Big) \\
&\quad + B_{p,m}^{(2,2)}\Big(\hat{V}_{p,m}^{(3,1)}(k)T_{p,m}^{(1,2)} + T_{p,m}^{(3,1)}\hat{V}_{p,m}^{(1,2)}(k)\Big) \\
&\quad + 3\Big(\hat{V}_{p,m}^{(1,2)}(k)T_{p,m}^{(1,1)}T_{p,m}^{(1,2)} + T_{p,m}^{(1,2)}\hat{V}_{p,m}^{(1,1)}(k)T_{p,m}^{(1,2)} + T_{p,m}^{(1,2)}T_{p,m}^{(1,1)}\hat{V}_{p,m}^{(1,2)}(k)\Big),
\end{aligned} \tag{3.4.20}$$

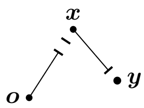
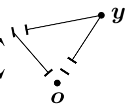
$$
\begin{aligned}
&\hat{\pi}_p^{(N)}(0,m) - \hat{\pi}_p^{(N)}(k,m) \\
&\le (N+1)\Bigg(\bigg(\hat{V}_{p,m}^{(1,1)}(k) + \frac{1}{2} B_{p,m}^{(2,2)} \hat{V}_{p,m}^{(1,3)}(k) + 2T_{p,m}^{(1,1)} \hat{V}_{p,m}^{(1,1)}(k) \bigg) \Big(2T_{p,m}^{(1,1)} \Big)^{N-1} \\
&\quad + 2(N-2) \bigg(T_{p,m}^{(1,1)} + \frac{1}{2} B_{p,m}^{(2,2)} T_{p,m}^{(1,3)} + \Big(T_{p,m}^{(1,1)} \Big)^2 \bigg) \hat{V}_{p,m}^{(1,1)}(k) \Big(2T_{p,m}^{(1,1)} \Big)^{N-2} \\
&\quad + 2 \bigg(T_{p,m}^{(1,1)} + \frac{1}{2} B_{p,m}^{(2,2)} T_{p,m}^{(1,3)} + \Big(T_{p,m}^{(1,1)} \Big)^2 \bigg) \Big(2T_{p,m}^{(1,1)} \Big)^{N-2} \hat{V}_{p,m}^{(1,1)}(k) \Bigg).
\end{aligned}
\tag{3.4.21}
$$

Sketch proof. We only deal with three examples of the bounds on the sum of a diagram multiplied by the factors m^t, t or $1 - \cos k \cdot x$ because one can easily calculate the other bounds on the analogy of such examples. The following proof is almost identical to the proof of [19, Lemma 5.3].

First, we consider the bounds (3.4.13)–(3.4.15). By the translation-invariance, for example,

$\sum_{\boldsymbol{x}}$  $m^{\tau(\boldsymbol{x})} = \sum_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{w}}$  $m^{\tau(\boldsymbol{y})} m^{\tau(\boldsymbol{x})-\tau(\boldsymbol{y})}$

$\le \sum_{\boldsymbol{w}} \Bigg(\sup_{\boldsymbol{y}} \sum_{\boldsymbol{x}}$  $m^{\tau(\boldsymbol{x})-\tau(\boldsymbol{y})} \Bigg) \Bigg(\sum_{\boldsymbol{y}}$  $m^{\tau(\boldsymbol{y})} \Bigg)$

$= \Bigg(\sup_{\boldsymbol{y}} \sum_{\boldsymbol{x}}$  $m^{\tau(\boldsymbol{x})-\tau(\boldsymbol{y})} \Bigg) \Bigg(\sum_{\boldsymbol{y}}$  $m^{\tau(\boldsymbol{y})} \Bigg)$

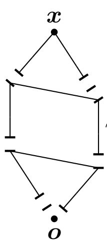
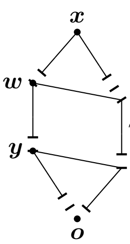
$$
\begin{aligned}
&= \bigg(\sup_{\boldsymbol{y}} \sum_{\boldsymbol{x}} \big(q_p^{\star 2} \star \varphi_p \big)(\boldsymbol{x}) (q_p \star \varphi_p)(\boldsymbol{x} - \boldsymbol{y}) m^{\tau(\boldsymbol{x})-\tau(\boldsymbol{y})} \bigg) \\
&\quad \times \bigg(\sum_{\boldsymbol{y}} \big(q_p^{\star 2} \star \varphi_p^{\star 2} \big)(\boldsymbol{y}) \big(q_p^{\star 2} \star \varphi_p \big)(\boldsymbol{y}) m^{\tau(\boldsymbol{y})} \bigg) \\
&\le B_{p,m}^{(2,1)} T_{p,m}^{(2,2)},
\end{aligned}
$$

which corresponds to the last term in the right hand side in (3.4.13). In the second equality, we used the translation invariance.

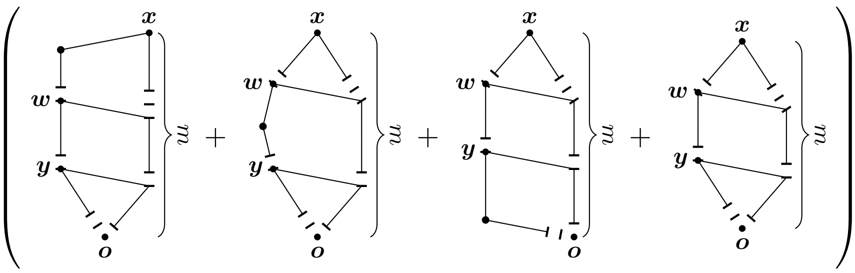
Next, we consider the bounds (3.4.16)–(3.4.18). Note that, by Lemma 3.2.1 and the equality $q_p(y,s) = 0$ for any $(y,s) \in \mathbb{L}^d \times \mathbb{Z}_+$ with $s \ne 1$,

$$
\begin{aligned}
(q_p \star \varphi_p)(x,t)t &\le \big(q_p \star \varphi_p^{\star 2} \big)(x,t), \\
\big(q_p^{\star 2} \star \varphi_p \big)(x,t)t &\le \big(q_p^{\star 2} \star \varphi_p^{\star 2} \big)(x,t) + \big(q_p \star \varphi_p^{\star 2} \big)(x,t).
\end{aligned}
$$

By using the above inequalities, for example,

$\sum_{\boldsymbol{x}}$  $m^{\tau(\boldsymbol{x})} \tau(\boldsymbol{x}) = \sum_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{w}}$  $m^{\tau(\boldsymbol{x})} \Big(\big(\tau(\boldsymbol{x}) - \tau(\boldsymbol{w}) \big) + \big(\tau(\boldsymbol{w}) - \tau(\boldsymbol{y}) \big) + \boldsymbol{y} \Big)$

$$\le \sum_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{w}}$$

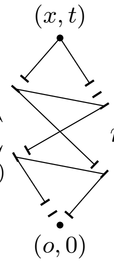


$$\le 3T^{(1,2)}_{p,m}T^{(1,1)}_{p,m}T^{(2,1)}_{p,m} + B^{(1,2)}_{p,m}T^{(1,1)}_{p,m}T^{(2,1)}_{p,m},$$

which corresponds to the 5th term in the right hand side in (3.4.17).

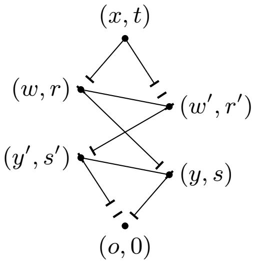
Finally, we consider (3.4.19)–(3.4.21). By the telescopic inequality (1.3.1), for example,

$$\sum_{(x,t)}$$



$$m^t(1-\cos k\cdot x)$$

$$\le 3 \sum_{\substack{(x,t),\\(y,s),(w,r),\\(y',s'),(w',r')}}$$



$$\times\, m^{t-r'}m^{r'-s'}m^{s'} \times \Big(\big(1-\cos k\cdot(x-w)\big) + \big(1-\cos k\cdot(w-y)\big) + \big(1-\cos k\cdot y\big)\Big)$$

$$\le 3\Big(\hat{V}^{(1,2)}_{p,m}(k)T^{(1,1)}_{p,m}T^{(1,2)}_{p,m} + T^{(1,2)}_{p,m}(k)\hat{V}^{(1,1)}_{p,m}T^{(1,2)}_{p,m} + T^{(1,2)}_{p,m}(k)T^{(1,1)}_{p,m}\hat{V}^{(1,2)}_{p,m}\Big),$$

which corresponds to the last term in the right hand side in (3.4.20). □

Taking the sum of the bounds in Lemma 3.4.3 over $N \in \mathbb{Z}_+$ immediately implies the next lemma.

Lemma 3.4.4 (Precise version of Lemma 3.4.1)**.** For $p < p_{\mathrm{c}}$, $m < m_p$ and $k \in \mathbb{T}^d$,

$$\begin{aligned}\hat{\Pi}^{\mathrm{even}}_p(0,m) &\le \hat{\pi}^{(2)}_p(0,m)\\ &\quad + \left(B^{(1,1)}_{p,m} + \frac{1}{2}B^{(2,2)}_{p,1}B^{(1,3)}_{p,m} + T^{(1,1)}_{p,m}B^{(1,1)}_{p,m}\right)\frac{\big(2T^{(1,1)}_{p,m}\big)^3}{1-\big(2T^{(1,1)}_{p,m}\big)^2},\end{aligned} \tag{3.4.22}$$

$$\begin{aligned}\hat{\Pi}^{\mathrm{odd}}_p(0,m) &\le \left|\hat{\pi}^{(0)}_p(0,m) - \hat{\pi}^{(1)}_p(0,m)\right|\\ &\quad + \left(B^{(1,1)}_{p,m} + \frac{1}{2}B^{(2,2)}_{p,1}B^{(1,3)}_{p,m} + T^{(1,1)}_{p,m}B^{(1,1)}_{p,m}\right)\frac{\big(2T^{(1,1)}_{p,m}\big)^2}{1-\big(2T^{(1,1)}_{p,m}\big)^2},\end{aligned} \tag{3.4.23}$$

$$
\begin{aligned}
&\sum_{(x,t)} \Big(\Pi_p^{\mathrm{even}}(x,t) + \Pi_p^{\mathrm{odd}}(x,t)\Big) m^t t \\
&\quad \le \sum_{(x,t)} \Big(\Big|\pi_p^{(0)}(x,t) - \pi_p^{(1)}(x,t)\Big| + \pi_p^{(2)}(x,t)\Big) m^t t \\
&\qquad + \bigg(T_{p,m}^{(1,1)} + \frac{1}{2}\big(2B_{p,m}^{(2,2)}B_{p,m}^{(1,3)} + B_{p,m}^{(2,2)}T_{p,m}^{(3,1)}\big) + 2\Big(T_{p,m}^{(1,1)}\Big)^2\bigg) \frac{\big(2T_{p,m}^{(1,1)}\big)^2}{1-2T_{p,m}^{(1,1)}} \\
&\qquad + \bigg(T_{p,m}^{(1,1)} + \frac{1}{2}B_{p,m}^{(2,2)}T_{p,m}^{(1,3)} + \Big(T_{p,m}^{(1,1)}\Big)^2\bigg) \frac{2T_{p,m}^{(1,1)}}{\big(1-2T_{p,m}^{(1,1)}\big)^2} \\
&\qquad + \bigg(T_{p,m}^{(1,1)} + \frac{1}{2}B_{p,m}^{(2,2)}T_{p,m}^{(1,3)} + \Big(T_{p,m}^{(1,1)}\Big)^2\bigg) \frac{\big(2T_{p,m}^{(1,1)}\big)^2}{1-2T_{p,m}^{(1,1)}},
\end{aligned} \tag{3.4.24}
$$

$$
\begin{aligned}
&\sum_{(x,t)} \Big(\Pi_p^{\mathrm{even}}(x,t) + \Pi_p^{\mathrm{odd}}(x,t)\Big) m^t (1-\cos k\cdot x) \\
&\quad \le \sum_{(x,t)} \Big|\pi_p^{(0)}(x,t) - \pi_p^{(1)}(x,t)\Big| m^t (1-\cos k\cdot x) + \Big(\hat\pi_p^{(2)}(0,m) - \hat\pi_p^{(2)}(k,m)\Big) \\
&\qquad + \bigg(\hat V_{p,m}^{(1,1)}(k) + \frac{1}{2}B_{p,m}^{(2,2)}\hat V_{p,m}^{(1,3)}(k) + 2T_{p,m}^{(1,1)}\hat V_{p,m}^{(1,1)}(k)\bigg) \frac{8\big(T_{p,m}^{(1,1)}\big)^2\big(2-3T_{p,m}^{(1,1)}\big)}{\big(1-2T_{p,m}^{(1,1)}\big)^2} \\
&\qquad + 2\bigg(T_{p,m}^{(1,1)} + \frac{1}{2}B_{p,m}^{(2,2)}T_{p,m}^{(1,3)} + \Big(T_{p,m}^{(1,1)}\Big)^2\bigg)\hat V_{p,m}^{(1,1)}(k) \frac{8T_{p,m}^{(1,1)}\big(1-T_{p,m}^{(1,1)}\big)}{\big(1-2T_{p,m}^{(1,1)}\big)^3} \\
&\qquad + 2\bigg(T_{p,m}^{(1,1)} + \frac{1}{2}B_{p,m}^{(2,2)}T_{p,m}^{(1,3)} + \Big(T_{p,m}^{(1,1)}\Big)^2\bigg)\hat V_{p,m}^{(1,1)}(k) \frac{4T_{p,m}^{(1,1)}\big(2-3T_{p,m}^{(1,1)}\big)}{\big(1-2T_{p,m}^{(1,1)}\big)^2}.
\end{aligned} \tag{3.4.25}
$$

