DOUBLE COVERINGS OF ARRANGEMENT COMPLEMENTS AND 2-TORSION IN MILNOR FIBER HOMOLOGY

MASAHIKO YOSHINAGA

Dedicated to the memory of Stefan Papadima

ABSTRACT. We prove that the mod 2 Betti numbers of double coverings of a complex hyperplane arrangement complement are combinatorially determined. The proof is based on a relation between the mod 2 Aomoto complex and the transfer long exact sequence.

Applying the above result to the icosidodecahedral arrangement (16 planes in the three dimensional space related to the icosidodecahedron), we conclude that the first homology of the Milnor fiber has non-trivial 2-torsion.

CONTENTS

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1. Introduction

An arrangement of hyperplanes is a finite collection of hyperplanes in a finite dimensional vector (affine, or projective) space. For a complex arrangement, we can associate several topological spaces: the complement, Milnor fiber, their covering spaces, boundary manifolds and so on. These provides important spaces in topology such as the classifying space of the pure braid (Artin) group and their subgroups.

Another important aspect of arrangement is the combinatorial structure. An arrangement defines the poset of subspaces expressed as intersections of hyperplanes. These subspaces can be also considered as (the closure of) strata of a stratification of the total space. As is naturally expected, the poset of intersections has a lot of information on the associated topological spaces. Indeed, some of topological invariant (e.g., cohomology ring of the complement [22]. See also §2 for more details) is determined by the poset of intersections, while some other can not be determined by the poset (e.g., the fundamental group [27]). Furthermore, there are many invariant whose relations to the poset of intersections are yet unclear.

Recently, Milnor fibers, and more generally, covering spaces of the arrangement complements received considerable amount of attention. See for [33] for survey on the topic. Among other results, we just recall some of recent results on the Milnor fiber of an arrangement:

- Explicit computations for interesting examples [2, 11, 20, 36].
- Upper/lower bounds of monodromy eigenspaces [3, 5, 25, 35, 36].
- Vanishing of non-trivial monodromy eigenspaces [4, 29, 30].
- Examples of arrangements with multiplicities whose first homology group of the Milnor fiber has torsion [6].
- Examples of arrangements whose higher degree homology of the Milnor fiber has torsion [9].
- Purely combinatorial description of monodromy eigenspaces for line arrangements which have only double and triple intersections. [10, 18, 26].

The purpose of this paper is to study the mod 2 homology and 2-torsion in the integral homology of covering spaces of arrangement complements. The main results of this paper are as follows. The first result is concerning mod 2 homology of the double covering.

Theorem 1.1. *(See Theorem 3.7 for more precise statement.) The mod* 2 *Betti numbers of double covers of arrangement complements are combinatorially determined.*

More precisely, we will obtain a combinatorial expression of the rank of mod 2 homology in terms of the mod 2 Aomoto complex.

The second result is about the Milnor fiber of the icosidodecahedral arrangement which is the arrangement of 16 planes in $\mathbb{R}^3$ associated to the icosidodecahedron. (See §4.1 for definition).

Theorem 1.2. *(See Theorem 4.3 (3).) Let $F_{\mathcal{A}_{\mathcal{ID}}}$ be the Milnor fiber of the icosidodecahedral arrangement $\mathcal{A}_{\mathcal{ID}}$. Then $H_1(F_{\mathcal{A}_{\mathcal{ID}}}, \mathbb{Z})$ has 2-torsion.*

The organization of this paper is as follows. In §2, we review several results concerning Milnor fiber of arrangements, intersection poset, Orlik-Solomon algebra, and Aomoto complex, which are necessary for stating results and proofs.

§3 summarizes results on abelian covering spaces. After recalling the transfer long exact sequence of a double covering in §3.1-§3.2, we prove, in §3.3, a key lemma (Lemma 3.2) which guarantees the first mod 2 Betti number does not decrease by taking certain ("$\mathbb{Z}_4$-liftable") double coverings. In §3.4, we prove the first main result, that is a combinatorial formula for the mod 2 Betti numbers of double coverings of an arrangement complement (Theorem 3.7).

In §4.1, we introduce the icosidodecahedral arrangement $\mathcal{A}_{\mathcal{ID}}$, and prove in §4.2 and §4.3 that the first homology of the Milnor fiber has 2-torsion.

Notation and Convention. We will use the following convention throughout the paper.

- In this paper, $\mathbb{Z}_n$ denotes the cyclic group (also a finite ring) $\mathbb{Z}/n\mathbb{Z}$.
- For a space X with the homotopy type of a finite CW complex, $b_i(X) = \operatorname{rank}_{\mathbb{Z}} H^i(X, \mathbb{Z})$ denotes the Betti number, and $\overline{b_i}(X) = \operatorname{rank}_{\mathbb{Z}_2} H^i(X, \mathbb{Z}_2)$ denotes the mod 2 Betti number.
- A *covering* $Y \longrightarrow X$ always means an unbranched covering. Unless otherwise stated, we assume X and Y are connected.
- We will assume that the base point $x_0 \in X$ is fixed when we consider the fundamental group $\pi_1(X) = \pi_1(X, x_0)$.
- For an element $\gamma \in \pi_1(X)$, denote by $[\gamma]$ its homology class. If $(Y, y_0) \longrightarrow (X, x_0)$ is a covering, $\tilde{\gamma}$ denotes the lifting of γ starting from the the base point y_0. Note that $\tilde{\gamma}$ is not necessarily a closed curve.

