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Proceedings of 46th Sapporo Symposium on Partial Differential Equations

Edited by
S.-I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, H. Kubo, H. Kuroda,
Y. Liu, T. Ozawa, T. Sakajo, and K. Tsutaya

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Preface

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 10 through August 12 in 2021 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school. Late Professor Taira Shirota started the symposium more than 40 years ago. Late Professor Kôji Kubota and late Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

Finally, it should be mentioned that some contributions in this volume are written in Japanese, since the symposium is held as a domestic one for preventing from the spread of COVID-19.

S.-I. Ei (Hokkaido University)
Y. Giga (The University of Tokyo)
N. Hamamuki (Hokkaido University)
S. Jimbo (Hokkaido University)
H. Kubo (Hokkaido University)
H. Kuroda (Hokkaido University)
Y. Liu (Hokkaido University)
T. Ozawa (Waseda University)
T. Sakajo (Kyoto University)
K. Tsutaya (Hirosaki University)

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G. Akagi (Tohoku University)

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K. Fujiwara (Nagoya University)

Necessary and sufficient condition for global existence of L^2 solutions for 1D periodic NLS with non-gauge invariant quadratic nonlinearity

The 46th Sapporo Symposium on Partial Differential Equations

第 46 回偏微分方程式論札幌シンポジウム

Period	August 10, 2021 – August 12, 2021
Organizers	Shuichi Jimbo, Hirotoshi Kuroda
Program Committee	Shin-Ichiro Ei, Yoshikazu Giga, Nao Hamamuki, Shuichi Jimbo, Hideo Kubo, Hirotoshi Kuroda, Yikan Liu, Tohru Ozawa, Takashi Sakajo, Kimitoshi Tsutaya
URL	http://www.math.sci.hokudai.ac.jp/sympo/sapporo/program210810.html

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Inverse problems for equations of fluid dynamics by Carleman estimates

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16:20-16:50　清水 雄貴（東京大学）Yuuki Shimizu (The University of Tokyo)
A current-valued solution of the Euler equation and its application

16:50-18:00　Discussion

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11:40-　Closing

Boundary regularity of minimizers of double phase functionals

Atsushi Tachikawa [*†]

(The 46th Sapporo Symposium on Partial Differential Equations)
August 2021

This talk is concerned with the boundary regularity for minimizers of functionals of double phase with variable exponents.

For $n, N \in \mathbb{N}$ with $n \geq 2, N \geq 1$, a domain $\Omega \subset \mathbb{R}^n$ and a function $u : \Omega \to \mathbb{R}^N$, we consider the variational problem for the functional defined by

$$\mathcal{F}(u; K) := \int_K F(x, u, Du)dx,$$

where K is a measurable set with $K \subset \Omega$ and $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ is a Carathéodory function, namely

- $F(x, u, \xi)$ is measurable in x for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{nN}$,
- $F(x, u, \xi)$ is continuous in (u, ξ) for almost every $x \in \Omega$.

Moreover, in many cases, we consider the following *growth condition* : for $q \geq p \geq 1$ and $\Lambda \geq \lambda > 0$ it holds that

$$\lambda|\xi|^p \leq F(x, u, \xi) \leq \Lambda(1 + |\xi|)^q \quad \text{for all } (x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}.$$

When $p = q$, the functional $\mathcal{F}$ is called to be of *standard growth* or *p-growth,* and when $q > p$, *non-standard growth* or *(p, q)-growth.* Non-standard growth functionals are of great interest to many researchers, since they have been introduced by Marcellini [12].

A very important and interesting class of (p, q)-growth functional is introduced by M.Colombo and G.Mingione [6, 7] (see also [4]). They introduced so-called *functionals of double phase* or *double phase functionals.* A typical model of them is given by

$$\mathcal{H}(u; K) := \int_K H(x, Du)dx, \quad H(x, \xi) := |\xi|^p + a(x)|\xi|^q \quad (1 < p, q), \tag{1}$$

where $a(x)$ is a non-negative function defined on Ω. Here and in what follows $Du = (D_\alpha u^i)_{1\leq\alpha\leq n, 1\leq i\leq N}$, $D_\alpha = \partial/\partial x^\alpha$, and $|\cdot|$ stands for the standard Euclidean norm, in addition, throughout this paper, we are assuming always that $q > p$. The integrand

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H changes the growth order; q-growth on $\{x \in \Omega\ ;\ a(x) > 0\}$, p-growth on $\{x \in \Omega\ ;\ a(x) = 0\}$.

Before the above mentioned papers by Colombo and Mingione, a different class of functionals with integrands which change their growth order iso-called $p(x)$-*growth functionals* or *functionals with variable exponent* is introduced by V.V.Zhikov [27, 28]. A typical example of $p(x)$-growth functionals is

$$\mathcal{D}_{p(x)}(u;K) := \int_K |Du|^{p(x)} dx.$$

Zhikov showed higher integrability results for the minimizers of $p(x)$-growth functionals with continuous $p(x)$. On the other hand, he gave examples showing that $p(x)$-growth functional with discontinuous exponent $p(x)$ can have a discontinuous minimizer.

In [6, 7, 5], other than $\mathcal{H}$, a double phase functional,

$$\mathcal{F}_1(u;K) := \int_K [f_1(Du) + a(x) f_2(Du)]\, dx,$$

with the condition

$$\lambda|\xi|^p \le f_1(\xi) \le \Lambda|\xi|^p, \quad \lambda|\xi|^q \le f_2(\xi) \le \Lambda|\xi|^q,$$

and more general one,

$$\mathcal{F}_2(u;K) := \int_K F(x,u,Du) dx,$$

with

$$\lambda\left(|\xi|^p + a(x)|\xi|^q\right) \le F(x,u,\xi) \le \Lambda\left(|\xi|^p + a(x)|\xi|^q\right)$$

are treated. Besides conditions on the convexity, continuity and sufficient differentiability of f, g and F, for $a(\cdot)$ it is assumed that

$$a(x) \ge 0 \ \text{ for all } x \in \Omega, \quad a(\cdot) \in C^{0,\alpha}(\overline{\Omega}) \ \text{ for some } \alpha \in (0,1]. \tag{2}$$

In [6], under the assumption

$$\frac{q}{p} < 1 + \frac{\alpha}{n}, \tag{3}$$

they showed $C^{1,\gamma}$-regularity of local minimizers of $\mathcal{F}_1$ for the case that $N = 1$, and of $\mathcal{H}$ for $N \ge 2$ respectively. With the methods presented in [6], it is possible to deal with $\mathcal{F}_1$ provided that f_1 and f_2 satisfy the following structural conditions

$$f_1(\xi) = \tilde{f}_1(|\xi|), f_2(\xi) = \tilde{f}_2(|\xi|) \ \text{ for } \ \tilde{f}_1, \tilde{f}_2 : [0,\infty) \to [0,\infty). \tag{4}$$

Remark 1. *In general, we call a functional that can be given by*

$$\mathcal{F}_{Uhl}(u;K) := \int_K \tilde{f}(x,u,|Du|) dx, \ \text{ for some } \ \tilde{f}\ ;\ \Omega \times \mathbb{R}^n \times \mathbb{R},$$

"a functional with Uhlenbeck structure".

In [7], assuming the boundedness of local minimizers, under the condition that

$$q \le p + \alpha \tag{5}$$

they showed the $C^{0,\gamma}$-, and $C^{1,\gamma}$-regularity for minimizers of $\mathcal{F}_2$ and of $\mathcal{F}_1$ respectively for the case that $N = 1$. For vector valued case, $N \ge 2$, assuming the strict bound $q < p + \alpha$ and the structural condition (4), they showed $C^{1,\gamma}$-regularity for local minimizers of $\mathcal{F}_1$. As mentioned in [7], the limiting case $q = p + \alpha$ can not be treated in the vector valued case.

In [5], they treat the functional of very general type $\mathcal{F}_2$ for the scalar-valued case, $N = 1$. Moreover, they succeeded to obtain regularity results also for borderline cases that $p/q = 1 + (\alpha/n)$ or $q = p + \alpha$ with boundedness condition on u.

In [4], another type of double phase functional,

$$\mathcal{P}_{\log}(u; K) := \int_K \left(|Du|^p + a(x)|Du|^p \log(e + |Du|)\right) dx,$$

is considered. They showed local Hölder continuity of minimizers u assuming that $a(\cdot)$ is *logarithmic continuous.* Under the Hölder continuity condition on $a(\cdot)$, it is also shown that Du is locally Hölder continuous.

For the scalar valued case, in [10], a wider class of functionals including $p(x)$-growth and double phase cases treated systematically in the setting of Musielak-Orlicz spaces.

For vector valued cases, as in standard growth problems, it is natural to expect the partial regularity of minimizers between manifolds. In fact, manifold constrained case is treated in [8]. They consider the functional of type $\mathcal{F}_2$ with the condition

$$F(x, u, \xi) = \tilde{F}(x, u, |\xi|),$$

and get partial regularity for minimizers of $\mathcal{F}_2$ in the class $W^{1,1}(\Omega; S^{N-1})$. In this case, since the target is compact, the maps under consideration are bounded, and therefore the condition on the exponents is $q < p + \alpha$.

Recently, some generalization of double phase functionals are considered. For example, in [9] regularity results for so-called *multi-phase* functionals are obtained. On the other hand, in [20] the double phase functional with variable exponents,

$$\mathcal{H}_{p(\cdot),q(\cdot)}(u; K) := \int_K \left(|Du|^{p(x)} + a(x)|Du|^{q(x)}\right) dx,$$

is considered and interior $C^{1,\gamma}$-regularity was obtained under the condition

$$p(\cdot), q(\cdot) \in C^{0,\sigma}, \quad 1 \le \frac{q(x)}{p(x)} < 1 + \frac{\beta}{n}, \quad (\beta = \min\{\alpha, \sigma\}).$$

Nowadays, there are many articles on nonstandard growth problems, especially on $p(x)$-growth and double phase problems. About $p(x)$-growth problems, see also, [1, 2, 3, 21, 18, 23, 26, 19, 25], and [22] for their applications. For general nonstandard problems, see Marcellini's papers [13, 14, 15, 16] etc. About regularity problems of various class there is a very useful survey paper [17].

As far as the author knows (at the time of preparing this paper), all known regularity results are on the interior regularity. The aim of this paper is to get boundary regularity results for the minimizers of double phase functional with variable exponents. We consider the minimizing problem with continuous Dirichlet boundary conditions on the boundary $\partial\Omega$ of a bounded domain Ω, and therefore, by virtue of maximum principle due to Leonetti-Siepe [11], the minimizers are bounded. So, the appropriate condition on exponents should be $q(x) < p(x) + \beta$, that corresponds to (5) for the constant exponents case.

When we treat boundary regularity, we need to transform the coordinates in order to flatten the boundary locally, and therefore we should consider the functionals with coefficient matrices deduced from the transformation. By this reason, in this paper, from the beginning we consider the functionals with $\left(\delta_{ij}g^{\alpha\beta}(x)D_\alpha u^i D_\beta u^j\right)^{1/2}$ in place of $|Du|$. Namely, writing $|\xi|_g := \left(\delta_{ij}g^{\alpha\beta}(x)\xi^i_\alpha\xi^j_\beta\right)^{1/2}$ for $\xi \in \mathbb{R}^{nN}$, we treat the functional defined by

$$\mathcal{G}(u;K) := \int_K G(x,Du)dx, \quad G(x,\xi) := |\xi|_g^{p(x)} + a(x)|\xi|_g^{q(x)}. \tag{6}$$

Here, we are assuming that $g(\cdot) = (g^{\alpha\beta}(\cdot))_{1\le\alpha,\beta\le n}$ is symmetric and uniformly continuous; namely

$$g^{\alpha\beta}(x) = g^{\beta\alpha}(x) \ \text{ at every } \ x \in \Omega, \quad \alpha,\beta = 1,\dots,n, \tag{7}$$

$$g^{\alpha\beta}(\cdot) \in C^0(\overline{\Omega}), \quad \alpha,\beta = 1,\dots,n. \tag{8}$$

We assume also that for some constants $0 < \lambda_g \le \Lambda_g$, g satisfies

$$\lambda_g|z|^2 \le g^{\alpha\beta}(x)z_\alpha z_\beta, \quad \max_{1\le\alpha,\beta\le 1}|g^{\alpha\beta}(x)| \le \Lambda_g, \quad \text{for all } \ (x,z) \in \Omega\times\mathbb{R}^N. \tag{9}$$

It is clear that the presence of the coefficient matrix g does not cause any difference of the features of $\mathcal{H}_{p(\cdot),q(\cdot)}$ and $\mathcal{G}$ under the above conditions.

For further convenience, let us write the modulus of continuity of g by ω_g, namely ω_g is a increasing continuous function with $\omega_g(0) = 0$ which satisfies

$$\omega_g(|x-y|) \ge \max_{1\le\alpha,\beta\le m}\left|g^{\alpha\beta}(x) - g^{\alpha\beta}(y)\right| \tag{10}$$

for all $x, y \in \Omega$.

About variable exponents $p(x)$ and $q(x)$ we consider the following conditions.

(i) For the exponents $p(x)$ and $q(x)$ we assume that

$$p(x), q(x) \in C^{0,\sigma}(\overline{\Omega}) \ \text{ for some } \ \sigma \in (0,1). \tag{11}$$

(ii) Putting

$$\beta := \min\{\alpha,\sigma\} \tag{12}$$

We assume that $p(x)$ and $q(x)$ satisfy

$$1 < p_0 \le p(x) \le q(x), \quad \sup_{x\in\Omega}(q(x)-p(x)) := \beta_0 < \beta \tag{13}$$

Let $\mathcal{F}(u,K)=\int_K F(x,Du)dx$ be one of the above functionals. For $w:\Omega\to\mathbb{R}^N$, if $F(x,Dw)$ is not integrable on K, we define the value of $\mathcal{F}(w;K)$ as $+\infty$. We say "$u\in W^{1,1}_{\rm loc}(\Omega;\mathbb{R}^N)$ *locally minimizes* $\mathcal{F}$" or *"u is a local minimizer of $\mathcal{F}$"*, if $F(x,Du)\in L^1_{\rm loc}(\Omega)$ and

$$\mathcal{F}(u;{\rm supp}\varphi)\le\mathcal{F}(u+\varphi;{\rm supp}\varphi)\text{ for all }\varphi\in W^{1,1}_0(\Omega;\mathbb{R}^N)\ \text{ with }{\rm supp}\varphi\Subset\Omega.$$

When Ω is bounded domain and we are considering the Dirichlet boundary condition

$$u=h\quad\text{on }\ \partial\Omega \tag{14}$$

for suitably given $h:\Omega\to\mathbb{R}^N$, we call $u\in h+W^{1,1}_0(\Omega;\mathbb{R}^N)$ is a *minimizer of $\mathcal{F}$ with the boundary condition (14)* if

$$\mathcal{F}(u;\Omega)\le\mathcal{F}(u+\varphi;\Omega)\quad\text{for all }\varphi\in W^{1,1}_0(\Omega;\mathbb{R}^N), \tag{15}$$

where

$$h+W^{1,1}_0(\Omega;\mathbb{R}^N):=\{w\in W^{1,1}(\Omega;\mathbb{R}^N)\ ;\ w-h\in W^{1,1}_0(\Omega;\mathbb{R}^N).$$

In [24], the following results are given.

Theorem 2. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with the boundary $\partial\Omega$ of class $C^{1,1}$ Suppose that $a(x)$, $p(x)$ and $q(x)$ satisfy (2), (11), (13) , and that g satisfies (8), (9). Let h be a function in the class $W^{1,s}(\Omega;\mathbb{R}^N)$ for some*

$$s>\max\left\{\sup_{x\in\Omega}q(x),\ n\cdot\sup_{x\in\Omega}\frac{q(x)}{p(x)}\right\}.$$

Then a minimizer $u\in W^{1,1}(\Omega;\mathbb{R}^N)$ of $\mathcal{G}(\,\cdot\,;\Omega)$ with the boundary condition (14) is in the class $C^{0,\gamma_0}(\overline{\Omega};\mathbb{R}^N)\cap C^{0,\gamma}_{\rm loc}(\Omega;\mathbb{R}^N)$ for any

$$\gamma_0\in\left(0,\,1-\frac{n}{s}\sup_{\Omega}\frac{q(x)}{p(x)}\right)$$

and $\gamma\in(0,1)$.

By a part of the proof of the above theorem, we see that a bounded local minimizer of $\mathcal{G}(\cdot;\Omega)$ is locally Hölder continuous. Moreover, when g is Hölder continuous, we can show the Hölder continuity of the derivatives of a local minimizer for $\mathcal{G}(\cdot;\Omega)$. More precisely we have the following:

Theorem 3. *Let $\mathcal{G}$ be as in Theorem 2 and let $u\in W^{1,1}\cap L^\infty(\Omega;\mathbb{R}^N)$ be a local minimizer of $\mathcal{G}(\cdot;\Omega)$. Then $u\in C^{0,\gamma}_{\rm loc}(\Omega;\mathbb{R}^N)$ for any $\gamma\in(0,1)$. Moreover, when g is Hölder continuous, we have that $Du\in C^{0,\zeta}_{\rm loc}(\Omega;\mathbb{R}^{nN})$ for some $\zeta\in(0,1)$.*

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Center manifold theory in L^2-framework for the motions of camphor boats with delta function

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1 Introduction

Spatiotemporal collective motions in chemical experiments with camphor have been investigated in [8]–[10]. It was shown in [8] and [9] that a unidirectional motion can be observed if we place a camphor boat in an annular water channel. Camphor boats constitute a system for changing the number of particles with simple interaction. In this system, we observe at least three different kinds of states. It was reported in [10] that if the number of camphor boats is small, *a homogeneous state* appears, that is, they move with a constant velocity and spatially disperse with the same spacing between the boats. On the other hand, if the number of boats is larger than some critical value, the velocities of the boats change with temporal oscillation, and the shock wave appears in the line of the boats, which is called *a jammed state*. Moreover, there is a mode in which their high-density groups of the boats are maintained over time like bird flocks and fish schools, which is called *a clustered state* in [5].

Various motions exhibited by camphor boats attracts mathematicians and have been studied analytically. We introduce the following mathematical model proposed in [8] for the self-sustaining motion of a single camphor boat:

$$\begin{cases} \dfrac{d^2x_\infty}{dt^2} = -\mu\dfrac{dx_\infty}{dt} + \gamma(u(x_\infty+\rho,t)) - \gamma(u(x_\infty-\rho,t)), \\ \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - \alpha u + f(x-x_\infty), \end{cases} \tag{1.1}$$

where α, μ, and ρ are positive constants. The first equation is a Newtonian equation of motion for a camphor boat with the surface tension of water given by $\gamma(u)$, which is assumed to be C^4 and monotonically decreasing function of u. In this model, a camphor scraping is regarded as a particle, and the position of a camphor boat is denoted by $x_\infty = x_\infty(t)$. The surface concentration, denoted by u, of a camphor molecular layer is assumed to obey the reaction-diffusion equation with the function $f(x)$ defined by

$$f(x) \equiv \begin{cases} 1, & -\rho < x < \rho, \\ 0, & \text{otherwise}, \end{cases}$$

which indicates that camphor molecules are supplied only from $(-\rho,\rho)$ where a camphor boat contacts the water surface.

The spontaneous motion of a camphor boat can be characterized by a traveling wave solution in (1.1) which has a form of $(x_\infty(t),u(x,t)) = (-ct, p_\infty(x+ct))$, where $c \geq 0$ is called *a wave speed*. As stated in [8], the function $p_\infty = p_\infty(z) \in H^2(\mathbb{R})$ exists under a suitable condition. By using this traveling wave solution, we rewrite (1.1). Let (x_∞,u) be a solution of (1.1). Put $z_\infty \equiv x_\infty + ct$, $y_\infty \equiv dz_\infty/dt$ and $z \equiv x + ct$. By

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setting $u(z,t) \to u(z,t) + p_\infty(z - z_\infty(t))$, (1.1) is replaced into

$$\begin{cases} \dfrac{dz_\infty}{dt} = y_\infty, & t > 0, \\ \dfrac{dy_\infty}{dt} = -\mu(y_\infty - c) + \gamma(p_\infty(\rho) + u(z_\infty + \rho, t)) - \gamma(p_\infty(-\rho) + u(z_\infty - \rho, t)), & t > 0, \\ \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial z^2} - c\dfrac{\partial u}{\partial z} - \alpha u + y_\infty p'_\infty(z - z_\infty), \qquad -\infty < z < \infty, & t > 0 \end{cases} \tag{1.2}$$

because we have

$$\frac{\partial}{\partial t}(p_\infty(z - z_\infty)) = -\frac{dz_\infty}{dt}p'_\infty(z - z_\infty) = -y_\infty p'_\infty(z - z_\infty).$$

We represent the right-hand side of (1.2) by $\mathcal{L}_\infty(z_\infty, y_\infty, u)$ for $(z_\infty, y_\infty, u) \in \mathbb{R}^2 \times H^2(\mathbb{R})$.

We make the folloing assumptions on the linearized operator $\mathcal{L}'_\infty(0)$ throughout this article:

(A1) The algebraic multiplicity of the zero eigenvalue of $\mathcal{L}'_\infty(0)$ is 1. Then, the generalized eigenspace associated to a zero eigenvalue is spanned by $\Phi = (-1, 0, 0)$.

(A2) No eigenvalue λ of $\mathcal{L}'_\infty(0)$ exists in $\mathbb{C}\backslash\Sigma$ except for $\lambda = 0$, where $\Sigma \equiv \{\lambda \in \mathbb{C} \mid -\lambda_0 \le \mathrm{Re}\lambda\} \cup \{\lambda \in \mathbb{C} \mid |\arg\lambda| \le \theta\}$ for some $\lambda_0 > 0$ and $\theta > \pi/2$. $\mathrm{Re}\lambda$ and $\arg\lambda$ denote the real part and argument of a complex value λ, respectively, where $\arg\lambda$ is assumed to satisfy $-\pi < \arg\lambda \le \pi$.

Under the assumptions (A1), (A2), it can be shown that the resolvent estimate

$$\|(\lambda I - \mathcal{L}'_\infty(0))^{-1}\| \le \frac{C}{|\lambda|} \tag{1.3}$$

holds true uniformly in $\lambda \in \Sigma$ with $\lambda \neq 0$. Thanks to the Riez-Schauder's theorem (see [7]), a zero eigenfunction Φ^* of the adjoint operator $(\mathcal{L}'_\infty(0))^*$ also exists. We can determine Φ^* uniquely by $\langle \Phi, \Phi^* \rangle_\infty = 1$. Thus, Φ^* is expressed as $\Phi^* = (-1, Y^*, \phi^*)$.

Now we consider the collective motion of $(N+1)$-camphor boats in a one-dimensional circuit of length L. For a function φ defined on $(-\infty, \infty)$, $\overline{\varphi}$ denotes an L-periodic function given by

$$\overline{\varphi}(z) \equiv \chi(z - nL)\varphi(z - nL), \quad \left(n - \frac{1}{2}\right)L < z < \left(n + \frac{1}{2}\right)L, \quad n \in \mathbb{Z}, \tag{1.4}$$

where $\chi \in C^\infty(\mathbb{R})$ is a cut-off function defined by

$$\chi(z) \equiv \begin{cases} 1, & |z| < \dfrac{L}{4}, \\ 0, & |z| > \dfrac{3L}{8}. \end{cases}$$

Let $p = p(z) \in H^2(\mathbb{T})$ be a solution of

$$p'' - cp' - \alpha p + \overline{f}(z) = 0. \tag{1.5}$$

Then our system is described by

$$\begin{cases} \dfrac{dz_i}{dt} = y_i, \\ \dfrac{dy_i}{dt} = -\mu(y_i - c) + \gamma\left(\displaystyle\sum_{j=0}^{N} p(z_i - z_j + \rho) + u(z_i + \rho, t)\right) - \gamma\left(\sum_{j=0}^{N} p(z_i - z_j - \rho) + u(z_i - \rho, t)\right), \\ \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial z^2} - c\dfrac{\partial u}{\partial z} - \alpha u + \displaystyle\sum_{i=0}^{N} y_i p'(z - z_i) \end{cases} \tag{1.6}$$

in $z \in \mathbb{T} \equiv \mathbb{R}/L\mathbb{Z}$ for $N \geq 0$ and $i = 0, \ldots, N$, where $z_i = z_i(t)$, $y_i = y_i(t)$, and $u = u(z,t) \in H^2(\mathbb{T})$. Here, we denote by $H^2(\mathbb{T})$ a functional space defined by the closure of $C^\infty(\mathbb{T})$ in $H^2(0,L)$, where $C^\infty(\mathbb{T})$ is a set of infinitely differentiable and L-periodic functions. For simplicity, we denote an operator associated with the right-hand sides of the first, second, and third equations of (1.6) by $\mathcal{L}(U) : X^1 \equiv \mathbb{R}^{2N+2} \times H^2(\mathbb{T}) \to X \equiv \mathbb{R}^{2N+2} \times L^2(\mathbb{T})$, where $U = (z_0, y_0, \ldots, z_N, y_N, u) \in X^1$ and $\mathcal{L}(U) \equiv (\mathcal{L}_0^z(U), \mathcal{L}_0^y(U), \ldots, \mathcal{L}_N^z(U), \mathcal{L}_N^y(U), \mathcal{L}^u(U))$ is explicitly given by

$$\begin{aligned}
&\mathcal{L}_i^z(U) \equiv y_i,\\
&\mathcal{L}_i^y(U) \equiv -\mu(y_i - c) + \gamma\left(\sum_{j=0}^{N} p(z_i - z_j + \rho) + u(z_i + \rho)\right) - \gamma\left(\sum_{j=0}^{N} p(z_i - z_j - \rho) + u(z_i - \rho)\right),\\
&\mathcal{L}^u(U) \equiv \frac{\partial^2 u}{\partial z^2} - c\frac{\partial u}{\partial z} - \alpha u + \sum_{i=0}^{N} y_i p'(z - z_i).
\end{aligned}$$

Let l be the position of the 0th particle and let us denote the distance between the i-th and the $(i+1)$-th particles by h_i for $i = 0, \ldots, N-1$. As we consider a one-dimensional circuit, h_N is defined by the distance between the N-th and the 0th particles, i.e., $h_N = L - \sum_{i=0}^{N-1} h_i$. Thus, we define a relative position of the i-th particle by $\overline{z}_i = \overline{z}_i(\mathbf{h}) \equiv \sum_{j=0}^{i-1} h_j$, where $\mathbf{h} \equiv (h_0, \ldots, h_{N-1})$. We consider $\overline{z}_0 \equiv 0$. Here, we assume that there is a sufficient distance from one particle to any other particle. In other words, h_i is assumed to satisfy $h_i > h^*$ for any $i = 0, \ldots, N$, where $h^* > 0$ is assumed to be sufficiently large.

In the same way as in (1.1), we can show that there exists a traveling wave solution of (1.6) associated with a homogeneous state. According to the result obtained in [10], the traveling wave solution is expected to be unstable for a large number N. To prove the instability of the traveling wave solution, it is necessary to determine an unstable eigenvalue. However, the linearized eigenvalue problem is too difficult to analyze theoretically even in the case of $N = 1$. From this point of view, we must reduce (1.6) and derive a simpler system. This is the motivation of our study.

Now we formally state our main results. In the statement of our result, we use several notation. Let λ_1, λ_2 be the roots of the canonical equation $\lambda^2 - c\lambda - \alpha = 0$, that is,

$$\lambda_1 = \frac{1}{2}(c - \sqrt{c^2 + 4\alpha}) < 0, \quad \lambda_2 = \frac{1}{2}(c + \sqrt{c^2 + 4\alpha}) > 0,$$

and the constants M_1, M_2 defined by

$$M_1 \equiv (\gamma^+ e^{\lambda_1 \rho} - \gamma^- e^{-\lambda_1 \rho})\frac{e^{\lambda_1 \rho} - e^{-\lambda_1 \rho}}{\lambda_1 \sqrt{c^2 + 4\alpha}}, \quad M_2 \equiv (\gamma^+ e^{\lambda_2 \rho} - \gamma^- e^{-\lambda_2 \rho})\frac{e^{\lambda_2 \rho} - e^{-\lambda_2 \rho}}{\lambda_2 \sqrt{c^2 + 4\alpha}}, \tag{1.7}$$

where $\gamma^\pm = \gamma'(p_\infty(\pm\rho))$. Owing to the interactions described by (1.6), the positions of the particles will be varied. Thus, the position l and the distance h_i are regarded as functions of time, denoted by $l(t)$ and $h_i(t)$, respectively. Consider $P(l, \mathbf{h}) \equiv \sum_{i=0}^{N} (\overline{z}_i + l) e_{2i+1}$, where e_i is the unit vector in $\mathbb{R}^{2N+3}$ given by $e_i \equiv (0, \ldots, 0, \overset{i}{\underset{\vee}{1}}, 0, \ldots, 0)$.

Main results

(1) If L, h^* are sufficiently large and the initial value U_0 of (1.6) is close to $P(l_0, \mathbf{h}_0)$ for some $l_0, \mathbf{h}_0$, then the solution $U(z,t)$ of (1.6) can be approximated by $P(l, \mathbf{h})$ for $t > 0$, where $(l, \mathbf{h}) = (l(t), \mathbf{h}(t))$.

(2) The dynamics of $(l, \mathbf{h})$ and $h_N = h_N(t) \equiv L - \sum_{i=0}^{N-1} h_i(t)$ is approximated by

$$\begin{aligned}
\frac{dl}{dt} &= Y^*[M_1 e^{\lambda_1 h_N} + M_2 e^{-\lambda_2 h_0}),\\
\frac{dh_i}{dt} &= Y^*[M_1(e^{\lambda_1 h_{i-1}} - e^{\lambda_1 h_i}) + M_2(e^{-\lambda_2 h_i} - e^{-\lambda_2 h_{i+1}})]
\end{aligned} \tag{H}$$

for $i = 0, \ldots, N$, where $h_{-1} = h_N$ and $h_{N+1} = h_0$.

Roughly speaking, our main results (1), (2) imply that the dynamics of $(N+1)$-camphor boats described in (1.6) can be reduced to a finite dimensional dynamical system (H) under (A1), (A2) if the length of the circuit is sufficiently large and each particle is sufficiently separated from other particles. The reduction process from (1.6) to (H) will be shown in the end of Section 3. The precise statement of our main result will be described in the forthcoming paper. Refer to our previous works [3], [4] for another type of a reduced system.

When we establish our main result (1), the key factor is the transformation from (1.6) into a system (3.5), (3.6) of $(l, \mathbf{h})$ and a function W by a homeomorphism map $\Pi(\mathbf{h})$. We construct $\Pi(\mathbf{h})$ in a standard way [1] (see Section 4). In the construction, we have to pay careful attention to the regularity of several functions. In a previous work [1], it is easy to study the regularity because the smoothness of functions is assumed. On the other hand, the lack of regularity of functions contained in (1.6) causes mathematical difficulties in the analysis. In particular, the adjoint operator of a linearized operator $L(\mathbf{h}) \equiv \mathcal{L}'(P(0,\mathbf{h}))$, denoted by $L(\mathbf{h})^*$, contains the Dirac delta functions. In a previous work [2], we have developed a center manifold approach in $(H^1)^*$-framework in order to overcome this mathematical difficulties. In the present study we establish a center manifold approach in L^2-framework after the replacement of the original variable $u(z,t)$ into $u(z,t) + \sum_{i=0}^{N} p(z - z_i(t))$. Our aim in this paper is to overcome those difficulties and develop a new center manifold theory for systems with delta functions. In the analysis it is essential to study the regularity of eigenfunctions $\Phi_i^*(\mathbf{h})$ $(i = 0, \dots, N)$ of $L(\mathbf{h})^*$ with respect to $\mathbf{h}$. The eigenfunctions shall be defined in Section 2

The rest of this paper is organized as follows. In Section 2 we introduce several notation which we use throughout this manuscript. Also we study spectra and resolvent operators of $L(\mathbf{h})$ and $L(\mathbf{h})^*$. Lemma 1 is essentially the same as in the previous works [1], [2]. As seen in Proposition 1, the fractional space induced by $L(\mathbf{h})$ can be embedded into usual functional spaces. Our subsequent discussions will focus entirely on the differentiabilities of eigenfunctions of $L(\mathbf{h})$ and $L(\mathbf{h})^*$ with respect to $\mathbf{h}$ (Lemmas 2, 3). In Section 3 we will explain the reduction process from (1.6) to system (3.5), (3.6) in a formal way. The reduced system (H) can be derived by the substitution of $W = 0$ into (3.5) and collecting the lowest-order terms with respect to $\mathbf{h}$. In the last section we will define $\Pi(\mathbf{h})$ and state the regularity (Lemma 7).

2 Preliminaries

Give $\hat{\mathbf{h}} = (\hat{h}_0, \dots, \hat{h}_{N-1}) \in \mathbb{R}^N$ and assume $\hat{h}_j > h^*$ for any $j = 0, \dots, N$, where $\hat{h}_N = L - \sum_{i=0}^{N-1} \hat{h}_i$ as described in Introduction. For $\rho_1 > 0$, we define

$$H(\hat{\mathbf{h}}, \rho_1) \equiv \{\mathbf{h} = (h_0, \dots, h_{N-1}) \in \mathbb{R}^N \mid \hat{h}_j < h_j < \hat{h}_j + \rho_1 \ (j = 0, \dots, N)\}.$$

Throughout this article, $\hat{\mathbf{h}}$ and ρ_1 are fixed. We define $\varepsilon = \varepsilon(\mathbf{h}) \equiv e^{\lambda_1 \min \mathbf{h}}$ for $\mathbf{h} \in H(\hat{\mathbf{h}}, \rho_1)$, where $\min \mathbf{h} \equiv \min\{h_0, \dots, h_N\}$. Obviously, ε is small if h^* is large. Let I be the identity map. The derivative of $W \in X$ with respect to z is denoted by $\partial W/\partial z = (0, \dots, 0, w'(z))$, where $w(z)$ is the $(2N+3)$-th component of W and $w'(z)$ denotes the derivative of w. Let δ_{z_0} be the Dirac delta function providing unit mass to the point z_0. The derivative of a function φ with respect to $\mathbf{h}$ is denoted by $\partial\varphi/\partial\mathbf{h}$. Throughout this paper we denote general positive constants independent of $L, h^*, \hat{\mathbf{h}}$ by C.

Generally, we denote a norm of a Banach space B by $\|\cdot\|_B$ and a usual Euclidian norm by $|\cdot|$. For $1 \le q \le \infty$ and $k \in \mathbb{N}$, let $L^q(\mathbb{R})$, $H^k(\mathbb{R})$ and $(H^1(\mathbb{R}))^*$ be a usual Lebesgue space, a Sobolev space on $\mathbb{R}$ and the dual space of $H^1(\mathbb{R})$, respectively. Analougously, $L^q(\mathbb{T})$, $H^k(\mathbb{T})$ and $(H^1(\mathbb{T}))^*$ are defined on $\mathbb{T}$. We denote $\langle\cdot,\cdot\rangle_{L^2(\mathbb{R})}$, $\langle\cdot,\cdot\rangle_{L^2(\mathbb{T})}$, and $\langle\cdot,\cdot\rangle$ by usual inner products in $L^2(\mathbb{R})$, $L^2(\mathbb{T})$, and X, respectively. The notation $\langle\cdot,\cdot\rangle$ is also used for the pairing between $\mathbb{R}^{2N+2} \times H^1(\mathbb{T})$ and $\mathbb{R}^{2N+2} \times (H^1(\mathbb{T}))^*$. We represent a usual operator norm simply by $\|\cdot\|$. Finally, we denote a Banach space consisting of Hölder continuous functions on $\mathbb{T}$ with a Hölder exponent κ by $C^\kappa(\mathbb{T})$, and define $C^{1,\kappa}(\mathbb{T}) \equiv \{\varphi \in C^1(\mathbb{T}) \mid \varphi' \in C^\kappa(\mathbb{T})\}$.

Next we obtain a resolvent estimate for $L(\mathbf{h})$. To begin with, we give an explicit representation of

$L(\mathbf{h}) \equiv (L_0^z(\mathbf{h}), L_0^y(\mathbf{h}), \dots, L_N^z(\mathbf{h}), L_N^y(\mathbf{h}), L^u(\mathbf{h}))$ as

$$\begin{aligned}
&L_i^z(\mathbf{h})U \equiv y_i,\\
&L_i^y(\mathbf{h})U \equiv -\mu y_i + \gamma_i^+ \left(\sum_{j\neq i} p_{ij}^+(z_i - z_j) + u(\overline{z}_i + \rho) \right) - \gamma_i^- \left(\sum_{j\neq i} p_{ij}^-(z_i - z_j) + u(\overline{z}_i - \rho) \right),\\
&L^u(\mathbf{h})U \equiv \frac{\partial^2 u}{\partial z^2} - c\frac{\partial u}{\partial z} - \alpha u + \sum_{i=0}^{N} y_i p'(z - \overline{z}_i)
\end{aligned} \tag{2.1}$$

for $i = 0, \dots, N$ and $U = (z_0, y_0, \dots, z_N, y_N, u) \in X^1$, where

$$\gamma_i^{\pm} = \gamma_i^{\pm}(\mathbf{h}) \equiv \gamma' \left(\sum_{j=0}^{N} p(\overline{z}_i - \overline{z}_j \pm \rho) \right), \quad p_{ij}^{\pm} = p_{ij}^{\pm}(\mathbf{h}) \equiv p'(\overline{z}_i - \overline{z}_j \pm \rho), \quad i, j = 0, \dots, N.$$

Since the proof of the following lemma is the same as in [1], [2], we omit the details.

Lemma 1. *Let $\mathbf{h} \in H(\hat{\mathbf{h}}, \rho_1)$. Then, there exist positive constants C_1, C_2 independent of $L, h^*, \hat{\mathbf{h}}, \mathbf{h}$ such that if h^* is sufficiently large, then $L(\mathbf{h})$ satisfies*

$$\|(\lambda I - L(\mathbf{h}))^{-1}\| \leq \frac{C_1}{|\lambda|}, \quad \|(\lambda I - L(\mathbf{h}))^{-1} U\|_{X^1} \leq C_1 \left(1 + \frac{1}{|\lambda|}\right) \|U\|_X \tag{2.2}$$

uniformly in $\lambda \in \Sigma$ with $C_2 \varepsilon^{1/8} \leq |\lambda|$ and $U \in X$.

As shown in Lemma 1, $-L(\hat{\mathbf{h}})$ is a sectorial operator, and the real parts of all spectra of $-L(\hat{\mathbf{h}}) + I$ have strictly positive values. Then we can define a fractional space by $X^\omega \equiv D((-L(\hat{\mathbf{h}}) + I)^\omega)$ for $\omega \in [0, 1)$, where $D((-L(\hat{\mathbf{h}}) + I)^\omega)$ is a domain of $(-L(\hat{\mathbf{h}}) + I)^\omega$. We denote the norm of X^ω by $\|\cdot\|_\omega$. Hereafter we fix $\omega \in (3/4, 1)$. Then, X^ω is imbedded into Sobolev and Hölder spaces as follows. We refer to [6] for the proof of the following proposition.

Proposition 1. *Set $\omega \in (3/4, 1)$ and let κ be a constant with $0 < \kappa < 2\omega - 3/2$. Then, X^ω is imbedded into $\mathbb{R}^{2N+2} \times H^1(\mathbb{T})$ and $\mathbb{R}^{2N+2} \times C^{1,\kappa}(\mathbb{T})$. More precisely, there is $C > 0$ independent of L such that*

$$\|U\|_{\mathbb{R}^{2N+2} \times H^1(\mathbb{T})} \leq C\|U\|_\omega, \quad \|U\|_{\mathbb{R}^{2N+2} \times C^{1,\kappa}(\mathbb{T})} \leq C\|U\|_\omega \tag{2.3}$$

uniformly in $L \geq 1$ and $U \in X^\omega$.

By using the resolvent operator of $L(\mathbf{h})$, a projection map $Q(\mathbf{h})$ can be introduced by the Dunford integral (see [7]) as

$$Q(\mathbf{h}) \equiv \frac{1}{2\pi i} \int_\Gamma (\lambda I - L(\mathbf{h}))^{-1} d\lambda,$$

where $\Gamma \equiv \{\lambda \in \mathbb{C} \mid |\lambda| = \lambda_*\} \subset \Sigma$ is a positively-oriented circle enclosing $\lambda = 0$ for $\lambda_* \in (0, \alpha)$ sufficiently small and independent of $L, h^*, \hat{\mathbf{h}}, \mathbf{h}$. Lemma 1 implies that Γ is included in the resolvent set of $L(\mathbf{h})$. Define $E(\mathbf{h}) \equiv Q(\mathbf{h})X$ and $E(\mathbf{h})^* \equiv Q(\mathbf{h})^* X$, where $Q(\mathbf{h})^*$ is the adjoint operator of $Q(\mathbf{h})$. We see that $E(\mathbf{h})$ is a $(N+1)$-dimensional space spanned by a basis which consists of (generalized) eigenfunctions of $L(\mathbf{h})$ approximated by Φ. We omit the details of the proof because it is almost literally the same as in [1], [2].

Lemma 2. *For $k = 0, \dots, N$, define $\hat{\Phi}_k \equiv -e_{2k+1}$ and $\Phi_k(\mathbf{h}) \equiv Q(\mathbf{h})\hat{\Phi}_k$. If h^* is sufficiently large, then $\{\Phi_k(\mathbf{h})\}_{k=0}^N$ is a basis of $E(\mathbf{h})$ for $\mathbf{h} \in H(\hat{\mathbf{h}}, \rho_1)$. Moreover, $\Phi_k(\mathbf{h})$ is twice continuously differentiable with respect to $\mathbf{h}$ and satisfies*

$$\begin{aligned}
&\|\Phi_k(\mathbf{h}) - \hat{\Phi}_k\|_{X^1} \leq C\varepsilon^{1/4}, \quad \left\| \frac{\partial \Phi_k}{\partial h_m}(\mathbf{h}) \right\|_{X^1} \leq C, \quad \left\| \frac{\partial^2 \Phi_k}{\partial h_m \partial h_n}(\mathbf{h}) \right\|_{\mathbb{R}^{2N+2} \times H^1(\mathbb{T})} \leq C,\\
&\left\| \frac{\partial^2 \Phi_k}{\partial h_m \partial h_n}(\mathbf{h}_1) - \frac{\partial^2 \Phi_k}{\partial h_m \partial h_n}(\mathbf{h}_2) \right\|_{\mathbb{R}^{2N+2} \times H^1(\mathbb{T})} \leq C|\mathbf{h}_1 - \mathbf{h}_2|
\end{aligned}$$

uniformly in $k = 0, \dots, N$, $m, n = 0, \dots, N-1$ *and* $\mathbf{h}, \mathbf{h}_1, \mathbf{h}_2 \in H(\hat{\mathbf{h}}, \rho_1)$ *for some* $C > 0$ *independent of* $L, h^*, \hat{\mathbf{h}}$.

Thanks to the Riez-Schauder's theorem (see [7]), $L(\mathbf{h})^*$ also has exactly $N+1$ eigenvalues close to 0. By the similar argument of Lemma 2, a basis of $E(\mathbf{h})^*$ is given by $\{\tilde{\Phi}_k^*(\mathbf{h})\}_{k=0}^N$, defined in the following lemma. We can verify the differentiability of $\tilde{\Phi}_k^*(\mathbf{h})$, which is the most important in our analysis. However, we need a somewhat different approach from the proof of Lemma 2 because delta functions of $L(\mathbf{h})^*$ cause the lack of regularity of $\tilde{\Phi}_k^*(\mathbf{h})$. We will describe the details of the proof of Lemma 3 in the forthcoming paper.