3.5 Diagrammatic bounds on the bootstrapping functions

In this section, using the lace expansion, we show upper bounds on $\{g_i(p,m)\}_{i=1}^3$ in terms of the lace expansion coefficients. Assume the absolute convergence of (3.3.2), which is proved in Section 3.8. Then, taking the Fourier-Laplace transform of both sides in (3.3.1) implies that

$$
\hat\varphi_p(k,z) = \frac{1+\hat\Pi_p(k,z)}{1-\hat q_p(k,z)\big(1+\hat\Pi_p(k,z)\big)}. \tag{3.5.1}
$$

An upper bound for $g_1(p,m)$. Note that $\hat\varphi_p(0,m) = \sum_{\boldsymbol{x}}(\delta_{\boldsymbol{o},\boldsymbol{x}} + \varphi_p(\boldsymbol{x})\mathbb{1}_{\{\boldsymbol{x}\neq\boldsymbol{o}\}})m^{\tau(\boldsymbol{x})} \ge 1$ since $m \ge 1$. This inequality and (3.5.1) yield

$$
\hat\varphi_p(0,m) = \frac{1+\hat\Pi_p(0,m)}{1-pm\big(1+\hat\Pi_p(0,m)\big)} \iff pm = \frac{1}{1+\hat\Pi_p(0,m)} - \hat\varphi_p(0,m)^{-1}.
$$

Recall $\Pi_p(\boldsymbol{x}) = \Pi_p^{\mathrm{even}}(\boldsymbol{x}) - \Pi_p^{\mathrm{odd}}(\boldsymbol{x})$, hence we obtain

$$
g_1(p,m) \le \frac{1}{1+\hat\Pi_p^{\mathrm{even}}(0,m) - \hat\Pi_p^{\mathrm{odd}}(0,m)} \le \frac{1}{1-\hat\Pi_p^{\mathrm{odd}}(0,m)}. \tag{3.5.2}
$$

An upper bound for $g_2(p,m)$. As preliminaries, we prove one proposition and two lemmas in the following. The next one plays the most important role. Notice that $\{ k \in \mathbb{T}^d \mid \hat{D}(k) \geq 0 \} = [-\pi/2, \pi/2]^d$ on $\mathbb{L}^d$.

Proposition 3.5.1. On the BCC lattice $\mathbb{L}^d$, it suffices to attend only the range $[-\pi/2,\pi/2]^d$ in order to calculate the supremum in $g_2(p,m)$ with respect to k:

$$g_2(p,m) = \sup_{\substack{k\in\mathbb{T}^d,\\ z\in\mathbb{C}\colon |z|\in\{1,m\}}} \frac{|\hat{\varphi}_p(k,z)|}{|\hat{S}_{\mu_p(z)}(k)|} = \sup_{\substack{k\in[-\pi/2,\pi/2]^d,\\ z\in\mathbb{C}\colon |z|\in\{1,m\}}} \frac{|\hat{\varphi}_p(k,z)|}{|\hat{S}_{\mu_p(z)}(k)|}.$$

Proof. We notice that

$$\begin{aligned} g_2(p,m) &= \sup_{\substack{k\in\mathbb{T}^d,\\ r\in\{1,m\},\\ \theta\in\mathbb{T}}} \frac{\left|\hat{\varphi}_p(k,r\mathrm{e}^{\mathrm{i}\theta})\right|}{|\hat{S}_{\mu_p(r\mathrm{e}^{\mathrm{i}\theta})}(k)|} \\ &= \left(\sup_{\substack{k\in[-\pi/2,\pi/2]^d,\\ r\in\{1,m\},\\ \theta\in\mathbb{T}}} \frac{\left|\hat{\varphi}_p(k,r\mathrm{e}^{\mathrm{i}\theta})\right|}{|\hat{S}_{\mu_p(r\mathrm{e}^{\mathrm{i}\theta})}(k)|} \right) \vee \left(\sup_{\substack{k\in\mathbb{T}^d\setminus[-\pi/2,\pi/2]^d,\\ r\in\{1,m\},\\ \theta\in\mathbb{T}}} \frac{\left|\hat{\varphi}_p(k,r\mathrm{e}^{\mathrm{i}\theta})\right|}{|\hat{S}_{\mu_p(r\mathrm{e}^{\mathrm{i}\theta})}(k)|} \right). \end{aligned} \tag{3.5.3}$$

Given some $k = (k_1,\dots,k_d) \in \mathbb{T}^d \setminus [-\pi/2,\pi/2]^d$, there exists $\tilde{k} = (\tilde{k}_1,\dots,\tilde{k}_d) \in [-\pi/2,\pi/2]^d$ such that, for $j = 1,\dots,d$,

$$\tilde{k}_j = \begin{cases} k_j & [-\pi/2 \leq k_j \leq \pi/2], \\ \pi - k_j & [\pi/2 < k_j \leq \pi], \\ -\pi - k_j & [-\pi \leq k_j < -\pi/2]. \end{cases}$$

The second and third lines are the reflection across the hyperplanes $k_j = \pi/2$ and $k_j = -\pi/2$, respectively. As we show in the following, it is possible to replace k in the second factor in (3.5.3) to $\tilde{k}$.

Fix $k \in \mathbb{T}^d \setminus [-\pi/2,\pi/2]^d$, $r \in \{1,m\}$ and $\theta \in \mathbb{T}$. Note that $\tilde{k}_j = k_j$ for $j = 1,\dots,d$ when $-\pi/2 \leq k_j \leq \pi/2$. Since $\varphi_p(x,t)$ is an even function with respect to x,

$$\begin{aligned} \hat{\varphi}_p(k,r\mathrm{e}^{\mathrm{i}\theta}) &= \sum_{(x,t)} \varphi_p(x,t) r^t \mathrm{e}^{\mathrm{i}\sum_{j=1}^d k_j x_j} \mathrm{e}^{\mathrm{i}\theta t} \\ &= \sum_{(x,t)} \varphi_p(x,t) r^t \mathrm{e}^{-\mathrm{i}\tilde{k}\cdot x} \mathrm{e}^{\pi\mathrm{i}\sum_{j=1}^d \mathbb{1}\{\tilde{k}_j\neq k_j\}\,\mathrm{sgn}(k_j)x_j} \mathrm{e}^{\mathrm{i}\theta t} \\ &= \sum_{(x,t)} \varphi_p(-x,t) r^t \mathrm{e}^{\mathrm{i}\tilde{k}\cdot x} (-1)^{-\sum_{j=1}^d \mathbb{1}\{\tilde{k}_j\neq k_j\}\,\mathrm{sgn}(k_j)x_j} \mathrm{e}^{\mathrm{i}\theta t} \\ &= \sum_{(x,t)} \varphi_p(x,t) r^t \mathrm{e}^{\mathrm{i}\tilde{k}\cdot x} (-1)^{\sum_{j=1}^d \mathbb{1}\{\tilde{k}_j\neq k_j\}\,\mathrm{sgn}(k_j)x_j} \mathrm{e}^{\mathrm{i}\theta t}, \end{aligned}$$

where $\mathrm{sgn}\colon \mathbb{R} \to \{-1,0,+1\}$ denotes the sign function. We divide into two cases that $\sum_{j=1}^d \mathbb{1}_{\{\tilde{k}_j\neq k_j\}}$ is either even or odd.

On the one hand, suppose that $\sum_{j=1}^d \mathbb{1}_{\{\tilde{k}_j\neq k_j\}}$ is even. Since all coordinates of some $x \in \mathbb{L}^d$ are either even or odd, $\sum_{j=1}^d \mathbb{1}_{\{\tilde{k}_j\neq k_j\}}\,\mathrm{sgn}(k_j)x_j$ is even. It is easy to see that

$$\hat{\varphi}_p(k,r\mathrm{e}^{\mathrm{i}\theta}) = \sum_{(x,t)} \varphi_p(x,t) r^t \mathrm{e}^{\mathrm{i}\tilde{k}\cdot x} \mathrm{e}^{\mathrm{i}\theta t} = \hat{\varphi}_p(\tilde{k},r\mathrm{e}^{\mathrm{i}\theta}).$$

On the other hand, suppose that $\sum_{j=1}^{d} \mathbb{1}_{\{\tilde{k}_j \neq k_j\}}$ is odd. Let $\mathbb{L}_{\mathrm{e}}^d = \{x \in \mathbb{L}^d \mid \forall j,\ x_j \text{ is even}\}$, $\mathbb{L}_{\mathrm{o}}^d = \{\, x \in \mathbb{L}^d \mid \forall j,\ x_j \text{ is odd}\,\}$, $\mathbb{Z}_{\mathrm{e}+}$ be the set of non-negative even numbers and $\mathbb{Z}_{\mathrm{o}+}$ be the set of positive odd numbers. Note that $\varphi_p(x,t) = \mathbb{P}_p((o,0) \to (x,t)) = 0$ when $(x,t) \neq (\mathbb{L}_{\mathrm{e}}^d \times \mathbb{Z}_{\mathrm{e}+}) \cup (\mathbb{L}_{\mathrm{o}}^d \times \mathbb{Z}_{\mathrm{o}+})$ due to the definition of the nearest-neighbor oriented percolation. Then, We rewrite $\hat{\varphi}_p(k, r\mathrm{e}^{\mathrm{i}\theta})$ as

$$\hat{\varphi}_p(k, r\mathrm{e}^{\mathrm{i}\theta}) = \sum_{\substack{x \in \mathbb{L}_{\mathrm{e}}^d, \\ t \in \mathbb{Z}_{\mathrm{e}+}}} \varphi_p(x,t) r^t \mathrm{e}^{\mathrm{i}\tilde{k}\cdot x} \mathrm{e}^{\mathrm{i}\theta t} - \sum_{\substack{x \in \mathbb{L}_{\mathrm{o}}^d, \\ t \in \mathbb{Z}_{\mathrm{o}+}}} \varphi_p(x,t) r^t \mathrm{e}^{\mathrm{i}\tilde{k}\cdot x} \mathrm{e}^{\mathrm{i}\theta t}. \tag{3.5.4}$$

Let $\tilde{\theta} = -\theta + \pi \operatorname{sgn}(\theta)$, which means the reflection of across points $\pi/2$ and $-\pi/2$ in one dimension. This definition is different from that of $\tilde{k}$ because even $\theta \in [-\pi/2, \pi/2]$ is reflected. By using $\tilde{\theta}$,

$$\mathrm{e}^{\mathrm{i}\theta t} = \mathrm{e}^{-\mathrm{i}\tilde{\theta} t} \mathrm{e}^{\pi \mathrm{i} \operatorname{sgn}(\theta) t} = (-1)^t \mathrm{e}^{-\mathrm{i}\tilde{\theta} t}. \tag{3.5.5}$$

Substituting (3.5.5) into (3.5.4), we obtain

$$\begin{aligned} \hat{\varphi}_p(k, r\mathrm{e}^{\mathrm{i}\theta}) &= \sum_{\substack{x \in \mathbb{L}_{\mathrm{e}}^d, \\ t \in \mathbb{Z}_{\mathrm{e}+}}} \varphi_p(x,t) r^t \mathrm{e}^{\mathrm{i}\tilde{k}\cdot x} \mathrm{e}^{-\mathrm{i}\tilde{\theta} t} + \sum_{\substack{x \in \mathbb{L}_{\mathrm{o}}^d, \\ t \in \mathbb{Z}_{\mathrm{o}+}}} \varphi_p(x,t) r^t \mathrm{e}^{\mathrm{i}\tilde{k}\cdot x} \mathrm{e}^{-\mathrm{i}\tilde{\theta} t} \\ &= \sum_{(x,t)} \varphi_p(x,t) r^t \mathrm{e}^{\mathrm{i}\tilde{k}\cdot x} \mathrm{e}^{-\mathrm{i}\tilde{\theta} t} = \hat{\varphi}_p(\tilde{k}, r\mathrm{e}^{-\mathrm{i}\tilde{\theta}}). \end{aligned}$$

As for θ, $-\tilde{\theta}$ takes a value in $\mathbb{T}$. It is possible to replace $-\tilde{\theta}$ in $\hat{\varphi}_p(\tilde{k}, r\mathrm{e}^{-\mathrm{i}\tilde{\theta}})$ to θ since it is the dummy variable of the supremimum in (3.5.3).