2. Generalities on hyperplane arrangements

In this section, we summarize notation and several results on hyperplane arrangements. See [23, 24] for more details.

Let $\mathcal{A} = \{H_1, H_2, \dots, H_n\}$ be a collection of affine hyperplanes in $\mathbb{C}^\ell$. We denote by $M(\mathcal{A}) = \mathbb{C}^\ell \smallsetminus \bigcup_{i=1}^n H_i$ the complement. Let $\overline{H_\infty}$ be a projective hyperplane in $\mathbb{CP}^\ell$. Using the identification $\mathbb{CP}^\ell = \mathbb{C}^\ell \sqcup \overline{H_\infty}$, $\mathcal{A}$ naturally induces an arrangement of $n+1$ projective hyperplanes

$\overline{\mathcal{A}} = \{\overline{H_1}, \dots, \overline{H_n}, \overline{H_\infty}\}$. A projective hyperplane $\overline{H_i}$ defines a linear hyperplane $\widetilde{H_i}$ in $\mathbb{C}^{\ell+1}$. The collection of these linear hyperplanes $\widetilde{\mathcal{A}} = \{\widetilde{H_1}, \dots, \widetilde{H_n}, \widetilde{H_\infty}\}$ is called the coning of $\mathcal{A}$ (which is also denoted by $c\mathcal{A}$). We can also define the opposite operation, which is called the deconing of $\widetilde{\mathcal{A}}$ with respect to $\widetilde{H_\infty}$ and denoted by $\mathcal{A} = d_{\widetilde{H_\infty}}\widetilde{\mathcal{A}}$.

There is a natural projection $p : M(c\mathcal{A}) \longrightarrow M(\mathcal{A})$. It is easily seen that $M(c\mathcal{A}) \simeq M(\mathcal{A}) \times \mathbb{C}^\times$. Let $\alpha_i : \mathbb{C}^{\ell+1} \longrightarrow \mathbb{C}$ be a non-zero linear form such that $\widetilde{H_i} = \alpha_i^{-1}(0)$ $(i = 1, \dots, n, \infty)$. Then $Q = \prod_i \alpha_i$ is a homogeneous polynomial of degree $n+1$. $F := \{x \in \mathbb{C}^{\ell+1} \mid Q(x) = 1\} \subset \mathbb{C}^{\ell+1}$ is called the Milnor fiber of $c\mathcal{A}$. Since Q is homogeneous, the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z} \simeq \{\zeta \in \mathbb{C} \mid \zeta^{n+1} = 1\}$ acts on F defined by the map $\mu : F \longrightarrow F, x \longmapsto \zeta \cdot x$. This action is nothing but the monodromy action of the fibration $Q : M(c\mathcal{A}) \longrightarrow \mathbb{C}^\times$. The monodromy map $\mu : F \longrightarrow F$ induces a linear map on homology $\mu_* : H_k(F, \mathbb{Z}) \longrightarrow H_k(F, \mathbb{Z})$. Since the map has finite order, the homology with coefficients in $\mathbb{C}$ is decomposed into the direct sum of eigenspaces,

$$H_k(F, \mathbb{C}) = \bigoplus_{\zeta^{n+1}=1} H_k(F, \mathbb{C})_\zeta,$$

where $H_k(F, \mathbb{C})_\zeta$ is the ζ-eigenspace of μ_*. Since $M(\mathcal{A})$ can be identified with the quotient $F/\langle\mu\rangle$, we have $H_k(F, \mathbb{C})_1 \simeq H_k(M(\mathcal{A}), \mathbb{C})$.

Given an affine arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$, non-empty intersections $X = \bigcap_{H\in\mathcal{S}} H$ $(\mathcal{S} \subset \mathcal{A})$ form a poset with respect to reverse inclusion, which is denoted by $L(\mathcal{A})$ and called the intersection poset. We say that $\mathcal{S} \subset \mathcal{A}$ does not intersect if $\bigcap_{H\in\mathcal{S}} H = \emptyset$. A subset $\mathcal{S} \subset \mathcal{A}$ is called dependent if $\bigcap_{H\in\mathcal{S}} H \neq \emptyset$ and $\operatorname{codim} \bigcap_{H\in\mathcal{S}} H < \#\mathcal{S}$. For $X \in L(\mathcal{A})$, we denote by $\mathcal{A}_X := \{H \in \mathcal{A} \mid H \supset X\}$ the localization of $\mathcal{A}$ at X.