Lemma 3. *For* $k = 0, \dots, N$, *define*

$$\hat{\Phi}_k^*(\mathbf{h}) \equiv -e_{2k+1} + Y^* e_{2k+2} + \overline{\phi^*}(z - \overline{z}_k) e_{2N+3}, \quad \tilde{\Phi}_k^*(\mathbf{h}) \equiv Q(\mathbf{h})^* \hat{\Phi}_k^*(\mathbf{h}).$$

If h^* *is sufficiently large, then* $\{\tilde{\Phi}_k^*(\mathbf{h})\}_{k=0}^N$ *is a basis of* $E(\mathbf{h})^*$. *Moreover,* $\tilde{\Phi}_k^*(\mathbf{h})$ *is twice continuously differentiable with respect to* $\mathbf{h}$, *and there is* $C > 0$ *independent of* $L, h^*, \hat{\mathbf{h}}$ *such that*

$$\|\tilde{\Phi}_k^*(\mathbf{h}) - \hat{\Phi}_k^*(\mathbf{h})\|_{X^1} \le C\varepsilon^{1/4}, \quad \left\| \frac{\partial \tilde{\Phi}_k^*}{\partial h_m}(\mathbf{h}) \right\|_X \le C, \quad \left\| \frac{\partial^2 \tilde{\Phi}_k^*}{\partial h_m \partial h_n}(\mathbf{h}) \right\|_{\mathbb{R}^{2N+2} \times (H^1(\mathbb{T}))^*} \le C,$$

$$\left| \left\langle \frac{\partial^2 \tilde{\Phi}_k^*}{\partial h_m \partial h_n}(\mathbf{h}_1) - \frac{\partial^2 \tilde{\Phi}_k^*}{\partial h_m \partial h_n}(\mathbf{h}_2), V \right\rangle \right| \le C\|V\|_\omega |\mathbf{h}_1 - \mathbf{h}_2|$$

uniformly in $\mathbf{h}, \mathbf{h}_1, \mathbf{h}_2 \in H(\hat{\mathbf{h}}, \rho_1)$, $n, m = 0, \dots, N-1$, *and* $V \in X^\omega$ *with* $3/4 < \omega < 1$.

Set $\Phi_k^*(\mathbf{h}) \equiv \sum_{j=0}^N a_{jk} \tilde{\Phi}_j^*(\mathbf{h})$ for constants a_{jk} in $j, k = 0, \dots, N$. Since $\Phi_k^*(\mathbf{h})$ is defined by a linear combination of $\{\tilde{\Phi}_i^*(\mathbf{h})\}$, it is obvious that $\Phi_k^*(\mathbf{h})$ is included in $E(\mathbf{h})^*$. From the definitions of $\hat{\Phi}_k$, $\hat{\Phi}_k^*(\mathbf{h})$, we see $\left\langle \hat{\Phi}_i, \hat{\Phi}_k^*(\mathbf{h}) \right\rangle = \delta_{ik}$, where δ_{ik} denotes the Kronecker delta. By choosing appropriate constants, $\Phi_i(\mathbf{h}), \Phi_k^*(\mathbf{h})$ also satisfy the orthogonal condition $\langle \Phi_i(\mathbf{h}), \Phi_k^*(\mathbf{h}) \rangle = \delta_{ik}$. Then $Q(\mathbf{h})$ is represented by $Q(\mathbf{h})U = \sum_{i=0}^N \langle U, \Phi_i^*(\mathbf{h}) \rangle \Phi_i(\mathbf{h})$, and $E^\perp(\mathbf{h}) = (I - Q(\mathbf{h}))X$ is characterized by

$$E^\perp(\mathbf{h}) = \{U \in X \mid \langle U, \Phi_i^*(\mathbf{h}) \rangle = 0 \quad (i = 0, \dots, N)\}.$$

3 Formal derivation of a reduced system

Our goal in this section is to reduce our system (1.6) into a lower-dynamical system (3.5), (3.6) in a formal way. To begin with, we state the existence and uniqueness of a solution in (1.6). We denote the initial value of (1.6) at $t = 0$ by U_0 and assume $U_0 \in X^\omega$ with $\omega \in (3/4, 1)$ throughout this section. Then it can be shown by a standard argument for analytical semigroups that the solution $U(z,t)$ of (1.6) exists uniquely and globally in time (see [6]). Moreover, it is also continuous in X^ω and differentiable in X with respect to t.

In the reductions process, it is important to decompose the system (1.6) into a new system (3.5), (3.6), We transform the solution $U(z,t)$ to $(l(t), \mathbf{h}(t), W(z,t))$ as

$$U(z,t) = P(l(t), \mathbf{h}(t)) + \Xi(l(t))\Pi(\mathbf{h}(t))W(z,t),$$

where $\Xi(l) : X \to X$ is a shift operator defined by $[\Xi(l)U](z) = U(z-l)$ and a homeomorphism $\Pi(\mathbf{h})$ will be defined in Section 4. For simplicity, we may represent $(l, \mathbf{h}, W) = (l(t), \mathbf{h}(t), W(z,t))$. It will be shown that $(l, \mathbf{h}, W)$ yields to (3.5), (3.6). In order to extract a lower-dimensional dynamical system from those equations, we apply a standard center manifold theorem in [6] by using an evolution operator $T(t, t_0; \mathbf{h})$ generated by an operator

$$\tilde{L}(\mathbf{h}) \equiv \Pi(\mathbf{h})^{-1} L(\mathbf{h}) \Pi(\mathbf{h}) : X^1 \cap E^\perp(\hat{\mathbf{h}}) \to E^\perp(\hat{\mathbf{h}}),$$

which is Lipschitz continuous with respect to $\mathbf{h}$, that is,

$$\|(\tilde{L}(\mathbf{h}_1) - \tilde{L}(\mathbf{h}_2))V\|_X \le C|\mathbf{h}_1 - \mathbf{h}_2|\|V\|_\omega$$

uniformly in $\mathbf{h}_1, \mathbf{h}_2 \in H(\hat{\mathbf{h}}, \rho_1)$ and $V \in X^\omega$ with $3/4 < \omega < 1$. According to Theorem 7.4.2 in [6], the Lipschitz continuity of $\tilde{L}(\mathbf{h})$ with respect to $\mathbf{h}$ contributes to the generation of $T(t, t_0; \mathbf{h})$.

Lemma 4. *Give t_*, t^* with $0 \le t_* < t^*$ arbitrarily. Let $\mathbf{h}(t) : [t_*, t^*] \to H(\hat{\mathbf{h}}, \rho_1)$ be a C^1 function such that $\sup_{t\in(t_*,t^*)} |d\mathbf{h}/dt| \le C\varepsilon$ for a constant $C > 0$ independent of $L, h^*, \hat{\mathbf{h}}, \mathbf{h}, t_*, t^*$. Then, for any $\beta_2 \in (0, \lambda_0)$, there are sufficiently large L, h^* and a positive constant C independent of $L, h^*, \hat{\mathbf{h}}, \mathbf{h}, t_*, t^*$ such that*

$$\begin{aligned}
&\|T(t, t_0; \mathbf{h})V\|_\omega \le C \max\left\{1, (t - t_0)^{-\omega}\right\} e^{-\beta_2(t-t_0)}\|V\|_X, \quad V \in E^\perp(\hat{\mathbf{h}}),\\
&\|T(t, t_0; \mathbf{h})V\|_\omega \le Ce^{-\beta_2(t-t_0)}\|V\|_\omega, \quad V \in X^\omega \cap E^\perp(\hat{\mathbf{h}})
\end{aligned} \tag{3.1}$$

uniformly in t, t_0 with $t_* \le t_0 \le t \le t^*$.

We rewrite (3.6) into an integral form by using the evolution operator and can prove the existence and uniqueness of a center manifold by applying a contraction mapping theorem. Actually, under a setting in [2], it is difficult to verify the Lipschitz continuity of the nonlinear terms, and then it seems that we cannot construct a center manifold by a standard theory. Therefore our approach in this manuscript overcomes the mathematical difficulties as seen in the previous work.

We formally derive governing equations of $l, \mathbf{h}, W$. Substituting $U(z,t) = P(l, \mathbf{h}) + \Xi(l)\Pi(\mathbf{h})W(z,t)$ into the system (1.6), we have

$$\frac{\partial U}{\partial t} = \Xi(l)\left(\frac{\partial P}{\partial l}(0, \mathbf{h}) - \frac{\partial}{\partial z}\Pi(\mathbf{h})W\right)\frac{dl}{dt} + \Xi(l)\left(\frac{\partial P}{\partial \mathbf{h}}(0, \mathbf{h}) + \frac{\partial}{\partial \mathbf{h}}\Pi(\mathbf{h})W\right)\frac{d\mathbf{h}}{dt} + \Xi(l)\Pi(\mathbf{h})\frac{\partial W}{\partial t}$$

and

$$\mathcal{L}(U) = \mathcal{L}(P(l, \mathbf{h}) + \Xi(l)\Pi(\mathbf{h})W) = \Xi(l)\mathcal{L}(P(0, \mathbf{h}) + \Pi(\mathbf{h})W).$$

Hence it follows that

$$\left(\frac{\partial P}{\partial l}(0, \mathbf{h}) - \frac{\partial}{\partial z}\Pi(\mathbf{h})W\right)\frac{dl}{dt} + \left(\frac{\partial P}{\partial \mathbf{h}}(0, \mathbf{h}) + \frac{\partial}{\partial \mathbf{h}}\Pi(\mathbf{h})W\right)\frac{d\mathbf{h}}{dt} + \Pi(\mathbf{h})\frac{\partial W}{\partial t} = \mathcal{L}(P(0, \mathbf{h}) + \Pi(\mathbf{h})W). \tag{3.2}$$

Multiplying $Q(\mathbf{h})$ and $\Pi(\mathbf{h})^{-1}(I - Q(\mathbf{h}))$ to (3.2), we decompose it into

$$Q(\mathbf{h})\left[\left(\frac{\partial P}{\partial l}(0, \mathbf{h}) - \frac{\partial}{\partial z}\Pi(\mathbf{h})W\right)\frac{dl}{dt} + \left(\frac{\partial P}{\partial \mathbf{h}}(0, \mathbf{h}) + \frac{\partial}{\partial \mathbf{h}}\Pi(\mathbf{h})W\right)\frac{d\mathbf{h}}{dt}\right] = Q(\mathbf{h})\mathcal{L}(P(0, \mathbf{h}) + \Pi(\mathbf{h})W) \tag{3.3}$$

and

$$\begin{aligned}
&\Pi(\mathbf{h})^{-1}(I - Q(\mathbf{h}))\left[\left(\frac{\partial P}{\partial l}(0, \mathbf{h}) - \frac{\partial}{\partial z}\Pi(\mathbf{h})W\right)\frac{dl}{dt} + \left(\frac{\partial P}{\partial \mathbf{h}}(0, \mathbf{h}) + \frac{\partial}{\partial \mathbf{h}}\Pi(\mathbf{h})W\right)\frac{d\mathbf{h}}{dt}\right] + \frac{\partial W}{\partial t}\\
&= \Pi(\mathbf{h})^{-1}(I - Q(\mathbf{h}))\mathcal{L}(P(0, \mathbf{h}) + \Pi(\mathbf{h})W).
\end{aligned} \tag{3.4}$$

The left-hand side of (3.3) can be represented as a linear combination of dl/dt and $d\mathbf{h}/dt$, that is, there are $(N+1)$-dimensional vectors $\mathbf{m}_i$ for $i = 0, \dots, N$ such that the left-hand side of (3.3) can be represented as

$$\mathbf{m}_i \cdot \left(\frac{dl}{dt}, \frac{d\mathbf{h}}{dt}\right) = \left\langle \left(\frac{\partial P}{\partial l}(0, \mathbf{h}) - \frac{\partial}{\partial z}\Pi(\mathbf{h})W\right)\frac{dl}{dt} + \left(\frac{\partial P}{\partial \mathbf{h}}(0, \mathbf{h}) + \frac{\partial}{\partial \mathbf{h}}\Pi(\mathbf{h})W\right)\frac{d\mathbf{h}}{dt}, \Phi_i^*(\mathbf{h})\right\rangle$$

for $i = 0, \dots, N$, where "$\cdot$" denotes the inner product in $\mathbb{R}^{N+1}$. The vector $\mathbf{m}_i = \mathbf{m}_i(\mathbf{h}, W)$ are defined in $\mathbf{h} \in H(\hat{\mathbf{h}}, \rho_1)$ and $W \in X^\omega$ with $3/4 < \omega < 1$. We define a $(N+1)\times(N+1)$ matrix $M = M(\mathbf{h}, W)$ such that the i-th row corresponds to $\mathbf{m}_{i-1}$. It is easy to see that M is close to an $(N+1)\times(N+1)$ lower triangular

matrix $M_0 = (m_{ik})_{i,k}$, where m_{ik} is defined by $m_{ik} = 1$ if $i \geq k$ and 0 if $i < k$ $(i, k = 1, \ldots, N+1)$. This fact implies that M is invertible. Moreover, M is uniformly bounded and Lipschitz continuous in $\mathbf{h} \in H(\hat{\mathbf{h}}, \rho_1)$. Set $H_i(\mathbf{h}, W) \equiv \langle \mathcal{L}(P(0, \mathbf{h}) + \Pi(\mathbf{h})W), \Phi_i^*(\mathbf{h}) \rangle$ for $i = 0, \ldots, N$. We see that $H_i(\mathbf{h}, W)$ is uniformly bounded and Lipschitz continuous in $\mathbf{h} \in H(\hat{\mathbf{h}}, \rho_1)$ and $W \in X^\omega$ with $\|W\|_\omega \leq 1$. Finally, define $\tilde{H}_i(\mathbf{h}, W)$ by

$$(\tilde{H}_0(\mathbf{h}, W), \ldots, \tilde{H}_N(\mathbf{h}, W)) \equiv M(\mathbf{h}, W)^{-1}(H_0(\mathbf{h}, W), \ldots, H_N(\mathbf{h}, W)).$$

Then $(l, \mathbf{h})$ yields to

$$\frac{dl}{dt} = \tilde{H}_0(\mathbf{h}, W), \quad \frac{d\mathbf{h}}{dt} = \mathbf{H}(\mathbf{h}, W) \equiv (\tilde{H}_1(\mathbf{h}, W), \ldots, \tilde{H}_N(\mathbf{h}, W)). \tag{3.5}$$

Substitution of the equation above into the left-hand side of (3.4) results in

$$\frac{\partial W}{\partial t} = \tilde{L}(\mathbf{h})W + \tilde{G}(\mathbf{h}, W), \tag{3.6}$$

where

$$\begin{aligned}\tilde{G}(\mathbf{h}, W) \equiv{}& \Pi(\mathbf{h})^{-1}R(\mathbf{h})(\mathcal{L}(P(0, \mathbf{h}) + \Pi(\mathbf{h})W)) \\ &+ \Pi(\mathbf{h})^{-1}R(\mathbf{h}) \left[\left(\frac{\partial P}{\partial l}(0, \mathbf{h}) - \frac{\partial}{\partial z}\Pi(\mathbf{h})W \right) \tilde{H}_0(\mathbf{h}, W) - \left(\frac{\partial P}{\partial \mathbf{h}}(0, \mathbf{h}) + \frac{\partial}{\partial \mathbf{h}}\Pi(\mathbf{h})W \right) \mathbf{H}(\mathbf{h}, W) \right].\end{aligned}$$

It follows from Lemma 4 that (3.5) and (3.6) exhibit the existence and uniqueness of a solution by a standard theory. Note that the variable l can be determined by the first differential equation (3.5). Therefore it is sufficient to consider the dynamics of $\mathbf{h}$ and W in the second equation of (3.5) and (3.6). Finally we follow a standard way as described in [1], [6], and construct of an attractive invariant manifold. As a result, we reduce the dynamics of $\mathbf{h}$ and W.

Finally we derive (H) from (3.5) and (3.6). We expand $\mathcal{L}(U)$ with respect to W as

$$\mathcal{L}(U) \sim \mathcal{L}(P(0, \mathbf{h})) + L(\mathbf{h})\Pi(\mathbf{h})W.$$

Due to $L(\mathbf{h})^*\Phi_i^*(\mathbf{h}) \in E(\mathbf{h})^*$ and $\Pi(\mathbf{h})W \in E^\perp(\mathbf{h})$, we see

$$\langle L(\mathbf{h})\Pi(\mathbf{h})W, \Phi_i^*(\mathbf{h}) \rangle = \langle \Pi(\mathbf{h})W, L(\mathbf{h})^*\Phi_i^*(\mathbf{h}) \rangle = 0.$$

Hence we obtain

$$H_i(\mathbf{h}, W) \sim \langle \mathcal{L}(P(0, \mathbf{h})), \Phi_i^*(\mathbf{h}) \rangle .$$

We next compute $\mathcal{L}(P(0, \mathbf{h}))$ directly and obtain $\mathcal{L}_i^z(P(0, \mathbf{h})) = \mathcal{L}^u(P(0, \mathbf{h})) = 0$,

$$\mathcal{L}_i^y(P(0, \mathbf{h})) = \gamma \left(\sum_{j=0}^{N} p(\overline{z}_i - \overline{z}_j + \rho) \right) - \gamma(p_\infty(\rho)) - \left(\gamma \left(\sum_{j=0}^{N} p(\overline{z}_i - \overline{z}_j - \rho) \right) - \gamma(p_\infty(-\rho)) \right).$$

Since p satisfies $|p(z)| \leq Ce^{\lambda_1|z|}$ in $|z| \leq L/2$, we can derive (H).

4 Homeomorphism

In the formal calculatations in the previous section, it is crucial to prove that that $\Pi(\mathbf{h})$ is differentiable with respect to $\mathbf{h}$ and its derivative is Lipschitz continuous in $\mathbf{h} \in H(\hat{\mathbf{h}}, \rho_1)$. In this section we define $\Pi(\mathbf{h})$ by the same way as in [1] and prove the Lipschitz continuity of $\partial\Pi(\mathbf{h})/\partial\mathbf{h}$.

Give $\mathbf{h} \in H(\hat{\mathbf{h}}, \rho_1)$. According to a usual theory for ordinary differential equations, we verify the existence and uniqueness of the solution $Y = Y(r; Y_0)$ in

$$\begin{cases} \dfrac{dY}{dr} = S(r;\mathbf{h})Y \equiv -\displaystyle\sum_{k=0}^{N} \left\langle Y, \frac{\partial \Phi_k^*}{\partial \mathbf{h}}(\mathbf{h}(r))(\mathbf{h}-\hat{\mathbf{h}}) \right\rangle \Phi_k(\mathbf{h}(r)), \quad 0 < r \le 1, \\ Y(0; Y_0) = Y_0 \end{cases} \tag{4.1}$$

due to Lemmas 2, 3, where $\mathbf{h}(r) \equiv (1-r)\hat{\mathbf{h}} + r\mathbf{h} \in H(\hat{\mathbf{h}}, \rho_1)$ for a parameter $r \in [0,1]$.

Lemma 5. *If $Y_0 \in E^{\perp}(\hat{\mathbf{h}})$, then $Y(r; Y_0) \in E^{\perp}(\mathbf{h}(r))$.*

Proof. We easily see

$$\frac{d}{dr}\langle Y(r;Y_0), \Phi_k^*(\mathbf{h}(r))\rangle = \left\langle \frac{dY}{dr}(r;Y_0), \Phi_k^*(\mathbf{h}(r)) \right\rangle + \left\langle Y(r;Y_0), \frac{d}{dr}(\Phi_k^*(\mathbf{h}(r))) \right\rangle = 0$$

because of the orthogonal conditions for $\Phi_i(\mathbf{h}), \Phi_k^*(\mathbf{h})$. Hence $\langle Y(r;Y_0), \Phi_k^*(\mathbf{h}(r))\rangle = 0$ for all $k = 0, \ldots, N$, that is, $Y(r;Y_0) \in E^{\perp}(\mathbf{h}(r))$. □

Using the solution of (4.1), we define $\Pi(r_0, \mathbf{h})$ by $\Pi(r_0, \mathbf{h})Y_0 \equiv Y(r_0; Y_0)$. We also define $\Pi(0, \mathbf{h}) = I$. Then we have the following lemma.

Lemma 6. *Set $r_0 \in [0,1]$ and $\mathbf{h} \in H(\hat{\mathbf{h}}, \rho_1)$. Let (A, B) be one of the pairs of Banach spaces given by*

$$(X, X), \quad (E^{\perp}(\hat{\mathbf{h}}), E^{\perp}(\mathbf{h})), \quad (X^{\omega}, X^{\omega}), \quad (E^{\perp}(\hat{\mathbf{h}}) \cap X^{\omega}, E^{\perp}(\mathbf{h}) \cap X^{\omega}).$$

Then $\Pi(r_0, \mathbf{h}) : A \to B$ is a homeomorphism.

Set $\Pi(\mathbf{h}) \equiv \Pi(1, \mathbf{h})$. We show the differentiability of $\Pi(\mathbf{h})$.

Lemma 7. *Assume $\omega \in (3/4, 1)$ and $Y_0 \in X^{\omega}$. Then $F(\mathbf{h}) \equiv \Pi(\mathbf{h})Y_0$ is differentiable with respect to $\mathbf{h} \in H(\hat{\mathbf{h}}, \rho_1)$. Moreover, there is a constant $C > 0$ independent of h^* such that*

$$\left\| \frac{\partial F}{\partial \mathbf{h}}(\mathbf{h}) \right\|_X \le C\|Y_0\|_{\omega}, \quad \left\| \frac{\partial F}{\partial \mathbf{h}}(\mathbf{h}_1) - \frac{\partial F}{\partial \mathbf{h}}(\mathbf{h}_2) \right\|_X \le C|\mathbf{h}_1 - \mathbf{h}_2|\|Y_0\|_{\omega}$$

uniformly in $\mathbf{h}, \mathbf{h}_1, \mathbf{h}_2 \in H(\hat{\mathbf{h}}, \rho_1)$ and $Y_0 \in X^{\omega}$.

Proof. Let $Y = Y(r; Y_0)$ be a solution of (4.1). We prove that Y is differentiable with respect to $\mathbf{h} = (h_0, \ldots, h_{N-1}) \in H(\hat{\mathbf{h}}, \rho_1)$ for $Y_0 \in X^{\omega}$. We differentiate the both sides of (4.1) by h_m for $m = 0, \ldots, N-1$ in a formal way, and then have

$$\begin{cases} \dfrac{dZ}{dr} = S(r;\mathbf{h})Z + S_1(r;\mathbf{h})Y, \quad 0 < r \le 1, \\ Z(0) = 0, \end{cases} \tag{4.2}$$

where $S_1(r;\mathbf{h})Y$ is defined by

$$\begin{aligned} S_1(r;\mathbf{h})Y \equiv &-\sum_{k=0}^{N} \left\langle Y, \left(\frac{\partial}{\partial h_m} \frac{\partial \Phi_k^*}{\partial \mathbf{h}} \right)(\mathbf{h}(r))r(\mathbf{h}-\hat{\mathbf{h}}) + \frac{\partial \Phi_k^*}{\partial h_m}(\mathbf{h}(r)) \right\rangle \Phi_k(\mathbf{h}(r)) \\ &- \sum_{k=0}^{N} \left\langle Y, \frac{\partial \Phi_k^*}{\partial \mathbf{h}}(\mathbf{h}(r))(\mathbf{h}-\hat{\mathbf{h}}) \right\rangle \frac{\partial \Phi_k}{\partial h_m}(\mathbf{h}(r))r. \end{aligned}$$

It follows from Lemma 3 that $\langle Y, (\partial^2 \Phi_k^*/\partial h_m \partial h_n)(\mathbf{h}) \rangle$ is Lipschitz continuous in $\mathbf{h}$ for $k = 0, \ldots, N$, $m, n = 0, \ldots, N-1$, $\mathbf{h} \in H(\hat{\mathbf{h}}, \rho_1)$ and any $Y \in X^\omega$. Since the solution $Y(r; Y_0) \in X^\omega$ in $0 \leq r \leq 1$ under the assumption, it is shown that (4.2) has a unique solution by a usual existence theorem in ordinary differential equations. This fact implies that $\partial Y(r; Y_0)/\partial h_m = Z(r)$ does exist and is a solution of (4.2). The inequalities in the statement can be readily obtained by the Lipschitz continuity of the right-hand side with respect to $\mathbf{h}$. □

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Inverse problems for equations of fluid dynamics by Carleman estimates

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Abstract

We consider several types of equations of fluid dynamics and one typical case is an inverse source problem for the Navier-Stokes equations:

$$\begin{cases} \partial_t v(x,t) - \kappa\Delta v + (v\cdot\nabla)v + \nabla p = F(x,t), \\ \operatorname{div} v(x,t) = 0 \quad \text{in } \Omega\times(0,T). \end{cases}$$

Here the viscosity $\kappa > 0$ is assumed to be a constant.

Inverse source problem *Let $\Gamma \subset \partial\Omega$ and $t_0 \in (0,T)$ be given. Then, determine a spatially varying factor of $F(x,t)$ in some subdomain of Ω by means of data $v|_{\Gamma\times(0,T)}$ and $v(\cdot,t_0)|_\Omega$.*

On the basis of Carleman estimates by Choulli, Imanuvilov, Puel and Yamamoto [2], we prove conditional stability for the inveres problem under constraint $\operatorname{div} F(x,t_0) = 0$ in Ω. This constraint is essential for the uniqueness because we do not request any data of the pressure p.

We discuss also other types of equations such as compressible viscosity equation (e.g., Imanuvilov and Yamamoto [5]).

For establishing the stability for inverse problems of determining spatially varying coefficients in partial differential equations, one main methodology is by Carleman estimate which Bukhgeim and Klibanov [1] established. Then by [1], Imanuvilov and Yamamoto [4] proved

the global Lipschitz stability for such an inverse problem. Recently in Huang, Imanuvilov and Yamamoto [3], we modify such conventional arguments and simplify for obtaining the stability for inverse problems. We apply it to similar inverse problems including the determination of spatially varying viscosity for several types of equations of fluid dynamics

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A game-theoretic approach to the asymptotic behavior of solutions to an obstacle problem for the mean curvature flow equation

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1 Introduction

We consider the following obstacle problem for the mean curvature flow equation.

$$\begin{cases} V = -\kappa, \text{ on } \partial D_t, \\ O_- \subset D_t, \end{cases} \tag{1.1}$$

where $\{D_t\}_{t\geq 0}$ is the unknown family of open sets in $\mathbb{R}^2$, V is the velocity of ∂D_t in the direction of its normal vector, κ is the curvature of ∂D_t, and O_- is a fixed open set in $\mathbb{R}^2$. Our main goal is to investigate the asymptotic behavior of solutions to (1.1) by the level set equation and its game-theoretic approximation.

1.1 Level set method

The level set method for surface evolution equations was first rigorously analyzed in [2, 3]. The basic idea of this method is to represent moving surfaces as level sets of auxiliary functions and to rewrite surface evolution equations by level set equations, whose unknown functions are the auxiliary functions. A great advantage is the point that viscosity solutions of level set equations follow the long time behavior of the moving surfaces even after topological change of surfaces. The level set method is applied to various surface evolution equations including the mean curvature flow equation. See also [5] in detail.

As a typical example, we first introduce the level set mean curvature flow equation.

$$\begin{cases} u_t(x,t) + F(Du(x), D^2u(x)) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{1.2}$$

where $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is the unknown function and $u_0 \in C(\mathbb{R}^n)$ is an initial datum. The function F is given by

$$F(Du, D^2u) = -|Du|\mathrm{div}\left(\frac{Du}{|Du|}\right).$$

Namely, F is the level-set mean curvature flow operator defined as

$$F(p,X) = -\mathrm{Tr}\left(\left(I - \frac{p\otimes p}{|p|^2}\right)X\right), \quad p \in \mathbb{R}^n \setminus \{0\},\ X \in \mathbb{S}^n, \tag{1.3}$$

where $p \otimes p = (p_i p_j)_{i,j=1}^n$ for a vector $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$ and $\mathbb{S}^n$ is the set of $n \times n$ real symmetric matrices. The initial value problem (1.2) is the level set equation corresponding to the surface evolution equation

$$V = -\kappa, \text{ on } \partial D_t. \tag{1.4}$$

Roughly speaking, the following are how to solve surface evolution equations such as (1.4) by the level set method.

(1) For a given open set D_0, take an initial function u_0 to satisfy $D_0 = \{x \in \mathbb{R}^n | u_0(x) > 0\}$.

(2) Solve the corresponding initial value problem such as (1.2) with u_0.

(3) Let $D_t = \{x \in \mathbb{R}^n | u(x,t) > 0\}$. Then, $\{\partial D_t\}_{t \geq 0}$ is the required moving surface.

There are two points we have to prove so that the above procedure works. First, we have to prove the long time existence and the uniqueness of viscosity solutions to the level set equation. Also, the uniqueness of moving surface is needed. Namely, we have to prove that if u_0 and v_0 satisfy $\{x \in \mathbb{R}^n | u_0(x) > 0\} = \{x \in \mathbb{R}^n | v_0(x) > 0\}$, then $\{x \in \mathbb{R}^n | u(x,t) > 0\} = \{x \in \mathbb{R}^n | v(x,t) > 0\}$, where u and v are respectively viscosity solutions to the level set equation with initial functions u_0 and v_0. Indeed, these points are proved for various surface evolution equations including the mean curvature flow equation.

1.2 Obstacle problem

Recently, obstacle problems for the mean curvature flow equation have been considered in [4, 7, 9, 13]. Obstacle problems are problems that have regions called obstacles which the solutions cannot exceed.

According to [4, 9], the level set method is still valid for (1.1). The corresponding level set equation is the following.

$$\begin{cases} u_t(x,t) + F(Du(x), D^2u(x)) = 0 & \text{in } \mathbb{R}^2 \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^2, \\ \Psi_-(x) \leq u(x,t) & \text{in } \mathbb{R}^2 \times (0,\infty), \end{cases} \tag{1.5}$$

where $\Psi_- \in Lip(\mathbb{R}^n)$ is a given obstacle function that satisfies $O_- = \{x \in \mathbb{R}^2 | \Psi_-(x) > 0\}$. The comparison principle for (1.5) is guaranteed, for instance, in the following assumption.

$$\text{For some } a \in \mathbb{R} \text{ and } R > 0,\ u_0 = a \text{ in } B^c(0,R), \tag{1.6}$$

where we let $a < 0$ to consider 0 level set of the solution.

1.3 Game interpretation

We employ the theory of viscosity solutions to solve (1.5). A notion of viscosity solutions originated from the optimal control theory. It is known that the value function of an optimal control problem or a differential game is characterized as the unique viscosity solution of the corresponding first order equation. See [1]. In this context, Kohn and Serfaty [10] present a deterministic game whose value functions u^ϵ ($\epsilon > 0$) approximate the unique viscosity solution u of the level set mean curvature

flow equation (1.2). We modify their game so that the limit of the value functions gives a solution of the following equation, which is more general than (1.5).

$$\begin{cases} u_t(x,t) + \nu|Du(x,t)| + F(Du(x), D^2u(x)) = f(x) & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n, \\ \Psi_-(x) \le u(x,t) \le \Psi_+(x) & \text{in } \mathbb{R}^n \times (0,\infty), \end{cases} \tag{1.7}$$

where $\Psi_+, \Psi_- \in Lip(\mathbb{R}^n)$ are given obstacle functions which satisfy $\Psi_- \le \Psi_+$ in $\mathbb{R}^n$. ν is a constant and f is a locally bounded function. The equation (1.7) without obstacle functions is first introduced in [6]. Also, it is considered in [8].

We use the game-theoretic approximation to study the asymptotic behavior of the solution. The value functions corresponding to (1.7) are written by three types of cost, running cost, terminal cost, and stopping cost. The running cost is the cost that is added at each time step of the game and depends on the current game positions. The terminal cost is the cost that is added when the game ends at the given terminal time t, depending on the last position of the game. Each player has the right to quit the game before the terminal time t. If either of the players enforces this right, the stopping cost is the entire game cost. An optimal control problems with only terminal cost is called *Mayer problem* and one with terminal cost and running cost is called *Bolza problem*. Also, one with stopping cost is called *optimal stopping time problem*.

The running cost and the terminal cost depend on f and u_0 respectively. Also, the stopping costs of each player depend on Ψ_- and Ψ_+ respectively. Such an interpretation of parameters of PDEs is well understood for first order equations; see [1]. See also [12], which deals with the optimal stopping time problem corresponding to the obstacle problem for the infinity Laplacian equation.

1.4 Literature

We give some other related works on the asymptotic behavior of solutions to obstacle problems for the mean curvature flow equation. [13] proves the convergence of moving surfaces in a situation that both initial surface and obstacles are given as periodic graphs. [7] proves the solution $u(x,t)$ to the problem (1.7) with driving force $\nu < 0$ converges as $t \to \infty$ to the stationary solution. The paper [7] also gives the result concerning to the shape of the stationary solution. However, it is limited to the case u_0 and Ψ_+ are spherical symmetric.

While [4, 9] and this manuscript consider given obstacle problems, they arise from many different situations. In [6], an obstacle problem naturally appears in the motion of the top and the bottom of the solution when the source term is a characteristic function though the equation has no obstacle functions. [11] shows that large exponent limit of power mean curvature flow equation is formulated by an obstacle problem involving 1-Laplacian.

2 Preliminaries

We denote the upper and lower semicontinous envelope of u by u^* and u_* respectively.

Definition 2.1 (Viscosity solution)**.** (1) A function u is a viscosity subsolution of (1.7) if it satisfies following conditions.

(a) $\Psi_-(x) \le u^*(x,t) \le \Psi_+(x)$ for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$.

(b) $u^*(x,t) \le u_0(x)$ for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$.

(c) Whenever $\phi(x,t)$ is smooth, $u^* - \phi$ has a local maximum at $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$ and $u^*(x_0,t_0) - \Psi_-(x_0) > 0$, we have

$$\phi_t(x_0,t_0) + \nu|D\phi(x_0,t_0)| + F_*(D\phi(x_0,t_0), D^2\phi(x_0,t_0)) \le f^*(x_0).$$

(2) A function u is a viscosity supersolution of (1.7) if it satisfies following conditions.

(a) $\Psi_-(x) \le u_*(x,t) \le \Psi_+(x)$ for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$.

(b) $u_*(x,t) \ge u_0(x)$ for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$.

(c) Whenever $\phi(x,t)$ is smooth, $u_* - \phi$ has a local minimum at $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$ and $u_*(x_0,t_0) < \Psi_+(x_0)$, we have

$$\phi_t(x_0,t_0) + \nu|D\phi(x_0,t_0)| + F^*(D\phi(x_0,t_0), D^2\phi(x_0,t_0)) \ge f_*(x_0).$$

(3) A function u is a viscosity solution of (1.7) if it is a viscosity subsolution and a viscosity supersolution of (1.7).

Definition 2.2. (1) Let D_0 and O_- be bounded open sets in $\mathbb{R}^2$. A family of open sets $\{D_t\}_{t\ge 0}$ is called an *open evolution* of (1.1) with initial data D_0 and obstacle O_- if the following hold.

(a) there exist $u_0 \in C(\mathbb{R}^2)$ and $\Psi_- \in Lip(\mathbb{R}^2)$ such that (1.6) holds for some $a < 0$, $D_0 = \{x \in \mathbb{R}^2 | u_0(x) > 0\}$, and $O_- = \{x \in \mathbb{R}^2 | \Psi_-(x) > 0\}$.

(b) there exist a viscosity solution u of (1.5) with u_0 and Ψ_- taken above that satisfies $D_t = \{x \in \mathbb{R}^2 | u(x,t) > 0\}$ for $t \ge 0$.

(2) Let E_0 be a bounded closed set and O_- be a bounded open set in $\mathbb{R}^2$. A family of closed sets $\{E_t\}_{t\ge 0}$ is called an *closed evolution* of (1.1) with initial data E_0 and obstacle O_- if the following hold.

(a) there exist $u_0 \in C(\mathbb{R}^2)$ and $\Psi_- \in Lip(\mathbb{R}^2)$ such that (1.6) holds for some $a < 0$, $E_0 = \{x \in \mathbb{R}^2 | u_0(x) \ge 0\}$, and $O_- = \{x \in \mathbb{R}^2 | \Psi_-(x) > 0\}$.

(b) there exist a viscosity solution u of (1.5) with u_0 and Ψ_- taken above that satisfies $E_t = \{x \in \mathbb{R}^2 | u(x,t) \ge 0\}$ for $t \ge 0$.

Remark 2.3. Open evolutions and closed evolutions of (1.1) uniquely exist since the long time existence and uniqueness of solutions to (1.5) are obtained from [4, 9].

To state the asymptotic behavior of $\{D_t\}_{t\ge 0}$ and $\{E_t\}_{t\ge 0}$, we prepare

$$\overline{\lim}_{t\to\infty} D_t := \cap_{\tau>0} \cup_{t>\tau} D_t,\ \underline{\lim}_{t\to\infty} D_t := \cup_{\tau>0} \cap_{t>\tau} D_t.$$

For a set A, we denote the smallest convex set which contains A by $Conv(A)$.

3 Results

3.1 Game interpretation and its convergence

We introduce the rule of the game corresponding to (1.7) with $\nu \geq 0$ and $n = 2$. The game is a deterministic two-person zero-sum game. For convenience, we name the first player Player I and the second player Player II. Player I's goal is to maximize the total cost of the game and Player II's goal is to minimize it.

Let $x \in \mathbb{R}^2$, $t > 0$, and $\epsilon > 0$ be the initial setting of the game. The game consists of $N = \lceil \frac{t}{\epsilon^2} \rceil$ rounds, where $\lceil r \rceil$ stands for the minimum integer that is no less than r. In each round i ($i = 1, 2, \cdots, N$), the following procedures are carried out.

(1) Player I decides whether to quit the game.

(2) Player II decides whether to quit the game.

(3) Player I chooses $v_i \in S^1$. (S^1 is the set of unit vectors in $\mathbb{R}^2$.)

(4) Player II chooses $w_i \in S^1$ and $b_i \in \{\pm 1\}$ after Player I's choice.

(5) Determine the next states as follows.

$$\begin{aligned} x_i &= x_{i-1} + \sqrt{2}\epsilon b_i v_i + \nu\epsilon^2 w_i, \\ c_i &= c_{i-1} + \epsilon^2 f(x_{i-1}). \end{aligned} \tag{3.1}$$

Here, $x_i \in \mathbb{R}^2$ and $c_i \in \mathbb{R}$ respectively denote game position and cost at the end of round i. The initial position $x_0 = x$ and the initial cost $c_0 = 0$. For convenience, we often interpret game position as Player I's position, and game cost as Player I's score.

If Player I quits the game at round i, the total cost is given by $\Psi_-(x_i)$. If Player II quits the game at round i, it is given by $\Psi_+(x_i)$. If both players go throughout N rounds of the game, it is given by $c_N + u_0(x_N)$. The value function $u^\epsilon(x,t)$ is defined inductively based on the following *Dynamic Programming Principle* and the initial condition.

$$u^\epsilon(x,t) = \max\{\Psi_-(x), \min\{\Psi_+(x), \sup_{v\in S^1} \inf_{\substack{w\in S^1 \\ b=\pm 1}} [u^\epsilon(x + \sqrt{2}\epsilon b v + \nu w \epsilon^2, t - \epsilon^2) + \epsilon^2 f(x)]\}\} \tag{3.2}$$

for $t > 0$.

$$u^\epsilon(x,t) = u_0(x) \tag{3.3}$$

for $t \leq 0$.

This $u^\epsilon(x,t)$ stands for the total cost given when Player I starts at x with terminal time t and both players take optimal choices.

To state the convergence of the value functions, we introduce the following half relaxed limits of the value functions.

$$\begin{aligned} \overline{u}(x,t) &:= \overline{\lim_{\substack{(y,s)\to(x,t) \\ \epsilon\searrow 0}}}\, u^\epsilon(y,s) := \inf_{\delta>0} \sup\{u^\epsilon(y,s) \mid |x-y| < \delta, |t-s| < \delta, 0 < \epsilon < \delta\}, \\ \underline{u}(x,t) &:= \underline{\lim_{\substack{(y,s)\to(x,t) \\ \epsilon\searrow 0}}}\, u^\epsilon(y,s) := \sup_{\delta>0} \inf\{u^\epsilon(y,s) \mid |x-y| < \delta, |t-s| < \delta, 0 < \epsilon < \delta\}. \end{aligned}$$

Theorem 3.1. *$\overline{u}$ and $\underline{u}$ are respectively viscosity sub- and supersolution of* (1.7). *Moreover,* $\overline{u}(x,0) = \underline{u}(x,0) = u_0(x)$ *for* $x \in \mathbb{R}^2$.

Remark 3.2. If the comparison principle holds for (1.7), then the convergence result implies that $\lim_{\epsilon \searrow 0} u^\epsilon(x,t) = u(x,t)$, where u is the unique continuous viscosity solution. Moreover, it is well known that this convergence is locally uniform one.

Remark 3.3. For the equation (1.7) without obstacles, a game interpretation and its convergence result are given in [8].

Remark 3.4. The Dynamic Programming Principle corresponding to (1.7) with $\nu < 0$ is given by

$$u^\epsilon(x,t) = \max\{\Psi_-(x), \min\{\Psi_+(x), \sup_{v,w \in S^1} \min_{b=\pm 1} [u^\epsilon(x + \sqrt{2}\epsilon b v + \nu w \epsilon^2, t - \epsilon^2) + \epsilon^2 f(x)]\}\}.$$

Namely, not Player II but Player I has the right to choose $w_i \in S^1$.

3.2 Asymptotic behavior of solutions

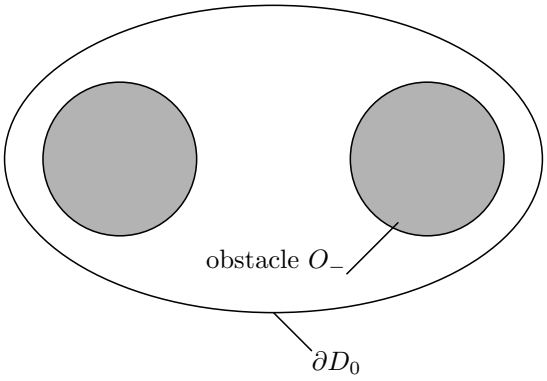

Figure 1: D_0 contains $Conv(O_-)$

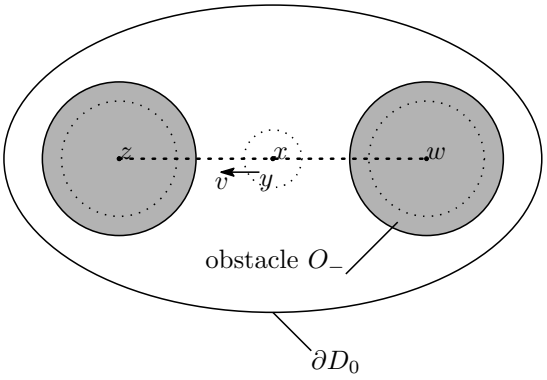

Figure 2: A strategy of Player I

We consider the asymptotic behavior of solutions to (1.1) by using a game-theoretic approximation introduced above.

Let us start with a simple situation. If O_- is two separated balls and D_0 contains $Conv(O_-)$ such as Figure 1, then it is naturally expected that D_t converges to $Conv(O_-)$. To prove this, we show that the value functions of the game $u^\epsilon(x,t)$ are positive in $Conv(O_-)$ and negative in $\overline{Conv(O_-)}^c$ for sufficiently large t. To prove the positivity of u^ϵ, we construct Player I's strategies of the game that attain positive game cost. To prove the negativity, we construct Player II's strategies that attain negative game cost.

In this manuscript, we only give a brief explanation of Player I's strategies. Notice that for (1.5), the game costs we have to consider are just the terminal cost and the stopping cost of Player I. Thus, the victory conditions of Player I are to quit the game in O_- or to reach D_0 at the terminal time t. It is the easiest situation for Player I when the initial position $y \in O_-$, because all he has to do is to quit the game at the first round. In the case of Figure 1, for $x \in Conv(O_-) \setminus O_-$, we can take $z, w \in O_-$ such that $x \in l_{z,w}$, where $l_{z,w}$ means the line segment with end points z and

w. If the initial position y is near x, it suffices for Player I to keep taking v parallel to $l_{z,w}$ until he reaches the domains near z or w, because he can stay near $l_{z,w}$ all the time by doing this. (Figure 2)

Considering additional game strategies, we obtain the convergence result for more general case as follows.

