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**Global in time stability with respect to the
rotational inertia for nonlocal extensible beam
equations**

(梁方程式の回転慣性に関する時間大域的安定性)

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Chapter 1

Introduction

1.1 Nonlocal extensible beam equation

In this thesis we consider the following models of extensible beams

$$(1 - \alpha\Delta)u_{tt} + \Delta^2 u - M\left(\int_{\Omega} |\nabla u|^2 dx\right)\Delta u = \mathcal{F}(x, u) + \mathcal{G}(u_t, \Delta u_t), \quad (1.1.1)$$

where M is a scalar function, $\mathcal{F}$ represents a forcing term and $\mathcal{G}$ represents a damping term. In the next paragraph we survey that model (1.1.1) is related to vibrations of nonlinear plates and extensible beams.

1.1.1 Background and previous works

The equation (1.1.1) has been extensively studied in different contexts. First we note that, for the linear case (that is, $M \equiv \mathcal{F} \equiv \mathcal{G} \equiv 0$), when $\alpha = 0$, the form of (1.1.1) is known as the Euler-Bernoulli beam equation, while for $\alpha > 0$, one has the so-called Rayleigh beam. The latter model was introduced for improving the former model by taking into account the effect of rotational inertia. Another context from which the equation (1.1.1) has arisen is the simplified “large deflection” model for a beam, by taking $\mathcal{F} = \mathcal{G} = 0$, $\alpha > 0$ and $M(s) = a + bs$, where a , b are constants related to the forces applied on the system. The large deflection model takes into account both in-plane and out-of-plane dynamics, and reflects the effect of stretching on bending. The equation (1.1.1) can be viewed as a simplification of the large deflection model when we suppose in-plane accelerations to be negligible. With respect to more detailed arguments, we refer to [44], [47]. Also, as another context, we can see the equation (1.1.1) as a simplification of the so-called scalar von Karman equation [13]. If one assumes that the second strain invariant is small, the scalar von Karman equation simplifies to what is known as the Berger plate equation, with same nonlocal nonlinear structure $M(s) = a + bs$.

In the case $\alpha = 0$, the model has been historically known as the Krieger or Krieger-Woinowsky beam,

$$u_{tt} + \Delta^2 u - M\left(\int_{\Omega} |\nabla u|^2 dx\right)\Delta u = \mathcal{F}(x, u) + \mathcal{G}(u_t, \Delta u_t). \quad (1.1.2)$$

(For example, see [32], [42].) The equation (1.1.2) has been extensively studied by many researchers since the model was proposed. In particular, the pioneering works related to extensible beams and plates ($\alpha \equiv \mathcal{F} \equiv \mathcal{G} \equiv 0$ and $M = a + bs$) were done by Ball [2], [3] and Dickey [15], [16] in which existence, uniqueness, regularity and stability of the solution was studied. Also, the equation (1.1.2) in more general setting in mathematical point of views has been treated by many authors, for instance, in [33], [6], [5], [24], [38], [7], [4], [31] and [43]. Our purpose in this thesis is to draw a comparison between the long-time behavior for dynamical systems generated by the specific type of the equation (1.1.1) with rotational inertia ($\alpha > 0$) and the one without rotational inertia ($\alpha = 0$).

More recently, the equation (1.1.2) also has been studied in the context of the piston-theory;

$$(1 - \alpha\Delta)u_{tt} + \Delta^2 u + \left(a + b \int_{\Omega} |\nabla u|^2 dx \right) \Delta u + \beta u_t = p + \nabla \cdot u.$$

This model is a non-gradient problem, owing to the non-dissipative lower order term $\nabla \cdot u$. Howell-Lasiecka-Webster [45] studied this model under the abstract setting ($\alpha = 0$) and showed the existence of global attractor and generalized exponential attractors, with clamped boundary condition, by taking $\mathcal{F} = p(x) - Lu$, $\mathcal{G} = -\beta u_t$ ($\beta > 0$) and $M(s) = as + b$ (L is an operator related to the lower order term and a, b are constants). They also established a proper exponential attractor by assuming the coefficient of damping β is large enough. Howell-Toundykov-Webster [47] considered the model with clamped-free boundary condition. They addressed both the case with rotational inertia ($\alpha > 0$) and the case without one ($\alpha = 0$), showed the existence of the solutions, by utilizing the semigroup-method in the former case and Galerkin-method in the later case. For the former, they proved the existence of global attractor. However, for the latter one, even the uniqueness is still an open problem.

We also would like to mention some works concerned with the stationary problems of extensible beams

$$\Delta^2 u - M\left(\int_{\Omega} |\nabla u|^2 dx\right) \Delta u = \mathcal{F}(x, u). \quad (1.1.3)$$

The stationary problems (1.1.3) have also been widely studied by several researchers and many results on nontrivial solutions were obtained. For example, we refer to [28], [29], [30], [40], [41], [56], [57], [58] and [59]. On the other hand, there are few works on relation between the long time behavior of the equation (1.1.1) and the solutions set of the equation (1.1.3). It is worth emphasizing here that it would be future work to consider difference of the long time dynamics associated with different boundary conditions like the clamped-free boundary condition and to investigate the relation between the long time behavior of the solution to (1.1.4)-(1.1.6) and the structure of the solutions set of (1.1.3).

1.1.2 Purpose of the thesis

Our main goal in this thesis is to examine the continuity of attractors when $\alpha \rightarrow 0$. That is, we show that the family of global attractors $\mathcal{A}_{\alpha}$ and exponential attractors

$\mathcal{A}_{\text{exp},\alpha}$ are continuous with respect to the parameter α in some sense. Specifically, we consider the model (1.1.1) which exhibits a nonlinear damping of the form

$$N\left(\int_{\Omega} |\nabla u|^2 dx\right) \Delta u_t,$$

where $N > 0$ is a nonlinear function. To our best knowledge, this kind of nonlocal damping was firstly used in Lange and Perla Menzala [48] as a type of nonlocal beam equations in $\Omega = \mathbb{R}^n$. Later Cavalcanti et al. [8] also dealt with the same type of damping for viscoelastic problem defined on a bounded or unbounded domain $\Omega \subset \mathbb{R}^n$ and showed exponential decay for the problem. More recently, Silva-Narciso [20] treated nonlinear fractional damping $N(\int_{\Omega} |\nabla u|^2 dx)(-\Delta)^{\theta} u_t$ ($0 \leq \theta \leq 1$) in the context of the Krieger-Woinowsky beam and showed the existence of a global attractor and an generalized exponential attractor to (1.1.2) with the supported boundary condition and the initial condition. Our work here is motivated by their result.

On the other hand, there exists another type of nonlocal nonlinear damping (so called *energy damping*)

$$\gamma \left[\int_{\Omega} (|\Delta u|^2 + |u_t|^2) dx \right]^q \Delta u_t,$$

where, $\gamma > 0$ and $q \geq 1$. Damping of the above type was introduced by Balakrishnan and Taylor [53] as a model for vibrating extensible beams with nonlocal nonlinear dissipative structure, which is intrinsically connected with the study of damping phenomena in flight structures. Recently, Silva-Narciso-Vicente [21] treated the above damping in higher-dimensional system and studied the global existence, uniqueness and stabilization with clamped boundary condition, by taking $\mathcal{F} = f(u)$ and $M \equiv \text{const.}$ As for related models and works, we also refer to [51], [52], [54] and [55].

The question for the stability of attractors naturally arises in the context of the relation between the Euler-Bernoulli beam model and the Rayleigh beam model. Also, this kind of problem has been considered from various viewpoints. Actually, besides rotational inertia, such a problem also appears in quasi-static problems in wave or plate vibrations and has been studied by many researchers. About the results for the von Karman equations, we refer to [13]. We also refer to the problem in the Krieger-Woinowski beam by Zelati [50]. In [13], the stability problems were well treated under the effect of internal dissipation and they showed the upper semicontinuity of global attractors. More recently, this problem was also studied under the effects of dissipation, one is geometrically constrained, the other is spatially localized. Moreover, upper semicontinuity was also obtained in [46]. In other words, we believe that it is interesting to consider the continuity problem with nonlinear damping. In particular, we concentrate upon here the construction of the exponential attractors for the equations (1.1.1) that are in the state space and continuous with respect to the rotational inertia parameter.

1.1.3 Plan of the thesis

Now we are in a position to state our problem explicitly. The work of this thesis is due to Niimura [22]. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following equation

$$(1 - \alpha \Delta)u_{tt} + \Delta^2 u - M\left(\int_{\Omega} |\nabla u|^2 dx\right) \Delta u - N\left(\int_{\Omega} |\nabla u|^2 dx\right) \Delta u_t + f(u) = h \quad (1.1.4)$$

with the simply supported boundary condition

$$u = \Delta u = 0 \quad \text{on } [0, \infty) \times \partial\Omega, \quad (1.1.5)$$

and initial conditions

$$u(\cdot, 0) = u_0(\cdot), \quad u_t(\cdot, 0) = u_1(\cdot) \quad \text{in } \Omega, \quad (1.1.6)$$

where $0 \leq \alpha \leq 1$, M and N are scalar functions specified later, $f(u)$ is a nonlinear source term and h is an external forcing term. For this problem, we shall show the well-posedness and the existence of the global attractor and a proper exponential attractor. Moreover, we wish to reveal the continuous dependance of the attractors on the parameter α in some sense. Basically, we follow the arguments in [20] for each α . Concerning the global attractor, we can apply to the abstract theory given in [12] to our problem. For the construction of exponential attractors, we invoke the arguments treated in [9]. However, for the construction of the family with continuous dependence on the parameter, we can not invoke their arguments as done in [9], so we construct the exponential attractors by modifying the discussion.

1.2 Notations

We first introduce notations concerning the function spaces and operators that will be used throughout the remainder of this thesis.

We denote by $L^2(\Omega)$ the set of square integrable functions with the usual L^2 -inner product $(\cdot, \cdot)$ and by $L^p(\Omega)$ the set of p -th power integrable functions with the usual $L^p(\Omega)$ -norm $\|\cdot\|_p$. We set $V = H^2(\Omega) \cap H_0^1(\Omega)$ with the inner product $(\Delta \cdot, \Delta \cdot)$ and the norm $\|\Delta \cdot\|_2$.

We define the operator $A : D(A) \rightarrow L^2(\Omega)$;

$$Au := \Delta^2 u \quad \text{with the domain} \quad D(A) := \{u \in H^4(\Omega) \mid u, \Delta u \in H_0^1(\Omega)\},$$

where Δ is the Laplace operator with the Dirichlet boundary condition. Obviously, A is self-adjoint in $D(A)$ and strictly positive on $D(A)$. Hence as is well-known A has the inverse operator with the domain $L^2(\Omega)$ and it is compact. Thus from the spectral theory there exists an orthonormal basis $(\omega_j)_{j \in \mathbb{N}}$ in $L^2(\Omega)$ composed by eigenfunctions of A such that

$$A\omega_j = \lambda_j \omega_j \quad \text{with} \quad 0 < \lambda_1 \leq \lambda_2 \leq \dots \quad \text{and} \quad \lambda_j \rightarrow \infty \quad \text{as} \quad j \rightarrow \infty.$$

Moreover, we can define the fractional powers $A^s, s \in \mathbb{R}$, of A with domains $D(A^s)$ being Hilbert spaces with the inner products and the norms defined by

$$(u, v)_{D(A^s)} = (A^s u, A^s v) \quad \text{and} \quad \|u\|_{D(A^s)} = \|A^s u\|_2, \quad u, v \in D(A^s).$$

The embedding $D(A^{s_1}) \hookrightarrow D(A^{s_2})$ is continuous if $s_1 \geq s_2$, is compact if $s_1 > s_2$, and it holds that

$$\|A^{s_2}u\|_2 \leq \lambda_1^{s_2-s_1} \|A^{s_1}u\|_2, \quad u \in D(A^{s_1}).$$

In particular, one has $D(A^0) = L^2(\Omega)$, $D(A^{1/4}) = H_0^1(\Omega)$, and

$$D(A^{1/2}) = H^2(\Omega) \cap H_0^1(\Omega) \quad \text{with} \quad A^{1/2}u = -\Delta u, \quad u \in D(A^{1/2}).$$

Then we can convert the concrete system (1.1.4) to an abstract evolutionary problem given by

$$\begin{aligned} (1 + \alpha A^{1/2})u_{tt} + Au + M(\|A^{1/4}u\|_2^2)A^{1/2}u + N(\|A^{1/4}u\|_2^2)A^{1/2}u_t + f(u) &= h, \\ (u(0), u_t(0)) &= (u_0, u_1). \end{aligned} \quad (1.2.1)$$

The long-time dynamics of (1.2.1) is considered on a Hilbert space $\mathcal{H}_\alpha := D(A^{1/2}) \times H_\alpha$ equipped with the norm

$$\|(u, v)\|_{\mathcal{H}_\alpha}^2 := \|A^{1/2}u\|_2^2 + \|v\|_{H_\alpha}^2,$$

where if $\alpha > 0$, H_α is the Hilbert space $H_0^1(\Omega)$ with the norm

$$\|v\|_{H_\alpha}^2 := \|v\|_2^2 + \alpha \|A^{1/4}v\|_2^2,$$

and if $\alpha = 0$, H_0 is the Hilbert space $L^2(\Omega)$. For the convenience of descriptions, we also introduce the notation $\mathcal{H}_{\alpha, \nu} := D(A^{\frac{1}{2} + \frac{\nu}{4}}) \times H_{\alpha, \nu}$ ($\nu \in \mathbb{R}$) equipped with the norm

$$\|(u, v)\|_{\mathcal{H}_{\alpha, \nu}}^2 := \|A^{\frac{1}{2} + \frac{\nu}{4}}u\|_2^2 + \|v\|_{H_{\alpha, \nu}}^2,$$

where if $\alpha > 0$, $H_{\alpha, \nu}$ is the Hilbert space $D(A^{(1+\nu)/4})$ with the norm

$$\|v\|_{H_{\alpha, \nu}}^2 := \|A^{\nu/4}v\|_2^2 + \alpha \|A^{(1+\nu)/4}v\|_2^2,$$

and if $\alpha = 0$, $H_{0, \nu}$ is the Hilbert space $D(A^{\nu/4})$.

The remainder of this thesis is organized as follows. In Chapter 2 we present our assumptions for well-posedness, state our results, and show that the problem (1.1.4)-(1.1.6) is well-posed. In Chapter 3 we first review the basic terminologies and definitions of infinite-dimensional dynamical systems, present assumptions for long-time dynamics analysis, examine the existence of the global attractors and construct exponential attractors. Chapter 4 is dedicated to the proof of the stability with respect to the rotational inertia. Finally, in Appendix we give the proofs of the statements on the infinite-dimensional dynamical systems we use in this thesis.

Chapter 2

Well-posedness for the nonlocal extensible beam equation

2.1 Assumptions and results

First of all, we list assumptions that we shall use for proving the well-posedness:

(H1) M and N are C^1 -functions on $[0, \infty)$ with

$$M(\tau) \geq 0 \quad \text{and} \quad N(\tau) > 0, \quad \forall \tau \geq 0. \quad (2.1.1)$$

(H2) $f : \mathbb{R} \rightarrow \mathbb{R}$ is a C^1 -function such that $f(0) = 0$, and

$$|f'(u)| \leq \sigma_1(1 + |u|^{p/2}), \quad \forall u \in \mathbb{R}, \quad (2.1.2)$$

for some constant $\sigma_1 > 0$, and the power p satisfying

$$p > 0 \quad \text{if} \quad 1 \leq n \leq 4 \quad \text{and} \quad 0 < p \leq \frac{8}{n-4} \quad \text{if} \quad n \geq 5. \quad (2.1.3)$$

Besides, let us suppose that there exists a constant $l_0 \geq 0$ such that

$$\tilde{f}(u) := \int_0^u f(s)ds \geq -\frac{\lambda_1}{8}|u|^2 - l_0, \quad \forall u \in \mathbb{R}. \quad (2.1.4)$$

Then the well-posedness of (1.1.4)-(1.1.6) is given by the following:

Theorem 2.1.1. *Let $T > 0$ be an arbitrary number, $h \in L^2(\Omega)$ and $0 \leq \alpha \leq 1$. Also, we assume (H1) and (H2). Then, we have:*

i. *If the initial data $(u_0, u_1) \in \mathcal{H}_{\alpha,2}$, then the problem (1.1.4)-(1.1.6) has a strong solution in the class*

$$u \in L^\infty(0, T; D(A)), \quad u_t \in L^\infty(0, T; H_{\alpha,2}) \quad u_{tt} \in L^\infty(0, T; D(A^{1/4})). \quad (2.1.5)$$

ii. *If the initial data $(u_0, u_1) \in \mathcal{H}_\alpha$, then the problem (1.1.4)-(1.1.6) has a weak solution in the class*

$$u \in L^\infty(0, T; D(A^{1/2})), \quad u_t \in L^\infty(0, T; H_\alpha) \cap L^2(0, T; D(A^{1/4})) \quad (2.1.6)$$

satisfying

$$(u, u_t) \in C([0, T]; \mathcal{H}_\alpha).$$

Moreover, the above weak solution has more regularity

$$u \in L^\infty(0, T; D(A^{3/4})), \quad u_t \in L^\infty(0, T; H_{\alpha,1}) \cap L^2(0, T; D(A^{1/2})) \quad (2.1.7)$$

for initial data $(u_0, u_1) \in \mathcal{H}_{\alpha,1}$.

iii. Both weak and strong solutions depend continuously on the initial data in $\mathcal{H}_\alpha$. More precisely, if $z = (u, u_t)$, $\tilde{z} = (\tilde{u}, \tilde{u}_t)$ are two solutions corresponding to the initial data $z_0 = (u_0, u_1)$, $\tilde{z}_0 = (\tilde{u}_0, \tilde{u}_1)$ lying in $\mathcal{H}_\alpha$, then

$$\|z(t) - \tilde{z}(t)\|_{\mathcal{H}_\alpha} \leq e^{CT} \|z_0 - \tilde{z}_0\|_{\mathcal{H}_\alpha}, \quad \forall t \in [0, T], \quad (2.1.8)$$

for some positive constant $C = C(\|z_0\|, \|\tilde{z}_0\|)$. In particular, the problem (1.1.4)-(1.1.6) has uniqueness.

2.2 Proof of Theorem 2.1.1

Let (ω_m) be the basis in $L^2(\Omega)$, W_m be the space generated by $\omega_1, \dots, \omega_m$, and for given C^2 -functions $y_{jm}(t)$, set

$$u^m(t) = \sum_{j=1}^m y_{jm}(t) \omega_j.$$

For $(u_0, u_1) \in D(A) \times D(A^{1/2})$, we consider the following problem:

$$\begin{aligned} & (u_{tt}^m, \omega) + \alpha(u_{tt}^m, A^{1/2}\omega) + (A^{1/2}u^m, A^{1/2}\omega) + M\left(\|A^{1/4}u^m\|_2^2\right)(A^{1/2}u^m, \omega) \\ & + N\left(\|A^{1/4}u^m\|_2^2\right)(u_t^m, A^{1/2}\omega) + (f(u^m), \omega) = (h, \omega), \quad \forall \omega \in W_m, \quad (2.2.1) \\ & u^m(0) = u_0^m \rightarrow u_0 \text{ in } D(A) \text{ and } u_t^m(0) = u_1^m \rightarrow u_1 \text{ in } D(A^{1/2}). \end{aligned}$$

By standard methods for ordinal differential equations, we can prove the existence of C^2 -class solutions to the approximate problem on some interval $[0, T_m)$ and this solution can be extended to the closed interval $[0, T]$ by using the first energy estimate (2.2.9) below.

