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Long time solution of some types of nonlinear wave equations

(種々の非線形波動方程式の長時間解について)

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Chapter 1

Introduction

1.1 Background and significance

The wave equation is a typical hyperbolic partial differential equation. We know that there are various wave phenomena in physics, such as spring, light-wave, water-wave, and so on. The wave equation often appears in the classical mathematical model to describe these wave phenomena, involving a wide range of fields, not only electromagnetics, fluid mechanics but also quantum mechanics and relativity. Therefore, the study of wave equation has important guiding significance and theoretical value in understanding reality and academic research. The basic characteristics of the wave equation are finite propagation velocity and zero curvature in the direction of generatrix. To a certain extent, it also has a dispersion property similar to the Schrödinger equation and other evolution equations.

The standard form of semilinear wave equation can be written as

$$\begin{cases} \Box u := (\partial_t^2 - \Delta)u = F(t, x, u, \partial u), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0, x) = f(x), \qquad u_t(0, x) = g(x). \end{cases} \tag{1.1}$$

Here, people usually focus on the local and global well-posedness, regularity, long-time behavior, and blow-up structure.

Considering (1.1), a typical form of this equation is that F only contain the powers of u, which means

$$\begin{cases} \Box u = F_p(u), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0, x) = f(x), \qquad u_t(0, x) = g(x), \end{cases} \tag{1.2}$$

where $F_p(u) = |u|^p$, $p > 1$, or some other disturbed form. We aim to investigate the relationship between the nonlinear power p, the smallness of the initial data (ε order), and

the longest existence time of the solution T_ε. This kind of problem is generally called the *Strauss* problem, or related problem of *Strauss* conjecture. For a more specific historical review and description, please refer to Wang-Yu [31].

The critical exponent of the standard *Strauss* problem is $p_S(n) = \frac{n+1+\sqrt{n^2+10n-7}}{2n-2}$, which is the positive root of equation

$$h_S(p;n) = 0, \qquad h_S(p;n) := (n-1)p^2 - (n+1)p - 2. \tag{1.3}$$

Meanwhile, as we said before, denote ε be the order of smallness of initial data, and define T_ε be the longest lifespan of the solution to (1.2). Then, roughly specking, when $p > \max(1, p_d(n))$, where $p_d(n) := \frac{2}{n-1}$, we have

$$\begin{cases} T_\varepsilon = \infty, & p > p_S; \\ \ln T_\varepsilon \approx \varepsilon^{-p(p-1)}, & p = p_S; \\ T_\varepsilon \approx \varepsilon^{\frac{2p(p-1)}{h_S(p;n)}}, & p < p_S. \end{cases}$$

In this article, we will discuss several classes of equation or system which related to the standard *Strauss* equation (1.2), and study the behavior of their long-time development. More specifically, we will firstly study a class of *Strauss* type systems in asymptotically flat space time (Section 1.2), and compare the result with that in flat space-time. Then, we study a kind of *Strauss* type equation with damping and potential (Section 1.3), and research the influence of damping term and potential term on critical index and existence time. After that, we study the semilinear equation with mixed force terms, $|u|^p$ and $|\partial_t u|^q$ (Section 1.4), to explore how such nonlinear form affects the evolution of the solution. Finally, we study a coupling system with above two type force terms (Section 1.5), and observe the relationship between this system and other systems which have only one type of force term.

1.2 *Strauss* type system on asymptotically flat space-time

With the development of wave equations in flat space-time, more and more attention has been paid to the study of wave equations on non flat manifolds. As we all know, there are many difficulties in the study of wave equations on manifolds. For example, in this situation, the fundamental solution is hard to obtain, the Fourier analysis is not easy to carry out, and the characteristic surface is not simple conical. These factors make that many obvious conclusions in flat space-time may not be valid for the general case.

However, for some special asymptotically flat manifolds, such as locally disturbed space-time, the region outside the Kerr black hole, and so on, a lot of progress has been made.

In this paper, we will also study the related *Strauss* problems on this kind of manifolds. Specifically, let $(M, \mathbf{g})$ be a $(3+1)$-dimensional asymptotically flat space-time manifold and $\Box_{\mathbf{g}} = \nabla^\mu \nabla_\mu$ be the associated *d'Alembertian* operator, where $M = \mathbb{R}^+ \times \mathbb{R}^3$ or $\mathbb{R}^+ \times (\mathbb{R}^3 \backslash \mathcal{K})$ with bounded smooth $\mathcal{K} \subset B(0, R_0)$ for some $R_0 > 0$. We will study the system

$$\Box_{\mathbf{g}} u_1 = F_{p_1}(u_2),\ \Box_{\mathbf{g}} u_2 = F_{p_2}(u_1),$$

where the power type nonlinearities $F_p \in C^2$ are assumed to satisfy

$$\sum_{0 \le j \le 2} |u|^j |\partial_u^j F_p(u)| \lesssim |u|^p \text{ for } |u| \ll 1. \tag{1.4}$$

Typical examples include $F_p(u) = \pm |u|^p$ and $F_p(u) = \pm |u|^{p-1} u$ $(p \ge 2)$.

For simplicity of presentation, we set $\iota \in \{1, 2\}$, $\hat{\iota} = 3 - \iota$, and rewrite the system as follows

$$\begin{cases} \Box_{\mathbf{g}} u_\iota = F_{p_\iota}(u_{\hat{\iota}}),\ (t, x) \in M, \\ (u_\iota, \partial_t u_\iota)|_{t=0} = (f_\iota, g_\iota), \end{cases} \qquad \iota = 1, 2, \tag{1.5}$$

with sufficiently nice and small initial data f_ι, g_ι.

This problem has been well discussed when $(M, \mathbf{g})$ is *Minkowski* space-time, see Del Santo-Georgiev-Mitidieri [8], Agemi-Kurokawa-Takamura [1]. Take $\sigma = \sigma_{SS}(p_1, p_2)$, where

$$\sigma_{SS}(p, q) := \max\left\{ \frac{p + 2 + q^{-1}}{pq - 1}, \frac{q + 2 + p^{-1}}{pq - 1} \right\} - 1. \tag{1.6}$$

It is well known that in this situation, roughly speaking,

$$\begin{cases} T_\varepsilon \approx \varepsilon^{-\frac{1}{\sigma}} & \sigma > 0, \\ \ln T_\varepsilon \approx \varepsilon^{-\min_\iota p_\iota (p_1 p_2 - 1)} & \sigma = 0, p_1 \ne p_2, \\ \ln T_\varepsilon \approx \varepsilon^{-p_\iota (p_\iota - 1)} & \sigma = 0, p_1 = p_2, \\ T_\varepsilon = \infty & \sigma < 0. \end{cases}$$

There are also some related discussions of the *Strauss* problem on asymptotically flat manifolds. First of all, it is obvious that weighted *Strichartz* estimate plays an important role in this kind of problems. Also, the weighted *Strichartz* estimate is closely related to the local energy estimate. Among many versions of local energy estimates, a sharp version

of uniform energy and (micro-localized) local energy estimates was established for wave operators with small, asymptotically flat metric perturbations, in Metcalfe-Tataru [27].

Based on this research, using trace theorem and some other technical tools, Metcalfe-Wang [23] obtain the global existence for the *Strauss* problem in the supercritical case ($p > 1 + \sqrt{2}$) for a large class of asymptotically flat space-time. Meanwhile, Wang [30] further combines the local energy estimate with the *KSS* estimate and the weighted trace theorem, and obtain the certain lower bound estimates of the lifespan in the subcritical and critical cases.

After that, Metcalfe-Spencer [26] applied similar discussion into system situation, and proved the global existence of solution for $\sigma < 0$ and $p_\iota > 2$. It is to be remarked that the case $p_1 = 2$ and $p_2 > 3.5$ was excluded in this paper, for which it is known to be admissible for global results on *Minkowski* space-time.

Based on previous work, using trace theorem, KSS estimate, interpolation and other technical tools, we reach a large class of weighted *Strichartz* estimate from local energy estimate. This estimate covers a wide range of indexes and is expected to be applied to many related problems. In particular, applying this estimate to (1.5), under Hypothesis (H1) and (H2), we prove that there exists a constant $c = c(p_1, p_2)$ such that (assuming $p_2 \geq p_1$)

$$T_\varepsilon \geq \begin{cases} c\varepsilon^{-\frac{1}{\sigma}} & \text{if } \sigma > 0, \\ \exp(c\varepsilon^{-2(p_1-1)}) & \text{if } \sigma = 0, p_1 > 2, \\ \exp(c\varepsilon^{-2+\delta}) & \text{if } \sigma = 0, p_1 = 2, \\ \infty & \text{if } \sigma < 0. \end{cases}$$

This conclusion covers all the known existence results in general asymptotically flat space-time. Meanwhile, we complete the result of Metcalfe-Spencer [26] with $p_\iota = 2$. Specially, reducing to the single equation situation, our result make some improvements on the requirement and result of Wang's [30]. At last, comparing with the results on flat space-time, it can be expected that our results are optimal in all non critical cases.

1.3 *Strauss* type equation with damping and potential

Next, we consider a class of semilinear wave equations with damping and potential.

$$\begin{cases} Pu := \partial_t^2 u + Dr^{-1}\partial_t u - \Delta u + Vr^{-2}u = |u|^p, \quad t > 0, r := |x|, \\ u(0,x) = \varepsilon f(x), \quad \partial_t u(0,x) = \varepsilon g(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{1.7}$$

Here $n \geq 2$, $D \geq 0$, $V \in \mathbb{R}$, $p > 1$ and $1 \gg \varepsilon > 0$. In recent years, the kind of *Strauss* equation with damping and potential has also caused a lot of discussions. However, there is not much progress on this kind of special wave equation (1.7) with scaling balance damping and potential. The main difficulty of studying this kind of equation is not only from the singularity of the equation itself at the origin, but also from that the influence of damping and potential are same order with the wave operator.

For (1.7) only with the damping term, which means $D > 0$, $V = 0$, one related work comes from Ikeda-Sobajima [16]. Roughly speaking, the authors proved the solution must blow-up when $p \leq p_S(n + D)$.

Another enlightening result came from Ikehata-Todorova-Yordanov [17]. When $V = 0$ and the coefficient of damping term is given by $(1 + |x|^2)^{-\alpha/2}$ $(0 \leq \alpha < 1)$, the critical exponent becomes $p_F(n - \alpha)$. $p_F(n) = 1 + 2/n$ stands for the *Fujita* exponent, which satisfies

$$h_F(p; n) = 0, \qquad h_F(p; n) := np - (n + 2).$$

The other characteristic of equation (1.7) is inverse square potential Vr^{-2}. Such a potential draws much attention from the viewpoint of mathematical physics. We know that when $V \geq -(n-2)^2/4$, the minimal operator $S_{\text{min}} = -\Delta + Vr^{-2}$ with domain $D(S_{\text{min}}) = C_0^\infty(\mathbb{R}^n \setminus \{0\})$ admits the *Friedrichs* extension. On the other hand, Burq-Planchon-Stalker [3] [4] made some discussion on the linear wave equation with potential. However, for the *Strauss* type equation with inverse square potential, there is no general result so far.

In this article, we give the blow-up result for the general case of (1.7). In order to reach the blow up results, we introduce the test function method. By using the dual property of wave equation and the asymptotic behavior of test function, we transform the original equation into an inequality with only time variables, and then show the blow up of the solution. When constructing test functions, we introduce a special kind of functions, hypergeometric functions. From this, we find a family of solutions of dual homogeneous equations with different asymptotic behaviors. In addition, compared with the previous test function method, we also introduce an additional time truncation. Such technique makes it easy to get the optimal results both in subcritical and critical cases. By choosing the suitable test function, we finally give an upper bound of the lifespan to the solution of (1.7). The result is as follows, considering

$$V \geq -(n-2)^2/4, \quad A := 2 + \sqrt{(n-2)^2 + 4V}, \quad 0 \leq D < A - 1. \tag{1.8}$$

Then, there exists a constant $C = C(p; n, A, D)$. When $(D + 3 - A)(A + n + 2) < 8$,

where $p_d(A-D) < p_F((n+A-2)/2) < p_S(n+D)$ with p_d, p_F, p_S defined above. We have

$$T_\varepsilon \le \begin{cases} C\varepsilon^{\frac{(p-1)}{h_F(p;(n+A-2)/2)}}, & p \le p_d(A-D); \\ C\varepsilon^{\frac{2p(p-1)}{h_S(p;n+D)}}, & p_d(A-D) < p < p_S(n+D); \\ \exp\left(C\varepsilon^{-p(p-1)}\right), & p = p_S(n+D). \end{cases}$$

When $(D+3-A)(A+n+2) = 8$, where $p_d(A-D) = p_F((n+A-2)/2) = p_S(n+D)$, we have

$$T_\varepsilon \le \begin{cases} C\varepsilon^{\frac{(p-1)}{h_F(p;(n+A-2)/2)}}, & p < p_F((n+A-2)/2)\,; \\ \exp\left(C\varepsilon^{-(p-1)}\right), & p = p_F((n+A-2)/2)\,. \end{cases}$$

When $(D+3-A)(A+n+2) > 8$, where $p_d(A-D) > p_F((n+A-2)/2) > p_S(n+D)$, we have

$$T_\varepsilon \le \begin{cases} C\varepsilon^{\frac{(p-1)}{h_F(p;(n+A-2)/2)}}, & p < p_F((n+A-2)/2)\,; \\ \exp\left(C\varepsilon^{-(p-1)}\right), & p = p_F((n+A-2)/2)\,. \end{cases}$$

It can be seen from these result that the blow-up phenomenon of (1.7) in relation both with the *Strauss* exponent and the *Fujita* exponent. This phenomenon is quite novel. It shows that there are two competing effects in this problem. For convenience, we call them *Strauss* effect and *Fujita* effect respectively.

We also discuss the long time existence of solution to (1.7) with $D = 0$ under spherical symmetric initial data. In this case, the discussion on the original equation is equivalent to the discussion on a $1+1$ dimensional equation (Eq. (3.21)), which can be regarded as the discussion on the spherical symmetric wave equation with 'fractional dimension'. However, in this framework, most results of wave equation will not be directly applicable for this problem. Therefore, we reestablish the solution space, fundamental solutions, related estimates and so on. Finally, we give the lower bound of the lifespan. That is, there exists a constant $c = c(p;n,A)$, when $(3-A)(A+n+2) < 8$, where $p_d(A) < p_F((n+A-2)/2) < p_S(n)$, we have

$$T_\varepsilon \ge \begin{cases} c\varepsilon^{\frac{(p-1)}{h_F(p;(n+A-2)/2)}}, & p < p_d(A); \\ c\varepsilon^{\frac{(p-1)}{h_F(p;(n+A-2)/2)}} \left|\ln\varepsilon\right|^{\frac{1}{h_F(p;(n+A-2)/2)}}, & p = p_d(A); \\ c\varepsilon^{\frac{2p(p-1)}{h_S(p;n)}}, & p_d(A) < p < p_S(n); \\ \exp\left(c\varepsilon^{p(1-p)}\right), & p = p_S(n); \\ \infty, & p > p_S(n). \end{cases}$$

when $(3-A)(A+n+2)=8$, where $p_d(A)=p_F((n+A-2)/2)=p_S(n)$, we have

$$T_\varepsilon \geq \begin{cases} c\varepsilon^{\frac{(p-1)}{h_F(p;(n+A-2)/2)}}, & p<p_d(A); \\ \exp\left(c\varepsilon^{\frac{1-p}{2}}\right), & p=p_d(A); \\ \infty, & p>p_d(A). \end{cases}$$

when $(3-A)(A+n+2)>8$, where $p_d(A)>p_F((n+A-2)/2)>p_S(n)$, we have

$$T_\varepsilon \geq \begin{cases} c\varepsilon^{\frac{(p-1)}{h_F(p;(n+A-2)/2)}},, & p<p_F((n+A-2)/2); \\ \exp(c\varepsilon^{1-p}), & p=p_F((n+A-2)/2); \\ \infty, & p>p_F((n+A-2)/2). \end{cases}$$

When $D=0$, comparing the existence result with the blow-up result, except for the critical case $p=P_d(A)$, the upper bound and the lower bound of the existence time are matched. This means they are both optimal. Thus, we also believe that the upper bound of the lifespan we reached is optimal when $p\neq p_d(A-D)$ in $D>0$.

It is well known that, *Fujita* exponent was first found in the study of heat equation. Comparing the results from Ikehata-Todorova-Yordanov's [17] and that in this article, such exponent seems to come from the influence of damping. However, when $D=0$, which means there is no damping, our result shows *Fujita* exponent still effect the long time behavior of solution. This may imply a deeper connection between the heat equation and the wave equation we concerned.

1.4 Equation with mixed power terms

Now we consider the semilinear wave equation with mixed power terms of u and $\partial_t u$. That is

$$\begin{cases} \Box u = |u|^p + |\partial_t u|^q, \\ u(0,x)=\varepsilon f(x), \qquad u_t(0,x)=\varepsilon g(x), \end{cases} \tag{1.9}$$

where $p>1$, $q>1$, and $x\in\mathbb{R}^n$. This equation can be regarded as an additional force disturbance to *Strauss* type equation. Noticing that the asymptotic behavior of equation with u and $\partial_t u$ forcing term are different, we expect to study such equation to understand how two different kinds of force will affect the development of the equation.

In space dimension $n=2,3$, equation (1.9) has already been studied by Hidano-Wang-Yokoyama [14] and Han-Zhou [12]. Generally speaking, the two kinds of force compete with each other, and the blow-up result is determined by the stronger one. When the two

effects are similar, the two effects will promote each other and obtain a new type blow-up result. As a conclusion, when $p, q \geq 2$ and $p > p_d(n)$, the optimal result of lifespan was obtained by the former paper, except a critical case $q \geq p = p_S$. Where, for the critical case, now we only know

$$\exp\big(c\varepsilon^{-(p_S-1)}\big) \leq T_\varepsilon \leq \exp\big(C\varepsilon^{-p_S(p_S-1)}\big)\,,\ n = 2, 3.$$

Here the upper bound came from the discussion similar to that of *Strauss* problem, which is considered to be the optimal upper bound of this problem. Also, the lower bound came from Hidano-Wang-Yokoyama [14]. In this paper, the author used an improved *Klainerman* vector field method to deal with two kinds of force terms simultaneously in the energy framework, and finally obtained the bound.

Proof in the energy framework is a intuitive method. However, for the *Strauss* critical problem, experience tells us that the precision of the lower bound depends on the form of the main estimate. To be more specific, most obtained lower bound of lifespan of the critical *Strauss* problem closely related to the exponent q in L_t^q, the norm of time for the force term in the key estimates. For example, the obtained bound $\exp\big(c\varepsilon^{-p_S(p_S-1)}\big)$ which comes from Lindblad-Sogge [24] coincides with $q = p_S$ in estimate (4.3), and the obtained bound $\exp\Big(c\varepsilon^{-(p_S-1)^2/2}\Big)$ which comes from Jiang-Wang-Yu [18] coincides with $q = (p_S - 1)/2$ in estimate (4.4).

In order to get a more precise lower bound, we use different estimate methods for different force terms. We use the estimate which gives the optimal bound of *Strauss* problem to deal with the power term of u, use the energy method and *Klainerman* field method to deal with the power term of $\partial_t u$, and use trace lemma, *Klainerman-Sobolev* inequality and other technical tools to connect them. Finally, when $n = 3$, we reach the optimal bound which is the same as the result in Lindblad-Sogge [24]. As for $n = 2$, the problem becomes more complicated as the classical wave equation itself.

As far as the authors are aware, even for the problem (1.2) with $p = p_S(2)$, the only available approach to prove the sharp lower bound is given in Zhou [32], which relies heavily on the fundamental solutions and seems inappropriate for the problems involving nonlinear term $|\partial_t u|^p$. Instead, the approach using space-time estimates is more robust for various nonlinear problems. Here, we adapt these estimates and reach the result which is the same as that in Jiang-Wang-Yu [18].

1.5 *Strauss-Glassey* coupled system

Finally, we discuss the coupled wave system,

$$\begin{cases} \Box u = |v|^q, \qquad \Box v = |\partial_t u|^p, \\ (u, \partial_t u)|_{t=0} = \varepsilon\big(f(x), g(x)\big), \ (v, \partial_t v)|_{t=0} = \varepsilon\big(\tilde{f}(x), \tilde{g}(x)\big), \end{cases} \tag{1.10}$$

where $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$, and ε positive and small enough. Compared with the 'competition' of the blow up effect caused by the combined power nonlinear term in the single equation in previous section, we expect that two kinds of forces term in the coupled system will lead to the 'equilibrium' of the blow up effect. Therefore, study of this type of problem can help us understand the influence of $\partial_t u$ power on the solution of the system.

To analyze this problem, we compare it with the following two related equations.

$$\Box u = |v|^q, \qquad \Box v = |u|^p, \qquad x \in \mathbb{R}^3; \tag{1.11}$$

$$\Box u = |\partial_t v|^q, \qquad \Box v = |\partial_t u|^p, \qquad x \in \mathbb{R}^3. \tag{1.12}$$

The critical curve of (1.11) is $\sigma_{SS}(p, q) = 0$, where σ_{SS} defined in (1.6). As for (1.12), its critical curve is

$$\sigma_{GG}(p, q) = 0, \qquad \sigma_{GG}(p, q) := \max\left\{ \frac{q+1}{pq-1}, \frac{p+1}{pq-1} \right\} - 1. \tag{1.13}$$

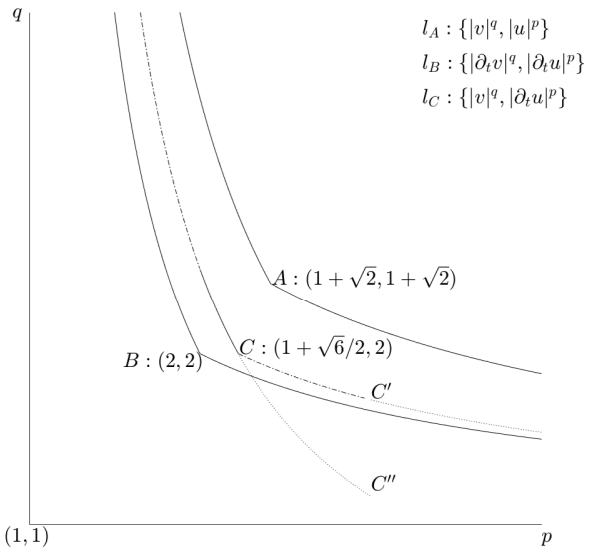
The optimality of the curve is given at least under the spherical symmetric initial data. The blow-up part can be found in Deng [9], while the existence part under spherical symmetric initial data can be found in Kubo-Kubota-Sunagawa [20]. Interested readers can read Hidano-Yokoyama [15] for more results on these two system.

For equation (1.10), Hidano-Yokoyama [15] found a curve

$$(p-1)(pq-1) = p+2, \qquad 1 < q, \qquad 1 < p < 3. \tag{1.14}$$

When (p, q) is below this curve, the solution of (1.10) must blow-up no matter how ε small is. On the contrary, when (p, q) is above the curve, and satisfied $2 < p < 3$, $2 < q$, it shows that there exists global solution for small spherical symmetric initial data.

Comparing the systems (1.10), (1.11) and (1.12), one should expect the critical curve of the former system will between that of the latter two. However, the curve (1.14) intersect with one of them (see the following graph, CC'' intersects with l_B). This inspire us to improve the result to $q < 2$.



Here l_A is the critical curve of (1.11), l_B is the critical curve of (1.12), CC'' is the curve (1.14) with $q < 2$. And we will show that CC' is the actual critical curve of (1.10) for $q < 2$, which is determined by the following equation

$$\sigma_{GS}(p, q) = 0, \qquad \sigma_{GS}(p, q) := \frac{p+2}{pq-1} - \frac{1}{pq} - 1. \tag{1.15}$$

In the research process, we first study the existence part. Formally, there are some similarities between equation (1.10) and equation (1.9). However, when $q < 2$, the low regularity of the force term makes common tools such as *Sobolev* embedding no longer applicable. So the similar methods cannot be used here. However, noted that if we focus on the radial symmetric wave equation, we know that three-dimensional wave equation can correspond to a one-dimensional wave equation in this situation. This means that the regularity of the solution can be raised properly, which makes the estimate possible.

For the three-dimensional homogeneous wave equation, the solution has a decay t^{-1} near the light cone. In Hidano-Yokoyama [15], the author establishes the weight function based on this decay, and reach the result for $q > 2$. However, after further study of the case of $q < 2$, we find that the decay of the solution of the equation near the light cone is different from that of the homogeneous equation, t^{-1}. This is essentially different from that of $q > 2$. Therefore, we choose the weight which is suitable for the case of $q < 2$. Through the detailed analysis and discussion of the basic solution, we finally get the global result above the CC' curve.

As for the blow up part, the method we used is similar to that in the section of 1.3 for general dimension $n \geq 3$. It is found that this kind of test function method is still suitable for system case and $\partial_t u$ power term. Finally, we give a blow up result in any dimension

$n \geq 3$. In particular, when $n = 3$, it matches the curve (1.15). This shows that (1.15) is indeed the critical curve of (1.10) with $q < 2$.

1.6 Symbols and article structure

We list here some notations which will be used in our article.

First, the *Einstein* summation convention is used, as well as the convention that Greek indices $\mu, \nu, \cdots$ range from 0 to n while Latin indices $i, j, \cdots$ will run from 1 to n. In this article, we often refer to some constants that do not depend on ε. When there is no ambiguity, we usually use C to refer them uniformly, even though constants may be different from line to line. We denote $x/0 = \infty$ for any $x > 0$ and $x/\infty = 0$ for any $-\infty < x < \infty$. For the writing convenience, $x \lesssim y$ and $y \gtrsim x$ mean $x \leq Cy$ for some $C > 0$. Similarly, $x \approx y$ means that $x \lesssim y \lesssim x$. We use $\mathcal{L}^p_r = L^p((0,\infty), r^{n-1}dr)$ stand for the weight L^p norm in radial direction. We use the notation: $\langle x \rangle := \sqrt{4 + |x|^2} \approx 1 + |x|$. Finally, for the derivative operator $\partial = \{\partial_\mu\}_{\mu=0}^n = (\partial_t, \partial_x)$, we set $\partial^{\leq k} f := \{\partial^\alpha f\}_{0 \leq \alpha \leq k}$, and similar generalization to other operators and applied to norm space.

In the following chapters, we will use the following structure. In the second to fifth chapters, we will discuss the related problems raised from Section 1.2 to Section 1.5 respectively. We will give the concrete conclusions we have obtained and detailed proofs.

Chapter 2

Long time behavior of *Strauss* type system in asymptotically flat space-time

2.1 Hypotheses, main conclusions and some notations

In this chapter, we study the equation (1.5) proposed in section 1.2. The results presented in this chapter was obtained by the author's joint work [6]. Before giving the main conclusion, we will give some assumptions used in this chapter. Note that these assumptions are naturally satisfied in many common models. For detailed discussion, see, for example, [30].

Hypothesis 1 (Space-time assumption)**.** *We shall assume that* $\mathbf{g} = g_{\mu\nu}(t,x)\,\mathrm{d}\,x^\mu\,\mathrm{d}\,x^\nu$ *is asymptotically flat in the following sense. Assume that* $\mathbf{g}$ *can be decomposed as*

$$\mathbf{g} = \mathbf{m} + \boldsymbol{\gamma}^1(t,r) + \boldsymbol{\gamma}^2(t,x), \tag{H1}$$

where $\mathbf{m}$ *denotes the* Minkowski *metric,* $\boldsymbol{\gamma}^1$ *is a radial long range perturbation,* $\boldsymbol{\gamma}^2$ *is a short range perturbation. Here* $\boldsymbol{\gamma}^1$*,* $\boldsymbol{\gamma}^2$ *satisfy*

$$\sum_k \|(1+|x|)^{|\alpha|+j-1}\partial^\alpha \boldsymbol{\gamma}^j_{\mu\nu}\|_{L^\infty_{t,x}(1+|x|\sim 2^k)} \le C_{\alpha,j}, \qquad j=1,2.$$

The long range perturbation is radial in the sense that when writing out the metric $\boldsymbol{\gamma}^1$ *in polar coordinates* $(t,x)=(t,r\omega)$ *with* $\omega\in S^{n-1}$*, we have*

$$\boldsymbol{\gamma}^1(t,r) = \boldsymbol{\gamma}^1_{tt}(t,r)\,\mathrm{d}\,t^2 + 2\boldsymbol{\gamma}^1_{tr}(t,r)\,\mathrm{d}\,t\,\mathrm{d}\,r + \boldsymbol{\gamma}^1_{rr}(t,r)\,\mathrm{d}\,r^2 + \boldsymbol{\gamma}^1_{\omega\omega}(t,r)\,\mathrm{d}\,\omega^2.$$

Hypothesis 2 (Local energy assumption)**.** *We assume that there exists an* $R_1 > R_0$*, such that for any solution to the linear equation* $\Box_{\mathbf{g}} u = F$*, we have the uniform energy estimates*

and weak local energy estimates, which is

$$\|\partial^{\leq k}u\|_{LE_{\mathbf{g}}} \lesssim \|\partial^{\leq 1}u(0,x)\|_{H^k_x} + \|F\|_{L^1_t H^k_x}, \tag{H2}$$

for any $k \geq 0$*, with*

$$\|u\|_{LE_{\mathbf{g}}} := \|\partial u\|_{L^\infty_t L^2_x} + \|\bar{\psi}_{R_1}\partial u\|_{l^{-\frac{1}{2}}_\infty L^2_t L^2_x} + \|u\|_{l^{-\frac{3}{2}}_\infty L^2_t L^2_x}, \tag{2.1}$$

where $\|f\|_{l^s_\infty L^2_t L^2_x} = \sup_{k\geq 0}\|2^{sk}f\|_{L^2_{t,x}(1+|x|\sim 2^k)}$*, and the smooth truncated function* $\bar{\psi}_{R_1}$ *equals* 0 *in* $r \leq R_1$*, and equals* 1 *in* $r \geq R_1 + 1$*.*

Hypothesis 3 (Static metric hypothesis)**.** *We assume that there exists a* $R_2 > R_0$*, such that*

$$\mathbf{g} = \mathbf{g}_{00}(x)\,\mathrm{d}\,t^2 + \mathbf{g}_{ij}(x)\,\mathrm{d}\,x^i\,\mathrm{d}\,x^j, \qquad |x| > R_2. \tag{H3}$$

Next, we give the main conclusions of this chapter. First of all, we present a kind of weighted Strichartz estimates for $n \geq 2$.

Theorem 2.1. *Let* $n \geq 2$*, we set*

$$A_\alpha(T) := \begin{cases} (T+2)^{-\alpha}, & \alpha \in (0, \frac{1}{2}], \\ \ln(T+2)^{-\frac{1}{2}}, & \alpha = 0, \\ 1, & \alpha < 0, \end{cases}$$

for $T \in (0,\infty)$*. Then for any* $\delta > 0$*,* $2 \leq o \leq q < \infty$*,* $\alpha \leq 1/q$ *and* $1/2 - 1/o \leq s \leq (n-\delta)(1/2-1/q)$*, we have*

$$\|u\|_{WS_{q,o,\alpha,s}} := A_{q\alpha/2}(T)^{\frac{2}{q}}\|r^{\frac{n}{2}+\alpha-\frac{1}{q}-\frac{n}{o}-s}\bar{\psi}_1 u\|_{L^q_t \mathcal{L}^o_r L^2_\omega} \lesssim \|u\|_{W^s\cap X^s}, \tag{2.2}$$

Here W^s *and* X^s *defined by (2.7), which are energy and local energy norm with regularity index* s*, and* $\bar{\psi}_1$ *also defined in the end of this section. Moreover, we have*

$$\|r^{\frac{n}{2}-\frac{1}{q}-\frac{n}{o}-s}u\|_{L^q_t \mathcal{L}^o_r L^2_\omega} \lesssim \|u\|_{W^s\cap X^s}, \tag{2.3}$$

with $2 \leq o, q \leq \infty$ *and* $1/2 - 1/o < s < n/2 - 1/q$*.*

The estimates in Theorem 2.1, together with Lemma 2.11, can deduce a series of important weighted *Strichartz* estimates, which, in turn, provide the following desired lower bound of lifespan to problem (1.5).

Theorem 2.2. *Let $n = 3$, $p_1, p_2 \geq 2$, and assume (H1) and (H2). Fix $\delta > 0$, set*

$$s_\iota := \frac{3}{2} + \sigma - \frac{2(p_\iota + 1)}{p_1 p_2 - 1}, \tag{2.4}$$

where $\sigma := \sigma_{SS}(p_1, p_2)$ defined in (1.6). Then, there exists $R, c, \varepsilon_0 > 0$ and

$$T_*(\varepsilon) := \begin{cases} c\varepsilon^{-\frac{1}{\sigma}} & \text{if } \sigma > 0, \\ \exp(c\varepsilon^{-2(p_1-1)}) & \text{if } \sigma = 0, p_1 > 2, \\ \exp(c\varepsilon^{-2+\delta}) & \text{if } \sigma = 0, p_1 = 2, \\ \infty & \text{if } \sigma < 0. \end{cases} \tag{2.5}$$

such that for any (f_ι, g_ι) satisfying

$$\sum_\iota \|Y^{\leq 2}(f_\iota, g_\iota)\|_{H^1_x \times L^2_x} + \|\bar{\psi}_R Y^{\leq 2} g_\iota\|_{\dot{H}^{s_\iota - 1}_x} = \varepsilon \leq \varepsilon_0 \ , \tag{2.6}$$

with additional condition $\mathrm{supp}(f_\iota, g_\iota) \subset B_R$ *when $p_{\hat{\iota}} = 2$, there is a unique solution u_ι of equation (1.5) in $M \cap ([0, T_*) \times \mathbb{R}^3)$, such that $\|\bar{\psi}_R Z^{\leq 2} u_\iota\|_{W^s \cap X^s} + \|Z^{\leq 2} u_\iota\|_{LE_{\mathbf{g}}} \lesssim \varepsilon$. At last, if (H3) is satisfied, the initial data do not need to be compactly supported.*

Remark 2.1. *By comparing with the result of* Minkowski *space, as discussed in Section 1.2, we know that the lower bound in our result is sharp in general for the subcritical and supercritical cases. But it is still possible to improve it in critical situation.*

When $F_{p_1} = F_{p_2}$ and $(f_1, g_1) = (f_2, g_2)$, we can also deduce the corollary of equation $\Box_{\mathbf{g}} u = F_p(u)$.

