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Differential Geometry of Statistical Submanifolds

(統計部分多様体の微分幾何学)

Naoto Satoh

Department of Mathematics
Graduate School of Science
Hokkaido University

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Introduction

In the 1980s, the notion of statistical manifolds has arisen from information geometry. The statistical manifold is a differential geometrical abstraction of the set of probability distributions. It is defined as a manifold M equipped with an affine connection ∇ and a Riemannian metric g . A statistical manifold (M, ∇, g) is naturally endowed with another affine connection called the dual connection denoted by ∇^* . It should be noted that statistical manifolds are reduced to Riemannian manifold if the attached connection ∇ coincides with the Levi-Civita connection ∇^g of g .

In Riemannian geometry, É. Cartan [16] showed that a Riemannian manifold (M, g) is locally determined by their Riemannian curvature tensor R^g and its behavior under ∇^g -parallel translation. The fact above indicates that the Riemannian curvature tensor has vital information about the Riemannian manifold. Using the Riemannian curvature tensor, we can define the Riemannian sectional curvature, one of the most important concepts in Riemannian geometry.

A natural question arises: what geometric invariants can characterize statistical manifolds? The answer is that a statistical manifold (M, ∇, g) is locally determined by its Riemannian curvature tensor R^g and the difference tensor $K = \nabla - \nabla^g$ and their behavior under ∇^g -parallel translations (Theorem 2.1.2). Therefore, we see that the Riemannian curvature tensor and the difference tensor have the information necessary to investigate local statistical structures. By using the Riemannian curvature tensor and the difference tensor, we can define the statistical curvature tensor as follow:

$$\begin{aligned} S(X, Y)Z &= \frac{1}{2}\{R^\nabla(X, Y)Z + R^{\nabla^*}(X, Y)Z\} \\ &= R^g(X, Y)Z + [K, K](X, Y)Z \end{aligned}$$

for any vector fields X, Y, Z , which is introduced by H. Furuhashi and I. Hasegawa [21] and B. Opozda [39]. Although R^∇ or R^{∇^*} cannot define a sectional curvature, we can introduce the statistical sectional curvature by using the statistical curvature tensor S on (M, ∇, g) . In the geometry of statistical manifolds, the statistical sectional curvature, the statistical Ricci tensor, and the statistical scalar curvature, each of three defined by using the statistical curvature tensor S , is the main intrinsic curvature invariants. However, there has been little study of statistical manifolds from intrinsic geometry's viewpoint by using statistical sectional curvature.

On the other hand, information geometry is a method for investigating statistical inference mechanisms, information loss, estimation, and so on, based on the differential

geometrical studies of the set of probability distributions. In information geometry, a set of probability distributions is regarded as a submanifold embedded in more general space, and the shape of the submanifolds is associated with statistical properties ([1], [2], [3]). Therefore, the study of submanifolds of statistical manifolds stands at the core of the study of information geometry.

It is a fundamental problem in the submanifold theory to establish some relationships between intrinsic invariants and extrinsic invariants of a submanifold. The first major contributor to the subject was L. Euler. For a surface M in the Euclidean space $\mathbb{E}^3$, the Euler inequality

$$\|\widehat{H}\|^2 \geq G$$

is fulfilled, where G is the (intrinsic) Gaussian curvature of M and $\|\widehat{H}\|^2$ is the (extrinsic) squared mean curvature of M . Furthermore, $\|\widehat{H}\|^2 = G$ at every point $x \in M$ if and only if M is totally umbilical, namely, a part of a plane or a part of a round sphere.

In 1979, P. Wintgen [49] generalized the equation above for any surfaces M in $\mathbb{E}^4$, and it is called the Wintgen inequality.

Whereas, in 1999, B. Y. Chen [18] generalized the Euler inequality and established a sharp relationship between the Ricci curvature and the squared mean curvature for arbitrary Riemannian submanifold in real space form known as the Chen inequality. Let (M, g) be an m -dimensional Riemannian submanifold in an $(m + p)$ -dimensional Riemannian manifold of constant Riemannian sectional curvature c . Then, for each unit tangent vector $X \in T_x M$, the relationship

$$\|\widehat{H}\|^2 \geq \frac{4}{m^2} \{\text{Ric}^g(X, X) - (m - 1)c\},$$

holds, where $\text{Ric}^g(X, X)$ is the Riemannian Ricci curvature for X . It is remarked that the Riemannian Ricci curvature for X is denoted by $\text{Ric}^g(X, X) = \sum_{i=2}^m \mathcal{K}^g(X \wedge e_i)$, where $\{X = e_1, e_2, \dots, e_m\}$ is an orthonormal frame of M , and $\mathcal{K}^g(X \wedge e_i)$ the Riemannian sectional curvature of a plane section spanned by $\{X, e_i\}$.

Since then, similar problems for different types of submanifolds in various ambient spaces have been studied, such as complex space forms [33] and Kenmotsu space forms [6] and so on.

In recent years, this inequality has been extended for statistical submanifolds in statistical manifolds and intensively studied by many researchers. In 2015, M. E. Aydin et al. [8] considered statistical submanifolds in statistical manifolds of constant curvature. Moreover, in 2017, they generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature [9]. In 2018, H. Aytimur and C. Özgür [12]

studied the case of statistical submanifolds in statistical manifolds of quasi-constant curvature and yielded similar inequalities. Moreover, A. Mihai and I. Mihai studied of statistical submanifolds in Hessian manifolds of constant Hessian curvature [34].

However, the problem seems to lie that one has not established inequality in the most straightforward and unified case. Moreover, it is hard to find researches about the statistical submanifolds such that the equality holds, although such submanifolds must play meaningful roles in many ways because they are generalized objects of a plane or a sphere in each setting.

In this thesis, we present some intrinsic notions by using statistical sectional curvature and investigate their properties. Furthermore, we establish superficial relationships between the main intrinsic invariants and the main extrinsic invariants of statistical submanifolds in statistical manifolds without additional assumptions for the ambient space.

Chapter 1 prepares some definitions and basic properties and formulas in affine connections, Riemannian metrics, statistical structures, and Hessian structures.

Chapter 2 considers the statistical diffeomorphisms, which preserve statistical structures between the same dimensional statistical manifolds, based on [41]. We explain that a statistical structure is locally characterized by its Riemannian curvature tensor and difference tensor and their behaviors under parallel translation (Theorem 2.1.2, [41]). Besides, we also show that a Hessian structure is locally determined by its Hessian curvature tensor and difference tensor and their behaviors under parallel translation (Theorem 2.2.1, [41]).

Chapter 3 deals with the statistical sectional curvature determined from the statistical curvature tensor field S , the arithmetic mean of curvature tensor fields with respect to the pair of dual connections. We also define the statistical Ricci tensor and statistical scalar curvature. We discuss statistical manifolds of constant statistical sectional curvature. Especially, in Example 3.2.5 ([42]), we construct a statistical manifold $(\mathbb{H}^m, \nabla = \nabla^g + K, g)$ which has constant statistical sectional curvature $c = \kappa(\alpha^2 - 1)$, where $\kappa > 0$ and $\alpha \in \mathbb{R}$, as follows:

$$\begin{aligned}\mathbb{H}^m &= \{y = (y^1, \dots, y^{m-1}, y^m) \in \mathbb{R}^m \mid y^m > 0\}, \\ g &= (\kappa)^{-1}(y^m)^{-2} \sum_{j=1}^m (dy^j)^2, \\ K(\partial_i, \partial_j) &= \delta_{ij}(-\alpha(y^m)^{-1} \partial_m + \alpha_i \partial_i), \\ K(\partial_i, \partial_m) &= K(\partial_m, \partial_i) = -\alpha(y^m)^{-1} \partial_i, \\ K(\partial_m, \partial_m) &= -2\alpha(y^m)^{-1} \partial_m,\end{aligned}$$

where $\partial_i = \frac{\partial}{\partial y^i}$ ($i = 1, \dots, m$), for $\alpha \in \mathbb{R}$, $\alpha_i \in C^\infty(\mathbb{H}^m)$ and $i, j = 1, \dots, m-1$. This example shows that for any constant $c \in (-\infty, \infty)$, we can construct a statistical manifold of constant

sectional curvature c on the hyperbolic space $(\mathbb{H}^m, g)$. Although the sign of curvature is one of the most important characteristics in Riemannian geometry, this example implies that the sign of curvature does not work in the geometry of statistical manifolds.

Furthermore, we introduce statistical solitons, and statistical Einstein manifolds as a generalization of statistical manifolds of constant statistical sectional curvature, and statistical Ricci solitons as statistical Einstein manifolds. We give their properties and prove the following:

Theorem 3.3.10. ([43]) *A statistical soliton is a statistical Ricci soliton.*

Chapter 4 treats statistical submanifolds in statistical manifolds and gives a generalized Euler inequality for a statistical submanifold, obtained in joint research [42] with H. Furuhashi and I. Hasegawa et al. For an m -dimensional statistical submanifold (M, ∇, g) in an $(m + p)$ -dimensional statistical manifold $(\tilde{M}, \tilde{\nabla}, \tilde{g})$, we show the following inequality holds for any point on (M, ∇, g) :

$$\|H\|^2 + \|H^*\|^2 \geq \frac{8}{m^2} \left\{ 2\text{Ric}^g(X, X) - L(X, X) - \sum_{j=2}^m \left(2\mathcal{K}^{\tilde{g}}(X \wedge e_j) - \tilde{\mathcal{K}}(X \wedge e_j) \right) \right\},$$

where $\{X = e_1, e_2, \dots, e_m\}$ is an orthonormal basis of $T_x M$, and Ric^g the Riemannian Ricci tensor, L the statistical Ricci tensor, $\mathcal{K}^{\tilde{g}}$ the Riemannian sectional curvature, $\tilde{\mathcal{K}}$ the statistical sectional curvature, $\|H\|^2$ and $\|H^*\|^2$ the squared mean curvature with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively.

By using the proposition above, we give a generalized Euler inequality as follows:

Theorem 4.2.8. ([42]) *For an m -dimensional statistical submanifold (M, ∇, g) in an $(m + p)$ -dimensional statistical manifold $(\tilde{M}, \tilde{\nabla}, \tilde{g})$, we have*

$$\|H\|^2 + \|H^*\|^2 \geq \frac{8}{m^3} \left(2\rho^g - \rho - 2 \sum_{1 \leq i < j \leq m} (2\mathcal{K}^{\tilde{g}}(e_i \wedge e_j) - \tilde{\mathcal{K}}(e_i \wedge e_j)) \right).$$

The equality holds at $x \in M$ if and only if either (i) $m \geq 3$ and x is a doubly autoparallel point or (ii) $m = 2$ and x is a doubly totally-umbilical point.

This theorem indicates that the equality characterizes doubly autoparallel statistical submanifolds. In section 4.3, we prove the existence theorem of doubly autoparallel statistical submanifold (Theorem 4.3.3).

Chapter 5 considers warped product statistical manifolds. We give a generalized Euler inequality for the case where the ambient space is the standard warped product statistical manifold $I \times_f \bar{N}(c)$ of a line and a statistical manifold of constant statistical sectional curvature.

Theorem 5.2.4. ([42]) *For an m -dimensional statistical submanifold (M, ∇, g) in an $(m + p)$ -dimensional standard warped product statistical manifold $I \times_f \bar{N}(c)$, we have*

$$\begin{aligned} \|H\|^2 + \|H^*\|^2 &\geq \frac{8}{m^3} \left\{ 2\rho^g - \rho - 4 \sum_{1 \leq i < j \leq m} \mathcal{K}^g(e_i \wedge e_j) \right. \\ &\quad \left. + m(m-1) \left(\frac{c}{f^2} - \left(\frac{f'}{f} \right)^2 \right) \right. \\ &\quad \left. - 2(m-1) \left(\frac{c}{f^2} - \left(\frac{f'}{f} \right)^2 + \frac{f''}{f} \right) \|T\|^2 \right\}. \end{aligned}$$

The equality holds at $x \in M$ if and only if either (i) $m \geq 3$ and x is a doubly autoparallel point or (ii) $m = 2$ and x is a doubly totally-umbilical point.

Furthermore, we consider a Kenmotsu statistical manifold, which is locally constructed as the warped product of a line and a Kähler manifold. We introduce the notion of statistical pseudo-Einstein manifolds and statistical pseudo-Ricci solutions on almost contact metric manifold by using statistical curvature tensor.

Theorem 5.3.10. ([43]) *A Kenmotsu statistical manifold $(M, \nabla, g, \phi, \xi, \eta)$ is statistical pseudo-Einstein if ξ defines a statistical pseudo-Ricci soliton.*

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Chapter 1

Preliminaries

Let M be an m -dimensional manifold, and all the objects are assumed to be smooth. Denote by $C^\infty(M)$ the set of all smooth real-valued functions on M . We denote by $\Gamma(E)$ the set of all smooth sections of a vector bundle $E \rightarrow M$. For example, $\Gamma(TM^{(p,q)})$ means the set of all the tensor fields on M of type (p, q) . Especially, $\Gamma(TM)$ means the set of all smooth vector fields on M .

1.1 Affine connections

Definition 1.1.1. An *affine connection* ∇ on a manifold M is a mapping $\nabla : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM)$ satisfying the following conditions:

- (1) $\nabla_{X_1+X_2}Y = \nabla_{X_1}Y + \nabla_{X_2}Y$,
- (2) $\nabla_{fX}Y = f\nabla_XY$,
- (3) $\nabla_X(Y_1 + Y_2) = \nabla_XY_1 + \nabla_XY_2$,
- (4) $\nabla_XfY = (Xf)Y + f\nabla_XY$

for $f \in C^\infty(M)$, $X, Y \in \Gamma(TM)$. ∇_XY is called the *covariant derivative* of Y with respect to X .

For a tensor field $F \in \Gamma(TM^{(0,p)})$ or $F \in \Gamma(TM^{(1,p)})$, we define the *covariant derivative* of F with respect to $X \in \Gamma(TM)$ as follow:

$$(\nabla_X F)(Y_1, \dots, Y_p) = \nabla_X(F(Y_1, \dots, Y_p)) - \sum_{i=1}^p F(Y_1, \dots, \nabla_X Y_i, \dots, Y_p).$$

The *torsion tensor* $T^\nabla \in \Gamma(TM^{(1,2)})$ of an affine connection ∇ defined by $T^\nabla = \nabla_XY - \nabla_YX - [X, Y]$. The *curvature tensor* $R^\nabla \in \Gamma(TM^{(1,3)})$ of an affine connection ∇ defined by $R^\nabla(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z$.

Definition 1.1.2. If $T^\nabla(X, Y) = 0$ for any $X, Y \in \Gamma(TM)$, ∇ is said to be *torsion-free* affine connection. If $T^\nabla(X, Y) = 0$ and $R^\nabla(X, Y)Z = 0$ for any $X, Y, Z \in \Gamma(TM)$, ∇ is said to be *flat* affine connection.

Let $c : I \rightarrow M$ be a smooth curve, and let $Y \in \Gamma(c^{-1}TM)$ be a vector field along c , that is, $Y(t) \in T_{c(t)}M$ for all $t \in I$. If $Y \in \Gamma(c^{-1}TM)$ satisfies $\nabla_{\dot{c}}Y = 0$, then Y is called the ∇ -parallel vector field along with c , where $\dot{c}(t)$ is the tangent vector field of c .

Let $(u^1, \dots, u^m)$ be a local coordinate system on an open subset $U \subset M$ and put

$$c^i(t) := u^i(c(t)) \quad (i = 1, \dots, m),$$

$$Y(t) = \sum_{i=1}^m Y^i(t) \left(\frac{\partial}{\partial u^i} \right)_{c(t)}.$$

Then $\nabla_{\dot{c}}Y = 0$ can be rewritten by

$$\frac{dY^k(t)}{dt} + \sum_{ij=1}^m \Gamma_{ij}^k(c(t)) \dot{c}^i(t) Y^j(t) = 0$$

for $k = 1, \dots, m$. This represents a system of linear first order ODEs. Therefore, for $a \in I$ and $y \in T_{c(a)}M$, there exists a unique parallel vector field Y along with c such that $Y(a) = y$. If $b \in I$ then the linear mapping $P_c^\nabla : T_{c(a)}M \ni y \mapsto Y(b) \in T_{c(b)}M$ is called ∇ -parallel translation along c from $c(a)$ to $c(b)$.

Given such a parallel translation P_c^∇ and a vector field Y in the neighborhood of the point $p = c(a)$, we can rewrite the covariant derivative of Y at $c(a)$ in the direction of the vector $x = \dot{c}(a)$ as

$$\nabla_x Y = \lim_{t \rightarrow a} \frac{(P_{c|_{[a,t]}}^\nabla)^{-1}(Y_{c(t)}) - Y_p}{t - a}, \quad (1.1)$$

where $P_{c|_{[a,t]}}^\nabla$ is ∇ -parallel translation along c from $c(a)$ to $c(t)$.

Let $(u^1, \dots, u^m)$ be a local coordinate system on an open subset $U \subset M$. The *Christoffel symbols* for the coordinate system are the functions Γ_{ij}^k on U such that $\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = \sum_{k=1}^m \Gamma_{ij}^k \frac{\partial}{\partial u^k}$ for $i, j = 1, 2, \dots, m$. It is known that ∇ is flat affine connection if and only if there exists a local coordinate system $(u^1, \dots, u^m)$ such that $\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = 0$ around any points $p \in M$. Then $(u^1, \dots, u^m)$ is said to be *affine coordinate system* of ∇ .

Let (M, ∇) be an m -dimensional manifold with a torsion-free affine connection ∇ , and let $\gamma : I \rightarrow M$ be a smooth curve with parameter $t \in I \subset \mathbb{R}$. We denote by $\Gamma(\gamma^{-1}TM)$ the

set of vector fields along γ on M . If the tangent vector fields $\dot{\gamma} \in \Gamma(\gamma^{-1}TM)$ is autoparallel relative to ∇ , that is,

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0, \quad (1.2)$$

we say that $\gamma(t)$ is a ∇ -geodesic.

Let $(u^1, \dots, u^m)$ be a local coordinate system on $U \subset M$ and write $\gamma^i(t) := u^i(\gamma(t))$ ($i = 1, \dots, m$). Then we can rewrite (1.2) as follows:

$$\ddot{\gamma}^k(t) + \sum_{i,j=1}^m \Gamma_{ij}^k(\gamma(t)) \dot{\gamma}^i(t) \dot{\gamma}^j(t) = 0$$

for $k = 1, \dots, m$. This constitutes a system of linear second order ODEs. Therefore for given $p \in M, v \in T_p M$, there exists a unique geodesic γ_v with a maximal open interval $I_v \subset \mathbb{R}$ with $0 \in I_v$ such that $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$.

If we set for $p \in M$

$$D_p = \{v \in T_p M ; 1 \in I_v\} \subset T_p M,$$

then D_p is the star-shaped neighborhood of the origin $0_p \in T_p M$. We can then define the ∇ -exponential map Exp_p^∇ on D_p as follows:

Definition 1.1.3. We define a mapping $\text{Exp}_p^\nabla : D_p \rightarrow M$ by

$$\text{Exp}_p^\nabla v = \gamma_v(1) \quad (v \in D_p),$$

and call it the ∇ -exponential map at p .

We then have $\text{Exp}_p^\nabla(tv) = \gamma_v(t)$ for any $t \in I_v$. Now if we identify $T_{0_p}(T_p M) = T_p M$, then the differential $(\text{Exp}_p^\nabla)_{*0_p}$ of Exp_p^∇ at 0_p turns out to be the identity map of $T_p M$. Thus, by the inverse function theorem, there exists a convex open neighborhood $\mathcal{U}_p \subset D_p$ of 0_p such that $\text{Exp}_p^\nabla : \mathcal{U}_p \rightarrow U := \text{Exp}_p^\nabla(\mathcal{U}_p)$ is a local diffeomorphism. If we take a basis $\{e_1, \dots, e_m\}$ of $T_p M$ and consider the correspondence

$$\text{Exp}_p^\nabla \left(\sum_{i=1}^m u^i e_i \right) \mapsto (u^1, \dots, u^m) \quad \left(\sum_{i=1}^m u^i e_i \in \mathcal{U}_p \right),$$

we obtain a local coordinate system around $p \in U$, which we call a ∇ -normal coordinate system.

Let $\gamma(t)$ a ∇ -geodesic on M . A vector field V along γ is said to be a ∇ -Jacobi-field along γ if it satisfies the ∇ -Jacobi equation

$$\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V + R^\nabla(V, \dot{\gamma}) \dot{\gamma} = 0. \quad (1.3)$$

The ∇ -Jacobi equation is a system of linear second order ODEs. Hence if each initial conditions $V(0)$ and $\nabla_{\dot{\gamma}}V(0)$ are given, there exists a unique vector field $V \in \Gamma(\gamma^{-1}TM)$ satisfying (1.3) and the initial conditions on the interval where γ is defined. For $p \in M$, let $\gamma : [0, l] \rightarrow M$ be a ∇ -geodesic with $\gamma(0) = p$, and take $\xi \in T_pM$. If V is a ∇ -Jacobi-field along γ such that $V(0) = 0_p$ and $(\nabla_{\dot{\gamma}}V)(0) = \xi$, then regarding $t\xi$ as in $T_{\gamma(0)}(T_pM)$ for each $t \in [0, l]$ we have

$$V(t) = (\text{Exp}_p^\nabla)_{*t\dot{\gamma}(0)}(t\xi) . \quad (1.4)$$

Therefore, we could say that a ∇ -Jacobi-field is nothing but a geometric expression of the differential of the ∇ -exponential map.

Let (M, ∇) and $(\bar{M}, \bar{\nabla})$ be two affine connected manifolds. Then a diffeomorphism $\varphi : M \rightarrow \bar{M}$ is said to be *affine connection preserving* if

$$\bar{\nabla}_{\varphi_*X}\varphi_*Y = \varphi_*\nabla_XY$$

for $X, Y \in \Gamma(TM)$.

1.2 Riemannian metrics

A *metric tensor* g on M is a symmetric non-degenerate $(0, 2)$ -tensor field on M . For all $p \in M$, g_p is a positive definite inner product on T_pM . A manifold M equipped with a Riemannian metric g is called a Riemannian manifold.

On a Riemannian manifold (M, g) there exists a unique affine connection ∇ such that

- (1) ∇ is torsion-free,
- (2) $\nabla g = 0$.

This unique affine connection ∇ is called the *Levi-Civita* connection of g and we denote by ∇^g . The Levi-Civita connection can be defined by the *Koszul formula*:

$$\begin{aligned} 2g(\nabla_X^g Y, Z) = & X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ & + g(Z, [X, Y]) + g(Y, [Z, X]) - g(X, [Y, Z]). \end{aligned}$$

Proposition 1.2.1. *Let (M, g) be a Riemannian manifold and $c : [a, b] \rightarrow M$ a smooth curve with parameter $t \in [a, b] \subset \mathbb{R}$. Then the ∇^g -parallel translation $P_c^{\nabla^g} : T_{c(a)}M \rightarrow T_{c(b)}M$ is a linear isometry.*

Definition 1.2.2. (1) For a Riemannian manifold (M, g) , we take $p \in M$ and a plane $\Pi = u \wedge v$ spanned by orthonormal vectors $u, v \in T_pM$. Then the following form

$$\mathcal{K}^g(\Pi) = \frac{g(R^{\nabla^g}(u, v)v, u)}{g(u, u)g(v, v) - g(u, v)^2}$$

is called the *Riemannian sectional curvature* of (M, g) for Π .

(2) Let (M, g) be a Riemannian manifold and $k \in \mathbb{R}$. (M, g) is said to be of *constant Riemannian sectional curvature* k at $p \in M$ if $\mathcal{K}^g(\Pi) = k$ for any plane $\Pi \subset T_p M$.

A Riemannian manifold (M, g) is of constant Riemannian sectional curvature k if and only if R^{∇^g} satisfies

$$R^{\nabla^g}(X, Y)Z = k\{g(Y, Z)X - g(X, Z)Y\}$$

for $X, Y, Z \in \Gamma(TM)$.

Let (M, g) and $(\bar{M}, \bar{g})$ be two Riemannian manifolds and let $\varphi : M \rightarrow \bar{M}$ be a smooth map. A mapping φ is said to be a *local isometry* if for each $p \in M$ the differential $(\varphi_*)_p$ is a linear isometry relative to $g_p, \bar{g}_{\varphi(p)}$, that is, $g_p(u, v) = \bar{g}_{\varphi(p)}((\varphi_*)_p(u), (\varphi_*)_p(v))$ for any $u, v \in T_p M$. If furthermore φ is a diffeomorphism of M onto $\bar{M}$, we say that φ is an *isometry*, and then we see

$$g(X, Y) = \bar{g}(\varphi_* X, \varphi_* Y)$$

for every $X, Y \in \Gamma(TM)$. In a local isometry case, it is to be noted that each point $p \in M$ has a neighborhood U such that $\varphi|_U$ is an isometry of U onto a neighborhood of $\varphi(p) \in \bar{M}$ because of the inverse function theorem.