From the intersection poset, the Orlik-Solomon algebra $A^\bullet_{\mathbb{Z}}(\mathcal{A})$ is defined as follows. Let $E = \bigoplus_{i=1}^n \mathbb{Z}e_i$ be the free abelian group generated by the symbols e_i corresponding to the hyperplanes H_i. Let $\wedge E = \mathbb{Z} \oplus E \oplus E^{\wedge 2} \oplus \cdots \oplus E^{\wedge n}$ be the exterior algebra on E. For given $\mathcal{S} = \{i_1, i_2, \dots, i_k\} \subset \mathcal{A}$ with $i_1 < \cdots < i_k$, let $e_{\mathcal{S}} := e_{i_1} \wedge \cdots \wedge e_{i_k}$, and define $\partial e_{\mathcal{S}} \in E^{\wedge(k-1)}$ by

$$\partial e_{\mathcal{S}} = \sum_{p=1}^k (-1)^{p-1} e_{i_1} \wedge \dots \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_k}.$$

Define $I(\mathcal{A})$ to be the ideal of $\wedge E$ generated by the following elements.

$$\{\partial e_{\mathcal{S}} \mid \mathcal{S} \text{ is dependent}\} \cup \{e_{\mathcal{S}} \mid \mathcal{S} \text{ does not intersect}\}.$$

The quotient ring $A^*_{\mathbb{Z}}(\mathcal{A}) := \wedge E/I(\mathcal{A})$ is called the Orlik-Solomon algebra. For any abelian group R, denote $A^*_R(\mathcal{A}) := A^*_{\mathbb{Z}}(\mathcal{A}) \otimes_{\mathbb{Z}} R$. Note that when R is a commutative ring, $A^*_R(\mathcal{A})$ becomes a graded commutative R-algebra.

Brieskorn's Lemma [23, Corollary 3.73] shows that the degree k component of the Orlik-Solomon algebra is

$$A_R^k(\mathcal{A}) \simeq \bigoplus_{\substack{X \in L(\mathcal{A}) \\ \operatorname{codim} X = k}} A_R^k(\mathcal{A}_X).$$

Orlik and Solomon [22] proved that each homology group $H_k(M(\mathcal{A}), \mathbb{Z})$ is torsion free and the cohomology ring is isomorphic to the Orlik-Solomon algebra. Namely, for any commutative ring R, we have an isormorphism

$$H^*(M(\mathcal{A}), R) \simeq A_R^*(\mathcal{A})$$

as graded commutative R-algebras.

The following will be used later.

Lemma 2.1. *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of n lines ($n \geq 3$) in $\mathbb{C}^2$ with unique intersection (Figure 1). Let R be an integral domain. Let $\eta = \sum_{i=1}^n a_i e_i$ and $\omega = \sum_{i=1}^n b_i e_i \in A_R^1(\mathcal{A})$. Then the following are equivalent.*

(a) $\eta \wedge \omega = 0$.
(b) $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = 0$ *or ω, η are linearly dependent (i. e., there are $c_1, c_2 \in R$, not both zero, such that $c_1\eta + c_2\omega = 0$).*

Proof. See [38, Proposition 2.1] (or [34, Lemma 3.1]). □

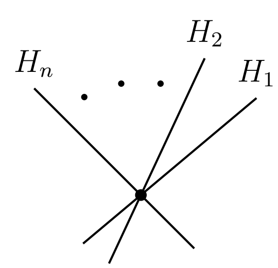

FIGURE 1. n lines with one intersection ($n \geq 3$)

For a given $\omega \in A_R^1(\mathcal{A})$, Orlik-Solomon algebra determines the cochain complex $(\mathcal{A}_R^\bullet(\mathcal{A}), \omega \wedge -)$, which is called the Aomoto complex. This complex plays crucial role in the computation of twisted cohomology groups [1, 12, 13, 19, 25, 31, 34, 38].

3. FINITE COVERINGS AND COMBINATORIAL STRUCTURES

3.1. **Finite abelian covering.** Let G be a finite abelian group. Recall that a group homomorphism $w : \pi_1(X) \longrightarrow G$ determines a finite abelian covering $p(w) : X^w \longrightarrow X$. Since G is abelian, we have

$$\operatorname{Hom}(\pi_1(X), G) = \operatorname{Hom}(H_1(X, \mathbb{Z}), G) \simeq H^1(X, G).$$

The element $w \in H^1(X, G)$ is called the characteristic class of the covering. Note that $\operatorname{Ker}(w : \pi_1(X) \to G)$ is isomorphic to $\pi_1(X^w)$ and $\pi_1(X)/\pi_1(X^w) \simeq \operatorname{Im}(w : \pi_1(X) \to G)$.

3.2. **Double covering and transfer exact sequence.** (See [15, §4.3] for details.) Now we consider the double covering $p : Y \longrightarrow X$ with the unique nontrivial deck transformation $\sigma : Y \longrightarrow Y$. Assume that Y is connected. Denote the characteristic class by $w \in H^1(X, \mathbb{Z}_2) = \operatorname{Hom}(\pi_1(X), \mathbb{Z}_2)$. Recall that the transfer map

$$\tau_* : H_k(X, \mathbb{Z}_2) \longrightarrow H_k(Y, \mathbb{Z}_2),$$

is given by $[\gamma] \longmapsto [p^{-1}(\gamma)]$. Denote by $\tau^* : H^k(Y, \mathbb{Z}_2) \longrightarrow H^k(X, \mathbb{Z}_2)$ the induced map between cohomology groups.

Example 3.1. Let us closely look at the transfer map τ_* in degree one. Let $\gamma \in \pi_1(X)$. Then either $w(\gamma) = 0$ or $\neq 0$.