2.2.1 Strong solutions

The First Energy: Taking $\omega = u_t^m$ in (2.2.1), we infer

$$\frac{d}{dt} E_\alpha^m(t) + N\left(\|A^{1/4}u^m(t)\|_2^2\right) \|A^{1/4}u_t^m(t)\|_2^2 = 0, \quad (2.2.2)$$

where we set $E_\alpha^m(t) = E_\alpha(u^m(t), u_t^m(t))$. Here the energy $E_\alpha(t) = E_\alpha(u(t), u_t(t))$ ($(u, u_t) \in \mathcal{H}_\alpha$) is defined by

$$E_\alpha(t) := \frac{1}{2} (\|A^{1/2}u(t)\|_2^2 + \|u_t(t)\|_2^2 + \alpha\|A^{1/4}u_t(t)\|_2^2 + \widetilde{M}(\|A^{1/4}u(t)\|_2^2)) + \int_\Omega (\tilde{f}(u(t)) - hu(t))dx, \quad (2.2.3)$$

where the definition of $\widetilde{M}$ is given by (3.2.2). Integrating (2.2.2) from 0 to t ($\leq T$) we get

$$E_\alpha^m(t) + \int_0^t N(\|A^{1/4}u^m(s)\|_2^2) \|A^{1/4}u_t^m(s)\|_2^2 ds = E_\alpha^m(0). \quad (2.2.4)$$

Now, using Young's inequality with $\varepsilon = \frac{\lambda_1}{8}$ and the condition (2.1.4) we have

$$\begin{aligned} \int_\Omega (\tilde{f}(u^m(t)) - hu^m(t))dx &\geq -\frac{\lambda_1}{8}\|u^m(t)\|_2^2 - l_0|\Omega| - \frac{1}{2}\left(\frac{\lambda_1}{4}\|u^m(t)\|_2^2 + \frac{4}{\lambda_1}\|h\|_2^2\right) \\ &\geq -\frac{1}{8}\|A^{1/2}u^m(t)\|_2^2 - l_0|\Omega| - \frac{1}{8}\|A^{1/2}u^m(t)\|_2^2 - \frac{2}{\lambda_1}\|h\|_2^2 \\ &= -\frac{1}{4}\|A^{1/2}u^m(t)\|_2^2 - \frac{2}{\lambda_1}\|h\|_2^2 - l_0|\Omega| \end{aligned} \quad (2.2.5)$$

hence

$$\frac{1}{4}(\|A^{1/2}u^m(t)\|_2^2 + \|u_t^m(t)\|_2^2 + \alpha\|A^{1/4}u_t^m(t)\|_2^2) \leq E_\alpha^m(t) + \frac{2}{\lambda_1}\|h\|_2^2 + l_0|\Omega|. \quad (2.2.6)$$

Compounding (2.2.4) with (2.2.6), we have

$$\|A^{1/2}u^m(t)\|_2^2 + \|u_t^m(t)\|_2^2 + \alpha\|A^{1/4}u_t^m(t)\|_2^2 \leq C_1 \quad \forall t \in [0, T], \quad m \in \mathbb{N}, \quad (2.2.7)$$

where $C_1 = C_1(\|A^{1/2}u_0\|_2, \|A^{1/4}u_1\|_2, \|h\|_2, |\Omega|, l_0) > 0$. Since $N(\tau) > 0$ and above the estimate, there exists a positive constant $N_0 = N_0(\|(u_0, u_1)\|_{\mathcal{H}_\alpha}) > 0$ such that $N(\|A^{1/4}u(t)\|_2^2) \geq N_0$ for any $t \in [0, T]$. Going back to (2.2.4) and combining this uniform boundedness with the easy relation $E_\alpha^m \leq E_1^m(t)$ ($t \geq 0$), we derive

$$E_\alpha^m(t) + N_0 \int_0^t \|A^{1/4}u_t^m(s)\|_2^2 ds \leq E_\alpha^m(0) \leq E_1^m(0). \quad (2.2.8)$$

Combining (2.2.6) with (2.2.8) we conclude

$$\|A^{1/2}u^m(t)\|_2^2 + \|u_t^m(t)\|_2^2 + \alpha\|A^{1/4}u_t^m(t)\|_2^2 + \int_0^t \|A^{1/4}u_t^m(s)\|_2^2 ds \leq C_1 \quad (2.2.9)$$

for any $t \in [0, T]$ and $m \in \mathbb{N}$, and some constant $C_1 > 0$ depending on the norm of the initial data in $\mathcal{H}_\alpha$.

The Second Energy: Differentiating (2.2.1) with respect to t and substituting $\omega = u_{tt}^m$, it holds that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \{ &\|A^{1/2}u_t^m(t)\|_2^2 + \|u_{tt}^m(t)\|_2^2 + \alpha\|A^{1/4}u_{tt}^m(t)\|_2^2 \\ &+ M(\|A^{1/4}u^m(t)\|_2^2) \|A^{1/4}u_t^m(t)\|_2^2 \} + N(\|A^{1/4}u^m\|_2^2) \|A^{1/4}u_{tt}^m\|_2^2 = \sum_{j=1}^4 I_j \end{aligned} \quad (2.2.10)$$

where

$$\begin{aligned}
I_1 &= -\left(\frac{d}{dt}N(\|A^{1/4}u^m(t)\|_2^2)\right)(A^{1/2}u_t^m(t), u_{tt}^m(t)), \\
I_2 &= -\left(\frac{d}{dt}M(\|A^{1/4}u^m(t)\|_2^2)\right)(A^{1/2}u^m(t), u_{tt}^m(t)), \\
I_3 &= -\int_{\Omega} f'(u^m(t))u_t^m(t)u_{tt}^m(t)dx, \\
I_4 &= \left(\frac{d}{dt}M(\|A^{1/4}u^m(t)\|_2^2)\right)\|A^{1/4}u_t^m(t)\|_2^2.
\end{aligned}$$

We shall estimate $I_1, \dots, I_4$. First of all, since M and N are C^1 -functions we have from (2.2.9) that

$$\max_{\tau \in [0, C_1]} \{|M'(\tau)|, |N(\tau)|, |N'(\tau)|\} =: C(\|(u_0, u_1)\|_{\mathcal{H}_\alpha}) < \infty,$$

where C_1 is the constant appeared in (2.2.9). In the following $C > 0$ denotes various constants which depends on the initial data in $\mathcal{H}_\alpha$, but not on $T > 0$. Using Young's inequality and the self-adjointness of operator $A^{1/4}$, we have

$$\begin{aligned}
|I_1| &= \left| \{N'(\|A^{1/4}u^m(t)\|_2^2)\} 2(A^{1/4}u^m(t), A^{1/4}u_t^m(t)) \} (A^{1/4}u_t^m(t), u_{tt}^m(t)) \right| \\
&\leq C \left(\|A^{1/4}u_t^m(t)\|_2^2 + \|u_{tt}^m(t)\|_2^2 \right)
\end{aligned}$$

and

$$\begin{aligned}
|I_2| &= \left| \{M'(\|A^{1/4}u^m(t)\|_2^2)\} 2(A^{1/4}u^m(t), A^{1/4}u_t^m(t)) \} (A^{1/2}u^m(t), u_{tt}^m(t)) \right| \\
&\leq C \|A^{1/4}u^m(t)\|_2 \|A^{1/4}u_t^m(t)\|_2 \|A^{1/2}u^m(t)\|_2 \|u_{tt}^m(t)\|_2 \\
&\leq C \left(\|A^{1/2}u_t^m(t)\|_2^2 + \|u_{tt}^m(t)\|_2^2 \right).
\end{aligned}$$

Further, utilizing the condition (2.1.2), generalized Hölder inequality with $\frac{p}{2(p+2)} + \frac{1}{p+2} + \frac{1}{2} = 1$, Young's inequality, the estimate (2.2.9) and the embedding $V = H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^{p+2}(\Omega)$, we infer

$$\begin{aligned}
|I_3| &\leq \sigma_1 \|(1 + |u^m(t)|^{p/2})u_t^m(t)u_{tt}^m(t)\|_1 \\
&\leq \sigma_1 \left(|\Omega|^{\frac{p}{2(p+2)}} + \|u^m(t)\|_{p+2}^{p/2} \right) \|u_t^m(t)\|_{p+2} \|u_{tt}^m(t)\|_2 \\
&\leq C \|A^{1/2}u_t^m(t)\|_2 \|u_{tt}^m(t)\|_2 \\
&\leq C \left(\|A^{1/2}u_t^m(t)\|_2^2 + \|u_{tt}^m(t)\|_2^2 \right).
\end{aligned}$$

It is easy to see that

$$|I_4| \leq C \|A^{1/2}u_t^m(t)\|_2^2$$

Since $N(\tau) > 0$, the estimate (2.2.9) implies that $N(\|A^{1/4}u^m(t)\|_2^2) \geq N_0 > 0$ for all $t \in [0, T]$, where $N_0 = N_0(\|(u_0, u_1)\|_{\mathcal{H}_\alpha})$ is a constant, so that we can derive $I_1 \geq N_0 \|A^{1/4}u_{tt}^m(t)\|_2^2$.

Using these five estimates in (2.2.10), there exists a constant $C > 0$ such that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left\{ \|A^{1/2} u_t^m(t)\|_2^2 + \|u_{tt}^m(t)\|_2^2 + \alpha \|A^{1/4} u_{tt}^m(t)\|_2^2 \right. \\
& \quad \left. + M(\|A^{1/4} u^m(t)\|_2^2) \|A^{1/4} u_t^m(t)\|_2^2 \right\} \\
& \leq C \left(\|A^{1/2} u_t^m(t)\|_2^2 + \|u_{tt}^m(t)\|_2^2 + \alpha \|A^{1/4} u_{tt}^m(t)\|_2^2 \right. \\
& \quad \left. + M(\|A^{1/4} u^m(t)\|_2^2) \|A^{1/4} u_t^m(t)\|_2^2 \right). \tag{2.2.11}
\end{aligned}$$

The estimates (2.2.9), (2.2.11) are sufficient to pass the limit in the approximate equation (2.2.1) to obtain a strong solution satisfying (2.2.1) and

$$\begin{aligned}
(1 + \alpha A^{1/2}) u_{tt} + Au + M\left(\|A^{1/4} u\|_2^2\right) A^{1/2} u + N\left(\|A^{1/4} u\|_2^2\right) A^{1/2} u_t + f(u) &= h \\
&\text{in } L^\infty(0, T; L^2(\Omega)).
\end{aligned}$$

2.2.2 Weak solutions

Let $z_0 = (u_0, u_1) \in \mathcal{H}_\alpha$. Then, since $D(A) \times D(A^{1/2})$ is dense in $\mathcal{H}_\alpha$, there exists $(u_0^k, u_1^k) \in D(A) \times D(A^{1/2})$ such that

$$(u_0^k, u_1^k) \rightarrow (u_0, u_1) \quad \text{in } \mathcal{H}_\alpha.$$

For each regular initial data (u_0^k, u_1^k) there exists a strong solution $u^k(t)$ satisfying the estimate (2.2.9). Also, taking the multiplier $A^{1/2} u_t^k$ in (2.2.1) it is easy to derive the following estimate

$$\|A^{3/4} u^k(t)\|_2^2 + \|A^{1/4} u_t^k(t)\|_2^2 + \alpha \|A^{1/2} u_t^k(t)\|_2^2 + \int_0^t \|A^{1/2} u_t(s)\|_2^2 ds \leq C,$$

for any $t \in [0, T]$ and $k \in \mathbb{N}$, where $C = C(\|z_0\|_{\mathcal{H}_{\alpha,1}}, h, |\Omega|, T)$.

Furthermore, the difference of strong solutions $w(t) := u^k(t) - u^l(t)$ satisfies the estimate

$$\begin{aligned}
& \|A^{1/2} w(t)\|_2^2 + \|w_t(t)\|_2^2 + \alpha \|A^{1/4} w_t(t)\|_2^2 \\
& \leq C \left(\|A^{1/2} w(0)\|_2^2 + \|w_t(0)\|_2^2 + \alpha \|A^{1/4} w_t(0)\|_2^2 \right) \tag{2.2.12}
\end{aligned}$$

for all $t \in [0, T]$ and some positive constant $C = C(\|z_0\|_{\mathcal{H}_\alpha}, T)$. This implies that

$$(u^k, u_t^k) \rightarrow (u(t), u_t(t)) \quad \text{in } C([0, T], \mathcal{H}_\alpha). \tag{2.2.13}$$

We omit the details of estimate (2.2.12) here, because they are identical to that concerning the continuous dependence presented in the below. These estimates are enough to conclude that the limit of approximate solutions satisfy (2.1.6) and the following weak formulation:

$$\begin{aligned}
& (u_{tt}, \omega) + \alpha(u_{tt}, A^{1/2} \omega) + (A^{1/2} u, A^{1/2} \omega) + M\left(\|A^{1/4} u\|_2^2\right) (A^{1/2} u, \omega) \\
& + N\left(\|A^{1/4} u\|_2^2\right) (u_t, A^{1/2} \omega) + (f(u), \omega) = (h, \omega), \quad \forall \omega \in V = H^2(\Omega) \cap H_0^1(\Omega). \tag{2.2.14}
\end{aligned}$$

2.2.3 Uniqueness of strong and weak solutions

Let $z = (u, u_t)$ and $\tilde{z} = (\tilde{u}, \tilde{u}_t)$ be two (strong or weak) solutions corresponding to the initial data $z_0 = (u_0, u_1)$ and $\tilde{z}_0 = (\tilde{u}_0, \tilde{u}_1)$, respectively. Putting $w = u - \tilde{u}$, we see that the function $(w, w_t) = z - \tilde{z}$ verifies

$$\begin{aligned} & (w_{tt}(t), \omega) + (A^{1/2}w(t), A^{1/2}\omega) + \alpha(A^{1/4}w_{tt}(t), A^{1/4}\omega) \\ & + N(\|A^{1/4}u(t)\|_2^2)(A^{1/4}w_t(t), A^{1/4}\omega) = -M(\|A^{1/4}u(t)\|_2^2)(A^{1/2}w(t), \omega) \\ & - \{N(\|A^{1/4}u(t)\|_2^2) - N(\|A^{1/4}\tilde{u}(t)\|_2^2)\}(A^{1/2}\tilde{u}_t(t), \omega) \\ & - \{M(\|A^{1/4}u(t)\|_2^2) - M(\|A^{1/4}\tilde{u}(t)\|_2^2)\}(A^{1/2}\tilde{u}(t), \omega) - (f(u(t)) - f(\tilde{u}(t)), \omega), \end{aligned} \quad (2.2.15)$$

with the initial data $(w(0), w_t(0)) = z_0 - \tilde{z}_0$, in the strong or weak sense.

We first deal with strong solutions. Substituting $\omega = w_t(t)$ in (2.2.15), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\{ \|A^{1/2}w(t)\|_2^2 + \|w_t(t)\|_2^2 + \alpha \|A^{1/4}w_t(t)\|_2^2 \right\} \\ & + N(\|A^{1/4}u(t)\|_2^2) \|A^{1/4}w_t(t)\|_2^2 = \sum_{j=1}^4 J_j, \end{aligned} \quad (2.2.16)$$

where

$$J_1 = -M(\|A^{1/4}u(t)\|_2^2)(A^{1/2}w(t), w_t(t)), \quad (2.2.17)$$

$$J_2 = -\{N(\|A^{1/4}u(t)\|_2^2) - N(\|A^{1/4}\tilde{u}(t)\|_2^2)\}(A^{1/4}\tilde{u}_t(t), A^{1/4}w_t(t)), \quad (2.2.18)$$

$$J_3 = -\{M(\|A^{1/4}u(t)\|_2^2) - M(\|A^{1/4}\tilde{u}(t)\|_2^2)\}(A^{1/2}\tilde{u}(t), w_t(t)), \quad (2.2.19)$$

$$J_4 = -(f(u(t)) - f(\tilde{u}(t)), w_t(t)). \quad (2.2.20)$$

Since $N(\tau) > 0$, the estimate (2.2.9) implies that $N(\|A^{1/4}u(t)\|_2^2) \geq N_0 > 0$ for all $t \in [0, T]$, where $N_0 = N_0(\|z_0\|_{\mathcal{H}_\alpha})$. Then (2.2.16) leads to

$$\frac{1}{2} \frac{d}{dt} \left\{ \|A^{1/2}w(t)\|_2^2 + \|w_t(t)\|_2^2 + \alpha \|A^{1/4}w_t(t)\|_2^2 \right\} + N_0 \|A^{1/4}w_t(t)\|_2^2 \leq \sum_{j=1}^4 J_j \quad (2.2.21)$$

Next we estimate J_1 , J_2 , J_3 , and J_4 . Analogously to the estimate of I_2 we have

$$|J_1| \leq C(\|A^{1/2}w(t)\|_2^2 + \|w_t(t)\|_2^2)$$

In the next two estimates we shall use the mean value theorem, Young's inequality and the estimate (2.2.9). Then we get

$$\begin{aligned} |J_2| & \leq C \left| \|A^{1/4}u(t)\|_2^2 - \|A^{1/4}\tilde{u}(t)\|_2^2 \right| \|A^{1/4}\tilde{u}_t(t)\|_2 \|A^{1/4}w_t(t)\|_2 \\ & \leq C [\|A^{1/4}u(t)\|_2 + \|A^{1/4}\tilde{u}(t)\|_2] \|A^{1/4}w(t)\|_2 \|A^{1/4}\tilde{u}_t(t)\|_2 \|A^{1/4}w_t(t)\|_2 \\ & \leq C \|A^{1/4}w(t)\|_2 \|A^{1/4}\tilde{u}_t(t)\|_2 \|A^{1/4}w_t(t)\|_2 \\ & \leq \epsilon \|A^{1/4}w_t(t)\|_2^2 + \frac{C^2}{4\epsilon} \|A^{1/4}\tilde{u}_t(t)\|_2^2 \|A^{1/2}w(t)\|_2^2 \end{aligned}$$

for any $\epsilon > 0$ and

$$\begin{aligned}
|J_3| &\leq C \left| \|A^{1/4}u(t)\|_2^2 - \|A^{1/4}\tilde{u}(t)\|_2^2 \right| \|A^{1/2}\tilde{u}(t)\|_2 \|w_t(t)\|_2 \\
&\leq C \left[\|A^{1/4}u(t)\|_2 + \|A^{1/4}\tilde{u}(t)\|_2 \right] \|A^{1/4}w(t)\|_2 \|A^{1/2}\tilde{u}(t)\|_2 \|w_t(t)\|_2 \\
&\leq C \left(\|A^{1/2}w(t)\|_2^2 + \|w_t(t)\|_2^2 \right).
\end{aligned}$$

From the condition (2.1.2), we can immediately see that there exists a constant $\sigma_0 > 0$ such that

$$|f(u) - f(v)| \leq \sigma_0(1 + |u|^{p/2} + |v|^{p/2})|u - v|, \quad \forall u, v \in \mathbb{R}, \quad (2.2.22)$$

and we can estimate J_4 likewise I_5

$$\begin{aligned}
|J_4| &\leq \sigma_0 \left(|\Omega|^{\frac{p}{2(p+2)}} + \|u(t)\|_{p+2}^{p/2} + \|\tilde{u}(t)\|_{p+2}^{p/2} \right) \|w(t)\|_{p+2} \|w_t(t)\|_2 \\
&\leq C \|A^{1/2}w(t)\|_2 \|w_t(t)\|_2 \\
&\leq C \left(\|A^{1/2}w(t)\|_2^2 + \|w_t(t)\|_2^2 \right).
\end{aligned}$$

Using these four estimates in (2.2.21) and taking $\epsilon > 0$ small enough, there exists a constant $C > 0$ such that

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left\{ \|A^{1/2}w(t)\|_2^2 + \|w_t(t)\|_2^2 + \alpha \|A^{1/4}w_t(t)\|_2^2 \right\} + N_0 \|A^{1/4}w_t(t)\|_2^2 \\
\leq C(1 + \|A^{1/4}\tilde{u}_t(t)\|_2^2) \left(\|A^{1/2}w(t)\|_2^2 + \|w_t(t)\|_2^2 + \alpha \|A^{1/4}w_t(t)\|_2^2 \right)
\end{aligned} \quad (2.2.23)$$

for any $t \in [0, T]$. From the estimate (2.2.9) the function $1 + \|A^{1/4}u_t(\cdot)\|_2^2$ is integrable on $[0, T]$. Then integrating (2.2.23) on $[0, t]$ and using Gronwall's inequality, we arrive at

$$\begin{aligned}
\|A^{1/2}w(t)\|_2^2 + \|w_t(t)\|_2^2 + \alpha \|A^{1/4}w_t(t)\|_2^2 + N_0 \int_0^t \|A^{1/4}w_t(s)\|_2^2 ds \\
\leq e^{C_T} \left(\|A^{1/2}w(0)\|_2^2 + \|w_t(0)\|_2^2 + \alpha \|A^{1/4}w_t(0)\|_2^2 \right)
\end{aligned} \quad (2.2.24)$$

for some positive constant $C_T = C_T(\|(u_0, u_1)\|_{\mathcal{H}_0})$. (2.2.24) shows the continuous dependence of strong solutions on the initial data in $\mathcal{H}_\alpha$.