Notice that an isometry $\varphi : (M, g) \rightarrow (\bar{M}, \bar{g})$ has the following properties:

- (1) $\varphi_*(\nabla_X^g Y) = \nabla_{\varphi_* X}^{\bar{g}} \varphi_* Y$ for all $X, Y \in \Gamma(TM)$.
- (2) Let P_c be the ∇^g -parallel translation from $T_{c(a)}M$ to $T_{c(b)}M$ along a smooth curve $c : [a, b] \rightarrow M$, and let $\bar{P}_{\varphi \circ c}$ be the $\nabla^{\bar{g}}$ -parallel translation from $T_{\varphi \circ c(a)}M$ to $T_{\varphi \circ c(b)}M$ along $\varphi \circ c$ in $\bar{M}$. Then $(\varphi_*)_{c(b)} \circ P_c = \bar{P}_{\varphi \circ c} \circ (\varphi_*)_{c(a)}$.
- (3) If γ is a ∇^g -geodesic in M , then $\varphi \circ \gamma$ is a $\nabla^{\bar{g}}$ -geodesic in $\bar{M}$.
- (4) Let Exp_p be the ∇^g -exponential map at $p \in M$ and $\bar{\text{Exp}}_{\varphi(p)}$ the $\nabla^{\bar{g}}$ -exponential map at $\varphi(p) \in \bar{M}$. Then $\varphi \circ \text{Exp}_p = \bar{\text{Exp}}_{\varphi(p)} \circ (\varphi_*)_p$, whenever the left-hand-side is defined.
- (5) Since φ preserves both brackets and covariant derivatives, φ preserves Riemannian curvature tensors: $\varphi_*(R^g(X, Y)Z) = R^{\bar{g}}(\varphi_* X, \varphi_* Y)\varphi_* Z$.

We see that a local isometry is uniquely determined by its differential map at a single point from the following proposition:

Proposition 1.2.3. *Let (M, g) and $(\bar{M}, \bar{g})$ be Riemannian manifolds, and $\varphi, \psi : M \rightarrow \bar{M}$ be local isometries. If there is a point $p \in M$ such that $(\varphi_*)_p = (\psi_*)_p$, then $\varphi = \psi$.*

Proof. Set $A = \{q \in M \mid (\varphi_*)_q = (\psi_*)_q\}$. By continuity, A is closed in M , and A is nonempty because A contains p . Thus it suffices to show that A is open. We assert that if $q \in A$ then

any ∇^g -normal neighborhood U of q is contained in A . If $q' \in U$ there exists a vector $v \in T_q M$ such that $\gamma_v(1) = \text{Exp}_q(v) = q'$. Hence

$$\begin{aligned}
\varphi(q') &= \varphi(\gamma_v(1)) \\
&= ((\overline{\text{Exp}}_{\varphi(q)} \circ (\varphi_*)_q \circ (\text{Exp}_q)^{-1})(\text{Exp}_q(v)) \\
&= \overline{\text{Exp}}_{\varphi(q)}((\varphi_*)_q(v)) \\
&= \overline{\gamma}_{(\varphi_*)_q(v)}(1) \\
&= \overline{\gamma}_{(\psi_*)_q(v)}(1) \\
&= \psi(\gamma_v(1)) \\
&= \psi(q').
\end{aligned}$$

Thus $\varphi = \psi$ on U . □

Definition 1.2.4. An *almost complex structure* on a manifold M is a tensor $J \in \Gamma(TM^{(1,1)})$ such that

$$J^2 = -I,$$

where I denotes the identity transformation. A manifold M with an almost complex structure J is called an *almost complex manifold*.

Every almost complex manifold is of even dimension and oriented.

In general, given an almost complex manifold (M, J) , we define the *Nijenhuis tensor* $N \in \Gamma(TM^{(1,2)})$ by

$$N(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]$$

for $X, Y \in \Gamma(TM)$. If $N(X, Y) = 0$ for all $X, Y \in \Gamma(TM)$, then J is called a *complex structure* and (M, J) is called a *complex manifold*.

Definition 1.2.5. Let (M, J) be an almost complex manifold (respectively, a complex manifold). A *Hermitian metric* on (M, J) is a Riemannian metric g such that

$$g(JX, JY) = g(X, Y)$$

for any $X, Y \in \Gamma(TM)$. An almost complex manifold with Hermitian metric (M, g, J) is called an *almost Hermitian manifold* (respectively, a *Hermitian manifold*).

On an almost Hermitian manifold (M, g, J) , we have

$$g(JX, Y) = g(J^2X, JY) = -g(X, JY)$$

for any $X, Y \in \Gamma(TM)$.

For an almost Hermitian manifold (M, g, J) , the *fundamental differential form* (or *Kähler form*) $\Omega \in \Gamma(TM^{(0,2)})$ is defined by

$$\Omega(X, Y) = g(X, JY)$$

for any $X, Y \in \Gamma(TM)$.

Definition 1.2.6. If a Hermitian manifold (M, g, J) with $\nabla^g J = 0$ (or $d\Omega = 0$) is called a *Kähler manifold* and the metric g is called a *Kähler metric*.

Next, let M be an odd-dimensional manifold.

Definition 1.2.7. Let $\phi \in \Gamma(TM^{(1,1)})$, $\xi \in \Gamma(TM)$, and $\eta \in \Gamma(T^*M)$, respectively. If ϕ, ξ and η satisfy the following condition:

$$\begin{aligned}\phi^2 X &= -X + \eta(X)\xi, \\ \phi\xi &= 0, \\ \eta(\phi X) &= 0, \\ \eta(\xi) &= 1\end{aligned}$$

for any $X \in \Gamma(TM)$, then (M, ϕ, ξ, η) is called an *almost contact manifold* and (ϕ, ξ, η) is called an *almost contact structure* on M .

Definition 1.2.8. Let (M, ϕ, ξ, η) be an almost contact manifold. If a Riemannian metric g on M satisfies

$$\begin{aligned}g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \\ \eta(X) &= g(X, \xi)\end{aligned}$$

for any $X, Y \in \Gamma(TM)$, then (M, g, ϕ, ξ, η) is called an *almost contact metric manifold* and (g, ϕ, ξ, η) is called an *almost contact metric structure* on M .

1.3 Statistical structures and Hessian structures

Let (M, g) be a Riemannian manifold, and ∇ an affine connection on M . Then we introduce the dual connection as follow.

Definition 1.3.1. We define the *dual connection* ∇^* of ∇ with respect to g by

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z) \quad (1.5)$$

for $X, Y, Z \in \Gamma(TM)$. Then ∇ and ∇^* are called *mutually dual connections* with respect to g .

From the definition of the dual connection, we see that $(\nabla^*)^* = \nabla$ holds. For any connection ∇ , $\widehat{\nabla} = \frac{1}{2}(\nabla + \nabla^*)$ satisfies $\widehat{\nabla}g = 0$.

Proposition 1.3.2. *If any two of the following conditions imply the others.*

- (1) ∇ is torsion-free.
- (2) ∇^* is torsion-free.
- (3) $\nabla g \in \Gamma(TM^{(0,3)})$ is totally symmetric, that is, the Codazzi equation $\nabla_X g(Y, Z) = \nabla_Y g(X, Z)$ holds for $X, Y, Z \in \Gamma(TM)$.
- (4) $\widehat{\nabla} = \frac{1}{2}(\nabla + \nabla^*)$ is the Levi-Civita connection of g , that is, $\widehat{\nabla} = \nabla^g$.

The $(0, 3)$ -tensor field $C = \nabla g$ is called *the cubic form* on M .

Hereafter, we consider the case where affine connections are torsion-free. Let us give the definition of statistical manifolds as follows.

Definition 1.3.3. Let M be a manifold endowed with a torsion-free connection ∇ and a Riemannian metric g . We say that (M, ∇, g) is a *statistical manifold* if the $(0, 3)$ -tensor field $C = \nabla g$ is totally symmetric. The pair (∇, g) is said to be the *statistical structure* on M .

The notion of statistical manifold was introduced by S. L. Lauritzen [32]. Originally, the triplet (M, g, C) is called a statistical manifold, where (M, g) is a Riemannian manifold and C is a totally symmetric $(0, 3)$ -tensor field on M .

Proposition 1.3.4. *Let (M, g) be a Riemannian manifold, C a totally symmetric $(0, 3)$ -tensor field on M . Set ∇ and ∇^* by*

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\nabla_X^g Y, Z) - \frac{1}{2}C(X, Y, Z), \\ g(\nabla_X^* Y, Z) &= g(\nabla_X^g Y, Z) + \frac{1}{2}C(X, Y, Z) \end{aligned}$$

for $X, Y, Z \in \Gamma(TM)$. Then ∇ and ∇^ are torsion-free mutually dual affine connections. Moreover ∇g and $\nabla^* g$ are symmetric $(0, 3)$ -tensor fields.*

The proposition above implies that a statistical manifold (M, ∇, g) can be defined by (M, g, C) . Hence these definitions are essentially equivalent.

For a given statistical structure (∇, g) on M , denote by ∇^* the dual connection of ∇ with respect to g . If (∇, g) is a statistical structure, so is (∇^*, g) , and we call (∇^*, g) the *dual statistical structure* on M . Furthermore, for a statistical structure (∇, g) , the Levi-Civita connection of g is characterized by $\nabla^g = \frac{1}{2}(\nabla + \nabla^*)$.

We remark that a statistical manifold with $C = 0$ is a Riemannian manifold with ∇^g . In this sense, the notion of statistical manifolds can be viewed as a generalization of the notion of Riemannian manifolds.

For a statistical manifold (M, ∇, g) , we can define a one-parameter family of torsion-free affine connections $\{\nabla^{(\alpha)}\}_{\alpha \in \mathbb{R}}$ by

$$\nabla^{(\alpha)} = \frac{1 + \alpha}{2} \nabla + \frac{1 - \alpha}{2} \nabla^*$$

with $\nabla^{(0)} = \nabla^g$, $\nabla^{(1)} = \nabla$, and $\nabla^{(-1)} = \nabla^*$. It is seen that $\nabla^{(\alpha)}$ and $\nabla^{(-\alpha)}$ are mutually dual with respect to g . The α -connection $\nabla^{(\alpha)}$ is written as follows:

$$g(\nabla_X^{(\alpha)} Y, Z) = g(\nabla_X^g Y, Z) - \frac{\alpha}{2} C(X, Y, Z)$$

for $X, Y, Z \in \Gamma(TM)$.

Let $(\mathcal{X}, \mathcal{A})$ be a measurable space. Suppose that $\mathcal{S}$ is a parametric model on $\mathcal{X}$, that is, $\mathcal{S}$ is a set of probability distribution on $(\mathcal{X}, \mathcal{A})$ parametrized by $\xi = (\xi^1, \dots, \xi^m) \in \Xi \subset \mathbb{R}^m$:

$$\mathcal{S} = \left\{ p(x; \xi) \mid \int_{\mathcal{X}} p(x; \xi) dx = 1, p(x; \xi) > 0, \xi \in \Xi \right\},$$

where $\Xi \subset \mathbb{R}^m$ is open, and the mapping $\xi \mapsto p(x; \xi)$ is one-to-one and smooth. We assume that integral and differential operators commutative. Then $\mathcal{S}$ is regarded as an m -dimensional manifold with a local coordinate system $(\xi^1, \dots, \xi^m)$. We define a symmetric matrix $[g_{ij}^F]_{i,j=1,\dots,m}$ by

$$\begin{aligned} g_{ij}^F(\xi) &= g^F(\partial_i, \partial_j) \\ &= E_p[(\partial_i l_\xi)(\partial_j l_\xi)] \\ &= \int_{\mathcal{X}} \left(\frac{\partial}{\partial \xi^i} \log p(x; \xi) \right) \left(\frac{\partial}{\partial \xi^j} \log p(x; \xi) \right) p(x; \xi) dx \end{aligned}$$

for $i, j = 1, 2, \dots, m$, where $\partial_i = \frac{\partial}{\partial \xi^i}$, $l_\xi = \log p(x; \xi)$ and $E_p[*]$ denotes the expectation with respect to the distribution $p(x; \xi)$. In addition, we assume that the integral operator with respect to x and the differential operator with respect to ξ is commutative, $g_{ij}^F(\xi)$ can be also written as

$$g_{ij}^F(\xi) = -E_p[\partial_i \partial_j l_\xi]. \quad (1.6)$$

If $[g_{ij}^F]_{i,j=1,\dots,m}$ is positive definite, it determines a Riemannian metric on $\mathcal{S}$. We then call g^F the *Fisher metric* on $\mathcal{S}$.

Then, the Levi-Civita connection $\nabla^{(0)}$ for g^F are expressed as

$$\begin{aligned} \Gamma_{ij,k}^{(0)}(\xi) &= E_p[(\partial_i \partial_j l_\xi)(\partial_k l_\xi)] + \frac{1}{2} E_p[(\partial_i l_\xi)(\partial_j l_\xi)(\partial_k l_\xi)], \\ \Gamma_{ij,k}^{(0)} &= g^F(\nabla_{\partial_i}^{(0)} \partial_j, \partial_k) = g^F\left(\sum_{l=1}^m \Gamma_{ij}^{(\alpha)l} \partial_l, \partial_k\right), \end{aligned}$$

where $E_p[(\partial_i l_\xi)(\partial_j l_\xi)(\partial_k l_\xi)]$, which is a totally symmetric $(0, 3)$ -tensor, is called the *Amari-Chentov cubic form* on $\mathcal{S}$ and denoted by $C^{AC}(\partial_i, \partial_j, \partial_k)$, for $i, j, k = 1, \dots, m$.

By using g^F and C^{AC} , for an arbitrary constant $\alpha \in \mathbb{R}$, we can define a torsion-free affine connection $\nabla^{(\alpha)}$ by

$$\begin{aligned} g(\nabla_{\partial_i}^{(\alpha)} \partial_j, \partial_k) &= g(\nabla_{\partial_i}^{(0)} \partial_j, \partial_k) - \frac{\alpha}{2} C^{AC}(\partial_i, \partial_j, \partial_k), \\ \Gamma_{ij,k}^{(\alpha)}(\xi) &= E_p[(\partial_i \partial_j l_\xi)(\partial_k l_\xi)] + \frac{1-\alpha}{2} E_p[(\partial_i l_\xi)(\partial_j l_\xi)(\partial_k l_\xi)]. \end{aligned}$$

Then $(\nabla^{(\alpha)}, g^F)$ is a statistical structure on a parameter space Ξ . Therefore $\mathcal{S}$ is regarded as a statistical manifold $(\Xi, \nabla^{(\alpha)}, g^F)$.

Example 1.3.5. ([3], [45]) (1) The family of 1-dimensional normal distributions $\mathcal{S}_{N(\mu, \sigma^2)}$ is a 2-dimensional statistical model:

$$\begin{aligned} \mathcal{X} &= \mathbb{R}, \quad \Xi = \{(\mu, \sigma) \mid (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}^+\}, \\ \mathcal{S}_{N(\mu, \sigma^2)} &= \left\{ p(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \mid (\mu, \sigma) \in \Xi \right\}. \end{aligned}$$

Then $\mathcal{S}_{N(\mu, \sigma^2)}$ with the Fisher metric g^F is of constant Riemannian sectional curvature $-\frac{1}{2}$, thus $\mathcal{S}_{N(\mu, \sigma^2)}$ can be regarded as the Poincaré half plane model of hyperbolic plane $(\mathbb{H}^2, g)$.

(2) The family of *Multivariate normal distributions* $\mathcal{S}_{N(\mu, \Sigma)}$ is an $\left(n + \frac{n(n+1)}{2}\right)$ -dimensional statistical model:

$$\mathcal{X} = \mathbb{R}^n, \quad \Xi = \{(\mu, \Sigma) \mid (\mu, \Sigma) \in \mathbb{R}^n \times \mathbb{R}^{n \times n}, \Sigma : \text{positive definite}\},$$

$$\mathcal{S}_{N(\mu, \Sigma)} = \left\{ p(x; \mu, \Sigma) = \frac{1}{(2\pi)^{n/2} (\det \Sigma)^{1/2}} \exp \left[-\frac{1}{2} (x - \mu)^t \Sigma^{-1} (x - \mu) \right] \mid (\mu, \Sigma) \in \Xi \right\}.$$

Let we consider the case where $\Sigma = \sigma^2 E_n$, that is,

$$\mathcal{S}_{N(\mu, \sigma^2 E_n)} = \left\{ p(x; \mu, \Sigma) = \frac{1}{(2\pi)^{n/2} \sigma^n} \prod_{i=1}^n \exp \left[-\frac{(x_i - \mu_i)^2}{2\sigma^2} \right] \mid (\mu, \sigma) \in \mathbb{R}^n \times \mathbb{R}^+ \right\},$$

where E_n is the identity matrix of size n . This statistical model $\mathcal{S}_{N(\mu, \sigma^2 E_n)}$ with the Fisher metric g^F is of constant Riemannian sectional curvature $-\frac{n}{2}$. Therefore $\mathcal{S}_{N(\mu, \sigma^2 E_n)}$ can be regarded as the upper half space model of hyperbolic space $(\mathbb{H}^{n+1}, g)$.

(3) The family of discrete positive probability distributions $\mathcal{S}^n$ for the sample space $\mathcal{X} = \{0, 1, \dots, n\}$ is an n -dimensional statistical model:

$$\mathcal{X} = \{0, 1, \dots, n\},$$

$$\Delta_n = \left\{ \xi = (\xi^0, \dots, \xi^n) \mid \xi^i > 0, \sum_{i=1}^n \xi^i < 1, \xi^0 = 1 - \sum_{i=1}^n \xi^i \right\} \subset (\mathbb{R}^+)^{n+1},$$

$$\mathcal{S}^n = \left\{ p(x; \xi) = \sum_{i=0}^n \xi_i \delta_i(x) \mid \xi \in \Delta_n \right\}.$$

The family of *multinomial distributions* can be regarded as a family of discrete positive probability distributions:

$$\text{Mult}(\{x_0, x_1, \dots, x_n\}; \{p_0, p_1, \dots, p_n\}, M) = \text{Prob}(X_0 = x_0, \dots, X_n = x_n)$$

$$= \frac{M!}{x_0! \dots x_n!} \prod_{i=0}^n p_i^{x_i},$$

where $\sum_{k=0}^n x_k = M$, $p_i > 0$ for $i = 0, 1, \dots, n$ and $\sum_{i=0}^n p_i = 1$. Then Δ_n with the Fisher metric g^F is of constant Riemannian sectional curvature $\frac{1}{4}$. Thus the family of discrete positive probability distributions $\mathcal{S}^n$ is isometric to the first domain in a sphere.

It is well-known that many important statistical models, for instance, (multivariate) normal distributions, multinomial distributions, Poisson distributions, and so on, are exponential families.

Definition 1.3.6. Let $(X, \mathcal{A})$ be a measurable space. A family of probability distributions $\mathcal{E} = \{p(x; \theta) \mid \theta \in \Theta\}$ on X is said to be an *exponential family* if there exist functions $C, F_1, \dots, F_m$ on X , and a function ψ on $\Theta \subset \mathbb{R}^m$, such that

$$p(x; \theta) = \exp \left[C(x) + \sum_{j=1}^m \theta^j F_j(x) - \psi(\theta) \right] \quad (\theta = (\theta^1, \dots, \theta^m)).$$

Then we say that $(\theta^1, \dots, \theta^m)$ are its *natural* or *canonical parameters*.

Example 1.3.7. The family of 1-dimensional normal distributions

$$\mathcal{S}_{N(\mu, \sigma^2)} = \left\{ p(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{(x - \mu)^2}{2\sigma^2} \right] \mid (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}^+ \right\}.$$

is an exponential family. Indeed, taking a variable transformation defined by

$$\theta^1 := \frac{\mu}{\sigma^2}, \quad \theta^2 := \frac{1}{2\sigma^2},$$

and put

$$F_1(x) := x, \quad F_2(x) := -x^2, \quad \psi(\theta) := \frac{(\theta^1)^2}{4\theta^2} + \frac{1}{2} \log \left(\frac{\pi}{\theta^2} \right),$$

we obtain

$$p(x; \theta) = \exp[\theta^1 F_1(x) + \theta^2 F_2(x) - \psi(\theta)],$$

$$\theta \in \Theta := \{(\theta^1, \theta^2) \in \mathbb{R}^2 \mid \theta^2 > 0\}.$$

Therefore the family of 1-dimensional normal distributions is an exponential family.

Definition 1.3.8. For a statistical structure (∇, g) , we define the *difference tensor field* $K = K^{(\nabla, g)} \in \Gamma(TM^{(1,2)})$ as

$$K_X Y = \nabla_X Y - \nabla_X^g Y$$

for any $X, Y \in \Gamma(TM)$. We often denote $K_X Y$ by $K(X, Y)$ as well.

The difference tensor K satisfies $K = \nabla^g - \nabla^* = \frac{1}{2}(\nabla - \nabla^*)$. We remark that if $K = 0$ then a statistical structure (∇, g) coincides with Riemannian structure. We can relate K to the cubic form $C = \nabla g$ as $C(X, Y, Z) = -2g(K(X, Y), Z)$. The following property holds.

Lemma 1.3.9. *For a statistical structure (∇, g) , the difference tensor $K \in \Gamma(TM^{(1,2)})$ satisfies*

$$K_X Y = K_Y X, \quad g(K_X Y, Z) = g(Y, K_X Z). \quad (1.7)$$

Conversely, for a Riemannian metric g if $K \in \Gamma(TM^{(1,2)})$ satisfies (1.7), the pair $(\nabla = \nabla^g + K, g)$ is a statistical structure.

For a statistical manifold (M, ∇, g) , we can consider the curvature tensor field R^∇ , R^{∇^g} , and R^{∇^*} . We denote R^∇ by R , R^{∇^g} by R^g , and R^{∇^*} by R^* for short. The following formulas are obtained by direct calculation.

Proposition 1.3.10. *For a statistical manifold (M, ∇, g) , the following formulas hold for $X, Y, Z, W \in \Gamma(TM)$:*

- (1) $R(X, Y)Z = R^g(X, Y)Z + (\nabla_X^g K)(Y, Z) - (\nabla_Y^g K)(X, Z) + [K_X, K_Y]Z,$
- (2) $R^*(X, Y)Z = R^g(X, Y)Z - (\nabla_X^g K)(Y, Z) + (\nabla_Y^g K)(X, Z) + [K_X, K_Y]Z,$
- (3) $R^g(X, Y)Z = R(X, Y)Z - (\nabla_X K)(Y, Z) + (\nabla_Y K)(X, Z) + [K_X, K_Y]Z,$
- (4) $(\nabla_X K)(Y, Z) - (\nabla_Y K)(X, Z) = 2[K_X, K_Y]Z + \frac{1}{2}\{R(X, Y)Z - R^*(X, Y)Z\},$
- (5) $g(R(X, Y)Z, W) = -g(Z, R^*(X, Y)W).$

Definition 1.3.11. Let (M, ∇, g) be a statistical manifold and $k \in \mathbb{R}$. (M, ∇, g) is said to be of constant curvature k if

$$R(X, Y)Z = k\{g(Y, Z)X - g(X, Z)Y\}$$

for $X, Y, Z \in \Gamma(TM)$.

Proposition 1.3.12. *If (M, ∇, g) is of constant curvature $k \in \mathbb{R}$ then (M, ∇^*, g) is also of constant curvature $k \in \mathbb{R}$.*

Let us consider an exponential family

$$\mathcal{E} = \left\{ p(x; \theta) = \exp \left(C(x) + \sum_{j=1}^m \theta^j F_j(x) - \psi(\theta) \right) \middle| \theta \in \Theta \right\}$$

on $\mathcal{X}$, where $\Theta \subset \mathbb{R}^m$. Then we have

$$\begin{aligned} \frac{\partial}{\partial \theta^i} \log p(x; \theta) &= \frac{\partial}{\partial \theta^i} \left[C(x) + \sum_{j=1}^m \theta^j F_j(x) - \psi(\theta) \right] \\ &= F_i(x) - \frac{\partial}{\partial \theta^i} \psi(\theta), \end{aligned}$$

thus

$$\frac{\partial^2 \log p}{\partial \theta^i \partial \theta^j}(x; \theta) = -\frac{\partial^2 \psi}{\partial \theta^i \partial \theta^j}(\theta). \quad (1.8)$$

From (1.6), the Fisher metric $g^F = [g_{ij}^F]_{i,j=1,\dots,m}$ and the α -connection $\nabla^{(\alpha)}$ on $\mathcal{E}$ are given by

$$g_{ij}^F = \partial_i \partial_j \psi, \quad \Gamma_{ij}^{(\alpha)k} = \frac{1-\alpha}{2} \partial_s g_{ij}^F (g^F)^{ks} \quad (i, j, k = 1, \dots, m)$$

where $\Gamma_{ij}^{(\alpha)k}$ is the Christoffel symbol with respect to $\nabla^{(\alpha)}$ and $\partial_i = \frac{\partial}{\partial \theta^i}$. Therefore, we obtain

$$R^{\nabla^{(\alpha)}}(\partial_i, \partial_j) \partial_k = \frac{(1-\alpha)(1+\alpha)}{4} \left(\partial_j g_{ks}^F \partial_i (g^F)^{st} - \partial_i g_{ks}^F \partial_j (g^F)^{st} \right) \partial_t$$

for $i, j, k, s, t = 1, \dots, m$. From the relation above, it follows that the exponential family $\mathcal{E}$ has flat affine connections $\nabla^{(\pm 1)}$.

Here we concentrate on statistical manifolds with flat affine connection. Let (M, ∇, g) be a statistical manifold with flat connection ∇ . From Proposition 1.3.10, we find ∇^* is also flat. Therefore, such a statistical manifold (M, ∇, g) is called a *dually flat manifold*.

