Doctoral Thesis

Study of Local Bulk Operator inside the Horizon of BTZ Black Hole

(BTZ ブラックホールのホライズン内部における局所的なバルク演算子に関する研究)

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Abstract

We study a quantum scalar field inside the horizon of the non-rotating BTZ black hole whose Penrose diagram consists of two exterior and two interior regions. As a preparation for quantization of a scalar field inside the horizon, we quantize a scalar field outside the horizon on a combined constant time slice in both two exterior regions by the canonical method. The scalar fields in each exterior region are expanded in terms of CFT operators on their own boundaries. We quantize a scalar field inside the horizon by the canonical method and impose the matching condition at the horizon on the normal modes of the scalar field. The explicit forms of the normal modes both near and deep inside the horizon are obtained. However, it turned out that the normal modes deep inside the horizon have undetermined functions. Thus the matching condition is not enough to determine the scalar field inside the horizon uniquely. In order to remove this ambiguity, we propose additional conditions at the region deep inside the horizon. By using the scalar field without the ambiguity, an integral form of a scalar propagator for points on both sides of the horizon is obtained. This propagator represented by integrals does not coincide with the propagator in the previous works.

Contents

1 Introduction

In the universe, it is known that there are four fundamental forces: the electromagnetic interaction, the weak interaction, the strong interaction and the gravitational interaction. The standard model (SM) of particle physics is a theory describing three of them except for the gravitational interaction. In 2012, the Higgs particle is observed in high energy experiment. The Higgs particle was last undiscovered particle which is predicted in the SM and it was confirmed that the SM is a theory describing the nature through this discovery. However, the SM does not describe the gravitational interaction and is not theory of everything.

In quantum field theory which construct the SM, the divergences at short distances are removed by mathematical techniques called renormalization. When the dimension of the coupling constant is not negative, the divergence is renormalizable. The gravitational theory is not renormalizable because the gravitational coupling constant has negative dimension. In order to construct a theory of quantum gravity, the non-renormalizability is problematic. The superstring theory is one of the strong candidates for quantum gravity that prevent this problem. In the superstring theory, point-like particles are replaced by one-dimensional objects called string. The length of the string l_s acts as a cutoff scale and the problematic divergence does not appear.

The superstring theory predicts a non-trivial equivalence between different theories. Maldacena discovered this prediction called the AdS/CFT correspondence [1]. The proposal is that Type IIB superstring theory on the product space of five-dimensional AdS spacetime AdS_5 and five-dimensional sphere S_5, where S_5 is compact manifold, is equivalent to $\mathcal{N} = 4$ super Yang-Mills theory on the four-dimensional boundary. The equivalence in this proposal is an example of the AdS/CFT, which is the duality between a gravitational theory in anti de-Sitter spacetime and a conformal field theory on the boundary. The correspondence is based on the holographic principle [2][3]. According to the holographic principle, information of gravitational theories in the bulk is encoded in that of non-gravitational theories on the boundary. The idea of this principle is inspired by the black hole thermodynamics, in particular the statement that the black hole entropy is given by the Bekenstein-Hawking formula [4]:

$$S_{\mathrm{BH}} = \frac{A}{4l_{\mathrm{pl}}^2} \tag{1.1}$$

where A is the area of the horizon and l_{pl} is the Planck length. The black hole entropy is proportional to the surface area, not the volume, which means that information inside the black hole is encoded in that on the surface area. In the AdS/CFT, a conformal field theory (CFT) lives on the boundary, or the surface, of an anti de-Sitter space (AdS). It is expected that the CFT on the boundary has completeness information in order to describe the quantum gravity in the bulk.

Since the Maldacena's discovery, scientists showed many evidences of the AdS/CFT. Some authors reported the study of the bulk structures from the perspective of the correspondence to understand quantum gravity [5][6][7][8][9]. One of the questions is how to define local operators or states in a theory of quantum gravity. Where a black hole is involved in the spacetime, the situation becomes to be more puzzling. There is a well known method to reconstruct bulk local operators which is called Hamilton-Kabat-Lifschytz-Lowe (HKLL) construction [5][10][11]. In the method, a bulk local operator is reconstructed as non-local CFT operators on the boundary. In [6][7], a bulk local state which is excited by a bulk local field is reconstructed by CFT primary states and its descendants on the boundary. This method is equivalent to the HKLL prescription. Goto and Takayanagi extended the method to construct bulk local states into a black hole interior [8]. Papadodimas and Raju proposed that their construction of bulk local operators is extended beyond a black hole horizon [9][12][13]. They proposed the matching condition requiring a scalar field behind the horizon to be smoothly connected with that outside the horizon. It is shown that the prescription works in the case of Minkowski-Rindler space. Understanding of the black hole interior is necessary to resolve the information paradox [14][15] and the firewall problem [8][9][12][13][16][17][18]. Papadodimas and Raju concluded that firewall at the black hole horizon is absent.

The information paradox and local bulk physics in the AdS/CFT correspondence was discussed by Mathur [15], Almheiri *et.al.* [16][17] and Polchinski [18]. These authors argued that the CFT does not contain operators with the right properties to play the role of local field operators behind the black hole horizon. This means that the CFT does not have a smooth horizon. It is important to check whether local bulk operators inside the horizon, which are quantized properly and are imposed suitable matching conditions at the horizon, are reconstructed as the CFT operators. There are not, however, so many examples where concrete calculations are possible. The case of Minkowski-Rindler space is shown in [9]. It is desirable to study the case of black hole spacetime.

In this paper, we will study the reconstruction of local bulk field in the BTZ black hole spacetime [19] according to the holographic principle and show whether the CFT on the boundary contains a description of the interior of BTZ black hole. In general curvature spacetime, the explicit form of normal mode of a scalar field is not available, and explicit construction of the scalar field is difficult. Fortunately, in the case of BTZ black hole spacetime the explicit form of the normal mode is available and it is possible to study the quantization of the scalar field inside the black hole and the smoothness between the scalar fields inside and outside the horizon. It is known that the metric tensor of BTZ space can be obtained from that of pure AdS_3 space by coordinate transformations [20]. The explicit form of normal mode in the BTZ space can be obtained by these transformations. And also by using the transformations and the shift of the complexified time direction, it will be possible to obtain two-point functions between points inside and outside the horizon from two-point functions for points outside the horizon [21]. The results can be expressed in terms of geodesic distance between these two points. However, it is not clear whether the correlation functions are precise observations until the quantization of the scalar fields is carried out explicitly. The roles of the time variable t and the radial variable r outside the horizon are interchanged inside the horizon, and it is not guaranteed that the correlation functions which are calculated by using the quantized scalar fields inside the horizon coincide with the correlation functions in [21]. Hence in order to check whether there are required procedures for obtaining the correct correlation functions in addition to the shift of t, it is necessary to carry out the quantization of scalar fields inside the horizon.

Our approach for quantization of scalar fields in the BTZ black hole spacetime is new one. The eternal AdS-Schwartzschild black hole is described by a tensor product state called the Thermo-Field Double (TFD) [22].

$$|\psi\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n/2} |n\rangle_L \otimes |n\rangle_R \tag{1.2}$$

This thermal state is a Hartle-Hawking-like state [23].

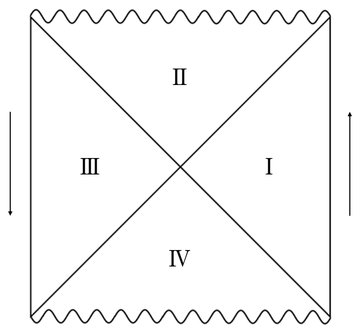

Fig. 1 Penrose diagram of an eternal black hole

The spacetime is represented by extended Penrose diagram and consist of two exterior and two interior regions (Fig. 1). The exterior regions are called Right and Left, or I and III. The interior regions are called Future and Past, or II and IV. In the region I, the time flows upwards, while in the region III the time flows downwards. The eternal AdS-Schwartzschild black hole has two boundaries in the Right and Left region. This means that the bulk in the eternal AdS-Schwartzschild black hole is reconstructed by two copies of the boundary CFT in the holographic principle. Therefore it is expected that the full Hilbert space $\mathcal{H}$ of the eternal black hole consist of two copies of the Hilbert space of the CFT $\mathcal{H}_L$ and $\mathcal{H}_R$ such as $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$. The TFD $|\psi\rangle \in \mathcal{H}$ (1.2) is a tensor product of the two boundary CFT states $|n\rangle_{L,R}$ which are defined by Euclidean path-integrals without any operator on the boundary. The TFD is defined as 'vacuum' state in the bulk. For quantization of a scalar field theory in curved spacetime, a scalar field is decomposed into a complete orthonormal set of positive-frequency modes $f_{\omega lm}$ on a Cauchy surface $\sum$ and the coefficients of the expansion are the annihilation operators $a_{\omega lm}$ [24]. The quantization of the theory is implemented by adopting the commutation relations between $a_{\omega lm}$ and the creation operator. Usually, the quantization in regions outside the horizon is implemented in the region I and Ⅲ separately. In this conventional case a scalar field is expanded in a single set of creation and annihilation operators in the region I, and another independent set in the region Ⅲ. Then the vacuum state is constructed in the tensor-product Hilbert space (1.2), and the scalar field on the full Hilbert space is connected at the horizon smoothly. In our approach a scalar field is constructed by the normal modes which are a complete orthonormal set on the combined constant-t slice in the both region I and Ⅲ (Fig. 2). The region I and Ⅲ of the Penrose diagram are distinct regions in the same space with common time. The time flows upward in the both I and Ⅲ. This prescription ensures the smoothness of the scalar field at the horizon automatically.

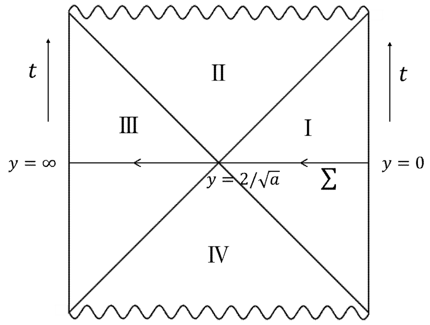

Fig. 2 A constant time slice Σ

We will quantize a scalar field theory in the eternal black hole in three-dimensional AdS space by imposing canonical commutation relations in a whole region of $t = 0$ Cauchy slice Σ obtained by combining the region I and Ⅲ. The scalar field is constructed as the same creation and annihilation operators in the both region I and Ⅲ in a single Hilbert space. And also the scalar field in the region Ⅱ will be studied. Because a particle in the region I and Ⅲ can fall into the region Ⅱ, the scalar field inside the horizon is constructed as the same operators in I and Ⅲ. Then it is expected that the states inside and outside the horizon will be smoothly entangled. It will be shown that normal modes inside the horizon which satisfy matching condition at the horizon are obtained.

Because the BTZ black hole is an eternal black hole solution in three-dimensional AdS spacetime, quantum field theory in the BTZ background is constructed as two-dimensional CFT according to $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. The metric is given by

$$ds^2 = \frac{1}{r^2 - a}dr^2 - (r^2 - a)dt^2 + r^2 d\phi^2 \tag{1.3}$$

where there is singularity at the horizon $r = \sqrt{a}$, and the spacetime is divided into the interior and the exterior. The normal modes in the BTZ background are obtained from those in the AdS_3 spacetime by suitable coordinate transformations. These normal modes turn out not to be eigenfunctions of energy and momentum. We quantize the scalar field in our prescription as shown above. The relation between the thermal vacuum $|0\rangle_\beta$ which is defined by the annihilation operator in our prescription and TFD (1.2) becomes clear by changing basis of the normal modes to that of eigenfunctions of energy and momentum. After the changing of basis, it is found that there are two sets of normal modes in each region I and Ⅲ. One is that of positive frequency normal modes and the other is that of negative frequency normal mode. The positive and negative frequency normal modes are associated with the annihilation operators $b_\pm$ respectively. The creation operator $b_+^\dagger$ creates a 'particle', and $b_-^\dagger$ creates a 'hole' [25]. It will be shown that the bulk operators $b_\pm$ are related to boundary operators $c_{R,L}$ by the Bogolyubov transformation. Then the operators in each region I and Ⅲ are identified as primary operators in those boundary CFT.

The thermal vacuum $|0\rangle_\beta$, which can be expressed in terms of the left and right boundary CFT operators, is shown to be the TFD (1.2). In this approach of quantization, the bulk scalar fields in each region I and Ⅲ are reconstructed as the CFT operators on their own boundary explicitly. Next we study normal mode expansion of scalar fields inside the horizon, which are smoothly connected with that of scalar fields outside the horizon. In this study, each normal mode are represented in terms of integral representations through the changing basis, and this representations make analysis of explicit asymptotic forms of these normal modes easy.

In the case of quantization in the region Ⅱ, the integral representations of normal modes are technically more helpful. As the normal modes in the region I and Ⅲ, we can obtain the normal modes in the region Ⅱ from that of AdS_3 by coordinate transformations. However, it will be shown that the normal modes for a scalar field behind the horizon are not obtained simply as this single set of the normal modes, which is related to one in AdS_3 spacetime by the coordinate transformations. In order to impose appropriate canonical commutation relations on the scalar field, we need to introduce a set of extra linearly independent mode and also two sets of modes which correspond to the non-normalizable modes in AdS_3 spacetime. We will assign certain liner combinations of these modes, instead of normal modes obtained from those of AdS_3, to operators in the scalar field. Furthermore it will also be shown that the Klein-Gordon inner product for these modes has discontinuity at the horizon. We will impose on the scalar field behind the horizon the matching conditions to smoothly connect the scalar fields on both sides of the horizon and the standard canonical commutation relation. Then the scalar field in the region Ⅱ is constructed as the operators which are the same ones in the region I and Ⅲ, and the liner combinations which are associated with these operators are smoothly connected with the normal modes in I and Ⅲ at the horizon. The quantized scalar field in the region Ⅱ are obtained, but are not uniquely determined. There exist some undetermined functions in the normal modes for the scalar field deep inside the horizon. Therefore unless appropriate additional conditions are imposed on the scalar theory inside the horizon, the interior of horizon will not be uniquely determined by the boundary CFTs. We will propose 'positive frequency' conditions for the normal modes deep inside the horizon and remove this ambiguity. Using this scalar fields without ambiguity, a new scalar propagator for two points inside and outside the horizon separately will be obtained. This propagator does not coincide with the results in [8] and [26].

This thesis is organized as follow. First, we will review some fundamentals of conformal field theory (CFT) and anti de-Sitter (AdS) space in Secs.2 and 3. In Sec.4, we will show how to translate the result of the calculation in AdS space into correlation function in conformal field theory. This is one of evidence for the duality, and called dictionary of the AdS/CFT correspondence. Conversely we will explain the method to construct free bulk fields in AdS space as operators in conformal field theory in Sec.5. The construction of free bulk fields inside the black hole horizon will be shown. These methods to describe the bulk as information of conformal field theory on the boundary are called bulk reconstruction. Sec.6 is another work of bulk reconstruction, and our original works are based on this work. We will review the near horizon behavior of bulk scalar fields on both sides of the horizon, which is smoothly connected at the horizon and is quantized by the canonical method. Sec.7 or later are our original works. In Secs.7 and 8, we will obtain the quantized scalar field outside the horizon. Then, Secs.9, 10 and 11 are devoted to construction of the quantized scalar field inside the horizon. Unlike the scalar field outside the horizon, the appropriate normal modes in the region Ⅱ can not be obtained from those in AdS_3 space by suitable transformations. Thus, we introduce additional sets of mode functions in Sec.9 and explain the necessity of these mode functions, which is related to the discontinuity of the inner product, in Sec.10. In Sec.11, we impose the matching condition, which is reviewed in Sec.6, and the standard canonical commutation relation on the scalar field in the region Ⅱ. And then we obtain the explicit form of the quantized scalar field with undetermined functions. In Sec.12, in order to remove this ambiguity we propose additional conditions at the region deep inside the horizon, and new propagator for two-points on both sides of the horizon, which does not coincide with other works, is obtained. In Sec.13, our original works are summarized and the new results which do not coincide with other works are discussed. There are some appendices A to F. This Thesis is based on [29].

2 Conformal Field Theory

In this section, we review the fundamental account of conformal field theory (CFT). For more details, see a standard text book such as [30].

2.1 Conformal Transformation

The conformal transformations of the coordinates are invertible maps $x \to x'$, which leave the metric tensor $g_{\mu\nu}$ invariant up to a coordinate dependent scale factor as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{dx^\rho}{dx'^\mu}\frac{dx^\sigma}{dx'^\nu}g_{\rho\sigma}(x) = \Lambda(x)g_{\mu\nu}(x). \tag{2.1}$$

The set of conformal transformations forms a group, and has the Poincaré group as a subgroup. When $\Lambda(x) = 1$ the group is identified with Poincaré group consisting of translations, rotations or boosts. The meaning of 'conformal' is that the transformation dose not change the angle between two arbitrary curves crossing each other at some point, despite a local dilatation.

In what follow for simplicity we assume that the conformal transformation is an infinitesimal deformation of the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, ..., 1)$. We will investigate the definition (2.1) on an infinitesimal transformation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$. The metric changes as follows at first order in ϵ.

$$\eta_{\mu\nu} \to \eta_{\mu\nu} - (\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) \tag{2.2}$$

Requiring that the transformation is conformal, the following equation will be obtained.

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = f(x)\eta_{\mu\nu} \tag{2.3}$$

The factor $f(x)$ is determined by taking the trace on both sides:

$$f(x) = \frac{2}{d}\partial_\rho\epsilon^\rho. \tag{2.4}$$

By applying an extra derivative ∂_ρ on (2.3), permuting the indices and taking a linear combination, we obtain

$$2\partial_\mu\partial_\nu\epsilon_\rho = \eta_{\mu\rho}\partial_\nu f + \eta_{\nu\rho}\partial_\mu f - \eta_{\mu\nu}\partial_\rho f. \tag{2.5}$$

Contracting with $\eta^{\mu\nu}$, this becomes

$$2\partial^2\epsilon_\mu = (2-d)\partial_\mu f. \tag{2.6}$$

Applying ∂_ν on this expression and ∂^2 on (2.3), we find

$$(2-d)\partial_\mu\partial_\nu f = \eta_{\mu\nu}\partial^2 f. \tag{2.7}$$

Finally, contracting with $\eta^{\mu\nu}$, we can obtain

$$(d-1)\partial^2 f = 0. \tag{2.8}$$

The explicit form of constraints for conformal transformations in d-dimension will be obtained from (2.3)-(2.8). Note that in the case of $d = 1$, there are no constraints for a factor $f(x)$. Then in one-dimension any transformations are conformal.

Let us consider the case of $d > 2$. The above equations (2.3)-(2.8) imply that

$$\partial_\mu\partial_\nu f = 0, \tag{2.9}$$
$$\partial^2 f = 0. \tag{2.10}$$

Then the function $f(x)$ is at most linear in the coordinates

$$f(x) = A + B_\mu x^\mu \tag{2.11}$$

where A and B_μ are constants. Substituting this expression into (2.5), it is found that ϵ_μ is at most quadratic in the coordinates. Therefore we have the general expression

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \tag{2.12}$$

where a_ν and $b_{\mu\nu}$ are constants and $c_{\mu\nu\rho}$ is symmetric constant under a permutation in the last two indices. The constant a_μ is free of constraint and correspond to an infinitesimal transformation. Substitution of this linear term into (2.3) yields

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} b^\lambda_\lambda \eta_{\mu\nu} \tag{2.13}$$

and this implies that $b_{\mu\nu}$ is the sum of a trace and an antisymmetric term

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu} \tag{2.14}$$

where $\alpha = b^\lambda_\lambda/d$ and $m_{\mu\nu} = -m_{\nu\mu}$. The trace term represents an infinitesimal scale transformation, while antisymmetric term represents an infinitesimal rotation. At last we substitute the quadratic term of (2.12) into (2.5) and obtain

$$c_{\mu\nu\rho} = \eta_{\mu\rho}b_\nu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu \tag{2.15}$$

where $b_\mu \equiv c^\rho_{\rho\mu}/d$. The corresponding infinitesimal transformation is

$$x'^\mu = x^\mu + 2(\boldsymbol{x}\cdot\boldsymbol{b})x^\mu - b^\mu\boldsymbol{x}^2 \tag{2.16}$$

which is called a special conformal transformation.

The finite transformations corresponding to the above are the follows.

$$\begin{aligned}
x'^\mu &= x^\mu + a^\mu && \text{translation} && (2.17)\\
x'^\mu &= \alpha x^\mu && \text{dilatation} && (2.18)\\
x'^\mu &= M^\mu_\nu x^\nu && \text{rotation} && (2.19)\\
x'^\mu &= \frac{x^\mu - b^\mu\boldsymbol{x}^2}{1 - 2\boldsymbol{b}\cdot\boldsymbol{x} + \boldsymbol{b}^2\boldsymbol{x}^2} && \text{special conformal transformation} && (2.20)
\end{aligned}$$

The generators of the above conformal group are given as follows.

$$\begin{aligned}
P_\mu &= -i\partial_\mu && \text{translation} && (2.21)\\
D &= -ix^\mu\partial_\mu && \text{dilatation} && (2.22)\\
L_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) && \text{rotation} && (2.23)\\
K_\mu &= -i(2x_\mu x^\nu\partial_\nu - \boldsymbol{x}^2\partial_\mu) && \text{special conformal transformation} && (2.24)
\end{aligned}$$

These generators obey the following commutation relations, which define the conformal algebra.

$$\begin{aligned}
[D, P_\mu] &= iP_\mu\\
[D, K_\mu] &= -iK_\mu\\
[K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D - L_{\mu\nu})\\
[K_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu)\\
[P_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu)\\
[L_{\mu\nu}, L_{\rho\sigma}] &= i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}) && (2.25)
\end{aligned}$$

In order to rewrite this commutation relation into a simpler form, the following generators are defined.

$$J_{\mu,\nu} = L_{\mu\nu}, \quad J_{-1,0} = D, \quad J_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu), \quad J_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu) \tag{2.26}$$

where $J_{\mu\nu} = -J_{\nu\mu}$ and $a, b \in \{-1, 0, 1, ..., d\}$. These new generators obey the $SO(1, d+1)$ algebra

$$[J_{ab}, J_{cd}] = i(\eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}) \tag{2.27}$$

where the diagonal metric η_{ab} is $\mathrm{diag}(-1, 1, 1, ..., 1)$ if the spacetime is Euclidean. Note that in the case of Lorentzian spacetime $SO(2, d)$ algebra is defined in the same way.

2.2 Conformal Invariance and Classical Field

We will consider how classical fields are affected by conformal transformations. Given an infinitesimal conformal transformation parametrized by ω, we seek a matrix representation T such that a classical field $\Phi(x)$ transforms as

$$\Phi'(x') = (1 - i\omega T)\Phi(x). \tag{2.28}$$

In order to find out the allowed form of these generators, we study the Poincaré group. Then we introduce a matrix representation $S_{\mu\nu}$ to define the action of infinitesimal Lorentz transformation on the field $\Phi(x)$

$$L_{\mu\nu}\Phi(0) = S_{\mu\nu}\Phi(0) \tag{2.29}$$

where $S_{\mu\nu}$ is the spin operator associated with the field $\Phi(x)$. Next using the commutation relations of the Poincaré group, we translate the generator $L_{\mu\nu}$ to a nonzero value of x as follows.

$$e^{i\boldsymbol{x}\cdot\boldsymbol{P}} L_{\mu\nu} e^{-i\boldsymbol{x}\cdot\boldsymbol{P}} = S_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu \tag{2.30}$$

This translation is calculated by use of the Hausdorff formula

$$e^{-A} B e^{A} = B + [B, A] + \frac{1}{2!}[[B, A], A] + \frac{1}{3!}[[[B, A], A], A] + \cdots \tag{2.31}$$

where A and B are two operators. This allows us to write the action of the generators as follows.

$$\begin{aligned} P_\mu \Phi(x) &= -i\partial_\mu \Phi(x) \\ L_{\mu\nu}\Phi(x) &= i(x_\mu \partial_\nu - x_\nu \partial_\mu)\Phi(x) + S_{\mu\nu}\Phi(x) \end{aligned} \tag{2.32}$$

In the same way we can obtain the action of dilatation and special conformal transformation for the field $\Phi(x)$.

$$\begin{aligned} D\Phi(x) &= (-ix^\nu \partial_\nu + \tilde{\Delta})\Phi(x) \\ K_\mu \Phi(x) &= (\kappa_\mu + 2x_\mu \tilde{\Delta} - x^\nu S_{\mu\nu} - 2ix_\mu x^\nu \partial_\nu + i\boldsymbol{x}^2 \partial_\mu)\Phi(x) \end{aligned} \tag{2.33}$$

Here $\tilde{\Delta}$ and κ_μ are the respective values of the generators D and K_μ at $x = 0$ as (2.29). For simplicity, we give the result only for spinless fields ($S_{\mu\nu} = 0$). Under the conformal transformation $x \to x'$, a spinless field $\phi(x)$ transforms as

$$\phi(x) \to \phi'(x) = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \phi(x) \tag{2.34}$$

where $|\partial x'/\partial x|$ is the Jacobian of the conformal transformation of the coordinates, and related to the scalar factor $\Lambda(x)$ of (2.1) as

$$\left|\frac{\partial x'}{\partial x}\right| = \Lambda(x)^{-d/2}. \tag{2.35}$$

A field transforming like the above is called quasi-primary field.

2.3 Conformal Transformation in $d = 2$

The conformal invariance in two-dimension is the special case. As we showed in Sec.4.1, when $d = 2$ there is only one constraint (2.10). Indeed, there exist an infinite variety of coordinate transformations, while the generators of conformal transformation in $d > 2$ consist of translation, dilatation, rotation and special conformal transformation. In this section we will consider the conformal transformation in two-dimension. We will use the Euclidean coordinates (x, τ) where a time direction is Wick rotated as $t = i\tau$.

At first, we introduce holomorphic and antiholomorphic coordinates $(z, \bar{z})$, respectively, as

$$z = x + i\tau, \quad \bar{z} = x - i\tau. \tag{2.36}$$

Then under changes of these coordinate systems $z \to w(z)$ and $\bar{z} \to \bar{w}(\bar{z})$, the covariant metric transforms as

$$ds^2 = dx^2 + d\tau^2 = dzd\bar{z} \to \frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}}dzd\bar{z} \tag{2.37}$$

where $|\partial w/\partial z|^2$ is the scalar factor.

Any holomorphic infinitesimal transformation are expressed by Laurent expansion around $z = 0$ such as

$$z' = z + \epsilon(z) = z + \sum_{n=-\infty}^{\infty} c_n z^{n+1}. \tag{2.38}$$

The effect of such a mapping on a spinless field $\phi(z, \bar{z})$ is

$$\phi(z, \bar{z}) \to \phi'(z', \bar{z}') = \phi(z', \bar{z}') - \epsilon(z')\partial'\phi(z', \bar{z}') - \bar{\epsilon}(\bar{z}')\bar{\partial}'\phi(z', \bar{z}') \tag{2.39}$$

where ∂ and $\bar{\partial}$ are differential operators with respect to z and $\bar{z}$, respectively. And then the variation of spinless field $\delta\phi$ is given by

$$\delta\phi = -\epsilon(z)\partial\phi - \bar{\epsilon}(\bar{z})\bar{\partial}\phi = \sum_{n=-\infty}^{\infty}(c_n l_n + \bar{c}_n \bar{l}_n)\phi(z, \bar{z}) \tag{2.40}$$

where we have introduced the generators

$$l_n = -z^{n+1}\partial_z, \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{2.41}$$

These generators obey the following commutation relations.

$$\begin{aligned} [l_n, l_m] &= (n-m)l_{n+m} \\ [\bar{l}_n, \bar{l}_m] &= (n-m)\bar{l}_{n+m} \\ [l_n, \bar{l}_m] &= 0 \end{aligned} \tag{2.42}$$

This algebra is called the Witt algebra. Each of these infinite-dimensional algebras contains a finite subalgebra generated by $l_{\pm 1}$ and l_0. This subalgebra is associated with the global conformal group.

Up to now we have not imposed the condition that conformal transformations be defined everywhere and be invertible. Strictly speaking, in order to form a group, the mapping must be invertible, and must map the whole plane into itself. Now we distinguish global conformal transformations, which satisfy these requirements, from local conformal transformations, which are not everywhere well-defined. The set of global conformal transformations forms the special conformal group. It turns out that the complete set of such mapping is

$$z \to f(z) = \frac{az + b}{cz + d} \tag{2.43}$$

where a, b, c, and d are complex constants and $ad - bc = 1$. The global conformal group in two-dimension is isomorphic to $SL(2, \mathbb{C})$.

2.4 Primary Field

In two-dimension the definition of quasi-primary fields applies also to fields with scaling dimension δ and planar spin s. We define the holomorphic conformal dimension and ots anti-holomorphic counterpart $(h, \bar{h})$ as

$$h = \frac{1}{2}(\Delta + s), \quad \bar{h} = \frac{1}{2}(\Delta - s). \tag{2.44}$$

Under a conformal transformation $z \to w(z)$ and $\bar{z} \to \bar{w}(\bar{z})$ a quasi-primary field transforms as

$$\phi'(w, \bar{w}) = \left(\frac{\partial w}{\partial z}\right)^{-h} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}). \tag{2.45}$$

This is a generalization of (2.34).

When we consider infinitesimal maps $w = z + \epsilon(z)$ and $\bar{w} = \bar{z} + \bar{\epsilon}(\bar{z})$, the variation of quasi-primary fields is

$$\delta\phi \equiv \phi'(z, \bar{z}) - \phi(z, \bar{z}) = -(h\phi\partial_z\epsilon + \epsilon\partial_z\phi) - (\bar{h}\phi\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}}\phi). \tag{2.46}$$

A field whose variation under any local conformal transformation in two-dimensions is given by (2.45) is called primary field. All primary fields are also quasi-primary, but the reverse is not true. A quasi-primary field in two-dimension transform as (2.45) for global conformal group $SL(2, \mathbb{C})$.

2.5 Radial Quantization

Until now we reviewed the fundamentals of classical conformal field theory. In order to consider quantum mechanics, it is necessary to quantize the field theory by appropriate method as the canonical quantization. In this section we will review radial quantization and operator formalism in conformal field theory.

The operator formalism distinguish a time direction from a space direction. We choose space direction along a circle centered at the origin and time direction to be orthogonal to the space. This choice of space and time leads to the radial quantization in two-dimensional conformal field theory. We consider an infinite cylinder with time t going from $-\infty$ and ∞ along an orthogonal direction of a circle and space being compactified with a coordinate x going from 0 to L, where the points $(0, t)$ and (L, t) are identified. If we continue to Euclidean space, the cylinder is described by a single complex coordinate $\xi = t + ix$. We use the following coordinate, and then the cylinder is mapped into the complex plane.

$$z = e^{2\pi\xi/L} \tag{2.47}$$

where a point at $t \to -\infty$ correspond to the origin $z = 0$.

We must assume the existence of a vacuum state $|0\rangle$. In addition for an interacting field ϕ, we assume that the Hilbert space is the same as for a free field. Then the asymptotic field is given by

$$\phi_{\text{in}} \propto \lim_{t \to -\infty} \phi(x, t) = \lim_{z, \bar{z} \to 0} \phi(z, \bar{z}). \tag{2.48}$$

This field reduces to a single operator, which creates a single asymptotic in-state by acting $|0\rangle$ as

$$|\phi_{\text{in}}\rangle = \lim_{z, \bar{z} \to 0} \phi(z, \bar{z}) |0\rangle . \tag{2.49}$$

On this Hilbert space in order to define an asymptotic out-state, we consider the Hermitian conjugate of the asymptotic in-state $|\phi\rangle_{\text{in}}$. In Euclidean space, the time $\tau = it$ must be reversed as $\tau \to -\tau$ upon Hermitian conjugation. In radial quantization, this correspond to the mapping $z \to 1/z^*$. This justifies the following definition of Hermitian conjugation

$$\phi^\dagger(z,\bar z) = \bar z^{-2h} z^{-2\bar h} \phi(1/\bar z, 1/z) \tag{2.50}$$

where ϕ is a primary field with dimensions h and $\bar h$. The prefactors on the right hand side is justified by a well-defined inner product with $|\phi_{\text{in}}\rangle$. The out-state is defined as

$$\langle\phi_{\text{out}}| = |\phi_{\text{in}}\rangle^\dagger . \tag{2.51}$$

A primary field $\phi(z,\bar z)$ is mode expanded as follows.

$$\phi(z,\bar z) = \sum_{m,n\in\mathbb{Z}} z^{-m-h} \bar z^{-n-\bar h} \phi_{m,n} \tag{2.52}$$

where $\phi_{m,n}$ are Laurent coefficients around $z = \bar z = 0$ given by

$$\phi_{m,n} = \frac{1}{2\pi i} \oint dz z^{m+h-1} \frac{1}{2\pi i} \oint d\bar z \bar z^{n+\bar h-1} \phi(z,\bar z). \tag{2.53}$$

A straightforward Hermitian conjugation give us

$$\phi^\dagger(z,\bar z) = \sum_{m,n\in\mathbb{Z}} \bar z^{-m-h} z^{-n-\bar h} \phi^\dagger_{m,n} \tag{2.54}$$

while from the definition (2.50) we obtain

$$\phi^\dagger(z,\bar z) = \sum_{m,n\in\mathbb{Z}} \bar z^{-m-h} z^{-n-\bar h} \phi_{-m,-n}. \tag{2.55}$$

Comparing these two expansions, we obtain the usual expression for Hermitian conjugate of modes.

$$\phi^\dagger_{m,n} = \phi_{-m,-n} \tag{2.56}$$

In order to obtain a well-defined in- and out-state, the vacuum must satisfy the following condition with $m > -h, n > -\bar h$

$$\phi_{m,n}|0\rangle = 0. \tag{2.57}$$

At last we confirm the notation by dropping the dependence of fields upon the antiholomorphic coordinate $\bar z$. Then the mode expansions (2.52) take the following simple form.

$$\begin{aligned} \phi(z) &= \sum_{m\in\mathbb{Z}} z^{-m-h} \phi_m \\ \phi_m &= \frac{1}{2\pi i} \oint dz z^{m+h-1} \phi(z) \end{aligned} \tag{2.58}$$

Note that the antiholomorphic dependence does not vanish. Since holomorphic and anti-holomorphic degrees of freedom are decoupled, we can restore the anti-holomorphic dependence when need.

2.6 Correlation Function

We will review correlation function in conformal field theory. Since the conformal invariance is strong constraint we can determine two-points or three-points function as conformal invariant quantity. In this section we consider two-points function for quasi-primary fields $\phi_1(z_1)$ and $\phi_2(z_2)$ with conformal dimension h_1 and h_2 in two-dimensional conformal field theory and determine the expression by use of the conformal invariance. In two-dimension the conformal invariance consists of translation, dilatation and inversion.

The two-points function is given by

$$\langle\phi_1(z_1)\phi_2(z_2)\rangle = \frac{1}{Z} \int \mathcal{D}\phi \phi_1(z_1)\phi_2(x_2) e^{-S[\phi]}. \tag{2.59}$$

The assumed conformal invariance of the action and of the functional integration measure leads to the following transformation of the correlation function.

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = \left|\frac{\partial z'}{\partial z}\right|_{z=z_1}^{h_1} \left|\frac{\partial z'}{\partial z}\right|_{z=z_2}^{h_2} \langle \phi_1(z_1')\phi_2(z_2')\rangle \tag{2.60}$$

We write the two-points function as

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = G(z_1, z_2) \tag{2.61}$$

and determine a function $G(z_1, z_2)$ by the conformal invariance. The translation invariance leads $G(z_1, z_2) = G(z_1 - z_2)$ which is invariance under a translation $z_i = z_i + a$. Consider scale transformation $z \to \lambda z$, we obtain

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = \lambda^{h_1+h_2} \langle \phi_1(\lambda z_1)\phi_2(\lambda z_2)\rangle . \tag{2.62}$$

Then the function G is expressed as

$$G(z_1 - z_2) = \frac{C_{12}}{(z_1 - z_2)^{h_1+h_2}} \tag{2.63}$$

where C_{12} is a constant coefficient. At last under inversion transformation $z \to -1/z$ the two-point function transforms as

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = z_1^{-2h_1} z_2^{-2h_2} \langle \phi_1(-1/z_1)\phi_2(-1/z_2)\rangle . \tag{2.64}$$

This leads a constraint $h_1 = h_2$ and $G(z_1 - z_2) = G(|z_1 - z_2|)$. The two-point function for quasi-primary operators is determined by global conformal transformations and the expression is given by

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = \frac{C_{12}\delta_{h_1,h_2}}{|z_1 - z_2|^{2h_1}}. \tag{2.65}$$

In the same way we can obtain two-point function in d-dimension by the conformal invariance which consist of translation, dilatation, rotation and special conformal transformation.

$$\langle \phi_1(\boldsymbol{x}_1)\phi_2(\boldsymbol{x}_2)\rangle = \frac{C_{12}\delta_{h_1,h_2}}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|^{2h_1}} \tag{2.66}$$

where $\boldsymbol{x}$ are d-dimensional vector.

3 Anti-de Sitter Space and BTZ Black Hole

In this section, we will review anti de-Sitter (AdS) space and BTZ black hole. First we will show the fundamentals of AdS space, the definition, the coordinate systems and the geodesic distance. Next we review the BTZ black hole spacetime. The BTZ black hole is an eternal black hole solution in three-dimensional AdS spacetime, thus we will show that the BTZ solution is obtained from AdS_3 space by suitable coordinate transformation and periodic identification. Finally we review the quantization of the scalar field in general curved space.

3.1 Difinition of AdS Space

AdS space is a maximally symmetric spacetime with negative curvature and it is a solution to the Einstein equation with a negative cosmological constant. The Einstein equation is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\Lambda g_{\mu\nu} \tag{3.1}$$

where G_N is the Newton constant and $\Lambda < 0$ for AdS space.

In $d+1$ dimension, AdS space (AdS_{d+1}) is defined by the hyperboloid

$$ds^2 = -dX_0^2 + \sum_{i=1}^{d} dX_i^2 - dX_{d+1}^2,$$

$$-X_0^2 + \sum_{i=1}^{d} X_i^2 - X_{d+1}^2 = -l^2 \tag{3.2}$$

embedded in the $d+2$ dimensional Minkowski space $M^{(2,d)}$ with AdS radius l. The definition is invariant under $SO(2,d)$, and AdS_{d+1} has the maximal number of spacetime symmetries $(d+1)(d+2)/2$. The Killing vectors in AdS_{d+1} space generate the AdS_{d+1} isometry $SO(2,d)$. These are the $2d$ boosts and the $d(d-1)/2+1$ rotations. The boosts are generated by the Killing vectors

$$J_{ab} = X_a \partial_b + X_b \partial_a \tag{3.3}$$

where $a \in \{0, d+1\}$ and $b \notin \{0, d+1\}$. The rotations are generated by the Killing vectors

$$L_{mn} = X_m \partial_n - X_n \partial_m \tag{3.4}$$

where $m, n \notin \{0, d+1\}$ or $m, n \in \{0, d+1\}$. And the group $SO(2,d)$ is isomorphic to conformal group in the d-dimensional Minkowski spacetime.