(1) Suppose $w(\gamma) \neq 0$, equivalently, $\gamma \notin \pi_1(Y) (\subset \pi_1(X))$. Then the lift $\widetilde{\gamma}$ is no longer a cycle. Note that $p^{-1}(\gamma)$ is the union of $\widetilde{\gamma}$ and $\sigma(\widetilde{\gamma})$, which is the lift of γ^2. Since $\gamma^2 \in \pi_1(Y) \subset \pi_1(X)$, we have $\tau_*([\gamma]) = [\gamma^2]$.
(2) Suppose $w(\gamma) = 0$. Then the lift $\widetilde{\gamma}$ is a cycle on Y. Hence $\tau_*([\gamma]) = [\widetilde{\gamma}] + [\sigma(\widetilde{\gamma})]$.

The transfer map fits into the following long exact sequence.

$$\cdots H^{k-1}(X, \mathbb{Z}_2) \xrightarrow{w\cup} H^k(X, \mathbb{Z}_2) \xrightarrow{p^*} H^k(Y, \mathbb{Z}_2) \xrightarrow{\tau^*} H^k(X, \mathbb{Z}_2) \xrightarrow{w\cup} H^{k+1}(X, \mathbb{Z}_2) \cdots . \tag{3.1}$$

3.3. **Mod 2 Betti number of $\mathbb{Z}_4$-liftable double coverings.** In this section, we prove an inequality between the mod 2 first Betti numbers of a double covering $Y \longrightarrow X$, which will play a crucial role later in §4.3.

Lemma 3.2. *Let $w : \pi_1(X) \longrightarrow \mathbb{Z}_4$ be a surjective homomorphism. By composing the canonical surjective homomorphism $\mathbb{Z}_4 \to \mathbb{Z}_2$, we obtain an epimorphism $\overline{w} : \pi_1(X) \longrightarrow \mathbb{Z}_2$. Consider the associated double coverings $X^w \longrightarrow X^{\overline{w}}$ and $X^{\overline{w}} \longrightarrow X$.*

(1) $\overline{w} \in H^1(X, \mathbb{Z}_2)$ satisfies $\overline{w} \cup \overline{w} = 0$.
(2) $\overline{b_1}(X^{\overline{w}}) \geq \overline{b_1}(X)$.

Proof. (1) Recall that the exact sequence of abelian groups $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$ induces the exact sequence

$$H^1(X, \mathbb{Z}_4) \xrightarrow{\varphi} H^1(X, \mathbb{Z}_2) \xrightarrow{\beta} H^2(X, \mathbb{Z}_2).$$

The first map φ sends w to $\overline{w}$. The second map $\beta : H^1(X, \mathbb{Z}_2) \to H^2(X, \mathbb{Z}_2)$ is the so-called Bockstein homomorphism, which is $\beta(x) = x \cup x$ [14, Section 4.L]. Since $\beta \circ \varphi = 0$, we have $\overline{w} \cup \overline{w} = 0$.

(2) Since $p^* : H^0(X, \mathbb{Z}_2) \simeq H^0(X^{\overline{w}}, \mathbb{Z}_2)$ is an isomorphism, the beginning of the transfer exact sequence is as follows.

$$0 \to H^0(X, \mathbb{Z}_2) \stackrel{\overline{w}\cup}{\to} H^1(X, \mathbb{Z}_2) \stackrel{p^*}{\to} H^1(X^{\overline{w}}, \mathbb{Z}_2) \stackrel{\tau^*}{\to} H^1(X, \mathbb{Z}_2) \stackrel{\overline{w}\cup}{\to} H^2(X, \mathbb{Z}_2).$$

We have

$$\operatorname{rank}_{\mathbb{Z}_2} H^1(X^{\overline{w}}, \mathbb{Z}_2) = \operatorname{rank}_{\mathbb{Z}_2} H^1(X, \mathbb{Z}_2) - 1 + \operatorname{rank}_{\mathbb{Z}_2} \operatorname{Im}(\tau^*).$$

Since $\overline{w} \in \operatorname{Ker}(\overline{w}\cup) = \operatorname{Im}(\tau^*)$, $\operatorname{rank}_{\mathbb{Z}_2} \operatorname{Im}(\tau^*) \geq 1$. This completes the proof. □

Remark 3.3. Without the assumption of $\mathbb{Z}_4$-liftability in Lemma 3.2 (2), the inequality between mod 2 Betti numbers does not hold in general. For example, the double covering $S^m \longrightarrow \mathbb{RP}^m$ does not satisfy the inequality for $m \geq 2$. Indeed $\operatorname{rank}_{\mathbb{Z}_2} H^1(S^m, \mathbb{Z}_2) = 0$ and $\operatorname{rank}_{\mathbb{Z}_2} H^1(\mathbb{RP}^m, \mathbb{Z}_2) = 1$.

Corollary 3.4. (1) *Let $X_k \to X_{k-1} \to \cdots \to X_0$ ($k \geq 2$) be a tower of double coverings of connected spaces (i.e., each $X_i \to X_{i-1}$ is a double covering) such that $X_k \to X_0$ is a cyclic $\mathbb{Z}_{2^k}$-covering. Then*

$$\overline{b_1}(X_{k-1}) \geq \overline{b_1}(X_{k-2}) \geq \cdots \geq \overline{b_1}(X_1) \geq \overline{b_1}(X_0).$$

(2) *Let $w : \pi_1(X) \longrightarrow \mathbb{Z}$ be a surjective homomorphism, and $\overline{w} : \pi_1(X) \longrightarrow \mathbb{Z}_{2^k}$ be the induced surjection ($k \geq 1$). Then*

$$\overline{b_1}(X^{\overline{w}}) \geq \overline{b_1}(X).$$

Proof. (1) is proved by induction on k using Lemma 3.2. (2) follows immediately from (1). □

3.4. **Double coverings of arrangement complements.** Now let us formulate a problem asking whether the Betti numbers of finite coverings of arrangement complements are combinatorially determined.