For an m -dimensional dually flat manifold (M, ∇, g) , there exists a ∇ -affine coordinate system $(\theta^1, \dots, \theta^m)$. In addition, there also exists a ∇^* -affine coordinate system $(\eta_1, \dots, \eta_m)$ such that

$$g\left(\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \eta_j}\right) = \delta_i^j$$

for $i, j = 1, \dots, m$. Then there exist functions ψ and ϕ on M such that

$$\frac{\partial \psi}{\partial \theta^i} = \eta_i, \quad \frac{\partial \phi}{\partial \eta_j} = \theta^j, \quad \psi(p) + \phi(p) - \sum_{i=1}^m \theta^i(p) \eta_i(p) = 0,$$

where $p \in M$. Furthermore, g is expressed by Hessian of a functions ψ and ϕ as follow:

$$[g_{ij}]_{i,j=1,\dots,m} = \left[\frac{\partial^2 \psi}{\partial \theta^i \partial \theta^j} \right]_{i,j=1,\dots,m}, \quad [g^{ij}]_{i,j=1,\dots,m} = \left[\frac{\partial^2 \phi}{\partial \eta_i \partial \eta_j} \right]_{i,j=1,\dots,m},$$

where $[g^{ij}]_{i,j=1,\dots,m}$ is the inverse matrix of $[g_{ij}]_{i,j=1,\dots,m}$, that is, ψ and ϕ are potential functions of g .

Definition 1.3.13. A statistical manifold (M, ∇, g) is called a *Hessian manifold* if ∇ is flat. Then (∇, g) is said to be a *Hessian structure* on M .

For an exponential family $\mathcal{E} = \{p(x; \theta) \mid \theta \in \Theta\}$ on $\mathcal{X}$, from (1.8), we have

$$\Gamma_{ij,k}^{(1)} = E_p \left[(\partial_i \partial_j l_\theta) (\partial_k l_\theta) \right] = 0.$$

This means that $\nabla^{(1)}$ is a flat connection and $(\theta^1, \dots, \theta^m)$ is a $\nabla^{(1)}$ -affine coordinate system. Moreover, ψ is the potential function of g , that is, $[g_{ij}]_{i,j=1,\dots,m} = \left[\frac{\partial^2 \psi}{\partial \theta^i \partial \theta^j} \right]_{i,j=1,\dots,m}$. Therefore, $(\nabla^{(1)}, g = \nabla^{(1)} d\psi)$ is a Hessian structure on a parameter space Θ .

Definition 1.3.14. Let $(\mathcal{X}, \mathcal{A})$ be a measurable space. A family of probability distribution $\mathcal{M} = \{p(x; \theta) \mid \theta \in \Theta\}$ on $\mathcal{X}$ is called a *mixture family* if there exist functions $C, F_1, \dots, F_m$ on $\mathcal{X}$ such that

$$p(x; \theta) = C(x) + \sum_{i=1}^m \theta^i F_i(x),$$

where $\Theta \subset \mathbb{R}^m$. Then $(\theta^1, \dots, \theta^m)$ is called its *mixture parameters*.

A mixture family is given by the mixture of $m+1$ probability distributions $\{p_0, p_1, \dots, p_m\}$ on $\mathcal{X}$ as follow:

$$\begin{aligned} p(x; \theta) &= \sum_{i=1}^m \theta^i p_i(x) + \left(1 - \sum_{i=1}^m \theta^i \right) p_0(x) \\ &= p_0(x) + \sum_{i=1}^m \theta^i \{p_i(x) - p_0(x)\}, \end{aligned}$$

where $(\theta^1, \dots, \theta^m)$ are subject to $0 < \theta^i < 1$ and $\sum_{i=1}^m \theta^i < 1$.

For a mixture family $\mathcal{M} = \{p(x; \theta) \mid \theta \in \Theta\}$ on $\mathcal{X}$ we get

$$\frac{\partial \log p}{\partial \theta^i}(x; \theta) = \frac{F_i(x)}{p(x; \theta)}, \quad \frac{\partial^2 \log p}{\partial \theta^i \partial \theta^j}(x; \theta) = -\frac{F_i(x) F_j(x)}{p(x; \theta)^2}.$$

Thus we find

$$\frac{\partial^2 \log p}{\partial \theta^i \partial \theta^j}(x; \theta) = -\frac{\partial \log p}{\partial \theta^i}(x; \theta) \frac{\partial \log p}{\partial \theta^j}(x; \theta).$$

Therefore we have

$$\begin{aligned} \Gamma_{ij,k}^{(-1)} &= E_p \left[(\partial_i \partial_j l_\theta) (\partial_k l_\theta) \right] + E_p \left[(\partial_i l_\theta) (\partial_j l_\theta) (\partial_k l_\theta) \right] \\ &= 0 \end{aligned}$$

It means that $\nabla^{(-1)}$ is a flat connection and $(\theta^1, \dots, \theta^m)$ is a $\nabla^{(-1)}$ -affine coordinate system.

Example 1.3.15. The family of discrete positive probability distributions $\mathcal{S}^n$ on finite set $\mathcal{X} = \{0, 1, \dots, n\}$ can be expressed as a mixture family.

Next we define the Hessian curvature tensor $Q \in \Gamma(TM^{(1,3)})$ on a Hessian manifold (M, ∇, g) as follows:

Definition 1.3.16. ([45]) (1) For a Hessian structure (∇, g) , we define the *Hessian curvature tensor* $Q \in \Gamma(TM^{(1,3)})$ by

$$Q(X, Y)Z := -(\nabla_X K)(Y, Z) = -(\nabla K)(Y, Z; X)$$

for $X, Y, Z \in \Gamma(TM)$.

(2) A Hessian manifold (M, ∇, g) is said to be of *constant Hessian sectional curvature* $c \in \mathbb{R}$ if

$$Q(X, Y)Z = -\frac{c}{2}\{g(X, Y)Z + g(X, Z)Y\}$$

holds for $X, Y, Z \in \Gamma(TM)$.

The Hessian curvature tensor Q plays a similar role to that of the Riemannian curvature tensor for a Kählerian metric. In fact, we can construct a Kählerian metric g^{TM} on the tangent bundle TM over a Hessian manifold (M, ∇, g) by the diagonal lifting (or the Sasakian lifting), and (TM, g^{TM}) is of constant holomorphic sectional curvature $-c$ if and only if (M, ∇, g) is of constant Hessian sectional curvature c .

From the following proposition, it can be seen that the Hessian curvature tensor Q carries more detailed information than the Riemannian curvature R^g .

Proposition 1.3.17. ([45]) *Let (M, ∇, g) be a Hessian manifold, and R^g the Riemannian curvature tensor for g . Then the following holds:*

$$R^g(X, Y)Z = -\frac{1}{2}\{Q(X, Y)Z - Q(Y, X)Z\}.$$

If furthermore (M, ∇, g) be of constant Hessian curvature c , then (M, ∇, g) has a Riemannian structure of constant Riemannian curvature $-\frac{c}{4}$.

Example 1.3.18. ([45]) (1) The family of 1-dimensional normal distributions $\mathcal{S}_{N(\mu, \sigma^2)}$, namely, $(\mathbb{H}^2, \nabla^{(-1)}, g^F)$ is a Hessian manifold of constant Hessian sectional curvature 2.

(2) The subfamily of Multivariate normal distributions $\mathcal{S}_{N(\mu, \sigma^2 E_n)}$, that is, $(\mathbb{H}^{n+1}, \nabla^{(-1)}, g^F)$ is a constant Hessian sectional curvature $\frac{2}{n}$.

(3) The family of discrete positive probability distribution $\mathcal{S}^n$ on finite set $\mathcal{X} = \{0, 1, \dots, n\}$, that is, $(\Delta_n, \nabla^{(1)}, g^F)$ has constant Hessian sectional curvature -1 .

Chapter 2

Statistical Deffeomorphisms

2.1 Statistical diffeomorphisms on statistical manifolds

This section devotes statistical diffeomorphisms.

In Riemannian geometry, isometry, a metric preserving diffeomorphism between Riemannian manifolds, plays important roles. Correspondingly, in the geometry of statistical manifolds, it is meaningful that we consider statistical structure preserving diffeomorphisms.

Definition 2.1.1. Let (M, ∇, g) and $(\bar{M}, \bar{\nabla}, \bar{g})$ be two statistical manifolds. A diffeomorphism $\varphi : M \rightarrow \bar{M}$ is called a *statistical diffeomorphism* if

$$\varphi_*(\nabla_X Y) = \bar{\nabla}_{\varphi_* X} \varphi_* Y, \quad g(X, Y) = \bar{g}(\varphi_* X, \varphi_* Y)$$

for any vector fields $X, Y \in \Gamma(TM)$. Then (M, ∇, g) and $(\bar{M}, \bar{\nabla}, \bar{g})$ are said to be *statistical diffeomorphic*.

Next, we state the setting of our theorems.

Let (M, ∇, g) and $(\bar{M}, \bar{\nabla}, \bar{g})$ be two statistical manifolds of the same dimension, and $p \in M, \bar{p} \in \bar{M}$. Let also a linear isometry $\Phi : T_p M \rightarrow T_{\bar{p}} \bar{M}$ be given. In addition, choose a convex open subset $\mathcal{U}_p \subset T_p M$ which determines normal coordinate neighborhoods $U := \text{Exp}_p(\mathcal{U}_p)$ and $\bar{U} := \bar{\text{Exp}}_{\bar{p}}(\Phi(\mathcal{U}_p))$ with respect to ∇^g and $\nabla^{\bar{g}}$ around p and $\bar{p}$, respectively.

Then we can define a diffeomorphism $\varphi : U \rightarrow \bar{U}$ by

$$\varphi := \bar{\text{Exp}}_{\bar{p}} \circ \Phi \circ (\text{Exp}_p)^{-1}. \quad (2.1)$$

Given a ∇^g -geodesic γ in U emanating from p , $\bar{\gamma} := \varphi \circ \gamma$ is a geodesic relative to $\nabla^{\bar{g}}$ in $\bar{U}$ emanating from $\bar{p}$. Denoting by q and $\bar{q} := \varphi(q)$ endpoints of γ and $\bar{\gamma}$, respectively. Then we define a linear isometry $\Phi_\gamma : T_q M \rightarrow T_{\bar{q}} \bar{M}$ by

$$\Phi_\gamma := \bar{P}_{\bar{\gamma}} \circ \Phi \circ (P_\gamma)^{-1}, \quad (2.2)$$

where $P_\gamma : T_p M \rightarrow T_q M$ and $\bar{P}_{\bar{\gamma}} : T_{\bar{p}} \bar{M} \rightarrow T_{\bar{q}} \bar{M}$ are parallel displacements relative to ∇^g and $\nabla^{\bar{g}}$ along γ and $\bar{\gamma}$, respectively. It is remarked that they are linear isometries.

Let us consider statistical diffeomorphisms under the setting above. Hereafter, we denote by R^g and $\bar{R}^g$ Riemannian curvature tensor fields, K and $\bar{K}$ difference tensor fields on (M, ∇, g) and $(\bar{M}, \bar{\nabla}, \bar{g})$, respectively. Our basics is the following theorem:

Theorem 2.1.2. *Let (M, ∇, g) and $(\bar{M}, \bar{\nabla}, \bar{g})$ be two statistical manifolds of the same dimension, and $p \in M$ and $\bar{p} \in \bar{M}$. Let $\Phi : T_p M \rightarrow T_{\bar{p}} \bar{M}$ be a linear isometry. Set φ and Φ_γ as (2.1), (2.2). If for any ∇^g -geodesic γ in U starting at p we have*

$$\Phi_\gamma(R^g(u, v)w) = \bar{R}^g(\Phi_\gamma(u), \Phi_\gamma(v))\Phi_\gamma(w), \quad (2.3)$$

$$\Phi_\gamma(K(u, v)) = \bar{K}(\Phi_\gamma(u), \Phi_\gamma(v)) \quad (2.4)$$

for any $u, v, w \in T_q M$, then φ is a statistical diffeomorphism from U onto $\bar{U}$.

Conversely, it is easy to see that a statistical diffeomorphism holds (2.3) and (2.4). Thus, we can say that this theorem gives us necessary and sufficient conditions for φ defined by (2.1) to be a statistical diffeomorphism.

In order to prove Theorem 2.1.2 we prepare the following lemma:

Lemma 2.1.3. *Let (M, ∇) and $(\bar{M}, \bar{\nabla})$ be two manifolds of the same dimension endowed with torsion-free affine connection, and $p \in M$ and $\bar{p} \in \bar{M}$. Let $\Phi : T_p M \rightarrow T_{\bar{p}} \bar{M}$ be a linear map. Set φ and Φ_γ as (2.1), (2.2) relative to ∇ and $\bar{\nabla}$. Suppose that for any ∇ -geodesic γ in U starting at p we have*

$$\Phi_\gamma(R(u, v)w) = \bar{R}(\Phi_\gamma(u), \Phi_\gamma(v))\Phi_\gamma(w)$$

for any $u, v, w \in T_q M$, where R and $\bar{R}$ are curvature tensor fields, respectively. Then φ_* coincides with Φ_γ at the endpoint of each γ .

Proof. Take any $q \in U$, $v \in T_q M$, and let $\gamma_q : [0, l] \rightarrow U$ be a ∇ -geodesic such that $\gamma_q(0) = p$, $\gamma_q(l) = q$. If $l = 0$, for any $v \in T_p M$,

$$\begin{aligned} (\varphi_*)_p(v) &= (\text{Exp}_{\bar{p}}^{\bar{\nabla}} \circ \Phi \circ (\text{Exp}_p^\nabla)^{-1})_{*p}(v) \\ &= (\text{Exp}_{\bar{p}}^{\bar{\nabla}} \circ \Phi)_{*0_p} (\text{Exp}_p^\nabla)^{-1}_{*p}(v) \\ &= (\text{Exp}_{\bar{p}}^{\bar{\nabla}})_{*0_{\bar{p}}} (\Phi_*)_{0_p}(v) \\ &= \Phi(v), \end{aligned}$$

we obtain $(\varphi_*)_p = \Phi$. Thus, we assume $l > 0$ hereafter. Bearing in mind that

$$(\text{Exp}_p^\nabla)_{*l\dot{\gamma}(0)} : T_{l\dot{\gamma}(0)}(T_p M) \cong T_p M \rightarrow T_q M$$

is a linear isomorphism, for any $v \in T_q M$ we choose a ∇ -Jacobi field V along γ such that

$$V(0) = 0_p, \quad (\nabla_{\dot{\gamma}} V)(0) = \frac{1}{l}(\text{Exp}_p^\nabla)_{*l\dot{\gamma}}^{-1}(v).$$

By using the formula (1.4), we obtain

$$V(l) = (\text{Exp}_p^\nabla)_{*l\dot{\gamma}(0)} \left(l \cdot \frac{1}{l} \cdot ((\text{Exp}_p^\nabla)_{*l\dot{\gamma}(0)})^{-1}(v) \right) = v.$$

Next we consider a $\bar{\nabla}$ -Jacobi field along $\bar{\gamma}_q := \varphi \circ \gamma_q$. For each $t \in [0, l]$ we define

$$\bar{V}(t) := \Phi_{\gamma_q|_{[0,t]}}(V(t)) = (P_{\bar{\gamma}_q|_{[0,t]}}^{\bar{\nabla}} \circ \Phi \circ (P_{\gamma_q|_{[0,t]}}^\nabla)^{-1})(V(t)), \quad (2.5)$$

where $P_{\gamma_q|_{[0,t]}}^\nabla : T_p M \rightarrow T_{\gamma_q(t)} M$ and $P_{\bar{\gamma}_q|_{[0,t]}}^{\bar{\nabla}} : T_{\bar{p}} \bar{M} \rightarrow T_{\bar{\gamma}_q(t)} \bar{M}$ are the parallel displacements relative to ∇ and $\bar{\nabla}$ along with γ_q and $\bar{\gamma}_q$, respectively. From property of parallel displacement, we see $\bar{V}$ is vector field along $\bar{\gamma}_q$. Moreover it satisfies the $\bar{\nabla}$ -Jacobi equation by assumption (2.1.3). Indeed,

$$\begin{aligned} \bar{\nabla}_{\dot{\bar{\gamma}_q}} \bar{\nabla}_{\dot{\bar{\gamma}_q}} \bar{V}(t) &= -R^{\bar{\nabla}}(\bar{V}(t), \dot{\bar{\gamma}_q}(t)) \dot{\bar{\gamma}_q}(t) \\ &= -\Phi_{\gamma_q|_{[0,t]}}(R^\nabla(V(t), \dot{\gamma}_q(t)) \dot{\gamma}_q(t)) \\ &= -\Phi_{\gamma_q|_{[0,t]}}(-\nabla_{\dot{\gamma}_q} \nabla_{\dot{\gamma}_q} V(t)) \\ &= \Phi_{\gamma_q|_{[0,t]}}(\nabla_{\dot{\gamma}_q} \nabla_{\dot{\gamma}_q} V(t)). \end{aligned}$$

Thus $\bar{V}$ is a $\nabla^{\bar{g}}$ -Jacobi field along $\bar{\gamma}_q$. From (2.5), we get

$$\bar{V}(0) = 0_{\bar{p}}, \quad (\bar{\nabla}_{\dot{\bar{\gamma}_q}} \bar{V})(0) = \Phi((\nabla_{\dot{\gamma}_q} V)(0)).$$

Applying the formula (1.4) to both V and $\bar{V}$, we obtain for each $t \in [0, l]$

$$V(t) = (\text{Exp}_p^\nabla)_{*t\dot{\gamma}_q(0)}(t(\nabla_{\dot{\gamma}_q} V)(0)),$$

and

$$\begin{aligned} \bar{V}(t) &= (\text{Exp}_{\bar{p}}^{\bar{\nabla}})_{*t\dot{\bar{\gamma}_q}(0)}(t(\bar{\nabla}_{\dot{\bar{\gamma}_q}} \bar{V})(0)) \\ &= (\text{Exp}_{\bar{p}}^{\bar{\nabla}})_{*t\dot{\bar{\gamma}_q}(0)} \Phi((t(\nabla_{\dot{\gamma}_q} V)(0)) \\ &= ((\text{Exp}_{\bar{p}}^{\bar{\nabla}})_{*t\dot{\bar{\gamma}_q}(0)} \circ \Phi \circ ((\text{Exp}_p^\nabla)_{*t\dot{\gamma}_q(0)})^{-1})(V(t)) \\ &= (\varphi_q)_{*\gamma_q(t)}(V(t)). \end{aligned}$$

Hence (2.5) implies

$$\Phi_{\gamma_q|_{[0,l]}}(V(t)) = (\varphi_q)_{*\gamma_q(t)}(V(t)) .$$

In particular, by setting $t = l$, we get $\Phi_{\gamma_q}(v) = (\varphi_*)_q(v)$. Since $v \in T_q M$ is a arbitrarily vector, we conclude $\Phi_{\gamma_q} = (\varphi_*)_q$ for any $q \in U$. $\square$

From this Lemma 2.1.3, we notice the following. In Riemannian case, Lemma 2.1.3 says as follows: If for any ∇^g -geodesic γ in U starting at p we have

$$\Phi_\gamma(R^g(u, v)w) = R^{\bar{g}}(\Phi_\gamma(u), \Phi_\gamma(v))\Phi_\gamma(w)$$

for any $u, v, w \in T_q M$, then φ_* coincides with Φ_γ at the endpoint of each γ .

Proof of Theorem 2.1.2. By using Lemma 2.1.3, φ_* coincides with Φ_γ at the endpoint of each γ . Thus we obtain

$$g(X, Y) = \bar{g}(\varphi_* X, \varphi_* Y)$$

for any $X, Y \in \Gamma(TU)$, and then we get $\nabla_{\varphi_* X}^{\bar{g}} \varphi_* Y = \varphi_*(\nabla_X^g Y)$. By assumption (2.4), we have

$$(\varphi_*)_q(K(u, v)) = \bar{K}((\varphi_*)_q(u), (\varphi_*)_q(v))$$

for any $u, v \in T_q M$. We now take $X, Y \in \Gamma(TU)$ such that $X_q = u, Y_q = v$, then at $q \in U$

$$\varphi_*(\nabla_X Y - \nabla_X^g Y) = \bar{\nabla}_{\varphi_* X} \varphi_* Y - \nabla_{\varphi_* X}^{\bar{g}} \varphi_* Y ,$$

thus

$$\bar{\nabla}_{\varphi_* X} \varphi_* Y = \varphi_*(\nabla_X Y) .$$

Since $q \in U$ is a arbitrarily point, we conclude $\bar{\nabla}_{\varphi_* X} \varphi_* Y = \varphi_*(\nabla_X Y)$. We have thus proved the theorem. $\square$

We notice that $\varphi := \overline{\text{Exp}}_{\bar{p}} \circ \Phi \circ (\text{Exp}_p)^{-1}$ is a unique statistical diffeomorphism from Proposition 1.2.3. If deference tensor fields K and $\bar{K}$ are vanish, Theorem 2.1.2 coincides with the Cartan-Ambrose-Hicks theorem:

Theorem 2.1.4. ([16]) *Let (M, g) and $(\bar{M}, \bar{g})$ be two Riemannian manifolds of the same dimension, and $p \in M$ and $\bar{p} \in \bar{M}$. Let $\Phi : T_p M \rightarrow T_{\bar{p}} \bar{M}$ be a linear isometry. Set φ and Φ_γ as (2.1), (2.2). Suppose that for any ∇^g -geodesic γ starting at p we have*

$$\Phi_\gamma(R^g(u, v)w) = R^{\bar{g}}(\Phi_\gamma(u), \Phi_\gamma(v))\Phi_\gamma(w)$$

for any $u, v, w \in T_q M$, where R^g and $R^{\bar{g}}$ are their Riemannian curvature tensor fields, respectively. Then φ is a local isometry around p and $\bar{p}$.

In a sense, this theorem means a Riemannian curvature tensor field locally determines a Riemannian metric is a crucial fact implying that space of constant Riemannian curvature is rich in isometries.

By using Lemma 1.3.10, we obtain the following corollary from Theorem 2.1.2.

Corollary 2.1.5. *Let (M, ∇, g) and $(\bar{M}, \bar{\nabla}, \bar{g})$ be two statistical manifolds of the same dimension, and $p \in M$ and $\bar{p} \in \bar{M}$. Let $\Phi : T_p M \rightarrow T_{\bar{p}} \bar{M}$ be a linear isometry. Set φ and Φ_γ as (2.1), (2.2). If for any ∇^g -geodesic γ in U starting at p we have*

$$\Phi_\gamma(R(u, v)w) = \bar{R}(\Phi_\gamma(u), \Phi_\gamma(v))\Phi_\gamma(w) , \quad (2.6)$$

$$\Phi_\gamma(K(u), K(v)) = \bar{K}(\Phi_\gamma(u), \Phi_\gamma(v)) , \quad (2.7)$$

$$\Phi_\gamma(\nabla K)(v, w; u) = (\bar{\nabla} \bar{K})(\Phi_\gamma(v), \Phi_\gamma(w); \Phi_\gamma(u)) \quad (2.8)$$

for all $u, v, w \in T_q M$, then φ is a statistical diffeomorphism from U onto $\bar{U}$.

Proof. By using Lemma 1.3.10 and assumption (2.6), (2.7), (2.8), we obtain

$$\begin{aligned} \Phi_\gamma(R^g(x, y)z) &= \Phi_\gamma(R(x, y)z) - \Phi_\gamma(\nabla K)(y, z; x) + \Phi_\gamma(\nabla K)(x, z; y) \\ &\quad + \Phi_\gamma(K(x, K(y, z))) - \Phi_\gamma(K(y, K(x, z))) \\ &= \bar{R}(\Phi_\gamma(x), \Phi_\gamma(y))\Phi_\gamma(z) \\ &\quad - (\bar{\nabla} \bar{K})(\Phi_\gamma(y), \Phi_\gamma(z); \Phi_\gamma(x)) + (\bar{\nabla} \bar{K})(\Phi_\gamma(x), \Phi_\gamma(z); \Phi_\gamma(y)) \\ &\quad + K(\Phi_\gamma(x), K(\Phi_\gamma(y), \Phi_\gamma(z))) - K(\Phi_\gamma(y), K(\Phi_\gamma(x), \Phi_\gamma(z))) \\ &= \bar{R}^g(\Phi_\gamma(x), \Phi_\gamma(y))\Phi_\gamma(z). \end{aligned}$$

As a direct consequence of this, we obtain (2.3). The given assumption (2.7) coincides with (2.3), thus completing the proof by virtue of Theorem 2.1.2. $\square$

In Theorem 2.1.2, we assume

$$\begin{aligned} P_\gamma(R^g(u, v)w) &= \bar{R}^g(P_\gamma(u), P_\gamma(v))P_\gamma(w) , \\ P_\gamma(K(u, v)) &= \bar{K}(P_\gamma(u), P_\gamma(v)) \end{aligned}$$

for any ∇^g -geodesics γ starting at p . When we consider the behavior of the Riemannian curvature field R^g and the difference tensor field K of a statistical manifold (M, ∇, g) , the simplest case is given by $R^g = 0$ and $K = 0$, where M is locally isometric to a Euclidean

space. The next simplest case is $\nabla^g R^g = 0$ and $\nabla^g K = 0$, that is, R^g and K are both parallel relative to ∇^g . These conditions are equivalent to the following:

$$\begin{aligned} P_c(R^g(u, v)w) &= R^{\bar{g}}(P_c(u), P_c(v))P_c(w), \\ P_c(K(u, v)) &= \bar{K}(P_c(u), P_c(v)) \end{aligned}$$

for all piecewise smooth curves c in M . These conditions are extension of assumptions of Theorem 2.1.2. We remark that $\nabla^g R^g = 0$ is known to be equivalent to the condition that (M, g) is Riemannian locally symmetric space. For example, a Riemannian manifold of constant Riemannian curvature is a Riemannian locally symmetric space. From the discussion above, we consider the following corollary:

Corollary 2.1.6. *In addition to the assumption of Theorem 2.1.2, we suppose $\nabla^g R^g = 0$, $\nabla^g K = 0$, $\nabla^{\bar{g}} R^{\bar{g}} = 0$ and $\nabla^{\bar{g}} \bar{K} = 0$. If Φ holds*

$$\Phi(R^g(u, v)w) = R^{\bar{g}}(\Phi(u), \Phi(v))\Phi(w), \quad (2.9)$$

$$\Phi(K(u, v)) = \bar{K}(\Phi(u), \Phi(v)) \quad (2.10)$$

for any $u, v, w \in T_p M$, then φ is a statistical diffeomorphism.