3.2 Coordinate Systems of AdS Space

The metric of AdS_{d+1} space can be written in different forms by considering different parametrization of the hyperboloid. In this section, we will review the three types of useful coordinate systems of AdS_{d+1} space namely global, Poincaré and Rindler coordinates.

Global Coordinates

The global coordinates of AdS_{d+1} are defined by

$$\begin{aligned} X_0 &= l \cosh\rho \cos\tau, \\ X_{d+1} &= l \cosh\rho \sin\tau, \\ X_i &= l \sinh\rho\ \Omega_i, \quad \sum_{i=1}^{d} \Omega_i^2 = 1 \end{aligned} \tag{3.5}$$

where Ω_i parametrizes a unit $d-1$-dimensional sphere S^{d-1}. The metric in the coordinates is given by

$$ds^2 = l^2 (d\rho^2 - \cosh^2\rho d\tau^2 + \sinh^2\rho\ d\Omega_{d-1}^2). \tag{3.6}$$

The coordinate system covers the entire hyperboloid once with $\rho \geq 0$ and $0 \leq \tau \leq 2\pi$. This is the reason that these coordinates are called global.

Poincaré Coordinates

The Poincaré coordinates of AdS_{d+1} are defined by

$$\begin{aligned} X_0 &= l \frac{1}{2y} \left(1 + y^2 - t^2 + \sum_{i}^{d-1} x_i^2 \right), \\ X_1 &= l \frac{t}{y}, \\ X_{d+1} &= l \frac{1}{2y} \left(1 - y^2 + t^2 - \sum_{i}^{d-1} x_i^2 \right), \\ X_i &= l \frac{x_{i-1}}{y},\ i = 2, ..., d. \end{aligned} \tag{3.7}$$

The metric in the coordinates is given by

$$ds^2 = \frac{l^2}{y^2}\left(dy^2 - dt^2 + \sum_i^{d-1} dx_i^2\right) \tag{3.8}$$

where $y \geq 0$ and $-\infty \leq t, x_i \leq \infty$. The coordinates system cover only half of the hyperboloid.

3.3 AdS Geodesic Distance

The geodesic distance is minimal distance between two points in curved space. In AdS space, the geodesic distance is calculated in the embedding space of AdS and then transformed into the expression in desired coordinate system.

Let us consider the $(d+2)$-dimensional embedding space (3.2). A point in this embedding space is $X^\mu(\mu = 1, ..., d+1)$ and the metric is $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, ..., 1, -1)$. Then $(d+1)$-dimensional AdS hyperboloid is defined as

$$X_\mu X^\mu = -l^2. \tag{3.9}$$

The geodesic distance in this space is calculated by the Lagrangian $\mathcal{L} = \eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu/2$ for a free scalar particle. We apply the Lagrange multipliers method with restriction given by the hyperboloid equation (3.9) in order to obtain the Lagrangian for the hyperboloid's geodesics, which is expressed as

$$\mathcal{L} = \frac{1}{2}\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu - \lambda(\tau)F(X^\mu) \tag{3.10}$$

where $F(X^\mu) = X_\mu X^\mu + l^2 = \eta_{\mu\nu}X^\mu X^\nu + l^2 = 0$ and λ is the Lagrange multiplier. Then geodesic equation are given by

$$\ddot{X}^\mu + 2\lambda X^\mu = 0. \tag{3.11}$$

In order to remove the multiplier we take the second derivative of the constraint F.

$$\ddot{F} = 2(\eta_{\mu\nu}\ddot{X}^\mu X^\nu + \eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu) = 0 \tag{3.12}$$

where $\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu$ is equal to the constants of motion for the geodesics as

$$\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu = \begin{cases} \mp 1, & \text{timelike/spacelike} \\ 0. & \text{null} \end{cases} \tag{3.13}$$

Using these relations, the geodesic equation (3.11) is rewritten as

$$\ddot{X}^\mu \pm \frac{1}{l^2}X^\mu = 0, \quad \text{timelike/spacelike} \tag{3.14}$$

$$\ddot{X}^\mu = 0. \quad \text{null} \tag{3.15}$$

Hence lightlike geodesics appear as straight lines in the embedding space.

The general solution for a spacelike geodesic is given by

$$X^\mu = m^\mu e^{\sqrt{\dot{X}^2}\tau} + n^\mu e^{-\sqrt{\dot{X}^2}\tau} \tag{3.16}$$

where m^μ and n^μ are constant vectors which obey

$$2m \cdot n = -l^2, \quad m^2 = n^2. \tag{3.17}$$

Then we can calculate the geodesic distance between two points $X(\tau_1)$ and $X(\tau_2)$ which are spacelike separated as follows.

$$X(\tau_1) \cdot X(\tau_2) = m \cdot n \left(e^{\sqrt{\dot{X}^2}(\tau_1 - \tau_2)} + e^{\sqrt{\dot{X}^2}(\tau_1 - \tau_2)} \right) \tag{3.18}$$

Using the fact that

$$d \equiv \int_{\tau_1}^{\tau_2} ds = \sqrt{\dot{X}^2}(\tau_1 - \tau_2), \tag{3.19}$$

we obtain the following expression for $X(\tau_1) \cdot X(\tau_2)$.

$$X(\tau_1) \cdot X(\tau_2) = -l^2 \cosh d \quad \text{spacelike separated} \tag{3.20}$$

In a similar way the geodesic distance for points timelike separated can be given by

$$X(\tau_1) \cdot X(\tau_2) = -l^2 \cos d. \quad \text{timelike separated} \tag{3.21}$$

Note that if $X(\tau_1) \cdot X(\tau_2) > l^2$ then it is not possible to connect the points by the geodesic distance.

Now we define the geodesic distance $\sigma(x|x')$ in terms of the embedding spacetime coordinates as

$$\sigma(x|x') = -\frac{X(x) \cdot X(x')}{l^2} \tag{3.22}$$

where x and x' are coordinates in AdS.

At last we will show the expression of geodesic distance $\sigma(x|x')$ in the global AdS coordinates and the Poincaré coordinates. For simplicity, we consider three-dimensional AdS space.

The three-dimensional global coordinate is defined by

$$\begin{aligned} X_0 &= l \cosh \rho \cos \tau, \\ X_3 &= l \cosh \rho \sin \tau, \\ X_1 &= l \sinh \rho \cos \phi, \\ X_2 &= l \sinh \rho \sin \phi. \end{aligned} \tag{3.23}$$

Substituting this definition into (3.22), the geodesic distance in the global AdS_3 is given by

$$\sigma(x|x') = \cosh \rho \cosh \rho' \cos(\tau - \tau') - \sinh \rho \sinh \rho' \cos(\phi - \phi'). \tag{3.24}$$

The three-dimensional Poincaré coordinate is defined by

$$\begin{aligned} X_0 &= l \frac{1}{2y}(1 + y^2 - t^2 + x^2), \\ X_1 &= l \frac{t}{y}, \\ X_2 &= l \frac{x}{y}, \\ X_3 &= l \frac{1}{2y}(1 - y^2 + t^2 - x^2). \end{aligned} \tag{3.25}$$

Substituting this definition into (3.22), the geodesic distance in the Poincaré AdS_3 is given by

$$\sigma(x|x') = \frac{|y - y'|^2 - |t - t'|^2 + |x - x'|^2}{2yy'} + 1. \tag{3.26}$$

3.4 BTZ Black Hole

The Bañados-Teitelboim-Zanelli (BTZ) black hole is the solution of the Einstein equation in $(2+1)$-dimensions with a negative cosmological constant $\Lambda = -1/l^2$. The solution which is discovered by Bañados *et.al.* is given by

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (d\phi + N^\phi dt)^2 \tag{3.27}$$

with

$$N^2(r) = -8GM + \frac{r^2}{l^2} + \frac{16G^2J^2}{r^2}, \tag{3.28}$$

$$N^\phi(r) = -\frac{4GJ}{r^2} \tag{3.29}$$

where $-\infty < t < \infty$, $0 < r < \infty$ and $0 \leq \phi \leq 2\pi$. And M and J are respectively identified with the mass and the angular momentum of the BTZ black hole.

The BTZ solution is also obtained from AdS_3 by periodic identification. AdS_3 is defined as the hyperboloid embedded in a four-dimensional flat space as follows:

$$\begin{aligned} ds^2 &= -dX_0^2 + dX_1^2 + dX_2^2 - dX_3^2, \\ -X_0^2 &+ X_1^2 + X_2^2 - X_3^2 = -l^2. \end{aligned} \tag{3.30}$$

We introduce two parameters r_+ and r_- $(r_+ > r_-)$, which correspond to the radii of the outer and inner horizons of the black hole, and classify AdS_3 into three regions. The coordinate transformations are given by

$$\begin{array}{lrlll} \text{Region 1.} & r_+ < r: & X_0 = \sqrt{\hat{r}^2}\cosh\hat{\phi}, & X_3 = \sqrt{\hat{r}^2 - l^2}\sinh\hat{t}, \\ & & X_1 = \sqrt{\hat{r}^2}\sinh\hat{\phi}, & X_2 = \sqrt{\hat{r}^2 - l^2}\cosh\hat{t}, \\ \text{Region 2.} & r_- < r < r_+: & X_0 = \sqrt{\hat{r}^2}\cosh\hat{\phi}, & X_3 = \sqrt{l^2 - \hat{r}^2}\sinh\hat{t}, \\ & & X_1 = \sqrt{\hat{r}^2}\sinh\hat{\phi}, & X_2 = \sqrt{l^2 - \hat{r}^2}\cosh\hat{t}, \\ \text{Region 3.} & r < r_-: & X_0 = \sqrt{-\hat{r}^2}\sinh\hat{\phi}, & X_3 = \sqrt{l^2 - \hat{r}^2}\sinh\hat{t}, \\ & & X_1 = \sqrt{-\hat{r}^2}\cosh\hat{\phi}, & X_2 = \sqrt{l^2 - \hat{r}^2}\cosh\hat{t} \end{array} \tag{3.31}$$

where

$$\hat{r}^2 = l^2\left(\frac{r^2 - r_-^2}{r_+^2 - r_-^2}\right), \quad \begin{pmatrix} \hat{t} \\ \hat{\phi} \end{pmatrix} = \frac{1}{l}\begin{pmatrix} r_+ & -r_- \\ -r_- & r_+ \end{pmatrix}\begin{pmatrix} t/l \\ \phi \end{pmatrix}. \tag{3.32}$$

By using the above coordinates, the AdS_3 metric (3.30) is transformed into the BTZ metric (3.27) with

$$N^2 = \frac{(r^2 - r_-^2)(r^2 - r_+^2)}{l^2 r^2}, \quad N^\phi = -\frac{r_+ r_-}{l r^2}, \tag{3.33}$$

$$8GMl^2 = r_+^2 + r_-^2, \quad 4GJl = r_+ r_-. \tag{3.34}$$

Then the metric becomes

$$ds^2 = -\frac{(r^2 - r_-^2)(r^2 - r_+^2)}{l^2 r^2} dt^2 + \frac{l^2 r^2}{(r^2 - r_-^2)(r^2 - r_+^2)} dr^2 + r^2 \left(d\phi - \frac{r_+ r_-}{l r^2} dt\right)^2. \tag{3.35}$$

The boundary of the BTZ black hole at a fixed time is a circle because ϕ has the periodicity $\phi \sim \phi + 2\pi$. In this paper, for simplicity we will consider the non-rotating BTZ black hole in three-dimensions with $r_- = 0$. Then the metric is given as follows:

$$ds^2 = -\frac{(r^2 - r_+^2)}{l^2}dt^2 + \frac{l^2}{(r^2 - r_+^2)}dr^2 + r^2 d\phi^2. \tag{3.36}$$

In this paper, the AdS length l will be set to unity.

3.5 Quantum Field Theory in Curved Space

In this section we will review the quantization of a free scalar field in curved space. We will start with Lagrangian density in curved space.

$$\mathcal{L}(x) = \frac{1}{2}\sqrt{-g(x)}\left[g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) - m^2\phi^2(x)\right] \tag{3.37}$$

where $\phi(x)$ is the scalar field with mass m, $g_{\mu\nu}(x)$ is the metric tensor in curved space and $g(x) = |\det g_{\mu\nu}(x)|$. The action is

$$S = \int \mathcal{L}(x)d^n x \tag{3.38}$$

where n is the spacetime dimension. Setting the variation of the action with respect to ϕ equal to zero we can obtain the scalar field equation.

$$\left[\Box + m^2\right]\phi(x) = 0 \tag{3.39}$$

where $\Box$ is given by

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right]. \tag{3.40}$$

The scalar product is defined by

$$(\phi_1, \phi_2) = -i\int_\Sigma d\Sigma^\mu\sqrt{-g_\Sigma(x)}\left[\phi_1(x)\partial_\mu\phi_2^*(x) - \phi_2^*(x)\partial_\mu\phi_1(x)\right] \tag{3.41}$$

where $d\Sigma^\mu = n^\mu d\Sigma$, n^μ is a future-directed unit vector orthogonal too the spacelike hypersurface Σ and $d\Sigma$ is the volume element in Σ. The hypersurface Σ is taken to be a Caucy surface in the spacetime and the value of (ϕ_1, ϕ_2) is independent of Σ.

There is a complete set of mode solutions $u_i(x)$ of (3.39) which are orthonormal in the product (3.41):

$$(u_i, u_j) = \delta_{ij}, \quad (u_i, u_j^*) = 0. \tag{3.42}$$

The index i represent the set of quantities necessary to label the modes. The scalar field ϕ is expanded as follows.

$$\phi(x) = \sum_i\left[a_i u_i(x) + a_i^\dagger u_i^*(x)\right] \tag{3.43}$$

The system is quantized in canonical quantization scheme by requiring the following equal time commutation relations:

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0. \tag{3.44}$$

As the case of Minkowski space the vacuum state may be defined by

$$a_i\left|0\right\rangle = 0. \tag{3.45}$$

In the case of curved spacetime, however, there is ambiguity in the formalism. In the Minkowski space there is a natural set of mode solutions which satisfy the following equation.

$$\frac{\partial}{\partial t}u_i^M(t, \boldsymbol{x}) = -i\omega u_i^M(t, \boldsymbol{x}) \tag{3.46}$$

where M denotes the mode solutions in the Minkowski space and this mode solution is said to be positive frequency ($\omega > 0$) with respect to t. This natural mode solutions are associated with the Poincarè group whose action leaves the Minkowski line element invariance. The vector $\partial/\partial t$ is a Killing vector of the Minkowski space, orthogonal to the spacelike hypersurface $t =$ constant, and the mode solutions u^M are eigenfunction of this Killing vector. Then the scalar field is expanded in terms of the positive frequency solutions multiplying annihilation operator a_i. The vacuum is invariant under the action of the Poincarè group.

In curved spacetime the Poincarè group is no longer a symmetry group of the spacetime. In general there will be no Killing vectors defining positive frequency mode solutions. Therefore let us consider a second complete orthonormal set of modes $\bar{u}_j(x)$. The scalar field ϕ is expanded in this set as follows.

$$\phi(x) = \sum_j \left[\bar{a}_j \bar{u}_j(x) + \bar{a}_j^\dagger \bar{u}_j^*(x)\right] \tag{3.47}$$

The new annihilation operators $\bar{a}_j$ define a new vacuum state.

$$\bar{a}_j \left|\bar{0}\right\rangle = 0 \tag{3.48}$$

As both sets of modes are complete, the new modes $\bar{u}_j$ can be expanded in terms of the old modes u_i:

$$\bar{u}_j = \sum_i \left(\alpha_{ji} u_i + \beta_{ji} u_i^*\right). \tag{3.49}$$

Conversely

$$u_i = \sum_j \left(\alpha_{ji} \bar{u}_j - \beta_{ji}^* \bar{u}_j^*\right). \tag{3.50}$$

These relations are known as the Bogolubov transformations. The matrices α_{ij}, β_{ij} are called Bogolubov coefficients, and by using (3.42) and the above relations they can be evaluated as

$$\alpha_{ij} = (\bar{u}_i, u_j), \quad \beta_{ij} = -(\bar{u}_i, u_j^*). \tag{3.51}$$

Equating the expansions of the scalar field (3.43) and (3.47) and using (3.50), (3.49) and (3.42), we can obtain the following relations.

$$a_i = \sum_j \left(\alpha_{ji} \bar{a}_j + \beta_{ji}^* \bar{a}_j^\dagger\right) \tag{3.52}$$

$$\bar{a}_j = \sum_i \left(\alpha_{ji} a_j - \beta_{ji} a_j^\dagger\right) \tag{3.53}$$

The Bogolubov coefficients have the following properties.

$$\sum_k \left(\alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^*\right) = \delta_{ij} \tag{3.54}$$

$$\sum_k \left(\alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk}\right) = 0 \tag{3.55}$$

Note that when $\beta_{ji} \neq 0$ in (3.52) the new vacuum $\left|\bar{0}\right\rangle$ will not be annihilated by a_i:

$$a_i \left|\bar{0}\right\rangle = \sum_j \left(\beta_{ji}^* \bar{a}_j^\dagger\right) \left|\bar{0}\right\rangle \neq 0. \tag{3.56}$$

in contrast to (3.48).

If we can obtain positive frequency modes u_i with respect to timelike Killing vector ξ, satisfying

$$\mathcal{Z}_\xi u_i = -i\omega u_i \quad (\omega > 0) \tag{3.57}$$

and $\bar{u}_j$ are a linear combination of u_i ($\beta_{ji} = 0$), the scalar field is expanded in terms of positive frequency modes multiplying the annihilation operators. Then the two set of modes u_i and $\bar{u}_j$ share a common vacuum state such as $a_i |0\rangle = \bar{a}_j |0\rangle = 0$.

4 AdS/CFT Correspondence

The anti-de Sitter/conformal field theory correspondence, or AdS/CFT, was proposed by Maldacena [1]. This is a conjecture that a $(d+1)$-dimensional quantum gravity in the bulk is encoded in a d-dimensional non-gravitational theory on the boundary, which is an example of holography. According to the holographic principle, it is expected that a quantum gravity is described in terms of language of non-gravitational theory. It is important that the AdS/CFT correspondence is a strong/weak duality. The non-gravitational theory with strong coupling is related to the gravitational theory with weak coupling and vice versa. For example, we can study the non-gravitational theory with strong coupling through the calculations of the gravitational theory with weak coupling.

Until now scientists have been studying this duality and discovered many evidence that the duality exists by calculating on both sides. Here we will review two convenient dictionaries which describe the equivalence of correlation functions on both sides. One of them is called GKPW dictionary, or 'differentiating' dictionary. In this dictionary the generating function of correlation functions in the conformal field theory is equivalence to the partition function of the semiclassical AdS gravity. Then the correlation functions of operators in the conformal field theory can be computed by 'differentiating' the partition function with respect to the sources and setting them to zero. The other dictionary is called BDHM dictionary, or 'exprapolate' dictionary. In this dictionary the correlation functions in conformal field theory can be obtained by 'extrapolating' the correlation functions in the bulk to the boundary. It will be shown that these dictionaries are equivalence.

4.1 GKPW Dictionary

Gubser, Klebanov, and Polyakov [31] and Witten [32] conjectured that a partition function on Euclidean AdS_{d+1} with a boundary value ϕ_0 is equal to a partition function on conformal field theory.

$$Z_{\mathrm{AdS}}(\phi_0) = Z_{\mathrm{CFT}} = \langle \exp \int \phi_0 \mathcal{O} \rangle \tag{4.1}$$

where a partition function Z_{AdS} is expressed as a classical gravitational action S

$$Z_{\mathrm{AdS}}(\phi_0) = \exp\left[-S(\phi_0)\right], \tag{4.2}$$

ϕ_0 is a boundary value of a massless scalar field ϕ and $\mathcal{O}$ is a operator on the conformal field theory. ϕ_0 is couple to a conformal operator $\mathcal{O}$, and then this partition function is a generating function of correlation function in the conformal field theory. The n-point functions is calculated by functional derivative of the generating function as follows.

$$\langle O(x_1)O(x_2)...O(x_n)\rangle_{\mathrm{CFT}} = \frac{1}{Z_{AdS}} \frac{\delta}{\delta\phi_0(x_1)} \frac{\delta}{\delta\phi_0(x_2)} ... \frac{\delta}{\delta\phi_0(x_n)} Z_{\mathrm{AdS}}(\phi_0) \tag{4.3}$$

For simplicity in order to confirm the above conjecture, let us consider a free scalar field ϕ with mass m in Euclidean AdS_{d+1} and calculate the two-point functions. The calculation will be carried out in Euclidean Poincarr̀ coordinates. The metric is given by

$$ds^2 = \frac{1}{y^2}(dy^2 + dt^2 + dx_1^2 + ... + dx_{d-1}^2) \tag{4.4}$$

where the time direction is Wick rotated. The action is given by

$$S(\phi) = \frac{1}{2} \int d^{d+1}x\sqrt{g}\left[(\partial_\mu\phi)^2 + m^2\phi^2\right]. \tag{4.5}$$

And the equation of motion is obtained by the principle of least action ($\delta S = 0$).

$$\frac{1}{\sqrt{g}}\left(\partial_\mu\sqrt{g}\partial^\mu\phi - m^2\phi\right) = 0 \tag{4.6}$$

This is the Klein-Gordon equation, and in Poincarè coordinate the K-G equation is expressed as

$$\left(\partial_y^2 - \frac{d-1}{y}\partial_y + \partial_t^2 + \partial_{x_1}^2 ... + \partial_{x_{d-1}}^2 - \frac{m^2}{y^2}\right)\phi = 0. \tag{4.7}$$

We will express the bulk scalar field ϕ as a integral form by using appropriate Green functions K and the boundary value ϕ_0.

$$\phi(y, \boldsymbol{x}_E) = \int d\boldsymbol{x}'_E K(y, \boldsymbol{x}_E - \boldsymbol{x}'_E)\phi_0(\boldsymbol{x}'_E) \tag{4.8}$$

where $\boldsymbol{x}_E$ is d-dimensional vector $(t, \boldsymbol{x})$. The Green function K satisfy the Klein-Gordon equation

$$\frac{1}{\sqrt{g}}\left(\partial_\mu\sqrt{g}\partial^\mu - m^2\right)K = 0 \tag{4.9}$$

and is required to be regular at $y \to 0$ and $\boldsymbol{x}_E \neq \boldsymbol{x}'_E$. Then the solution is given by

$$K(y, \boldsymbol{x}_E - \boldsymbol{x}'_E) = \frac{y^\Delta}{(y^2 + |\boldsymbol{x}_E - \boldsymbol{x}'_E|^2)^\Delta} \tag{4.10}$$

where $\Delta = (d + \sqrt{d^2 + 4m^2})/2$.

By substituting (4.8) into the action (4.5) and integrating by parts, we will obtain a surface integral term

$$S(\phi) = \frac{1}{2}\int d\boldsymbol{x}_E y^{-d+1}\phi_0(\boldsymbol{x}_E)\partial_y\phi(y, \boldsymbol{x}_E)\Big|_{y=0} = \frac{d}{2}\int d\boldsymbol{x}_E d\boldsymbol{x}'_E \frac{\phi_0(\boldsymbol{x}_E)\phi_0(\boldsymbol{x}'_E)}{|\boldsymbol{x}_E - \boldsymbol{x}'_E|^{2\Delta}}. \tag{4.11}$$

Then the generating function is given by

$$Z_{\mathrm{AdS}} = \exp\left[-\frac{d}{2}\int d\boldsymbol{x}_E d\boldsymbol{x}'_E \frac{\phi_0(\boldsymbol{x}_E)\phi_0(\boldsymbol{x}'_E)}{|\boldsymbol{x}_E - \boldsymbol{x}'_E|^{2\Delta}}\right] \tag{4.12}$$

and by differentiating this with respect to the boundary values ϕ_0, we will obtain the two-point function of operators in conformal field theory with conformal weight Δ

$$\langle\mathcal{O}(\boldsymbol{x}_E)\mathcal{O}(\boldsymbol{x}'_E)\rangle_{\mathrm{CFT}} \sim \frac{1}{|\boldsymbol{x}_E - \boldsymbol{x}'_E|^{2\Delta}}. \tag{4.13}$$

4.2 BDHM Dictionary

Banks, Douglas, Horowitz and Martinec (BDHM) proposed the another method to calculate the correlation function in conformal field theory [33]. They calculated the bulk correlators and pulled them to the boundary. The leading behavior of the bulk correlators can be extracted to give correlators of the operators dual to the bulk fields

$$\langle\mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle_{\mathrm{CFT}} = \lim_{y\to 0} y^{-n\Delta}\langle\phi(y, x_1)...\phi(y, x_n)\rangle_{\mathrm{bulk}}. \tag{4.14}$$

The equivalence for the two dictionaries will be given by the explicit form of bulk correlators. In this section we will consider a free scalar field ϕ with mass m in AdS_{d+1} and obtain the same two-point function of operators in conformal field theory (4.13).

Let us consider the quantization of a scalar field in Lorentzian Poincarè. The equation of motion for a scalar field is given by

$$\left(\partial_y^2 - \frac{d-1}{y}\partial_y - \partial_t^2 + \partial_{x_1}^2 ... + \partial_{x_{d-1}}^2 - \frac{m^2}{y^2}\right)\phi = 0. \tag{4.15}$$

Separating variables in these coordinates the solution to this equation of motion is given by

$$\phi(y,t,\boldsymbol{x}) = y^{d/2}\chi(qy)e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{4.16}$$

where $q^2 = \boldsymbol{k}^2 - \omega^2 > 0$ and χ satisfy the following equation.

$$\partial_z^2\chi(z) + \frac{1}{z}\partial_z\chi(z) - \left(1 + \frac{\nu^2}{z^2}\right)\chi(z) = 0 \tag{4.17}$$

where $z = qy$ and $\nu = \sqrt{d^2+4m^2}$. The solution of this equation (4.17) is modified Bessel function. Replacing a function f with a modified Bessel function of the second kind $K_\nu(ky)$ which is regular at $z\to\infty$, we can obtain the scalar field as follows.

$$\phi(y,t,\boldsymbol{x}) = \int \frac{d\omega d^{d-1}k}{(2\pi)^d}\left(a(\boldsymbol{k})y^{d/2}K_\nu(ky)e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}} + h.c.\right) \tag{4.18}$$

We consider the canonical quantization by imposing the following equal time commutation relations

$$\begin{aligned}&[\phi(y,t,\boldsymbol{x}),\phi(y',t,\boldsymbol{x}')] = 0\\ &[\phi(y,t,\boldsymbol{x}),\pi(y',t,\boldsymbol{x}')] = i\delta(y-y')\delta(\boldsymbol{x}-\boldsymbol{x}')\end{aligned} \tag{4.19}$$

where $\pi = \sqrt{-g}(-g^{tt})\partial_t\phi$. Then the following commutation relations are also satisfied.

$$[a(\boldsymbol{k}),a^\dagger(\boldsymbol{k}')] = \delta(\boldsymbol{k}-\boldsymbol{k}') \tag{4.20}$$

We will define the vacuum $|0\rangle$ which is annihilated by $a(\boldsymbol{k})$.

$$a(\boldsymbol{k})\,|0\rangle = 0 \tag{4.21}$$

The bulk two-point function for free scalar fields is given by

$$G_{\text{bulk}}(y,t,\boldsymbol{x}|y',t',\boldsymbol{x}') = \langle 0|\,\phi(y,t,\boldsymbol{x})\phi(y',t',\boldsymbol{x}')\,|0\rangle\,. \tag{4.22}$$

Using (4.19) we can obtain the following equation for G_{bulk}.

$$\left(\Box + m^2\right)G_{\text{bulk}}(y,t,\boldsymbol{x}|y',t',\boldsymbol{x}') = \frac{1}{\sqrt{-g}}\delta(y-y')\delta(t-t')\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{4.23}$$

where the scalar Laplacian is given by

$$\Box = -y^2\partial_y^2 + (d-1)y\partial_y + y^2\partial_t^2 - y^2\partial_{x_1}^2 ... - y^2\partial_{x_{d-1}}^2. \tag{4.24}$$

This is a hypergeometric equation and the solution is given by

$$G_{\text{bulk}}(y,t,\boldsymbol{x}|y',t',\boldsymbol{x}') = G_{\text{bulk}}(\xi) = C(\Delta,d)\xi^\Delta F\left(\frac{\Delta}{2},\frac{\Delta+1}{2};\Delta-\frac{d}{2}+1;\xi^2\right) \tag{4.25}$$

where F is the hypergeometric function and the overall normalization constant is defined by

$$C(\Delta,d) = \frac{2^{-\Delta}\Gamma(\Delta)}{(2\Delta-d)\pi^{d/2}\Gamma(\Delta-d/2)} \tag{4.26}$$

and ξ, which is the AdS invariant distance between two points, is defined by

$$\xi(y,t,\boldsymbol{x}|y',t',\boldsymbol{x}') \equiv \frac{2yy'}{y^2+y'^2-(t-t')^2+|\boldsymbol{x}-\boldsymbol{x}'|^2}. \tag{4.27}$$

G_{bulk} is the scalar Green function. Note that because AdS_{d+1} has $SO(2,d)$ isometry, it is expected that the scalar green function can be rewritten in terms of $SO(2,d)$ invariant quantity such as (4.27). Taking the boundary limit $y,y'\to 0$, the above expression reduce to

$$G_{\text{bulk}}(y,t,\boldsymbol{x}|y',t',\boldsymbol{x}') \to \frac{1}{\xi^{\Delta}} \sim \frac{(yy')^{\Delta}}{(-(t-t')^2+|\boldsymbol{x}-\boldsymbol{x}'|^2)^{\Delta}}. \tag{4.28}$$

At last we will Wick rotate the result and conclude that the boundary behavior of the bulk correlator for free scalar fields is proportional to the correlator for operators dual to the bulk fields.

$$G_{\text{bulk}}(y,\boldsymbol{x}_E|y',\boldsymbol{x}'_E)\big|_{y,y'\to 0} \sim y^{\Delta}y'^{\Delta}\left\langle \mathcal{O}(\boldsymbol{x}_E)\mathcal{O}(\boldsymbol{x}'_E)\right\rangle \sim \frac{(yy')^{\Delta}}{|\boldsymbol{x}_E-\boldsymbol{x}'_E|^{2\Delta}} \tag{4.29}$$

This is consistent with the result of GKPW dictionary (4.13).

5 Bulk Reconstruction

In the dictionaries reviewed in the previous section, we obtained the correlation function of the operators in the conformal field theory in terms of the bulk fields. On the other hand, there are some methods to construct the bulk field in AdS spacetime as the operators in the conformal field theory on the boundary, which are so-called bulk reconstruction.

Hamilton *et.al.* proposed that a scalar bulk operator in AdS spacetime can be expressed as local operators in the conformal field theory on the finite boundary region [5]. Miyaji *et.al.* proposed that a scalar bulk state can be expressed by Ishibashi state which is a linear combination of CFT primary state and the descendant state [6]. First, we will review these two methods of bulk reconstruction. And next, we will review works by Goto and Takayanagi, which is extension of the second method [8]. They showed that we can obtain the bulk state in the BTZ black hole and also obtain the correlation functions, which are consistent with the previous other works [21][26].

5.1 HKLL Construction

There is a well known method to reconstruct bulk local operators, which is so-called Hamilton-Kabat-Lifschytz-Lowe(HKLL) construction [5][10][11]. Their proposal is that in AdS/CFT correspondence the bulk local field is encoded on the boundary by CFT operators. According to BDHM dictionary, a bulk field with normalizable fall-off near the boundary behaves as

$$\phi(y,x) \sim y^{\Delta}\phi_0(x) \tag{5.1}$$

where y is a radial coordinate which vanishes at the boundary. The bulk field can be expressed in terms of the boundary field ϕ_0 multiplied the appropriate kernel K so called a smearing function.

$$\phi(y,x) = \int dx' K(x'|y,x)\phi_0(x') \tag{5.2}$$

$\phi_0(x)$ corresponds to a local operator $\mathcal{O}(x)$ in the boundary CFT.

$$\phi_0(x) \leftrightarrow \mathcal{O}(x) \tag{5.3}$$

Hence AdS/CFT implies that the bulk local field is constructed as non-local boundary operotors, which mean CFT operators smeared over a boundary region.

$$\phi(y,x) \leftrightarrow \int dx' K(x'|y,x)\mathcal{O}(x') \tag{5.4}$$

The bulk-to-bulk correlation function is equal to correlation functions of corresponding non-local operators in the boundary CFT.

$$\langle \phi(y_1,x_1)\phi(y_2,x_2)\rangle = \int dx'_1 dx'_2 K(x'_1|y_1,x_1)K(x'_2|y_2,x_2)\left\langle \mathcal{O}(x'_1)\mathcal{O}(x'_2)\right\rangle \tag{5.5}$$

In this section we review the construction of the smearing function K. In general the smearing function is not determined uniquely and there is the ambiguity of the integral region in terms of the smeared boundary operators (5.2). In order to eliminate this ambiguity in the expression of the bulk local field we will impose the locality, or causality, on the bulk local field as usual in quantum field theory.

$$[\phi(y,x),\phi(y',x')]=0 \tag{5.6}$$

where these bulk points are spacelike separated. The locality condition (5.6) is satisfied when we construct the bulk local field with the smearing function which is non-zero only at the spacelike separation between the bulk point (y,x) and the boundary point of smeared operator x' (5.2).

In the following we will consider a scalar field in Lorentzian Poincaré AdS_{d+1} which satisfy the locality condition, and construct the smearing function. The metric is given by

$$ds^2=\frac{l^2}{y^2}(dy^2-dt^2+d\boldsymbol{x}^2) \tag{5.7}$$

where we set the AdS radius as $l=1$, and $0<y<\infty$, $-\infty<t<\infty$ and $\boldsymbol{x}$ is $(d-1)$-dimensional vector. The scalar field which satisfy the Klein-Gordon equation $(\Box-m^2)\phi=0$ is expanded in a complete set of modes as follows.

$$\phi(y,t,\boldsymbol{x})=\int_{\omega>|\boldsymbol{k}|}d\omega d^{d-1}\boldsymbol{k}a_{\omega,\boldsymbol{k}}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}}y^{d/2}J_\nu(\sqrt{\omega^2-\boldsymbol{k}^2}y) \tag{5.8}$$

where J_ν is a Bessel function with $\nu=\Delta-d/2$, and Δ is the conformal dimension on the boundary CFT satisfying $\Delta=d/2+\sqrt{d^2/4+m^2l^2}$. The boundary field is given by

$$\begin{aligned}\phi_0(t,\boldsymbol{x})&=\lim_{y\to0}\frac{1}{y^\Delta}\phi(y,t,\boldsymbol{x})\\&=\frac{1}{2^\nu\Gamma(\nu+1)}\int_{\omega>|\boldsymbol{k}|}d\omega d^{d-1}\boldsymbol{k}a_{\omega,\boldsymbol{k}}e^{-i\omega t}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}(\omega^2-\boldsymbol{k}^2)^{\nu/2}.\end{aligned} \tag{5.9}$$

Note that

$$a_{\omega,\boldsymbol{k}}=\frac{2^\nu\Gamma(\nu+1)}{(2\pi)^d(\omega^2-\boldsymbol{k}^2)^{\nu/2}}\int dtd^{d-1}\boldsymbol{x}e^{i\omega t}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\phi_0(t,\boldsymbol{x}). \tag{5.10}$$

By substituting (5.10) back into the bulk field (5.8), we obtain the expression of the bulk field in terms of the boundary field as follows.

$$\phi(y,t,\boldsymbol{x})=\int dt'd^{d-1}\boldsymbol{x}'K(t',\boldsymbol{x}'|y,t,\boldsymbol{x})\phi_0(t',\boldsymbol{x}') \tag{5.11}$$

where

$$K(t',\boldsymbol{x}'|y,t,\boldsymbol{x})=\frac{2^\nu\Gamma(\nu+1)}{(2\pi)^d}\int_{\omega>|\boldsymbol{k}|}d\omega d^{d-1}\boldsymbol{k}e^{-i\omega(t-t')}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\frac{y^{d/2}J_\nu(\sqrt{\omega^2-\boldsymbol{k}^2}y)}{(\omega^2-\boldsymbol{k}^2)^{\nu/2}}. \tag{5.12}$$

The integral representation for the smearing function K was evaluated in [5][10]. However, we obtain the smearing function with support on the entire boundary in the Poincaré coordinate, and could not show the locality condition of the bulk local fields and correct bulk to boundary correlation functions. In order to manifest these properties of the bulk local field, we will require the compact support on the smearing function K. The compact support is obtained by the complexification of the boundary spacial coordinate $\boldsymbol{x}$ [11]. This prescription is the main point in HKLL construction, and then we can obtain the bulk local fields whose dual boundary operators are spacelike separated.