Problem 3.5. *Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^\ell$. Let G be a finite abelian group, and R be an abelian group. Let $w \in A^1_G(\mathcal{A}) \simeq \operatorname{Hom}(\pi_1(M(\mathcal{A})), G)$. Describe the cohomology groups $H^k(M(\mathcal{A})^w, R)$ (or their ranks) in terms of $L(\mathcal{A})$ and $w \in A^1_G(\mathcal{A})$.*

Example 3.6. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of n hyperplanes and $G = \mathbb{Z}_{n+1}$ be the cyclic group of order $(n+1)$. Let $w := e_1 + e_2 + \cdots + e_n \in A^1_{\mathbb{Z}_{n+1}}(\mathcal{A})$. Then the corresponding covering $M(\mathcal{A})^w$ is homeomorphic to the Milnor fiber F of the coning $c\mathcal{A}$.

Problem 3.5 is widely open. Indeed, many research problems are related to Problem 3.5 [8, 16, 17, 28]. For example, the notion of characteristic variety is deeply related to the computation of $\operatorname{rank}_{\mathbb{Z}} H^k(M(\mathcal{A})^w, \mathbb{Z})$. See [32, 33] for surveys of the topic.

As a special case of Problem 3.5, we now prove that the mod 2 Betti numbers of the double covering $M(\mathcal{A})^w$ of an arrangement complement are combinatorially determined.

Theorem 3.7. *Let $w \in A^1_{\mathbb{Z}_2}(\mathcal{A}), w \neq 0$. Then the k-th mod 2 Betti number ($k \geq 0$) of the double covering $M(\mathcal{A})^w$ is expressed as follows.*

$$\overline{b_k}(M(\mathcal{A})^w) = b_k(M(\mathcal{A})) + \operatorname{rank}_{\mathbb{Z}_2} H^k(A^\bullet_{\mathbb{Z}_2}(\mathcal{A}), w \wedge -). \tag{3.2}$$

Proof. Denote $\mathcal{Z}^k := \operatorname{Ker}(w\wedge : A^k_{\mathbb{Z}_2}(\mathcal{A}) \to A^{k+1}_{\mathbb{Z}_2}(\mathcal{A}))$ and $\mathcal{B}^k := \operatorname{Im}(w\wedge : A^{k-1}_{\mathbb{Z}_2}(\mathcal{A}) \to A^k_{\mathbb{Z}_2}(\mathcal{A}))$. From the transfer long exact sequence, we have the following exact sequence.

$$0 \longrightarrow \mathcal{B}^k \longrightarrow A^k_{\mathbb{Z}_2}(\mathcal{A}) \longrightarrow H^k(M(\mathcal{A})^w, \mathbb{Z}_2) \longrightarrow \mathcal{Z}^k \longrightarrow 0.$$

Since $\mathcal{Z}^k/\mathcal{B}^k \simeq H^k(A^\bullet_{\mathbb{Z}_2}(\mathcal{A}), w \wedge -)$,

$$\begin{aligned}\operatorname{rank}_{\mathbb{Z}_2} H^k(M(\mathcal{A})^w, \mathbb{Z}_2) &= \operatorname{rank}_{\mathbb{Z}_2} A^k_{\mathbb{Z}_2}(\mathcal{A}) + \operatorname{rank}_{\mathbb{Z}_2} \mathcal{Z}^k - \operatorname{rank}_{\mathbb{Z}_2} \mathcal{B}^k \\ &= b_k(M(\mathcal{A})) + \operatorname{rank}_{\mathbb{Z}_2} H^k(A^\bullet_{\mathbb{Z}_2}(\mathcal{A}), w \wedge -).\end{aligned}$$

□

Remark 3.8. Note that Theorem 3.7 gives only mod 2 Betti numbers. It is not clear whether we can combinatorially describe Betti numbers or (mod 2) cohomology ring structure of the double covering.

4. 2-TORSIONS IN MILNOR FIBER HOMOLOGY

4.1. Icosidodecahedral arrangement $\mathcal{A}_{\mathcal{ID}}$.

Definition 4.1. The icosidodecahedral arrangement $\mathcal{A}_{\mathcal{ID}}$ is the coning of the 15 affine lines in Figure 2. Namely, $\mathcal{A}_{\mathcal{ID}}$ consists of 16 planes in $\mathbb{R}^3$. (Another deconing of $\mathcal{A}_{\mathcal{ID}}$ is depicted in Figure 3.)

Let us briefly comment on the naming "Icosidodecahedral arrangement". Actually, $\mathcal{A}_{\mathcal{ID}}$ can be constructed from the icosidodecahedron as follows, which seems to be the most symmetric realization of $\mathcal{A}_{\mathcal{ID}}$.