Proof. Since $\nabla^g R^g = 0$, $\nabla^g K = 0$, for any ∇^g -geodesic γ in U emanating from p we get

$$\begin{aligned} P_\gamma^g(R^g(u, v)w) &= R^g(P_\gamma^g(u), P_\gamma^g(v))P_\gamma^g(w), \\ P_\gamma^g(K(u, v)) &= \bar{K}(P_\gamma^g(u), P_\gamma^g(v)) \end{aligned}$$

for any $u, v, w \in T_q M$, where q is the endpoint of γ . Similarly, from $\nabla^{\bar{g}} R^{\bar{g}} = 0$, $\nabla^{\bar{g}} \bar{K} = 0$, we obtain

$$\begin{aligned} P_{\bar{\gamma}}^{\bar{g}}(R^{\bar{g}}(\bar{u}, \bar{v})\bar{w}) &= R^{\bar{g}}(P_{\bar{\gamma}}^{\bar{g}}(\bar{u}), P_{\bar{\gamma}}^{\bar{g}}(\bar{v}))P_{\bar{\gamma}}^{\bar{g}}(\bar{w}), \\ P_{\bar{\gamma}}^{\bar{g}}(K(\bar{u}, \bar{v})) &= \bar{K}(P_{\bar{\gamma}}^{\bar{g}}(\bar{u}), P_{\bar{\gamma}}^{\bar{g}}(\bar{v})) \end{aligned}$$

for any $\bar{u}, \bar{v}, \bar{w} \in T_{\bar{q}} \bar{M}$, where $\bar{\gamma} := \varphi \circ \gamma$ is the $\nabla^{\bar{g}}$ -geodesic from $\bar{p}$ to $\bar{q} := \varphi(q)$. Therefore, from assumptions (2.9) and (2.10), we obtain (2.3), (2.4). By using Theorem 2.1.2 we can conclude that φ is a statistical diffeomorphism. $\square$

In general, $\varphi : M \rightarrow \bar{M}$ is said to be a local diffeomorphism if for each $p \in M$ there exists a neighborhood U such that $\varphi|_U : U \rightarrow \varphi(U)$ is a diffeomorphism. Furthermore, φ is a statistical structure preserving, we say that φ is a local statistical diffeomorphism. When K and $\bar{K}$ are both vanish, φ is a local isometry.

Proposition 2.1.7. *Let (M, ∇, g) and $(\overline{M}, \overline{\nabla}, \overline{g})$ be two statistical manifolds, and $\varphi : M \rightarrow \overline{M}$ a local isometry. For any $p \in M$, if $(\varphi_*)_p$ holds*

$$(\varphi_*)_p(K(u, v)) = \overline{K}((\varphi_*)_p(u), (\varphi_*)_p(v)) \quad (2.11)$$

for any $u, v \in T_p M$, then φ is a local statistical diffeomorphism.

Proof. Since φ is a local isometry, for any $p \in M$ we obtain

$$(\varphi_*)_p(R^g(u, v)w) = \overline{R}^g((\varphi_*)_p(u), (\varphi_*)_p(v))(\varphi_*)_p(w)$$

for any $u, v, w \in T_p M$. Take a ∇^g -normal neighborhood $U \subset M$ around p such that $\overline{U} := \varphi(U) \subset \overline{M}$ is a $\overline{\nabla}^g$ -normal neighborhood around $\varphi(p)$. In addition, for any $q \in U$ we take a ∇^g -geodesic γ from p to q . Let P_γ be the ∇^g -parallel translation from p to q along γ , and $\overline{P}_{\varphi \circ \gamma}$ be the $\overline{\nabla}^g$ -parallel translation from $\varphi(p)$ to $\varphi(q)$ along $\varphi \circ \gamma$ in $\overline{U}$. Since $(\varphi_*)_q \circ P_\gamma = \overline{P}_{\varphi \circ \gamma} \circ (\varphi_*)_p$, we see that $\Phi_\gamma = (\varphi_*)_q$. Thus we obtain

$$\Phi_\gamma(R^g(u, v)w) = \overline{R}^g(\Phi_\gamma(u), \Phi_\gamma(v))\Phi_\gamma(w)$$

for any $u, v, w \in T_q M$. In addition by using assumption (2.11), we have

$$\Phi_\gamma(K(u, v)) = \overline{K}(\Phi_\gamma(u), \Phi_\gamma(v)) .$$

By virtue of Theorem 2.1.2, we can conclude φ is a local statistical diffeomorphism. $\square$

2.2 Statistical diffeomorphisms on Hessian manifolds

In the preceding section, we thought of the local existence of statistical diffeomorphism between statistical manifolds and showed Theorem 2.1.2. This section treats Hessian manifolds, which are the important subclass of statistical manifolds, and considers the local existence of statistical diffeomorphism between Hessian manifolds. We denote by Q and $\overline{Q}$ Hessian curvature tensor fields on M and $\overline{M}$, respectively.

Theorem 2.2.1. *Let (M, ∇, g) and $(\overline{M}, \overline{\nabla}, \overline{g})$ be two Hessian manifolds of the same dimension, and $p \in M$ and $\overline{p} \in \overline{M}$. Let $\Phi : T_p M \rightarrow T_{\overline{p}} \overline{M}$ be a linear isometry. Set φ and Φ_γ as (2.1), (2.2). If for any ∇^g -geodesic γ in U starting at p we have*

$$\begin{aligned} \Phi_\gamma(K(u, v)) &= \overline{K}(\Phi_\gamma(u), \Phi_\gamma(v)) , \\ \Phi_\gamma(Q(u, v)w) &= \overline{Q}(\Phi_\gamma(u), \Phi_\gamma(v))\Phi_\gamma(w) \end{aligned}$$

for all $u, v, w \in T_q M$, then φ is a statistical diffeomorphism from U onto $\overline{U}$.

Proof. By Definition 1.3.13, we see R and $\bar{R}$ vanish. Since $Q(X, Y)Z = -(\nabla K)(Y, Z; X)$, we obtain Theorem 2.2.1 from Corollary 2.1.5. $\square$

Next we consider the case where Hessian manifolds are of constant Hessian sectional curvature.

Corollary 2.2.2. *Let (M, ∇, g) and $(\bar{M}, \bar{\nabla}, \bar{g})$ be two Hessian manifolds of the same dimension and of constant Hessian sectional curvature $c \in \mathbb{R}$. Now let $p \in M$ and $\bar{p} \in \bar{M}$ be two any fixed points, and φ an isometry between their neighborhoods U and $\bar{U}$ such that $\varphi(p) = \bar{p}$. If for any $q \in U$ we have*

$$(\varphi_*)_q(K(u, v)) = \bar{K}((\varphi_*)_q(u), (\varphi_*)_q(v))$$

for all $u, v \in T_q M$, then φ is a statistical diffeomorphism.

Proof. From Proposition 1.3.17, both (M, ∇, g) and $(\bar{M}, \bar{\nabla}, \bar{g})$ are of constant Riemannian curvature $-c/4 \in \mathbb{R}$. Now it is to be noted that a Riemannian manifold of constant Riemannian curvature is a Riemannian locally symmetric space, that is, $\nabla^g R^g = 0$. Hence, by using Theorem 2.1.4, we see that for any points $p \in M$ and $\bar{p} \in \bar{M}$ there exists an isometry φ between their neighborhoods. By virtue of Proposition 2.1.7, we arrive at the conclusion. $\square$

Example 2.2.3. We define $((\mathbb{R}^+)^m, \nabla, g)$ by

$$\begin{aligned} (\mathbb{R}^+)^m &:= \{y = (y^1, \dots, y^m) \in \mathbb{R}^m \mid y^1 > 0, \dots, y^m > 0\}, \\ \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} &:= -\frac{1}{y^j} \delta_{ij} \frac{\partial}{\partial y^j}, \\ g &:= g^E|_{(\mathbb{R}^+)^m} = \sum_{i=1}^m (dy^i)^2, \end{aligned}$$

where $g^E = \sum_{i,j=1}^n \delta_{ij} dy^i dy^j$. Then $((\mathbb{R}^+)^m, \nabla, g)$ is a Hessian manifold of constant Hessian sectional curvature 0. The triplet $(\mathbb{R}^m, D, \tilde{g} = Dd(\sum_{i=1}^m e^{2x^i}/4))$ is also a Hessian manifold of constant Hessian sectional curvature 0, where D is the standard connection. We define $\varphi : (\mathbb{R}^+)^m \rightarrow \mathbb{R}^m$ by

$$\varphi(y^1, \dots, y^m) := (\log y^1, \dots, \log y^m),$$

then φ is a statistical diffeomorphism.

Let us recall that the subfamily of the multivariate normal distributions $\mathcal{S}_{N(\mu, \sigma^2 E_n)}$ can be regarded as the upper half model of hyperbolic space $(\mathbb{H}^{n+1}, g)$. Especially, in the case when $n = 1$, this is the family of 1-dimensional normal distribution $\mathcal{S}_{N(\mu, \sigma^2)}$, which can be regarded as the Poincaré half plane model of hyperbolic plane $(\mathbb{H}^2, g)$.

The following theorem states the statistical diffeomorphisms on the family of normal distributions.

Theorem 2.2.4. *Let $(\mathbb{H}^m, \nabla^{(-1)}, g)$ is a Hessian manifold of constant Hessian sectional curvature, which defined as*

$$\mathbb{H}^m = \{y = (y^1, \dots, y^{m-1}, y^m) \in \mathbb{R}^m \mid y^m > 0\},$$

$$g = (\kappa)^{-1} (y^m)^{-2} \sum_{j=1}^m (dy^j)^2.$$

and

$$\begin{aligned} \nabla^{(-1)}_{\partial_i} \partial_j &= 2(y^m)^{-2} \delta_{ij} \partial_m, \\ \nabla^{(-1)}_{\partial_i} \partial_m &= \nabla^{(-1)}_{\partial_m} \partial_i = 0, \\ \nabla^{(-1)}_{\partial_m} \partial_m &= (y^m)^{-1} \partial_m, \end{aligned}$$

where $\partial_i = \frac{\partial}{\partial y^i}$ for $i, j = 1, \dots, m-1$.

Then a diffeomorphism $\varphi : (\mathbb{H}^m, \nabla^{(-1)}, g) \rightarrow (\mathbb{H}^m, \nabla^{(-1)}, g)$ is a statistical diffeomorphism if and only if

$$\varphi(y) = (c \sum_{j=1}^{m-1} a_j^1 y^j + b^1, \dots, c \sum_{j=1}^{m-1} a_j^{m-1} y^j + b^{m-1}, cy^m), \quad (2.12)$$

where $A = [a_j^i]_{i,j=1,\dots,m-1} \in O(m-1)$, $b \in \mathbb{R}^{m-1}$, $c > 0$.

Proof. When φ be a statistical diffeomorphism, the following hold:

$$\nabla_{\varphi_* \partial_s}^{(-1)} \varphi_* \partial_t - \varphi_* (\nabla_{\partial_s}^{(-1)} \partial_t) = 0, \quad (2.13)$$

$$\nabla_{\varphi_* \partial_s}^g \varphi_* \partial_t - \varphi_* (\nabla_{\partial_s}^g \partial_t) = 0, \quad (2.14)$$

where $s, t = 1, \dots, m$. We note that the Levi-Civita connection of g is given by

$$\begin{aligned} \nabla_{\partial_i}^g \partial_j &= (y^m)^{-1} \delta_{ij} \partial_m, \\ \nabla_{\partial_i}^g \partial_m &= \nabla_{\partial_m}^g \partial_i = -(y^m)^{-1} \partial_i, \\ \nabla_{\partial_m}^g \partial_m &= -(y^m)^{-1} \partial_m, \end{aligned}$$

where $i, j = 1, \dots, m-1$. Now we write $\varphi(y) = (\varphi^1(y), \dots, \varphi^m(y))$ for $y \in \mathbb{H}^m$. From (2.13) and (2.14) with direct calculation, we obtain

$$\frac{\partial^2 \varphi^k}{\partial y^i \partial y^j}(y) = 0, \quad (2.15)$$

$$\frac{\partial^2 \varphi^m}{\partial y^m \partial y^m}(y) = 0, \quad (2.16)$$

$$\frac{\partial \varphi^m}{\partial y^m}(y) = \frac{1}{y^m} \varphi^m(y), \quad (2.17)$$

$$\sum_{l=1}^{m-1} \frac{\partial \varphi^l}{\partial y^i}(y) \frac{\partial \varphi^l}{\partial y^j}(y) = \frac{1}{(y^m)^2} (\varphi^m)^2 \delta_{ij}, \quad (2.18)$$

where $i, j, k = 1, \dots, m-1$. Indeed, (2.15) and (2.16) are obtained from (2.13) or (2.14). On the other hand, (2.17) and (2.18) are yielded by combining (2.13) with (2.14).

From (2.15) we see $\varphi^k(y)$ is a linear function, that is,

$$\varphi^k(y) = \sum_{i=1}^{m-1} d_i^k y^i + b^k,$$

where $d_i^k, b^k \in \mathbb{R}$ ($k = 1, \dots, m-1$). By using both (2.16) and (2.17), we get

$$\varphi^m(y) = cy^m,$$

where $c > 0$. Moreover, from (2.18), we obtain

$$\sum_{l=1}^{m-1} a_i^l a_j^l = \delta_{ij}, \quad (2.19)$$

where $a_i^l := d_i^l/c$. Expression (2.19) can be restated that $A = [a_j^i]_{i,j=1,\dots,m-1} \in O(m-1)$. For the above reasons, we reach (2.12). Conversely, we set φ as (2.12), we see that φ is a statistical diffeomorphism by computation. $\square$

In particular, when $n = 2$, we have

$$\varphi(y^1, y^2) = (\pm cy^1 + b, cy^2), \quad (2.20)$$

where $b, c \in \mathbb{R}, c > 0$. Therefore, we see that a statistical diffeomorphism on statistical models of one-dimensional normal distributions is given in (2.20).

It is known that the routes of ∇^g -geodesics in $\mathbb{H}^2$ are the constant speed parametrizations of all vertical lines y^1 constant (for a while, we name them L-type ∇^g -geodesics) and

all semicircles with centers on the y^1 -axis (C-type ∇^g -geodesics), and L-type ∇^g -geodesics and C-type ∇^g -geodesics can translate each other by isometric transformations. However, in the case of statistical diffeomorphisms, L-type ∇^g -geodesics and C-type ∇^g -geodesics cannot translate each other.

Chapter 3

Statistical sectional curvature

3.1 Statistical sectional curvature

The curvature tensor field for ∇ is defined as

$$R^\nabla(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

for $X, Y, Z \in \Gamma(TM)$. We denote R^∇ by R , R^{∇^*} by R^* , and R^{∇^g} by R^g for short.

Definition 3.1.1. For a statistical structure (∇, g) on M , we define

$$S(X, Y)Z = \frac{1}{2}\{R(X, Y)Z + R^*(X, Y)Z\}$$

for $X, Y, Z \in \Gamma(TM)$, and call $S \in \Gamma(TM^{(1,3)})$ the *statistical curvature* tensor field of (∇, g) .

Theorem 2.1.2 indicates that the statistical curvature tensor field is a suitable curvature tensor field for statistical manifolds because the statistical curvature tensor field can be rewritten using the Riemannian curvature tensor field R^g and the difference tensor field K as follows:

Proposition 3.1.2. *The following holds:*

$$S(X, Y)Z = R^g(X, Y)Z + [K, K](X, Y)Z$$

for $X, Y, Z \in \Gamma(TM)$, where $[K, K]$ is defined by

$$[K, K](X, Y)Z = [K_X, K_Y]Z = K_X(K_Y Z) - K_Y(K_X Z).$$

Remark 3.1.3. If a statistical manifold (M, ∇, g) with $K = 0$, that is, (M, ∇, g) is a Riemannian manifold, the statistical curvature tensor field S coincides with the Riemannian curvature tensor R^g . Thus, the statistical curvature tensor field is a generalization of the Riemannian curvature tensor field for statistical manifolds.

The statistical curvature tensor satisfies the following formulas:

Proposition 3.1.4. *For a statistical manifold (M, ∇, g) , the following formulas hold for $X, Y, Z, W \in \Gamma(TM)$:*

- (1) $g(S(X, Y)Z, W) = -g(S(Y, X)Z, W),$
- (2) $g(S(X, Y)Z, W) = -g(S(X, Y)W, Z),$
- (3) $S(X, Y)Z + S(Y, Z)X + S(Z, X)Y = 0,$
- (4) $g(S(X, Y)Z, W) = g(S(Z, W)X, Y).$

Then, the sectional curvature by using the statistical sectional curvature tensor field can be defined as follows:

Definition 3.1.5. Let (M, ∇, g) be a statistical manifold. For $x \in M$ and a 2-dimensional plane section $\Pi = u \wedge v$ spanned by $u, v \in T_x M$, we define the *statistical sectional curvature* of (M, ∇, g) for $u \wedge v$ by

$$\mathcal{K}(u \wedge v) = \frac{g(S(u, v)v, u)}{g(u, u)g(v, v) - g(u, v)^2}.$$

Remark 3.1.6. If ∇ is the Levi-Civita connection for g , the definition of the statistical sectional curvature $\mathcal{K}$ coincides with the Riemannian sectional curvature $\mathcal{K}^g$. Thus, the statistical sectional curvature is a generalization of the Riemannian sectional curvature for statistical manifolds.

The *Ricci curvature tensor* with respect to ∇ , denoted by Ric^∇ is defined by

$$\text{Ric}^\nabla(Y, Z) = \text{tr}\{X \mapsto R^\nabla(X, Y)Z\}.$$

We denote Ric^∇ by Ric , Ric^{∇^*} by Ric^* and Ric^{∇^g} by Ric^g respectively, for short.

For a statistical manifold (M, ∇, g) , we can also define the Ricci curvature tensor relative to the statistical curvature tensor field S as follows.

Definition 3.1.7. Let (M, ∇, g) be an m -dimensional statistical manifold and $S \in \Gamma(TM^{(1,3)})$ the statistical curvature tensor field.

(1) The *statistical Ricci curvature tensor*, denoted by L , is defined as

$$L(Y, Z) = \text{tr}\{X \mapsto S(X, Y)Z\}.$$

(2) The statistical scalar curvature, denoted by $\rho^{(\nabla, g)}$, is defined as

$$\rho^{(\nabla, g)} = \text{tr}_g L = \sum_{i=1}^m L(e_i, e_i),$$

where $\{e_1, \dots, e_m\}$ is an orthonormal basis of $T_x M$ with respect to g for $x \in M$.

Remark 3.1.8. (1) For a statistical manifold (M, ∇, g) , it is easy to see that $\rho^{(\nabla, g)} = \rho^\nabla = \rho^{\nabla^*}$, where

$$\rho^\nabla = \text{tr}_g \text{Ric}, \quad \rho^{\nabla^*} = \text{tr}_g \text{Ric}^*.$$

We denote $\rho^{(\nabla, g)}$ by ρ for short. We remark that, in general, $\rho \neq \rho^g$, where

$$\rho^g = \text{tr}_g \text{Ric}^g.$$

(2) The scalar curvatures can be rewritten by using the sectional curvatures as follows:

$$\begin{aligned} \rho &= \sum_{i,j=1}^m g(S(e_i, e_j)e_j, e_i) = 2 \sum_{1 \leq i < j \leq m} \mathcal{K}(e_i \wedge e_j), \\ \rho^g &= \sum_{i,j=1}^m g(R^g(e_i, e_j)e_j, e_i) = 2 \sum_{1 \leq i < j \leq m} \mathcal{K}^g(e_i \wedge e_j). \end{aligned}$$

3.2 Constant statistical sectional curvature

In this section, we consider statistical manifolds of constant statistical sectional curvature.

Definition 3.2.1. A statistical manifold (M, ∇, g) is said to be of *constant statistical sectional curvature* $k \in \mathbb{R}$ if the statistical sectional curvature $\mathcal{K}(\Pi)$ is k for any plane sections $\Pi \subset T_x M$ at each $x \in M$.

Proposition 3.2.2. A statistical manifold is of constant statistical sectional curvature $k \in \mathbb{R}$ if and only if

$$S(X, Y)Z = k\{g(Y, Z)X - g(X, Z)Y\}$$

for $X, Y, Z \in \Gamma(TM)$.

From Proposition 1.3.12, we find that if a statistical manifold (M, ∇, g) is of constant curvature $k \in \mathbb{R}$ with respect to ∇ , then it is of constant statistical sectional curvature $k \in \mathbb{R}$. Therefore, statistical manifolds of constant statistical sectional curvature are more general geometric objects than statistical manifolds of constant curvature.

Example 3.2.3. Let $(\mathbb{R}^m, g^{\mathbb{B}})$ be an m -dimensional Euclidean space. Denote $\partial_i = \frac{\partial}{\partial x^i}$ for short. We define $K \in \Gamma(T\mathbb{R}^{m(1,2)})$ by

$$\begin{aligned} K(\partial_m, \partial_m) &= \lambda \partial_m, \\ K(\partial_i, \partial_j) &= 0 \quad ((i, j) \neq (m, m)), \end{aligned}$$

where λ is a non-vanishing function on $\mathbb{R}^m$. Then, the triplet $(\mathbb{R}^m, \nabla = \nabla^{g^{\mathbb{B}}} + K, g^{\mathbb{B}})$ is a statistical manifold of constant statistical sectional curvature 0, but not of constant curvature.

Example 3.2.4. The triplet $((\mathbb{R}^+)^m, \nabla = \nabla^{g^0} + K, g^0)$ defined by

$$\begin{aligned} (\mathbb{R}^+)^m &= \{x = (x^1, \dots, x^m) \in \mathbb{R}^m \mid x^i > 0, i = 1, \dots, m\}, \\ g^0 &= \sum_{i=1}^m (dx^i)^2 = g^{\mathbb{B}}|_{(\mathbb{R}^+)^m}, \\ K(\partial_i, \partial_j) &= -\delta_{ij}(x^j)^{-1} \partial_j \quad (i, j = 1, \dots, m) \end{aligned}$$

is a statistical manifold of constant curvature 0, that is, a Hessian manifold.

The following statistical manifold is an geometric generalization of the statistical manifold with respect to α -connection on the hyperbolic space $(\mathbb{H}^m, \nabla^{(\alpha)}, g)$.

Example 3.2.5. Let $(\mathbb{H}^m, g)$ be the hyperbolic space of constant curvature $-\kappa$ ($\kappa > 0$) defined by

$$\begin{aligned} \mathbb{H}^m &= \{y = (y^1, \dots, y^{m-1}, y^m) \in \mathbb{R}^m \mid y^m > 0\}, \\ g &= (\kappa)^{-1} (y^m)^{-2} \sum_{j=1}^m (dy^j)^2. \end{aligned}$$

We denote $\partial_i = \frac{\partial}{\partial y^i}$ ($i = 1, \dots, m$). Define $K \in \Gamma(T\mathbb{H}^{m(1,2)})$ by

$$\begin{aligned} K(\partial_i, \partial_j) &= \delta_{ij}(-\alpha(y^m)^{-1} \partial_m + \alpha_i \partial_i), \\ K(\partial_i, \partial_m) &= K(\partial_m, \partial_i) = -\alpha(y^m)^{-1} \partial_i, \\ K(\partial_m, \partial_m) &= -2\alpha(y^m)^{-1} \partial_m \end{aligned}$$

for $\alpha \in \mathbb{R}$, $\alpha_i \in C^\infty(\mathbb{H}^m)$ and $i, j = 1, \dots, m-1$. Then, $(\mathbb{H}^m, \nabla = \nabla^g + K, g)$ is a statistical manifold of constant statistical sectional curvature $\kappa(\alpha^2 - 1)$.

This can be seen in the following way. By the definition of the difference tensor K , we obtain

$$[K, K](X, Y)Z = \kappa\alpha^2\{g(Y, Z)X - g(X, Z)Y\}$$

for any $X, Y, Z \in \Gamma(T\mathbb{H}^m)$. Thus it follows that

$$\begin{aligned} S(X, Y)Z &= R^g(X, Y)Z + [K, K](X, Y)Z \\ &= \kappa(\alpha^2 - 1)\{g(Y, Z)X - g(X, Z)Y\} \end{aligned}$$

or any $X, Y, Z \in \Gamma(T\mathbb{H}^m)$. We remark that if there exists $i \in \{1, \dots, m-1\}$ such that $\alpha_i \neq 0$, $(\mathbb{H}^m, \nabla = \nabla^g + K, g)$ is non-constant curvature with respect to ∇ for any $\alpha \in \mathbb{R}$.

However if $\alpha_i = 0$ for all $i = 1, \dots, m-1$, then ∇ coincides with the α -connection $\nabla^{(\alpha)}$, and $(\mathbb{H}^m, \nabla, g)$ is of constant curvature $\kappa(\alpha^2 - 1)$ with respect to $\nabla^{(\alpha)}$. In particular, in the case when $\alpha = -1$ and $\alpha_i = 0$, $(\mathbb{H}^m, \nabla^{(-1)}, g)$ is a Hessian manifold of constant Hessian sectional curvature.

Denote $c = \kappa(\alpha^2 - 1)$ and $\mathbb{H}^m(c) = (\mathbb{H}^m, \nabla, g)$ for short. This example shows that for any constant $c \in (-\infty, \infty)$, we can construct a statistical manifold of constant sectional curvature c on the hyperbolic space $(\mathbb{H}^m, g)$. It is interpreted from this example that the sign of curvature does not work in the geometry of statistical manifolds, though it is one of the most essential characteristics in Riemannian geometry.