Corollary 2.3. *Let $n = 3$, $p \geq 2$ and assume (H1) and (H2). Set*

$$s := \frac{1}{2} - \frac{1}{p},$$

then, there exists $R, c, \varepsilon_0 > 0$ and

$$T_*(\varepsilon) := \begin{cases} c\varepsilon^{-p(p-1)/(1+2p-p^2)} & 2 \leq p < 1 + \sqrt{2}, \\ \exp(c\varepsilon^{-2\sqrt{2}}) & p = 1 + \sqrt{2}, \\ \infty & p > 1 + \sqrt{2} \ . \end{cases}$$

such that for any (f, g) satisfying

$$\|Y^{\leq 2}(f, g)\|_{H^1_x \times L^2_x} + \|\bar{\psi}_R Y^{\leq 2} g\|_{\dot{H}^{s-1}_x} = \varepsilon \leq \varepsilon_0,$$

and $\mathrm{supp}(f,g)\subset B_R$ *if* $p=2$*, there is a unique solution* u *of*

$$\begin{cases}\Box_{\mathbf{g}}u=F_p(u),\ (t,x)\in M\ ,\\ (u,\partial_t u)|_{t=0}=(f,g)\ ,\end{cases}$$

in $M\cap([0,T_*)\times\mathbb{R}^3)$*, such that* $\|\bar{\psi}_R Z^{\leq 2}u\|_{W^s\cap X^s}+\|Z^{\leq 2}u\|_{LE_{\mathbf{g}}}\lesssim\varepsilon$*.*

Remark 2.2. *For* $p\in(2,1+\sqrt{2}]$*, this result removes the technical assumption (H3) for the corresponding result in [30]. In particular, we have the improved lower bound of the lifespan*

$$T_\varepsilon\geq\exp(c\varepsilon^{-2\sqrt{2}}).$$

for the critical wave equations (with $p=1+\sqrt{2}$*), posed on Kerr space-times with small angular momentum* $a\ll M$*.*

Remark 2.3. *The same results of Theorem 2.2 can be applied for general operators* $P=\Box_{\mathbf{m}}+a^{\mu\nu}\partial_\mu\partial_\nu+b^\mu\partial_\mu+c$ *at least for* $p_2\geq p_1>2$*, where* a,b,c *have sufficient decay and regularity. The only difference in the proof is that the admissible range of* s *in Lemma 2.11 shrinks to* $s\in(-1,0)$ *if there exists zero order term in* P*.*

At the end of this section, we give some notations used in this chapter.

$$\begin{aligned}&\partial:=\{\partial_\mu\}=(\partial_t,\partial_x),\qquad &&\Omega:=\{\Omega_{jk}=x^k\partial_j-x^j\partial_k\},\\ &Y:=\{\partial_x,\Omega\},\qquad &&Z:=\{\partial,\Omega\}.\end{aligned}$$

Next, we shall use some auxiliary functions. Let $\phi\in C_0^\infty(\mathbb{R})$ with $\mathrm{supp}\,\phi\subset[1/2,2]$, $\phi_j(x)=\phi(|x|/2^j)$ and $\sum_{j\in\mathbb{Z}}\phi_j^2=1$ for any $0\neq x\in\mathbb{R}^n$. Also, we define S_j to be the corresponding homogeneous Littlewood-Paley projection. We also fix a class of smooth functions ψ_R satisfying $0\leq\psi_R\leq 1$ and

$$\psi_R(x):=\begin{cases}1, & r\leq R,\\ 0, & r\geq R+1,\end{cases}$$

meanwhile, we denote $\bar{\psi}_R=1-\psi_R$.

Then, we denote the shorthand for some norm spaces,

$$\|f(x)\|_{l_q^s(A)}:=\|2^{js}\phi_j(x)f(x)\|_{l^q_{j\in\mathbb{Z}}A}\ ,$$

and $l_q^s(A)$ is defined similarly with the inhomogeneous dyadic decomposition. We also denote the $W^s=W_2^s$ norm and $X^s=X_2^s$ norm by

$$\begin{aligned}&\|f\|_{W_p^s}=\|2^{ks}S_kf\|_{l^p_{k\in\mathbb{Z}}L_t^\infty L_x^2},\qquad &&\|f\|_{X_p^s}=\|2^{ks}S_kf\|_{l^p_{k\in\mathbb{Z}}X_k},\\ &\|f\|_{X_k}=2^{\frac{k}{2}}\|f\|_{L^2(A_{\leq -k})}+\sup_{j>-k}\||x|^{-1/2}f\|_{L^2(A_j)},\end{aligned}\tag{2.7}$$

with $A_j = \mathbb{R}_+ \times \{|x| \sim 2^j\}$, $A_{\leq -k} = \cup_{j \leq -k} A_j$, $j, k \in \mathbb{Z}$. Moreover, we usually omit $T > 0$ in the norm when it is clear from the context that the norm is taken for $t \in [0, T]$ for the given $T > 0$.

2.2 Proof of Theorem 2.1

We begin with a sketch for the proof of Theorem 2.1. First of all, we record a property of X^s space.

Proposition 2.4 (Lemma 1 of [27]). *Let $n \geq 2$, we have*

$$\|u\|_{l_\infty^{-\frac{1}{2}} L_t^2 L_x^2} \lesssim \|u\|_{X^0}, \tag{2.8}$$

and when $0 < s < (n-1)/2$ we have

$$\|r^{-\frac{1}{2}-s} u\|_{L_t^2 L_x^2} \lesssim \|u\|_{X^s}. \tag{2.9}$$

hen, based on the well known *KSS* type estimates, for $\alpha \leq 1/2$ we get

$$A_\alpha(T) \| \langle r \rangle^{-\frac{1}{2}+\alpha} u \|_{L_t^2 L_x^2} \lesssim \|u\|_{L_t^\infty L_x^2 \cap l_\infty^{-\frac{1}{2}} L_{t,x}^2} \lesssim \|u\|_{W^0 \cap X^0}. \tag{2.10}$$

Here the last inequality comes from (2.8) and *Minkowski* inequality, and the first inequality follows from the typical *KSS* type estimate, see, e.g., [18] Appendix 7.2, [25] for its proof.

To prove (2.2), we need the following trace estimates.

Lemma 2.5 (Trace estimates). *Let $n \geq 2$, for $2 \leq o \leq \infty$ and $1/2 - 1/o \leq s < n/2$, we have*

$$\|r^{\frac{n}{2}-\frac{n}{o}-s} u\|_{\mathcal{L}_r^o L_\omega^2} \lesssim \begin{cases} \|u\|_{\dot{B}_{2,1}^{\frac{1}{2}}} & o = \infty, s = \frac{1}{2}, \\ \|u\|_{\dot{H}_x^s} & else. \end{cases} \tag{2.11}$$

Now, to combine (2.10) with (2.11), we need an interpolation property.

Claim 2.6. *Consider $s_0 \neq s_1$, set $s = (1-\theta)s_0 + \theta s_1$ with $\theta \in (0,1)$, for complex interpolation, we have*

$$W^s \cap X^s = [W^{s_0} \cap X^{s_0}, W^{s_1} \cap X^{s_1}]_\theta, \tag{2.12}$$

and for real interpolation, we have

$$W_p^s \cap X_p^s = (W_{p_0}^{s_0} \cap X_{p_0}^{s_0}, W_{p_1}^{s_1} \cap X_{p_1}^{s_1})_{\theta,p}, \qquad \forall p, p_0, p_1 \in [1, \infty]. \tag{2.13}$$

Meanwhile, these results are also correct if we consider the W^s or X^s norm separately.

Now, based on (2.11) with L_t^∞ norm on both sides, and interpolation with (2.10), we get (2.2). As for (2.3), we need a similar trace estimate in X^s space. Firstly we have a weighted trace estimate.

Lemma 2.7. *Let $n \geq 2$, $o \in [2, \infty]$, $\alpha \in (1/2 - 1/o, 1)$ and $\beta \in (\alpha - n/2, n/2)$. Then we have*

$$\|r^{\frac{n}{2}-\frac{n}{o}-\alpha+\beta} u\|_{\mathcal{L}_r^o L_\omega^2} \lesssim \|r^\beta D^\alpha u\|_{L_x^2}. \tag{2.14}$$

Then combining Lemma 2.7 with Proposition 2.4 we obtain the trace estimate in X^s space.

Lemma 2.8. *Let $n \geq 2$, for $2 \leq o \leq \infty$ and $1/2 - 1/o < s < (n-1)/2$, we have*

$$\|r^{\frac{n-1}{2}-\frac{n}{o}-s} u\|_{L_t^2 \mathcal{L}_r^o L_\omega^2} \lesssim \|u\|_{X^s}. \tag{2.15}$$

Finally, interpolating (2.11) and (2.15) we get (2.3) which finishes the sketch. In the following subsections, we give the detailed proof for each part.

2.2.1 Proof of Lemma 2.5

Firstly, when $o = \infty$, (2.11) is the well known trace estimate, see, e.g., Fang-Wang [11] and references therein. When $o = 2$, it is the *Hardy*'s inequality and so we have

$$\begin{cases} \|r^{\frac{n}{2}-s_0} u\|_{\mathcal{L}_r^\infty L_\omega^2} \lesssim \|u\|_{\dot{B}_{2,1}^{s_0}}, \ s_0 \in [1/2, n/2)\ , \\ \|r^{-s_1} u\|_{\mathcal{L}_r^2 L_\omega^2} \lesssim \|u\|_{\dot{H}_x^{s_1}}, \ s_1 \in [0, n/2). \end{cases} \tag{2.16}$$

The estimate for $s = 1/2 - 1/o$ is obtained in [22] Theorem 2.10, see also [14] Proposition 2.2 for an alternative proof based on interpolation and trace estimate. In what follows, we give a unified proof for the case of $o \in (2, \infty)$, in spirit of [14].

Let $2 < o < \infty$ and $1/2 - 1/o \leq s < n/2$, we set $\theta = 2/o \in (0, 1)$, $s_0 = (n - os + nos)/(no - o + 2)$ and $s_1 = (n - no/2 + nos)/(no - o + 2)$. Then a routine calculation shows that $s_0 > s_1$, $s_0 \in [1/2, n/2)$, $s_1 \in [0, n/2)$, and

$$\frac{1-\theta}{\infty} + \frac{\theta}{2} = \frac{1}{o}, \qquad (1-\theta)s_0 + \theta s_1 = s.$$

With help of these parameters, we shall use the real interpolation with parameters (θ, o) to give the proof, for which we record the following facts:

- [2, Theorem 3.7.1]: $(A_0, A_1)'_{\theta,q} = (A_0', A_1')_{\theta,q'}$ for $1 \leq q < \infty$, $A_0 \cap A_1$ is dense in A_0 and A_1;

- [2, Theorem 5.5.1]: $(L^{p_0}_{(\rho_0\,\mathrm{d}x)}(A), L^{p_1}_{(\rho_1\,\mathrm{d}x)}(A))_{\theta,p} = L^{p}_{(\rho\,\mathrm{d}x)}(A)$ for $1 \le p_0, p_1 < \infty$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $\rho = \rho_0^{p(1-\theta)/p_0}\rho_1^{p\theta/p_1}$;
- [2, Theorem 6.4.5]: $(\dot{B}^{s_0}_{p,q_0}, \dot{B}^{s_1}_{p,q_1})_{\theta,q} = \dot{B}^{s}_{p,q}$ for $s_0 \neq s_1$, $s = (1-\theta)s_0 + \theta s_1$ and $1 \le p, q_0, q_1, q \le \infty$.

More specifically, defining $\|v\|_{A_0} := \|r^{s_0-n/2}v\|_{\mathcal{L}^1_r L^2_\omega}$ and $\|v\|_{A_1} := \|r^{s_1}v\|_{\mathcal{L}^2_r L^2_\omega}$, by (2.16) and the first fact, we conclude

$$\|u\|_{(A_0,A_1)'_{\theta,o'}} = \|u\|_{(A_0',A_1')_{\theta,o}} \lesssim \|u\|_{(\dot{B}^{s_0}_{2,1},\dot{H}^{s_1}_x)_{\theta,o}}.$$

Now, for the left hand side (LHS), with $\rho_0 := r^{s_0-n/2+n-1}$ and $\rho_1 := r^{2s_1+n-1}$, we set

$$\rho := r^{o'(\frac{n}{o}+s-\frac{n}{2})+n-1} = \rho_0^{o'(1-\theta)}\rho_1^{\frac{o'\theta}{2}}.$$

then by the second fact with $p_0 = 1$, $p_1 = 2$, $p = o'$ and $A = L^2_\omega$, we have

$$\|v\|_{(A_0,A_1)_{\theta,o'}} = \|r^{\frac{n}{o}+s-\frac{n}{2}}v\|_{\mathcal{L}^{o'}_r L^2_\omega} \Rightarrow \|u\|_{(A_0,A_1)'_{\theta,o'}} = \|r^{\frac{n}{2}-\frac{n}{o}-s}u\|_{\mathcal{L}^{o}_r L^2_\omega}.$$

For the right hand side (RHS), noticing that $s_1 = \dot{B}^{s_1}_{2,2}$, by the third fact we get

$$\|u\|_{(\dot{B}^{s_0}_{2,1},\dot{H}^{s_1}_x)_{\theta,o}} = \|u\|_{\dot{B}^{s}_{2,o}} \lesssim \|u\|_{\dot{H}^s_x}$$

Summing up, we finish the proof.

2.2.2 Proof of Claim 2.6

Similarly to [28], we define the co-retraction and retraction by

$$Qf = \left(\mathbf{1}_{l=k}\phi\left(\frac{\langle 2^k x\rangle}{2^j}\right)S_k f\right)_{j\in\mathbb{N},k\in\mathbb{Z},l\in\mathbb{Z}},$$

$$Q : X^s_p \to {}^{s}_{p(k)}l^{-\frac{1}{2}}_{\infty(j)}L^{2\ \frac{1}{2}}_{t,x1(l)} \qquad Q : W^s_p \to {}^{s}_{p(k)}L^{\infty}_t L^{20\ \ 0}_{x1(j)1(l)},$$

$$R(a_{jkl})_{j\in\mathbb{N},k\in\mathbb{Z},l\in\mathbb{Z}} = \sum_{k\in Z} S_k \sum_{j\ge 0}\phi\left(\frac{\langle 2^k x\rangle}{2^j}\right)a_{jkk},$$

$$R : {}^{s}_{p(k)}l^{-\frac{1}{2}}_{\infty(j)}L^{2\ \frac{1}{2}}_{t,x1(l)} \to X^s_p \qquad R : {}^{s}_{p(k)}L^{\infty}_t L^{20\ \ 0}_{x1(j)1(l)} \to W^s_p,$$

then we obtain $RQf = f$ for $f \in W^s_p \cap X^s_p$. Set $A = l^{-\frac{1}{2}}_\infty L^{2\ \frac{1}{2}}_{t,x1} \cap L^\infty_t L^{200}_{x11}$, it is obvious that $W^s_p \cap X^s_p$ norm is equivalent to ${}^{s}_{p}(A)$ norm. By [2, Theorem 5.6.3] and [2, Theorem 5.6.1], we have

$$[{}^{s_0}_{2}(A), {}^{s_1}_{2}(A)]_\theta = {}^{s}_{2}(A),\ ({}^{s_0}_{p_0}(A), {}^{s_1}_{p_1}(A))_{\theta,p} = {}^{s}_{p}(A), \qquad 1 \le p_0, p_1, p \le \infty,$$

provided that $s_0 \neq s_1$, $0 < \theta < 1$ and $s = (1-\theta)s_0 + \theta s_1$. This completes the proof, in view of [2, Theorem 6.4.2].

2.2.3 Proof of Eq.(2.2)

When $q = 2$, we have $o = 2$, $s = 0$, and then (2.2) follows directly from (2.10). It remains to consider $2 < q < \infty$, for which we set $\alpha_0 = q\alpha/2$, $o_1 = o(q-2)/(q-o)$, $s_1 = qs/(q-2)$. Similarly to the proof in subsection 2.2.1, we have $\alpha_0 \le 1/2$, $2 \le o_1 \le \infty$ and $1/2 - 1/o_1 \le s_1 < n/2$, by the conditions on q, o, α and s. Moreover, for $\theta = 1 - 2/q \in (0,1)$, we have

$$\frac{1-\theta}{2} + \frac{\theta}{\infty} = \frac{1}{q}, \qquad \frac{1-\theta}{2} + \frac{\theta}{o_1} = \frac{1}{o}, \qquad (1-\theta)0 + \theta s_1 = s.$$

Firstly when $q = o$ and $s = 1/2 - 1/o$, where $o_1 = \infty$ and $s_1 = 1/2 - 1/o_1$, by (2.10) and (2.11) we obtain

$$\begin{cases} A_{\alpha_0}(T)\|r^{-\frac{1}{2}+\alpha_0}\bar{\psi}_1 u\|_{L_t^2\mathcal{L}_r^2 L_\omega^2} \lesssim \|u\|_{W^0\cap X^0}, \\ \|r^{\frac{n-1}{2}}\bar{\psi}_1 u\|_{L_t^\infty \mathcal{L}_r^\infty L_\omega^2} \lesssim \|u\|_{L_t^\infty \dot{B}^{\frac{1}{2}}_{2,1}} \lesssim \|u\|_{W_1^{\frac{1}{2}}\cap X_1^{\frac{1}{2}}}. \end{cases}$$

Recall the following facts

- [2, Theorem 3.7.1]: $(A_0, A_1)'_{\theta,q} = (A_0', A_1')_{\theta,q'}$ for $1 \le q < \infty$, $A_0 \cap A_1$ is dense in A_0 and A_1;
- [29, 1.18.4 eq.(3)]: $(L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,p} = L^p((A_0,A_1)_{\theta,p})$ for $1 \le p_0, p_1 < \infty$, and $1/p = (1-\theta)/p_0 + \theta/p_1$;
- [2, Theorem 5.5.1]: $(L^{p_0}_{(\rho_0\,\mathrm{d}x)}(A), L^{p_1}_{(\rho_1\,\mathrm{d}x)}(A))_{\theta,p} = L^p_{(\rho\,\mathrm{d}x)}(A)$ for $1 \le p_0, p_1 < \infty$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $\rho = \rho_0^{p(1-\theta)/p_0}\rho_1^{p\theta/p_1}$,

we could use real interpolation with $(1-2/q, q)$ for LHS to obtain

$$\text{LHS of } (2.2) \lesssim \|u\|_{(W^0\cap X^0, W_1^{\frac{1}{2}}\cap X_1^{\frac{1}{2}})_{\theta,q}}.$$

Applying (2.13) for RHS, we conclude

$$\text{LHS of } (2.2) \lesssim \|u\|_{W_q^{\frac{1}{2}-\frac{1}{q}}\cap X_q^{\frac{1}{2}-\frac{1}{q}}} \lesssim \|u\|_{W_2^{\frac{1}{2}-\frac{1}{q}}\cap X_2^{\frac{1}{2}-\frac{1}{q}}} = \text{RHS of } (2.2),$$

where we have used the fact $l^2 \subset l^q$ in the last inequality.

For the remaining case when $q > o$ or $s > 1/2 - 1/o$, we have $o_1 < \infty$ or $1/2 - 1/o_1 < s_1$. By (2.10) and (2.11), we obtain

$$\begin{cases} A_{\alpha_0}(T)\|r^{-\frac{1}{2}+\alpha_0}\bar{\psi}_1 u\|_{L_t^2\mathcal{L}_r^2 L_\omega^2} \lesssim \|u\|_{W^0\cap X^0}, \\ \|r^{\frac{n}{2}-\frac{n}{o_1}-s_1}\bar{\psi}_1 u\|_{L_t^\infty \mathcal{L}_r^{o_1} L_\omega^2} \lesssim \|u\|_{W^{s_1}\cap X^{s_1}}. \end{cases}$$

To apply complex interpolation with $\theta = 1 - 2/q$, we use (2.12) for RHS, and record the following facts for LHS:

- [2, Corollary 4.5.2]: $[A_0, A_1]'_\theta = [A'_0, A'_1]_\theta$ for $A_0 \cap A_1$ is dense in A_0 and A_1 and at least one of the spaces A_0 and A_1 is reflexive;

- [2, Theorem 5.1.2]: $[L^{p_0}(A_0), L^{p_1}(A_1)]_\theta = L^p([A_0, A_1]_\theta)$ for $1 \le p_0, p_1 < \infty$, and $1/p = (1-\theta)/p_0 + \theta/p_1$;

- [2, Theorem 5.5.3]: $[L^{p_0}_{(\rho_0 \,\mathrm{d}x)}(A), L^{p_1}_{(\rho_1 \,\mathrm{d}x)}(A)]_\theta = L^p_{(\rho \,\mathrm{d}x)}(A)$ for $1 \le p_0, p_1 < \infty$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $\rho = \rho_0^{p(1-\theta)/p_0} \rho_1^{p\theta/p_1}$.

This gives us (2.2) and completes the proof.

2.2.4 Proof of Lemma 2.7

For given $n \ge 2$, $o \in [2, \infty]$, $\alpha \in (1/2 - 1/o, 1)$ and $\beta \in (\alpha - n/2, n/2)$, we set

$$\alpha_0 = \frac{\alpha o + 2}{o+2} \in (\frac{1}{2}, 1)\ ,\ \alpha_1 = \frac{2\alpha o - o + 2}{o+2} \in (0,1)\ ,$$

$$\beta_0 = \frac{n}{2} - \frac{n-\alpha_0}{n-\alpha}(\frac{n}{2} - \beta) \in (-\frac{n}{2} + \alpha_0, \frac{n}{2})\ ,$$

$$\beta_1 = \frac{n}{2} - \frac{n-\alpha_1}{n-\alpha}(\frac{n}{2} - \beta) \in (-\frac{n}{2} + \alpha_1, \frac{n}{2})\ .$$

Let $\theta = 2/o$, we have $(1-\theta)\alpha_0 + \theta\alpha_1 = \alpha$, $(1-\theta)\beta_0 + \theta\beta_1 = \beta$. Recall that by [30, (4.10)], we have

$$\|r^\beta D^\alpha u\|_{L^2_x} \approx \|r^\beta 2^{\alpha k} S_k u\|_{l^2_{k\in\mathbb{Z}} L^2_x} \approx \|2^{\beta j + \alpha k} \phi_j S_k u\|_{l^2_{j\in\mathbb{Z}} l^2_{k\in\mathbb{Z}} L^2_x}, \tag{2.17}$$

for any $|\beta| < n/2$. Then by weighted trace estimate of [30, Lemma 4.2] and weighted *Hardy-Littlewood-Sobolev* estimates of Stein-Weiss, we obtain

$$\begin{cases} \|r^{\frac{n}{2}-\alpha_0+\beta_0} u\|_{\mathcal{L}^\infty_r L^2_\omega} \lesssim \|r^{\beta_0} D^{\alpha_0} u\|_{L^2} \lesssim \|2^{\beta_0 j + \alpha_0 k} \phi_j S_k u\|_{l^2_{j\in\mathbb{Z}} l^2_{k\in\mathbb{Z}} L^2_x}\ , \\ \|r^{-\alpha_1+\beta_1} u\|_{\mathcal{L}^2_r L^2_\omega} \lesssim \|r^{\beta_1} D^{\alpha_1} u\|_{L^2} \lesssim \|2^{\beta_1 j + \alpha_1 k} \phi_j S_k u\|_{l^2_{j\in\mathbb{Z}} l^2_{k\in\mathbb{Z}} L^2_x}\ , \end{cases}$$

where $D = \sqrt{-\Delta}$. Here we should mention that the first inequality was stated for $n \ge 3$ and $\alpha_0 \in (1/2, 1]$ in [30, Lemma 4.2]. However, we observe that the same proof also apply in the situation of $n = 2$ and $\alpha_0 \in (1/2, 1)$.

We remark that we can do complex interpolation for RHS, if we follow a similar proof as in subsection 2.2.2. For LHS, we record the following facts:

- [2, Corollary 4.5.2]: $[A_0, A_1]'_\theta = [A'_0, A'_1]_\theta$ for $A_0 \cap A_1$ is dense in A_0 and A_1 and at least one of the spaces A_0 and A_1 is reflexive;

- [2, Theorem 5.5.3]: $[L^{p_0}_{(\rho_0\,\mathrm{d}x)}(A), L^{p_1}_{(\rho_1\,\mathrm{d}x)}(A)]_\theta = L^{p}_{(\rho\,\mathrm{d}x)}(A)$ for $1 \le p_0, p_1 < \infty$, $1/p = (1-\theta)/p_0 + \theta/p_1$ and $\rho = \rho_0^{p(1-\theta)/p_0}\rho_1^{p\theta/p_1}$.

Thus, by complex interpolation with $\theta = 2/o$, together with (2.17), we get

$$\|r^{\frac{n}{2}-\frac{n}{o}-\alpha+\beta}u\|_{\mathcal{L}^o_r L^2_\omega} \lesssim \|2^{\beta j+\alpha k}\phi_j S_k u\|_{l^2_{j\in\mathbb{Z}}l^2_{k\in\mathbb{Z}}L^2_x} \lesssim \|r^\beta D^\alpha u\|_{L^2_x}\,.$$

This completes the proof.

2.2.5 Proof of Lemma 2.8

We follow the similar proof of [30, Lemma 5.5]. Take $\delta > 0$ and small enough such that $s > 1/2 - 1/o + \delta$, then we conclude

$$\|r^{\frac{n-1}{2}-\frac{n}{o}-s}u\|_{L^2_t\mathcal{L}^o_r L^2_\omega} \lesssim \|r^{-s-\frac{1}{o}+\delta}D^{\frac{1}{2}-\frac{1}{o}+\delta}u\|_{L^2_t L^2_x} \lesssim \|D^{\frac{1}{2}-\frac{1}{o}+\delta}u\|_{X^{s-\frac{1}{2}+\frac{1}{o}-\delta}} \lesssim \|u\|_{X^s},$$

where we have used (2.14) in the first inequality and (2.9) in the second inequality.

2.2.6 Proof of (2.3)

The proof of (2.3) is very similar to the proof of (2.2). Let $o_0 = o_1 = o$ and $\theta = 2/q$, we set

$$s_0 = s - \frac{2os - o + 2}{2o - 2q + oq - noq} \in (\frac{1}{2} - \frac{1}{o_0}, \frac{n}{2})\,,$$
$$s_1 = s + \frac{2os - o + 2}{4o - 4q + 2oq - 2noq}(q-2) \in (\frac{1}{2} - \frac{1}{o_1}, \frac{n-1}{2})\,,$$

such that $s_0(1-\theta) + s_1\theta = s$. By (2.11) and (2.15) we obtain

$$\begin{cases} \|r^{\frac{n}{2}-\frac{n}{o_0}-s_0}u\|_{L^\infty_t\mathcal{L}^{o_0}_r L^2_\omega} \lesssim \|u\|_{W^{s_0}\cap X^{s_0}}\,, \\ \|r^{\frac{n-1}{2}-\frac{n}{o_1}-s_1}u\|_{L^2_t\mathcal{L}^{o_1}_r L^2_\omega} \lesssim \|u\|_{W^{s_1}\cap X^{s_1}}\,. \end{cases}$$

Using complex interpolation with $\theta = 2/q$, we get (2.3).

2.3 The proof of Theorem 2.2

2.3.1 Preparation

In order to facilitate our later proof, we give some lemmas which will be used in this section.

Proposition 2.9 (Lemma 4.3 in [30]). *Let $n \geq 2$, $R \geq 3$ and $k = \lfloor (n+2)/2 \rfloor$ be the integral part of $(n+2)/2$. We have*

$$\|r^b u\|_{L_r^q L_\omega^\infty (r \geq R+1)} \lesssim \|r^{b-(n-1)/p+(n-1)/q} Y^{\leq k} u\|_{L_r^p L_\omega^2 (r \geq R)} \tag{2.18}$$

for any $b \in \mathbb{R}$, $2 \leq p \leq q \leq \infty$. Moreover, for any $2 \leq p \leq q \leq q_2 < \infty$, $b \in \mathbb{R}$, $s \in \mathbb{Z}$ with $s \geq n/2 - n/q_2$, we have

$$\|r^b u\|_{L_r^q L_\omega^{q_2} (r \geq R+1)} \lesssim \|r^{b-(n-1)/p+(n-1)/q} Y^{\leq s} u\|_{L_r^p L_\omega^2 (r \geq R)}. \tag{2.19}$$

Proposition 2.10 (Theorem 5.9 in [30]). *Let $n \geq 3$, and assume (H1) and (H2). Then we have*

$$\|Z^{\leq k} u\|_{LE_{\mathbf{g}}} \lesssim \|Y^{\leq k} u(0,x)\|_{H_x^1} + \|Y^{\leq k} \partial_t u(0,x)\|_{L_x^2} + \|Z^{\leq k} F\|_{L_t^1 L_x^2},$$

for any u satisfies $\Box_{\mathbf{g}} u = F$ and any $k \geq 0$.

Lemma 2.11. *For $n = 3$, consider*

$$P = \partial_t^2 - \Delta + \gamma^{\mu\nu}(t,x)\partial_\mu \partial_\nu + b^\mu(t,x)\partial_\mu,$$

with $\gamma^{\mu\nu} = \gamma^{\nu\mu} \in C^2$, $b^\mu \in C^1$, and

$$\|\gamma\|_{{}_0^1 L_{t,x}^\infty} + \|(\partial\gamma, b)\|_{{}_1^1 L_{t,x}^\infty} + \|(\partial^2\gamma, \partial b)\|_{{}_1^2 L_{t,x}^\infty} \leq \delta\,.$$

Then there exists $\delta > 0$ so that we have

$$\|\partial u\|_{W^s \cap X^s} \lesssim \|\partial u(0)\|_{\dot{H}_x^s} + \|Pu\|_{(W^{-s} \cap X^{-s})'}, \tag{2.20}$$

for any $|s| < 1$.

Proof. This lemma is a simple modification of [27, Theroem 1]. For the convenience of the reader, we adopt the notation in [27] and give a sketch of the proof. By [27, Eq.(50)] we have

$$\|\partial u\|_{L_t^\infty L_x^2 \cap X_k} \lesssim \|\partial u(0)\|_{L_x^2} + \|P_{(k)} u\|_{L_t^1 L_x^2 + X_k'}.$$

Then after summation we obtain

$$\begin{aligned}
\|\partial u\|_{W^s \cap X^s}^2 &\lesssim \sum_k 2^{2sk} \|\partial S_k u\|_{L_t^\infty L_x^2 \cap X_k}^2 \\
&\lesssim \sum_k 2^{2sk} \Big(\|S_k \partial u(0)\|_{L_x^2}^2 + \|P_{(k)} S_k u\|_{L_t^1 L_x^2 + X_k'}^2 \Big) \\
&\lesssim \|\partial u(0)\|_{H_x^s}^2 + \|Pu\|_{(W^{-s} \cap X^{-s})'}^2 + \delta \|\partial u\|_{X^s}^2,
\end{aligned}$$

which gives us (2.20) provided that δ is small. □

Corollary 2.12. *Let $n = 3$, $s > 0$ and q, o, α, s satisfy the condition of (2.2). Assume (H1), then there exists $K \geq R_0$ such that*

$$\|u\|_{WS_{q,o,\alpha,s}\cap L_t^\infty \dot{H}_x^s} \lesssim \|f\|_{\dot{H}_x^s} + \|g\|_{\dot{H}_x^{s-1}} + \|F\|_{(W^{1-s}\cap X^{1-s})'}\,, \tag{2.21}$$

for any u satisfying $\Box_{\mathbf{g}} u = F$ with initial data (f, g) on $\{t = 0\}$ and vanishing in the region $\{r < K\}$.

Proof. Based on (H1), we set $\tilde{\mathbf{g}} = \psi_{K-1}\mathbf{m} + \bar{\psi}_{K-1}\mathbf{g}$ with $K \geq R_0 + 1$ such that $\tilde{\mathbf{g}}$ satisfies the assumption in Lemma 2.11. Since we are assuming that u vanishes in the region $\{r < K\}$, we have $\Box_{\mathbf{g}} u = \Box_{\tilde{\mathbf{g}}} u$. Then, by (2.2) and (2.20), we get

$$\|u\|_{WS_{q,o,\alpha,s}\cap L_t^\infty \dot{H}_x^s} \lesssim \|u\|_{W^s\cap X^s} \lesssim \|\nabla u\|_{W^{s-1}\cap X^{s-1}} \lesssim \|f\|_{\dot{H}_x^s} + \|g\|_{\dot{H}_x^{s-1}} + \|F\|_{(W^{1-s}\cap X^{1-s})'}\,,$$

where we have assumed $s > 0$ so that $|s - 1| < 1$ to apply (2.20). □

2.3.2 Proof of Theorem 2.2 with $p_1 > 2$

Now we begin the proof of Theorem 2.2. To close the estimates we define $WS_\iota = WS_{q_\iota,o_\iota,\alpha_\iota,s_\iota}$ and $\widetilde{WS}_\iota = WS_{\tilde{q}_\iota,\tilde{o}_\iota,\tilde{\alpha}_\iota,s_\iota}$ with s_ι defined in (2.4) and

$$\begin{aligned} q_\iota :=& 2p_{\hat{\iota}}, \quad o_\iota := 2p_{\hat{\iota}} - p_1, \quad \alpha_\iota := \frac{\sigma(p_{\hat{\iota}} - 1)}{p_{\hat{\iota}}}, \\ \tilde{q}_\iota :=& p_{\hat{\iota}}, \quad \tilde{o}_\iota := p_{\hat{\iota}}, \qquad\quad \tilde{\alpha}_\iota := s_\iota - \frac{3}{2} + \frac{3}{p_{\hat{\iota}}}. \end{aligned} \tag{2.22}$$

Then we define

$$\|u\|_{U_\iota^m} = \|\bar{\psi}_R Z^{\leq m} u\|_{WS_\iota \cap \widetilde{WS}_\iota} + \|Z^{\leq m} u\|_{LE_{\mathbf{g}}}, \tag{2.23}$$

$$\|F\|_{V_\iota^m} = \|\bar{\psi}_R^{p_\iota} r^{3-s_\iota-\frac{3p_\iota}{o_{\hat{\iota}}}} Z^{\leq m} F\|_{L_t^2 \mathcal{L}_r^{\frac{o_{\hat{\iota}}}{p_\iota}} L_\omega^2} + \|Z^{\leq m} F\|_{L_t^1 L_x^2}. \tag{2.24}$$

Here we mention that $s_\iota > 0$ as $p_1 > 2$, the indices defined in (2.22) satisfy $\tilde{\alpha}_\iota < 0$ and the condition of (2.2). Moreover, the indices $q = 2$, $o = (o_{\hat{\iota}}/p_\iota)'$, $s = 1 - s_\iota$ satisfy the condition of (2.3), and

$$3 - s_\iota - \frac{3p_\iota}{o_{\hat{\iota}}} = p_\iota\left(\frac{3}{2} + \alpha_{\hat{\iota}} - \frac{1}{q_{\hat{\iota}}} - \frac{3}{o_{\hat{\iota}}} - s_{\hat{\iota}}\right). \tag{2.25}$$

Next, we present the linear estimates, which will be proved in subsection 2.3.3.