In what follow for simplicity we will consider the case of AdS_3 and a concrete expression of the smearing function K with the compact support. The bulk field in AdS_3 is given by

$$\phi(y,t,x)=\frac{2^\nu\Gamma(\nu+1)}{4\pi^2}\int_{\omega>|k|}d\omega dk\frac{yJ_\nu(\sqrt{\omega^2-k^2}y)}{(\omega^2-k^2)^{\nu/2}}\left(\int dt'dx'e^{-i\omega(t-t')}e^{ik(x-x')}\phi_0(t',x')\right). \tag{5.13}$$

First we perform the t' and x' integrals and the result is given by

$$\phi(t,x,z) = 2^{\nu}\Gamma(\nu+1)\int d\omega dk e^{-i\omega t}e^{ikx}\frac{zJ_{\nu}(\sqrt{\omega^2-k^2}z)}{(\omega^2-k^2)^{\nu/2}}\phi_0(\omega,k) \tag{5.14}$$

where $\phi_0(\omega,k)$ is the Fourier transform of the boundary field. Next we use the following integrals:

$$\int_0^{2\pi} d\theta e^{-ir\omega\sin\theta - kr\cos\theta} = 2\pi J_0(r\sqrt{\omega^2-k^2}), \tag{5.15}$$

$$\int_0^1 drr(1-r^2)^{\nu-1}J_0(br) = 2^{\nu-1}\Gamma(\nu)b^{-\nu}J_{\nu}(b). \tag{5.16}$$

Then we obtain

$$\frac{J_{\nu}(\sqrt{\omega^2-k^2}y)}{(\omega^2-k^2)^{\nu/2}} = \frac{1}{\pi(2y)^{\nu}\Gamma(\nu)}\int_{T^2+X^2<y^2} dTdX(y^2-T^2-X^2)^{\nu-1}e^{-i\omega T}e^{-kX} \tag{5.17}$$

where we define $T = r\sin\theta$ and $X = r\cos\theta$. By inserting this into (5.13), the bulk local field is expressed as

$$\begin{aligned}\phi(y,t,x) &= \frac{\nu}{\pi}\int_{T^2+X^2<y^2} dTdX\left(\frac{y^2-T^2-X^2}{y}\right)^{\nu-1}\int d\omega dk e^{-i\omega(t+T)}e^{ik(x+iX)}\phi_0(\omega,k)\\ &= \frac{\Delta-1}{\pi}\int_{T^2+X^2<y^2} dTdX\left(\frac{y^2-T^2-X^2}{y}\right)^{\Delta-2}\phi_0(t+T,x+iX)\end{aligned} \tag{5.18}$$

where $\nu = \Delta - 1$ and we identify the Fourier form in the first line as $\phi_0(t+T,x+iX)$. We can also express this result in terms of the AdS invariant distance as follows.

$$\phi(y,t,x) = \frac{\Delta-1}{\pi}\int_{T^2+X^2<y^2} dTdX \lim_{y'\to 0}\left[2y'\sigma(y,t,x|y',t+T,x+iX)\right]^{\Delta-2}\phi_0(t+T,x+iX) \tag{5.19}$$

where σ is a geodesic distance defined by

$$\sigma(y,t,x|y',t',x') = \frac{1}{2yy'}\left(y^2+y'^2-(t-t')^2+(x-x')^2\right). \tag{5.20}$$

In this form it becomes clear that the bulk local field is expressed in terms of the integral over a finite boundary region which is spacelike separated from the bulk point (y,t,x). The boundary spacial coordinate x is complexified, and the integral region is a disk of radius z in the real T and the imaginary X. At last the boundary field $\phi_0(t+T,x+iX)$ is replaced by a dual CFT operator $\mathcal{O}(t+T,x+iX)$ (5.3).

We will refer to the locality condition of the bulk local field in the expression of (5.18). We consider two bulk local fields ϕ^1 and ϕ^2 and each integral region $\mathcal{A}_1$ and $\mathcal{A}_2$ where the smeared boundary fields ϕ_0^1 and ϕ_0^2 are located. Now ϕ^1 and ϕ^2 are spacelike separated and the integral region $\mathcal{A}_1$ entirely overlap $\mathcal{A}_2$. Therefore the bolk local field ϕ_1 is also spacelike separated from the smeared boundary fields ϕ_0^2, in other words ϕ^1 commute with ϕ_0^2. Then by using (5.2) the locality condition is evaluated as follows.

$$[\phi(y_1,x_1),\phi(y_2,x_2)] = \int dx_2' K(x_2'|y_2,x_2)[\phi(y_1,x_1),\phi_0(x_2')] = 0 \tag{5.21}$$

Note that this locality condition can be guaranteed since (5.18) gives the correct bulk to boundary correlation function.

The HKLL method seems to define the bulk local fields constructed as the dual boundary CFT operators. However, in order to know the smearing function we need to solve the bulk equation of motion. We implicitly assume the existance of bulk geometry, thus it is not satisfactory for understanding the emergence of the bulk geometry. Furthermore the main point of the HKLL construction is the complexification of the spacial direction on the boundary, but it is not a convenient prescription in ordinary quantum field theory. In next section we review another method to reconstruct the bulk local operator without solving the equation of motion.

5.2 Reconstruction of Bulk Local State

In this section we will construct a bulk local state as a excitated CFT state on the boundary. Miyaji *et. al.* proposed the method of the construction by using Ishibashi states which is a liner combination of CFT primary state and the descendant states [6]. Nakayama and Ooguri generalize this construction and show that the construction of bulk local state is equivalent to HKLL construction [7]. Furthermore Goto and Takayanagi proposed to reconstruct bulk local state in the BTZ black hole background [8].

At first we review the construction of the bulk local states in global AdS space. We consider $(d+1)$-dimensional AdS gravity and its dual CFT in d-dimension. A bulk local operator $\hat{\psi}$ is an operator in the Hilbert space of CFT and is a local probe of the bulk AdS geometry. As a local probe, it must depend on the position $(\rho, t, \vec{x})$ in AdS with the following metric.

$$ds^2 = d\rho^2 - \cosh^2 \rho dt^2 + \sinh^2 \rho d\vec{x}^2 \tag{5.22}$$

where we set the AdS radius to be 1 and $\vec{x}$ is parametrization of the sphere S^{d-1}, $\vec{x}^2 = 1$.

The only requirement which we impose on $\hat{\psi}$ is that the actions of the conformal symmetry and the bulk isometry on $\hat{\psi}$ are compatible.

$$[J, \hat{\psi}(\rho, t, \vec{x})] = i\mathcal{L}_{\mathcal{J}}\hat{\psi}(\rho, t, \vec{x}) \tag{5.23}$$

where J is a generator of the conformal symmetry and $\mathcal{L}_{\mathcal{J}}$ is the Lie derivative in the AdS coordinate with respect to the Killing vector field $\mathcal{J}$ corresponding to J. When $\hat{\psi}$ is a scalar field in AdS, the Lie derivative acts as

$$\mathcal{L}_{\mathcal{J}}\hat{\psi} = \mathcal{J}^\mu \partial_\mu \hat{\psi}. \tag{5.24}$$

The $SO(2,d)$ conformal generators on the cylinder $\mathbb{R} \times S^{d-1}$ which is the boundary of global AdS$_{d+1}$ space can be organized as $J = (H, M_{ab}, P_a, K_a)$ where $a, b = 1, ..., d$ H is Hamiltonian, M_{ab} rotation, P_a translation and K_a special conformal transformation.

Let us consider the implication of the requirement (5.23) when the bulk local operator $\hat{\psi}$ is located at the origin of AdS space where $\rho = 0$ on the $t = 0$ slice. The isometry of this point is $SO(1,d)$ which is subgroup of the AdS isomery $SO(2,d)$ and is generated by M_{ab} and $P_a + K_a$. The action by these generator keep the origin of AdS space invariant, thus the bulk local operator $\hat{\psi}$ satisfy

$$[M_{ab}, \hat{\psi}(0)] = 0, \tag{5.25}$$

$$[P_a + K_a, \hat{\psi}(0)] = 0. \tag{5.26}$$

Acting $\hat{\psi}(0)$ on the conformal invariant vacuum $|0\rangle$, we obtain the state $|\psi(0)\rangle \equiv \hat{\psi}(0)|0\rangle$ which satisfy

$$M_{ab}|\psi(0)\rangle = 0, \tag{5.27}$$

$$(P_a + K_a)|\psi(0)\rangle = 0. \tag{5.28}$$

In the following for simplicity we consider three-dimensional global AdS space and construct a bulk local state as Ishibashi state. The metric is given by

$$ds^2 = d\rho^2 - \cosh^2 \rho dt^2 + \sinh^2 \rho d\phi^2 \tag{5.29}$$

where $0 < \rho < \infty$, $-\infty < t < \infty$, $0 < \phi < 2\pi$ and (t, ϕ) are the boundary coordinates of global AdS$_3$. The global AdS$_3$ has an isometry $SO(2,2) = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ which is generated by Virasoro generators $(L_0, L_{\pm 1})$ and $(\bar{L}_0, \bar{L}_{\pm 1})$. Then the condition of bulk local state located at the origin (5.30) is rewritten as follows.

$$\big(L_0 - \bar{L}_0\big)|\psi(0)\rangle = \big(L_{\pm 1} - \bar{L}_{\mp 1}\big)|\psi(0)\rangle = 0 \tag{5.30}$$

The solution of these equations can be constructed by primary states $|\mathcal{O}\rangle = \mathcal{O}|0\rangle$ dual to the bulk local operator $\hat{\psi}$, which satisfy

$$L_1|\mathcal{O}\rangle = \bar{L}_1|\mathcal{O}\rangle = 0, \tag{5.31}$$

$$L_0|\mathcal{O}\rangle = \bar{L}_0|\mathcal{O}\rangle = \frac{\Delta}{2}|\mathcal{O}\rangle. \tag{5.32}$$

It is written as Ishibashi state with respect to $SL(2,\mathbb{R})$ algebra.

$$|\psi(0)\rangle = \sum_{k=0}^{\infty}(-1)^k \frac{\Gamma(\Delta)}{k!\Gamma(k+\Delta)} L_{-1}^k \bar{L}_{-1}^k |\mathcal{O}\rangle \tag{5.33}$$

The bulk local state located at the origin of AdS space is constructed by a liner combination of a primary state and its descendant states. Note that in this method we do not need to solve bulk equations of motion for representing bulk local fields and we only use the properties of conformal field theory. Hence since the bulk local field is reconstructed from conformal field theory, the existance of the dual AdS gravity is expected.

The bulk local states located at the different bulk point are constructed by $SL(2,\mathbb{R})$ transformation generator $g(\rho,t,\phi)$, which is obtained from the following correspondence between Killing vectors in AdS and generators of conformal symmetry, or Virasoro generators.

$$\begin{aligned}
L_0 &= i\partial_{x^+} \\
L_{\pm1} &= ie^{\pm ix^+}\left[\mp\frac{i}{2}\partial_\rho + \frac{\cosh 2\rho}{\sinh 2\rho}\partial_{x^+} - \frac{1}{\sinh 2\rho}\partial_{x^-}\right] \\
\bar{L}_0 &= i\partial_{x^-} \\
\bar{L}_{\pm1} &= ie^{\pm ix^-}\left[\mp\frac{i}{2}\partial_\rho + \frac{\cosh 2\rho}{\sinh 2\rho}\partial_{x^-} - \frac{1}{\sinh 2\rho}\partial_{x^+}\right]
\end{aligned} \tag{5.34}$$

where $x^\pm = t \pm \phi$. The expression for a bulk local states at a bulk point (ρ,t,ϕ) is given by

$$|\psi(\rho,t,\phi)\rangle = e^{i(L_0+\bar{L}_0)t}e^{i(L_0-\bar{L}_0)\phi}e^{-\rho(L_1-L_{-1}+\bar{L}_1-\bar{L}_{-1})/2}|\psi(0)\rangle \equiv g(\rho,t,\phi)|\psi(0)\rangle. \tag{5.35}$$

Here the transformation generator $g(\rho,t,\phi)$ shift the points of bulk local operator and the dual primary operator on the boundary from these origins, thus we can obtain the map between the positions between the bulk and the boundary. This state is an eigenstate of the conformal Casimir operator $C = L_0^2 - (L_1L_{-1} + L_{-1}L_1) + (L \to \bar{L})$ with the eigenvalue $\Delta(\Delta-2)$. Substituting Killing vector (5.34) into the Casimir operator C, we can confirm that C is identical with the d'Alembertian in the global AdS coordinate. Therefore the eigenvalue equation where Casimir operator acts on the bulk local states (5.35) can be interpreted as the Klein-Gordon equation with mass $m^2 = \Delta(\Delta-2)$. The bulk local operator satisfy the K-G equation on the AdS background.

$$\left(\Box + m^2\right)\hat{\psi} = 0 \tag{5.36}$$

We will refer to the definition of the conformal invariant vacuum $|0\rangle$ and the map between points of the bulk local operator and those of the dual primary operator. The vacuum state $|0\rangle$ is defined by the path-integral along the Euclidean time τ without any operator insertions. We Wick-rotate the boundary coordinate $t = -i\tau$ in $t < 0$. Therefore the conditions of bulk local state at the origin (5.30) are located at the equator of the Euclidean S^2 which is related to the $t = 0$ slice of the Lorentzian $\mathbb{R}\times S$ by the Wick-rotation. And according to the map obtained by $SL(2,\mathbb{R})$ transformation generator, the bulk local operator is dual to the primary operator on the Euclidean boundary S^2. We constructed the bulk local state as the summation of primary state and its descendant states in terms of the dual primary operator, which is inserted at the Euclidean boundary point corresponding to the bulk position through the map. If we can obtain the appropriate map, it is possible to systematically reconstruct bulk local states on the other background geometry from primary state on the dual CFT. Goto and Takayanagi proposed to reconstruct the bulk local state in the BTZ black hole background [8].

At last we will confirm the equivalence with the HKLL construction in the Poincaré coordinate. The bulk local states in Poincaré (y, t, x) is given by

$$|\psi(y,t,x)\rangle = \sum_{k=0}^{\infty}(-1)^k y^{\Delta+2k}\frac{\Gamma\Delta}{k!\Gamma(k+\Delta)}(L^h_{-1})^k(\bar{L}^h_{-1})^k\mathcal{O}(t,x)\,|0\rangle \tag{5.37}$$

which is satisfy the following relation by the $SL(2,\mathbb{R})$ transformation

$$|\psi(y,t,x)\rangle = g(y,t,x)\,|\psi(0)\rangle = e^{i(L^h_{-1}+\bar{L}^h_{-1})t}e^{i(L^h_{-1}-\bar{L}^h_{-1})x}y^{L^h_0+\bar{L}^h_0}\,|\psi(0)\rangle \tag{5.38}$$

and the following conditions of the state located at the origin

$$\left(L^h_0 - \bar{L}^h_0\right)|\psi(0)\rangle = \left(L^h_{\pm1} + \bar{L}^h_{\mp1}\right)|\psi(0)\rangle = 0. \tag{5.39}$$

Here L^h is a representation of Virasoro generator in the different basis from L and L^h can be written by the Killing vectors in the Poincaré AdS as follows.

$$\begin{aligned}
L^h_0 &= -\frac{y}{2}\partial_y - x^+\partial_{x^+}\\
L^h_{-1} &= i\partial_{x^+}\\
L^h_{+1} &= -ix^+y\partial_y - i(x^+)^2\partial_{x^+} - iy^2\partial_{x^-}\\
L^h_0 &= -\frac{y}{2}\partial_y - x^+\partial_{x^-}\\
L^h_{-1} &= i\partial_{x^-}\\
L^h_{+1} &= -ix^-y\partial_y - i(x^-)^2\partial_{x^-} - iy^2\partial_{x^+}
\end{aligned} \tag{5.40}$$

In HKLL construction the scalar fields in the Poincaré are expressed as

$$\psi^{\mathrm{HKLL}}(y,t,x) = \frac{\Delta-1}{\pi}\int_{t'^2+x'^2\le y^2}dt'dx'\left(\frac{y^2-t'^2-x'^2}{y}\right)^{\Delta-2}\mathcal{O}(t+t',x+ix'). \tag{5.41}$$

Expanding in Taylor series of a primary operator, this expression becomes

$$\begin{aligned}
\psi^{\mathrm{HKLL}}(y,t,x) &= \frac{\Delta-1}{\pi}\int_{t'^2+x'^2\le y^2}dt'dx'\left(\frac{y^2-t'^2-x'^2}{y}\right)^{\Delta-2}\sum_{m,n}\frac{1}{m!n!}(x'^+)^m(x'^-)^n\partial^m_{x^+}\partial^n_{x^-}\mathcal{O}(x^+,x^-)\\
&= \sum_{k=0}^{\infty}y^{\Delta+2k}\frac{\Gamma(\Delta)}{k!\Gamma(k+\Delta)}\partial^k_{x^+}\partial^k_{x^-}\mathcal{O}(x^+,x^-)
\end{aligned} \tag{5.42}$$

where $x^\pm = t \pm x$. Since $\partial^k_{x^+}\partial^k_{x^-} = -(L^h_{-1})^k(\bar{L}^h_{-1})^k$, we can conclude the equivalence

$$\psi^{\mathrm{HKLL}}(y,t,x)\,|0\rangle = |\psi(y,t,x)\rangle\,. \tag{5.43}$$

5.3 Bulk Scalar Field and Correlator in BTZ Black Hole Background

The reconstruction of bulk local operator in the BTZ black hole was discussed in the method which we reviewed in the previous sections [8][11]. In this section we will review the construction of bulk local state in the BTZ black hole, and also the two-point correlation functions.

The simple form of correlation function in the global AdS space in three-dimension was proposed by Ichinose and Satoh [26].

$$G_{\mathrm{AdS}}(\boldsymbol{x}|\boldsymbol{x}') = \frac{1}{2\sqrt{\sigma^2-1}(\sigma+\sqrt{\sigma^2-1})^{\Delta-1}} \tag{5.44}$$

where $\boldsymbol{x} = (\rho, t, \phi)$ and $\sigma = \sigma(\boldsymbol{x}|\boldsymbol{x}')$ is the geodesic distance in the global AdS space in three-dimension. They quantized the scalar field in the global AdS space and obtained this correlation function by the solution of the Klein-Gordon equation $(\Box_{\mathrm{AdS}_3} + m^2)\Psi = 0$ for a free massive scalar Ψ. And also they obtained the correlation function in the BTZ black hole by replacing σ for the geodesic distance of AdS-Rindler coordinates $\sigma_{\mathrm{Rindler}}$ and using the method of mirror image as follows.

$$G_{\mathrm{BTZ}}(\boldsymbol{x}|\boldsymbol{x}') = \sum_n G_{\mathrm{Rindler}}(\rho, t, \phi + 2\pi n|\rho', t', \phi') \tag{5.45}$$

where we impose the identification $\phi \sim \phi + 2\pi L$ on the AdS-Rindler coordinates in order to obtain the BTZ black hole coordinates.

In the construction of bulk local state in the BTZ black hole is also based on the locally equivalence between the AdS-Rindler metric and the BTZ black hole metric. We will start to construct bulk local states in the AdS-Rindler space. Since the Rindler coordinates is related to the global AdS coordinate through analytic continuations, we can obtain the bulk local states in the AdS-Rindler space from those in the global AdS space by replacements of the bulk coordinates and the $SL(2,\mathbb{R})$ generators. And then we will translate the bulk local states in the AdS-Rindler space to those in the BTZ black hole and obtain the two-point correlation function in the BTZ which is consistent with the result of other works [21][26].

AdS-Rindler Coordinates

Rindler coordinate consists of four wedges, right, left, future and past. The metric of the right wedge is given by

$$\begin{aligned} ds^2 &= R^2(d\rho_r^2 - \sinh^2\rho_r dt_r^2 + \cosh^2\rho_r d\phi_r^2) \\ &= \frac{R^2}{r^2 - R^2}dr^2 - (r^2 - R^2)dt_r^2 + r^2 d\phi_r^2 \end{aligned} \tag{5.46}$$

where $r = R\cosh\rho_r$ and (ρ_r, t_r, ϕ_r) are the coordinates of the right wedge, which are take values in $\rho_r > 0 (r > R), -\infty < t_r < \infty, -\infty < \phi_r < \infty$. Note that this right Rindler metric is the same as the global AdS metric with (5.29) with a replacement of the coordinates, $t \to -i\phi_r$ and $\phi \to -it_r$. This left wedge is related to the right wedge by the analytic continuation of the left time coordinate $t_l = -t_r + i\pi$. And the metric of the future wedge is given by

$$\begin{aligned} ds^2 &= R^2(-d\rho_f^2 + \sin^2\rho_f dt_f^2 + \cos^2\rho_f d\phi_f^2) \\ &= -\frac{R^2}{R^2 - r^2}dr^2 + (R^2 - r^2)dt_f^2 + r^2 d\phi_f^2 \end{aligned} \tag{5.47}$$

where $r = R\cos\rho_f$ and (t_f, ρ_f, ϕ_f) are the coordinates of the future wedge, which are take values in $\rho_f > 0 (r < R), -\infty < t_f < \infty, -\infty < \phi_f < \infty$. This future wedge is related to the right wedge by the analytic continuation $t_f = t_r + i\pi/2$.

Thermo-Field Double and $SL(2,\mathbb{R})$ generators

In the Rindler coordinate, the vacuum CFT state on right wedge boundary is defined by the Euclidean path-integrals over $-\pi < \tau_r (= it_r) < 0$ and $-\infty < \phi_r < \infty$. And the Hilbert space of the CFT on $\tau_r = 0$ is decomposed as the direct product of the Hilbert space of the left CFT $\mathcal{H}_l$ and the right CFT $\mathcal{H}_r$.

$$\mathcal{H}_{\mathrm{CFT}} \simeq \mathcal{H}_l \otimes \mathcal{H}_r \tag{5.48}$$

Thus the 'vacuum' state is expressed as a thermofield double state.

$$|\Psi_{\mathrm{TFD}}\rangle = \sum_E e^{-\beta E/2} |E\rangle_l \otimes |E\rangle_r \tag{5.49}$$

where $|E\rangle_{l,r}$ represent energy eigenstates of each CFT. Note that this has a $SL(2,\mathbb{R})$ symmetry as follows.

$$L_n^R \left|\Psi_{\mathrm{TFD}}\right\rangle = \bar{L}_n^R \left|\Psi_{\mathrm{TFD}}\right\rangle = 0 \Leftrightarrow \left(L_n^r - (-1)^n L_{-n}^l\right) \left|\Psi_{\mathrm{TFD}}\right\rangle = \left(\bar{L}_n^r - (-1)^n \bar{L}_{-n}^l\right) \left|\Psi_{\mathrm{TFD}}\right\rangle = 0 \qquad (5.50)$$

where $n = 0, \pm 1$ and the $SL(2,\mathbb{R})$ generators are expressed as differential operators in the right and left wedge.

$$\begin{aligned}
L_0^{r,l} &= -\partial_{x^+} \\
L_{\pm 1}^{r,l} &= -\frac{\sqrt{r^2-R^2}}{2r} e^{\pm x^+} \left[\mp r\partial_r + \frac{2r^2-R^2}{r^2-R^2}\partial_{x^+} - \frac{R^2}{r^2-R^2}\partial_{x^-}\right] \\
\bar{L}_0^{r,l} &= -\partial_{x^-} \\
\bar{L}_{\pm 1}^{r,l} &= -\frac{\sqrt{r^2-R^2}}{2r} e^{\pm x^-} \left[\mp r\partial_r + \frac{2r^2-R^2}{r^2-R^2}\partial_{x^-} - \frac{R^2}{r^2-R^2}\partial_{x^+}\right]
\end{aligned} \qquad (5.51)$$

Noto that the $SL(2,\mathbb{R})$ symmetry is not globally defined because of the existence of the horizon, while this symmetry is recovered on the Hilbert space which is the product of the left and right CFTs.

Bulk Local States in AdS-Rindler Space

Then the bulk local state in the AdS-Rindler space is given by

$$\left|\psi(0)\right\rangle_{\mathrm{Rindler}} = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\Delta)}{k!\Gamma(k+\Delta)} (L_{-1}^R)^k (\bar{L}_{-1}^R)^k \mathcal{O}(0, -\pi/2) \left|\Psi_{\mathrm{TFD}}\right\rangle . \qquad (5.52)$$

This state satisfy the condition at the origin of the AdS space

$$\left(L_0^R - \bar{L}_0^R\right) \left|\psi(0)\right\rangle_{\mathrm{Rindler}} = \left(L_{\pm 1}^R + \bar{L}_{\mp 1}^R\right) \left|\psi(0)\right\rangle_{\mathrm{Rindler}} = 0. \qquad (5.53)$$

The $SL(2,\mathbb{R})$ transformation generators which move the bulk point from the origin of the AdS space are given by

$$\begin{aligned}
\left|\psi(\rho_r, t_r, \phi_r)\right\rangle_{\mathrm{Rindler}} &= g(\rho_r, t_r, \phi_r) \left|\psi(0)\right\rangle_{\mathrm{Rindler}} \\
&= e^{(L_0^R - \bar{L}_0^R)t_r} e^{(L_0^R + \bar{L}_0^R)\phi_r} e^{-\rho_r (L_1^R - L_{-1}^R + \bar{L}_1^R - \bar{L}_{-1}^R)/2} \left|\psi(0)\right\rangle_{\mathrm{Rindler}} .
\end{aligned} \qquad (5.54)$$

Note that this expression is obtained by replacements of the $SL(2,\mathbb{R})$ generators $L \to L^R, \bar{L} \to \bar{L}^R$ and of the coordinates $t \to -i\phi_r, \phi \to -it_r$.

Correlation Functions in AdS-Rindler and in BTZ Black Hole

The two-point functions of the bulk local states in the global AdS space (5.35) become bulk-to-bulk propagators in the global coordinate [34]

$$\begin{aligned}
\langle \psi(\rho, t, \phi)\psi(\rho', t', \phi')\rangle &= \left\langle \psi(0)\right| g^{-1}(\rho, t, \phi) g(\rho', t', \phi') \left|\psi(0)\right\rangle \\
&= \frac{1}{2\sqrt{\sigma^2-1}(\sigma+\sqrt{\sigma^2-1})^{\Delta-1}}
\end{aligned} \qquad (5.55)$$

where

$$\sigma(\boldsymbol{x}|\boldsymbol{x}') = \cosh\rho\cosh\rho'\cos(t-t') - \sinh\rho\sinh\rho'\cos(\phi-\phi'). \qquad (5.56)$$

This result is exactly equivalent with (5.44). On the other hand, the two-point functions of the bulk local states in the right wedge in the AdS-Rindler space (5.54) is expressed as follows.

$$\begin{aligned}\langle\Psi_{\mathrm{TFD}}|\,\psi(\rho_r,t_r,\phi_r)\psi(\rho'_r,t'_r,\phi'_r)\,|\Psi_{\mathrm{TFD}}\rangle &= {}_{\mathrm{Rindler}}\,\langle\psi(0)|\,g^{-1}(\rho_r,t_r,\phi_r)g(\rho'_r,t'_r,\phi'_r)\,|\psi(0)\rangle_{\mathrm{Rindler}}\\ &= \langle\psi(0)|\,g^{-1}(\rho_r,-i\phi_r,-it_r)g(\rho'_r,-i\phi'_r,-it'_r)\,|\psi(0)\rangle \end{aligned} \tag{5.57}$$

where the second line is the expression in the global coordinate because the bulk local states in each patch are related by the replacements of the $SL(2,\mathbb{R})$ generators and of the coordinates $t\to -i\phi_r, \phi\to -it_r$. Therefore we can obtain the two-point function in the AdS-Rindler coordinate

$$G_{\mathrm{Rindler}}(\rho,t,\phi|\rho',t',\phi') = \frac{1}{2\sqrt{\sigma^2-1}(\sigma+\sqrt{\sigma^2-1})^{\Delta-1}} \tag{5.58}$$

where

$$\sigma(\boldsymbol{x}_r|\boldsymbol{x}'_r) = \cosh\rho_r\cosh\rho'_r\cosh(\phi_r-\phi'_r) - \sinh\rho_r\sinh\rho'_r\cosh(t_r-t'_r). \tag{5.59}$$

Note that $\sigma=\sigma(\boldsymbol{x}_r|\boldsymbol{x}'_r)$ is an geodesic distance in the right wedge in the AdS-Rindler coordinates which is obtained from one in the global AdS space by the same replacements $t\to -i\phi_r, \phi\to -it_r$. This result is consistent with the works in [21] and [26].

As proposed in [21], by analytic continuation of the time direction it is possible to obtain two-point functions between points inside and outside the horizon from two-point functions for points outside the horizon. The coordinates in the future wedge is related to those in the right wedge by the analytic continuation $t_f = t_r + i\pi/2$ and $r = R\cosh\rho_r = R\cos\rho_f$, which is shown in (5.47). Therefore the geodesic distance between points in the right wedge and in the future wedge is given by

$$\sigma(\boldsymbol{x}_r|\boldsymbol{x}'_f) = \cosh\rho_r\cos\rho'_f\cosh(\phi_r-\phi'_f) - \sinh\rho_r\sin\rho'_f\sinh(t_r-t'_f). \tag{5.60}$$

Substituting this geodesic distance into (5.58) as $\sigma=\sigma(\boldsymbol{x}_r|\boldsymbol{x}'_f)$, we can obtain two-point function between points in the right wedge and in the future wedge (Fig. 3).

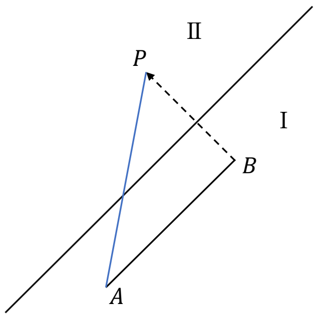

Fig. 3 Two-point function between points A and P is obtained from two-point function between points A and B. The point B can enter from the right wedge to the future wedge by analytic continuation $t_f = t_r + i\pi/2$.

In the same way, as examples the geodesic distances are given by

$$\sigma(\boldsymbol{x}_f|\boldsymbol{x}'_f) = \cos\rho_r\cos\rho'_f\cosh(\phi_f-\phi'_f) - \sin\rho_r\sin\rho'_f\cosh(t_f-t'_f), \tag{5.61}$$

$$\sigma(\boldsymbol{x}_l|\boldsymbol{x}'_r) = \cosh\rho_l\cosh\rho'_r\cosh(\phi_l-\phi'_r) + \sinh\rho_l\sinh\rho'_r\cosh(t_l-t'_r) \tag{5.62}$$

where we use the relation between the right and left wedges as $t_l = t_r + i\pi$ and $r = R\cosh\rho_r = R\cosh\rho_l$. We can obtain two-point functions in terms of these geodesic distances. Then the two-point function in the BTZ black hole is constructed by the method of images. The expression is the same as (5.45).

$$G_{\mathrm{BTZ}}(\boldsymbol{x}|\boldsymbol{x}') = \sum_n G_{\mathrm{Rindler}}(\boldsymbol{x}_n|\boldsymbol{x}')$$
$$= \sum_n \frac{1}{2\sqrt{\sigma_n^2 - 1}(\sigma_n + \sqrt{\sigma_n^2 - 1})^{\Delta - 1}} \tag{5.63}$$

where

$$\boldsymbol{x}_n \equiv \boldsymbol{x}|_{\phi \to \phi + 2\pi n}, \quad \sigma_n(\boldsymbol{x}|\boldsymbol{x}') \equiv \sigma(\boldsymbol{x}_n|\boldsymbol{x}'). \tag{5.64}$$

As an example, the geodesic distance between points in the right wedge and in the future wedge in the BTZ black hole is given by

$$\sigma_n(\boldsymbol{x}_r|\boldsymbol{x}'_f) = \cosh\rho_r \cos\rho'_f \cosh(\phi_r - \phi'_f + 2\pi n) - \sinh\rho_r \sin\rho'_f \sinh(t_r - t'_f). \tag{5.65}$$

Bulk Reconstruction inside the Horizon and the Canonical Quantization

As we have seen so far, other groups showed that the bulk local states inside the horizon, or in future wedge, and the two-point functions for points on both sides of the horizon are obtained by analytic continuation. The two-point functions between points inside and outside the horizon were constructed from the two-point functions for both points outside the horizons by the shift of time coordinate t. And these two-point functions were expressed in terms of the geodesic distance in the AdS-Rindler space. However, these results have not been obtained by the method of canonical quantization on quantum field theory in the AdS black hole background. Since the roles of the time and radial coordinates outside the horizon are interchanged inside the horizon, it is not clear that the two-point functions obtained by analytic continuation can also be obtained as the results of the canonical quantization. Ichinose and Satoh have carried out the quantization of the scalar field in the global AdS background. Therefore two-point functions in the global AdS space can be obtained as the results of the canonical quantization. However, the quantization is implemented on the constant time slice in the global AdS space which is different with that in the AdS-Rindler space. In order to confirm whether the bulk reconstruction by analytic continuation in the AdS black hole background is correct or not, it is necessary to implement the quantization in the AdS-Rindler coordinates and compare the result by the quantization with that by analytic continuation.

6 Papadodimas and Raju Proposal to Construct a Scalar Field

In this section we review a prescription for quantization of matter fields inside the horizon of an eternal black hole, which was proposed by Papadodimas and Raju [9][12][13]. Let us consider two-side black hole which consists of four regions I, II, III and IV. In order to describe the region inside the horizon as information outside the horizon, they proposed the matching conditions which are required to obtain a smooth exterior and interior geometry for a eternal black hole in AdS/CFT. First we focus on the horizon between the region I and II. According to their analysis in order to quantize a scalar field on both sides of horizon, while the Schwartzchild left movers cross the horizon between the region I and II smoothly, the Schwartzchild right movers do not. Their main claim is that to obtain a smooth interior we need another operator, which play the role of right movers behind the horizon and are entangled with the operator outside the horizon. In eternal AdS black hole these right movers originate from a left asymptotic region, or the region III. The scalar field in the region III is expanded in terms of the operators in the region I and III.

Their prescription works in Fourier space, as opposed to the position space prescriptions of HKLL construction reviewed in the previous section. We will start by considering empty AdS$_{d+1}$ space and free field $\mathcal{O}(t, \boldsymbol{x})$ in the boundary CFT. The Fourier modes of $\mathcal{O}(t, \boldsymbol{x})$ are defined as usual

$$\mathcal{O}_{\omega,\boldsymbol{k}} = \int dt d^{d-1}\boldsymbol{x} \mathcal{O}(t, \boldsymbol{x}) e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{6.1}$$

Here $\boldsymbol{k}$ and $\boldsymbol{x}$ have the $(d-1)$-dimensional spatial components which are correspond to the spacial variable in the d-dimensional boundary CFT.

6.1 Geometry of Eternal AdS Black Hole

The metric of the eternal AdS_{d+1} black hole is given by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_{d-1}^2 \tag{6.2}$$

where

$$f(r) = r^2 + 1 - \frac{16\pi GM}{(d-1)\Omega_{d-1} r^{d-2}}. \tag{6.3}$$

Here Ω_{d-1} is the volume of a unit $(d-1)$-sphere and we have set the radius of AdS to 1. The horizon is at $r = r_0$ which is defined by $f(r_0) = 0$ and the boundary is at $r = \infty$. It is convenient to introduce tortoise coordinate r_* by $dr_*/dr = -f^{-1}(r)$ where the horizon is at $r_* \to -\infty$ and the boundary is at $r_* \to 0$.

By introducing the Kruskal coordinates (U, V) it will become clear that four regions I, II, III and IV are smoothly connected. In the region I the metric in the Kruskal coordinete is defined by

$$ds^2 = \left(\frac{\beta}{2\pi}\right)^2 \frac{f(r)}{UV} dUdV + r^2 \Omega_{d-1}^2 \tag{6.4}$$

where

$$U = -e^{2\pi(r_* - t)/\beta}, \quad V = e^{2\pi(r_* + t)/\beta}. \tag{6.5}$$

By coordinate transformation (6.5) from the metric of the eternal AdS_{d+1} black hole (6.2) the metric in the Kruskal coordinate (6.4) is obtained. Here the variables U and V takes values in $U < 0, V > 0$ in the region I. The future horizon is at $U \to 0, V =$ constant, and the past horizon $V \to 0, U =$ constant. On the other hand in the region III the Kruskal coordinate is defined by

$$U = e^{2\pi(r_* - t)/\beta}, \quad V = -e^{2\pi(r_* + t)/\beta} \tag{6.6}$$

where the metric is the same as (6.4) and the variables U and V takes values in $U > 0, V < 0$. In the metric (6.4) it is clear that these horizons are smooth.

6.2 Scalar Field outside and inside Horizon and Matching Condition

In order to show the quantization of a scalar field on the eternal AdS_{d+1} black hole background we will consider the union of two smaller slices, the Cauchy slices in the region I and III.

For the reconstruction of bulk local observables in the region I it is important to identify the modes of the scalar field in the region I with Fourier modes of the boundary operator.The bulk local operator in the region I is constructed as follows.

$$\phi^{\mathrm{I}}_{\mathrm{CFT}}(r, t, \boldsymbol{x}) = \int_{\omega > 0} \frac{d\omega d^{d-1}\boldsymbol{k}}{(2\pi)^d} \left[\mathcal{O}_{\omega,\boldsymbol{k}} f_{\omega,\boldsymbol{k}}(r, t, \boldsymbol{x}) + \mathcal{O}^\dagger_{\omega,\boldsymbol{k}} f^*_{\omega,\boldsymbol{k}}(r, t, \boldsymbol{x})\right] \tag{6.7}$$

where $f_{\omega,\boldsymbol{k}}(r, t, \boldsymbol{x}) = e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} \psi_{\omega,\boldsymbol{k}}(r)$ which are the mode solutions of K-G equation, and $f_{\omega,\boldsymbol{k}}(r, t, \boldsymbol{x})$ behave as r^Δ at $r \to 0$. It is also possible to normalize the modes so that they are canonical with respect to the Klein-Gordon norm. The normalized modes satisfy

$$\hat{f}_{\omega,\boldsymbol{k}}(r, t, \boldsymbol{x}) = e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\psi}_{\omega,\boldsymbol{k}}(r) \tag{6.8}$$

where

$$\hat{\psi}_{\omega,\boldsymbol{k}}(r) \xrightarrow{r \to r_0} c_h \left(e^{-i\delta_{\omega,\boldsymbol{k}}} e^{-i\omega r_*} + e^{i\delta_{\omega,\boldsymbol{k}}} e^{i\omega r_*}\right) \tag{6.9}$$

and also $\hat{f}_{\omega,\boldsymbol{k}}(r,t,\boldsymbol{x})$ are normalized so that $\hat{\psi}_{\omega,\boldsymbol{k}} \sim r^{\Delta}$ as $r \to 0$. The overall factor c_h is the near horizon normalization, which is automatically fixed by requiring the modes (6.8) to satisfy the canonical K-G norm. The mode $e^{-i\omega(t+r_*)}$ is correspond to left mover, while the mode $e^{-i\omega(t-r_*)}$ right mover. Hence the normalized mode $\hat{f}_{\omega,\boldsymbol{k}}(r,t,\boldsymbol{x})$ near the horizon is expressed by a liner combination of left and right movers. Then the quantized scalar field can be expanded in terms of normalized modes $\hat{f}_{\omega,\boldsymbol{k}}$ as follows.

$$\phi^{\mathrm{I}}_{\mathrm{CFT}}(r,t,\boldsymbol{x}) = \int_{\omega>0} \frac{d\omega d^{d-1}\boldsymbol{k}}{(2\pi)^d} \frac{1}{\sqrt{2\omega}} \left[a_{\omega,\boldsymbol{k}} \hat{f}_{\omega,\boldsymbol{k}}(r,t,\boldsymbol{x}) + a^{\dagger}_{\omega,\boldsymbol{k}} \hat{f}^{*}_{\omega,\boldsymbol{k}}(r,t,\boldsymbol{x}) \right] \tag{6.10}$$

Note that because the normalization factor relating $\mathcal{O}_{\omega,\boldsymbol{k}}$ to $a_{\omega,\boldsymbol{k}}$ is precisely the inverse of that relating $f_{\omega,\boldsymbol{k}}$ to $\hat{f}_{\omega,\boldsymbol{k}}$, the two expressions (6.7) and (6.10) are consistent. Hence $a_{\omega,\boldsymbol{k}}$ is the boundary operator in region I. The normalized operators $a_{\omega,\boldsymbol{k}}$ satisfy the standard commutation relations

$$[a_{\omega,\boldsymbol{k}}, a^{\dagger}_{\omega',\boldsymbol{k}'}] = \delta(\omega-\omega')\delta^{d-1}(\boldsymbol{k}-\boldsymbol{k}'). \tag{6.11}$$

In the region III we have similar expression of a scalar field as follows.

$$\phi^{\mathrm{III}}_{\mathrm{CFT}}(r,t,\boldsymbol{x}) = \int_{\omega>0} \frac{d\omega d^{d-1}\boldsymbol{k}}{(2\pi)^d} \frac{1}{\sqrt{2\omega}} \left[\tilde{a}^{\dagger}_{\omega,\boldsymbol{k}} \hat{f}_{\omega,\boldsymbol{k}}(r,t,\boldsymbol{x}) + \tilde{a}_{\omega,\boldsymbol{k}} \hat{f}^{*}_{\omega,\boldsymbol{k}}(r,t,\boldsymbol{x}) \right] \tag{6.12}$$

Note that the scalar field is expanded by $\tilde{a}_{\omega,\boldsymbol{k}}$ which are the boundary operators in the region III, and these operators satisfy the standard commutation relation as (6.11). And while we parametrize the points in the region III by the same set of coordinates $(r,t,\boldsymbol{x})$, they are distinct points from those in the region I because of (6.5) and (6.6).