3.3 Statistical Einstein manifolds and statistical Ricci solitons

In this section, we will introduce *statistical Einstein manifolds* and *statistical Ricci solitons* for statistical manifolds as statistical counterparts of Einstein manifolds and Ricci solitons in Riemannian geometry.

Definition 3.3.1. Let (M, ∇, g) be a statistical manifold, L the statistical Ricci tensor.

(1) For a unit vector field $X \in \Gamma(TM)$, $L(X, X)$ is called the *statistical Ricci curvature* of X .

(2) A statistical manifold (M, ∇, g) is said to be *statistical Einstein manifold* if the statistical Ricci curvature $L(X, X)$ is constant for any unit vector field $X \in \Gamma(TM)$.

Remark 3.3.2. (1) For an m -dimensional statistical manifold (M, ∇, g) , denote $\mathcal{K}(\Pi)$ the statistical sectional curvature of a plane section Π . Choose an orthonormal frame $\{e_1, \dots, e_m\}$ of TM such that $e_1 = X$. Then the statistical Ricci curvature of X can be written as

$$L(X, X) = \sum_{i=2}^m \mathcal{K}(X \wedge e_i). \quad (3.1)$$

(2) If ∇ is the Levi-Civita connection for g , the definition of the statistical Einstein manifold coincides with the one for the Einstein manifold in Riemannian geometry.

Proposition 3.3.3. *For a statistical manifold (M, ∇, g) , the statistical Ricci tensor $L \in \Gamma(TM^{(0,2)})$ is symmetric, that is, we have*

$$L(Y, Z) = L(Z, Y)$$

for all $X, Y \in \Gamma(TM)$.

Proof. By definition and Proposition 3.1.4, we have

$$\begin{aligned} L(Y, Z) &= \text{tr} \{X \mapsto S(X, Y)Z\} \\ &= \text{tr}_g \{(X, W) \mapsto g(S(X, Y)Z, W)\} \\ &= \text{tr}_g \{(X, W) \mapsto g(S(Z, W)X, Y)\} \\ &= \text{tr}_g \{(X, W) \mapsto -g(S(W, Z)X, Y)\} \\ &= \text{tr}_g \{(X, W) \mapsto g(S(W, Z)Y, X)\} \\ &= \text{tr} \{W \mapsto S(W, Z)Y\} \\ &= L(Z, Y). \end{aligned}$$

□

Remark 3.3.4. (1) Let (M, ∇) be an m -dimensional manifold with an affine connection. We say that ∇ is *locally equiaffine* if around each point $x \in M$ there is a parallel volume element, that is, a non-vanishing m -form ω such that $\nabla\omega = 0$. It is well-known that a (∇, ω) is local equiaffine structure on M if and only if the Ricci tensor field with respect to ∇ is symmetric (See [36]).

(2) A statistical manifold (M, ∇, g) has an equiaffine structure with respect to ∇ if and only if (M, ∇, g) has an equiaffine structure with respect to ∇^* . In other words, for a statistical manifold (M, ∇, g) , Ric is symmetric if and only if Ric^* is symmetric (See [15]). If we do not consider volume elements on M , a statistical manifold (M, ∇, g) with symmetric Ricci tensor with respect to ∇ (equivalently, or ∇^*) is called *Ricci-symmetric* statistical manifold (See [44]).

By considering the statistical Ricci tensor L , we can investigate a broader class including locally equiaffine or Ricci-symmetric statistical manifolds.

Proposition 3.3.5. *Let (M, ∇, g) be an m -dimensional statistical manifold, L the statistical Ricci tensor, and ρ the statistical scalar curvature. If (M, ∇, g) is a statistical Einstein manifold, then L is of the form*

$$L = \lambda g,$$

where $\lambda = \frac{\rho}{m}$ is constant.

Proof. If the statistical Ricci tensor L satisfies $L = \lambda g$ for some $\lambda \in \mathbb{R}$, we have

$$L(X, X) = \lambda$$

for any unit vector field $X \in \Gamma(TM)$. Thus, (M, ∇, g) is a statistical Einstein manifold.

Conversely, we assume that (M, ∇, g) is a statistical Einstein manifold, that is, there exists a constant λ such that $L(X, X) = \lambda$ holds for any unit vector $X \in \Gamma(TM)$. Then, for any non-vanishing vector field $Y \in \Gamma(TM)$, we get

$$L(Y, Y) = \|Y\|^2 L\left(\frac{Y}{\|Y\|}, \frac{Y}{\|Y\|}\right) = \lambda \|Y\|^2,$$

where $\|Y\|^2 = g(Y, Y)$. The equation above holds for $Y = 0$, thus it holds for any $Y \in \Gamma(TM)$. From Proposition 3.3.3, for any $Y, Z \in \Gamma(TM)$ we obtain

$$\begin{aligned} \lambda \|Y + Z\|^2 &= L(Y + Z, Y + Z) \\ &= L(Y, Y) + 2L(Y, Z) + L(Z, Z) \\ &= \lambda \|Y\|^2 + 2L(Y, Z) + \lambda \|Z\|^2 \end{aligned}$$

Thus we get

$$\begin{aligned} L(Y, Z) &= \frac{\lambda}{2} (\|Y + Z\|^2 - \|Y\|^2 - \|Z\|^2) \\ &= \lambda g(Y, Z). \end{aligned}$$

Moreover, we find

$$\rho = \sum_{i=1}^m L(e_i, e_i) = m\lambda,$$

where the set $\{e_1, \dots, e_m\}$ is the orthonormal frame with respect to g on TM . Thus we get $\lambda = \frac{\rho}{m}$. $\square$

Proposition 3.3.6. *If a statistical manifold (M, ∇, g) is of constant statistical sectional curvature, then (M, ∇, g) is a statistical Einstein manifold.*

Proof. Assume M is m -dimensional manifold and choose an orthonormal normal frame $\{e_1, \dots, e_m\}$ of TM such that $e_1 = X$. A statistical manifold (M, ∇, g) is of constant statistical sectional curvature k , we have

$$\mathcal{K}(X \wedge e_i) = k$$

for any $i = 2, \dots, m$. Thus, for any unit vector field $X \in \Gamma(TM)$, the following holds:

$$L(X, X) = \sum_{i=2}^m \mathcal{K}(X \wedge e_i) = (m-1)k.$$

Since $(m-1)k$ is constant, (M, ∇, g) is a statistical Einstein manifold. $\square$

Next, we will introduce the notion of *statistical Ricci solitons* for a natural generalization of statistical Einstein manifolds.

Definition 3.3.7. Let (M, ∇, g) be a statistical manifold. A smooth vector field $V \in \Gamma(TM)$ is said to define a *statistical Ricci soliton* if it satisfies the following equation:

$$L + \frac{1}{2} \mathcal{L}_V g = \lambda g,$$

where $\mathcal{L}_V g$ is the Lie-derivative of the Riemannian metric g with respect to V , L is the statistical Ricci tensor, and λ is a constant. We call the vector field V the *potential vector field of the statistical Ricci soliton*. We denote a statistical manifold with such a vector field by (M, ∇, g, V) and call it a statistical Ricci soliton.

Remark 3.3.8. (1) A statistical Ricci soliton (M, ∇, g, V) with λ is called *shrinking*, *steady*, or *expanding* according to $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively.

(2) A statistical Ricci soliton with $V = 0$ or Killing vector field is a statistical Einstein manifold. In this sense, a statistical Ricci soliton is regarded as a natural generalization of a statistical Einstein manifold.

(3) If ∇ is the Levi-Civita connection for g , the definition of the statistical Ricci soliton coincides with that of the Ricci soliton

$$\text{Ric}^g + \frac{1}{2} \mathcal{L}_V g = \lambda g,$$

where Ric^g is the Ricci tensor associated to g .

Another generalization of the Ricci soliton for a statistical manifold by A. N. Siddiqui et al. [44] is *statistical soliton* on a Ricci-symmetric statistical manifold as follows:

A smooth vector field V on a statistical manifold (M, ∇, g) is said to define a *statistical soliton* if ∇ is Ricci-symmetric and if

$$\nabla V + Q = \lambda I, \quad (3.2)$$

$$\nabla^* V + Q^* = \lambda I \quad (3.3)$$

hold, where $g(Q(X), Y) = \text{Ric}(X, Y)$, $g(Q^*(X), Y) = \text{Ric}^*(X, Y)$, I denotes the identity endomorphism.

Lemma 3.3.9. *Let (M, ∇, g, V) be a statistical soliton. We have*

$$g(\nabla_X V, Y) = g(X, \nabla_Y V),$$

$$g(\nabla_X^* V, Y) = g(X, \nabla_Y^* V),$$

$$(\mathcal{L}_V g)(X, Y) = g(\nabla_X V, Y) + g(X, \nabla_Y^* V) = g(\nabla_X^* V, Y) + g(X, \nabla_Y V)$$

for any $X, Y \in \Gamma(TM)$.

Proof. From (3.2), we obtain

$$\begin{aligned} g(\nabla_X V, Y) &= \lambda g(X, Y) - g(Q(X), Y) \\ &= \lambda g(X, Y) - g(X, Q(Y)) \\ &= g(X, \nabla_Y V) \end{aligned}$$

for all $X, Y \in \Gamma(TM)$. Similarly, by using (3.3), we get

$$g(\nabla_X^* V, Y) = g(X, \nabla_Y^* V)$$

for all $X, Y \in \Gamma(TM)$.

From the definition of the Lie derivative of a tensor field, we have

$$\begin{aligned} (\mathcal{L}_V g)(X, Y) &= Vg(X, Y) - g([V, X], Y) - g(X, [V, Y]) \\ &= g(\nabla_V X, Y) + g(X, \nabla_V^* Y) - g(\nabla_V X - \nabla_X V, Y) - g(X, \nabla_V^* Y - \nabla_Y^* V) \\ &= g(\nabla_X V, Y) + g(X, \nabla_Y^* V) \\ &= g(\nabla_X^* V, Y) + g(X, \nabla_Y V) \end{aligned}$$

for any $X, Y \in \Gamma(TM)$. Obviously, since g is symmetric, we also get

$$(\mathcal{L}_V g)(X, Y) = g(\nabla_X^* V, Y) + g(X, \nabla_Y V)$$

for any $X, Y \in \Gamma(TM)$. □

Theorem 3.3.10. *A statistical soliton is a statistical Ricci soliton.*

Proof. Let (M, ∇, g, V) be a statistical soliton with λ . From (3.2) and (3.3), we have

$$\begin{aligned}\text{Ric}(X, Y) + g(\nabla_X V, Y) &= \lambda g(X, Y), \\ \text{Ric}^*(X, Y) + g(\nabla_X^* V, Y) &= \lambda g(X, Y)\end{aligned}$$

for any $X, Y \in \Gamma(TM)$. By using Lemma 3.3.9, we see that

$$\begin{aligned}\lambda g(X, Y) &= \frac{1}{2} \{\text{Ric}(X, Y) + \text{Ric}^*(X, Y)\} + \frac{1}{2} \{g(\nabla_X V, Y) + g(\nabla_X^* V, Y)\} \\ &= L(X, Y) + \frac{1}{2} \{g(\nabla_X V, Y) + g(X, \nabla_Y^* V)\} \\ &= L(X, Y) + \frac{1}{2} (\mathcal{L}_V g)(X, Y)\end{aligned}$$

holds for any $X, Y \in \Gamma(TM)$, that is, (M, ∇, g, V) is a statistical Ricci soliton. □

Example 3.3.11. Let $(\mathbb{H}^m, \nabla, g)$ be the upper half space of constant statistical sectional curvature $c = \kappa(\alpha^2 - 1)$ with parameters $\{\kappa, \alpha, \alpha_1, \dots, \alpha_{m-1}\}$ as in Example 3.2.5. Then $(\mathbb{H}^m, \nabla, g)$ is statistical Einstein manifold. Indeed,

$$L(X, X) = (m - 1)c$$

holds for any unit vector field $X \in \Gamma(TM)$. Moreover, for any Killing vector field V , $(\mathbb{H}^m, \nabla, g, V)$ is a statistical Ricci soliton. We find the following:

- (1) If $\alpha = \pm 1$ then $(\mathbb{H}^m, \nabla, g, V)$ is a steady statistical Ricci soliton.
- (2) If $|\alpha| > 1$ then $(\mathbb{H}^m, \nabla, g, V)$ is a shrinking statistical Ricci soliton.
- (3) If $|\alpha| < 1$ then $(\mathbb{H}^m, \nabla, g, V)$ is an expanding statistical Ricci soliton.

We remark that if there exists $i \in \{1, \dots, m - 1\}$ such that $\alpha_i \neq 0$, $(\mathbb{H}^m, \nabla, g)$ does not have any statistical soliton. However, if $\alpha_i = 0$ for all $i = 1, \dots, m - 1$, then $(\mathbb{H}^m, \nabla, g, V)$ is a statistical soliton.

Chapter 4

Statistical submanifolds and geometric inequalities

4.1 Statistical submanifolds

From the viewpoint of information geometry, characteristics of statistical estimators are determined entirely by shapes of corresponding submanifolds in a set of probability distributions ([3]). Therefore, the study of submanifolds in a statistical manifold is a great interest in information geometry. In this section, we will review the basic notion of statistical submanifolds in statistical manifolds.

Let $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ be an $(m+p)$ -dimensional statistical manifold. Let M be an m -dimensional submanifold of $\tilde{M}$ and g the induced metric on M . We often denote $\tilde{g}$ and g by $\langle , \rangle$ if there is no danger of confusion.

Lemma 4.1.1. *We define an affine connection ∇ on M by*

$$\nabla_X Y = (\tilde{\nabla}_X Y)^\top,$$

where $()^\top$ denotes the orthogonal projection of the tangent space of $\tilde{M}$ with respect to $\tilde{g}$, that is, $\langle \nabla_X Y, Z \rangle = \langle \tilde{\nabla}_X Y, Z \rangle$ for $X, Y, Z \in \Gamma(TM)$. Then (M, ∇, g) becomes a statistical manifold, and this (∇, g) is called the induced statistical structure on M

Definition 4.1.2. We say that (M, ∇, g) is a statistical submanifold in $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ if (∇, g) is the induced statistical structure on M .

Let $T^\perp M$ be the normal bundle of M in $\tilde{M}$ with respect to $\tilde{g}$. We define the *second fundamental form* of M for $\tilde{\nabla}$ by

$$h(X, Y) = (\tilde{\nabla}_X Y)^\perp$$

for $X, Y \in \Gamma(TM)$ where $(\)^\perp$ denotes the orthogonal projection of $(\)$ on the normal space of M with respect to $\tilde{g}$. Since $\tilde{\nabla}$ is of torsion free, $h \in \Gamma(T^\perp M \otimes TM^{(0,2)})$ is symmetric.

We define the *shape operator* A and the *normal connection* D for $\tilde{\nabla}$, respectively, by

$$A_\xi X = -(\tilde{\nabla}_X \xi)^\top, \quad D_X \xi = (\tilde{\nabla}_X \xi)^\perp$$

for $\xi \in \Gamma(T^\perp M)$ and $X \in \Gamma(TM)$. In the same fashion, we define these objects for the dual connection $\tilde{\nabla}^*$ and the Levi-Civita connection ∇^g . Summing up, we obtain the following formulas.

Proposition 4.1.3. *Let (M, ∇, g) be a statistical submanifold in $(\tilde{M}, \tilde{\nabla}, \tilde{g})$.*

(1) *The Gauss and Weingarten formulas are written by*

$$\begin{cases} \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \\ \tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \end{cases}$$

$$\begin{cases} \tilde{\nabla}_X^* Y = \nabla_X^* Y + h^*(X, Y), \\ \tilde{\nabla}_X^* \xi = -A_\xi^* X + D_X^* \xi, \end{cases}$$

$$\begin{cases} \nabla_X^g Y = \nabla_X^g Y + \hat{h}(X, Y), \\ \nabla_X^g \xi = -\hat{A}_\xi X + \hat{D}_X \xi \end{cases}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$.

(2) *The following hold for $X, Y, Z \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(T^\perp M)$:*

$$\begin{aligned} X\langle Y, Z \rangle &= \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X^* Z \rangle, \\ \langle h(X, Y), \xi \rangle &= \langle A_\xi^* X, Y \rangle, \quad \langle h^*(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle, \\ X\langle \xi, \eta \rangle &= \langle D_X \xi, \eta \rangle + \langle \xi, D_X^* \eta \rangle. \end{aligned}$$

Let $\{e_1, \dots, e_m\}$ be an orthonormal frame on TM and $\{\xi_1, \dots, \xi_p\}$ an orthonormal frame on $T^\perp M$. Then we call $\{e_1, \dots, e_m, \xi_1, \dots, \xi_p\}$ an adapted orthonormal frame of M in $\tilde{M}$.

Definition 4.1.4. The mean curvature vector fields of M in $\widetilde{M}$ are defined by

$$\begin{aligned} H &= \frac{1}{m} \text{tr}_g h = \frac{1}{m} \sum_{i=1}^m h_{ii} = \frac{1}{m} \sum_{\beta=1}^p \left(\sum_{i=1}^m h_{ii}^\beta \right) \xi_\beta, \\ H^* &= \frac{1}{m} \text{tr}_g h^* = \frac{1}{m} \sum_{i=1}^m h_{ii}^* = \frac{1}{m} \sum_{\beta=1}^p \left(\sum_{i=1}^m h_{ii}^{*\beta} \right) \xi_\beta, \\ \widehat{H} &= \frac{1}{m} \text{tr}_g \widehat{h} = \frac{1}{m} \sum_{i=1}^m \widehat{h}_{ii} = \frac{1}{m} \sum_{\beta=1}^p \left(\sum_{i=1}^m \widehat{h}_{ii}^\beta \right) \xi_\beta, \end{aligned}$$

where

$$\begin{aligned} h_{ij} &= h(e_i, e_j), & h_{ij}^\beta &= \langle h(e_i, e_j), \xi_\beta \rangle, \\ h_{ij}^* &= h^*(e_i, e_j), & h_{ij}^{*\beta} &= \langle h^*(e_i, e_j), \xi_\beta \rangle, \\ \widehat{h}_{ij} &= \widehat{h}(e_i, e_j), & \widehat{h}_{ij}^\beta &= \langle \widehat{h}(e_i, e_j), \xi_\beta \rangle \end{aligned}$$

for $1 \leq i, j \leq m$ and $1 \leq \beta \leq p$.

Definition 4.1.5. (1) A statistical submanifold M is said to be *totally-umbilical with respect to $\widetilde{\nabla}$* if $h = H \otimes g$. Further, M is said to be *doubly totally-umbilical* if $h = H \otimes g$ and $h^* = H^* \otimes g$ hold.

(2) A statistical submanifold M is said to be *autoparallel with respect to $\widetilde{\nabla}$* if h vanishes identically. Moreover, M is said to be *doubly autoparallel* if $h = 0$ and $h^* = 0$.

Remark 4.1.6. (1) A doubly autoparallel statistical submanifold is obviously doubly totally-umbilical, so the class of doubly autoparallel statistical submanifolds is subclass of that of doubly totally-umbilical.

(2) A submanifold $M \subset (\widetilde{M}, \widetilde{\nabla})$ is said to be *totally geodesic with respect to $\widetilde{\nabla}$* if every geodesic curves with respect to $\widetilde{\nabla}$ that is tangent to $T_x M$ lies in M . When $\widetilde{\nabla}$ is torsion-free, M is totally geodesic with respect to $\widetilde{\nabla}$ if and only if M is autoparallel with respect to $\widetilde{\nabla}$. Thus, a doubly autoparallel statistical submanifold M is also said to be doubly totally geodesic.

(3) We remark that the following hold:

$$\widehat{h} = \frac{1}{2}(h + h^*), \tag{4.1}$$

$$\widehat{H} = \frac{1}{2}(H + H^*). \tag{4.2}$$

Thus, we see that if M is doubly autoparallel (resp. doubly totally-umbilical) then M is autoparallel (resp. totally-umbilical) with respect to ∇^g .

We denote

$$\|H\|^2 = \langle H, H \rangle, \quad \|h\|^2 = \sum_{i,j=1}^m \langle h_{ij}, h_{ij} \rangle,$$

$$\langle h, h^* \rangle = \sum_{i,j=1}^m \langle h_{ij}, h_{ij}^* \rangle.$$

We denote $R^{\bar{\nabla}}$ by $\widetilde{R}$, $R^{\bar{\nabla}^*}$ by $\widetilde{R}^*$, $R^{\bar{\nabla}^g}$ by $\widetilde{R}^g$, R^D by $R^\perp$ and R^{D^*} by $R^{\perp*}$, respectively. Moreover, we define $\widetilde{S}$ and $S^\perp$ by

$$\widetilde{S} = \frac{1}{2} (\widetilde{R} + \widetilde{R}^*), \quad S^\perp = \frac{1}{2} (R^\perp + R^{\perp*}).$$

Proposition 4.1.7. *Let (M, ∇, g) be a statistical submanifold in $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$. The following equations of Gauss, Codazzi and Ricci hold for $X, Y, Z \in \Gamma(TM)$ and $\xi \in \Gamma(T^\perp M)$:*

$$\begin{aligned} (\widetilde{R}(X, Y)Z)^\top &= R(X, Y)Z - A_{h(Y, Z)}X + A_{h(X, Z)}Y, \\ (\widetilde{R}(X, Y)Z)^\perp &= (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z), \\ (\widetilde{R}(X, Y)\xi)^\top &= (\bar{\nabla}_Y A)_\xi X - (\bar{\nabla}_X A)_\xi Y, \\ (\widetilde{R}(X, Y)\xi)^\perp &= R^\perp(X, Y)\xi - h(X, A_\xi Y) + h(Y, A_\xi X), \end{aligned}$$

where

$$\begin{aligned} (\bar{\nabla}_X h)(Y, Z) &= D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \\ (\bar{\nabla}_X A)_\xi Y &= \nabla_X(A_\xi Y) - A_{D_X \xi} Y - A_\xi \nabla_X Y. \end{aligned}$$

In the same setting, we have equations of Gauss, Codazzi and Ricci for $\widetilde{R}^*$ and $\widetilde{R}^g$ as well.

Proposition 4.1.8. *For a statistical submanifold (M, ∇, g) in $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$, we obtain*

$$\begin{aligned} 2(\widetilde{S}(X, Y)Z)^\top &= 2S(X, Y)Z - A_{h(Y, Z)}X + A_{h(X, Z)}Y - A_{h^*(Y, Z)}^*X + A_{h^*(X, Z)}^*Y, \\ 2(\widetilde{S}(X, Y)Z)^\perp &= (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z) + (\bar{\nabla}_X^* h^*)(Y, Z) - (\bar{\nabla}_Y^* h^*)(X, Z), \\ 2(\widetilde{S}(X, Y)\xi)^\top &= (\bar{\nabla}_Y A)_\xi X - (\bar{\nabla}_X A)_\xi Y + (\bar{\nabla}_Y^* A^*)_ \xi X - (\bar{\nabla}_X^* A^*)_ \xi Y, \\ 2(\widetilde{S}(X, Y)\xi)^\perp &= 2S^\perp(X, Y)\xi - h(X, A_\xi Y) + h(Y, A_\xi X) - h^*(X, A_\xi^* Y) + h^*(Y, A_\xi^* X) \end{aligned}$$

for $X, Y, Z \in \Gamma(TM)$, $\xi \in \Gamma(T^\perp M)$, where S and $\widetilde{S}$ are statistical curvature tensor for (∇, g) and $(\widetilde{\nabla}, \widetilde{g})$, respectively.

Remark 4.1.9. We can rewrite the Gauss equation for statistical curvature as follows:

$$\begin{aligned} 2\langle \widetilde{S}(X, Y)Z, W \rangle &= 2\langle S(X, Y)Z, W \rangle \\ &\quad - \langle h(Y, Z), h^*(X, W) \rangle + \langle h(X, Z), h^*(Y, W) \rangle \\ &\quad - \langle h^*(Y, Z), h(X, W) \rangle + \langle h^*(X, Z), h(Y, W) \rangle. \end{aligned}$$

Moreover, it is easy to see the following:

$$\begin{aligned} \rho &= \sum_{i,j=1}^m \langle \widetilde{S}(e_i, e_j)e_j, e_i \rangle + m^2 \langle H, H^* \rangle - \sum_{i,j=1}^m \langle h(e_i, e_j), h^*(e_i, e_j) \rangle \\ &= 2 \sum_{1 \leq i < j \leq m} \widetilde{\mathcal{K}}(e_i \wedge e_j) + m^2 \langle H, H^* \rangle - \sum_{i,j=1}^m \langle h(e_i, e_j), h^*(e_i, e_j) \rangle, \end{aligned} \quad (4.3)$$

where $\widetilde{\mathcal{K}}$ is the statistical sectional curvature of $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$.

4.2 Doubly autoparallel and geometric inequalities

Doubly autoparallel statistical submanifolds play important roles, because projections with respect to $\nabla^{(\alpha)}$ on a doubly autoparallel statistical submanifold are uniquely determine for all $\alpha \in \mathbb{R}$ ([37], [38]). Thus, the discussion turns into the characterization of doubly autoparallel statistical submanifolds in this section.

The notion of doubly autoparallel (or doubly totally-umbilical) statistical submanifolds is the generalization of the notion of planes (or sphere) in $\mathbb{E}^3$, because a doubly autoparallel (or doubly totally-umbilical) statistical submanifold M in the Euclidean space $\mathbb{E}^3$ is nothing but a part of a planar surface (or sphere).