The same conclusion holds for weak solutions by using the density argument. In fact, if we consider the initial data $z_0 = (u_0, u_1)$, $\tilde{z}_0 = (\tilde{u}_0, \tilde{u}_1) \in \mathcal{H}_\alpha$, then similarly to (2.2.13) there exist sequences of strong solutions $z^k = (u^k, u_t^k)$ and $\tilde{z}^k = (\tilde{u}^k, \tilde{u}_t^k)$ such that

$$(z^k, \tilde{z}^k) \rightarrow (z, \tilde{z}) \quad \text{in } C([0, T], \mathcal{H}_\alpha \times \mathcal{H}_\alpha).$$

The difference $z^k - \tilde{z}^k := (w^k, w_t^k)$ satisfies (2.2.24) for each $k \in \mathbb{N}$, hence the estimate (2.1.8) holds for the difference of weak solutions $z - \tilde{z}$ after passing the limit as $k \rightarrow \infty$.

Particularly, we have uniqueness of both strong and weak solutions. This completes the proof of Theorem 2.1.1. $\square$

Remark 2.2.1. The well-posedness of the problem (1.2.1) provide the family of evolution operators $S_\alpha(t) : \mathcal{H}_\alpha \rightarrow \mathcal{H}_\alpha$ defined by

$$S_\alpha(t)(u_0, u_1) = (u(t), u_t(t)), \quad t \geq 0, \quad (2.2.25)$$

where (u, u_t) is the unique weak solution of (1.2.1). S_α are nonlinear C_0 -semi-groups, and is locally Lipschitz continuous on the space $\mathcal{H}_\alpha$. Hence, the problem (1.2.1) generates a dynamical system $(\mathcal{H}_\alpha, S_\alpha(t))$ and we study the asymptotic behavior of solutions for the problem (1.2.1) through the dynamical system $(\mathcal{H}_\alpha, S_\alpha(t))$.

Chapter 3

Stability and long-time dynamics

In this chapter we show the existence of attractors and clarifying their properties. First of all, we introduce the definition of the dynamical system and present a couple of notions of the dynamical systems theory in infinite-dimensional case.

3.1 Review on dynamical systems

In this thesis dynamical system is taken to mean the pair of objects $(X, S(t)) = (X, S(t))_{t \in \mathbb{T}}$ consisting of a complete metric space X and a family $\{S(t)\}_{t \in \mathbb{T}}$ of continuous mappings of the space X into itself with the properties

$$S(t + \tau) = S(t) \circ S(\tau), \quad \forall t, \tau \in \mathbb{T}, \quad S(0) = id_X,$$

where $\mathbb{T}$ coincides with either a set $\mathbb{R}_{\geq 0}$ of nonnegative real numbers or a set $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$. If $\mathbb{T} = \mathbb{R}_{\geq 0}$, we also assume that the mapping $t \mapsto S(t)x$ is continuous from $\mathbb{R}_{\geq 0}$ into X for every $x \in X$. Then the space X is called a **phase space** (or a state space). If $\mathbb{T} = \mathbb{Z}_{\geq 0}$, then the dynamical system is called **discrete** (or the system with discrete time).

Next, we give the definitions of **global attractor**, **minimal attractor**, **fractal dimension**, **exponential attractor** and **unstable manifold**.

Definition 3.1.1. *Let X be a complete linear metric space. A bounded set $A \subset X$ is said to be a **global attractor** of the dynamical system $(X, S(t))$ if and only if the following properties hold:*

- i. A is an invariant set; that is, $S(t)A = A, \quad \forall t \geq 0$.*
- ii. A is uniformly attracting; that is, for all bounded set $D \subset X$*

$$\lim_{t \rightarrow \infty} h_X(S(t)D, A) = 0,$$

where $h_X(A, B) = \sup_{x \in A} \text{dist}_X(x, B)$ is the Hausdorff semidistance.

Definition 3.1.2. *Let X be a complete linear metric space. A bounded set $A_{\min} \subset X$ is said to be a **minimal attractor** of the dynamical system $(X, S(t))$ if and only if the following properties hold:*

- i. $A_{\min}$ is a positively invariant set; that is $S(t)A_{\min} \subseteq A_{\min}$, $\forall t \geq 0$
- ii. $A_{\min}$ attracts every point x in X ; that is,

$$\lim_{t \rightarrow \infty} \text{dist}_X(S(t)x, A_{\min}) = 0 \quad \text{for any } x \in X;$$

- iii. $A_{\min}$ is minimal; that is, $A_{\min}$ has no proper subsets possessing the above properties.

Definition 3.1.3. Let X be a complete linear metric space and K be a compact set in X . The **fractal** (box-counting) **dimension** $\dim_f K$ of K is defined by the formula

$$\dim_f K := \limsup_{\epsilon \rightarrow 0} \frac{\log n(K, \epsilon)}{\log(1/\epsilon)},$$

where, $n(K, \epsilon)$ is the minimal number of closed sets of the radius ϵ that cover K .

Definition 3.1.4. Let X be a complete linear metric space. A compact set $A_{\text{exp}} \subset X$ is said to be a **exponential attractor** of the dynamical system $(X, S(t))$ if and only if A_{exp} is a positively invariant set of finite fractal dimension and for every bounded set $D \subset X$ there exists positive constants t_D , C_D and γ_D such that

$$h_X(S(t)D, A_{\text{exp}}) \leq C_D \cdot e^{-\gamma_D(t-t_D)}, \quad \forall t \geq t_D.$$

Definition 3.1.5. Let Y be a subset of the phase space X of the dynamical system $(X, S(t))$. Then the **unstable manifold** $\mathcal{M}^u(Y)$ emanating from Y is defined as the set of points $x \in X$ such that there exists a trajectory $\gamma = \{S(t)x = u(t) : t \in \mathbb{R}\}$ with the properties

$$\lim_{t \rightarrow -\infty} \text{dist}_X(u(t), Y) = 0.$$

Here, we introduce some important concepts we shall need for proving the statement on stability properties:

Definition 3.1.6. Let S be a semiflow on a Banach space X .

- i. A closed set $B \subset X$ is said to be **absorbing** for $(X, S(t))$ if and only if for any bounded set $D \subset X$ there exists $t_0(D)$ such that $S(t)D \subset B$ for all $t \geq t_0(D)$.
- ii. $(X, S(t))$ is said to be **dissipative** if and only if it possesses a bounded absorbing set B .
- iii. $(X, S(t))$ is said to be **asymptotically compact** if and only if there exists an attracting compact set K ; that is for any bounded set D we have

$$\lim_{t \rightarrow \infty} h_X(S(t)D, K) = 0.$$

Definition 3.1.7. Let $B \subseteq X$ be a positively invariant set of a dynamical system $(X, S(t))$.

- i. The continuous functional Φ defined on B is said to be the **Lyapunov function** for the dynamical system $(X, S(t))$ on B if and only if the function $t \mapsto \Phi(S(t)z)$ is a non-increasing function for any $z \in B$.

- ii. The Lyapunov function Φ is said to be **strict** on B if and only if for $z \in B$, the equation $\Phi(S(t)z) = \Phi(z)$ for all $t > 0$ implies that $S(t)z = z$ for all $t > 0$; that is, z is a stationary point of $(X, S(t))$.
- iii. The dynamical system $(X, S(t))$ is said to be **gradient** if and only if there exists a strict Lyapunov function for $(X, S(t))$ on the whole phase space X .

The concept introduced below is one of the form so-called **quasi-stable** or **compact perturbation of contraction operator**. In order to state the definition of the concept, we assume the following structure:

Assumption 3.1.8. Let X_1 and X_2 be reflexive Banach spaces such that X_1 is compactly embedded in X_2 . We endow the space $X := X_1 \times X_2$ with the norm

$$\|x\|_X^2 := \|y_1\|_{X_1}^2 + \|y_2\|_{X_2}^2, \quad x = (y_1, y_2).$$

We assume that $(X, S(t))$ is a dynamical system with the evolution operator of the form

$$S(t)x_0 = x(t) = (y(t), y_t(t)), \quad x_0 = (y_1, y_2) \in X,$$

where the function $y(\cdot)$ possesses the property

$$y \in C(\mathbb{R}_{\geq 0}; X_1) \cap C^1(\mathbb{R}_{> 0}; X_2). \quad (3.1.1)$$

Then the concept **quasi-stable** is given in the following:

Definition 3.1.9. (See [13]) Let Assumption 3.1.8 be valid. Then the dynamical system $(X, S(t))$ is said to be **quasi-stable** on a closed bounded set $B \subset X$ if and only if there exist nonnegative functions $a(\cdot), b(\cdot)$ and $c(\cdot)$ on $[0, \infty)$ such that (i) $a(\cdot)$ and $c(\cdot)$ are locally bounded on $[0, \infty)$, (ii) $b(\cdot)$ possesses the property $\lim_{t \rightarrow \infty} b(t) = 0$, (iii) for every $x^1, x^2 \in B$ and $T > 0$ the relations

$$\|S(T)x^1 - S(T)x^2\|_X \leq a(T)\|x^1 - x^2\|_X \quad (3.1.2)$$

and

$$\|S(T)x^1 - S(T)x^2\|_X \leq b(T)\|x^1 - x^2\|_X + c(T) \cdot \sup_{0 \leq t \leq T} \|y^1(t) - y^2(t)\|_{X_2} \quad (3.1.3)$$

hold.

Remark 3.1.10. The concept “quasi-stable” means a decomposition of the semiflow S into a contraction part and compact part, roughly speaking. Actually, if we define the map $\phi_T : B \rightarrow X_T$ as

$$(\phi_T x)(\cdot) := y(\cdot) \in X_T := C([0, T]; X_2), \quad x \in B,$$

then we can see that the image of B through ϕ_T is compact in X_T . Indeed, for a given sequence of initial data $x_0^i \in B$, we write $S(t)x_0^i = x^i(t) = (y^i(t), y_t^i(t))$ ($t \in [0, T]$). Since B is bounded and (3.1.2), the sequence $(x^i)_{i \in \mathbb{N}} \subset C([0, T]; X)$ is bounded. Consequently, the sequence $(y^i)_{i \in \mathbb{N}}$ is bounded and equicontinuous in X_T . Combining these properties with the compactness of the embedding $X_1 \hookrightarrow X_2$, there exists a subsequence still denoted by (y^i) such that

$$(y^i) \text{ converges strongly in } X_T.$$

Thus $\phi_T(B)$ is compact in X_T .

3.2 Assumptions and results

Now, we add assumptions for establishing stability properties of solutions:

(H3) There exists a constant $l_1 \geq 0$ such that

$$\tilde{f}(u) \leq f(u)u + \frac{\lambda_1}{8}|u|^2 + l_1, \quad \forall u \in \mathbb{R}. \quad (3.2.1)$$

(H4) There exists a constant $l_2 \geq 0$ such that

$$\widetilde{M}(\tau) := \int_0^\tau M(s)ds \leq 2M(\tau)\tau + \frac{\lambda_1^{1/2}}{4}\tau + 2l_2, \quad \forall \tau \geq 0. \quad (3.2.2)$$

Then we can state the stability properties of solutions as follows:

Theorem 3.2.1. *Let us assume that the hypothesis of Theorem 2.1.1 holds. Besides, we suppose that (H3) and (H4) hold. Then we have*

- i. *The dynamical system $\mathcal{H}_\alpha$ generated from the problem (1.1.4)-(1.1.6) possesses the global attractor $\mathcal{A}_\alpha \subset \mathcal{H}_\alpha$ and it is compact and connected.*
- ii. *The global attractor $\mathcal{A}_\alpha$ is an unstable manifold $\mathcal{A}_\alpha = \mathcal{M}^u(\mathcal{N})$, emanating from the set $\mathcal{N}$ consisting of stationary points of S_α , namely,*

$$\mathcal{N} = \{(u, 0) \in \mathcal{H}_\alpha \mid Au + M(\|A^{1/4}u\|_2^2)A^{1/2}u + f(u) = h\}.$$

- iii. *There exists the minimal attractor $\mathcal{A}_{\min}$ to the dynamical system $\mathcal{H}_\alpha$ generated from the problem (1.1.4)-(1.1.6), which is the set of the stationary points, that is, $\mathcal{A}_{\min} = \mathcal{N}$.*

In addition, we consider the following slightly stronger requirement on p than (2.1.3):

(H5) Instead of (2.1.3) from (H2) we consider either of the conditions below

$$p > 0 \quad \text{if} \quad 1 \leq n \leq 4 \quad \text{or else} \quad 0 < p < \frac{8}{n-4} \quad \text{if} \quad n \geq 5. \quad (3.2.3)$$

Then we also have the following:

Theorem 3.2.2. *Let us assume that the hypothesis of Theorem 2.1.1 and Theorem 3.2.1 hold. Besides, we suppose that (H5) holds. Then we have*

- i. *The compact global attractor $\mathcal{A}_\alpha$ has a finite fractal dimension.*
- ii. *Any full trajectory $(u(t), u_t(t))$ that belongs to the global attractor $\mathcal{A}_\alpha$ enjoys the following regularity properties,*

$$(u_t, u_{tt}) \in L^\infty(\mathbb{R}; \mathcal{H}_\alpha).$$

Specifically, there exists $R_1 > 0$ such that

$$\|A^{1/2}u_t(t)\|_2^2 + \|u_{tt}(t)\|_2^2 + \alpha\|A^{1/4}u_{tt}\|_2^2 \leq R_1^2, \quad t \in \mathbb{R}, \quad (3.2.4)$$

where R_1 does not depend on α . In addition, the global attractor $\mathcal{A}_\alpha$ lies in $\mathcal{H}_{\alpha,2}$.

- iii. *The dynamical system $(\mathcal{H}_\alpha, S_\alpha(t))$ possesses a proper fractal exponential attractor $\mathcal{A}_{\exp, \alpha}$.*

3.3 Proof of Theorem 3.2.1

We show the existence of attractors and their structures by using the following criteria:

Proposition 3.3.1. (See [13]) *Suppose that a dynamical system $(X, S(t))$ possesses the following properties:*

- i. The dynamical system $(X, S(t))$ is dissipative.*
- ii. The dynamical system $(X, S(t))$ is asymptotically compact.*

Then the system $(X, S(t))$ possesses the global attractor and it is compact and connected.

Proposition 3.3.2. (See [13]) *Let a dynamical system $(X, S(t))$ possess a compact global attractor A . Assume also that the Lyapunov function Φ exists on A . Then*

- i. $A = \mathcal{M}^u(\mathcal{N})$, where $\mathcal{N}$ is the set of stationary points of the dynamical system.*
- ii. $A_{\min} = \mathcal{N}$, where $A_{\min}$ is the minimal attractor of the dynamical system $(X, S(t))$.*

Let us show the existence of the global attractor of $(\mathcal{H}_\alpha, S_\alpha(t))$ according to the Proposition 3.3.1. First we show the dissipativity of dynamical system $(\mathcal{H}_\alpha, S_\alpha(t))$.

Dissipativity of the dynamical system $(\mathcal{H}_\alpha, S_\alpha(t))$

Proposition 3.3.3. *Let us assume that $(u_0, u_1) \in \mathcal{H}_\alpha$ and the hypotheses of Theorem 2.1.1 and 3.2.1 hold. Then, if $z = (u, u_t)$ is a weak solution corresponding to the initial data $z_0 = (u_0, u_1)$, we have*

$$\|z(t)\|_{\mathcal{H}_\alpha}^2 \leq K_1 e^{-\delta t} + K_2, \quad \forall t > 0, \quad (3.3.1)$$

for some positive constants $K_1 = K_1(\|z_0\|_{\mathcal{H}_\alpha})$ and $K_2 = K_2(h, l_0, l_1, |\Omega|)$ and a small constant $\delta > 0$.

Proof. We deal with only strong solutions, because the same conclusion follows easily for weak solutions by using the density argument.

The strong solution $z = (u, u_t)$ satisfies the following estimates

$$\begin{aligned} \frac{1}{4} \|z(t)\|_{\mathcal{H}_\alpha}^2 &= \frac{1}{4} \left(\|A^{1/2}u(t)\|_2^2 + \|u_t(t)\|_2^2 + \alpha \|A^{1/4}u_t(t)\|_2^2 \right) \\ &\leq E_\alpha(t) + \frac{2}{\lambda_1} \|h\|_2^2 + l_0 |\Omega|, \end{aligned} \quad (3.3.2)$$

and

$$\frac{d}{dt} E_\alpha(t) + N \left(\|A^{1/4}u(t)\|_2^2 \right) \|A^{1/4}u_t(t)\|_2^2 = 0,$$

where the energy E_α is given in (2.2.3). From $N(\tau) > 0$ and the estimate (2.2.9), we get

$$\frac{d}{dt} E_\alpha(t) \leq -N_0 \|A^{1/4}u_t(t)\|_2^2, \quad (3.3.3)$$

for some positive constant $N_0 > 0$ depending on the initial data in $\mathcal{H}_\alpha$. Now we define for any $\epsilon > 0$ a perturbed energy

$$E_{\alpha,\epsilon}(t) := E_\alpha(t) + \epsilon \Psi_\alpha(t) \quad \text{with} \quad \Psi_\alpha(t) := (u_t(t), u(t)) + \alpha(A^{1/4}u_t(t), A^{1/4}u(t)).$$

In the following we use C_0, C_1, C_2 to denote several positive constants appearing in the estimates. Firstly, we claim that there exists a constant $C_0 > 0$ such that

$$\frac{d}{dt} \Psi_\alpha(t) \leq -E_\alpha(t) + C_0 \|A^{1/4}u_t(t)\|_2^2 + l_1 |\Omega| + l_2. \quad (3.3.4)$$

In fact, taking derivative of the function $\Psi_\alpha(t)$, using the weak formulation (2.2.1), adding and subtracting $E_\alpha(t)$ into the resulting expression, we obtain

$$\frac{d}{dt} \Psi_\alpha(t) = -E_\alpha(t) + \frac{3}{2} \left(\|u_t(t)\|_2^2 + \alpha \|A^{1/4}u_t(t)\|_2^2 \right) - \frac{1}{2} \|A^{1/2}u(t)\|_2^2 + \sum_{j=1}^3 L_j, \quad (3.3.5)$$

where

$$\begin{aligned} L_1 &= \frac{1}{2} \widetilde{M} \left(\|A^{1/4}u(t)\|_2^2 \right) - M \left(\|A^{1/4}u(t)\|_2^2 \right) \|A^{1/4}u(t)\|_2^2, \\ L_2 &= -N \left(\|A^{1/4}u(t)\|_2^2 \right) (A^{1/4}u_t(t), A^{1/4}u(t)), \\ L_3 &= \int_{\Omega} \tilde{f}(u(t)) dx - (f(u(t)), u(t)). \end{aligned}$$

Now we estimate L_1, L_2 and L_3 . From the condition (3.2.2) and embedding $D(A^{1/2}) \hookrightarrow D(A^{1/4})$, we get

$$|L_1| \leq \frac{1}{8} \|A^{1/2}u(t)\|_2^2 + l_2.$$

Using Young's inequality and the embedding $D(A^{1/2}) \hookrightarrow D(A^{1/4})$, we obtain

$$\begin{aligned} |L_2| &\leq N \left(\|A^{1/4}u(t)\|_2^2 \right) \|A^{1/4}u_t(t)\|_2 \|A^{1/2}u(t)\|_2 \\ &\leq \frac{1}{8} \|A^{1/2}u(t)\|_2^2 + C_0 \|A_t^{1/4}(t)\|_2^2. \end{aligned}$$

From the condition (3.2.1) we see that

$$|L_3| \leq \frac{1}{8} \|A^{1/2}u(t)\|_2^2 + l_1 |\Omega|.$$

Inserting these last three estimates in (3.3.5) and using the embedding $D(A^{1/4}) \hookrightarrow L^2(\Omega)$, then we get (3.3.4).