We now turn to the expansion of a scalar field behind the horizon. We focus on the region II which is parametrized by $(r,t,\boldsymbol{x})$ with $r_0 < r < \infty$ and $-\infty < t < \infty$. In these coordinate the singularity is at $r \to \infty$. By imposing the smooth continuity on the scalar field at the horizon between I and II and also between III and II, the scalar field in the region II is expanded as follows.

$$\phi^{\mathrm{II}}_{\mathrm{CFT}}(r,t,\boldsymbol{x}) = \int_{\omega>0} \frac{d\omega d^{d-1}\boldsymbol{k}}{(2\pi)^d} \frac{1}{\sqrt{2\omega}} \left[a_{\omega,\boldsymbol{k}} \hat{g}^{(1)}_{\omega,\boldsymbol{k}}(r,t,\boldsymbol{x}) + \tilde{a}^{\dagger}_{\omega,\boldsymbol{k}} \hat{g}^{(2)}_{\omega,\boldsymbol{k}}(r,t,\boldsymbol{x}) + h.c. \right] \tag{6.13}$$

where

$$\hat{g}^{(1,2)}_{\omega,\boldsymbol{k}}(r,t,\boldsymbol{x}) = e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\chi}^{(1,2)}_{\omega,\boldsymbol{k}}(r). \tag{6.14}$$

Here the modes (6.14) are solutions of the K-G equation and satisfy the matching conditions which impose the smooth continuity on the scalar field at the horizon. Note that the scalar field in the region II is expanded by the boundary CFT operators in the region I and III. This mean that the modes in the region II are constructed in the region I and III and come from these regions. The scalar field in the region II is defined by two CFTs. It is known that an eternal AdS black hole can be described by two CFTs as fallows.

$$|\psi\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n/2} |n\rangle_L \otimes |n\rangle_R \tag{6.15}$$

The tensor product state (6.15) is called Thermo-Field Double (TFD) state [22][23]. This fact is consistent with the expression of the scalar field (6.16), and then the whole region in an eternal AdS black hole is described by TFD state.

We will expand the scalar field in the region II near the horizon as follows.

$$\begin{aligned} &\phi^{\mathrm{II}}_{\mathrm{CFT}}(r,t,\boldsymbol{x}) \\ &\xrightarrow{r\to r_0} \int_{\omega>0} \frac{d\omega d^{d-1}\boldsymbol{k}}{(2\pi)^d} \frac{1}{\sqrt{2\omega}} \left[a_{\omega,\boldsymbol{k}} e^{-i\delta_{\omega,\boldsymbol{k}}} e^{-i\omega(t+r_*)} c_h e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \tilde{a}_{\omega,\boldsymbol{k}} e^{-i\delta_{\omega,\boldsymbol{k}}} e^{i\omega(t-r_*)} c_h^{*} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} + h.c. \right] \end{aligned} \tag{6.16}$$

Notice the following points.

1. The mode in the region I corresponding to the left mover $e^{-i\omega(t+r_*)} = V^{i\beta\omega/(2\pi)}$ cross the horizon. Because of the continuity of this mode $e^{-i\omega(t+r_*)}$, the operator $a_{\omega,\boldsymbol{k}}$ in the region II must be the same as the operator in the region I.
2. We introduce another operator $\tilde{a}_{\omega,\boldsymbol{k}}$ multiplying the right mover mode $e^{i\omega(t-r_*)}$ in the region II. This right mover mode $e^{i\omega(t-r_*)}$ is origin from the region III, the mode in the region III $e^{i\omega(t-r_*)} = U^{i\beta\omega/(2\pi)}$ cross the horizon between the region III and II. Hence the operator $\tilde{a}_{\omega,\boldsymbol{k}}$ in the region II must be the same as the operator in the region III.
3. Note that the time coordinate in the region II is r_*, and the operator multiplying $e^{i\omega(t-r_*)}$ is 'annihilation' operator. This is because this mode function has positive frequency with respect to t, while it has negative frequency with respect to r_*. The scalar field is expanded by the mode functions whose number is double that in the region I (or III). This was to be expected since the region II is causally connected with the both region I and III and the information of the both region I and III enters the region II.

As mentioned above the left and right movers are respectively origin from the region I and III, and satisfy the matching conditions at each crossing horizons. We now summarize the matching conditions in the Kruskal-Szeckeres coordinates U and V, which was introduced in Section. Each regions is defined by as follows.

$$U_{\mathrm{I}} = -e^{2\pi(r_*-t)/\beta}, \quad V_{\mathrm{I}} = e^{2\pi(r_*+t)/\beta} \tag{6.17}$$

$$U_{\mathrm{II}} = e^{2\pi(r_*-t)/\beta}, \quad V_{\mathrm{II}} = e^{2\pi(r_*+t)/\beta} \tag{6.18}$$

$$U_{\mathrm{III}} = e^{2\pi(r_*-t)/\beta}, \quad V_{\mathrm{III}} = -e^{2\pi(r_*+t)/\beta} \tag{6.19}$$

$$U_{\mathrm{IV}} = -e^{2\pi(r_*-t)/\beta}, \quad V_{\mathrm{IV}} = -e^{2\pi(r_*+t)/\beta} \tag{6.20}$$

Near the horizon between the region I and II the mode functions in the region I is expressed as (6.8) and (6.9), and also expressed in terms of Kruskal coordinates as follows.

$$\hat{f}_{\omega,\boldsymbol{k}}(r,t,\boldsymbol{x}) \xrightarrow{r\to r_0} c_h e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \left(e^{-i\delta_{\omega,\boldsymbol{k}}} V_{\mathrm{I}}^{-i\omega\beta/(2\pi)} + e^{i\delta_{\omega,\boldsymbol{k}}} (-U_{\mathrm{I}})^{i\omega\beta/(2\pi)} \right) \tag{6.21}$$

Here the mode crossing the horizon ($U \to 0$ and $V = \mathrm{const}$) is the first term with $V_{\mathrm{I}}^{-i\omega\beta/(2\pi)}$. Therefore the matching condition of (6.16) at the horizon between the region I and II is given by

$$\hat{g}^{(1)}_{\omega,\boldsymbol{k}}(r,t,\boldsymbol{x}) \xrightarrow[U\to 0]{r\to r_0} \hat{f}_{\omega,\boldsymbol{k}}(r,t,\boldsymbol{x})\big|_{U\to 0} \approx c_h e^{i\boldsymbol{k}\cdot\boldsymbol{x}} e^{-i\delta_{\omega,\boldsymbol{k}}} V_{\mathrm{I}}^{-i\omega\beta/(2\pi)} \tag{6.22}$$

where the term with $(-U_{\mathrm{I}})^{i\omega\beta/(2\pi)}$ is discarded due to the Riemann-Lebesgue lemma. Similarly the matching condition of (6.16) at the horizon between the region III and II ($V \to 0$ and $U = \mathrm{const}$) is given by

$$\hat{g}^{(2)}_{\omega,\boldsymbol{k}}(r,t,\boldsymbol{x}) \xrightarrow[V\to 0]{r\to r_0} \hat{f}_{\omega,\boldsymbol{k}}(r,t,\boldsymbol{x})\big|_{V\to 0} \approx c_h e^{i\boldsymbol{k}\cdot\boldsymbol{x}} e^{i\delta_{\omega,\boldsymbol{k}}} U_{\mathrm{III}}^{i\omega\beta/(2\pi)}. \tag{6.23}$$

Note that non-vanishing term is $U_{\mathrm{III}}^{i\omega\beta/(2\pi)}$.

These matching condition (6.22) and (6.23) guarantee the scalar fields at the horizon to be smooth. The construction of a scalar field inside the horizon by Papadodimas and Raju is regular as we cross the horizon. The scalar field inside the horizon is constructed by the same operators in the region I and III. It is expected that we obtain the description inside the black hole in terms of CFT's information.

6.3 Short Distance Behavior of Correlation Function

We expect that two-point scalar function is given by

$$\langle \phi(x_1)\phi(x_2) \rangle \approx \frac{1}{[g^{\mu\nu}(x_1-x_2)_\mu (x_1-x_2)_\nu]^{(d-1)/2}} \tag{6.24}$$

where the geodesic distance l_{12} between x_1 and x_2 is small in comparison to the scale of curvature $l_{12} \ll \beta^{-1}$.

Now we consider the correlation function for differentiated scalar fields in the Kruskal coordinate, or in the U-V plane.

$$\langle\partial_{V_1}\phi(U_1,V_1,\Omega_1)\partial_{V_2}\phi(U_2,V_2,\Omega_2)\rangle = \partial_{V_1}\partial_{V_2}\frac{1}{(-\kappa(U_1-U_2)(V_1-V_2)+\Omega_{12}^2)^{(d-1)/2}} \tag{6.25}$$

where κ is a constant and Ω_{12}^2 is defined as the distance between the points Ω_1 and Ω_2. Taking $(U_1-U_2)\to 0$, the right hand side is evaluated as delta function in the coordinates Ω_1 and Ω_2.

$$\lim_{U_1-U_2\to 0}\langle\partial_{V_1}\phi(U_1,V_1,\Omega_1)\partial_{V_2}\phi(U_2,V_2,\Omega_2)\rangle = \kappa_N\frac{1}{(V_1-V_2)^2}\delta^{d-1}(\Omega_1-\Omega_2) \tag{6.26}$$

where κ_N is a normalization constant. We also have

$$\lim_{V_1-V_2\to 0}\langle\partial_{U_1}\phi(U_1,V_1,\Omega_1)\partial_{U_2}\phi(U_2,V_2,\Omega_2)\rangle = \kappa_N\frac{1}{(U_1-U_2)^2}\delta^{d-1}(\Omega_1-\Omega_2). \tag{6.27}$$

Next we confirm what this result implies for the correlation functions of operators in the eternal AdS$_{d+1}$ black hole. Consider again the region near the horizon of a black hole. We have the expansion of the scalar field outside the horizon in tortoise coordinate as

$$\begin{aligned}\phi(r_*,t,\boldsymbol{x}) &\xrightarrow[U\to 0]{} \int_{\omega>0}\frac{d\omega d^{d-1}\boldsymbol{k}}{(2\pi)^d}\frac{1}{\sqrt{2\omega}}\left[a_{\omega,\boldsymbol{k}}e^{-i\omega t+i\boldsymbol{k}\cdot\boldsymbol{x}}c_h(e^{-i\delta_{\omega,\boldsymbol{k}}}e^{-i\omega r_*}+e^{i\delta_{\omega,\boldsymbol{k}}}e^{i\omega r_*})+h.c.\right]\\ &=\int_{\omega>0}\frac{d\omega d^{d-1}\boldsymbol{k}}{(2\pi)^d}\frac{1}{\sqrt{2\omega}}\left[a_{\omega,\boldsymbol{k}}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}c_h\left\{e^{-i\delta_{\omega,\boldsymbol{k}}}(-U)^{i\omega\beta/(2\pi)}+e^{i\delta_{\omega,\boldsymbol{k}}}(V)^{-i\omega\beta/(2\pi)}\right\}+h.c.\right].\end{aligned} \tag{6.28}$$

Then the two-point function with both points outside the horizon is given by

$$\begin{aligned}&\langle\partial_{U_1}\phi(U_1,V_1,\Omega_1)\partial_{U_2}\phi(U_2,V_2,\Omega_2)\rangle\\ &=\frac{\beta^2c_h^2}{8\pi^2U_1U_2}\int_{\omega>0}\frac{d\omega d^{d-1}\boldsymbol{k}}{(2\pi)^{2d}}\omega e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}\left[(N_{\omega,\boldsymbol{k}}+1)\left(\frac{U_1}{U_2}\right)^{i\omega\beta/(2\pi)}+N_{\omega,\boldsymbol{k}}\left(\frac{U_2}{U_1}\right)^{i\omega\beta/(2\pi)}+h.c.\right]\end{aligned} \tag{6.29}$$

where we have defined the two point expectation value

$$\langle a_{\omega,\boldsymbol{k}}^\dagger,a_{\omega',\boldsymbol{k}'}\rangle = N_{\omega,\boldsymbol{k}}\delta(\omega-\omega')\delta(\boldsymbol{k}-\boldsymbol{k}'). \tag{6.30}$$

Note that this correlation function converge although there is the prefactor $1/(U_1U_2)$. Now we must have

$$N_{\omega,\boldsymbol{k}} = \frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}. \tag{6.31}$$

The integral can be evaluated as

$$\int_0^\infty d\omega\omega\left[\frac{1}{1-e^{-\beta\omega}}\left(\frac{U_1}{U_2}\right)^{i\omega\beta/(2\pi)}+\frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}\left(\frac{U_2}{U_1}\right)^{i\omega\beta/(2\pi)}\right] = -\frac{U_1U_2}{\beta(U_1-U_2)^2}. \tag{6.32}$$

The correlation function is rewritten as

$$\langle\partial_{U_1}\phi(U_1,V_1,\Omega_1)\partial_{U_2}\phi(U_2,V_2,\Omega_2)\rangle = -\frac{\beta c_h^2}{(2\pi)^{d+3}}\frac{1}{(U_1-U_2)^2}\delta(\boldsymbol{x}_1-\boldsymbol{x}_2). \tag{6.33}$$

This result is consistent with (6.27) and then we find the following relations.

$$\langle a_{\omega,\boldsymbol{k}},a_{\omega',\boldsymbol{k}'}^\dagger\rangle = \frac{1}{1-e^{-\beta\omega}}\delta(\omega-\omega')\delta^{d-1}(\boldsymbol{k}-\boldsymbol{k}') \tag{6.34}$$

$$\langle a^\dagger_{\omega,\boldsymbol{k}}, a_{\omega',\boldsymbol{k}'}\rangle = \frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}\delta(\omega-\omega')\delta^{d-1}(\boldsymbol{k}-\boldsymbol{k}') \tag{6.35}$$

In the same way using (6.28) and (6.16), we consider the correlation function with one point outside the horizon and another point behind the horizon, and obtain the following relation between operators $a_{\omega,\boldsymbol{k}}$ and $\tilde{a}_{\omega,\boldsymbol{k}}$.

$$\langle a_{\omega,\boldsymbol{k}}, \tilde{a}_{\omega',\boldsymbol{k}'}\rangle = \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}\delta(\omega-\omega')\delta^{d-1}(\boldsymbol{k}-\boldsymbol{k}'); \quad \langle a_{\omega,\boldsymbol{k}}, \tilde{a}^\dagger_{\omega',\boldsymbol{k}'}\rangle = 0, \tag{6.36}$$

$$\langle a^\dagger_{\omega,\boldsymbol{k}}, \tilde{a}^\dagger_{\omega',\boldsymbol{k}'}\rangle = \frac{e^{-\beta\omega/2}}{1-e^{-\beta\omega}}\delta(\omega-\omega')\delta^{d-1}(\boldsymbol{k}-\boldsymbol{k}'); \quad \langle a^\dagger_{\omega,\boldsymbol{k}}, \tilde{a}_{\omega',\boldsymbol{k}'}\rangle = 0, \tag{6.37}$$

And considering the case where both points are inside the horizon, the following results are obtained.

$$\langle \tilde{a}_{\omega,\boldsymbol{k}}, \tilde{a}^\dagger_{\omega',\boldsymbol{k}'}\rangle = \frac{1}{1-e^{-\beta\omega}}\delta(\omega-\omega')\delta^{d-1}(\boldsymbol{k}-\boldsymbol{k}') \tag{6.38}$$

$$\langle \tilde{a}^\dagger_{\omega,\boldsymbol{k}}, \tilde{a}_{\omega',\boldsymbol{k}'}\rangle = \frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}\delta(\omega-\omega')\delta^{d-1}(\boldsymbol{k}-\boldsymbol{k}') \tag{6.39}$$

These thermal expectation values of the number operator characterize the vacuum for quantized scalar field in AdS black hole background, or Hartle-Hawking state.

6.4 The Simple Case: Quantization in Minkowski-Rindler Space

Finally we will review the quantization of a scalar field in the Minkowski-Rinlder coordinate as the simple case of Papadodimas and Raju construction. We start with the $(d+1)$-dimensional Minkowski space

$$ds^2 = dr^2 - dt^2 + d\boldsymbol{x}^2 \tag{6.40}$$

and consider a massless scalar field obeying

$$\Box\phi = 0. \tag{6.41}$$

Region I

First we expand the field in terms of modes in the region I. The Rindler coordinates $(\sigma, \tau, \boldsymbol{x})$ are defined as $r = \sigma\cosh\tau$ and $t = \sigma\sinh\tau$. Then the metric is given by

$$ds^2 = d\sigma^2 - \sigma^2 d\tau^2 + d\boldsymbol{x}^2. \tag{6.42}$$

The scalar field in the region I is expanded as

$$\phi(\sigma,\tau,\boldsymbol{x}) = \int_{\omega>0}\frac{d\omega d^{d-1}\boldsymbol{k}}{(2\pi)^d}\frac{1}{\sqrt{2\omega}}\left[a_{\omega,\boldsymbol{k}}e^{-i\omega\tau+i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{2K_{i\omega}(|\boldsymbol{k}|\sigma)}{\Gamma(i\omega)} + h.c.\right]. \tag{6.43}$$

Note that the Bessel function $K_{i\omega}(|\boldsymbol{k}|\sigma)$ is real.

We use the Kruskal coordinate

$$\begin{aligned} U &= t - r = -\sigma e^{-\tau}, \\ V &= t + r = \sigma e^{\tau} \end{aligned} \tag{6.44}$$

where $U < 0, V > 0$. Considering the field expansion near the horizon between the region I and II we obtain

$$\phi \xrightarrow[U\to 0]{} \int_{\omega>0} \frac{d\omega d^{d-1}\boldsymbol{k}}{(2\pi)^d}\frac{1}{\sqrt{2\omega}}\left[a_{\omega,\boldsymbol{k}}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\left\{V^{-i\omega}e^{-i\delta}+(-U)^{i\omega}e^{i\delta}\right\}+a^\dagger_{\omega,\boldsymbol{k}}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\left\{V^{i\omega}e^{i\delta}+(-U)^{-i\omega}e^{-i\delta}\right\}\right] \tag{6.45}$$

where the phase shift is defined as

$$e^{i\delta}=\left(\frac{|\boldsymbol{k}|}{2}\right)^{i\omega}\frac{\Gamma(-i\omega)}{|\Gamma(i\omega)|}. \tag{6.46}$$

Due to the Riemann-Lebesgue lemma, when $U\to 0$ terms like $U^{-i\omega}$ can be discarded. Then we have

$$\phi \xrightarrow[U\to 0]{} \int_{\omega>0} \frac{d\omega d^{d-1}\boldsymbol{k}}{(2\pi)^d}\frac{1}{\sqrt{2\omega}}\left[a_{\omega,\boldsymbol{k}}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}V^{-i\omega}e^{-i\delta}+a^\dagger_{\omega,\boldsymbol{k}}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}V^{i\omega}e^{i\delta}\right]. \tag{6.47}$$

Region III

The coordinates are $r=\sigma\cosh\tau$ and $t=-\sigma\sinh\tau$. Here the Rindler time τ runs opposite of the Minkowski time t.
The scalar field in the region III is expanded as

$$\phi(\sigma,\tau,\boldsymbol{x})=\int_{\omega>0} \frac{d\omega d^{d-1}\boldsymbol{k}}{(2\pi)^d}\frac{1}{\sqrt{2\omega}}\left[\tilde{a}_{\omega,\boldsymbol{k}}e^{i\omega\tau-i\boldsymbol{k}\cdot\boldsymbol{x}}\frac{2K_{i\omega}(|\boldsymbol{k}|\sigma)}{\Gamma(i\omega)}+h.c.\right]. \tag{6.48}$$

Note that we have defined the modes $\tilde{a}_{\omega,\boldsymbol{k}}$ to multiply the function which is the conjugate of the one in the region I. The Kruskal coordinates are

$$\begin{aligned} U &= \sigma e^{-\tau}, \\ V &= -\sigma e^{\tau} \end{aligned} \tag{6.49}$$

where $U>0, V<0$. The horizon between the region III and II correspond to $V\to 0$. The scalar field near the horizon is given by

$$\phi \xrightarrow[V\to 0]{} \int_{\omega>0} \frac{d\omega d^{d-1}\boldsymbol{k}}{(2\pi)^d}\frac{1}{\sqrt{2\omega}}\left[\tilde{a}_{\omega,\boldsymbol{k}}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}U^{-i\omega}e^{-i\delta}+\tilde{a}^\dagger_{\omega,\boldsymbol{k}}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}U^{i\omega}e^{i\delta}\right]. \tag{6.50}$$

Region II

We choose the Rindler coordinates as $r=\sigma\sinh\tau$ and $t=\sigma\cosh\tau$. In terms of the Kruskal coordinates we have

$$\begin{aligned} U &= \sigma e^{-\tau}, \\ V &= \sigma e^{\tau} \end{aligned} \tag{6.51}$$

where $U,V>0$ and the horizon between I and II is at $\tau\to+\infty$.

The general expansion in the region II is expressed as

$$\phi(\sigma,\tau,\boldsymbol{x})=\int_{\omega>0} \frac{d\omega d^{d-1}\boldsymbol{k}}{(2\pi)^d}\frac{1}{\sqrt{2\omega}}\left[A_{\omega,\boldsymbol{k}}e^{-i\omega\tau+i\boldsymbol{k}\cdot\boldsymbol{x}}J_{i\omega}(|\boldsymbol{k}|\sigma)+B_{\omega,\boldsymbol{k}}e^{-i\omega\tau+i\boldsymbol{k}\cdot\boldsymbol{x}}J_{-i\omega}(|\boldsymbol{k}|\sigma)+h.c.\right]. \tag{6.52}$$

When $\sigma\to 0$ the Bessel functions J is evaluated as

$$J_{i\omega}(|\boldsymbol{k}|\omega)\approx\frac{1}{\Gamma(1+i\omega)}\left(\frac{|\boldsymbol{k}|\omega}{2}\right)^{i\omega}+\cdots. \tag{6.53}$$

Now we compare the expansion closed to the horizon between I and II. When $U \to 0$ we keep only the terms which depend on V and we find

$$\phi \xrightarrow[U\to 0]{} \int_{\omega>0} \frac{d\omega d^{d-1}\boldsymbol{k}}{(2\pi)^d} \frac{1}{\sqrt{2\omega}} \left[\frac{B_{\omega,\boldsymbol{k}}}{\Gamma(1-i\omega)} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \left(\frac{|\boldsymbol{k}|}{2}\right)^{-i\omega} V^{-i\omega} e^{-i\delta} + h.c. \right]. \tag{6.54}$$

Comparing with the expansion (6.47) we find that

$$B_{\omega,\boldsymbol{k}} = \frac{\Gamma(1-i\omega)\Gamma(i\omega)}{|\Gamma(i\omega)|} a_{\omega,\boldsymbol{k}} = -i\sqrt{\frac{\pi\omega}{\sinh \pi\omega}} a_{\omega,\boldsymbol{k}}. \tag{6.55}$$

While, considering the horizon between II and III we obtain

$$\phi \xrightarrow[V\to 0]{} \int_{\omega>0} \frac{d\omega d^{d-1}\boldsymbol{k}}{(2\pi)^d} \frac{1}{\sqrt{2\omega}} \left[\frac{A_{\omega,\boldsymbol{k}}}{\Gamma(1+i\omega)} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \left(\frac{|\boldsymbol{k}|}{2}\right)^{i\omega} U^{i\omega} e^{-i\delta} + h.c. \right]. \tag{6.56}$$

Comparing with the expansion (6.50) we find that

$$A_{\omega,\boldsymbol{k}} = \frac{\Gamma(1+i\omega)\Gamma(-i\omega)}{|\Gamma(i\omega)|} \tilde{a}^{\dagger}_{\omega,\boldsymbol{k}} = i\sqrt{\frac{\pi\omega}{\sinh \pi\omega}} \tilde{a}^{\dagger}_{\omega,\boldsymbol{k}}. \tag{6.57}$$

Then the scalar field in the region II is expanded as

$$\phi(\sigma,\tau,\boldsymbol{x}) = \int_{\omega>0} \frac{d\omega d^{d-1}\boldsymbol{k}}{(2\pi)^d} \frac{1}{\sqrt{2\omega}} \sqrt{\frac{\pi\omega}{\sinh \pi\omega}} \left[i\tilde{a}^{\dagger}_{\omega,\boldsymbol{k}} e^{-i\omega\tau + i\boldsymbol{k}\cdot\boldsymbol{x}} J_{i\omega}(|\boldsymbol{k}|\sigma) - i a_{\omega,\boldsymbol{k}} e^{-i\omega\tau + i\boldsymbol{k}\cdot\boldsymbol{x}} J_{-i\omega}(|\boldsymbol{k}|\sigma) + h.c. \right]. \tag{6.58}$$

Let us consider the quantization for the scalar field in the region II. We define the mode functions as follows.

$$\psi_{\omega,\boldsymbol{k}} = e^{-i\omega\tau + i\boldsymbol{k}\cdot\boldsymbol{x}} J_{-i\omega}(|\boldsymbol{k}|\sigma) \tag{6.59}$$

$$\tilde{\psi}_{\omega,\boldsymbol{k}} = e^{-i\omega\tau + i\boldsymbol{k}\cdot\boldsymbol{x}} J_{i\omega}(|\boldsymbol{k}|\sigma) \tag{6.60}$$

The Klein-Gordon inner product for $f_{\omega,\boldsymbol{k}}$ and $g_{\omega',\boldsymbol{k}'}$ is explicitly written by

$$(f_{\omega,\boldsymbol{k}}, g_{\omega',\boldsymbol{k}'}) = i\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\boldsymbol{x}\sigma \left(f^{*}_{\omega,\boldsymbol{k}} \partial_\sigma g_{\omega',\boldsymbol{k}'} - g_{\omega',\boldsymbol{k}'} \partial_\sigma f^{*}_{\omega,\boldsymbol{k}} \right). \tag{6.61}$$

Then the K-G inner product for mode functions in the region II are given by

$$\begin{aligned}
(\psi_{\omega,\boldsymbol{k}}, \psi_{\omega',\boldsymbol{k}'}) &= \frac{2}{\pi}(2\pi)^d \sinh \pi\omega \delta(\omega-\omega')\delta(\boldsymbol{k}-\boldsymbol{k}'), \\
(\tilde{\psi}_{\omega,\boldsymbol{k}}, \tilde{\psi}_{\omega',\boldsymbol{k}'}) &= -\frac{2}{\pi}(2\pi)^d \sinh \pi\omega \delta(\omega-\omega')\delta(\boldsymbol{k}-\boldsymbol{k}'), \\
(\tilde{\psi}_{\omega,\boldsymbol{k}}, \psi_{\omega',\boldsymbol{k}'}) &= 0.
\end{aligned} \tag{6.62}$$

Note that we can calculate these inner products taking both $\sigma \to 0$ and $\sigma \to \infty$, and we can obtain the same results in either case. Therefore these inner products (6.62) are satisfied in the whole region II. The results are consistent with the standard commutation relations.

$$[a_{\omega,\boldsymbol{k}}, a_{\omega',\boldsymbol{k}'}] = \delta(\omega-\omega')\delta(\boldsymbol{k}-\boldsymbol{k}') \tag{6.63}$$

$$[\tilde{a}_{\omega,\boldsymbol{k}}, \tilde{a}_{\omega',\boldsymbol{k}'}] = \delta(\omega-\omega')\delta(\boldsymbol{k}-\boldsymbol{k}') \tag{6.64}$$

We quantized the scalar field in the region II in the Minkowski-Rindler space.

7 Normal Modes of Real Scalar Field and Klein-Gordon Inner Product

At the beginning of our original works, we will quantize a scalar field theory outside the horizon in the BTZ black hole background by using a new method. The normal modes of a real scalar field in the BTZ background are obtained by solving the equation of motion in [26][35]. On the other hand, in this paper the normal modes will be obtained from those in pure AdS_3 space by suitable coordinate transformations. In this section, these normal modes will be shown and the quantization will be carried out. It will be clear that these normal modes are consistent with the solution of the equation of motion by changing the basis of the normal modes in the next section.

In [26][35], the normal mode functions of a real scalar field in the BTZ black hole background are obtained by solving the equation of motion. For simplicity, we will only consider the case of the BTZ black hole without angular momentum in this paper. The metric is given by (3.36)

$$ds^2 = \frac{1}{r^2 - a} dr^2 - (r^2 - a) dt^2 + r^2 d\varphi^2 \tag{7.1}$$

where the radial coordinate $r \geq 0$ and the angular coordinate φ has periodic identification $\varphi \sim \varphi + 2\pi$. And here a is identified as the mass of the black hole M and the horizon is located at $r = \sqrt{a}$. The classical equation of motion for a scalar field $\phi(r, t, \varphi)$ with mass m is given by

$$(r^2 - a)\partial_r^2 \phi + \frac{1}{r}(3r^2 - a)\partial_r \phi - \frac{1}{r^2 - a}\partial_t^2 \phi + \frac{1}{r^2}\partial_\varphi^2 \phi - m^2 \phi = 0. \tag{7.2}$$

Through separation of variables the normal mode functions of the scalar field are written as

$$\phi_{\omega n}(r, t, \varphi) = e^{-i\omega t + in\varphi} f_{\omega n}(r) \ (n \in \mathbb{Z}). \tag{7.3}$$

We impose normalizability of the solution near the boundary ($r \sim \infty$), $f_{\omega n}(r) \sim r^{\nu+1}$, and obtain those solutions

$$f_{n\omega}(r) = (u-1)^{\alpha} u^{-\alpha - \Delta/2} F\left(\alpha + \beta + \frac{\Delta}{2}, \alpha - \beta + \frac{\Delta}{2}, \Delta; \frac{1}{u}\right) \tag{7.4}$$

where ${}_pF_q(a, b, c; z)$ is a hypergeometric function. And here $\nu = \sqrt{1 + m^2}$ and $\Delta = 1 + \nu$ is a scaling dimension of primary operator in the dual boundary CFT. In this paper only the case of $\nu \neq$ integer will be considered. The variable u is defined by $u = r^2/a$ and the parameters α and β are

$$\alpha = \pm i \frac{\omega}{2\sqrt{a}}, \quad \beta = \pm i \frac{n}{2\sqrt{a}}. \tag{7.5}$$

In what follows we will show the normal modes, which are obtained from those in pure AdS_3 by coordinate transformations, and the quantization. The quantization is implemented in the following form of metric for the outer region of non-rotating BTZ black hole.

$$ds^2 = \frac{1}{y^2} dy^2 - \frac{1}{y^2}\left(1 - \frac{a}{4}y^2\right)^2 dt^2 + \frac{1}{y^2}\left(1 + \frac{a}{4}y^2\right)^2 dx^2 \tag{7.6}$$

This metric is obtained from the BTZ metric (7.1) by a transformation of the radial coordinate r,

$$r = \frac{1}{y} + \frac{a}{4} y \tag{7.7}$$

where $r \geq \sqrt{a}$ and the horizon $r = \sqrt{a}$ corresponds to $y = 2/\sqrt{a}$. And here $y = 0$ and $y = \infty$ correspond to $r = \infty$, or the boundary where the CFT lives. Maximally extended spacetime of the BTZ black hole is divided into four regions, I, II, III, IV. The outer regions are I (right; $0 < y \leq 2/\sqrt{a}$) and III (left; $y \geq 2/\sqrt{a}$), and the inner regions are II (future) and IV (past). Notice that the spacial variable x in a planar BTZ black hole (7.6) takes value in $-\infty < x < \infty$, while the spacial variable φ in a spherical BTZ (7.1) takes value in $0 < \varphi < 2\pi$. In this paper, the quantization of a scalar field is carried out by using the normal modes in a planar BTZ. The result for the case of a spherical BTZ, where the spacial variable is a circle, will be obtained by using the method of mirror images. For example, two point functions of scalar fields in a spherical BTZ are given by those in a planar BTZ by the method of mirror images in [8][26].

The normal modes of a scalar field in the BTZ background are obtained from those in pure AdS_3 space by suitable coordinate transformations. The detail of the transformations is shown in Appendix A. The normal modes for a scalar field in the region I are given by

$$\begin{aligned}\Phi^{\mathrm{I}}_{\omega,k}(y,t,x) \equiv \frac{4\sqrt{a}y}{4+ay^2}e^{\sqrt{a}x}J_\nu\left(\sqrt{\omega^2-k^2}\frac{4\sqrt{a}y}{4+ay^2}e^{\sqrt{a}x}\right)&\\ \times\exp\left[i\frac{4-ay^2}{4+ay^2}e^{\sqrt{a}x}(k\cosh\sqrt{a}t-\omega\sinh\sqrt{a}t)\right]&\end{aligned} \tag{7.8}$$

where $\omega \geq 0$ and $|k| \leq \omega$. The superscript I on Φ shows that these are the normal modes in region I.

First, we will compute the Klein-Gordon (K-G) inner product for these normal modes in region I. The K-G inner product for functions $f(y,t,x)$ and $g(y,t,x)$ in region I is defined as follows:

$$(f,g)_{\mathrm{I}} \equiv i\int_0^{2/\sqrt{a}}dy\int_{-\infty}^{\infty}dx\frac{1}{y}\frac{4+ay^2}{4-ay^2}\left[f^*\partial_t g - g\partial_t f^*\right] \tag{7.9}$$

where the integral region in I for y is $0 \leq y \leq 2/\sqrt{a}$. The subscript I for the inner product on the left hand side means that the integration is to be carried out in the region I. The integrand which is expressed by the radial coordinate y is $\sqrt{-g}(-g^{tt})$. Because this inner product does not depend on t, we will compute the inner product at $t=0$ for simplicity.

By setting $t=0$ and using the normal modes in the region I (7.8) the inner product $(\Phi^{\mathrm{I}}_{\omega,k},\Phi^{\mathrm{I}}_{\omega',k'})_{\mathrm{I}}$ is given by

$$\begin{aligned}(\Phi^{\mathrm{I}}_{\omega,k},\Phi^{\mathrm{I}}_{\omega',k'})_{\mathrm{I}} = (\omega+\omega')\int_0^{2/\sqrt{a}}dy\int_{-\infty}^{\infty}dx\frac{16a^{\frac{3}{2}}y}{(4+ay^2)^2}e^{3\sqrt{a}x}e^{i(k'-k)\frac{4-ay^2}{4+ay^2}e^{\sqrt{a}x}}&\\ \times J_\nu\left(\sqrt{\omega^2-k^2}\frac{4\sqrt{a}y}{4+ay^2}e^{\sqrt{a}x}\right)J_\nu\left(\sqrt{\omega'^2-k'^2}\frac{4\sqrt{a}y}{4+ay^2}e^{\sqrt{a}x}\right).&\end{aligned} \tag{7.10}$$

After rescaling $y \to y/\sqrt{a}$, we translate x by $\rho = e^{\sqrt{a}x}$. Then the inner product becomes

$$(\Phi^{\mathrm{I}}_{\omega,k},\Phi^{\mathrm{I}}_{\omega',k'})_{\mathrm{I}} = (\omega+\omega')\int_0^{2}dy\int_0^{\infty}d\rho\rho^2\frac{16y}{(4+y^2)^2}e^{i(k'-k)\frac{4-y^2}{4+y^2}\rho}J_\nu\left(\mu\frac{4y}{4+y^2}\rho\right)J_\nu\left(\mu'\frac{4y}{4+y^2}\rho\right) \tag{7.11}$$

where we defined

$$\mu \equiv \sqrt{\omega^2-k^2}, \quad \mu' \equiv \sqrt{\omega'^2-k'^2}. \tag{7.12}$$

In order to implement the integration, we define new integration variables in place of ρ and y.

$$z = \frac{4y\rho}{4+y^2}, \quad \omega = \frac{4-y^2}{4+y^2}\rho \tag{7.13}$$

These variables take values in $z, \omega \geq 0$ and the above integral (7.12) is evaluated as

$$\begin{aligned}(\Phi^{\mathrm{I}}_{\omega,k},\Phi^{\mathrm{I}}_{\omega',k'})_{\mathrm{I}} &= (\omega+\omega')\int_0^{\infty}dz\int_0^{\infty}d\omega z J_\nu(\mu z)J_\nu(\mu' z)e^{-i(k-k')\omega}\\ &= -\frac{i}{\mu}(\omega+\omega')\delta(\mu-\mu')\frac{1}{k-k'-i\epsilon}\\ &= \frac{1}{\mu}(\omega+\omega')\delta(\mu-\mu')\left(\pi\delta(k-k')-i\frac{\mathrm{P}}{k-k'}\right)\end{aligned} \tag{7.14}$$

where P stands for a principal-value prescription and the following Fourier-Bessel formula is used.

$$\int_0^\infty dx x J_\nu(ax) J_\nu(bx) = \frac{1}{a}\delta(a-b) \quad (a, b > 0) \tag{7.15}$$

This inner product in the region I is a projection matrix.