In classical result, it is known that (a part of) a plane or sphere in the Euclidean spaces $\mathbb{E}^3 = (\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ is characterized by the equality case of the Euler inequality as follow:

Theorem 4.2.1. (The Euler inequality) *Surfaces M in the Euclidean spaces $\mathbb{E}^3 = (\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ satisfy*

$$\|\widehat{H}\|^2 \geq G$$

for any point, where G is the Gaussian curvature of M and $\|\widehat{H}\|^2$ is the squared mean curvature of M . Furthermore, $\|\widehat{H}\|^2 = G$ holds everywhere on M if and only if M is totally umbilical, that is, M is (a part of) a plane or sphere in $\mathbb{E}^3$.

We note that $G = \frac{1}{2}\rho^{\langle \cdot, \cdot \rangle}$, where $\rho^{\langle \cdot, \cdot \rangle}$ is the scalar curvature on M , thus we can represent the Euler inequality as follow:

$$\|\widehat{H}\|^2 \geq \frac{1}{2}\rho^{\langle \cdot, \cdot \rangle}. \quad (4.4)$$

We will generalize the inequality above (4.4) for statistical submanifolds and characterize doubly autoparallel statistical submanifolds.

Let (M, ∇, g) be an m -dimensional statistical submanifold in an $(m + p)$ -dimensional statistical manifold $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$. Let $\{e_1, \dots, e_m, \xi_1, \dots, \xi_p\}$ be an adapted orthonormal frame of M in $\widetilde{M}$. We obtain the following.

Lemma 4.2.2.

$$\begin{aligned} \rho = 2\rho^g + 2 \sum_{1 \leq i < j \leq m} \left(\widetilde{\mathcal{K}}(e_i \wedge e_j) - 2\mathcal{K}^g(e_i \wedge e_j) \right) \\ - \frac{m^2}{2} (\|H\|^2 + \|H^*\|^2) + \frac{1}{2} (\|h\|^2 + \|h^*\|^2). \end{aligned} \quad (4.5)$$

Proof. Using (4.1) and (4.2), we obtain

$$\langle h, h^* \rangle = 2\|\widehat{h}\|^2 - \frac{1}{2} (\|h\|^2 + \|h^*\|^2) \quad (4.6)$$

and

$$\langle H, H^* \rangle = 2\|\widehat{H}\|^2 - \frac{1}{2} (\|H\|^2 + \|H^*\|^2). \quad (4.7)$$

Substituting (4.6) and (4.7) in (4.3), we obtain the conclusion. $\square$

Next, let X be a unit tangent vector of M at $x \in M$. We now use an adapted orthonormal frame $\{e_1, \dots, e_m, \xi_1, \dots, \xi_p\}$ of M in $\widetilde{M}$ such that $e_1 = X$ at $x \in M$. Then the following is a fundamental inequality (cf., [18]).

Lemma 4.2.3. *We have*

$$\|h\|^2 \geq \frac{m^2}{2}\|H\|^2 + 2 \sum_{2 \leq i < j \leq m} (\|h_{ij}\|^2 - \langle h_{ii}, h_{jj} \rangle).$$

The equality holds if and only if $2h_{11} = mH$ and $h_{1j} = 0$ hold at $x \in M$ for $j = 2, \dots, m$.

Proof.

$$\begin{aligned}
\|h\|^2 &= \sum_{\beta=1}^p \sum_{i,j=1}^m (h_{ij}^\beta)^2 \\
&= \frac{1}{2} \sum_{\beta=1}^p \{ (h_{11}^\beta + (h_{22}^\beta + \cdots + h_{mm}^\beta))^2 \\
&\quad + (h_{11}^\beta - (h_{22}^\beta + \cdots + h_{mm}^\beta))^2 \} \\
&\quad + 2 \sum_{\beta=1}^p \left\{ \sum_{j=2}^m (h_{1j}^\beta)^2 + \sum_{2 \leq i < j \leq m} ((h_{ij}^\beta)^2 - h_{ii}^\beta h_{jj}^\beta) \right\} \\
&\geq \frac{1}{2} \sum_{\beta=1}^p (h_{11}^\beta + h_{22}^\beta + \cdots + h_{mm}^\beta)^2 \\
&\quad + 2 \sum_{\beta=1}^p \sum_{2 \leq i < j \leq m} ((h_{ij}^\beta)^2 - h_{ii}^\beta h_{jj}^\beta) \\
&= \frac{m^2}{2} \|H\|^2 + 2 \sum_{2 \leq i < j \leq m} \{ \|h_{ij}\|^2 - \langle h_{ii}, h_{jj} \rangle \}.
\end{aligned} \tag{4.8}$$

From (4.8) we see that the equality case holds if and only if

$$h_{11} = h_{22} + \cdots + h_{mm} \quad \text{and} \quad h_{1j} = 0$$

for $j = 2, \dots, m$. □

Proposition 4.2.4. *For an m -dimensional statistical submanifold (M, ∇, g) in an $(m + p)$ -dimensional statistical manifold $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$, we have*

$$\begin{aligned}
\|H\|^2 + \|H^*\|^2 &\geq \frac{8}{m^2} \{ 2\text{Ric}^g(X, X) - L(X, X) \\
&\quad - \sum_{j=2}^m (2\mathcal{K}^{\widetilde{g}}(X \wedge e_j) - \widetilde{\mathcal{K}}(X \wedge e_j)) \},
\end{aligned}$$

where $\{X = e_1, e_2, \dots, e_m\}$ is an orthonormal basis of $T_x M$.

The equality holds at $x \in M$ if and only if $2h_{11} = mH$, $2h_{11}^* = mH^*$ and $h_{1j} = 0$, $h_{1j}^* = 0$ hold at $x \in M$ for $j = 2, \dots, m$.

Proof. By Lemma 4.2.3 for h and h^* , it follows that

$$\begin{aligned}
\|h\|^2 + \|h^*\|^2 &\geq \frac{m^2}{2} (\|H\|^2 + \|H^*\|^2) \\
&\quad + 2 \sum_{\beta=1}^p \sum_{2 \leq i < j \leq m} \{ (h_{ij}^\beta)^2 - h_{ii}^\beta h_{jj}^\beta \} + \{ (h_{ij}^{*\beta})^2 - h_{ii}^{*\beta} h_{jj}^{*\beta} \} \\
&= \frac{m^2}{2} (\|H\|^2 + \|H^*\|^2) \\
&\quad + 2 \sum_{\beta=1}^p \sum_{2 \leq i < j \leq m} \{ (h_{ij}^\beta + h_{ij}^{*\beta})^2 - (h_{ii}^\beta + h_{ii}^{*\beta})(h_{jj}^\beta + h_{jj}^{*\beta}) \\
&\quad \quad \quad + (h_{ii}^\beta h_{jj}^{*\beta} - h_{ij}^\beta h_{ij}^{*\beta}) + (h_{ii}^{*\beta} h_{jj}^\beta - h_{ij}^{*\beta} h_{ij}^\beta) \} \\
&= \frac{m^2}{2} (\|H\|^2 + \|H^*\|^2) \\
&\quad + 8 \sum_{2 \leq i < j \leq m} \{ \|\widehat{h}_{ij}\|^2 - \langle \widehat{h}_{ii}, \widehat{h}_{jj} \rangle \} - 4 \sum_{2 \leq i < j \leq m} \{ \langle h_{ij}, h_{ij}^* \rangle - \langle h_{ii}, h_{jj}^* \rangle \} \\
&= \frac{m^2}{2} (\|H\|^2 + \|H^*\|^2) \tag{4.9} \\
&\quad - 8 \sum_{2 \leq i < j \leq m} \{ \langle R^g(e_i, e_j)e_j, e_i \rangle - \langle \widetilde{R}^g(e_i, e_j)e_j, e_i \rangle \} \\
&\quad + 4 \sum_{2 \leq i < j \leq m} \{ \langle S(e_i, e_j)e_j, e_i \rangle - \langle \widetilde{S}(e_i, e_j)e_j, e_i \rangle \}.
\end{aligned}$$

Substituting (4.9) into (4.5), we obtain

$$\begin{aligned}
\rho &\geq 2\rho^g + 2 \sum_{1 \leq i < j \leq m} \left(\widetilde{\mathcal{K}}(e_i \wedge e_j) - 2\mathcal{K}^g(e_i \wedge e_j) \right) \\
&\quad - \frac{m^2}{4} (\|H\|^2 + \|H^*\|^2) \\
&\quad - 4 \sum_{2 \leq i < j \leq m} \left(\langle R^g(e_i, e_j)e_j, e_i \rangle - \langle \widetilde{R}^g(e_i, e_j)e_j, e_i \rangle \right) \\
&\quad + 2 \sum_{2 \leq i < j \leq m} \left(\langle S(e_i, e_j)e_j, e_i \rangle - \langle \widetilde{S}(e_i, e_j)e_j, e_i \rangle \right).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\|H\|^2 + \|H^*\|^2 &\geq \frac{4}{m^2} \left\{ 2\rho^g - \rho + 2 \sum_{1 \leq i < j \leq m} \left(\widetilde{\mathcal{K}}(e_i \wedge e_j) - 2\mathcal{K}^g(e_i \wedge e_j) \right) \right. \\
&\quad - 4 \sum_{2 \leq i < j \leq m} \left(\langle R^g(e_i, e_j)e_j, e_i \rangle - \langle \widetilde{R}^g(e_i, e_j)e_j, e_i \rangle \right) \\
&\quad \left. + 2 \sum_{2 \leq i < j \leq m} \left(\langle S(e_i, e_j)e_j, e_i \rangle - \langle \widetilde{S}(e_i, e_j)e_j, e_i \rangle \right) \right\} \\
&= \frac{8}{m^2} \left\{ 2\text{Ric}^g(X, X) - L(X, X) \right. \\
&\quad \left. - 2 \sum_{j=2}^m \mathcal{K}^g(X \wedge e_j) + \sum_{j=2}^m \widetilde{\mathcal{K}}(X \wedge e_j) \right\}.
\end{aligned}$$

The equality condition follows from the one in Lemma 4.2.3 for h and h^* . $\square$

Proposition 4.2.5. *Let $\widetilde{M}(c) = (\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be an $(m + p)$ -dimensional statistical manifold of constant statistical sectional curvature c . For an m -dimensional statistical submanifold (M, ∇, g) in $\widetilde{M}(c)$, we have*

$$\begin{aligned}
\|H\|^2 + \|H^*\|^2 &\geq \frac{8}{m^2} \left\{ 2\text{Ric}^g(X, X) - L(X, X) \right. \\
&\quad \left. - 2 \sum_{j=2}^m \mathcal{K}^g(X \wedge e_j) + (m-1)c \right\},
\end{aligned}$$

where $\{X = e_1, e_2, \dots, e_m\}$ is an orthonormal basis of $T_x M$.

The equality holds at $x \in M$ if and only if $2h_{11} = mH$, $2h_{11}^* = mH^*$ and $h_{1j} = 0$, $h_{1j}^* = 0$ hold at $x \in M$ for $j = 2, \dots, m$.

Proof. Since $\widetilde{M}(c)$ is of constant statistical sectional curvature c , we obtain

$$\begin{aligned}
\sum_{j=2}^m \widetilde{\mathcal{K}}(X \wedge e_j) &= \sum_{j=2}^m \langle \widetilde{S}(X, e_j)e_j, X \rangle \\
&= \sum_{j=2}^m c \left(\langle e_j, e_j \rangle \langle X, X \rangle - \langle X, e_j \rangle \langle e_j, X \rangle \right) \\
&= (m-1)c.
\end{aligned}$$

Therefore, from Proposition 4.2.4, we can reach the conclusion. $\square$

Remark 4.2.6. In the case where $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ is of constant curvature c , a similar inequality is obtained in [8].

Corollary 4.2.7. Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be an $(m + p)$ -dimensional Hessian manifold of constant Hessian curvature k . For an m -dimensional statistical submanifold (M, ∇, g) in $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$, we have

$$\|H\|^2 + \|H^*\|^2 \geq \frac{8}{m^2} \left(2\text{Ric}^g(X, X) - L(X, X) + \frac{(m-1)k}{2} \right),$$

for a unit tangent vector $X \in T_x M$.

The equality holds at $x \in M$ if and only if $2h_{11} = mH$, $2h_{11}^* = mH^*$ and $h_{1j} = 0$, $h_{1j}^* = 0$ hold at $x \in M$ for $j = 2, \dots, m$, where $\{X = e_1, e_2, \dots, e_m\}$ is an orthonormal basis of $T_x M$.

A similar inequality is obtained in [34].

Proof. We remark that a Hessian manifold $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ of constant Hessian curvature k is a statistical manifold of constant curvature 0 such that $\widetilde{g}$ is of constant sectional curvature $-k/4$.

Thus, we have

$$\begin{aligned} \sum_{j=2}^m \mathcal{K}^{\widetilde{g}}(X \wedge e_j) &= \sum_{j=2}^m \langle R^{\widetilde{g}}(X, e_j)e_j, X \rangle \\ &= - \sum_{j=2}^m \frac{k}{4} (\langle e_j, e_j \rangle \langle X, X \rangle - \langle X, e_j \rangle \langle e_j, X \rangle) \\ &= - \frac{(m-1)k}{4}. \end{aligned}$$

By using Proposition 4.2.4, we get the conclusion. $\square$

Theorem 4.2.8. For an m -dimensional statistical submanifold (M, ∇, g) in an $(m + p)$ -dimensional statistical manifold $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$, we have

$$\|H\|^2 + \|H^*\|^2 \geq \frac{8}{m^3} \left(2\rho^g - \rho - 2 \sum_{1 \leq i < j \leq m} (2\mathcal{K}^{\widetilde{g}}(e_i \wedge e_j) - \widetilde{\mathcal{K}}(e_i \wedge e_j)) \right). \quad (4.10)$$

The equality holds at $x \in M$ if and only if either (i) $m \geq 3$ and x is a doubly autoparallel point or (ii) $m = 2$ and x is a doubly totally-umbilical point.

Proof. Consider the case where Proposition 4.2.4 holds for all unit tangent vectors at $x \in M$, we obtain

$$m(\|H\|^2 + \|H^*\|^2) \geq \frac{8}{m^2} \left(2\rho^g - \rho - 2 \sum_{1 \leq i < j \leq m} (2\mathcal{K}^g(e_i \wedge e_j) - \widetilde{\mathcal{K}}(e_i \wedge e_j)) \right).$$

If the equality of (4.10) holds, then the equality case of Proposition 4.2.4 holds for all unit tangent vectors at $x \in M$. Thus, we have

$$h_{ij} = 0, \quad i \neq j, \quad (4.11)$$

$$h_{ij}^* = 0, \quad i \neq j, \quad (4.12)$$

$$2h_{ii} = h_{11} + \cdots + h_{mm}, \quad i = 1, \dots, m, \quad (4.13)$$

$$2h_{ii}^* = h_{11}^* + \cdots + h_{mm}^*, \quad i = 1, \dots, m. \quad (4.14)$$

In the case of $m \geq 3$, conditions (4.11)-(4.14) imply that $h = h^* = 0$. It is easy to show that $h = H \otimes g$, $h^* = H^* \otimes g$ if $m = 2$. $\square$

Corollary 4.2.9. *Let (M, ∇, g) be an m -dimensional statistical submanifold in an $(m + p)$ -dimensional statistical manifold $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$.*

(1) *If $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ is of constant statistical sectional curvature c , then*

$$\|H\|^2 + \|H^*\|^2 \geq \frac{8}{m^3} \left(2\rho^g - \rho - 4 \sum_{1 \leq i < j \leq m} \mathcal{K}^g(e_i \wedge e_j) + m(m-1)c \right).$$

(2) *If $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ is a Hessian manifold of constant Hessian curvature k , then we have*

$$\|H\|^2 + \|H^*\|^2 \geq \frac{8}{m^3} \left(2\rho^g - \rho + \frac{m(m-1)k}{2} \right).$$

(3) *Each equality holds at $x \in M$ if and only if either (i) $m \geq 3$ and x is a doubly autoparallel point or (ii) $m = 2$ and x is a doubly totally-umbilical point.*

Example 4.2.10. (1) Let $(\mathbb{R}^3, \nabla, g^{\mathbb{B}})$ be a 3-dimensional statistical manifold as in Example 3.2.3. A surface M in $(\mathbb{R}^3, \nabla, g^{\mathbb{B}})$ is doubly totally-umbilical if and only if there exists $c^i \in \mathbb{R}$ such that M is a part of $\{x^3 = c^3\}$ or a plane containing $\{(x^1, x^2, x^3) = (c^1, c^2, x^3) \mid x^3 \in \mathbb{R}\}$. Such an M satisfies the equality condition of Theorem 4.2.8.

(2) Let $((\mathbb{R}^+)^3, \nabla, g^0)$ be as in Example 3.2.4. A surface M in $((\mathbb{R}^+)^3, \nabla, g^0)$ is doubly totally-umbilical if and only if either (i) M is a part of $\{x^1 = c^1\}, \{x^2 = c^2\}, \{x^3 = c^3\}$ for some $c^j \in \mathbb{R}^+$, (ii) M is a part of a plane containing the x^j -axis, or (iii) M is a part of a sphere $\{(x^1)^2 + (x^2)^2 + (x^3)^2 = r^2\}$ for some $r > 0$. Such an M satisfies the equality condition of Theorem 4.2.8.

4.3 Existence of doubly autoparallel submanifolds

In this section, we are interested in the existence of doubly autoparallel statistical submanifolds. The existence theorem of autoparallel submanifolds in a Riemannian manifold is due to É. Cartan and R. Hermann (See [16], [25]). The following theorem is obtained by N. J. Hicks (See [26], [40]).

Theorem 4.3.1. *Let $(\tilde{M}, \tilde{\nabla})$ be a manifold with a torsion-free connection. Choose $p \in \tilde{M}$, $V \subset T_p \tilde{M}$ a subspace, $\tilde{\mathcal{U}}_p \subset T_p \tilde{M}$ a convex open subset which determines normal coordinate neighborhoods $\tilde{U} = \text{Exp}_p(\tilde{\mathcal{U}}_p)$ with respect to $\tilde{\nabla}$ around p , and let $M = \text{Exp}_p(\tilde{\mathcal{U}}_p \cap V)$ with induced connection ∇ from $(\tilde{M}, \tilde{\nabla})$. Denote γ_v the $\tilde{\nabla}$ -geodesic through p whose tangent vector at p is v , where $\text{Exp}_p(v) = \gamma_v(1)$. Let P_{γ_v} be the $\tilde{\nabla}$ -parallel translating along γ_v from p to $\gamma_v(1)$. (M, ∇) is an autoparallel submanifold with respect to $\tilde{\nabla}$ if and only if*

$$\tilde{R}(P_{\gamma_v}(v), P_{\gamma_v}(x))P_{\gamma_v}(y) \in P_{\gamma_v}(V).$$

holds for any $v, x, y \in \tilde{\mathcal{U}}_p \cap V$.

Note that then $P_{\gamma_v}(V)$ coincides with $T_{\gamma_v(1)}M$.

For generalization of the above theorem to statistical manifolds, we prepare the following lemma:

Lemma 4.3.2. *Let (M, ∇, g) be a statistical submanifold in $(\tilde{M}, \tilde{\nabla}, \tilde{g})$, and $\tilde{\nabla}^{(\alpha)}$ the α -connection on $(\tilde{M}, \tilde{\nabla}, \tilde{g})$. Then, the following properties are equivalent:*

- (1) *A statistical submanifold M is doubly autoparallel.*
- (2) *For $\alpha_1, \alpha_2 \in \mathbb{R}$ ($\alpha_1 \neq \alpha_2$), a statistical submanifold M is autoparallel with respect to $\tilde{\nabla}^{(\alpha_1)}$ and $\tilde{\nabla}^{(\alpha_2)}$.*
- (3) *For any $\alpha \in \mathbb{R}$, a statistical submanifold M is autoparallel with respect to $\tilde{\nabla}^{(\alpha)}$.*

Proof. If (M, ∇, g) is a doubly autoparallel statistical submanifold in $(\tilde{M}, \tilde{\nabla}, \tilde{g})$, then $\tilde{\nabla}_X Y, \tilde{\nabla}_X^* Y \in \Gamma(TM)$ for all $X, Y \in \Gamma(TM)$. Since $\tilde{\nabla}^{(\alpha)}$ is defined as

$$\tilde{\nabla}_X^{(\alpha)} Y = \frac{1 + \alpha}{2} \tilde{\nabla}_X Y + \frac{1 - \alpha}{2} \tilde{\nabla}_X^* Y \quad (X, Y \in \Gamma(T\tilde{M})),$$

it is easy to see that M is autoparallel with respect to $\alpha \in \mathbb{R}$ for any $\alpha \in \mathbb{R}$ if M is doubly autoparallel. Conversely, if M is autoparallel with respect to $\tilde{\nabla}^{(\alpha)}$ for any $\alpha \in \mathbb{R}$, then it is obviously that (1) and (2) hold. Moreover, if M is autoparallel with respect to $\tilde{\nabla}^{(\alpha_1)}$ and $\tilde{\nabla}^{(\alpha_2)}$ ($\alpha_1, \alpha_2 \in \mathbb{R}$ $\alpha_1 \neq \alpha_2$), we get

$$\tilde{\nabla}_X^{(\alpha_1)} Y = \frac{1 + \alpha_1}{2} \tilde{\nabla}_X^{(1)} Y + \frac{1 - \alpha_1}{2} \tilde{\nabla}_X^{(-1)} Y, \quad (4.15)$$

$$\tilde{\nabla}_X^{(\alpha_2)} Y = \frac{1 + \alpha_2}{2} \tilde{\nabla}_X^{(1)} Y + \frac{1 - \alpha_2}{2} \tilde{\nabla}_X^{(-1)} Y \quad (4.16)$$

for any $X, Y \in \Gamma(TM)$. Hence we find if M is autoparallel with respect to $\widetilde{\nabla}^{(\alpha_1)}$ and $\widetilde{\nabla}^{(\alpha_2)}$ ($\alpha_1, \alpha_2 \in \mathbb{R}$ $\alpha_1 \neq \alpha_2$), M is doubly autoparallel. Thus we see the equivalence of (1), (2) and (3). $\square$

Theorem 4.3.3. *Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be a statistical manifold. Take $p \in \widetilde{M}$, $V \subset T_p \widetilde{M}$ a subspace, $\widetilde{\mathcal{U}}_p \subset T_p \widetilde{M}$ a convex open subset which determines normal coordinate neighborhoods $\widetilde{U} = \text{Exp}_p(\widetilde{\mathcal{U}}_p)$ with respect to $\nabla^{\widetilde{g}}$ around p , and let $M = \text{Exp}_p(\widetilde{\mathcal{U}}_p \cap V)$ with induced statistical structure (∇, g) from $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$. Denote γ_v the $\nabla^{\widetilde{g}}$ -geodesic through p whose tangent vector at p is v , where $\text{Exp}_p(v) = \gamma_v(1)$. Let P_{γ_v} be the $\nabla^{\widetilde{g}}$ -parallel translating along γ_v from p to $\gamma_v(1)$. M is a doubly autoparallel statistical submanifold if and only if*

$$R^{\widetilde{g}}(P_{\gamma_v}(v), P_{\gamma_v}(x))P_{\gamma_v}(y) \in P_{\gamma_v}(V), \quad (4.17)$$

$$\widetilde{K}(P_{\gamma_v}(x), P_{\gamma_v}(y)) \in P_{\gamma_v}(V) \quad (4.18)$$

holds for any $v, x, y \in \widetilde{\mathcal{U}}_p \cap V$, where $\widetilde{K}$ is the difference tensor field on $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$.

Proof. If M is a doubly autoparallel, its construction implies $P_{\gamma_v}(V) = T_{\gamma_v(1)}M$ for every $v \in \widetilde{\mathcal{U}}_p \cap V$. Therefore, M satisfies (4.17) and (4.18).

From Theorem 4.3.1 and (4.17), M is an autoparallel submanifold with respect to $\nabla^{\widetilde{g}}$. Furthermore, we get $P_{\gamma_v}(V)$ coincides with $T_{\gamma_v(1)}M$, so the set of all $P_{\gamma_v}(V)$ coincides with TM . By assumption (4.18), we see that $\widetilde{K}(X, Y) \in \Gamma(TM)$ for all $X, Y \in \Gamma(TM)$. Thus, we have

$$K(X, Y) = \widetilde{K}(X, Y)$$

for all $X, Y \in \Gamma(TM)$, where K is the difference tensor field on (M, ∇, g) . Since M is an autoparallel submanifold with respect to $\nabla^{\widetilde{g}}$, we obtain

$$\nabla_X Y = \widetilde{\nabla}_X Y$$

for all $X, Y \in \Gamma(TM)$. Therefore (M, ∇, g) is autoparallel with respect to $\widetilde{\nabla} = \widetilde{\nabla}^{(1)}$. By using lemma 4.3.2, we can conclude M is doubly autoparallel. $\square$

Chapter 5

Warped product statistical manifolds

5.1 Standard warped product statistical manifolds

This section treats standard warped product statistical manifolds. Warped product manifolds are one of the most meaningful generalizations of Riemannian product manifolds. The notion of the warped product manifolds is introduced in [14] for constructing a larger class of manifolds of negative Riemannian curvature.

Definition 5.1.1. Let (B, g_B) and (F, g_F) be Riemannian manifolds and $f \in C^\infty(B, \mathbb{R}^+)$. The warped product manifold $B \times_f F$ is the product manifold $B \times F$ endowed with the following Riemannian metric:

$$g = \pi_B^*(g_B) + (f \circ \pi_B)^2 \pi_F^*(g_F),$$

where $\pi_B : B \times F \rightarrow B$ and $\pi_F : B \times F \rightarrow F$ are projections.