Lemma 2.13. *For $2 < p_1 \leq p_2$, assume (H1) and (H2), there exists $R > 0$ big enough such that for any $m \geq 0$*

$$\begin{aligned} \|u_\iota\|_{U_\iota^m} \lesssim& \|Y^{\leq m} f_\iota\|_{H_x^1} + \|Y^{\leq m} g_\iota\|_{L_x^2} + \|\bar{\psi}_R Y^{\leq m} g_\iota\|_{\dot{H}_x^{s_\iota - 1}} \\ &+ \|\bar{\psi}_R Z^{\leq m-1} F_\iota(0, x)\|_{\dot{H}_x^{s_\iota-1}} + \|F_\iota\|_{V_\iota^m}, \end{aligned} \tag{2.26}$$

where u_ι is the solution of

$$\Box_{\mathbf{g}} u_\iota = F_\iota, \qquad u_\iota(0,x) = f_\iota(x), \qquad \partial_t u_\iota(0,x) = g_\iota(x).$$

On the other hand, we have the following nonlinear estimates, to be proved in subsection 2.3.4.

Lemma 2.14. *For $2 < p_1 \le p_2$, we have*

$$\|F_\iota(u_{\hat\iota})\|_{V_\iota^2} \lesssim A_{q_{\hat\iota}\alpha_{\hat\iota}/2}(T)^{-\frac{2p_\iota}{q_{\hat\iota}}} \|u_{\hat\iota}\|_{U_{\hat\iota}^2}^{p_\iota}, \tag{2.27}$$

$$\|F_\iota(u_{\hat\iota}) - F_\iota(v_{\hat\iota})\|_{V_\iota^0} \lesssim A_{q_{\hat\iota}\alpha_{\hat\iota}/2}(T)^{-\frac{2p_\iota}{q_{\hat\iota}}} \|(u_{\hat\iota}, v_{\hat\iota})\|_{U_{\hat\iota}^2}^{p_\iota-1} \|u_{\hat\iota} - v_{\hat\iota}\|_{U_{\hat\iota}^0}. \tag{2.28}$$

Equipped with Lemma 2.13 and Lemma 2.14, it is a standard procedure to prove Theorem 2.2 with $p_1, p_2 > 2$. Let T_ε be given in (2.5) and c, ε small enough such that $A_{q_{\hat\iota}\alpha_{\hat\iota}/2}(T_\varepsilon)^{-2p_\iota/q_{\hat\iota}} \varepsilon^{p_\iota - 1} \ll 1$, we could iterate in $X_k := \{(u_1,u_2) : \max_\iota \|u_\iota\|_{U_\iota^k} \le \varepsilon\}$, with the time interval $[0, T_\varepsilon]$. By (2.26), (2.27) and (2.28), we know that the sequence is well defined in X_2 and contractive in X_0. Then we obtain the fixed point $(u_1,u_2) \in X_2$ solving (1.5). Again, by (2.26) and (2.28), we know the solution is unique, which finishes the proof.

2.3.3 Proof of Lemma 2.13

Without loss of generality we omit the index ι. For the WS and $\widetilde{WS}$ norm defined in (2.23), set $R \ge K$ big enough such that $w := \bar\psi_R Z^{\le m} u$ satisfies Corollary 2.12, we obtain

$$\|w\|_{WS\cap\widetilde{WS}} \lesssim \|\partial w(0,x)\|_{\dot H_x^{s-1}} + \|\Box_{\mathbf{g}} w\|_{(W^{1-s}\cap X^{1-s})'}.$$

Similar to the proof of [30, Lemma 5.6.] we have

$$\Box_{\mathbf{g}} w = [\Box_{\mathbf{g}}, \bar\psi_R] Z^{\le m} u + \bar\psi_R [\Box_{\mathbf{g}}, Z^{\le m}] u + \bar\psi_R Z^{\le m} F. \tag{2.29}$$

Now, we can control each term in $\Box_{\mathbf{g}} w$ separately. For the first term in (2.29), with the help of the duality of (2.9), we see

$$\begin{aligned}
\|[\Box_{\mathbf{g}}, \bar\psi_R] Z^{\le m} u\|_{(W^{1-s}\cap X^{1-s})'} &\lesssim \|\chi_{r\in[R,R+1]} \mathcal{O}(\partial^{\le m+1} u)\|_{(W^{1-s}\cap X^{1-s})'} \\
&\lesssim \|r^{\frac{3}{2}-s} \chi_{r\in[R,R+1]} \mathcal{O}(\partial^{\le m+1} u)\|_{L_t^2 L_x^2} \\
&\lesssim \|\partial^{\le m+1} u\|_{L_t^2 L_x^2(r\in[R,R+1])}.
\end{aligned}$$

For the second term in (2.29), we have [30, eq.(5.25)]

$$\bar\psi_R [\Box_{\mathbf{g}}, Z^{\le m}] u = \partial_x(\beta_1 \bar\psi_R \partial Z^{\le m-1} u) + \beta_2 \bar\psi_R \partial Z^{\le m-1} u + \beta_1 \bar\psi_R Z^{\le m-1} F, \tag{2.30}$$

with [30, eq.(5.24)]

$$\|\beta_1\|_{l_1^1 L_t^\infty L_x^\infty} + \||\partial\beta_1| + |\beta_2|\|_{l_1^2 L_t^\infty L_x^\infty} \lesssim 1.$$

Since we always have $0 < s < 1$, with the help of the duality of (2.8), the first term in (2.30) with $(W^{1-s} \cap X^{1-s})'$ norm can be controlled by

$$\begin{aligned}
&\|\partial_x(\beta_1\bar{\psi}_R\partial Z^{\leq m-1}u)\|_{(X^0)'\cap(X^1)'}\\
\lesssim&\|\partial_x(\beta_1\bar{\psi}_R\partial Z^{\leq m-1}u)\|_{l_1^{\frac{1}{2}}L_t^2L_x^2} + \|\beta_1\bar{\psi}_R\partial Z^{\leq m-1}u\|_{l_1^{\frac{1}{2}}L_t^2L_x^2}\\
\lesssim&\|\bar{\psi}_R\partial Z^{\leq m}u\|_{l_\infty^{-\frac{1}{2}}L_t^2L_x^2} + \|\partial^{\leq m}u\|_{L_t^2L_x^2(r\in[R,R+1])}.
\end{aligned}$$

With the help of the duality of (2.9), the second term in (2.30) with $(W^{1-s} \cap X^{1-s})'$ norm can be controlled by

$$\|\beta_2\bar{\psi}_R\partial Z^{\leq m-1}u\|_{l_2^{\frac{3}{2}-s}L_t^2L_x^2} \lesssim \|\bar{\psi}_R\partial Z^{\leq m-1}u\|_{l_\infty^{-\frac{1}{2}}L_t^2L_x^2}.$$

By duality of (2.3) and *Minkowski* inequality, the third term in (2.30) together with the last term in (2.29) with $(W^{1-s} \cap X^{1-s})'$ norm can be controlled by

$$\begin{aligned}
&\|(\bar{\psi}_R - \bar{\psi}_R^p)(\beta_1 Z^{\leq m-1}F + Z^{\leq m}F)\|_{l_2^{1-s}L_t^1L_x^2}\\
&+ \|r^{3-s-\frac{3p}{o}}\bar{\psi}_R^p(\beta_1 Z^{\leq m-1}F + Z^{\leq m}F)\|_{L_t^2\mathcal{L}_r^{\frac{o}{p}}L_\omega^2}\\
\lesssim&\|\partial^{\leq m}F\|_{L_t^1L_x^2(r\in[R,R+1])} + \|r^{3-s-\frac{3p}{o}}\bar{\psi}_R^p Z^{\leq m}F\|_{L_t^2\mathcal{L}_r^{\frac{o}{p}}L_\omega^2}.
\end{aligned}$$

In summary, recall the definition of $\|\cdot\|_{LE_{\mathbf{g}}}$ in (2.1), we conclude

$$\|u\|_{U^m} \lesssim \|\partial^{\leq 1}Z^{\leq m}u(0,x)\|_{\dot{H}_x^{s-1}} + \|Z^{\leq m}u\|_{LE_{\mathbf{g}}} + \|F\|_{V^m}.$$

Then we get Lemma 2.13 from Proposition 2.10 and a direct calculation as [30].

2.3.4 Proof of Lemma 2.14

Here, we only give the proof of (2.27) and omit the similar proof of (2.28). Noticing that

$$|Z^{\leq 2}F_\iota(u_{\hat{\iota}})| \lesssim |u_{\hat{\iota}}|^{p_\iota-1}|Z^{\leq 2}u_{\hat{\iota}}| + |u_{\hat{\iota}}|^{p_\iota-2}|Zu_{\hat{\iota}}|^2 \equiv I_\iota + II_\iota,$$

which will be handled separately.

To control the first semi-norm in (2.24), we apply *Sobolev*'s inequality on sphere to get

$$\begin{aligned}
\|Z^{\leq 2}F_\iota(u_{\hat{\iota}})\|_{L_\omega^2} &\lesssim \|u_{\hat{\iota}}\|_{L_\omega^\infty}^{p_\iota-1}\|Z^{\leq 2}u_{\hat{\iota}}\|_{L_\omega^2} + \|u_{\hat{\iota}}\|_{L_\omega^\infty}^{p_\iota-2}\|Z^{\leq 1}u_{\hat{\iota}}\|_{L_\omega^4}^2\\
&\lesssim \|Z^{\leq 2}u_{\hat{\iota}}\|_{L_\omega^2}^{p_\iota}.
\end{aligned}$$

By (2.25), (2.2) and *Hölder's* inequality, we see that

$$\begin{aligned}\|\bar{\psi}_R^{p_\iota} r^{3-s_\iota-\frac{3p_\iota}{o_{\hat{\iota}}}} Z^{\leq 2} F_\iota(u_{\hat{\iota}})\|_{L_t^2 \mathcal{L}_r^{\frac{o_{\hat{\iota}}}{p_\iota}} L_\omega^2} \lesssim & \|\bar{\psi}_R r^{\frac{3-s_\iota-\frac{3p_\iota}{o_{\hat{\iota}}}}{p_\iota}} Z^{\leq 2} u_{\hat{\iota}}\|_{L_t^{2p_\iota} \mathcal{L}_r^{o_{\hat{\iota}}} L_\omega^2}^{p_\iota} \\ = & \|\bar{\psi}_R r^{\frac{3}{2}+\alpha_{\hat{\iota}}-\frac{1}{q_{\hat{\iota}}}-\frac{3}{o_{\hat{\iota}}}-s_{\hat{\iota}}} Z^{\leq 2} u_{\hat{\iota}}\|_{L_t^{q_{\hat{\iota}}} \mathcal{L}_r^{o_{\hat{\iota}}} L_\omega^2}^{p_\iota} \\ \lesssim & A_{q_{\hat{\iota}}\alpha_{\hat{\iota}}/2}(T)^{-\frac{2p_\iota}{q_{\hat{\iota}}}} \|u_{\hat{\iota}}\|_{U_{\hat{\iota}}^2}^{p_\iota}.\end{aligned}$$

To control the second semi-norm in (2.24), we begin by dealing with I_ι. For the part $r \geq R+2$, we observe that

$$\begin{aligned}\|I_\iota\|_{L_t^1 \mathcal{L}_r^2 L_\omega^2(r\geq R+2)} \lesssim & \|r^{\frac{1}{p_\iota(p_\iota-1)}} u_{\hat{\iota}}\|_{L_t^{p_\iota} \mathcal{L}_r^{\frac{2p_\iota(p_\iota-1)}{(p_\iota-2)}} L_\omega^\infty(r\geq R+2)}^{p_\iota-1} \|r^{-\frac{1}{p_\iota}} \bar{\psi}_R Z^{\leq 2} u_{\hat{\iota}}\|_{L_t^{p_\iota} \mathcal{L}_r^{p_\iota} L_\omega^2} \\ \lesssim & \|r^{-\frac{1}{p_\iota}} \bar{\psi}_R Z^{\leq 2} u_{\hat{\iota}}\|_{L_t^{p_\iota} \mathcal{L}_r^{p_\iota} L_\omega^2}^{p_\iota} = \|\bar{\psi}_R Z^{\leq 2} u_{\hat{\iota}}\|_{\widetilde{WS}_{\hat{\iota}}}^{p_\iota} \lesssim \|u_{\hat{\iota}}\|_{U_{\hat{\iota}}^2}^{p_\iota},\end{aligned}$$

where we have applied (2.18) in the second inequality. Meanwhile, for the spatial compact region $r \leq R+2$, by *Sobolev* embedding $\dot{H}_x^2 \cap \dot{H}_x^1 \subset L_x^\infty$, we see

$$\begin{aligned}\|I_\iota\|_{L_t^1 L_x^2(r\leq R+2)} \lesssim & \|u_{\hat{\iota}}\|_{(L_t^2 \cap L_t^\infty) L_x^\infty(r\leq R+2)}^{p_\iota-1} \|\partial^{\leq 2} u_{\hat{\iota}}\|_{L_t^2 L_x^2(r\leq R+2)} \\ \lesssim & \|\partial^{\leq 2} u_{\hat{\iota}}\|_{LE_{\mathbf{g}}}^{p_\iota} \lesssim \|u_{\hat{\iota}}\|_{U_{\hat{\iota}}^2}^{p_\iota}.\end{aligned}$$

The treatment of II_ι is similar. For the spatial compact region, we obtain

$$\begin{aligned}\|II_\iota\|_{L_t^1 L_x^2(r\leq R+2)} \lesssim & \|u_{\hat{\iota}}\|_{L_t^\infty L_x^\infty(r\leq R+2)}^{p_\iota-2} \|\partial u_{\hat{\iota}}\|_{L_t^2 L_x^4(r\leq R+2)}^2 \\ \lesssim & \|\partial^{\leq 2} u_{\hat{\iota}}\|_{LE_{\mathbf{g}}}^{p_\iota} \lesssim \|u_{\hat{\iota}}\|_{U_{\hat{\iota}}^2}^{p_\iota},\end{aligned}$$

by $\dot{H}_x^2 \cap \dot{H}_x^1 \subset L_x^\infty$ and $H_x^1 \subset L_x^4$. For the remaining region $r \geq R+2$, we deal with two cases separately: $p_\iota \leq 4$ and $p_\iota > 4$. At first, when $p_\iota \leq 4$, with the help of (2.18) and (2.19), we have

$$\begin{aligned}\|II_\iota\|_{L_t^1 \mathcal{L}_r^2 L_\omega^2(r\geq R+2)} \lesssim & \|r^{\frac{1}{p_\iota}} u_{\hat{\iota}}\|_{L_t^{p_\iota} \mathcal{L}_r^\infty L_\omega^\infty(r\geq R+2)}^{p_\iota-2} \|r^{\frac{1}{p_\iota}-\frac{1}{2}} Z u_{\hat{\iota}}\|_{L_t^{p_\iota} \mathcal{L}_r^4 L_\omega^4(r\geq R+2)}^2 \\ \lesssim & \|r^{-\frac{1}{p_\iota}} \bar{\psi}_R Z^{\leq 2} u_{\hat{\iota}}\|_{L_t^{p_\iota} \mathcal{L}_r^{p_\iota} L_\omega^2}^{p_\iota} = \|\bar{\psi}_R Z^{\leq 2} u_{\hat{\iota}}\|_{\widetilde{WS}_{\hat{\iota}}}^{p_\iota} \lesssim \|u_{\hat{\iota}}\|_{U_{\hat{\iota}}^2}^{p_\iota}.\end{aligned}$$

When $p_\iota > 4$, by (2.18), we get

$$\begin{aligned}\|II_\iota\|_{L_t^1 \mathcal{L}_r^2 L_\omega^2(r\geq R+2)} \lesssim & \|r^{\frac{2}{p_\iota(p_\iota-2)}} u_{\hat{\iota}}\|_{L_t^{p_\iota} \mathcal{L}_r^{\frac{2p_\iota(p_\iota-2)}{p_\iota-4}} L_\omega^\infty(r\geq R+2)}^{p_\iota-2} \|r^{-\frac{1}{p_\iota}} Z u_{\hat{\iota}}\|_{L_t^{p_\iota} \mathcal{L}_r^{p_\iota} L_\omega^{p_\iota}(r\geq R+2)}^2 \\ \lesssim & \|r^{-\frac{1}{p_\iota}} \bar{\psi}_R Z^{\leq 2} u_{\hat{\iota}}\|_{L_t^{p_\iota} \mathcal{L}_r^{p_\iota} L_\omega^2}^{p_\iota} = \|\bar{\psi}_R Z^{\leq 2} u_{\hat{\iota}}\|_{\widetilde{WS}_{\hat{\iota}}}^{p_\iota} \lesssim \|u_{\hat{\iota}}\|_{U_{\hat{\iota}}^2}^{p_\iota}.\end{aligned}$$

In summary, we finish the proof of (2.27), as we have $A_{q_{\hat{\iota}}\alpha_{\hat{\iota}}/2}(T) \leq 1$ by definition.

2.3.5 Proof of Theorem 2.2 with $p_1 = 2$

Firstly we consider the case $p_2 > p_1 = 2$. The main difference here is that, s_2 defined in (2.4) is zero but Corollary 2.12 requires $s_\iota > 0$.

To overcome the difficulty, we follow the idea of [23]. The key observation is that, although the homogeneous estimate is hard to prove or even fail to hold, we could still prove the inhomogeneous estimate in (2.21) with $s = 0$. Actually, by (2.20) with $s = 0$, if v has vanishing initial data at $t = T$, we obtain, for $t \in [0, T]$,

$$\|v\|_{W^1 \cap X^1} \lesssim \|\Box_{\tilde{\mathbf{g}}} v\|_{(W^0 \cap X^0)'},$$

where $\tilde{\mathbf{g}}$ is the same as that in the proof of Corollary 2.12. Then, by duality for the wave operator $\Box_{\tilde{\mathbf{g}}}$, we still have the desired inhomogeneous estimate

$$\|u\|_{W^0 \cap X^0} \lesssim \|\Box_{\tilde{\mathbf{g}}} u\|_{(W^1 \cap X^1)'} = \|\Box_{\mathbf{g}} u\|_{(W^1 \cap X^1)'} ,$$

provided that u has vanishing initial data at $t = 0$ and vanishes in a region $\{r < R + 1\}$ with $R \gg 1$.

Therefore, we require $\mathrm{supp}(f_2, g_2) \subset B_R$ to ensure that $\bar{\psi}_R Z^{\leq 2} u_2$ has zero initial data.

Another difference in $p_1 = 2$ is that in this situation, $q = 2$, $s = 1$ no longer satisfy the condition of (2.3), so we modify the definition of q_ι in (2.22) to $q_\iota := (2 - \delta) p_{\hat{\iota}}$ and correspondingly modify (2.24) to

$$\|F\|_{V_\iota^m} = \|\bar{\psi}_R^{p_\iota} r^{\frac{7}{2} - \frac{1}{2-\delta} - s_\iota - \frac{3p_\iota}{o_{\hat{\iota}}}} Z^{\leq m} F\|_{L_t^{2-\delta} \mathcal{L}_r^{\frac{o_{\hat{\iota}}}{p_\iota}} L_\omega^2} + \|Z^{\leq m} F\|_{L_t^1 L_x^2} .$$

Then $q = (2 - \delta)'$, $s = 1$ satisfy the condition of (2.3), and we obtain

$$\frac{7}{2} - \frac{1}{2-\delta} - s_\iota - \frac{3p_\iota}{o_{\hat{\iota}}} = p_\iota \left(\frac{3}{2} + \alpha_{\hat{\iota}} - \frac{1}{q_{\hat{\iota}}} - \frac{3}{o_{\hat{\iota}}} - s_{\hat{\iota}}\right).$$

The rest of the proof is almost the same, and we leave the details for the interested reader.

As for the second case $p_1 = p_2 = 2$, we shall require $\mathrm{supp}(f_j, g_j) \subset B_R$, and then the proof is similar.

Finally, if we additionally assume (H3), we could recover the homogeneous estimate in (2.21) with $s = 0$, as in [30, eq. (6.13)]. With the help of this observation, we could prove the same result, without the compact requirement.

Chapter 3

Long time behavior of *Strauss* type equation with damping and potential

3.1 Main results

In this chapter, we study the long-time existence of solution of the equation (1.7) proposed in section 1.3 under condition (1.8). Firstly, we give the blow up results of the solution, which obtained by the author's joint work [7]. For this purpose, we establish the space of the solution for blow up problem.

Definition 3.1. *Assume that (1.8) is satisfied and* $\operatorname{supp}(f,g) \subset B(0,1)$ *where* $B(x,r)$ *stands for the ball centered at* x *with radials* r*. For* $f \in H^1(\mathbb{R}^n)$*,* $g \in L^2(\mathbb{R}^n)$*, we call* u *a well-posed solution of (1.7) in* $[0,T]\times\mathbb{R}^n$ *if*

$$\begin{aligned} &u \in C^1([0,T];L^2(\mathbb{R}^n)) \cap C([0,T];H^1(\mathbb{R}^n)),\\ &|u|^p \in L^1(0,T;H^{-1}(\mathbb{R}^n)), \quad \operatorname{supp} u \subset B(0,1+t), \end{aligned}$$

when $n \geq 3$*,* u *should satisfy the integral equation*

$$\begin{aligned} &\int_{\mathbb{R}^n} \varepsilon\big(g + Dr^{-1}f\big)\,\Phi(0)\,\mathrm{d}\,x + \int_0^T\int_{\mathbb{R}^n} |u|^p\Phi\,\mathrm{d}\,x\,\mathrm{d}\,t\\ =&\int_0^T\int_{\mathbb{R}^n} \big(\nabla u\cdot\nabla\Phi + Vr^{-2}u\Phi\big)\,\mathrm{d}\,x\,\mathrm{d}\,t - \int_0^T\int_{\mathbb{R}^n}\big(\partial_t u + Dr^{-1}u\big)\,\partial_t\Phi\,\mathrm{d}\,x\,\mathrm{d}\,t \end{aligned}$$

for any $\Phi \in \cap_{k=0}^1 C_0^k((-\infty,T);H^{1-k}(\mathbb{R}^n))$*, when* $n=2$*,* u *should satisfy above equation for* Φ *with additional requirement* $r^{-1-\delta}\Phi \in L^2(\mathbb{R}^2)(\delta>0)$*.*

For convenience, we define $\rho = \frac{A-n}{2}$ in this chapter, then we have the result

Theorem 3.1. *If there exists some u which is the solution in the sense of Definition 3.1 with maximum lifespan $T_{\varepsilon,A,B}(n,p)$ and the initial data (f,g) satisfies*

$$g(x)+Dr^{-1}f(x)\geq 0, \tag{3.1}$$

but not vanish, then there exist some constant C. When $(D+3-A)(A+n+2)<8$, where $p_d(A-D)<p_F((n+A-2)/2)<p_S(n+D)$, we have

$$T_\varepsilon\leq\begin{cases}C\varepsilon^{\frac{(p-1)}{h_F(p;(n+A-2)/2)}}, & p\leq p_d(A-D)\\ C\varepsilon^{\frac{2p(p-1)}{h_S(p;n+D)}}, & p_d(A-D)<p<p_S(n+D);\\ \exp\left(C\varepsilon^{-p(p-1)}\right), & p=p_S(n+D).\end{cases} \tag{3.2}$$

When $(D+3-A)(A+n+2)=8$, where $p_d(A-D)=p_F((n+A-2)/2)=p_S(n+D)$, we have

$$T_\varepsilon\leq\begin{cases}C\varepsilon^{\frac{(p-1)}{h_F(p;(n+A-2)/2)}}, & p<p_F((n+A-2)/2)\,;\\ \exp\left(C\varepsilon^{-(p-1)}\right), & p=p_F((n+A-2)/2)\,.\end{cases} \tag{3.3}$$

When $(D+3-A)(A+n+2)>8$, where $p_d(A-D)>p_F((n+A-2)/2)>p_S(n+D)$, we have

$$T_\varepsilon\leq\begin{cases}C\varepsilon^{\frac{(p-1)}{h_F(p;(n+A-2)/2)}}, & p<p_F((n+A-2)/2)\,;\\ \exp\left(C\varepsilon^{-(p-1)}\right), & p=p_F((n+A-2)/2)\,.\end{cases} \tag{3.4}$$

Remark 3.1. *When $D=0$, we do not need the requirement $\frac{n+\rho}{n+\rho-1}<p$ because no singularity comes from the damping.*

Next, we consider the long time existence for the equation with $D=0$. Here we assume the initial data is spherical symmetric, and rewrite the equation as

$$\begin{cases}\partial_t^2U-\Delta U+Vr^{-2}U=|U|^p, \quad r=|x|,\ x\in\mathbb{R}^n;\\ U(0,x)=\varepsilon U_0(r), \quad U_t(0,x)=\varepsilon U_1(r);\end{cases} \tag{3.5}$$

with $p>1$, $n\geq 2$, $0<\varepsilon\ll 1$ and $V\geq -(n-2)^2/4$ is a constant. When we discuss the existence of solutions, we use the solution space

Definition 3.2. *We call U is a weak solution of (3.5) in $[0,T)\times\mathbb{R}^n$ if U satisfies*

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n}|U|^p\Phi\,\mathrm{d}x\,\mathrm{d}t=&\int_0^T\int_{\mathbb{R}^n}U\left(\partial_t^2-\Delta+\frac{V}{r^2}\right)\Phi\,\mathrm{d}x\,\mathrm{d}t\\ &-\varepsilon\int_{\mathbb{R}^n}(U_1\Phi(0,x)-U_0\partial_t\Phi(0,x))\,\mathrm{d}x,\end{aligned} \tag{3.6}$$

for any $\Phi(t,x)\in r^{\frac{A-n}{2}}C_0^\infty((-\infty,T)\times\mathbb{R}^n)$.

The rationality of such definition will be discussed in Section 3.4. For the simplicity of presentation, we adopt some notations $p_S = p_S(n)$, $p_F = p_F((n+A-2)/2)$, $p_d = p_d(A)$, and

$$p_t := \frac{n+A}{n-1}, \qquad p_{conf} := \frac{n+3}{n-1}, \qquad p_m := \frac{n+1}{n-1}, \qquad p_M := \begin{cases} \frac{n+1}{n-A} & n > A \\ \infty & n \le A \end{cases}.$$

Theorem 3.2. *When $2 \le n$, $2 \le A \le 3$ and $p_m < p < p_M$. Assume the initial data satisfies*

$$\|r^{\frac{n-A+2}{2}} U_0'(r)\|_{L_r^\infty} + \|r^{\frac{n-A}{2}} U_0(r)\|_{L_r^\infty} + \|r^{\frac{n-A+2}{2}} U_1(r)\|_{L_r^\infty} < \infty, \tag{3.7}$$

and supported in $[0,1)$. Then there exists an $\varepsilon_0 > 0$, such that when $\varepsilon < \varepsilon_0$, there exists a constant $c = c(p;n,A)$. So that, there is a weak solution U of (3.5) on $[0,T_*) \times \mathbb{R}^n$ with

$$r^{\frac{A-n}{2}} U \in L^\infty_{loc;t,x}([0,T_*) \times \mathbb{R}^n).$$

Where, $(3-A)(A+n+2) < 8$, where $p_d < p_F < p_S$, we have

$$T_* = \begin{cases} c\varepsilon^{\frac{(p-1)}{h_F(p;(n+A-2)/2)}}, & p < p_d; \\ c\varepsilon^{\frac{(p-1)}{h_F(p;(n+A-2)/2)}} |\ln \varepsilon|^{\frac{1}{h_F(p;(n+A-2)/2)}}, & p = p_d; \\ c\varepsilon^{\frac{2p(p-1)}{h_S(p;n)}}, & p_d < p < p_S; \\ \exp\left(c\varepsilon^{p(1-p)}\right), & p = p_S; \\ \infty, & p > p_S. \end{cases} \tag{3.8}$$

When $(3-A)(A+n+2) = 8$, where $p_d = p_F = p_S$, we have

$$T_* = \begin{cases} c\varepsilon^{\frac{(p-1)}{h_F(p;(n+A-2)/2)}}, & p < p_d; \\ \exp\left(c\varepsilon^{\frac{1-p}{2}}\right), & p = p_d; \\ \infty, & p > p_d. \end{cases} \tag{3.9}$$

When $(3-A)(A+n+2) > 8$, where $p_d > p_F > p_S$, we have

$$T_* = \begin{cases} c\varepsilon^{\frac{(p-1)}{h_F(p;(n+A-2)/2)}}, & p < p_F; \\ \exp(c\varepsilon^{1-p}), & p = p_F; \\ \infty, & p > p_F. \end{cases} \tag{3.10}$$

Theorem 3.3. *When $2 \le n$, $3 \le A$ and $1 < p$. Assume $1 < p \le p_m$, the initial data (U_0, U_1) satisfies*

$$\|r^{\frac{n-1}{2}} U_0(r)\|_{L^p_r} + \|r^{\frac{n+1}{2}} U_1(r)\|_{L^p_r} < \infty, \tag{3.11}$$

and supported in $[0,1)$. Then there exists an $\varepsilon_0 > 0$, such that when $\varepsilon < \varepsilon_0$, (3.5) has a weak solution U in $[0,T_*] \times \mathbb{R}^n$ verifying

$$\|(1+t)^{\frac{(n-1)p-n-1}{2p}} r^{\frac{n+1}{2p}} U\|_{L^\infty_t L^p_r([0,T_*]\times\mathbb{R}^n)} < \infty,$$

with T_* defined in (3.8).

Assume $p_m \le p < p_S$ and the initial data satisfies (3.11). Then there exists an $\varepsilon_0 > 0$, such that when $\varepsilon < \varepsilon_0$, (3.5) has a weak solution in $[0,T_*] \times \mathbb{R}^n$ verifying

$$\|t^{\frac{(n-1)p-n-1}{2p}} r^{\frac{n+1}{2p}} U\|_{L^\infty_t L^p_r([0,T_*]\times\mathbb{R}^n)} < \infty,$$

with T_* defined in (3.8).

Assume $p = p_S$, if (3.11) hold for $p = p_S$ and some $p > p_S$. Then there exists an $\varepsilon_0 > 0$, such that when $\varepsilon < \varepsilon_0$, (3.5) has a weak solution U in $[0,T_*] \times \mathbb{R}^n$ verifying

$$\|t^{\frac{(n-1)p_S-n-1}{2p_S}} r^{\frac{n+1}{2p_S}} U\|_{L^\infty_t L^{p_S}_r([0,T_*]\times\mathbb{R}^n)} < \infty,$$

with T_* defined in (3.8).

Assume $p > p_S$ and the initial data satisfies

$$\|r^{\frac{n-1}{2}} U_0(r)\|_{L^q_r} + \|r^{\frac{n+1}{2}} U_1(r)\|_{L^q_r} < \infty, \qquad q := \frac{2(p-1)}{(n+1)-(n-1)p}.$$

Then there exists an $\varepsilon_0 > 0$, such that when $\varepsilon < \varepsilon_0$, (3.5) has a weak solution U in $\mathbb{R}_+ \times \mathbb{R}^n$ verifying

$$\|r^{\frac{n+1}{2p}} U\|_{L^{pq}_t L^p_r(\mathbb{R}_+\times\mathbb{R}^n)} + \|r^{\frac{n-1}{2}} U\|_{L^\infty_t L^q_r(\mathbb{R}_+\times\mathbb{R}^n)} < \infty.$$

3.2 Special solutions of the conjugate equation deduced by (1.7)

In this section, firstly we will construct a family of special solutions to

$$P^*\Psi = \partial_t^2\Psi - Dr^{-1}\partial_t\Psi - \Delta\Psi + Vr^{-2}\Psi = 0, \tag{3.12}$$

here P^* is the time-space conjugate operator of P. Then, we will discuss the properties of such Ψ, and use them as the test functions in the next section.