The above discussions also hold for $\hat{S}_{\mu_p(r\mathrm{e}^{\mathrm{i}\theta})}(k)$ because $Q_1(x,t)$ satisfies the same properties "$Q_1(x,t)$ is an even function with respect to x" and "$Q_1(x,t) = D^{*t}(x)\mathbb{1}_{\{t \in \mathbb{Z}_+\}} = 0$ when $(x,t) \neq (\mathbb{L}_{\mathrm{e}}^d \times \mathbb{Z}_{\mathrm{e}+}) \cup (\mathbb{L}_{\mathrm{o}}^d \times \mathbb{Z}_{\mathrm{o}+})$" as $\hat{\varphi}_p(k, r\mathrm{e}^{\mathrm{i}\theta})$. Therefore,

$$g_2(p,m) = \Bigg(\sup_{\substack{k \in [-\pi/2, \pi/2]^d, \\ r \in \{1, m\}, \\ \theta \in \mathbb{T}}} \frac{\big|\hat{\varphi}_p(k, r\mathrm{e}^{\mathrm{i}\theta})\big|}{|\hat{S}_{\mu_p(r\mathrm{e}^{\mathrm{i}\theta})}(k)|} \Bigg) \vee \Bigg(\sup_{\substack{\tilde{k} \in [-\pi/2, \pi/2]^d, \\ r \in \{1, m\}, \\ \theta \in \mathbb{T}}} \frac{\big|\hat{\varphi}_p(\tilde{k}, r\mathrm{e}^{\mathrm{i}\theta})\big|}{|\hat{S}_{\mu_p(r\mathrm{e}^{\mathrm{i}\theta})}(\tilde{k})|} \Bigg).$$

This completes the proof of Proposition 3.5.1. □

Lemma 3.5.2. For every $k \in \mathbb{T}^d$ satisfying $\hat{D}(k) \geq 0$, $r \in [0,1]$ and $\theta \in \mathbb{T}$,

$$\Big|1 - r\mathrm{e}^{\mathrm{i}\theta} \hat{D}(k)\Big| \geq \frac{1}{2}\bigg(\frac{|\theta|}{\pi} + 1 - \hat{D}(k)\bigg).$$

Proof. Let $\xi \equiv \hat{D}(k)$ and

$$\begin{aligned} f_r(\xi, \theta) &\equiv 4\Big|1 - r\mathrm{e}^{\mathrm{i}\theta}\xi\Big|^2 - \bigg(\frac{|\theta|}{\pi} + 1 - \xi\bigg)^2 \\ &= \underbrace{(4r^2 - 1)\xi^2 + \frac{2}{\pi}\xi|\theta| + 2\xi - 8r\xi\cos\theta}_{(a)} + \underbrace{3 - \bigg(\frac{\theta^2}{\pi^2} + \frac{2}{\pi}|\theta|\bigg)}_{(b)}. \end{aligned}$$

$f_r(\xi, \theta)$ is an even function with respect to θ. It suffices to show that $f_r(\xi, \theta) \geq 0$ for $\xi \in [0,1]$ and $\theta \in [0, \pi]$. Clearly, the part (b) in $f_r(\xi, \theta)$ is monotonically decreasing with respect to θ and non-negative. On the other hand, since the cosine function is monotonically decreasing for such θ, the part (a) in $f_r(\xi, \theta)$ is monotonically increasing with respect to (ξ, θ). It takes the minimum value 0 at $(\xi, \theta) = (0, 0)$. This completes the proof of Lemma 3.5.2. □

The next lemma is almost identical to Lemma 3.5.2, but the ranges of k are different from each other.

Lemma 3.5.3. For every $k \in \mathbb{T}^d$ and $\mu \in \mathbb{C}$ satisfying $|\mu| \le 1$,

$$\Big|1 - \mathrm{e}^{\mathrm{i}\arg\mu}\hat{D}(k)\Big| \le 2\Big|1 - \mu\hat{D}(k)\Big|. \tag{3.5.6}$$

Proof. Let $r = |\mu| \in [0,1]$ and $\theta = \arg\mu \in \mathbb{T}$. We show that

$$\begin{aligned} f(\theta) &\equiv 4\Big|1 - r\mathrm{e}^{\mathrm{i}\theta}\hat{D}(k)\Big|^2 - \Big|1 - \mathrm{e}^{\mathrm{i}\theta}\hat{D}(k)\Big|^2 \\ &= 3 + \big(4r^2 - 1\big)\hat{D}(k)^2 - 2(4r-1)\hat{D}(k)\cos\theta \\ &\ge 0. \end{aligned}$$

By the symmetry of the cosine function, we restrict the range of θ to $[0,\pi]$. Since

$$f'(\theta) = 2(4r-1)\hat{D}(k)\sin\theta \begin{cases} \ge 0 & [r \ge 1/4 \quad \text{and} \quad \hat{D}(k) \ge 0], \\ \le 0 & [r < 1/4 \quad \text{and} \quad \hat{D}(k) \ge 0], \\ \le 0 & [r \ge 1/4 \quad \text{and} \quad \hat{D}(k) < 0], \\ \ge 0 & [r < 1/4 \quad \text{and} \quad \hat{D}(k) < 0], \end{cases}$$

$f(\theta)$ is monotonic with respect to θ whenever r and k are fixed. Thus, $f(\theta)$ takes the minimum value at $\theta = 0$, $\pi/2$ or π. When $\theta = 0$ or π, we need to show $3 + \big(4x^2 - 1\big)y^2 \pm 2(4x-1)y \ge 0$ for all $(x,y) \in [0,1]\times[-1,1]$. When $\theta = \pi/2$, we need to show $3 + \big(4x^2-1\big)y^2 \ge 0$ for all $(x,y) \in [0,1]\times[-1,1]$. We omit proofs of these inequalities because it is not hard to see them by standard calculus of derivatives. Therefore, $f(\theta) \ge 0$. $\square$

Now, we start to show an upper bound on $g_2(p,m)$. Applying the same method as [26, Lemma 8.11], we rewrite $\hat{\varphi}_p$ as

$$\begin{aligned} \frac{\hat{\varphi}_p(k,z)}{\hat{S}_{\mu_p(z)}(k)} = \frac{1 + \hat{\Pi}_p(k,z)}{1 + \hat{\Pi}_p(0,|z|)} + \frac{\hat{\varphi}_p(k,z)}{1 + \hat{\Pi}_p(0,|z|)} \\ \times \Big(\hat{\Pi}_p(0,|z|)\big(1 - \mathrm{e}^{\mathrm{i}\theta}\hat{D}(k)\big) + \hat{q}_p(k,z)\big(\hat{\Pi}_p(k,z) - \hat{\Pi}_p(0,|z|)\big)\Big), \end{aligned}$$

where $\theta = \arg z$. By Proposition 3.5.1, the triangle inequality and the bootstrapping hypotheses $g_i(p,m) \le K_i$ for $i = 1,2$,

$$\begin{aligned} g_2(p,m) \le \sup_{\substack{k \in [-\frac{\pi}{2},\frac{\pi}{2}]^d, \\ z \in \mathbb{C}\colon |z| \in \{1,m\}}} &\left(\frac{1 + |\hat{\Pi}_p(k,z)|}{1 + \hat{\Pi}_p(0,|z|)} + \frac{K_2\hat{S}_{\mu_p(z)}(k)}{1 + \hat{\Pi}_p(0,|z|)}\bigg(\hat{\Pi}_p(0,|z|)\Big|1 - \mathrm{e}^{\mathrm{i}\theta}\hat{D}(k)\Big|\right. \\ &\left.\left. + p|z|\underbrace{\big|\hat{D}(k)\big|}_{\le 1}\Big(\big|\hat{\Pi}_p(0,|z|) - \hat{\Pi}_p(0,z)\big| + \big|\hat{\Pi}_p(0,z) - \hat{\Pi}_p(k,z)\big|\Big)\right)\right) \\ \le \sup_{\substack{k \in [-\frac{\pi}{2},\frac{\pi}{2}]^d, \\ z \in \mathbb{C}\colon |z| \in \{1,m\}}} &\left(\frac{1 + |\hat{\Pi}_p(k,z)|}{1 + \hat{\Pi}_p(0,|z|)} + \frac{K_2}{1 + \hat{\Pi}_p(0,|z|)}\bigg(\hat{\Pi}_p(0,|z|)\frac{|1 - \mathrm{e}^{\mathrm{i}\theta}\hat{D}(k)|}{|1 - \mu_p(z)\hat{D}(k)|}\right. \\ &\left.\left. + K_1\sum_{(x,t)}\Big(\Big|\pi_p^{(0)}(x,t) - \pi_p^{(1)}(x,t)\Big| + \sum_{N=0}^{\infty}\pi_p^{(N)}(x,t)\Big)|z|^t \cdot \frac{|1 - \mathrm{e}^{\mathrm{i}\theta t}| + 1 - \cos k\cdot x}{|1 - \mu_p(z)\hat{D}(k)|}\right)\right). \end{aligned} \tag{3.5.7}$$

By Lemma 3.5.2–3.5.3 and the inequality $\left|1-\mathrm{e}^{\mathrm{i}\theta t}\right| = 2|\sin(\theta t/2)| \le t|\theta|$,

$$\begin{aligned}
g_2(p,m) \le & \sup_{\substack{k\in[-\frac{\pi}{2},\frac{\pi}{2}]^d,\\ z\in\mathbb{C}\colon |z|\in\{1,m\}}} \Bigg(\frac{1+|\hat\Pi_p(k,z)|}{1+\hat\Pi_p(0,|z|)} + \frac{K_2}{1+\hat\Pi_p(0,|z|)} \Bigg(2\hat\Pi_p(0,|z|) \\
& + 2K_1 \sum_{(x,t)} \Big(\Big|\pi_p^{(0)}(x,t) - \pi_p^{(1)}(x,t)\Big| + \sum_{N=0}^{\infty} \pi_p^{(N)}(x,t) \Big) |z|^t \cdot \frac{t|\theta| + 1 - \cos k\cdot x}{\pi^{-1}|\theta| + 1 - \hat D(k)} \Bigg)\Bigg) \\
= & \sup_{\substack{k\in[-\frac{\pi}{2},\frac{\pi}{2}]^d,\\ z\in\mathbb{C}\colon |z|\in\{1,m\}}} \Bigg(\frac{1+|\hat\Pi_p(k,z)|}{1+\hat\Pi_p(0,|z|)} + \frac{K_2}{1+\hat\Pi_p(0,|z|)} \Bigg(2\hat\Pi_p(0,|z|) \\
& + \frac{2K_1}{\pi^{-1}|\theta| + 1 - \hat D(k)} \Bigg(\pi \sum_{(x,t)} \Big(\Big|\pi_p^{(0)}(x,t) - \pi_p^{(1)}(x,t)\Big| + \sum_{N=0}^{\infty} \pi_p^{(N)}(x,t) \Big) |z|^t t \pi^{-1}|\theta| \\
& + \sum_{(x,t)} \Big(\Big|\pi_p^{(0)}(x,t) - \pi_p^{(1)}(x,t)\Big| + \sum_{N=0}^{\infty} \pi_p^{(N)}(x,t) \Big) |z|^t (1-\cos k\cdot x) \Bigg)\Bigg)\Bigg).
\end{aligned}$$

By the inequality $(a_1\phi + b_1\psi)/(a_2\phi+b_2\psi) \le (a_1 \vee b_1)/(a_2 \wedge b_2)$ for some $a_1, b_1, a_2, b_2, \phi, \psi \in \mathbb{R}$, we arrive in

$$\begin{aligned}
g_2(p,m) \le & \frac{1+\hat\Pi_p^{\mathrm{even}}(0,m) + \hat\Pi_p^{\mathrm{odd}}(0,m)}{1-\hat\Pi_p^{\mathrm{odd}}(0,m)} + \frac{K_2}{1-\hat\Pi_p^{\mathrm{odd}}(0,m)} \Bigg(2\hat\Pi_p^{\mathrm{even}}(0,m) \\
& + 2K_1 \Bigg(\pi \sum_{(x,t)} \Big(\Pi_p^{\mathrm{even}}(x,t) + \Pi_p^{\mathrm{odd}}(x,t) \Big) m^t t \Bigg) \\
& \vee \Bigg(\sum_{(x,t)} \Big(\Pi_p^{\mathrm{even}}(x,t) + \Pi_p^{\mathrm{odd}}(x,t) \Big) m^t \frac{1-\cos k\cdot x}{1-\hat D(k)} \Bigg)\Bigg).
\end{aligned} \tag{3.5.8}$$