Theorem 5.1.2. ([14]) *Let $M = B \times_f F$ be a warped product manifold. Denote by c_F is the Riemannian sectional curvature of F . Then the warped product $M = B \times_f F$ has constant Riemannian sectional curvature $c < 0$ if the following conditions holds:*

- (1) $\dim B = 1$ or B is of negative Riemannian curvature $c < 0$;
- (2) f is strictly convex;
- (3) $\dim F = 1$ or $c_F < 0$ if f has a minimum, or $c_F \leq 0$ if f does not have a minimum.

We remark that warped product manifolds can, of course, have positive curvature.

Based on the theorem above, we consider warped product manifolds $M = B \times_f F$ with $\dim B = 1$. It is significant to investigate this class because it is related to Sasakian manifolds and Kenmotsu manifolds. We treat Kenmotsu manifolds in section 5.3.

In [47], L. Todjihounde introduced a dualistic structure on warped product manifolds.

Let $B = (I, \nabla^I, (dt)^2)$ and $F = (\bar{N}, \bar{\nabla}, \bar{g})$ be statistical manifolds, where I is an open interval in $\mathbb{R}$, $\bar{N}$ an $(m + p - 1)$ -dimensional manifold, and $f : I \rightarrow \mathbb{R}^+$ a smooth positive function on I .

We put $\tilde{M} = I \times_f \bar{N}$. Let $\mathcal{L}(I)$ and $\mathcal{L}(\bar{N})$ be the set of all horizontal and vertical lifts on I and $\bar{N}$, respectively.

Denote by $\partial_0 = \frac{\partial}{\partial t}$ for $t \in I$ the unit tangent vector field of I . From (1.5), it is easy to see that

$$\nabla_{\partial_0}^I \partial_0 = \nu(t) \partial_0, \quad \nabla_{\partial_0}^{I*} \partial_0 = -\nu(t) \partial_0,$$

where $\nu \in C^\infty(I)$.

Definition 5.1.3. For two statistical manifolds $(I, \nabla^I, (dt)^2)$ and $(\bar{N}, \bar{\nabla} = \nabla^{\bar{g}} + \bar{K}, \bar{g})$ and $f \in C^\infty(I, \mathbb{R}^+)$, we define $(\tilde{M}, \tilde{\nabla} = \nabla^{\tilde{g}} + \tilde{K}, \tilde{g})$ by

$$\begin{aligned} \tilde{M} &= I \times \bar{N}, \\ \tilde{g} &= \langle \cdot, \cdot \rangle = (dt)^2 + f^2 \bar{g}, \\ \tilde{K}(\partial_0, \partial_0) &= \nu(t) \partial_0, \\ \tilde{K}(\partial_0, \bar{X}) &= \tilde{K}(\bar{X}, \partial_0) = 0, \\ \tilde{K}(\bar{X}, \bar{Y}) &= \bar{K}(\bar{X}, \bar{Y}) \end{aligned}$$

for $\bar{X}, \bar{Y} \in \mathcal{L}(\bar{N})$. Then $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ is a statistical manifold. We call it the *standard warped product statistical manifold* of $(I, \nabla^I, (dt)^2)$ and $(\bar{N}, \bar{\nabla}, \bar{g})$ with warping function $f \in C^\infty(I, \mathbb{R}^+)$, and denote it by $(\tilde{M}, \tilde{\nabla}, \tilde{g}) = I \times_f (\bar{N}, \bar{\nabla}, \bar{g})$.

See [23] and [47] for a general definition for warped product. We denote $f' = \frac{df}{dt}$, $f'' = \frac{d^2 f}{dt^2}$ and define

$$\bar{X} = \tilde{X} - \langle \tilde{X}, \partial_0 \rangle \partial_0 \in \mathcal{L}(\bar{N})$$

for $\tilde{X} \in \Gamma(T\tilde{M})$.

Proposition 5.1.4. Let $(\tilde{M}, \tilde{\nabla}, \tilde{g}) = I \times_f (\bar{N}, \bar{\nabla}, \bar{g})$ be a statistical warped product manifold. We denote by $\bar{S}$ and $\tilde{S}$ the statistical curvature tensor fields of $(\bar{N}, \bar{g})$ and $(\tilde{N}, \tilde{g})$,

respectively. Then we have the following formula:

$$\begin{aligned}\widetilde{S}(\widetilde{X}, \widetilde{Y})\widetilde{Z} &= \bar{S}(\bar{X}, \bar{Y})\bar{Z} - \left(\frac{f'}{f}\right)^2 \{\langle \bar{Y}, \bar{Z} \rangle \bar{X} - \langle \bar{X}, \bar{Z} \rangle \bar{Y}\} \\ &\quad + \left(\left(\frac{f'}{f}\right)^2 - \frac{f''}{f}\right) \{\langle \bar{Y}, \bar{Z} \rangle \langle \bar{X}, \partial_0 \rangle \partial_0 - \langle \bar{X}, \bar{Z} \rangle \langle \bar{Y}, \partial_0 \rangle \partial_0 \\ &\quad + \langle \bar{Y}, \partial_0 \rangle \langle \bar{Z}, \partial_0 \rangle \bar{X} - \langle \bar{X}, \partial_0 \rangle \langle \bar{Z}, \partial_0 \rangle \bar{Y}\}\end{aligned}\tag{5.1}$$

for $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Gamma(T\widetilde{M})$.

Proof. From Definition 5.1.3, it is easy to see that the following hold for $\bar{X}, \bar{Y}, \bar{Z} \in \mathcal{L}(\bar{N})$:

$$\begin{aligned}\widetilde{\nabla}_{\partial_0} \partial_0 &= \nu(t) \partial_0, \\ \widetilde{\nabla}_{\partial_0} \bar{X} &= \widetilde{\nabla}_{\bar{X}} \partial_0 = \frac{f'}{f} \bar{X}, \\ \widetilde{\nabla}_{\bar{X}} \bar{Y} &= \bar{\nabla}_{\bar{X}} \bar{Y} - \frac{f'}{f} \bar{g}(\bar{X}, \bar{Y}) \partial_0,\end{aligned}$$

and

$$\begin{aligned}\widetilde{\nabla}_{\partial_0}^* \partial_0 &= -\nu(t) \partial_0, \\ \widetilde{\nabla}_{\partial_0}^* \bar{X} &= \widetilde{\nabla}_{\bar{X}}^* \partial_0 = \frac{f'}{f} \bar{X}, \\ \widetilde{\nabla}_{\bar{X}}^* \bar{Y} &= \bar{\nabla}_{\bar{X}}^* \bar{Y} - \frac{f'}{f} \bar{g}(\bar{X}, \bar{Y}) \partial_0,\end{aligned}$$

where $\nu \in C^\infty$. Thus, we get

$$\begin{aligned}\widetilde{R}(\partial_0, \partial_0) \partial_0 &= 0, \\ \widetilde{R}(\bar{X}, \partial_0) \partial_0 &= \frac{\nu f' - f''}{f} \bar{X}, \\ \widetilde{R}(\bar{X}, \bar{Y}) \partial_0 &= 0, \\ \widetilde{R}(\partial_0, \partial_0) \bar{X} &= 0, \\ \widetilde{R}(\partial_0, \bar{X}) \bar{Y} &= -\frac{\nu f' + f''}{f} \langle \bar{X}, \bar{Y} \rangle \partial_0, \\ \widetilde{R}(\bar{X}, \bar{Y}) \bar{Z} &= \bar{R}(\bar{X}, \bar{Y}) \bar{Z} - \left(\frac{f'}{f}\right)^2 \{\langle \bar{Y}, \bar{Z} \rangle \bar{X} - \langle \bar{X}, \bar{Z} \rangle \bar{Y}\},\end{aligned}$$

and

$$\begin{aligned}
\widetilde{R}^*(\partial_0, \partial_0)\partial_0 &= 0, \\
\widetilde{R}^*(\bar{X}, \partial_0)\partial_0 &= -\frac{\nu f' + f''}{f}\bar{X}, \\
\widetilde{R}^*(\bar{X}, \bar{Y})\partial_0 &= 0, \\
\widetilde{R}^*(\partial_0, \partial_0)\bar{X} &= 0, \\
\widetilde{R}^*(\partial_0, \bar{X})\bar{Y} &= \frac{\lambda f' - f''}{f}\langle \bar{X}, \bar{Y} \rangle \partial_0, \\
\widetilde{R}^*(\bar{X}, \bar{Y})\bar{Z} &= R^*(\bar{X}, \bar{Y})\bar{Z} - \left(\frac{f'}{f}\right)^2 \{\langle \bar{Y}, \bar{Z} \rangle \bar{X} - \langle \bar{X}, \bar{Z} \rangle \bar{Y}\}.
\end{aligned}$$

From these equations, we directly obtain

$$\begin{aligned}
\widetilde{S}(\partial_0, \partial_0)\partial_0 &= 0, \\
\widetilde{S}(\bar{X}, \partial_0)\partial_0 &= -\frac{f''}{f}\bar{X}, \\
\widetilde{S}(\bar{X}, \bar{Y})\partial_0 &= 0, \\
\widetilde{S}(\partial_0, \partial_0)\bar{X} &= 0, \\
\widetilde{S}(\partial_0, \bar{X})\bar{Y} &= -\frac{f''}{f}\langle \bar{X}, \bar{Y} \rangle \partial_0, \\
\widetilde{S}(\bar{X}, \bar{Y})\bar{Z} &= \bar{S}(\bar{X}, \bar{Y})\bar{Z} - \left(\frac{f'}{f}\right)^2 \{\langle \bar{Y}, \bar{Z} \rangle \bar{X} - \langle \bar{X}, \bar{Z} \rangle \bar{Y}\}.
\end{aligned}$$

Therefore, we get (5.1). □

Theorem 5.1.5. *Let $\bar{N}(c) = (\bar{N}, \bar{\nabla}, \bar{g})$ be an $(m + p - 1)$ -dimensional statistical manifold of constant statistical sectional curvature c . Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}) = I \times_f \bar{N}(c)$ be an $(m + p)$ -dimensional standard warped product statistical manifold. Then we have*

$$\begin{aligned}
\widetilde{S}(\widetilde{X}, \widetilde{Y})\widetilde{Z} &= \left(\frac{c}{f^2} - \left(\frac{f'}{f}\right)^2\right) \{\langle \widetilde{Y}, \widetilde{Z} \rangle \widetilde{X} - \langle \widetilde{X}, \widetilde{Z} \rangle \widetilde{Y}\} \\
&\quad - \left(\frac{c}{f^2} - \left(\frac{f'}{f}\right)^2 + \frac{f''}{f}\right) \{\langle \widetilde{Y}, \widetilde{Z} \rangle \langle \widetilde{X}, \partial_0 \rangle \partial_0 - \langle \widetilde{X}, \widetilde{Z} \rangle \langle \widetilde{Y}, \partial_0 \rangle \partial_0 \\
&\quad \quad + \langle \widetilde{Y}, \partial_0 \rangle \langle \widetilde{Z}, \partial_0 \rangle \widetilde{X} - \langle \widetilde{X}, \partial_0 \rangle \langle \widetilde{Z}, \partial_0 \rangle \widetilde{Y}\}
\end{aligned}$$

for $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Gamma(T\widetilde{M})$.

From the theorem above, we find we can construct warped product statistical manifolds of constant statistical sectional curvature as follows:

Theorem 5.1.6. *Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}) = I \times_f \widetilde{N}(c)$ be a standard warped product statistical manifold and $\lambda \in \mathbb{R}$ non-zero constant.*

(1) *If $c = 0$ and $f(t) = \exp(\sqrt{|k|}t)$ for non-positive constant k on $I = \mathbb{R}$, then $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ is of constant statistical sectional curvature k .*

(2) *If $c < 0$ and $f(t) = \cosh(\sqrt{|k|}t)$ on $I = \mathbb{R}$, then $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ is of constant statistical sectional curvature c .*

(3) *If $c > 0$ and $\lambda > 0$ and $f(t) = \cos(\sqrt{c}t)$ on $I = (-\frac{\pi}{2\lambda}, \frac{\pi}{2\lambda})$, then $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ is of constant statistical sectional curvature c .*

(4) *If $c > 0$ and $f(t) = \sqrt{c}t$ on $I = \mathbb{R}^+$, then $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ is of constant statistical sectional curvature 0.*

Proof. (1) If $f = \exp(\lambda t)$ on $I = \mathbb{R}$, then we have

$$\begin{aligned}\frac{c}{f^2} - \left(\frac{f'}{f}\right)^2 &= \frac{c}{f^2} - \lambda^2, \\ \frac{c}{f^2} - \left(\frac{f'}{f}\right)^2 + \frac{f''}{f} &= \frac{c}{f^2}.\end{aligned}$$

Hence, if $c = 0$ and $\lambda = \sqrt{|k|}$ for non-positive constant k , then $(\widetilde{\nabla}, \widetilde{g})$ is of constant statistical sectional curvature k .

(2) If $f(t) = \cosh(\lambda t)$ on $I = \mathbb{R}$, then we have

$$\begin{aligned}\frac{c}{f^2} - \left(\frac{f'}{f}\right)^2 &= \frac{1}{f^2}(c + \lambda^2) - \lambda^2, \\ \frac{c}{f^2} - \left(\frac{f'}{f}\right)^2 + \frac{f''}{f} &= \frac{1}{f^2}(c + \lambda^2).\end{aligned}$$

Thus, if $c < 0$ and $\lambda = \sqrt{|c|}$, then $(\widetilde{\nabla}, \widetilde{g})$ is of constant statistical sectional curvature c .

(3) If $\lambda > 0$ and $f(t) = \cos(\lambda t)$ on $I = (-\frac{\pi}{2\lambda}, \frac{\pi}{2\lambda})$, then we have

$$\begin{aligned}\frac{c}{f^2} - \left(\frac{f'}{f}\right)^2 &= \frac{1}{f^2}(c - \lambda^2) + \lambda^2, \\ \frac{c}{f^2} - \left(\frac{f'}{f}\right)^2 + \frac{f''}{f} &= \frac{1}{f^2}(c - \lambda^2).\end{aligned}$$

Hence, if $c > 0$ and $\lambda = \sqrt{c}$, then $(\widetilde{\nabla}, \widetilde{g})$ is of constant statistical sectional curvature c .

(4) If $\lambda > 0$ and $f(t) = \lambda t$ on $I = \mathbb{R}^+$, then we have

$$\begin{aligned}\frac{c}{f^2} - \left(\frac{f'}{f}\right)^2 &= \frac{1}{t^2} \left(\frac{c}{\lambda^2} - 1\right), \\ \frac{c}{f^2} - \left(\frac{f'}{f}\right)^2 + \frac{f''}{f} &= \frac{1}{t^2} \left(\frac{c}{\lambda^2} - 1\right).\end{aligned}$$

Thus, if $c > 0$ and $\lambda = \sqrt{c}$, then it follow that $\widetilde{S} \equiv 0$. □

5.2 Statistical submanifolds in warped product statistical manifolds

This section treats statistical submanifolds in standard warped product statistical manifolds and discusses geometric inequalities characterizing doubly autoparallel.

Let $\bar{N}(c) = (\bar{N}, \bar{\nabla}, \bar{g})$ be an $(m + p - 1)$ -dimensional statistical manifold of constant statistical sectional curvature c , and $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}) = I \times_f \bar{N}(c)$ the $(m + p)$ -dimensional standard warped product statistical manifold of $(I, \nabla^I, (dt)^2)$ and $\bar{N}(c)$ with warping function f . Let (M, ∇, g) be an m -dimensional statistical submanifold in $I \times_f \bar{N}(c)$.

Definition 5.2.1. For a submanifold M in $I \times_f \bar{N}(c)$, we denote by $T \in \Gamma(TM)$ the tangent part of $\partial_0 \in \mathcal{L}(I)$, which is given by

$$T = \sum_{i=1}^m \langle \partial_0, e_i \rangle e_i,$$

where $\{e_1, \dots, e_m\}$ be an orthonormal tangent frame of M in $\widetilde{M}$.

Proposition 5.2.2. *Let (M, ∇, g) be an m -dimensional statistical submanifold in $I \times_f \bar{N}(c)$. Then the scalar curvature ρ of M satisfies*

$$\begin{aligned}\rho \geq & m(m-1) \left(\frac{c}{f^2} - \left(\frac{f'}{f}\right)^2 \right) \\ & - 2(m-1) \left(\frac{c}{f^2} - \left(\frac{f'}{f}\right)^2 + \frac{f''}{f} \right) \|T\|^2 \\ & + m^2 \langle H, H^* \rangle - \|h\| \|h^*\|.\end{aligned}$$

The equality holds at $x \in M$ if and only if $h = 0$ or $h^ = 0$ or there exists a positive constant μ such that $h^* = \mu h$.*

Proof. From Proposition 5.1.5 and (4.3), we obtain

$$\begin{aligned}\rho &= m(m-1) \left(\frac{c}{f^2} - \left(\frac{f'}{f} \right)^2 \right) \\ &\quad - 2(m-1) \left(\frac{c}{f^2} - \left(\frac{f'}{f} \right)^2 + \frac{f''}{f} \right) \|T\|^2 \\ &\quad + m^2 \langle H, H^* \rangle - \sum_{i,j=1}^m \langle h_{ij}, h_{ij}^* \rangle.\end{aligned}$$

By using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\rho &\geq m(m-1) \left(\frac{c}{f^2} - \left(\frac{f'}{f} \right)^2 \right) \\ &\quad - 2(m-1) \left(\frac{c}{f^2} - \left(\frac{f'}{f} \right)^2 + \frac{f''}{f} \right) \|T\|^2 \\ &\quad + m^2 \langle H, H^* \rangle - \|h\| \|h^*\|,\end{aligned}$$

where the equality holds for $h = 0$ or $h^* = 0$ or $h = \mu h^*$ ($\mu \in \mathbb{R}^+$). $\square$

Proposition 5.2.3. *For an m -dimensional statistical submanifold (M, ∇, g) in $I \times_f \bar{N}(c)$, it follows that*

$$\begin{aligned}\|H\|^2 + \|H^*\|^2 &\geq \frac{8}{m^2} \left\{ 2\text{Ric}^g(X, X) - L(X, X) - 2 \sum_{j=2}^m \mathcal{K}^{\tilde{g}}(X \wedge e_j) \right. \\ &\quad + (m-1) \left(\frac{c}{f^2} - \left(\frac{f'}{f} \right)^2 \right) \\ &\quad \left. - \left(\frac{c}{f^2} - \left(\frac{f'}{f} \right)^2 + \frac{f''}{f} \right) (\|T\|^2 + (m-2) \langle X, T \rangle^2) \right\}\end{aligned}$$

for a unit tangent vector $X \in T_x M$.

The equality holds at $x \in M$ if and only if $2h_{11} = mH$, $2h_{11}^* = mH^*$ and $h_{1j} = 0$, $h_{1j}^* = 0$ hold at $x \in M$ for $j = 2, \dots, m$, where $\{X = e_1, e_2, \dots, e_m\}$ is an orthonormal basis of $T_x M$.

Proof. From Proposition 5.1.5, we obtain

$$\begin{aligned}\sum_{j=2}^m \tilde{\mathcal{K}}(X \wedge e_j) &= \sum_{j=1}^m \langle \tilde{S}(X, e_j) e_j, X \rangle \\ &= (m-1) \left(\frac{c}{f^2} - \left(\frac{f'}{f} \right)^2 \right) \\ &\quad - \left(\frac{c}{f^2} - \left(\frac{f'}{f} \right)^2 + \frac{f''}{f} \right) (\|T\|^2 - (m-2) \langle X, T \rangle^2).\end{aligned}$$

From Proposition 4.2.4, we reach the conclusion. $\square$

Theorem 5.2.4. *For an m -dimensional statistical submanifold (M, ∇, g) in an $(m+p)$ -dimensional standard warped product statistical manifold $I \times_f \tilde{N}(c)$, we have*

$$\begin{aligned}\|H\|^2 + \|H^*\|^2 &\geq \frac{8}{m^3} \left\{ 2\rho^g - \rho - 4 \sum_{1 \leq i < j \leq m} \tilde{\mathcal{K}}^g(e_i \wedge e_j) \right. \\ &\quad + m(m-1) \left(\frac{c}{f^2} - \left(\frac{f'}{f} \right)^2 \right) \\ &\quad \left. - 2(m-1) \left(\frac{c}{f^2} - \left(\frac{f'}{f} \right)^2 + \frac{f''}{f} \right) \|T\|^2 \right\}.\end{aligned}$$

The equality holds at $x \in M$ if and only if either (i) $m \geq 3$ and x is a doubly autoparallel point or (ii) $m = 2$ and x is a doubly totally-umbilical point.

Proof. Since $\tilde{N}(c)$ is of constant statistical sectional curvature c , we have

$$\begin{aligned}2 \sum_{1 \leq i < j \leq m} \tilde{\mathcal{K}}(e_i \wedge e_j) &= \sum_{j=1}^m \langle \tilde{S}(e_i, e_j) e_j, e_i \rangle \\ &= m(m-1) \left(\frac{c}{f^2} - \left(\frac{f'}{f} \right)^2 \right) \\ &\quad - 2(m-1) \left(\frac{c}{f^2} - \left(\frac{f'}{f} \right)^2 + \frac{f''}{f} \right) \|T\|^2.\end{aligned} \tag{5.2}$$

Thus, by substituting (5.2) to Theorem 4.2.8, we obtain the inequality.

As in the proof of Theorem 4.2.8, if the equality case of Proposition 5.2.3 holds identically for all unit tangent vectors at $x \in M$, then we have $h = h^* = 0$ if $m \geq 3$. It is easy to see that $h = g \otimes H$, $h^* = g \otimes H^*$ if $m = 2$. $\square$

Example 5.2.5. We consider a standard warped product statistical manifold

$$(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}) = \mathbb{R} \times_{\cosh(\lambda t)} \mathbb{H}^{m+p-1}(c), \quad m \geq 2 \text{ and } p \geq 1,$$

where λ is constant and $\mathbb{H}^{m+p-1}(c)$ is in Example 3.2.5 of parameters $\kappa, \alpha, \alpha_1, \dots, \alpha_{m+p-2}$ with $c = \kappa(\alpha^2 - 1)$, which is written as

$$\widetilde{M} = \{(t, y^1, \dots, y^{m+p-1}) \in \mathbb{R}^{m+p} \mid y^{m+p-1} > 0\}$$

by using the global coordinates. We denote $\partial_0 = \frac{\partial}{\partial t}$ and $\partial_i = \frac{\partial}{\partial y^i}$ for $i = 1, \dots, m+p-1$.

If $\lambda = 0$, the above ambient space is a product statistical manifold. If $\lambda = \sqrt{\kappa}$, then $(\widetilde{M}, \widetilde{g})$ is actually the hyperbolic space of constant sectional curvature $-\kappa$. Furthermore, by Theorem 5.1.6, if $\lambda = \sqrt{\kappa}$ and $\alpha = 0$, then $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ is $\mathbb{H}^{m+p}(-\kappa)$ constructed in Example 3.2.5 with $\alpha = 0$.

(1) For constants $(k^1, \dots, k^p) \in \mathbb{R}^{p-1} \times \mathbb{R}^+$, we consider

$$M = \{(x^1, \dots, x^m, k^1, \dots, k^p) \in \widetilde{M} \mid (x^1, \dots, x^m) \in \mathbb{R}^m\}.$$

Then, M satisfies the equality condition of Proposition 5.2.2 at each point if and only if $|\alpha| \leq 1$. On the other hand, M does not satisfy the equality condition of Theorem 5.2.4. In fact, we obtain

$$\begin{aligned} h(e_1, e_i) &= h^*(e_1, e_i) = 0 \quad (i = 1, \dots, m), \\ h(e_i, e_j) &= \delta_{ij}(1 + \alpha)\kappa^{\frac{1}{2}} \cosh^{-1}(\lambda t)\xi_p \quad (i, j = 2, \dots, m), \\ h^*(e_i, e_j) &= \delta_{ij}(1 - \alpha)\kappa^{\frac{1}{2}} \cosh^{-1}(\lambda t)\xi_p \quad (i, j = 2, \dots, m), \end{aligned}$$

where $\{e_1, \dots, e_m, \xi_1, \dots, \xi_p\}$ is an adapted orthonormal frame given by

$$\begin{aligned} e_1 &= \partial_0, \\ e_i &= \kappa^{\frac{1}{2}} k^p \cosh^{-1}(\lambda t) \partial_{i-1} \quad (i = 2, \dots, m), \\ \xi_\beta &= \kappa^{\frac{1}{2}} k^p \cosh^{-1}(\lambda t) \partial_{m+\beta-1} \quad (\beta = 1, \dots, p). \end{aligned}$$

(2) For constants $(k^1, \dots, k^p) \in \mathbb{R}^p$, the submanifold

$$M = \{(x^1, \dots, x^{m-1}, k^1, \dots, k^p, x^m) \in \widetilde{M} \mid (x^1, \dots, x^m) \in \mathbb{R}^{m-1} \times \mathbb{R}^+\}$$

is doubly autoparallel. This means that M satisfies the equality conditions of Proposition 5.2.2 and Theorem 5.2.4.