The basic idea is to reshape (3.12) and seek solutions only depend on t and r. We consider $\Psi = r^\rho \Phi$, then we have

Lemma 3.4. *Assume Ψ is spherically symmetric. Then $P^*\Psi(t,x)=0$ if and only if $\Phi(t,x)$ satisfy the equation*

$$\partial_t^2\Phi - Dr^{-1}\partial_t\Phi - \partial_r^2\Phi - (n-1+2\rho)r^{-1}\partial_r\Phi = 0, \quad t>0, \quad r>0. \tag{3.13}$$

Proof. We only need to replace $\Psi(t,x)$ by $r^\rho\Phi(t,x)$ and multiply $r^{-\rho}$ on both sides. For the last two terms in $P^*\Phi$, we have

$$\begin{aligned} r^{-\rho}\partial_r(r^\rho\Phi) &= \rho r^{-1}\Phi + \partial_r\Phi, \\ r^{-\rho}\partial_r^2(r^\rho\Phi) &= \rho(\rho-1)r^{-2}\Phi + 2\rho r^{-1}\partial_r\Phi + \partial_r^2\Phi, \\ r^{-\rho}(-\Delta + Vr^{-2})\Psi &= -r^{-\rho}(\partial_r^2 + (n-1)r^{-1}\partial_r - Vr^{-2})(r^\rho\Phi) \\ &= -\left(\partial_r^2 + (n-1+2\rho)r^{-1}\partial_r\right)\Phi, \end{aligned}$$

where we used the relation between ρ, n, V in the last equality. Adding the first two terms, we finish the proof. □

Now, we seek a family of homogeneous solutions of (3.13) with the form

$$\Phi_\beta(t,x;\lambda) = (t+r+\lambda)^{-\beta}\phi\left(\frac{2r}{t+r+\lambda}\right), \quad \beta\in\mathbb{R}, \lambda\geq 0.$$

Lemma 3.5. *Let Φ_β be the one defined above. Then Φ_β satisfies (3.13) in $Q_\lambda := \{(t,x) : t+\lambda > |x|\}$ if and only if*

$$0 = z(1-z)\phi''(z) + (\gamma - (\alpha+\beta+1)z)\phi'(z) - \alpha\beta\phi(z), \quad z\in(0,1) \tag{3.14}$$

with the notation

$$\alpha = \frac{n+D-1+2\rho}{2}, \quad \gamma = n-1+2\rho, \tag{3.15}$$

which will be used through the paper.

Proof. This lemma is verified by some calculations and similar to the proof in [16], but for the reader's convenience, we show the details. It is obvious that we only need to prove the case $\lambda=0$. Setting $z=2r(t+r)^{-1}$, we have

$$\partial_t z = -2r(t+r)^{-2}, \quad \partial_r z = 2(t+r)^{-1} - 2r(t+r)^{-2} = 2t(t+r)^{-2},$$

then we get

$$\begin{aligned} \partial_t\Phi_\beta =& -\beta(t+r)^{-\beta-1}\phi - 2r(t+r)^{-\beta-2}\phi', \\ \partial_t^2\Phi_\beta =& \beta(\beta+1)(t+r)^{-\beta-2}\phi + 4(\beta+1)r\phi'(t+r)^{-\beta-3} + 4r^2(t+r)^{-\beta-4}\phi'', \\ \partial_r\Phi_\beta =& -\beta(t+r)^{-\beta-1}\phi + 2t(t+r)^{-\beta-2}\phi', \\ \partial_r^2\Phi_\beta =& \beta(\beta+1)(t+r)^{-\beta-2}\phi - 4(\beta+1)t\phi'(t+r)^{-\beta-3} + 4t^2(t+r)^{-\beta-4}\phi''. \end{aligned}$$

Then

$$\begin{aligned}0 =&\left(\partial_t^2 - Dr^{-1}\partial_t - \partial_r^2 - (n-1+2\rho)r^{-1}\partial_r\right)\Phi_\beta\\ =&4(r-t)(t+r)^{-\beta-3}\phi'' + (2Dr - 2(n-1+2\rho)t + 4(\beta+1)r)r^{-1}(t+r)^{-\beta-2}\phi'\\ &+ (D\beta + (n-1+2\rho)\beta)r^{-1}(t+r)^{-\beta-1}\phi.\end{aligned}$$

We multiply above equation with $(t+r)^{\beta+2}$ in both sides, using the fact $tr^{-1} = 2z^{-1} - 1$ and we get

$$\begin{aligned}0 =&4\frac{1-tr^{-1}}{1+tr^{-1}}\phi'' + (2D - 2(n-1+2\rho)tr^{-1} + 4(\beta+1))\phi'\\ &+ (D\beta + (n-1+2\rho)\beta)\left(tr^{-1}+1\right)\phi\\ =&4(z-1)\phi'' + \left(2D + 2(n-1+2\rho) - 4(n-1+2\rho)z^{-1} + 4(\beta+1)\right)\phi'\\ &+ (D\beta + (n-1+2\rho)\beta)\left(2z^{-1}\right)\phi\\ =&-4z^{-1}\left(z(1-z)\phi'' + \left(n-1+2\rho - \left(\frac{n+D-1+2\rho}{2} + \beta + 1\right)z\right)\phi'\right.\\ &\left.- \frac{D\beta + (n-1+2\rho)\beta}{2}\phi\right)\\ =&-4z^{-1}(z(1-z)\phi''(z) + (\gamma - (\alpha+\beta+1)z)\phi'(z) - \alpha\beta\phi(z)),\end{aligned}$$

which finishes the proof. □

Noticing (3.13) is a hypergeometric function. Its solution is called hypergeometric function. Then we have

Corollary 3.6. *When $D < n-1+2\rho$, the equation (3.14) has a special solution*

$$\phi(z) = F(\alpha, \beta, \gamma; z),$$

where $F(\alpha,\beta,\gamma;z)$ is the hypergeometric function given by

$$F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 s^{\alpha-1}(1-s)^{\gamma-\alpha-1}(1-zs)^{-\beta}\,\mathrm{d}\,t$$

for $\gamma > \alpha > 0$.

For more properties of hypergeometric differential equations and hypergeometric functions, we refer the readers to, e.g., [21]. Now, we are going to discuss some properties of such $\Psi_\beta = r^\rho\Phi_\beta$.

Lemma 3.7. *For any $\beta \in \mathbb{R}$ and $(t,x) \in Q_\lambda$ with Q_λ defined in Lemma 3.5, we have*

$$\partial_t\Psi_\beta(t,x;\lambda) = -\beta\Psi_{\beta+1}(t,x;\lambda). \tag{3.16}$$

Moreover, for every $(t,x)\in Q_\lambda$, we have

$$\Psi_\beta(t,x;\lambda)\approx r^\rho(\lambda+t)^{-\beta}\cdot\begin{cases}1, & \beta<\gamma-\alpha,\\ 1-\ln\big(1-\frac{r}{\lambda+t}\big), & \beta=\gamma-\alpha,\\ \big(1-\frac{r}{\lambda+t}\big)^{\gamma-\alpha-\beta}, & \beta>\gamma-\alpha.\end{cases}\tag{3.17}$$

Proof. This lemma is similar to [16, Lemma 3.2] but we give the details here. Firstly for (3.16), with $z=\frac{2r}{t+r+\lambda}$ we have

$$\begin{aligned}\partial_t\Psi_\beta(t,x;\lambda)&=r^\rho(t+r+\lambda)^{-\beta-1}(-\beta-z\partial_z)F(\alpha,\beta,\gamma;z)\,,\\ \Psi_{\beta+1}(t,x)&=r^\rho(t+r+\lambda)^{-\beta-1}F(\alpha,\beta+1,\gamma;z)\,.\end{aligned}$$

Using the properties

$$\begin{aligned}\partial_zF(\alpha,\beta,\gamma;z)&=\frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1;z)\,,\\ \frac{\alpha z}{\gamma}F(\alpha+1,\beta+1,\gamma+1;z)&=F(\alpha,\beta+1,\gamma;z)-F(\alpha,\beta,\gamma;z)\end{aligned}$$

of hypergeometric functions, see e.g. [21, Section 9.2], we get (3.16) for any $n\geq 1$. For (3.17), we only need to estimate $F(\alpha,\beta,\gamma;z)$ here. Firstly we notice that $1-z\approx 1-\frac{r}{\lambda+t}$ by its expression. Also, by the expression of F, it is obvious that $F\geq C>0$ for some C when $0<z<1$. When $0<z\leq 1/2$, for $0<s<1$ we have $(1-zs)^{-\beta}\approx 1$ so we get the result. As for $z>1/2$, we have

$$\begin{aligned}F(\alpha,\beta,\gamma;z)&\approx\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-zt)^{-\beta}\,\mathrm{d}\,t\\ &=\int_{1/2}^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-zt)^{-\beta}\,\mathrm{d}\,t+O(1)\\ &\approx\int_{1/2}^1(1-t)^{\gamma-\alpha-1}(1-zt)^{-\beta}\,\mathrm{d}\,t+O(1).\end{aligned}$$

Using the change of variable $t=1-(1-z)s$, we continue the calculation and get

$$\begin{aligned}F(\alpha,\beta,\gamma;z)&\approx(1-z)^{\gamma-\alpha-\beta}\int_0^{(1-z)^{-1}/2}s^{\gamma-\alpha-1}(1+zs)^{-\beta}\,\mathrm{d}\,s+O(1)\\ &\approx(1-z)^{\gamma-\alpha-\beta}\int_1^{(1-z)^{-1}/2}s^{\gamma-\alpha-\beta-1}\,\mathrm{d}\,s+O(1).\end{aligned}$$

By a fundamental calculation of this integral with different β, we finish our proof of (3.17).

□

3.3 Proof of the blow-up phenomenon

In this section we use Ψ_β defined in the last section to consider the upper bound of the lifespan of solutions to (1.7) and its dependence of ε under the condition (1.8).

3.3.1 Preliminaries for showing blowup phenomenon

To begin with, we introduce a cut-off function $\eta(t) \in C_0^\infty((-\infty, 1))$ satisfies

$$\eta(t) = \begin{cases} 1, & 0 \le s \le \frac{1}{2}, \\ \text{non-increasing}, & \frac{1}{2} < s < 1 \end{cases},$$

and $\eta_T(t) := \eta(t/T)$ with arbitrary $T > 0$.

Now, we want to multiply a test function to both sides of equation (1.7) and use integration by parts. But, instead of multiplying some ψ satisfies $P^*\psi = 0$ as many proofs do, we further multiply $\eta_T(t)^{2p'}$ on both sides, then we will get that

Lemma 3.8. *Let u be a solution of (1.7) in the sense of Definition 3.1 with maximum lifespan T_ε, then for any $1 \le T < T_\varepsilon$, we have*

$$\varepsilon C_{f,g}(\lambda, \beta_1) + \int_0^T G_{\beta_1}(t;T)\,\mathrm{d}\,t \lesssim \left(\int_{T/2}^T G_\beta(t;T)\,\mathrm{d}\,t\right)^{\frac{1}{p}} S(T;\beta_1,\beta), \tag{3.18}$$

for any $\beta, \beta_1 \in \mathbb{R}$ and $\lambda > 1$,where

$$\begin{aligned} C_{f,g}(\lambda,\beta) :=& \int_{\mathbb{R}^n} \big(g(x) + Dr^{-1}f(x)\big)\,\Psi_\beta(0,x;\lambda) - f(x)\partial_t\Psi_\beta(0,x;\lambda)\,\mathrm{d}\,x, \\ G_\beta(t;T) :=& \int_{\mathbb{R}^n} |u|^p \eta_T^{2p'} \Psi_\beta\,\mathrm{d}\,x, \\ S(T;\beta_1,\beta) :=& T^{-2}\left(\int_{T/2}^T \int_{B(0,1+t)} \Psi_{\beta_1}^{p'} \Psi_\beta^{-\frac{p'}{p}}\,\mathrm{d}\,x\,\mathrm{d}\,t\right)^{\frac{1}{p'}} \\ &+ T^{-1}\left(\int_{T/2}^T \int_{B(0,1+t)} r^{-p'} \Psi_{\beta_1}^{p'} \Psi_\beta^{-\frac{p'}{p}}\,\mathrm{d}\,x\,\mathrm{d}\,t\right)^{\frac{1}{p'}} \\ &+ T^{-1}\left(\int_{T/2}^T \int_{B(0,1+t)} |\partial_t\Psi_{\beta_1}|^{p'} \Psi_\beta^{-\frac{p'}{p}}\,\mathrm{d}\,x\,\mathrm{d}\,t\right)^{\frac{1}{p'}}. \end{aligned}$$

Proof. To begin with, for any given $\beta_1 \in \mathbb{R}$, we want to choose $\eta_T^{2p'}\Psi_{\beta_1}$ to be the test function. Noticing that $\eta_T^{2p'} \in C_0^\infty((-\infty, T))$ and $\operatorname{supp}_x u \subset B(0, 1+t)$, we only need to show that Ψ_{β_1} belongs to $C^k(\mathbb{R}; H^{1-k}(B(0,1+t))$ for $k = 0, 1$ for $n \ge 3$.

By (3.17) with $\lambda > 1$, we get that

$$\Psi_{\beta_1} \lesssim C_t r^{\rho},$$

in $B(0, 1+t)$ with some constant C_t depending on t. Since $2\rho + n - 1 > 1$, its easy to see that $\|\Psi_{\beta_1}\|_{L^2_x(B(0,t+1))}$ is bounded by some constant depending on t. The same estimate also holds for $\|\partial_t \Psi_{\beta_1}\|_{L^2_x(B(0,t+1))}$ because of (3.16). As for the space derivatives, since Ψ_{β_1} is spherically symmetric, we only need to consider $\partial_r \Psi_{\beta_1}$. However, the worst singularity will be $r^{\rho-1}$ with $2(\rho-1) + n - 1 > -1$. So we also have that $\|\partial_r \Psi_{\beta_1}\|_{L^2_x(B(0,t+1))}$ is bounded by some constant depending on t. Hence, we have shown that Ψ_{β_1} belongs to the function space for test functions. When $n = 2$, the procedure is similar so we omit it.

Using Definition 3.1 and making the integration by parts again, we find that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \varepsilon(g + Dr^{-1}f)\eta_T^{2p'}(0)\Psi_{\beta_1}(0)\,\mathrm{d}\,x + \int_0^T \int_{\mathbb{R}^n} |u|^p \eta_T^{2p'} \Psi_{\beta_1}\,\mathrm{d}\,x\,\mathrm{d}\,t \\
=& \int_0^T \int_{\mathbb{R}^n} \Big(\nabla u \cdot \nabla\Big(\eta_T^{2p'}\Phi_{\beta_1}\Big) + V r^{-2} u\, \eta_T^{2p'}\Phi_{\beta_1}\Big)\,\mathrm{d}\,x\,\mathrm{d}\,t \\
&- \int_0^T \int_{\mathbb{R}^n} \big(\partial_t u + Dr^{-1}u\big)\,\partial_t\Big(\eta_T^{2p'}\Phi_{\beta_1}\Big)\,\mathrm{d}\,x\,\mathrm{d}\,t \\
=& \int_0^T \int_{\mathbb{R}^n} u P^*\Big(\eta_T^{2p'}\Psi_{\beta_1}\Big)\,\mathrm{d}\,x\,\mathrm{d}\,t - \int_{\mathbb{R}^n} u\,\partial_t\Big(\eta_T^{2p'}\Psi_{\beta_1}\Big)\,\mathrm{d}\,x\Big|_{t=0}^{T}.
\end{aligned}$$

Noticing that $\eta_T(T) = \partial_t \eta_T(T) = \partial_t \eta_T(0) = 0$ and $\eta_T(0) = 1$, we can simplify the above identity as

$$\varepsilon C_{f,g}(\lambda, \beta_1) + \int_0^T G_{\beta_1}(t;T)\,\mathrm{d}\,t = \int_0^T \int_{\mathbb{R}^n} u P^*\Big(\eta_T^{2p'}\Psi_{\beta_1}\Big)\,\mathrm{d}\,x\,\mathrm{d}\,t.$$

Since $P^*\Psi_{\beta_1} = 0$ and η_T only depends on t, we have

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^n} u P^*\Big(\eta_T^{2p'}\Psi_{\beta_1}\Big)\,\mathrm{d}\,x\,\mathrm{d}\,t =& \int_0^T \int_{\mathbb{R}^n} u\Psi_{\beta_1}\partial_t^2\Big(\eta_T^{2p'}\Big)\,\mathrm{d}\,x\,\mathrm{d}\,t \\
&- \int_0^T \int_{\mathbb{R}^n} u\Psi_{\beta_1} Dr^{-1}\partial_t\Big(\eta_T^{2p'}\Big)\,\mathrm{d}\,x\,\mathrm{d}\,t \\
&+ \int_0^T \int_{\mathbb{R}^n} 2u\partial_t\Psi_{\beta_1}\partial_t\Big(\eta_T^{2p'}\Big)\,\mathrm{d}\,x\,\mathrm{d}\,t.
\end{aligned}$$

For the first part, noticing that $2p' - 2 = \frac{2p'}{p}$, supp $u(t,\cdot) \subset B(0, t+1)$ and for every $k \geq 1$, $\eta_T^{(k)}(t)\Big|_{0\leq t\leq T/2} = 0$ and $\Big|\eta_T^{(k)}(t)\Big|_{0\leq t\leq T} \lesssim C_k$ with some constant C_k, we use the chain rule

and get that

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbb{R}^n} u\Psi_{\beta_1}\partial_t^2\left(\eta_T^{2p'}\right)\mathrm{d}\,x\,\mathrm{d}\,t\right|\\
\lesssim& T^{-2}\int_{T/2}^T\int_{B(0,t+1)}\left|u\Psi_{\beta_1}\eta_T''\eta_T^{2p'-1}\right|+\left|u\Psi_{\beta_1}\eta'^2\eta_T^{2p'-2}\right|\mathrm{d}\,x\,\mathrm{d}\,t\\
\lesssim& T^{-2}\left(\int_{T/2}^T\int_{B(0,t+1)}|u|^p\Psi_\beta\eta_T^{2p'}\,\mathrm{d}\,x\,\mathrm{d}\,t\right)^{\frac{1}{p}}\left(\int_{T/2}^T\int_{B(0,t+1)}\Psi_{\beta_1}^{p'}\Psi_\beta^{-\frac{p'}{p}}\,\mathrm{d}\,x\,\mathrm{d}\,t\right)^{\frac{1}{p'}}.
\end{aligned}$$

Estimates for the rest two parts go similarly, so we omit them and finish the proof. □

To use the inequality above, we need to ask $C_{f,g}(\lambda,\beta)>0$ for some λ, for this purpose we give the following lemma.

Lemma 3.9. *For any given β, if $f\in H^1(\mathbb{R}^n)$, $g\in L^2(\mathbb{R}^n)$ with compact support and*

$$g(x)+Dr^{-1}f(x)\geq 0,$$

but not vanishes, then there exists a $\lambda_\beta>1$ such that for any $\lambda>\lambda_\beta$, $C_{f,g}(\lambda,\beta)>0$.

Proof. Based on Lemma 3.7, when $\lambda>1$, $t=0$ and $r<1$, we have

$$\Psi_\beta(0,x;\lambda)\approx r^\rho\lambda^{-\beta},\qquad \partial_t\Psi_\beta(0,x;\lambda)\lesssim r^\rho\lambda^{-\beta-1}.$$

Noticing that $r^{\rho-1}\in L^2_{loc}(\mathbb{R}^n)$, so all integrates appeared in $C_{f,g}(\lambda,\beta)$ is finite. Then, this lemma is just an obvious conclusion. □

Next, we want to use Lemma 3.8 to estimate $\int G_\beta(t;T)\,\mathrm{d}\,t$ for some β. So we need to estimate the $S(T;\beta_1,\beta)$.

Here we mention that when we want to estimate the relation between $\int_{T/2}^T G_\beta(t;T)\,\mathrm{d}\,t$ and the initial data, we should let $S(T;\beta_1,\beta)$ be small. And when we want to estimate the relation between $\int_{T/2}^T G_\beta(t;T)\,\mathrm{d}\,t$ and $\int_0^T G_\beta(t;T)\,\mathrm{d}\,t$, we should choose $\beta_1=\beta$. Now we give the following lemma.

Lemma 3.10. *Assume $T\geq 1$, $\lambda>1$, $p>\frac{n+\rho}{n+\rho-1}$ and $\beta<\gamma-\alpha$. When $\beta_1=0$, we have that*

$$S(T;0,\beta)\lesssim T^{\frac{\rho+n+1}{p'}-2+\frac{\beta}{p}}.$$

When $\beta_1>\gamma-\alpha-\frac{1}{p}$, we have that

$$S(T;\beta_1,\beta)\lesssim T^{\frac{\rho+n}{p'}-\frac{n-A+2\rho+1}{2}+\frac{\beta}{p}}.$$

When $\beta_1 = \beta \neq 0$, we have that

$$S(T;\beta,\beta) \lesssim T^{\frac{\rho+n}{p'}-\beta-2+\frac{\beta}{p}} S_0(T)^{\frac{1}{p'}},$$
$$S_0(T) := \begin{cases} T, & \beta < \gamma-\alpha-\frac{1}{p}, \\ T\ln(T+1), & \beta = \gamma-\alpha-\frac{1}{p}, \end{cases}$$

Proof. We begin with the $\beta_1 = 0$ situation, for the first term in $S(T;0,\beta)$, when $\beta < \gamma-\alpha$ we calculate

$$\begin{aligned}\int_{B(0,t+1)} \Psi_0^{p'} \Psi_\beta^{-\frac{p'}{p}} \,\mathrm{d}\,x \lesssim& \int_{B(0,t+1)} r^\rho (t+\lambda)^{(-\beta)\left(-\frac{p'}{p}\right)} \,\mathrm{d}\,x \\ \lesssim& (t+1)^{\beta\frac{p'}{p}} \int_0^{t+1} r^{\rho+n-1} \,\mathrm{d}\,r \\ \approx& (t+1)^{\rho+n+\beta\frac{p'}{p}}.\end{aligned}$$

Then we have that

$$\begin{aligned}T^{-2}\left(\int_{T/2}^T \int_{B(0,t+1)} \Psi_0^{p'} \Psi_\beta^{-\frac{p'}{p}} \,\mathrm{d}\,x\,\mathrm{d}\,t\right)^{\frac{1}{p'}} \lesssim& T^{-2}\left(\int_{T/2}^T (t+1)^{\rho+n+\beta\frac{p'}{p}} \,\mathrm{d}\,t\right)^{\frac{1}{p'}} \\ \lesssim& T^{\frac{\rho+n+1}{p'}-2+\frac{\beta}{p}}.\end{aligned}$$

Although the estimate for the second term goes similarly, we mention that we need to require $p > \frac{n+\rho}{n+\rho-1}$ such that there is no singularity in the r-integral. As for the third term, it vanishes since $\partial_t \Psi_0 = 0$ by (3.16). Now we finish the proof for this situation $\beta_1 = 0$.

When we consider $\beta_1 > \gamma-\alpha-\frac{1}{p}$, the estimate for the first two terms is almost the same as before. As for the third term, we use Lemma 3.7 and calculate

$$\begin{aligned}&\int_{B(0,t+1)} |\partial_t \Psi_{\beta_1}|^{p'} \Psi_\beta^{-\frac{p'}{p}} \,\mathrm{d}\,x \\ \lesssim& \int_{B(0,t+1)} r^\rho (t+\lambda)^{(-\beta_1-1)p'-(-\beta)\frac{p'}{p}} \left(1-\frac{r}{\lambda+t}\right)^{(\gamma-\alpha-\beta_1-1)p'} \,\mathrm{d}\,x \\ \lesssim& (t+1)^{\rho+n-1+(-\beta_1-1)p'+\beta\frac{p'}{p}} \int_0^{t+1} \left(1-\frac{r}{\lambda+t}\right)^{(\gamma-\alpha-\beta_1-1)p'} \,\mathrm{d}\,r.\end{aligned}$$

Based on the assumption on β_1, we have

$$\int_0^{t+1} \left(1-\frac{r}{\lambda+t}\right)^{(\gamma-\alpha-\beta_1-1)p'} \,\mathrm{d}\,r \lesssim (t+1)^{-(\gamma-\alpha-\beta_1-1)p'}.$$

Here we mention that $\alpha = \frac{n+D-1+2\rho}{2}$, $\gamma = n-1+2\rho$, then we get the result by a simple calculation.

Next, we consider $\beta_1 = \beta \neq 0$, similarly to the above, we only consider the third term. Here when $\beta < \gamma - \alpha - 1$, we have

$$\int_0^{t+1} 1 \,\mathrm{d}\, r \lesssim (t+1) \equiv S_0(t+1).$$

When $\beta = \gamma - \alpha - 1$, we have

$$\int_0^{t+1} \left(1 - \ln\left(1 - \frac{r}{t+\lambda}\right)\right)^{p'} \mathrm{d}\, r \lesssim (t+1) \equiv S_0(t+1).$$

When $\gamma - \alpha - 1 < \beta < \gamma - \alpha - \frac{1}{p}$, we have

$$\int_0^{t+1} \left(1 - \frac{r}{\lambda + t}\right)^{(\gamma-\alpha-\beta-1)p'} \mathrm{d}\, r \lesssim (t+1) \equiv S_0(t+1).$$

When $\beta = \gamma - \alpha - \frac{1}{p}$, we have

$$\int_0^{t+1} \left(1 - \frac{r}{\lambda + t}\right)^{(\gamma-\alpha-\beta-1)p'} \mathrm{d}\, r \lesssim (t+1)\ln(t+2) \equiv S_0(t+1).$$

Then we get the results same as before. □

To use (3.8), we still need to show the relation between $\int_{T/2}^T G_\beta(t;T)\,\mathrm{d}\, t$ and $\int_0^T G_\beta(t;T)\,\mathrm{d}\, t$. Since $G_\beta(t,T) \geq 0$, we obviously have the former less than or equal to the latter. Also, we can transform the relationship between the two into the relationship between an auxiliary function and its differential, by introducing the auxiliary function.

Lemma 3.11. *or any given $\beta \in \mathbb{R}$, if we define*

$$H_\beta(t) := \int_0^T \int_{t/2}^t G_\beta(s;t)\,\mathrm{d}\, s \frac{\mathrm{d}\, t}{t},$$

then

$$T H'_\beta(T) = \int_{T/2}^T G_\beta(t;T)\,\mathrm{d}\, t, \qquad H_\beta(t) \leq \int_0^T G_\beta(t;T)\,\mathrm{d}\, t.$$

Proof. The first equality is obvious, so we only focus on the inequality. Here we exchange the order of integration, then

$$\begin{aligned} H_\beta(T) &= \int_0^T \int_s^{\min\{2s,T\}} \frac{1}{t} G_\beta(s;t)\,\mathrm{d}\, t\,\mathrm{d}\, s \\ &\leq \int_0^T \int_s^{2s} \frac{1}{t} G_\beta(s;T)\,\mathrm{d}\, t\,\mathrm{d}\, s \leq \int_0^T G_\beta(s;T)\,\mathrm{d}\, s. \end{aligned}$$

This finishes the proof. □

3.3.2 Proof of Theorem 3.1

Based on (3.18) and Lemma 3.10, we find that the different choices of β_1 lead to different kind of lower bounds of $\int_{T/2}^T G_\beta(t;T)\,\mathrm{d}\,t$. To start with, we fix $\beta < \gamma - \alpha - \frac{1}{p}$.

Firstly we choose $\beta_1 > \gamma - \alpha - \frac{1}{p}$ and show the blow-up result corresponding to the *Strauss* exponent. Using Lemma 3.8, Lemma 3.9 and Lemma 3.10, under their condition we have

$$\varepsilon \lesssim T^{\frac{\rho+n}{p'}-\frac{n-A+2\rho+1}{2}+\frac{\beta}{p}}\left(\int_{T/2}^T G_\beta(t;T)\,\mathrm{d}\,t\right)^{\frac{1}{p}},$$

$$\int_0^T G_\beta(t;T)\,\mathrm{d}\,t \lesssim T^{\frac{\rho+n+1}{p'}-\beta-2+\frac{\beta}{p}}\left(\int_{T/2}^T G_\beta(t;T)\,\mathrm{d}\,t\right)^{\frac{1}{p}},$$

respectively. Mixing them and we get that

$$\varepsilon^p T^{n+\rho-\frac{n+A-1}{2}p-\beta} \lesssim \int_{T/2}^T G_\beta(t;T)\,\mathrm{d}\,t \lesssim T^{n+\rho-1-\frac{2}{p-1}-\beta}, \tag{3.19}$$

which means

$$T \lesssim \varepsilon^{\frac{2p(p-1)}{h_S(n+D,p)}}$$

for any $p < p_S(n+A)$. Because this relation must be satisfied for any $1 \le T < T_\varepsilon$, we finish the proof of $p_d < p < p_S$ case in (3.2).

Then, we choose $\beta_1 = 0$ and show the blow-up result corresponding to the *Fujita* exponent. Using these lemma again, under their condition we have

$$\varepsilon \lesssim T^{\frac{\rho+n+1}{p'}-2+\frac{\beta}{p}}\left(\int_{T/2}^T G_\beta(t;T)\,\mathrm{d}\,t\right)^{\frac{1}{p}},$$

$$\int_0^T G_\beta(t;T)\,\mathrm{d}\,t \lesssim T^{\frac{\rho+n+1}{p'}-\beta-2+\frac{\beta}{p}}\left(\int_{T/2}^T G_\beta(t;T)\,\mathrm{d}\,t\right)^{\frac{1}{p}}$$

respectively. Mixing them and we get that

$$\varepsilon^p T^{(n+\rho-1)(1-p)+2-\beta} \lesssim \int_{T/2}^T G_\beta(t;T)\,\mathrm{d}\,t \lesssim T^{n+\rho-1-\frac{2}{p-1}-\beta},$$

which means

$$T \lesssim \varepsilon^{\frac{p-1}{h_F(n-1+\rho,p)}}$$

for any $p < p_F(n-1+\rho)$. Similarly to the above we finish the proof of $p < p_F$, $p \le p_d$ case of (3.2), (3.3), (3.4).

Next, we are going to consider the 'critical' situation. As before, we begin with the estimate related to *Strauss* exponent. When $p = p_S(n+A)$, we have

$$n+\rho-\frac{n+A-1}{2}p = n+\rho-1-\frac{2}{p-1} = \gamma-\alpha-\frac{1}{p}.$$

Then from (3.19) we can only get that

$$\varepsilon^p T^{\gamma-\alpha-\frac{1}{p}-\beta} \lesssim \int_{T/2}^{T} G_\beta(t;T)\,\mathrm{d}\,t \lesssim T^{\gamma-\alpha-\frac{1}{p}-\beta}. \tag{3.20}$$

This suggests us to choose $\beta = \gamma - \alpha - \frac{1}{p}$ and treat the integral of $G_\beta(t,T)$ more precisely. Using Lemma 3.10 again and under above condition we get

$$\begin{aligned}\varepsilon &\lesssim \left(\int_{T/2}^{T} G_\beta(t;T)\,\mathrm{d}\,t\right)^{\frac{1}{p}},\\ \int_0^T G_\beta(t;T)\,\mathrm{d}\,t &\lesssim \left(\int_{T/2}^{T} G_\beta(t;T)\,\mathrm{d}\,t\right)^{\frac{1}{p}} \ln(T+1)^{\frac{1}{p'}}.\end{aligned}$$

Here we use H_β defined in Lemma 3.11 and we get that for any $T \geq 2$

$$\varepsilon^p \lesssim T H'_\beta(T), \qquad H_\beta(T)^p \ln(T)^{1-p} \lesssim T H'_\beta(T).$$

We introduce $s = \ln(T)$ and $I(s) = H(T)$ where $H'(T) = T^{-1} I'(s)$, then above ODI system equivalents to

$$\varepsilon^p \lesssim I'(s), \qquad I(s)^p s^{1-p} \lesssim I'(s).$$

This is a classical ODI system which must blow-up in finite time, using the scaling method we know that $I(s)$ must blow up before $C\varepsilon^{-p(p-1)}$ with some constant C, by the relation between s and T we finish the proof of $p = p_S$ case of (3.2).

For the $p = p_F(n-1+\rho)$ situation, similarly we have

$$(n+\rho-1)(1-p)+2 = n+\rho-1-\frac{2}{p-1} = 0.$$

Since $\gamma - \alpha > 0$, we choose $\beta = 0$ and get

$$\begin{aligned}\varepsilon &\lesssim \left(\int_{T/2}^{T} G_\beta(t;T)\,\mathrm{d}\,t\right)^{\frac{1}{p}},\\ \int_0^T G_\beta(t;T)\,\mathrm{d}\,t &\lesssim \left(\int_{T/2}^{T} G_\beta(t;T)\,\mathrm{d}\,t\right)^{\frac{1}{p}}.\end{aligned}$$

Similar as before, with $s = \ln(T)$ and $I(s) = H(T)$ we get

$$\varepsilon^p \lesssim I'(s), \qquad I(s)^p \lesssim I'(s).$$

Using scaling method again we know that $I(s)$ must blow up before $C\varepsilon^{-(p-1)}$ with some constant C, by the relation between s and T we finish the proof for $p = p_F$ case of (3.3) and (3.4).

3.4 The definition and properties of the solution of equation (3.5)

In this section, we will transform (3.5) to an equivalent equation (3.21), and explain the rationality of Definition 3.2. Then we will give the homogeneous solution of this equation, and analyze the properties of the solution.

3.4.1 The definition of weak solution

Here we notice that the initial data of (3.5) are spherically symmetric. This suggests the solution U is also spherically symmetric. If we set $u(t,r) := r^{\frac{n-A}{2}}U(t,x)$, a formally calculation shows u satisfied the equation

$$\begin{cases} \partial_t^2 u - \Delta_A u = r^{\frac{(A-n)p+n-A}{2}}|u|^p, \quad r \in \mathbb{R}_+, \\ u(0,x) = \varepsilon r^{\frac{n-A}{2}} U_0(r), \quad u_t(0,x) = \varepsilon r^{\frac{n-A}{2}} U_1(r), \end{cases} \tag{3.21}$$

where $\Delta_A := \partial_r^2 + (A-1)r^{-1}\partial_r$. We pause here and consider its linear form equation

$$\begin{cases} \partial_t^2 u - \Delta_A u = F(t,r), \quad r \in \mathbb{R}_+ \\ u(0,r) = f(r), \quad u_t(0,r) = g(r), \end{cases} \tag{3.22}$$

with f, g, F good enough. When $A \in \mathbb{Z}^+$, (3.22) can be considered as an A-dimensional spherically symmetric wave equation, where $u(t,r)$ is a classical solution in $[0,T]$ if $u(t,r)$ satisfies (3.22) and $u(t,|x|) \in C^2([0,T]\times\mathbb{R}^A)$. Thus, for general situation, we should define u is a classical solution of (3.22) in $[0,T]$ if $u \in C^2([0,T]\times\mathbb{R}_+)$, $\partial_r u(t,0) = 0$ and u satisfies (3.22) point wise.