The icosidodecahedron (Figure 4) is a polyhedron which is commonly obtained as vertex truncations of the icosahedron and the dodecahedron (by truncating mid points of edges). An icosidodecahedron has 32 faces (20 triangles and 12 pentagons), 60 edges and 30 vertices (Figure 4). We can choose 10 edges to form the equator of the polyhedron, which are lying on a plane in $\mathbb{R}^3$. Similarly, we obtain 6 planes in total (they correspond to the blue lines $H_{11}, \dots, H_{15}, H_{16}$ in Figures 2 and 3).

Each pentagonal face of the icosidodecahedron has five diagonals. There are 60 such diagonals in all. If we choose appropriately consecutive six of them, they are lying on a plane (the red diagonals in Figure 4). We obtain

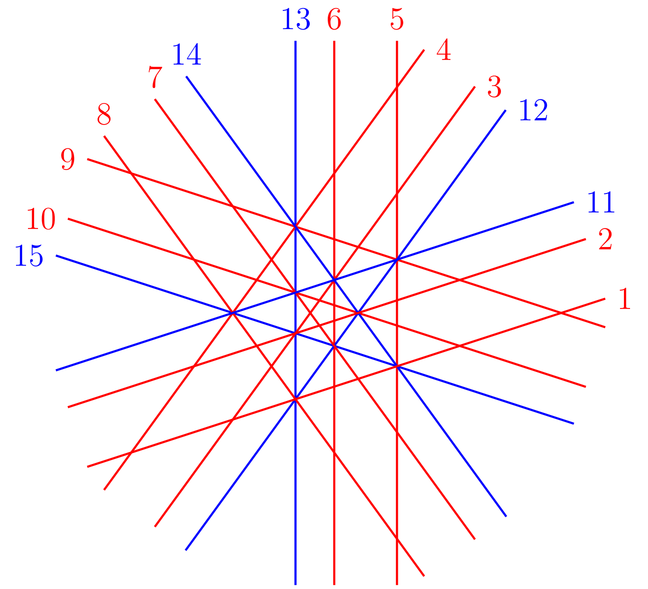

FIGURE 2. A deconing $d_{H_{16}}\mathcal{A}_{\mathcal{ID}}$ of the icosidodecahedral arrangement (with respect to a blue line at infinity H_{16}). The coloring expresses the nontrivial cocycle in the mod 2 Aomoto complex (§4.2).

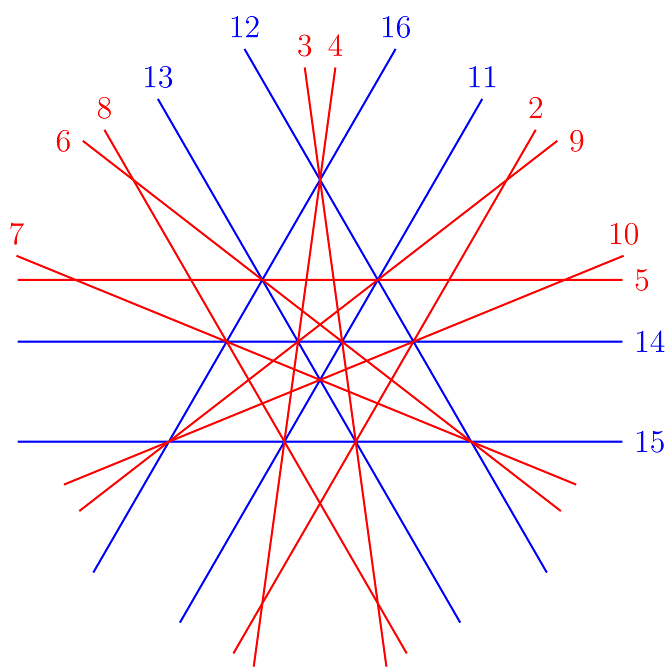

FIGURE 3. Another deconing $d_{H_1}\mathcal{A}_{\mathcal{ID}}$ of the icosidodecahedral arrangement (with respect to a red line H_1 at infinity)

in this manner 10 planes consisting of diagonals (Figure 5). As a result, we have an arrangement of 16 planes in $\mathbb{R}^3$, which is isomorphic to $\mathcal{A}_{\mathcal{ID}}$.

For computations, we use mainly the deconing from Figure 2.

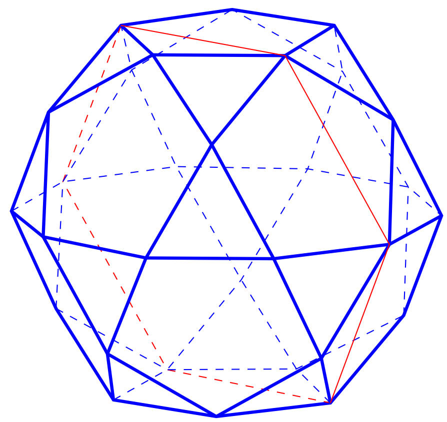

FIGURE 4. Icosidodecahedron (blue) and a diagonal plane (red)