Choosing $\epsilon > 0$ small enough such that $\epsilon \leq \frac{N_0}{C_0}$, then we have

$$\frac{d}{dt} E_{\alpha,\epsilon}(t) \leq -\epsilon E_\alpha(t) + \epsilon(l_1 |\Omega| + l_2), \quad \forall t > 0. \quad (3.3.6)$$

On the other hand, using Young's inequality and the estimate (2.2.9), there exists a constant $C_1 > 0$ such that

$$|E_{\alpha,\epsilon}(t) - E_\alpha(t)| \leq \epsilon C_1 (E_\alpha(t) + \|h\|_2^2 + |\Omega|), \quad \forall t > 0, \quad \forall \epsilon > 0. \quad (3.3.7)$$

Let us take and fix $\epsilon > 0$ small enough such that $\epsilon \leq \min\{\frac{N_0}{C_0}, \frac{1}{2C_1}\}$. Then the estimate (3.3.7) implies

$$-\frac{1}{2}(\|h\|_2^2 + |\Omega|) + \frac{1}{2}E_\alpha(t) \leq E_{\alpha,\epsilon}(t) \leq \frac{3}{2}E_\alpha(t) + \frac{1}{2}(\|h\|_2^2 + |\Omega|) \quad (3.3.8)$$

and combining (3.3.6) with (3.3.8), we have

$$E_\alpha(t) \leq 3E_\alpha(0)e^{-\frac{2}{3}\epsilon t} + C, \quad \forall t > 0, \quad (3.3.9)$$

where $C = C(\|h\|_2, |\Omega|) > 0$. Therefore, again using (2.2.6) we conclude that (3.3.1) holds true. $\square$

Remark 3.3.4. From Proposition 3.3.3, we immediately see that

$$B_\alpha := \{z \in \mathcal{H}_\alpha \mid \|z\|_{\mathcal{H}_\alpha} \leq K_2 + \delta'\}$$

is absorbing set of the system $(\mathcal{H}_\alpha, S_\alpha(t))$ and

$$\sup_{z \in B_\alpha} \sup_{t \geq 0} \|S_\alpha(t)z\|_{\mathcal{H}_\alpha} \leq C, \quad (3.3.10)$$

where δ' is an arbitrary positive constant, $K_2 = K_2(h, l_0, l_1, l_2, |\Omega|)$ and C is a positive constant which does not depend on $t \geq 0$ and $\alpha \in [0, 1]$.

Existence of the attractors and its structures

There remains to prove the asymptotically compactness. For showing the property, we use the other equivalent description below:

Proposition 3.3.5. (See [11]) Assume that X is a Banach space and $(X, S(t))$ is a dissipative dynamical system. Then the following assertions are equivalent.

- i. $(X, S(t))$ is asymptotically compact.
- ii. There exists a decomposition $S(t) = S^1(t) + S^2(t)$, where $S^1(t)$ is continuous mapping in X such that

$$r_D(t) = \sup\{\|S^1(t)x\|_X : x \in D\} \rightarrow 0 \quad (t \rightarrow \infty)$$

for every bounded set D and $S^2(t)$ is uniformly compact for large t ; that is, for any bounded set D there exists $t_0 = t_0(D)$ such that the set $\bigcup_{\tau \geq t_0} S^2(\tau)D$ is relatively compact in X .

The semigroup decomposition. Now we decompose the solution $S_\alpha(t)z$ with initial data $z \in B_\alpha$ into the sum

$$S_\alpha(t)z = S_\alpha^1(t)z + S_\alpha^2(t)z \quad (3.3.11)$$

where

$$S_\alpha^1(t)z = (\xi(t), \xi_t(t)) \quad \text{and} \quad S_\alpha^2(t)z = (\zeta(t), \zeta_t(t))$$

respectively solve the problems

$$\begin{aligned} \xi_{tt} + \alpha A^{1/2} \xi_{tt} + A\xi + M(\|A^{1/4}u\|_2^2)A^{1/2}\xi + N(\|A^{1/4}u\|_2^2)A^{1/2}\xi_t &= 0, \\ (\xi(0), \xi_t(0)) &= z, \end{aligned} \quad (3.3.12)$$

and

$$\begin{aligned} \zeta_{tt} + \alpha A^{1/2} \zeta_{tt} + A\zeta + M(\|A^{1/4}u\|_2^2)A^{1/2}\zeta + N(\|A^{1/4}u\|_2^2)A^{1/2}\zeta_t + f(u) &= h, \\ (\zeta(0), \zeta_t(0)) &= 0. \end{aligned} \quad (3.3.13)$$

This decomposition gives the asymptotic regularity of S_α , for initial data $z \in B_\alpha$. First, we begin from the exponential decay of S_α^1 .

Proposition 3.3.6. *For every $\alpha \in [0, 1]$, there exists $\delta > 0$ (independent of α) such that*

$$\|S_\alpha^1(t)z\|_{\mathcal{H}_\alpha} \leq Ce^{-\delta t}\|z\|_{\mathcal{H}_\alpha}.$$

Proof. Here, the generic positive constant C may depend on the radius B_α . The proof is similar to the one of Proposition 3.3.3, so we will give a quick sketch. One defines the functionals

$$E_\alpha^1(t) := \frac{1}{2} \left\{ \|A^{1/2}\xi(t)\|_2^2 + \|\xi_t(t)\|_2^2 + \alpha \|A^{1/4}\xi_t(t)\|_2^2 + M\left(\|A^{1/4}u(t)\|_2^2\right) \|A^{1/4}\xi(t)\|_2^2 \right\}$$

and

$$E_{\alpha,\epsilon}^1(t) := E_\alpha^1(t) + \epsilon \Psi_\alpha^1(t) \quad \text{with} \quad \Psi_\alpha^1(t) := (\xi_t(t), \xi(t)) + \alpha(A^{1/4}\xi_t(t), A^{1/4}\xi(t))$$

for any $\epsilon > 0$. The former functional satisfies the relation below

$$\frac{d}{dt} E_\alpha^1(t) + N\left(\|A^{1/4}u(t)\|_2^2\right) \|A^{1/4}\xi_t(t)\|_2^2 = L_1^1, \quad (3.3.14)$$

where

$$L_1^1 = \left(\frac{1}{2} \frac{d}{dt} M\left(\|A^{1/4}u(t)\|_2^2\right) \right) (A^{1/2}\xi(t), \xi_t(t)).$$

Since $S_\alpha(t)z = (u(t), u_t(t))$ is uniformly bounded by the inequality (2.2.9), we can estimate L_1^1

$$\begin{aligned} |L_1^1| &\leq |M'(\|A^{1/4}u(t)\|_2^2)| (A^{1/4}u(t), A^{1/4}u_t(t)) (A^{1/2}\xi(t), \xi_t(t))| \\ &\leq C \|u_t(t)\|_2 \|A^{1/2}\xi(t)\|_2 \|\xi_t(t)\|_2 \\ &\leq \epsilon' \|A^{1/4}\xi_t(t)\|_2^2 + \frac{C}{4\epsilon'} \|u_t(t)\|_2^2 \|A^{1/2}\xi(t)\|_2^2. \end{aligned}$$

Inserting the estimate in (3.3.14), we have

$$\frac{d}{dt} E_\alpha^1(t) \leq (\epsilon' - N_0) \|A^{1/4}\xi_t(t)\|_2^2 + C_{\epsilon'} \|u_t(t)\|_2^2 E_\alpha^1(t). \quad (3.3.15)$$

Now we consider the derivative of the latter functional. Likewise the estimation of the derivative of function $\Psi_\alpha(t)$, we obtain the following

$$\frac{d}{dt}\Psi_\alpha^1(t) \leq E_\alpha^1(t) - \frac{1}{2}\|A^{1/2}\xi(t)\|_2^2 + \frac{3}{2}\left(\|\xi_t(t)\|_2^2 + \alpha\|A^{1/4}\xi_t(t)\|_2^2\right) + L_2^1 \quad (3.3.16)$$

where

$$L_2^1 = N\left(\|A^{1/4}u(t)\|_2^2\right)(A^{1/4}\xi_t(t), A^{1/4}\xi(t)).$$

Using the same argument in L_2 , we get

$$\begin{aligned} |L_2^1| &\leq N\left(\|A^{1/4}u(t)\|_2^2\right)\|A^{1/4}\xi_t(t)\|_2^2\|A^{1/4}\xi(t)\|_2^2 \\ &\leq \frac{1}{2}\|A^{1/2}\xi(t)\|_2^2 + C\|A^{1/4}\xi_t(t)\|_2^2. \end{aligned} \quad (3.3.17)$$

Combining the estimates (3.3.15) and (3.3.17), and choosing $\epsilon, \epsilon' > 0$ small enough, then the derivative of the function $E_{\alpha,\epsilon}^1$ is controlled by

$$\frac{d}{dt}E_{\alpha,\epsilon}^1(t) \leq -\delta E_\alpha^1(t), \quad \forall t > 0.$$

Similarly to the proof of the Proposition 3.3.3, we arrive at the desire estimate. $\square$

Next we show the boundedness of $S_\alpha^2(t)z$ in more regular space.

Proposition 3.3.7. *For every $\alpha \in [0, 1]$, we have the estimate:*

$$\sup_{t \geq 0} \|S_\alpha^2(t)z\|_{\mathcal{H}_{\alpha,1}} \leq C.$$

Proof. Set

$$E_\alpha^2(t) := \|A^{3/4}\zeta(t)\|_2^2 + \|A^{1/4}\zeta_t(t)\|_2^2 + \alpha\|A^{1/2}\zeta(t)\|_2^2$$

and

$$\begin{aligned} E_{\alpha,\epsilon}^2(t) &:= E_\alpha^2(t) + \epsilon\{(A^{1/4}\zeta(t), A^{1/4}\zeta_t(t)) + \alpha(A^{1/2}\zeta(t), A^{1/2}\zeta_t(t))\} \\ &\quad + \frac{\epsilon}{2}N\left(\|A^{1/4}u(t)\|_2^2\right)\|A^{1/2}\zeta(t)\|_2^2 + M\left(\|A^{1/4}u(t)\|_2^2\right)\|A^{1/2}\zeta(t)\|_2^2, \end{aligned}$$

where $\epsilon > 0$ to be determined. By easy calculation, we have

$$E_{\alpha,\epsilon}^2(t) \geq \left\{1 - \frac{\epsilon m}{2}\right\} E_\alpha^2(t) \quad \text{where} \quad m := \max\left\{\frac{1}{\lambda_1}(1 + \alpha\lambda_1^{1/2}), 1\right\}$$

and we can immediately see that $E_\alpha^2(t) \leq 2E_{\alpha,\epsilon}^2(t)$ for $\epsilon \leq m^{-1}$.

Next, we multiply (3.3.13) by $\epsilon A^{1/2}\zeta + 2A^{1/2}\zeta_t$ ($\epsilon \leq m^{-1}$), we get

$$\begin{aligned} \frac{d}{dt}E_{\alpha,\epsilon}^2(t) + \epsilon E_{\alpha,\epsilon}^2(t) - 2\epsilon(\|A^{1/4}\zeta_t(t)\|_2^2 + \|A^{1/2}\zeta_t(t)\|_2^2) \\ + 2N\left(\|A^{1/4}u(t)\|_2^2\right)\|A^{1/2}\zeta_t(t)\|_2^2 = \sum_{k=1}^6 L_k^2, \end{aligned}$$

where

$$\begin{aligned}
L_1^2 &= \left(\frac{d}{dt} M(\|A^{1/4}u(t)\|_2^2) \right) \|A^{1/2}\zeta(t)\|_2^2, \\
L_2^2 &= \left(\frac{\epsilon}{2} \frac{d}{dt} N(\|A^{1/4}u(t)\|_2^2) \right) \|A^{1/2}\zeta(t)\|_2^2, \\
L_3^2 &= \epsilon^2 \left\{ \frac{1}{2} N(\|A^{1/4}u(t)\|_2^2) \|A^{1/2}\zeta(t)\|_2^2 + (A^{1/2}\zeta(t), \zeta_t(t)) \right\}, \\
L_4^2 &= \epsilon(h - f(u(t)), A^{1/2}\zeta(t)), \\
L_5^2 &= 2(h - f(u(t)), A^{1/2}\zeta_t(t)) + \epsilon^2 \alpha(A^{1/2}\zeta(t), A^{1/2}\zeta_t(t)).
\end{aligned}$$

It is easy to estimate these terms. Indeed, from the estimates (2.2.9) and (3.3.10), we learn that

$$|L_i^2| \leq C_{B_\alpha, \epsilon, M, N, f, h} \quad \text{for each } j = 1, 2, 3, 4$$

and

$$|L_5^2| \leq \epsilon \|A^{1/2}\zeta(t)\|_2^2 + C_{B_\alpha, \epsilon, f, h}$$

for any $\epsilon > 0$. Substituting the above estimates in (3.3.18), we arrive at

$$\frac{d}{dt} E_{\alpha, \epsilon}^2(t) + \epsilon E_{\alpha, \epsilon}^2(t) + (N_0 - C_{\lambda_1} \epsilon) \|A^{1/2}\zeta(t)\|_2^2 \leq C_{B_\alpha, \epsilon, M, N, f, h}.$$

Since $E_\alpha^2(t) \leq 2E_{\alpha, \epsilon}^2(t)$ for $\epsilon \leq m^{-1}$ and $E_{\alpha, \epsilon}^2(0) = 0$, choosing $\epsilon > 0$ small enough so that $\epsilon := \min\{\frac{1}{m}, \frac{N_0}{C_{\lambda_1}}\}$, we can obtain the desired estimate by applying the Gronwall's inequality. $\square$

Remark 3.3.8. *The decomposition indicates the existence of an exponential attracting compact set. Now, till the end of the thesis, we will set*

$$\mathbb{B}_{\alpha, \nu}(r) := \{z \in \mathcal{H}_{\alpha, \nu} \mid \|z\|_{\mathcal{H}_{\alpha, \nu}} \leq r\}.$$

Proposition 3.3.6 and Proposition 3.3.7 show that S_α admits a decomposition

$$S_\alpha(t)z = S_\alpha^1(t)z + S_\alpha^2(t)z,$$

for every $z \in B_\alpha$. S_α^1 is exponentially stable, while

$$S_\alpha^2(t)z \subset \mathbb{B}_{\alpha, 1}(r_0)$$

for some $r_0 > 0$ independent of α . The embedding of $\mathcal{H}_{\alpha, 1}$ into $\mathcal{H}_\alpha$ is compact, so $\mathbb{B}_{\alpha, 1}(r_0)$ is a compact subset of $\mathcal{H}_\alpha$. This means that the closed ball $\mathbb{B}_{\alpha, 1}(r_0)$ is uniformly exponential attracting set for $(\mathcal{H}_\alpha, S_\alpha(t))$, namely, for every bounded set $B \subset \mathcal{H}_\alpha$,

$$h_{\mathcal{H}_\alpha}(S_\alpha(t)B, \mathbb{B}_{\alpha, 1}(r_0)) \leq C_{r_0} e^{-\delta(t-t_{B_\alpha})}. \quad (3.3.18)$$

holds, where $t_{B_\alpha} > 0$ is a time B_α such that $S_\alpha(t)B \subset B_\alpha$ for every $t \geq t_{B_\alpha}$.

Remark 3.3.9. *It is worth noting that, from the well-posedness result (2.1.7), any semitrajectory $(u(t), u_t(t))$ emanating from $\mathbb{B}_{\alpha, 1}(r_0)$ has more regularity as in (2.1.7). Hence, in the same way as in the proof of Proposition 3.3.6 and Remark 3.3.4, There exists a ball $\mathbb{B}_{\alpha, 1}(R_0)$ ($R_0 > r_0$) such that $\mathbb{B}_{\alpha, 1}(R_0)$ is an absorbing set for the system $(\mathcal{H}_{\alpha, 1}, S_\alpha(t))$.*

We conclude this section with comment on the existence and the structure of the global attractor and minimal attractor. The decomposition instantly leads the existence of the global attractor and minimal attractor as in Proposition 3.3.1. It is also easy to check that the energy E_α is a Lyapunov function for the dynamical system $(\mathcal{H}_\alpha, S_\alpha(t))$ and we can immediately see that $(\mathcal{H}_\alpha, S_\alpha(t))$ is gradient. Thus we are able to get the conclusions on the structures of the global attractor and minimal attractor from Proposition 3.3.2.

3.4 Proof of Theorem 3.2.2

Our aim in this section is to prove the properties of the global attractor as in Theorem 2.1.1 and the existence of an exponential attractor, for all $\alpha \in [0, 1]$. Before we get into the main topic, we introduce the criterion we use for showing the properties of the global attractor.

Proposition 3.4.1. *Assume that the dynamical system $(X, S(t))$ possesses a compact global attractor A and is quasi-stable on A . Then the attractor A has a finite fractal dimension.*

Proposition 3.4.2. *Let the assumption 3.4.1 be in force. Besides, we assume that the function $c(\cdot)$ satisfies the property $c_\infty := \sup_{t>0} c(t) < \infty$. Then any full trajectory $\{(y(t), y_t(t)) \mid t \in \mathbb{R}\}$ that belongs to the global attractor enjoys the following regularity properties:*

$$y_t \in L^\infty(\mathbb{R}; X_1), \quad y_{tt} \in L^\infty(\mathbb{R}; X_2).$$

Moreover, there exists $R > 0$ such that

$$\|y_t(t)\|_{X_1}^2 + \|y_{tt}(t)\|_{X_2}^2 \leq R^2, \quad t \in \mathbb{R},$$

where R depends on the constant c_∞ and the diameter of the global attractor.

For establishing the existence, we derive the following important inequality, which leads to the quasi-stability of the dynamical system $(\mathcal{H}_\alpha, S_\alpha(t))$.

3.4.1 Quasi-stability of the dynamical system $(\mathcal{H}_\alpha, S_\alpha(t))$

Proposition 3.4.3. *Let the assumptions of Theorem 2.1.1 and 3.2.2 be in force. Given a bounded set $B \subset (\mathcal{H}_\alpha, S_\alpha(t))$ we consider two weak solutions $z^1 = (u, u_t)$, $z^2 = (v, v_t)$ corresponding to initial data $z_0^1 = (u_0, u_1)$, $z_0^2 = (v_0, v_1)$ lying in B . Then the following inequality holds:*

$$\|z^1(t) - z^2(t)\|_{\mathcal{H}_\alpha}^2 \leq C e^{-\delta t} \|z_0^1 - z_0^2\|_{\mathcal{H}_\alpha}^2 + C \int_0^t e^{-\delta(t-s)} \|w(s)\|_2^2 ds, \quad (3.4.1)$$

for all $t > 0$, where $C = C_B$ and $\delta = \delta_B$ are positive constants, and $w = u - v$.