Here we give a quick proof to show that such classical solution is unique. When $f, g, F = 0$, multiplying $r^{A-1}u_t$ to both sides of (3.22) and integrate them in $\Omega := \{(t,r) : 0 < t < T, 0 < r < R + T - t\}$, we see

$$0 = \frac{1}{2}\int_0^R r^{A-1}\left(u_t^2 + u_r^2\right)\mathrm{d}\, r\Big|_{t=T} + \frac{1}{2\sqrt{2}} \int\limits_{\substack{t+r=T+R \\ 0<t<T}} r^{A-1}(u_t - u_r)^2 \,\mathrm{d}\,\sigma_{t,r},$$

which gives the uniqueness. After the discussion of the classical solution, we naturally define u is the weak solution of (3.22) if u satisfies

$$\begin{aligned} \int_0^T \int_0^\infty F\phi r^{A-1}\,\mathrm{d}\, r\,\mathrm{d}\, t = &\int_0^T \int_0^\infty u\left(\partial_t^2 - \Delta_A\right)\phi r^{A-1}\,\mathrm{d}\, r\,\mathrm{d}\, t \\ &- \int_0^\infty (g\phi(0,r) - f\partial_t\phi(0,r))r^{A-1}\,\mathrm{d}\, r, \end{aligned} \tag{3.23}$$

for any $\phi(t,r) \in C_0^\infty((-\infty,T)\times\mathbb{R})$ with $\partial_r^{1+2k}\phi(t,0) = 0$ where $k \in \mathbb{N}$. Also, set $u = r^{\frac{n-A}{2}}U$, $\phi = r^{\frac{n-A}{2}}\Phi$, $f = \varepsilon r^{\frac{n-A}{2}}U_0$, $g = \varepsilon r^{\frac{n-A}{2}}U_1$ and $F = r^{\frac{(A-n)p+n-A}{2}}|u|^p$. It is obvious that u satisfies (3.23) is equivalent to that U satisfies (3.6) with $\Phi(t,x) \in r^{\frac{A-n}{2}}C_0^\infty((-\infty,T)\times\mathbb{R}^n)$, which lead to the definition of weak solution to (3.5).

3.4.2 The formula of classical solution

In this section we are going to give the formula of solution to (3.22). We will show that

Lemma 3.12. *We denote u_g, u_f and u_F be the classical solution of (3.22) with only $g \neq 0$, $f \neq 0$ and $F \neq 0$ separately. Then, the classical solution of (3.22) is $u = u_f + u_g + u_F$ with*

$$\begin{aligned}
u_g =& r^{\frac{1-A}{2}}\int_0^{t+r}\rho^{\frac{A-1}{2}}g(\rho)I_A(\mu)\,\mathrm{d}\,\rho, \quad \mu = \frac{r^2+\rho^2-t^2}{2r\rho},\\
u_f =& r^{\frac{1-A}{2}}\partial_t\int_0^{t+r}\rho^{\frac{A-1}{2}}g(\rho)I_A(\mu)\,\mathrm{d}\,\rho, \quad \mu = \frac{r^2+\rho^2-t^2}{2r\rho},\\
u_F =& r^{\frac{1-A}{2}}\int_0^{t}\int_0^{t-s+r}\rho^{\frac{A-1}{2}}F(s,\rho)I_A(\mu)\,\mathrm{d}\,\rho\,\mathrm{d}\,s, \quad \mu = \frac{r^2+\rho^2-(t-s)^2}{2r\rho},\\
I_A(\mu) :=& \frac{2^{\frac{1-A}{2}}}{\Gamma\left(\frac{A-1}{2}\right)}\int_{-1}^{1}\mathscr{X}_+^{\frac{1-A}{2}}(\lambda-\mu)\sqrt{1-\lambda^2}^{A-3}\,\mathrm{d}\,\lambda.
\end{aligned}$$

Remark 3.2. *Considering $\mu = \frac{r^2+\rho^2-t^2}{2r\rho}$. When $r \geq t$, we have*

$$\mu|_{0<\rho<r-t} > \mu|_{\rho=r-t} = 1 > \mu|_{r-t<\rho<t+r} < \mu|_{\rho=t+r} = 1 < \mu|_{t+r<\rho},$$

and when $r \leq t$, we have

$$\mu|_{0<\rho<t-r} < \mu|_{\rho=t-r} = -1 < \mu|_{t-r<\rho<t+r} < \mu|_{\rho=t+r} = 1 < \mu|_{t+r<\rho}.$$

To show Lemma 3.12, we need to explore some properties of $I_A(\mu)$.

Lemma 3.13. *For $A > 1$ and $I_A(\mu)$ defined in Lemma 3.12, we have*

$$I_A(1-) = \frac{1}{2}, \qquad I_A(\mu) = 0 \ for\ \mu > 1. \tag{3.24}$$

Moreover, for $A \notin 1+2\mathbb{Z}^+$ with some positive constants C_0 and C_1 depend on A, we have

$$|\partial_\mu^m I_A(\mu)| \lesssim (1-\mu)^{\frac{1-A}{2}-m}, \qquad \mu \leq -2, m = 0,1; \tag{3.25}$$

$$I_A(\mu) = C_0 \ln\frac{3}{|1+\mu|} + O(1), \qquad -2 < \mu < 1; \tag{3.26}$$

$$\partial_\mu I_A(\mu) = C_1(1+\mu)^{-1} + O\left(\ln\frac{3}{|1+\mu|}\right), \qquad -2 < \mu < 1. \tag{3.27}$$

On the other hand, for $A \in 1+2\mathbb{Z}^+$, we have

$$I_A(\mu) = 0, \qquad \mu < -1; \tag{3.28}$$

$$|\partial_\mu^m I_A(\mu)| \lesssim 1, \qquad -1 < \mu < 1,\ m = 0, 1. \tag{3.29}$$

3.4.3 Proof of Lemma 3.12

Here we only show the proof of $u = u_g$ with $f = F = 0$, the other formulas can be demonstrated by a direct calculation and *Duhamel*'s principle.

Before the proof, we more give a classical expressions of u_g constructed in Lemma 3.12. Substituting $I_A(\mu)$ into u_g, noticing $\mathcal{X}_+^\alpha$ is an α-degree homogeneous distribution, we find

$$u_g = \frac{1}{\Gamma\left(\frac{A-1}{2}\right)} \int_0^\infty \int_{-1}^1 g(\rho) \mathcal{X}_+^{\frac{1-A}{2}} \left(t^2 - r^2 - \rho^2 + 2r\rho\lambda\right) \rho^{A-1} \sqrt{1-\lambda^2}^{A-3} \,\mathrm{d}\,\lambda \,\mathrm{d}\,\rho.$$

When $A = 1 + 2k + 2\theta$ with $k \in \mathbb{N}$ and $\theta \in (0, 1)$. Changing the variable

$$\rho = \sqrt{r^2 + \tilde{\rho}^2 - 2r\tilde{\rho}\tilde{\lambda}} \qquad \lambda = \frac{r - \tilde{\rho}\tilde{\lambda}}{\sqrt{r^2 + \tilde{\rho}^2 - 2r\tilde{\rho}\tilde{\lambda}}},$$

we calculate

$$\begin{aligned} u_g =& \frac{1}{\Gamma(k+\theta)} \int_0^\infty \int_{-1}^1 g\left(\sqrt{r^2 + \tilde{\rho}^2 - 2r\tilde{\rho}\tilde{\lambda}}\right) \\ & \qquad \mathcal{X}_+^{-k-\theta}\left(t^2 - \tilde{\rho}^2\right) \tilde{\rho}^{A-1} \sqrt{1-\tilde{\lambda}^2}^{A-3} \,\mathrm{d}\,\tilde{\lambda} \,\mathrm{d}\,\tilde{\rho} \\ =& \frac{1}{\Gamma(k+\theta)} \left(\frac{\partial_t}{2t}\right)^k \int_0^\infty \int_{-1}^1 g\left(\sqrt{r^2 + \rho^2 - 2r\rho\lambda}\right) \\ & \qquad \mathcal{X}_+^{-\theta}\left(t^2 - \rho^2\right) \rho^{A-1} \sqrt{1-\lambda^2}^{A-3} \,\mathrm{d}\,\lambda \,\mathrm{d}\,\rho. \end{aligned}$$

Set $\rho = t\sigma$, considering the support of $\mathcal{X}_+^{-\theta}$ we finally reach

$$u_g = \frac{1}{\Gamma(k+\theta)\,\Gamma(1-\theta)} \left(\frac{\partial_t}{2t}\right)^k t^{1+2k} \int_0^1 \int_{-1}^1 \frac{g\left(\sqrt{r^2 + t^2\sigma^2 - 2rt\sigma\lambda}\right)}{(1-\sigma^2)^\theta} \sigma^{A-1} \sqrt{1-\lambda^2}^{A-3} \,\mathrm{d}\,\lambda \,\mathrm{d}\,\sigma.$$

Now we begin the proof. Here we only need to deal with the case $A \notin \mathbb{Z}^+$. Firstly, using the expression we just obtained, we can easily find $u \in C^2(\mathbb{R}^2_+)$ while $g \in C_0^\infty((0,\infty))$. We can also calculate that

$$\begin{aligned} \partial_r u =& \frac{1}{\Gamma(k+\theta)\,\Gamma(1-\theta)} \left(\frac{\partial_t}{2t}\right)^k t^{1+2k} \int_0^1 \int_{-1}^1 \frac{r - t\sigma\lambda}{\sqrt{r^2 + t^2\sigma^2 - 2rt\sigma\lambda}} \frac{g'\left(\sqrt{r^2 + t^2\sigma^2 - 2rt\sigma\lambda}\right)}{(1-\sigma^2)^\theta} \\ & \qquad \sigma^{A-1} \sqrt{1-\lambda^2}^{A-3} \,\mathrm{d}\,\lambda \,\mathrm{d}\,\sigma. \end{aligned}$$

Let $r = 0$, since the integrand is an odd function of λ, such u satisfies the boundary requirement.

To check the initial conditions we temporarily use the original expression. Using Lemma 3.13 we know $I_A(\mu) = 0$ when $\mu > 1$, which happens when $\rho < r - t < 0$. Then for any $r > t > 0$ we have

$$\begin{aligned}
u(t,r) =& r^{\frac{1-A}{2}} \int_{r-t}^{t+r} \rho^{\frac{A-1}{2}} g(\rho) I_A(\mu) \,\mathrm{d}\,\rho \\
u_t(t,r) =& r^{\frac{1-A}{2}} \left((t+r)^{\frac{A-1}{2}} g(t+r) + (r-t)^{\frac{A-1}{2}} g(r-t) \right) I_A(1-) \\
&+ r^{\frac{1-A}{2}} \int_{r-t}^{t+r} \rho^{\frac{A-1}{2}} g(\rho) \frac{-t}{r\rho} I_A'(\mu) \,\mathrm{d}\,\rho.
\end{aligned}$$

Set $t = 0$ and using Lemma 3.13 again we find $u(0,r) = 0$ and $u_t(0,r) = g(r)$.

Finally, we check u satisfies the inner equation. By a calculation trick

$$\partial_t^2 \left(\frac{\partial_t}{t} \right)^k t^{1+2k} = \left(\frac{\partial_t}{t} \right)^{k+1} t^{2k+2} \partial_t,$$

we calculate that

$$\begin{aligned}
\partial_t^2 u =& \frac{2}{\Gamma(k+\theta)\,\Gamma(1-\theta)} \left(\frac{\partial_t}{2t} \right)^{k+1} \left(t^{2k+2} w_1 \right), \\
w_1 :=& \int_0^1 \int_{-1}^1 \frac{t\sigma^2 - r\sigma\lambda}{\sqrt{r^2 + t^2\sigma^2 - 2rt\sigma\lambda}} \frac{g'\left(\sqrt{r^2 + t^2\sigma^2 - 2rt\sigma\lambda}\right)}{(1-\sigma^2)^{\theta}} \sigma^{A-1} \sqrt{1-\lambda^2}^{A-3} \,\mathrm{d}\,\lambda \,\mathrm{d}\,\sigma.
\end{aligned}$$

A similar process as deducing the classical expression of u also shows

$$\begin{aligned}
u =& \frac{1}{\Gamma(k+\theta)\,\Gamma(2-\theta)} \left(\frac{\partial_t}{2t} \right)^{k+1} \left(t^{2k+2} \tilde{w}_2 \right), \\
\tilde{w}_2 :=& t \int_0^1 \int_{-1}^1 \frac{g\left(\sqrt{r^2 + t^2\sigma^2 - 2rt\sigma\lambda}\right)}{(1-\sigma^2)^{\theta-1}} \sigma^{A-1} \sqrt{1-\lambda^2}^{A-3} \,\mathrm{d}\,\lambda \,\mathrm{d}\,\sigma.
\end{aligned}$$

Then, we see

$$\begin{aligned}
\partial_r \tilde{w}_2 =& t \int_0^1 \int_{-1}^1 \frac{r - t\sigma\lambda}{\sqrt{r^2 + t^2\sigma^2 - 2rt\sigma\lambda}} \frac{g'\left(\sqrt{r^2 + t^2\sigma^2 - 2rt\sigma\lambda}\right)}{(1-\sigma^2)^{\theta-1}} \sigma^{A-1} \sqrt{1-\lambda^2}^{A-3} \,\mathrm{d}\,\lambda \,\mathrm{d}\,\sigma \\
=& - \int_0^1 \int_{-1}^1 \frac{\partial_\lambda g\left(\sqrt{r^2 + t^2\sigma^2 - 2rt\sigma\lambda}\right)}{(1-\sigma^2)^{\theta-1}} \sigma^{A-2} \sqrt{1-\lambda^2}^{A-1} \,\mathrm{d}\,\lambda \,\mathrm{d}\,\sigma \\
& - \int_0^1 \int_{-1}^1 \lambda \frac{\partial_\sigma g\left(\sqrt{r^2 + t^2\sigma^2 - 2rt\sigma\lambda}\right)}{(1-\sigma^2)^{\theta-1}} \sigma^{A-1} \sqrt{1-\lambda^2}^{A-3} \,\mathrm{d}\,\lambda \,\mathrm{d}\,\sigma.
\end{aligned}$$

Using integration by parts, we get

$$\partial_r \tilde{w}_2 = -2(1-\theta) \int_0^1 \int_{-1}^1 \lambda \frac{g\left(\sqrt{r^2 + t^2\sigma^2 - 2rt\sigma\lambda}\right)}{(1-\sigma^2)^{\theta}} \sigma^{A} \sqrt{1-\lambda^2}^{A-3} \,\mathrm{d}\,\lambda \,\mathrm{d}\,\sigma.$$

Thus we have

$$\begin{aligned}\partial_r u =&\frac{2}{\Gamma(k+\theta)\,\Gamma(1-\theta)}\left(\frac{\partial_t}{2t}\right)^{k+1}\left(t^{2k+2}w_2\right)\\ w_2 :=&-\int_0^1\int_{-1}^1 \lambda\frac{g\left(\sqrt{r^2+t^2\sigma^2-2rt\sigma\lambda}\right)}{(1-\sigma^2)^\theta}\sigma^A\sqrt{1-\lambda^2}^{A-3}\,\mathrm{d}\,\lambda\,\mathrm{d}\,\sigma.\end{aligned}$$

Taking the derivation again, we also have

$$\begin{aligned}\partial_r^2 u =&\frac{2}{\Gamma(k+\theta)\,\Gamma(1-\theta)}\left(\frac{\partial_t}{2t}\right)^{k+1}\left(t^{2k+2}w_3\right)\\ w_3 :=&\int_0^1\int_{-1}^1 \frac{t\sigma^2\lambda^2-r\sigma\lambda}{\sqrt{r^2+t^2\sigma^2-2rt\sigma\lambda}}\frac{g'\left(\sqrt{r^2+t^2\sigma^2-2rt\sigma\lambda}\right)}{(1-\sigma^2)^\theta}\sigma^{A-1}\sqrt{1-\lambda^2}^{A-3}\,\mathrm{d}\,\lambda\,\mathrm{d}\,\sigma.\end{aligned}$$

Mixing $\partial_t^2 u$, $\partial_r^2 u$ and $\partial_r u$ together, we finally calculate

$$\begin{aligned}&(\partial_t^2-\partial_r^2-(A-1)r^{-1}\partial_r)u\\ =&\frac{2r^{-1}}{\Gamma(k+\theta)\,\Gamma(1-\theta)}\left(\frac{\partial_t}{2t}\right)^{k+1}\left(t^{2k+2}(rw_1-rw_3-(A-1)w_2)\right)\end{aligned}$$

where

$$\begin{aligned}rw_1-rw_3=&-\int_0^1\int_{-1}^1\frac{\partial_\lambda g\left(\sqrt{r^2+t^2\sigma^2-2rt\sigma\lambda}\right)}{(1-\sigma^2)^\theta}\sigma^A\sqrt{1-\lambda^2}^{A-1}\,\mathrm{d}\,\lambda\,\mathrm{d}\,\sigma\\ =&(A-1)w_2.\end{aligned}$$

This finishes the proof.

3.4.4 Proof of Lemma 3.13

We begin with the second half of (3.24), it is obviously since $\mathcal{X}_+^{\frac{1-A}{2}}(\lambda-\mu)=0$ for $\lambda\in[-1,1]$ and $\mu>1$. As for other results, we need to divide A into two cases.

Firstly, we consider the case $A=1+2k+2\theta$ with $k\in\mathbb{N}$ and $0<\theta<1$. By definition we see when $\mu\le-2$ and $\lambda\in[-1,1]$, we have

$$\partial_\mu^m\mathcal{X}_+^{\frac{1-A}{2}}(\lambda-\mu)\approx(1-\mu)^{\frac{1-A}{2}-m},$$

which gives (3.25). When $-1<\mu<1$, I_A has the formula

$$\begin{aligned}I_A(\mu)=&\frac{2^{-k-\theta}}{\Gamma(k+\theta)}\int_{-1}^1\partial_\lambda^k\mathcal{X}_+^{-\theta}(\lambda-\mu)\,(1-\lambda^2)^{k+\theta-1}\,\mathrm{d}\,\lambda\\ =&\frac{2^{-k-\theta}}{\Gamma(k+\theta)}\int_{-1}^1\mathcal{X}_+^{-\theta}(\lambda-\mu)\,(-\partial_\lambda)^k(1-\lambda^2)^{k+\theta-1}\,\mathrm{d}\,\lambda\\ =&\frac{2^{-k-\theta}}{\Gamma(k+\theta)\,\Gamma(1-\theta)}\int_\mu^1(\lambda-\mu)^{-\theta}\sum_{j=0}^{k/2}C_{j,k,\theta}\lambda^{k-2j}(1-\lambda^2)^{j+\theta-1}\,\mathrm{d}\,\lambda\end{aligned}$$

with some constants $C_{j,k,\theta}$. Let μ close to 1, we have

$$\begin{aligned}&\lim_{\mu\to 1-}\int_\mu^1(\lambda-\mu)^{-\theta}\lambda^{k-2j}(1-\lambda^2)^{j+\theta-1}\,\mathrm{d}\,\lambda\\ =&2^{j+\theta-1}\lim_{\mu\to 1-}\int_\mu^1(\lambda-\mu)^{-\theta}(1-\lambda)^{j+\theta-1}\,\mathrm{d}\,\lambda.\end{aligned}$$

Set $\lambda=(1-\mu)\sigma+\mu$, it goes to

$$2^{j+\theta-1}\lim_{\mu\to 1-}(1-\mu)^j\int_0^1\sigma^{-\theta}(1-\sigma)^{j+\theta-1}\,\mathrm{d}\,\sigma.$$

This means we only need to consider $j=0$, where

$$C_{0,k,\theta}=\begin{cases}1, & k=0;\\ 2^k(k+\theta-1)(k+\theta-2)\cdots\theta, & k>0.\end{cases}$$

Adding together, we give the first half of (3.24) in this case by calculating

$$\lim_{\mu\to 1-}I_A(\mu)=\frac{1}{2\Gamma(\theta)\,\Gamma(1-\theta)}\int_0^1\sigma^{-\theta}(1-\sigma)^{\theta-1}\,\mathrm{d}\,\sigma=\frac{1}{2}.$$

Then we let μ close to -1, without lose of generality we assume $\mu<-1/2$, then

$$\begin{aligned}&\int_\mu^1(\lambda-\mu)^{-\theta}\lambda^{k-2j}(1-\lambda^2)^{j+\theta-1}\,\mathrm{d}\,\lambda\\ =&\int_\mu^0(\lambda-\mu)^{-\theta}\lambda^{k-2j}(1-\lambda^2)^{j+\theta-1}\,\mathrm{d}\,\lambda+\int_0^1(\lambda-\mu)^{-\theta}\lambda^{k-2j}(1-\lambda^2)^{j+\theta-1}\,\mathrm{d}\,\lambda.\end{aligned}$$

The estimate of second integral is trivially. For the first integral we introduce the change of variable $\lambda=\sigma(1+\mu)-1$. Then the integral equals to

$$\begin{aligned}&(1+\mu)^j\int_1^{(1+\mu)^{-1}}(\sigma-1)^{-\theta}\sigma^{j+\theta-1}f(\sigma(1+\mu))\,\mathrm{d}\,\sigma\\ =&(1+\mu)^j\int_2^{(1+\mu)^{-1}}\sigma^{j-1}\,\mathrm{d}\,\sigma+(1+\mu)^j\int_1^2(\sigma-1)^{-\theta}\sigma^{j+\theta-1}\,\mathrm{d}\,\sigma\\ &+(1+\mu)^j\int_2^{(1+\mu)^{-1}}\left((\sigma-1)^{-\theta}-\sigma^{-\theta}\right)\sigma^{j+\theta-1}\,\mathrm{d}\,\sigma\\ &+(1+\mu)^j\int_1^{(1+\mu)^{-1}}(\sigma-1)^{-\theta}\sigma^{j+\theta-1}(f(\sigma(1+\mu))-f(0))\,\mathrm{d}\,\sigma,\end{aligned}$$

where

$$f(s):=(s-1)^{k-2j}\,(2-s)^{j+\theta-1},\qquad \|f'(s)\|_{L^\infty_s(0,1)}<\infty.$$

Adding together, we find

$$\begin{aligned}\int_{\mu}^{1}(\lambda-\mu)^{-\theta}\lambda^{k-2j}(1-\lambda^2)^{j+\theta-1}\,\mathrm{d}\,\lambda =& C_j(1+\mu)^j\int_{2}^{(1+\mu)^{-1}}\sigma^{j-1}\,\mathrm{d}\,\sigma+O(1)\\ =& C_j'\ln\frac{3}{1+\mu}+O(1),\end{aligned}$$

which gives (3.26) for $-1<\mu$. Using the same decomposition, for the derivation we similarly find

$$\partial_\mu\int_{\mu}^{1}(\lambda-\mu)^{-\theta}\lambda^{k-2j}(1-\lambda^2)^{j+\theta-1}\,\mathrm{d}\,\lambda=C_j''(1+\mu)^{-1}+O\left(\ln\frac{3}{1+\mu}\right),$$

which gives (3.27) for $-1<\mu$.

To get another part of (3.26), we only need to control $I_A(\mu)-I_A(-2-\mu)$ for $-3/2<\mu<-1$. Here, for $-2\le\mu<-1$, I_A has the formula

$$I_A(\mu)=\frac{2^{-k-\theta}}{\Gamma(k+\theta)\,\Gamma(1-\theta)}\int_{-1}^{1}(\lambda-\mu)^{-\theta}\sum_{j=0}^{k/2}C_{j,k,\theta}\lambda^{k-2j}(1-\lambda^2)^{j+\theta-1}\,\mathrm{d}\,\lambda.$$

Thus, to show (3.26), we only need to estimate

$$\begin{aligned}&\int_{-1}^{1}(\lambda-\mu)^{-\theta}\lambda^{k-2j}(1-\lambda^2)^{j+\theta-1}\,\mathrm{d}\,\lambda-\int_{-\mu-2}^{1}(\lambda+\mu+2)^{-\theta}\lambda^{k-2j}(1-\lambda^2)^{j+\theta-1}\,\mathrm{d}\,\lambda\\ =&\int_{-1}^{-\mu-2}(\lambda-\mu)^{-\theta}\lambda^{k-2j}(1-\lambda^2)^{j+\theta-1}\,\mathrm{d}\,\lambda\\ &+\int_{-\mu-2}^{0}\left((\lambda-\mu)^{-\theta}-(\lambda+\mu+2)^{-\theta}\right)\lambda^{k-2j}(1-\lambda^2)^{j+\theta-1}\,\mathrm{d}\,\lambda\\ &+\int_{0}^{1}\left((\lambda-\mu)^{-\theta}-(\lambda+\mu+2)^{-\theta}\right)\lambda^{k-2j}(1-\lambda^2)^{j+\theta-1}\,\mathrm{d}\,\lambda\\ \equiv& J_1+J_2+J_3.\end{aligned}$$

For J_1, we change the variable $\lambda=-\sigma(1+\mu)+\mu$, then we have

$$\begin{aligned}J_1=&(-1-\mu)^j\int_{1}^{2}\sigma^{-\theta}(\sigma-1)^{j+\theta-1}(-\sigma(1+\mu)+\mu)^{k-2j}\left(1-\mu+\sigma(1+\mu)\right)^{j+\theta-1}\,\mathrm{d}\,\sigma\\ |J_1|\lesssim&(-1-\mu)^j\int_{1}^{2}(\sigma-1)^{j+\theta-1}\,\mathrm{d}\,\sigma\lesssim 1.\end{aligned}$$

For J_2, we also set $\lambda=-\sigma(1+\mu)+\mu$, then we have

$$\begin{aligned}J_2=&(-1-\mu)^j\int_{2}^{\frac{1}{-1-\mu}+1}\left(\sigma^{-\theta}-(\sigma-2)^{-\theta}\right)(\sigma-1)^{j+\theta-1}(-\sigma(1+\mu)+\mu)^{k-2j}\\ &\left(1-\mu+\sigma(1+\mu)\right)^{j+\theta-1}\,\mathrm{d}\,\sigma\\ |J_2|\lesssim&(-1-\mu)^j\int_{3}^{\frac{1}{-1-\mu}+1}(\sigma-2)^{-\theta-1}(\sigma-1)^{j+\theta-1}\,\mathrm{d}\,\sigma+1\lesssim 1.\end{aligned}$$

Finally, the estimate of J_3 is trivially, adding together we finish (3.26). Using the same method and estimating $\partial_\mu J_1$, $\partial_\mu J_2$, $\partial_\mu J_3$, we also get (3.27).

Next, we consider the case $A = 1 + 2k$ with $k \in \mathbb{Z}^+$. In this case we have

$$\operatorname{supp} \mathcal{X}_+^{\frac{1-A}{2}}(x) = \operatorname{supp} \delta^{(k-1)}(x) = \{0\},$$

which gives (3.28). On the other hand, when $-1 < \mu < 1$, we have

$$\begin{aligned} I_A(\mu) =& \frac{2^{-k}}{\Gamma(k)} \int_{-1}^{1} \partial_\lambda^{k-1} \delta(\lambda - \mu) \, (1-\lambda^2)^{k-1} \, \mathrm{d}\, \lambda \\ =& \frac{2^{-k}}{\Gamma(k)} \int_{-1}^{1} \delta(\lambda - \mu) \, (-\partial_\lambda)^{k-1} (1-\lambda^2)^{k-1} \, \mathrm{d}\, \lambda \\ =& \frac{2^{-k}}{\Gamma(k)} \sum_{j=0}^{(k-1)/2} C'_{j,k} \mu^{k-1-2j} (1-\mu^2)^j. \end{aligned}$$

This gives (3.29). Here we also find $C'_{0,k} = 2^{k-1}(k-1)!$, which implies the first half of (3.24) in this case. Now we finish the proof.

3.4.5 Additional discussion of weak solutions

In light of the frame we took is slightly different from the normal one, we will discuss a bit more of the weak solution. We will show that when

$$r^{A-1} f \in L^1_{loc;r}, \; r^{A-1} g \in L^1_{loc;r}, \; r^{A-1} F \in L^1_{loc;t,r}, \; r^{A-1} u \in L^1_{loc;t,r} \tag{3.30}$$

with u calculated by Lemma 3.12, u satisfies (3.23).

To show this result, we divide u to u_g, u_f and u_F. We begin with u_g part, noticing $I_A(\mu) = 0$ while $\mu > 1$, in this case we have

$$\begin{aligned} &\int_0^T \int_0^\infty u\big(\partial_t^2 - \Delta_A\big) \, \phi r^{A-1} \, \mathrm{d}\, r \, \mathrm{d}\, t \\ =& \int_0^T \int_0^\infty \int_0^\infty r^{\frac{A-1}{2}} \rho^{\frac{A-1}{2}} g(\rho) I_A\left(\frac{r^2+\rho^2-t^2}{2r\rho}\right) \big(\partial_t^2 - \Delta_A\big) \, \phi(t,r) \, \mathrm{d}\, \rho \, \mathrm{d}\, r \, \mathrm{d}\, t. \end{aligned}$$

Set $t = T - s$, swap r and ρ then exchange the order of integration, it goes to

$$\int_0^\infty \int_0^T \int_0^\infty r^{\frac{A-1}{2}} \rho^{\frac{A-1}{2}} g(r) I_A\left(\frac{r^2+\rho^2-(T-s)^2}{2r\rho}\right) \big(\partial_s^2 - \Delta_A\big) \, \phi(T-s,\rho) \, \mathrm{d}\, \rho \, \mathrm{d}\, s \, \mathrm{d}\, r.$$

Here $\phi(T-s,\rho)$ has zero initial data at $s = 0$ and regular enough, by expression of u_F deduced in Lemma 3.12, we know

$$\begin{aligned} &r^{\frac{1-A}{2}} \int_0^T \int_0^\infty \rho^{\frac{A-1}{2}} I_A\left(\frac{r^2+\rho^2-(T-s)^2}{2r\rho}\right) \big(\partial_s^2 - \Delta_A\big) \, \phi(T-s,\rho) \, \mathrm{d}\, \rho \, \mathrm{d}\, s \\ =& \phi(T-s,r)\big|_{s=T} = \phi(0,r). \end{aligned}$$

This gives (3.23) with $f = F = 0$. The proof of u_f and u_F parts is much similar, we leave them to the reader.

3.5 Long time existence for $A \in [2, 3]$

In this section, we will consider the case $A \in [2, 3]$, and show the proof of Theorem 3.2. Without lose of generality we assume $n \neq A$, otherwise $V = 0$ then (3.5) reduced to the basic *Strauss* type wave equation.

By the discussion in last section, we begin to study the equation (3.21) and (3.22).

3.5.1 Estimate for homogeneous solution

In this subsection, we will give an estimate of the homogeneous solution to (3.22).

Lemma 3.14. *Assume* $\operatorname{supp}(f, g) \subset [0, 1)$. *We have*

$$|u_f + u_g| \lesssim \langle t+r\rangle^{\frac{1-A}{2}} \langle t-r\rangle^{\frac{1-A}{2}} \left(\|\rho g(\rho)\|_{L^\infty_\rho} + \|f(\rho)\|_{L^\infty_\rho} + \|\rho f'(\rho)\|_{L^\infty_\rho} \right).$$

Here and through the paper, we use $\langle a\rangle$ *stands for* $\sqrt{|a|^2+4}$.

Proof of Lemma 3.14. Here we define

$$\Omega_0 := \{\rho : 0 < \rho < t - r\}, \qquad \Omega_1 := \{\rho : |t-r| < \rho < \min(1, t+r)\}.$$

Firstly we consider u_g with $A \in [2, 3)$. For u_g, by Lemma 3.12 we have

$$u_g(t, r) = r^{\frac{1-A}{2}} \left(\int_{\Omega_0} + \int_{\Omega_1} \right) I_A(\mu) \rho^{\frac{A-1}{2}} g(\rho) \, \mathrm{d}\rho \equiv J_0 + J_1.$$

Part 1: $t + r \leq 4$.

In this part, we have $\langle t+r\rangle \approx \langle t-r\rangle \approx 1$ and $r \lesssim 1$. In the region of Ω_0 where $\mu < -1$, by (3.25) and (3.26) we see

$$|I_A(\mu)| \lesssim (-1-\mu)^{\frac{1-A}{2}} = \left(\frac{2r\rho}{(t+r+\rho)(t-r-\rho)} \right)^{\frac{A-1}{2}} \lesssim r^{\frac{A-1}{2}} (t-r-\rho)^{\frac{1-A}{2}}.$$

Then we have

$$\begin{aligned} |J_0| &\lesssim \int_0^{t-r} (t-r-\rho)^{\frac{1-A}{2}} \rho^{\frac{A-3}{2}} \rho |g(\rho)| \, \mathrm{d}\rho \\ &\lesssim \|\rho g(\rho)\|_{L^\infty} \lesssim \langle t+r\rangle^{\frac{1-A}{2}} \langle t-r\rangle^{\frac{1-A}{2}} \|\rho g(\rho)\|_{L^\infty}. \end{aligned}$$

In the region of Ω_1 where $\mu > -1$, noting $A < 3$, we know that

$$\ln \frac{3}{|1+\mu|} \lesssim |1+\mu|^{\frac{A-3}{2}}, \qquad -2 < \mu < 1.$$

Then, by (3.26) we see

$$|I_A(\mu)| \lesssim (1+\mu)^{\frac{A-3}{2}} = \left(\frac{2r\rho}{(t+r+\rho)(r+\rho-t)}\right)^{\frac{3-A}{2}} \lesssim \rho^{\frac{3-A}{2}}(r+\rho-t)^{\frac{A-3}{2}}.$$

Thus we get

$$\begin{aligned}|J_1| \lesssim & r^{\frac{1-A}{2}} \int_{t-r}^{t+r} (r+\rho-t)^{\frac{A-3}{2}} \rho |g(\rho)| \,\mathrm{d}\,\rho \\ \lesssim & \|\rho g(\rho)\|_{L^\infty} \lesssim \langle t+r\rangle^{\frac{1-A}{2}} \langle t-r\rangle^{\frac{1-A}{2}} \left\|\rho^{\frac{A-1}{2}} g(\rho)\right\|_{L^\infty}.\end{aligned}$$

Part 2: $t+r \geq 4$, $-1 \leq t-r \leq 2$.