Next, in order to compute the inner product in the region III we use the following normal modes in the region III ($y > 2/\sqrt{a}$).

$$\begin{aligned}\Phi^{\mathrm{III}}_{\omega,k}(y,t,x) &\equiv \Phi^{\mathrm{I}}_{\omega,k}(y,-t,x) = \left[\Phi^{\mathrm{I}}_{\omega,-k}(y,t,x)\right]^* \\ &= \frac{4\sqrt{a}y}{4+ay^2} e^{\sqrt{a}x} J_\nu\left(\sqrt{\omega^2-k^2}\frac{4\sqrt{a}y}{4+ay^2}e^{\sqrt{a}x}\right) \\ &\qquad \times \exp\left[i\frac{4-ay^2}{4+ay^2}e^{\sqrt{a}x}(k\cosh\sqrt{a}t + \omega\sinh\sqrt{a}t)\right]\end{aligned} \tag{7.16}$$

Here the direction of time t in the region III is flipped compared to that in Penrose diagram (Fig. 1) and t is assumed to flow upwards (Fig. 2). These two set of normal modes $\Phi^{\mathrm{I}}_{\omega,k}(y,t,x)$ and $\Phi^{\mathrm{III}}_{\omega,k}(y,t,x)$ form a complete set of orthonormal functions in the full outer regions ($0 < y < \infty$), although the two regions are causally disconnected.

$$\Phi_{\omega,k}(y,t,x) \equiv \begin{cases} \Phi^{\mathrm{I}}_{\omega,k}(y,t,x)(0 < y < 2/\sqrt{a}) \\ \Phi^{\mathrm{III}}_{\omega,k}(y,t,x)(2/\sqrt{a} < y) \end{cases} \tag{7.17}$$

At the horizon and $t = 0$ these modes are smoothly connected. The K-G inner product in the region III is defined as follows:

$$(f,g)_{\mathrm{III}} \equiv i\int_{2/\sqrt{a}}^\infty dy \int_{-\infty}^\infty dx \frac{1}{y}\frac{4+ay^2}{ay^2-4}\left[f^*\partial_t g - g\partial_t f^*\right] \tag{7.18}$$

where the integral region in III for y is $2/\sqrt{a} \le y \le \infty$. The subscript III for the inner product on the left hand side means that the integration is to be carried out in the region III. The factor $\sqrt{-g}(-g^{tt})$ which is expressed by y is changed compared with the K-G inner product in the region I (7.9) because the integral region for y in III is $2/\sqrt{a} < y$. After rescaling $y \to y/\sqrt{a}$ and the change of variable $\rho = e^{\sqrt{a}x}$, the inner product in the region III is given by

$$(\Phi^{\mathrm{III}}_{\omega,k}, \Phi^{\mathrm{III}}_{\omega',k'})_{\mathrm{III}} = (\omega+\omega')\int_2^\infty dy \int_0^\infty d\rho\rho^2 \frac{16y}{(4+y^2)^2} e^{i(k'-k)\frac{4-y^2}{4+y^2}\rho} J_\nu\left(\mu\frac{4y}{4+y^2}\rho\right) J_\nu\left(\mu'\frac{4y}{4+y^2}\rho\right). \tag{7.19}$$

This coincide with a complex conjugate of the inner product in the region I (7.11). The sum of these inner product in the region I and III is a delta function.

$$(\Phi_{\omega,k}, \Phi_{\omega',k'}) \equiv (\Phi^{\mathrm{I}}_{\omega,k}, \Phi^{\mathrm{I}}_{\omega',k'})_{\mathrm{I}} + (\Phi^{\mathrm{III}}_{\omega,k}, \Phi^{\mathrm{III}}_{\omega',k'})_{\mathrm{III}} = 4\pi\delta(\omega-\omega')\delta(k-k') \tag{7.20}$$

As for the other inner products, it can be shown that

$$(\Phi^{\mathrm{I}*}_{\omega,k}, \Phi^{\mathrm{I}}_{\omega',k'})_{\mathrm{I}} = -(\Phi^{\mathrm{III}*}_{\omega,k}, \Phi^{\mathrm{III}}_{\omega',k'})_{\mathrm{III}} = \frac{-i}{\mu}\delta(\mu-\mu')\frac{\omega-\omega'}{k+k'-i\epsilon} \tag{7.21}$$

and $(\Phi^*_{\omega,k}, \Phi_{\omega',k'}) = 0$.

The scalar field $\Phi(y,t,x)$ is expanded into normal modes (7.8) and (7.16). We expand Φ in the region I by using $\Phi^{\mathrm{I}}_{\omega,k}$

$$\Phi(y,t,x) = \int_0^\infty \frac{d\omega}{2\sqrt{\pi}}\int_{-\omega}^\omega dk \left[a(\omega,k)\Phi^{\mathrm{I}}_{\omega,k}(y,t,x) + a^\dagger(\omega,k)\Phi^{\mathrm{I}*}_{\omega,k}(y,t,x)\right], \tag{7.22}$$

while we expand Φ in the region III by using $\Phi^{\mathrm{III}}_{\omega,k}$

$$\Phi(y,t,x) = \int_0^\infty \frac{d\omega}{2\sqrt{\pi}} \int_{-\omega}^{\omega} dk \left[a(\omega,k)\Phi^{\mathrm{III}}_{\omega,k}(y,t,x) + a^\dagger(\omega,k)\Phi^{\mathrm{III}*}_{\omega,k}(y,t,x) \right]. \tag{7.23}$$

These scalar fields are constructed as the single set of operators, $a(\omega,k)$, $a^\dagger(\omega,k)$ in the both region I and III. By using the K-G inner products, the creation and annihilation operators are expressed as

$$a(\omega,k) = \frac{1}{2\sqrt{\pi}} \left[(\Phi^{\mathrm{I}}_{\omega,k}, \Phi)_{\mathrm{I}} + (\Phi^{\mathrm{III}}_{\omega,k}, \Phi)_{\mathrm{III}} \right], \tag{7.24}$$

$$a^\dagger(\omega,k) = \frac{-1}{2\sqrt{\pi}} \left[(\Phi^{\mathrm{I}*}_{\omega,k}, \Phi)_{\mathrm{I}} + (\Phi^{\mathrm{III}*}_{\omega,k}, \Phi)_{\mathrm{III}} \right]. \tag{7.25}$$

The commutation relations of these operators,

$$\begin{aligned} [a(\omega,k), a^\dagger(\omega',k')] &= \delta(\omega-\omega')\delta(k-k'), \\ [a(\omega,k), a(\omega',k')] &= [a^\dagger(\omega,k), a^\dagger(\omega',k')] = 0, \end{aligned} \tag{7.26}$$

are obtained by imposing the canonical commutation relations (CCR's):

$$[\Phi(y,t,x), \Pi(y',t,x')] = i\delta(y-y')\delta(x-x') \tag{7.27}$$

where (y,t,x) and (y',t,x') are in the both region I or III, and

$$[\Phi(y,t,x), \Pi(y',t,x')] = 0 \tag{7.28}$$

when these two points are separated by the horizon. Furthermore,

$$[\Phi(y,t,x), \Phi(y',t,x')] = [\Pi(y,t,x), \Pi(y',t,x')] = 0 \tag{7.29}$$

must hold for any separation of these two points. Here a canonical momentum field Π is defined as $\Pi = \sqrt{-g}(-g^{tt})\partial_t\Phi$. We also checked that these CCR's (7.27)-(7.28) are satisfied for $0 < y, y' < \infty$ by using the mode expansions (7.22)-(7.23) and the commutation relations (7.26).

The above annihilation operator $a(\omega,k)$ defines a vacuum $|\Psi_{\mathrm{TFD}}\rangle_\beta$.

$$a(\omega,k)\,|\Psi_{\mathrm{TFD}}\rangle_\beta = 0 \quad (\omega \geq 0, |k| \leq \omega) \tag{7.30}$$

Here β is an inverse temperature defined as $\beta = 2\pi/\sqrt{a}$. As will be shown in Sec.8.2, this is the Thermo-Field Double state. This vacuum is invariant under t and x transformations. Under these transformations the normal modes are transformed as

$$\Phi^{\mathrm{I}}_{\omega,k}(y,t+\epsilon,x) = \Phi^{\mathrm{I}}_{\omega',k'}(y,t,x), \tag{7.31}$$

$$\Phi^{\mathrm{I}}_{\omega,k}(y,t,x+\epsilon) = e^{\sqrt{a}\epsilon}\Phi^{\mathrm{I}}_{\omega'',k''}(y,t,x) \tag{7.32}$$

where $\omega' = \omega\cosh\sqrt{a}\epsilon + k\sinh\sqrt{a}\epsilon$, $k' = k\cosh\sqrt{a}\epsilon - \omega\sinh\sqrt{a}\epsilon$, $\omega'' = \omega e^{\sqrt{a}\epsilon}$ and $k'' = ke^{\sqrt{a}\epsilon}$. Same transformations are also valid for $\Phi^{\mathrm{III}}_{\omega,k}$. Therefore, $a(\omega,k)$ and $a^\dagger(\omega,k)$ must be liner representations of the two transformations, and the vacuum in (7.30) respects translation invariance of the vacuum.

8 Change of Basis of Normal Modes and Boundary Limit of a Scalar Field

In previous section we implemented quantization of a scalar field in the both region I and III in a single Hilbert space and showed that the scalar field is expanded by a single set of operators $a(\omega,k)$ and $a^\dagger(\omega,k)$ in these regions. $a(\omega,k)$ annihilated the vacuum $|\Psi_{\mathrm{TFD}}\rangle_\beta$ and was associated with normal mode functions in the both region I and III, which are not the eigenfunctions of energy and momentum. In this section, the basis of the normal modes will be changed to that of eigenfunctions of energy and momentum. Then the scalar field is expanded by new operators $b_\pm$ which are identified with the doubled operators in the thermo-field dynamics of Takahashi and Umezawa [25][36]. The operators $b_\pm$ annihilate the vacuum $|\Psi_{\rm TFD}\rangle_\beta$. Moreover in each boundary limit in the region I and III, it will be shown that these boundary CFT operators $c_{R,L}$ are related to $b_\pm$ by the Bogolubov transformation and the vacuum $|\Psi_{\rm TFD}\rangle_\beta$ is identified with the Hertle-Hawking state, or the Thermo-Field Double state, which consist of CFT grand states, $|0\rangle_R$ annihilated by c_R and $|0\rangle_L$ annihilated by c_L.

8.1 Change of Basis of Normal Modes into Eigenfunctions of Energy and Momentum

Before the change of the basis, we replace ω and k as

$$\omega = \mu\cosh\zeta, \quad k = \mu\sinh\zeta \quad (\mu \geq 0, -\infty < \zeta < \infty). \tag{8.1}$$

Then it is easy to show that ζ in (7.8) and (7.16) has the same roles as $\sqrt{a}t$ because $k\cosh\sqrt{a}t - \omega\sinh\sqrt{a}t = \mu\sinh(\zeta - \sqrt{a}t)$. We implement the Fourier transformations as follows:

$$\Phi^{\rm I}_{E,p}(y,t,x) = \int_{-\infty}^{\infty} d\zeta \int_0^\infty d\mu e^{-i\frac{E}{\sqrt{a}}\zeta - i\frac{p}{\sqrt{a}}\ln\mu}\Phi^{\rm I}_{\omega.k}(y,t,x), \tag{8.2}$$

$$\Phi^{\rm III}_{E,p}(y,t,x) = \int_{-\infty}^{\infty} d\zeta \int_0^\infty d\mu e^{-i\frac{E}{\sqrt{a}}\zeta - i\frac{p}{\sqrt{a}}\ln\mu}\Phi^{\rm III}_{\omega.k}(y,t,x). \tag{8.3}$$

This new mode $\Phi^{\rm I}_{E,p}(y,t,x)$ is a periodic function of t with an imaginary period $i2\pi/\sqrt{a} \equiv i\beta$, as (7.8) is. Furthermore, we carry out the following shift of integration variables, ζ and $\ln\mu$.

$$\zeta \to \zeta + \sqrt{a}t \tag{8.4}$$

$$\ln\mu \to \ln\mu - \sqrt{a}x \tag{8.5}$$

The normal mode becomes

$$\Phi^{\rm I}_{E,p}(y,t,x) = e^{-iEt+ipx}\int_{-\infty}^{\infty} d\zeta \int_0^\infty d\mu e^{-i\frac{E}{\sqrt{a}}\zeta - i\frac{p}{\sqrt{a}}\ln\mu}\frac{4\sqrt{a}y}{4+ay^2}J_\nu\left(\mu\frac{4\sqrt{a}y}{4+ay^2}\right)e^{i\mu\frac{4-ay^2}{4+ay^2}\sinh\zeta}. \tag{8.6}$$

Now energy E and momentum p are diagonalized. This normal mode will coincide with one of (7.3). In this representation, the imaginary periodicity of t is lost. This means that the move of the contour of ζ in the imaginary direction is not allowed. At the horizon $y = 2/\sqrt{a}$ (8.6) is proportional to $\delta(E)$ and vanishes for $E \neq 0$. In the AdS$_3$/CFT$_2$ correspondence and in the leading order of $1/N$ expansion, the eigenvalues E and p for a scalar field are given by $E = \Delta + 2n + |m|$ and $p = m$, such that $n = 0, 1, ...$ and $m = 0, \pm1, \pm2, ...$ However, in the large N limit, the energy eigenvalue become continuous [9]. In this paper, a continuous spectrum of energy and momentum eigenvalues will be adopted.

The new normal mode in the region III is similarly given by

$$\Phi^{\rm III}_{E,p}(y,t,x) = e^{iEt+ipx}\int_{-\infty}^{\infty} d\zeta \int_0^\infty d\mu e^{-i\frac{E}{\sqrt{a}}\zeta - i\frac{p}{\sqrt{a}}\ln\mu}\frac{4\sqrt{a}y}{4+ay^2}J_\nu\left(\mu\frac{4\sqrt{a}y}{4+ay^2}\right)e^{i\mu\frac{4-ay^2}{4+ay^2}\sinh\zeta}. \tag{8.7}$$

Here the time t in the region III is defined to flow upwards as opposed to the usual choice for the Penrose diagram. The coefficient of iEt in the first exponent is flipped with reference to that in (8.6).

Then the K-G inner product will be calculated by using (7.20), (8.2) and (8.3). The inner product is given by

$$\begin{aligned}
&(\Phi^{\rm I}_{E,p},\Phi^{\rm I}_{E',p'})_{\rm I} + (\Phi^{\rm III}_{E,p},\Phi^{\rm III}_{E',p'})_{\rm III} \\
&\quad = \int_{-\infty}^{\infty} d\zeta \int_0^\infty d\mu e^{i\frac{E}{\sqrt{a}}\zeta + i\frac{p}{\sqrt{a}}\ln\mu}\int_{-\infty}^{\infty} d\zeta' \int_0^\infty d\mu' e^{-i\frac{E'}{\sqrt{a}}\zeta - i\frac{p'}{\sqrt{a}}\ln\mu}\frac{4\pi}{\mu}\delta(\zeta-\zeta')\delta(\mu-\mu') \\
&\quad = 4\pi a\delta(E-E')\delta(p-p')
\end{aligned} \tag{8.8}$$

where a formula $\delta(\omega-\omega')\delta(k-k')=\mu^{-1}\delta(\zeta-\zeta')\delta(\mu-\mu')$ is used.

The scalar field is expanded in terms of these modes (8.6) and (8.7). In the region I, the scalar field is given by

$$\Phi^{\mathrm{I}}(y,t,x)=\frac{1}{4\pi\sqrt{\pi a}}\int_{-\infty}^{\infty}dp\int_0^{\infty}dE\Big[b_+(E,p)\Phi^{\mathrm{I}}_{E,p}(y,t,x)+b_+^\dagger(E,p)\Phi^{\mathrm{I}*}_{E,p}(y,t,x)$$
$$+b_-(E,p)\Phi^{\mathrm{I}}_{-E,-p}(y,t,x)+b_-^\dagger(E,p)\Phi^{\mathrm{I}*}_{-E,-p}(y,t,x)\Big] \tag{8.9}$$

where $b_\pm(E,p)$ are annihilation operators of positive-(negative-)frequency normal modes. Note that the negative-frequency normal mode $\Phi^{\mathrm{I}}_{-E,-p}$ is associated with the annihilation operator $b_-(E,p)$. This means that the creation operator $b_-^\dagger$ creates a 'hole', while $b_+^\dagger$ creates a 'particle' [25][36].

Similarly, in the region Ⅲ the scalar field is given by

$$\Phi^{\mathrm{III}}(y,t,x)=\frac{1}{4\pi\sqrt{\pi a}}\int_{-\infty}^{\infty}dp\int_0^{\infty}dE\Big[b_+(E,p)\Phi^{\mathrm{III}}_{E,p}(y,t,x)+b_+^\dagger(E,p)\Phi^{\mathrm{III}*}_{E,p}(y,t,x)$$
$$+b_-(E,p)\Phi^{\mathrm{III}}_{-E,-p}(y,t,x)+b_-^\dagger(E,p)\Phi^{\mathrm{III}*}_{-E,-p}(y,t,x)\Big] \tag{8.10}$$

where the annihilation operator b_+ is associated with the negative-frequency mode, and b_- is associated with the positive-frequency mode.

The relation between $a(\omega,k)$ and $b_\pm(E,p)$ is given by

$$a(\omega,k)=\frac{1}{2\pi\sqrt{a}\mu}\int_0^{\infty}dE\int_{-\infty}^{\infty}dp\left[e^{-\frac{i}{\sqrt{a}}E\zeta-\frac{i}{\sqrt{a}}p\ln\mu}b_+(E,p)+e^{\frac{i}{\sqrt{a}}E\zeta+\frac{i}{\sqrt{a}}p\ln\mu}b_-(E,p)\right]. \tag{8.11}$$

Hermitian conjugate of this equation gives the relation for $a^\dagger(\omega,k)$. $b_\pm(E,p)$ also annihilate the vacuum $|\Psi_{\mathrm{TFD}}\rangle_\beta$ as (7.30). The operators $b_\pm$ and $b_\pm^\dagger$ satisfy the standard commutation relations,

$$[b_+(E,p),b_+^\dagger(E',p')]=[b_-(E,p),b_-^\dagger(E',p')]=\delta(E-E')\delta(p-p'). \tag{8.12}$$

The other commutators vanish.

8.2 Boundary Limit of a Scalar Field and the Dual Boundary Operator

In what follows we will consider the boundary limit of the scalar field Φ^{I} and Φ^{III}, and identify the CFT operators on the boundaries in the region I and Ⅲ. Some details are summarized in Appendix B.

According to the BDHM dictionary, in $y\to 0$ limit the normal mode $\Phi^{\mathrm{I}}_{E,p}$ (8.6) is given by

$$\Phi^{\mathrm{I}}_{E,p}\to y^{\Delta}e^{-iEt+ipx}g(E,p) \tag{8.13}$$

where $g(E,p)$ is a function defined by

$$g(E,p)=2^{-\nu}a^{\Delta/2}(\Gamma(\Delta))^{-1}\int_{-\infty}^{\infty}d\zeta\int_0^{\infty}d\mu\mu^{\nu}\exp\left[-i\frac{E}{\sqrt{a}}\zeta-i\frac{p}{\sqrt{a}}\ln\mu+i\mu\sinh\zeta\right]. \tag{8.14}$$

Then in $y\to 0$ limit, the scalar field in the region I is expressed as follows.

$$\begin{aligned}\Phi^{\mathrm{I}}(y,t,x)\to\ &\frac{1}{4\pi\sqrt{\pi a}}y^{\Delta}\int_0^{\infty}dE\int_{-\infty}^{\infty}dp\\ &\left[|g(E,p)|^2-|g(-E,-p)|^2\right]^{1/2}\left[e^{-iEt+ipx}2^{-\frac{i}{\sqrt{a}}p}c_R(E,p)+e^{iEt-ipx}2^{\frac{i}{\sqrt{a}}p}c_R^\dagger(E,p)\right]\\ &\equiv y^{\Delta}O_R(t,x)\end{aligned} \tag{8.15}$$

Here $O_R(t,x)$ is a CFT operator on the right boundary, and define the boundary CFT operator $c_R(E,p)$ by Fourier transformation [9]. The relation between $c_R(E,p)$ and $b_\pm(E,p)$ is given by

$$c_R(E,p) = 2^{\frac{i}{\sqrt{a}}p}\left[b_+(E,p)g(E,p) + b_-^\dagger(E,p)g^*(-E,-p)\right]\left[|g(E,p)|^2 - |g(-E,-p)|^2\right]^{-1/2}. \tag{8.16}$$

Hermitian conjugate of this equation gives $c_R^\dagger$. These operators satisfy the standard commutation relations.

$$[c_R(E,p), c_R^\dagger(E',p')] = \delta(E-E')\delta(p-p') \tag{8.17}$$

Similarly, in the boundary limit $y \to \infty$ the normal mode $\Phi^{\mathrm{III}}_{E,p}$ (8.7) is given by

$$\Phi^{\mathrm{III}}_{E,p} \to (4/a)^\Delta y^{-\Delta} e^{iEt+ipx}\tilde{g}(E,p) \tag{8.18}$$

where $\tilde{g}(E,p)$ is a function defined by

$$\tilde{g}(E,p) = 2^{-\nu}a^{\Delta/2}(\Gamma(\Delta))^{-1}\int_{-\infty}^{\infty} d\zeta \int_0^\infty d\mu\mu^\nu \exp\left[i\frac{E}{\sqrt{a}}\zeta - i\frac{p}{\sqrt{a}}\ln\mu + i\mu\sinh\zeta\right]. \tag{8.19}$$

Then in $y \to \infty$ limit, the scalar field in the region III is expressed as follows.

$$\begin{aligned}\Phi^{\mathrm{III}}(y,t,x) \to \frac{1}{4\pi\sqrt{\pi a}}(4/a)^\Delta y^{-\Delta}\int_0^\infty dE \int_{-\infty}^\infty dp \\ \left[|\tilde{g}(E,p)|^2 - |\tilde{g}(-E,-p)|^2\right]^{1/2}\left[e^{iEt+ipx}2^{\frac{i}{\sqrt{a}}p}c_L(E,p) + e^{-iEt-ipx}2^{-\frac{i}{\sqrt{a}}p}c_L^\dagger(E,p)\right]\end{aligned} \tag{8.20}$$

The relation between the left boundary CFT operator $c_L(E,p)$ and $b_\pm(E,p)$ is given by

$$c_L(E,p) = 2^{-\frac{i}{\sqrt{a}}p}\left[b_+^\dagger(E,p)\tilde{g}^*(E,p) + b_-(E,p)\tilde{g}(-E,-p)\right]\left[|\tilde{g}(-E,-p)|^2 - |\tilde{g}(E,p)|^2\right]^{-1/2}. \tag{8.21}$$

Hermitian conjugate of this equation gives $c_R^\dagger$. These operators satisfy the standard commutation relations.

$$[c_L(E,p), c_L^\dagger(E',p')] = \delta(E-E')\delta(p-p') \tag{8.22}$$

Moreover, because the region I and III are causally disconnected, the scalar fields in the both regions commute with each other. Therefore c_R and $c_R^\dagger$ commute with c_L and $c_L^\dagger$.

Next, we will define a grand state of boundary CFT's, $|0\rangle = |0\rangle_L \otimes |0\rangle_R$, and show the Hartle-Hawking state, or the Thermo-Field Double state, which is characterized by the thermal expectation value of operators c_R and c_L. The grand state of CFT's is annihilated by c_R and c_L.

$$c_R\,|0\rangle_R = c_L\,|0\rangle_L = 0 \tag{8.23}$$

The relation for operators (8.16) and (8.21) can be simplified by using the results in Appendix B. These relations are the Bogolubov transformations as follows.

$$c_R(E,p) = \frac{1}{\sqrt{1-e^{-\beta E}}}\left(b_+(E,p) + e^{-\beta E/2}b_-^\dagger(E,p)\right) \tag{8.24}$$

$$c_L(E,p) = \frac{1}{\sqrt{1-e^{-\beta E}}}\left(b_-(E,p) + e^{-\beta E/2}b_+^\dagger(E,p)\right) \tag{8.25}$$

$$c_R^\dagger c_R - c_L^\dagger c_L = b_+^\dagger b_+ - b_-^\dagger b_- \tag{8.26}$$

By using (8.23) and $b_\pm$ which are the solutions of (8.24) and (8.25), it can be shown that the vacuum state $|\Psi_{\mathrm{TFD}}\rangle_\beta$ annihilated by $b_\pm$ is the Hartle-Hawking state, or the Thermo-Field Double state.

$$|\Psi_{\rm TFD}\rangle_\beta = \frac{1}{\sqrt{Z(\beta)}}\exp\left[\int_0^\infty dE\int_{-\infty}^\infty dp e^{-(\beta/2)E}c_L^\dagger(E,p)c_R^\dagger(E,p)\right]|0\rangle_L\otimes|0\rangle_R \tag{8.27}$$

The entangled state is characterized by the thermal expectation values of the number operators, which are the Bose-Einstein distribution.

$$\begin{aligned}
{}_\beta\langle\Psi_{\rm TFD}|\, c_R^\dagger(E,p)c_R(E',p')\,|\Psi_{\rm TFD}\rangle_\beta &= {}_\beta\langle\Psi_{\rm TFD}|\, c_L^\dagger(E,p)c_L(E',p')\,|\Psi_{\rm TFD}\rangle_\beta \\
&= \frac{1}{e^{\beta E}-1}\delta(E-E')\delta(p-p') \\
{}_\beta\langle\Psi_{\rm TFD}|\, c_R(E,p)c_R^\dagger(E',p')\,|\Psi_{\rm TFD}\rangle_\beta &= {}_\beta\langle\Psi_{\rm TFD}|\, c_L(E,p)c_L^\dagger(E',p')\,|\Psi_{\rm TFD}\rangle_\beta \\
&= \frac{1}{1-e^{-\beta E}}\delta(E-E')\delta(p-p')
\end{aligned} \tag{8.28}$$

We also have the following thermal averages.

$$\begin{aligned}
{}_\beta\langle\Psi_{\rm TFD}|\, c_R(E,p)c_L(E',p')\,|\Psi_{\rm TFD}\rangle_\beta &= {}_\beta\langle\Psi_{\rm TFD}|\, c_L(E,p)c_R(E',p')\,|\Psi_{\rm TFD}\rangle_\beta \\
&= \frac{e^{-\beta E/2}}{1-e^{-\beta E}}\delta(E-E')\delta(p-p')
\end{aligned} \tag{8.29}$$

There are similar averages for $c_L^\dagger$ and $c_R^\dagger$. The Thermo-Field Hamiltonian is given by

$$\begin{aligned}
H_{TF} &= \int_0^\infty dE\int_{-\infty}^\infty dpE\left[b_+^\dagger(E,p)b_+(E,p) - b_-^\dagger(E,p)b_-(E,p)\right] \\
&= \int_0^\infty dE\int_{-\infty}^\infty dpE\left[c_R^\dagger(E,p)c_R(E,p) - c_L^\dagger(E,p)c_L(E,p)\right] \equiv H_R - H_L.
\end{aligned} \tag{8.30}$$

Therefore we implemented the quantization of a scalar field in a single Hilbert space which is constructed by combining the region I and Ⅲ, and after some calculation we obtained the Hartle-Hawking state as a 'vacuum' which is annihilated by the operators $b_\pm(E,p)$ for the scalar field.

In addition we can obtain the scalar field expressed by the boundary CFT operators $c_{R,L}(E,p)$, which is bulk reconstruction of the scalar field outside the horizon. By using the asymptotic form of the normal modes in the region I (8.13) and Appendix B it can be shown that the normal mode in the region I satisfy the following equation in $y\to 0$ limit.

$$\Phi^{\rm I}_{-E,-p} - e^{-\beta E/2}\Phi^{\rm I*}_{E,p} = O(y^{\Delta+1}) \tag{8.31}$$

Because $\Phi^{\rm I}_{E,p}$ is a solution of the K-G equation and the allowed asymptotic leading powers of y are $1\pm\nu$, the left hand side vanishes identically.

$$\Phi^{\rm I}_{-E,-p} - e^{-\beta E/2}\Phi^{\rm I*}_{E,p} = 0 \tag{8.32}$$

The above formula (8.32) and the Bogolubov transformation (8.24)-(8.25) simplify the scalar field in the region I (8.9) as follows.

$$\Phi^{\rm I}(y,t,x) = \frac{1}{4\pi\sqrt{\pi a}}\int_{-\infty}^\infty dp\int_0^\infty dE\sqrt{1-e^{-\beta E}}\left[c_R(E,p)\Phi^{\rm I}_{E,p}(y,t,x) + h.c.\right] \tag{8.33}$$

Similarly, in the region Ⅲ we obtain the formula in $y\to\infty$ limit

$$\Phi^{\rm III}_{-E,-p} - e^{\beta E/2}\Phi^{\rm III*}_{E,p} = 0 \tag{8.34}$$

and the scalar field expressed by the boundary CFT operators $c_L(E,p)$

$$\Phi^{\text{III}}(y,t,x) = \frac{1}{4\pi\sqrt{\pi a}} \int_{-\infty}^{\infty} dp \int_0^{\infty} dE \sqrt{1-e^{-\beta E}} \left[c_L(E,p) \Phi^{\text{III}}_{-E,-p}(y,t,x) + h.c. \right]. \tag{8.35}$$

The scalar field in the region I (8.33) is expressed by only CFT operator c_R on the right boundary, while the scalar field in the region III (8.35) only c_L on the left boundary. The operator c_R exists only in the region I and c_L exists only in the region III. Hence by changing the basis of the normal modes to the eigenfunctions of energy and momentum the operators in the region I and those in the region III are decoupled. (8.33) and (8.35) are the bulk reconstruction of a scalar field outside the horizon in the AdS-Rindler space.

9 Normal Modes of a Real Scalar Field behind the Horizon

We obtained the scalar field outside the horizon which is expanded by the boundary CFT operators multiplying normal modes and the Klein-Gordon inner product for these normal modes are normalized appropriately. In the following sections, we will construct the quantized scalar field behind the horizon. According to the method by Papadodimas and Raju, it is necessary to identify the normal modes in the region II multiplying the same operators in the region I and III, which are connected by the matching condition [9]. In addition to analysis of the matching condition, we will explicitly calculate the K-G inner product for the normal modes in the whole region II. In this section we will consider solutions to the Klein-Gordon equation behind the horizon and calculate the K-G inner products. As the mode functions in the region I and III, the mode functions in the region II are obtained from those in pure AdS_3 space by coordinate transformations. However, the K-G inner product for these mode functions is not positive definite. It will be necessary to introduce additional modes (9.13) for the quantization. In this paper we will only consider the region II, and the region IV can be similarly studied.

The metric behind the horizon can be obtained from (7.1) by coordinate transformations.

$$\eta = \frac{2}{r}(1 \pm \frac{1}{\sqrt{a}}\sqrt{a-r^2}) \tag{9.1}$$

Then the metric tensor behind the horizon is given by

$$ds^2 = \frac{a(a\eta^2-4)^2}{(a\eta^2+4)^2}dt^2 - \frac{16a}{(a\eta^2+4)^2}d\eta^2 + \frac{16a^2\eta^2}{(a\eta^2+4)^2}dx^2. \tag{9.2}$$

Here in the region II the upper sign of (9.1) is used and η takes values in $\eta \geq 2/\sqrt{a}$, while in the region IV the lower sign of (9.1) is used and η takes values in $0 \leq \eta \leq 2/\sqrt{a}$. Note that in the region II and IV the variable η is a time variable, while x is a spacial one (Fig. 4). In this paper we only consider the case of a scalar field in the region II.

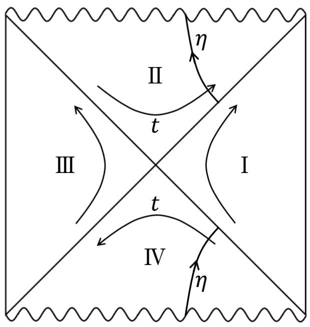

Fig. 4 A time direction η in regions II and IV

In the same as the mode functions in the region I are given by (7.8), one set of mode functions in the region II is obtained from those in AdS_3 space by suitable coordinate transformations. The details will be shown in Appendix A and the mode functions in the region II are given by

$$\Phi^{\mathrm{II}(\nu)}_{\omega,k}(t,\eta,x) = \frac{4+a\eta^2}{4\sqrt{a}\eta} e^{\sqrt{a}x} J_\nu\left(\mu \frac{4+a\eta^2}{4\sqrt{a}\eta} e^{\sqrt{a}x}\right) \exp\left[i\frac{a\eta^2-4}{4\sqrt{a}\eta} e^{\sqrt{a}x} \mu \cosh\left(\zeta - \sqrt{a}t\right)\right] \tag{9.3}$$

where $\omega = \mu\cosh\zeta$ and $k = \mu\sinh\zeta$. Then the eigenfunctions of energy and momentum corresponding to t and x translations are given by the Fourier transformation of (9.3) with respect to ζ and $\ln\mu$.

$$\begin{aligned}\Phi^{\mathrm{II}}_{E,p}(t,\eta,x) &= \int_{-\infty}^{\infty} d\zeta \int_0^\infty d\mu e^{-i\frac{E}{\sqrt{a}}\zeta - i\frac{p}{\sqrt{a}}\ln\mu} \Phi^{\mathrm{II}}_{\omega.k}(t,\eta,x) \\ &= e^{-iEt+ipx} \int_{-\infty}^{\infty} d\zeta \int_0^\infty d\mu e^{-i\frac{E}{\sqrt{a}}\zeta - i\frac{p}{\sqrt{a}}\ln\mu} \frac{4+a\eta^2}{4\sqrt{a}\eta} J_\nu\left(\mu\frac{4+a\eta^2}{4\sqrt{a}\eta}\right) \exp\left[i\frac{a\eta^2-4}{4\sqrt{a}\eta}\mu\cosh\zeta\right]\end{aligned} \tag{9.4}$$

Here E and p are energy and momentum eigenvalues conjugate to t and x, and take values in $-\infty < E, p < \infty$. This solution can also be represented as a suitable linear combination of a hypergeometric functions as in [26]. This set of mode functions, however, does not form a complete set of linearly independent functions. In (C.5) of Appendix C it is shown that the following relation holds for $\eta > 2/\sqrt{a}$.

$$\left(\Phi^{\mathrm{II}(\nu)}_{E,p}(t,\eta,x)\right)^* = e^{-\pi i(\nu+1)} e^{\beta p/2} \Phi^{\mathrm{II}(\nu)}_{-E,-p}(t,\eta,x) \tag{9.5}$$

where note that E is a component of spatial momentum. We need additional set of mode functions to obtain a complete set of linearly independent functions.

Now we will introduce a new set of mode functions.

$$\begin{aligned}\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}(t,\eta,x) = e^{-iEt+ipx} \int_{-\infty}^{\infty} d\zeta \int_0^\infty d\mu e^{-i\frac{E}{\sqrt{a}}\zeta + i\frac{p}{\sqrt{a}}\ln\mu} \frac{4+a\eta^2}{4\sqrt{a}\eta} \\ \times J_\nu\left(\mu\frac{4+a\eta^2}{4\sqrt{a}\eta}\right) \exp\left[i\frac{a\eta^2-4}{4\sqrt{a}\eta}\mu\cosh\zeta\right]\end{aligned} \tag{9.6}$$

This is related to (9.4) by a relation[*1]

$$\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}(t,\eta,x) = \Phi^{\mathrm{II}(\nu)}_{E,-p}(t,\eta,-x). \tag{9.7}$$

The Klein-Gordon (K-G) inner products for these modes in the region II, $\eta \geq 2/\sqrt{a}$, is defined for functions $f(t,\eta,x)$ and $g(t,\eta,x)$ as follows.

$$(f,g)_{\mathrm{II}} \equiv i\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \, \frac{a\eta(a\eta^2-4)}{a\eta^2+4} \left[f^*\partial_\eta g - g\partial_\eta f^*\right] \tag{9.8}$$

where note that in the region II time variable is η and the integration is carried out with respect to spacial variables t and x. The η dependent factor in the integrand is $\sqrt{-g}(-g^{\eta\eta})$. The subscript II for the inner product on the left hand side means that the integration is to be carried out in the region II. This inner product does not depend on η owing to the K-G equation, which is correct at least in the region away from the horizon. We will refer to the discontinuity of the inner product in the next section. The inner products for the mode functions in the region II will be computed in the limit $\eta \to +\infty$. They are evaluated in Appendix C. Some of the results for $\eta > 2/\sqrt{a}$ are

$$(\Phi^{\mathrm{II}(\nu)}_{E,p}, \Phi^{\mathrm{II}(\nu)}_{E',p'})_{\mathrm{II}} = \frac{-8\pi^3 a\exp(\beta p/2)}{\sinh(\beta p/2)} \delta(E-E')\delta(p-p'), \tag{9.9}$$

$$(\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}, \tilde{\Phi}^{\mathrm{II}(\nu)}_{E',p'})_{\mathrm{II}} = \frac{8\pi^3 a\exp(-\beta p/2)}{\sinh(\beta p/2)} \delta(E-E')\delta(p-p'), \tag{9.10}$$

$$(\Phi^{\mathrm{II}(\nu)}_{E,p}, \tilde{\Phi}^{\mathrm{II}(\nu)}_{E',p'})_{\mathrm{II}} = 0. \tag{9.11}$$

[*1] In the (ω,k) basis of (9.3), a relation $\tilde{\Phi}^{\mathrm{II}(\nu)}_{\omega,k}(t,\eta,x) = (\omega'^2 - k'^2)\Phi^{\mathrm{II}(\nu)}_{\omega',k'}(t,\eta,-x)$ with $(\omega',k') = (\omega/(\omega^2-k^2), k/(\omega^2-k^2))$ holds.

These inner products are not positive definite *e.g.* (9.9) is negative for $p > 0$ and positive for $p < 0$. Furthermore, (9.4) behaves as $\Phi^{\mathrm{II}(\nu)}_{E,p} \propto \eta^{ip/\sqrt{a}} \exp(-iEt + ipx)$ at large η. Similarly (9.6) behaves as $\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p} \propto \eta^{-ip/\sqrt{a}} \exp(-iEt + ipx)$. By using these properties of mode functions it can be shown that

$$(\Phi^{\mathrm{II}(\nu)*}_{E,-p}, \tilde{\Phi}^{\mathrm{II}(\nu)}_{-E',p'})_{\mathrm{II}} = 0 \tag{9.12}$$

for $-\infty < E, E', p, p' < \infty$.