An upper bound for $g_3(p,m)$. We use a modified version of [41, Lemma 5.7]. Let

$$\begin{aligned}
\hat A(l,z) &= \frac{1}{1-\hat q_p(l,z)\big(1+\hat\Pi_p(l,z)\big)} = \frac{1}{1-\hat a(l,z)}, \\
a(x,t) &= q_p(x,t) + (q_p \star \Pi_p)(x,t), \\
\hat a^{\cos}(l,k;z) &= \sum_{(x,t)} a(x,t)\cos(l\cdot x)\cos(k\cdot x) z^t, \\
\hat a^{\sin}(l,k;z) &= \sum_{(x,t)} a(x,t)\sin(l\cdot x)\sin(k\cdot x) z^t.
\end{aligned}$$

Then,

$$\begin{aligned}
\Big|\hat\Delta_k\big(\hat q_p(l,z)\hat\varphi_p(l,z)\big)\Big| &= \Bigg|\hat\Delta_k\Bigg(\frac{1}{1-\hat q_p(l,z)\big(1+\hat\Pi_p(l,z)\big)} - 1\Bigg)\Bigg| = \Big|\hat\Delta_k \hat A(l,z)\Big| \\
&\le \frac{|\hat A(l+k,z)| + |\hat A(l-k,z)|}{2}\Big|\hat A(l,z)\Big| |\hat a(l,z) - \hat a^{\cos}(l,k;z)| \\
&\qquad + \Big|\hat A(l,z)\Big|\Big|\hat A(l+k,z)\Big|\Big|\hat A(l-k,z)\Big| \big|\hat a^{\sin}(l,k;z)\big|^2.
\end{aligned}$$

By the triangle inequality and arithmetic of the trigonometric functions, combining

$$|\hat{a}(l,z) - \hat{a}^{\cos}(l,k;z)| \le \sum_{(x,t)} |a(x,t)||z|^t(1-\cos k\cdot x)$$

and

$$\begin{aligned}
\left|\hat{a}^{\sin}(l,k;z)\right|^2 &\le \left(\sum_{(x,t)} |a(x,t)||z|^t \sin^2 l\cdot x\right)\left(\sum_{(x,t)} |a(x,t)||z|^t \sin^2 k\cdot x\right) && (3.5.9)\\
&= \left(\sum_{(x,t)} |a(x,t)||z|^t \frac{1-\cos(2l\cdot x)}{2}\right)\left(\sum_{(x,t)} |a(x,t)||z|^t (1-\cos^2 k\cdot x)\right) && (3.5.10)\\
&\le \left(\sum_{(x,t)} |a(x,t)||z|^t \big(1-\cos(2l\cdot x)\big)\right)\left(\sum_{(x,t)} |a(x,t)||z|^t (1-\cos k\cdot x)\right)
\end{aligned}$$

leads to

$$\begin{aligned}
\left|\hat{\Delta}_k \hat{A}(l,z)\right| \le \Big(\widehat{|a|}(0,|z|) - \widehat{|a|}(k,|z|)\Big)\Bigg(&\frac{|\hat{A}(l+k,z)| + |\hat{A}(l-k,z)|}{2}\left|\hat{A}(l,z)\right|\\
&+ \left|\hat{A}(l,z)\right|\left|\hat{A}(l+k,z)\right|\left|\hat{A}(l-k,z)\right|\Big(\widehat{|a|}(0,|z|) - \widehat{|a|}(2l,|z|)\Big)\Bigg) \qquad (3.5.11)
\end{aligned}$$

(cf., Lemma 1.3.2). By the telescopic inequality (1.3.1),

$$\begin{aligned}
&\widehat{|a|}(0,|z|) - \widehat{|a|}(k,|z|) = \sum_{(x,t)} |q_p(x,t) + (q_p \star \Pi_p)(x,t)||z|^t(1-\cos k\cdot x)\\
&\le \sum_{(x,t)} q_p(x,t)|z|^t(1-\cos k\cdot x)\\
&\quad + \sum_{(x,t)}\sum_{(y,s)} q_p(y,s)|\Pi_p(x-y,t-s)||z|^t \underbrace{\Big(1-\cos k\cdot\big(y+(x-y)\big)\Big)}_{\le 2(1-\cos k\cdot y)+2(1-\cos k\cdot(x-y))}\\
&\le p|z|\big(1-\hat{D}(k)\big)\left(1 + 2\widehat{|\Pi_p|}(0,|z|) + 2\cdot\frac{\widehat{|\Pi_p|}(0,|z|) - \widehat{|\Pi_p|}(k,|z|)}{1-\hat{D}(k)}\right)\\
&\le K_1\big(1-\hat{D}(k)\big)\Bigg(1 + 2\hat{\Pi}_p^{\mathrm{even}}(0,|z|) + 2\hat{\Pi}_p^{\mathrm{odd}}(0,|z|)\\
&\quad + 2\cdot\frac{\hat{\Pi}_p^{\mathrm{even}}(0,|z|) - \hat{\Pi}_p^{\mathrm{even}}(k,|z|)}{1-\hat{D}(k)} + 2\cdot\frac{\hat{\Pi}_p^{\mathrm{odd}}(0,|z|) - \hat{\Pi}_p^{\mathrm{odd}}(k,|z|)}{1-\hat{D}(k)}\Bigg).
\end{aligned}$$

Substituting this bound into (3.5.11), and applying the same method as the proof of Lemma 2.5.1, we obtain

$$\begin{aligned}
g_3(p,m) \le \left(1 \vee \frac{g_2(p,m)}{1-\hat{\Pi}_p^{\mathrm{even}}(0,m) - \hat{\Pi}_p^{\mathrm{odd}}(0,m)}\right)^3 K_1^2&\\
\times \Bigg(1 + 2\Big(\hat{\Pi}_p^{\mathrm{even}}(0,m) + \hat{\Pi}_p^{\mathrm{odd}}(0,m)\Big)&\\
+ 2\left\|\frac{\big(\hat{\Pi}_p^{\mathrm{even}}(0,m) + \hat{\Pi}_p^{\mathrm{odd}}(0,m)\big) - \big(\hat{\Pi}_p^{\mathrm{even}}(\bullet,m) + \hat{\Pi}_p^{\mathrm{odd}}(\bullet,m)\big)}{1-\hat{D}(\bullet)}\right\|_\infty\Bigg)^2&. \qquad (3.5.12)
\end{aligned}$$

3.6 Diagrammatic bounds in terms of random-walk quantities

In this section, we bound the basic diagrams $B_{p,m}^{(\lambda,\rho)}$, $T_{p,m}^{(\lambda,\rho)}$ and $V_{p,m}^{(\lambda,\rho)}(k)$ above and prove Proposition 3.1.3. We define the random-walk quantities as

$$\varepsilon_i^{(\nu)} = \sum_{n=\nu}^{\infty} D^{*2n}(o) \times \begin{cases} 1 & [i=1], \\ (n-\nu+1) & [i=2] \end{cases} \tag{3.6.1}$$

for $\nu \in \mathbb{Z}_+$.

Lemma 3.6.1. For $\mu, \rho \in \mathbb{N}$, $p \in [0, p_c)$ and $m \in [1, m_p)$,

$$B_{p,m}^{(\lambda,\rho)} \leq K_1^{\lambda+\rho} K_2^2 \varepsilon_1^{(\lfloor(\lambda+\rho)/2\rfloor)}, \tag{3.6.2}$$

$$T_{p,m}^{(\lambda,\rho)} \leq \sqrt{2} K_1^{\lambda+\rho} K_2^3 \varepsilon_2^{(\lfloor(\lambda+\rho)/2\rfloor)}, \tag{3.6.3}$$

$$\left\| \frac{\hat{V}_{p,m}^{(\lambda,\rho)}}{1-\hat{D}} \right\|_\infty \leq \begin{cases} K_1^2 \|D\|_\infty + K_1^3 K_2 \sqrt{\varepsilon_1^{(2)}} + \left\| \dfrac{\hat{V}_{p,m}^{(2,1)}}{1-\hat{D}} \right\|_\infty & [\lambda = \rho = 1], \\ \lambda(\lambda-1) K_1^{\lambda+\rho} K_2^2 \varepsilon_1^{(\lfloor(\lambda+\rho-1)/2\rfloor)} & \\ \quad + \lambda K_1^{\lambda+\rho-1} K_2 K_3 \big(\sqrt{2}+4\big) \varepsilon_2^{(\lfloor(\lambda+\rho-1)/2\rfloor)} & [\lambda \geq 2 \text{ or } \rho \geq 2]. \end{cases} \tag{3.6.4}$$

Proof of (3.6.2) *and* (3.6.3) *in Lemma 3.6.1.* We first show (3.6.2). By the inverse Fourier-Laplace transform and the bootstrapping hypotheses,

$$\begin{aligned} B_{p,m}^{(\lambda,\rho)} &= \sup_{(x,t)} \left| \int_{\mathbb{T}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \hat{q}_p(k, \mathrm{e}^{\mathrm{i}\theta})^\lambda \hat{\varphi}_p(k, \mathrm{e}^{\mathrm{i}\theta}) m^\rho \hat{q}_p(k, \mathrm{e}^{-\mathrm{i}\theta})^\rho \hat{\varphi}_p(k, m\mathrm{e}^{-\mathrm{i}\theta}) \right| \\ &\leq \int_{\mathbb{T}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \left| \hat{q}_p(k, \mathrm{e}^{\mathrm{i}\theta}) \right|^\lambda \left| \hat{\varphi}_p(k, \mathrm{e}^{\mathrm{i}\theta}) \right| m^\rho \left| \hat{q}_p(k, \mathrm{e}^{-\mathrm{i}\theta}) \right|^\rho \left| \hat{\varphi}_p(k, m\mathrm{e}^{-\mathrm{i}\theta}) \right| \\ &= p^{\lambda+\rho} m^\rho \int_{\mathbb{T}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \left| \hat{D}(k) \right|^{\lambda+\rho} \left| \hat{\varphi}_p(k, \mathrm{e}^{\mathrm{i}\theta}) \right| \left| \hat{\varphi}_p(k, m\mathrm{e}^{-\mathrm{i}\theta}) \right| \\ &\leq K_1^{\lambda+\rho} K_2^2 \int_{\mathbb{T}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \left| \hat{D}(k) \right|^{\lambda+\rho} \left| \hat{S}_{\mu_p(\mathrm{e}^{\mathrm{i}\theta})}(k) \right| \left| \hat{S}_{\mu_p(m\mathrm{e}^{-\mathrm{i}\theta})}(k) \right|. \end{aligned}$$

Applying Schwarz's inequality and the residue theorem to the integral with respect to θ, we obtain

$$\begin{aligned} B_{p,m}^{(\lambda,\rho)} &\leq K_1^{\lambda+\rho} K_2^2 \int_{\mathbb{T}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \left| \hat{D}(k) \right|^{\lambda+\rho} \left(\int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \left| \hat{S}_{\mu_p(\mathrm{e}^{\mathrm{i}\theta})}(k) \right|^2 \right)^{1/2} \left(\int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \left| \hat{S}_{\mu_p(m\mathrm{e}^{-\mathrm{i}\theta})}(k) \right|^2 \right)^{1/2} \\ &= K_1^{\lambda+\rho} K_2^2 \int_{\mathbb{T}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \left| \hat{D}(k) \right|^{\lambda+\rho} \left(\frac{1}{1-\mu_p(1)^2 \hat{D}(k)^2} \right)^{1/2} \left(\frac{1}{1-\mu_p(m)^2 \hat{D}(k)^2} \right)^{1/2} \\ &\leq K_1^{\lambda+\rho} K_2^2 \int_{\mathbb{T}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\hat{D}(k)^{2\lfloor\frac{\lambda+\rho}{2}\rfloor}}{1-\hat{D}(k)^2} = K_1^{\lambda+\rho} K_2^2 \varepsilon_1^{(\lfloor(\lambda+\rho)/2\rfloor)}. \end{aligned}$$

In the equality in the above, we have regarded an integral $(2\pi)^{-1} \int_{-\pi}^{\pi} f(\exp(\mathrm{i}\theta))\,\mathrm{d}\theta$ for some function f as the line integral over the unit circle on $\mathbb{C}$: $(2\pi\mathrm{i})^{-1} \int_{\{z||z|=1\}} f(z)/z\,\mathrm{d}z$. In the last line, we have used $\mu_p(m) \leq 1$ for every $m \geq 0$, and the trivial inequality $\hat{D}(k)^{\lambda+\rho} \leq \hat{D}(k)^{2\lfloor\frac{\lambda+\rho}{2}\rfloor}$, which is led to by $\hat{D}(k) \leq 1$ for all $k \in \mathbb{T}^d$.