In this part, we have $\langle t-r\rangle \approx 1$ and $r \approx \langle t+r\rangle$. In the region of Ω_0 where $\mu < -1$, we have

$$|I_A(\mu)| \lesssim (-1-\mu)^{\frac{1-A}{2}} = \left(\frac{2r\rho}{(t+r+\rho)(t-r-\rho)}\right)^{\frac{A-1}{2}} \lesssim \rho^{\frac{A-1}{2}}(t-r-\rho)^{\frac{1-A}{2}}.$$

Then we get

$$\begin{aligned}|J_0| \lesssim & \langle t+r\rangle^{\frac{1-A}{2}} \int_0^{t-r} (t-r-\rho)^{\frac{1-A}{2}} \rho^{A-2} \rho |g(\rho)| \,\mathrm{d}\,\rho \\ \lesssim & \langle t+r\rangle^{\frac{1-A}{2}} (t-r)^{\frac{A-1}{2}} \|\rho g(\rho)\|_{L^\infty} \lesssim \langle t+r\rangle^{\frac{1-A}{2}} \langle t-r\rangle^{\frac{1-A}{2}} \|\rho g(\rho)\|_{L^\infty}.\end{aligned}$$

In the region of Ω_1 where $\mu > -1$, we also have

$$|I_A(\mu)| \lesssim (1+\mu)^{\frac{A-3}{2}} = \left(\frac{2r\rho}{(t+r+\rho)(r+\rho-t)}\right)^{\frac{3-A}{2}} \lesssim \rho^{\frac{3-A}{2}}(r+\rho-t)^{\frac{A-3}{2}}.$$

Then we see

$$\begin{aligned}|J_1| \lesssim & \langle t+r\rangle^{\frac{1-A}{2}} \int_{t-r}^{2} (r+\rho-t)^{\frac{A-3}{2}} \rho |g(\rho)| \,\mathrm{d}\,\rho \\ \lesssim & \langle t+r\rangle^{\frac{1-A}{2}} (r+2-t)^{\frac{A-1}{2}} \|\rho g(\rho)\|_{L^\infty} \lesssim \langle t+r\rangle^{\frac{1-A}{2}} \langle t-r\rangle^{\frac{1-A}{2}} \|\rho g(\rho)\|_{L^\infty}.\end{aligned}$$

Part 3: $t+r \geq 4$, $t-r \geq 2$.

In this part, we have $t+r \gtrsim \langle t+r\rangle$ and $t-r-1 \gtrsim \langle t-r\rangle$. Here we only need to consider Ω_0 where $\mu < -1$, again we see

$$\begin{aligned}|I_A(\mu)| \lesssim & (-1-\mu)^{\frac{1-A}{2}} = \left(\frac{2r\rho}{(t+r+\rho)(t-r-\rho)}\right)^{\frac{A-1}{2}} \\ \lesssim & r^{\frac{A-1}{2}} \rho^{\frac{A-1}{2}} \langle t+r\rangle^{\frac{1-A}{2}} \langle t-r\rangle^{\frac{1-A}{2}}.\end{aligned}$$

Then we have

$$\begin{aligned}|u_g| \lesssim & \langle t+r\rangle^{\frac{1-A}{2}} \langle t-r\rangle^{\frac{1-A}{2}} \int_0^1 \rho^{A-1}|g(\rho)| \,\mathrm{d}\,\rho \\ \lesssim & \langle t+r\rangle^{\frac{1-A}{2}} \langle t-r\rangle^{\frac{1-A}{2}} \|\rho g(\rho)\|_{L^\infty}.\end{aligned}$$

Mixing together, we finish this part.

Next we consider u_f, for simplicity we denote $u_{g=\phi}$ stands for u_g with $g=\phi$. By Lemma 3.12 and the classical expression we showed in 3.4.3, we know

$$\begin{aligned}u_f =&\partial_t(u_{g=f}) \\ =&\frac{1}{\Gamma\left(\frac{A-1}{2}\right)\Gamma\left(\frac{3-A}{2}\right)}\partial_t\left(t\int_0^1\int_{-1}^1 \frac{f\left(\sqrt{r^2+t^2\sigma^2-2rt\sigma\lambda}\right)}{(1-\sigma^2)^{\frac{A-1}{2}}}\sigma^{A-1}\sqrt{1-\lambda^2}^{A-3}\,\mathrm{d}\,\lambda\,\mathrm{d}\,\sigma\right) \\ \lesssim & t\int_0^1\int_{-1}^1 \frac{|t\sigma^2-r\sigma\lambda|}{\sqrt{r^2+t^2\sigma^2-2rt\sigma\lambda}}\frac{\left|f'\left(\sqrt{r^2+t^2\sigma^2-2rt\sigma\lambda}\right)\right|}{(1-\sigma^2)^{\frac{A-1}{2}}}\sigma^{A-1}\sqrt{1-\lambda^2}^{A-3}\,\mathrm{d}\,\lambda\,\mathrm{d}\,\sigma \\ &+\int_0^1\int_{-1}^1 \frac{\left|f\left(\sqrt{r^2+t^2\sigma^2-2rt\sigma\lambda}\right)\right|}{(1-\sigma^2)^{\frac{A-1}{2}}}\sigma^{A-1}\sqrt{1-\lambda^2}^{A-3}\,\mathrm{d}\,\lambda\,\mathrm{d}\,\sigma \\ \equiv & H_1+H_2.\end{aligned}$$

Since $A\in[2,3)$, we can easily found that $H_2\lesssim\|f\|_{L^\infty}$. Meanwhile, we find that

$$H_2 = t^{-1}u_{g=|f|}\lesssim t^{-1}\langle t+r\rangle^{\frac{1-A}{2}}\langle t-r\rangle^{\frac{1-A}{2}}\|f(\rho)\|_{L^\infty_\rho}.$$

Dividing t greater or smaller than 1, we finish the estimate of H_2. Next we consider H_1, noticing that when $\lambda\in[-1,1]$, we have

$$\left(\frac{t\sigma^2-r\sigma\lambda}{\sqrt{r^2+t^2\sigma^2-2rt\sigma\lambda}}\right)^2 = \frac{r^2\sigma^2\lambda^2+t^2\sigma^4-2rt\sigma^3\lambda}{r^2+t^2\sigma^2-2rt\sigma\lambda}\le\sigma^2\le 1.$$

Then we calculate

$$H_1\le u_{g=|f'|}\lesssim\langle t+r\rangle^{\frac{1-A}{2}}\langle t-r\rangle^{\frac{1-A}{2}}\|\rho f'(\rho)\|_{L^\infty_\rho}.$$

Adding together and we finish this part.

When $A=3$, the estimate for u_g is similarly. We only need to mention that in this case we have (3.28) and (3.29) instead of (3.25) and (3.26). As for u_f, we easily calculate

$$u_f=\frac{(t+r)f(t+r)-(t-r)f(|t-r|)}{2r}$$

with supp $u_f\subset\{(t,r):|t-r|\le 1\}$. When $r\le t\le 2$, by mean value theorem we see

$$|u_f|=\left|\frac{(t+r)f(t+r)-(t-r)f(t-r)}{2r}\right|\le\|\partial_\rho(\rho f(\rho))\|_{L^\infty_\rho}.$$

When $t \le r \le 3$, we see

$$|u_f| \le |f(t+r)| + |f(r-t)| \lesssim \|f(\rho)\|_{L^\infty_\rho}.$$

Finally when $2r \ge t \ge 2$, we see

$$|u_f| \le \frac{|(t+r)f(t+r)| + \big||t-r|f(|t-r|)\big|}{2t} \lesssim \|\rho f(\rho)\|_{L^\infty_\rho} \le \|f(\rho)\|_{L^\infty_\rho}.$$

Mixing together, we finish the proof. □

3.5.2 Estimate for non-homogeneous solution

In this subsection, we will omit the initial data and give the estimate of solution to the nonlinear equation (3.21). For simplicity, we shift the time variable and consider the equation

$$\begin{cases} (\partial_t^2 - \partial_r^2 - (A-1)r^{-1}\partial_r)\, v(t,r) = r^{\frac{(A-n)p+n-A}{2}}|u|^p \\ v(3,r) = 0, \qquad v_t(3,r) = 0. \end{cases} \tag{3.31}$$

Lemma 3.15. *Defining* $\Omega := \{(t,r) \in [3,\infty) \times \mathbb{R}_+ : t > r+2\}$ *and* $\Lambda_{t,r} := \{(s,\rho) \in \Omega : s+\rho < t+r, s-\rho < t-r\}$. *We will show that, if* v *solves the equation (3.31) with* $\operatorname{supp} u \subset \Omega$, *then for any* $(t,r) \in \Omega$ *and* $k = 1,2,3$ *we have*

$$\langle t+r\rangle^{\frac{A-1}{2}} |v| \lesssim N_k(t-r)\|\omega_k u\|^p_{L^\infty_{s,\rho}(\Lambda_{t,r})}. \tag{3.32}$$

Here

$$\omega_k(t,r) := \langle t+r\rangle^{\frac{A-1}{2}} \beta_k(t-r), \qquad k = 1,2,3;$$

$$\beta_k(t,r) := \begin{cases} \langle t-r\rangle^{\frac{A-1}{2}}, & k=1,\ for\ p_m < p < p_M, \\ \langle t-r\rangle^{\frac{(n-1)p-n-1}{2}}, & k=2,\ for\ p_m < p < p_t, \\ \langle t-r\rangle^{\frac{A-1}{2}} \ln^{-1}\langle t-r\rangle, & k=3,\ for\ p = p_t > p_d; \end{cases}$$

$$N_1(t-r) := \begin{cases} \langle t-r\rangle^{\frac{1-A}{2}}, & p > \max(p_d, p_t) \text{ or } p = p_d > p_t \text{ or } p_F < p < p_d, \\ \langle t-r\rangle^{\frac{1-A}{2}} \ln\langle t-r\rangle, & p = p_t > p_d \text{ or } p = p_F < p_d, \\ \langle t-r\rangle^{\frac{(-n+1)p+n+1}{2}}, & p_t > p > p_d, \\ \langle t-r\rangle^{\frac{(-n-A+2)p+n+3}{2}}, & p < \min(p_d, p_F), \\ \langle t-r\rangle^{\frac{(-n-A+2)p+n+3}{2}} \ln\langle t-r\rangle, & p = p_d < p_t, \\ \langle t-r\rangle^{\frac{1-A}{2}} \ln^2\langle t-r\rangle, & p = p_d = p_t; \end{cases}$$

$$N_2(t-r) := \begin{cases} \langle t-r\rangle^{\frac{(-n+1)p+n+1}{2}} & p_S < p < p_t, \\ \langle t-r\rangle^{\frac{(-n+1)p+n+1}{2}} \ln\langle t-r\rangle & p = p_S < p_t, \\ \langle t-r\rangle^{\frac{(-n+1)p^2+2p+n+3}{2}} & p_m < p < \min(p_S, p_t); \end{cases}$$

$$N_3(t-r) := \langle t-r\rangle^{\frac{1-A}{2}} \ln\langle t-r\rangle, \qquad p = p_t > p_d.$$

Remark 3.3. *Here we remark that for any $A \in [2,3]$, a quick calculation shows*

$$\begin{cases} p_d < p_F < p_S < p_t, & n < \frac{A^2-A+2}{3-A}; \\ p_d = p_F = p_S = p_t, & n = \frac{A^2-A+2}{3-A}; \\ p_d > p_F > p_S > p_t, & n > \frac{A^2-A+2}{3-A}. \end{cases}$$

Proof of Lemma 3.15. Using Lemma 3.12, we find

$$v = r^{\frac{1-A}{2}} \int_\Lambda I_A(\mu) \rho^{\frac{(A-n)p+n-1}{2}} |u|^p \,\mathrm{d}\,\rho \,\mathrm{d}\, s,$$

with $\mu := \frac{r^2+\rho^2-(t-s)^2}{2r\rho}$. To reach (3.32), we calculate

$$|\langle t+r\rangle^{\frac{A-1}{2}} v| \le r^{\frac{1-A}{2}} \langle t+r\rangle^{\frac{A-1}{2}} \|\omega_k u\|^p_{L^\infty_{s,\rho}(\Lambda)} \int_\Lambda |I_A(\mu)| \rho^{\frac{(A-n)p+n-1}{2}} \omega_k^{-p} \,\mathrm{d}\,\rho \,\mathrm{d}\, s.$$

Thus, we only need to show that

$$J_{ij;k} := r^{\frac{1-A}{2}} \langle t+r\rangle^{\frac{A-1}{2}} \int_{\Lambda_{ij}} |I_A(\mu)| \rho^{\frac{(A-n)p+n-1}{2}} \omega_k^{-p} \,\mathrm{d}\,\rho \,\mathrm{d}\, s \lesssim N_k(t-r)$$

for $i=1,2,3$, $j=1,2$ with

$$\begin{aligned}
\Lambda_{11} :=&\{(s,\rho)\in\Lambda : s+\rho\in(t-r,t+r), \rho\le s/2\};\\
\Lambda_{12} :=&\{(s,\rho)\in\Lambda : s+\rho\in(t-r,t+r), \rho\ge s/2\};\\
\Lambda_{21} :=&\{(s,\rho)\in\Lambda : s+\rho\in\left(\frac{t-r}{2},t-r\right), \rho\le s/2\};\\
\Lambda_{22} :=&\{(s,\rho)\in\Lambda : s+\rho\in\left(\frac{t-r}{2},t-r\right), \rho\ge s/2\};\\
\Lambda_{31} :=&\{(s,\rho)\in\Lambda : s+\rho\in\left(3,\frac{t-r}{2}\right), \rho\le s/2\};\\
\Lambda_{32} :=&\{(s,\rho)\in\Lambda : s+\rho\in\left(3,\frac{t-r}{2}\right), \rho\ge s/2\}.
\end{aligned}$$

We will firstly consider $A\in[2,3)$, and estimate each part separately.

Part 1: $r\le t/2$.

In this part we have $t-r\approx t+r$. In the region of Λ_{11} we have

$$s-\rho\approx s+\rho, \tag{3.33}$$

using (3.26) we also have

$$|I_A(\mu)|\lesssim(1+\mu)^{\frac{A-3}{2}}=\left(\frac{2r\rho}{(r+\rho+t-s)(r+\rho-t+s)}\right)^{\frac{3-A}{2}}\lesssim\left(\frac{\rho}{r+\rho-t+s}\right)^{\frac{3-A}{2}}. \tag{3.34}$$

Noticing that p satisfies $p<p_M$, then

$$\begin{aligned}
J_{11}\lesssim& r^{\frac{1-A}{2}}\langle t-r\rangle^{\frac{A-1}{2}}\int_{\Lambda_{11}}\rho^{\frac{(-n+A)p+n-A+2}{2}}\langle s+\rho\rangle^{-\frac{A-1}{2}p}\beta^{-p}(s+\rho)(r+\rho-t+s)^{\frac{A-3}{2}}\,\mathrm{d}\,\rho\,\mathrm{d}\,s\\
\lesssim& r^{\frac{1-A}{2}}\langle t-r\rangle^{\frac{(-A+1)p+A-1}{2}}\beta^{-p}(t-r)\int_{t-r}^{t+r}\int_{(t-r)/3}^{t-r}(\xi-\eta)^{\frac{(-n+A)p+n-A+2}{2}}(\xi-(t-r))^{\frac{A-3}{2}}\,\mathrm{d}\,\eta\,\mathrm{d}\,\xi\\
\lesssim& r^{\frac{1-A}{2}}\langle t-r\rangle^{\frac{(-n+1)p+n+3}{2}}\beta^{-p}(t-r)\int_{t-r}^{t+r}(\xi-(t-r))^{\frac{A-3}{2}}\,\mathrm{d}\,\xi\\
\lesssim&\langle t-r\rangle^{\frac{(-n+1)p+n+3}{2}}\beta^{-p}(t-r)
\end{aligned}$$

In the region of Λ_{12} we have

$$\rho\approx s+\rho \tag{3.35}$$

and (3.34). Then

$$\begin{aligned}
J_{12}\lesssim& r^{\frac{1-A}{2}}\langle t-r\rangle^{\frac{A-1}{2}}\int_{\Omega_2}\langle s+\rho\rangle^{\frac{(-n+1)p+n-A+2}{2}}\beta^{-p}(s-\rho)(r+\rho-t+s)^{\frac{A-3}{2}}\,\mathrm{d}\,\rho\,\mathrm{d}\,s\\
\lesssim& r^{\frac{1-A}{2}}\langle t-r\rangle^{\frac{(-n+1)p+n+1}{2}}\int_{t-r}^{t+r}(\xi-(t-r))^{\frac{A-3}{2}}\,\mathrm{d}\,\xi\int_2^{(t+r)/3}\beta^{-p}(\eta)\,\mathrm{d}\,\eta\\
\lesssim&\langle t-r\rangle^{\frac{(-n+1)p+n+1}{2}}\int_2^{t-r}\beta^{-p}(\eta)\,\mathrm{d}\,\eta
\end{aligned}$$

In the region of Λ_{21} we have (3.33), using (3.25) and (3.26) we also have

$$|I_A(\mu)| \lesssim (-1-\mu)^{\frac{1-A}{2}} = \left(\frac{2r\rho}{(t+r-s+\rho)(t-r-s-\rho)}\right)^{\frac{A-1}{2}} \lesssim \left(\frac{r}{t-r-s-\rho}\right)^{\frac{A-1}{2}}. \tag{3.36}$$

Noticing $p < P_M$, we calculate

$$\begin{aligned}
J_{21} \lesssim & \langle t-r\rangle^{\frac{A-1}{2}} \int_{\Lambda_{21}} \rho^{\frac{(A-n)p+n-1}{2}} \langle s+\rho\rangle^{\frac{(-A+1)p}{2}} \beta^{-p}(s+\rho)(t-r-s-\rho)^{\frac{1-A}{2}} \,\mathrm{d}\rho\,\mathrm{d}s \\
\lesssim & \langle t-r\rangle^{\frac{(-A+1)p+A-1}{2}} \beta^{-p}(t-r) \int_{(t-r)/2}^{t-r} \int_{\xi/3}^{\xi} (\xi-\eta)^{\frac{(A-n)p+n-1}{2}} (t-r-\xi)^{\frac{1-A}{2}} \,\mathrm{d}\eta\,\mathrm{d}\xi \\
\lesssim & \langle t-r\rangle^{\frac{(-n+1)p+n+A}{2}} \beta^{-p}(t-r) \int_{(t-r)/2}^{t-r} (t-r-\xi)^{\frac{1-A}{2}} \,\mathrm{d}\xi \\
\lesssim & \langle t-r\rangle^{\frac{(-n+1)p+n+3}{2}} \beta^{-p}(t-r).
\end{aligned}$$

In the region of Λ_{22} we have (3.35) and (3.36), then

$$\begin{aligned}
J_{22} \lesssim & \langle t-r\rangle^{\frac{A-1}{2}} \int_{\Lambda_{22}} \langle s+\rho\rangle^{\frac{(-n+1)p+n-1}{2}} \beta^{-p}(s-\rho)(t-r-s-\rho)^{\frac{1-A}{2}} \,\mathrm{d}\rho\,\mathrm{d}s \\
\lesssim & \langle t-r\rangle^{\frac{(-n+1)p+n+A-2}{2}} \int_{(t-r)/2}^{t-r} (t-r-\xi)^{\frac{1-A}{2}} \,\mathrm{d}\xi \int_{2}^{(t-r)/3} \beta^{-p}(\eta) \,\mathrm{d}\eta \\
\lesssim & \langle t-r\rangle^{\frac{(-n+1)p+n+1}{2}} \int_{2}^{(t-r)/3} \beta^{-p}(\eta) \,\mathrm{d}\eta
\end{aligned}$$

In the region of Λ_{31}, again we have (3.33) and

$$|I_A(\mu)| \lesssim (-1-\mu)^{\frac{1-A}{2}} = \left(\frac{2r\rho}{(t+r-s+\rho)(t-r-s-\rho)}\right)^{\frac{A-1}{2}} \lesssim \left(\frac{r\rho}{(t-r)^2}\right)^{\frac{A-1}{2}}. \tag{3.37}$$

Since $p < p_M$, we see

$$\begin{aligned}
J_{31} \lesssim & \langle t-r\rangle^{\frac{A-1}{2}} \int_{\Lambda_{31}} \rho^{\frac{(-n+A)p+n+A-2}{2}} \langle s+\rho\rangle^{\frac{(-A+1)p}{2}} \beta^{-p}(s+\rho)(t-r)^{-A+1} \,\mathrm{d}\rho\,\mathrm{d}s \\
\lesssim & \langle t-r\rangle^{\frac{1-A}{2}} \int_{3}^{(t-r)/2} \int_{\xi/3}^{\xi} (\xi-\eta)^{\frac{(-n+A)p+n+A-2}{2}} \langle \xi\rangle^{\frac{(-A+1)p}{2}} \beta^{-p}(\xi) \,\mathrm{d}\eta\,\mathrm{d}\xi \\
\lesssim & \langle t-r\rangle^{\frac{1-A}{2}} \int_{3}^{(t-r)/2} \langle \xi\rangle^{\frac{(-n+1)p+n+A}{2}} \beta^{-p}(\xi) \,\mathrm{d}\xi
\end{aligned}$$

In the region of Λ_{32} we have (3.35) and (3.37), then

$$\begin{aligned}
J_{32} \lesssim & \langle t-r\rangle^{\frac{A-1}{2}} \int_{\Lambda_{32}} \langle s+\rho\rangle^{\frac{(-n+1)p+n+A-2}{2}} \beta^{-p}(s-\rho)(t-r)^{-A+1} \,\mathrm{d}\rho\,\mathrm{d}s \\
\lesssim & \langle t-r\rangle^{\frac{1-A}{2}} \int_{3}^{(t-r)/2} \int_{2}^{\xi/3} \langle \xi\rangle^{\frac{(-n+1)p+n+A-2}{2}} \beta^{-p}(\eta) \,\mathrm{d}\eta\,\mathrm{d}\xi
\end{aligned}$$

Part 2: $r \geq t/2$.

In this part we have $r \approx t + r$. In the region of Λ_{11}, we have (3.33) and (3.34). Using $p < p_M$ we deduce

$$\begin{aligned}
J'_{11} \lesssim & \int_{\Lambda_{11}} \rho^{\frac{(-n+A)p+n-A+2}{2}} \langle s+\rho\rangle^{\frac{(-A+1)p}{2}} \beta^{-p}(s+\rho)(r+\rho-t+s)^{\frac{A-3}{2}} \,\mathrm{d}\rho\,\mathrm{d}s \\
\lesssim & \langle t-r\rangle^{\frac{(-A+1)p}{2}} \beta^{-p}(t-r) \int_{t-r}^{3(t-r)} \int_{(t-r)/3}^{t-r} (\xi-\eta)^{\frac{(-n+A)p+n-A+2}{2}} (\xi-(t-r))^{\frac{A-3}{2}} \,\mathrm{d}\eta\,\mathrm{d}\xi \\
\lesssim & \langle t-r\rangle^{\frac{(-n+1)p+n-A+4}{2}} \beta^{-p}(t-r) \int_{t-r}^{3(t-r)} (\xi-(t-r))^{\frac{A-3}{2}} \,\mathrm{d}\xi \\
\lesssim & \langle t-r\rangle^{\frac{(-n+1)p+n+3}{2}} \beta^{-p}(t-r)
\end{aligned}$$

In the region of Λ_{12}, we have (3.35) and (3.34). Using $p > p_m$ we find

$$\begin{aligned}
J'_{12} \lesssim & \int_{\Lambda_{12}} \langle s+\rho\rangle^{\frac{(-n+1)p+n-A+2}{2}} \beta^{-p}(s-\rho)(r+\rho-t+s)^{\frac{A-3}{2}} \,\mathrm{d}\rho\,\mathrm{d}s \\
\leq & \int_{t-r}^{t+r} \langle\xi\rangle^{\frac{(-n+1)p+n-A+2}{2}} (\xi-(t-r))^{\frac{A-3}{2}} \,\mathrm{d}\xi \int_2^{t-r} \beta^{-p}(\eta)\,\mathrm{d}\eta \\
\lesssim & \left(\langle t-r\rangle^{\frac{(-n+1)p+n+1}{2}} + \int_{3(t-r)}^{t+r} \langle\xi\rangle^{\frac{(-n+1)p+n-1}{2}} \,\mathrm{d}\xi \right) \int_2^{t-r} \beta^{-p}(\eta)\,\mathrm{d}\eta \\
\lesssim & \langle t-r\rangle^{\frac{(-n+1)p+n+1}{2}} \int_2^{t-r} \beta^{-p}(\eta)\,\mathrm{d}\eta
\end{aligned}$$

In the region of Λ_{21}, we have (3.33) and

$$|I_A(\mu)| \lesssim (-1-\mu)^{\frac{1-A}{2}} = \left(\frac{2r\rho}{(t+r-s+\rho)(t-r-s-\rho)} \right)^{\frac{A-1}{2}} \lesssim \left(\frac{\rho}{t-r-s-\rho} \right)^{\frac{A-1}{2}}. \tag{3.38}$$

By $p < p_M$ we see

$$\begin{aligned}
J'_{21} \lesssim & \int_{\Lambda_{21}} \rho^{\frac{(-n+A)p+n+A-2}{2}} \langle s+\rho\rangle^{\frac{(-A+1)p}{2}} \beta^{-p}(s+\rho)(t-r-s-\rho)^{\frac{1-A}{2}} \,\mathrm{d}\rho\,\mathrm{d}s \\
\leq & \langle t-r\rangle^{\frac{(-A+1)p}{2}} \beta^{-p}(t-r) \int_{(t-r)/2}^{t-r} \int_{\xi/3}^{\xi} (\xi-\eta)^{\frac{(-n+A)p+n+A-2}{2}} (t-r-\xi)^{\frac{1-A}{2}} \,\mathrm{d}\eta\,\mathrm{d}\xi \\
\lesssim & \langle t-r\rangle^{\frac{(-n+1)p+n+A}{2}} \beta^{-p}(t-r) \int_{(t-r)/2}^{t-r} (t-r-\xi)^{\frac{1-A}{2}} \,\mathrm{d}\xi \\
\lesssim & \langle t-r\rangle^{\frac{(-n+1)p+n+3}{2}} \beta^{-p}(t-r).
\end{aligned}$$

In the region of Λ_{22}, we have (3.35) and (3.38), then

$$\begin{aligned}J'_{22}\lesssim&\int_{\Lambda_{22}}\langle s+\rho\rangle^{\frac{(-n+1)p+n+A-2}{2}}\beta^{-p}(s-\rho)(t-r-s-\rho)^{\frac{1-A}{2}}\,\mathrm{d}\,\rho\,\mathrm{d}\,s\\ \leq&\langle t-r\rangle^{\frac{(-n+1)p+n+A-2}{2}}\int_{(t-r)/2}^{t-r}(t-r-\xi)^{\frac{1-A}{2}}\,\mathrm{d}\,\xi\int_2^{(t-r)/3}\beta^{-p}(\eta)\,\mathrm{d}\,\eta\\ \lesssim&\langle t-r\rangle^{\frac{(-n+1)p+n+1}{2}}\int_2^{(t-r)/3}\beta^{-p}(\eta)\,\mathrm{d}\,\eta.\end{aligned}$$

In the region of Λ_{31}, we have

$$|I_A(\mu)|\lesssim(-1-\mu)^{\frac{1-A}{2}}=\left(\frac{2r\rho}{(t+r-s+\rho)(t-r-s-\rho)}\right)^{\frac{A-1}{2}}\lesssim\left(\frac{\rho}{t-r}\right)^{\frac{A-1}{2}}.\tag{3.39}$$

and (3.33). Since $p<p_M$, we have

$$\begin{aligned}J'_{31}\lesssim&\int_{\Lambda_{31}}\rho^{\frac{(-n+A)p+n+A-2}{2}}\langle s+\rho\rangle^{\frac{(-A+1)p}{2}}\beta^{-p}(s+\rho)(t-r)^{\frac{1-A}{2}}\,\mathrm{d}\,\rho\,\mathrm{d}\,s\\ \lesssim&\langle t-r\rangle^{\frac{1-A}{2}}\int_3^{(t-r)/2}\int_{\xi/3}^{\xi}(\xi-\eta)^{\frac{(-n+A)p+n+A-2}{2}}\langle\xi\rangle^{\frac{(-A+1)p}{2}}\beta^{-p}(\xi)\,\mathrm{d}\,\eta\,\mathrm{d}\,\xi\\ \lesssim&\langle t-r\rangle^{\frac{1-A}{2}}\int_3^{(t-r)/2}\langle\xi\rangle^{\frac{(-n+1)p+n+A}{2}}\beta^{-p}(\xi)\,\mathrm{d}\,\xi.\end{aligned}$$

In the region of Λ_{32}, we have (3.35) and (3.39), then

$$\begin{aligned}J'_{32}\lesssim&\int_{\Lambda_{32}}\langle s+\rho\rangle^{\frac{(-n+1)p+n+A-2}{2}}\beta^{-p}(s-\rho)(t-r)^{\frac{1-A}{2}}\,\mathrm{d}\,\rho\,\mathrm{d}\,s\\ \lesssim&\langle t-r\rangle^{\frac{1-A}{2}}\int_3^{(t-r)/2}\int_2^{\xi/3}\langle\xi\rangle^{\frac{(-n+1)p+n+A-2}{2}}\beta^{-p}(\eta)\,\mathrm{d}\,\eta\,\mathrm{d}\,\xi.\end{aligned}$$

We will focus on the estimate of J_{32}, the other terms can be estimate in a same discussion. Now, for $J_{32;1}$ with $p_m<p<p_M$, we have

$$J_{32;1}\lesssim\langle t-r\rangle^{\frac{1-A}{2}}\int_3^{(t-r)/2}\int_2^{\xi/3}\langle\xi\rangle^{\frac{(-n+1)p+n+A-2}{2}}\langle\eta\rangle^{-\frac{A-1}{2}p}\,\mathrm{d}\,\eta\,\mathrm{d}\,\xi.$$

When $p>p_d$, we have

$$J_{32;1}\lesssim\langle t-r\rangle^{\frac{1-A}{2}}\int_3^{(t-r)/2}\langle\xi\rangle^{\frac{(-n+1)p+n+A-2}{2}}\,\mathrm{d}\,\xi\lesssim N_1(t-r).$$

When $p=p_d$, we have

$$J_{32;1}\lesssim\langle t-r\rangle^{\frac{1-A}{2}}\int_3^{(t-r)/2}\langle\xi\rangle^{\frac{(-n+1)p+n+A-2}{2}}\ln\xi\,\mathrm{d}\,\xi\lesssim N_1(t-r).$$

When $p < p_d$, we have

$$J_{32;1} \lesssim \langle t-r\rangle^{\frac{1-A}{2}} \int_3^{(t-r)/2} \langle \xi\rangle^{\frac{(-n-A+2)p+n+A}{2}} \,\mathrm{d}\,\xi \lesssim N_1(t-r).$$

This leads to the result of (3.32) with $k=1$. For $J_{32;2}$ with $p_m < p < p_t$, we have

$$\begin{aligned} J_{32;2} \lesssim & \langle t-r\rangle^{\frac{1-A}{2}} \int_3^{(t-r)/2} \langle \xi\rangle^{\frac{(-n+1)p+n+A-2}{2}} \,\mathrm{d}\,\xi \int_2^{(t-r)/6} \langle \eta\rangle^{\frac{(-n+1)p^2+(n+1)p}{2}} \,\mathrm{d}\,\eta \\ \lesssim & \langle t-r\rangle^{\frac{(-n+1)p+n+1}{2}} \int_2^{(t-r)/6} \langle \eta\rangle^{\frac{(-n+1)p^2+(n+1)p}{2}} \,\mathrm{d}\,\eta \lesssim N_2(t-r). \end{aligned}$$

This finish the case $k=2$. Finally for $J_{32;3}$ with $p = p_t > p_d$, we have

$$\begin{aligned} J_{32;3} \lesssim & \langle t-r\rangle^{\frac{1-A}{2}} \int_3^{(t-r)/2} \int_2^{\xi/3} \langle \xi\rangle^{-1} \langle \eta\rangle^{\frac{(-A+1)p}{2}} \ln^p \eta \,\mathrm{d}\,\eta \,\mathrm{d}\,\xi \\ \lesssim & \langle t-r\rangle^{\frac{1-A}{2}} \int_3^{(t-r)/2} \langle \xi\rangle^{-1} \,\mathrm{d}\,\xi \lesssim \langle t-r\rangle^{\frac{1-A}{2}} \ln\langle t-r\rangle \lesssim N_3(t-r). \end{aligned}$$

Now we finish the proof for $A \in [2,3)$. The case $A=3$ is much simpler, using (3.28) we only need to consider Λ_{11} and Λ_{12}. By (3.29) and a similar process we get the desired estimate. □

3.5.3 Long time existence

In this subsection, we will construct a Cauchy sequence to approach the desired solution. We set $u_{-1}=0$ and u_{j+1} be the solution of the equation

$$\begin{cases} \partial_t^2 u_{j+1} - \Delta_A u_{j+1} = r^{\frac{(A-n)p+n-A}{2}} |u_j|^p, \quad r \in \mathbb{R}_+, \\ u_{j+1}(0,x) = \varepsilon r^{\frac{n-A}{2}} U_0(r), \quad \partial_t u_{j+1}(0,x) = \varepsilon r^{\frac{n-A}{2}} U_1(r). \end{cases} \tag{3.40}$$

By Lemma 3.14 and Lemma 3.15, we see

$$\begin{aligned} \langle t+r\rangle^{\frac{A-1}{2}} |u_{j+1}| \leq & \varepsilon C_0 \langle t-r\rangle^{\frac{1-A}{2}} \Psi + C_0 N_k(t-r) \|\omega_k u_j\|^p_{L^\infty_{s,\rho}(\Lambda_{t,r})}, \\ \Psi := & \|r^{\frac{n-A+2}{2}} U_0'(r)\|_{L^\infty_r} + \|r^{\frac{n-A}{2}} U_0(r)\|_{L^\infty_r} + \|r^{\frac{n-A+2}{2}} U_1(r)\|_{L^\infty_r}, \end{aligned}$$

with $k=1,2,3$ and C_0 great enough. Here

$$\begin{aligned} \Lambda_{t,r} := & \{(s,\rho) \in \Omega : s+\rho < t+r, s-\rho < t-r\}, \\ \Omega := & \{(t,r) \in \mathbb{R}^2_+ : t > r-1\}. \end{aligned}$$

Meanwhile, noticing that for any $p>1$, we have

$$||a|^p - |b|^p| \lesssim |a-b| \max(|a|,|b|)^{p-1}.$$

A little modification of the process in last subsection also shows

$$\begin{aligned}&\langle t+r\rangle^{\frac{A-1}{2}}\,|u_{j+1}-u_j|\\ \leq& C_1N_k(t-r)\|\omega_k(u_j-u_{j-1})\|_{L^\infty_{s,\rho}(\Lambda_{t,r})}\max_{l\in\{j,j-1\}}\|\omega_k u_l\|^{p-1}_{L^\infty_{s,\rho}(\Lambda_{t,r})},\end{aligned}$$

For $C_1\geq C_0$ great enough. Now, we are prepared to give the proof of Theorem 3.2.