In addition to $\Phi^{\mathrm{II}(\nu)}_{E,p}$ and $\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}$ we will also introduce new mode functions, or basis functions, $\Phi^{\mathrm{II}(-\nu)}_{E,p}$ and $\tilde{\Phi}^{\mathrm{II}(-\nu)}_{E,p}$, which are obtained from $\Phi^{\mathrm{II}(\nu)}_{E,p}$ and $\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}$, respectively, by replacing Bessel function $J_\nu(z)$ by $J_{-\nu}(z)$. Thus there are four types of basis functions:

$$\mathcal{V} = \left\{ \Phi^{\mathrm{II}(\nu)}_{E,p}, \quad \Phi^{\mathrm{II}(-\nu)}_{E,p}, \quad \tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}, \quad \tilde{\Phi}^{\mathrm{II}(-\nu)}_{E,p} \right\}. \tag{9.13}$$

Although usually, only two of these functions are chosen to be a basis of linearly independent functions, we will consider a vector space $\mathcal{V}$ of functions spanned by this set of the mode functions, and expand the scalar field inside the horizon into these modes. One of the reason is that in curved spacetime without time-translation symmetry positive- and negative-frequency solutions cannot be defined and it is not known beforehand which normal modes are to be assigned to annihilation operators in a scalar field expansion. Thus the number of independent basis functions must be doubled. Another reason is that it will be shown in the next section that the inner products of these functions are not independent of η at the horizon, *i.e.*, take distinct values at the horizon from those for $\eta > 2/\sqrt{a}$. Therefore it is necessary to keep all of the modes (9.13) as the basis in the analysis of matching conditions at the horizon. Furthermore, $\Phi^{\mathrm{II}(\nu)}_{E,p}$ and $\Phi^{\mathrm{II}(\nu)*}_{-E,-p}$ are not linearly independent as shown above. Thus the mode functions with $p > 0$ and those with $p < 0$ must be treated separately. The above four mode functions (9.13) are all proportional to $e^{-iEt+ipx}$.

The scalar field must satisfy matching conditions at the horizon [9] as well as normalization condition of the inner products for the normal modes. The inner products for the normal modes, which are used to expand the scalar field, must be smooth in the whole interior region of the black hole background.

10 Discontinuity of the Inner Product in η at the Horizon

We will show that although the inner product (9.8) for the mode functions is independent of time η for $\eta > 2/\sqrt{a}$, it change discontinuously at $\eta = 2/\sqrt{a}$. This is the reason that in the previous section we introduced the vector space $\mathcal{V}$ (9.13) spanned by four sets of mode functions. Only inner products of appropriate linear combinations of these mode functions will be continuous at the horizon, which will be shown in the next section. In this section we will refer to origin of the discontinuity and explain difference of our approach from that of Papadodimas and Raju.

Singularity of Klein-Gordon Inner Product

The inner product for f and g (9.8) satisfies the following equation owing to the K-G equation.

$$\begin{aligned} \frac{d}{d\eta}(f,g)_{\mathrm{II}} = i \int dt \int dx \partial_t \left[g^{tt}\sqrt{-g}(f^* \partial_t g - g \partial_t f^*) \right] \\ + i \int dt \int dx \partial_x \left[g^{xx}\sqrt{-g}(f^* \partial_x g - g \partial_x f^*) \right] \end{aligned} \tag{10.1}$$

Here the factor $g^{tt}\sqrt{-g}$ in the first line is given by

$$g^{tt}\sqrt{-g} = \frac{16a\eta}{(a\eta^2+4)(a\eta^2-4)}. \tag{10.2}$$

This has a singularity exactly at the location of the horizon $\eta = 2/\sqrt{a}$. Because the integrand is a total derivative, the right hand side will vanish for $\eta > 2/\sqrt{a}$. Then the inner products are independent of η as long as $\eta > 2/\sqrt{a}$. Just at the horizon, however, special care must be taken. This problem can be studied by computing the inner product of the mode functions as $\eta \to \infty$ and at $\eta = 2/\sqrt{a}$, separately, and showing that the results of the two cases are same or distinct.[*2]

Inner products of (9.13) at $\eta > 2/\sqrt{a}$ are given in (C.9)-(C.16). These are computed by using the asymptotic behaviors (C.4) for $\eta \to \infty$. Inner products of the four modes in (9.13) can also be computed in a region near the horizon. In (D.6) a behavior of $\Phi^{\mathrm{II}(\nu)}_{E,p}$ near the horizon is presented. By using this result some inner products of (9.13) are computed in Appendix D. The results are completely different from those in (C.9)-(C.16). Hence the inner products of the mode functions have discontinuities at $\eta = 2/\sqrt{a}$.

Discontinuity of Linear Dependence of Mode Functions in V

We will study linear dependence of the four types of mode functions (9.13). In the $\eta \to \infty$ limit, $\Phi^{\mathrm{II}(\nu)}_{E,p}$ behaves as (C.4). This shows that there exist following two relations in this limit.

$$\Phi^{\mathrm{II}(-\nu)}_{E,p} - e^{-\pi i\nu} D(E,p)\Phi^{\mathrm{II}(\nu)}_{E,p} = 0 \tag{10.3}$$

$$\tilde{\Phi}^{\mathrm{II}(-\nu)}_{E,p} - e^{-\pi i\nu} D^*(E,p)\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p} = 0 \tag{10.4}$$

Here $D(E,p)$ is defined in (C.17). It can be shown that inner products of the left hand sides of (10.3) and (10.4) with the four functions in (9.13) all vanish, as long as $\eta > 2/\sqrt{a}$. Therefore in the region at $\eta > 2/\sqrt{a}$ there are only two linearly independent modes.

On the contrary by using the behavior (D.6) of $\Phi^{\mathrm{II}(\nu)}_{E,p}$ near the horizon, it can be shown that the following relation holds near the horizon.

$$\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p} = 2^{2ip/\sqrt{a}} \frac{\Gamma\left(\frac{1}{2}(1+\nu+i\frac{E+p}{\sqrt{a}})\right)\Gamma\left(\frac{1}{2}(1+\nu-i\frac{E-p}{\sqrt{a}})\right)}{\Gamma\left(\frac{1}{2}(1+\nu-i\frac{E+p}{\sqrt{a}})\right)\Gamma\left(\frac{1}{2}(1+\nu+i\frac{E-p}{\sqrt{a}})\right)} \Phi^{\mathrm{II}(\nu)}_{E,p} \tag{10.5}$$

Another relation obtained by a replacement $\nu \to -\nu$ also holds. In this case there are only two linearly independent modes which are inconsistent with the linearly dependent modes at $\eta > 2/\sqrt{a}$. Hence the linear dependence of mode functions is discontinuous at the horizon. It is then unavoidable to use all the four modes altogether behind the horizon. Thus we will work in the vector space of mode functions $\mathcal{V}$.

Difference of our approach from that of Papadodimas and Raju

Our construction of a scalar field behind the horizon is based on the proposal of Papadodimas and Raju [9]. Here we explain the difference of our method to construct a scalar field behind the horizon from that by Papadodimas and Raju. Especially, we compare the analysis results of scalar fields constructed by each method.

In order to construct a scalar field behind the horizon we impose two conditions. One is the matching condition of the scalar field at the horizon, which was proposed by Papadodimas and Raju. The matching condition is the boundary condition at the horizon so that the scalar field behind the horizon is smoothly connected to that outside the horizon. In their proposal, by using the matching condition the scalar field behind and near the horizon in a eternal AdS black hole background was constructed. Then it was shown that two-point function crossing the horizon for the differentiated scalar fields has the short distance behavior in terms of the geodesic distance (6.27), which we reviewed in Sec.6.3. As shown in Appendix F, we can also calculate the two-point function crossing the horizon for the differentiated scalar fields, which are obtained in our construction, and then our two-point function crossing the horizon also satisfy the same short distance behavior (6.27). Therefore our analysis of the scalar field near the horizon is consistent with that by Papadodimas and Raju.

The another condition is the orthonormality of the Klein-Gordon inner product for the mode functions of the scalar field behind the horizon. The orthonormality of the K-G inner product is the condition to carry out the canonical quantization, and then the quantized scalar field is described by the same

[*2] In the case of Rindler space it can be shown that there is also a singularity at the horizon in an equation similar to (10.1). In this case, however, inner products of mode functions are not discontinuous at the horizon.

quantum theory outside the horizon. We will explicitly calculate the K-G inner products for the modes (9.13) in the whole region behind the horizon, and impose the orthonormality of the inner product on the mode functions of the scalar field behind the horizon. Since the region of analysis by Papadodimas and Raju is only near the horizon, the calculation of the inner product deep inside the horizon is the difference between our analysis and that by Papadodimas and Raju. In our analysis of the inner product, it is found that the inner products for the modes (9.13) have discontinuity at the horizon. In order to obtain orthonormal inner product for the mode functions in the whole region II, we must impose the orthonormality of the inner product on the linear combinations of the modes (9.13) and identify these linear combinations as the mode functions of the scalar field behind the horizon. This analysis will be shown in Sec.11. What is important in the difference between our method and that by Papadodimas and Raju is that the region of our analysis is not only near the horizon, but deep inside the horizon.

Before we will show the specific analysis in the next section, we stress some problems, when constructing a scalar field in the region II by the above conditions, as follows.

1. The discontinuity in the Klein-Godon inner product for mode functions in $\mathcal{V}$ at the horizon makes it difficult to obtain the orthonormal K-G inner product in the whole region inside the horizon.
2. The discontinuity of the linear dependence of mode functions in $\mathcal{V}$ at the horizon makes it difficult to obtain the mode functions of the scalar field in the region II which is smoothly connected to the normal modes outside the horizon.
3. In order to show whether the scalar field, which solve the both problems 1. and 2. , exists or not, we must carry out complicated analysis.

If the inner products at $\eta = 2/\sqrt{a}$ were used, the quantum theory deep inside the horizon might not be the same theory outside the horizon. If the inner products for $\eta > 2/\sqrt{a}$ were used, then the quantum theory inside the horizon might be discontinuous from that outside the horizon. In order to construct the quantum theory in the whole region II as information outside the horizon, we must solve the problem 1. and obtain the orthonormal inner product without discontinuity. In addition, we need to assign the creation and annihilation operators in the region II, which are the same operators in the region I and III, to the appropriate normal modes in the region II. If these normal modes were discontinuous from those outside, the quantum theory inside the horizon might be discontinuous from that outside the horizon. We must solve the problem 2. and obtain the normal modes in the region II which is smoothly connected to those outside the horizon.

In the following analysis we will start by using the inner products for $\eta > 2/\sqrt{a}$, (C.9)-(C.16), and solve the matching conditions and the orthonormality conditions. It will be shown in Secs.11 and 12 that when these conditions are satisfied, special linear combinations of mode functions, $\psi^{(i)}_{E,p}$ (11.2) defined below, have appropriate inner products and are continuous at the horizon. We will show that the above problems are solved.

11 Quantization Behind the Horizon

Behind the horizon the scalar field will be expanded into linear combinations of the modes (9.13) introduced above. For the horizon to be smooth, the operators behind the horizon must be the same as those in the region I and III, and the mode functions multiplied by these operators on both sides of the horizon must be smoothly connected. As shown in the previous section, however, there are some problems related to the discontinuities of the inner product and the linear dependence for mode functions behind the horizon. In order to solve these problems the matching condition for normal modes and the continuous inner product in the whole region II will be studied. We will show that the mode function solution is obtained and these inner products are continuous. All above problems will be solved. Furthermore it will be found that the solution contains parameters $F^{(1)}_n(E,p)$, $F^{(2)}_n(E,p)$ $(n = 1, 2)$ and it is not unique.

11.1 Normal Mode Expansion of a Scalar Field in Region II

At first we will consider the construction of a scalar field in the region II and the Klein-Gordon inner product at $\eta > 2/\sqrt{a}$. A scalar field in the region II will be expanded in terms of the operators $c_R(E,p)$ and $c_L(E,p)$, which are the same as those in the region I and III, for the smooth connection of the normal modes at the horizons [9]. In order to find appropriate normal modes multiplied by these operators, let us consider the exponentials $e^{\pm iEt\pm ipx}$ in the normal modes. The exponentials $e^{\pm iEt\pm ipx}$ in the normal modes on both sides of the horizon, which are multiplied by the same operators, will coincide with each other. Then normal mode expansion of a scalar field in the region II is given by

$$\begin{aligned}\Phi^{\mathrm{II}}(t,\eta,x) = \frac{1}{4\pi\sqrt{\pi a}}\int_0^\infty dE\int_0^\infty dp\ \Big[&c_R(E,-p)\psi^{(1)}_{E,-p} + c_R(E,p)\psi^{(2)}_{E,p} \\ &+c_L(E,-p)\psi^{(3)}_{E,-p} + c_L(E,p)\psi^{(4)}_{E,p} + h.c.\Big].\end{aligned} \tag{11.1}$$

Here $c_R(E,p)$ and $c_L(E,p)$ with $E \geq 0$ and $-\infty < p < +\infty$ are two sets of annihilation operators which are the same as those in the region I and III. Note that this expression (11.1) is consistent with the fact that the eternal black hole is described by the Thermo-Field Double state (8.27). As the BTZ black hole is described by two CFTs, the scalar field is defined by two set of CFT operators. In the above expansion of a scalar field, the integration region for p is divided into two, $p > 0$ and $p < 0$, because $\Phi^{\mathrm{II}(\nu)}_{E,p}$ and $\Phi^{\mathrm{II}(\nu)}_{-E,-p}$ are related to each other by complex conjugation. $\psi^{(i)}_{E,p}(t,\eta,x)$ are normal mode functions defined by

$$\begin{aligned}
\psi^{(1)}_{E,-p} &= N^{(1)}_{E,p}\Big[\gamma^{(1)}_1(E,p)\Phi^{\mathrm{II}(\nu)}_{E,-p} + \gamma^{(1)}_2(E,p)\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,-p} + \gamma^{(1)}_3(E,p)\Phi^{\mathrm{II}(-\nu)}_{E,-p} + \gamma^{(1)}_4(E,p)\tilde{\Phi}^{\mathrm{II}(-\nu)}_{E,-p}\Big],\\
\psi^{(2)}_{E,p} &= N^{(2)}_{E,p}\Big[\gamma^{(2)}_1(E,p)\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p} + \gamma^{(2)}_2(E,p)\Phi^{\mathrm{II}(\nu)}_{E,p} + \gamma^{(2)}_3(E,p)\tilde{\Phi}^{\mathrm{II}(-\nu)}_{E,p} + \gamma^{(2)}_4(E,p)\Phi^{\mathrm{II}(-\nu)}_{E,p}\Big],\\
\psi^{(3)}_{E,-p} &= N^{(3)}_{E,p}\Big[\gamma^{(3)}_1(E,p)\tilde{\Phi}^{\mathrm{II}(\nu)}_{-E,p} + \gamma^{(3)}_2(E,p)\Phi^{\mathrm{II}(\nu)}_{-E,p} + \gamma^{(3)}_3(E,p)\tilde{\Phi}^{\mathrm{II}(-\nu)}_{-E,p} + \gamma^{(3)}_4(E,p)\Phi^{\mathrm{II}(-\nu)}_{-E,p}\Big],\\
\psi^{(4)}_{E,p} &= N^{(4)}_{E,p}\Big[\gamma^{(4)}_1(E,p)\Phi^{\mathrm{II}(\nu)}_{-E,-p} + \gamma^{(4)}_2(E,p)\tilde{\Phi}^{\mathrm{II}(\nu)}_{-E,-p} + \gamma^{(4)}_3(E,p)\Phi^{\mathrm{II}(-\nu)}_{-E,-p} + \gamma^{(4)}_4(E,p)\tilde{\Phi}^{\mathrm{II}(-\nu)}_{-E,-p}\Big]
\end{aligned} \tag{11.2}$$

where $E \geq 0$ and $p \geq 0$. Comparing (8.33) and (11.1) $\psi^{(i)}_{E,\mp p}(i = 1, 2)$ should behave as $e^{-iEt\mp ipx}$ and be smoothly connected such as $\psi^{(1)}_{E,-p} = \sqrt{1-e^{-\beta E}}\Phi^{\mathrm{I}}_{E,-p}$ and $\psi^{(2)}_{E,p} = \sqrt{1-e^{-\beta E}}\Phi^{\mathrm{I}}_{E,p}$ at the horizon between the region I and II. Similarly comparing (8.35) and (11.1) $\psi^{(i)}_{E,\mp p}(i = 3, 4)$ should behave as $e^{-iEt\pm ipx}$ and be smoothly connected such as $\psi^{(3)}_{E,-p} = \sqrt{1-e^{-\beta E}}\Phi^{\mathrm{III}}_{-E,p}$ and $\psi^{(4)}_{E,p} = \sqrt{1-e^{-\beta E}}\Phi^{\mathrm{III}}_{-E,-p}$ at the horizon between the region III and II. Functions $N^{(i)}_{E,p}$ and $\gamma^{(i)}_n(E,p)$ $(i = 1 \sim 4)$ should satisfy the above connection to mode functions outside the horizons.

The constraints to $\gamma^{(i)}_n(E,p)$ $(i = 1, 3)$ are given by orthogonality of the mode functions as $(\psi^{(3)*}_{E,-p}, \psi^{(1)}_{E',-p'}) = 0$. This is guaranteed by requiring $\gamma^{(3)}_1/\gamma^{(1)}_1 = \gamma^{(3)}_2/\gamma^{(1)}_2$ and $\gamma^{(3)}_3/\gamma^{(1)}_3 = \gamma^{(3)}_4/\gamma^{(1)}_4$. We will impose the following constraint.

$$\gamma^{(3)}_n(E,p) = \gamma^{(1)}_n(E,p) \quad (n = 1 \sim 4) \tag{11.3}$$

Although these constraints are a bit stronger than necessary, it is possible to show that there exist the solutions to matching conditions. Similarly we will require $(\psi^{(4)*}_{E,p}, \psi^{(2)}_{E',p'}) = 0$ and impose the following constraints.

$$\gamma^{(4)}_n(E,p) = \gamma^{(2)}_n(E,p) \quad (n = 1 \sim 4) \tag{11.4}$$

On the other hand orthogonality of $\psi^{(i)}_{E,\pm p}$ and $\psi^{(j)}_{E',\pm p'}$ with $i \neq j$ is ensured by the orthogonality properties of $e^{\pm iEt\pm ipx}$.

The normal modes $\psi^{(i)}_{E,\mp p}$ defined as above should have 'norm's normalized to unity as

$$(\psi^{(i)}_{E,\pm p}, \psi^{(i)}_{E',\pm p'})_{\mathrm{II}} = 16\pi^3 a\delta(E-E')\delta(p-p') \quad (i = 1 \sim 4). \tag{11.5}$$

Then operators c_R and c_L in (11.1) become to be consistent with the standard commutation relations (8.17) and (8.22). The norms of the normal modes $\psi^{(i)}_{E,\mp p}$ $(i = 1 \sim 4)$ are computed by using the results in Appendix C as

$$(\psi^{(i)}_{E,\mp p},\psi^{(i)}_{E',\mp p'})_{\mathrm{II}} = 8\pi^3 a\,[\sinh(\beta p/2)]^{-1}\left|N^{(i)}_{E,p}\right|^2\delta(E-E')\delta(p-p')$$
$$\times\left[e^{-\beta p/2}\left|\gamma^{(i)}_1+e^{-\pi i\nu}D^*(E,p)\gamma^{(i)}_3\right|^2-e^{\beta p/2}\left|\gamma^{(i)}_2+e^{-\pi i\nu}D(E,p)\gamma^{(i)}_4\right|^2\right]. \quad (11.6)$$

where $D(E,p)$ is defied by (C.17). In general these norms are not positive definite. Let us consider to impose $\gamma^{(i)}_2 = \gamma^{(i)}_4 = 0$. In this case the normal modes $\psi^{(i)}_{E,-p}$ are defined only by $\Phi^{\mathrm{II}(\nu)}_{E,p}$ and $\Phi^{\mathrm{II}(-\nu)}_{E,p}$. Although then (11.6) could be made positive definite, a solution to the matching conditions of the mode functions at horizons would not satisfy appropriate normalization of the inner products (11.5). Another prescription for restriction, $\gamma^{(i)}_3 = \gamma^{(i)}_4 = 0$, does not work, either. In what follows, solutions which satisfy suitable positive norms (11.5) will be found.

11.2 Matching Conditions

In the maximally extended Penrose diagram the black hole spacetime is expressed in terms of the Kruskal-Szeckeres coordinates, U and V (Fig. 5). The matching condition is expressed in terms of the Kruskal-Szeckeres coordinates. Each region has its own U, V coordinates.

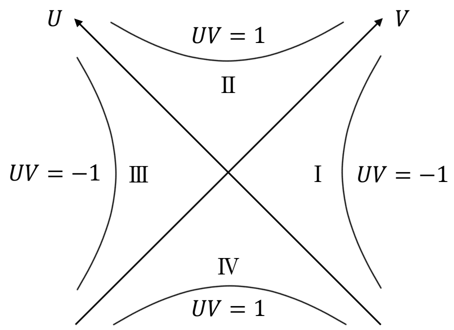

Fig. 5 Kruskal-Szeckeres coordinates

In the region I these coordinates are given by

$$U_{\mathrm{I}} = -e^{\sqrt{a}(r_*-t)} = -e^{-\sqrt{a}t}\frac{2-\sqrt{a}y}{2+\sqrt{a}y}, \qquad V_{\mathrm{I}} = e^{\sqrt{a}(r_*+t)} = e^{\sqrt{a}t}\frac{2-\sqrt{a}y}{2+\sqrt{a}y}. \quad (11.7)$$

Here r_* is the tortoise coordinate. In the region II they are given by

$$U_{\mathrm{II}} = e^{-\sqrt{a}t}\frac{\sqrt{a}\eta-2}{\sqrt{a}\eta+2}, \qquad V_{\mathrm{II}} = e^{\sqrt{a}t}\frac{\sqrt{a}\eta-2}{\sqrt{a}\eta+2}. \quad (11.8)$$

Similarly, in the region III they are

$$U_{\mathrm{III}} = e^{\sqrt{a}t}\frac{\sqrt{a}y-2}{\sqrt{a}y+2}, \qquad V_{\mathrm{III}} = -e^{-\sqrt{a}t}\frac{\sqrt{a}y-2}{\sqrt{a}y+2}. \quad (11.9)$$

Here the time direction in the region III is flipped compared to that in the usual maximally extended Penrose diagram to make the positive direction of t upward. The near-horizon behaviors of mode functions are either $V^{\pm iE/\sqrt{a}}$ or $U^{\pm iE/\sqrt{a}}$. We can confirm this by the above definition of the Kruskal coordinates and the near-horizon behaviors of $\Phi^{\mathrm{I}}_{E,p}$, $\Phi^{\mathrm{III}}_{E,p}$, $\Phi^{\mathrm{II}(\pm\nu)}_{E,p}$ and $\tilde{\Phi}^{\mathrm{II}(\pm\nu)}_{E,p}$ which are presented in Appendix D.

The matching conditions are depend on the following analysis. Near the horizon between the region I and II ($U=0, V=$ constant), it is important to keep the only terms depend on V. When $U\to 0$ terms like $U^{-iE/\sqrt{a}}$ can be discarded, because of the Riemann-Lebesgue lemma. This means that the only left mover in the region I enter into the region II. Therefore near the horizon $\psi^{(1)}_{E,-p}$ and $\psi^{(2)}_{E,p}$ multiply c_R must behave as $V^{-iE/\sqrt{a}}$. Similarly near the horizon between the region III and II ($V=0, U=$ constant), the only terms depend on U are kept and terms like $V^{-iE/\sqrt{a}}$ are discarded. Near the horizon $\psi^{(3)}_{E,-p}$ and $\psi^{(4)}_{E,p}$ multiply c_L must behave as $U^{-iE/\sqrt{a}}$. The coefficient $\gamma_n^{(1)}$, $\gamma_n^{(2)}$ and $N^{(1,2)}_{E,p}$ in (11.2) must determined based on this analysis.

The matching conditions are expressed as follows.

- At the horizon between the region I and II ($U=0, V=$ constant)

$$\begin{aligned}\psi^{(1)}_{E,-p} &= N^{(1)}_{E,p}e^{-ipx}\left[\gamma_1^{(1)}C^{\nu}(E,-p)+\gamma_2^{(1)}C^{\nu}(E,p)+\gamma_3^{(1)}C^{-\nu}(E,-p)+\gamma_4^{(1)}C^{-\nu}(E,p)\right]V^{-iE/\sqrt{a}}\\ &= \sqrt{1-e^{-\beta E}}e^{-ipx}2^{ip/\sqrt{a}}\frac{\Gamma\left(\frac{1}{2}(1+\nu-i\frac{E-p}{\sqrt{a}})\right)}{\Gamma\left(\frac{1}{2}(1+\nu+i\frac{E-p}{\sqrt{a}})\right)}e^{\beta E/4}\Gamma\left(i\frac{E}{\sqrt{a}}\right)(2V)^{-iE/\sqrt{a}}\\ &= b_1\, e^{-ipx}V^{-iE/\sqrt{a}}\end{aligned}\tag{11.10}$$

$$\begin{aligned}\psi^{(2)}_{E,p} &= N^{(2)}_{E,p}e^{ipx}\left[\gamma_1^{(2)}C_1^{\nu}(E,-p)+\gamma_2^{(2)}C^{\nu}(E,p)+\gamma_3^{(2)}C^{-\nu}(E,-p)+\gamma_4^{(2)}C_4^{-\nu}(E,p)\right]V^{-iE/\sqrt{a}}\\ &= \sqrt{1-e^{-\beta E}}e^{ipx}2^{-ip/\sqrt{a}}\frac{\Gamma\left(\frac{1}{2}(1+\nu-i\frac{E+p}{\sqrt{a}})\right)}{\Gamma\left(\frac{1}{2}(1+\nu+i\frac{E+p}{\sqrt{a}})\right)}e^{\beta E/4}\Gamma\left(i\frac{E}{\sqrt{a}}\right)(2V)^{-iE/\sqrt{a}}\\ &= b_2\, e^{ipx}V^{-iE/\sqrt{a}}\end{aligned}\tag{11.11}$$

- At the horizon between the region II and III ($V=0, U=$ constant)

$$\begin{aligned}\psi^{(3)}_{E,-p} &= N^{(3)}_{E,p}e^{ipx}\left[\gamma_1^{(1)}C^{\nu}(E,-p)+\gamma_2^{(1)}C^{\nu}(E,p)+\gamma_3^{(1)}C^{-\nu}(E,-p)+\gamma_4^{(1)}C^{-\nu}(E,p)\right]U^{-iE/\sqrt{a}}\\ &= \sqrt{1-e^{-\beta E}}e^{ipx}2^{-ip/\sqrt{a}}\frac{\Gamma\left(\frac{1}{2}(1+\nu-i\frac{E+p}{\sqrt{a}})\right)}{\Gamma\left(\frac{1}{2}(1+\nu+i\frac{E+p}{\sqrt{a}})\right)}e^{\beta E/4}\Gamma\left(i\frac{E}{\sqrt{a}}\right)(2U)^{-iE/\sqrt{a}}\\ &= b_3\, e^{ipx}U^{-iE/\sqrt{a}}\end{aligned}\tag{11.12}$$

$$\begin{aligned}\psi^{(4)}_{E,p} &= N^{(4)}_{E,p}e^{-ipx}\left[\gamma_1^{(2)}C^{\nu}(E,-p)+\gamma_2^{(2)}C^{\nu}(E,p)+\gamma_3^{(2)}C^{-\nu}(E,-p)+\gamma_4^{(2)}C^{-\nu}(E,p)\right]U^{-iE/\sqrt{a}}\\ &= \sqrt{1-e^{-\beta E}}e^{-ipx}2^{ip/\sqrt{a}}\frac{\Gamma\left(\frac{1}{2}(1+\nu-i\frac{E-p}{\sqrt{a}})\right)}{\Gamma\left(\frac{1}{2}(1+\nu+i\frac{E-p}{\sqrt{a}})\right)}e^{\beta E/4}\Gamma\left(i\frac{E}{\sqrt{a}}\right)(2U)^{-iE/\sqrt{a}}\\ &= b_4\, e^{-ipx}U^{-iE/\sqrt{a}}\end{aligned}\tag{11.13}$$

The conditions for vanishing of terms in $\psi^{(1,2)}$ proportional to $(-U)^{iE/\sqrt{a}}$ at the horizon between the region I and II are

$$\gamma_1^{(1)}C^{\nu}(-E,-p)+\gamma_2^{(1)}C^{\nu}(-E,p)+\gamma_3^{(1)}C^{-\nu}(-E,-p)+\gamma_4^{(1)}C^{-\nu}(-E,p)=0,\tag{11.14}$$
$$\gamma_1^{(2)}C^{\nu}(-E,-p)+\gamma_2^{(2)}C^{\nu}(-E,p)+\gamma_3^{(2)}C^{-\nu}(-E,-p)+\gamma_4^{(2)}C^{-\nu}(-E,p)=0.\tag{11.15}$$

Exactly the same conditions are obtained for vanishing of terms in $\psi^{(3,4)}$ proportional to $(-V)^{iE/\sqrt{a}}$ at the horizon between the region III and II. Here $C^{\pm\nu}$ are defined by

$$C^{\pm\nu}(E,p)=e^{-\beta E/4}\Gamma\left(i\frac{E}{\sqrt{a}}\right)2^{-i\frac{E+p}{\sqrt{a}}}\frac{\Gamma\left(\frac{1}{2}(1\pm\nu-i\frac{E+p}{\sqrt{a}})\right)}{\Gamma\left(\frac{1}{2}(1\pm\nu+i\frac{E+p}{\sqrt{a}})\right)}.\tag{11.16}$$

Note that then $\psi^{(3)}_{E,-p}$ and $\psi^{(4)}_{E,p}$ are only proportional to $U^{-iE/\sqrt{a}}$ and vanish at the horizon between I and II where $U = 0$. If $\psi^{(3)}_{E,-p}$ and $\psi^{(4)}_{E,p}$ have terms proportional to $V^{-iE/\sqrt{a}}$, at the horizon between I and II there are mode functions which are proportional to $V^{-iE/\sqrt{a}}$ and are multiplied by c_L. This mode functions can not be smoothly connected to mode functions in the region I, which are multiplied by c_R. Similarly $\psi^{(1)}_{E,-p}$ and $\psi^{(2)}_{E,p}$ are only proportional to $V^{-iE/\sqrt{a}}$ and vanish at the horizon between III and II where $V = 0$. Hence (11.14) is the condition for $\psi^{(3)}_{E,-p}$ and $\psi^{(4)}_{E,p}$ at the horizon between I and II, and (11.15) is the condition for $\psi^{(1)}_{E,-p}$ and $\psi^{(2)}_{E,p}$ at the horizon between III and II.

11.3 Solutions

By solving (11.10)-(11.15) the following solution for $\gamma_n^{(i)}$ are obtained. Details are given in Appendix E.

$$\gamma_1^{(1)} = K\, e^{-\pi i\nu}\left[e^{\beta p/2}e^{i\delta_1} - D^*(E,p)(e^{-\beta p/2}e^{i\delta_2} - J^{(1)})\right] \tag{11.17}$$

$$\gamma_3^{(1)} = K\left[-e^{\beta p/2}e^{i\delta_1} + e^{-\beta p/2}e^{i\delta_2} - J^{(1)}\right] \tag{11.18}$$

Here

$$K \equiv \frac{\left[e^{\beta(E-p)/2} + e^{\pi i\nu}\right]\left[e^{\beta(E+p)/2} + e^{-\pi i\nu}\right]}{4ie^{\beta E/2}\sinh(\beta p/2)e^{-\pi i\nu}\sin\pi\nu}, \tag{11.19}$$

$$J^{(1)} \equiv \frac{b_1}{N^{(1)}_{E,p}}\frac{Ee^{-\beta E/2}e^{-\pi i\nu}}{2\pi\sqrt{a}}C^{\nu}(-E,p)\left[e^{\beta(E-p)/2} + e^{\pi i\nu}\right]. \tag{11.20}$$

b_1 is defined in (E.7), and

$$\gamma_2^{(1)} = -\frac{M_1}{M_2}\gamma_1^{(1)} + \frac{C^{-\nu}(-E,p)}{M_2(E,p)}\frac{b_1}{N^{(1)}_{E,p}}, \tag{11.21}$$

$$\gamma_4^{(1)} = \frac{M_3}{M_2}\gamma_3^{(1)} - \frac{C^{\nu}(-E,p)}{M_2(E,p)}\frac{b_1}{N^{(1)}_{E,p}}. \tag{11.22}$$

Here $M_{1,2,3}$ are defined in (E.4)-(E.6). Solutions for $\gamma_n^{(2)}$ $(n = 1 \sim 4)$ are also presented in Appendix E.

By using these results norms of $\psi^{(i)}_{E,\mp p}(i = 1 \sim 4)$ are found to be

$$(\psi^{(i)}_{E,\mp p}, \psi^{(i)}_{E',\mp p'})_{\mathrm{II}} = 16\pi^3 a\left|N^{(i)}_{E,p}\right|^2\delta(E-E')\delta(p-p'). \tag{11.23}$$

If $N^{(i)}_{E,p}$ are phase factors, (11.5) hold. These norms are positive-definite and normalized to unity. Due to the pre-factor $1/(4\pi\sqrt{\pi a})$ in the scalar field (11.1), the operators $c_{R,L}(E,p)$ must satisfy

$$[c_R(E,p), c_R^\dagger(E',p')] = \delta(E-E')\delta(p-p') \tag{11.24}$$

and a similar relation for c_L and $c_L^\dagger$. These relations agree with (8.17) and (8.22), and then the scalar field is constructed in terms of the same operators outside the horizons.

The above solution, however, contains phase factors $e^{i\delta_i(E,p)}$ and $N^{(i)}_{E,p}$ $(i = 1 \sim 4)$. Furthermore, it is possible to obtain more general solutions which satisfy the matching conditions and the condition of normalization of normal modes. Let $F_1^{(i)}(E,p)$ and $F_2^{(i)}(E,p)$ $(i = 1,2)$ be real functions of E and p which satisfy for $E, p > 0$

$$e^{-\beta p/2}F_1^{(i)}(E,p) - e^{\beta p/2}F_2^{(i)}(E,p) = e^{\beta p/2} - e^{-\beta p/2},$$
$$F_1^{(i)}(E,p) \geq 0,\ F_2^{(i)}(E,p) \geq 0 \quad (i = 1,2). \tag{11.25}$$

These functions $F_n^{(i)}$ can be parametrized as

$$F_1^{(i)}(E,p) = \left(e^{\beta p} - 1\right) \cosh^2 \alpha^{(i)}(E,p) + e^{\beta p/2} W^{(i)}(E,p), \tag{11.26}$$
$$F_2^{(i)}(E,p) = \left(1 - e^{-\beta p}\right) \sinh^2 \alpha^{(i)}(E,p) + e^{-\beta p/2} W^{(i)}(E,p) \tag{11.27}$$

where $\alpha^{(i)}(E,p) \geq 0$ and $W^{(i)}(E,p) \geq 0$ $(i = 1, 2)$.

Then $\gamma_1^{(1)}$ and $\gamma_3^{(1)}$ are defined to be solutions to the following equations.

$$\left|\gamma_1^{(1)} + \gamma_3^{(1)} e^{-\pi i \nu} D^*(E,p)\right|^2 = F_1^{(1)}(E,p) \tag{11.28}$$
$$\left|\gamma_1^{(1)} + \gamma_3^{(1)} e^{-\pi i \nu} + J^{(1)}\right|^2 = F_2^{(1)}(E,p) \tag{11.29}$$

If $F_1^{(1)} = e^{\beta p}$, $F_2^{(1)} = e^{-\beta p}$, these agree with the results mentioned above. In the general case the phase factors $e^{i\delta_{1,2}}$ in the above results are replaced by $e^{i\delta_{1,2}}\sqrt{F_{1,2}^{(1)}}$. In the same way by defining $\gamma_{1,3}^{(2)}$ in terms of $F_{1,2}^{(2)}(E,p)$ as in (11.28) and (11.29) the solutions for $\gamma_n^{(2)}$ $(n = 1 \sim 4)$ are obtained. These solutions depend on phase factors $e^{i\delta_{1,2,3,4}}$ and functions $F_{1,2}^{(1,2)}(E,p)$. Thus there are too many possibilities for quantum theory of the scalar field inside the horizon.

11.4 Continuity of Inner Products for $\psi_{E,p}^{(i)}$

In Sec.10 it was noticed that the inner products of the basis functions (9.13) are discontinuous in η at $\eta = 2/\sqrt{a}$. Difficulties to construct a scalar field in the region II was expected. We solved some of them in the previous section, where it was shown that special linear combinations of modes functions satisfy the matching conditions and the appropriate normalization condition for $\eta > 2/\sqrt{a}$. In this section it will be studied whether the inner products of $\psi_{E,p}^{(i)}$ in (11.2) are continuous or not, if the solutions for $\gamma_n^{(i)}$ are substituted into (11.2). It will be shown that for the solution to the matching condition the inner products of the normal modes at the horizon is smoothly connected to those far from the horizon.

Let us consider $\psi_{E,-p}^{(1)}$. By substitution of (11.21) and (11.22) it is found that

$$\begin{aligned}\psi_{E,-p}^{(1)} = N_{E,p}^{(1)}\, \gamma_1^{(1)} \left[\Phi_{E,-p}^{\mathrm{II}(\nu)} - \frac{M_1}{M_2} \tilde{\Phi}_{E,-p}^{\mathrm{II}(\nu)}\right] + N_{E,p}^{(1)}\, \gamma_3^{(1)} \left[\Phi_{E,-p}^{\mathrm{II}(-\nu)} + \frac{M_3}{M_2} \tilde{\Phi}_{E,-p}^{\mathrm{II}(-\nu)}\right] \\ + \frac{b_1}{M_2}\, C^{-\nu}(-E,p) \left[\tilde{\Phi}_{E,-p}^{\mathrm{II}(\nu)} - \frac{C^{\nu}(-E,p)}{C^{-\nu}(-E,p)}\, \tilde{\Phi}_{E,-p}^{\mathrm{II}(-\nu)}\right].\end{aligned} \tag{11.30}$$

At the horizon those factors in the first line which multiply $\gamma_1^{(1)}$ and $\gamma_3^{(1)}$, respectively, vanish due to the relation (10.5). In fact, the latter relation can be rewritten as $\Phi_{E,-p}^{\mathrm{II}(\nu)} = [C^{\nu}(-E,-p)/C^{\nu}(-E,p)]\, \tilde{\Phi}_{E,-p}^{\mathrm{II}(\nu)} = [M_1/M_2]\, \tilde{\Phi}_{E,-p}^{\mathrm{II}(\nu)}$. Thus $\psi_{E,-p}^{(1)}$ is independent of $\gamma_{E,-p}^{(1)}$, $\gamma_{E,-p}^{(3)}$ and $N_{E,p}^{(1)}$ at the horizon. When the inner product of the last term of (11.30) is computed at the horizon by using (D.9)-(D.13), we obtain at $\eta = 2/\sqrt{a}$

$$(\psi_{E,-p}^{(1)}, \psi_{E',-p'}^{(1)})_{\mathrm{H}} = 16\pi^3 a\, \delta(E - E')\delta(p - p'). \tag{11.31}$$

Here H stands for horizon. It can be shown that the other inner products, $(\psi_{E,p}^{(2)}, \psi_{E',p'}^{(2)})_{\mathrm{H}}$, $(\psi_{E,-p}^{(3)}, \psi_{E',-p'}^{(3)})_{\mathrm{H}}$ and $(\psi_{E,p}^{(4)}, \psi_{E',p'}^{(4)})_{\mathrm{H}}$ have the same forms as above at the horizon. Thus, although inner products of $\Phi_{E,p}^{\mathrm{II}(\pm\nu)}$ and $\tilde{\Phi}_{E,p}^{\mathrm{II}(\pm\nu)}$ are discontinuous at the horizon $\eta = 2/\sqrt{a}$, the inner products of 'the special linear combinations $\psi_{E,p}^{(i)}$' are continuous at the horizon. This ensure that the horizon is smooth for the scalar field, hence the prescription of [9] works and there are no firewalls for a scalar field at the horizon.