Definition 3.5. Let O_- and D_0 be sets in $\mathbb{R}^n$ that satisfy $O_- \subset D_0$. Let $\mathscr{O}$ be the family of connected components of O_-. We say that connected components $A \in \mathscr{O}$ and $B \in \mathscr{O}$ are *directly connected* through D_0 if $l_{x,y} \subset D_0$ for some $x \in A$ and $y \in B$. We say that connected components A and B are *connected* through D_0 if there is such a sequence $\{O_i\}_{i=0}^N \subset \mathscr{O}$ that $O_0 = A$, $O_N = B$, and O_i and O_{i+1} $(i = 0, 1, \cdots, N-1)$ are directly connected through D_0.

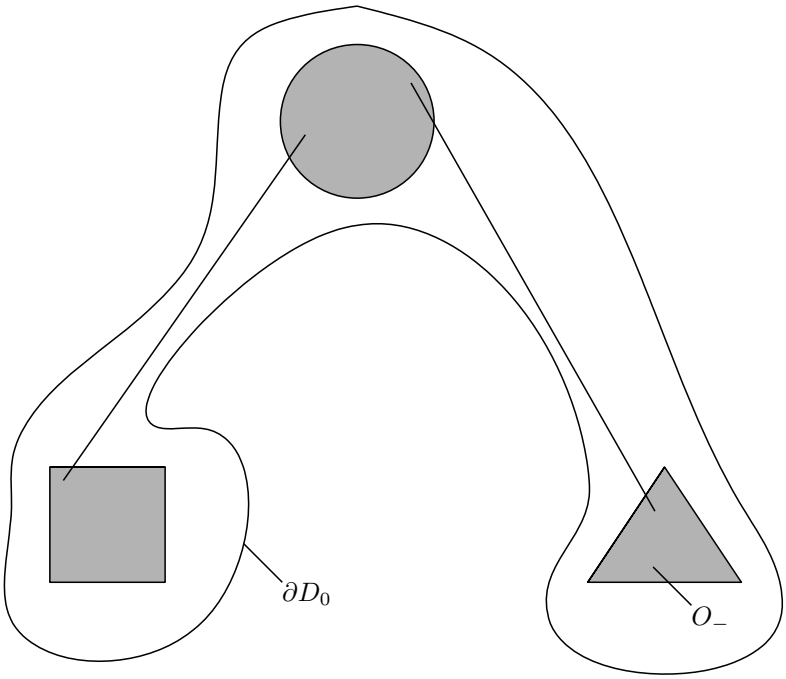

Figure 3: D_0 and O_- that satisfy the assumptions of Theorem 3.6

Theorem 3.6. *Let D_0 and O_- be open sets in $\mathbb{R}^2$ that satisfy $O_- \subset D_0$. Let $E_0 = \overline{D_0}$. Assume that any two connected components of O_- are connected through D_0. (See Figure 3.) Let $\{D_t\}_{t\geq 0}$ and $\{E_t\}_{t\geq 0}$ be respectively the open evolution of* (1.1) *with D_0 and O_- and the closed evolution of* (1.1) *with E_0 and O_-. Then,*

$$Conv(O_-) \subset \underline{\lim}_{t\to\infty} D_t \subset \overline{\lim}_{t\to\infty} D_t \subset \overline{Conv(O_-)}$$

and

$$Conv(O_-) \subset \underline{\lim}_{t\to\infty} E_t \subset \overline{\lim}_{t\to\infty} E_t \subset \overline{Conv(O_-)}.$$

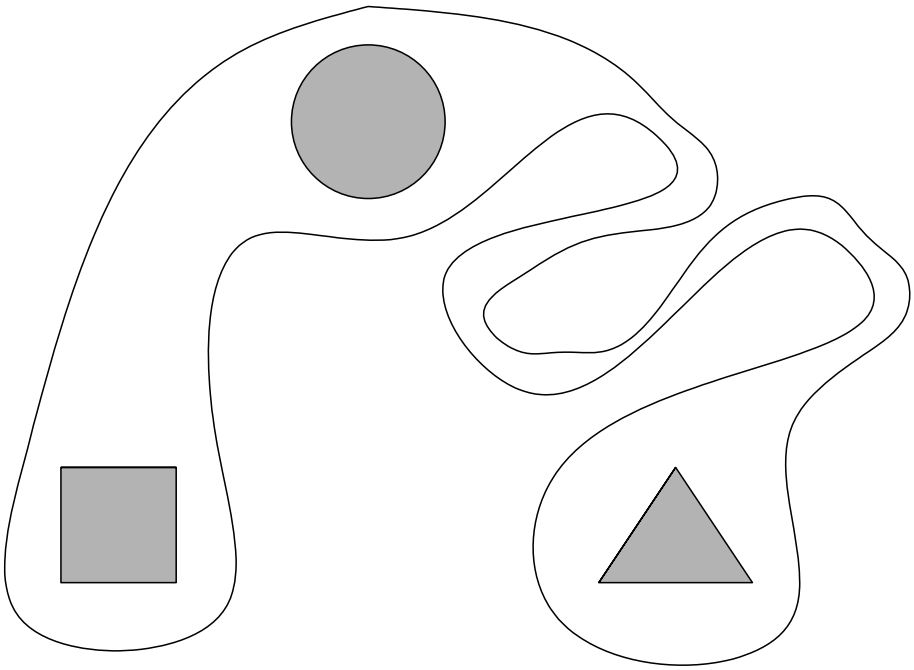

Figure 4: D_0 and O_- that do not satisfy the assumptions of Theorem 3.6

Remark 3.7. The assumptions of Theorem 3.6 are technical, because we expect that the same conclusion holds for cases such as Figure 4. However, the condition that there is a connected component O of D_0 that satisfies $O_- \subset O$ is crucial. Otherwise, for example, when $O_- = B((3,0),1) \cup B((-3,0),1)$ and $D_0 = B((3,0),2) \cup B((-3,0),2)$, the same conclusion does not hold.

4 Future works

There remains many unanswered questions.

- For cases such as Figure 4, does the same conclusion as Theorem 3.6 hold?
- How about double obstacles case?
- How about higher dimensional case?
- How about the case obstacles move?

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Mathematical reformulation of the Kolmogorov-Richardson energy cascade in terms of vortex stretching

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In this study, we investigate the energy cascade picture [9, 2] in turbulence of an incompressible fluid with uniform density and kinematic viscosity ν under periodic boundary conditions in three orthogonal directions. When Re is sufficiently high, turbulence is composed of vortices with various length scales. We emphasize that these vortices are coherent in time and space. More precisely, the band-pass filter (i.e. the Littlewood-Paley decomposition) of the Fourier components of the vorticity reveals a clear hierarchical structures of coherent tubular vortices (see Fig. 2 in Ref. [4]). Tubular vortices at a given length (i.e. a given thickness) are stretched and amplified in straining motions around (pairs of) larger tubular vortices. Here, it is important that this generation process of smaller vortices by larger ones occurs locally in scale. In other words, this sustaining process of turbulence is consistent with the classical Kolmogorov theory (i.e. the Kolmogorov-Richardson energy cascade [2]).

The main purpose of the present study is to mathematically describe the feature of solutions of the Navier-Stokes equation on the basis of these physical pictures of turbulence. More concretely, with the aid of the DNS of turbulence in a periodic cube (the details of which are given in Sec. 1), we propose a mathematical description of the energy cascading process in developed turbulence (Sec. 2). Then, we derive the Kolmogorov $-5/3$ power law of the energy spectrum by employing this energy cascade picture, without directly using the Kolmogorov hypotheses.

1 Numerical simulations

In this section, we summarize the numerical method, and we briefly explain the results of our DNS (for the detail, see [10]).

1.1 Numerical method

We numerically integrate the Navier-Stokes equation in $\mathbb{T}^3 := (\mathbb{R}/2\pi\mathbb{Z})^3$:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + f, \\ \nabla \cdot u = 0, \\ u(t=0) = u_0 \end{cases} \tag{1}$$

with the external force,

$$f = (-\sin x \cos y, +\cos x \sin y, 0). \tag{2}$$

Here, $u : [0,\infty) \times \mathbb{T}^3 \to \mathbb{R}^3$ and $p : [0,\infty) \times \mathbb{T}^3 \to \mathbb{R}$ denote the velocity and pressure of the fluid, respectively. We use the standard Fourier spectral method. Here, we numerically integrate, by the fourth order Runge-Kutta-Gill scheme, the vorticity equation in terms of the Fourier component $\widehat{\omega}(K)$ of vorticity, which is defined by

$$\omega(x) = \sum_{K \in \mathbb{Z}^3} \widehat{\omega}(K) e^{iK\cdot x} \quad \text{with} \quad (\mathcal{F}u)(K) = \widehat{u}(K) := \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} u(x) e^{-iK\cdot x}\, dx. \tag{3}$$

[1]This is a joint work with Susumu Goto (Osaka Univ.) and Tomonori Tsuruhashi (Univ. of Tokyo).

Note that we need to integrate only two components of $\widehat{\omega}(K)$ because of the solenoidal condition $K \cdot \widehat{\omega} = 0$. For numerical efficiency, we use the fast Fourier transform to evaluate the nonlinear term in the vorticity equation in the real space. Here, we remove the aliasing errors by the phase shift method.

1.2 Energy transfer due to vortex stretching (four wavenumber ranges)

There are four distinct wavenumber ranges (or equivalently, scale ranges) in each of which the energy transfer due to vortex stretching exhibits different characteristics. To show this, we define the energy transfer due to vortex stretching by

$$T(K, K') := \frac{1}{K^2} \int_{\mathbb{T}^3} \omega(K) \cdot \nabla u(K') \cdot \omega(K)\, dx \tag{4}$$

where

$$\omega(K) := \mathcal{F}^{-1}\Big[\chi_{[K/\sqrt{2},\sqrt{2}K)}\, \mathcal{F}[\omega]\Big] \tag{5}$$

and

$$u(K) := \mathcal{F}^{-1}\Big[\chi_{[K/\sqrt{2},\sqrt{2}K)}\, \mathcal{F}[u]\Big]\,. \tag{6}$$

We plotted $T(K, K', t)$ in turbulence at $R_\lambda \approx 280$ (Run 2-1K) in Figure 1 in [10].

For a low wavenumber range ($K \in [0.0020\eta^{-1}, 0.0087\eta^{-1}]$), where the effect of the external force is dominant. It is clear in the figure that the Fourier modes at K acquire energy from smaller wavenumber modes ($K' < K$) and transfer the energy to larger wavenumber modes ($K < K'$).

Secondly, we look at the inertial range. The figure shows the energy transfer in the wavenumber range $K \in [0.0105\eta^{-1}, 0.026\eta^{-1}]$. The figure also shows that the energy transfers from lower ($K' < K$) to higher $K' > K$ wavenumber modes. In contrast to the lower wavenumber range, the contribution from about the half (more precisely, $K' = 0.58K$) wavenumber modes (i.e. 1.7 times larger scales) transfer the energy to K than the forcing modes. This is also consistent with the observation in the same figure that the energy at the wavenumber K transfers most to a higher wavenumber mode at $K' = 1.7K$ (i.e. 0.58 times smaller scales). It is further important to observe that all the curves for different K collapse almost perfectly. This means that the energy transfer due to vortex stretching occurs in a self-similar manner, and that the energy is conserved by this cascading process. In other words, the viscous effect is negligible in this wavenumber range (i.e. the inertial range). We may verify the self-similarity (see Figure 2 in [10]), where we show $K'_{\max}/K$ and $K'_{\min}/K$ as functions of K. Here, $K'_{\max}$ and $K'_{\min}$ denote the wavenumbers which attain the maximum and minimum of the energy transfer:

$$K'_{\max}(K, t) := \operatorname*{argmax}_{K'} T(K, K', t) \quad \text{and} \quad K'_{\min}(K, t) := \operatorname*{argmin}_{K'} T(K, K', t)\,. \tag{7}$$

We can see in Figure 2 in [10] that $K'_{\max} \sim K$ and $K'_{\min} \sim K$ in the wavenumber range $K \in [0.01\eta^{-1}, 0.04\eta^{-1}]$.

Thirdly, we look at the dissipation range. The figure shows the energy transfer in a higher wavenumber range: $K \in [0.031\eta^{-1}, 0.16\eta^{-1}]$. The behaviors of the energy transfer are qualitatively different from those in the inertial range. (i) There is no collapse of the curves because the amount of energy transfer decreases with wavenumber, since the energy is dissipated due to viscosity. (ii) the valleys of the energy transfer located always at $K' = 1.7K$, whereas the wavenumber K' giving the maximum behaves in a non-trivial manner (see Figure 2 in [10]). This implies the disruption of the self-similarity, which is also reasonable because of the viscous effects.

Fourthly, we look at the far dissipation range. The figure shows the energy transfer in $K \in [0.19\eta^{-1}, 0.33\eta^{-1}]$. The behaviors of the energy transfer in this high wavenumber

range is further different from others. Though the Fourier modes acquire the energy form larger scales, almost no energy transfers to higher wavenumbers.

Thus, the energy transfer behaves differently in the four wavenumber ranges. In summary, we can describe the scale-local energy transfer once we define it by using the enstrophy production rate as in (4).

2 Reformulation of the Kolmogorov-Richardson energy cascade

In this section, with the aid of the DNS result in Sec. 1, we describe the Kolmogorov-Richardson energy cascade in terms of vortex stretching. In what follows, we employ $\mathbb{R}^3$ which is as an approximation domain of $\mathbb{T}^3$. Also we simply write $\int f := \int_{\mathbb{R}^3} f(x)dx$. Let u be a smooth solution to the 3D-Navier-Stokes equation (1) (but replace $\mathbb{T}^3$ to $\mathbb{R}^3$), and ω be the corresponding vorticity. Let

$$\bar{\omega}_K(t,x) := \mathcal{F}_\xi^{-1}[\chi_{A_K}(\xi)\hat{\omega}(t,\xi)](x),$$

where (coarse-graining in an annulus, Littlewood-Paley decomposition)

$$A_K := \{\xi \in \mathbb{R}^3 : K/\sqrt{\alpha} \le |\xi| \le \sqrt{\alpha}K\}$$

and $\alpha > 1$ is a prescribed constant expressing the ratio of adjacent scales (see Definition 1). In what follows we take wavenumber K from $\alpha^{\mathbb{Z}}$. Also, in the following arguments, we assume that the spatially integrated value of any quantity is independent of time because the size of the domain is much larger than the correlation length (the integral length) of the flow, that is,

$$\frac{d}{dt}\|\bar{\omega}_K(t)\|_{L^2}^2 = 0 \quad \text{for any} \quad K \quad \text{and} \quad t \ge 0. \tag{8}$$

Our specific purpose is to approximate the following vortex stretching term:

$$\int (\omega \cdot \nabla)u \cdot \bar{\omega}_K.$$

Note that $\omega = \sum_{K\in\alpha^{\mathbb{Z}}} \bar{\omega}_K$ and $u = \sum_{K\in\alpha^{\mathbb{Z}}} \bar{u}_K$. First, we propose a new hypothesis. For the original version of Kolmogorov hypothesis, see [6], for a recasted version, see [3, Section 6].

Outline of our hypothesis. The crucial point of our hypothesis is that there exists a universal vortex stretching/compressing mechanism, independent of the energy input rate ϵ (which will appear later). More precisely, a pair of tubular vortices $\widetilde{W}_{j,K}$ are stretching several tubular vortices in the adjacent smaller scale, or, several tubular vortices are compressing a pair of tubular vortices in the adjacent larger scale. Also, supports of these tubular vortices (with compact supports) are disjoint. We conjecture that such vortex stretching/compressing in adjacent two scales are the dominant event, so, in our cascade picture, we exclude multiscale events. Note that, due to the uncertainty principle for the Fourier transform, we expect that the three-wave interaction of this adjacent scale event is rather nonlocal (c.f. [8]).

A pair of tubular vortices. Let $\widetilde{W}_{j,K} \in C_c^\infty(\mathbb{R}^3)$ represents a pair of "normalized" tubular vortices, where j and K are important indexes, will be explained later. Then $W_{j,K}(x)$ is defined by

$$W_{j,K}(x) := \mathcal{F}^{-1}\left[\chi\mathcal{F}[\widetilde{W}_{j,K}]\right](x),$$

where χ is a characteristic function such that

$$\chi(\xi) = \begin{cases} 1 & \frac{1}{\sqrt{\alpha}} \le |\xi| \le \sqrt{\alpha}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\operatorname{supp} \mathcal{F}[W_{j,K}] \subset A_1$ and $W_{j,K}$ is decaying (not compactly supported) in the physical space, due to the uncertainty principle. We assume that

$$\int_{\mathbb{R}^3} |W_{j,K}(x)|^2 dx = \gamma$$

for some prescribed (non-dimensional) constant γ which is independent of all parameters. For the sake of simplicity, here, we assume $W_{j,K} \simeq \widetilde{W}_{j,K}$. This means, the corresponding error term is negligible (for the precise treatment of the error term, see [10]).

Decomposition of vorticity field. In the inertial range, we employ the following hypothesis. For any fixed t, assume that $\bar{\omega}_K$ is mainly expressed by $\{W_{j,K}\}_{j,K}$, and is independent of ν, as follows:

$$\bar{\omega}_K(t,x) = gK^{-H+1} \left(\sum_{j=1}^{cK^D} W_{j,K}(Kx) \right), \tag{9}$$

where $c, g, H, D \in \mathbb{R}_{\ge 0}$. Rigorously, there must exist a small perturbation (in order to recover divergence-free) depending on ν, but we neglect it. H and D express the Hölder exponent and the fractal dimension respectively. Rigorously, we need to require cK^D to be integer. In this case we slightly approximate c and D. This D is ideally determined by physical experiments, in particular, determined by α and $|\Omega^+_{j,K}|$, which will appear later.

The external force. For fixed $c > 0$, we assume that, for any energy input rate ϵ and wavenumber K_f, we can construct an external force f such that $\operatorname{supp} \hat{f} \subset A_{K_f}$ and

$$\frac{1}{K_f^2} \int \overline{(\nabla \times f)}_{K_f} \cdot \omega = \epsilon,$$

where ω is a solution to the Navier-Stokes (vorticity) equation.

Hierarchy of tubular vortices. Let $K_\pm = \alpha^{\mp 1} K$ and let us assume that a pair of tubular vortices $\widetilde{W}_{J,K_+}$ stretch (through Biot-Savart law) $\widetilde{W}_{j,K}$ ($j \in^\exists \Omega^+_{J,K} \subset \{1, 2, \cdots, cK^D\}$) and assume disjointness $\Omega^+_{J,K} \cap \Omega^+_{J',K} = \emptyset$ ($J \neq J'$). In the forcing scale, the external force $\nabla \times f$ stretches $\widetilde{W}_{j,K_f}$ ($j \in^\exists \Omega^+_{K_f} \subset \{1, 2, \cdots, c(K_f)^D\}$). We give the precise definition of "stretching" later (see Definition 3). Let

$$\Omega^+_K := \bigcup_{J=1}^{cK_+^D} \Omega^+_{J,K}.$$

If $\widetilde{W}_{J,K_+}$ does not stretch anything, then we regard $\Omega^+_{J,K}$ as an empty set. To the contrary, assume that, for any $j \in (\Omega^+_K)^c := \{1, 2, \cdots, cK^D\} \setminus \Omega^+_K$, $\widetilde{W}_{j,K}$ is compressed by $\widetilde{W}_{j',K_-}$ ($j' \in^\exists \Omega^-_{j,K} \subset \mathbb{N}$) through Biot-Savart law. We also give the precise definition of "compressing" later (see Definition 4). Note that $|\Omega^+_K| + |(\Omega^+_K)^c| = K^D$.

Definition 1. (Clarification of the fractal dimention D.) In this new hypothesis, we crucially assume α and $|\Omega^+_{j,K}| (= |\Omega^-_{j,K}| = |\Omega^+_{K_F}|)$ are non-dimensional prescribed constants (ideally determined by physical experiments), that is, independent of K and j. Rigorously

this $|\Omega^+_{j,K}|$ is integer for each time t, but here we are implicitly spatially averaging it, to be a positive real number. We conjecture that this $|\Omega^+_{j,K}|$ is determined by an unstable mode (which is independent of any type of disturbance) induced by the elliptical instability (see Subsection 2.2 in [5], see also [7]). More precisely, the elliptical instability originates from the parametric excitation of Kelvin modes in the vortex cores in each tubular vortices $\{\widetilde{W}_{j,K}\}_{j,K}$. Then we can define the fractal dimension D from $|\Omega^+_{j,K}|$ and α as follows:

$$D := \frac{\log |\Omega^+_{j,K}|}{\log \alpha^{-1}} \quad \text{in other words,} \quad |\Omega^+_{j,K}| = \frac{K^D}{K_+^D} = \alpha^{-D}.$$

Assume compatibility condition on the number of tubular vortices between adjacent scales:

$$|(\Omega^+_K)^c| \left|\Omega^+_{j,K_-}\right| = |\Omega^+_{K_-}|. \tag{10}$$

Let $\tau_+(K)$ and $\tau_-(K)$ be ratios of the number of tubular vortices, being stretched and being compressed, respectively:

$$\tau_+(K) := \frac{|\Omega^+_K|}{K^D} \quad \text{and} \quad \tau_-(K) := \frac{|(\Omega^+_K)^c|}{K^D}.$$

Due to the self-similarity, we reasonably assume $\tau_\pm$ are independent of K.

Lemma 2. *We have*

$$\tau_\pm(K) = \frac{1}{2}.$$

In particular, $\tau_\pm$ are independent of ϵ.

Vortex stretching/compressing relation vs spatial and statistical independence.
By the definition of "spatial and statistical independence" (see [10]), we see that

$$\int (\omega \cdot \nabla) u \cdot \bar{\omega}_K \simeq \int (\bar{\omega}_K \cdot \nabla) \bar{u}_{K_+} \cdot \bar{\omega}_K + \int (\bar{\omega}_K \cdot \nabla) \bar{u}_{K_-} \cdot \bar{\omega}_K.$$

Normalization of vortex stretching/compressing.
Let $\omega^*_K(x) := \bar{\omega}_K(x)/(gK^{-H+1})$ and let $\omega^{**}_K(x) := \omega^*_K(K^{-1}x)$. Then we see that

$$\begin{aligned}
&\int (\bar{\omega}_K \cdot \nabla) \bar{u}_{K_+} \cdot \bar{\omega}_K + \int (\bar{\omega}_K \cdot \nabla) \bar{u}_{K_-} \cdot \bar{\omega}_K \\
=& g^3 K^{-3(H-1)} \left(\alpha_+^{-H+1} \int (\omega^*_K \cdot \nabla) u^*_{K_+} \cdot \omega^*_K + \alpha_-^{-H+1} \int (\omega^*_K \cdot \nabla) \cdot u^*_{K_-} \cdot \omega^*_K \right) \\
=& g^3 K^{-3(H-1)} K^{-3} \left(\alpha_+^{-H+1} \int (\omega^{**}_K \cdot \nabla) u^{**}_{K_+} \cdot \omega^{**}_K + \alpha_-^{-H+1} \int (\omega^{**}_K \cdot \nabla) \cdot u^{**}_{K_-} \cdot \omega^{**}_K \right),
\end{aligned}$$

where (let $\nabla U_{J,K} := \nabla(\nabla \Delta^{-1} \times W_{J,K})$)

$$\int (\omega^{**}_K \cdot \nabla) u^{**}_{K_+} \cdot \omega^{**}_K = \sum_J \sum_{j \in \Omega^+_{J,K}} \int (W_{j,K} \cdot \nabla) U_{J,K_+} \cdot W_{j,K},$$

$$\int (\omega^{**}_K \cdot \nabla) u^{**}_{K_-} \cdot \omega^{**}_K = \sum_{j \in (\Omega^+_K)^c} \sum_{j' \in \Omega^-_{j,K}} \int (W_{j,K} \cdot \nabla) U_{j',K_-} \cdot W_{j,K}.$$

Let $S_\pm$ and $P_\pm$ be the strain and rotation tensors of $\nabla u^{**}_{K_\pm}$ respectively. Then we can rewrite the following vortex stretching/compressing terms:

$$\int ((\omega^{**}_K \cdot \nabla) u^{**}_{K_\pm}) \cdot \omega^{**}_K = \int (\omega^{**}_K)^T (P_\pm + S_\pm) \cdot \omega^{**}_K = \int (\omega^{**}_K)^T S_\pm \cdot \omega^{**}_K,$$

where $(\omega_K^{**})^T$ is the row vector of ω_K^{**}. By the direct calculation, the rotation tensor part disappears. Let $\widetilde{S}_{J,K_\pm}$ be the rate-of-strain tensor of $\nabla(\nabla\Delta^{-1}\times\widetilde{W}_{J,K_\pm})$.

Definition of vortex stretching/compressing and approximate parameters

- The vortex stretching case:

In this case, a pair of "normalized" tubular vortices $\widetilde{W}_{j,K}$ ($j\in\Omega_{J,K}^+$) are stretched by $\widetilde{S}_{J,K_+}$ and align with the eigenvector of $\widetilde{S}_{J,K_+}$ of the first positive eigenvalue. We now give the definition of "vortex stretching" as below.

Definition 3. (Vortex stretching.) For $x\in\operatorname{supp}\widetilde{W}_{j,K}$, there are eigenvalues $\lambda_i(x)$ and eigenvectors $e_i(x)$ ($i=1,2,3$) of $\widetilde{S}_{J,K_+}(x)$ such that $\lambda_1(x)>0$, $|\lambda_1(x)|>|\lambda_2(x)|\geq|\lambda_3(x)|$, $|e_1(x)|=|e_2(x)|=|e_3(x)|=1$ and

$$e_1(x)\cdot\frac{\widetilde{W}_{j,K}(x)}{|\widetilde{W}_{j,K}(x)|}=1\quad\text{in other words,}\quad(\widetilde{W}_{j,K})^T\widetilde{S}_{J,K_+}\cdot\widetilde{W}_{j,K}=\lambda_1\widetilde{W}_{j,K}\cdot\widetilde{W}_{j,K}$$

for $x\in\operatorname{supp}\widetilde{W}_{j,K}$.

From the above observation, we significantly assume the following:

$$\int W_{j,K}^T S_{J,K_+}\cdot W_{j,K}\simeq\Lambda_+\int W_{j,K}\cdot W_{j,K}.$$

for $j\in\Omega_K^+$ and some approximate parameter $\Lambda_+>0$ independent of ϵ.

- The vortex compressing case:

In this case, a pair of tubular vortices $\widetilde{W}_{j,K}$ are compressed by $\widetilde{S}_{j',K_-}$ ($j'\in\Omega_{j,K}^-$), and align (but opposite direction) with the eigenvector of the rate-of-strain tensor $\widetilde{S}_{j',K_-}$ of the first positive eigenvalue. Now we give the definition of "vortex compressing" as below.

Definition 4. (Vortex compressing.) For $x\in\operatorname{supp}\widetilde{W}_{j,K}$, there are eigenvalues $\lambda_i(x)$ and eigenvectors $e_i(x)$ ($i=1,2,3$) of $\widetilde{S}_{j',K_-}(x)$ ($j'\in\Omega_{j,K}^-$) such that $\lambda_1(x)>0$, $|\lambda_1(x)|>|\lambda_2(x)|\geq|\lambda_3(x)|$, $|e_1(x)|=|e_2(x)|=|e_3(x)|=1$ and

$$e_1(x)\cdot\frac{\widetilde{W}_{j,K}(x)}{|\widetilde{W}_{j,K}(x)|}=-1\quad\text{in other words,}\quad(\widetilde{W}_{j,K})^T\sum_{j'\in\Omega_{j,K}^-}\widetilde{S}_{j',K_-}\cdot\widetilde{W}_{j,K}=-\lambda_1\widetilde{W}_{j,K}\cdot\widetilde{W}_{j,K}$$

for $x\in\operatorname{supp}\widetilde{W}_{j,K}$.

From the above observation, we significantly assume the following:

$$\int W_{j,K}^T\sum_{j'\in\Omega_{j,K}^-}S_{j',K_-}\cdot W_{j,K}\simeq-\Lambda_-\int W_{j,K}\cdot W_{j,K}$$

for $j\in(\Omega_K^+)^c$ and some approximate parameter $\Lambda_->0$ independent of ϵ.

By applying the above approximations, we can derive Kolmogorov's $-5/3$ law.

Theorem 5. *Assume* (8) *and approximations described above. Taking* $\nu\to0$*, then we have Kolmogorov's* $-5/3$ *law:*

$$E_{LP}(K):=K^{-3}\|\bar{\omega}_K\|_{L^2}^2\simeq c\gamma\epsilon^{2/3}K^{-5/3}K^{-\frac{1}{3}(3-D)}\tag{11}$$

for $K\geq K_f$.

Proof. See [10] (also, see Sec. 2 in [1] for the definition of the Littlewood-Paley spectra).

3 Conclusion

With the aid of DNS of forced turbulence in a periodic domain, we derived Kolmogorov's $-5/3$ power law of the energy spectrum of statistically stationary turbulence. We emphasize that our derivation does not require Kolmogorov's hypotheses, but instead it requires a universal mechanism of the vortex stretching and compressing. Though the present study lacks the discussion on the origin of this universality, the recent study by McKeown *et al.* [7] may provide with it; they investigated how the elliptical instability led to the persistence of the turbulent energy cascade through the local interactions of vortices over a hierarchy of scales.

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Mathematical modeling for grain boundary motion with dynamic lattice misorientations and triple junction drag

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結晶粒界の運動を記述する数理モデルの導出とそのモデルの数学解析は, Mullins, Herring [9, 16, 17] の研究を契機として大きく発展した. 特に, Mullins 方程式と呼ばれる, 粒界の曲率をとり入れた方程式は, 平均曲率流, 曲線短縮流方程式を典型例に持つこともあり非線形解析, 幾何学的変分問題との関係もあいまって, 深い研究が進んでいる. ところで, 結晶粒界の問題にたちかえったとき, その運動を決定する因子は結晶粒界の形状のみならず, 粒界を構成する結晶の性質や結晶粒界間の相互作用も考えるべきである. しかしながら, Mullins 方程式の持つ曲率の非線形性の解析が困難であること, 曲率の効果が粒界の運動において支配的であると考えられていたことから, それらの効果を数学的に, 論理的に整合性のある方法で (consistent な方法で) 研究されてはいないように思われる. 本稿では, [7, 8, 15] の結果をもとに, 粒界における結晶格子方位差 (lattice misorientation) と三重点 (triple junction) の影響を取り入れた, 発展方程式の変分原理を用いた導出手法とその数学解析について述べる.

1. 発展方程式の導出

本節では, 平面上の結晶粒界の運動を記述することを考える. 結晶粒界エネルギーをもとにして, エネルギーが消散することを課したときの結晶格子方位差と三重点の効果をとり入れた発展方程式の導出を行う.

3 つの結晶粒界のそれぞれの端点の 1 つが三重点を形成していることを仮定する (図 1). 結晶粒界エネルギーを

$$E(\Gamma_t^{(j)}, \Delta^{(j)}\alpha\,;\,t) = \sum_{j=1}^{3} \int_{\Gamma_t^{(j)}} \sigma(\Delta^{(j)}\alpha)\, d\mathcal{H}^1 = \sum_{j=1}^{3} \sigma(\Delta^{(j)}\alpha)|\Gamma_t^{(j)}| \tag{1}$$

で与える. $\Gamma_t^{(j)}$ は結晶粒界を表し, 固定点 $\boldsymbol{x}^{(j)}$ と三重点 $\boldsymbol{a}(t)$ を端点に持つとする. $\mathcal{H}^1$ は 1 次元 Hausdorff 測度, $|\Gamma_t^{(j)}|$ は $\Gamma_t^{(j)}$ の長さとする. $\alpha^{(j)}(t)$ は, 結晶の格子方位の角度を表す, t を独立変数とするスカラー値関数とし, $\Delta^{(j)}\alpha(t) = \alpha^{(j-1)}(t) - \alpha^{(j)}(t)$ は結晶粒界を構成する結晶の格子方位の差を表すものとする. ただし, $\alpha^{(0)} = \alpha^{(3)}$ と定める. $\sigma : \mathbb{R} \to \mathbb{R}$ は表面張力を表し, 格子方位

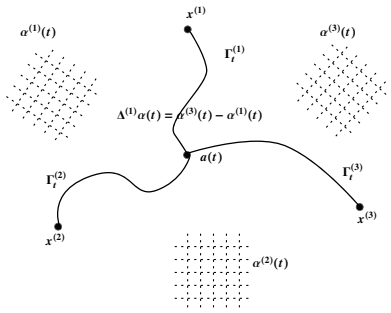

図 1. 結晶粒界の模式図

本講演の内容は, Yekaterina Epshteyn (The University of Utah), Chun Liu (Illinois Institute of Technology), 高棹 圭介氏 (京都大学理学研究科) との共同研究に基づく.

差に依存する非負の関数とする. この設定のもとで, エネルギーが消散する, すなわち $\frac{dE}{dt} \le 0$ がみたされるような $\Gamma_t^{(j)}$, $\alpha^{(j)}$, $\boldsymbol{a}$ の発展方程式を導出しよう.

注意 1. 表面張力 σ は $\Gamma_t^{(j)}$ の法方向 $\boldsymbol{n}^{(j)}$ に依存してもよい. このとき, 曲率の影響が異方的となる結晶粒界の問題となる. 本稿では簡単のため, σ が $\boldsymbol{n}^{(j)}$ に依存しない, すなわち, 曲率の効果が等方的な問題のみ扱う. 異方性を持つ方程式の導出については, [8, Section 2] をみよ.

注意 2. 方程式をたてるときは, 何が未知で何が既知であるかは本質的ではないが, とりあえず, 本節では $\boldsymbol{x}^{(j)}$, σ は既知, $\Gamma_t^{(j)}$, $\alpha^{(j)}$, $\boldsymbol{a}$ は未知と考えてよい. ただし, 応用上は $\Gamma_t^{(j)}$, $\alpha^{(j)}$, $\boldsymbol{a}$ を与えて, σ や $\boldsymbol{x}^{(j)}$ の条件を求めることも重要である. 実験結果から, σ や $\boldsymbol{x}^{(j)}$ がどうあるべきかを求めることも必要だからである.

エネルギー E が消散する方程式系を導出するために, E の消散レート $\frac{dE}{dt}$ を計算する. 結晶粒界 $\Gamma_t^{(j)}$ を $\Gamma_t^{(j)} : \boldsymbol{\xi}^{(j)}(s,t), (0 \le s \le 1), \boldsymbol{\xi}^{(j)}(1,t) = \boldsymbol{a}(t), \boldsymbol{\xi}^{(j)}(0,t) = \boldsymbol{x}^{(j)}$ と媒介変数表示する. ただし, s は弧長パラメータではないことに注意する. $d\mathcal{H}^1\lfloor_{\Gamma_t^{(j)}} = |\boldsymbol{\xi}_s^{(j)}(s,t)|ds$ だから

$$
\begin{aligned}
\frac{d}{dt}E(t) &= \frac{d}{dt}\sum_{j=1}^{3}\sigma(\Delta^{(j)}\alpha(t))\int_0^1 |\boldsymbol{\xi}_s^{(j)}(s,t)|\,ds \\
&= \sum_{j=1}^{3}\left(\sigma(\Delta^{(j)}\alpha(t))\int_0^1 \frac{\boldsymbol{\xi}_s^{(j)}(s,t)}{|\boldsymbol{\xi}_s^{(j)}(s,t)|}\cdot\boldsymbol{\xi}_{st}^{(j)}(s,t)\,ds + \sigma'(\Delta^{(j)}\alpha(t))|\Gamma_t^{(j)}|(\Delta^{(j)}\alpha(t))_t\right)
\end{aligned}
\tag{2}
$$

となる. 部分積分により

$$
\int_0^1 \frac{\boldsymbol{\xi}_s^{(j)}(s,t)}{|\boldsymbol{\xi}_s^{(j)}(s,t)|}\cdot\boldsymbol{\xi}_{st}^{(j)}(s,t)\,ds = -\int_0^1 \left(\frac{\boldsymbol{\xi}_s^{(j)}(s,t)}{|\boldsymbol{\xi}_s^{(j)}(s,t)|}\right)_s\cdot\boldsymbol{\xi}_t^{(j)}(s,t)\,ds + \left[\frac{\boldsymbol{\xi}_s^{(j)}(s,t)}{|\boldsymbol{\xi}_s^{(j)}(s,t)|}\cdot\boldsymbol{\xi}_t^{(j)}(s,t)\right]_{s=0}^{1}
$$

が得られる. $\boldsymbol{\xi}_s^{(j)}$ が $\Gamma_t^{(j)}$ の接ベクトル場になることから Frenet-Serret の公式より

$$
\left(\frac{\boldsymbol{\xi}_s^{(j)}(s,t)}{|\boldsymbol{\xi}_s^{(j)}(s,t)|}\right)_s = \kappa^{(j)}(s,t)\boldsymbol{n}^{(j)}(s,t), \quad \boldsymbol{n}^{(j)} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\boldsymbol{\xi}_s^{(j)}
$$

である. $\Gamma_t^{(j)}$ の法速度 $v_n^{(j)}$ は $v_n^{(j)} = \boldsymbol{\xi}_t^{(j)}\cdot\boldsymbol{n}^{(j)}/|\boldsymbol{n}^{(j)}|$ で与えられることから

$$
\int_0^1 \left(\frac{\boldsymbol{\xi}_s^{(j)}(s,t)}{|\boldsymbol{\xi}_s^{(j)}(s,t)|}\right)_s\cdot\boldsymbol{\xi}_t^{(j)}(s,t)\,ds = \int_0^1 \kappa^{(j)}(s,t)v_n^{(j)}(s,t)|\boldsymbol{n}^{(j)}(s,t)|\,ds = \int_{\Gamma_t^{(j)}}\kappa^{(j)}v_n^{(j)}\,d\mathcal{H}^1 \tag{3}
$$

となる. 他方, $\boldsymbol{\xi}^{(j)}(1,t) = \boldsymbol{a}(t), \boldsymbol{\xi}^{(j)}(0,t) = \boldsymbol{x}^{(j)}$ となるから

$$
\left[\frac{\boldsymbol{\xi}_s^{(j)}(s,t)}{|\boldsymbol{\xi}_s^{(j)}(s,t)|}\cdot\boldsymbol{\xi}_t^{(j)}(s,t)\right]_{s=0}^{1} = \frac{\boldsymbol{\xi}_s^{(j)}(1,t)}{|\boldsymbol{\xi}_s^{(j)}(1,t)|}\cdot\boldsymbol{a}_t(t) \tag{4}
$$

となる. 次に

$$
\sum_{j=1}^{3}\sigma'(\Delta^{(j)}\alpha(t))|\Gamma_t^{(j)}|(\Delta^{(j)}\alpha(t))_t = \sum_{j=1}^{3}\left(\sigma'(\Delta^{(j+1)}\alpha(t))|\Gamma_t^{(j+1)}| - \sigma'(\Delta^{(j)}\alpha(t))|\Gamma_t^{(j)}|\right)\alpha_t^{(j)}(t) \tag{5}
$$

と変形する. (2) に (3), (4) と (5) を代入すると

$$
\begin{aligned}
\frac{d}{dt}E(t) = &-\sum_{j=1}^{3}\int_{\Gamma_t^{(j)}} \sigma(\Delta^{(j)}\alpha(t))\kappa^{(j)}v_n^{(j)}\,d\mathcal{H}^1 + \sum_{j=1}^{3}\frac{\boldsymbol{\xi}_s^{(j)}(1,t)}{|\boldsymbol{\xi}_s^{(j)}(1,t)|}\cdot\boldsymbol{a}_t(t)\\
&+\sum_{j=1}^{3}\left(\sigma'(\Delta^{(j+1)}\alpha(t))|\Gamma_t^{(j+1)}| - \sigma'(\Delta^{(j)}\alpha(t))|\Gamma_t^{(j)}|\right)\alpha_t^{(j)}(t)
\end{aligned}
\tag{6}
$$

が得られる. 従って, エネルギー (1) が消散することを保証するためには, 正定数 $\mu,\gamma,\eta>0$ を用いて

$$
\begin{cases}
v_n^{(j)} = \mu\sigma(\Delta^{(j)}\alpha)\kappa^{(j)}, \quad \text{on } \Gamma_t^{(j)},\ t>0, \quad j=1,2,3,\\
\alpha_t^{(j)} = -\gamma\left(\sigma'(\Delta^{(j+1)}\alpha(t))|\Gamma_t^{(j+1)}| - \sigma'(\Delta^{(j)}\alpha(t))|\Gamma_t^{(j)}|\right), \quad j=1,2,3,\\
\boldsymbol{a}_t = -\eta\displaystyle\sum_{j=1}^{3}\frac{\boldsymbol{\xi}_s^{(j)}(1,t)}{|\boldsymbol{\xi}_s^{(j)}(1,t)|}, \quad t>0
\end{cases}
\tag{7}
$$

とすればよい. 逆にこの方程式系 (7) を課せばエネルギー消散評価式

$$
\frac{d}{dt}E(t) = -\frac{1}{\mu}\sum_{j=1}^{3}\int_{\Gamma_t^{(j)}}|v_n^{(j)}|^2\,d\mathcal{H}^1 - \frac{1}{\gamma}\sum_{j=1}^{3}|\alpha_t^{(j)}(t)|^2 - \frac{1}{\eta}|\boldsymbol{a}_t(t)|^2
$$

が得られる.

注意 3. 上記による方程式の導出手法は最大消散原理, Onsager の原理と呼ばれている [2, 18].

エネルギー (1) の平衡状態をみるために $(\Gamma^{(j)},\alpha)$ がエネルギー最小であるとする. $\Gamma^{(j)}:\boldsymbol{\xi}^{(j)}(s)$, $(0\le s\le 1)$, $\boldsymbol{\xi}^{(j)}(1)=\boldsymbol{a}$, $\boldsymbol{\xi}^{(j)}(0)=\boldsymbol{x}^{(j)}$ と媒介変数表示する. $\boldsymbol{\phi}^{(j)}(0)=\boldsymbol{0}$ をみたす滑らかなベクトル場 $\boldsymbol{\phi}^{(j)}(s)\in\mathbb{R}^2$ と $\boldsymbol{\psi}=(\psi^{(1)},\psi^{(2)},\psi^{(3)})\in\mathbb{R}^3$ に対して

$$
\frac{\partial}{\partial\varepsilon_2}\frac{\partial}{\partial\varepsilon_1}\bigg|_{\varepsilon_1=\varepsilon_2=0}\sum_{j=1}^{3}\int_0^1\sigma(\Delta^{(j)}\alpha+\varepsilon_2\Delta^{(j)}\psi)|\boldsymbol{\xi}_s^{(j)}(s)+\varepsilon_1\boldsymbol{\phi}_s^{(j)}(s)|\,ds = 0
\tag{8}
$$

を考える. すると (8) の右辺は先と同じような計算により,

$$
\begin{aligned}
&\sum_{j=1}^{3}\left(\sigma'(\Delta^{(j)}\alpha)(\psi^{(j-1)}-\psi^{(j)})|\Gamma_t^{(j)}| + \int_0^1\sigma(\Delta^{(j)}\alpha)\frac{\boldsymbol{\xi}_s^{(j)}(s)}{|\boldsymbol{\xi}_s^{(j)}(s)|}\cdot\boldsymbol{\phi}_s^{(j)}\,ds\right)\\
&=\sum_{j=1}^{3}\left(\sigma'(\Delta^{(j+1)}\alpha)|\Gamma_t^{(j+1)}| - \sigma'(\Delta^{(j)}\alpha)|\Gamma_t^{(j)}|\right)\psi^{(j)} - \sum_{j=1}^{3}\int_{\Gamma^{(j)}}\sigma(\Delta^{(j)}\alpha)\left(\frac{\boldsymbol{\xi}_s^{(j)}(s)}{|\boldsymbol{\xi}_s^{(j)}(s)|}\right)_s\cdot\boldsymbol{\phi}^{(j)}\,d\mathcal{H}^1\\
&\quad+\sum_{j=1}^{3}\sigma(\Delta^{(j)}\alpha)\frac{\boldsymbol{\xi}_s^{(j)}(1)}{|\boldsymbol{\xi}_s^{(j)}(1)|}\cdot\boldsymbol{\phi}^{(j)}(1)
\end{aligned}
$$

が得られる. $\boldsymbol{\phi}^{(j)}(s)$, $\boldsymbol{\psi}$ は任意だから

$$
\begin{cases}
0 = \sigma(\Delta^{(j)}\alpha)\kappa^{(j)}, \quad \text{on } \Gamma^{(j)}, \quad j=1,2,3,\\
0 = \sigma'(\Delta^{(j+1)}\alpha)|\Gamma^{(j+1)}| - \sigma'(\Delta^{(j)}\alpha)|\Gamma^{(j)}|, \quad j=1,2,3,\\
\boldsymbol{0} = \displaystyle\sum_{j=1}^{3}\frac{\boldsymbol{\xi}_s^{(j)}(1)}{|\boldsymbol{\xi}_s^{(j)}(1)|},
\end{cases}
\tag{9}
$$

が得られる. このことにより, (7) の第 1 式, 第 2 式はエネルギー E の勾配流とみなせる. (9) の第 3 式, 三重点における関係はエネルギーのつりあいの式であり, Herring 条件と呼ばれている. この条件はエネルギーの平衡状態を考えるときの**必要条件**である. しかし, 発展問題を考えるうえでは, Herring 条件は**必要条件ではない**. すなわち, 我々が何らかの条件を課さなければならない. [8] では, 三重点に Herring 条件ではなく, 動的条件を課す問題を提唱した. (7) で緩和極限 $\eta \to \infty$ とすれば, Herring 条件が得られることに注意する. 平均曲率流方程式に Herring 条件を課したネットワーク解の数学解析は古典解, 弱解の双方で研究が進んでいる [11, 12, 13, 14].