(3) For constants $(k^1, \dots, k^p) \in \mathbb{R}^p$, the submanifold

$$M = \{(k^1, \dots, k^p, x^1, \dots, x^m) \in \tilde{M} \mid (x^1, \dots, x^m) \in \mathbb{R}^{m-1} \times \mathbb{R}^+\}$$

is doubly totally-umbilical. Then, M satisfies the equality condition of Proposition 5.2.2 at each point. Furthermore, M satisfies the equality condition of Theorem 5.2.4 at each point if and only if either (i) $m \geq 3$ and $\lambda k^1 = 0$ or (ii) $m = 2$. In fact, we obtain

$$h(e_i, e_j) = h^*(e_i, e_j) = -\delta_{ij} \lambda \tanh(\lambda k^1) \xi_1 \quad (i, j = 1, \dots, m),$$

where $\{e_1, \dots, e_m, \xi_1, \dots, \xi_p\}$ is an adapted orthonormal frame given by

$$\begin{aligned} e_i &= \kappa^{\frac{1}{2}} x^m \cosh^{-1}(\lambda k^1) \partial_{i+p-1} \quad (i = 1, \dots, m), \\ \xi_1 &= \partial_0, \\ \xi_\beta &= \kappa^{\frac{1}{2}} x^m \cosh^{-1}(\lambda k^1) \partial_{\beta-1} \quad (\beta = 2, \dots, p). \end{aligned}$$

(4) For constants $(k^1, \dots, k^p) \in \mathbb{R}^{p-1} \times \mathbb{R}^+$ ($p \geq 2$), the submanifold

$$M = \{(k^1, x^1, \dots, x^m, k^2, \dots, k^p) \in \tilde{M} \mid (x^1, \dots, x^m) \in \mathbb{R}^m\}$$

is doubly totally-umbilical. Then, M satisfies the equality condition of Proposition 5.2.2 at each point if and only if either (i) $\lambda k^1 = 0$ and $|\alpha| \leq 1$ or (ii) $\lambda k^1 \neq 0$ and $\alpha = 0$. Furthermore, M satisfies the equality condition of Theorem 5.2.4 at each point if and only if $m = 2$. Actually, we obtain

$$\begin{aligned} h(e_i, e_j) &= \delta_{ij} \{(1 + \alpha) \kappa^{\frac{1}{2}} \cosh^{-1}(\lambda k^1) \xi_p - \lambda \tanh(\lambda k^1) \xi_1\}, \\ h^*(e_i, e_j) &= \delta_{ij} \{(1 - \alpha) \kappa^{\frac{1}{2}} \cosh^{-1}(\lambda k^1) \xi_p - \lambda \tanh(\lambda k^1) \xi_1\} \end{aligned}$$

for $i, j = 1, \dots, m$, where $\{e_1, \dots, e_m, \xi_1, \dots, \xi_p\}$ is an adapted orthonormal frame given by

$$\begin{aligned} e_i &= \kappa^{\frac{1}{2}} k^p \cosh^{-1}(\lambda k^1) \partial_i \quad (i = 1, \dots, m), \\ \xi_1 &= \partial_0, \\ \xi_\beta &= \kappa^{\frac{1}{2}} k^p \cosh^{-1}(\lambda k^1) \partial_{m+\beta-1} \quad (\beta = 2, \dots, p). \end{aligned}$$

(5) For constants $(k^1, \dots, k^{p-1}) \in \mathbb{R}^{p-1}$, we consider

$$M = \{(x^1, x^1, x^2, \dots, x^{m-1}, k^1, \dots, k^{p-1}, x^m) \in \widetilde{M} \mid (x^1, \dots, x^m) \in \mathbb{R}^{m-1} \times \mathbb{R}^+\}.$$

Then, M satisfies the equality condition of Proposition 5.2.2 at each point if and only if either (i) $\lambda = 0$, $\nu = \alpha_1$ and $|\alpha| \leq 1$ or (ii) $\lambda \neq 0$, $\nu = \alpha_1$ and $\alpha = 0$. Nevertheless, M does not satisfy the equality condition of Theorem 5.2.4. In fact, we have

$$\begin{aligned} h(e_1, e_1) &= q^{-1} r^{-\frac{1}{2}} \{\alpha_1 - \nu + \lambda(1 + q) \tanh(\lambda t)\} \xi_1, \\ h^*(e_1, e_1) &= q^{-1} r^{-\frac{1}{2}} \{-\alpha_1 + \nu + \lambda(1 + q) \tanh(\lambda t)\} \xi_1, \\ h(e_1, e_m) &= -q^{-\frac{1}{2}} r^{-\frac{1}{2}} \kappa^{\frac{1}{2}} \cosh^{-1}(\lambda t) (1 - \alpha) \xi_1, \\ h^*(e_1, e_m) &= -q^{-\frac{1}{2}} r^{-\frac{1}{2}} \kappa^{\frac{1}{2}} \cosh^{-1}(\lambda t) (1 + \alpha) \xi_1, \\ h(e_i, e_i) &= h^*(e_i, e_i) = r^{-\frac{1}{2}} \lambda \tanh(\lambda t) \xi_1 \quad (i = 2, \dots, m), \\ h(e_i, e_j) &= h^*(e_i, e_j) = 0 \quad ((i, j) = \text{otherwise}), \end{aligned}$$

where

$$\begin{aligned} q &= 1 + \kappa^{-1} (x^m)^{-2} \cosh^2(\lambda t), \\ r &= 1 + \kappa (x^m)^2 \cosh^{-2}(\lambda t) \end{aligned}$$

and $\{e_1, \dots, e_m, \xi_1, \dots, \xi_p\}$ is an adapted orthonormal frame given by

$$\begin{aligned} e_1 &= q^{-\frac{1}{2}} (\partial_0 + \partial_1), \\ e_i &= \kappa^{\frac{1}{2}} x^m \cosh^{-1}(\lambda t) \partial_i \quad (i = 2, \dots, m-1), \\ e_m &= \kappa^{\frac{1}{2}} x^m \cosh^{-1}(\lambda t) \partial_{m+p-1}, \\ \xi_1 &= r^{-\frac{1}{2}} (-\partial_0 + \kappa (x^m)^2 \cosh^{-2}(\lambda t) \partial_1), \\ \xi_\beta &= \kappa^{\frac{1}{2}} x^m \cosh^{-1}(\lambda t) \partial_{m+\beta-2} \quad (\beta = 2, \dots, p). \end{aligned}$$

5.3 Statistical pseudo-Ricci solitons on Kenmotsu statistical manifolds

Regarding notions of hyperbolic space and warped product, the Kenmotsu manifold is one of the most important objects because a typical example of a Kenmotsu manifold is an odd-dimensional hyperbolic space, and a Kenmotsu manifold is locally constructed as the warped product manifold of a line and a Kähler manifold.

Let us recall that an almost contact metric manifold (M, g, ϕ, ξ, η) satisfies

$$\begin{aligned}\phi^2 X &= -X + \eta(X)\xi, \\ \phi\xi &= 0, \\ \eta(\phi X) &= 0, \\ \eta(\xi) &= 1, \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \\ \eta(X) &= g(X, \xi)\end{aligned}$$

for any $X, Y \in \Gamma(TM)$ as in Definition 1.2.8.

Definition 5.3.1. ([27]) An almost contact metric manifold (M, g, ϕ, ξ, η) is said to be Kenmotsu manifold if the following conditions

$$\begin{aligned}(\nabla_X^g \phi)Y &= -\eta(Y)\phi X + g(\phi X, Y)\xi, \\ \nabla_X^g \xi &= X - \eta(X)\xi\end{aligned}\tag{5.3}$$

hold for any $X, Y \in \Gamma(TM)$.

Proposition 5.3.2. ([27]) Let $(\bar{N}, \bar{g}, J)$ be a Kähler manifold and λ a non-zero constant. Let $f(t) = \lambda \exp(t)$ be a function on a line $\mathbb{R}$. Then the warped product manifold $\tilde{M} = \mathbb{R} \times_f \bar{N}$ has a Kenmotus structure $(\tilde{g}, \phi, \xi, \eta)$ as follow:

$$\begin{aligned}\tilde{g} &= (dt)^2 + \lambda^2 \exp(2t)\bar{g}, \\ \phi\tilde{X} &= J\bar{X}, \quad \phi\xi = 0, \\ \xi &= \frac{\partial}{\partial t}, \quad \eta(\tilde{X}) = g(\tilde{X}, \xi)\end{aligned}$$

for any $\tilde{X} \in \Gamma(T\tilde{N})$, $\tilde{X} \in \Gamma(T\tilde{M})$.

Conversely, for any point x in a Kenmotsu manifold $(\tilde{M}, \tilde{g}, \phi, \xi, \eta)$, some neighborhood U_x of $x \in \tilde{M}$ is identified with a warped product manifold $(-\varepsilon, \varepsilon) \times_f \bar{N}$ such that $(-\varepsilon, \varepsilon)$ is an open interval, $f(t) = \lambda \exp(t)$ and $\bar{N}$ is a Kähler manifold.

The notion of Kenmotsu statistical manifolds is the statistical counterpart of a Kenmotsu manifold and defined as follow:

Definition 5.3.3. ([23]) Let (M, g, ϕ, ξ, η) be a Kenmotsu manifold and $(\nabla = \nabla^g + K, g)$ a statistical structure on M . If $(\nabla, g, \phi, \xi, \eta)$ satisfies the following condition

$$K(X, \phi Y) = -\phi K(X, Y) \quad (5.4)$$

for any $X, Y \in \Gamma(TM)$, then $(M, \nabla, g, \phi, \xi, \eta)$ is said to be a *Kenmotsu statistical manifold* and $(\nabla, g, \phi, \xi, \eta)$ is called a *Kenmotsu statistical structure* on M .

We remark that $(\nabla, g, \phi, \xi, \eta)$ a Kenmotsu statistical structure on M , so is $(\nabla^*, g, \phi, \xi, \eta)$.

Theorem 5.3.4. ([23]) Let $(M, \nabla = \nabla^g + K, g)$ be a statistical manifold, and (g, ϕ, ξ, η) an almost contact metric structure on M . $(\nabla, g, \phi, \xi, \eta)$ is a Kenmotsu statistical structure on M if and only if

$$\nabla_X(\phi Y) - \phi \nabla_X^* Y = -\eta(Y)\phi X + g(\phi X, Y), \quad (5.5)$$

$$\nabla_X \xi = X - \{\eta(X) - \mu(X)\}\xi \quad (5.6)$$

for any $X, Y \in \Gamma(TM)$, where $\mu(X) = -\eta(\nabla_X^* \xi) = \eta(\nabla_X \xi) = \eta(K(\xi, \xi))\eta(X)$.

The following proposition indicates that a Kenmotsu statistical manifold is constructed as the warped product of a line and a holomorphic statistical manifold, where a holomorphic statistical manifold is defined as follow:

Let (M, g, J) be a Kähler manifold. A triplet $(\nabla = \nabla^g + K, g, J)$ is called a *holomorphic statistical structure* on M if (∇, g) is a statistical structure on M , and K and J satisfy

$$K(X, JY) = -JK(X, Y)$$

for any $X, Y \in \Gamma(TM)$. Then (M, ∇, g, J) is called a *holomorphic statistical manifold*.

Proposition 5.3.5. ([23]) Let $(\bar{N}, \bar{\nabla} = \nabla^{\bar{g}} + \bar{K}, \bar{g}, J)$ be a holomorphic statistical manifold, and $(\tilde{M} = \mathbb{R} \times \bar{N}, \tilde{g}, \phi, \xi)$ the Kenmotsu manifold as in Proposition 5.3.2. For any $v \in C^\infty(\tilde{M})$, define $\tilde{K} \in \Gamma(T\tilde{M}^{(1,2)})$ by

$$\tilde{K}(\xi, \xi) = v\xi, \quad (5.7)$$

$$\tilde{K}(\xi, V) = \tilde{K}(U, \xi),$$

$$\tilde{K}(U, V) = \bar{K}(U, V)$$

for any $U, V \in \Gamma(T\bar{N})$. Then $(\tilde{\nabla} = \nabla^{\tilde{g}} + \tilde{K}, \tilde{g}, \phi, \xi, \eta)$ is a Kenmotsu statistical structure on $\tilde{M}$.

It is a remark that the warped product statistical manifold above is more general than the standard warped product statistical manifold as in Definition 5.1.3 since ν is a smooth function on $\tilde{M}$ as (5.7).

Let $(\tilde{M}, \tilde{\nabla}, \tilde{g}, \phi, \xi, \eta)$ be a Kenmotsu statistical manifold constructed as the warped product of a line and a holomorphic statistical manifold $(\bar{N}, \bar{\nabla} = \nabla^{\bar{g}} + \bar{K}, \bar{g})$ as in Proposition 5.3.5. Then the following relations hold for any $\bar{X}, \bar{Y}, \bar{Z} \in \Gamma(T\bar{N})$ (See [23]):

$$\begin{aligned}\tilde{S}(\bar{X}, \bar{Y})\bar{Z} &= \bar{S}(\bar{X}, \bar{Y})\bar{Z} - \tilde{g}(\bar{Y}, \bar{Z})\bar{X} + \tilde{g}(\bar{X}, \bar{Z})\bar{Y}, \\ \tilde{S}(\bar{X}, \bar{Y})\xi &= 0, \\ \tilde{S}(\bar{X}, \xi)\xi &= -\bar{X},\end{aligned}$$

where $\tilde{S}$ denote the statistical curvature tensor field for $(\tilde{\nabla}, \tilde{g})$ on $\tilde{M}$, and $\bar{S}$ the one for $(\bar{\nabla}, \bar{g})$ on $\bar{N}$. Furthermore, if a Kenmotsu statistical manifold is of *constant ϕ statistical sectional curvature c* , that is,

$$\begin{aligned}\tilde{S}(\bar{X}, \bar{Y})\bar{Z} &= \frac{c-3}{4} \{ \tilde{g}(\bar{Y}, \bar{Z})\bar{X} - \tilde{g}(\bar{X}, \bar{Z})\bar{Y} \} \\ &+ \frac{c+1}{4} \{ \tilde{g}(\phi\bar{Y}, \bar{Z})\phi\bar{X} - \tilde{g}(\phi\bar{X}, \bar{Z})\phi\bar{Y} - 2\tilde{g}(\phi\bar{X}, \bar{Y})\phi\bar{Z} \\ &\quad - \tilde{g}(\bar{Y}, \xi)\tilde{g}(\bar{Z}, \xi)\bar{X} + \tilde{g}(\bar{X}, \xi)\tilde{g}(\bar{Z}, \xi)\bar{Y} \\ &\quad + \tilde{g}(\bar{Y}, \xi)\tilde{g}(\bar{Z}, \bar{X})\xi + \tilde{g}(\bar{X}, \xi)\tilde{g}(\bar{Z}, \bar{Y})\xi \}\end{aligned}$$

for any $\bar{X}, \bar{Y}, \bar{Z} \in \Gamma(T\bar{M})$, then $c = -1$, and $(\bar{N}, \bar{\nabla}, \bar{g}, J)$ is of constant holomorphic sectional curvature 0.

In the submanifold theory, it is known that any real hypersurface in a non-flat complex space form is not Einstein (See [29]). Furthermore, a real hypersurface in a non-flat complex space form does not admit a Ricci soliton whose potential vector field is the characteristic vector field (See [19]). Accordingly, the notion of pseudo-Einstein and that of pseudo-Ricci solitons are necessary on almost contact metric manifolds.

Let us recall that an almost contact metric g satisfies

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for any $X, Y \in \Gamma(TM)$. S. Tanno [46] introduced the notion of *pseudo-Einstein* on almost contact metric manifolds (M, g, ϕ, ξ, η) . If the following equation

$$\text{Ric}^g = \lambda g + \omega \eta \otimes \eta$$

holds, where Ric^g is the Ricci tensor associated to g , and λ and ω are constants, then (M, g, ϕ, ξ, η) is called a *pseudo-Einstein manifold*. If $\omega = 0$ in the equation above, it is an Einstein manifold.

Moreover, as a generalization of Ricci solitons, the notion of *pseudo-Ricci solitons* was introduced by J. T. Cho and M. Kimura [19]. A smooth vector field V is said to define a *pseudo-Ricci soliton* on almost contact metric manifold (M, g, ϕ, ξ, η) if it satisfies the following equation

$$\text{Ric}^g + \frac{1}{2}\mathcal{L}_V g = \lambda g + \omega\eta \otimes \eta,$$

where $\mathcal{L}_V g$ is the Lie-derivative of the Riemannian metric g with respect to V , Ric^g is the Ricci tensor associated to g , and λ and ω are constants. We call the vector field V the *potential vector field of the pseudo-Ricci soliton*.

For an almost contact metric manifold with a statistical structure, we introduce the *statistical pseudo-Einstein manifold* as follow:

Definition 5.3.6. Let $(M, \nabla, g, \phi, \xi, \eta)$ be an almost contact metric manifold with a statistical structure. If the statistical Ricci tensor satisfies

$$L = \lambda g + \omega\eta \otimes \eta,$$

where λ and ω are constants, then $(M, \nabla, g, \phi, \xi, \eta)$ is called a *statistical pseudo-Einstein manifold*.

Remark 5.3.7. (1) We notice that if $\omega = 0$ in the equation above, $(M, \nabla, g, \phi, \xi, \eta)$ is an statistical Einstein manifold.

(2) If a statistical structure (∇, g) is trivial, that is, $\nabla = \nabla^g$, then a statistical pseudo-Einstein manifold is a pseudo-Einstein manifold.

Correspondingly, we introduce *statistical pseudo-Ricci solitons* as a natural generalization of statistical pseudo-Einstein manifolds.

Definition 5.3.8. Let $(M, \nabla, g, \phi, \xi, \eta)$ be an almost contact metric manifold with a statistical structure. A vector field V is said to define a *statistical pseudo-Ricci soliton* on $(M, \nabla, g, \phi, \xi, \eta)$ if it satisfies the following equation:

$$L + \frac{1}{2}\mathcal{L}_V g = \lambda g + \omega\eta \otimes \eta,$$

where λ and ω are constants. We call V the potential vector field of the statistical pseudo-Ricci soliton.

Remark 5.3.9. We remark that $\omega = 0$, then V defines a statistical Ricci soliton. A statistical pseudo-Einstein manifold is considered as a statistical pseudo-Ricci soliton with zero or Killing vector field V .

Theorem 5.3.10. *A Kenmotsu statistical manifold $(M, \nabla, g, \phi, \xi, \eta)$ is statistical pseudo-Einstein if ξ defines a statistical pseudo-Ricci soliton.*

Proof. For any $X \in \Gamma(TM)$, we get

$$\begin{aligned}\nabla_X \xi &= X - \{\eta(X) - \mu(X)\}\xi, \\ \nabla_X^* \xi &= X - \{\eta(X) + \mu(X)\}\xi.\end{aligned}$$

Thus, we get

$$\begin{aligned}(\mathcal{L}_\xi g)(X, Y) &= g(\nabla_X \xi, Y) + g(X, \nabla_Y^* \xi) \\ &= g(X - \{\eta(X) - \mu(X)\}\xi, Y) + g(X, Y - \{\eta(Y) + \mu(Y)\}\xi) \\ &= 2g(X, Y) - 2\eta(X)\eta(Y)\end{aligned}$$

for any $X, Y \in \Gamma(TM)$. Therefore, if $(M, \nabla, g, \phi, \xi, \eta)$ is a statistical pseudo-Ricci soliton, then we have

$$L(X, Y) = -(\lambda - 1)g(X, Y) + (\omega + 1)\eta(X)\eta(Y), \quad (5.8)$$

which implies that $(M, \nabla, g, \phi, \xi, \eta)$ is a statistical pseudo-Einstein manifold. $\square$

Example 5.3.11. We set

$$M = \{(t, x^1, \dots, x^m, x^{m+1}, \dots, x^{2m}) \in \mathbb{R}^{2m+1}\},$$

where $(t, x^1, \dots, x^m, x^{m+1}, \dots, x^{2m})$ are the standard coordinates in $\mathbb{R}^{2m+1}$. Then we take the vector fields $\{e_0 = \xi, e_1, \dots, e_m, e_{m+1}, \dots, e_{2m}\}$ as

$$e_0 = \xi = \frac{\partial}{\partial t}, \quad e_i = \exp(-t) \frac{\partial}{\partial x^i} \quad (i = 1, \dots, 2m).$$

The Riemannian metric g is given by

$$g(e_i, e_j) = \delta_{ij} \quad (i, j = 0, 1, \dots, 2m),$$

that is, $\{e_0 = \xi, e_1, \dots, e_m, e_{m+1}, \dots, e_{2m}\}$ is an orthonormal frame on M with respect to g . Then, let η be the 1-form defined by

$$\eta(X) = g(X, \xi)$$

for any $X \in \Gamma(TM)$.

Let ϕ be the $(1, 1)$ -tensor field on M defined by

$$\phi(e_0) = \phi(\xi) = 0, \quad \phi(e_i) = e_{i+m}, \quad \phi(e_{m+i}) = -e_i$$

for $i = 1, \dots, m$.

For any $X, Y \in \Gamma(TM)$ and constant $\nu \in C^\infty(M)$, we set the difference tensor field K by

$$K(X, Y) = \nu\eta(X)\eta(Y)\xi.$$

Then, $(M, \nabla = \nabla^g + K, g, \phi, \xi, \eta)$ is a $(2m + 1)$ -dimensional Kenmotsu statistical manifold. Moreover, it is a statistical Einstein manifold. Indeed, it is of constant ϕ -sectional statistical sectional curvature -1 and

$$L(X, Y) = -2mg(X, Y)$$

holds for all $X, Y \in \Gamma(TM)$. Obviously, it admits an expanding statistical Ricci soliton.

We generalize Definitions 5.3.6 and 5.3.8 for a statistical submanifold in a Kenmotsu statistical manifold, even if it does not admit an almost metric structure.

Definition 5.3.12. (1) Let (M, ∇, g) be a statistical submanifold in a Kenmotsu statistical manifold $(\tilde{M}, \tilde{\nabla}, \tilde{g}, \phi, \xi, \eta)$. M is called a *statistical pseudo-Einstein submanifold* if the statistical Ricci tensor L is of the form

$$L = \lambda g + \omega\eta \otimes \eta,$$

where λ and ω are constants.

(2) Let (M, ∇, g) be a statistical submanifold of a Kenmotsu statistical manifold $(\tilde{M}, \tilde{\nabla}, \tilde{g}, \phi, \xi, \eta)$. A vector field V on M is said to define a *statistical pseudo-Ricci soliton on M in $\tilde{M}$* if it satisfies

$$L + \frac{1}{2}\mathcal{L}_V g = \lambda g + \omega\eta \otimes \eta,$$

for constants λ and ω .

Theorem 5.3.13. *Let (M, ∇, g) be a statistical submanifold of a Kenmotsu statistical manifold $(\tilde{M}, \tilde{\nabla}, \tilde{g}, \phi, \xi, \eta)$. Suppose that ξ is tangent to M and defines a statistical pseudo-Ricci soliton on M in $\tilde{M}$. Then M is a statistical pseudo-Einstein submanifold.*

Proof. For any $X \in \Gamma(TM)$,

$$\widetilde{\nabla}_X \xi = \nabla_X \xi + h(X, \xi)$$

holds. On the other hand, we have

$$\widetilde{\nabla}_X \xi = X - \{\eta(X) - \mu(X)\}\xi$$

for any $X, Y \in \Gamma(TM)$. Therefore we get

$$\nabla_X \xi = X - \{\eta(X) - \mu(X)\}\xi.$$

In the same way, we obtain

$$\nabla_X^* \xi = X - \{\eta(X) + \mu(X)\}\xi.$$

Thus, we get

$$\begin{aligned} (\mathcal{L}_\xi g)(X, Y) &= g(\nabla_X \xi, Y) + g(X, \nabla_Y^* \xi) \\ &= g(X - \{\eta(X) - \mu(X)\}\xi, Y) + g(X, Y - \{\eta(Y) + \mu(Y)\}\xi) \\ &= 2g(X, Y) - 2\eta(X)\eta(Y) \end{aligned}$$

for any $X, Y \in \Gamma(TM)$. Therefore, if (M, ∇, g, ξ) is a statistical pseudo-Ricci soliton, then we have

$$L(X, Y) = -(\lambda - 1)g(X, Y) + (\omega + 1)\eta(X)\eta(Y), \quad (5.9)$$

which implies that (M, ∇, g) is a statistical pseudo-Einstein submanifold. $\square$

Corollary 5.3.14. *Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, \phi, \xi, \eta)$ be a Kenmotsu statistical manifold such that ξ is the potential vector field of statistical pseudo-Ricci soliton, and (M, ∇, g) be a statistical submanifold admitting a statistical pseudo-Ricci soliton of ξ with $\omega = -1$. Then (M, ∇, g) is a statistical Einstein manifold.*

Proof. From (5.9), it is easy to see that

$$L(X, Y) = -(\lambda - 1)g(X, Y),$$

for any $X, Y \in \Gamma(TM)$. Thus, (M, ∇, g) is a statistical Einstein manifold. $\square$

Corollary 5.3.15. *Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, \phi, \xi)$ be a Kenmotsu statistical manifold such that ξ is the potential vector field of statistical Ricci soliton, and (M, ∇, g) be a statistical submanifold admitting a statistical Ricci soliton of ξ . Then (M, ∇, g) is a statistical pseudo-Einstein manifold.*

Proof. From (5.9), it is easy to see that

$$L(X, Y) = -(\lambda - 1)g(X, Y) + \eta(X)\eta(Y),$$

for any $X, Y \in \Gamma(TM)$. Thus, (M, ∇, g) is a statistical pseudo-Einstein manifold. $\square$

Remark 5.3.16. For constants $(k_1, \dots, k_{2m-n+1}) \in \mathbb{R}^{2m-n+1}$ ($n < 2m + 1$), we consider

$$M = \{(x^1, \dots, x^n, k_1, \dots, k_{2m-n+1}) \in \widetilde{M} \mid (x^1, \dots, x^n) \in \mathbb{R}^n\},$$

where $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, \phi, \xi, \eta)$ be a $(2m + 1)$ -dimensional Kenmotsu statistical manifold as in Example 5.3.11. In the setting above, if $\xi = \frac{\partial}{\partial x^1}$ satisfies

$$L(X, Y) + (\mathcal{L}_\xi g)(X, Y) = \lambda g(X, Y) + \omega \eta(X)\eta(Y)$$

for any $X, Y \in \Gamma(TM)$, then M is a statistical pseudo-Einstein manifold.

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