On the other hand $\psi^{(i)}_{E,p}$ values depend on $\gamma^{(i)}_{E,p}$ values away from the horizon, because equations which relate $\Phi^{\mathrm{II}(\nu)}_{E,p}$ and $\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}$ are not valid there. Instead (10.3) and (10.4) hold. By using (E.13), (E.14), (E.1) and (E.3) the following expression is obtained away from the horizon.

$$\psi^{(1)}_{E,-p} = N^{(1)}_{E,p}\left[\sqrt{F^{(1)}_1(E,p)}e^{i\delta_1}\Phi^{\mathrm{II}(\nu)}_{E,-p} + \sqrt{F^{(1)}_2(E,p)}e^{i\delta_2}\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,-p}\right] \tag{11.32}$$

This has the correct K-G norm (11.5) owing to (11.25), (C.9) and (C.12). Similarly, the other normal modes are found to be

$$\psi^{(2)}_{E,p} = N^{(2)}_{E,p}\left[\sqrt{F^{(2)}_1(E,p)}e^{i\delta_3}\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p} + \sqrt{F^{(2)}_2(E,p)}e^{i\delta_4}\Phi^{\mathrm{II}(\nu)}_{E,p}\right], \tag{11.33}$$

$$\psi^{(3)}_{E,-p} = N^{(3)}_{E,p}\left[\sqrt{F^{(1)}_1(E,p)}e^{i\delta_1}\tilde{\Phi}^{\mathrm{II}(\nu)}_{-E,p} + \sqrt{F^{(1)}_2(E,p)}e^{i\delta_2}\Phi^{\mathrm{II}(\nu)}_{-E,p}\right], \tag{11.34}$$

$$\psi^{(4)}_{E,p} = N^{(4)}_{E,p}\left[\sqrt{F^{(2)}_1(E,p)}e^{i\delta_3}\Phi^{\mathrm{II}(\nu)}_{-E,-p} + \sqrt{F^{(2)}_2(E,p)}e^{i\delta_4}\tilde{\Phi}^{\mathrm{II}(\nu)}_{-E,-p}\right]. \tag{11.35}$$

These also have the correct K-G norm (11.5). The near horizon and deep horizon normal modes are expressed as (11.30) and these are connected. The near horizon normal mode, which is identified as the last term of (11.30), is unique, while the deep horizon normal mode depends on arbitrary functions $F^{(1)}_n(E,p)$ $(n = 1,2)$ and is not unique. In general curved spacetime, since there is no isometry for time direction, it is not possible to distinguish between positive- and negative-frequency solutions and the normal modes solution is not unique. Indeed, inside the horizon there is no isometry for the time variable η and the number of independent modes is doubled. Therefore solutions which contain continuous parameters can exist. This means that the matching condition [9] is not sufficient for determining the quantum scalar field inside the horizon uniquely. It is necessary to impose yet additional conditions to carry out bulk reconstruction. In the next section a natural choice of these parameters will be proposed.

Because the mode functions $\psi^{(i)}_{E,p}$ $(i = 1 \sim 4)$ are defined for either $p \geq 0$ or $p \leq 0$, these must satisfy continuity conditions at $p = 0$.

$$\psi^{(1)}_{E,p=0} = \psi^{(2)}_{E,p=0} \tag{11.36}$$

$$\psi^{(3)}_{E,p=0} = \psi^{(4)}_{E,p=0} \tag{11.37}$$

It can be shown that if $W^{(i)}(E,p=0) \neq 0$ $(i = 1,2)$, where $W^{(i)}$ is defined in (11.26) and (11.27), the following relations must be satisfied.

$$e^{i(\delta_1(E,p=0)-\delta_2(E,p=0))} = e^{-i(\delta_3(E,p=0)-\delta_4(E,p=0))} \tag{11.38}$$

$$\frac{N^{(1)}_{E,p=0}}{N^{(3)}_{E,p=0}} = \frac{N^{(2)}_{E,p=0}}{N^{(4)}_{E,p=0}} \tag{11.39}$$

In contrast, it can be shown that at the horizon the above conditions (11.36) and (11.37) are satisfied without constraints.

12 Determination of $F^{(1)}_n$, $F^{(2)}_n$ and the Propagator for Two Points on Both Sides of the Horizon

As was seen in the previous section it is not enough to determine the normal modes behind the horizon uniquely by only the matching condition. If the ambiguity of the scalar field cannot be removed, holographic correspondence of the interior to the boundary CFTs is not realized. It is necessary to impose additional conditions. In this section we will propose a set of conditions for removing the ambiguity. Then using the mode expansion of the scalar field inside the horizon, a scalar field propagator with one point inside the horizon and the other outside will be derived in an integral form.

Additional Conditions for a Scalar Field

Let us consider the normal mode away from the horizon (11.32)-(11.35). From the asymptotic formulas (C.4) and (C.7) for $\Phi^{\mathrm{II}(\nu)}_{E,p}$ and $\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}$, respectively, it is noticed that the first term of (11.32) behaves as $\eta^{-ip/\sqrt{a}}$ and the second one as $\eta^{+ip/\sqrt{a}}$. Each term of (11.33), (11.34) and (11.35) also behaves in the same way. In these four ψ terms, p is a positive number, and the term $\eta^{-ip/\sqrt{a}} = e^{-i(p/\sqrt{a})\ln\eta}$ may be regarded as a 'positive frequency' term in the large η asymptotic limit. Therefore we will set the following conditions so that all normal modes (11.32)- (11.35) are 'positive frequency' modes.

$$F^{(1)}_1(E,p) = F^{(2)}_1(E,p) = e^{\beta p} - 1 \tag{12.1}$$
$$F^{(1)}_2(E,p) = F^{(2)}_2(E,p) = 0 \tag{12.2}$$

Note that these conditions satisfy (11.25). These are conditions for the normal modes deep inside the horizon. For simplicity by setting $N^{(i)}_{E,p} = 1$ $(i = 1 \sim 4)$ and $\delta_n(E,p) = 0$ $(n = 1 \sim 4)$, we obtain

$$\psi^{(1)}_{E,-p} = \sqrt{e^{\beta p} - 1}\,\Phi^{\mathrm{II}(\nu)}_{E,-p}, \tag{12.3}$$
$$\psi^{(2)}_{E,p} = \sqrt{e^{\beta p} - 1}\,\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}, \tag{12.4}$$
$$\psi^{(3)}_{E,-p} = \sqrt{e^{\beta p} - 1}\tilde{\Phi}^{\mathrm{II}(\nu)}_{-E,p}, \tag{12.5}$$
$$\psi^{(4)}_{E,p} = \sqrt{e^{\beta p} - 1}\,\Phi^{\mathrm{II}(\nu)}_{-E,-p}. \tag{12.6}$$

This is the simplest set of normal modes. Then the mode expansion (11.1) contains only these 'positive frequency' normal modes and their complex conjugates. We propose to adopt the above conditions for F terms and remove the ambiguity as (12.3)-(12.6).

It is possible to show that the ambiguity associated with $F^{(i)}_n(E,p)$ is related to the Bogolubov transformations. Under the Bogolubov transformations operators $c_R(E,p)$ and $c_L(E,p)$ and their Hermitian conjugates (h.c.) are transformed into new operators, $d_R(E,p)$ and $d_L(E,p)$ and their h.c. By adjusting suitable transformations the mode expansion for the scalar field (11.1) is transformed to

$$\begin{aligned}\Phi^{\mathrm{II}}(t,\eta,x) = \frac{1}{4\pi\sqrt{\pi a}}\int_0^\infty dE\int_0^\infty dp\,\sqrt{e^{\beta p}-1}\Big[&d_R(E,-p)\Phi^{\mathrm{II}(\nu)}_{E,-p} + d_R(E,p)\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}\\ &+d_L(E,p)\Phi^{\mathrm{II}(\nu)}_{-E,-p} + d_L(E,-p)\tilde{\Phi}^{\mathrm{II}(\nu)}_{-E,p} + h.c.\Big].\end{aligned} \tag{12.7}$$

The above expansion is a result of four sets of transformations. For example,

$$\begin{aligned}d_R(E,-p) = (e^{\beta p}-1)^{-1/2}\Big[&c_R(E,-p)N^{(1)}\sqrt{F^{(1)}_1}e^{i\delta_1}\\ &+c^\dagger_L(E,-p)N^{(3)*}\sqrt{F^{(1)}_2}e^{\beta p/2}e^{-i\delta_2}e^{-i\pi(\nu+1)}\Big]\end{aligned} \tag{12.8}$$

and $d^\dagger_R(E,-p)$ are given by the Hermitian conjugate of the above equation. The ambiguity of the solution which satisfies the matching condition coincides with the parameters of these transformations as (12.8). In curved spacetime without isometry for the time direction, the mode expansion of a scalar field is not unique, and then the mode functions of each expansion are related with each other by the Bogolubov transformation, which is reviewed in Sec.3.5. Therefore origin of the ambiguity of the scalar field is the degree of freedom for the Bogolubov transformations in the region behind the horizon. Now $F^{(i)}_n$ terms have disappeared from (12.7). New operators, d_R and d_L, do not have the thermal averages such as (8.28) in the state $|\Psi_{\mathrm{TFD}}\rangle_\beta$, unless[*3] imposing (12.1).

[*3] Even if eqs (12.1) are not satisfied, the mode expansion (12.7) defines a quantum scalar theory at finite temperature inside the horizon, because the scalar field (11.1) is expressed in terms of $c_{L,R}$'s.

Correlation Function on Both Sides of the Horizon

With the choice (12.3)-(12.6) a propagator for two points in the region I and II, respectively, are given by

$$
\begin{aligned}
&{}_\beta\langle\Psi_{\mathrm{TFD}}|\Phi^{\mathrm{I}}(y_1,t_1,x_1)\Phi^{\mathrm{II}}(t_2,\eta_2,x_2)|\Psi_{\mathrm{TFD}}\rangle_\beta \\
&= \frac{1}{16\pi^3 a}\int_0^\infty dp\int_0^\infty dE\,\sqrt{\frac{e^{\beta p}-1}{1-e^{-\beta E}}}\Big[\Phi^{\mathrm{I}}_{E,p}\,\tilde{\Phi}^{\mathrm{II}(\nu)*}_{E,p}+\Phi^{\mathrm{I}}_{E,-p}\,\Phi^{\mathrm{II}(\nu)*}_{E,-p}+e^{-\beta E}\,\Phi^{\mathrm{I}*}_{E,p}\,\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}+e^{-\beta E}\,\Phi^{\mathrm{I}*}_{E,-p}\,\Phi^{\mathrm{II}(\nu)}_{E,-p} \\
&+e^{-\beta E/2}\,\Phi^{\mathrm{I}}_{E,p}\,\Phi^{\mathrm{II}(\nu)}_{-E,-p}+e^{-\beta E/2}\,\Phi^{\mathrm{I}}_{E,-p}\,\tilde{\Phi}^{\mathrm{II}(\nu)}_{-E,p}+e^{-\beta E/2}\,\Phi^{\mathrm{I}*}_{E,p}\,\Phi^{\mathrm{II}(\nu)*}_{-E,-p}+e^{-\beta E/2}\,\Phi^{\mathrm{I}*}_{E,-p}\,\tilde{\Phi}^{\mathrm{II}(\nu)*}_{-E,p}\Big]. \qquad (12.9)
\end{aligned}
$$

This is obtained by using (8.28), (8.33) and (11.1). An explicit calculation of the integral on the right-hand side is not carried out yet. It is, however, possible to argue that this propagator will not be the same function as those obtained in [8] and [26]. In fact, if the propagator is a function only of the geodesic distance, it will take the form

$$
\begin{aligned}
G^{\mathrm{I,II}} &= \frac{1}{16\pi^3 a}\int_{-\infty}^\infty dp\int_0^\infty dE\,\Big[\Phi^{\mathrm{I}}_{E,p}\,\Phi^{\mathrm{II}(\nu)*}_{E,p}+\Phi^{\mathrm{I}}_{-E,-p}\,\Phi^{\mathrm{II}(\nu)*}_{-E,-p}\Big] \\
&= \frac{1}{16\pi^3 a}\int_{-\infty}^\infty dp\int_0^\infty dE\,\Big[\Phi^{\mathrm{I}}_{E,p}\,\Phi^{\mathrm{II}(\nu)*}_{E,p}+e^{-\pi i(\nu+1)}e^{-\beta(E+p)/2}\Phi^{\mathrm{I}*}_{Ep}\,\Phi^{\mathrm{II}(\nu)}_{E,p}\Big]. \qquad (12.10)
\end{aligned}
$$

This is because (12.10) can be rewritten as

$$
G^{\mathrm{I,II}} = \int_0^\infty \frac{d\omega}{4\pi}\int_{|k|\le\omega} dk\,\Phi^{\mathrm{I}}_{\omega,k}(y_1,t_1,x_1)\Phi^{\mathrm{II}(\nu)*}_{\omega,k}(t_2,\eta_2,x_2) \qquad (12.11)
$$

where $\Phi^{\mathrm{I}}_{\omega,k}$ and $\Phi^{\mathrm{II}(\nu)}_{\omega,k}$ are defined in (7.8) and (9.3), respectively. The result is [26][37]

$$
G^{\mathrm{I,II}} = \frac{1}{2\pi}\rho^{(\nu+1)/2}/(1-\rho) \qquad (12.12)
$$

where

$$
\rho = \frac{1}{\xi^2}\left[1-\sqrt{1-\xi^2}\right]^2 \equiv e^{-2\sigma}, \qquad (12.13)
$$

$$
\xi = \frac{4\sqrt{a}y_1(a\eta_2^2+4)}{4\sqrt{a}\eta_2(4+ay_1^2)\cosh\left(\sqrt{a}(x_1-x_2)\right)-(4-ay_1^2)(a\eta_2^2-4)\sinh\left(\sqrt{a}(t_1-t_2)\right)}. \qquad (12.14)
$$

Here σ is a geodesic distance between the two points.

Because the normal modes in the region II, $\psi^{(i)}_{E,p}$, contain $\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}$ in addition to $\Phi^{\mathrm{II}(\nu)}_{E,p}$, the propagator (12.9) inevitably contains this additional mode, and the propagator cannot be transformed into the integral form (12.11).[*4] At the very least, this propagator (12.9) will not be obtained from that in AdS_3 by coordinate transformations and not coincide with the result in other works [8][26]. Even for choices of $F^{(1)}_{1,2}$ and $F^{(2)}_{1,2}$ other than (12.1), this observation will also be valid. Furthermore, the propagator in terms of the geodesic distance (12.10) is not correct as the propagator for points in the region I and II. (12.10) is obtained from the propagator for two points in the region I,

$$
G^{\mathrm{I,I}} = \frac{1}{16\pi^3 a}\int_{-\infty}^\infty dp\int_0^\infty dE\,\left[\Phi^{\mathrm{I}}_{E,p}\,\Phi^{\mathrm{I}*}_{E,p}+\Phi^{\mathrm{I}}_{-E,-p}\,\Phi^{\mathrm{I}*}_{-E,-p}\right]. \qquad (12.15)
$$

By coordinate transformations (A.7) and (A.23), and also by analytic continuation $t_2\to t_2-i\beta/4$, the integrand of (12.15) is replaced to that of (12.10). The propagator outside the horizon (12.15) is calculated

[*4] See the footnote 2 for the (ω,k) representation of $\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}$.

by (8.33) and the thermal expectation values in terms of c_R (8.28). Therefore, the propagator (12.10) does not contain the degree of freedom with respect to a scalar particle in the region Ⅲ. If the integral for the propagator could be computed explicitly, these claims would be established. This will be left for study in the future. It is also interesting to study the propagator with both points inside the horizon.

Finally, the two-point function between fields on both sides of the horizon in the BTZ black hole background which is periodic for $\varphi \to \varphi + 2\pi$ is obtained by replacing x by φ and summing over images of (12.9) obtained by a shift of φ_1.

$$G^{\mathrm{I},\mathrm{II}}_{\mathrm{BTZ}}(y_1, t_1, \varphi_1 | t_2, \eta_2, \varphi_2) = \sum_{n=-\infty}^{\infty} {}_{\beta}\langle \Psi_{TFD}| \, \Phi^{\mathrm{I}}(y_1, t_1, \varphi_1 + 2\pi n) \Phi^{\mathrm{II}}(t_2, \eta_2, \varphi_2) \, |\Psi_{TFD}\rangle_{\beta} \tag{12.16}$$

13 Summary and Future Works

In this thesis, we studied the quantization of a free scalar field inside and outside the horizon in the BTZ black hole background. We carried out the quantization of a scalar field in the AdS-Rindler space, and obtained the propagator in the BTZ black hole by the method of mirror images through the locally equivalence between the AdS-Rindler space and the BTZ black hole. In previous works by other groups, the bulk propagator was also constructed through this equivalence of spacetime in the same way, while the scalar field in the AdS-Rindler space was obtained from those in pure AdS_3 space by suitable coordinate transformations. The quantization was carried out in global AdS space [26]. Therefore it is unknown whether the results of the quantization in the AdS-Rindler space are consistent with the previous works. In what follows we will summarize the quantization for the explicit forms of scalar fields in the BTZ black hole and the results, and will confirm whether there is consistency with other works or not.

We started from the construction of a scalar field outside the horizon. The normal modes of a scalar field outside the horizon in the BTZ black hole background are obtained from those of AdS_3 spacetime by suitable coordinate transformations. These normal modes turn out to be not eigenfunctions of energy and momentum. We quantized a scalar field on the same constant t slice Σ in the both region I and Ⅲ where the Klein-Gordon inner products for these normal modes are calculated. By changing the basis of the normal modes to that of eigenfunctions of energy and momentum, it is found that the scalar field in the region I and Ⅲ is constructed as the CFT primary operators on each boundary and the vacuum state is identified as the TFD state (1.2). We expressed the bulk reconstruction outside the horizon.

The scalar field behind the horizon in the eternal BTZ black hole is quantized by imposing the matching condition [9] and the orthonormality of the Klein-Gordon inner products on mode functions. The Klein-Gordon inner products for these mode functions are calculated on constant time η slice. The matching condition guarantee the scalar fields behind the horizon to be smoothly connected to those outside the horizon, and the orthonormality of K-G inner products in the whole region II imply that the scalar field satisfy correct equal time commutation relations behind the horizon. It is found that the scalar field is expanded in terms of the normal modes which are expressed as linear combinations of $\Phi^{\mathrm{II}(\nu)}_{E,p}$, $\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}$, $\Phi^{\mathrm{II}(-\nu)}_{E,p}$ and $\tilde{\Phi}^{\mathrm{II}(-\nu)}_{E,p}$. These normal modes just inside the horizons are expressed in terms of $\Phi^{\mathrm{II}(\nu)}_{E,p}$ and $\Phi^{\mathrm{II}(-\nu)}_{E,p}$, while deep inside the horizon they are expressed in terms of $\Phi^{\mathrm{II}(\nu)}_{E,p}$ and $\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}$. It is, however, shown that there are still undetermined functions $F^{(i)}_n(E,p)$ in the scalar field deep inside the horizon. The matching condition at the horizon is not sufficient to determine a scalar theory inside the horizons of the BTZ black hole uniquely. $F^{(i)}_n(E,p)$ is appeared in the normal modes deep inside the horizon, and then we imposed additional conditions (12.1) so that these normal modes become 'positive frequency' modes.

We obtained the explicit form of the scalar field behind the horizon with ambiguity which is removed by the 'positive frequency' conditions for the normal modes deep inside the horizon. In contrast, Papadodimas and Raju proposed to construct the scalar field behind the horizon by imposing the matching condition which is the boundary condition at the horizon [9]. They confirmed that for short distance two-point function with one point inside the horizon and another point outside the horizon is given by the geodesic distance. We also calculated the behavior of two-point function across the horizon for short distance by using the normal modes which satisfy the matching condition (11.10)-(11.13), and then obtained the same result exactly. Therefore our results are consistent with Papadodimas and Raju works near the horizon region. Furthermore taking the standard commutation relation on the scalar field into account, we extended the analysis into the region deep inside the horizon. Then it is found that the scalar field behind the horizon is expressed in terms of the extra modes which are not obtained from the normal modes in pure AdS_3 space by coordinate transformations and the normal modes in the scalar field inside and near the horizon is distinct from those deep inside the horizon. Hence the matching condition is not sufficient for bulk reconstruction inside the horizon.

With the use of the quantized scalar field behind the horizon without ambiguity, a scalar propagator between two points on opposite sides of the horizon, respectively, is obtained. This propagator is not consistent with the propagator in [8][21][26], because the mode functions (11.32)-(11.35) are composed of both $\Phi^{\mathrm{II}(\nu)}_{E,p}$ and $\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}$. If the prescription of the ambiguity behind the horizon in Sec.12 is appropriate, then the CFT on the boundary have the information to describe the bulk fields in the whole region of the BTZ black hole. It will be possible to obtain information on the structure of the interior of the horizon by studying the propagator (12.9). In Sec.10, however, it is shown that there are discontinuities of both $\Phi^{\mathrm{II}(\nu)}_{E,p}$ and $\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}$ at the horizon. If the propagator (12.9) does not coincide with those in other works [8][21][26], this might imply that the propagator (12.9) have some non-analyticity at the horizon. The analysis of the integral representation for the propagator to solve this question is future work.

The result of this paper will have important implications on the quantization of matter fields in higher-dimensional AdS black hole backgrounds. As it was found in the BTZ black hole in this paper, normal modes of a scalar field inside and near the horizon of higher-dimensional black holes may be distinct from those deep inside the horizon. It is expect that the normal modes deep inside the horizon have new properties. Goto and Takayanagi also proposed the bulk local operators inside the horizon of the BTZ black hole. In the higher-dimensional AdS black hole, however, their method does not work since there are not the analytic continuations in order to obtain the propagator, such as $(t, \phi) \to (-i\phi_r, -it_r)$ in the BTZ black hole. Since there are severe curvature singularities behind the horizon in the higher-dimensional black holes, it will not be easy to analyze the whole interior region. In addition, ambiguity of normal modes deep inside the horizon, as shown in this paper, might occur. Then in the higher dimension we will study the matching condition at the horizon for quantized scalar field, and the analysis may be possible only in the region of small curvature. It might be necessary to cope with the ambiguity of the normal mode expansion. In such a case it would be necessary to come up with a new principle to remove this ambiguity to comply with the holographic principle. Because the singularity is stronger than that in the three-dimension, it is not clear whether it is appropriate to impose the conditions at the region deep inside the horizon. This issue needs to be studied further.

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A Connection of Normal Modes of a Scalar Field in BTZ Black Hole to those in AdS$_3$ Space

The metric for massless BTZ black hole is given by

$$ds^2 = \frac{1}{u^2}\left(du^2 + dw^+ dw^-\right). \tag{A.1}$$

Here AdS length is set to unity. u is a radial coordinate and $w^+ = \chi + \tau$ and $w^- = \chi - \tau$ and τ is a time, and χ the spatial one. We perform the following coordinate transformations $(u, w^+, w^-) \to (y, z^+, z^-)$ in the above equation [19][37][38].

$$u = \frac{4y(f'(z^+)\bar{f}'(z^-))^{3/2}}{4f'(z^+)\bar{f}'(z^-) + y^2 f''(z^+)\bar{f}''(z^-)} \tag{A.2}$$

$$w^+ = f(z^+) - \frac{2y^2(f'(z^+))^2 \bar{f}''(z^-)}{4f'(z^+)\bar{f}'(z^-) + y^2 f''(z^+)\bar{f}''(z^-)} \tag{A.3}$$

$$w^- = \bar{f}(z^-) - \frac{2y^2(\bar{f}'(z^-))^2 f''(z^+)}{4f'(z^+)\bar{f}'(z^-) + y^2 f''(z^+)\bar{f}''(z^-)} \tag{A.4}$$

Then we obtain a new metric.

$$ds^2 = \frac{1}{y^2}(dy^2 + dz^+ dz^-) - \frac{1}{2}S[f, z^+](dz^+)^2 - \frac{1}{2}S[\bar{f}, z^-](dz^-)^2 + y^2\frac{1}{4}S[f, z^+]S[\bar{f}, z^-]dz^+ dz^- \tag{A.5}$$

Here $S[f, z]$ is a Schwarzian derivative.

$$S[f, z] = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{A.6}$$

Let us next consider the transformations generated by $f(z^+) = e^{\sqrt{a}z^+}$ and $\bar{f}(z^-) = e^{\sqrt{a}z^-}$.

$$\begin{aligned}
u &= \frac{4\sqrt{a}y}{4 + ay^2} e^{\frac{1}{2}\sqrt{a}(z^+ + z^-)} \\
w^+ &= \frac{4 - ay^2}{4 + ay^2}\, e^{\sqrt{a}z^+} \\
w^- &= \frac{4 - ay^2}{4 + ay^2}\, e^{\sqrt{a}z^-}
\end{aligned} \tag{A.7}$$

Here $a = 4GM$ is a parameter related to the black hole mass M, and G is a 3d Newton constant. Then the metric is transformed to

$$ds^2 = \frac{1}{y^2}dy^2 + \frac{1}{4}\left(a(dz^+)^2 + a(dz^-)^2\right) + \left(\frac{1}{y^2} + \frac{1}{16}a^2 y^2\right) dz^+ dz^-. \tag{A.8}$$

When we define $z^+ = x + t$, $z^- = x - t$ and

$$r \equiv \frac{1}{y} + \frac{a}{4}y, \tag{A.9}$$

then the metric reads

$$ds^2 = \frac{1}{r^2 - a}dr^2 - (r^2 - a)dt^2 + r^2 dx^2. \tag{A.10}$$

Because $r \geq \sqrt{a}$, (A.8) describes the space-time outside the black hole. (A.9) is solved for y as

$$y = \frac{2}{a}(r \pm \sqrt{r^2 - a}). \tag{A.11}$$

The horizon is located at $r = r_+ = \sqrt{a}$. Each $0 < y \leq 2/\sqrt{a}$ and $2/\sqrt{a} \leq y < \infty$ correspond to the exterior region $r \geq \sqrt{a}$.

Maximally extended Schwarzschild space time consists of four regions: right (I) region and left (III) one, which are spacetimes outside the horizon. Future (II) and past (IV) one, which are behind the horizon. The normal modes of a scalar field in the BTZ background are obtained from those in pure AdS_3 space (A.1) by coordinate transformations.

A classical equation of motion for a real scalar field ϕ with mass m in (A.1) is given by

$$\partial_u^2 \phi - \frac{1}{u}\partial_u \phi + (\partial_\chi^2 - \partial_\tau^2)\phi - \frac{m^2}{u^2}\phi = 0. \tag{A.12}$$

By separation of variables mode functions of this scalar field will be obtained in the form

$$\phi_{\omega k}(u, \tau, \chi) = e^{-i\omega\tau + ik\chi} f_{\omega k}(u) \tag{A.13}$$

where ω and k are constants, and $f_{\omega k}(u)$ is a solution to the following equation

$$f''_{\omega k}(u) - \frac{1}{u} f'_{\omega k}(u) + \left(\omega^2 - k^2 - \frac{m^2}{u^2}\right) f_{\omega k}(u) = 0. \tag{A.14}$$

By BDHM dictionary [33] the scalar field dual to the 2d CFT must satisfy a boundary condition $\phi \sim u^{1+\nu}$ near the boundary $u \to 0$, where $\nu = \sqrt{1+m^2}$, and for non-integer ν, this determines $f_{\omega k}(u) = u J_\nu(\sqrt{\omega^2 - k^2}\, u)$. The mode functions are then given by

$$\phi_{\omega k}(u, \tau, \chi) = e^{-i\omega\tau + ik\chi} u J_\nu(\sqrt{\omega^2 - k^2}\, u) \quad (\omega \geq 0,\ -\infty < k < \infty). \tag{A.15}$$

Here $J_\nu(z)$ is a Bessel function of the ν-th order.

In the region I the map connecting the uniformization coordinates and those of black hole space-time is given by (A.7). In this region y takes values in $0 < y \leq 2/\sqrt{a}$. Hence the mode functions in the region I are

$$\Phi^{\mathrm{I}}_{\omega,k}(y,t,x) \equiv \frac{4\sqrt{a}y}{4+ay^2} e^{\sqrt{a}x} J_\nu\left(\mu \frac{4\sqrt{a}y}{4+ay^2} e^{\sqrt{a}x}\right) \exp\left[i \frac{4-ay^2}{4+ay^2} e^{\sqrt{a}x}(k\cosh\sqrt{a}t - \omega \sinh\sqrt{a}t)\right]. \tag{A.16}$$

Here $\mu = \sqrt{\omega^2 - k^2}$.

In the region III we need to make a shift $t \to t - i\beta/2$ in (A.7). However, y must be mapped to the region $2\pi/\sqrt{a} \leq y$ by a transformation $y \to 4/(ay)$. Then the relation (A.7) is unchanged. To obtain the normal modes in the region III the direction of time must be flipped: $t \to -t$ in (A.7). Then the normal modes in the region III are given by

$$\Phi^{\mathrm{III}}_{\omega,k}(y,t,x) = \frac{4\sqrt{a}y}{4+ay^2} e^{\sqrt{a}x} J_\nu\left(\mu \frac{4\sqrt{a}y}{4+ay^2} e^{\sqrt{a}x}\right) \exp\left[i \frac{4-ay^2}{4+ay^2} e^{\sqrt{a}x}(k\cosh\sqrt{a}t + \omega \sinh\sqrt{a}t)\right]. \tag{A.17}$$

To describe the interior of the horizon, we introduce a new radial coordinate $\eta (> 0)$ by

$$y = \frac{8\eta}{a\eta^2 + 4} + i\frac{2(a\eta^2 - 4)}{\sqrt{a}(a\eta^2 + 4)}. \tag{A.18}$$

Note that this is a complex transformation. Then the metric is transformed to

$$ds^2 = \frac{a(a\eta^2 - 4)^2}{(a\eta^2+4)^2} dt^2 - \frac{16a}{(a\eta^2+4)^2} d\eta^2 + \frac{16a^2\eta^2}{(a\eta^2+4)^2} dx^2. \tag{A.19}$$

Now t is a space-like variable and η the time-like one. Note that $\eta = 0$ and $\eta = \infty$ are singularities. This metric describes the region behind the horizon of the spacetime (A.10), This can also be confirmed by

$$r^2 - a = -a\Big(\frac{a\eta^2 - 4}{a\eta^2 + 4}\Big)^2 \leq 0. \tag{A.20}$$

The relation between η and r is also two-fold,

$$\eta = \frac{2}{r}\Big(1 \pm \frac{1}{\sqrt{a}}\sqrt{a - r^2}\Big). \tag{A.21}$$

To go to the region II the transformation (A.18) is used. The relation between η and r is given by

$$\eta = \frac{2}{r}\Big(1 + \frac{1}{\sqrt{a}}\sqrt{a - r^2}\Big). \tag{A.22}$$

The range of η is $2/\sqrt{a} < \eta < \infty$. In addition the shift $t \to t - i\beta/4$ must be carried out in order to make coordinates real-valued. Here β is the inverse temperature. Hence we have

$$\begin{aligned} u &= \frac{1}{4\sqrt{a}}\left(a\eta + \frac{4}{\eta}\right) e^{\frac{1}{2}\sqrt{a}(z^+ + z^-)}, \\ w^+ &= -i\frac{4 - ay^2}{4 + ay^2} e^{\sqrt{a}z^+} = -\frac{a\eta^2 - 4}{4\sqrt{a}\eta} e^{\sqrt{a}z^+}, \\ w^- &= i\frac{4 - ay^2}{4 + ay^2} e^{\sqrt{a}z^-} = \frac{a\eta^2 - 4}{4\sqrt{a}\eta} e^{\sqrt{a}z^-}. \end{aligned} \tag{A.23}$$

Mode functions in the region II are given by

$$\Phi^{\mathrm{II}}_{\omega,k}(t,\eta,x) = \frac{4 + a\eta^2}{4\sqrt{a}\eta}\, e^{\sqrt{a}x} J_\nu\left(\mu\frac{4 + a\eta^2}{4\sqrt{a}\eta} e^{\sqrt{a}x}\right) \exp\left[i\frac{a\eta^2 - 4}{4\sqrt{a}\eta} e^{\sqrt{a}x}(\omega\cosh\sqrt{a}t - k\sinh\sqrt{a}t)\right]. \tag{A.24}$$

Finally, in region IV we need to shift $t \to t - 3i\beta/4$ in the results (A.23).

$$\begin{aligned} u &= \frac{1}{4\sqrt{a}}\left(a\eta + \frac{4}{\eta}\right) e^{\frac{1}{2}\sqrt{a}(z^+ + z^+)} \\ w^+ &= i\frac{4 - ay^2}{4 + ay^2} e^{\sqrt{a}z^+} = \frac{a\eta^2 - 4}{4\sqrt{a}\eta} e^{\sqrt{a}z^+} \\ w^- &= -i\frac{4 - ay^2}{4 + ay^2} e^{\sqrt{a}z^-} = -\frac{a\eta^2 - 4}{4\sqrt{a}\eta} e^{\sqrt{a}z^-} \end{aligned} \tag{A.25}$$

Region for η is $0 \leq \eta \leq 2/\sqrt{a}$.

B Functions $g(E,p)$ and $\tilde{g}(E,p)$

In this appendix some formulas related to the functions $g(E,p)$ and $\tilde{g}(E,p)$ are presented. $g(E,p)$ is defined in (8.14). By using a representation of the modified Bessel function of the second kind,

$$K_\nu(x) = \frac{1}{2}\, e^{\nu\pi i/2} \int_{-\infty}^{\infty} e^{-ix\sinh t + \nu t} dt, \tag{B.1}$$

this function is evaluated as

$$g(E,p) = 2^{1-\nu} a^{\Delta/2} \left[\Gamma(\Delta)\right]^{-1} e^{\frac{\pi}{2\sqrt{a}}E} \int_0^\infty \mu^{\nu - \frac{i}{\sqrt{a}}p} K_{\frac{iE}{\sqrt{a}}}(\mu) d\mu. \tag{B.2}$$

Furthermore by using a formula

$$\int_0^\infty x^{\mu-1}\,K_\nu(ax)dx = 2^{\mu-2}a^{-\mu}\,\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\mu+\nu}{2}\right), \tag{B.3}$$

which is valid for $\mathrm{Re}\mu > |\mathrm{Re}\nu|$, the following equation is finally obtained.

$$g(E,p) = \frac{a^{\Delta/2}}{\Gamma(\Delta)} e^{\frac{\pi}{2\sqrt{a}}E}\,2^{-\frac{ip}{\sqrt{a}}}\,\Gamma\left(\frac{1}{2}(\Delta - i\frac{p}{\sqrt{a}} - i\frac{E}{\sqrt{a}})\right)\Gamma\left(\frac{1}{2}(\Delta - i\frac{p}{\sqrt{a}} + i\frac{E}{\sqrt{a}})\right) \tag{B.4}$$

Then the following quantity related to the normalization factor is positive semi-definite.

$$\begin{aligned} &|g(E,p)|^2 - |g(-E,-p)|^2 \\ &= \frac{a^\Delta}{(\Gamma(\Delta))^2}\left(e^{\beta E/2} - e^{-\beta E/2}\right)\left|\Gamma\left(\frac{1}{2}(\Delta - i\frac{p}{\sqrt{a}} - i\frac{E}{\sqrt{a}})\right)\right|^2\left|\Gamma\left(\frac{1}{2}(\Delta - i\frac{p}{\sqrt{a}} + i\frac{E}{\sqrt{a}})\right)\right|^2 \end{aligned} \tag{B.5}$$

In a similar fashion the following results can be derived for the function $\tilde{g}(E,p)$ (8.19).

$$\tilde{g}(E,p) = \frac{a^{\Delta/2}}{\Gamma(\Delta)} e^{-\frac{\pi}{2\sqrt{a}}E}\,2^{-\frac{ip}{\sqrt{a}}}\,\Gamma\left(\frac{1}{2}(\Delta - i\frac{p}{\sqrt{a}} - i\frac{E}{\sqrt{a}})\right)\Gamma\left(\frac{1}{2}(\Delta - i\frac{p}{\sqrt{a}} + i\frac{E}{\sqrt{a}})\right) \tag{B.6}$$

$$\begin{aligned} &|\tilde{g}(-E,-p)|^2 - |\tilde{g}(E,p)|^2 \\ &= \frac{a^\Delta}{(\Gamma(\Delta))^2}\left(e^{\beta E/2} - e^{-\beta E/2}\right)\left|\Gamma\left(\frac{1}{2}(\Delta - i\frac{p}{\sqrt{a}} - i\frac{E}{\sqrt{a}})\right)\right|^2\left|\Gamma\left(\frac{1}{2}(\Delta - i\frac{p}{\sqrt{a}} + i\frac{E}{\sqrt{a}})\right)\right|^2 \end{aligned} \tag{B.7}$$

C Inner Products of $\Phi^{\mathrm{II}(\pm\nu)}_{E,p}$ and $\tilde{\Phi}^{\mathrm{II}(\pm\nu)}_{E,p}$ for $\eta > 2/\sqrt{a}$

In this appendix the result for the inner product (9.9), $(\Phi^{\mathrm{II}(\nu)}_{E,p}, \Phi^{\mathrm{II}(\nu)}_{E',p'})_{\mathrm{II}}$, will be derived, and inner products of various mode functions will be presented. For this purpose the asymptotic form of $\Phi^{\mathrm{II}(\nu)}_{E,p}$ as $\eta \to +\infty$ is necessary. In this limit $\Phi^{\mathrm{II}(\nu)}_{E,p}$ (9.4) behaves after rescaling $\mu \to 4\mu/(\sqrt{a}\eta)$ as

$$\Phi^{\mathrm{II}(\nu)}_{E,p} \xrightarrow{\eta\to+\infty} e^{-iEt+ipx}\left(\frac{\sqrt{a}}{4}\eta\right)^{i\frac{p}{\sqrt{a}}}\int_{-\infty}^\infty d\zeta e^{-i\frac{E}{\sqrt{a}}\zeta}\int_0^\infty d\mu\mu^{-i\frac{p}{\sqrt{a}}}J_\nu(\mu)e^{i\mu\cosh\zeta}. \tag{C.1}$$

Here ζ integral is carried out by using a formula

$$\int_{-\infty}^\infty d\zeta e^{-\nu\zeta - z\cosh\zeta}d\zeta = 2K_\nu(z). \tag{C.2}$$

Then μ integration is performed for $Re(a \pm ib) > 0$, $Re(\nu - \lambda + 1) > |Re\mu|$ by [39].

$$\begin{aligned} &\int_0^\infty x^{-\lambda}K_\mu(ax)J_\nu(bx)dx \\ &= \frac{b^\nu\Gamma(\frac{\nu-\lambda+\mu+1}{2})\Gamma(\frac{\nu-\lambda-\mu+1}{2})}{2^{\lambda+1}a^{\nu-\lambda+1}\Gamma(\nu+1)}F(\frac{\nu-\lambda-\mu+1}{2}, \frac{\nu-\lambda+\mu+1}{2}; \nu+1; -\frac{b^2}{a^2}) \end{aligned} \tag{C.3}$$

The following result is obtained.

$$\Phi^{\mathrm{II}(\nu)}_{E,p} \xrightarrow{\eta\to+\infty} e^{\pi i(\nu+1)/2} e^{-iEt+ipx}\, 2^{-i\frac{p}{\sqrt{a}}} e^{\beta p/4} \frac{\Gamma\left(\frac{1}{2}(\nu+1+i\frac{E-p}{\sqrt{a}})\right)\Gamma\left(\frac{1}{2}(\nu+1-i\frac{E+p}{\sqrt{a}})\right)}{\Gamma(\nu+1)}$$

$$\times F\left(\frac{1}{2}(\nu+1+i\frac{E-p}{\sqrt{a}}), \frac{1}{2}(\nu+1-i\frac{E+p}{\sqrt{a}}); \nu+1; 1\right)(\sqrt{a}\eta/4)^{ip/\sqrt{a}}$$

$$= e^{\pi i(\nu+1)/2} e^{-iEt+ipx}\, 2^{-i\frac{p}{\sqrt{a}}} e^{\beta p/4}\, \Gamma\left(\frac{ip}{\sqrt{a}}\right) \frac{\Gamma\left(\frac{1}{2}(\nu+1+i\frac{E-p}{\sqrt{a}})\right)\Gamma\left(\frac{1}{2}(\nu+1-i\frac{E+p}{\sqrt{a}})\right)}{\Gamma\left(\frac{1}{2}(\nu+1-i\frac{E-p}{\sqrt{a}})\right)\Gamma\left(\frac{1}{2}(\nu+1+i\frac{E+p}{\sqrt{a}})\right)}(\sqrt{a}\eta/4)^{ip/\sqrt{a}} \tag{C.4}$$

This asymptotics also shows that $\Phi^{\mathrm{II}(\nu)}_{E,p}$ satisfies the following complex conjugation rule.

$$\left(\Phi^{\mathrm{II}(\nu)}_{E,p}\right)^* = e^{-\pi i(\nu+1)}\, e^{\beta p/2}\, \Phi^{\mathrm{II}(\nu)}_{-E,-p} \tag{C.5}$$

This shows that $\Phi^{\mathrm{II}(\nu)}_{E,p}$ does not form a complete set of orthonormal basis of functions in the region II.