注意 4. 定数 $\mu, \gamma, \eta > 0$ は緩和時間パラメータといい, 系が平衡状態に近づく速さの指標となっている. このパラメータにより, 我々の系は多重スケール問題となっている.

2. 結晶格子方位差と動的三重点

結晶格子方位差と動的三重点の相互作用をみるために, 緩和極限 $\mu \to \infty$ を考えた常微分方程式系を考える. $\kappa^{(j)} = 0$ となり, $\Gamma_t^{(j)}$ は線分となるから $\boldsymbol{\xi}^{(j)}(s,t) = \boldsymbol{x}^{(j)} + s(\boldsymbol{a}(t) - \boldsymbol{x}^{(j)})$ と表示すると

$$
\text{(10)} \qquad \begin{cases} \alpha_t^{(j)} = -\gamma \left(\sigma'(\Delta^{(j+1)}\alpha(t))|\boldsymbol{a}(t) - \boldsymbol{x}^{(j+1)}| - \sigma'(\Delta^{(j)}\alpha(t))|\boldsymbol{a}(t) - \boldsymbol{x}^{(j)}| \right), \quad j = 1,2,3, \\ \boldsymbol{a}_t = -\eta \displaystyle\sum_{j=1}^{3} \frac{\boldsymbol{a}(t) - \boldsymbol{x}^{(j)}}{|\boldsymbol{a}(t) - \boldsymbol{x}^{(j)}|}, \quad t > 0 \end{cases}
$$

が得られる. この常微分方程式により定まる結晶粒界モデルの適切性, 長時間挙動を考察する.

2.1. 適切性.

この節では, 系の時間局所適切性を示す. 常微分方程式の解の存在定理によれば, (10) の時間局所解の存在がわかる. しかし, $\boldsymbol{a}$ が三重点であることを示すためには, $\boldsymbol{a}$ が $\boldsymbol{x}^{(j)}$ から作られる三角形の内側にあることなど, $\boldsymbol{a}$ の位置に対する先験評価も必要である. このことを踏まえて以下の仮定をおく.

$$
\text{(11)} \qquad \left| \sum_{j=1,\, i\neq j}^{3} \frac{\boldsymbol{x}^{(j)} - \boldsymbol{x}^{(i)}}{|\boldsymbol{x}^{(j)} - \boldsymbol{x}^{(i)}|} \right| > 1, \quad i = 1,2,3.
$$

この仮定 (11) のもと, $f(\boldsymbol{a}) := \sum_{i=1}^{3} |\boldsymbol{a} - \boldsymbol{x}^{(i)}|$ $(\boldsymbol{a} \in \mathbb{R}^2)$ は最小値を持ち, 最小値を達成する点 $\boldsymbol{a} = \boldsymbol{a}_{\mathrm{FT}}$ は唯一つであり, $\boldsymbol{a}_{\mathrm{FT}}$ はどの $\boldsymbol{x}^{(i)}$ とも等しくなく,

$$
\nabla f(\boldsymbol{a}_{\mathrm{FT}}) = \sum_{i=1}^{3} \frac{\boldsymbol{a}_{\mathrm{FT}} - \boldsymbol{x}^{(i)}}{|\boldsymbol{a}_{\mathrm{FT}} - \boldsymbol{x}^{(i)}|} = \boldsymbol{0}
$$

をみたす.

注意 5. 一般に $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(N)} \in \mathbb{R}^2$ に対して, $f(\boldsymbol{a}) := \sum_{i=1}^{N} |\boldsymbol{a} - \boldsymbol{x}^{(i)}|$ $(\boldsymbol{a} \in \mathbb{R}^2)$ の最小値を達成する点を Fermat-Torricelli 点という [1].

表面張力 σ は C^3 級で,

(1) (正値性) 正定数 $C_1 > 0$ が存在して $\theta \in \mathbb{R}$ に対して $\sigma(\theta) \geq C_1$;
(2) (凸性) $\theta \in \mathbb{R}$ に対して $\theta\sigma'(\theta) \geq 0$;
(3) $\sigma'(\theta) = 0$ と $\theta = 0$ は同値

を仮定する. この仮定のもとで, σ は $\theta = 0$ で最小となることに注意する.

Fermat-Torricelli 点の近くに $\boldsymbol{a}$ がある限り, その $\boldsymbol{a}$ は三重点となる. このことにより, Fermat-Torricelli 点を原点とみたてて, 次の存在定理を得る.

定理 6 (局所存在定理 [8, Theorem 4.1, Theorem 6.3])**.** $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(3)} \in \mathbb{R}^2$, $\boldsymbol{a}_0 \in \mathbb{R}^2$, $\alpha_0^{(j)} \in \mathbb{R}$ は (11) と

$$|\boldsymbol{a}_0 - \boldsymbol{a}_{\mathrm{FT}}| < \frac{1}{2}|\boldsymbol{a}_{\mathrm{FT}} - \boldsymbol{x}^{(j)}| \tag{12}$$

をみたすとする. このとき (10) に初期条件 $\alpha^{(j)}(0) = \alpha_0^{(j)}, \boldsymbol{a}(0) = \boldsymbol{a}_0$ を与えた初期値問題の時間局所解が唯一つ存在する.

証明のアイデアは, (10) を積分方程式に書き換えて

$$X_T := \left\{ (\alpha^{(j)}, \boldsymbol{a}) \in C([0,T]) : \sup_{0\le t\le T} |\alpha^{(j)}(t)| \le C, \sup_{0\le t\le T} |\boldsymbol{a}(t) - \boldsymbol{a}_{\mathrm{FT}}| \le C \right\}$$

上で縮小写像を構成することである. 一意性は Lipschitz 連続性と Gronwall 不等式による. 詳細は [8] をみよ.

2.2. **解の性質.** この節では, 解の先験評価について述べる. まず, 導出の方法からエネルギー消散が得られる.

命題 7 (エネルギー消散 [8, Proposition 5.1])**.** $(\alpha^{(j)}, \boldsymbol{a})$ を $0 \le t \le T$ 上の (10) の解とする. このとき, $0 \le t \le T$ に対して, 次が成り立つ:

$$E(t) + \frac{1}{\gamma}\sum_{j=1}^{3}\int_0^t \left|\frac{d\alpha^{(j)}}{dt}(\tau)\right|^2 d\tau + \frac{1}{\eta}\int_0^t \left|\frac{d\boldsymbol{a}}{dt}(\tau)\right|^2 d\tau = E(0).$$

次に σ の凸性を用いて, 結晶格子方位 $\alpha^{(j)}$, 結晶格子方位差 $\Delta^{(j)}\alpha$ に対する最大値原理を述べる.

命題 8 (結晶格子方位に対する最大値原理 [8, Proposition 5.3])**.** $(\alpha^{(j)}, \boldsymbol{a})$ を $0 \le t \le T$ 上の (10) の解とする. このとき, $0 \le t \le T$ に対して, 次が成り立つ:

$$\sum_{j=1}^{3}|\alpha^{(j)}(t)|^2 + 2\gamma\sum_{j=1}^{3}\int_0^t \left(\sigma'(\Delta\alpha^{(j)}(\tau))|\boldsymbol{a}(\tau) - \boldsymbol{x}^{(j)}|\right)\Delta\alpha^{(j)}(\tau)\,d\tau = \sum_{j=1}^{3}|\alpha_0^{(j)}|^2. \tag{13}$$

証明. (10) の第一式に $\alpha^{(j)}$ をかけて, $j = 1,2,3$ について和をとると, $\Delta^{(j)}\alpha = \alpha^{(j-1)} - \alpha^{(j)}$ に注意して

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\sum_{j=1}^{3}|\alpha^{(j)}|^2\right) &= -\gamma\sum_{j=1}^{3}\left(\sigma'(\Delta\alpha^{(j+1)})|\boldsymbol{a} - \boldsymbol{x}^{(j+1)}| - \sigma'(\Delta\alpha^{(j)})|\boldsymbol{a} - \boldsymbol{x}^{(j)}|\right)\alpha^{(j)} \\
&= -\gamma\sum_{j=1}^{3}\left(\sigma'(\Delta\alpha^{(j)})|\boldsymbol{a} - \boldsymbol{x}^{(j)}|\right)(\alpha^{(j-1)} - \alpha^{(j)}) \\
&= -\gamma\sum_{j=1}^{3}\left(\sigma'(\Delta\alpha^{(j)})|\boldsymbol{a} - \boldsymbol{x}^{(j)}|\right)\Delta\alpha^{(j)}
\end{aligned}$$

を得る. 両辺 t で積分することにより, (13) を得る.

注意 9. σ の凸性の仮定 $\theta\sigma'(\theta) \ge 0$ を用いると

$$\sum_{j=1}^{3}|\alpha^{(j)}(t)|^2 \le \sum_{j=1}^{3}|\alpha_0^{(j)}|^2 \tag{14}$$

が得られる. すなわち, 結晶格子方位の上からの評価が得られる.

結晶格子方位 $\alpha^{(j)}$ に対する最大値原理と同様に, 結晶格子方位差 $\Delta^{(j)}\alpha$ に対する最大値原理も得られる. 証明は [8] をみよ.

命題 10 (結晶格子方位差に対する最大値原理 [8, Proposition 5.4])**.** $(\alpha^{(j)}, \boldsymbol{a})$ を $0 \le t \le T$ 上の (10) の解とする. このとき, $0 \le t \le T$ に対して, 次が成り立つ:

$$\sum_{j=1}^{3}(\Delta^{(j)}\alpha(t))^2 + 6\gamma\sum_{j=1}^{3}\int_0^t \sigma_\theta(\Delta^{(j)}\alpha(\tau))|\boldsymbol{a}(\tau)-\boldsymbol{x}^{(j)}|\Delta^{(j)}\alpha(\tau)\,d\tau = \sum_{j=1}^{3}(\Delta^{(j)}\alpha(0))^2.$$

次に三重点についての先験評価を示す. そのために

$$C_2 := \inf\left\{\sum_{j=1}^{3}|\boldsymbol{a}-\boldsymbol{x}^{(j)}| : j=1,2,3 \text{ が存在して } |\boldsymbol{a}-\boldsymbol{a}_{\mathrm{FT}}| \ge \frac{1}{2}|\boldsymbol{a}_{\mathrm{FT}}-\boldsymbol{x}^{(j)}|\right\} \tag{15}$$

とおく. $\boldsymbol{a}_{\mathrm{FT}}$ が連続関数 $f(\boldsymbol{a}) = \sum_{j=1}^{3}|\boldsymbol{x}^{(j)}-\boldsymbol{a}| \quad (\boldsymbol{a}\in\mathbb{R}^2)$ の最小値を達成する点だから

$$0 < \sum_{j=1}^{3}|\boldsymbol{a}_{\mathrm{FT}}-\boldsymbol{x}^{(j)}| = f(\boldsymbol{a}_{\mathrm{FT}}) < C_2$$

となる. この定数 C_2 を用いると, 三重点 $\boldsymbol{a}$ の先験評価が得られる.

命題 11 (三重点の有界性 [7, Lemma 4.1])**.** 初期値 $(\alpha_0^{(j)}, \boldsymbol{a}_0)$ は

$$E(0) = \sum_{j=1}^{3}\sigma(\Delta^{(j)}\alpha_0)|\boldsymbol{a}_0-\boldsymbol{x}^{(j)}| < \sigma(0)C_2 \tag{16}$$

をみたすとする. $(\alpha^{(j)}, \boldsymbol{a})$ を $0 \le t \le T$ 上の (10) の解とする. このとき, $0 \le t \le T$ と $j = 1,2,3$ に対して, 次が成り立つ:

$$|\boldsymbol{a}(t)-\boldsymbol{a}_{\mathrm{FT}}| < \frac{1}{2}|\boldsymbol{a}_{\mathrm{FT}}-\boldsymbol{x}^{(j)}|.$$

証明. 背理法で示す. $0 \le t_1 \le T$ と $j = 1,2,3$ が存在して $|\boldsymbol{a}(t_1)-\boldsymbol{a}_{\mathrm{FT}}| \ge \frac{1}{2}|\boldsymbol{a}_{\mathrm{FT}}-\boldsymbol{x}^{(j)}|$ をみたすとする. すると, σ が $\theta = 0$ で最小になること, (15), 命題 7 (エネルギー消散) より

$$C_2 \le \sum_{j=1}^{3}|\boldsymbol{a}(t_1)-\boldsymbol{x}^{(j)}| \le \frac{1}{\sigma(0)}\sum_{j=1}^{3}\sigma(\Delta^{(j)}\alpha_0)|\boldsymbol{a}_0-\boldsymbol{x}^{(j)}| = \frac{1}{\sigma(0)}E(0)$$

となり (16) に矛盾する.

注意 12. (16) を仮定すると

$$\sum_{j=1}^{3}|\boldsymbol{a}_0-\boldsymbol{x}^{(j)}| \le \frac{1}{\sigma(0)}\sum_{j=1}^{3}\sigma(\Delta^{(j)}\alpha_0)|\boldsymbol{a}_0-\boldsymbol{x}^{(j)}| < C_2$$

となるから, $|\boldsymbol{a}_0-\boldsymbol{a}_{\mathrm{FT}}| < \frac{1}{2}|\boldsymbol{a}_{\mathrm{FT}}-\boldsymbol{x}^{(j)}|$ がすべての $j = 1,2,3$ について成り立つ. 従って, 時間局所解の存在の仮定 (12) は自動的にみたされる.

2.3. **時間大域解と長時間挙動.** 結晶格子方位差と三重点に対する先験評価により, 解の大域存在を示すことができる.

定理 13 (時間大域解 [7, Theorem 4.1])**.** $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \boldsymbol{x}^{(3)} \in \mathbb{R}^2$, $\boldsymbol{a}_0 \in \mathbb{R}^2$, $\alpha_0^{(j)} \in \mathbb{R}^3$ は (11) と (12), (16) をみたすとする. このとき (10) に初期条件 $\alpha^{(j)}(0) = \alpha_0^{(j)}, \boldsymbol{a}(0) = \boldsymbol{a}_0$ を与えた初期値問題の時間大域解が唯一つ存在する.

次に, 三重点 $\boldsymbol{a}$ と結晶格子方位 $\alpha^{(j)}$ の長時間挙動を調べる. $\alpha^{(j)}$ には束縛条件

$$\frac{d}{dt}(\alpha^{(1)}(t) + \alpha^{(2)}(t) + \alpha^{(3)}(t)) = 0 \tag{17}$$

があるので, 結晶格子方位差の消散評価とくみあわせると

$$\alpha^{(j)}(t) \to \alpha_\infty = \frac{\alpha_0^{(1)} + \alpha_0^{(2)} + \alpha_0^{(3)}}{3}, \qquad t \to \infty$$

となることが推測できる. このとき, $\Delta^{(j)}\alpha_\infty = 0$ だから, 三重点は, $\boldsymbol{x}^{(j)}$ から定まる Fermat-Torricelli 点に収束する. すなわち,

$$\boldsymbol{a}(t) \to \boldsymbol{a}_{\mathrm{FT}}, \qquad t \to \infty$$

となることが予想できる. この予想は三重点の初期状態が $\boldsymbol{a}_{\mathrm{FT}}$ に近いときに正しい. すなわち, 次の指数安定性が得られる.

定理 14 ([7, Theorem 5.1])**.** 正定数 $\varepsilon > 0$ が存在して, もし, $\sum_{j=1}^3 |\alpha_0^{(j)} - \alpha_\infty| + |\boldsymbol{a}_0 - \boldsymbol{a}_{\mathrm{FT}}| < \varepsilon$ ならば, 正定数 $C_3, \lambda > 0$ が存在して, 対応する (10) の大域解 $(\alpha^{(j)}, \boldsymbol{a})$ は十分大きな $t > 0$ に対して

$$\sum_{j=1}^3 |\alpha^{(j)}(t) - \alpha_\infty^{(j)}| + |\boldsymbol{a}(t) - \boldsymbol{a}_{\mathrm{FT}}| \le C_3 e^{-\lambda t} \tag{18}$$

をみたす.

証明は方程式 (10) の右辺の線形化作用素の固有値を解析することによる. (10) の $(\alpha_\infty^{(j)}, \boldsymbol{a}_{\mathrm{FT}})$ まわりの線形化問題は

$$\begin{cases} \dfrac{d\boldsymbol{\alpha}_L}{dt} = -\gamma\sigma''(0)\mathbb{B}_\infty\boldsymbol{\alpha}_L, \quad \mathbb{B}_\infty = \begin{pmatrix} |\boldsymbol{b}_\infty^{(1)}| + |\boldsymbol{b}_\infty^{(2)}| & -|\boldsymbol{b}_\infty^{(2)}| & -|\boldsymbol{b}_\infty^{(1)}| \\ -|\boldsymbol{b}_\infty^{(2)}| & |\boldsymbol{b}_\infty^{(2)}| + |\boldsymbol{b}_\infty^{(3)}| & -|\boldsymbol{b}_\infty^{(3)}| \\ -|\boldsymbol{b}_\infty^{(1)}| & -|\boldsymbol{b}_\infty^{(3)}| & |\boldsymbol{b}_\infty^{(3)}| + |\boldsymbol{b}_\infty^{(1)}| \end{pmatrix}, \\ \dfrac{d\boldsymbol{a}_L}{dt} = -\eta\sigma(0)\displaystyle\sum_{j=1}^3 \left(\frac{1}{|\boldsymbol{b}_\infty^{(j)}|}\left(I - \frac{\boldsymbol{b}_\infty^{(j)}}{|\boldsymbol{b}_\infty^{(j)}|} \otimes \frac{\boldsymbol{b}_\infty^{(j)}}{|\boldsymbol{b}_\infty^{(j)}|} \right) \right) \boldsymbol{a}_L =: -\eta\sigma(0) L_{\boldsymbol{a}} \boldsymbol{a}_L, \\ \displaystyle\sum_{j=1}^3 \frac{\boldsymbol{b}_\infty^{(j)}}{|\boldsymbol{b}_\infty^{(j)}|} = \boldsymbol{0}, \end{cases}$$

となる. ただし, $\boldsymbol{b}_\infty^{(j)} = \boldsymbol{a}_{\mathrm{FT}} - \boldsymbol{x}^{(j)}$ であり, $\otimes$ はベクトルのテンソル積である. $L_{\boldsymbol{a}}$ は正定値であることに対して, $\mathbb{B}_\infty$ は半正定値であり, 退化している, すなわち 0 固有値を持つことに注意する. この固有値は束縛条件 (17) によるものであり, 解 $\alpha^{(j)}$ に対して

$$(\alpha^{(1)}(t), \alpha^{(2)}(t), \alpha^{(3)}(t)) - \alpha_\infty(1,1,1) \in (\mathrm{Ker}\,\mathbb{B}_\infty)^\perp = \langle (1,1,1) \rangle^\perp$$

が成り立つ. 従って, $\alpha_p^{(j)}(t) := \alpha^{(j)}(t) - \alpha_\infty$ とおくと $\alpha_p^{(j)}(t)$ に対する指数減衰を得ることができて, (18) を得ることができる. 詳細は [7] をみよ.

3. 曲率流方程式と結晶格子方位差

本節では, 曲率と結晶格子方位差の関係をグラフ解をもとに調べる. 結晶粒界 Γ_t は $\Gamma_t := \{(x,u(x,t)) : 0 < x < 1\}$ と $u(x,t)$ でグラフ表示されているとする. 結晶粒界エネルギー E を

$$E := \int_{\Gamma_t} \sigma(\Delta\alpha(t)) d\mathcal{H}^1 = \sigma(\Delta\alpha(t))|\Gamma_t|$$

で定める. Γ_t に周期境界条件を課して, 最大消散原理を用いると

$$\begin{cases} \dfrac{u_t}{\sqrt{1+|u_x|^2}} = \mu\sigma(\Delta\alpha)\left(\dfrac{u_x}{\sqrt{1+|u_x|^2}}\right)_x, & 0 < x < 1,\ t > 0, \\ \alpha_t = -\gamma\sigma_\alpha(\alpha)|\Gamma_t|, & t > 0 \end{cases} \tag{19}$$

なる曲率流方程式が得られる. この曲率流方程式を解析することにより, 結晶粒界の運動における曲率と結晶格子方位差の関係を調べる.

以下, 表面張力 σ は $C^{1,1}$ 級で,

(1) (正値性) 正定数 $C_1 > 0$ が存在して $\theta \in \mathbb{R}$ に対して $\sigma(\theta) \geq C_1$;
(2) (凸性) $\theta \in \mathbb{R}$ に対して $\theta\sigma'(\theta) \geq 0$;
(3) $\sigma'(\theta) = 0$ と $\theta = 0$ は同値

を仮定する.

3.1. 適切性.

(19) の初期値-周期境界値問題に対する可解性について述べる.

定理 15 ([15, Theorem 4.5])**.** 初期値 u_0 を周期境界条件をみたす $[0,1)$ 上の Lipschitz 関数, $\Delta\alpha_0 \in \mathbb{R}$ を定数とする. すると, $\varepsilon > 0$ と $0 < \beta < 1$ に対して, 初期条件 $(u(\cdot,0),\Delta\alpha(0)) = (u_0,\Delta\alpha_0)$ をみたす (19) の一意解 $(u,\Delta\alpha) \in (C(\overline{[0,1)\times[0,\infty)}) \cap C^{2,\beta}([0,1)\times(\varepsilon,\infty))) \times (C([0,\infty)) \cap C^{1,1}(\varepsilon,\infty))$ が存在する.

証明の方針は, u を固定した常微分方程式の解を $(\Delta\alpha)_u$ とおくときに

$$\frac{u_t}{\sqrt{1+|u_x|^2}} = \mu\sigma((\Delta\alpha)_u)\left(\frac{u_x}{\sqrt{1+|u_x|^2}}\right)_x$$

に対する先験評価を構成し, 適切な関数空間上での固定点を探す問題に帰着させることである. この先験評価において, 解 u の勾配評価を導くことが鍵となる. $\sigma \equiv 1$ のとき, 後ろ向き熱核を用いた Huisken の単調性公式が知られており, 単調性公式から勾配評価を導くことができる [3, 4, 5, 6, 10]. (19) は時間依存するモビリティを持つため, 通常の後ろ向き熱核では単調性公式を導くことができない. そこで,

$$\rho(X,t) := \frac{1}{(4\pi(\Sigma(t_0)-\Sigma(t)))^{\frac{1}{2}}} \exp\left(-\frac{|X-X_0|^2}{4(\Sigma(t_0)-\Sigma(t))}\right), \quad 0 < t < t_0, \quad X \in \mathbb{R}^2, \tag{20}$$

とおく. ただし,

$$\Sigma(t) := \mu\int_0^t \sigma(\Delta\alpha(\tau))\, d\tau \tag{21}$$

である.

注意 16. 熱伝導率が $t > 0$ に依存する熱方程式

$$\frac{\partial u}{\partial t}(x,t) = k'(t)\Delta u(x,t), \quad x \in \mathbb{R}^d,\ t > 0, \tag{22}$$

を考える. 変数変換 $s = k(t)$ を考えると

$$\frac{\partial u}{\partial s}(x,s) = \Delta u(x,s) \quad x \in \mathbb{R}^d,\ s > 0.$$

となるから, (22) の基本解は

$$\frac{1}{(4\pi s)^{d/2}} \exp\left(-\frac{|x|^2}{4s}\right) = \frac{1}{(4\pi k(t))^{d/2}} \exp\left(-\frac{|x|^2}{4k(t)}\right)$$

で与えられる. (21) を用いたのは $k = \Sigma$ とおいたことによる.

(20) で定義される ρ は

$$\frac{\partial u}{\partial t}(x,t) = \mu\sigma(\Delta\alpha(t))\Delta u(x,t), \quad x \in \mathbb{R}^2,\ t > 0,$$

の 1 次元後ろ向き熱核となっていることに注意する. この ρ を用いて次の単調性公式を得た. 定理を述べるために記号を準備する.

連続微分可能な関数 $f := f(x,y,t) : [0,1] \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$ が admissible であるとは, 任意の $y \in \mathbb{R}$ と $t \geq 0$ に対して, $f(0,y,t) = f(1,y,t)$ と $f_x(0,y,t) = f_x(1,y,t)$ が成り立つことをいう. (19) の解 u に対して, $\boldsymbol{n} = \frac{1}{\sqrt{1+|u_x|^2}}(-u_x, 1)$ をグラフ Γ_t の上向きの単位法線ベクトル, $\kappa = (\frac{u_x}{\sqrt{1+|u_x|^2}})_x$ をグラフ Γ_t の曲率とする.

定理 17 ([15, Theorem 3.1])**.** $(u, \Delta\alpha)$ を (19) の解とする. このとき, 任意の $X_0 \in \mathbb{R}^2$ と $t_0 > 0$, admissible な関数 $f : [0,1] \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$ に対して

$$\begin{aligned} &\frac{d}{dt}\int_{\Gamma_t} f\rho\sigma(\Delta\alpha(t))\, d\mathcal{H}^1 \\ &\quad = \int_{\Gamma_t} (f_t - \mu\sigma(\Delta\alpha(t))\Delta_{\Gamma_t} f + \mu\sigma(\Delta\alpha(t))\kappa(Df\cdot\boldsymbol{n}))\rho\sigma(\Delta\alpha(t))\, d\mathcal{H}^1 \\ &\quad - \mu\sigma(\Delta\alpha(t))\int_{\Gamma_t}\left(f\rho\left(-\kappa + \frac{D\rho\cdot\boldsymbol{n}}{\rho}\right)^2\sigma(\Delta\alpha(t))\right) d\mathcal{H}^1 - \frac{1}{\gamma|\Gamma_t|}\int_{\Gamma_t} f\rho(\Delta\alpha)_t^2\, d\mathcal{H}^1 \end{aligned} \tag{23}$$

が成り立つ. ただし, $\rho = \rho_{(X_0,t_0)}$ は (20) で与えられた後ろ向き熱核である.

(23) は Γ_t に対する単調性公式という. (23) の右辺第 2 項と第 3 項は非正だから, 第 1 項が非正であることがわかれば, $f\rho\sigma(\Delta\alpha(t))$ の積分の時間変数に関する単調性が得られる. このときに, ρ が (20) で与えられた後ろ向き熱核であることから, $t \to t_0$ としたときに, $f(X_0,t_0)\sigma(\Delta\alpha(t_0))$ の評価が得られる.

$v = \sqrt{1+u_x^2}$ は admissible な関数となるため, 定理 17 で $f = v$ とおけば

$$\frac{d}{dt}\int_{\Gamma_t} v\rho\sigma(\Delta\alpha(t)) \leq \int_{\Gamma_t}(v_t - \mu\sigma(\Delta\alpha(t))\Delta_{\Gamma_t} v + \mu\sigma(\Delta\alpha(t))\kappa(Dv\cdot\boldsymbol{n}))\rho\sigma(\Delta\alpha(t)) \tag{24}$$

が得られる. v に対して直接計算により

$$v_t - \mu\sigma\Delta_{\Gamma_t} v + \mu\sigma(Dv\cdot\boldsymbol{\kappa}) = -\mu\sigma v\kappa^2 - 2\mu\sigma\frac{v_x^2}{v^3}$$

となるから, (24) で $t \to t_0$ とすると, 次が得られる. 証明の詳細は [15] をみよ.

定理 18 ([15, Theorem 4.2])**.** $(u, \Delta\alpha)$ を (19) の解とし, $v := \sqrt{1+u_x^2}$ とおく. このとき, 任意の $0 < x_0 < 1$ と $t_0 > 0$ に対して次が成り立つ:

$$v(x_0, t_0) \le \frac{\sigma(\Delta\alpha(0))}{C_1} \sup_{0<x<1} v^2(x, 0).$$

3.2. **長時間挙動.** 最後に (19) の大域解に対する長時間挙動について述べる. 平滑化効果により, 初期値 u_0 は十分に滑らかであると仮定してよい.

定理 19 ([15, Theorem 5.1])**.** 初期値 u_0 を周期境界条件をみたす $[0,1)$ 上の Lipschitz 関数, $\Delta\alpha_0 \in \mathbb{R}$ を定数とする. $(u, \Delta\alpha)$ を初期条件 $(u(\cdot,0), \Delta\alpha(0)) = (u_0, \Delta\alpha_0)$ をみたす (19) の大域解とする. このとき, ある定数 u_∞ が存在して, $\|u_\infty - u\|_{C^2([0,1))}$ は 0 に指数減衰する. さらに, 曲率 κ も $[0,1)$ 上一様に 0 へ指数減衰する.

定理 19 を証明するために, (19) に対するエネルギー減衰評価を導く.

命題 20 ([15, Proposition 5.2])**.** $(u, \Delta\alpha)$ を (19) の解とするとき, 次が成り立つ:

$$\frac{d}{dt}|\Gamma_t| + \mu\sigma(\Delta\alpha(t)) \int_{\Gamma_t} \kappa^2 = 0, \tag{25}$$

ただし, $\kappa = \left(\frac{u_x}{\sqrt{1+u_x^2}} \right)_x$ は Γ_t の曲率である.

この評価は, 方程式の導出方法から自明に得られる. 証明は [15] をみよ. (25) を両辺 t で積分して, 曲率の二乗の非負値性を使えば, 次が得られる.

系 21 ([15, Corollary 5.2])**.** $(u, \Delta\alpha)$ を (19) の解とすると, $t > 0$ に対して, $|\Gamma_t| \le |\Gamma_0|$ が成り立つ.

(25) の曲率の時空積分が有界となることから次が得られる.

系 22 ([15, Corollary 5.3])**.** $(u, \Delta\alpha)$ を (19) の解とする. このとき, 部分列 $\{t_j\}_{j=1}^\infty$ が存在して, $t_j \to \infty$ かつ

$$\kappa(x, t_j) \to 0, \quad j \to \infty$$

がほとんどすべての $x \in (0,1)$ について成り立つ.

さらなる長時間挙動の解析のために, 高階エネルギー評価を用いる. $(u, \Delta\alpha)$ が (19) の古典解ならば,

$$u_t = \frac{\mu\sigma(\Delta\alpha)}{1+|u_x|^2} u_{xx}.$$

となるので, x について微分すれば

$$\begin{aligned}
u_{xt} &= \frac{\mu\sigma(\Delta\alpha)}{1+|u_x|^2} u_{xxx} - \frac{2\mu\sigma(\Delta\alpha)}{(1+|u_x|^2)^2} u_x u_{xx}^2, \\
u_{xxt} &= \frac{\mu\sigma(\Delta\alpha)}{1+|u_x|^2} u_{xxxx} - \frac{6\mu\sigma(\Delta\alpha)}{(1+|u_x|^2)^2} u_x u_{xx} u_{xxx} - \frac{2\mu\sigma(\Delta\alpha)}{(1+|u_x|^2)^2} u_{xx}^3 + \frac{8\mu\sigma(\Delta\alpha)}{(1+|u_x|^2)^3} u_x^2 u_{xx}^3, \\
u_{xxxt} &= \frac{\mu\sigma(\Delta\alpha)}{1+|u_x|^2} u_{xxxxx} - \frac{8\mu\sigma(\Delta\alpha)}{(1+|u_x|^2)^2} u_x u_{xx} u_{xxxx} - \frac{12\mu\sigma(\Delta\alpha)}{(1+|u_x|^2)^2} u_{xx}^2 u_{xxx} - \frac{6\mu\sigma(\Delta\alpha)}{(1+|u_x|^2)^2} u_x u_{xxx}^2 \\
&\quad + \frac{48\mu\sigma(\Delta\alpha)}{(1+|u_x|^2)^3} u_x^2 u_{xx}^2 u_{xxx} + \frac{24\mu\sigma(\Delta\alpha)}{(1+|u_x|^2)^3} u_x u_{xx}^4 - \frac{48\mu\sigma(\Delta\alpha)}{(1+|u_x|^2)^4} u_x^3 u_{xx}^4
\end{aligned} \tag{26}$$

が得られる.

命題 23 ([15, Proposition 5.6])**.** $(u, \Delta\alpha)$ を (19) の古典解とする. このとき, 正定数 $C_4 > 0$ が存在して

$$\int_0^1 |u_x(x,t)|^2\,dx \le e^{-C_4 t}\int_0^1 |u_{0x}(x)|^2\,dx \tag{27}$$

が $t > 0$ に対して成り立つ.

証明. (27) の左辺を t について微分して, (26) を用いると

$$\begin{aligned}\frac{d}{dt}\int_0^1 |u_x(x,t)|^2\,dx &= 2\int_0^1 u_x(x,t)u_{xt}(x,t)\,dx\\ &= 2\int_0^1 \frac{\mu\sigma(\Delta\alpha)}{1+|u_x|^2}u_x u_{xxx}\,dx - 4\int_0^1 \frac{\mu\sigma(\Delta\alpha)}{(1+|u_x|^2)^2}u_x^2 u_{xx}^2\,dx\end{aligned}$$

となる. 部分積分により

$$\int_0^1 \frac{\mu\sigma(\Delta\alpha)}{1+|u_x|^2}u_x u_{xxx}\,dx = -\int_0^1 \frac{\mu\sigma(\Delta\alpha)}{1+|u_x|^2}u_{xx}^2\,dx + 2\int_0^1 \frac{\mu\sigma(\Delta\alpha)}{(1+|u_x|^2)^2}u_x^2 u_{xx}^2\,dx$$

だから

$$\frac{d}{dt}\int_0^1 |u_x(x,t)|^2\,dx = -2\int_0^1 \frac{\mu\sigma(\Delta\alpha)}{1+|u_x|^2}u_{xx}^2\,dx.$$

となる. σ の正値性, 定理 18 と Poincaré 不等式を組み合わせると, 定数 $C_5 > 0$ が存在して

$$\int_0^1 \frac{\mu\sigma(\Delta\alpha)}{1+|u_x|^2}u_{xx}^2\,dx \ge C_5\int_0^1 |u_x(t,x)|^2\,dx,$$

とできる. Gronwall の不等式により (27) を得る.

命題 23 と同様の結果を u_{xx}, u_{xxx} に対して構成することで, u や κ の指数減衰を得ることができる. 詳細は [15, Section 5] をみよ.

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Optimal control for shape memory alloys of the one-dimensional Frémond model

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1 Introduction

We consider optimal control problems for a simplified version of the mathematical model proposed by Michel Frémond to describe the thermomechanical evolution of a shape memory alloy.

Shape memory materials are very important and interesting ones. Indeed, at a fixed temperature, materials are deformed by traction, for instance. However, some materials totally recover their original shape just by thermal means. This is the so-called shape memory effect. The phenomenon of shape memory can be interpreted as the effect of a structural austenite-martensite phase transition (cf. [2, 12]).

Michel Frémond [13] originally introduced the three-dimensional model by taking the absolute temperature θ, the displacement u of the body, and the volumetric proportions of austenite and martensite as state variables. For simplicity, we assume that only two martensitic variants are present beside one austenite and indicate the respective proportions as β_1, β_2, and β_A, respectively. Moreover, we assume that the phases possibly coexist at each point of the body, that no overlapping between different phases can occur, and that no void appears in the mixture. Then, the phase proportions are regarded as thermodynamic quantities and are constrained to fulfill the obvious relation:

$$0 \le \beta_1,\ \beta_2,\ \beta_A \le 1, \qquad \beta_1 + \beta_2 + \beta_A = 1. \tag{1.1}$$

For the detailed presentation of the Frémond model, we refer to [9], where the related mathematical problem has been shown to be well-posed.

In this paper, we consider the one-dimensional setting of shape memory material (e.g., a metallic wire) that occupies the region $\Omega := (0, 1)$. In order to eliminate β_A in (1.1), we use the new variables χ_1 and χ_2, defined by:

$$\chi_1 := \beta_1 + \beta_2, \qquad \chi_2 := \beta_1 - \beta_2. \tag{1.2}$$

Then, we investigate the following simplified Frémond system of a shape memory alloy:

$$(L_0\theta - L_1\chi_1)_t - h\theta_{xx} = a_0 f(t,x) \ \text{ in } Q := (0,T)\times\Omega, \tag{1.3}$$

$$\mu_0 \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}_t - \mu_1 \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}_{xx} + \partial I_K(\chi_1,\chi_2) \ni \begin{pmatrix} l(\theta^c-\theta)\\ -\beta a_1 g(t)\alpha(\theta) + \beta\alpha(\theta)^2\chi_2\end{pmatrix} \ \text{ in } Q, \tag{1.4}$$

$$-h\theta_x(t,0) + k(\theta(t,0) - a_2\gamma_0(t)) = h\theta_x(t,1) + k(\theta(t,1) - a_3\gamma_1(t)) = 0, \quad t\in(0,T), \tag{1.5}$$

$$(\chi_i)_x(t,0) = (\chi_i)_x(t,1) = 0, \quad t\in(0,T),\ i=1,2, \tag{1.6}$$

$$\theta(0,x) = \theta_0(x), \quad \chi_i(0,x) = \chi_{i,0}(x), \quad x\in\Omega,\ i=1,2. \tag{1.7}$$

Note that the initial boundary value problem (SMA):=$\{$(1.3), (1.4), (1.5), (1.6), (1.7)$\}$ is based on energy balance and phase dynamics, while the (longitudinal) displacement u, which plays the major role in the momentum balance, does not appear explicitly in (SMA). Indeed, the momentum balance equation in the quasi-stationary form reads $\sigma_x = 0$, where the stress σ is related to the strain u_x, the temperature θ, and the phase proportions by the following constitutive relation:

$$\sigma = u_x + \beta\alpha(\theta)\chi_2.$$

Since the boundary conditions $u(t,0) = 0$ (one end of the wire fixed) and $\sigma(1,t) = \beta a_1 g(t)$ (external traction prescribed) are usually considered, we observe that in the one-dimensional case, σ and u can be completely determined in terms of data and other unknowns (see especially the papers [5, 10] dealing with the one dimensional problem). That is the reason why in the dynamics of χ_i, $i = 1,2$, written above, one finds the complicate expression $-\beta a_1 g(t)\alpha(\theta) + \beta\alpha(\theta)^2\chi_2$ instead of $-\alpha(\theta)u_x$. Here, note that, in our system, L_0, L_1, h, k, μ_0 μ_1, l, β, and θ^c are positive coefficients with proper physical meaning; in particular, θ^c represents a critical temperature. In addition, a_0, a_1, a_2, and a_3 are fixed real numbers. Furthermore, $f : Q \to \mathbb{R}$ stands for a known source term, while g, γ_0, γ_1 are given functions defined on the finite time interval $[0,T]$. The nonlinearity α acting on temperature values is a smooth nonnegative decreasing function, vanishing on the interval $[\theta^{Cu}, +\infty)$ for a certain fixed temperature (the so-called Curie point) $\theta^{Cu} > \theta^c$. Actually, among the properties of α, in our analysis we just use the fact that $\alpha \in W^{2,\infty}(\mathbb{R})$.

Note from (1.1) (cf. (1.2)) that the Frémond model (SMA) has a non-smooth constraint ∂I_K in (1.4), representing the subdifferential of the indicator function I_K of the plane triangle K:

$$K := \left\{(\xi,\eta)\in\mathbb{R}^2\ ;\ 0\le\xi\le 1,\ |\eta|\le\xi\right\}. \tag{1.8}$$

The set K is convex, and contains the admissible phase proportions. We also notice that

$$I_K(\chi_1,\chi_2) := \begin{cases} 0, & \text{if } (\chi_1,\chi_2)\in K,\\ \infty, & \text{otherwise.}\end{cases} \tag{1.9}$$

An updated and detailed presentation of the Frémond model and related system of equations and conditions, applying to the multidimensional case as well, is provided in [3,4] and [14, Chapter 13]. We also point out [3, 4] for existence and uniqueness results in the

three-dimensional situation: here, the various nonlinear terms arising in the derivation of the model are taken into account. For a list of related references as well as for a survey of previous mathematical work, we address the reader to [6, 10]. The large time behavior of solutions is investigated in [8, 10, 11] in connection with the convergence to steady-state solutions, global attractors, and so on. However, the study of the optimal control problem for the Frémond model has not been reported to date, up to our knowledge. The reason for that is, in our opinion, due to the difficulties created by the presence of the plane triangle set K.

Recently, in [7] we deal with the followig optimal control problem (OP) of (SMA):

Problem (OP): Find a quadruplet of control functions $(f^*, g^*, \gamma_0^*, \gamma_1^*) \in \mathcal{U}_{ad}^M$ such that

$$J(f^*, g^*, \gamma_0^*, \gamma_1^*) = \inf_{(f,g,\gamma_0,\gamma_1)\in\mathcal{U}_{ad}^M} J(f, g, \gamma_0, \gamma_1),$$

where $(f^*, g^*, \gamma_0^*, \gamma_1^*)$ is called an optimal control for (OP); here, putting

$$\mathcal{U} := L^2(0,T;L^2(\Omega)) \times H^1(0,T) \times H^1(0,T) \times H^1(0,T),$$

$\mathcal{U}_{ad}^M$ is the control space specified by

$$\mathcal{U}_{ad}^M := \left\{ (f, g, \gamma_0, \gamma_1) \in \mathcal{U} \;\middle|\; \begin{array}{c} |g|_{H^1(0,T)} \le M, \\ |\gamma_i|_{H^1(0,T)} \le M,\ i = 0, 1 \end{array} \right\} \tag{1.10}$$

for some fixed positive number M, and $J(f, g, \gamma_0, \gamma_1)$ is the cost functional defined by

$$\begin{aligned} J(f, g, \gamma_0, \gamma_1) \;:=\; & \frac{c_0}{2}\int_0^T |(\theta - \theta_d)(t)|^2_{L^2(\Omega)} dt + \frac{c_1}{2}\int_0^T |(\chi_1 - \chi_{1,d})(t)|^2_{L^2(\Omega)} dt \\ & + \frac{c_2}{2}\int_0^T |(\chi_2 - \chi_{2,d})(t)|^2_{L^2(\Omega)} dt \\ & + \frac{m_0}{2}\int_0^T a_0^2 |f(t)|^2_{L^2(\Omega)} dt + \frac{m_1}{2}\int_0^T a_1^2 |g(t)|^2 dt \\ & + \frac{m_2}{2}\int_0^T a_2^2 |\gamma_0(t)|^2 dt + \frac{m_3}{2}\int_0^T a_3^2 |\gamma_1(t)|^2 dt, \end{aligned} \tag{1.11}$$

where $(f, g, \gamma_0, \gamma_1) \in \mathcal{U}_{ad}^M$ denotes the generic control and the triplet of functions (θ, χ_1, χ_2) yields the unique solution to the state problem (SMA) with the source term $(f, g, \gamma_0, \gamma_1)$. We also point out that the given constants c_0, c_1, c_2, m_0, m_1, m_2, m_3 are nonnegative, and $\theta_d \in L^2(0,T;L^2(\Omega))$, $\chi_{1,d} \in L^2(0,T;L^2(\Omega))$, $\chi_{2,d} \in L^2(0,T;L^2(\Omega))$ represent the known desired target profiles.