Part 1: $\max(p_t,p_F)<p$.

In this part, we choose $k=1$. For $(t,r)\in\Omega$ we find

$$\begin{aligned}\omega_1|u_{j+1}|\leq&\varepsilon C_1\Psi+C_1\|\omega_1u_j\|^p_{L^\infty_{s,\rho}(\Lambda_{t,r})},\\ \omega_1|u_{j+1}-u_j|\leq&C_1\|\omega_1(u_j-u_{j-1})\,\|_{L^\infty_{s,\rho}(\Lambda_{t,r})}\max_{l\in\{j,j-1\}}\|\omega_1u_l\|^{p-1}_{L^\infty_{s,\rho}(\Lambda_{t,r})}.\end{aligned}$$

Taking the $L^\infty_{t,r}(\Omega)$ norm on both sides and we get

$$\begin{aligned}\|\omega_1u_{j+1}\|_{L^\infty_{t,r}(\Omega)}\leq&\varepsilon C_1\Psi+C_1\|\omega_1u_j\|^p_{L^\infty_{t,r}(\Omega)},\\ \|\omega_1(u_{j+1}-u_j)\|_{L^\infty_{t,r}(\Omega)}\leq&C_1\|\omega_1(u_j-u_{j-1})\,\|_{L^\infty_{t,r}(\Omega)}\max_{l\in\{j,j-1\}}\|\omega_1u_l\|^{p-1}_{L^\infty_{t,r}(\Omega)}.\end{aligned}$$

For any $\varepsilon>0$ satisfies $(2\varepsilon C_1\Psi)^p<\varepsilon\Psi$. We find

$$\|\omega_1u_j\|_{L^\infty_{t,r}(\Omega)}\leq 2\varepsilon C_1\Psi$$

holds for any j since $u_{-1}=0$. Meanwhile it also gives us

$$\begin{aligned}\|\omega_1(u_{j+1}-u_j)\|_{L^\infty_{t,r}(\Omega)}\leq&C_1(2\varepsilon C_1\Psi)^{p-1}\,\|\omega_1(u_j-u_{j-1})\,\|_{L^\infty_{t,r}(\Omega)}\\ \leq&\frac{1}{2}\|\omega_1(u_j-u_{j-1})\,\|_{L^\infty_{t,r}(\Omega)}.\end{aligned}$$

This means $\{u_j\}$ is a Cauchy sequence in its space. Set the limitation as u, its easy to check u and $F=r^{\frac{(A-n)p+n-A}{2}}|u|^p$ satisfied (3.30). Thus we get the desired weak solution.

Part 2: $p_S<p_t=p$.

In this part we take $k=3$, for $(t,r)\in\Omega$ we find

$$\begin{aligned}\omega_3|u_{j+1}|\leq&\varepsilon C_1\ln^{-1}\langle t-r\rangle\,\Psi+C_1\|\omega_3u_j\|^p_{L^\infty_{s,\rho}(\Lambda_{t,r})},\\ \omega_3|u_{j+1}-u_j|\leq&C_1\|\omega_3(u_j-u_{j-1})\,\|_{L^\infty_{s,\rho}(\Lambda_{t,r})}\max_{l\in\{j,j-1\}}\|\omega_3u_l\|^{p-1}_{L^\infty_{s,\rho}(\Lambda_{t,r})}.\end{aligned}$$

Noticing $\langle t-r\rangle\geq 2$, by a similarly process as above we get the desired solution.

Part 3: $p_S<p<p_t$.

In this part we take $k=2$, for $(t,r)\in\Omega$ we find

$$\begin{aligned}\omega_2|u_{j+1}|\leq&\varepsilon C_1\langle t-r\rangle^{\frac{(n-1)p-n-A}{2}}\,\Psi+C_1\|\omega_2u_j\|^p_{L^\infty_{s,\rho}(\Lambda_{t,r})},\\ \omega_2|u_{j+1}-u_j|\leq&C_1\|\omega_2(u_j-u_{j-1})\,\|_{L^\infty_{s,\rho}(\Lambda_{t,r})}\max_{l\in\{j,j-1\}}\|\omega_2u_l\|^{p-1}_{L^\infty_{s,\rho}(\Lambda_{t,r})}.\end{aligned}$$

Since $p < p_t$, we have $\frac{(n-1)p-n-A}{2} \leq 0$, by a similarly process again we get the desired solution

Part 4: $p < \min(p_d, p_F)$.

In this case, we choose $T_* = T_*(\varepsilon)$ satisfies $\varepsilon^{p-1} T_*^{\frac{(-n-A+2)p+n+A+2}{2}} = a$ with a fixed later. Here we define $\Omega_* := \Omega \cap \{t : t < T_*\}$ and choose $k = 1$, then for $(t, r) \in \Omega_*$ we find

$$\begin{aligned}
\omega_1|u_{j+1}| \leq & \varepsilon C_1 \Psi + C_1 \langle t-r \rangle^{\frac{(-n-A+2)p+n+A+2}{2}} \|\omega_1 u_j\|^p_{L^\infty_{s,\rho}(\Lambda_{t,r})}, \\
\omega_1|u_{j+1} - u_j| \leq & C_1 \langle t-r \rangle^{\frac{(-n-A+2)p+n+A+2}{2}} \|\omega_1 (u_j - u_{j-1})\|_{L^\infty_{s,\rho}(\Lambda_{t,r})} \\
& \cdot \max_{l \in \{j, j-1\}} \|\omega_1 u_l\|^{p-1}_{L^\infty_{s,\rho}(\Lambda_{t,r})}.
\end{aligned}$$

Taking the $L^\infty_{t,r}(\Omega_*)$ norm on both sides, when $T_* \geq 3$ we get

$$\begin{aligned}
\|\omega_1 u_{j+1}\|_{L^\infty_{t,r}(\Omega_*)} \leq & \varepsilon C_2 \Psi + C_2 T_*^{\frac{(-n-A+2)p+n+A+2}{2}} \|\omega_1 u_j\|^p_{L^\infty_{t,r}(\Omega_*)}, \\
\|\omega_1 (u_{j+1} - u_j)\|_{L^\infty_{t,r}(\Omega_*)} \leq & C_2 T_*^{\frac{(-n-A+2)p+n+A+2}{2}} \|\omega_1 (u_j - u_{j-1})\|_{L^\infty_{t,r}(\Omega_*)} \\
& \cdot \max_{l \in \{j, j-1\}} \|\omega_1 u_l\|^{p-1}_{L^\infty_{t,r}(\Omega_*)},
\end{aligned}$$

with some C_2 great enough. Considering ε such that $T_*(\varepsilon) \geq 3$ and defining $a = (2C_2)^{-p} \Psi^{1-p}$. We find

$$\begin{aligned}
\|\omega_1 u_j\|_{L^\infty_{t,r}(\Omega_*)} \leq & 2\varepsilon C_2 \Psi, \\
\|\omega_1 (u_{j+1} - u_j)\|_{L^\infty_{t,r}(\Omega_*)} \leq & \frac{1}{2} \|\omega_1 (u_j - u_{j-1})\|_{L^\infty_{t,r}(\Omega_*)}
\end{aligned}$$

holds for any j. Then we get the desired solution.

Part 5: $p = p_F < p_d$.

In this case, we choose T_* satisfies $\varepsilon^{p-1} \ln T_* = a$ with a fixed later, Ω_* as above and $k = 1$, for $(t, r) \in \Omega_*$ we find

$$\begin{aligned}
\omega_1|u_{j+1}| \leq & \varepsilon C_1 \Psi + C_1 \ln\langle t-r \rangle \|\omega_1 u_j\|^p_{L^\infty_{s,\rho}(\Lambda_{t,r})}, \\
\omega_1|u_{j+1} - u_j| \leq & C_1 \ln\langle t-r \rangle \|\omega_1 (u_j - u_{j-1})\|_{L^\infty_{s,\rho}(\Lambda_{t,r})} \max_{l \in \{j, j-1\}} \|\omega_1 u_l\|^{p-1}_{L^\infty_{s,\rho}(\Lambda_{t,r})}.
\end{aligned}$$

Similar as above, taking ε such that $T_*(\varepsilon) \geq 3$, defining $a := (2C_2)^{-p} \Psi^{1-p}$ and C_2 great enough, we get the Cauchy sequence $\{u_j\}$ and the desired solution.

Part 6: $p = p_d < p_F$.

In this case, we choose T_* satisfies

$$\varepsilon^{p-1} T_*^{\frac{(-n-A+2)p+n+A+2}{2}} \ln T_* = a \tag{3.41}$$

and Ω_* same as above. Taking $k=1$, for $(t,r)\in\Omega_*$ we find

$$\begin{aligned}\omega_1|u_{j+1}| \leq& \varepsilon C_1\Psi + C_1\langle t-r\rangle^{\frac{(-n-A+2)p+n+A+2}{2}}\ln\langle t-r\rangle\,\|\omega_1 u_j\|^p_{L^\infty_{s,\rho}(\Lambda_{t,r})},\\ \omega_1|u_{j+1}-u_j| \leq& C_1\langle t-r\rangle^{\frac{(-n-A+2)p+n+A+2}{2}}\ln\langle t-r\rangle\,\|\omega_1(u_j-u_{j-1})\|_{L^\infty_{s,\rho}(\Lambda_{t,r})}\\ &\cdot\max_{l\in\{j,j-1\}}\|\omega_1 u_l\|^{p-1}_{L^\infty_{s,\rho}(\Lambda_{t,r})}.\end{aligned}$$

Choosing ε such that $T_*(\varepsilon)\geq 3$, $a=(2C_2)^{-p}\Psi^{1-p}$ and C_2 great enough, we get the Cauchy sequence $\{u_j\}$ and the desired solution. To finish Theorem 3.2 of this part, we introduce the following claim and postpone the proof to the end of this section.

Claim 3.16. *Assume T_* satisfies $\varepsilon^{p-1}T_*^{\frac{-h_F(p)}{2}}\ln T_* = a$ with some constant a, then there exists two constant c_1, c_2 such that $c_1\varepsilon^{\frac{2(p-1)}{h_F(p)}}\ln(\varepsilon)^{\frac{2}{h_F(p)}}\leq T_*\leq c_2\varepsilon^{\frac{2(p-1)}{h_F(p)}}\ln(\varepsilon)^{\frac{2}{h_F(p)}}$ for ε small enough.*

Part 7: $p=p_d=p_F$.

In this case, we choose T_* satisfies $\varepsilon^{p-1}\ln^2 T_* = a$ and Ω_* same as above. Taking $k=1$, for $(t,r)\in\Omega_*$ we find

$$\begin{aligned}\omega_1|u_{j+1}| \leq& \varepsilon C_1\Psi + C_1\ln^2\langle t-r\rangle\,\|\omega_1 u_j\|^p_{L^\infty_{s,\rho}(\Lambda_{t,r})},\\ \omega_1|u_{j+1}-u_j| \leq& C_1\ln^2\langle t-r\rangle\,\|\omega_1(u_j-u_{j-1})\|_{L^\infty_{s,\rho}(\Lambda_{t,r})}\\ &\cdot\max_{l\in\{j,j-1\}}\|\omega_1 u_l\|^{p-1}_{L^\infty_{s,\rho}(\Lambda_{t,r})}.\end{aligned}$$

Choosing ε such that $T_*(\varepsilon)\geq 3$, $a=(2C_2)^{-p}\Psi^{1-p}$ and C_2 great enough, we get the Cauchy sequence $\{u_j\}$ and finish the proof.

Part 8: $p_d<p<p_S$.

In this case, we choose T_* satisfies $\varepsilon^{p(p-1)}T_*^{\frac{(-n+1)p^2+(n+1)p+2}{2}} = a$ with a fixed later and Ω_* as above. Moreover, we separate the region Ω_* to

$$\begin{aligned}\Omega_{*1} :=& \Omega_*\cap\{(t,r):\langle t-r\rangle\leq(b\varepsilon)^{\frac{2(p-1)}{(n-1)p-n-A}}\},\\ \Omega_{*2} :=& \Omega_*\cap\{(t,r):\langle t-r\rangle\geq(b\varepsilon)^{\frac{2(p-1)}{(n-1)p-n-A}}\}\end{aligned}$$

with b fixed later. Firstly we take $k=1$, for $(t,r)\in\Omega_{*1}$ we find

$$\begin{aligned}\omega_1|u_{j+1}| \leq& \varepsilon C_1\Psi + C_1\langle t-r\rangle^{\frac{(-n+1)p+n+A}{2}}\,\|\omega_1 u_j\|^p_{L^\infty_{s,\rho}(\Lambda_{t,r})},\\ \omega_1|u_{j+1}-u_j| \leq& C_1\langle t-r\rangle^{\frac{(-n+1)p+n+A}{2}}\,\|\omega_1(u_j-u_{j-1})\|_{L^\infty_{s,\rho}(\Lambda_{t,r})}\\ &\cdot\max_{l\in\{j,j-1\}}\|\omega_1 u_l\|^{p-1}_{L^\infty_{s,\rho}(\Lambda_{t,r})}.\end{aligned}$$

Taking the $L^\infty_{t,r}(\Omega_{*1})$ norm on both sides we get

$$\begin{aligned}\|\omega_1 u_{j+1}\|_{L^\infty_{t,r}(\Omega_{*1})} \leq& \varepsilon C_1\Psi + C_1(b\varepsilon)^{1-p}\|\omega_1 u_j\|^p_{L^\infty_{t,r}(\Omega_{*1})},\\ \|\omega_1(u_{j+1}-u_j)\|_{L^\infty_{t,r}(\Omega_{*1})} \leq& C_1(b\varepsilon)^{1-p}\|\omega_1(u_j-u_{j-1})\|_{L^\infty_{t,r}(\Omega_{*1})}\\ &\cdot \max_{l\in\{j,j-1\}}\|\omega_1 u_l\|^{p-1}_{L^\infty_{t,r}(\Omega_{*1})}.\end{aligned}$$

Choosing b such that $(2C_1)^p b^{1-p}\Psi^{p-1}=1$, we find

$$\begin{aligned}\|\omega_1 u_j\|_{L^\infty_{t,r}(\Omega_{*1})} \leq& 2\varepsilon C_1\Psi,\\ \|\omega_1(u_{j+1}-u_j)\|_{L^\infty_{t,r}(\Omega_{*1})} \leq& \frac{1}{2}\|\omega_1(u_j-u_{j-1})\|_{L^\infty_{t,r}(\Omega_{*1})}\end{aligned} \tag{3.42}$$

holds for any j. On the other hand, for $(t,r)\in\Omega_{*2}$, we separate $\Lambda_{t,r}$ into $\Lambda_{t,r,1}:=\Lambda_{t,r}\cap\Omega_{*1}$ and $\Lambda_{t,r,2}:=\Lambda_{t,r}\cap\Omega_{*2}$. Taking $k=1$ in the former region and $k=2$ in the latter region, a simple modification shows for $(t,r)\in\Omega_{*2}$,

$$\begin{aligned}\omega_2|u_{j+1}| \leq& \varepsilon C_1\langle t-r\rangle^{\frac{(n-1)p-n-A}{2}}\Psi + C_1\|\omega_1 u_j\|^p_{L^\infty_{s,\rho}(\Lambda_{t,r,1})}\\ &+ C_1\langle t-r\rangle^{\frac{(-n+1)p^2+(n+1)p+2}{2}}\|\omega_2 u_j\|^p_{L^\infty_{s,\rho}(\Lambda_{t,r,2})}\\ \omega_2|u_{j+1}-u_j| \leq& C_1\|\omega_1(u_j-u_{j-1})\|_{L^\infty_{s,\rho}(\Lambda_{t,r,1})}\max_{l\in\{j,j-1\}}\|\omega_1 u_l\|^{p-1}_{L^\infty_{s,\rho}(\Lambda_{t,r,1})}\\ &+ C_1\langle t-r\rangle^{\frac{(-n+1)p^2+(n+1)p+2}{2}}\|\omega_2(u_j-u_{j-1})\|_{L^\infty_{s,\rho}(\Lambda_{t,r,2})}\\ &\cdot \max_{l\in\{j,j-1\}}\|\omega_2 u_l\|^{p-1}_{L^\infty_{s,\rho}(\Lambda_{t,r,2})}\end{aligned}$$

Taking the $L^\infty_{t,r}(\Omega_{*2})$ norm on both sides, when $T_*\geq 3$ we get

$$\begin{aligned}\|\omega_2 u_{j+1}\|_{L^\infty_{t,r}(\Omega_{*2})} \leq& C_2(\varepsilon\Psi)^p + C_2T_*^{\frac{(-n+1)p^2+(n+1)p+2}{2}}\|\omega_2 u_j\|^p_{L^\infty_{t,r}(\Omega_{*2})},\\ \|\omega_2(u_{j+1}-u_j)\|_{L^\infty_{t,r}(\Omega_{*2})} \leq& C_2(\varepsilon\Psi)^{p-1}\|\omega_1(u_j-u_{j-1})\|_{L^\infty_{t,r}(\Omega_{*1})}\\ &+ C_2T_*^{\frac{(-n+1)p^2+(n+1)p+2}{2}}\|\omega_2(u_j-u_{j-1})\|_{L^\infty_{t,r}(\Omega_{*2})}\\ &\cdot \max_{l\in\{j,j-1\}}\|\omega_2 u_l\|^{p-1}_{L^\infty_{t,r}(\Omega_{*2})},\end{aligned}$$

with some C_2 great enough. Choosing ε such that $T_*(\varepsilon)\geq 3$ and $a=(2C_2)^{-p}\Psi^{p(1-p)}$, we find

$$\begin{aligned}\|\omega_2 u_{j+1}\|_{L^\infty_{t,r}(\Omega_{*2})} \leq& 2C_2(\varepsilon\Psi)^p,\\ \|\omega_2(u_{j+1}-u_j)\|_{L^\infty_{t,r}(\Omega_{*2})} \leq& C_2(\varepsilon\Psi)^{p-1}\|\omega_1(u_j-u_{j-1})\|_{L^\infty_{t,r}(\Omega_{*1})}\\ &+ \frac{1}{2}\|\omega_2(u_j-u_{j-1})\|_{L^\infty_{t,r}(\Omega_{*2})},\end{aligned}$$

holds for any j. Adding (3.42), we find $\{u_j\}$ is a Cauchy sequence in its space, then we get the desired solution.

Part 9: $p_d < p = p_S$.

In this part, we choose T_* satisfies $\varepsilon^{p(p-1)} \ln T_* = a$ and Ω_{*1}, Ω_{*2} as above with a, b fixed latter. Similarly, for $(t,r) \in \Omega_{*1}$ we find

$$\begin{aligned}
\omega_1|u_{j+1}| \leq & \varepsilon C_1 \Psi + C_1\langle t-r\rangle^{\frac{(-n+1)p+n+A}{2}} \|\omega_1 u_j\|^p_{L^\infty_{s,\rho}(\Lambda_{t,r})},\\
\omega_1|u_{j+1}-u_j| \leq & C_1\langle t-r\rangle^{\frac{(-n+1)p+n+A}{2}} \|\omega_1(u_j-u_{j-1})\|_{L^\infty_{s,\rho}(\Lambda_{t,r})}\\
& \cdot \max_{l\in\{j,j-1\}} \|\omega_1 u_l\|^{p-1}_{L^\infty_{s,\rho}(\Lambda_{t,r})}.
\end{aligned}$$

For $(t,r) \in \Omega_{*2}$ we find

$$\begin{aligned}
\omega_2|u_{j+1}| \leq & \varepsilon C_1\langle t-r\rangle^{\frac{(n-1)p-n-A}{2}} \Psi + C_1\|\omega_1 u_j\|^p_{L^\infty_{s,\rho}(\Lambda_{t,r,1})}\\
& + C_1\langle t-r\rangle^{\frac{(-n+1)p^2+(n+1)p+2}{2}} \|\omega_2 u_j\|^p_{L^\infty_{s,\rho}(\Lambda_{t,r,2})}\\
\omega_2|u_{j+1}-u_j| \leq & C_1\|\omega_1(u_j-u_{j-1})\|_{L^\infty_{s,\rho}(\Lambda_{t,r,1})} \max_{l\in\{j,j-1\}} \|\omega_1 u_l\|^{p-1}_{L^\infty_{s,\rho}(\Lambda_{t,r,1})}\\
& + C_1 \ln\langle t-r\rangle \|\omega_2(u_j-u_{j-1})\|_{L^\infty_{s,\rho}(\Lambda_{t,r,2})}\\
& \cdot \max_{l\in\{j,j-1\}} \|\omega_2 u_l\|^{p-1}_{L^\infty_{s,\rho}(\Lambda_{t,r,2})}.
\end{aligned}$$

Taking b satisfies $(2C_1)^p b^{1-p}\Psi^{p-1} = 1$, choosing ε such that $T_*(\varepsilon) \geq 3$, $a = (2C_2)^{-p}\Psi^{p(1-p)}$ and C_2 great enough, we get the Cauchy sequence $\{u_j\}$ and the desired solution.

Before the end of this section, we show the proof of Claim 3.16. Following (3.41), we can easily found

$$\varepsilon^{\frac{2(p-1)}{h_F(p)}-\delta} \lesssim T_* \lesssim \varepsilon^{\frac{2(p-1)}{h_F(p)}}$$

for any $0 < \delta \ll 1$ and ε small enough. This suggests us to define

$$S(\varepsilon) := \varepsilon^{-\frac{2(p-1)}{h_F(p)}} T_*, \qquad \varepsilon^{-\delta} \lesssim S \lesssim 1.$$

Then, (3.41) goes to

$$a = S^{\frac{-h_F(p)}{2}} \ln\left(\varepsilon^{\frac{2(p-1)}{h_F(p)}} S\right) \approx S^{\frac{-h_F(p)}{2}} \ln(\varepsilon)\,,$$

and then

$$S \approx \ln^{\frac{2}{h_F(p)}}(\varepsilon), \qquad T_* \approx \varepsilon^{\frac{2(p-1)}{h_F(p)}} \ln^{\frac{2}{h_F(p)}}(\varepsilon),$$

which finishes the proof.

3.6 Long time existence for $A \in [3, \infty)$

In this section, we will consider the case $A \in [3, \infty)$, and show the proof of Theorem 3.3. Again, we only need to consider the equation (3.21).

3.6.1 Estimate for linear solution

In this subsection, we will construct some prior estimates of the solution to the linear equation (3.22). Firstly we give the following argument.

Lemma 3.17. *Let u be the solution of (3.22). We have*

$$\left\| r^{\frac{A-1}{2}} u \right\|_{L_t^\infty L_r^q} \lesssim \|r^{\frac{A+1}{2}} g\|_{L_r^q} + \|r^{\frac{A-1}{2}} f\|_{L_r^q} + \|r^{\frac{A+1}{2}} F\|_{L_t^q L_r^1}. \tag{3.43}$$

for any $1 < q < \infty$, and

$$\left\| r^{\frac{A-1}{2}-\alpha} u \right\|_{L_t^\sigma L_r^p} \lesssim \|r^{\frac{A+1}{2}} g\|_{L_r^q} + \|r^{\frac{A-1}{2}} f\|_{L_r^q} + \|r^{\frac{A+1}{2}} F\|_{L_t^q L_r^1}, \tag{3.44}$$

provided that

$$1 < \frac{q}{p} < \frac{\sigma}{p} < \infty, \qquad \alpha p = 1 - \frac{p}{q} + \frac{p}{\sigma}. \tag{3.45}$$

We also have a modification result.

Lemma 3.18. *Let u be the solution of (3.22). If $1 < p \le (n+1)/(n-1)$, then*

$$\begin{aligned} &(T+1)^{\frac{(n-1)p-n-1}{2p}} \left\| r^{\frac{(-n+A)p+n+1}{2p}} u(T,\cdot) \right\|_{L_r^p(r<T+1)} \\ \lesssim &\|r^{\frac{A+1}{2}} g\|_{L_r^p} + \|r^{\frac{A-1}{2}} f\|_{L_r^p} + \|r^{\frac{A+1}{2}} F\|_{L_t^p L_r^1(t<T)}. \end{aligned} \tag{3.46}$$

Also, if $(n+1)/(n-1) \le p \le p_S$, then

$$\begin{aligned} &T^{\frac{(n-1)p-n-1}{2p}} \left\| r^{\frac{(-n+A)p+n+1}{2p}} u(T,\cdot) \right\|_{L_r^p} \\ \lesssim &\|r^{\frac{A+1}{2}+\frac{1}{p}} g\|_{L_r^\infty} + \|r^{\frac{A-1}{2}+\frac{1}{p}} f\|_{L_r^\infty} + T^{\frac{1}{p}} \|r^{\frac{A+1}{2}} F\|_{L_t^\infty L_r^1(T/4<t<T)} \\ &+ \|r^{\frac{A+1}{2}} g\|_{L_r^p} + \|r^{\frac{A-1}{2}} f\|_{L_r^p} + \|r^{\frac{A+1}{2}} F\|_{L_t^p L_r^1(t<T/4)} \end{aligned} \tag{3.47}$$

Proof of Lemma 3.17. When $A \in \mathbb{Z}^+$, this result is exactly the same with the Theorem 4.7 of [24] since we can take κ arbitrary close to 2. So we only deal with the non-integer case.

By Lemma 3.12 and Lemma 3.13, for $\mu = \frac{r^2+\rho^2-t^2}{2r\rho}$ and $\delta > 0$ arbitrary small we have

$$\begin{aligned} r^{\frac{A-1}{2}}|u_g| \lesssim & \int_{|r-t|}^{t+r}(1+\mu)^{-\delta}\rho^{\frac{A-1}{2}}|g(\rho)|\,\mathrm{d}\,\rho \\ & + \int\limits_{0\le\rho\le t-r}(1-\mu)^{\frac{1-A}{2}+\delta}|1+\mu|^{-\delta}\rho^{\frac{A-1}{2}}|g(\rho)|\,\mathrm{d}\,\rho \\ \lesssim & \int_{|r-t|}^{t+r}(1+\mu)^{-\delta}\rho^{-1}|\rho^{\frac{A+1}{2}}g(\rho)|\,\mathrm{d}\,\rho \\ & + \int\limits_{0\le\rho\le t-r}(1-\mu)^{-1}|1+\mu|^{-\delta}\rho^{-1}|\rho^{\frac{A+1}{2}}g(\rho)|\,\mathrm{d}\,\rho. \end{aligned} \tag{3.48}$$

Using Proposition 2.7 and Proposition 4.4 in [24], we get (3.43) for $u = u_g$. Through a simple calculation, we also find

$$r^{\frac{A-1}{2}}u_f = \frac{1}{2}h(t+r) - \int_0^{t+r}\frac{t}{r\rho}I_A'(\mu)h(\rho)\,\mathrm{d}\,\rho, \qquad h(\rho) := r^{\frac{A-1}{2}}f(\rho).$$

Using Lemma 3.13 again and noticing $A > 3$, for $\delta > 0$ arbitrary small we have

$$\begin{aligned} r^{\frac{A-1}{2}}|u_f| \lesssim & \frac{1}{2}|h(t+r)| + \int_{|t-r|}^{t+r}\frac{t}{r\rho}|1+\mu|^{-\delta}|h(\rho)|\,\mathrm{d}\,\rho \\ & + \int\limits_{0\le\rho\le t-r}\frac{t}{r\rho}(1-\mu)^{-1}|1+\mu|^{-\delta}|h(\rho)|\,\mathrm{d}\,\rho \\ & + \left|\int_0^{t+r}\frac{t}{r\rho}(1+\mu)^{-1}h(\rho)\,\mathrm{d}\,\rho\right| + \int\limits_{0\le\rho\le t-r}\frac{t}{r\rho}|1-\mu|^{-1}|h(\rho)|\,\mathrm{d}\,\rho \\ \equiv & K_1 + K_2 + K_3 + K_4 + K_5. \end{aligned} \tag{3.49}$$

Here we easily find that

$$\|K_1\|_{L_t^\infty L_r^q} \lesssim \|h\|_{L_r^q}, \qquad \|r^{-\alpha}K_1\|_{L_t^\sigma L_r^p} \lesssim \|h\|_{L_r^q}.$$

Where the later one is the consequence of *Hardy-Littlewood* inequality provided that (3.45).

Meanwhile, adopting Proposition 2.5 in [24] to deal with K_2, and adopting Proposition 4.4 in [24] for K_3 and K_5, we find both of them have the same control as K_1.

As for K_4, noticing

$$\frac{t}{r\rho}(1+\mu)^{-1} = \frac{1}{r+\rho-t} - \frac{1}{t+r+\rho},$$

we can separate the K_4 to

$$\begin{aligned} K_4 \lesssim & \left|\int_0^\infty \frac{1}{\rho+r-t}h(\rho)\,\mathrm{d}\,\rho\right| + \int_{t+r}^\infty \frac{1}{\rho+r-t}|h(\rho)|\,\mathrm{d}\,\rho \\ & + \frac{1}{t+r}\int_0^{t+r}|h(\rho)|\,\mathrm{d}\,\rho \\ \equiv & K_{4,1}(t-r) + K_{4,2}(t,r) + K_{4,3}(t+r). \end{aligned}$$

Because *Hilbert*-transform is bounded in L^q space for $1 < q < \infty$, adding *Hardy-Littlewood* inequality we get the control of $K_{4,1}(t-r)$. Also, using the boundedness of *Hardy-Littlewood* maximal function in L^q space, adding *Hardy-Littlewood* inequality again we get a same estimate of $K_{4,3}(t+r)$.

Finally, for $K_{4,2}(t,r)$, we have

$$K_{4,2}(t,r) = \int_r^\infty \frac{1}{\rho+r}|h(\rho+t)|\,\mathrm{d}\,\rho \le \int_r^\infty \rho^{-1}|h(\rho+t)|\,\mathrm{d}\,\rho.$$

Then, using *Hardy's* inequality we find

$$\|K_{4,2}\|_{L_t^\infty L_r^q} \lesssim \|h\|_{L_r^q}.$$

On the other hand, for any $G(t,r)$ with $\|G\|_{L_t^{\sigma'}L_r^{p'}} \le 1$ we see

$$\begin{aligned}
&\int_0^\infty\int_0^\infty r^{-\alpha}K_{4,2}(t,r)G(t,r)\,\mathrm{d}\,r\,\mathrm{d}\,t\\
=&\int_0^\infty\int_t^\infty\int_0^{\rho-t}\frac{r^{-\alpha}}{\rho+r-t}|h(\rho)|G(t,r)\,\mathrm{d}\,r\,\mathrm{d}\,\rho\,\mathrm{d}\,t\\
\lesssim&\int_0^\infty\int_t^\infty|h(\rho)|\left\|\frac{r^{-\alpha}}{\rho+r-t}\right\|_{L_r^p(0,\rho-t)}\|G\|_{L_r^{p'}}\,\mathrm{d}\,\rho\,\mathrm{d}\,t\\
\approx&\int_0^\infty\int_t^\infty|h(\rho)||\rho-t|^{-\alpha-1+\frac{1}{p}}\|G\|_{L_r^{p'}}\,\mathrm{d}\,\rho\,\mathrm{d}\,t\\
\lesssim&\left\|\int_t^\infty|h(\rho)||\rho-t|^{-\alpha-1+\frac{1}{p}}\,\mathrm{d}\,\rho\right\|_{L_t^\sigma}\\
\lesssim&\|h\|_{L_r^q},
\end{aligned}$$

where in the last step we use the *Hardy-Littlewood* inequality. Now, we find

$$\|r^{-\alpha}K_{4,2}\|_{L_t^\sigma L_r^p} \le \sup_{\|G\|_{L_t^{\sigma'}L_r^{p'}}\le 1}\left\langle r^{-\alpha}K_{4,2},G\right\rangle \lesssim \|h\|_{L_r^q}.$$

Adding all together, we get (3.43) for $u = u_f$. Finally, under $\mu = \frac{r^2+\rho^2-(t-s)^2}{2r\rho}$ we have

$$r^{\frac{A-1}{2}}u_F = \int_0^t\int_0^{r+t-s} I_A(\mu)\rho^{\frac{A-1}{2}}F(s,\rho)\,\mathrm{d}\,\rho\,\mathrm{d}\,s.$$

Using Lemma 3.13 and noticing $A > 3$, for $\delta > 0$ arbitrary small we have

$$\begin{aligned}
r^{\frac{A-1}{2}}|u_F| \lesssim&\int_0^t\int_{|r-t+s|}^{r+t-s}(1+\mu)^{-\delta}\rho^{-1}|G(s,\rho)|\,\mathrm{d}\,\rho\,\mathrm{d}\,s\\
&+\iint\limits_{\substack{0\le s\le t\\ 0\le\rho\le t-s-r}}(1-\mu)^{-1}|1+\mu|^{-\delta}\rho^{-1}|G(s,\rho)|\,\mathrm{d}\,\rho\,\mathrm{d}\,s
\end{aligned}\tag{3.50}$$

with $G(s,\rho) := \rho^{\frac{A+1}{2}}F(s,\rho)$. Because δ is arbitrary, using Proposition 4.5 in [24], we get (3.43) for $u = u_F$. Adding together, we finish the proof. □

Proof of Lemma 3.18. The proof of Lemma 3.18 is most the same with the proof of Theorem 6.4 in [24] so we skip it. The only difference is that we don't need to deal with the spherical variable and we can choose δ in (3.48), (3.49), (3.50) arbitrary small, which both better than the original one. □

3.6.2 Long time existence for $1 < p < p_{conf}$

In this subsection, we will give the proof of Theorem 3.3. The main process of proof is most the same as that of [24]. So we only prove global existence in $p_S < p < p_{conf}$ and long time existence in $p_m \le p < p_S$ to show such processes fit our frame.