Similarly,

$$
\begin{aligned}
T_{p,m}^{(\lambda,\rho)} &= \sup_{(x,t)} \left| \int_{\mathbb{T}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \hat{q}_p(k,\mathrm{e}^{\mathrm{i}\theta})^\lambda \hat{\varphi}_p(k,\mathrm{e}^{\mathrm{i}\theta})^2 m^\rho \hat{q}_p(k,\mathrm{e}^{-\mathrm{i}\theta})^\rho \hat{\varphi}_p(k, m\mathrm{e}^{-\mathrm{i}\theta}) \right| \\
&\le \int_{\mathbb{T}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \left| \hat{q}_p(k,\mathrm{e}^{\mathrm{i}\theta}) \right|^\lambda \left| \hat{\varphi}_p(k,\mathrm{e}^{\mathrm{i}\theta}) \right|^2 m^\rho \left| \hat{q}_p(k,\mathrm{e}^{-\mathrm{i}\theta}) \right|^\rho \left| \hat{\varphi}_p(k, m\mathrm{e}^{-\mathrm{i}\theta}) \right| \\
&= p^{\lambda+\rho} m^\rho \int_{\mathbb{T}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \left| \hat{D}(k) \right|^{\lambda+\rho} \left| \hat{\varphi}_p(k,\mathrm{e}^{\mathrm{i}\theta}) \right|^2 \left| \hat{\varphi}_p(k, m\mathrm{e}^{-\mathrm{i}\theta}) \right| \\
&\le K_1^{\lambda+\rho} K_2^3 \int_{\mathbb{T}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \left| \hat{D}(k) \right|^{\lambda+\rho} \left| \hat{S}_{\mu_p(\mathrm{e}^{\mathrm{i}\theta})}(k) \right|^2 \left| \hat{S}_{\mu_p(m\mathrm{e}^{-\mathrm{i}\theta})}(k) \right| \\
&\le K_1^{\lambda+\rho} K_2^3 \int_{\mathbb{T}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \left| \hat{D}(k) \right|^{\lambda+\rho} \left(\int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \left| \hat{S}_{\mu_p(\mathrm{e}^{\mathrm{i}\theta})}(k) \right|^4 \right)^{1/2} \left(\int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \left| \hat{S}_{\mu_p(m\mathrm{e}^{-\mathrm{i}\theta})}(k) \right|^2 \right)^{1/2} \\
&= K_1^{\lambda+\rho} K_2^3 \int_{\mathbb{T}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \left| \hat{D}(k) \right|^{\lambda+\rho} \left(\frac{1+\mu_p(1)^2 \hat{D}(k)^2}{\left(1-\mu_p(1)^2 \hat{D}(k)^2\right)^3} \right)^{1/2} \left(\frac{1}{1-\mu_p(m)^2 \hat{D}(k)^2} \right)^{1/2} \\
&\le \sqrt{2} K_1^{\lambda+\rho} K_2^3 \int_{\mathbb{T}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\hat{D}(k)^{2\lfloor \frac{\lambda+\rho}{2} \rfloor}}{\left(1-\hat{D}(k)^2\right)^2} \\
&= \sqrt{2} K_1^{\lambda+\rho} K_2^3 \sum_{n=0}^\infty (n+1) \int_{\mathbb{T}^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \hat{D}(k)^{2\left(n+\lfloor \frac{\lambda+\rho}{2} \rfloor\right)} = \sqrt{2} K_1^{\lambda+\rho} K_2^3 \varepsilon_2^{(\lfloor(\lambda+\rho)/2\rfloor)}.
\end{aligned}
$$

This completes the proof of (3.6.3). □

To prove (3.6.4), we need the next lemma.

Lemma 3.6.2. For every $k \in \mathbb{T}^d$ and $\mu \in \mathbb{C}$ satisfying $|\mu| \le 1$,

$$
1 - \hat{D}(2k) \le 2\left(1 - \hat{D}(k)^2\right). \tag{3.6.5}
$$

Proof. Note that

$$
\begin{aligned}
2\left(1-\hat{D}(k)^2\right) - \left(1-\hat{D}(2k)\right) &= 1 - 2\prod_{j=1}^d \cos^2 k_j + \prod_{j=1}^d \cos(2k_j) \\
&= 1 - 2\prod_{j=1}^d \cos^2 k_j + \prod_{j=1}^d \left(2\cos^2 k_j - 1\right)
\end{aligned}
$$

on $\mathbb{L}^d$. Let $h_d(\xi_1,\ldots,\xi_d) \equiv 1 - 2\prod_{j=1}^d \xi_j + \prod_{j=1}^d (2\xi_j - 1)$. It suffices to show that $h_d(\xi_1,\ldots,\xi_d) \ge 0$ for every $(\xi_1,\ldots,\xi_d) \in [0,1]^d$. To do so, we use the method of induction.

When $d = 1$, $h_1(\xi_1) = 0$ for every ξ_1. Next, we assume that $h_d(\xi_1,\ldots,\xi_d) \ge 0$ holds in some dimension d. Fix ξ_j for each $j = 1,\ldots,d$. Then, $h_{d+1}(\xi_1,\ldots,\xi_{d+1})$ is regarded as a linear function with respect to ξ_{d+1} and hence takes the minimum value at either the boundary point 0 or 1. When $\xi_{d+1} = 0$, clearly,

$$
h_{d+1}(\xi_1,\ldots,\xi_d,0) = 1 - \prod_{j=1}^d (2\xi_j - 1) \ge 0.
$$

When $\xi_{d+1} = 1$, by the induction hypothesis,

$$
h_{d+1}(\xi_1,\ldots,\xi_d,1) = 1 - 2\prod_{j=1}^d \xi_j + \prod_{j=1}^d (2\xi_j - 1) \ge 0.
$$

Therefore, $h_d(\xi_1,\dots,\xi_d)\ge 0$ holds in every dimensions d. $\square$

Proof of (3.6.4) *in Lemma 3.6.1 when $\lambda\ge 2$ and $\rho\ge 2$.* Recall the notation $\boldsymbol{x}=(x,\tau(\boldsymbol{x}))\in\mathbb{L}^d\times\mathbb{Z}_+$. By the telescopic inequality (1.3.1),

$$
\begin{aligned}
\hat{V}^{(\lambda,\rho)}_{p,m}(k) =\ & \sup_{\boldsymbol{x}}\sum_{\boldsymbol{y}_\lambda}\Big(q_p^{\star\lambda}\star\varphi_p\Big)(\boldsymbol{y}_\lambda)(1-\cos k\cdot y)\Big(m^\rho q_p^{\star\rho}\star\varphi_p^{(m)}\Big)(\boldsymbol{y}_\lambda-\boldsymbol{x})\\
=\ & \sup_{\boldsymbol{x}}\sum_{\boldsymbol{y}_1,\dots,\boldsymbol{y}_\lambda}\left(\prod_{i=1}^{\lambda-1}q_p(\boldsymbol{y}_i-\boldsymbol{y}_{i-1})\right)(q_p\star\varphi_p)(\boldsymbol{y}_\lambda-\boldsymbol{y}_{\lambda-1})\\
&\times\left(1-\cos\left(k\cdot\sum_{j=1}^{\lambda}(y_j-y_{j-1})\right)\right)\Big(m^\rho q_p^{\star\rho}\star\varphi_p^{(m)}\Big)(\boldsymbol{y}_\lambda-\boldsymbol{x})\\
\le\ & \lambda\sup_{\boldsymbol{x}}\sum_{j=1}^{\lambda-1}\sum_{\boldsymbol{y}_1,\dots,\boldsymbol{y}_\lambda}\left(\prod_{i\ne j}q_p(\boldsymbol{y}_i-\boldsymbol{y}_{i-1})\right)(q_p\star\varphi_p)(\boldsymbol{y}_\lambda-\boldsymbol{y}_{\lambda-1})\\
&\times q_p(\boldsymbol{y}_j-\boldsymbol{y}_{j-1})\Big(1-\cos\big(k\cdot(y_j-y_{j-1})\big)\Big)\Big(m^\rho q_p^{\star\rho}\star\varphi_p^{(m)}\Big)(\boldsymbol{y}_\lambda-\boldsymbol{x})\\
&+\lambda\sup_{\boldsymbol{x}}\sum_{\boldsymbol{y}_1,\dots,\boldsymbol{y}_\lambda}(q_p\star\varphi_p)(\boldsymbol{y}_\lambda-\boldsymbol{y}_{\lambda-1})\Big(1-\cos\big(k\cdot(y_\lambda-y_{\lambda-1})\big)\Big)\\
&\times\left(\prod_{i=1}^{\lambda-1}q_p(\boldsymbol{y}_i-\boldsymbol{y}_{i-1})\right)\Big(m^\rho q_p^{\star\rho}\star\varphi_p^{(m)}\Big)(\boldsymbol{y}_\lambda-\boldsymbol{x})\\
=\ & \sup_{\boldsymbol{x}}\sum_{\boldsymbol{y}_{j-1},\boldsymbol{y}_j,\boldsymbol{y}_\lambda}q_p^{\star(j-1)}(\boldsymbol{y}_{j-1})q_p(\boldsymbol{y}_j-\boldsymbol{y}_{j-1})\Big(1-\cos\big(k\cdot(y_j-y_{j-1})\big)\Big)\\
&\times\Big(q_p^{\star(\lambda-j)}\star\varphi_p\Big)(\boldsymbol{y}_\lambda-\boldsymbol{y}_j)\Big(m^\rho q_p^{\star\rho}\star\varphi_p^{(m)}\Big)(\boldsymbol{y}_\lambda-\boldsymbol{x})\\
&+\lambda\sup_{\boldsymbol{x}}\sum_{\boldsymbol{y}_{\lambda-1},\boldsymbol{y}_\lambda}q_p^{\star(\lambda-1)}(\boldsymbol{y}_{\lambda-1})\\
&\times(q_p\star\varphi_p)(\boldsymbol{y}_\lambda-\boldsymbol{y}_{\lambda-1})\Big(1-\cos\big(k\cdot(y_\lambda-y_{\lambda-1})\big)\Big)\Big(m^\rho q_p^{\star\rho}\star\varphi_p^{(m)}\Big)(\boldsymbol{y}_\lambda-\boldsymbol{x})
\end{aligned}
$$

where $\boldsymbol{y}_0=\boldsymbol{o}$. By the inverse Fourier-Laplace transform,

$$
\begin{aligned}
&\hat{V}^{(\lambda,\rho)}_{p,m}(k)\\
&\le\lambda(\lambda-1)\int_{\mathbb{T}^d}\frac{\mathrm{d}^dl}{(2\pi)^d}\int_{\mathbb{T}}\frac{\mathrm{d}\theta}{2\pi}\Big|\hat{q}_p(l,\mathrm{e}^{\mathrm{i}\theta})\Big|^{\lambda-1}\Big|\hat{\Delta}_k\hat{q}_p(l,\mathrm{e}^{\mathrm{i}\theta})\Big|\Big|\hat{\varphi}_p(l,\mathrm{e}^{\mathrm{i}\theta})\Big|m^\rho\Big|\hat{q}_p(l,\mathrm{e}^{-\mathrm{i}\theta})\Big|^\rho\Big|\hat{\varphi}_p(l,m\mathrm{e}^{-\mathrm{i}\theta})\Big|\\
&\quad+\lambda\int_{\mathbb{T}^d}\frac{\mathrm{d}^dl}{(2\pi)^d}\int_{\mathbb{T}}\frac{\mathrm{d}\theta}{2\pi}\Big|\hat{q}_p(l,\mathrm{e}^{\mathrm{i}\theta})\Big|^{\lambda-1}\Big|\hat{\Delta}_k\Big(\hat{q}_p(l,\mathrm{e}^{\mathrm{i}\theta})\hat{\varphi}_p(l,\mathrm{e}^{\mathrm{i}\theta})\Big)\Big|m^\rho\Big|\hat{q}_p(l,\mathrm{e}^{-\mathrm{i}\theta})\Big|^\rho\Big|\hat{\varphi}_p(l,m\mathrm{e}^{-\mathrm{i}\theta})\Big|\\
&\le\lambda(\lambda-1)\big(1-\hat{D}(k)\big)K_1^{\lambda+\rho}K_2^2\underbrace{\int_{\mathbb{T}^d}\frac{\mathrm{d}^dl}{(2\pi)^d}\int_{\mathbb{T}}\frac{\mathrm{d}\theta}{2\pi}\Big|\hat{D}(l)\Big|^{\lambda+\rho-1}\Big|\hat{S}_{\mu_p(\mathrm{e}^{\mathrm{i}\theta})}(l)\Big|\Big|\hat{S}_{\mu_p(m\mathrm{e}^{-\mathrm{i}\theta})}(l)\Big|}_{\le\varepsilon_1^{(\lfloor(\lambda+\rho-1)/2\rfloor)}}\\
&\quad+\lambda K_1^{\lambda+\rho-1}K_2K_3\int_{\mathbb{T}^d}\frac{\mathrm{d}^dl}{(2\pi)^d}\int_{\mathbb{T}}\frac{\mathrm{d}\theta}{2\pi}\Big|\hat{D}(l)\Big|^{\lambda+\rho-1}\Big|\hat{U}_{\mu_p(\mathrm{e}^{\mathrm{i}\theta})}(k,l)\Big|\Big|\hat{S}_{\mu_p(m\mathrm{e}^{-\mathrm{i}\theta})}(l)\Big|. \qquad (3.6.6)
\end{aligned}
$$