Note that if the constant a_0 is equal to 0, then (OP) is a boundary control problem. Similarly, if $a_1 = a_2 = a_3 = 0$, then (OP) reduces to a distributed control problem with the heat source as control. In addition, we remark that γ_0 (resp. γ_1) denotes the outside temperature control function at $x = 0$ (resp. $x = 1$).

There is a vast amount of literature on optimal control problems for variational inequalities, phase transitions problems, and so on. However, to the best of our knowledge,

no result is available for the optimal control analyisis of problems like (SMA), probably because of the triangular shape of K and the non-smooth nonlinearity of the two-components constraints $\partial I_K(\chi_1, \chi_2)$ in (1.4). In [7], an effective approximating approach to the optimal controls of our control problem (OP) was presented. Therefore, in this paper, we briefly mention the existence of an optimal control for (OP), approximation procedures for (SMA) and (OP), and so on.

1.1 Notations and basic assumptions

We first state the notations that are used throughout this paper.

For any reflexive Banach space B, we denote by $|\cdot|_B$ the norm of B, and denote by B' the dual space of B. Additionally, we denote by $\langle\cdot,\cdot\rangle_{B',B}$ the duality pairing between B' and B. Furthermore, for a positive integer $m \in \mathbb{N}$, we use the product space B^m:

$$B^m := \prod_{i=1}^{m} B = \overbrace{B \times B \times \cdots \times B}^{m\text{-factors}}$$

with the norm:

$$|z|_{B^m} := \sum_{i=1}^{m} |z_i|_B \quad \text{for } z = (z_1, z_2, \cdots, z_m) \in B^m.$$

In particular, we put $H := L^2(\Omega)$ with the usual real Hilbert structure, and denote by $(\cdot,\cdot)_H$ the inner product in H, for simplicity. In addition, let V be the Sobolev space $H^1(\Omega)$ with the inner product and norm:

$$(z, w)_V := (z_x, w_x)_H + \frac{k}{h}\left(z(0)w(0) + z(1)w(1)\right) \quad \text{for any } z, w \in V,$$

and

$$|z|_V := \left\{ |z_x|_H^2 + \frac{k}{h}\left(|z(0)|^2 + |z(1)|^2\right) \right\}^{1/2} \quad \text{for any } z \in V,$$

which are equivalent to the standard inner product and norm of $H^1(\Omega)$.

Next, we give some assumptions on data. Throughout this paper, we assume the following conditions (A1)–(A4):

(A1) $T > 0$, $L_0 > 0$, $L_1 > 0$, $h > 0$, $k > 0$, $\mu_0 > 0$, $\mu_1 > 0$, $l > 0$, $\beta > 0$, $\theta^c > 0$, $c_0 \geq 0$, $c_1 \geq 0$, $c_2 \geq 0$, $m_0 \geq 0$, $m_1 \geq 0$, $m_2 \geq 0$, $m_3 \geq 0$, and $M > 0$ are fixed constants. In addition, a_0, a_1, a_2, a_3 are fixed real numbers.

(A2) $\alpha \in W^{2,\infty}(\mathbb{R})$.

(A3) $\theta_0 \in V$, and $\chi_{i,0} \in V$ $(i = 1, 2)$ with $(\chi_{1,0}, \chi_{2,0}) \in K$, a.e. in Ω.

(A4) $\theta_d \in L^2(0, T; H)$, $\chi_{1,d} \in L^2(0, T; H)$, $\chi_{2,d} \in L^2(0, T; H)$ are the given desired target profiles.

2 Main results

We begin by defining the notion of solutions to (SMA). To this end, given $\gamma_i \in H^1(0,T)$, $i=1,2$, we define γ by putting (cf. [11, (2.4)]):

$$\gamma(t,x) := \frac{k}{2h+k}(a_3\gamma_1(t) - a_2\gamma_0(t))x + \frac{ha_3\gamma_1(t) + (h+k)a_2\gamma_0(t)}{2h+k}, \quad (t,x) \in Q. \tag{2.1}$$

It is easy to check that $\gamma \in W^{1,2}(0,T;H^2(\Omega))$ solves the boundary value problem

$$\begin{cases} \gamma_{xx}(t,x) = 0 \ \text{ for any } (t,x) \in Q, \\ -h\gamma_x(t,0) + k(\gamma(t,0) - a_2\gamma_0(t)) = h\gamma_x(t,1) + k(\gamma(t,1) - a_3\gamma_1(t)) = 0, \ t \in (0,T). \end{cases} \tag{2.2}$$

Definition 2.1. *Let $\theta_0 \in V$ and $\chi_{i,0} \in V$ $(i=1,2)$. Then, a triplet of functions (θ, χ_1, χ_2) is called a solution to* (SMA)*, or* (SMA;$\theta_0, \chi_{1,0}, \chi_{2,0}, f, g, \gamma_0, \gamma_1$) *when the data are specified, on $[0,T]$, if the following conditions are satisfied:*

(S1) $\theta \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V)$.

(S2) $\chi_i \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V)$, $i=1,2$.

(S3) *For all $z \in V$ and a.a. $t \in (0,T)$,*

$$(L_0\theta_t(t) - L_1(\chi_1)_t(t), z)_H + h(\theta(t) - \gamma(t), z)_V = (a_0 f(t), z)_H.$$

(S4) *There is a pair of functions $(\xi_1, \xi_2) \in (L^2(0,T;H))^2$ such that*

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \partial I_K(\chi_1, \chi_2), \quad \text{a.e. in } Q$$

and

$$\begin{aligned} &\sum_{i=1}^{2} \{\mu_0((\chi_i)_t(t), z_i)_H + \mu_1((\chi_i)_x(t), (z_i)_x)_H + (\xi_i(t), z_i)_H\} \\ &= l\,(\theta^c - \theta(t), z_1)_H + \left(-\beta a_1 g(t)\alpha(\theta(t)) + \beta\alpha(\theta(t))^2\chi_2(t), z_2\right)_H \\ &\qquad \text{for any } (z_1, z_2) \in V \times V \text{ and a.a. } t \in (0,T). \end{aligned}$$

(S5) $\theta(0) = \theta_0$ *in H, and* $\chi_i(0) = \chi_{i,0}$ *in H, $i=1,2$.*

Here, we recall the known result of the existence-uniqueness and boundedness of solutions to (SMA).

Proposition 2.1 (cf. [11, Theorems 2.1 and 2.2]). *Suppose that assumptions* (A1), (A2), *and* (A3) *hold. Let* $f \in L^2(0,T;H)$, $g \in H^1(0,T)$, $\gamma_0 \in H^1(0,T)$, *and* $\gamma_1 \in H^1(0,T)$. *Then, there is a unique solution* (θ, χ_1, χ_2) *to* (SMA;$\theta_0, \chi_{1,0}, \chi_{2,0}, f, g, \gamma_0, \gamma_1$) *on* $[0,T]$ *in the sense of Definition 2.1. In addition, there is a positive constant* N_1, *independent of* f, g, γ_0, γ_1, θ_0, $\chi_{1,0}$, *and* $\chi_{2,0}$, *such that the following estimate holds:*

$$\begin{aligned} &\sup_{t\in[0,T]} |\theta(t)|_V^2 + \sum_{i=1}^{2} \sup_{t\in[0,T]} |\chi_i(t)|_V^2 + \int_0^T |\theta_t(t)|_H^2 dt + \sum_{i=1}^{2} \int_0^T |(\chi_i)_t(t)|_H^2 dt \\ \le\ & N_1 \Big(|\theta_0|_V^2 + |\chi_{1,0}|_V^2 + |\chi_{2,0}|_V^2 + a_2^2|\gamma_0(0)|^2 + a_3^2|\gamma_1(0)|^2 \\ &\qquad + a_0^2 |f|_{L^2(0,T;H)}^2 + a_1^2 |g|_{L^2(0,T)}^2 + a_2^2 |\gamma_0|_{H^1(0,T)}^2 + a_3^2 |\gamma_1|_{H^1(0,T)}^2 + 1 \Big) \end{aligned} \tag{2.3}$$

We now state the first main result of this paper, which is concerned with the existence of an optimal control for (OP).

Theorem 2.1. *Suppose that assumptions* (A1), (A2), (A3), *and* (A4) *hold. Then, the problem* (OP) *has at least one optimal control* $(f^*, g^*, \gamma_0^*, \gamma_1^*) \in \mathcal{U}_{ad}^M$, *namely,*

$$J(f^*, g^*, \gamma_0^*, \gamma_1^*) = \inf_{(f,g,\gamma_0,\gamma_1)\in\mathcal{U}_{ad}^M} J(f, g, \gamma_0, \gamma_1).$$

Note that it is very difficult to show the necessary conditions for (OP) directly, because of the constraint $\partial I_K(\chi_1, \chi_2)$ in (1.4). Therefore, by investigating approximating problems for (SMA) and (OP), we show a limiting optimality system for (OP). To this end, we consider the following smooth function $\widehat{K}^\varepsilon$ on $\mathbb{R}^2$ for each $\varepsilon \in (0,1]$:

(A5) For each $\varepsilon \in (0,1]$, the function $\widehat{K}^\varepsilon$ is convex and non-negative on $\mathbb{R}^2$ such that $\widehat{K}^\varepsilon \in C^2(\mathbb{R}^2)$, $\partial_i\partial_j \widehat{K}^\varepsilon \in W^{1,\infty}(\mathbb{R}^2)$ $(i,j=1,2)$,

$$K^\varepsilon := \Big\{ (z_1, z_2) \in \mathbb{R}^2 \ ;\ \widehat{K}^\varepsilon(z_1, z_2) = 0 \Big\} \supset K,$$

$$|\partial_i\partial_j \widehat{K}^\varepsilon(z_1, z_2)| \le \frac{1}{\varepsilon} \ \text{ for any } i,j = 1,2, \text{ and any } (z_1, z_2) \in \mathbb{R}^2, \tag{2.4}$$

and

$$\widehat{K}^\varepsilon \text{ converges to } I_K \text{ on } \mathbb{R}^2 \text{ in the sense of Mosco [15] as } \varepsilon \to 0,$$

where $\partial_i \widehat{K}^\varepsilon(z_1, z_2)$ is the partial derivative of $\widehat{K}^\varepsilon(z_1, z_2)$ with respect to the variable z_i $(i = 1,2)$, namely, $\partial_i := \partial/\partial z_i$.

Now, for each $\varepsilon \in (0,1]$, we present the following approximating state system for (SMA), denoted by (SMA)$^\varepsilon$:

Problem (SMA)$^\varepsilon$.

$$(L_0\theta^\varepsilon - L_1\chi_1^\varepsilon)_t - h\theta_{xx}^\varepsilon = a_0 f(t,x) \ \text{ in } Q = (0,T)\times\Omega, \tag{2.5}$$

$$\mu_0 \begin{pmatrix} \chi_1^\varepsilon \\ \chi_2^\varepsilon \end{pmatrix}_t - \mu_1 \begin{pmatrix} \chi_1^\varepsilon \\ \chi_2^\varepsilon \end{pmatrix}_{xx} + \nabla \widehat{K}^\varepsilon(\chi_1^\varepsilon, \chi_2^\varepsilon) = \begin{pmatrix} l(\theta^c - \theta^\varepsilon) \\ -\beta a_1 g(t)\alpha(\theta^\varepsilon) + \beta\alpha(\theta^\varepsilon)^2 \chi_2^\varepsilon \end{pmatrix} \quad \text{in } Q, \tag{2.6}$$

$$-h\theta_x^\varepsilon(t,0) + k(\theta^\varepsilon(t,0) - a_2\gamma_0(t)) = h\theta_x^\varepsilon(t,1) + k(\theta^\varepsilon(t,1) - a_3\gamma_1(t)) = 0, \ \ t \in (0,T), \tag{2.7}$$

$$(\chi_i^\varepsilon)_x(t,0) = (\chi_i^\varepsilon)_x(t,1) = 0, \quad t \in (0,T), \ i = 1,2, \tag{2.8}$$

$$\theta^\varepsilon(0,x) = \theta_0(x), \quad \chi_i^\varepsilon(0,x) = \chi_{i,0}(x), \quad x \in \Omega, \ i = 1,2. \tag{2.9}$$

In the rest, we denote (SMA)$^\varepsilon$ by (SMA;$\theta_0, \chi_{1,0}, \chi_{2,0}, f, g, \gamma_0, \gamma_1$)$^\varepsilon$ when the data of the initial value $\theta_0, \chi_{1,0}, \chi_{2,0}$ and the control functions f, g, γ_0, γ_1 are specified. Note that for each $\varepsilon \in (0,1]$, the constraint $\partial I_K(\chi_1, \chi_2)$ as in (1.4) is approximated by $\nabla \widehat{K}^\varepsilon(\chi_1^\varepsilon, \chi_2^\varepsilon)(= \partial \widehat{K}^\varepsilon(\chi_1^\varepsilon, \chi_2^\varepsilon))$. In a similar way to Proposition 2.1, we immediately get the solvability of the approximating state problem (SMA)$^\varepsilon$ ($\varepsilon \in (0,1]$).

Next, for each $\varepsilon \in (0,1]$ and $\delta \geq 0$, we present an approximating optimal control problem for (OP), denoted by (OP)$_\delta^\varepsilon$, as follows:

Problem (OP)$_\delta^\varepsilon$. Find an optimal control $(f^{*,\varepsilon}, g^{*,\varepsilon}, \gamma_0^{*,\varepsilon}, \gamma_1^{*,\varepsilon}) \in \mathcal{U}_{ad}^M$, namely,

$$J_\delta^\varepsilon(f^{*,\varepsilon}, g^{*,\varepsilon}, \gamma_0^{*,\varepsilon}, \gamma_1^{*,\varepsilon}) = \inf_{(f,g,\gamma_0,\gamma_1) \in \mathcal{U}_{ad}^M} J_\delta^\varepsilon(f, g, \gamma_0, \gamma_1),$$

where $J_\delta^\varepsilon(f, g, \gamma_0, \gamma_1)$ is the cost functional defined by

$$\begin{aligned} J_\delta^\varepsilon(f, g, \gamma_0, \gamma_1) \ := \ & \frac{c_0}{2} \int_0^T |(\theta^\varepsilon - \theta_d)(t)|_H^2 dt + \frac{c_1}{2} \int_0^T |(\chi_1^\varepsilon - \chi_{1,d})(t)|_H^2 dt \\ & + \frac{c_2}{2} \int_0^T |(\chi_2^\varepsilon - \chi_{2,d})(t)|_H^2 dt \\ & + \frac{m_0}{2} \int_0^T a_0^2 |f(t)|_H^2 dt + \frac{m_1}{2} \int_0^T a_1^2 |g(t)|^2 dt \\ & + \frac{m_2}{2} \int_0^T a_2^2 |\gamma_0(t)|^2 dt + \frac{m_3}{2} \int_0^T a_3^2 |\gamma_1(t)|^2 dt \\ & + \frac{\delta}{2} \int_0^T |(f - f^*)(t)|_H^2 dt + \frac{\delta}{2} \int_0^T |(g - g^*)(t)|^2 dt \\ & + \frac{\delta}{2} \int_0^T |(\gamma_0 - \gamma_0^*)(t)|^2 dt + \frac{\delta}{2} \int_0^T |(\gamma_1 - \gamma_1^*)(t)|^2 dt. \end{aligned} \tag{2.10}$$

Here, $(f, g, \gamma_0, \gamma_1) \in \mathcal{U}_{ad}^M$ is the control and the triplet of functions $(\theta^\varepsilon, \chi_1^\varepsilon, \chi_2^\varepsilon)$ is the unique solution to the state problem (SMA;$\theta_0, \chi_{1,0}, \chi_{2,0}, f, g, \gamma_0, \gamma_1$)$^\varepsilon$. Moreover, as in (1.11), $\theta_d \in L^2(0,T;H)$, $\chi_{1,d} \in L^2(0,T;H)$, $\chi_{2,d} \in L^2(0,T;H)$ are the given desired target profiles, while $(f^*, g^*, \gamma_0^*, \gamma_1^*) \in \mathcal{U}_{ad}^M$ is any fixed optimal control for (OP) obtained in Theorem 2.1.

We now state the second main result of this paper, which is concerned with the existence of an optimal control for (OP)$_\delta^\varepsilon$ for each $\varepsilon \in (0,1]$ and $\delta \geq 0$.

Theorem 2.2. *Suppose that assumptions* (A1)–(A5) *hold. Let* $\varepsilon \in (0,1]$, $\delta \geq 0$, *and let* $(f^*, g^*, \gamma_0^*, \gamma_1^*) \in \mathcal{U}_{ad}^M$ *be a chosen optimal control for* (OP) *given by Theorem 2.1. Then, the*

approximating problem $(\mathrm{OP})^{\varepsilon}_{\delta}$ *has at least one optimal control* $(f^{*,\varepsilon}, g^{*,\varepsilon}, \gamma_0^{*,\varepsilon}, \gamma_1^{*,\varepsilon}) \in \mathcal{U}^M_{ad}$, *namely,*

$$J^{\varepsilon}_{\delta}(f^{*,\varepsilon}, g^{*,\varepsilon}, \gamma_0^{*,\varepsilon}, \gamma_1^{*,\varepsilon}) = \inf_{(f,g,\gamma_0,\gamma_1)\in\mathcal{U}^M_{ad}} J^{\varepsilon}_{\delta}(f, g, \gamma_0, \gamma_1).$$

The following third main result of the paper is concerned with the relationship between (OP) and $(\mathrm{OP})^{\varepsilon}_{\delta}$.

Theorem 2.3. *Suppose that all the assumptions of Theorem 2.2 hold. Then, the following two statements hold.*

(I) *Let* $\delta = 0$, $\varepsilon \in (0,1]$, *and let* $(f^{*,\varepsilon}, g^{*,\varepsilon}, \gamma_0^{*,\varepsilon}, \gamma_1^{*,\varepsilon}) \in \mathcal{U}^M_{ad}$ *be an optimal control for the approximating problem* $(\mathrm{OP})^{\varepsilon}_0$. *In addition, let* $(\theta^{*,\varepsilon}, \chi_1^{*,\varepsilon}, \chi_2^{*,\varepsilon})$ *be the unique solution to the state problem* $(\mathrm{SMA};\theta_0, \chi_{1,0}, \chi_{2,0}, f^{*,\varepsilon}, g^{*,\varepsilon}, \gamma_0^{*,\varepsilon}, \gamma_1^{*,\varepsilon})^{\varepsilon}$ *on* $[0,T]$. *Then, there exist a subsequence* $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \{\varepsilon\}_{\varepsilon\in(0,1]}$, *a quadruplet of functions* $(f^{**}, g^{**}, \gamma_0^{**}, \gamma_1^{**}) \in \mathcal{U}^M_{ad}$, *and the unique solution* $(\theta^{**}, \chi_1^{**}, \chi_2^{**})$ *to* $(\mathrm{SMA};\theta_0, \chi_{1,0}, \chi_{2,0}, f^{**}, g^{**}, \gamma_0^{**}, \gamma_1^{**})$ *on* $[0,T]$ *such that* $(f^{**}, g^{**}, \gamma_0^{**}, \gamma_1^{**})$ *is an optimal control for* (OP), $\varepsilon_n \to 0$, *and*

$$f^{*,\varepsilon_n} \to f^{**} \quad \text{weakly in } L^2(0,T;H), \tag{2.11}$$

$$g^{*,\varepsilon_n} \to g^{**} \quad \text{weakly in } H^1(0,T), \text{ and in } L^2(0,T), \tag{2.12}$$

$$\gamma_i^{*,\varepsilon_n} \to \gamma_i^{**} \quad \text{weakly in } H^1(0,T), \text{ and in } L^2(0,T), \ (i=0,1), \tag{2.13}$$

$$(\theta^{*,\varepsilon_n}, \chi_1^{*,\varepsilon_n}, \chi_2^{*,\varepsilon_n}) \to (\theta^{**}, \chi_1^{**}, \chi_2^{**}) \quad \text{in } (C([0,T];H))^3 \tag{2.14}$$

as $n \to \infty$.

(II) *Let* $\delta > 0$, $\varepsilon \in (0,1]$, *and let* $(f^{*,\varepsilon}, g^{*,\varepsilon}, \gamma_0^{*,\varepsilon}, \gamma_1^{*,\varepsilon}) \in \mathcal{U}^M_{ad}$ *be an optimal control for the approximating problem* $(\mathrm{OP})^{\varepsilon}_{\delta}$. *Let* $(f^*, g^*, \gamma_0^*, \gamma_1^*) \in \mathcal{U}^M_{ad}$ *be an optimal control for* (OP) *obtained in Theorem 2.1. In addition, let* $(\theta^{*,\varepsilon}, \chi_1^{*,\varepsilon}, \chi_2^{*,\varepsilon})$ *and* $(\theta^*, \chi_1^*, \chi_2^*)$ *be the unique solution to the state problem* $(\mathrm{SMA};\theta_0, \chi_{1,0}, \chi_{2,0}, f^{*,\varepsilon}, g^{*,\varepsilon}, \gamma_0^{*,\varepsilon}, \gamma_1^{*,\varepsilon})^{\varepsilon}$ *and* $(\mathrm{SMA};\theta_0, \chi_{1,0}, \chi_{2,0}, f^*, g^*, \gamma_0^*, \gamma_1^*)$ *on* $[0,T]$, *respectively. Then, there exist a subsequence* $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \{\varepsilon\}_{\varepsilon\in(0,1]}$ *such that* $\varepsilon_n \to 0$,

$$f^{*,\varepsilon_n} \to f^* \quad \text{in } L^2(0,T;H), \tag{2.15}$$

$$g^{*,\varepsilon_n} \to g^* \quad \text{weakly in } H^1(0,T), \text{ and in } L^2(0,T), \tag{2.16}$$

$$\gamma_i^{*,\varepsilon_n} \to \gamma_i^* \quad \text{weakly in } H^1(0,T), \text{ and in } L^2(0,T), \ (i=0,1), \tag{2.17}$$

$$(\theta^{*,\varepsilon_n}, \chi_1^{*,\varepsilon_n}, \chi_2^{*,\varepsilon_n}) \to (\theta^*, \chi_1^*, \chi_2^*) \quad \text{in } (C([0,T];H))^3 \tag{2.18}$$

as $n \to \infty$.

Note that we can show the Gâteaux differentiability of the cost functional $J^{\varepsilon}_{\delta}(\cdot,\cdot,\cdot,\cdot)$, since the approximate state system $(\mathrm{SMA})^{\varepsilon}$ has the smooth constraint $\nabla\widehat{K}^{\varepsilon}(\cdot,\cdot)$ (cf. (2.6)). Thus, we can show the optimality condition for $(\mathrm{OP})^{\varepsilon}_{\delta}$.

However, in general, it is difficult to show the necessary condition of the optimal control for (OP), since the subdifferential $\partial I_K(\cdot,\cdot)$ in (1.4) is not smooth. Thus, by using the approximating problems $(\mathrm{OP})^{\varepsilon}_{\delta}$, more precisely, through the limiting observation of $(\mathrm{OP})^{\varepsilon}_{\delta}$ as $\varepsilon \to 0$, we can prove the following final main result of this paper, which is concerned with the necessary condition of the optimal control for (OP).

Theorem 2.4. *Suppose that all the assumptions of Theorem 2.2 hold. Let the quadruplet* $(f^*, g^*, \gamma_0^*, \gamma_1^*) \in \mathcal{U}_{ad}^M$ *be any optimal control for* (OP) *obtained in Theorem 2.1. Let* $(\theta^*, \chi_1^*, \chi_2^*)$ *be the unique solution to the state problem* (SMA;$\theta_0, \chi_{1,0}, \chi_{2,0}, f^*, g^*, \gamma_0^*, \gamma_1^*$) *on* $[0,T]$. *In addition, let us set:*

$$W := \{\zeta \in H^1(Q) \ ; \ \zeta(0,x) = 0, \ \ a.a. \ x \in \Omega\}.$$

Then, there are the functions

$$p \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V), \quad q_i \in L^2(0,T;V) \cap L^\infty(0,T;H) \ \ (i=1,2),$$

and the elements $\varsigma_i \in W'$ $(i=1,2)$ *satisfying the following:*

$$\begin{aligned} -L_0 p_t - h p_{xx} + l q_1 + \beta a_1 g^*(t) \alpha'(\theta^*) q_2 - 2\beta \alpha'(\theta^*)\alpha(\theta^*)\chi_2^*(t) q_2 \\ = c_0(\theta^* - \theta_d) \quad in \ Q, \end{aligned} \tag{2.19}$$

$$-hp_x(t,0) + kp(t,0) = hp_x(t,1) + kp(t,1) = 0, \quad t \in (0,T), \tag{2.20}$$

$$\begin{aligned} &\int_0^T (\mu_0 q_1(t), \zeta_t(t))_H dt + \int_0^T (L_1 p_t(t), \zeta(t))_H dt + \int_0^T (\mu_1 (q_1)_x(t), \zeta_x(t))_H dt \\ &+ \langle \varsigma_1, \zeta \rangle_{W',W} = c_1 \int_0^T (\chi_1^*(t) - \chi_{1,d}(t), \zeta(t))_H dt \quad for \ all \ \zeta \in W, \end{aligned} \tag{2.21}$$

$$\begin{aligned} &\int_0^T (\mu_0 q_2(t), \zeta_t(t))_H dt + \int_0^T (\mu_1 (q_2)_x(t), \zeta_x(t))_H dt + \langle \varsigma_2, \zeta \rangle_{W',W} \\ &\qquad - \int_0^T (\beta \alpha(\theta^*(t))^2 q_2(t), \zeta(t))_H dt \\ &= c_2 \int_0^T (\chi_2^*(t) - \chi_{2,d}(t), \zeta(t))_H dt \quad for \ all \ \zeta \in W, \end{aligned} \tag{2.22}$$

$$p(T,x) = 0, \quad x \in \Omega. \tag{2.23}$$

In addition, (p, q_1, q_2) *satisfies the following inequality:*

$$\begin{aligned} &\int_0^T a_0((a_0 m_0 f^* + p)(t), (\check{f} - f^*)(t))_H dt \\ &\quad + \int_0^T a_1(a_1 m_1 g^*(t) - (\beta\alpha(\theta^*(t)), q_2(t))_H)(\check{g} - g^*)(t) dt \\ &\quad + \int_0^T a_2(a_2 m_2 \gamma_0^*(t) + kp(t,0))(\check{\gamma}_0 - \gamma_0^*)(t) dt \\ &\quad + \int_0^T a_3(a_3 m_3 \gamma_1^*(t) + kp(t,1))(\check{\gamma}_1 - \gamma_1^*)(t) dt \\ &\geq 0, \quad \forall (\check{f}, \check{g}, \check{\gamma}_0, \check{\gamma}_1) \in \mathcal{U}_{ad}^M. \end{aligned} \tag{2.24}$$

Remark 2.1. *The identities* (2.21) *and* (2.22) *can be regarded as some variational forms of*

$$-\mu_0 (q_1)_t + L_1 p_t - \mu_2 (q_1)_{xx} + \varsigma_1 = c_1(\chi_1^* - \chi_{1,d}),$$

and

$$-\mu_0 (q_2)_t - \mu_1 (q_2)_{xx} + \varsigma_2 - \beta\alpha(\theta^*)^2 q_2 = c_2(\chi_2^* - \chi_{2,d})$$

in the distribution sense, respectively.

3 Key ideas of analysis

In this section, we give the key to proving Theorems 2.1–2.4.

Note that (SMA) can be reformulated to the following system of abstract evolution equations:

$$\theta_t(t) - \frac{L_1}{L_0}(\chi_1)_t(t) + \partial\varphi^t(\theta(t)) \ni \frac{a_0}{L_0}f(t) \ \text{ in } H \ \text{ for } t \in (0,T), \tag{3.1}$$

$$\frac{d}{dt}(\chi_1(t),\chi_2(t)) + \partial\psi(\chi_1(t),\chi_2(t)) + G^t_{\theta(t)}(\chi_1(t),\chi_2(t)) \ni (0,0) \ \text{ in } H\times H \ \text{ for } t \in (0,T), \tag{3.2}$$

$$\theta(0) = \theta_0 \text{ in } H, \text{ and } \ (\chi_1(0),\chi_2(0)) = (\chi_{1,0},\chi_{2,0}) \text{ in } H\times H. \tag{3.3}$$

Here, $\partial\varphi^t(\cdot)$ is the subdifferential of a time-dependent convex function $\varphi^t(\cdot)$ on H for each $t \in [0,T]$, defined by

$$\varphi^t(z) := \begin{cases} \dfrac{h}{2L_0}|z-\gamma(t)|_V^2, & \text{if } z \in V, \\ \infty, & \text{otherwise,} \end{cases} \tag{3.4}$$

where γ is the function defined in (2.1). In addition, $\partial\psi(\cdot,\cdot)$ is the subdifferential of a convex function ψ on $H\times H$, defined by:

$$\psi(z_1,z_2) := \begin{cases} \dfrac{\mu_1}{2\mu_0}\displaystyle\sum_{i=1}^{2}\int_\Omega |(z_i)_x(x)|^2dx + \frac{1}{\mu_0}\int_\Omega I_K(z_1(x),z_2(x))dx, & \text{if } z_i \in V, i=1,2, \\ \infty, \quad \text{otherwise.} \end{cases}$$

Furthermore, $G^t_{\theta(t)}(\cdot,\cdot)$ is a time-dependent operator on $H\times H$ for each $\theta \in C([0,T];H)$ and $t \in [0,T]$, defined by

$$G^t_{\theta(t)}(z_1,z_2) := \frac{1}{\mu_0}(-l(\theta^c-\theta(t)), \beta a_1 g(t)\alpha(\theta(t)) - \beta\alpha(\theta(t))^2 z_2) \quad \text{for any } (z_1,z_2) \in H\times H. \tag{3.5}$$

Then, we can prove Proposition 2.1. Indeed, we can show the existence of a solution to (3.1)–(3.3), hence (SMA), by applying the abstract theory of nonlinear evolution equations and by employing the fixed point argument for continuous operators in compact convex sets (e.g., Schauder's fixed point theorem). In addition, by the standard monotone arguments as in [11, Theorem 2.1], we can show the uniqueness of solutions to (SMA). Moreover, multiplying (1.3) by $(1/L_1)(\theta(t)-\gamma(t))$ and $\theta_t(t)-\gamma_t(t)$, multiplying (1.4) by $(\mu_1/l\mu_0)(\chi_1(t),\chi_2(t))$ and $(1/l)((\chi_1)_t(t),(\chi_2)_t(t))$, using Young's inequality, and integrating in time, we can get the a priori estimate (2.3).

Furthermore, we can prove Theorem 2.1 by applying the abstract convergence theorem of nonlinear evolution equations (cf. [1]) and by taking a minimizing sequence $\{(f_n,g_n,\gamma_{0,n},\gamma_{1,n})\}_{n\in\mathbb{N}} \subset \mathcal{U}^M_{ad}$ such that

$$\lim_{n\to\infty} J(f_n,g_n,\gamma_{0,n},\gamma_{1,n}) = \inf_{(f,g,\gamma_0,\gamma_1)\in\mathcal{U}^M_{ad}} J(f,g,\gamma_0,\gamma_1).$$

For the detailed proof, we refer to [7, Theorem 2.1].

By a similar argument to (SMA), we can construct the unique solution $(\theta^\varepsilon, \chi_1^\varepsilon, \chi_2^\varepsilon)$ to $(\mathrm{SMA})^\varepsilon$ on $[0,T]$. Indeed, the approximating problem $(\mathrm{SMA})^\varepsilon$ is reformulated to the following system of abstract evolution equations (cf. (3.1)–(3.3)):

$$\theta_t^\varepsilon(t) - \frac{L_1}{L_0}(\chi_1^\varepsilon)_t(t) + \partial\varphi^t(\theta^\varepsilon(t)) \ni \frac{a_0}{L_0} f(t) \ \text{ in } H \ \text{ for } t \in (0,T), \tag{3.6}$$

$$\frac{d}{dt}(\chi_1^\varepsilon(t), \chi_2^\varepsilon(t)) + \partial\psi^\varepsilon(\chi_1^\varepsilon(t), \chi_2^\varepsilon(t)) + G^t_{\theta^\varepsilon(t)}(\chi_1^\varepsilon(t), \chi_2^\varepsilon(t)) = (0,0) \ \text{ in } H \times H \ \text{ for } t \in (0,T), \tag{3.7}$$

$$\theta^\varepsilon(0) = \theta_0 \text{ in } H, \ \text{ and } \ (\chi_1^\varepsilon(0), \chi_2^\varepsilon(0)) = (\chi_{1,0}, \chi_{2,0}) \text{ in } H \times H, \tag{3.8}$$

where $\varphi^t(\cdot)$ and $G^t_{(\cdot)}(\cdot,\cdot)$ are same ones defined by (3.4) and (3.5), respectively. In addition, for each $\varepsilon \in (0,1]$, ψ^ε is a proper, l.s.c., and convex function on $H \times H$, defined by:

$$\psi^\varepsilon(z_1, z_2) := \begin{cases} \dfrac{\mu_1}{2\mu_0} \displaystyle\sum_{i=1}^{2} \int_\Omega |(z_i)_x(x)|^2 dx + \frac{1}{\mu_0} \int_\Omega \widehat{K}^\varepsilon(z_1(x), z_2(x)) dx, & \text{if } z_i \in V, \ i = 1,2, \\ \infty, \quad \text{otherwise}, \end{cases}$$

where $\widehat{K}^\varepsilon(\cdot,\cdot)$ is the function in the assumption (A5).

Therefore, by the slight modification of the proof of Proposition 2.1 (cf. [11, Theorems 2.1 and 2.2]), we can show that $(\mathrm{SMA})^\varepsilon$ has a unique $(\theta^\varepsilon, \chi_1^\varepsilon, \chi_2^\varepsilon)$. For the detailed arguments, we refer to [11, Theorems 2.1 and 2.2]. Moreover, by a similar arguments to Theorem 2.1, we can prove Theorem 2.2.

Furthermore, we observe from assumption (A5) that

$$\psi^\varepsilon \to \psi \ \text{ on } H \times H \text{ in the sense of Mosco [15] } \text{ as } \varepsilon \to 0.$$

Therefore, applying the abstract convergence theorem of evolution equations (cf. [1]), we conclude that the solution to $(\mathrm{SMA})^\varepsilon$ converges to one to (SMA) on $[0,T]$ in some appropriate sense, as $\varepsilon \to 0$. Taking account of this fact, we can prove Theorems 2.3.

Finally, note that we can show the optimality condition for the approximating problem $(\mathrm{OP})^\varepsilon_\delta$, since the cost functional J^ε_δ admits the Gâteaux derivative at any $(f, g, \gamma_0, \gamma_1) \in \mathcal{U}$. Thus, through the limiting observation of approximating problems $(\mathrm{OP})^\varepsilon_\delta$ as $\varepsilon \to 0$, we can prove Theorem 2.4 concerning the necessary condition of the optimal control for (OP). For the detailed proof, we refer to [7, Theorem 2.5].

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Solution of hyperbolic equations through energy preserving approximations

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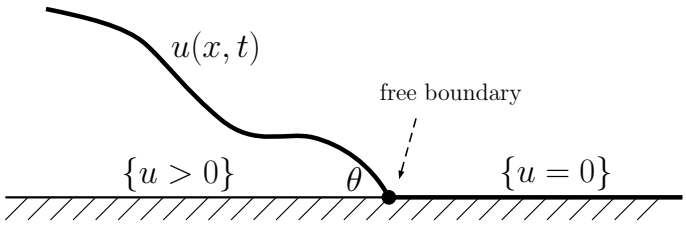
1 Introduction

In this article, we present a new type of approximation of solutions to hyperbolic free boundary problems with the desirable feature of energy preservation, which is important in both mathematical analysis and numerical implementation of such problems.

Hyperbolic free boundary problems appear as model equations of phenomena involving interaction of lower dimensional objects with obstacles, where oscillations due to inertia are not negligible. Examples of such phenomena include bouncing of shells such as balls [9], peeling of membranes such as adhesive tapes from a surface [19], droplets or bubbles moving on obstacles [13], or paper feeding rollers in printers [2].



As a motivation, let us derive a governing equation for a simplified model of removing adhesive tape from solid surface. For simplicity, assume that a tape is uniformly glued to a horizontal solid surface region $\Omega \subset \mathbb{R}^d$, and that its boundary is lifted with a prescribed velocity. Further assume that the tension τ of the tape is constant throughout the evolution, that the adhesion force has a potential $-\alpha$ and that the shape of the tape can be represented by the graph of a scalar function $u(x,t)$. Then the energy of the system where the tape is partially detached from the surface can be written as

$$\int_\Omega \left\{ \tau \left(\sqrt{1+|\nabla u|^2} - 1 \right) \chi_{u>0} - \alpha \left(1 - \chi_{u>0}\right) \right\} dx,$$

where the first term expresses the elastic extension energy of the tape in the detached region $\{u>0\} = \{x \in \Omega;\ u(t,x) > 0\}$ with its characteristic function $\chi_{u>0}$, and the

second term is the adhesion energy proportional to the area $\int_\Omega (1-\chi_{u>0})\,dx$ of the region where the tape is attached to the surface.

Approximating $\sqrt{1+|\nabla u|^2}-1 \approx \frac{1}{2}|\nabla u|^2$ for small contact angles θ and omitting a constant term $-\alpha|\Omega|$ yields the energy functional

$$I[u] = \int_\Omega \left(\frac{\tau}{2}|\nabla u|^2 + \alpha\chi_{u>0}\right)\,dx. \tag{1}$$

The stationary minimization problem for this functional was first studied by Alt and Caffarelli in [3] and later in [4, 16, 17, 18]. In [3] it was shown that a minimizer of (1) exists in $\{u \in L^1_{\text{loc}}(\Omega);\ \nabla u \in L^2(\Omega), u = u_0 \text{ on } \partial\Omega\}$ and satisfies

$$\Delta u = 0 \quad \text{in } \{u>0\}, \qquad |\nabla u|^2 = 2\alpha/\tau \quad \text{on } \partial\{u>0\},$$

while Δu is a Radon measure supported on the free boundary $\partial\{u>0\}$ and absolutely continuous with respect to the $(d-1)$-dimensional Hausdorff measure. Regularity of the free boundary is also discussed.

A corresponding parabolic evolution problem was subsequently investigated in [7]. Here, we are interested in the hyperbolic version which can be formulated by considering stationary points of the action integral corresponding to potential energy (1), namely,

$$\widetilde{J}[u] = \int_0^T \int_\Omega \left(\frac{\rho}{2}(u_t)^2\chi_{u>0} - \frac{\tau}{2}|\nabla u|^2 - \alpha\chi_{u>0}\right)\,dx\,dt.$$

The characteristic function in the kinetic energy term models zero reflection constant. A stationary point of this action, if it exists, is expected to satisfy the wave equation in the detached region $\{u>0\}$, while this region is going to change in time and, therefore, free boundary will be present. The second order derivatives of the solution will then be measures supported in the free boundary, making the analysis of the hyperbolic problem challenging. Moreover, in view of further applications, such as sliding droplets, one would like to impose global constraints on the motion, e.g., the condition of volume being constant in time, naturally leading to taking a variational approach to the problem. This article summarizes the progress, though yet small one, that has been made in this direction.

2 Definition of solution

We discuss possible notions of solutions. To make formulas shorter, we first divide $\widetilde{J}$ by $\tau/2$, set $Q^2 = 2\alpha/\tau$ and scale time, so that the action functional can be written as

$$J[u] = \int_0^T \int_\Omega \left(u_t^2\chi_{u>0} - |\nabla u|^2 - Q^2\chi_{u>0}\right)\,dx\,dt. \tag{2}$$

We start with formally calculating the first variation of J. Assuming that it is possible to take test functions φ supported in $\{u>0\}$, we find

$$\int_0^T \int_{\Omega\cap\{u>0\}} (\nabla u\cdot\nabla\varphi - u_t\varphi_t)\,dx\,dt = 0 \qquad \forall\varphi \in C_0^\infty(\Omega\times(0,T)\cap\{u>0\}),$$

which means that any stationary point u satisfies the wave equation $u_{tt} - \Delta u = 0$ in $\{u > 0\}$. In addition, u must be nonnegative, and thus equal to zero outside of $\{u > 0\}$, as immediately follows from the maximum principle for the Laplacian. On the other hand, computing the inner variation $\frac{d}{d\epsilon} J(u \circ \tau_\epsilon^{-1})|_{\epsilon=0} = 0$, where the map $\tau_\epsilon : \Omega \times (0,T) \to \Omega \times (0,T)$ is defined by $\tau_\epsilon(x,t) := (x,t) + \epsilon\eta(x,t)$ with $\eta \in C_0^\infty(\Omega \times (0,T); \mathbb{R}^d \times \mathbb{R})$ and ϵ sufficiently small, we arrive at the condition $|\nabla u|^2 - u_t^2 = Q^2$ on $\partial\{u > 0\}$ [20]. Hence, we have derived the formulation

$$\begin{cases} u_{tt} - \Delta u &= 0 \quad \text{in} \quad \Omega \times (0,T) \cap \{u > 0\}, \\ |\nabla u|^2 - u_t^2 &= Q^2 \quad \text{on} \quad \Omega \times (0,T) \cap \partial\{u > 0\}. \end{cases} \tag{3}$$

This together with appropriate initial and boundary conditions and regularity assumptions on the solution can be considered as the definition of a strong solution. In [20] it is also formally shown that the above conditions can be expressed by a single equation with a positive Radon measure term:

$$\chi_{\overline{u>0}} u_{tt} - \Delta u = -\frac{Q^2}{\sqrt{|\nabla u|^2 + u_t^2}} \mathscr{H}^d \llcorner \partial\{u > 0\}.$$

It was proved in [10] that if $d = 1$ and certain natural regularity and compatibility conditions are satisfied, then the formulation (3) admits a unique solution $u \in C^2(\Omega \times (0,T) \cap \{u > 0\})$ whose free boundary is a C^1-curve in space-time. This supports the correctness of the formulation (3).

However, since it is hard to expect that strong solutions can be obtained in more general settings, we would like to derive a suitable weak formulation. Unfortunately, this is difficult when the singular term $Q^2\chi_{u>0}$ is present, and therefore we consider its regularization in the same manner as employed in the study of free boundary problems of elliptic type. In particular, we let β_ε be a smooth function satisfying

$$\beta_\varepsilon(s) \begin{cases} = 0 & s \le 0, \\ \in [0, 2/\varepsilon] & 0 < s < \varepsilon \\ = 0 & \varepsilon \le s, \end{cases} \quad \text{and} \quad |\beta_\varepsilon'(s)| \le \frac{C}{\varepsilon^2} \text{ for some } C > 1$$

with $\displaystyle\int_0^\varepsilon \beta_\varepsilon(s) ds = 1$. We then have $B_\varepsilon(u) := \displaystyle\int_0^u \beta_\varepsilon(s)\, ds \to \chi_{u>0}$ pointwise as $\varepsilon \to 0$, yielding a smooth approximation of $\chi_{u>0}$. Replacing the term $Q^2\chi_{u>0}$ in (2) by the term $\frac{Q^2}{2} B_\varepsilon(u)$ then leads to the regularized action

$$J_\varepsilon[u] = \int_0^T \int_\Omega \left(u_t^2 \chi_{u>0} - |\nabla u|^2 - \frac{Q^2}{2} B_\varepsilon(u) \right) dx\, dt, \tag{4}$$

and the corresponding strong formulation

$$\chi_{\overline{u>0}} u_{tt} - \Delta u = -\frac{Q^2}{2} \beta_\varepsilon(u).$$

Assuming the existence of a solution u^ε to this problem equipped with initial and boundary conditions, and its convergence in a suitable topology to a function v, a formal calculation (see, e.g., [20]) shows that the limit function satisfies the free boundary condition $|\nabla v|^2 - v_t^2 = Q^2$ on $\partial\{v > 0\}$. In this sense, (4) is a good approximation of the nonsmooth problem (2).

We can now define weak solutions.