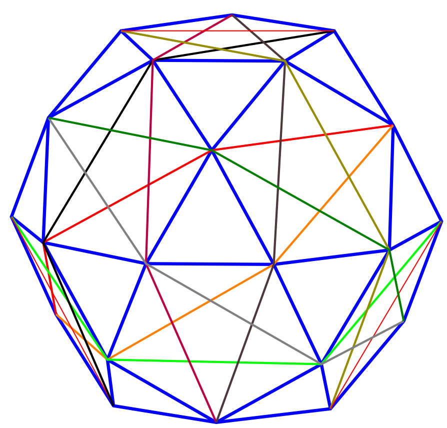

FIGURE 5. Icosidodecahedral arrangement

4.2. **Mod 2 Aomoto complex of icosidodecahedral arrangement.** Let $d\mathcal{A}_{\mathcal{ID}} := d_{H_{16}}\mathcal{A}_{\mathcal{ID}}$ be the affine arrangement in Figure 2. Let $w_2 := e_1 + \cdots + e_{15} \in A^1_{\mathbb{Z}_2}(d\mathcal{A}_{\mathcal{ID}})$. For a subset $\mathcal{S} \subset d\mathcal{A}_{\mathcal{ID}}$, let $e_{\mathcal{S}} := \sum_{H_i \in \mathcal{S}} e_i$. Obviously, every element in $A^1_{\mathbb{Z}_2}(d\mathcal{A}_{\mathcal{ID}})$ can be expressed as $e_{\mathcal{S}}$, where $\mathcal{S}$ is a subset of $d\mathcal{A}_{\mathcal{ID}}$.

Proposition 4.2. $\operatorname{rank}_{\mathbb{Z}_2} H^1(A^{\bullet}_{\mathbb{Z}_2}(d\mathcal{A}_{\mathcal{ID}}), w_2 \wedge -) = 1.$

Proof. Let $\mathcal{S}_0 := \{1, 2, \ldots, 10\}$ (red in Figure 2) and $\mathcal{S}_1 := \{11, 12, 13, 14, 15\}$ (blue in Figure 2). Note that at each intersection p, the localization $(d\mathcal{A}_{\mathcal{ID}})_p$ consists of either

- two lines from $\mathcal{S}_0$, or
- two from $\mathcal{S}_0$ and two from $\mathcal{S}_1$.

This property, together with Lemma 2.1, enables us to conclude

$$w_2 \wedge e_{\mathcal{S}_0} = w_2 \wedge e_{\mathcal{S}_1} = 0.$$

Therefore, $\operatorname{rank}_{\mathbb{Z}_2} H^1(A^\bullet_{\mathbb{Z}_2}(d\mathcal{A}_{\mathcal{ID}}), w_2 \wedge -) \geq 1$.

Next we show that $[e_{\mathcal{S}_0}] = [e_{\mathcal{S}_1}]$ is the unique nonzero cohomology class. Suppose $w_2 \wedge e_{\mathcal{S}} = 0$ for some $\mathcal{S} \subset d\mathcal{A}_{\mathcal{ID}}$. If $\mathcal{S} \cap \mathcal{S}_0 \neq \emptyset$, choose an $i \in \mathcal{S} \cap \mathcal{S}_0$, then by Lemma 2.1, all j such that i and j intersect at a double point must be contained in $\mathcal{S}$. Thus, if $\mathcal{S} \cap \mathcal{S}_0 \neq \emptyset$, $\mathcal{S} \supset \mathcal{S}_0$. By replacing $e_{\mathcal{S}}$ by $e_{\mathcal{S}} + w_2 = e_{d\mathcal{A}_{\mathcal{ID}} \smallsetminus \mathcal{S}}$, we may assume $\mathcal{S} \cap \mathcal{S}_0 = \emptyset$, in other words, $\mathcal{S} \subset \mathcal{S}_1$. Again by Lemma 2.1, $\mathcal{S}$ must be either $\mathcal{S}_1$ or $\emptyset$. This completes the proof. $\square$

4.3. **Milnor fiber of icosidodecahedral arrangement.**

Theorem 4.3. *Let $F_{\mathcal{A}_{\mathcal{ID}}}$ be the Milnor fiber of the icosidodecahedral arrangement $\mathcal{A}_{\mathcal{ID}}$. Then,*

(1) $\operatorname{rank}_{\mathbb{Z}} H^1(F_{\mathcal{A}_{\mathcal{ID}}}, \mathbb{Z}) = 15$.
(2) $\operatorname{rank}_{\mathbb{Z}_2} H^1(F_{\mathcal{A}_{\mathcal{ID}}}, \mathbb{Z}_2) \geq 16$.
(3) *The integral first homology group $H_1(F_{\mathcal{A}_{\mathcal{ID}}}, \mathbb{Z})$ has 2-torsion.*

Proof. (1) Recall that $H_1(F_{\mathcal{A}_{\mathcal{ID}}}, \mathbb{Z})$ admits $\mathbb{Z}_{16}$ action by monodromy. Then the homology with complex coefficients has eigenspace decomposition

$$H^1(F_{\mathcal{A}_{\mathcal{ID}}}, \mathbb{C}) = \bigoplus_{\lambda^{16}=1} H^1(F_{\mathcal{A}_{\mathcal{ID}}}, \mathbb{C})_\lambda.$$