In the second inequality, we have used

$$
\Big|\hat{\Delta}_k\hat{D}(l)\Big|=\left|\sum_x D(x)(1-\cos k\cdot x)\mathrm{e}^{\mathrm{i}l\cdot x}\right|\le\sum_x D(x)(1-\cos k\cdot x)=1-\hat{D}(k).
$$

The second term in the right most in (3.6.6) contains two types of integrals:

$$\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \Big|\hat{D}(l)\Big|^{\lambda+\rho-1} \Big|\hat{S}_{\mu_p(\mathrm{e}^{\mathrm{i}\theta})}(l \pm k)\Big| \Big|\hat{S}_{\mu_p(\mathrm{e}^{\mathrm{i}\theta})}(l)\Big| \Big|\hat{S}_{\mu_p(m\mathrm{e}^{-\mathrm{i}\theta})}(l)\Big|, \tag{3.6.7}$$

$$\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \Big|\hat{D}(l)\Big|^{\lambda+\rho-1} \Big|\hat{S}_{\mu_p(\mathrm{e}^{\mathrm{i}\theta})}(l + k)\Big| \Big|\hat{S}_{\mu_p(\mathrm{e}^{\mathrm{i}\theta})}(l - k)\Big| \Big|\hat{S}_{\mu_p(\mathrm{e}^{\mathrm{i}\theta})}(l)\Big| \Big|\hat{S}_{\mu_p(m\mathrm{e}^{-\mathrm{i}\theta})}(l)\Big| \big(1 - \hat{D}(2l)\big). \tag{3.6.8}$$

The integral (3.6.7) is bounded above as, by Schwarz's inequality,

$$\begin{aligned}
(3.6.7) \le{}& \int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \Big|\hat{D}(l)\Big|^{\lambda+\rho-1} \left(\int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \Big|\hat{S}_{\mu_p(\mathrm{e}^{\mathrm{i}\theta})}(l \pm k)\Big|^2 \right)^{1/2} \\
&\times \left(\int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \Big|\hat{S}_{\mu_p(\mathrm{e}^{\mathrm{i}\theta})}(l)\Big|^4 \right)^{1/4} \left(\int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \Big|\hat{S}_{\mu_p(m\mathrm{e}^{-\mathrm{i}\theta})}(l)\Big|^4 \right)^{1/4} \\
={}& \int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \Big|\hat{D}(l)\Big|^{\lambda+\rho-1} \left(\frac{1}{1 - \mu_p(1)^2 \hat{D}(l \pm k)^2} \right)^{1/2} \\
&\times \left(\frac{1 + \mu_p(1)^2 \hat{D}(l)^2}{\big(1 - \mu_p(1)^2 \hat{D}(l)^2\big)^3} \right)^{1/4} \left(\frac{1 + \mu_p(m)^2 \hat{D}(l)^2}{\big(1 - \mu_p(m)^2 \hat{D}(l)^2\big)^3} \right)^{1/4} \\
\le{}& \sqrt{2} \int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \left(\frac{|\hat{D}(l)|^{(\lambda+\rho-1)/2}}{1 - \hat{D}(l \pm k)^2} \right)^{1/2} \left(\frac{|\hat{D}(l)|^{3(\lambda+\rho-1)/2}}{\big(1 - \hat{D}(l)^2\big)^3} \right)^{1/2}.
\end{aligned}$$

By Hölder's inequality and Schwarz's inequality again,

$$\begin{aligned}
(3.6.7) \le{}& \sqrt{2} \left(\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{|\hat{D}(l)|^{\lambda+\rho-1}}{\big(1 - \hat{D}(l \pm k)^2\big)^2} \right)^{1/4} \left(\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{|\hat{D}(l)|^{\lambda+\rho-1}}{\big(1 - \hat{D}(l)^2\big)^2} \right)^{3/4} \\
\le{}& \sqrt{2} \left(\sum_{n=0}^{\infty} (n+1) \int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \hat{D}(l)^{2\lfloor \frac{\lambda+\rho-1}{2} \rfloor} \hat{D}(l \pm k)^{2n} \right)^{1/4} \\
&\times \left(\sum_{n=0}^{\infty} (n+1) \int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \hat{D}(l)^{2\left(n + \lfloor \frac{\lambda+\rho-1}{2} \rfloor\right)} \right)^{3/4} \\
={}& \sqrt{2} \left(\sum_{n=0}^{\infty} (n+1) \int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \hat{D}(l)^{2\lfloor \frac{\lambda+\rho-1}{2} \rfloor} \sum_x D^{*2n}(x) \mathrm{e}^{\mathrm{i}(l \pm k)\cdot x} \right)^{1/4} \left(\varepsilon_2^{(\lfloor (\lambda+\rho-1)/2 \rfloor)} \right)^{3/4} \\
={}& \sqrt{2} \left(\sum_{n=0}^{\infty} (n+1) \sum_x D^{*2\lfloor \frac{\lambda+\rho-1}{2} \rfloor}(x) D^{*2n}(x) \mathrm{e}^{\pm \mathrm{i}k\cdot x} \right)^{1/4} \left(\varepsilon_2^{(\lfloor (\lambda+\rho-1)/2 \rfloor)} \right)^{3/4} \\
\le{}& \sqrt{2} \varepsilon_2^{(\lfloor (\lambda+\rho-1)/2 \rfloor)}.
\end{aligned}$$

The integral (3.6.8) is bounded above as, by Schwarz's inequality,

$$\begin{aligned}
(3.6.8) \le{}& \int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \Big|\hat{D}(l)\Big|^{\lambda+\rho-1} \left(\int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \Big|\hat{S}_{\mu_p(\mathrm{e}^{\mathrm{i}\theta})}(l + k)\Big|^4 \right)^{1/4} \left(\int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \Big|\hat{S}_{\mu_p(\mathrm{e}^{\mathrm{i}\theta})}(l - k)\Big|^4 \right)^{1/4} \\
&\times \left(\int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \Big|\hat{S}_{\mu_p(\mathrm{e}^{\mathrm{i}\theta})}(l)\Big|^4 \right)^{1/4} \left(\int_{\mathbb{T}} \frac{\mathrm{d}\theta}{2\pi} \Big|\hat{S}_{\mu_p(m\mathrm{e}^{-\mathrm{i}\theta})}(l)\Big|^4 \right)^{1/4} \big(1 - \hat{D}(2l)\big)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \Big|\hat{D}(l)\Big|^{\lambda+\rho-1} \left(\frac{1+\mu_p(1)^2\hat{D}(l+k)^2}{\big(1-\mu_p(1)^2\hat{D}(l+k)^2\big)^3} \right)^{1/4} \left(\frac{1+\mu_p(1)^2\hat{D}(l-k)^2}{\big(1-\mu_p(1)^2\hat{D}(l-k)^2\big)^3} \right)^{1/4} \\
&\quad \times \left(\frac{1+\mu_p(1)^2\hat{D}(l)^2}{\big(1-\mu_p(1)^2\hat{D}(l)^2\big)^3} \right)^{1/4} \left(\frac{1+\mu_p(m)^2\hat{D}(l)^2}{\big(1-\mu_p(m)^2\hat{D}(l)^2\big)^3} \right)^{1/4} \big(1-\hat{D}(2l)\big) \\
&\le 2 \int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \left(\frac{|\hat{D}(l)|^{3(\lambda+\rho-1)/2}}{\big(1-\hat{D}(l+k)^2\big)^3} \right)^{1/4} \left(\frac{|\hat{D}(l)|^{3(\lambda+\rho-1)/2}}{\big(1-\hat{D}(l-k)^2\big)^3} \right)^{1/4} \\
&\quad \times \left(\frac{|\hat{D}(l)|^{(\lambda+\rho-1)/2}}{\big(1-\hat{D}(l)^2\big)^3} \right)^{1/2} \big(1-\hat{D}(2l)\big).
\end{aligned}
$$

By Lemma 3.6.2,

$$
(3.6.8) \le 4 \int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \left(\frac{|\hat{D}(l)|^{3(\lambda+\rho-1)/2}}{\big(1-\hat{D}(l+k)^2\big)^3} \right)^{1/4} \left(\frac{|\hat{D}(l)|^{3(\lambda+\rho-1)/2}}{\big(1-\hat{D}(l-k)^2\big)^3} \right)^{1/4} \left(\frac{|\hat{D}(l)|^{(\lambda+\rho-1)/2}}{1-\hat{D}(l)^2} \right)^{1/2}.
$$

By Hölder's inequality and Schwarz's inequality again,

$$
\begin{aligned}
(3.6.8) &\le 4 \left(\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{|\hat{D}(l)|^{(\lambda+\rho-1)/2} \cdot |\hat{D}(l)|^{(\lambda+\rho-1)/2}}{\big(1-\hat{D}(l+k)^2\big)\big(1-\hat{D}(l-k)^2\big)} \right)^{3/4} \left(\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{|\hat{D}(l)|^{\lambda+\rho-1}}{\big(1-\hat{D}(l)^2\big)^2} \right)^{1/4} \\
&\le 4 \left(\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{|\hat{D}(l)|^{\lambda+\rho-1}}{\big(1-\hat{D}(l+k)^2\big)^2} \right)^{3/8} \left(\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{|\hat{D}(l)|^{\lambda+\rho-1}}{\big(1-\hat{D}(l-k)^2\big)^2} \right)^{3/8} \\
&\quad \times \left(\int_{\mathbb{T}^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{|\hat{D}(l)|^{\lambda+\rho-1}}{\big(1-\hat{D}(l)^2\big)^2} \right)^{1/4} \\
&\le 4\varepsilon_2^{(\lfloor(\lambda+\rho-1)/2\rfloor)}.
\end{aligned}
$$

(3.6.6) and the upper bounds on (3.6.7) and (3.6.8) imply (3.6.4) when $\lambda \ge 2$ and $\rho \ge 2$. □

Proof of (3.6.4) *in Lemma 3.6.1 when* $\lambda = 1$ *and* $\rho = 1$. By the trivial inequality (3.2.1), note that $\varphi_p(\boldsymbol{x}) = \delta_{\boldsymbol{o},\boldsymbol{x}} + \varphi_p(\boldsymbol{x})\mathbb{1}_{\{\boldsymbol{o}\neq\boldsymbol{x}\}} \le \delta_{\boldsymbol{o},\boldsymbol{x}} + (q_p \star \varphi_p)(\boldsymbol{x})$ for every $\boldsymbol{x} \in \mathbb{L}^d \times \mathbb{Z}_+$. Then, we obtain

$$
\begin{aligned}
\hat{V}_{p,m}^{(1,1)}(k) &= \sup_{\boldsymbol{x}} \sum_{\boldsymbol{y}} (q_p \star \varphi_p)(\boldsymbol{y})(1-\cos k\cdot y)\Big(mq_p \star \varphi_p^{(m)}\Big)(\boldsymbol{y}-\boldsymbol{x}) \\
&\le \sup_{\boldsymbol{x}} \sum_{\boldsymbol{y}} q_p(\boldsymbol{y})(1-\cos k\cdot y) m q_p(\boldsymbol{y}-\boldsymbol{x}) \\
&\quad + \sup_{\boldsymbol{x}} \sum_{\boldsymbol{y}} q_p(\boldsymbol{y})(1-\cos k\cdot y)\Big(m^2 q_p^{\star 2} \star \varphi_p^{(m)}\Big)(\boldsymbol{y}-\boldsymbol{x}) \\
&\quad + \sup_{\boldsymbol{x}} \sum_{\boldsymbol{y}} \big(q_p^{\star 2} \star \varphi_p\big)(\boldsymbol{y})(1-\cos k\cdot y)\Big(mq_p \star \varphi_p^{(m)}\Big)(\boldsymbol{y}-\boldsymbol{x}) \\
&\le K_1^2 \|D\|_\infty \big(1-\hat{D}(k)\big) + K_1 L_{p,m}^{(0,2)}\big(1-\hat{D}(k)\big) + \hat{V}_p^{(2,1)}(k,m).
\end{aligned}
$$