Proof. First of all we fix a bounded set $B \subset \mathcal{H}_\alpha$ and consider two weak solutions $z^1 = (u, u_t)$, $z^2 = (v, v_t)$ with the initial data $z_0^1, z_0^2 \in B$, that is $\|z_0^1\|_{\mathcal{H}_\alpha}, \|z_0^2\|_{\mathcal{H}_\alpha} \leq R$, where $R > 0$ depends on the size of B . Putting the difference $z^1 - z^2 = (w, w_t)$

and proceeding exactly as in the proof of the a priori estimates we get the following inequality

$$\frac{1}{2} \frac{d}{dt} \left\{ \|A^{1/2}w(t)\|_2^2 + \|w_t(t)\|_2^2 + \alpha \|A^{1/4}w_t(t)\|_2^2 \right\} + N_R \|A^{1/4}w_t(t)\|_2^2 \leq \sum_{j=1}^4 J_j, \quad (3.4.2)$$

for some constant $N_R > 0$, where we use the global estimate (2.2.9) and $N(\tau) > 0$. Here, the expressions for $J_j, j = 1, 2, 3, 4$, are the same ones given in (2.2.17)-(2.2.20). Using the relation

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left[M \left(\|A^{1/4}u(t)\|_2^2 \right) \|A^{1/4}w(t)\|_2^2 \right] \\ = M' \left(\|A^{1/4}u(t)\|_2^2 \right) (A^{1/2}u(t), u_t(t)) \|A^{1/4}w(t)\|_2^2 - J_1, \end{aligned}$$

we can rewrite (3.4.2) as follows:

$$\frac{d}{dt} F_\alpha(t) + N_R \|A^{1/4}w_t(t)\|_2^2 \leq J_2 + J_3 + J_4 + J_5, \quad (3.4.3)$$

where we set

$$\begin{aligned} F_\alpha(t) \\ := \frac{1}{2} \left\{ \|A^{1/2}w(t)\|_2^2 + \|w_t(t)\|_2^2 + \alpha \|A^{1/4}w_t(t)\|_2^2 + M \left(\|A^{1/4}u(t)\|_2^2 \right) \|A^{1/4}w(t)\|_2^2 \right\} \end{aligned} \quad (3.4.4)$$

and

$$J_5 := M' \left(\|A^{1/4}u(t)\|_2^2 \right) (A^{1/2}u(t), u_t(t)) \|A^{1/4}w(t)\|_2^2. \quad (3.4.5)$$

Next we shall estimate the right-hand side of (3.4.3). To simplify the notation we use C_R to denote various positive constants depending on $R > 0$, but not on time. Firstly, since $M, N \in C^1([0, \infty))$, then the estimate (2.2.9) implies

$$\max_{\tau \in [0, C_R]} \{|M(\tau)|, |M'(\tau)|, |N(\tau)|, |N'(\tau)|\} < \infty.$$

Likewise the proof of the a priori estimate, applying the mean value theorem, Young's inequality with $\epsilon > 0$, the estimate (2.2.9) and the embedding $D(A^{1/2}) \hookrightarrow D(A^{1/4}) \hookrightarrow L^2(\Omega)$, we obtain

$$\begin{aligned} |J_2| &\leq C_R [\|A^{1/4}u(t)\|_2 + \|A^{1/4}v(t)\|_2] \|A^{1/4}w(t)\|_2 \|A^{1/4}v_t(t)\|_2 \|A^{1/4}w_t(t)\|_2 \\ &\leq C_R \|A^{1/2}w(t)\|_2 \|A^{1/4}v_t(t)\|_2 \|A^{1/4}w_t(t)\|_2 \\ &\leq \epsilon' \|A^{1/4}w_t(t)\|_2^2 + C_{R, \epsilon'} \|A^{1/4}v_t(t)\|_2^2 \|A^{1/2}w(t)\|_2^2 \end{aligned}$$

for any $\epsilon' > 0$. Adding the interpolation inequality, we have

$$\begin{aligned} |J_3| &\leq C_R [\|A^{1/4}u(t)\|_2 + \|A^{1/4}v(t)\|_2] \|A^{1/4}w(t)\|_2 \|A^{1/2}v(t)\|_2 \|w_t(t)\|_2 \\ &\leq C_R \|w(t)\|_2^{1/2} \|A^{1/2}w(t)\|_2^{1/2} \|A^{1/4}w_t(t)\|_2 \\ &\leq (C_{R, \epsilon'} \|w(t)\|_2 + \epsilon' \|A^{1/2}w(t)\|_2) \|A^{1/4}w_t(t)\|_2 \\ &\leq \epsilon' \|A^{1/4}w_t(t)\|_2^2 + \epsilon' \|A^{1/2}w(t)\|_2^2 + C_{R, \epsilon'} \|w(t)\|_2^2. \end{aligned}$$

Using the same conditions in the proof of the uniqueness, Young's inequality and Nirenberg-Gagliard's inequality, we have

$$\begin{aligned}
|J_4| &\leq \sigma_0(|\Omega|^{\frac{p}{2(p+2)}} + \|u(t)\|_{p+2}^{p/2} + \|v(t)\|_{p+2}^{p/2}) \|w(t)\|_{p+2} \|w_t(t)\|_2 \\
&\leq C_R \|w(t)\|_{p+2} \|A^{1/4} w_t(t)\|_2 \\
&\leq C_{R,\vartheta} \|A^{1/2} w(t)\|_2^\vartheta \|w(t)\|_2^{1-\vartheta} \|A^{1/4} w_t(t)\|_2 \\
&\leq (\epsilon' \|A^{1/2} w(t)\|_2 + C_{R,\vartheta,\epsilon'} \|w(t)\|_2) \|A^{1/4} w_t(t)\|_2 \\
&\leq \epsilon' \|A^{1/4} w_t(t)\|_2^2 + \epsilon' \|A^{1/2} w(t)\|_2^2 + C_{R,\epsilon'} \|w(t)\|_2^2,
\end{aligned}$$

where $\vartheta = \frac{n}{4} \left(1 - \frac{2}{p+2}\right)$. In addition, from the global estimate (2.2.9) and interpolation inequality, we get immediately

$$|J_5| \leq \epsilon' \|A^{1/2} w(t)\|_2^2 + C_{R,\epsilon'} \|w(t)\|_2^2.$$

Substituting these four estimates in (3.4.3) and choosing $\epsilon' > 0$ small enough, we see that there exists constants $N_R, C_R > 0$ such that

$$\begin{aligned}
\frac{d}{dt} F_\alpha(t) &\leq -N_R \|A^{1/4} w_t(t)\|_2^2 + C_R \|A^{1/4} v_t(t)\|_2^2 \|A^{1/2} w(t)\|_2^2 \\
&\quad + \epsilon' \|A^{1/2} w(t)\|_2^2 + C_R \|w(t)\|_2^2, \quad t > 0.
\end{aligned} \tag{3.4.6}$$

Now we define the functional

$$F_{\alpha,\eta}(t) := F_\alpha(t) + \eta \Phi_\alpha(t) \quad \text{with} \quad \Phi_\alpha(t) := (w_t(t), w(t)) + \alpha(A^{1/4} w_t(t), A^{1/4} w(t))$$

where $\eta > 0$ will be fixed later. We first show that there exists a constant $C_R, D_R > 0$ such that

$$\frac{d}{dt} \Phi_\alpha(t) \leq -F_\alpha(t) + C_R \|A^{1/4} w_t(t)\|_2^2 - D_R \|A^{1/2} w(t)\|_2^2 + \|w(t)\|_2^2, \quad t > 0. \tag{3.4.7}$$

Indeed, taking derivative of $\Phi_\alpha(t)$, using the weak formulation for w , adding and subtracting $F_\alpha(t)$ in the resulting expression and neglecting unnecessary terms, we arrive at

$$\frac{d}{dt} \Phi_\alpha(t) \leq -F_\alpha(t) + \frac{3}{2} \left(\|w_t(t)\|_2^2 + \alpha \|A^{1/4} w_t(t)\|_2^2 \right) - \frac{1}{2} \|A^{1/2} w(t)\|_2^2 + \sum_{j=1}^4 K_j, \tag{3.4.8}$$

where

$$\begin{aligned}
K_1 &= -N \left(\|A^{1/4} u(t)\|_2^2 \right) (A^{1/4} w(t), A^{1/4} w_t(t)), \\
K_2 &= -\left\{ M \left(\|A^{1/4} u(t)\|_2^2 \right) - M \left(\|A^{1/4} v(t)\|_2^2 \right) \right\} (A^{1/4} v(t), A^{1/4} w(t)), \\
K_3 &= -\left\{ N \left(\|A^{1/4} u(t)\|_2^2 \right) - N \left(\|A^{1/4} v(t)\|_2^2 \right) \right\} (v_t(t), A^{1/2} w(t)), \\
K_4 &= -(f(u(t)) - f(v(t)), w(t)).
\end{aligned}$$

First we estimate K_1 . From the estimate (2.2.9) and the uniform boundedness of N , Young's inequality with $\epsilon > 0$ and the embedding $D(A^{1/2}) \hookrightarrow D(A^{1/4})$ we get

$$|K_1| \leq C_R \|A^{1/4} w_t(t)\|_2 \|A^{1/2} w(t)\|_2$$

$$\leq \epsilon \|A^{1/2}w(t)\|_2^2 + C_{R,\epsilon} \|A^{1/4}w(t)\|_2^2.$$

Proceeding almost the same way as J_1, J_2 , and J_4 , but replacing the function w_t by w , we derive

$$\begin{aligned} |K_2| &\leq \epsilon \|A^{1/2}w(t)\|_2^2 + C_{R,\epsilon} \|w(t)\|_2^2, \\ |K_3| &\leq \epsilon \|A^{1/2}w(t)\|_2^2 + C_{R,\epsilon} \|w(t)\|_2^2, \\ |K_4| &\leq C_R \|w(t)\|_{p+2} \|w(t)\|_2 \\ &\leq \epsilon \|A^{1/2}w(t)\|_2^2 + C_{R,\epsilon} \|w(t)\|_2^2. \end{aligned}$$

for any $\epsilon > 0$ and some positive constant C_R . Going back to (3.4.8) and inserting these four estimates, we result that (3.4.7) holds, after choosing $\epsilon > 0$ small enough and using the embedding $D(A^{1/4}) \hookrightarrow L^2(\Omega)$.

Combining (3.4.6) with (3.4.7), noting that $\|A^{1/2}w(t)\|_2^2 \leq F_\alpha(t)$, and taking $\eta, \epsilon' > 0$ small enough such that $\eta < \frac{N_R}{C_R}$ and $\epsilon' < D_R$, there exists a constant $C_R > 0$ such that

$$\frac{d}{dt} F_{\alpha,\eta}(t) \leq -\eta F_\alpha(t) + C_R \|A^{1/4}v_t(t)\|_2^2 F_\alpha(t) + C_R \|w(t)\|_2^2, \quad (3.4.9)$$

for all $t > 0$ $\epsilon' < D_R$ and $\eta < \frac{N_R}{C_R}$.

On the other hand, by taking $C_1 := \max\{1, 1/\lambda_1\} > 0$, it is readily to see that

$$|F_{\alpha,\eta}(t) - F_\alpha(t)| \leq \eta C_1 F_\alpha(t), \quad \forall t \geq 0, \forall \eta > 0. \quad (3.4.10)$$

Then taking and fixing $\eta > 0$ such that $\eta \leq \min\{\frac{1}{2C_1}, \frac{N_R}{C_R}\}$, (3.4.10) implies that

$$\frac{1}{2} F_\alpha(t) \leq F_{\alpha,\eta}(t) \leq \frac{3}{2} F_\alpha(t), \quad \forall t \geq 0. \quad (3.4.11)$$

Compounding (3.4.9) with (3.4.11) we get

$$\frac{d}{dt} F_{\alpha,\eta}(t) \leq \phi_\eta(t) F_{\alpha,\eta}(t) + C_R \|w(t)\|_2^2, \quad t > 0, \quad (3.4.12)$$

where we define

$$\phi_\eta(t) := -\frac{\eta}{3} + C_R \|A^{1/4}v_t(t)\|_2^2$$

Applying Gronwall's inequality, we deduce that

$$F_{\alpha,\eta}(t) \leq e^{\int_0^t \phi_\eta(s) ds} \left(F_{\alpha,\eta}(0) + C_R \int_0^t e^{\int_0^s \phi_\eta(\xi) d\xi} \|w(s)\|_2^2 ds \right). \quad (3.4.13)$$

Moreover, from the estimate (2.2.9) we also have

$$\int_0^t \phi_\eta(s) ds = -\frac{\eta}{3} t + C_R \int_0^t \|A^{1/4}v_t(t)\|_2^2 ds \leq -\frac{\eta}{3} t + \widetilde{C}_R,$$

for some positive constant $\widetilde{C}_R$. Thus (3.4.13) leads to

$$F_{\alpha,\eta}(t) \leq C_R F_{\alpha,\eta}(0) e^{-\frac{\eta}{3} t} + C_R \int_0^t e^{-\frac{\eta}{3}(t-s)} \|w(s)\|_2^2 ds, \quad (3.4.14)$$

for all $t > 0$, and some constant $C_R > 0$. Lastly, from (3.4.4) we note that there exists a constant $C_R > 0$ such that

$$\|z_1(t) - z_2(t)\|_{\mathcal{H}_\alpha}^2 \leq F_\alpha(t) \leq (1 + C_R)\|z_1(t) - z_2(t)\|_{\mathcal{H}_\alpha}^2, \quad \forall t \geq 0. \quad (3.4.15)$$

Therefore, combining (3.4.11) with (3.4.14)-(3.4.15), we get the conclusion that the stability inequality (3.4.1) holds true for some constants $C_R > 0$ and $\delta_R > 0$. The proof of Proposition 3.4.3 is complete. $\square$

We conclude this subsection with comment on the properties of the global attractor stated in Theorem 3.2.2. From the inequality (3.4.1), we can see the quasi-stability of the dynamical system $(\mathcal{H}_\alpha, S_\alpha(t))$. Indeed, we are able to rewrite the inequality (3.4.1) as follows:

$$\begin{aligned} & \|S_\alpha(T)z^1 - S_\alpha(T)z^2\|_{\mathcal{H}_\alpha} \\ & \leq C_B e^{-\delta T} \|z^1 - z^2\|_{\mathcal{H}_\alpha} + C_B \int_0^T e^{-\frac{\eta}{3}(T-t)} dt \cdot \sup_{0 \leq t \leq T} \|u(t) - v(t)\|_{L^2(\Omega)}. \end{aligned} \quad (3.4.16)$$

From the inequality, we immediately see that the dynamical system $(\mathcal{H}_\alpha, S_\alpha(t))$ is quasi-stable and satisfies the condition in Proposition 3.4.2. Therefore we can obtain the conclusion on the finite dimensionality and regularity of the global attractor $\mathcal{A}_\alpha$ from Proposition 3.4.1 and Proposition 3.4.2

3.4.2 Construction of exponential attractors

In this subsection, we consider the existence of exponential attractors. Mentioning beforehand, the way of the construction of exponential attractors used in the following is essentially based on the method given in [9]. However, for the constructing exponential attractors with continuous dependence on the parameter α , we need to grasp the construction (the detail will be treated in the next section). So, along with the reader's convenience, we will introduce their method in brief.

For the beginning, we recall that each system $(\mathcal{H}_\alpha, S_\alpha(t))$ possesses an exponentially attracting set $\mathbb{B}_{\alpha,1}(R)$ ($R = R_0$) which is compact in $\mathcal{H}_\alpha$ and hereinafter, we denote the ball by $\mathbb{B}_\alpha$. Hence, it is sufficient to consider the dynamical system $(\mathcal{H}_\alpha, \mathbb{B}_\alpha, S_\alpha)$ and we show that each system $(\mathcal{H}_\alpha, \mathbb{B}_\alpha, S_\alpha)$ possesses an exponential attractor.

From now on, we consider a discrete dynamical system $(\mathcal{H}_\alpha, \mathbb{B}_\alpha, S_\alpha^m)$, where $S_\alpha^m := S_\alpha(mT) = (u^\alpha(mT), u^\alpha(mT))$ and $T > 0$ is a constant determined in the following. From the inequality (3.4.16), the flow S_α satisfies

$$\|S_\alpha z^1 - S_\alpha z^2\|_{\mathcal{H}_\alpha} \leq C_R e^{-\delta_R T} \|z^1 - z^2\|_{\mathcal{H}_\alpha} + C_R \|\phi_{T,\alpha} z^1 - \phi_{T,\alpha} z^2\|_{Z_T}, \quad (3.4.17)$$

where $Z_T := C([0, T]; L^2(\Omega))$ and $\phi_{T,\alpha} : \mathbb{B}_\alpha \rightarrow Z_T$ is the map defined as $\phi_{T,\alpha} z := u^\alpha(\cdot) \in Z_T$ ($z \in \mathbb{B}_\alpha$). Then we choose $T = T_0 > 0$ so that $C_R e^{-\delta T_0} =: q < 1/2$ holds and define constant $0 < \nu < 1$ as $\nu := 2C_R(q + \rho L)$, where $\rho > 0$ satisfies $0 < \rho < (1 - 2C_R q)/2LC_R$. Under the above condition, we can construct an exponential attractor for the system $(\mathcal{H}_\alpha, \mathbb{B}_\alpha, S_\alpha^m)$ as a collection of the central points of a finite

covering balls for each $S_\alpha^m \mathbb{B}_\alpha$. That is, for each $k = 0, 1, 2, \dots$, there exists a finite covering of $S_\alpha^k \mathbb{B}_\alpha$ such that

$$S_\alpha^k \mathbb{B}_\alpha \subset \bigcup_{i=1}^{N_\rho^k} \overline{B}^{\mathcal{H}_\alpha}(\tilde{z}_{k,i}; r_k), \quad (3.4.18)$$

where $\tilde{z}_{k,i}$ are the central points $\tilde{z}_{k,i} := S_\alpha^k z_{k,i} \in S_\alpha^k \mathbb{B}_\alpha$, $1 \leq i \leq N_\rho^k$ and $r_k = R\nu^k$. Then we have the following assertion on the existence of an exponential attractor:

Proposition 3.4.4. *A set defined below is an exponential attractor for the system $(\mathcal{H}_\alpha, \mathbb{B}_\alpha, S_\alpha^m)$:*

$$\mathcal{A}_{\text{exp}, \alpha}^* := \overline{\bigcup_{k,l=0}^{\infty} S_\alpha^l E_\alpha^k}^{\mathcal{H}_\alpha},$$

where $E_\alpha^k := \{S_\alpha^k z_{k,i} \mid z_{k,i} \in \mathbb{B}_\alpha, i = 1, 2, \dots, N_\rho^k\}$.

Proof. Indeed, it is easy to see that $\mathcal{A}_{\text{exp}, \alpha}^*$ is a compact positively invariant set. We also have that

$$\text{dist}_{\mathcal{H}_\alpha}(S_\alpha^n z, \mathcal{A}_{\text{exp}, \alpha}^*) \leq \text{dist}_{\mathcal{H}_\alpha}(S_\alpha^n z, E_\alpha^n) \leq R\nu^n, \quad n = 0, 1, 2, \dots,$$

for every $z \in \mathbb{B}_\alpha$. Hence the set $\mathcal{A}_{\text{exp}, \alpha}^*$ attracts $\mathbb{B}_\alpha$ exponentially fast.

Now we check the finite dimensionality of the set $\mathcal{A}_{\text{exp}, \alpha}^*$. For estimating the fractal dimension we note that, from the remark 3.3.9, there exists the time $t_R > 0$ such that the ball $\mathbb{B}_\alpha$ is absorbed by itself and we put $l_0 := [t_R] + 1$, where $[t_R]$ is an integer part of the number t_R . Then, it is easy to see that

$$\mathcal{A}_{\text{exp}, \alpha}^* \subset \bigcup_{k+l < n} S_\alpha^k E_\alpha^l \cup \left(\bigcup_{i=0}^{l_0-1} S_\alpha^{n+i} \mathbb{B}_\alpha \right), \quad \text{for any } n = l_0, l_0 + 1, \dots$$

Thus

$$n(\mathcal{A}_{\text{exp}, \alpha}^*, r_n) \leq \sum_{k+l < n} n(S_\alpha^k E_\alpha^l, r_n) + \sum_{i=0}^{l_0-1} n(S_\alpha^{n+i} \mathbb{B}_\alpha, r_n).$$

From the definition of the set E_α^l , we find that

$$\text{Card} S_\alpha^k E_\alpha^l = N_\rho^l$$

and

$$\begin{aligned} \sum_{k+l < n} n(S_\alpha^k E_\alpha^l, r_n) &= \sum_{k=0}^{n-1} \sum_{l=0}^{n-k-1} \text{Card} S_\alpha^k E_\alpha^l \\ &= \sum_{k=0}^{n-1} \sum_{l=0}^{n-k-1} N_\rho^l \\ &= \frac{N_\rho^{n+1} - nN_\rho + n}{(1 - N_\rho)^2}. \end{aligned}$$

On the other hand, the set $S_\alpha^n \mathbb{B}_\alpha$ possesses the covering as in (3.4.18), so it is clear that $n(S_\alpha^n \mathbb{B}_\alpha, r_n) \leq N_\rho^n$ and

$$n(S_\alpha^{n+i} \mathbb{B}, r_n) \leq n(S_\alpha^{n+i} \mathbb{B}_\alpha, r_{n+i}) \leq N_\rho^{n+i}, \quad (\forall i = 0, 1, \dots, l_0 - 1).$$

Then we have that

$$\sum_{i=0}^{l_0-1} n(S_\alpha^{n+i} \mathbb{B}, r_n) \leq \sum_{i=0}^{l_0-1} N_\rho^{n+i} = \frac{N_\rho^n (1 - N_\rho^{l_0})}{1 - N_\rho}.$$

Combining these estimates, we get that

$$n(\mathcal{A}_{\text{exp}, \alpha}^*, r_n) \leq N_\rho^{n+1} + n + N_\rho^{n+l_0} = N_\rho^n (1 + N_\rho^{l_0}) + n$$

and

$$\log n(\mathcal{A}_{\text{exp}, \alpha}^*, r_n) \leq n \log N_\rho + \log(1 + N_\rho^{l_0}) + \log n.$$

Now, for any $\epsilon < r_{l_0}$, we can find $n = n_\epsilon$ such that

$$r_{n_\epsilon} \leq \epsilon < r_{n_\epsilon-1}$$

and

$$\begin{aligned} \log n(\mathcal{A}_{\text{exp}, \alpha}^*, \epsilon) &\leq \log n(\mathcal{A}_{\text{exp}, \alpha}^*, r_{n_\epsilon}) \\ &\leq n_\epsilon \log N_\rho + \log(1 + N_\rho^{l_0}) + \log n_\epsilon. \end{aligned}$$

From the inequality $\epsilon < r_{n_\epsilon-1} = R\nu^{n_\epsilon-1}$, the integer n_ϵ is estimates as

$$n_\epsilon < 1 + \frac{\log \frac{R}{\epsilon}}{\log \frac{1}{\nu}}$$

Hence

$$\dim_f \mathcal{A}_{\text{exp}, \alpha}^* \leq N_\rho \cdot \frac{1}{\log \frac{1}{\nu}}.$$

Therefore the fractal dimension of the set $\mathcal{A}_{\text{exp}, \alpha}^*$ is certainly finite. $\square$

Below, we shall go on with the procedure for constructing the above covering.