Each eigenspace $H^1(F_{\mathcal{A}_{\mathcal{ID}}}, \mathbb{C})_\lambda$ is known to be isomorphic to $H^1(M(d\mathcal{A}_{\mathcal{ID}}), \mathcal{L}_\lambda)$ [7], where $\mathcal{L}_\lambda$ is the complex rank one local system on $M(d\mathcal{A}_{\mathcal{ID}})$ which has monodromy λ along the meridian of each line $H \in d\mathcal{A}_{\mathcal{ID}}$ in $\mathbb{C}^2$. In particular, the 1-eigen space is $H^1(F_{\mathcal{A}_{\mathcal{ID}}}, \mathbb{C})_1 \simeq H^1(M(d\mathcal{A}_{\mathcal{ID}}), \mathbb{C}) \simeq \mathbb{C}^{15}$. For $\lambda \neq 1$, there are several practical ways to check that $H^1(F_{\mathcal{A}_{\mathcal{ID}}}, \mathbb{C})_\lambda = 0$. One of the methods is to apply the result by Esnault-Schechtman-Viehweg [12] and Schechtman-Varchenko-Terao [31]. Let $a_1, \dots, a_{16} \in \mathbb{C}$. Assume the following three conditions.

(P1) $\exp\left(2\pi\sqrt{-1} \cdot a_i\right) = \lambda$,
(P2) $\sum_{i=1}^{16} a_i = 0$.
(P3) Let $\overline{\mathcal{A}_{\mathcal{ID}}}$ be the induced projective arrangement of 16 lines. For each quadruple intersection $p \in \mathbb{CP}^2$ of $\overline{\mathcal{A}_{\mathcal{ID}}}$, the sum $\sum_{H_i \ni p} a_i$ is not contained in $\mathbb{Z}_{>0}$.

(In some literature, such a local system is called admissible [21].) Then [12, 31] asserts that

$$H^k(M(d\mathcal{A}_{\mathcal{ID}}), \mathcal{L}_\lambda) \simeq H^k(A^\bullet_{\mathbb{C}}(d\mathcal{A}_{\mathcal{ID}}), \eta \wedge -), \tag{4.1}$$

for $k \geq 0$, where $\eta = \sum_{i=1}^{15} a_i e_i \in \mathcal{A}^1_{\mathbb{C}}(d\mathcal{A}_{\mathcal{ID}})$.

Here let us illustrate how to prove $H^1(M(d\mathcal{A}_{\mathcal{ID}}), \mathcal{L}_\lambda) = 0$ for $\lambda = -1$. We can choose $a_1, \dots, a_{16}$ as

$$\begin{aligned} a_1 = a_2 = \cdots = a_9 = a_{10} &= \frac{1}{2}, \\ a_{11} = a_{12} = \cdots = a_{15} &= -\frac{1}{2} \\ a_{16} &= -\frac{5}{2}. \end{aligned}$$

Then at each quadruple point, the sum of a_i's is either 0 or -2. Thus the conditions (P1), (P2) and (P3) are verified. Set $\eta := \sum_{i=1}^{15} a_i e_i \in A^1_{\mathbb{C}}(d\mathcal{A}_{\mathcal{ID}})$. We can check $H^k(A^\bullet_{\mathbb{C}}(d\mathcal{A}_{\mathcal{ID}}), \eta\wedge -) = 0$ by arguments similar to the proof of Proposition 4.2. The vanishing of the eigenspaces for other eigenvalues λ can be proved in a similar way.

We can obtain the same result also by using resonant band algorithms formulated in [36, 37].

(2) follows from Theorem 3.7, Proposition 4.2 and Corollary 3.4.

(3) By universal coefficient theorem and (1), if $H_1(F_{\mathcal{A}_{\mathcal{ID}}}, \mathbb{Z})$ does not have 2-torsion, $H_1(F_{\mathcal{A}_{\mathcal{ID}}}, \mathbb{Z}_2) \simeq H_1(F_{\mathcal{A}_{\mathcal{ID}}}, \mathbb{Z}) \otimes \mathbb{Z}_2$ has rank 15 over $\mathbb{Z}_2$. This contradicts (2). □

Remark 4.4. The proof of Theorem 4.3 works more generally. Let $\mathcal{A}$ be an arrangement in $\mathbb{C}^3$. Suppose

- $\#\mathcal{A}$ is a power of 2.
- The first cohomology of the mod 2 Aomoto complex of the deconing of $\mathcal{A}$ does not vanish.
- The first cohomology of the Milnor fiber $H^1(F_{\mathcal{A}}, \mathbb{C})$ does not have nontrivial monodromy eigenspaces.

Then we can conclude $H_1(F_{\mathcal{A}}, \mathbb{Z})$ has 2-torsion.

Remark 4.5. Proposition 4.2 and Theorem 4.3 (1) show that $\mathcal{A}_{\mathcal{ID}}$ is a counterexample to a conjectures in [26, Conjecture 1.9]. More precisely, the equality $\beta_2 = e_2$ in [26] does not hold for $\mathcal{A}_{\mathcal{ID}}$.

Remark 4.6. Enrique Artal-Bartolo communicated to us that he checked by computer that $H_1(F_{\mathcal{A}_{\mathcal{ID}}}, \mathbb{Z}) \simeq \mathbb{Z}^{15} \oplus \mathbb{Z}_2$. It would be a challenging problem to develop a method which can check the result theoretically. The following are also interesting problems.

(1) Describe the mod 2 Betti numbers of the Milnor fibers of arrangements.
(2) Describe the mod 2 cohomology rings of double covers of arrangement complements.

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