Because the inner product does not depend on η as far as $\eta > 2/\sqrt{a}$, it can be evaluated in the limit $\eta \to +\infty$ by using (C.4). The following results are obtained.

$$(\Phi^{\mathrm{II}(\nu)}_{E,p}, \Phi^{\mathrm{II}(\nu)}_{E',p'})_{\mathrm{II}} = \frac{-8\pi^3 a}{\sinh(\beta p/2)} e^{\beta p/2} \delta(E-E')\delta(p-p') \tag{C.6}$$

Similarly, formulas for $\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}$ (9.6) can be derived. Asymptotic form for (9.6) in the limit $\eta \to \infty$ is given by

$$\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p} \xrightarrow{\eta\to+\infty}$$

$$e^{\pi i(\nu+1)/2} e^{-iEt+ipx}\, 2^{i\frac{p}{\sqrt{a}}} e^{-\beta p/4}\, \Gamma\left(\frac{-ip}{\sqrt{a}}\right) \frac{\Gamma\left(\frac{1}{2}(\nu+1+i\frac{E+p}{\sqrt{a}})\right)\Gamma\left(\frac{1}{2}(\nu+1-i\frac{E-p}{\sqrt{a}})\right)}{\Gamma\left(\frac{1}{2}(\nu+1-i\frac{E+p}{\sqrt{a}})\right)\Gamma\left(\frac{1}{2}(\nu+1+i\frac{E-p}{\sqrt{a}})\right)}(\sqrt{a}\eta/4)^{-ip/\sqrt{a}}. \tag{C.7}$$

Inner products of (9.6) are given by

$$(\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}, \tilde{\Phi}^{\mathrm{II}(\nu)}_{E',p'})_{\mathrm{II}} = \frac{8\pi^3 a}{\sinh(\beta p/2)} e^{-\beta p/2} \delta(E-E')\delta(p-p'). \tag{C.8}$$

In the remainder of this appendix, inner products of $\Phi^{\mathrm{II}(\pm\nu)}_{E,p}$ and $\tilde{\Phi}^{\mathrm{II}(\pm\nu)}_{E,p}$ are presented.

$$(\Phi^{\mathrm{II}(\nu)}_{E,p}, \Phi^{\mathrm{II}(\nu)}_{E',p'})_{\mathrm{II}} = (\Phi^{\mathrm{II}(-\nu)}_{E,p}, \Phi^{\mathrm{II}(-\nu)}_{E',p'})_{\mathrm{II}} = -8\pi^3 a \frac{e^{\beta p/2}}{\sinh \beta p/2} \delta(E-E')\delta(p-p') \tag{C.9}$$

$$(\Phi^{\mathrm{II}(\nu)}_{E,p}, \Phi^{\mathrm{II}(-\nu)}_{E',p'})_{\mathrm{II}} = -8\pi^3 a \frac{e^{\beta p/2}}{\sinh \beta p/2} e^{-\pi i\nu} D(E,p) \delta(E-E')\delta(p-p') \tag{C.10}$$

$$(\Phi^{\mathrm{II}(-\nu)}_{E,p}, \Phi^{\mathrm{II}(\nu)}_{E',p'})_{\mathrm{II}} = -8\pi^3 a \frac{e^{\beta p/2}}{\sinh \beta p/2} e^{\pi i\nu} D^*(E,p) \delta(E-E')\delta(p-p') \tag{C.11}$$

$$(\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}, \tilde{\Phi}^{\mathrm{II}(\nu)}_{E',p'})_{\mathrm{II}} = (\tilde{\Phi}^{\mathrm{II}(-\nu)}_{E,p}, \tilde{\Phi}^{\mathrm{II}(-\nu)}_{E',p'})_{\mathrm{II}} = 8\pi^3 a \frac{e^{-\beta p/2}}{\sinh \beta p/2} \delta(E-E')\delta(p-p') \tag{C.12}$$

$$(\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}, \tilde{\Phi}^{\mathrm{II}(-\nu)}_{E',p'})_{\mathrm{II}} = 8\pi^3 a \frac{e^{-\beta p/2}}{\sinh \beta p/2} e^{-\pi i\nu} D^*(E,p) \delta(E-E')\delta(p-p') \tag{C.13}$$

$$(\tilde{\Phi}^{\mathrm{II}(-\nu)}_{E,p}, \tilde{\Phi}^{\mathrm{II}(\nu)}_{E',p'})_{\mathrm{II}} = 8\pi^3 a \frac{e^{-\beta p/2}}{\sinh \beta p/2} e^{\pi i\nu} D(E,p) \delta(E-E')\delta(p-p') \tag{C.14}$$

Here $-\infty < E, E', p, p' < \infty$. On the other hand we have

$$(\Phi^{\mathrm{II}(\nu)}_{E,p}, \tilde{\Phi}^{\mathrm{II}(\nu)}_{E',p'})_{\mathrm{II}} = (\Phi^{\mathrm{II}(-\nu)}_{E,p}, \tilde{\Phi}^{\mathrm{II}(-\nu)}_{E',p'})_{\mathrm{II}} = 0, \tag{C.15}$$

$$(\Phi^{\mathrm{II}(\nu)}_{E,p}, \tilde{\Phi}^{\mathrm{II}(-\nu)}_{E',p'})_{\mathrm{II}} = (\Phi^{\mathrm{II}(-\nu)}_{E,p}, \tilde{\Phi}^{\mathrm{II}(\nu)}_{E',p'})_{\mathrm{II}} = 0. \tag{C.16}$$

Here $D(E,p)$ is defined by

$$\begin{aligned} D(E,p) &= \frac{\Gamma(\frac{1+\nu}{2} + i\frac{E+p}{2\sqrt{a}})\Gamma(\frac{1+\nu}{2} - i\frac{E-p}{2\sqrt{a}})\Gamma(\frac{1-\nu}{2} + i\frac{E-p}{2\sqrt{a}})\Gamma(\frac{1-\nu}{2} - i\frac{E+p}{2\sqrt{a}})}{\Gamma(\frac{1+\nu}{2} + i\frac{E-p}{2\sqrt{a}})\Gamma(\frac{1+\nu}{2} - i\frac{E+p}{2\sqrt{a}})\Gamma(\frac{1-\nu}{2} + i\frac{E+p}{2\sqrt{a}})\Gamma(\frac{1-\nu}{2} - i\frac{E-p}{2\sqrt{a}})} \\ &= \frac{\cosh(\beta E/2) + \cos\pi\nu\cosh(\beta p/2) + i\sin\pi\nu\sinh(\beta p/2)}{\cosh(\beta E/2) + \cos\pi\nu\cosh(\beta p/2) - i\sin\pi\nu\sinh(\beta p/2)}. \end{aligned} \tag{C.17}$$

It can be checked that $D(E,-p) = D(E,p)^*$ and $D(-E,p) = D(E,p)$.

D Near-Horizon Behavior of Normal Modes

The near horizon behavior of the mode function $\Phi^{\mathrm{II}(\nu)}_{E,p}$ in the region II, (9.4), are obtained as follows. For $\eta \sim 2/\sqrt{a}$, this mode asymptotes to

$$\Phi^{\mathrm{II}(\nu)}_{E,p}(t,\eta,x) \sim e^{-iEt+ipx}\int_{-\infty}^{\infty} d\zeta \int_0^{\infty} d\mu e^{-i\frac{E}{\sqrt{a}}\zeta - i\frac{p}{\sqrt{a}}\ln\mu} J_\nu(\mu)\exp\left[i\frac{\sqrt{a}\eta - 2}{2}\mu\cosh\zeta\right]. \tag{D.1}$$

By using formula (C.2) the integration over ζ in (D.1) is carried out.

$$\Phi^{\mathrm{II}(\nu)}_{E,p}(t,\eta,x) \sim 2e^{-iEt+ipx}\int_0^{\infty} d\mu\mu^{-ip/\sqrt{a}} J_\nu(\mu) K_{iE/\sqrt{a}}\left(-i(\sqrt{a}\eta - 2)\mu/2\right) \tag{D.2}$$

The near horizon behavior is then estimated by using

$$\begin{aligned} K_\nu(z) &= \frac{\pi}{2\sin\nu\pi}\left[I_{-\nu}(z) - I_\nu(z)\right] \\ &\xrightarrow{z\to 0} \frac{\pi}{2\sin\nu\pi}\left[\left(\frac{z}{2}\right)^{-\nu}\frac{1}{\Gamma(1-\nu)} - \left(\frac{z}{2}\right)^{\nu}\frac{1}{\Gamma(1+\nu)}\right] \end{aligned} \tag{D.3}$$

as

$$\begin{aligned} \Phi^{\mathrm{II}(\nu)}_{E,p}(t,\eta,x) \sim\; & e^{-iEt+ipx}e^{-\beta E/4}\Gamma\left(i\frac{E}{\sqrt{a}}\right)\left(\frac{\sqrt{a}\eta - 2}{4}\right)^{-iE/\sqrt{a}}\int_0^{\infty} d\mu\mu^{-i(E+p)/\sqrt{a}} J_\nu(\mu) \\ & + e^{-iEt+ipx}e^{\beta E/4}\Gamma\left(-i\frac{E}{\sqrt{a}}\right)\left(\frac{\sqrt{a}\eta - 2}{4}\right)^{iE/\sqrt{a}}\int_0^{\infty} d\mu\mu^{i(E-p)/\sqrt{a}} J_\nu(\mu). \end{aligned} \tag{D.4}$$

Finally by using an integration formula

$$\int_0^{\infty} x^{\mu-1} J_\nu(ax)dx = 2^{\mu-1}a^{-\mu}\frac{\Gamma((\mu+\nu)/2)}{\Gamma((\nu-\mu)/2+1)}, \tag{D.5}$$

the following behavior is obtained.

$$\Phi^{\mathrm{II}(\nu)}_{E,p}(t,\eta,x) \sim e^{-iEt+ipx}e^{-\beta E/4}\Gamma\left(i\frac{E}{\sqrt{a}}\right)\left(\frac{\sqrt{a}\eta-2}{4}\right)^{-iE/\sqrt{a}}2^{-i(E+p)/\sqrt{a}}\frac{\Gamma\left(\frac{1}{2}(1+\nu-i\frac{E+p}{\sqrt{a}})\right)}{\Gamma\left(\frac{1}{2}(1+\nu+i\frac{E+p}{\sqrt{a}})\right)}$$
$$+ e^{-iEt+ipx}e^{\beta E/4}\Gamma\left(-i\frac{E}{\sqrt{a}}\right)\left(\frac{\sqrt{a}\eta-2}{4}\right)^{iE/\sqrt{a}}2^{i(E-p)/\sqrt{a}}\frac{\Gamma\left(\frac{1}{2}(1+\nu+i\frac{E-p}{\sqrt{a}})\right)}{\Gamma\left(\frac{1}{2}(1+\nu-i\frac{E-p}{\sqrt{a}})\right)} \quad \text{(D.6)}$$

This result can also be obtained by using (C.3) in (D.2).

Another mode function, $\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p}$ defined by (9.6) is related to (9.4) by a relation (9.7), and the corresponding formula near the horizon can be obtained by using these relations.

The near horizon behavior of the normal mode in the region I, (8.6), can be obtained in a similar way by using (D.3) and $K_\nu(z) = \frac{1}{2}e^{\nu\pi i/2}\int_{-\infty}^{\infty} dt\, e^{-\nu t+iz\sinh t}$.

$$\Phi^{\mathrm{I}}_{E,p}(y,t,x) \sim e^{-iEt+ipx}e^{\beta E/4}2^{-i\frac{p}{\sqrt{a}}}$$
$$\times\left[\Gamma\left(\frac{iE}{\sqrt{a}}\right)\frac{\Gamma(\frac{1+\nu}{2}-i\frac{E+p}{2\sqrt{a}})}{\Gamma(\frac{1+\nu}{2}+i\frac{E+p}{2\sqrt{a}})}\left\{\frac{1}{2}(2-\sqrt{a}y)\right\}^{-i\frac{E}{\sqrt{a}}}\right.$$
$$\left.+\Gamma\left(\frac{-iE}{\sqrt{a}}\right)\frac{\Gamma(\frac{1+\nu}{2}+i\frac{E-p}{2\sqrt{a}})}{\Gamma(\frac{1+\nu}{2}-i\frac{E-p}{2\sqrt{a}})}\left\{\frac{1}{2}(2-\sqrt{a}y)\right\}^{i\frac{E}{\sqrt{a}}}\right] \quad \text{(D.7)}$$

The normal mode in the region III, (8.7), is also obtained.

$$\Phi^{\mathrm{III}}_{E,p}(y,t,x) \sim e^{iEt+ipx}e^{-\beta E/4}2^{-i\frac{p}{\sqrt{a}}}$$
$$\times\left[\Gamma\left(\frac{iE}{\sqrt{a}}\right)\frac{\Gamma(\frac{1+\nu}{2}-i\frac{E+p}{2\sqrt{a}})}{\Gamma(\frac{1+\nu}{2}+i\frac{E+p}{2\sqrt{a}})}\left\{\frac{1}{2}(\sqrt{a}y-2)\right\}^{-i\frac{E}{\sqrt{a}}}\right.$$
$$\left.+\Gamma\left(\frac{-iE}{\sqrt{a}}\right)\frac{\Gamma(\frac{1+\nu}{2}+i\frac{E-p}{2\sqrt{a}})}{\Gamma(\frac{1+\nu}{2}-i\frac{E-p}{2\sqrt{a}})}\left\{\frac{1}{2}(\sqrt{a}y-2)\right\}^{i\frac{E}{\sqrt{a}}}\right] \quad \text{(D.8)}$$

In what follows some inner products of the four mode functions near the horizon are presented. Inner products evaluated at the horizon will be denoted with subscript H.

$$(\Phi^{\mathrm{II}(\nu)}_{E,p},\Phi^{\mathrm{II}(\nu)}_{E',p'})_{\mathrm{H}} = (\Phi^{\mathrm{II}(-\nu)}_{E,p},\Phi^{\mathrm{II}(-\nu)}_{E',p'})_{\mathrm{H}} = -16\pi^3 a\delta(E-E')\delta(p-p') \quad \text{(D.9)}$$
$$(\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p},\tilde{\Phi}^{\mathrm{II}(\nu)}_{E',p'})_{\mathrm{H}} = (\tilde{\Phi}^{\mathrm{II}(-\nu)}_{E,p},\tilde{\Phi}^{\mathrm{II}(-\nu)}_{E',p'})_{\mathrm{H}} = -16\pi^3 a\delta(E-E')\delta(p-p') \quad \text{(D.10)}$$
$$(\Phi^{\mathrm{II}(\nu)}_{E,p},\Phi^{\mathrm{II}(-\nu)}_{E',p'})_{\mathrm{H}} = -16\pi^3 ae^{\pi i\nu}\frac{1+e^{\beta p}+e^{\beta p/2}(e^{\beta E/2}+e^{-\beta E/2})}{e^{\beta p}+e^{2\pi i\nu}+e^{\pi i\nu}e^{\beta p/2}(e^{\beta E/2}+e^{-\beta E/2})}$$
$$\times\delta(E-E')\delta(p-p') \quad \text{(D.11)}$$
$$(\tilde{\Phi}^{\mathrm{II}(\nu)}_{E,p},\tilde{\Phi}^{\mathrm{II}(-\nu)}_{E',p'})_{\mathrm{H}} = -16\pi^3 ae^{\pi i\nu}\frac{1+e^{-\beta p}+e^{-\beta p/2}(e^{\beta E/2}+e^{-\beta E/2})}{e^{-\beta p}+e^{2\pi i\nu}+e^{\pi i\nu}e^{-\beta p/2}(e^{\beta E/2}+e^{-\beta E/2})}$$
$$\times\delta(E-E')\delta(p-p') \quad \text{(D.12)}$$
$$(\Phi^{\mathrm{II}(\nu)}_{E,p},\tilde{\Phi}^{\mathrm{II}(\nu)}_{E',p'})_{\mathrm{H}} = 8\pi^2\sqrt{a}E2^{2ip/\sqrt{a}}|\Gamma(iE/\sqrt{a})|^2(e^{-\beta E/2}-e^{\beta E/2})$$
$$\times\frac{\Gamma(\frac{1}{2}(1+\nu+i\frac{E+p}{\sqrt{a}}))\Gamma(\frac{1}{2}(1+\nu-i\frac{E-p}{\sqrt{a}}))}{\Gamma(\frac{1}{2}(1+\nu-i\frac{E+p}{\sqrt{a}}))\Gamma(\frac{1}{2}(1+\nu+i\frac{E-p}{\sqrt{a}}))}\delta(E-E')\delta(p-p') \quad \text{(D.13)}$$

E Solutions to the Matching Conditions

In this Appendix appropriate solutions to the matching conditions (11.10)-(11.15) will be obtained. First (11.10) and (11.14) are analyzed. When $\gamma_4^{(1)}$ is eliminated from these equations, it is found that $\gamma_3^{(1)}$ also disappears, and we have

$$\gamma_2^{(1)} = -\gamma_1^{(1)} \frac{M_1}{M_2} + \frac{b_1}{N_{E,p}^{(1)} M_2} C^{-\nu}(-E,p) \tag{E.1}$$

where M_1 and M_2 are defined in (E.4) and (E.5) below. This is because the following relations hold

$$C^{-\nu}(-E,p)C^{-\nu}(E,-p) = \left|\Gamma\left(i\frac{E}{\sqrt{a}}\right)\right|^2 = C^{-\nu}(-E,-p)C^{-\nu}(E,p). \tag{E.2}$$

Similarly, when $\gamma_2^{(1)}$ is eliminated from (11.10) and (11.12), then $\gamma_1^{(1)}$ also dropps out.

$$\gamma_4^{(1)} = \gamma_3^{(1)} \frac{M_3}{M_2} - \frac{b_1}{N_{E,p}^{(1)} M_2} C^{\nu}(-E,p) \tag{E.3}$$

where M_3 is defined in (E.6). Here

$$M_1 \equiv C^{-\nu}(-E,p)C^{\nu}(E,-p) - C^{-\nu}(E,p)C^{\nu}(-E,-p), \tag{E.4}$$
$$M_2 \equiv C^{-\nu}(-E,p)C^{\nu}(E,p) - C^{-\nu}(E,p)C^{\nu}(-E,p), \tag{E.5}$$
$$M_3 \equiv C^{\nu}(-E,p)C^{-\nu}(E,-p) - C^{\nu}(E,p)C^{-\nu}(-E,-p). \tag{E.6}$$

b_1 is defined by removing $e^{-ipx}V^{-iE/\sqrt{a}}$ from the right-hand side of (11.10).

$$b_1 \equiv \sqrt{1-e^{-\beta E}}\, 2^{i(p-E)/\sqrt{a}} \frac{\Gamma(\frac{1+\nu}{2} - i\frac{E-p}{2\sqrt{a}})}{\Gamma(\frac{1+\nu}{2} + i\frac{E-p}{2\sqrt{a}})} e^{\beta E/4}\Gamma\left(i\frac{E}{\sqrt{a}}\right) \tag{E.7}$$

In order to make the norm (11.6) with $i=1$ take the form (11.23), it is assumed that $\gamma_1^{(1)}$ and $\gamma_3^{(1)}$ satisfy the following equations.

$$\left|\gamma_1^{(1)} + \gamma_3^{(1)} e^{-\pi i\nu} D^*(E,p)\right|^2 = F_1^{(1)}(E,p), \tag{E.8}$$
$$\left|\gamma_1^{(1)} + \gamma_3^{(1)} e^{-\pi i\nu} + J^{(1)}\right|^2 = F_2^{(1)}(E,p) \tag{E.9}$$

where $F_{1,2}^{(1)}(E,p)$ are functions which satisfy

$$e^{-\beta p/2}F_1^{(1)}(E,p) - e^{\beta p/2}F_2^{(1)}(E,p) = e^{\beta p/2} - e^{-\beta p/2}, \tag{E.10}$$
$$F_1^{(1)}(E,p) \geq 0, \quad F_2^{(1)}(E,p) \geq 0 \quad (E,p \geq 0) \tag{E.11}$$

where

$$J^{(1)} \equiv \frac{b_1}{N_{E,p}^{(1)}} \frac{Ee^{-\beta E/2}e^{-\pi i\nu}}{2\pi\sqrt{a}} C^{\nu}(-E,p)\left[e^{\beta(E-p)/2} + e^{\pi i\nu}\right]. \tag{E.12}$$

Then it can be shown that (11.23) with $i=1$ holds. First, note that (E.8) and (E.9) can be rewritten as

$$\gamma_1^{(1)} + \gamma_3^{(1)} e^{-\pi i\nu} D^*(E,p) = (F_1^{(1)})^{1/2} e^{i\delta_1}, \tag{E.13}$$
$$\gamma_1^{(1)} + \gamma_3^{(1)} e^{-\pi i\nu} = (F_2^{(1)})^{1/2} e^{i\delta_2} - J^{(1)}. \tag{E.14}$$

Here $\delta_{1,2}(E,p)$ are arbitrary real constants. These equations are solved for $\gamma_{1,3}^{(1)}$ as

$$\gamma_1^{(1)} = K\, e^{-\pi i\nu}\left[(F_1^{(1)})^{1/2}e^{i\delta_1} - D^*(E,p)((F_2^{(1)})^{1/2}e^{i\delta_2} - J^{(1)})\right], \tag{E.15}$$

$$\gamma_3^{(1)} = K\left[-(F_1^{(1)})^{1/2}e^{i\delta_1} + (F_2^{(1)})^{1/2}e^{i\delta_2} - J^{(1)}\right] \tag{E.16}$$

where

$$K \equiv \frac{\left[e^{\beta(E-p)/2} + e^{\pi i\nu}\right]\left[e^{\beta(E+p)/2} + e^{-\pi i\nu}\right]}{4ie^{\beta E/2}\sinh(\beta p/2)e^{-\pi i\nu}\sin\pi\nu}. \tag{E.17}$$

Now by using (E.1) and (E.3) $\gamma_2^{(1)}$ and $\gamma_4^{(1)}$ are also evaluated explicitly. The results are given in (11.21) and (11.22). It is now straightforward to show that

$$\begin{aligned}(\psi^{(1)}_{E,-p}, \psi^{(1)}_{E',-p'}) = (\psi^{(3)}_{E,-p}, \psi^{(3)}_{E',-p'}) = \left|N^{(1,3)}_{E,p}\right|^2 (e^{\beta p/2} - e^{-\beta p/2})^{-1} H(E,p)\\ \times 16\pi^3 a\delta(E-E')\delta(p-p')\end{aligned} \tag{E.18}$$

where

$$H(E,p) = e^{-\beta p/2}F_1^{(1)}(E,p) - e^{\beta p/2}F_2^{(1)}(E,p) = e^{\beta p/2} - e^{-\beta p/2} \tag{E.19}$$

by using (C.9)-(C.16). Then it is necessary to set $|N^{(1)}_{E,p}| = |N^{(3)}_{E,p}| = 1$ in order to enforce a correct normalization of the norms.

Also by comparing (11.10) and (11.12) it is found that

$$\frac{b_1}{N^{(1)}_{E,p}} = \frac{b_3}{N^{(3)}_{E,p}} \tag{E.20}$$

where b_3 is defined by removing $e^{ipx}U^{-iE/\sqrt{a}}$ from the right-hand side of (11.12)

$$b_3 \equiv \sqrt{1-e^{-\beta E}}\,2^{-i(p+E)/\sqrt{a}}\frac{\Gamma(\frac{1+\nu}{2} - i\frac{E+p}{2\sqrt{a}})}{\Gamma(\frac{1+\nu}{2} + i\frac{E+p}{2\sqrt{a}})}e^{\beta E/4}\Gamma\left(i\frac{E}{\sqrt{a}}\right). \tag{E.21}$$

Then (E.7) and (E.20) imply that $N^{(1,3)}_{E,p}$ must have the forms

$$N^{(1)}_{E,p} = 2^{i(p-E)/\sqrt{a}}\frac{\Gamma(\frac{1+\nu}{2} - i\frac{E-p}{2\sqrt{a}})}{\Gamma(\frac{1+\nu}{2} + i\frac{E-p}{2\sqrt{a}})}e^{i\delta_N}, \tag{E.22}$$

$$N^{(3)}_{E,p} = 2^{-i(p+E)/\sqrt{a}}\frac{\Gamma(\frac{1+\nu}{2} - i\frac{E+p}{2\sqrt{a}})}{\Gamma(\frac{1+\nu}{2} + i\frac{E+p}{2\sqrt{a}})}e^{i\delta_N} \tag{E.23}$$

where $\delta_N(E,p)$ is an arbitrary real number.

Exactly the same way, solution to the matching conditions for $\psi^{(2)}_{E,p}$ and $\psi^{(4)}_{E,p}$ are solved and $\gamma_n^{(2)}$ ($n = 1 \sim 4$) are obtained. By using (11.11) and (11.15) $\gamma_2^{(2)}$ and $\gamma_4^{(2)}$ are expressed in terms of $\gamma_1^{(2)}$ and $\gamma_3^{(2)}$ as

$$\gamma_2^{(2)} = -\gamma_1^{(2)}\frac{M_1}{M_2} + \frac{b_2}{N^{(2)}_{E,p}}\frac{C^{-\nu}(-E,p)}{M_2}, \tag{E.24}$$

$$\gamma_4^{(2)} = \gamma_3^{(2)}\frac{M_3}{M_2} - \frac{b_2}{N^{(2)}_{E,p}}\frac{C^{\nu}(-E,p)}{M_2}. \tag{E.25}$$

In order to make the inner product of $\psi^{(2)}_{E,p}$ positive definite and normalized to unity the following two conditions are imposed

$$\left|\gamma_1^{(2)} + \gamma_3^{(2)} e^{-\pi i\nu} D^*(E,p)\right|^2 = F_1^{(2)}(E,p), \tag{E.26}$$

$$\left|\gamma_1^{(2)} + \gamma_3^{(2)} e^{-\pi i\nu} + J^{(2)}\right|^2 = F_2^{(2)}(E,p). \tag{E.27}$$

Here $F^{(2)}_{1,2}(E,p)$ satisfty the same conditions as those for $F^{(1)}_{1,2}$. These equations are solved as

$$\gamma_1^{(2)} = Ke^{-\pi i\nu}\left[(F_1^{(2)})^{1/2}e^{i\delta_3} - D^*(E,p)((F_2^{(2)})^{1/2}e^{i\delta_4} - J^{(2)})\right], \tag{E.28}$$

$$\gamma_3^{(2)} = K\left[-(F_1^{(2)})^{1/2}e^{i\delta_3} + (F_2^{(2)})^{1/2}e^{i\delta_4} - J^{(2)}\right]. \tag{E.29}$$

Here $e^{i\delta_3(E,p)}$ and $e^{i\delta_4(E,p)}$ are phase factors and

$$J^{(2)} = \frac{b_2}{N^{(2)}_{E,p}}\frac{Ee^{-\beta E/2}e^{-\pi i\nu}}{2\pi\sqrt{a}}C^\nu(-E,p)\left[e^{\beta(E-p)/2} + e^{\pi i\nu}\right]. \tag{E.30}$$

Then (E.24) and (E.25) determine $\gamma_2^{(2)}$ and $\gamma_4^{(2)}$, and it can be shown that $(\psi^{(2)}_{E,p}, \psi^{(2)}_{E',p'}) = (\psi^{(4)}_{-E,-p}, \psi^{(4)}_{-E',-p'}) = 16\pi^3 a|N^{(2,4)}_{E,p}|^2\delta(E-E')\delta(p-p')$ by using (C.9)-(C.16). $N^{(2,4)}_{E,p}$ are phase factors which satisfy $b_2/N^{(2)}_{E,p} = b_4/N^{(4)}_{E,p}$. Since $b_4 = b_1$ and $b_3 = b_2$, it follows that $N^{(2)}_{E,p}/N^{(4)}_{E,p} = b_2/b_4 = b_3/b_1 = N^{(3)}_{E,p}/N^{(1)}_{E,p}$, and we obtain $N^{(2)}_{E,p} = N^{(3)}_{E,p}e^{i(\delta'_N-\delta_N)}$ and $N^{(4)}_{E,p} = N^{(1)}_{E,p}e^{i(\delta'_N-\delta_N)}$, where $\delta'_N(E,p)$ is a real constant.

F Short Distance Behavior of Correlation Function

In Sec.6.3, we reviewed the short distance behavior of the correlation function for differentiated scalar fields. We expect that two-point scalar function is given by

$$\langle\phi(x_1)\phi(x_2)\rangle \approx \frac{1}{[g^{\mu\nu}(x_1-x_2)_\mu(x_1-x_2)_\nu]^{(d-1)/2}} \tag{F.1}$$

where the geodesic distance l_{12} between x_1 and x_2 is small in comparison to the scale of curvature $l_{12} \ll \beta^{-1}$. Then the correlation function for differentiated scalar fields in the Kruskal coordinate satisfy the following short distance behavior.

$$\langle\partial_{U_1}\phi(U_1,V_1,\Omega_1)\partial_{U_2}\phi(U_2,V_2,\Omega_2)\rangle = \kappa_N\frac{1}{(U_1-U_2)^2}\delta^{d-1}(\Omega_1-\Omega_2) \tag{F.2}$$

where κ_N is a normalization constant and Ω_{12}^2 is defined as the distance between the points Ω_1 and Ω_2. We showed that the correlation function for the scalar fields outside the horizon (6.28) with (6.34)-(6.35), which is given by Papadpsimas and Raju, satisfy the short distance behavior (F.2).

In this section we will confirm that our scalar field behind the horizon (11.1) satisfy the same short distance behavior (F.2). Using the normal modes at the horizons (11.10)-(11.13), we can obtain differentiated scalar fields in the region II as follows.

$$\partial_U\Phi^{\mathrm{II}}(U,V,x) = \frac{1}{4\pi\sqrt{\pi a}U}\int_0^\infty dEdp\left[-i\frac{E}{\sqrt{a}}\left\{c_L(E,-p)\psi^{(3)}_{E,-p} + c_L(E,p)\psi^{(4)}_{E,p}\right\} + h.c.\right] \tag{F.3}$$

where

$$\psi^{(3)}_{E,-p} = \sqrt{1-e^{-\beta E}} e^{ipx} 2^{-ip/\sqrt{a}} \frac{\Gamma(\frac{1+\nu}{2} - i\frac{E+p}{2\sqrt{a}})}{\Gamma(\frac{1+\nu}{2} + i\frac{E+p}{2\sqrt{a}})} e^{\beta E/4} \Gamma\left(i\frac{E}{\sqrt{a}}\right)(2U)^{-iE/\sqrt{a}},$$

$$\psi^{(4)}_{E,p} = \sqrt{1-e^{-\beta E}} e^{-ipx} 2^{ip/\sqrt{a}} \frac{\Gamma(\frac{1+\nu}{2} - i\frac{E-p}{2\sqrt{a}})}{\Gamma(\frac{1+\nu}{2} + i\frac{E-p}{2\sqrt{a}})} e^{\beta E/4} \Gamma\left(i\frac{E}{\sqrt{a}}\right)(2U)^{-iE/\sqrt{a}}. \tag{F.4}$$

The differentiated scalar field just outside the horizon is obtain by (8.33) and the normal modes at the horizon (D.7).

$$\partial_U \Phi^{\mathrm{I}} = \frac{1}{4\pi\sqrt{\pi a}U} \int_{-\infty}^{\infty} dp \int_0^{\infty} dE \sqrt{1-e^{-\beta E}} \left[i\frac{E}{\sqrt{a}} \Big\{ c_R(E,-p)\Phi^{\mathrm{I}}_{E,-p} + c_R(E,p)\Phi^{\mathrm{I}}_{E,p} \Big\} + h.c. \right] \tag{F.5}$$

where

$$\Phi^{\mathrm{I}}_{E,p} = e^{ipx} 2^{-ip/\sqrt{a}} \frac{\Gamma(\frac{1+\nu}{2} + i\frac{E-p}{2\sqrt{a}})}{\Gamma(\frac{1+\nu}{2} - i\frac{E-p}{2\sqrt{a}})} e^{\beta E/4} \Gamma\left(-i\frac{E}{\sqrt{a}}\right)(-2U)^{iE/\sqrt{a}}. \tag{F.6}$$

Then the correlation function for differentiated scalar fields just inside and outside the horizon is expressed as

$$\begin{aligned}
&\langle \partial_{U_1}\Phi^{\mathrm{I}}(U_1,V_1,x_1)\partial_{U_2}\Phi^{\mathrm{II}}(U_2,V_2,x_2)\rangle \\
&= \frac{1}{16\pi^3 a^2 U_1 U_2} \int_0^{\infty} dE dp \int_0^{\infty} dE' dp' E E' \Big[\langle c_R(E,-p) c_L(E',-p')\rangle \, \Phi^{\mathrm{I}}_{E,-p}\psi^{(3)}_{E',-p'} \\
&\qquad\qquad + \langle c_R(E,p) c_L(E',p')\rangle \, \Phi^{\mathrm{I}}_{E,p}\psi^{(4)}_{E',p'} + h.c. \Big].
\end{aligned} \tag{F.7}$$

Substituting the thermal averages (8.29) and the normal modes (F.4) and (F.6),

$$\begin{aligned}
&\langle \partial_{U_1}\Phi^{\mathrm{I}}(U_1,V_1,x_1)\partial_{U_2}\Phi^{\mathrm{II}}(U_2,V_2,x_2)\rangle \\
&= \frac{1}{8\pi^2 a\sqrt{a} U_1 U_2} \int_0^{\infty} dE dp E \frac{e^{-\beta E/2}}{1-e^{-\beta E}} \left[e^{-ip(x_1-x_2)} \left(-\frac{U_1}{U_2}\right)^{i\frac{E}{\sqrt{a}}} + e^{ip(x_1-x_2)} \left(-\frac{U_1}{U_2}\right)^{i\frac{E}{\sqrt{a}}} + c.c. \right] \\
&= \frac{\delta(x_1-x_2)}{2\pi a\sqrt{a} U_1 U_2} \int_0^{\infty} dE E \frac{e^{-\beta E}}{1-e^{-\beta E}} \left[\left(\frac{U_1}{U_2}\right)^{i\frac{E}{\sqrt{a}}} + \left(\frac{U_2}{U_1}\right)^{i\frac{E}{\sqrt{a}}} \right]
\end{aligned} \tag{F.8}$$

The integral at the last line is evaluated as (6.32), and then we can obtain the correlation function for differentiated scalar field as follows.

$$\langle \partial_{U_1}\Phi^{\mathrm{I}}(U_1,V_1,x_1)\partial_{U_2}\Phi^{\mathrm{II}}(U_2,V_2,x_2)\rangle = -\frac{1}{2\pi a\sqrt{a}\beta} \frac{1}{(U_1-U_2)^2} \delta(x_1-x_2) \tag{F.9}$$

This result is consistent with (F.2).

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