Proof. We prove the relation (3.4.18) utilizing the inductive argument. The case $k = 0$ holds clearly. Now we assume that the relation (3.4.18) holds for k . Then, it is easy to see that

$$S_\alpha^{k+1} \mathbb{B}_\alpha = S_\alpha(S_\alpha^k \mathbb{B}_\alpha) \subset \bigcup_{i=1}^{N_\rho^k} S_\alpha \overline{B}^{\mathcal{H}_\alpha}(\tilde{z}_{k,i}; R\nu^k)$$

and it is sufficient to prove that, for each set $S_\alpha \overline{B}^{\mathcal{H}_\alpha}(\tilde{z}_{k,i}; R\nu^k)$, there exists a covering such that

$$S_\alpha \overline{B}^{\mathcal{H}_\alpha}(\tilde{z}_{k,i}; R\nu^k) \subset \bigcup_{j=1}^{N_\rho} \overline{B}^{\mathcal{H}_\alpha}(\tilde{z}_{k+1,j}; R\nu^{k+1}).$$

For constructing such a covering, we consider the image of $S_\alpha \bar{B}^{\mathcal{H}_\alpha}(\tilde{z}_{k,i}; R\nu^k)$ through $\phi_{T,\alpha}$. From the continuity (2.1.8), it is clear that

$$\phi_{T,\alpha}(S_\alpha \bar{B}^{\mathcal{H}_\alpha}(\tilde{z}_{k,i}; R\nu^k)) \subset V^T(\phi_{T,\alpha}\tilde{z}_{k,i}; RL\nu^k),$$

where

$$V^T(\phi_{T,\alpha}\tilde{z}_{k,i}; RL\nu^k) := \{x \in Z_T \mid \|x - \phi_{T,\alpha}\tilde{z}_{k,i}\|_{Z_T} \leq RL\nu^k, \ x \text{ is equicontinuous, } \|A^{1/2}x\|_{Z_T} \leq C\}$$

and $L = L(R, T)$ is the Lipschitz constant of the map $\phi_{T,\alpha}$. Since $V^T(\phi_{T,\alpha}\tilde{z}_{k,i}; RL\nu^k)$ and $\phi_{T,\alpha}(S_\alpha \bar{B}(\tilde{z}_{k,i}; R\nu^k))$ are compact in Z_T^1 , there exists a covering balls such that

$$V^T(\phi_{T,\alpha}\tilde{z}_{k,i}; RL\nu^k) \cap \phi_{T,\alpha}(S_\alpha \bar{B}(\tilde{z}_{k,i}; R\nu^k)) \subset \bigcup_{j=1}^{N_\rho} U^T(RL\rho\nu^k; \phi_{T,\alpha}w_{k+1,j}),$$

where $U^T(RL\rho\nu^k; \phi_{T,\alpha}w_{k+1,j}) := \{w \in S_\alpha \bar{B}(\tilde{z}_{k,i}; R\nu^k) \mid \|\phi_{T,\alpha}w - \phi_{T,\alpha}w_{k+1,j}\|_{Z_T} \leq RL\rho\nu^k\}$ and $w_{k+1,j} = S_\alpha^k z_{k+1,j}$. Then, from the inequality (3.4.16), we see that, for any $z \in S_\alpha \bar{B}(\tilde{z}_{k,i}; R\nu^k)$, there exists j ($1 \leq j \leq N_\rho$) such that

$$\|S_\alpha z - S_\alpha w_{k+1,j}\|_{\mathcal{H}_\alpha} \leq 2C_R q R\nu^k + 2C_R RL\rho\nu^k \leq R\nu^{k+1}$$

and this implies that $S_\alpha \bar{B}^{\mathcal{H}_\alpha}(\tilde{z}_{k,i}; R\nu^k)$ is covered with $\bigcup_{j=1}^{N_\rho} \bar{B}^{\mathcal{H}_\alpha}(\tilde{z}_{k+1,j}; R\nu)$. This holds for every $i = 1, 2, \dots, N_\rho^k$, hence the statement for $k+1$ is also true. $\square$

Thanks to the way of the construction of the discrete exponential attractor, one can see that

$$\mathcal{A}_{\text{exp},\alpha} := \overline{\bigcup_{0 \leq t \leq T} S_\alpha(t) \mathcal{A}_{\text{exp},\alpha}^*}^{\mathcal{H}_\alpha}$$

is a compact positively invariant set with respect to $S_\alpha(\cdot)$. Adding the Hölder continuity on time (we show it in the following), we have $\dim_f^{\mathcal{H}_\alpha} \mathcal{A}_{\text{exp},\alpha} \leq c(\dim_f^{\mathbb{R}}[0, T] + \dim_f^{\mathcal{H}_\alpha} \mathcal{A}_{\text{exp},\alpha}^*) < \infty$. We also have

$$h_{\mathcal{H}_\alpha}(S_\alpha(t)\mathbb{B}_\alpha, \mathcal{A}_{\text{exp},\alpha}) \leq Ce^{-\gamma t}, \quad t \geq 0$$

for some $\gamma = \gamma_{\mathbb{B}_\alpha} > 0$. Thus $\mathcal{A}_{\text{exp},\alpha}$ is an exponential attractor.

We would like to finish this section by checking the mapping $t \mapsto S_\alpha(t)z$ is Hölder continuous in $\mathcal{H}_\alpha$ for every $z \in \mathcal{A}_{\text{exp},\alpha}^*$. In order to do it, we estimate u_{tt} for given $z \in \mathbb{B}_\alpha$.

Lemma 3.4.5. *Let us assume that $z_0 \in \mathbb{B}_\alpha$ and the hypotheses Theorems 2.1.1 and 3.2.1 hold. Then, for weak solution (u, u_t) corresponding to the initial data $z_0 \in \mathbb{B}_\alpha$, we have*

$$\int_t^{t+T} \|A^{-1/4}u_{tt}(s)\|_2^2 ds + \alpha \int_t^{t+T} \|u_{tt}(s)\|_2^2 ds \leq C(R, T) \quad (\forall t \geq 0).$$

¹See Remark 3.1.10

Proof. Multiply (1.2.1) by $A^{-1/2}u_{tt}$, we get

$$\begin{aligned} & \|A^{-1/4}u_{tt}(t)\|_2^2 + \alpha\|u_{tt}(t)\|_2^2 \\ = & (h, A^{-1/2}u_{tt}(t)) - (A^{3/4}u(t), A^{-1/4}u_{tt}(t)) - (f(u(t)), A^{-1/2}u_{tt}(t)) \\ & - M(\|A^{1/4}u(t)\|_2^2)(A^{1/2}u(t), A^{-1/2}u_{tt}(t)) - N(\|A^{1/4}u(t)\|)(A^{1/2}u(t), A^{-1/2}u_{tt}(t)). \end{aligned}$$

In view of the Remark 3.3.9, we end up with the inequality

$$\frac{1}{2} \left(\|A^{-1/4}u_{tt}(t)\|_2^2 + \alpha\|u_{tt}(t)\|_2^2 \right) \leq C_R + N_0\|A^{1/2}u_t(t)\|_2^2.$$

Integrating over $(t, t+T)$, we obtain

$$\int_t^{t+T} \|A^{-1/4}u_{tt}(s)\|_2^2 ds + \alpha \int_t^{t+T} \|u_{tt}(s)\|_2^2 ds \leq C(R, T).$$

□

Now we show the Hölder continuity of the mapping $t \mapsto S_\alpha(t)z$. For given $z = (u_0, u_1) \in \mathcal{A}_{\text{exp}, \alpha}^* \subset \mathbb{B}_\alpha$, we can see from Lemma 3.4.5 that

$$(u_t, u_{tt}) \in L^2(0, T; D(A^{1/2})) \times L^2(0, T; H_{\alpha, -1}), \quad \forall T > 0.$$

Hence we have

$$\begin{aligned} \|S_\alpha(t_2)z_0 - S_\alpha(t_1)z_0\|_{\mathcal{H}_{\alpha, -1}} & \leq \int_{t_1}^{t_2} \left\| \frac{d}{ds} S_\alpha(s)z_0 \right\|_{\mathcal{H}_{\alpha, -1}} ds \\ & \leq \left(\int_0^T \|(u_t(s), u_{tt}(s))\|_{\mathcal{H}_{\alpha, -1}}^2 ds \right)^{1/2} |t_2 - t_1|^{1/2} \\ & \leq C(R, T)|t_2 - t_1|^{1/2}, \end{aligned}$$

for every $t_1, t_2 \in [0, T]$, so $t \mapsto S_\alpha(t)z_0$ is Hölder continuous in $\mathcal{H}_{\alpha, -1}$. Utilizing the boundedness of $\mathcal{A}_{\text{exp}, \alpha}^* \subset \mathbb{B}_\alpha$ in $\mathcal{H}_{\alpha, 1}$ and interpolation inequality, we find

$$\begin{aligned} \|S_\alpha(t_2)z_0 - S_\alpha(t_1)z_0\|_{\mathcal{H}_\alpha} & \leq C_s \|S_\alpha(t_2)z_0 - S_\alpha(t_1)z_0\|_{\mathcal{H}_{\alpha, 1}}^{1/2} \|S_\alpha(t_2)z_0 - S_\alpha(t_1)z_0\|_{\mathcal{H}_{\alpha, -1}}^{1/2} \\ & \leq C_{s, R} \|S_\alpha(t_2)z_0 - S_\alpha(t_1)z_0\|_{\mathcal{H}_{\alpha, -1}}^{1/2} \\ & \leq C_{s, R, T} |t_2 - t_1|^{1/4}, \quad \forall t_1, t_2 \in [0, T]. \end{aligned}$$

Thus $t \mapsto S_\alpha(t)z_0$ is Hölder continuous in $\mathcal{H}_\alpha$.

Chapter 4

Stability of attractors with respect to the rotational inertia

In this chapter we show the result on the stability of attractors with respect to the rotational inertia.

4.1 Main Theorem

Theorem 4.1.1. *Let the assumptions of Theorem 3.2.2 be in force. Then we have*

- i. The family of attractors $\mathcal{A}_\alpha$ for $(\mathcal{H}_\alpha, S_\alpha(t))$ is upper semi-continuous at the point 0, that is,*

$$h_{\mathcal{H}_0}(\mathcal{A}_\alpha, \mathcal{A}_0) \equiv \sup_{y \in \mathcal{A}_\alpha} \inf_{z \in \mathcal{A}_0} \|y - z\|_{\mathcal{H}_0} \rightarrow 0$$

as $\alpha \rightarrow 0+$.

- ii. There exist exponential attractors $\mathcal{A}_{\text{exp}, \alpha}$ for $(\mathcal{H}_\alpha, S_\alpha(t))$ such that the estimate*

$$H_{\mathcal{H}_0}(\mathcal{A}_{\text{exp}, \alpha}, \mathcal{A}_{\text{exp}, 0}) \equiv \max\{h_{\mathcal{H}_0}(\mathcal{A}_{\text{exp}, \alpha}, \mathcal{A}_{\text{exp}, 0}), h_{\mathcal{H}_0}(\mathcal{A}_{\text{exp}, 0}, \mathcal{A}_{\text{exp}, \alpha})\} \leq C\alpha^\kappa \quad (4.1.1)$$

holds with some exponent $0 < \kappa < 1$ and constant $C > 0$.

4.2 Upper semicontinuity of the global attractors $\mathcal{A}_\alpha$ when $\alpha \rightarrow 0$

In order to show the upper semicontinuity of attractors, we use the following criterion:

Proposition 4.2.1. (See [13]) *Assume that a dynamical system $(X, S_\alpha(t))$ possesses a compact global attractor A_α for every $\alpha \in [0, 1]$. Assume that the following conditions hold:*

- There exists a compact set $K \subset X$ such that $A_\alpha \subset K$ for all $\alpha \in [0, 1]$.*
- If $\alpha_k \rightarrow 0$, $x_k \in A_{\alpha_k}$ and $x_k \rightarrow x_0$, then $S_{\alpha_k}(t_0)x_k \rightarrow S(t_0)x_0$ for some $t_0 > 0$.*

Then the family of attractors A_α is upper semicontinuous at the point $\alpha = 0$, that is,

$$h_X(A_\alpha, A) \equiv \sup_{y \in A_\alpha} \inf_{z \in A} \|y - z\|_X \rightarrow 0$$

as $\alpha \rightarrow 0+$.

In the remainder of this section we check that the system $(\mathcal{H}_\alpha, S_\alpha(t))$ satisfies the conditions of this proposition.

Lemma 4.2.2. *Let $T > 0$ be an arbitrary number. Assume that the hypotheses of Theorems 2.1.1 and 3.2.1 hold. Let $z^\alpha(t)$, $\alpha \in (0, 1]$ be a solution to (1.2.1) ($\alpha > 0$) with initial data $z_0^\alpha = (u_0^\alpha, u_1^\alpha) \in \mathbb{B}_\alpha = \mathbb{B}_\alpha(R_0)$ and $z^0(t)$ solve (1.2.1) ($\alpha = 0$) with $z_0^0 = (u_0^0, u_1^0) \in \mathbb{B}_0 = \mathbb{B}_0(R_0)$. Then*

$$\|z^\alpha(t) - z^0(t)\|_{\mathcal{H}_\alpha} \leq C_{r_0, T} \|z_0^\alpha - z_0^0\|_{\mathcal{H}_{0, -1}}^{1/2} + \alpha^{1/4} C_{r_0, T}, \quad \forall t \in [0, T], \quad \forall \alpha \in (0, 1]. \quad (4.2.1)$$

In particular, if $z_0^\alpha \rightarrow z_0^0$ in $\mathcal{H}_0$ as $\alpha \rightarrow 0+$, then

$$\lim_{\alpha \rightarrow 0+} \sup_{t \in [0, T]} \|z^\alpha(t) - z^0(t)\|_{\mathcal{H}_\alpha} = 0 \quad \text{for any } T > 0. \quad (4.2.2)$$

Proof. Set $w(t) := u^\alpha(t) - u^0(t)$. Then we have

$$\begin{aligned} & \alpha A^{1/2} u_{tt}^\alpha + w_{tt} + Aw + N\left(\|A^{1/4}u\|_2^2\right) A^{1/2} w_t \\ &= -M\left(\|A^{1/4}u\|_2^2\right) A^{1/2} w - \left(N\left(\|A^{1/4}u^\alpha\|_2^2\right) - N\left(\|A^{1/4}u^0\|_2^2\right)\right) A^{1/2} u_t^0 \\ & \quad - \left(M\left(\|A^{1/4}u^\alpha\|_2^2\right) - M\left(\|A^{1/4}u^0\|_2^2\right)\right) A^{1/2} u^0 - (f(u^\alpha) - f(u^0)), \end{aligned}$$

Multiplying this by $A^{-1/2}w_t$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\{ \|A^{1/4}w(t)\|_2^2 + \|A^{-1/4}w_t(t)\|_2^2 \right\} + \alpha (A^{-1/4}u_{tt}^\alpha(t), A^{1/4}w_t(t)) \\ & \quad + N(\|A^{1/4}u(t)\|_2^2) \|w_t(t)\|_2^2 = \sum_{j=1}^4 \tilde{J}_j, \end{aligned}$$

where the expressions for $\tilde{J}_j$ ($j = 1, 2, 3, 4$) are almost the same ones given in (2.2.17)-(2.2.20), but replacing the function w_t by $A^{-1/2}w_t$. As in the proof of the uniqueness, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\{ \|A^{1/4}w(t)\|_2^2 + \|A^{-1/4}w_t(t)\|_2^2 \right\} + \alpha (A^{-1/4}u_{tt}^\alpha(t), A^{1/4}w_t(t)) \\ & \quad + N_0 \|w_t(t)\|_2^2 \leq C(1 + \|A^{1/4}u_t^0(t)\|_2^2) (\|A^{1/4}w(t)\|_2^2 + \|A^{-1/4}w_t(t)\|_2^2), \end{aligned} \quad (4.2.3)$$

where the C and N_0 are constants. Integrating (4.2.3) on $[0, t]$ and using Gronwall's inequality we arrive at

$$\begin{aligned} & \|A^{1/4}w(t)\|_2^2 + \|A^{-1/4}w_t(t)\|_2^2 + N_0 \int_0^t \|w_t(s)\|_2^2 ds \\ & \leq e^{Ct} \left(\|A^{1/4}w(0)\|_2^2 + \|A^{-1/4}w_t(0)\|_2^2 + \alpha \int_0^t \|A^{-1/4}u_{tt}^\alpha(s)\|_2^2 ds + \alpha \int_0^t \|A^{1/4}w_t(s)\|_2^2 ds \right). \end{aligned}$$

Since we take the each initial datum z_0^α from the bounded set $\mathbb{B}_\alpha \subset \mathcal{H}_{\alpha,1}$ and from Lemma 3.4.5, we see that $\int_0^t \|A^{-1/4}u_{tt}^\alpha(s)\|_2^2 ds, \int_0^t \|A^{1/4}w_t(s)\|_2^2 ds < \infty$. Hence, utilizing the boundedness of $\mathbb{B}_\alpha$ in $\mathcal{H}_{\alpha,1}$ and interpolation inequality, we find

$$\begin{aligned} \|z^\alpha(t) - z^0(t)\|_{\mathcal{H}_\alpha} &\leq \|z^\alpha(t) - z^0(t)\|_{\mathcal{H}_0} + \sqrt{\alpha}R_0 \\ &\leq \|z^\alpha(t) - z^0(t)\|_{\mathcal{H}_{0,1}}^{1-\frac{1}{2}} \|z^\alpha(t) - z^0(t)\|_{\mathcal{H}_{0,-1}}^{1/2} + \sqrt{\alpha}R_0 \\ &\leq C_{r_0,T} \|z_0^\alpha - z_0^0\|_{\mathcal{H}_{0,-1}}^{1/2} + \alpha^{1/4}C_{r_0,T}. \end{aligned}$$

□

From the continuity of the semiflows S_α on the parameter α , we can conclude the upper semicontinuity of $\mathcal{A}_\alpha$. Indeed, from Theorem 3.2.2 every attractor $\mathcal{A}_\alpha$ is included in a bounded set $K \subset D(A) \times D(A^{1/2})$, that is,

$$\|Au^\alpha\|_2^2 + \|A^{1/2}v^\alpha\|_2^2 \leq R_1,$$

where R_1 does not depend on α and t . Since $D(A) \times D(A^{1/2})$ is compactly embedded into $D(A^{1/2}) \times L^2(\Omega)$, the set K is compact in $D(A^{1/2}) \times L^2(\Omega)$. Thus we are able to obtain the conclusion of Proposition 4.2.1.