This completes the proof of (3.6.4) when $\lambda = 1$ and $\rho = 1$. □

3.7 Numerically computing the random-walk quantities

In this section, we show the expressions of the upper bounds on (3.6.1) to compute their numerical values. We use Stirling's formula due to the property (1.1.1) of the BCC lattice, which helps us to obtain high precise estimates. Now, fix a sufficiently large number N. For $\nu \in \mathbb{N}$,

$$
\begin{aligned}
\varepsilon_1^{(\nu)} &= \sum_{n=\nu}^{\infty} D^{*2n}(o) \\
&\le \sum_{n=\nu}^{\nu+N-1} D^{*2n}(o) + \sum_{n=\nu+N}^{\infty} \frac{1}{(\pi n)^{d/2}} \\
&\le \sum_{n=0}^{N-1} D^{*2(n+\nu)}(o) + \frac{1}{\pi^{d/2}(\nu+N)^{d/2}} + \frac{1}{\pi^{d/2}} \int_{\nu+N}^{\infty} s^{-d/2}\,\mathrm{d}s \\
&\le \sum_{n=0}^{N-1} D^{*2(n+\nu)}(o) + \frac{1}{\pi^{d/2}(\nu+N)^{d/2}} + \frac{2}{\pi^{d/2}\,(d-2)}\,(\nu+N)^{(2-d)/2}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\varepsilon_2^{(\nu)} &= \sum_{n=\nu}^{\infty} (n-\nu+1)\,D^{*2n}(o) \\
&\le \sum_{n=\nu}^{\nu+N-1} (n-\nu+1)\,D^{*2n}(o) + \sum_{n=\nu+N}^{\infty} (n-\nu+1)\,\frac{1}{(\pi n)^{d/2}} \\
&\le \sum_{n=0}^{N-1} (n+1)\,D^{*2(n+\nu)}(o) + \frac{1}{\pi^{d/2}(\nu+N)^{d/2-1}} \\
&\quad + \frac{1}{\pi^{d/2}} \int_{\nu+N}^{\infty} s^{1-d/2}\,\mathrm{d}s - \frac{\nu-1}{\pi^{d/2}} \int_{\nu+N}^{\infty} s^{-d/2}\,\mathrm{d}s \\
&\le \sum_{n=0}^{N-1} (n+1)\,D^{*2(n+\nu)}(o) + \frac{1}{\pi^{d/2}(\nu+N)^{(d-2)/2}} \\
&\quad + \frac{2}{\pi^{d/2}\,(d-4)}\,(\nu+N)^{(4-d)/2} - \frac{2\,(\nu-1)}{\pi^{d/2}\,(d-2)}\,(\nu+N)^{(2-d)/2}.
\end{aligned}
$$

When $N = 500$, we obtain numerical values in Table 3.1 by directly computing these expressions.

Table 3.1: Numerical values of random walk quantities on the BCC lattice.

d	$\varepsilon_1^{(1)}$	$\varepsilon_1^{(2)}$	$\varepsilon_2^{(1)}$	$\varepsilon_2^{(2)}$
3	$3.932\,160 \times 10^{-1}$	$2.682\,160 \times 10^{-1}$	∞	∞
4	$1.186\,367 \times 10^{-1}$	$5.613\,669 \times 10^{-2}$	∞	∞
5	$4.682\,556 \times 10^{-2}$	$1.557\,556 \times 10^{-2}$	$1.125\,787 \times 10^{-1}$	$6.575\,313 \times 10^{-2}$
6	$2.046\,078 \times 10^{-2}$	$4.835\,778 \times 10^{-3}$	$3.223\,235 \times 10^{-2}$	$1.177\,158 \times 10^{-2}$
7	$9.405\,986 \times 10^{-3}$	$1.593\,486 \times 10^{-3}$	$1.235\,360 \times 10^{-2}$	$2.947\,612 \times 10^{-3}$
8	$4.450\,665 \times 10^{-3}$	$5.444\,148 \times 10^{-4}$	$5.302\,886 \times 10^{-3}$	$8.522\,211 \times 10^{-4}$
9	$2.143\,604 \times 10^{-3}$	$1.904\,782 \times 10^{-4}$	$2.410\,377 \times 10^{-3}$	$2.667\,729 \times 10^{-4}$
10	$1.044\,317 \times 10^{-3}$	$6.775\,369 \times 10^{-5}$	$1.132\,063 \times 10^{-3}$	$8.774\,675 \times 10^{-5}$

3.8 Completing the bootstrapping arguments

In this section, we set

$$d = 9.0000, \quad K_1 = 1.0020, \quad K_2 = 1.0500, \quad K_3 = 1.2500. \tag{3.8.1}$$

Proof of Proposition 3.1.2. Clearly, $g_1(0,1) = 0$. By the definitions, $\hat{\varphi}_0(k,z) = \sum_{(x,t)}(\delta_{o,x}\delta_{0,t} + \mathbb{P}_0((o,0) \to (x,t) \neq (o,0)))\mathrm{e}^{\mathrm{i}k\cdot x}z^t = 1$, $\mu_0(z) = (1-\hat{\varphi}_0(0,|z|))\mathrm{e}^{\mathrm{i}\arg z} = 0$ and hence $\hat{S}_{\mu_0(z)}(k) = 1$. These lead to $g_2(0,1) = 1$ and $g_3(0,1) = 0$. Therefore, the initial conditions hold for (3.8.1). $\square$

Proof of Proposition 3.1.3. From Lemma 3.6.1 and Table 3.1, the basic diagrams are bounded above as

$$\begin{aligned}
B_{p,m}^{(2,2)} &\leq 2.116\,88 \times 10^{-4}, \\
B_{p,m}^{(0,2)} &\leq 2.372\,79 \times 10^{-3}, \\
T_{p,m}^{(2,2)} &\leq 4.402\,47 \times 10^{-4}, \\
T_{p,m}^{(1,1)} &\leq 3.961\,90 \times 10^{-3}, \\
\frac{\hat{V}_{p,m}^{(2,2)}(k)}{1-\hat{D}(k)} &\leq 3.922\,76 \times 10^{-2}.
\end{aligned}$$

The 4th inequality in the above implies the sufficient condition of Lemma 3.4.1. In addition, from Lemma 3.4.1,

$$\begin{aligned}
\hat{\Pi}_p^{\mathrm{even}}(0,m) &\leq 1.000\,00 \times 10^{-5}, \\
\hat{\Pi}_p^{\mathrm{odd}}(0,m) &\leq 1.158\,44 \times 10^{-4}, \\
\sum_{(x,t)}\Big(\Pi_p^{\mathrm{even}}(x,t) + \Pi_p^{\mathrm{odd}}(x,t)\Big)m^t t &\leq 4.000\,00 \times 10^{-4}, \\
\sum_{(x,t)}\Big(\Pi_p^{\mathrm{even}}(x,t) + \Pi_p^{\mathrm{odd}}(x,t)\Big)m^t \frac{1-\cos k\cdot x}{1-\hat{D}(k)} &\leq 2.100\,00 \times 10^{-2},
\end{aligned}$$

which imply the absolute convergence of the alternative series (3.3.2). Then, by (3.5.2), (3.5.8) and (3.5.12), we obtain

$$\begin{aligned}
g_1(p,m) &\leq 1.0002 < K_1, \\
g_2(p,m) &\leq 1.0445 < K_2, \\
g_3(p,m) &\leq 1.2433 < K_3.
\end{aligned}$$

This completes the proof of Proposition 3.1.3. $\square$

3.9 Further discussion

We have been able to prove convergence of the lace expansion for oriented percolation on $\mathbb{L}^{d\geq 9} \times \mathbb{Z}_+$ in full detail. As compared to the analysis on $\mathbb{Z}^{d\gg 4} \times \mathbb{Z}_+$ as Nguyen and Yang [33], the current analysis identifies an upper bound 9 on the critical dimension d_{c}. To begin with, since [33] contains a wrong calculation as we describe in Item (2) below, their bound corresponding to (3.5.8) is failed for any dimensions even if d is sufficiently large. To solve the problem, it suffices to prove only Proposition 3.5.1, Lemma 3.6.2 and the numerical values in Table 3.1 on $\mathbb{Z}^d$, whose proofs depend on the structure of the BCC lattice $\mathbb{L}^d$. Although proving Proposition 3.5.1 on $\mathbb{Z}^d$ is not succeeded, the others are completed. Appendix A deals with them.

To go down to the desired spatial 5 dimensions, we must improve our analysis in various aspects. Some of the ideas are summarized as follows.

(1) In Lemma 3.4.1, we paid attention to the coefficient 1/2 of the diagram containing four q_p's. If one isolates the diagrams containing six q_p's and specify their coefficients, then one can obtain better upper bounds than those in this thesis. The upper bounds on $B_{p,m}^{(\lambda,\rho)}$ and $T_{p,m}^{(\lambda,\rho)}$ in Lemma 3.6.1 may be helpful for the improvement. However, although their random-walk counterparts, such as $\sum_{\boldsymbol{x}}(q_p^{\star\lambda}\star\varphi_p^{\star 2})(\boldsymbol{x})(q_p^{\star\rho}\star\varphi_p)(\boldsymbol{x})$, are decreasing in λ and ρ, the bound on $T_{p,m}^{(\lambda,\rho)}$ may attain the minimum at some $\lambda_*,\rho_*\in\mathbb{N}$, due to the exponentially growing factor $(p(m\vee 1))^{\lambda+\rho}$. That is why we did not use $T_{p,m}^{(\lambda_*,\rho_*)}$ in the thesis.

(2) Notice that, if we do not use Proposition 3.5.1 and naively apply the triangle inequality as in (3.5.7),

$$\frac{\big|\hat{\Pi}_p(0,|z|)-\hat{\Pi}_p(k,z)\big|}{\big|1-\mathrm{e}^{\mathrm{i}\theta}\hat{D}(k)\big|}\le\frac{\big|\hat{\Pi}_p(0,|z|)-\hat{\Pi}_p(0,z)\big|+\big|\hat{\Pi}_p(0,z)-\hat{\Pi}_p(k,z)\big|}{\big|1-\mathrm{e}^{\mathrm{i}\theta}\hat{D}(k)\big|} \tag{3.9.1}$$

appears in an upper bound on $g_2(p,m)$, but this bound becomes infinite for some situations. When $\hat{D}(k)=-1$ and $\theta=\pm\pi$, its denominator equals 0 even though its numerator is non-zero. Thanks to Proposition 3.5.1, we can avoid this problem. Although one may consider to use the left hand side in (3.9.1) without using the triangle inequality, the way does not work currently because we do not have any nice methods to decompose diagrams with complex numbers. Recall the decomposition in the proof of Lemma 3.4.3 in which we use Lemma 1.3.4.

Incidentally, Nguyen and Yang wrote "$\ldots 1-e^{it}\hat{D}(k)$ is bounded below by a positive constant for $\{(k,t)\in[-\pi,\pi]^d\times[-\pi,\pi]:|(k,t)|>\varepsilon\}\ldots$" below [33, (13)]. However, it is impossible for the nearest-neighbor model because it can occur that $\hat{D}(k)=-1$ when $k_i=\pm\pi$ for some $i=1,\ldots,d$. That is a reason why we need Proposition 3.5.1. The spread-out model does not cause such problem due to $1+\hat{D}_L(k)>0$ with $L\gg 1$ in [33, Proof of Lemma 10], where D_L is the random-walk transition probability for the spread-out model. This inequality implies that $|1-\mathrm{e}^{\mathrm{i}\theta}\hat{D}(k)|>\mathrm{const.}(\|k\|_2L^2+\theta^2)$, hence the naive bound (3.9.1) works for sufficiently large L. If we consider the oriented percolation whose staying probability is positive (considered in, e.g., [18]), then the naive bound also works for such "nearest-neighbor" model because, in this case, D turns to the lazy random-walk transition probability D_{lazy}. By taking the staying probability as a sufficiently large value, $1+\hat{D}_{\mathrm{lazy}}(k)>0$ holds. In this thesis, we would like to prove (3.5.8) for the so-called nearest-neighbor model, and it is the technical reason at all to use the lazy random walk, so that we did not use it.

(3) The proof of Proposition 3.5.1 depends on the structure of the BCC lattice $\mathbb{L}^d$. It suffices for $\mathbb{L}^d$ to attend only the range of k to $[-\pi/2,\pi/2]^d$ due to reflections in perpendicular hyperplanes to the axes, whereas we need extra efforts to prove the same claim for $\mathbb{Z}^d$ due to lack of symmetry. The shape of the domain of k such that $\hat{D}(k)\ge 0$ for $\mathbb{Z}^d$, that is $\{k\in\mathbb{T}^d\mid d^{-1}\sum_{j=1}^d\cos k_j\ge 0\}$ is not as simple as $\{k\in\mathbb{T}^d\mid\prod_{j=1}^d\cos k_j\ge 0\}$ for $\mathbb{L}^d$.