Definition 2.1. A **weak solution** is defined as a function $u \in H^1((0,T);L^2(\Omega)) \cap L^\infty((0,T);H_0^1(\Omega))$ satisfying the following equality for all test functions $\phi \in C_0^\infty(\Omega \times [0,T) \cap \{u>0\})$:

$$\int_0^T \int_\Omega \left(-u_t\phi_t + \nabla u \cdot \nabla \phi + \frac{Q^2}{2}\beta_\varepsilon(u)\phi \right) dx\,dt - \int_\Omega v_0\phi(x,0)\,dx = 0.$$

Moreover, we require that $u \equiv 0$ is satisfied outside of $\{u>0\}$, and that u fulfills the initial condition $u(0,x) = u_0(x)$ and boundary condition in the sense of traces.

On the other hand, authors of [5] define weak solution in the following way:

Definition 2.2. A **weak solution of type II** is defined as a function $u \in L^\infty(0,T;H^1(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$ with $u(t,x) \geq 0$ for a.e. $(t,x) \in (0,T)\times\Omega$, such that there exist weak left and right derivatives $u_t^\pm$ on $[0,T]$ and such that for all $\phi \in W^{1,\infty}(0,T;L^2(\Omega)) \cap L^1(0,T;H^1(\Omega))$ with $\phi \geq 0$, spt $\phi \subset [0,T)$, we have

$$\int_0^T \int_\Omega \left(-u_t\phi_t + \nabla u \cdot \nabla \phi + \frac{Q^2}{2}\beta_\varepsilon(u)\phi \right) dx\,dt - \int_\Omega v_0\phi(x,0)\,dx \geq 0,$$

and initial conditions are satisfied in weak sense.

One observes that Definition 2.2 introduces a weaker notion of solution than Definition 2.1 in the sense that it only requires the subsolution property on the whole of $(0,T)\times\Omega$. In fact, the relation between the definitions and the uniqueness of their respective solutions are still open. Nevertheless, existence of solutions of both types was obtained using the same method of approximations through semi-discretization in time. Before explaining this method, we give a brief account of obtained results. As mentioned above, existence of unique strong solution in one dimension was proved in [10] and the result was extended in [14, 15] to global and periodic solutions. A method for approximation of solutions in arbitrary dimension was devised in [22] and further developed in [11, 24] and in [23, 8] also applied to a problem with volume constraint. This series of studies succeeded in constructing a reasonable approximation for a weak solution in the sense of Definition 2.1 but the weak solution itself was obtained again only in the one-dimensional case. On the other hand, the papers [5, 6] take the strategy of Definition 2.2 and generalize the semidiscretization method to nonlocal problems (with fractional Laplacian) and to semilinear problems in a vector-valued setting, obtaining weak solutions of type II in arbitrary dimension.

3 Discrete Morse flow method

3.1 Simple example: the wave equation

We first explain the idea of the construction of weak solution on an example of an initial-boundary value problem for the linear wave equation:

$$\begin{aligned} u_{tt}(t,x) &= \Delta u(t,x) && \text{in } Q_T = (0,T)\times\Omega, \\ u(t,x) &= 0 && \text{on } (0,T)\times\partial\Omega, \\ u(0,x) &= u_0(x) && \text{in } \Omega, \\ u_t(0,x) &= v_0(x) && \text{in } \Omega, \end{aligned} \tag{5}$$

where $u_0 \in H_0^1(\Omega)$ and $v_0 \in H_0^1(\Omega)$.

We fix a natural number $N > 0$ that determines the time step $h = T/N$ and set $u_{-1}(x) = u_0(x) - hv_0(x)$. Function u_0 corresponds to the approximate solution at time level $t = 0$, while function u_{-1} is the approximate solution at time level $t = -h$. We define the approximate solution u_n on a further time levels $t = nh$ for $n = 1, 2, \ldots, N$, to be the minimizer of the following functional in $H_0^1(\Omega)$:

$$J_n[u] = \int_\Omega \frac{|u - 2u_{n-1} + u_{n-2}|^2}{2h^2}\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx. \tag{6}$$

We observe that the second term of the functional is lower-semicontinuous with respect to the weak convergence in $H^1(\Omega)$ and the first term is continuous in $L^2(\Omega)$. The existence of minimizers then follows immediately from the fact that the functionals are bounded from below for each $n = 1, 2, \ldots, N$. This is a crucial advantage over the continuous version of this functional, the Lagrangian.

As the next step, we define the approximate solutions $\bar{u}^h$ and u^h through an interpolation of the minimizers $\{u_n\}_{n=-1}^N$ in time. The interpolation is given by

$$\begin{aligned} \bar{u}^h(t,x) &= u_n(x), \\ u^h(t,x) &= \frac{t-(n-1)h}{h}u_n(x) + \frac{nh-t}{h}u_{n-1}(x), \end{aligned} \tag{7}$$

for $t \in ((n-1)h, nh], n = 0, \ldots, N$.

Since u_n is a minimizer of J_n, the first variation of J_n at u_n vanishes. Thus, for any $\varphi \in H_0^1(\Omega)$ we have

$$0 = \frac{d}{d\varepsilon}J_n[u_n + \varepsilon\varphi]|_{\varepsilon=0} = \int_\Omega \frac{u_n - 2u_{n-1} + u_{n-2}}{h^2}\varphi\,dx + \int_\Omega \nabla u_n \nabla\varphi\,dx. \tag{8}$$

Using the definition of $\bar{u}^h$ and u^h in (7), this can be rewritten as

$$\int_\Omega \Big[\frac{u_t^h(t) - u_t^h(t-h)}{h}\varphi + \nabla\bar{u}^h(t)\nabla\varphi\Big]\,dx = 0 \qquad \text{for a.e. } t \in (0,T) \quad \forall\varphi \in H_0^1(\Omega).$$

We note that the above relation holds also when multiplied by any function $\tilde{\varphi} \in C([0,T])$. Hence, integrating over the time interval $(0,T)$ and using a standard density argument, we arrive at

$$\int_0^T\int_\Omega \Big[\frac{u_t^h(t) - u_t^h(t-h)}{h}\varphi + \nabla\bar{u}^h\nabla\varphi\Big]\,dx\,dt = 0 \qquad \forall\varphi \in L^2(0,T;H_0^1(\Omega)). \tag{9}$$

Now, we would like to take the time step h to zero. To be able to do so, uniform estimate on the approximate solutions is needed.

Lemma 3.1. For $h \le 1$, the following energy estimate holds:

$$\|u_t^h(t)\|_{L^2(\Omega)} + \|\nabla\bar{u}^h(t)\|_{L^2(\Omega)} \le C_E \qquad \text{for a.e. } t \in (0,T), \tag{10}$$

where constant C_E depends on H^1-norms of the initial data but is independent of h.

We omit the details of the proof which simply sets $\varphi := u_n - u_{n-1}$ in (8) and obtains the desired estimate by summing over n and applying Young's inequality. Basic calculations lead also to the following estimates:

$$\|\bar{u}^h(t) - u^h(t)\|_{L^2(\Omega)} \le h\|u_t^h(t)\|_{L^2(\Omega)} \qquad \text{for a.e. } t \in (0,T),$$
$$\|u^h\|^2_{L^2(Q_T)} \le \|\bar{u}^h\|^2_{L^2(Q_T)} + \frac{h}{2}\|u_0\|^2_{L^2(\Omega)},$$
$$\|\nabla u^h\|^2_{L^2(Q_T)} \le \|\nabla \bar{u}^h\|^2_{L^2(Q_T)} + \frac{h}{2}\|\nabla u_0\|^2_{L^2(\Omega)}.$$

These estimates together with standard compactness arguments yield a common subsequence of $\{u^h\}$ and $\overline{u}^h$ and a function $u \in H^1(Q_T)$ such that

$$\begin{aligned} \bar{u}^h &\to u && \text{strongly in } L^2(Q_T), \\ \nabla \bar{u}^h &\rightharpoonup \nabla u && \text{weakly in } (L^2(Q_T))^d, \\ u_t^h &\rightharpoonup u_t && \text{weakly in } L^2(Q_T). \end{aligned}$$

Now, we can pass to limit in (9) as $h \to 0+$ to obtain

$$\int_0^T \int_\Omega (-u_t\varphi_t + \nabla u \nabla \varphi)\, dx\, dt - \int_\Omega v_0\varphi(0,x)\, dx = 0 \quad \forall \varphi \in C_0^\infty([0,T)\times\Omega). \tag{11}$$

As the space of functions from $H^1(Q_T)$ with zero trace on $(\{0\}\times\Omega)\cup([0,T]\times\partial\Omega)$ is a closed linear subspace of $H^1(Q_T)$ and, therefore, weakly closed by Mazur's theorem, we conclude that u satisfies initial and boundary conditions in the sense of traces (since u also belongs to $C([0,T];L^2(\Omega))$, the initial condition is satisfied even in the strong sense).

To summarize, we have proved through the discrete Morse flow approximations that there exists a weak solution $u \in H^1(Q_T)$ to problem (5) in the sense of Definition 2.1.

3.2 Application to constrained free boundary problem

Next, we apply the method described above to a nontrivial volume-constrained free boundary problem asking for stationary points of the functional

$$J_\varepsilon[u] = \int_0^T \int_\Omega \left(u_t^2\chi_{u>0} - |\nabla u|^2 - B_\varepsilon(u)\right)\, dx\, dt \tag{12}$$

in the space

$$\mathcal{K} = \left\{u \in H_0^1(\Omega);\ u \ge 0,\ \int_\Omega u\, dx = V\right\}$$

under given initial conditions u_0, v_0.

We present only an outline of the main steps and refer to [21] for details.

1. **Formulation of approximate discrete problem.** Set $h = T/N$, define u_{-1} as above and for each $n = 1, 2, \ldots, N$, find a minimizer u_n of the functional

$$J_n[u] = \int_\Omega \frac{u - 4u_{n-1} + 2u_{n-2}}{2h^2} u\chi_{u>0}\, dx + \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \int_\Omega B_\varepsilon(u)\, dx \tag{13}$$

in the function set

$$\mathcal{K}_V = \left\{u \in H_0^1(\Omega);\ \int_\Omega u\chi_{u>0}\, dx = V\right\}.$$

2. **Existence of minimizers.** For given $u_{n-1}, u_{n-2} \in \mathcal{K}_V$, there exists a nonnegative minimizer u_n of the functional J_n in $\mathcal{K}_V$. This follows from boundedness from below and lower semicontinuity of J_n.

3. **Energy estimate.** There is a constant C_E independent of n and h, so that minimizers u_n satisfy
$$\Big\|\frac{u_n - u_{n-1}}{h}\Big\|^2_{L^2(\Omega)} + \|\nabla u_n\|^2_{L^2(\Omega)} \le C_E. \tag{14}$$
This is derived by passing to the limit as $\theta \to 0+$ in
$$0 \le \frac{1}{\theta}\big(J_n[(1-\theta)u_n + \theta u_{n-1}] - J_n[u_n]\big),$$
summing over n and applying discrete Gronwall's lemma.

4. **Subsolution property.** Any nonnegative minimizer u of J_n satisfies the following inequality for arbitrary nonnegative function $\zeta \in C_0^\infty(\Omega)$:
$$\begin{aligned}&\int_\Omega \Big\{\frac{u - 2u_{n-1} + u_{n-2}}{h^2}\chi_{u>0}\zeta + \nabla u\nabla\zeta + \beta_\varepsilon(u)\zeta\Big\}\,dx \\ &-\Big(\int_\Omega \Big\{\frac{u - 2u_{n-1} + u_{n-2}}{h^2}u + |\nabla u|^2 + \beta_\varepsilon(u)u\Big\}\,dx\Big)\Big(\int_\Omega \zeta\,dx\Big) \le 0.\end{aligned} \tag{15}$$
This is proved by considering $J_n[\psi_\delta] - J_n[u] \ge 0$ for $\psi_\delta = \frac{u-\delta\zeta}{I_\delta}\chi_{u-\delta\zeta>0}$ (here I_δ is chosen so that the volume constraint is satisfied), performing several technical estimates and finally taking $\delta \to 0+$.

5. **L^∞-boundedness.** If $u_0, v_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ then minimizers fulfil the bound $\|u_n\|_{L^\infty(\Omega)} \le M < \infty$, where M depends on h.
The property is proven by a standard elliptic technique of [12] but we have to consider a test function belonging to $\mathcal{K}_V$, i.e.,
$$\psi_\delta(u) = \frac{u - \delta(u-k)^+}{\int_\Omega (u - \delta(u-k)^+)\,dx}, \qquad \delta \to 0+,$$
where $(u-k)^+ = \max\{u-k, 0\}$.

6. **Hölder continuity.** For any compact subset $\tilde\Omega \subset\subset \Omega$, there exists a positive constant α (depending on h) with $0 < \alpha < 1$, such that minimizers u_n belong to $C^\alpha(\tilde\Omega)$.
This is achieved by showing that minimizer belongs to the space $\mathcal{B}_2(\Omega, M, \gamma, d)$ introduced by de Giorgi, which by the results in [12] implies the desired regularity.

7. **Euler-Lagrange equation.** By use of the above continuity result, we can choose the support of test functions in the set $\{u_n > 0\}$. Then the first variation formula of $J_n[u]$ is
$$\begin{aligned}&\int_\Omega \Big(\frac{u_n - 2u_{n-1} + u_{n-2}}{h^2}\phi + \nabla u_n\nabla\phi + \beta_\varepsilon(u_n)\phi\Big)\,dx = \int_\Omega \lambda_n\phi\,dx \\ &\qquad\qquad \forall\phi \in C_0^\infty(\Omega \cap \{u_n > 0\}), \\ &\int_\Omega \nabla u_n\nabla\phi\,dx = 0 \qquad \forall\phi \in C_0^\infty(\Omega \cap \{u_n \le 0\}^\circ).\end{aligned}$$

Here,

$$\lambda_n = \frac{1}{V}\int_\Omega \Big(\frac{u_n - 2u_{n-1} + u_{n-2}}{h^2}u_n + |\nabla u_n|^2 + \beta_\varepsilon(u_n)u_n\Big)\,dx$$

is the Lagrange multiplier coming from the volume constraint. From the second identity, we can conclude that $u_n \equiv 0$ outside the set $\{u_n > 0\}$.

8. **Approximate weak solution.** The interpolated functions u^h satisfy

$$\int_h^T \int_\Omega \Big(\frac{u_t^h(t) - u_t^h(t-h)}{h}\phi + \nabla \bar{u}^h \nabla\phi + \beta_\varepsilon(\bar{u}^h)\phi\Big)\,dx\,dt = \int_h^T \int_\Omega \bar{\lambda}^h \phi\,dx\,dt, \qquad \forall \phi \in C_0^\infty([0,T)\times\Omega \cap \{u^h > 0\}), \tag{16}$$
$$u^h \equiv 0 \quad \text{in} \quad Q_T \setminus \{u^h > 0\},$$

and the initial conditions $u^h(0) = u_0$, $u^h(-h) = u_0 - hv_0$.

9. **Convergent subsequence.** Rewritten in terms of $\bar{u}^h$ and u^h, the energy estimate (14) becomes

$$\|u_t^h(t)\|^2_{L^2(\Omega)} + \|\nabla \bar{u}^h(t)\|^2_{L^2(\Omega)} \le C \qquad \text{for a.e. } t \in (0,T). \tag{17}$$

This allows for extracting a convergent subsequence in certain topology.

10. **Existence of weak solution in one dimension.** We call $u \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega))$ a weak solution if $u(0,x) = u_0(x)$ in Ω and u satisfies

$$\int_0^T \int_\Omega (-u_t\phi_t + \nabla u \nabla\phi + \beta_\varepsilon(u)\phi)\,dx\,dt - \int_\Omega v_0\phi(0,x)\,dx \tag{18}$$
$$= \frac{1}{V}\int_0^T \int_\Omega \big(-u_t(u\Phi)_t + |\nabla u|^2\Phi + \beta_\varepsilon(u)u\Phi\big)\,dx\,dt - \frac{1}{V}\int_\Omega u_0 v_0 \Phi(0)\,dx$$
$$\forall \phi \in C_0^\infty([0,T)\times\Omega \cap \{u > 0\}),$$
$$u \equiv 0 \text{ outside } \{u > 0\}, \tag{19}$$

where $\Phi(t) = \int_\Omega \phi(t,x)\,dx$.
Unfortunately, the bound (17) is not sufficient to pass to the limit $h \to 0$ in (16) since $\operatorname{supp}\phi \subset \{u > 0\}$ does not imply $\operatorname{supp}\phi \subset \{u^h > 0\}$ for small h. However, this property can be proved if $\dim(\Omega) = 1$ by extracting a uniformly convergent subsequence based on Arzelà-Ascoli theorem. This makes the limit $h \to 0$ possible and leads to a weak solution in accordance with the above definition.

On the other hand, note that to obtain a weak solution in the sense of Definition 2.2 is possible independently of spatial dimension of the problem because the energy estimate (17) in principle contains necessary information to take $h \to 0$ in the approximate subsolution inequality (15).

4 Energy preserving approximations

Energy estimates (10) and (17) enabled us to extract a convergent subsequence, leading to a reasonable candidate for solution. On the other hand, these estimates are unsatisfactory in the sense that when no dissipative terms are present, the solution should conserve its energy, hence an "energy equality" is expected to hold. This is essential in two ways. First, numerical solutions obtained via discrete Morse flow approximation excessively dissipate energy, which is undesirable in applications. Second, if energy preservation is known, one might be able to make progress in theoretical analysis of the free boundary problems in terms of existence and uniqueness of energy-preserving solutions.

We have addressed this problem in [1] for a free boundary problem but here we explain the main idea on the simple example of linear wave equation. The key point is to modify the discrete functional as follows:

$$J_n^{CN}[u] := \int_\Omega \frac{|u-2u_{n-1}+u_{n-2}|^2}{2h^2}\,dx + \frac{1}{4}\int_\Omega |\nabla u + \nabla u_{n-2}|^2\,dx. \tag{20}$$

As in the standard discrete Morse flow, we can prove that minimizers exist and satisfy

$$\int_\Omega \frac{u_n-2u_{n-1}+u_{n-2}}{h^2}\phi + \nabla\Big(\frac{u_n+u_{n-2}}{2}\Big)\cdot\nabla\phi\,dx = 0$$

for any $\phi \in H_0^1(\Omega)$. This is the weak form of the discretized modified wave equation

$$\frac{u-2u_{n-1}+u_{n-2}}{h^2} = \Delta\left(\frac{u+u_{n-2}}{2}\right),$$

which is why we call the functional (20) "of Crank-Nicolson type". This functional enjoyes the following energy conservation property.

Theorem 4.1. The following quantity is independent of $k \geq 1$:

$$E_k := \left\|\frac{u_k-u_{k-1}}{h}\right\|^2_{L^2(\Omega)} + \frac{1}{2}(\|\nabla u_k\|^2_{L^2(\Omega)} + \|\nabla u_{k-1}\|^2_{L^2(\Omega)}).$$

Proof. For $n = 2, ..., k$, since

$$0 = \frac{d}{d\theta}J_n^{CN}[u_n+\theta(u_{n-2}-u_n)]\Big|_{\theta=0},$$

after calculation, we have

$$0 = \int_\Omega \Big[\frac{(u_{n-1}-u_{n-2})^2-(u_n-u_{n-1})^2}{h^2} + \frac{1}{2}|\nabla u_{n-2}|^2 - \frac{1}{2}|\nabla u_n|^2\Big]\,dx.$$

Summing over $n = 2, ..., k$, we arrive at

$$\int_\Omega \Big[\frac{1}{h^2}(u_1-u_0)^2 - \frac{1}{h^2}(u_k-u_{k-1})^2 + \frac{1}{2}|\nabla u_0|^2 + \frac{1}{2}|\nabla u_1|^2 - \frac{1}{2}|\nabla u_{k-1}|^2 - \frac{1}{2}|\nabla u_k|^2\Big]\,dx = 0,$$

which means

$$\begin{aligned} E_1 &= \int_\Omega \Big[\frac{1}{h^2}(u_1-u_0)^2 + \frac{1}{2}|\nabla u_0|^2 + \frac{1}{2}|\nabla u_1|^2\Big]\,dx \\ &= \int_\Omega \Big[\frac{1}{h^2}(u_k-u_{k-1})^2 + \frac{1}{2}|\nabla u_{k-1}|^2 + \frac{1}{2}|\nabla u_k|^2\Big]\,dx = E_k, \end{aligned}$$

completing the proof.

The quantity E_k corresponds the time-discretized variant of the energy $\int_\Omega \left(u_t^2 + |\nabla u|^2\right)\, dx$ and hence the modified scheme preserves energy at the discrete level. This was essentially employed in a practical application to printers [2]. However, exploiting this useful property theoretically remains as our next research goal.

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A free boundary problem of reaction diffusion equation with a multi-stable type nonlinearity in high space dimensions

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This talk is based on joint works [14, 15] with Dr.Yuki Kaneko(Japan Women's University) and Professor Yoshio Yamada(Waseda University). In this talk we consider the following free boundary problem of a reaction diffusion equation:

$$\begin{cases} u_t = \Delta u + f(u), & t > 0,\ 0 < r < h(t), \\ u_r(t,0) = u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_r(t,h(t)), & t > 0, \\ h(0) = h_0,\ u(0,r) = u_0(r), & 0 \le r \le h_0, \end{cases} \tag{1}$$

where $r = |x|$ for $x \in \mathbb{R}^N$ $N > 1$, $r = h(t)$ denotes a free boundary and is to be determined together with $u(t,r)$, $\Delta u = u_{rr} + \frac{N-1}{r} u_r$, μ and h_0 are given positive constants. Nonlinearity f is a C^1 function satisfying

$$f(0) = 0,\ f(K) = 0 \text{ and } f(u) < 0 \text{ for } u > K \tag{2}$$

for some positive number $K > 0$. For any given $h_0 > 0$, u_0 belongs to $\mathscr{H}(h_0)$, where

$$\mathscr{H}(h_0) := \{\phi \in C^2([0,h_0]) : \phi'(0) = \phi(h_0) = 0,\ \phi(r) > 0 \text{ in } [0,h_0)\}.$$

For any given $h_0 > 0$ and $u_0 \in \mathscr{H}(h_0)$ we call a pair $(u(t,r), h(t))$ is a classical solution of (1) on the time interval $[0,T]$ if $h \in C^1([0,T])$, $u \in C^{1,2}(D_T)$ and (u,h) satisfies all the identities in (1) pointwisely, where

$$D_T := \{(t,r) : t \in (0,T], r \in [0,h(t)]\}.$$

The existence, uniqueness and some properties of solutions were obtained by [3] and [7] and are summarized in the following theorem.

Theorem 1 ([7]). *Suppose that f satisfies (2) and let $h_0 > 0$ and $u_0 \in \mathscr{H}(h_0)$. Then for any $\alpha \in (0,1)$, problem (1) admits a unique solution $(u,h) = (u(t,r), h(t))$ defined for all $t > 0$ satisfying*

$$u \in C^{\frac{1+\alpha}{2}, 1+\alpha}(\overline{\Omega}) \cap C^{1+\frac{\alpha}{2}, 2+\alpha}(\Omega),\ h \in C^{1+\frac{\alpha}{2}}([0,\infty)),$$

where $\Omega = \{(t,r) : t \in (0,\infty), r \in [0,h(t)]\}$. Moreover (u,h) satisfies the following properties:

(i) *There exist positive constants C_1 and C_2, which depend only on the initial data, such that*

$$0 < u(t,r) \le C_1 \quad \text{and} \quad 0 < h'(t) \le \mu C_2 \quad \text{for all} \quad t > 0 \quad \text{and} \quad r \in [0, h(t))$$

(ii) $u_r(t,r) < 0$ *for all* $t > 0$ *and* $r \in [h_0, h(t)]$.

Theorem 2 ([7]). *Let $(u(t,r), h(t))$ be any solution to (1) and let $h_\infty := \lim_{t\to\infty} h(t) \in (0,\infty]$. Either $h_\infty < \infty$ and $\lim_{t\to\infty} \|u(t,\cdot)\|_{L^\infty([0,h(t)])} = 0$; or $h_\infty = \infty$ and $\lim_{t\to\infty} u(t,r) = v(r)$ locally uniformly for $r \in [0,\infty)$ holds, where $v(r)$ satisfies*

$$v'' + \frac{N-1}{r} v' + f(v) = 0 \text{ for } \ r > 0, \quad v'(0) = 0 \tag{3}$$

and either v is a constant function or $v'(r) < 0$ for $r \ge h_0$; in the former case, the constant is necessarily a nonnegative zero of f.

This problem was introduced first by Du and Lin [5], when $N = 1$ and $f(u) = u(a - bu)$ (a, $b > 0$), as a population model which describes the spreading of a new or invasive species. They showed that, as $t \to \infty$, either spreading ($h(t) \to \infty$, $u(t,\cdot) \to a/b$ locally uniformly in $[0,\infty)$) or vanishing ($h(t) \to h_\infty < \infty$, $\|u(t,\cdot)\|_{C([0,h(t)])} \to 0$) occurs. This result is called the spreading-vanishing dichotomy. Since their work, this result have been extended by lots of researches (see for example [6, 10, 12, 17]). For $N \ge 2$, Du and Guo [3] considered logistic type nonlinearity, Du, Lou and Zhou [7] studied (1) with quite general nonlinearities, in particular, monostable and bistable nonlinearities. See [11] for more studies of (1) with $N \ge 2$

In this talk, we consider f satisfying the following conditions:

(F1) $f \in C^1([0,\infty))$, $f(0) = f(u_1^*) = f(u_2^*) = f(u_3^*) = 0$ with $0 < u_1^* < u_2^* < u_3^*$, $f'(0) > 0$, $f'(u_1^*) < 0$ and $f'(u_3^*) < 0$;

(F2) $f(u) > 0$ for $u \in (0,u_1^*) \cup (u_2^*, u_3^*)$, $f(u) < 0$ for $u \in (u_1^*, u_2^*) \cup (u_3^*, \infty)$ and $\displaystyle\int_{u_1^*}^{u_3^*} f(u)\,du > 0$;

(F3) $f(u)/(u - \overline{u}_2)$ is non-increasing for $u \in (\overline{u}_2, u_3^*)$, where $\overline{u}_2 \in (u_2^*, u_3^*)$ is a uniquely determined number by $\displaystyle\int_{u_1^*}^{\overline{u}_2} f(u)\,du = 0$;

(F4) $\displaystyle\lim_{u \searrow u_2^*} f(u)/(u - u_2^*)^\kappa \in (0,\infty]$ for some $\kappa \in (0,\infty)$ if $N = 2$, and $\kappa = N/(N-2)$ if $N > 2$.

The nonlinearity which satisfies (F1)–(F4) is one of *positive bistable nonlinearity*. The typical example of f satisfying (F1) and (F2) is given by

$$f(u) = ku\left(1 - \frac{u}{q}\right) - \frac{u^2}{1+u^2}$$

with positive parameters k and q being in certain parameter range. For $N \geq 2$ we impose additional condition (F3) and (F4) (see the paragraph followed by Theorem A).

When $N = 1$ and f satisfies (F1), (F2) and $f'(u_2^*) > 0$, Kawai and Yamada [18] proved that asymptotic behaviors of solutions are classified into four cases, in particular, they discovered multiple spreading phenomenon corresponding to stable equilibrium points of f. Here we remark that $f|_{[0,u_1^*]}$ is a monostable nonlinearity while $f|_{[u_1^*,u_3^*]}$ is a bistable nonlinearity.

First result of this talk is about classification of the asymptotic behavior of solutions.

Theorem A. *Suppose that f satisfies* (F1)–(F4). *Let $(u(t,r), h(t))$ be any solution to* (1). *Then exactly one of the following four cases occurs:*

(i) **Vanishing :** $\lim_{t\to\infty} h(t) < \sqrt{\lambda_1/f'(0)}$ *and* $\lim_{t\to\infty} \|u(t,\cdot)\|_{C([0,h(t)])} = 0$, *where λ_1 is the first eigenvalue to the following eigenvalue problem with usual Laplacian Δ:*

$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } B_1(0) := \{x \in \mathbb{R}^N : |x| < 1\}, \\ \varphi = 0 & \text{on } \partial B_1(0), \end{cases}$$

where $B_\rho(0) = \{x \in R^N : |x| < \rho\}$ and $\partial B_\rho(0)$ denotes the boundary of $B_\rho(0)$;

(ii) **Small spreading :** $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,\cdot) = u_1^*$ *uniformly on* $[0,R]$ *for any $R > 0$;*

(iii) **Big spreading:** $\lim_{t\to\infty} h(t) < \infty$ *and* $\lim_{t\to\infty} u(t,\cdot) = u_3^*$ *uniformly on* $[0,R]$ *for any $R > 0$;*

(iv) **Transition :** $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,\cdot) = V_{\rm dec}$ *uniformly on* $[0,R]$ *for any $R > 0$, where $V_{\rm dec} = V_{\rm dec}(r)$ is a unique positive decreasing function satisfying*

$$V'' + \frac{N-1}{r}V + f(V) = 0 \quad in \quad (0,\infty), \quad V'(0) = 0 \quad and \quad \lim_{r\to\infty} V(r) = u_1^*.$$

Moreover, if (u_σ, h_σ) is the solution to (1) *for $u_0 = \sigma\phi$ with $\sigma > 0$ and any fixed $\phi \in \mathscr{H}(h_0)$, then there exist two threshold numbers σ_1^* and σ_2^* such that (u_σ, h_σ) satisfies the following: For $\sigma \in (\sigma_2^*, \infty)$ the big spreading occurs, for $\sigma \in (\sigma_1^*, \sigma_2^*)$ the small spreading occurs, for $\sigma = \sigma_2^*$ the transition occurs and for $\sigma \in [0, \sigma_1^*]$ the vanishing occurs.*

To classify the asymptotic behavior of solutions to (1) it is important to investigate corresponding stationary problem. When $N = 1$ we can use a phase plane analysis to study the stationary problem. However when $N \geq 2$ we can not use this method. Conditions (F3) and (F4) are used to guarantee the uniqueness of the *ground state solution* $V_{\rm dec}$ (see [19, 20]). We also note that $f'(u_2^*) > 0$ implies (F4).

We next give a result about the spreading speed of the free boundary, when the small spreading, big spreading or the transition occurs. To investigate the spreading speed of the free boundary, we have to consider the following *semi-wave problem*:

$$(\mathrm{SWP})_{u^*} \begin{cases} q_{zz} - cq_z + f(q) = 0, \ q(z) > 0 \ \text{ for } \ z > 0. \\ q(0) = 0, \ \mu q_z(0) = c, \ \lim_{z\to\infty} q(z) = u^*, \end{cases}$$

where $u^* \in \mathbb{R}$ is a positive and stable equilibrium of f. Since $f|_{[0,u_1^*]}$ is a monostable nonlinearity, it follows from [6, Proposition 1.9] that $(\mathrm{SWP})_{u_1^*}$ admits a unique solution pair $(c_{\mathrm{S}}, q_{\mathrm{S}})$ for any $\mu > 0$. Moreover, $c_{\mathrm{S}} = c_{\mathrm{S}}(\mu)$ is monotone increasing in μ, $\lim_{\mu\to 0} c_{\mathrm{S}}(\mu) = 0$ and $\lim_{\mu\to\infty} c_{\mathrm{S}}(\mu) = c_0^{\mathrm{S}}$ holds, where c_0^{S} is the minimal speed of traveling waves determined by $f|_{[0,u_1^*]}$.

The solvability of $(\mathrm{SWP})_{u_3^*}$ is delicate and it was shown in [18] that one of the following two cases holds true:

(Case A) For every $\mu > 0$, there exists a unique solution pair $(c, q) = (c_{\mathrm{B}}, q_{\mathrm{B}})$ to $(\mathrm{SWP})_{u_3^*}$.

(Case B) There exists a positive number μ^* such that $(\mathrm{SWP})_{u_3^*}$ admits a unique solution $(c, q) = (c_{\mathrm{B}}, q_{\mathrm{B}})$ for $\mu \in (0, \mu^*)$ and no solution for every $\mu \in [\mu^*, \infty)$.

It was also shown in [18] that if $(c_{\mathrm{B}}, q_{\mathrm{B}})$ exists, then $c_{\mathrm{S}} < c_{\mathrm{B}}$ holds. To study when Case B occurs for $(\mathrm{SWP})_{u_3^*}$, we recall results for the following *traveling wave problem*:

$$(\mathrm{TWP})_{[u_*,u^*]} \begin{cases} \mathcal{Q}_{zz} - c\mathcal{Q}_z + f(\mathcal{Q}) = 0 \quad \text{for} \quad -\infty < z < \infty, \\ \lim\limits_{z\to-\infty} \mathcal{Q}(z) = u_*, \ \mathcal{Q}(0) = (u_* + u^*)/2, \ \lim\limits_{z\to\infty} \mathcal{Q}(z) = u^*, \end{cases}$$

where u_*, $u^* \in \{0, u_1^*, u_3^*\}$ with $u_* < u^*$.

Since $f|_{[0,u_1^*]}$ is monostable and $f|_{[u_1^*,u_3^*]}$ is bistable, classical works of Aronson and Weinberger [1, 2] tell us

- there exists a unique positive number c_0^{S} such that $(\mathrm{TWP})_{[0,u_1^*]}$ has a unique solution if and only if $c \geq c_0^{\mathrm{S}}$
- there exists a unique positive number c_1^{B} such that $(\mathrm{TWP})_{[u_1^*,u_3^*]}$ has a unique solution if and only if $c = c_1^{\mathrm{B}}$.

We denote by $(c_1^{\mathrm{B}}, \mathcal{Q}_1^{\mathrm{B}})$ the unique solution pair to $(\mathrm{TWP})_{[u_1^*,u_3^*]}$.

By the phase plane analysis of Kawai and Yamada [18], one can obtain a necessary and sufficient condition for Case B by using above positive numbers c_0^{S} and c_1^{B}.

Proposition 3. *Case B occurs if and only if $c_1^{\mathrm{B}} < c_0^{\mathrm{S}}$. Moreover threshold number μ^* in Case B is uniquely determined by $c_{\mathrm{S}}(\mu^*) = c_1^{\mathrm{B}}$ and it holds that*

$$c_{\mathrm{S}}(\mu) > c_1^{\mathrm{B}} \quad \textit{for} \quad \mu > \mu^*, \tag{4}$$

where $c_{\mathrm{S}}(\mu)$ is the speed of semi-wave q_{S} and determined by $(\mathrm{SWP})_{u_1^*}$.

Next theorem is about asymptotic speed of $h(t)$ for the solutions which satisfies $\lim_{t\to\infty} h(t) = \infty$.

Theorem B. *Suppose that f satisfies* (F1)–(F4) *and let (u, h) be the solution.*

(i) *If the small spreading occurs for (u, h), then*

$$\lim_{t\to\infty} \frac{h(t)}{t} = c_{\mathrm{S}},$$

where c_{S} is the number determined by unique solution pair to $(\mathrm{SWP})_{u_1^*}$.

(ii) *If the transition occurs for* (u,h)*,*

$$\lim_{t\to\infty}\frac{h(t)}{t}=c_{\rm S}.$$

(iii) *If the big spreading occurs for* (u,h)*, then the following properties holds true :*
In Case A,

$$\lim_{t\to\infty}\frac{h(t)}{t}=c_{\rm B},$$

and in Case B,

$$\lim_{t\to\infty}\frac{h(t)}{t}=\begin{cases} c_{\rm B} & \text{when } \mu<\mu^*,\\ c_{\rm S} & \text{when } \mu\geq\mu^*,\end{cases}$$

where $c_{\rm B}$ *is the number determined by unique solution pair to* $(\mathrm{SWP})_{u_3^*}$ *and* μ^* *is the number given in Case B(or Proposition 3).*

Finally I will consider asymptotic profile of $u(t,\,\cdot\,)$ on $[0,h(t)]$ and sharp estimates of $h(t)$ as $t\to\infty$ for small and big spreading solutions For the case $N=1$, the asymptotic profiles of such spreading solutions have been established in [10, 16, 13]. However, as far as I know, for the case $N>1$, the asymptotic profile of spreading solution was studied only in the work of Du, Matsuzawa and Zhou [11] where problem (1) with a monostable, bistable or combustion type nonlinearity was considered. Moreover, in [11], a striking difference between the case $N=1$ and the case $N>1$ was shown; in particular, in the case $N>1$ a logarithmic term appears in a sharp estimate of $h(t)$ and u converges to the semi-wave with a logarithmic shifting for spreading solutions (u,h) as $t\to\infty$. Such a logarithmic shifting was discovered by Uchiyama [21] in the corresponding Cauchy problem.

Once the solvability of $(\mathrm{SWP})_{u_1^*}$ and $(\mathrm{SWP})_{u_3^*}$ has been established, one can follow the arguments in [11] to obtain the following result.

Proposition 4. *Suppose that* f *satisfies* (F1)–(F4)*. The following properties hold true:*

(i) *If* (u,h) *is any small spreading solution, then there exists* $R_{\rm S}\in\mathbb{R}$ *such that*

$$\lim_{t\to\infty}\left[h(t)-\{c_{\rm S}t-(N-1)c_{{\rm S},*}\log t\}\right]=R_{\rm S},\quad \lim_{t\to\infty}h'(t)=c_{\rm S} \tag{5}$$

and

$$\lim_{t\to\infty}\sup_{r\in[0,h(t)]}|u(t,r)-q_{\rm S}(h(t)-r)|=0,$$

where $(c_{\rm S},q_{\rm S})$ *is the unique solution pair to* $(\mathrm{SWP})_{u_1^*}$ *and* $c_{{\rm S},*}=c_{{\rm S},*}(\mu)$ *is given by*

$$c_{{\rm S},*}=c_{{\rm S},*}(\mu)=\frac{1}{c_{\rm S}}\left\{1+\frac{c_{\rm S}}{\mu^2\int_0^\infty q_{\rm S}'(z)^2e^{-c_{\rm S}z}dz}\right\}^{-1}.$$

(ii) *Suppose that* $(\mathrm{SWP})_{u_3^*}$ *has a unique solution pair* $(c_{\mathrm{B}}, q_{\mathrm{B}})$. *If* (u,h) *is any big spreading solution, then there exists* $R_{\mathrm{B}} \in \mathbb{R}$ *such that*

$$\lim_{t\to\infty}[h(t) - \{c_{\mathrm{B}}t - (N-1)c_{\mathrm{B},*}\log t\}] = R_{\mathrm{B}}, \quad \lim_{t\to\infty} h'(t) = c_{\mathrm{B}} \tag{6}$$

and

$$\lim_{t\to\infty}\sup_{r\in[0,h(t)]}|u(t,r) - q_{\mathrm{B}}(h(t)-r)| = 0.$$

where $c_{\mathrm{B},*} = c_{\mathrm{B},*}(\mu)$ *is given by*

$$c_{\mathrm{B},*} = c_{\mathrm{B},*}(\mu) = \frac{1}{c_{\mathrm{B}}}\left\{1 + \frac{c_{\mathrm{B}}}{\mu^2\int_0^\infty q_{\mathrm{B}}'(z)^2 e^{-c_{\mathrm{B}}z}dz}\right\}^{-1}.$$

Proposition 4 tells us when $(\mathrm{SWP})_{u^*}$ has the solution pair, asymptotic profiles of small spreading solutions and big spreading solutions are described by using the corresponding semi-waves. Moreover when $N > 1$, sharp estimates of $h(t)$ in (5) and (6) contain a logarithmic term. The appearance of such a logarithmic term is called *logarithmic shifting*.

For the case where $(\mathrm{SWP})_{u_3^*}$ does not have solution, the profile of u for which the big spreading occurs has been unclear. If $N = 1$ we proved in [13] that the big spreading solution approaches a so-called propagating terrace connecting 0 and u_3^*.

The third main theorem is about purpose of the a sharp estimate of $h(t)$ and a sharp asymptotic profile of u over the interval $[0, h(t)]$ for a big spreading solution (u,h) when $(\mathrm{SWP})_{u_3^*}$ has no solution. However, it is necessary to determine the exact coefficients of all logarithmic shiftings which appear in the propagating terrace. In fact we intend to show that u approaches a propagating terrace which consists of a semi-wave and a traveling wave with logarithmic shifts by determining all the coefficients of the logarithmic shiftings.

Theorem C. *Assume that* f *satisfies Case B in addition to* (F1)–(F4). *Let* μ *satisfy* (4) *and let* (u,h) *be any big spreading solution of* (1). *Then there exists* $R_{\mathrm{S}} \in \mathbb{R}$ *such that*

$$\lim_{t\to\infty}[h(t) - \{c_{\mathrm{S}}t - (N-1)c_{\mathrm{S},*}\log t\}] = R_{\mathrm{S}}, \quad \lim_{t\to\infty} h'(t) = c_{\mathrm{S}}. \tag{7}$$

Moreover, there exists $R_{\mathrm{B}} \in \mathbb{R}$ *and* $L_* > \max\{(N-1)c_{\mathrm{S},*}, (N-1)/c_1^{\mathrm{B}}\}$ *such that for any* $L \geq L_*$

$$\lim_{t\to\infty}\sup_{r\in[c_{\mathrm{S}}t - L\log t, h(t)]}|u(t,r) - q_{\mathrm{S}}(h(t)-r)| = 0 \tag{8}$$

and

$$\lim_{t\to\infty}\sup_{r\in[0,c_{\mathrm{S}}t - L\log t]}\left|u(t,r) - \mathcal{Q}_1^{\mathrm{B}}\left(c_1^{\mathrm{B}}t - \frac{N-1}{c_1^{\mathrm{B}}}\log t + R_{\mathrm{B}} - r\right)\right| = 0. \tag{9}$$

Here $(c_{\mathrm{S}}, q_{\mathrm{S}})$ *is the unique solution pair to* $(\mathrm{SWP})_{u_1^*}$, $c_{\mathrm{S},*} = c_{\mathrm{S},*}(\mu)$ *is given by*

$$c_{\mathrm{S},*} = c_{\mathrm{S},*}(\mu) = \frac{1}{c_{\mathrm{S}}}\left\{1 + \frac{c_{\mathrm{S}}}{\mu^2\int_0^\infty q_{\mathrm{S}}'(z)^2 e^{-c_{\mathrm{S}}z}dz}\right\}^{-1}$$

and $(c_1^{\mathrm{B}}, \mathcal{Q}_1^{\mathrm{B}})$ *is the unique solution pair to* $(\mathrm{TWP})_{[u_1^*, u_3^*]}$.

This theorem means that if (4) holds and (u,h) is any big spreading solution, then u converges to the propagating terrace connecting 0 and u_3^* which consists of semi-wave $q_{\rm S}$ with speed $c_{\rm S}$ and traveling wave $\mathcal{Q}_1^{\rm B}$ with speed $c_1^{\rm B}$ and accompanies logarithmic shiftings in both waves. We call the solution (u,h) to (1) which satisfies (7), (8) and (9), a *radial terrace solution* to a general free boundary problem given later in (10)(See Du and Matano [8] where the notion of a radial terrace solution is introduced for the Cauchy problem). Moreover we should remark that the radial terrace solution in this theorem has two kinds of logarithmic shiftings. In fact, the logarithmic term in (8) with (7) coincides with the one for free boundary problem (see [11]) and the logarithmic shifting in (9) is the same as the one which appears in the Cauchy problem (see [21]).

Here we should remark that problem (1) is a radially symmetric setting of the following general free boundary problem:

$$\begin{cases} v_t = \displaystyle\sum_{i=1}^{N} v_{x_i x_i} + f(v), & t>0,\ x\in\Omega(t),\\ v=0,\ v_t=\mu|\nabla_x v|^2, & t>0,\ x\in\Gamma(t),\\ v(0,x)=v_0(x), & x\in\Omega_0, \end{cases} \tag{10}$$

where $\Omega(t)\subset\mathbb{R}^N$ is bounded by the free boundary $\Gamma(t)(=\partial\Omega(t))$ with $\Omega(0)=\Omega_0$, which is a bounded domain with smooth boundary $\partial\Omega_0$. The relation between (1) and (10) is given by $v_0(x)=u_0(|x|)$, $v(t,x)=u(t,|x|)$ with $\Omega(t)=\{x\in\mathbb{R}^N : |x|<h(t)\}$. Du and Guo [4] defined a weak solution to (10) and proved that (10) has a unique weak solution $v(t,x)$ defined for all $t>0$.