4.3 Continuity of exponential attractors when $\alpha \rightarrow 0$

At the end of this chapter, we verify the existence of the family of exponential attractors $\{\mathcal{A}_{\text{exp},\alpha}\}_{\alpha \in [0,1]}$ possessing the Hölder continuity with respect to the parameter α . In order to do it, we define the scaling operators $j_\alpha : \mathcal{H}_{0,1} \rightarrow \mathcal{H}_{\alpha,1}$ as

$$j_\alpha z_0 = j_\alpha(u_0, u_1) := (u_0, (1 + \sqrt{\alpha}A^{1/4})^{-1}u_1).$$

Then the scaling operators satisfy the following properties:

- $j_\alpha \mathbb{B}_0 \subset \mathbb{B}_\alpha$.
- $\|j_\alpha z - z\|_{\mathcal{H}_0} \leq \sqrt{\alpha}R$ (where $R > 0$ is the radius of the ball $\mathbb{B}_0$).

We can show the above properties by using the relation below

$$\|v\|_2^2 \leq \|v\|_2^2 + \alpha \|A^{1/4}v\|_2^2 \leq \|(1 + \sqrt{\alpha}A^{1/4})v\|_2^2 \quad (v \in D(A^{1/4})).$$

Moreover, the distance between $\mathbb{B}_0$ and $\mathbb{B}_\alpha$ is estimated as

$$H_{\mathcal{H}_0}(\mathbb{B}_0, \mathbb{B}_\alpha) \leq \sqrt{\alpha}R. \quad (4.3.1)$$

Indeed, it is sufficient to check that $h_{\mathcal{H}_0}(\mathbb{B}_0, \mathbb{B}_\alpha) \leq \sqrt{\alpha}R$ since $\mathbb{B}_\alpha$ is included in $\mathbb{B}_0$. Taking z from $\mathbb{B}_0$ arbitrary and fix it, then we see that

$$\text{dist}_{\mathcal{H}_0}(z, \mathbb{B}_\alpha) \leq \|z - j_\alpha z\|_{\mathcal{H}_0} \leq \sqrt{\alpha}R.$$

The element z is arbitrary, so the estimate surely holds.

From now on, we get into the proof of Main Theorem. The construction below is based on the arguments in [9], but we need to modify the process of the estimating of the difference between $\mathcal{A}_{\text{exp},\alpha}$ and $\mathcal{A}_{\text{exp},0}$. So, along with the reader's convenience, we show the process of the construction in the following.

Proof. We recall that the exponential attractor for the system $(\mathcal{H}_0, \mathbb{B}_0, S_0^m)$ is constructed as follows:

$$\mathcal{A}_{\text{exp},0}^* = \overline{\bigcup_{k,l=0}^{\infty} S_0^l E_0^k}^{\mathcal{H}_\alpha}, \quad (4.3.2)$$

where $E_0^k = \{S_0^k z_{k,i} \mid z_{k,i} \in \mathbb{B}_0, i = 1, 2, \dots, N_\rho^k\}$, $\rho > 0$ and N_ρ are constants satisfying the conditions appeared in the subsection 3.4.2. More precisely, the above set is constructed as a collection of the central points of finite covering balls for each $S_0^k \mathbb{B}_0$; that is, under the conditions (3.4.17), we are able to construct an $R\nu^k$ -radius covering of $S_0^k \mathbb{B}_0$ by a finite number of closed balls of $\mathcal{H}_{0,1}$ with center points in $S_0^k \mathbb{B}_0$ as follows

$$S_0^k \mathbb{B}_0 \subset \bigcup_{i=1}^{N_\rho^k} \overline{B}^{\mathcal{H}_0}(\tilde{z}_{k,i}; R\nu^k), \quad (4.3.3)$$

where $R > 0$ is the radius of the ball $\mathbb{B}_0$ and $\tilde{z}_{k,i}$ are the central points $\tilde{z}_{k,i} := S_0^k z_{k,i} \in S_0^k \mathbb{B}_0$, $1 \leq i \leq N_\rho^k$. Then the set defined in (4.3.2) gives an exponential attractor for $(\mathcal{H}_0, \mathbb{B}_0, S_0^m)$.

Now, we are going to construct an exponential attractor for $(\mathcal{H}_\alpha, \mathbb{B}_\alpha, S_\alpha^m)$, $0 < \alpha \leq 1$, on the basis of the set $\bigcup_{k=0}^\infty E_0^k$. From the property of the scaling operators and Proposition 5.1, $S_\alpha^k \mathbb{B}_\alpha$ ($k \in \mathbb{N}$) has a covering as follows:

$$S_\alpha^k \mathbb{B}_\alpha \subset \bigcup_{i=1}^{N_\rho^k} \overline{B}^{\mathcal{H}_0}(\tilde{z}_{k,i}^\alpha; R\nu^k + \tilde{L}^k \alpha^{1/4}), \quad (4.3.4)$$

where $\tilde{z}_{k,i}^\alpha = S_\alpha^k j_\alpha z_{k,i}$, $\tilde{L} = \tilde{L}(R, T)$ is a constant depending on R and $T > 0$. Indeed, for any $w = S_\alpha^k z \in S_\alpha^k \mathbb{B}_\alpha$, there exists $z_{k,i} \in \mathbb{B}_0$ ($\mathbb{B}_\alpha \subset \mathbb{B}_0$) such that the difference $S_\alpha^k z - S_\alpha^k j_\alpha z_{k,i}$ on the topology $\mathcal{H}_\alpha$ is estimated as below

$$\begin{aligned} & \|S_\alpha^k z - S_\alpha^k j_\alpha z_{k,i}\|_{\mathcal{H}_\alpha} \\ & \leq \|S_\alpha^k z - S_0^k z\|_{\mathcal{H}_\alpha} + \|S_0^k z - S_0^k z_{k,i}\|_{\mathcal{H}_\alpha} + \|S_0^k z_{k,i} - S_\alpha^k j_\alpha z_{k,i}\|_{\mathcal{H}_\alpha} \\ & \leq R\nu^k + \tilde{L}^k \alpha^{1/4}. \end{aligned}$$

So, right-hand-side of (4.3.4) is a covering balls of $S_\alpha^k \mathbb{B}_\alpha$ and let us construct an exponential attractor for the system $(\mathcal{H}_\alpha, \mathbb{B}_\alpha, S_\alpha^m)$ by modifying the above covering (4.3.4). Now, we define the integer m_α as the largest one satisfying $\tilde{L}^k \alpha^{1/4} \leq R\nu^k$. Then, it is clear that

$$S_\alpha^k \mathbb{B}_\alpha \subset \bigcup_{i=1}^{N_\rho^k} \overline{B}^{\mathcal{H}_\alpha}(\tilde{z}_{k,i}^\alpha; 2R\nu^k) \quad (0 \leq k \leq m_\alpha).$$

For $k > m_\alpha$, we redo the procedure of constructing appropriate covering balls for each $S_\alpha^k \mathbb{B}_\alpha$ by restarting the same argument handled in the subsection 4.3.2 from the balls $\overline{B}^{\mathcal{H}_\alpha}(\tilde{z}_{m_\alpha,i}^\alpha; 2R\nu^{m_\alpha})$ ($1 \leq i \leq N_\rho^{m_\alpha}$). Then we are able to construct inductively for every $m_\alpha < k < \infty$, a $2R\nu^k$ -radius covering of $S_\alpha^k \mathbb{B}_\alpha$ such that

$$S_\alpha^k \mathbb{B}_\alpha \subset \bigcup_{i=1}^{N_\rho^k} \overline{B}^{\mathcal{H}_\alpha}(\tilde{z}_{k,i}^\alpha; 2R\nu^k), \quad (\tilde{z}_{k,i}^\alpha = S_\alpha^k z'_{k,i} \in S_\alpha^k \mathbb{B}_\alpha).$$

We redefine E_α^k as

$$E_\alpha^k := \begin{cases} \{S_\alpha^k z_{k,i} \mid z_{k,i} \in \mathbb{B}_0, i = 1, 2, \dots, N_\rho^k\} & (0 \leq k \leq m_\alpha) \\ \{S_\alpha^k z'_{k,i} \mid z'_{k,i} \in \mathbb{B}_\alpha, i = 1, 2, \dots, N_\rho^k\} & (m_\alpha < k < \infty) \end{cases}$$

and we obtain a new exponential attractor $\mathcal{A}_{\text{exp},\alpha}^*$ as follows:

$$\mathcal{A}_{\text{exp},\alpha}^* := \overline{\bigcup_{k,l=0}^{\infty} S_\alpha^k E_\alpha^l}^{\mathcal{H}_\alpha}.$$

We can verify the above set $\mathcal{A}_{\text{exp},\alpha}^*$ satisfies the definition of exponential attractor by the same way in subsection 3.4.2, so we only check (4.1.1).

Now, we recall that, from the remark 3.3.9, there exists the time $t_{R_0} > 0$ such that the ball $\mathbb{B}_0$ is absorbed by itself and we put $l_0 := [t_{R_0}] + 1$, where $[t_{R_0}]$ is an integer part of the number t_{R_0} . Then, it is easy to see

$$\mathcal{A}_{\text{exp},\alpha}^* \subset \bigcup_{k+l \leq m_\alpha} S_\alpha^k E_\alpha^l \cup \left(\bigcup_{i=1}^{l_0} S_\alpha^{m_\alpha+i} \mathbb{B}_\alpha \right)$$

and the semi-distance $h_{\mathcal{H}_0}(\mathcal{A}_{\text{exp},\alpha}^*, \mathcal{A}_{\text{exp},0}^*)$ is estimated from above by

$$\begin{aligned} & h_{\mathcal{H}_0} \left\{ \left(\bigcup_{k+l \leq m_\alpha} S_\alpha^k E_\alpha^l \right) \cup \left(\bigcup_{i=1}^{l_0} S_\alpha^{m_\alpha+i} \mathbb{B}_\alpha \right), \mathcal{A}_{\text{exp},0}^* \right\} \\ &= \max_{k,l,i} \{ h_{\mathcal{H}_0}(S_\alpha^k E_\alpha^l, \mathcal{A}_{\text{exp},\alpha}^*), h_{\mathcal{H}_0}(S_\alpha^{m_\alpha+i} \mathbb{B}_\alpha, \mathcal{A}_{\text{exp},0}^*) \}. \end{aligned} \quad (4.3.5)$$

So, in order to prove the statement, it is sufficient to estimate the right-hand-side of (4.3.5), for each k, l, i . From the definition of the set E_α^l , it is easy to see that

$$\begin{aligned} h_{\mathcal{H}_0}(S_\alpha^k E_\alpha^l, \mathcal{A}_{\text{exp},0}^*) &\leq h_{\mathcal{H}_0}(S_\alpha^k E_\alpha^l, S_0^k E_0^l) \\ &\leq \tilde{L}^{k+l} \alpha^{1/4} \quad (k+l \leq m_\alpha). \end{aligned} \quad (4.3.6)$$

By using triangle inequality, (4.2.1) and (4.3.1), we have

$$\begin{aligned} h_{\mathcal{H}_0}(S_\alpha^l \mathbb{B}_\alpha, \mathcal{A}_{\text{exp},\alpha}^*) &\leq h_{\mathcal{H}_0}(S_\alpha^l \mathbb{B}_\alpha, S_0^l \mathbb{B}_0) + h_{\mathcal{H}_0}(S_0^l \mathbb{B}_0, \mathcal{A}_{\text{exp},0}^*) \\ &\leq R \tilde{L}^l \alpha^{1/4} + R \nu^l \\ &\leq 2R \tilde{L}^l \alpha^{1/4} \quad (m_\alpha + 1 \leq l \leq m_\alpha + l_0). \end{aligned} \quad (4.3.7)$$

Combining (4.3.6) with (4.3.7), we get

$$h_{\mathcal{H}_0} \left\{ \left(\bigcup_{k+l \leq m_\alpha} S_\alpha^k E_\alpha^l \right) \cup \left(\bigcup_{i=1}^{l_0} S_\alpha^{m_\alpha+i} \mathbb{B}_\alpha \right), \mathcal{A}_{\text{exp},0}^* \right\} \leq C_{R,T} \tilde{L}^{m_\alpha} \alpha^{1/4}.$$

Since $m_\alpha = \lfloor \frac{-\log \alpha^{1/4}}{\log L - \log \nu} \rfloor$, we obtain that $\tilde{L}^{m_\alpha} \alpha^{1/4} \leq C_{R,T} \alpha^\kappa$ with the exponent $\kappa = \frac{-\log \nu}{4(\log L - \log \nu)}$.

The converge estimate $h_{\mathcal{H}_0}(\mathcal{A}_{\text{exp},0}^*, \mathcal{A}_{\text{exp},\alpha}^*) \leq C_{R,T} \alpha^\kappa$ is also verified in a similar way. Thus, we conclude that (4.1.1) holds true. $\square$

Chapter 5

Appendix

In this appendix we give the proof of propositions of the infinite-dynamical systems we used in this thesis.

5.1 Proof of Proposition 3.3.1

This section deal with the existence theory of global attractors. Before getting into the proof, we give a lemma on the property of ω -limit set.

Lemma 5.1.1. *Let a dynamical system $(X, S(t))$ be asymptotically compact. Then for any bounded set D of X the ω -limit set $\omega(D)$ is a nonempty compact invariant set.*

Proof. Let us prove that the ω -limit set $\omega(D)$ is nonempty. Since the dynamical system $(X, S(t))$ is asymptotically compact, for any sequence $\{t_n\}$ tending to infinity the set $\{S(t_n)x_n \mid x_n \in D, n = 1, 2, \dots\}$ is relatively compact. Indeed, for an attracting compact set K , there exists a subsequence n_k such that

$$\text{dist}_X(S(t_{n_k})x_{n_k}, K) < \frac{1}{k}.$$

This means that there exists an element $x \in K$ such that $S(t_{n_k})x_{n_k} \rightarrow x$ ($k \rightarrow \infty$), thus $\omega(D)$ is nonempty.

Now we shall prove the invariance of ω -limit set. Let $x \in \omega(D)$, then there exist sequence $\{t_n\}$ tending to infinity and $\{x_n\} \subset D$ such that $S(t_n)x_n \rightarrow x$. The mapping $S(t)$ is continuous, so we have

$$S(t + t_n)x_n = S(t)S(t_n)x_n \rightarrow S(t)x, \quad (n \rightarrow \infty).$$

This implies that $S(t)x \in \omega(D)$, hence

$$S(t)\omega(D) \subset \omega(D) \quad (t > 0).$$

We shall prove the reverse inclusion. Let $x \in \omega(D)$ and $t > 0$. Then there exist sequences $\{x_n\} \subset D$ and $\{t_n\}$ ($t_n \rightarrow \infty$) such that $S(t_n)x_n \rightarrow x$. Let us consider the sequence $y_n = S(t_n - t)x_n$ ($t_n \geq t$). The asymptotic compactness implies that there exist an element $z \in K$ such that

$$S(t_n - t)x_n \rightarrow z \quad (n \rightarrow \infty).$$

Therefore $z \in \omega(D)$. Moreover,

$$S(t)z = \lim_{n \rightarrow \infty} S(t)S(t_n - t)x_n = \lim_{n \rightarrow \infty} S(t_n)x_n = x.$$

Thus $x \in S(t)\omega(D)$ and the invariance of the set is proved.

Lastly, we shall prove the compactness of the set $\omega(D)$. Let $\{y_n\}$ be a sequence in $\omega(D)$. Then we can find $t_n \geq n$ and $x_n \in D$ such that $\text{dist}_X(y_n, S(t_n)x_n) < \frac{1}{n}$. As said above, the property of asymptotic compactness enable us to find an element y and a sequence $\{n_k\}$ such that

$$S(t_{n_k})x_{n_k} \rightarrow y \quad (k \rightarrow \infty).$$

This implies that $y \in \omega(D)$ and $y_{n_k} \rightarrow y$. This means that $\omega(D)$ is a closed and compact set in X . $\square$

Now we show Proposition 3.3.1 based on the above Lemma.

Proof. Let B be a bounded absorbing set of the dynamical system $(X, S(t))$. Then the ω -limit set $\omega(B)$ is a global attractor. Indeed, assume the contrary, then the value $h_X(S(t)B, \omega(B))$ does not tend to zero as $t \rightarrow \infty$. This means that there exists $\delta > 0$ and a sequence $t_n : t_n \rightarrow \infty$ such that

$$h_X(S(t_n)B, \omega(B)) \geq \delta.$$

Therefore, there exists an element $x_n \in B$ such that

$$\text{dist}_X(S(t_n)x_n, \omega(B)) \geq \delta, \quad n = 1, 2, \dots \quad (5.1.1)$$

As before, a convergent subsequence $\{S(t_{n_k})x_{n_k}\}$ can be extracted from the sequence $\{S(t_n)x_n\}$ and the following

$$z \equiv \lim_{k \rightarrow \infty} S(t_{n_k})x_{n_k} \in \omega(B)$$

holds. This contradicts the estimate (5.1.1). Thus $\omega(B)$ uniformly attracts the absorbing set B and is a global attractor.

Now we shall prove the connectedness of the attractor by contradiction. Assume that the attractor A is not a connected set. Then there exists a pair of open sets U_1 and U_2 such that

$$U_i \cap A \neq \emptyset, \quad i = 1, 2, \quad A \subset U_1 \cup U_2, \quad U_1 \cap U_2 \neq \emptyset.$$

Let $c(A) = \text{conv}(A)$ be a convex hull of the set A ; that is,

$$c(A) := \left\{ \sum_{i=1}^N \lambda_i x_i \mid x_i \in A, \lambda_i \geq 0, \sum_{i=1}^N \lambda_i = 1, N = 1, 2, \dots \right\}.$$

It is clear that $c(A)$ is a bounded connected set and $A \subset c(A)$. The continuity of the mapping $S(t)$ implies that the set $S(t)c(A)$ is also connected. Also, we see immediately that $A = S(t)A \subset S(t)c(A)$. Therefore, $U_i \cap S(t)c(A) \neq \emptyset$, $i = 1, 2$. Hence, for any $t > 0$ the pair U_1, U_2 cannot cover $S(t)c(A)$. It follows that there exists a sequence of points $y_n = S(t_n)x_n \in S(t_n)c(A)$ such that $y_n \notin U_1 \cup U_2$. The asymptotic compactness of the dynamical system enable us to extract a subsequence $n_k : k = 1, 2, \dots$ such that $y_{n_k} = S(t_{n_k})x_{n_k}$ tends to an element $y \in X$ as $k \rightarrow \infty$. It is obvious that $y \notin U_1 \cup U_2$ and $y \in \omega(c(A))$. These relations contradict one another since $\omega(c(A)) \subset \omega(B) = A \subset U_1 \cup U_2$. Therefore, the proof of Proposition 3.3.1 is complete. $\square$

5.2 Proof of Proposition 3.3.2

In this section we deal with the structure theory on global attractors and minimal attractors. Before getting into the main topic, we refer to the basic fact on the properties of global attractors.

Proposition 5.2.1. *Let $(X, S(t))$ be a dynamical system and let A be a global attractor for the dynamical system $(X, S(t))$. Then the following properties are satisfied:*

- i. (Uniqueness) A is uniquely determined.*
- ii. (Maximality) A is maximal among bounded closed invariant sets.*
- iii. An element z_0 belongs to the global attractor A if and only if there exists a bounded full trajectory $\gamma = \{z(t) \mid t \in \mathbb{R}\}$ such that $z(0) = z_0$.*

Proof. The proofs of Item i and ii are easy, so we only prove Item iii. In general, the following fact holds:

A bounded set $K \subset X$ is invariant $\Leftrightarrow$ for each $w \in K$ there exists a full trajectory $\gamma = \{w(t) \mid t \in \mathbb{R}\}$ such that $w(0) = w$ and $\gamma \subset K$.

Actually, first we shall assume that K is invariant. Since $S(1)K = K$, we see that for each $w_0 \in K$ there is a $w_1 \in K$ with $S(t)w_1 \in K$, for $0 \leq t \leq 1$, and $S(1)w_1 = w_0$. By iterating this and using the induction, one finds a sequence $w_i \in K$ with $S(t)w_i \in K$, for $0 \leq t \leq 1$, and $S(t)w_i = w_{i-1}$, for $i \in \mathbb{N}$. We now define the function $w(\cdot) : \mathbb{R} \rightarrow K$ by

$$w(t) := \begin{cases} S(t)w_0 & (t \geq 0) \\ S(t+i)w_i & (-i \leq t < -(i-1)) \quad (i = 1, 2, \dots). \end{cases} \quad (5.2.1)$$

This trajectory is full and included in K . Thus “ $\Rightarrow$ ” is certainly true.