(4) To obtain the upper bound on $\hat{V}_{p,m}^{(\lambda,\rho)}(k)$ in Lemma 3.6.1, we used Lemma 3.6.2. Its proof depends on the structure of $\mathbb{L}^d$ as well as Proposition 3.5.1, but it is not hard to prove this on $\mathbb{Z}^d$ similarly. See Appendix A for the details. Thanks to this lemma, we could show the finiteness of $\hat{V}_{p,m}^{(\lambda,\rho)}(k)$ for $d>4$, cf., [33, Remark 5] in which they proved the finiteness of the similar quantity for only $d>8$.

(5) If one do not use the Laplace transform, then our result may be improved. We could not avoid using bad triangle inequality estimates due to complex numbers, e.g., not splitting contributions of Π_p^{even} and Π_p^{odd} in the second half in Lemma 3.4.1. In this end, we may apply the inductive approach to the lace expansion [28, 29] for the spread-out model if we can extend it to the nearest-neighbor model.

(6) It is considered that not to use the infrared bound (1.2.3) and to directly prove the triangle condition (1.2.1) yield some improvement. The range $\mathbb{T}^d$ of k in the infrared bound may be too strong because its approximation becomes bad out of the neighborhood of $(0, \dots, 0) \in \mathbb{T}^d$. This approach may be also effective to prove the mean-field behavior on $\mathbb{Z}^d$ because (3.9.1) appears in the calculations to derive the infrared bound.

Appendix A

Topics on the simple cubic lattice

In this Appendix, we show a progress to prove Theorem 1.2.1 for oriented percolation on $\mathbb{Z}\times\mathbb{Z}_+$. Only Proposition 3.5.1, Lemma 3.6.2 and the numerical values in Table 3.1 depend on the structure of the BCC lattice. Although we do not have any proof of Proposition 3.5.1 on $\mathbb{Z}^d$, the others are shown in the following.

A.1 Proof of Lemma 3.6.2

Note that

$$\begin{aligned}
2\Big(1-\hat{D}(k)^2\Big)-\Big(1-\hat{D}(2k)\Big) &= 1-\frac{2}{d^2}\left(\sum_{j=1}^{d}\cos k_j\right)^2+\frac{1}{d}\sum_{j=1}^{d}\cos(2k_j)\\
&= \frac{2}{d}\sum_{j=1}^{d}\cos^2 k_j-\frac{2}{d^2}\left(\sum_{j=1}^{d}\cos k_j\right)^2
\end{aligned}$$

on $\mathbb{Z}^d$. Let $h_d(\xi_1,\dots,\xi_d)\equiv 2d^{-1}\sum_{j=1}^{d}\xi_j^2-2d^{-2}(\sum_{j=1}^{d}\xi_j)^2$. It suffices to show that $h_d(\xi_1,\dots,\xi_d)\ge 0$ for every $(\xi_1,\dots,\xi_d)\in[-1,1]^d$. Notice that the definition of h_d and the range of ξ_j's are different from those in the proof of (3.6.5) for $\mathbb{L}^d$. To do so, we use the method of induction.

When $d=1$, $h_1(\xi_1)=0$ for every ξ_1. Next, we assume that $h_d(\xi_1,\dots,\xi_d)\ge 0$ holds in some dimension d. Fix ξ_j for each $j=1,\dots,d$. By the induction hypothesis,

$$\begin{aligned}
h_{d+1}(\xi_1,\dots,\xi_{d+1}) &= \frac{2}{d+1}\left(\xi_{d+1}^2+\sum_{j=1}^{d}\xi_j^2-\frac{1}{d+1}\xi_{d+1}^2\right.\\
&\qquad\left.-\frac{2}{d+1}\left(\sum_{j=1}^{d}\xi_j\right)\xi_{d+1}-\frac{1}{d+1}\left(\sum_{j=1}^{d}\xi_j\right)^2\right)\\
&\ge \frac{2}{d(d+1)^2}\left(d^2\xi_{d+1}^2-2d\left(\sum_{j=1}^{d}\xi_j\right)\xi_{d+1}+\left(\sum_{j=1}^{d}\xi_j\right)^2\right)\\
&\ge \frac{2}{d(d+1)^2}\left(d\xi_{d+1}-\sum_{j=1}^{d}\xi_j\right)^2\ge 0.
\end{aligned}$$

Therefore, $h_d(\xi_1,\dots,\xi_d)\ge 0$ holds in every dimensions d.

A.2 Numerically computing the random-walk quantities

We rewrite $\varepsilon_i^{(\nu)}$ for $i,\nu\in\mathbb{Z}_+$ as, by the inverse Fourier transform,

$$\begin{aligned}
\varepsilon_i^{(\nu)} &= \int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\hat{D}(k)^{2\nu}}{\big(1-\hat{D}(k)^2\big)^i} = \int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\big(1-\big(1-\hat{D}(k)^2\big)\big)^\nu}{\big(1-\hat{D}(k)^2\big)^i} \\
&= \int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{\big(1-\hat{D}(k)^2\big)^i}\sum_{j=0}^{\nu}\binom{\nu}{j}(-1)^j\Big(1-\hat{D}(k)^2\Big)^j \\
&= \sum_{j=0}^{i}\binom{\nu}{j}(-1)^j\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{\big(1-\hat{D}(k)^2\big)^{i-j}} + \sum_{j=i+1}^{\nu}\binom{\nu}{j}(-1)^j\int\frac{\mathrm{d}^dk}{(2\pi)^d}\Big(1-\hat{D}(k)^2\Big)^{j-i} \\
&= \sum_{j=0}^{i}\binom{\nu}{j}(-1)^j\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{\big(1-\hat{D}(k)^2\big)^{i-j}} + \sum_{j=i+1}^{\nu}\sum_{\lambda=0}^{j-i}\binom{\nu}{j}\binom{j-i}{\lambda}(-1)^{\lambda+j}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\hat{D}(k)^{2\lambda},
\end{aligned} \tag{A.2.1}$$

where, if $m<n$, then $\binom{m}{n}=0$ and $\sum_{i=n}^m f(i)=0$, $\forall f\colon\mathbb{Z}\to\mathbb{R}$.

In the first term in (A.2.1), we need to calculate only two types of integrals: $i-j=1,2$. When $i-j=1$,

$$\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{1-\hat{D}(k)^2} = \int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{\big(1-\hat{D}(k)\big)\big(1+\hat{D}(k)\big)} = \frac{1}{2}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\bigg(\frac{1}{1-\hat{D}(k)}+\frac{1}{1+\hat{D}(k)}\bigg).$$

Similarly, when $i-j=2$,

$$\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{\big(1-\hat{D}(k)^2\big)^2} = \frac{1}{4}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\left(\frac{1}{\big(1-\hat{D}(k)\big)^2}+\frac{1}{1-\hat{D}(k)}+\frac{1}{1+\hat{D}(k)}+\frac{1}{\big(1+\hat{D}(k)\big)^2}\right).$$

By these expressions, it suffices for us to estimate the integrals of $(1\pm\hat{D}(k))^{-\lambda}$ for each $\lambda\in\mathbb{N}$. In fact, we can rewrite them as following: in general,

$$\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{e}^{-\mathrm{i}k\cdot x}}{\big(1\pm\hat{D}(k)\big)^\lambda} = d^\lambda\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{e}^{-\mathrm{i}k\cdot x}}{\big(\sum_{j=1}^d(1\pm\cos k_j)\big)^\lambda},$$

by the identity $A^{-\lambda}=\int_0^\infty s^{\lambda-1}\exp(-As)\mathrm{d}s/(\lambda-1)!$,

$$\begin{aligned}
&= \frac{d^\lambda}{(\lambda-1)!}\int_{[-\pi,\pi]^d}\frac{\mathrm{d}^dk}{(2\pi)^d}\int_0^\infty \mathrm{d}s\, s^{\lambda-1}\mathrm{e}^{-\mathrm{i}k\cdot x-s\sum_{j=1}^d(1\pm\cos k_j)} \\
&= \frac{d^\lambda}{(\lambda-1)!}\int_0^\infty \mathrm{d}s\, s^{\lambda-1}\prod_{j=1}^d\int_{-\pi}^{\pi}\frac{\mathrm{d}k_j}{2\pi}\mathrm{e}^{-\mathrm{i}k_jx_j-s(1\pm\cos k_j)},
\end{aligned}$$

since the cosine is an even function,

$$\begin{aligned}
&= \frac{d^\lambda}{(\lambda-1)!}\int_0^\infty \mathrm{d}s\, s^{\lambda-1}\prod_{j=1}^d\int_{-\pi}^{\pi}\frac{\mathrm{d}k_j}{2\pi}\mathrm{e}^{-s(1\pm\cos k_j)}\cos(k_j|x_j|) \\
&= \frac{d^\lambda}{(\lambda-1)!}\int_0^\infty \mathrm{d}s\, s^{\lambda-1}\prod_{j=1}^d\mathrm{e}^{-s}I_{|x_j|}(\mp s),
\end{aligned} \tag{A.2.2}$$

where $I_{|x_j|}(s)$ is the modified Bessel function of the first kind. In particular, since $I_n(-s) = (-1)^n I_\nu(s)$ for each $n \in \mathbb{Z}$, we obtain

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{1 - \hat{D}(k)^2} = d \int_0^\infty \mathrm{d}s \, \big(\mathrm{e}^{-s} I_0(s)\big)^d$$

and

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\big(1 - \hat{D}(k)^2\big)^2} = \frac{d^2}{2} \int_0^\infty \mathrm{d}s \, s \big(\mathrm{e}^{-s} I_0(s)\big)^d + \frac{d}{2} \int_0^\infty \mathrm{d}s \, \big(\mathrm{e}^{-s} I_0(s)\big)^d.$$

In the second term in (A.2.1), we use the $1/d$ expansions in [25, Appendix A.2]. Note that $\hat{D}(k) = d^{-1} \sum_{j=1}^d \cos k_j$ on $\mathbb{Z}^d$ and

$$\int_{-\pi}^{\pi} \cos^n \theta \, \mathrm{d}\theta = \frac{(n-1)!!}{n!!} \times \begin{cases} 2\pi & [n \text{ is even}], \\ 0 & [n \text{ is odd}] \end{cases}$$

for $n \in \mathbb{Z}_+$. Applying the multinomial expansion and the above formula, we arrive in, for $\lambda \in \mathbb{N}$,

$$\begin{aligned} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \hat{D}(k)^{2\lambda} &= \frac{1}{d^{2\lambda}} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\sum_{j=1}^d \cos k_j \right)^{2\lambda} \\ &= \frac{1}{d^{2\lambda}} \sum_{\substack{j_1, \dots, j_d \in \mathbb{Z}_+ \\ (j_1 + \dots + j_d = 2\lambda)}} \binom{2\lambda}{j_1, \dots, j_d} \prod_{\nu=1}^d \frac{(j_\nu - 1)!!}{j_\nu !!} \mathbb{1}_{\{j_\nu \text{ is even}\}} \\ &= \begin{cases} \dfrac{1}{2d} & [\lambda = 1], \\ \dfrac{3}{(2d)^2} - \dfrac{3}{(2d)^3} & [\lambda = 2], \\ \dfrac{15}{(2d)^3} - \dfrac{45}{(2d)^4} + \dfrac{40}{(2d)^5} & [\lambda = 3], \\ \dots \end{cases} \end{aligned}$$

where $\binom{2\lambda}{j_1, \dots, j_d} = (2\lambda)!/(j_1! \cdots j_d!)$ is a multinomial coefficient.

Finally, applying some numerical integration method and a numerical estimation of the Bessel function to (A.2.2), we obtain numerical values in Table A.1.

Table A.1: Numerical values of random walk quantities on the simple cubic lattice.

d	$\varepsilon_1^{(1)}$	$\varepsilon_1^{(2)}$	$\varepsilon_2^{(1)}$	$\varepsilon_2^{(2)}$
3	$5.163\,861 \times 10^{-1}$	$3.497\,194 \times 10^{-1}$	∞	∞
4	$2.394\,672 \times 10^{-1}$	$1.144\,672 \times 10^{-1}$	∞	∞
5	$1.563\,082 \times 10^{-1}$	$5.630\,813 \times 10^{-2}$	$3.893\,167 \times 10^{-1}$	$2.330\,086 \times 10^{-1}$
6	$1.169\,634 \times 10^{-1}$	$3.363\,004 \times 10^{-2}$	$1.985\,923 \times 10^{-1}$	$8.162\,887 \times 10^{-2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
713	$7.027\,418 \times 10^{-4}$	$1.479\,460 \times 10^{-6}$	$7.042\,265 \times 10^{-4}$	$1.484\,673 \times 10^{-6}$
714	$7.017\,555 \times 10^{-4}$	$1.475\,313 \times 10^{-6}$	$7.032\,360 \times 10^{-4}$	$1.480\,504 \times 10^{-6}$
715	$7.007\,719 \times 10^{-4}$	$1.471\,183 \times 10^{-6}$	$7.022\,483 \times 10^{-4}$	$1.476\,352 \times 10^{-6}$

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