Du, Matano and Wang [9] obtained the regularity of the free boundary $\Gamma(t)$ and the following general result concerning asymptotic behavior of $\Omega(t)$.

Theorem 5 ([9, Theorem 1.2])**.** *$\Omega(t)$ is expanding in the sense that $\overline{\Omega_0}\subset\Omega(t)\subset\Omega(s)$ if $0<t<s$. Moreover, $\Omega_\infty:=\cup_{t>0}\Omega(t)$ is either the entire space $\mathbb{R}^N$, or it is a bounded set. Furthermore, when $\Omega_\infty=\mathbb{R}^N$, for any large t, $\Gamma(t)$ is a smooth closed hypersurface in $\mathbb{R}^N$, and there exists a continuous function $M(t)$ such that*

$$\Gamma(t)\subset\left\{x : M(t)-\frac{d_0}{2}\pi\le|x|\le M(t)\right\};$$

and when Ω_∞ is bounded, $\lim_{t\to\infty}\|v(t,\cdot)\|_{L^\infty(\Omega(t))}=0$, where $d_0=\mathrm{diam}(\Omega_0)$.

Theorem 5 suggests that problem (10) can be reduced to a simpler situation, that is, radially symmetric setting. In fact, Du, Lou and Zhou [7] showed that, for the case where f is monostable or bistable nonlinearity, if spreading happens, then $M(t)$ in Theorem 5 can be estimated by using $h(t)$ determined by (1) with a suitable initial data in the following form:

$$|M(t)-h(t)|\le C \quad \text{for all large } t>0$$

for some constant $C>0$. In this sense, to obtain sharper estimates of $h(t)$ than those in Theorem B for spreading solutions is indispensable for the better understanding of large-time behaviors of solutions to (10). Problem (10) with a positive bistable nonlinearity will be discussed elsewhere.

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A current-valued solution of the Euler equation and its application

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Abstract

Besides the Euler flow, there is another mathematical model that describes 2D incompressible and inviscid fluid flows, called the point vortex dynamics, which is derived formally from the Euler flow. Since the point vortex dynamics is a finite-dimensional Hamiltonian system, it is mathematically easier to handle this than the Euler flow. For this reason, the point vortex dynamics is of importance in applications, since it is utilized as simple models for fluid phenomena with localized vortex structures. In the meantime, since the point vortex dynamics is derived formally from the Euler flow, it still remains open whether the insight gained by using the point vortex dynamics is also applicable to the Euler flow as well. Therefore, in order to solve this problem, we need to establish a space of solutions to the Euler equation that contains the pair of the fluid velocity and the pressure for the point vortex dynamics. However, since the vorticity of the point vortex dynamics is given by a linear combination of delta functions, due to the singularity of the vorticity and the nonlinearity of the Euler equation, the problem of mathematical justification is considered to be a difficult one in the analysis of the 2D Euler equation. Although there have been many studies to date on the indirect characterization of the point vortex dynamics by approximate sequences of weak solutions to the Euler equation, the problem remains open in terms of constructing a space of solutions that directly includes the point vortex dynamics. In this talk, we solve the problem in a geometric way, rather than in an analytic way.

1 Introduction

The motion of incompressible and inviscid fluids in the Euclidean plane is governed by the Euler equation,

$$\partial_t v_t + (v_t \cdot \nabla) v_t = -\operatorname{grad} p_t, \quad \operatorname{div} v_t = 0, \tag{1}$$

The fluid velocity field $v_t \in \mathfrak{X}(\mathbb{R}^2)$ and the scalar pressure $p_t \in C(\mathbb{R}^2)$ are the unknowns and a solution of the Euler equation (1) is called an Euler flow. Applying the curl operator to the Euler equation, we obtain the following equation for the vorticity $\omega_t = \operatorname{curl} v_t = \partial_1 v_t^2 - \partial_2 v_t^1$.

$$\partial_t \omega_t + (v_t \cdot \nabla) \omega_t = 0,$$

which is called the vorticity equation. Since the fluid velocity is divergence free, the velocity becomes a Hamiltonian vector field, i.e., $v_t = -\mathcal{J}\mathrm{grad}\,\psi_t = (\partial_2\psi_t, -\partial_1\psi_t)$ for a Hamiltonian $\psi_t \in C(\mathbb{R}^2)$, where $\mathcal{J}$ is the symplectic matrix. The vorticity of the Hamiltonian vector field $v_t = -\mathcal{J}\mathrm{grad}\,\psi_t = (\partial_2\psi_t, -\partial_1\psi_t)$ satisfies the Poisson equation:

$$\omega_t = -\,\mathrm{curl}\,\mathcal{J}\mathrm{grad}\,\psi_t = -\triangle\psi_t.$$

Solving the Poisson equation, we obtain the Biot-Savart law:

$$v_t = -\,\mathcal{J}\mathrm{grad}\langle G_H, \omega_t\rangle,$$

where G_H is the Green function for the Laplacian $-\triangle$. The vorticity equation is now written only in terms of the vorticity:

$$\partial_t\omega_t + (-\,\mathcal{J}\mathrm{grad}\langle G_H, \omega_t\rangle \cdot \nabla)\omega_t = 0, \tag{2}$$

Applying the divergence operator to the Euler equation, the pressure is also determined by the Poisson equation:

$$-\triangle p_t = \mathrm{div}(v_t\cdot\nabla)v_t.$$

Hence, the fluid velocity v_t and pressure p_t are explicitly presented by the vorticity:

$$v_t = -\,\mathcal{J}\mathrm{grad}\langle G_H, \omega_t\rangle, \quad p_t = \langle G_H, \mathrm{div}(\mathcal{J}\mathrm{grad}\langle G_H, \omega_t\rangle \cdot \nabla)\,\mathcal{J}\mathrm{grad}\langle G_H, \omega_t\rangle\rangle. \tag{3}$$

As a result, solving the vorticity equation (2), we obtain an Euler flow (v_t, p_t) that is defined by (3).

Note that the formulae (3) still make sense in the sense of distributions. Let us consider the case where the vorticity Ω_t is given by a linear combination of delta functions centered at $q_n(t)$ over real coefficients $\Omega_t = \sum_{n=1}^N \Gamma_n\delta_{q_n(t)}$. Replacing ω_t by Ω_t in the equation (3), we formally obtain a velocity field $V_t \in \mathfrak{X}(\mathbb{R}^2 \setminus \{q_n(t)\}_{n=1}^N)$ and a pressure $\Pi_t \in C(\mathbb{R}^2 \setminus \{q_n(t)\}_{n=1}^N)$. However, since the pair (V_t, Π_t) is not of well-known class $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ of weak solutions [4], it is no longer an Euler flow in a regular sense, which implies that we can not determine the evolution of the vorticity. Instead, Helmholtz proposed that $q_n(t)$ be determined by the following ordinary differential equation [8].

$$\begin{aligned}\dot{q}_n &= \lim_{q\to q_n}\left[-\,\mathcal{J}\mathrm{grad}\langle G_H, \Omega_t\rangle + \mathcal{J}\mathrm{grad}\langle G_H, \Gamma_n\delta_{q_n(t)}\rangle(q)\right] \\ &= -\,\mathcal{J}\mathrm{grad}\sum_{\substack{m=1\\ m\neq n}}^N \Gamma_m G_H(q_n, q_m) \equiv v_n(q_n).\end{aligned} \tag{4}$$

The equation (4) is called the point vortex equation, and its solution is called the point vortex dynamics. By substituting ω_t for Ω_t in the sense of distributions in (3), the velocity $V_t \in \mathfrak{X}^\infty(\mathbb{R}^2 \setminus \{q_n(t)\})$ and the pressure $\Pi_t \in C^\infty(\mathbb{R}^2 \setminus \{q_n(t)\})$ for the point vortex dynamics are formally derived.

The point vortex dynamics is easier to deal with mathematically than the Euler flow, since it is a finite-dimensional dynamical system. For this reason, it is utilized as simple models for fluid phenomena with localized vortex structures [8]. Meanwhile, since point vortex dynamics is derived formally from the Euler flow, it still remains open whether the insight obtained by using

point vortex dynamics is also applicable to the Euler flow. Thus, it is of significance for applications to establish a solution space for the Euler equation that contains the pair (V_t, Π_t) of the fluid velocity and the pressure for the point vortex dynamics. For the pair (V_t, Π_t) to be an Euler flow, the time evolution of the vorticity Ω_t is to satisfy the vorticity equation. However, due to the singularity of Ω_t and the nonlinearity of the vorticity equation, the problem of mathematical justification is considered to be a difficult one in the analysis of the 2D Euler equation [4, 5, 6]. Although there have been many studies to date on the indirect characterization of point vortex dynamics by approximate sequences of weak solutions to the Euler equation, the problem remains open in terms of constructing a space of solutions that directly includes point vortex dynamics.

Point vortex dynamics is generalized to the case where point vortices are affected by a background field. The point vortex dynamics is sometimes considered in the presence of the velocity field $X_t \in \mathfrak{X}^r(\mathbb{R}^2)$ of an Euler flow, in which the evolution of $q_n(t)$ is governed by the following equation.

$$\dot{q}_n(t) = \beta_X X_t(q_n(t)) + \beta_\omega v_n(q_n(t)), \quad n = 1, \dots N$$

for a given parameter $(\beta_X, \beta_\omega) \in \mathbb{R}^2$. The velocity field X_t is called the background field. An experimental study confirm the importance of background fields in two-dimensional turbulence [10]. In the same way as in the absence of a background field, a velocity field $V_t \in \mathfrak{X}(\mathbb{R}^2 \setminus \{q_n(t)\}_{n=1}^N)$ and a pressure $\Pi_t \in C(\mathbb{R}^2 \setminus \{q_n(t)\}_{n=1}^N)$ are defined:

$$V_t(q) = X_t(q) - \mathcal{J}\mathrm{grad} \sum_{n=1}^{N} \Gamma_n G_H(q, q_n(t)), \quad P_t = \langle G_H, \mathrm{div}(V_t \cdot \nabla) V_t \rangle.$$

Even if the regularity of the background field is simply of class C^r, it remains open whether the pair is a weak solution of the Euler equation as it is open in the case with no background field.

The purpose of this paper is to justify the point vortex dynamics in the background field as an Euler flow. The problem is originated in the analysis of the 2D Euler equation, but it is solved by a geometric method in this paper. We will provide a weak formulation of the Euler equation in the space of de Rham currents, and show that the point vortex dynamics in the background field is a current-valued solution of the Euler equation. The de Rham current is a continuous linear functional defined on the space of differential forms, which is used mainly in geometric analysis and geometric measure theory. Roughly speaking, currents are locally differential forms over distribution coefficients, which are characterized as a weak extension of differential forms. We then rewrite the notions involved in the analysis of the Euler equation, such as the divergence operator and the curl operator, in terms of differential forms.

The weak formulation obtained in this procedure is applicable not only to the plane but also to curved surfaces in general. Taking into account that point vortex dynamics in a rotational field on a sphere has been adopted as a mathematical model of geophysical fluids to investigate the effect of Coriolis force on non-viscous fluid motion [7], It is of significance for applications to justify the point vortex dynamics in the background field as an Euler flow for curved surfaces. For the generalization to curved surfaces, however, we refer to the original paper [9]. In this paper, we focus on the planar case in order to provide the essence of what is discussed to the reader who is not familiar with the notions of differential geometry. Therefore, we note that the main results presented in this paper hold true for general curved surfaces as well.

This paper is organized in the following sections. In section 2, weak formulations by currents of the notions involved in the analysis of the Euler equation are presented. Since the weak formulations by currents are obtained as weak extensions of differential forms, these notions are rewritten in terms of differential forms in Section 2.1, and then their weak formulations by currents are carried out in Section 2.2. In Section 3, current-valued solutions of the Euler equation and its regular-singular decomposition are formulated. The main result is stated in Section 4, and its applications are discussed in Section 5.

2 Weak Formulation of Vector Calculus

2.1 Reformulations in Differential Forms

Let $\mathfrak{X}^r(\mathbb{R}^2)$ be the space of all vector fields on $\mathbb{R}^2$ of class C^r and $\Omega^p_{[r]}(\mathbb{R}^2)$ be the space of all p-forms on $\mathbb{R}^2$ of class C^r. We denote $\Omega^p_{[\infty]}(\mathbb{R}^2)$ briefly by $\Omega^p(\mathbb{R}^2)$. Let $\mathcal{D}^p(\mathbb{R}^2)$ be the space of all smooth p-forms with compact support.

The curl operator $\operatorname{curl} : X = (X^1, X^2) \in \mathfrak{X}^r(\mathbb{R}^2) \to \operatorname{curl} X \in C^{r-1}(\mathbb{R}^2)$ and the divergence operator $\operatorname{div} : X = (X^1, X^2) \in \mathfrak{X}^r(\mathbb{R}^2) \to \operatorname{div} X \in C^{r-1}(\mathbb{R}^2)$ are defined as follows:

$$\operatorname{curl} X = \partial_1 X^2 - \partial_2 X^1, \quad \operatorname{div} X = \partial_1 X^1 + \partial_2 X^2.$$

A vector field $X \in \mathfrak{X}^r(\mathbb{R}^2)$ is said to be incompressible (or divergence free) if $\operatorname{div} X = 0$. Given $X \in \mathfrak{X}^r(\mathbb{R}^2)$, we call the function $\omega = \operatorname{curl} X$ the vorticity. A vector field $X = (X^1, X^2) \in \mathfrak{X}^r(\mathbb{R}^2)$ is called a gradient vector field if there exists a function $\phi \in C^{r+1}(\mathbb{R}^2)$ such that $X = \operatorname{grad} \phi = (\partial_1 \phi, \partial_2 \phi)$. A vector field $X = (X^1, X^2) \in \mathfrak{X}^r(\mathbb{R}^2)$ is called a Hamiltonian vector field if there exists a function $\psi \in C^{r+1}(\mathbb{R}^2)$ such that $X = -\mathcal{J}\operatorname{grad} \psi = (\partial_2 \psi, -\partial_1 \psi)$. where $\mathcal{J}$ is the matrix defined by

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The function ψ is then called the Hamiltonian or the stream function. Since every incompressible vector field $X \in \mathfrak{X}^r(\mathbb{R}^2)$ is a Hamiltonian vector field $X = -\mathcal{J}\operatorname{grad} \psi$, the vorticity $\omega = \operatorname{curl} X$ and the stream function ψ satisfy

$$\omega = \operatorname{curl}(-\mathcal{J}\operatorname{grad} \psi) = -\triangle \psi.$$

Accordingly, we obtain the Biot-Savart law:

$$X = -\mathcal{J}\operatorname{grad}\langle G_H, \omega \rangle,$$

where $\langle G_H, \omega \rangle$ is the convolution of the Green's function $G_H \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2 \setminus \Delta)$ and the vorticity ω.

An isomorphism between vector fields and differential 1 forms plays an essential role in rewriting those facts on vector calculus in terms of differential forms, and is called the musical isomorphism. Note that every vector field $X \in \mathfrak{X}^r(\mathbb{R}^2)$ is conventionally written as $X = {}^{\mathsf{T}}(X^1, X^2)$, and identified with a vector-valued function. In the theory of manifolds, it is rewritten as $X = X^1\partial_1 + X^2\partial_2$ by using an orthogonal basis (∂_1, ∂_2) of the tangent space of $\mathbb{R}^2$. Here, the orthogonal basis is identified with the partial derivative. In contrast, differential 1-form α is written as by using an orthogonal basis $(\mathrm{d}\, x^1, \mathrm{d}\, x^2)$ of

the cotangent space of $\mathbb{R}^2$. Let us remember that the cotangent space is defined as the dual vector space of the tangent space and $\mathrm{d}\,x^i$ is the dual basis of ∂_i. The dual basis is now identified with the line element. As a result, by replacing the basis ∂_i with the basis $\mathrm{d}\,x^i$, we obtain a 1 form $X^\flat = X^1\,\mathrm{d}\,x^1 + X^2\,\mathrm{d}\,x^2 \in \Omega^1_{[r]}(\mathbb{R}^2)$. The operator $\flat : X \in \mathfrak{X}^r(\mathbb{R}^2) \to X^\flat \in \Omega^1_{[r]}(\mathbb{R}^2)$ is called the flat operator, and the corresponding 1 form is called the velocity form. Conversely, for each 1-form $\alpha = \alpha_1\,\mathrm{d}\,x^1 + \alpha_2\,\mathrm{d}\,x^2 \in \Omega^1_{[r]}(\mathbb{R}^2)$, a vector field is defined by $\alpha_\sharp = \alpha_1\partial_1 + \alpha_2\partial_2 \in \mathfrak{X}^r(\mathbb{R}^2)$. The operator $\sharp : \alpha \in \Omega^1_{[r]}(\mathbb{R}^2) \to \alpha_\sharp \in \mathfrak{X}^r(\mathbb{R}^2)$ is called the sharp operator, and the corresponding vector field is called the dual vector field. These two operators are isomorphisms between the space of vector fields $\mathfrak{X}^r(\mathbb{R}^2)$ and the space of 1-forms $\Omega^1_{[r]}(\mathbb{R}^2)$, which together are called musical isomorphisms. The inner product (α, β) of 1 forms $\alpha = \alpha_1\,\mathrm{d}\,x^1 + \alpha_2\,\mathrm{d}\,x^2, \beta = \beta_1\,\mathrm{d}\,x^1 + \beta_2\,\mathrm{d}\,x^2 \in \Omega^1_{[r]}(\mathbb{R}^2)$ is defined by $(\alpha, \beta) = \alpha_1\beta_1 + \alpha_2\beta_2$.

The differential operators used in this paper consist of two basic operators: the (exterior) differential operator d and the Hodge-$*$ operator. For each p-form $\alpha \in \Omega^p_{[r]}(\mathbb{R}^2)(p = 0, 1, 2)$, the differential operator $\mathrm{d} : \alpha \in \Omega^p_{[r]}(\mathbb{R}^2) \to \mathrm{d}\,\alpha \in \Omega^{p-1}_{[r-1]}(\mathbb{R}^2)$ and the Hodge-$*$ operator $* : \alpha \in \Omega^p_{[r]}(\mathbb{R}^2) \to *\alpha \in \Omega^{2-p}_{[r]}(\mathbb{R}^2)$ are defined by

$$\mathrm{d}\,\alpha = \begin{cases} \partial_1\alpha_0\,\mathrm{d}\,x^1 + \partial_2\alpha_0\,\mathrm{d}\,x^2, & \text{if } \alpha = \alpha_0 \in \Omega^0_{[r]}(\mathbb{R}^2), \\ (\partial_1\alpha_2 - \partial_2\alpha_1)\,\mathrm{d}\,x^1 \wedge \mathrm{d}\,x^2, & \text{if } \alpha = \alpha_1\,\mathrm{d}\,x^1 + \alpha_2\,\mathrm{d}\,x^2 \in \Omega^1_{[r]}(\mathbb{R}^2), \\ 0, & \text{if } \alpha = \alpha_{12}\,\mathrm{d}\,x^1 \wedge \mathrm{d}\,x^2 \in \Omega^2_{[r]}(\mathbb{R}^2), \end{cases}$$

$$*\alpha = \begin{cases} \alpha_0\,\mathrm{d}\,x^1 \wedge \mathrm{d}\,x^2, & \text{if } \alpha = \alpha_0 \in \Omega^0_{[r]}(\mathbb{R}^2), \\ -\alpha_2\,\mathrm{d}\,x^1 + \alpha_1\,\mathrm{d}\,x^2, & \text{if } \alpha = \alpha_1\,\mathrm{d}\,x^1 + \alpha_2\,\mathrm{d}\,x^2 \in \Omega^1_{[r]}(\mathbb{R}^2), \\ \alpha_{12}, & \text{if } \alpha = \alpha_{12}\,\mathrm{d}\,x^1 \wedge \mathrm{d}\,x^2 \in \Omega^2_{[r]}(\mathbb{R}^2). \end{cases}$$

Moreover, by using the differential operator and the Hodge-$*$ operator, the codifferential operator $\delta = *\,\mathrm{d}\,* : \Omega^p_{[r]}(\mathbb{R}^2) \to \Omega^{p-1}_{[r-1]}(\mathbb{R}^2)$ and the Hodge Laplacian $\triangle = \delta\,\mathrm{d} + \mathrm{d}\,\delta : \Omega^p_{[r]}(\mathbb{R}^2) \to \Omega^p_{[r-2]}(\mathbb{R}^2)$ are defined.

Using the above notion of differential forms, the curl operator and the divergence operator are thus rewritten as

$$\operatorname{curl} X = *\,\mathrm{d}\,X^\flat, \quad \operatorname{div} X = \delta X^\flat$$

since

$$\begin{aligned} *\,\mathrm{d}\,X^\flat &= *\,\mathrm{d}(X^1\,\mathrm{d}\,x^1 + X^2\,\mathrm{d}\,x^2) = *(\partial_1 X^2 - \partial_2 X^1)\,\mathrm{d}\,x^1 \wedge \mathrm{d}\,x^2 \\ &= \partial_1 X^2 - \partial_2 X^1, \\ \delta X^\flat &= *\,\mathrm{d}\,*(X^1\,\mathrm{d}\,x^1 + X^2\,\mathrm{d}\,x^2) = *\,\mathrm{d}(-X^2\,\mathrm{d}\,x^1 + X^1\,\mathrm{d}\,x^2) \\ &= *(\partial_1 X^1 + \partial_2 X^2)\,\mathrm{d}\,x^1 \wedge \mathrm{d}\,x^2 = \partial_1 X^1 + \partial_2 X^2. \end{aligned}$$

Similarly, since

$$\mathrm{d}\,\phi = \partial_1\phi\,\mathrm{d}\,x^1 + \partial_2\phi\,\mathrm{d}\,x^2, \quad *\,\mathrm{d}\,\psi = -\partial_2\psi\,\mathrm{d}\,x^1 + \partial_1\psi\,\mathrm{d}\,x^2,$$

we obtain

$$\operatorname{grad}\phi = (\mathrm{d}\,\phi)_\sharp, \quad \mathcal{J}\operatorname{grad}\psi = (*\,\mathrm{d}\,\psi)_\sharp.$$

The Euler equation is now presented as follows:

$$\partial_t v^\flat + (*\,\mathrm{d}\,v^\flat) * v^\flat + \mathrm{d}\,|v^\flat|^2/2 = -\,\mathrm{d}\,p. \tag{5}$$

It is easy to see that the equation (5) is equivalent to the Euler equation since

$$\begin{aligned}
\partial_t v^\flat &= \partial_t v^1 \,\mathrm{d}\, x^1 + \partial_t v^2 \,\mathrm{d}\, x^2, \\
(*\,\mathrm{d}\, v^\flat) * v^\flat &= (-v^2\partial_1 v^2 + v^2\partial_2 v^1)\,\mathrm{d}\, x^1 + (v^1\partial_1 v^2 - v^1\partial_2 v^1)\,\mathrm{d}\, x^2, \\
\mathrm{d}\,|v|^2/2 &= (v^1\partial_1 v^1 + v^2\partial_1 v^2)\,\mathrm{d}\, x^1 + (v^1\partial_2 v^1 + v^2\partial_2 v^2)\,\mathrm{d}\, x^2, \\
\mathrm{d}\, p &= \partial_1 p\,\mathrm{d}\, x^1 + \partial_2 p\,\mathrm{d}\, x^2.
\end{aligned}$$

Although the equation (5) is introduced heuristically in this paper, the derivation of the equation (5) from the Euler equation is based on an idea in differential geometry, the symmetric-antisymmetric decomposition of the Levi-Civita connection [9]. It is still open whether a similar presentation can be obtained for the three-dimensional Euler equation.

2.2 Weak Formulation by Currents

We begin with a brief review of the basics of currents. For details, we refer the reader to [1]. A p-current is defined as a continuous linear functional over $\mathbb{R}$ on $\mathcal{D}^{2-p}(\mathbb{R}^2)$. Let $T[\phi]$ denote the coupling of the p-current T and $\phi \in \mathcal{D}^{2-p}(\mathbb{R}^2)$. The elements of $\mathcal{D}^p(\mathbb{R}^2)$ are often called test forms. The space of all p-currents on $\mathbb{R}^2$ is denoted by $\mathcal{D}'_p(\mathbb{R}^2)$. In the same manner as distributions, calculations on currents are defined via calculations on test forms. The differential operator $\mathrm{d} : T \in \mathcal{D}'_p(\mathbb{R}^2) \to \mathrm{d}\,T \in \mathcal{D}'_{p+1}(\mathbb{R}^2)$ for currents is defined as $\mathrm{d}\,T[\phi] = (-1)^{p+1}T[\mathrm{d}\,\phi]$, and the Hodge-$*$ operator $* : T \in \mathcal{D}'_p(\mathbb{R}^2) \to *T \in \mathcal{D}'_p(\mathbb{R}^2)$ is defined as $*T[\phi] = (-1)^{p(2-p)}T[*\phi]$. As for the codifferential operator and the Hodge Laplacian for currents, it is immediately obtained from the composition of the differential operator and the Hodge-$*$ operator.

For instance, the space $\mathcal{D}'_0(\mathbb{R}^2)$ corresponds to the space of distributions on $\mathbb{R}^2$. In particular, for $p \in \mathbb{R}^2$, we define *the delta current*, say $\delta_p \in \mathcal{D}'_0(\mathbb{R}^2)$, by $\delta_p[\phi] = *\phi(p)$. This is the counterpart of the delta function in the theory of distributions. For each p-form $\alpha \in \Omega^p_{[r]}(\mathbb{R}^2)$, it is naturally identified with a p-current when we define a functional $I(\alpha)$ on $\mathcal{D}^{2-p}(\mathbb{R}^2)$ as

$$I(\alpha)[\phi] = \int_{\mathbb{R}^2} \alpha \wedge \phi.$$

Since the topology of the space of currents is weak-$*$ topology, the inclusion map $I : \Omega^p_{[r]}(\mathbb{R}^2) \to \mathcal{D}'_p(\mathbb{R}^2)$ is now an embedding map. Therefore, currents are characterized as a weak extension of differential forms.

Let us formulate a notion of currents corresponding to singular integral operators. We first establish the space of all singular integral kernels. Namely, we call a p-form on a nonempty open subset $U \subset \mathbb{R}^2$ *local p-form*. Let $\Omega^p_{[r]\mathrm{loc}}(\mathbb{R}^2)$ denote the space of all local p-form. In the same manner as $L^p_{\mathrm{loc}}(\mathbb{R}^2)$, the topology of $\Omega^p_{[r]\mathrm{loc}}(\mathbb{R}^2)$ can be defined. For each local p-form $\alpha \in \Omega^p_{[r]\mathrm{loc}}(\mathbb{R}^2)$, there exists a maximal open subset U such that $\alpha \in \Omega^p_{[r]}(U)$. Then, the closed subset $\mathsf{S}^r(\alpha) = \mathbb{R}^2 \setminus U$ is called the singular support of α. We next define a singular integral operator as an integral operator whose integral kernel is local p-form. A p-current $T \in \mathcal{D}'_p(\mathbb{R}^2)$ is said to be of class C^r, if there exists a local p-form $\alpha_T \in \Omega^p_{[r]\mathrm{loc}}(\mathbb{R}^2)$ such that for every $\phi \in \mathcal{D}^{2-p}(\mathbb{R}^2 \setminus \mathsf{S}^r(\alpha_T))$, $T[\phi] = I(\alpha_T)[\phi]$. Let $\mathcal{D}'^r_p(\mathbb{R}^2)$ denote the space of all p-currents of class C^r. For each $T \in \mathcal{D}'^r_p(\mathbb{R}^2)$, the subset $\mathsf{S}^r(T) = \mathsf{S}^r(\alpha_T)$ and the local p-form α_T

is called the singular support of T and the density of T. Owing to the fundamental lemma of calculus of variation, the density is uniquely determined. Thus, the map $K : T \in \mathcal{D}'^r_p(\mathbb{R}^2) \to K(T) = \alpha_T \in \Omega^p_{[r]\mathrm{loc}}(\mathbb{R}^2)$ is well-defined and called the derivative. Obviously, the restriction $K|_{\Omega^p_{[r]}(\mathbb{R}^2)}$ is the identity map on $\Omega^p_{[r]}(\mathbb{R}^2)$. For example, for each $p \in \mathbb{R}^2$, the delta current δ_p is of class C^∞ since $\delta_p = I(0)$ in $\mathbb{R}^2 \setminus \{p\}$. We thus obtain $\mathsf{S}^\infty(\delta_p) = \{p\}$ and $K(\delta_0) = 0$.

We define the principal value p.v. : $T \in \Omega^p_{[r]\mathrm{loc}}(\mathbb{R}^2) \to \mathrm{p.v.}\, T \in \mathcal{D}'^r_p(\mathbb{R}^2)$ by

$$\mathrm{p.v.}\, T[\phi] = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2 \setminus B_\varepsilon(\mathsf{S}(T))} T \wedge \phi$$

for each $\phi \in \mathcal{D}^{2-p}(\mathbb{R}^2)$ if the limit exists. The domain of p.v., say Dom(p.v.), is defined by the space of p-currents in which the limit exists for every $\phi \in \mathcal{D}^{2-p}(\mathbb{R}^2)$. Let us introduce the operator $\mathfrak{L}$ by $\mathfrak{L} = \mathrm{d}\,\mathrm{p.v.} : T \in \Omega^p_{[r]\mathrm{loc}}(\mathbb{R}^2) \to \mathfrak{L}\,T \in \mathcal{D}'_{p+1}(\mathbb{R}^2)$, which is called the *localizing operator*. For each $\phi \in \mathcal{D}^{p+1}(\mathbb{R}^2)$, we have

$$\mathfrak{L}\,T[\phi] = \mathrm{d}\,\mathrm{p.v.}\, T[\phi] = (-1)^{p+1}\,\mathrm{p.v.}\, T[\mathrm{d}\,\phi].$$

The domain of $\mathfrak{L}$, $\mathrm{Dom}(\mathfrak{L})$, is the space of p-currents T in which $\mathrm{p.v.}\, T[\mathrm{d}\,\phi]$ is well-defined for every $\phi \in \mathcal{D}^{1-p}(\mathbb{R}^2)$. If $T \in \mathrm{Dom}(\mathfrak{L})$ satisfies $\mathrm{d}\,T = 0$ in $\mathbb{R}^2 \setminus \mathsf{S}(T)$, then we obtain

$$\mathfrak{L}\,T[\phi] = (-1)^{p+1} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2 \setminus B_\varepsilon(\mathsf{S}(T))} T \wedge \mathrm{d}\,\phi = -\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(\mathsf{S}(T))} T \wedge \phi,$$

since $(-1)^{p+1} T \wedge \mathrm{d}\,\phi = -\,\mathrm{d}(T \wedge \phi) + \mathrm{d}\,T \wedge \phi$. Hence, $\mathfrak{L}\,T$ is determined by the asymptotic behavior of T near the singular support $\mathsf{S}(T)$. The name of *localizing* operator is named after this property.

In light of the above basics for currents, let us apply it to a weak formulation of vector analysis in terms of currents. What remains to be done is basically to replace the part of the velocity form with a 1-current. The divergence of a 1 current $\alpha \in \mathcal{D}'_1(\mathbb{R}^2)$ is defined by $\delta\alpha \in \mathcal{D}'_0(\mathbb{R}^2)$, based on the presentation of the divergence operator by the velocity form: $\mathrm{div}\, X = \delta X^\flat$. But for the curl operator $\mathrm{curl}\, X = *\,\mathrm{d}\, X^\flat$, the principal value is used so that the vorticity can be calculated from the density of the current. Namely, the vorticity of a local 1 form $\alpha \in \mathrm{Dom}(\mathfrak{L})$ is defined by $*\,\mathfrak{L}\,\alpha \in \mathcal{D}'_0(\mathbb{R}^2)$. Let us remember that $\Omega^1_{[r]}(\mathbb{R}^2)$ is identified with $\mathfrak{X}^r(\mathbb{R}^2)$ by the sharp operator $\sharp : \alpha \in \Omega^1_{[r]}(\mathbb{R}^2) \to \alpha_\sharp \in \mathfrak{X}^r(\mathbb{R}^2)$, satisfying $\alpha[X] = g(\alpha_\sharp, X)$ for all $X \in \mathfrak{X}^r(\mathbb{R}^2)$. As 1-form generates, every C^r 1-current T generates a vector field allowing singularities in $\mathsf{S}(T)$. For each $T \in \mathcal{D}'^r_1(\mathbb{R}^2)$, we define $T_\sharp \in \mathfrak{X}^r(\mathbb{R}^2 \setminus \mathsf{S}^r(T))$ by $T_\sharp = K(T)_\sharp$. As an example, for each $\psi \in \mathcal{D}'^r_0(\mathbb{R}^2)$, we define $\mathcal{J}\mathrm{grad}\,\psi \in \mathfrak{X}^{r-1}(\mathbb{R}^2 \setminus \mathsf{S}^{r-1}(\mathrm{d}\,\psi))$ by $\mathcal{J}\mathrm{grad}\,\psi = K(*\,\mathrm{d}\,\psi)_\sharp$. The vector field $\mathcal{J}\mathrm{grad}\,\psi$ stands for the Hamiltonian vector field induced from the Hamiltonian ψ with singularities in $\mathsf{S}(\psi)$. We will use this notion when we take a vector field generated by point vortices. In what follows, for a given $T \in \mathcal{D}'^r_p(\mathbb{R}^2)$, we abbreviate $\mathsf{S}^r(T)$ to $\mathsf{S}(T)$. All currents are of class C^r and the singular support consists of a finite set of points in this paper.

3 Current-Valued Solutions of the Euler Equation

In Section 2.1, we saw that the equation (5) is an equivalent presentation to the Euler equation. Before we replace differential forms in (5) with currents, we

need to deal with the nonlinear term carefully in order to avoid multiplication of currents. Let us recall multiplication of local p-forms is still valid and a local p-form is converted to a current by taking the principle value. Hence, the Euler equation is reformulated for $\alpha_t \in \mathcal{D}'^r_1(\mathbb{R}^2)$ and $p_t \in \Omega^0_{[r]\mathrm{loc}}(\mathbb{R}^2)$ as follows.

$$\partial_t \,\mathrm{p.v.}\, K(\alpha_t) + \mathrm{p.v.}\{K(*\,\mathrm{d}\,\alpha_t)K(*\alpha_t) + \mathrm{d}\,|K(\alpha_t)|^2/2\} = -\,\mathrm{p.v.}\,\mathrm{d}\,p_t, \qquad (6)$$

in $\mathcal{D}'_1(\mathbb{R}^2)$, if each of terms is contained in Dom(p.v.). When we focus on evolution of vorticity, the vorticity equation is useful rather than the Euler equation. Let us remember that the vorticity equation is obtained by applying the curl operator curl $= *\,\mathrm{d}$ to the Euler equation. Based on this, applying the differential operator d to (6), we obtain the vorticity equation corresponding to (6) if each density in (6) is contained in $\mathrm{Dom}(\mathrm{d}\,\mathrm{p.v.}) = \mathrm{Dom}(\mathfrak{L})$.

Definition 3.1. *A pair of time-dependent currents $(\alpha_t, p_t) \in \mathcal{D}'^r_1(\mathbb{R}^2)\times\Omega^0_{[r]\mathrm{loc}}(\mathbb{R}^2)$ is called a weak Euler flow, if the following conditions are satisfied:*

1. *$K(\alpha_t) \in \mathrm{Dom}(\mathfrak{L})$ and $K(*\,\mathrm{d}\,\alpha_t)K(*\alpha_t) + \mathrm{d}\,|K(\alpha_t)|^2/2 \in \mathrm{Dom}(\mathfrak{L})$;*
2. *$\mathrm{d}\,p_t \in \mathrm{Dom}(\mathfrak{L})$;*
3. *$\partial_t\,\mathfrak{L}\,K(\alpha_t)+\mathfrak{L}\left\{K(*\,\mathrm{d}\,\alpha_t))K(*\alpha_t) + \mathrm{d}\,|K(\alpha_t)|^2/2\right\} = -\,\mathfrak{L}\,\mathrm{d}\,p_t$ in $\mathcal{D}'_2(\mathbb{R}^2)$;*
4. *$\delta\alpha_t = 0$ in $\mathcal{D}'_1(\mathbb{R}^2)$;*

In particular, we call the third condition the weak Euler equation and α_t the velocity current.

Let us decompose a weak Euler flow into a regular part and a singular part, assuming that the regular part is given by a classical solution of the Euler equation. Let us fix a weak Euler flow $(\alpha_t, p_t) \in \mathcal{D}'^r_1(\mathbb{R}^2) \times \Omega^0_{[r]\mathrm{loc}}(\mathbb{R}^2)$ and a classical Euler flow $(X_t, P_t) \in \mathfrak{X}^r(\mathbb{R}^2) \times C^r(\mathbb{R}^2)$:

$$\partial_t X_t^\flat + (*\,\mathrm{d}\,X_t^\flat) * X_t^\flat + \mathrm{d}\,|X_t^\flat|^2/2 = -\,\mathrm{d}\,P_t.$$

Unless otherwise stated, the subscript t is omitted hereafter. Writing $u = \alpha - X^\flat \in \mathcal{D}'^r_1(\mathbb{R}^2)$, $\omega = *\,\mathrm{d}\,u \in \mathcal{D}'^r_0(\mathbb{R}^2)$ and $\omega_X = *\,\mathrm{d}\,X^\flat$, we obtain

$$\begin{aligned}
\partial_t\,\mathfrak{L}\,K(\alpha) - \partial_t\,\mathfrak{L}\,X^\flat &= \partial_t\,\mathfrak{L}\,K(u)\\
\mathfrak{L}\,K(*\,\mathrm{d}\,\alpha)K(*\alpha) - \mathfrak{L}\,\omega_X * X^\flat &= \mathfrak{L}\,K(\omega+\omega_X)K(*(u+X^\flat)) - \mathfrak{L}\,\omega_X * X^\flat\\
&= \mathfrak{L}\{(\omega_X + K(\omega))K(*u) + K(\omega) * X^\flat\},\\
\mathfrak{L}\,\mathrm{d}\,|K(\alpha)|^2 - \mathfrak{L}\,\mathrm{d}\,|X|^2 &= \mathfrak{L}\,\mathrm{d}(X^\flat + K(u), X^\flat + K(u)) - \mathfrak{L}\,\mathrm{d}\,|X^\flat|^2\\
&= \mathfrak{L}\,\mathrm{d}(2X^\flat + K(u), K(u)).
\end{aligned}$$

Hence, the weak Euler equation (Definition 3.1-3) is reduced to

$$\partial_t\,\mathfrak{L}\,K(u)+\mathfrak{L}\{(\omega_X+K(\omega))K(*u)+K(\omega)*X^\flat+\mathrm{d}(2X^\flat+K(u),K(u))/2\} = -\,\mathfrak{L}\,\mathrm{d}(p-P). \qquad (7)$$

We examine closely the role of each term in the equation (7). When the relative vorticity ω is given by a linear combination of delta currents, for instance, a straightforward calculation of currents shows that

$$\mathfrak{L}\{(\omega_X + K(\omega))K(*u) + K(\omega) * X^\flat\} = 0$$

is actually valid [9]. Furthermore, if the pressure term vanishes under some assumptions, the equation (7) can be written only in terms of the velocity

current. Based on this, we introduce a model for the pressure that $\mathfrak{L}p$ is killed out with a linear combination of singular terms $\mathfrak{L}\,\mathrm{d}\,g(X^\flat, K(u))$ and $\mathfrak{L}\,\mathrm{d}\,|K(u)|^2/2$, that is,

$$p = P + (2\beta_X - 1)(X^\flat, K(u)) + (2\beta_\omega - 1)|K(u)|^2/2$$

for some $(\beta_X, \beta_\omega) \in \mathbb{R}^2$. This mathematical model can be interpreted that the singular behavior of the pressure is balanced with the interaction energy density $g(X^\flat, K(u))$ and the kinetic energy density $|K(u)|^2/2$ with a growth rate (β_X, β_ω). Summarizing the above, we propose the following regular-singular decomposition of a weak Euler-Arnold flow.

Definition 3.2. *A weak Euler flow $(\alpha_t, p_t) \in \mathcal{D}'^r_1(\mathbb{R}^2) \times \Omega^0_{[r]\mathrm{loc}}(\mathbb{R}^2)$ is said to be C^r decomposable $(r \geq 1)$, if there exists a classical Euler flow $(X_t, P_t) \in \mathfrak{X}^r(\mathbb{R}^2) \times C^r(\mathbb{R}^2)$ and $(\beta_X, \beta_\omega) \in \mathbb{R}^2$ such that for each time t,*

$$p_t = P_t + (2\beta_X - 1)(X_t^\flat, K(u_t)) + (2\beta_\omega - 1)|K(u_t)|^2/2,$$

where $u_t = \alpha_t - X_t$. Then we call X_t a background field of α_t, u_t a relative velocity current and (β_X, β_ω) a growth rate of p_t.

Let us note that the C^r decomposability of the weak Euler flow guarantees the existence of the decomposition but there is no mention of the uniqueness of the decomposition. Hence, when we study a C^r decomposable weak Euler flow $(\alpha_t, p_t) \in \mathcal{D}'^r_1(\mathbb{R}^2) \times \Omega^0_{[r]\mathrm{loc}}(\mathbb{R}^2)$, we need to fix a classical Euler flow $(X_t, P_t) \in \mathfrak{X}^r(\mathbb{R}^2) \times C^r(\mathbb{R}^2)$ and a parameter $(\beta_X, \beta_\omega) \in \mathbb{R}^2$ such that the velocity field X_t is a background field of α_t and the parameter (β_X, β_ω) is a growth rate of p_t. If a weak Euler-Arnold flow (α_t, p_t) is C^r decomposable $(r \geq 1)$, the equation (7) is written without the pressure term as follows.

$$\partial_t\, \mathfrak{L}\, K(u) + \mathfrak{L}\{(\omega_X + K(\omega))K(*u) + K(\omega) * X^\flat + \mathrm{d}(2\beta_X X^\flat + \beta_\omega K(u), K(u))\} = 0.$$

4 Main Results

Let us fix $N, r \in \mathbb{Z}_{\geq 1}$, $(\Gamma_n)_{n=1}^N \in (\mathbb{R} \setminus \{0\})^N$ and a C^r one-parameter family $\Phi : t \in [0, T] \to \Phi_t \in \mathrm{Diff}^r(Q_N)$ in what follows, where

$$Q_N = (\mathbb{R}^2)^N \setminus \{(q_n)_{n=1}^N \in (\mathbb{R}^2)^N \,|\, \exists i, j,\ q_i = q_j\}.$$

Let us denote by $(q_n(t))_{n=1}^N = \Phi_t((q_n(0))_{n=1}^N)$ an orbit of Φ. In what follows, we focus on a time-dependent 0-current $\sum_{n=1}^N \Gamma_n \delta_{q_n(t)} \in \mathcal{D}'_0(\mathbb{R}^2)$ and call it a *singular vorticity of point vortices placed on* $\{q_n(t)\}_{n=1}^N$. Let us recall the vector field $v_n \in \mathfrak{X}^\infty(\mathbb{R}^2 \setminus \{q_m\}_{m \neq n})$ that is defined by

$$v_n(q_n) = -\mathcal{J}\mathrm{grad}_{q_n} \sum_{m \neq n}^N \Gamma_m G_H(q_n, q_m),$$

and the point vortex equation in the background field

$$\dot{q}_n(t) = \beta_X X_t(q_n(t)) + \beta_\omega v_n(q_n(t)), \quad n = 1, \ldots N, \tag{8}$$

where $(\beta_X, \beta_\omega) \in \mathbb{R}^2$ is a given parameter and X_t is the fluid velocity of a given Euler flow. First, we show that if the singular part of the vorticity of the C^r-decomposable weak Euler flow is given by the singular vorticity of point vortices, then $q_n(t)$ is a solution of the point vortex equation in the background field (8).

Theorem 4.1. *Let $(\alpha_t, p_t) \in \mathcal{D}'^r_1(\mathbb{R}^2) \times \Omega^0_{[r]\mathrm{loc}}(\mathbb{R}^2)$ be a C^r-decomposable weak Euler flow ($r \geq 1$). Fix a background field X_t of α_t, a growth rate (β_X, β_ω) of p_t. Suppose the relative vorticity ω_t is a singular vorticity of point vortices placed on $\{q_n(t)\}_{n=1}^N$. Then, $q_n(t)(n = 1, \ldots, N)$ is a solution of the point vortex equation in the background field* (8).