Next we assume that for every $w \in K$ there is a full trajectory γ_w passing through w with $\gamma_w \subset K$. Then from the hypothesis it is obvious that for any w , one has $S(t)\gamma_w \subset K$, for each $t \geq 0$. Thus $S(t)K \subset K$. On the other hand, for any $w \in K$, one has $w = S(t)S(-t)w \in S(t)K$. Therefore K is invariant.

Combining this fact with the maximality of the global attractor, we see Proposition 5.2.1 is true. $\square$

Now, we prove the structure theory of global attractors and minimal attractors based on the above properties. First of all, we show the following inclusion relation which holds in general.

Lemma 5.2.2. *Let $(X, S(t))$ be a dynamical system and let $\mathcal{N}$ be the set of the stationary points of the dynamical system $(X, S(t))$ possessing the global attractor A . Then $\mathcal{M}^u(\mathcal{N}) \subset A$*

Proof. It is obvious that the set $\mathcal{N} = \{z \mid S(t)z = z, t > 0\}$ lies in the attractor A and thus it is bounded. Let $z_0 \in \mathcal{M}^u(\mathcal{N})$. Then there exists a trajectory $\gamma_{z_0} = \{z(t) \mid t \in \mathbb{R}\}$ such that $z(0) = z_0$ and

$$\text{dist}_X(z(t), \mathcal{N}) \rightarrow 0, \quad t \rightarrow -\infty.$$

Therefore if we choose $s > 0$ large enough, then $D_s := \{z(\tau) \mid \tau \leq -s\}$ is bounded. Hence the set $S(t)D_s$ tends to the attractor of the system as $t \rightarrow \infty$. However, $z_0 \in S(t)D_s$ for $t \geq s$. Therefore,

$$\text{dist}(z_0, A) \leq h_X(S(t)D_s, A) \rightarrow 0, \quad t \rightarrow \infty.$$

This implies that $z_0 \in A$. □

We next show the reverse inclusion in the following.

Proof. Let $z \in A$. Let us consider a full trajectory γ passing through z . Since $\gamma \subset A$, the α -limit set;

$$\alpha(\gamma) := \bigcap_{t \leq 0} \overline{\bigcup_{\tau \leq t} S(\tau)z}$$

of trajectory γ is nonempty and compact. In order to prove this theorem, we show the following:

$$\alpha(\gamma) \subset \mathcal{N}. \tag{5.2.2}$$

If (5.2.2) holds, we get the desire conclusion by using the contradiction argument as in the proof of Proposition 3.3.1. From now on let us show that the Lyapunov function Φ is constant on $\alpha(\gamma)$. Indeed, if $\tilde{z} \in \alpha(\gamma)$, then there exists a sequence $\{t_j\}_{j \in \mathbb{N}}$ tending to $-\infty$ as $j \rightarrow \infty$ such that

$$\lim_{j \rightarrow \infty} z(t_j) = \tilde{z}.$$

Consequently

$$\Phi(\tilde{z}) = \lim_{j \rightarrow \infty} \Phi(z(t_j)).$$

Since γ is compact, Φ is bounded from above on γ and by the monotonicity of Φ along the trajectory, we have

$$\Phi(\tilde{z}) = \lim_{j \rightarrow \infty} \Phi(z(t_j)) = \sup_{\tau < 0} \Phi(z(\tau)).$$

Thus, for all $t > 0$, we have

$$\Phi(S(t)\tilde{z}) = \lim_{j \rightarrow \infty} \Phi(S(t)z(t_j)) = \sup_{\tau < 0} \Phi(z(\tau)) = \Phi(\tilde{z}).$$

Therefore Φ is constant on $\alpha(\gamma)$. □

Finally, we give the proof of $A_{\min} = \mathcal{N}$.

Proof. It is obvious that $\mathcal{N}$ is invariant. We verify that $\mathcal{N}$ attracts every element of X . Let $z \in X$ and $\gamma = \gamma(z) = \{z(t) \mid t \in \mathbb{R}_{\geq 0}, z(0) = z\}$. Since the dynamical system possesses the compact global attractor A , there exists a sequence $\{t_i\}_{i \in \mathbb{N}}$, $t_i \rightarrow \infty$ as $i \rightarrow \infty$ such that

$$\text{dist}(S(t_i)z, A) \leq \frac{1}{i}, \quad 0 < t_i \leq t_{i+1}.$$

From the compactness of A , there exists a subsequence still denoted by $\{t_i\}$ such that $\lim_{i \rightarrow \infty} S(t_i)z$ exists. Thus γ is compact and $\omega(\gamma)$ is nonempty, and compact. In a similar way to the argument of the proof of $A \subset \mathcal{M}^u(\mathcal{N})$, we obtain that

$$\omega(\gamma) \subset \mathcal{N}$$

and it follows

$$\lim_{i \rightarrow \infty} \text{dist}_X(S(t_i)z, \mathcal{N}) = 0.$$

Finally, it is easy to show that $A_{\min}$ includes $\mathcal{N}$. Indeed, let us take z from $\mathcal{N}$, then

$$\lim_{t \rightarrow \infty} \text{dist}(S(t)z, A_{\min}) = 0.$$

On the other hand $S_\alpha(t)z = z$, $\forall t > 0$. Thus $\text{dist}_X(z, A_{\min}) = 0$ and this implies that $\mathcal{N} \subset A_{\min}$. Combining this inclusion relation with the minimality of $A_{\min}$, we conclude that $\mathcal{N} = A_{\min}$. $\square$

5.3 Proof of Proposition 3.3.5

Proof. (i $\Rightarrow$ ii) Since the distance to the compact set is reached on some element, for any $x \in X$ and $t > 0$ there exists an element $y := S^2(t)x \in K$ such that

$$\text{dist}_X(S(t)x, K) = \text{dist}_X(S(t)x, S^2(t)x).$$

Hence, we take $S^1(t)x := S(t)x - S^2(t)x$, it is easy to see that each part of the decomposition satisfies all the requirements.

(ii $\Rightarrow$ i) Let us define the set K as follows:

$$K := \bigcup_{t \geq t_B} S^2(t)B,$$

where $B \subset X$ is a bounded absorbing set for the system $(X, S(t))$ and $t_B > 0$ is the time appeared in the hypothesis. Then, it is easy to see that K is an attracting compact set for the dynamical system. Indeed, for any $x \in B$ and $t \geq t_B$, we have

$$\text{dist}_X(S(t)x, K) \leq \text{dist}_X(S(t)x, S^2(t)x) = \|S^1(t)x\|_X \rightarrow 0 \quad (t \rightarrow \infty).$$

The set $B \subset X$ is bounded and absorbing, hence we obtain the conclusion. $\square$

5.4 Proof of Proposition 3.4.1

In this section we give the proof of the Proposition 3.4.1. Before giving the proof, we introduce some important notion we need.

Definition 5.4.1. Let X be a Banach space with the norm $\|\cdot\|_X$ and B be a bounded set in X . A subset U in B is said to be (X, ϵ) -**distinguishable** if $\|x - x'\|_X > \epsilon$ for any $x, x' \in U$, $x \neq x'$. We denote $m_X(B, \epsilon)$ by the maximal number of elements of an (X, ϵ) -distinguishable subset of B .

We derive the finite dimensionality from the following general result.

Lemma 5.4.2. Let X be a Banach space and M be a compact set in X . Assume that there exists a map $V : M \rightarrow X$ such that

$$i. \quad M \subseteq VM$$

ii. There exist a Banach space Z and a Lipschitz mapping $K : M \rightarrow Z$ under which the image of M is compact in Z such that

$$\|Vx_1 - Vx_2\|_X \leq \eta\|x_1 - x_2\|_X + \|Kx_1 - Kx_2\|_Z \quad (x_1, x_2 \in M),$$

where $\eta \in (0, 1)$ is a constant.

Then M is a finite fractal dimensional set and the dimension is estimated as follow:

$$\dim_f M \leq \left[\log \frac{2}{1 + \eta} \right]^{-1} \cdot \log m_Z(B_{4L_K/(1+\eta)}(0), 1), \quad (5.4.1)$$

where $L_K > 0$ is the Lipschitz constant for K :

$$\|Kx_1 - Kx_2\|_Z \leq L_K\|x_1 - x_2\|_X \quad (x_1, x_2 \in M). \quad (5.4.2)$$

Proof. Assume that $\{F_i \mid i = 1, 2, \dots, N(M, \epsilon)\}$ is a minimal covering of M by its closed subsets with a diameter equal or less than 2ϵ . Let $0 < \delta < 1$ and $U_i \subset KF_i$ be a maximal $(Z, \delta\epsilon)$ -distinguishable subset. Then there is a collection of balls $B_1^i, \dots, B_{n_i}^i$ ($n_i = m_Z(KF_i, \delta\epsilon)$) such that

$$F_i \subset \bigcup_{j=1}^{n_i} B_j^i, \quad B_j^i \equiv \{x \in F_i \mid \|Kx - Kx_j^i\|_Z \leq \delta\epsilon, \quad Kx_j^i \in U_i\}.$$

Therefore

$$VM \subset \bigcup_{i=1}^{N(M, \epsilon)} \bigcup_{j=1}^{n_i} VB_j^i.$$

For $x, x' \in B_j^i$, we have

$$\|Vx - Vx'\|_X \leq \eta\|x - x'\|_X + \|Kx - Kx_j^i\|_Z + \|Kx_j^i - Kx'\|_Z$$

¹ $N(M, \epsilon)$ is the minimal number of closed sets of the diameter 2ϵ that cover M .

$$\leq 2(\eta + \delta)\epsilon.$$

Thus $\text{diam}_X VB_j^i \leq 2(\eta + \delta)\epsilon$ for any $\epsilon > 0$ and $0 < \delta < 1$. Therefore

$$N(VM, (\eta + \delta)\epsilon) \leq \sum_{i=1}^{N(M, \epsilon)} n_i. \quad (5.4.3)$$

Now, we shall show that $n_i = m_Z(KF_i, \delta\epsilon)$ is estimated as follows:

$$m_Z(KF_i, \delta\epsilon) \leq m_Z(B_{2L_K/\delta}(0), 1). \quad (5.4.4)$$

Indeed, by the Lipschitz condition (5.4.2), we see that

$$\text{diam}_Z KF_i = \sup_{x, x' \in F_i} \|Kx - Kx'\|_Z \leq L_K \|x - x'\|_X \leq 2\epsilon L_K$$

and there exists $z_0 = Kx_0 \in KF_i$ such that

$$KF_i \subset B_{2\epsilon L_K}(z_0) = \{z \in Z \mid \|z - z_0\|_Z \leq 2\epsilon L_K\}.$$

Hence we obtain that

$$\begin{aligned} m_Z(KF_i, \delta\epsilon) &\leq m_Z(B_{2\epsilon L_K}(z_0), \delta\epsilon) \\ &= \text{Card}\{z_j \in B_{2\epsilon L_K}(z_0) \mid \|z_j - z_k\|_Z > \delta\epsilon, j \neq k\} \\ &= \text{Card}\{z_j \in B_{2\epsilon L_K}(0) \mid \|z_j - z_k\|_Z > \delta\epsilon, j \neq k\} \\ &= \text{Card}\{z_j \in B_{2L_K/\delta}(0) \mid \|z_j - z_k\|_Z > 1, j \neq k\} \\ &= m_Z(B_{2L_K/\delta}(0), 1). \end{aligned}$$

Therefore the estimate (5.4.4) truly holds.

From the estimate (5.4.4) and the assumption $M \subseteq VM$, if we choose $0 < \delta < 1 - \eta$ in (5.4.3), we obtain that

$$\log N(M, q\epsilon) \leq \log m_Z(B_{2L_K/\delta}(0), 1) + \log N(M, \epsilon)$$

for any $\epsilon > 0$, where $q = \eta + \delta < 1$. Let $\epsilon_n := q^n \epsilon_0$ for some $\epsilon_0 > 0$. It is obvious that

$$\log N(M, \epsilon_n) \leq n \cdot m_Z(B_{2L_K/\delta}(0), 1) + \log N(M, \epsilon_0), \quad n = 1, 2, \dots \quad (5.4.5)$$

Now for any $\epsilon < \epsilon_0$ we can find $n = n_\epsilon$ and $\epsilon' \in [\epsilon_1, \epsilon_0)$ such that

$$\epsilon_{n+1} \leq \epsilon < \epsilon_n, \quad \epsilon = q^n \epsilon'.$$

Hence we have that

$$\begin{aligned} \log N(M, \epsilon) &\leq n_\epsilon \cdot m_Z(B_{2L_K/\delta}(0), 1) + \log N(M, \epsilon') \\ &\leq n_\epsilon \cdot m_Z(B_{2L_K/\delta}(0), 1) + \log N(M, \epsilon_1). \end{aligned}$$

By the formula of the fractal dimension², we obtain that

$$\dim_f M \leq m_Z(B_{2L_K/\delta}(0), 1) \cdot \limsup_{\epsilon \rightarrow 0} \frac{n_\epsilon}{\log(1/\epsilon)}.$$

²In this argument we use the other formulation of the fractal dimension; that is,

$$\dim_f A \stackrel{\text{def}}{=} \limsup_{\epsilon \rightarrow 0} \frac{\log n(A, \epsilon)}{\log \frac{1}{\epsilon}} \stackrel{\text{iff}}{=} \limsup_{\epsilon \rightarrow 0} \frac{\log N(A, \epsilon)}{\log \frac{1}{\epsilon}}$$

It follows from (5.4) that

$$n_\epsilon = \frac{\log(\epsilon'/\epsilon)}{\log(1/q)} \leq \frac{\log(\epsilon_0/\epsilon)}{\log(1/q)}.$$

Therefore

$$\dim_f M \leq m_Z(B_{2L_K/\delta}(0), 1) \cdot \frac{1}{\log(1/q)}.$$

Taking $\delta = (1 - \eta)/2$, we obtain the estimate (5.4.2). The proof of Lemma 5.4.2 is complete. $\square$

Thanks to the above general result, we are able to show the finite dimensionality of the global attractor $\mathcal{A}_\alpha$. In fact, the semi-group $S_\alpha(t) : \mathcal{H}_\alpha \rightarrow \mathcal{H}_\alpha$ satisfies the condition in Lemma 5.4.2 for some $t > 0$ large enough. Thus we can immediately apply the result with $V = S_\alpha(t)$ and obtain the conclusion.

5.5 Proof of Proposition 3.4.2

In this section we show the regularity property on the global attractor.

Proof. Let us take the trajectories γ^1 and γ^2 from the global attractor:

$$\gamma^1 = \{x^1(t) \mid x^1(0) = x_0^1 \in A, t \in \mathbb{R}\}, \quad \gamma^2 = \{x^2(t) \mid x^2(0) = x_0^2 \in A, t \in \mathbb{R}\}$$

From the (3.1.3), we have

$$\|S(t)x_0^1 - S(t)x_0^2\|_X \leq b(t)\|x_0^1 - x_0^2\|_X + c(t) \sup_{0 \leq \tau \leq t} \|y^1(\tau) - y^2(\tau)\|_{X_2}.$$

Now, we replace x_0^1 (resp. x_0^2) with $S(s)x_0^1$ (resp. $S(t)x_0^2$) and $S(t)x_0^1$ with $S(t-s)x_0^1$ (resp. $S(t-s)x_0^2$) ($s \leq t$). Then we have

$$\begin{aligned} & \|S(t)x_0^1 - S(t)x_0^2\|_X \\ & \leq b(t-s)\|S(s)x_0^1 - S(s)x_0^2\|_X + c(t-s) \sup_{s \leq \tau \leq t} \|y^1(\tau) - y^2(\tau)\|_{X_2} \end{aligned} \quad (5.5.1)$$

for all $s \leq t$, $s, t \in \mathbb{R}$. By letting $s \rightarrow -\infty$ in (5.5.1), we have

$$\|S(t)x_0^1 - S(t)x_0^2\|_X \leq c_\infty \sup_{-\infty \leq \tau \leq t} \|y^1(\tau) - y^2(\tau)\|_{X_2} \quad (5.5.2)$$

for every $t \in \mathbb{R}$ and for every couple of full trajectories γ^1 and γ^2 taken from the attractor. Now we fix a trajectory γ and we consider the shifted trajectory $\gamma_h := \{x(t+h) \mid t \in \mathbb{R}\}$ for $0 < |h| < 1$. Applying (5.5.2) for this pair of trajectories γ and γ_h , we get

$$\|x^h(t)\|_X = \|y^h(t)\|_{X_1} + \|y_t^h(t)\|_{X_2} \leq c_\infty \sup_{-\infty \leq \tau \leq t} \|y^h(\tau)\|_{X_2} \quad (5.5.3)$$

where $x^h(t) = (y^h(t), y_t^h(t))$, $y^h(t) = \{y(t+h) - y(t)\} \cdot h^{-1}$. Using the mean value theorem, we have

$$\sup_{-\infty \leq \tau \leq t} \|y^h(\tau)\|_{X_2} \leq \sup_{-\infty \leq \tau \leq t} \|y_t(\tau)\|_{X_2} \leq C_A.$$

Thus passing with the limit on $h \rightarrow 0$ in (5.5.3), we obtain

$$\|y_t(t)\|_{X_1} + \|y_{tt}(t)\|_{X_2} \leq C_{A, c_\infty}.$$

Therefore, we get the conclusion on the regularity of trajectories from the attractor. $\square$

5.6 Proof of Proposition 4.2.1

This section gives the general result of the upper semi continuity of global attractors.

Proof. First, we shall show the following:

$$\text{If } \alpha_k \rightarrow 0, z_k \in A_{\alpha_k} \text{ and } z_k \rightarrow z_0, \text{ then } z_0 \in A. \quad (5.6.1)$$

Indeed, let $\gamma^k = \{z_k(t) \mid t \in \mathbb{R}\}$ be a trajectory of the dynamical system $(X_{\alpha_k}, S_{\alpha_k}(t))$ passing through the element z_k ($z_k(0) = z_0$). Using the standard diagonal process, it is easy to find that there exists a subsequence $\{k_i\}_{i \in \mathbb{N}}$ and a sequence of elements $\{w_j\} \subset K$ such that

$$\lim_{i \rightarrow \infty} z_{k(i)}(-jt_0) = w_j \text{ in } X \quad (\text{for all } j = 0, 1, 2, \dots)$$

where $w_0 = z_0$, and $t_0 > 0$ is the number appeared in the assumption of Proposition 4.2.1. Under the assumptions, we see that

$$S_{\alpha_k}(t_0)z_k \rightarrow S(t_0)z_0 \text{ in } X$$

as $k \rightarrow \infty$. We shall prove that this property gives us

$$w_{j-l} = S(lt_0)w_j \text{ in } X \quad (5.6.2)$$

for all $j = 1, 2, \dots$ and $l = 1, 2, \dots, j$. First let us check it for the case $l = 1$;

$$w_{j-1} = \lim_{i \rightarrow \infty} z_{k(i)}(-(j-1)t_0) = \lim_{i \rightarrow \infty} S_{\alpha_{k_i}}(t_0)(z_{k(i)}(-jt_0)) = S(t_0)w_{j-1}.$$

Inductively, we can prove

$$w_{j-l} = S(lt_0)w_j.$$

Thus the relation (5.6.2) holds true.

Let us define the function $z(\cdot) : \mathbb{R} \rightarrow X$ by

$$z(t) := \begin{cases} S(t)w_0 & (t \geq 0) \\ S(t + jt_0)w_j & (-jt_0 \leq t < -(j-1)t_0) \quad (j = 1, 2, \dots). \end{cases} \quad (5.6.3)$$

Then, $z(\cdot)$ yields a full trajectory γ passing through the point z_0 . It is clear that the trajectory γ is bounded. From the properties of global attractor, it wholly belongs to A . Hence the statement (5.6.1) is proved.

The property (5.6.1) leads the upper semi-continuity of the global attractors. Actually, if the statement does not hold, then there exists a sequence $\alpha_k \rightarrow 0$ and a sequence $x_k \in A_{\alpha_k}$ such that $\text{dixt}_X(x_k, A_0) \geq \delta$ for some $\delta > 0$. However, the sequence x_k lies in the compact set K , there exists subsequence x_{k_j} such that $x_{k_j} \rightarrow x_0$ as $j \rightarrow \infty$. Owing to the property (5.6.1), $x_0 \in A_0$ but this contradicts the hypothesis. Proposition 4.2.1 is proved. $\square$

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