Conversely, if $q_n(t)$ is a solution of the point vortex equation in a background field (8), then there exists a C^r-decomposable weak Euler flow such that the singular part of the vorticity is given by a linear combination of delta currents centered at $q_n(t)$.

Theorem 4.2. *Fix a classical Euler flow $(X_t, P_t) \in \mathfrak{X}^r(\mathbb{R}^2) \times C^r(\mathbb{R}^2)$ and $(\beta_X, \beta_\omega) \in \mathbb{R}^2$. Let $\omega_t \in \mathcal{D}'_0(\mathbb{R}^2)$ be a singular vorticity of point vortices placed on $\{q_n(t)\}_{n=1}^N$. Define a time-dependent current $u_t \in \mathcal{D}'^\infty_1(\mathbb{R}^2)$ by $u_t = -*\mathrm{d}\langle G_H, \omega_t\rangle$. Suppose $q_n(t)(n = 1, \ldots, N)$ is a solution of the point vortex equation in the background field X_t* (8). *Then, the following pair of time-dependent currents α_t and p_t defines a C^r-decomposable weak Euler flow.*

$$\alpha_t = X_t^\flat + u_t \in \mathcal{D}'^r_1(\mathbb{R}^2),$$
$$p_t = P_t + (2\beta_X - 1)(X_t^\flat, K(u_t)) + (2\beta_\omega - 1)|K(u_t)|^2/2 \in \Omega^0_{[r]\mathrm{loc}}(\mathbb{R}^2).$$

We refer the reader to the paper [9] for the proof of the theorem.

5 Applications

First, Theorem 4.2 shows that the point vortex dynamics in the background field justified as a C^r-decomposable weak Euler flow although it is a mathematical model derived formally from the Euler flow. Next, from Theorem 4.1, we deduce that the parameter (β_X, β_ω) is not only a dynamical parameter in the point vortex equation in the background field, but also a physical parameter in terms of the growth rate of pressure. With these perspectives in mind, we revisit the following examples of point vortex dynamics in the background field.

Two identical point vortices in a linear shear Two identical point vortices in a linear shear is used as a model of the vortex merger in [10]. The vortex merger is characterized as a fundamental process of the inverse cascade in 2D turbulence. In the process, vortices with similarly small scales are affected by the shear flow induced from the surrounding vortices. As a result, small vortices combine to form a vortex with large scale. As a simple model for the vortex merger, two identical point vortices in a linear shear is adopted. To use the notations of this paper, let us set $N = 2$, $\Gamma_1 = \Gamma_2 = \gamma$ and the linear shear $X = (cy, 0)$ where $y = (z - \bar{z})/2\mathrm{i}$. Based on [10], the evolution equation of two identical point vortices in a linear shear placed on $\{q_n(t)\}_{n=1}^2$ is given by

$$\dot{q}_n(t) = X(q_n) + v_n(q_n). \tag{9}$$

Since the equation (9) is a Hamiltonian system, plotting the Hamiltonian contours, we see that the topology of the contours changes around the threshold $\mu = c\xi_0^2/\gamma$ where $\xi_0 \in \mathbb{R}$ is the initial distance between two point vortices. Obviously, for the case where $\mu = 0$, there is no background field and two point vortices moves in a circle without changing the distance. When $\mu < 0$, two

point vortices are in periodic motion and the distance between them becomes shorter, which implies the vortex merger occurs. Otherwise, the distance is longer. This qualitative understanding is confirmed to be consistent with experimental studies in [10]. In other words, it is pointed out that the sign of μ, especially the orientation of the background field, determines the occurrence of vortex merger.

As an application of Theorem 4.1, let us take a C^∞-decomposable weak Euler flow (α_t, p_t) such that the background field is the linear shear X and the growth rate is $(\beta_X, \beta_\omega) \in \mathbb{R}^2$ and the relative vorticity is a singular vorticity of point vortices placed on $\{q_n(t)\}_{n=1}^2$ with $\Gamma_1 = \Gamma_2 = \gamma$. From Theorem 4.1, we can deduce that $q_n(t)$ satisfies the point vortex equation in the background field (8). Then, we can easily see that $q_n(t)$ comes from a Hamiltonian system and the Hamiltonian contours are the same as those given in [10] by changing parameters from c to $c\beta_X$ and from γ to $\gamma\beta_\omega$. In the similar manner, instead of μ, we obtain the threshold $\mu' = c\beta_X\xi_0^2/\gamma\beta_\omega$. Moreover, the same criterion is valid, that is, the vortex merger occurs if $\mu' < 0$. Note that Theorem 4.1 tells us that the parameter (β_X, β_ω) is not just a dynamical parameter contained in the point vortex equation, but also a physical parameter derived from the pressure of the Euler flow. From this point of view, we conclude that besides the orientation of the background field, the sign of the growth rate of the pressure determines the occurrence of vortex merger.

N-point vortices on a surface in an irrotational flow. In the second case, let us take a C^∞-decomposable weak Euler flow (α_t, p_t) such that the background field is an irrotational field $X \in \mathfrak{X}^\infty(\mathbb{R}^2)$ and the relative vorticity is a singular vorticity of point vortices placed on $\{q_n(t)\}_{n=1}^N$. The irrotational field is the fluid velocity of a steady Euler-Arnold flow $(X, P) \in \mathfrak{X}^\infty(\mathbb{R}^2) \times C^\infty(\mathbb{R}^2)$ and the pressure $P \in C^\infty(\mathbb{R}^2)$ satisfies the Bernoulli law: $P = -|X|^2/2$. Let us fix the growth rate $(\beta_X, \beta_\omega) \in \mathbb{R}^2$ and the parameter $(\Gamma_n)_{n=1}^N \in (\mathbb{R} \setminus \{0\})^N$. For simplicity, we ignore the interaction between the background field and point vortices, that is, we assume $\beta_X = 0$. Then owing to Theorem 4.1, $q_n(t)$ and p_t satisfy

$$\begin{cases} \dot{q}_n = \beta_\omega v_n(q_n), \\ p_t = -|\alpha_t|^2 + \beta_\omega |u_t|^2. \end{cases} \tag{10}$$

Let us focus on two cases $\beta_\omega = 0$ and 1. In the case $\beta_\omega = 0$, it follows from $\dot{q}_n(t) = 0$ that each of point vortices does not move. Hence, we deduce that (α_t, p_t) is a steady solution. Moreover, the pressure satisfies a generalization of the Bernoulli law $p = -|\alpha|^2/2$ to the case with non-moving point vortices. On the other hand, for $\beta_\omega = 1$ we obtain the conventional point vortex equation and the pressure satisfies a modified Bernoulli law $p = -|\alpha|^2/2 + |u|^2$. As a corollary, the Bernoulli law is generalized to the case where the presence of moving or non-moving point vortices is taken into account as follows.

Corollary 5.1 (Generalized Bernoulli law with non-moving point vortices)**.** *If a C^∞-decomposable weak Euler flow (α_t, p_t) on a surface satisfies that the background field of α_t is an irrotational field $X \in \mathfrak{X}^\infty(\mathbb{R}^2)$ and that the pressure is given by $p_t = -|\alpha_t|^2/2$ and that the relative vorticity is a singular vorticity of point vortices, then (α_t, p_t) is a steady solution of the weak Euler equations.*

Corollary 5.2 (Modified Bernoulli law with moving point vortices)**.** *If a C^∞-decomposable weak Euler flow (α_t, p_t) on a surface satisfies that the background*

field of α_t is an irrotational field and that $p_t = -|\alpha_t|^2/2 + |u_t|^2$ and that the relative vorticity is a singular vorticity of point vortices placed on $\{q_n(t)\}_{n=1}^N$, then for every $n \in \{1, \dots, N\}$, $q_n(t)$ is a solution of the point vortex equation:

$$\dot{q}_n(t) = v_n(q_n(t)).$$

Let us finally discuss the role of the growth rate β_ω in the motion of point vortices and the pressure given in (10). Denoting $Q_n(t)$ by the solution of $\dot{Q}_n = v_n(Q_n)$, the solution $q_n(t)$ in (10) can be written as $q_n(t) = Q_n(\beta_\omega t)$. Letting $\beta_\omega \to 0$, we see that q_n moves quite slowly on the orbit of Q_n. In this sense, β_ω stands for the flexibility of the motion of point vortices besides the growth rate of the pressure relative to the kinetic energy density. We notice that as $\beta_\omega \to 0$ the pressure p_t converges to $-|\alpha_t|^2/2$ in $\Omega^0_{[r]\mathrm{loc}}(\mathbb{R}^2)$, which is consistent with the generalized steady Bernoulli law. From this we can observe that point vortices are frozen if the pressure p_t is sufficiently close to the generalized Bernoulli law. As a consequence, we conclude that β_ω describes the the flexibility of the motion of point vortices and the growth rate of the pressure and that point vortices are slower to move as the pressure is sufficiently close to the generalized Bernoulli law.

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TRAVELING WAVE DYNAMICS FOR CONSTRAINED ALLEN-CAHN EQUATIONS

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Abstract. This talk is concerned with a joint work with C. Kuehn (TU München) and K.-I. Nakamura (Kanazawa University) in [2], where traveling wave dynamics for a one-dimensional fully nonlinear Allen-Cahn type equation involving the positive-part function is considered.

This talk is based on a joint work [2] with C. Kuehn (TU München) and K.-I. Nakamura (Kanazawa University). We consider the Cauchy problem for a one-dimensional fully nonlinear Allen-Cahn type equation involving the positive-part function,

$$\partial_t u = \big(\partial_x^2 u - f(u)\big)_+ \ \text{ in } \mathbb{R}\times(0,+\infty), \quad u|_{t=0} = u_0 \ \text{ in } \mathbb{R}, \tag{1}$$

where $(\,\cdot\,)_+ = \max\{\cdot, 0\} \geq 0$ and f is a function in $\mathbb{R}$ satisfying

$$\begin{cases} f(a_\pm) = f(a_0) = 0, \quad f'(a_\pm) > 0, \\ f > 0 \ \text{ in } (a_-, a_0), \quad f < 0 \ \text{ in } (a_0, a_+) \end{cases} \tag{2}$$

for some $-\infty < a_- < a_0 < a_+ < +\infty$ and

$$f \in C^2(\mathbb{R}), \quad \hat{f} \geq 0, \quad f' \geq -\lambda \tag{3}$$

for some primitive function $\hat{f}$ of f and a constant $\lambda > 0$. We restrict ourselves to *traveling wave solutions* for (1), that is,

$$u(x,t) = \phi(x - x_0 - ct) \tag{4}$$

with $x_0 \in \mathbb{R}$, a velocity constant c and a monotone profile function ϕ. Traveling wave dynamics has been a topic of great interest in the field of reaction-diffusion equations such as the Allen-Cahn equation,

$$\partial_t u - \partial_x^2 u + f(u) = 0 \ \text{ in } \mathbb{R}\times\mathbb{R}_+. \tag{5}$$

Substituting (4) to the equation above, one can derive

$$-\phi'' + f(\phi) = c\phi' \ \text{ in } \mathbb{R}, \tag{6}$$

which is often called a *profile equation*. In the celebrated papers [5] and [7], traveling waves were studied for the so-called Fisher-KPP equation whose reaction term is classified as a *mono-stable* reaction, and then, a bi-stable reaction such as (5) was also studied in [6]. Thereafter there have been made many contributions devoted to this field. In particular, existence, uniqueness (up to translation) as well as exponential stability of traveling wave solutions are proved for bi-stable reactions in [4]. The existence and uniqueness are proved through the use of phase-plane analysis, and the proof of the exponential stability relies on a classical parabolic regularity theory,

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a sub- and supersolution method and the theory of Dynamical Systems (e.g., ω-limit set and linearization technique). In [3], the stability result is extended to nonlocal reaction-diffusion equations by developing a sub- and supersolution method and by excluding the other ingredients of the proof in [4].

In order to construct a traveling wave solution for (1), one may simply substitute $u(x,t) = \phi(x-ct)$ to (1), and then, the profile equation reads,

$$-c\phi' = \big(\phi'' - f(\phi)\big)_+ \quad \text{in } \mathbb{R}, \tag{7}$$

which is still severely nonlinear. Hence we shall introduce the following auxiliary obstacle problem as an alternative profile equation:

$$-c\phi' - \phi'' + \partial I_{[\alpha,+\infty)}(\phi) + f(\phi) \ni 0 \quad \text{in } \mathbb{R} \tag{8}$$

for any $\alpha \in (a_-, a_0)$. Here $\partial I_{[\alpha,+\infty)}$ denotes the subdifferential operator (with domain $D(\partial I_{[\alpha,+\infty)}) = [\alpha, +\infty)$) defined by

$$\partial I_{[\alpha,+\infty)}(s) = \begin{cases} \{0\} & \text{if } s > \alpha, \\ (-\infty, 0] & \text{if } s = \alpha, \\ \emptyset & \text{if } s < \alpha \end{cases} \tag{9}$$

of the indicator function $I_{[\alpha,+\infty)}$ supported over $[\alpha, +\infty)$, that is,

$$I_{[\alpha,+\infty)}(s) = \begin{cases} 0 & \text{if } s \geq \alpha, \\ +\infty & \text{otherwise.} \end{cases}$$

We further impose the asymptotic conditions,

$$\lim_{\xi\to-\infty} \phi(\xi) = \alpha, \quad \lim_{\xi\to+\infty} \phi(\xi) = a_+, \quad \phi' \geq 0 \quad \text{in } \mathbb{R} \tag{10}$$

and

$$\lim_{\xi\to\pm\infty} \phi'(\xi) = 0, \tag{11}$$

on the profile ϕ. Indeed, due to (2), all $u \equiv \alpha \in (a_-, a_0)$ are constant steady-states of (1), even though they are not critical points of the free energy. Hence we shall construct an entire solution (of the form (4)), which is a heteroclinic orbit connecting $u(\cdot,t) \equiv \alpha \in (a_-, a_0)$ at $t = -\infty$ and $u(\cdot,t) \equiv a_+$ at $t = +\infty$.

Our result reads,

Theorem 1 (Existence of traveling wave solutions [2])**.** *Assume that* (2) *with* $f \in C^1(\mathbb{R})$ *holds. Let* α *be a constant satisfying*

$$\alpha \in (a_-, a_0), \quad \int_\alpha^{a_+} f(z)\,\mathrm{d}z < 0. \tag{12}$$

Then the strongly irreversible Allen-Cahn equation (1) *admits a traveling wave solution* $u(x,t) = \phi(x-ct)$ *for some* $c \in \mathbb{R}$*. More precisely, there exists* c *such that* (7) *possesses a solution* ϕ *satisfying* (10) *and* (11)*. Moreover, the following conditions hold true*:

(i) $\phi \in W^{2,\infty}(\mathbb{R})$, $\phi' \in L^2(\mathbb{R})$, ϕ *solves* (8) *in* $\mathbb{R}$,
(ii) $\alpha \leq \phi < a_+$ *and* $0 \leq \phi' \leq C_1$ *in* $\mathbb{R}$, $-C_2 \leq c < 0$,

(iii) *if* $\phi(s) > \alpha$ *at* $s \in \mathbb{R}$, *then* $\phi'(s) > 0$

for some constants $C_1, C_2 > 0$ *depending only on* α *and* f. *Furthermore, there exists* $s_* \in \mathbb{R}$ *such that*

$$\phi(s) = \alpha \quad \text{on } (-\infty, s_*], \quad \phi(s) > \alpha \quad \text{in } (s_*, +\infty). \tag{13}$$

In particular, $\phi \in C^2((s_*, +\infty))$, $\phi' > 0$ *in* $(s_*, +\infty)$ *and* ϕ *solves* (6) *in* $(s_*, +\infty)$.

We here emphasize that the constrained Allen-Cahn equation (1) admits a one-parameter family of *degenerate* traveling wave solutions (even identified when coinciding up to translation), although the traveling wave solution for the classical one (5) can be uniquely (up to translation) determined (under (2)) and it is strictly monotone. Furthermore, the velocity of traveling waves for (1) can be negative, even though the potential $\hat{f}$ is balanced, i.e., $\hat{f}(a_+) = \hat{f}(a_-)$, under which the velocity of the traveling wave for (5) must be zero.

Remark 2. (i) As for the case $\alpha = a_-$, any solution ϕ of (8) with $\alpha = a_-$ solves (6) in $\mathbb{R}$, i.e., it is also the traveling wave (profile) for (5). Then ϕ is strictly monotone.

(ii) We emphasize that $\phi \notin C^2(\mathbb{R})$ under the assumption (12). Indeed, one can observe that ϕ loses C^2 regularity at s_* as follows:

$$\lim_{s \nearrow s_*} \phi''(s) = 0 \quad \text{and} \quad \lim_{s \searrow s_*} \phi''(s) = f(\alpha) > 0.$$

On the other hand, ϕ is smooth elsewhere.

We further observe that

Proposition 3 (Unique determination of velocity and profile [2])**.** *For each* α *satisfying the assumptions of Theorem 1,*

(i) *the velocity* c_α *is uniquely determined*;
(ii) *the profile function* ϕ_α *is uniquely* (*up to translation*) *determined.*

Moreover, c_α *is strictly decreasing in* $\alpha \in (a_-, a_0)$, *and furthermore, the map* $\alpha \mapsto (c_\alpha, \phi_\alpha)$ *is continuous from* (a_-, a_0) *to* $\mathbb{R} \times C(\mathbb{R})$.

In what follows, we denote by c_α and ϕ_α the velocity and unique profile function for each α, and moreover, we translate the profile function ϕ_α so that $s_* = 0$ in (13), that is,

$$\phi_\alpha \equiv \alpha \ \text{ in } (-\infty, 0] \ \text{ and } \ \phi_\alpha > \alpha \ \text{ in } (0, +\infty), \tag{14}$$

by translation.

Let us next discuss traveling wave dynamics of more general solutions to (1). To this end, combining local-energy methods with a gradient flow theory, we first exhibit well-posedness for (1) in a certain setting suitable for considering traveling wave solutions.

Theorem 4 (Existence of L^2_{loc} solutions [2])**.** *Assume* (3) *with* $f(0) = 0$. *Let* $T > 0$ *and assume that*

$$u_0 \in H^2_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \tag{15}$$

and there exist constants $\gamma_\pm \in \mathbb{R}$ *such that*

$$\gamma_- \le \inf_{x\in\mathbb{R}} u_0(x) \le \sup_{x\in\mathbb{R}} u_0(x) \le \gamma_+ \quad \textit{and} \quad \pm f(\gamma_\pm) \ge 0. \tag{16}$$

Then (1) *admits a unique bounded* L^2_{loc} *solution* $u = u(x,t)$ *on* $[0,T]$.

The next theorem will guarantee C^1 regularity both in space and time of $u(x,t)$ on $\mathbb{R}\times\mathbb{R}_+$, and this fact will enable us to determine the singularity of $u(x,t)$ at the free boundary point

$$r(t) := \sup\{r \in \mathbb{R} \colon u(x,t) = \alpha \text{ for all } x \le r\}, \tag{17}$$

provided that $r(0)$ is finite. More precisely, $u(x,t)$ loses C^2 regularity at $x = r(t)$ in a similar fashion to traveling waves, i.e.,

$$\lim_{x\to r(t)-0} \partial_x^2 u(x,t) = 0 \quad \text{and} \quad \lim_{x\to r(t)+0} \partial_x^2 u(x,t) = f(\alpha) > 0,$$

for any $t > 0$.

Theorem 5 (C^1 regularity [2])**.** *Assume* (3) *with* $f(0) = 0$, (15) *and* (16). *Then* (1) *admits a unique bounded* L^2_{loc}*-solution* $u = u(x,t)$ *and satisfies*

$$u \in W^{1,\infty}(0,T;L^2(-R,R)) \cap W^{1,2}(0,T;H^1(-R,R)),$$
$$\sqrt{t}\partial_t u \in L^\infty(0,T;H^1(-R,R)), \quad \sqrt{t}\partial_t^2 u \in L^2(0,T;L^2(-R,R))$$

for any $T > 0$ *and* $R > 0$. *In particular,* u *belongs to* $C^1(\mathbb{R}\times(0,+\infty))$.

Now, we are ready to discuss stability of traveling wave solutions constructed in Theorem 1. We shall discuss sufficient conditions for initial data to prove the convergence of L^2_{loc} solutions for (1) to traveling waves and their convergence rates. We first observe that

Proposition 6 (Instability of traveling waves in $L^\infty(\mathbb{R})$ [2])**.** *For each* $\alpha \in (a_-, a_0)$, *the traveling wave solution* $\phi_\alpha(x - c_\alpha t)$ *of* (1) *is unstable in* $L^\infty(\mathbb{R})$.

This proposition suggests that one needs to impose a more restrictive assumption on initial data than the classical Allen-Cahn (see [4, 3]) to prove convergence. Here we assume instead of (12) that

$$\alpha \in (a_-, a_0), \quad \int_{a_-}^{a_+} f(z)\,\mathrm{d}z \le 0\,; \tag{18}$$

and moreover, let us introduce the following assumptions for initial data u_0:

$(\mathbf{H})_\alpha$**:** it holds that

$$\begin{cases} u_0 \in H^2_{\mathrm{loc}}(\mathbb{R}), \quad \liminf_{x\to+\infty} u_0(x) > a_0, \\ a_- \le \inf_{x\in\mathbb{R}} u_0(x) \le \sup_{x\in\mathbb{R}} u_0(x) < a_+, \end{cases} \tag{19}$$

and furthermore, there exists $\xi_1 \in \mathbb{R}$ such that

$$u_0(x) = \alpha \quad \text{for all } x \le \xi_1. \tag{20}$$

Our main result is then stated as follows:

Theorem 7 (Exponential convergence to degenerate traveling waves [2])**.** *In addition to* (2), (3) *and* (18)*, assume that* $(\mathbf{H})_\alpha$ *and* $f'(\alpha) > 0$ *hold. Let* $u = u(x,t)$ *be an* L^2_{loc} *solution to* (1) *in* $\mathbb{R} \times \mathbb{R}_+$*. Then there exist constants* $K, \kappa > 0$ *and* $x_0 \in \mathbb{R}$ *such that*

$$\|u(\cdot,t) - \phi_\alpha(\cdot - c_\alpha t - x_0)\|_{L^\infty(\mathbb{R})} \le K\mathrm{e}^{-\kappa t} \quad \text{for all } \ t \ge 0.$$

Moreover, let $r(t)$ *be defined by* (17)*. Then* $r(t) - c_\alpha t$ *converges to* x_0 *at the rate of* $O(\mathrm{e}^{-\frac{\kappa t}{2}})$ *as* $t \to +\infty$*.*

In the theorem stated above, the condition $f'(\alpha) > 0$ is assumed in addition to (18). Under (2), one can always ensure that

$$A := \big\{\alpha \in (a_-, a_0) \colon f'(\alpha) > 0\big\} \neq \emptyset.$$

Indeed, by $f'(a_-) > 0$ along with the continuity of f', the set A involves any $\alpha \ge a_-$ close enough to a_-.

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Necessary and sufficient condition for global existence of L^2 solutions for 1D periodic NLS with non-gauge invariant quadratic nonlinearity

藤原 和将

1 概要

本講演では, 次の周期境界条件に於ける非線型 Schrödinger 方程式の初期値問題の時間大域可解性を扱う.

$$\begin{cases} i\partial_t u + \Delta u = |u|^2, & t \in (0,T), \quad x \in \mathbb{T}, \\ u(0,x) = \phi(x), & x \in \mathbb{T}. \end{cases} \tag{1}$$

i は虚数単位であり, $T > 0$, $\Delta = \partial_x^2$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ とする. 本講演の目的は, (1) の $L^2(\mathbb{T})$ に於ける時間大域可解性に対する $\phi \in L^2(\mathbb{T})$ の必要充分条件を与える事である.

(1) の解の挙動を考えるに先立って, 周期境界条件での自由 Schrödinger 方程式の初期値問題の解の挙動は, $\mathbb{R}$ 上の自由 Schrödinger 方程式の初期値問題の解の挙動とは異なる事に注意する. $\mathbb{R}$ 上の自由 Schrödinger 方程式の解は, 任意の $t > 0$ に対して評価式

$$\|e^{it\Delta}\phi\|_{L^p(\mathbb{R})} \leq (4\pi t)^{-(1/2-1/p)} \|\phi\|_{L^{p'}(\mathbb{R})} \tag{2}$$

を満たす (例えば [5] を参照). 但し, $p \geq 2$ であり p' は p の Hölder 共役指数とした. (2) は基本解の具体形を用いて示す事ができる. (2) により, $\mathbb{R}$ 上の自由 Schrödinger 方程式の解は, 時間発展に際して初期時刻とは異なる可積分性を有する事が分かる. 特に, (2) による初期時刻に於ける特異性の評価を用いる事で, (1) の様な非線型初期値問題の時間局所解が構成できる事はよく知られている. 一方で, 周期境界条件に於ける自由 Schrödinger 方程式の解に対しては, (2) の様な評価式は成立しない. 例えば, $f \in L^2(\mathbb{T})$ を任意の $p > 2$ に対して, $f \notin L^p(\mathbb{T})$ となる様に取る. 次に $\phi = e^{-i\Delta} f$ とすれば,

$$\sup_{q \in [1,2]} \|\phi\|_{L^q(\mathbb{T})} \leq \sqrt{2\pi} \|\phi\|_{L^2(\mathbb{T})} = \sqrt{2\pi} \|f\|_{L^2(\mathbb{T})}$$

であるが,

$$e^{i\Delta}\phi = f \notin \bigcup_{p>2} L^p(\mathbb{T})$$

であるから, (2) の様に時間発展に際して初期値の特異性が緩和されるとは限らない. 従って, $L^\infty(\mathbb{T})$ に埋蔵されない Sobolev 空間 H^s $(s \leq 1/2)$ に対する (1) の時間局所解の構成には, (2) とは異なる評価が必要となる. Bourgain[1] は $(t,x) \in \mathbb{T} \times \mathbb{T}$ に於ける自由 Schrödinger 方程式の解が評価式

$$\|e^{it\Delta}\phi\|_{L^4(\mathbb{T}\times\mathbb{T})} \leq \sqrt{2} \|\phi\|_{L^2(\mathbb{T})} \tag{3}$$

を満たす事を示した. (3) は時間周期解に対する評価であるが, (3) により (1) は $L^2(\mathbb{T})$ に於いて時間局所に適切である事が示される. (3) は関係式

$$|e^{it\Delta}\phi(x)|^2 = \sum_{k\in\mathbb{Z}} \left(\sum_{\ell\in\mathbb{Z}} \overline{\hat{\phi}(k)}\hat{\phi}(k+\ell)e^{-2ik\ell t} \right) e^{ikx-ik^2t} \tag{4}$$

と Plancherel の定理から従う. ここで $\hat{\phi}(k)$ は ϕ の周波数 k に対する Fourier 係数とした.

註 1.1. 評価式 (3) を用いることで, より一般の冪乗型の非線型項に対しても $L^2(\mathbb{T})$ に於ける時間局所適切性が示される. 具体的には, 冪が 1 以上 3 以下であればよい. 一方で, (3) の左辺を $L^6(\mathbb{T})$ に変更する事は出来ない. また, (3) は多次元に於いては成立しない. この為 $n \geq 2$ に対して, (1) を $\mathbb{T}^n$ で考察した場合, (1) が $L^2(\mathbb{T}^n)$ に於いて時間局所適切か否かは不明である.

(1) の時間大域可解性に就いては, 非線型項の非負値性の為, 否定的な観点から考えらえてきた. 先行研究を紹介する為に, (1) を一般化した初期値問題

$$\begin{cases} i\partial_t u + \Delta u = |u|^p, & t \in (0,T), \quad x \in \mathbb{T}^n, \\ u(0,x) = \phi(x), & x \in \mathbb{T}^n. \end{cases} \tag{5}$$

を考える. 但し, $n \geq 1$ は整数で, $p > 1$ は実数とする. 王 [4] は, $\phi \in L^1(\mathbb{T}^n)$ が条件

$$\mathrm{Im} \int_{\mathbb{T}^n} \phi(x)dx < 0 \tag{6}$$

を満たすならば, (5) は時間大域的な弱解は存在しない事を示した. (5) の時間大域的な弱解とは, $u \in L^p(0,\infty; L^p(\mathbb{T}^n))$ が, 任意の $T > 0$ と $\psi \in C([0,T]\times\mathbb{T}^n) \cap C^\infty((0,T)\times\mathbb{T}^n)$ で $\psi(T,x) = 0$ となる試験関数に対して, 弱方程式

$$\begin{aligned} &\int_0^t \int_{\mathbb{T}^n} u(t,x)(-i\partial_t\psi(t,x) + \Delta\psi(t,x))dx\ dt \\ &= i\int_{\mathbb{T}^n} \phi(x)\psi(0,x)dx + \int_0^t \int_{\mathbb{T}^n} |u(t,x)|^p\psi(t,x)dx\ dt \end{aligned} \tag{7}$$

を満たす事を云う. 王は, (7) が成立しないような試験関数の列を見出すことで, 時間大域的弱解の非存在を証明した. この手法は試験関数法と呼ばれる手法で, 周期境界条件に於ける Schrödinger 方程式に限らず, $|u|^p$ を非線型項とする非線型問題に広く用いられる汎用的な手法である. 特に, 仮定 (6) は非線型項の非負値性に呼応した仮定であり, 試験関数法を用いる上では必須の仮定である. 講演者と小澤 [2] は, 試験関数法を若干直接的な常微分方程式の議論に変更し, 仮定 (6) を拡張した. 具体的には, $\phi \in L^1(\mathbb{T}^n)$ が条件

$$\mathrm{Im} \int_{\mathbb{T}^n} \phi(x)dx < 0 \quad \text{or} \quad \mathrm{Re} \int_{\mathbb{T}^n} \phi(x)dx \neq 0 \tag{8}$$

を満たすならば, $C^1((0,\infty); L^2(\mathbb{T}^n)) \cap C([0,\infty); (H^2 \cap L^{2p})(\mathbb{T}^n))$ の元で (5) を $L^2(\mathbb{T}^n)$ の意味で満たすものは存在しない事を示した. 大域解が存在しない事を見る為に, 先ず仮定 (8) の下では複素定数 α で $\mathrm{Re}\,\alpha > 0$ 且つ

$$\mathrm{Im}\left(\alpha \int_{\mathbb{T}^n} \phi(x)dx \right) < 0$$

を満たすものが存在する事に注意する. (5) が解 $u \in C^1((0,\infty); L^2) \cap C([0,\infty); H^2 \cap L^{2p})$ を有するとする. ここに,

$$M(t) = \mathrm{Im}\left(\alpha \int_{\mathbb{T}^n} u(t,x)dx \right)$$

と置けば, (5) より評価

$$\frac{d}{dt}M(t) \geq C(\mathrm{Re}\,\alpha)M(t)^p$$

が従う事が分かる. この評価は, $u \in C([0,\infty); H^2)$ となる事に矛盾する. この様に, 仮定 (8) の下では, 解が時間大域的に延長されない事が解の Fourier-0 モードに着目した常微分方程式の議論で見る事が出来る. 更に, (8) は $\int_{\mathbb{T}^n} \phi(x)dx$ に対して, 解の時間大域可解性を見る上では精密な条件となっている. 例えば, 実定数 μ_0 に対して, 初期値を $\phi(x) = i\mu_0$ で与える. この時, (5) の解は

$$u(t,x) = \begin{cases} i(\frac{t}{p-1} + \mu_0^{-\frac{1}{p-1}})^{-p+1} & \text{if} \quad \mu_0 > 0, \\ 0 & \text{if} \quad \mu_0 = 0, \\ -i(|\mu_0|^{-\frac{1}{p-1}} - \frac{t}{p-1})^{-p+1} & \text{if} \quad \mu_0 < 0,\ t < (p-1)|\mu_0|^{-\frac{1}{p-1}} \end{cases} \tag{9}$$

で与えられる. $\mu_0 \geq 0$ の時, ϕ は条件 (8) を満足しないので, (8) を初期値が満たさない場合は, 一般に時間大域解の存在が否定されない事が (9) から分かる. 一方で, (8) を満たさない初期値でも解が時間大域的に延長されない場合があり, (8) は (5) の時間大域可解性の為の精密な条件とはなっていない.

そこで本研究では, (1) の時間大域可解性を $L^2(\mathbb{T})$ の枠組みで検討した. ここで解が $L^2(\mathbb{T})$ の意味で時間大域的とは, 解が $C([0,\infty); L^2(\mathbb{T})) \cap C^1((0,\infty); H^{-2}(\mathbb{T}))$ の元で, $H^{-2}(\mathbb{T})$ の意味で (1) を満たす事とする. この設定の下, 次の精密な条件を得た.

定理 1.1 (藤原-Georgiev, preprint)**.** (1) の解が $L^2(\mathbb{T})$ の意味で時間大域的である事と非負定数 μ_0 によって初期値が $\phi = i\mu_0$ で与えられる事は同値である.

定理 1.1 により, (1) の $L^2(\mathbb{T})$ の意味での大域解は (9) によって与えられる多項式に限られる. 定理 1.1 は, これまでの研究と異なり, 解や初期値の振動の影響を加味する事によって, 精密な時間大域可解性の条件を与えている. 実際, これまでの試験関数法や常微分方程式の議論では, 非線型項の正値性を本質的に用いる為, 解の振動の影響を検討する事は困難であった. 一方本研究では, Bourgain の手法に倣い解を自由 Schrödinger 方程式の解で分解し, 自己相互作用に於ける共鳴部の影響を評価した. この評価により, 非共鳴部の相互作用によって, 僅かな振動でさえも解の時間大域的な延長を許さない事が判明した. 特に, Bourgain の議論によって初期値問題 (1) の $L^2(\mathbb{T})$ 値時間局所解は, 縮小写像の議論によって構成される事から, 解が時間大域的でない場合は解の $L^2(\mathbb{T})$ ノルムが有限時刻で爆発する事が分かる.

次の章では, (1) の共鳴部と非共鳴部への分解の方法と共鳴部や振動項の先験的評価に就いて概説する. 第 3 章では, 定理 1.1 の証明を与える.

2 主定理の証明の準備

この章では, (1) の共鳴部と非共鳴部への分解を考える.

$\phi \in L^2(\mathbb{T})$ とし, u を (1) の $L^2(\mathbb{T})$ 値解とする. 整数 k に対して, $u_k : [0,T) \to \mathbb{C}$ を

$$u(t) = \sum_{k\in\mathbb{Z}} u_k(t)e^{ikx - ik^2t}$$

となる様に定める. ここに, e^{ikx-ik^2t} は周波数 k に対する自由 Schrödinger 方程式の解である. $u(t) \in L^2(\mathbb{T})$ に対して, $\{u_k(t)\}_{k\in\mathbb{Z}}$ は

$$u_k(t) = \hat{u}(t,k)e^{ik^2t}$$

によって一意的定まる事に注意する. この分解と関係式 (4) により, 非線型項 $|u|^2$ も自由 Schrödinger 方程式の解によって分解される. 今, (1) の両辺を自由 Schrödinger 方程式の解によって分解し, 周波数毎に係数を比較する事で, (1) は無限次元連立常微分方程式系

$$\begin{cases} \frac{d}{dt}u_k(t) = -i\sum_{\ell\in\mathbb{Z}} \overline{u_\ell(t)}u_{k+\ell}(t)e^{-2ik\ell t}, & t>0, \quad k\in\mathbb{Z}, \\ u_k(0) = \hat{\phi}(k), & k\in\mathbb{Z} \end{cases} \tag{10}$$

に書き換えられる. この常微分方程式系を, 初期値の仮定 (8) の下で更に変形する. (1) の両辺を積分し, 両辺の虚部に注目すると,

$$\frac{d}{dt}\mathrm{Re}\int_{\mathbb{T}^n} u(t,x)dx = 0$$

が任意の $t>0$ に対して成立する. 実数値関数 μ を

$$\mu(t) = -i\int_{\mathbb{T}^n} u(t,x)dx$$

によって定めれば, (10) は

$$\begin{cases} \frac{d}{dt}\mu(t) = -\mu(t)^2 - \sum_{\ell\neq 0}|u_\ell(t)|^2, & t>0, \\ \frac{d}{dt}u_k(t) = -\mu(t)u_k(t) - i\sum_{\ell\neq 0}\overline{u_\ell(t)}u_{k+\ell}(t)e^{-2ik\ell t}, & t>0, \quad k\in\mathbb{Z}\backslash\{0\}, \\ \mu(0) = -i\hat{\phi}(0), & \\ u_k(0) = \hat{\phi}(k), & k\in\mathbb{Z}\backslash\{0\} \end{cases} \tag{11}$$

と変形される. 無限次元連立常微分方程式系 (11) を簡略化する為に,

$$\nu(t) = \sum_{k\neq 0}|u_k(t)|^2$$

と置く. (11) の第二式から, ν は関係式

$$\frac{d}{dt}\nu(t) = -2\mu(t)\nu(t) + 2R(t), \tag{12}$$

を満たす. 但し,

$$R(t) = \mathrm{Im}\left(\sum_{k\neq 0}\sum_{\ell\neq 0}\overline{u_k(t)u_\ell(t)}u_{\ell+k}(t)e^{-2ik\ell t}\right) \tag{13}$$

とした. 簡単の為, $\mu_0=\mu(0)$, $\nu_0=\nu(0)$ とする. この R を (11) の共鳴部と呼び, その他の部分を非共鳴部と呼ぶことにする. (1) の解の挙動は (11) の非共鳴部によって主導的に決定される. 以下共鳴部 R を剰余項と見做す為の評価を用意する.

先ず共鳴部 R の級数は, 各時刻に於いて収束するとは限らない事に注意する. 然し次の意味で, 解の $L^2(\mathbb{T})$ ノルムによって R は制御される.

補題 2.1. *$T,\mu_0,\nu_0>0$ とし, μ と $\{u_k\}_{k\in\mathbb{Z}}\in C([0,T);\ell^2)$ を (11) の解とする.*

$$\inf_{t\in(0,T)}\mu(t) > 0$$

を仮定する. このとき, 2 つの正定数 A と B が存在し, 任意の $t\in(0,T)$ に対して, 評価

$$\begin{aligned}\left|\int_0^t R(\tau)d\tau\right| &\leq A(\mu(t)\nu(t) + \nu^{3/2}(t) + \mu_0\nu_0 + \nu_0^{3/2}) \\ &\quad + B\int_0^t\left(\mu(\tau)\nu^{3/2}(\tau) + \nu^2(\tau)\right)d\tau\end{aligned}$$

が成立する.

R の級数の各項に就いて積分を行えば,

$$\int_0^t \overline{u_k(\tau)u_\ell(\tau)}u_{\ell+k}(\tau)e^{-2ik\ell\tau}d\tau$$
$$= \frac{i}{2k\ell}\left[\overline{u_k(\tau)u_\ell(\tau)}u_{\ell+k}(\tau)e^{-2ik\ell\tau}\right]_0^t - \frac{i}{2k\ell}\int_0^t \frac{d}{d\tau}\left(\overline{u_k(\tau)u_\ell(\tau)}u_{\ell+k}(\tau)\right)e^{-2ik\ell\tau}d\tau$$

となり, 級数の収束の為に必要な k と ℓ の減衰が得られ, 補題 2.1 が示される. 補題 2.1 より, ν に関する次の先験的評価を得る.

系 2.2. 補題 *2.1* の仮定の下, μ_0 と ν_0 が充分に小さい正数ならば, 評価

$$\delta := \inf_{t\in[0,T)}\left[\nu_0 - 2A\big(\mu(t)\nu(t) + \nu(t)^{3/2} + \mu_0\nu_0 + \nu_0^{3/2}\big)\right] > 0$$

が成立する.

次の章で, 上記 2 つの評価を用いて定理 1.1 を示す.

3 主定理の証明

初期値 $\phi \in L^2(\mathbb{T})$ が条件 (8) を満たす場合は, 解は時間大域的に存在しない. また, 初期値 ϕ が純虚数の定数である場合は, (9) の通りである. 従って, $\mathrm{Re}\,\hat{\phi}(0) = 0$ で $\mathrm{Im}\,\hat{\phi}(0) = \mu_0 > 0$ である場合を示せばよい. $T = T(\phi)$ を $L^2(\mathbb{T})$ 値解 u の存在最大時刻とする. u_k, μ を前章の様に定めると, (11) から $T_0 \in (0, T]$ が存在して,

$$\lim_{t\nearrow T_0}\mu(t) = 0$$

となる. 加えて, 次の 2 つのうちどちらか一方が成立する.

場合-1 $T_0 = T = \infty$ であり, $\lim_{t\nearrow T_0}\nu(t) = 0$ となる. 即ち, 解は $L^2(\mathbb{T})$ の意味で時間大域的である.

場合-2 $T_0 < T$ であり, $\liminf_{t\nearrow T_0}\nu(t) > 0$ となる. この時, 解は時刻 $t \in (T_0, T)$ に於いて条件 (8) を満足するので, 解は $L^2(\mathbb{T})$ の意味で時間大域的に延長されない.

以下, **場合-2** のみが成立する事を背理法で示す.

場合-1 が成立するとする. この時, $\lim_{t\to\infty}(\mu(t), \nu(t)) = (0, 0)$ であるから, 一般性を失う事なく μ_0 と ν_0 は系 2.2 が適用可能な程度に小さいものとして良い. 従って, 補題 2.1 と系 2.2 より, ν は次の様に評価される.

$$\nu(t) = \nu_0 + \int_0^t -\mu(\tau)\nu(\tau) + 2R(\tau)d\tau$$
$$\geq \delta - 2\int_0^t \mu(\tau)\nu(\tau) + B\mu(\tau)\nu(\tau)^{3/2} + B\nu(\tau)^2 d\tau. \tag{14}$$

(14) を見かけ上 ν に就いて閉じた評価とする為に, 変数変換

$$[0, T_0) \ni t \mapsto s = \mu_0 - \mu(t) \in [0, \mu_0)$$

を考える. $V(s) = \nu(t)$ とすると, 評価式 (14) により,

$$V(s) \geq \delta - 2\int_0^s \frac{(\mu_0 - \sigma)V(\sigma) + B(\mu_0 - \sigma)V(\sigma)^{3/2} + BV(\sigma)^2}{(\mu_0 - \sigma)^2 + V(\sigma)}d\sigma$$
$$\geq \delta - \int_0^s 2\frac{(\mu_0 - \sigma)V(\sigma)}{(\mu_0 - \sigma)^2 + V(\sigma)} + 3BV(\sigma)d\sigma \tag{15}$$

が成立する. 但し, 最後の評価は Young の不等式によって

$$(\mu_0-\sigma)V(\sigma)^{1/2}\le\frac{1}{2}\Big((\mu_0-\sigma)^2+V(\sigma)\Big)$$

と計算される事を用いた. $\liminf_{t\nearrow T_0}\nu(t)=\liminf_{s\nearrow\mu_0}V(s)$ を評価する為に, $g\in C([0,\mu_0))\cap C^1((0,\mu_0))$ を,

$$g(s)=\frac{(\mu_0-s)^2}{W(C_1(\mu_0-s)^2)}e^{3B(\mu_0-s)}$$

と置く. ここで W は Lambert の W 関数で $W(s)e^{W(s)}=s$ となる関数であり, C_1 は δ と μ_0 によって定まる充分大きな定数とする. g は常微分不等式

$$\frac{d}{ds}g(s)\le-2\frac{(\mu_0-s)g(s)}{(\mu_0-s)^2+g(s)}-3Bg(s)$$

を $s\in(0,\mu_0)$ に対して満足する. 更に, $g(0)\le\delta$ であり, $\liminf_{s\nearrow\mu_0}g(s)>0$ である. 従って, 積分不等式に対する比較原理から

$$\liminf_{t\nearrow T_0}\nu(t)=\liminf_{s\nearrow\mu_0}V(s)>\liminf_{s\nearrow\mu_0}g(s)>0$$

を得る. これは, **場合-1** が成立するとした仮定に矛盾する. 従って, 初期値 ϕ が定数でなく, 条件 (8) を満足しない場合は, 解は L^2 の意味で時間大域的ではなく, 有限時刻で爆発する.

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