Similar to the last section, we will construct a Cauchy sequence to approach the weak solution. We set $u_{-1} = 0$ and u_{j+1} be the solution of the equation (3.40).

We are going to use the (3.44), where we set

$$q = \frac{2(p-1)}{(n+3)-(n-1)p}, \qquad \sigma = pq, \qquad \alpha = \frac{(-n+A)p+n+1}{2p}.$$

Here $p < p_{conf}$ so $0 < q < \infty$ and $p > p_S$ so $q > p$. Then, we conclude

$$\begin{aligned}
\left\| r^{\frac{(-n+A)p+n+1}{2p}} u_{j+1} \right\|_{L_t^\sigma L_r^p} \le & \varepsilon C_0 \Psi + C_0 \| r^{\frac{(-n+A)p+n+1}{2}} |u_j|^p \|_{L_t^q L_r^1} \\
\le & \varepsilon C_0 \Psi + C_0 \| r^{\frac{(-n+A)p+n+1}{2p}} u_j \|_{L_t^{pq} L_r^p}^p \\
\Psi := & \left\| r^{\frac{n+1}{2}} U_1 \right\|_{L_r^q} + \| r^{\frac{n-1}{2}} U_0 \|_{L_r^q}
\end{aligned}$$

for some C_0 great enough. Then, for any ε satisfies $(2\varepsilon C_0 \Psi)^p < \varepsilon \Psi$ we find

$$\left\| r^{\frac{(-n+A)p+n+1}{2p}} u_j \right\|_{L_t^\sigma L_r^p} \le 2\varepsilon C_0 \Psi$$

holds for any $j \ge 0$. By this result and (3.44) we also find

$$\begin{aligned}
& \left\| r^{\frac{(-n+A)p+n+1}{2p}} (u_{j+1} - u_j) \right\|_{L_t^\sigma L_r^p} \\
\le & C_0 \| r^{\frac{(-n+A)p+n+1}{2}} (|u_j|^p - |u_{j-1}|^p) \|_{L_t^q L_r^1} \\
\le & C_1 \| r^{\frac{(-n+A)p+n+1}{2p}} (u_j - u_{j-1}) \|_{L_t^{pq} L_r^p} \max_{k \in \{j, j-1\}} \| r^{\frac{(-n+A)p+n+1}{2p}} u_k \|_{L_t^{pq} L_r^p}^{p-1} \\
\le & C_1 (2\varepsilon C_1 \Psi)^{p-1} \| r^{\frac{(-n+A)p+n+1}{2p}} (u_j - u_{j-1}) \|_{L_t^{pq} L_r^p}
\end{aligned}$$

with some $C_1 \ge C_0$ great enough. Now, for any $0 < \varepsilon$ with $C_1 (2\varepsilon C_1 \Psi)^{p-1} = 1/2$, we find $\{u_j\}$ is a Cauchy sequence in its space. Set the limitation as u, using (3.43) we can

also find $\|r^{\frac{A-1}{2}}u\|_{L_t^\infty L_r^q} \le 2\varepsilon C_2\Psi$ with some C_2. To check it is the weak solution of (3.21) indeed, we need to show (3.30). For any compact set $K_t \times K_x \subset \mathbb{R}_+ \times \mathbb{R}^n$, we find

$$\begin{aligned}
\left\| r^{\frac{(-n+A)p+n+A}{2}-1}|u|^p \right\|_{L^1_{t,r}(K_t\times K_x)} \le & C(K_t)\left\| r^{\frac{(-n+A)p+n+1}{2p}}u\right\|^p_{L_t^\sigma L_r^p}\left\| r^{\frac{A-3}{2}}\right\|_{L_r^\infty(K_x)} \\
\le & (2\varepsilon C_0\Psi)^p C(K_t,K_x) < \infty \\
\|r^{A-1}u\|_{L^1_{t,r}(K_t\times K_x)} \le & C(K_t)\left\| r^{\frac{A-1}{2}}u\right\|_{L_t^\infty L_r^q}\left\| r^{\frac{A-1}{2}}\right\|_{L_r^{q'}(K_x)} \\
\le & (2\varepsilon C_2\Psi) C(K_t,K_x) < \infty.
\end{aligned}$$

This finish the proof.

Next we consider $(n+1)/(n-1) \le p < p_S$. Set

$$A_j(T) := T^{\frac{(n-1)p-n-1}{2p}}\left\| r^{\frac{(-n+A)p+n+1}{2p}}u_j(T,\cdot)\right\|_{L_r^p},$$

using (3.47) we see

$$\begin{aligned}
A_{j+1}(T) \le & \varepsilon C_0\Psi + C_0T^{\frac{1}{p}}\|r^{\frac{(-n+A)p+n+1}{2}}|u_j|^p\|_{L_t^\infty L_r^1(T/4<t<T)} \\
& + C_0\|r^{\frac{(-n+A)p+n+1}{2}}|u_j|^p\|_{L_t^pL_r^1(t<T/4)} \\
\le & \varepsilon C_0\Psi + C_0T^{\frac{1}{p}}\|t^{\frac{(-n+1)p+n+1}{2p}}A_j(t)\|^p_{L_t^\infty(T/4<t<T)} \\
& + C_0\|t^{\frac{(-n+1)p+n+1}{2p}}A_j(t)\|^p_{L_t^{p^2}(t<T/4)}
\end{aligned}$$

for some C_0 great enough and

$$\Psi := \|r^{\frac{n+1}{2}+\frac{1}{p}}U_1\|_{L_r^\infty} + \|r^{\frac{n-1}{2}+\frac{1}{p}}U_0\|_{L_r^\infty} + \|r^{\frac{u+1}{2}}U_1\|_{L_r^p} + \|r^{\frac{n-1}{2}}U_0\|_{L_r^p}.$$

While $\sup_{0\le T\le T_*} A_j(T) \le 2\varepsilon C_0\Psi$ holds for some j and T_* defined in (3.8) with c fixed later, we find

$$\begin{aligned}
\sup_{0\le T\le T_*} A_{j+1}(T) \le & \varepsilon C_0\Psi + C_1(2\varepsilon C_0\Psi)^pT_*^{\frac{(-n+1)p^2+(n+1)p+2}{2p}} \\
\le & \varepsilon C_0\Psi + \varepsilon C_1(2C_0\Psi)^pc^{\frac{-h_S(p;n)}{2p}}
\end{aligned}$$

with some C_1. Choosing c small enough such that $C_1(2C_0\Psi)^pc^{\frac{-h_S(p;n)}{2p}} \le C_0\Psi$ and noticing $A_{-1}(T) \equiv 0$, we find such dominate holds for any j. A similar process also shows

$$\begin{aligned}
\sup_{0\le T\le T_*} B_{j+1}(T) \le & \frac{1}{2}\sup_{0\le T\le T_*} B_j(T), \\
B_j(T) := & T^{\frac{(n-1)p-n-1}{2p}}\left\| r^{\frac{(-n+A)p+n+1}{2p}}(u_j-u_{j-1})(T,\cdot)\right\|_{L_r^p}.
\end{aligned}$$

Now we check (3.30). For any compact set $K_t \times K_x \subset \mathbb{R}_+ \times \mathbb{R}^n$, we find

$$\begin{aligned}
\left\| r^{\frac{(-n+A)p+n+A}{2}-1}|u|^p \right\|_{L^1_{t,r}(K_t\times K_x)} \leq & \left\| t^{\frac{(n-1)p-n-1}{2p}} r^{\frac{(-n+A)p+n+1}{2p}} u \right\|^p_{L^\infty_t L^p_r} \\
& \left\| t^{\frac{(-n+1)p+n+1}{2}} r^{\frac{A-3}{2}} \right\|_{L^1_t L^\infty_r} \\
\leq & (2\varepsilon C_1 \Psi)^p C(K_t, K_x) < \infty, \\
\|r^{A-1}u\|_{L^1_{t,r}(K_t\times K_x)} \leq & \left\| t^{\frac{(n-1)p-n-1}{2p}} r^{\frac{(-n+A)p+n+1}{2p}} u \right\|_{L^\infty_t L^p_r} \\
& \left\| t^{\frac{(-n+1)p+n+1}{2p}} r^{\frac{(n+A-2)p-n-1}{2p}} \right\|_{L^1_t L^{p'}_r} \\
\leq & (2\varepsilon C_1 \Psi)^p C(K_t, K_x) < \infty.
\end{aligned}$$

This finish the proof.

Chapter 4

Long time existence of wave equation with mixed power terms

4.1 Main results and notations

In this chapter, we study the long time existence of equation (1.9) proposed in 1.4. The results presented in this chapter was obtained by the author's joint work [5]. First of all, we give our main conclusion, when the dimension $n = 3$, we have

Theorem 4.1. *Let $n = 3$, $p = p_S(3) = 1 + \sqrt{2}$ and $q \geq p$. Suppose that the data (f, g) satisfy*

$$\Lambda := \|\langle x\rangle^5 \partial_x^{\leq 3} f\|_{L_x^\infty} + \|\langle x\rangle^5 \partial_x^{\leq 2} g\|_{L_x^\infty} < \infty. \tag{4.1}$$

then there exists an $\varepsilon_0(\Lambda, q) > 0$ and a constant $c > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, (1.9) has a unique solution $u \in C^0\big(\big[0, \bar{T}_\varepsilon\big]; H^3(\mathbb{R}^3)\big) \cap C^1\big(\big[0, \bar{T}_\varepsilon\big]; H^2(\mathbb{R}^3)\big)$ where

$$\bar{T}_\varepsilon := \exp\big(c\varepsilon^{-p_S(p_S-1)}\big).$$

Remark 4.1. *As we discussed in Section 1.4, this bound matches the upper bound, so it is optimal.*

Turning to the 2 dimensional case, we have

Theorem 4.2. *Assume $p = p_S(2) = (3 + \sqrt{17})/2$ and $q \geq p$. Considering (1.9) with data (f, g) satisfying (4.1), we have a similar existence result with small data, as in Theorem 4.1, with $\bar{T}_\varepsilon$ reqlaced by*

$$\tilde{T}_\varepsilon := \exp\Big(c\varepsilon^{-(p_S-1)^2/2}\Big).$$

Remark 4.2. *This result improve the result $\exp\big(c\varepsilon^{-(p_S-1)}\big)$ which is proposed in [14].*

We also introduce some notations we will use in this chapter. We will use some kinds of special vector fields. There are spatial rotation: $\Omega_{jk} = x^k\partial_j - x^j\partial_k$, Lorentz boost: $\Omega_{0k} = t\partial_k + x^k\partial_t$ and scaling: $L_0 = t\partial_t + x^i\partial_i$. Set $\Gamma = \{\partial_\mu, \Omega_{\mu\nu}, L_0\}$ be the well-known *Klainerman* vector fields. For such vector fields, we have

$$\begin{aligned} &[\partial_\mu, \Gamma_\alpha]\, f = C^\beta_{\mu\alpha}\partial_\beta f, \qquad [\Gamma_\alpha, \Gamma_\beta]\, f = C^\sigma_{\alpha\beta}\Gamma_\sigma f, \\ &[\Gamma_\alpha, \Box]\, f = C_\alpha\, \Box f, \qquad \Big(\alpha, \beta = 0, 1, \cdots, (n^2+3n+2)/2\Big) \end{aligned} \tag{4.2}$$

where $[X, Y]$ denotes the commutator $XY - YX$. And all coefficients are belong to C_b^∞.

4.2 The proof of Theorem 4.1 and Theorem 4.2

4.2.1 Preliminaries

We firstly list some lemmas to be used later.

Lemma 4.3 (Local existence)**.** *When $n \le 3$. For the equation*

$$\begin{cases} \partial_t^2 u - \Delta u = F(u, \partial u), \\ u(0,x) = f(x), u_t(0,x) = g(x). \end{cases}$$

If $(f, g) \in H^3 \times H^2$, $F \in C^2$ and $F(0,0) = 0$ then there is a $T > 0$, depending on the norm of the data, so that this Cauchy problem has a unique solution satisfying

$$\|\partial^{\le 3} u(t,\cdot)\|_{L^2_x} < \infty, \quad 0 \le t \le T.$$

Also, if T_* is the supremum over all such times T, then either $T_* = \infty$ or

$$|\partial^{\le 3} u| \notin L^\infty_t L^2_x (0 \le t < T_*).$$

Proof. The result is classical, for a proof, see, e.g., Chapter 12 of [10]. □

Lemma 4.4 (Theorem 6.4 of [24])**.** *When $n = 3$, suppose u solves the linear equation*

$$\begin{cases} \Box u(t,x) = F(t,x), \\ u(0,x) = f(x), u_t(0,x) = g(x) \end{cases}$$

in $[0, T_*) \times \mathbb{R}^3$. Then there exists a constant C, such that $\forall\, T < T_*$

$$\begin{aligned} T^{1/p_S^2}\|u(T,\cdot)\|_{\mathcal{L}^{p_S}_r L^3_\omega} \le & C\Big(\|r^{1-2/p_S}\partial_\omega^{\le 1} f\|_{\mathcal{L}^{p_S}_r L^{3/2}_\omega} + \|r^{1+1/p_S}\partial_\omega^{\le 1} f\|_{\mathcal{L}^\infty_r L^{3/2}_\omega} \\ & + \|r^{2-2/p_S} g\|_{\mathcal{L}^{p_S}_r L^{3/2}_\omega} + \|r^{2+1/p_S} g\|_{\mathcal{L}^\infty_r L^{3/2}_\omega} \\ & + \|F\|_{L^{p_S}_t \mathcal{L}^1_r L^{3/2}_\omega (t<T/4)} + T^{1/p_S}\|F\|_{L^\infty_t \mathcal{L}^1_r L^{3/2}_\omega (T/4<t<T)} \Big). \end{aligned} \tag{4.3}$$

Lemma 4.5 (Theorem 1.4 of [18]). *When $n = 2$, suppose u solves the linear equation*

$$\begin{cases} \Box u(t,x) = F(t,x), \\ u(0,x) = f(x), u_t(0,x) = g(x) \end{cases}$$

in $[0,T_*)\times\mathbb{R}^2$. Let $p_* = \frac{p_S-1}{2}$, $s_c = 1-1/p, b > \frac{1}{p_* p_S} + \frac{1}{p_S}$, $X^b \equiv H_\omega^{s_c+\delta,b} \times H_\omega^{s_c-1+\delta,b} (\delta > 0)$, where

$$H_\omega^{s,b} = \{u \in H^s : \left\|(1-\Delta_\omega)^{b/2} u\right\|_{H^s} < \infty\}, \qquad \Delta_\omega = \sum_{1\le i<j\le n} \Omega_{ij}^2.$$

Then there exists a constant C, such that $\forall\, T < T_*$

$$\|u\|_{L_t^{p_* p_S} \mathcal{L}_r^{p_S} L_\omega^2 (t<T)} \le C(\ln(2+T))^{1/(p_* p_S)} \Big(\|(f,g)\|_{X^b} + \|F\|_{L_t^{p_*} \mathcal{L}_r^1 L_\omega^2} \Big). \tag{4.4}$$

Lemma 4.6 (Energy inequality). *For any n, suppose u solves the linear equation*

$$\begin{cases} \Box u(t,x) = F(t,x), \\ u(0,x) = f(x), u_t(0,x) = g(x). \end{cases}$$

We have, $\forall\, T > 0$

$$\|\partial u(T,\cdot)\|_{L_x^2} \le \|\partial u(0,\cdot)\|_{L_x^2} + \|F\|_{L_t^1 L_x^2 (t<T)}$$

Since the property of commutator (4.2), we also have for any $k \in \mathbb{Z}^+$

$$\|\partial \Gamma^{\le k} u(T,\cdot)\|_{L_x^2} \le C_k \big(\|\partial \Gamma^{\le k} u(0,\cdot)\|_{L_x^2} + \|\Gamma^{\le k} F\|_{L_t^1 L_x^2 (t<T)} \big) .$$

Lemma 4.7 (*Klainerman-Sobolev* inequalities). *Let $v \in C^\infty(\mathbb{R}^{1+n})$ vanish when $|x|$ is large, $1 \le q < \infty$ and $k > n/p$, then if $t > 0$*

$$\langle t+r\rangle^{(n-1)/p} \langle t-r\rangle^{1/p} |v(t,x)| \le C \|\Gamma^{\le k} v(t,\cdot)\|_{L_x^p}. \tag{4.5}$$

Otherwise if $1 \le p < q < \infty$ and $k \ge n/p - n/q$, then

$$\langle t\rangle^{(n-1)(1/p-1/q)} \|v(t,\cdot)\|_{L_x^q} \le C \|\Gamma^{\le k} v(t,\cdot)\|_{L_x^p}. \tag{4.6}$$

Proof. The inequality (4.5) is the well-known *Klainerman-Sobolev* inequality, which is proved in [19]. Heuristically, the inequality (4.6) can be viewed as a consequence of (4.5). We refer [13, (3.9)] for a complete proof. □

Lemma 4.8 (Sobolev embedding). *Let $x \in \mathbb{R}^n$, and $1 \le p < q \le \infty$, then*

$$\|f(x)\|_{L_x^q} \le C\|\partial^{\le k} f(x)\|_{L_x^p}, \tag{4.7}$$

where $k \ge n/p - n/q$ if $q < \infty$, or $k > n/p$ if $q = \infty$. When it comes to S^{n-1}, with $1 \le p < q \le \infty$, then

$$\|f(\omega)\|_{L_\omega^q} \le C\|\Omega^{\le k} f(\omega)\|_{L_\omega^q}, \tag{4.8}$$

where $k \ge (n-1)/p - (n-1)/q$ if $q < \infty$, or $k > (n-1)/p - (n-1)/q$ if $q = \infty$. For mixed-norm, we have

$$\|f(x)\|_{\mathcal{L}_r^q L_\omega^4} \le C\|\partial^{\le k} f(x)\|_{\mathcal{L}_r^p L_\omega^2}, \tag{4.9}$$

where $2 \le p \le q \le 4$ and $k \ge n/4$.

Proof. Here inequality (4.7) is known as *Sobolev* embedding, and inequality (4.8) just the same inequality on sphere. Finally the inequality (4.9) is a simple Corollary of lemma 3.2 in [28] and normal *Sobolev* embedding in a small ball contains origin point. □

Claim 4.9. *We claim that there exists a constant C_0 depends on Λ, n but not on ε if ε is small enough, then all of the initial norms we will use can be bounded by $C_0\varepsilon$.*

The proof of Claim 4.9 need some delicate calculation, so we will postpone it to the end of this section. For now, we are ready to prove the Theorem 4.1 and Theorem 4.2.

4.2.2 Proof of Theorem 4.1(n=3)

By Lemma 4.3, we only need to prove that if $T_\varepsilon \le \bar{T}_\varepsilon$, then $\|\partial^{\le 3} u\|_{L_t^\infty L_x^2(t<T_\varepsilon)} \le M < \infty$, which gives the contradiction.

Firstly, we want to prove that for some suitable C_M, c, ε_0 to be fixed later, if $0 \le T < T_\varepsilon \le \bar{T}_\varepsilon$ then

$$\begin{aligned} A(T) &= \langle T\rangle^{1/p_S^2} \|\Gamma^{\le 2} u(T,\cdot)\|_{\mathcal{L}_r^{p_S} L_\omega^3} &&\le C_M\varepsilon, \\ B(T) &= \|\partial\Gamma^{\le 2} u(T,\cdot)\|_{L_x^2} &&\le C_M\varepsilon. \end{aligned} \tag{4.10}$$

With the help of Claim 4.9, we choose C_M satisfying that $A(0), B(0) \le C_M\varepsilon/4$. Take $X = \{T \in [0, T_\varepsilon) : A(t), B(t) \le C_M\varepsilon, \forall\, t \in [0,T]\}$. We want to prove $X = [0, T_\varepsilon)$. Consequently, it would suffice to show that, if T is as above, then equation (4.10) implies that $A(T), B(T) \le C_M\varepsilon/2$. Because of Lemma 4.3, we assume $T > 1$ without loss of generality.

Part 1: Estimate of $A(t)$.

By Lemma 4.4, we obtain

$$\begin{aligned} A(T) \leq\ & C_M\varepsilon/4 + C\big\|\Gamma^{\leq 2}|u|^{p_S}\big\|_{L_t^{p_S}\mathcal{L}_r^1 L_\omega^{3/2}(t<T/4)} \\ & + C\big\|\Gamma^{\leq 2}|\partial u|^{q}\big\|_{L_t^{p_S}\mathcal{L}_r^1 L_\omega^{3/2}(t<T/4)} \\ & + CT^{1/p_S}\big\|\Gamma^{\leq 2}|u|^{p_S}\big\|_{L_t^{\infty}\mathcal{L}_r^1 L_\omega^{3/2}(T/4<t<T)} \\ & + CT^{1/p_S}\big\|\Gamma^{\leq 2}|\partial u|^{q}\big\|_{L_t^{\infty}\mathcal{L}_r^1 L_\omega^{3/2}(T/4<t<T)}. \end{aligned} \tag{4.11}$$

First we consider the spatial norm part of $|u|$ terms, by *Hölder's* inequality and Lemma 4.8, we see

$$\begin{aligned} \big\|\Gamma^{\leq 2}|u|^{p_S}\big\|_{\mathcal{L}_r^1 L_\omega^{3/2}} &\leq C\big\||\Gamma^{\leq 2}u||u|^{p_S-1}\big\|_{\mathcal{L}_r^1 L_\omega^{3/2}} + C\big\||\Gamma^{\leq 1}u|^2|u|^{p_S-2}\big\|_{\mathcal{L}_r^1 L_\omega^{3/2}} \\ &\leq C\big\|\Gamma^{\leq 2}u\big\|_{\mathcal{L}_r^{p_S} L_\omega^3}\|u\|^{p_S-1}_{\mathcal{L}_r^{p_S} L_\omega^{3\sqrt{2}}} + C\big\|\Gamma^{\leq 1}u\big\|^2_{\mathcal{L}_r^{p_S} L_\omega^3}\|u\|^{p_S-2}_{\mathcal{L}_r^{p_S} L_\omega^{\infty}} \\ &\leq C\big\|\Gamma^{\leq 2}u\big\|^{p_S}_{\mathcal{L}_r^{p_S} L_\omega^3} \\ &\leq C\Big(\langle t\rangle^{-1/p_S^2} A(t)\Big)^{p_S} \\ &\leq CC_M^{p_S}\varepsilon^{p_S}\langle t\rangle^{-1/p_S}. \end{aligned}$$

Then, we consider the spatial norm part of $|\partial u|$ terms, by *Hölder's* inequality and Lemma 4.7, Lemma 4.8, we have

$$\begin{aligned} \big\|\Gamma^{\leq 2}|\partial u|^{q}\big\|_{\mathcal{L}_r^1 L_\omega^{3/2}} &\leq C\big\||\Gamma^{\leq 2}\partial u||\partial u|^{q-1}\big\|_{\mathcal{L}_r^1 L_\omega^{3/2}} + C\big\||\Gamma^{\leq 1}\partial u|^2|\partial u|^{q-2}\big\|_{\mathcal{L}_r^1 L_\omega^{3/2}} \\ &\leq C\big\|\Gamma^{\leq 2}\partial u\big\|_{\mathcal{L}_r^2 L_\omega^2}\|\partial u\|_{\mathcal{L}_r^2 L_\omega^6}\|\partial u\|^{q-2}_{\mathcal{L}_r^\infty L_\omega^\infty} + C\big\|\Gamma^{\leq 1}\partial u\big\|^2_{\mathcal{L}_r^2 L_\omega^3}\|\partial u\|^{q-2}_{\mathcal{L}_r^\infty L_\omega^\infty} \\ &\leq C\big\|\Gamma^{\leq 2}\partial u\big\|^2_{L_x^2}\|\partial u\|^{q-2}_{L_x^\infty} \\ &\leq C\big\|\Gamma^{\leq 2}\partial u\big\|^q_{L_x^2}\left(\langle t\rangle^{-1}\right)^{q-2} \\ &\leq CC_M^q\varepsilon^q\langle t\rangle^{2-q}. \end{aligned}$$

Back to equation (4.11), assume $\varepsilon_0 < 1/C_M$. Since $q \geq p_S = 1+\sqrt{2}$, $0<\varepsilon<\varepsilon_0$, we conclude

$$\begin{aligned} A(T) &\leq C_M\varepsilon/4 + CC_M^{p_S}\varepsilon^{p_S}\left(1+\left(\int_0^{T/4}\langle t\rangle^{-1}\,\mathrm{d}\,t\right)^{1/p_S}\right) \\ &\quad + CC_M^q\varepsilon^q\left(\langle T\rangle^{p_S-q} + \left(\int_0^{T/4}\langle t\rangle^{-p_S(q-2)}\,\mathrm{d}\,t\right)^{1/p_S}\right) \\ &\leq C_M\varepsilon\big(1/4 + CC_M^{p_S-1}\varepsilon^{p_S-1}(\ln(T+2))^{\,1/p_S} + CC_M^{q-1}\varepsilon^{q-1}\big) \\ &\leq C_M\varepsilon\big(1/3 + CC_M^{p_S-1}c^{1/p_S} + CC_M^{p_S-1}\varepsilon_0^{p_S-1}\big). \end{aligned}$$

Part 2: Estimate of $B(t)$.

For this part, by the Lemma 4.6, we know that

$$B(T) \le C_M\varepsilon/4 + C\big\|\Gamma^{\le 2}|u|^{p_S}\big\|_{L_t^1L_x^2(t<T)} + C\big\|\Gamma^{\le 2}|\partial u|^q\big\|_{L_t^1L_x^2(t<T)}\,. \tag{4.12}$$

For the $|u|$ term, by *Hölder's* inequality and Lemma 4.7, we have

$$\begin{aligned}
\big\|\Gamma^{\le 2}|u|^{p_S}\big\|_{L_x^2} &\le C\big\|\Gamma^{\le 2}u\big\|_{L_x^{p_S}}\|u\|_{L_x^a}^{p_S-1} + C\big\|\Gamma^{\le 1}u\big\|_{L_x^4}^2\|u\|_{L_x^\infty}^{p_S-2}\\
&\le C\big\|\Gamma^{\le 2}u\big\|_{L_x^{p_S}}^{p_S}\Big(\langle t\rangle^{-2(1/p_S-1/a)(p_S-1)} + \langle t\rangle^{-2(1/p_S-1/4)\cdot 2-2(1/p_S)(p_S-2)}\Big)\\
&\le C\big\|\Gamma^{\le 2}u\big\|_{L_x^{p_S}}^{p_S}\langle t\rangle^{-1}\\
&\le C\Big(\langle t\rangle^{-1/p_S^2}A(t)\Big)^{p_S}\langle t\rangle^{-1}\\
&\le CC_M^{p_S}\varepsilon^{p_S}\langle t\rangle^{1-p_S}\,,
\end{aligned}$$

where $a = (p_S-1)/(1/2-1/p_S)$.

For the $|\partial u|$ term, by *Hölder's* inequality and Lemma 4.7 again, we know

$$\begin{aligned}
\big\|\Gamma^{\le 2}|\partial u|^q\big\|_{L_x^2} &\le C\big\|\Gamma^{\le 2}\partial u\big\|_{L_x^2}\|\partial u\|_{L_x^\infty}^{q-1} + C\big\|\Gamma^{\le 1}\partial u\big\|_{L_x^4}^2\|\partial u\|_{L_x^\infty}^{q-2}\\
&\le C\big\|\Gamma^{\le 2}\partial u\big\|_{L_x^2}^{q}\Big(\langle t\rangle^{-2(1/2)(q-1)} + \langle t\rangle^{-2(1/2-1/4)2-2(1/2)(q-2)}\Big)\\
&\le CC_M^q\varepsilon^q\langle t\rangle^{1-q}\,.
\end{aligned}$$

Back to (4.12), we obtain

$$B(T) \le C_M\varepsilon\big(1/4 + CC_M^{p_S-1}\varepsilon^{p_S-1} + CC_M^{q-1}\varepsilon^{q-1}\big) \le C_M\varepsilon\big(1/4 + CC_M^{p_S-1}\varepsilon_0^{p_S-1}\big)\,.$$

Part 3: The boundness of $A(t), B(t)$.

We choose ε_0 and the constant c in $\bar{T}_\varepsilon$ small enough, such that $A(T), B(T) \le C_M\varepsilon/2$, which completes the proof.

4.2.3 Proof of Theorem 4.2(n=2)

Similar to last subsection, we need to prove that for some suitable C_M, c, ε_0 to be fixed hereafter, if $0 \le T < T_\varepsilon \le \tilde{T}_\varepsilon$ then

$$\begin{aligned}
A(T) = \|\Gamma^{\le 2}u\|_{L_t^{p_*p_S}\mathcal{L}_r^{p_S}L_\omega^2(t<T)} &\le C_M\varepsilon^{1/p_S},\\
B(T) = \|\partial\Gamma^{\le 2}u(T,\cdot)\|_{L_x^2} &\le C_M\varepsilon,
\end{aligned} \tag{4.13}$$

where $p_* = (p_S-1)/2$ as in Lemma 4.5. By Claim 4.9, we fix C_M such that $B(0) \le C_M\varepsilon/4$. We need to show that equation (4.13) implies $A(T) \le C_M\varepsilon^{1/p_S}/2$ and $B(T) \le C_M\varepsilon/2$ for above T and T_ε.

Part 1: Estimate of $A(t)$.

By Lemma 4.5, and $T < \tilde{T}_\varepsilon$, we get

$$\begin{aligned} A(T) \leq cC_M\varepsilon/4^{1/p_S} &+ cC\varepsilon^{1/p_S-1}\left\|\Gamma^{\leq 2}|u|^{p_S}\right\|_{L_t^{p_*}\mathcal{L}_r^1L_\omega^2(t<T)} \\ &+ cC\varepsilon^{1/p_S-1}\left\|\Gamma^{\leq 2}|\partial u|^q\right\|_{L_t^{p_*}\mathcal{L}_r^1L_\omega^2(t<T)}. \end{aligned} \tag{4.14}$$

For the second term, similar to $n = 3$, we have

$$\begin{aligned} \left\|\Gamma^{\leq 2}|u|^{p_S}\right\|_{L_t^{p_*}\mathcal{L}_r^1L_\omega^2} &\leq C\left\||\Gamma^{\leq 2}u||u|^{p_S-1}\right\|_{L_t^{p_*}\mathcal{L}_r^1L_\omega^2} + C\left\||\Gamma^{\leq 1}u|^2|u|^{p_S-2}\right\|_{L_t^{p_*}\mathcal{L}_r^1L_\omega^2} \\ &\leq C\left\|\Gamma^{\leq 2}u\right\|^{p_S}_{L_t^{p_*p_S}\mathcal{L}_r^{p_S}L_\omega^2} \\ &\leq CC_M^{p_S}\varepsilon, \end{aligned}$$

and

$$\begin{aligned} \left\|\Gamma^{\leq 2}|\partial u|^q\right\|_{\mathcal{L}_r^1L_\omega^2} &\leq C\left(\left\|\Gamma^{\leq 2}\partial u\right\|_{\mathcal{L}_r^2L_\omega^2}\|\partial u\|_{\mathcal{L}_r^2L_\omega^\infty} + \left\|\Gamma^{\leq 1}\partial u\right\|^2_{\mathcal{L}_r^2L_\omega^4}\right)\|\partial u\|^{q-2}_{\mathcal{L}_r^\infty L_\omega^\infty} \\ &\leq C\left\|\Gamma^{\leq 2}\partial u\right\|^2_{L_x^2}\|\partial u\|^{q-2}_{L_x^\infty} \\ &\leq CC_M^q\varepsilon^q\langle t\rangle^{1-q/2}. \end{aligned}$$

Since $q \geq p_S$, $(1 - p_S/2)p_* = -1$, $T < \tilde{T}_\varepsilon$, we get

$$\begin{aligned} A(T) &\leq cC_M\varepsilon^{1/p_S} + cCC_M^{p_S}\varepsilon^{1/p_S-1+1} + cCC_M^q\varepsilon^{1/p_S-1+q}\left(\int_0^T\langle t\rangle^{(1-q/2)p_*}\,\mathrm{d}\,t\right)^{1/p_*} \\ &\leq cC_M\varepsilon^{1/p_S} + cCC_M^{p_S}\varepsilon^{1/p_S-1+1} + cCC_M^q\varepsilon^{1/p_S-1+q}(\ln(T+2))^{1/p_*} \\ &\leq C_M\varepsilon^{1/p_S}(c + cCC_M^{p_S-1}). \end{aligned}$$

Part 2: Estimate of $B(t)$.

In this part, by applying the energy inequality, we obtain

$$B(T) \leq C_M\varepsilon/4 + C\left\|\Gamma^{\leq 2}|u|^{p_S}\right\|_{L_t^1L_x^2(t<T)} + C\left\|\Gamma^{\leq 2}|\partial u|^q\right\|_{L_t^1L_x^2(t<T)}.$$

For the second term, by *Hölder's* inequality, Lemma 4.7 and Lemma 4.8, we have

$$\begin{aligned} \left\|\Gamma^{\leq 2}|u|^{p_S}\right\|_{L_t^1L_x^2} &= \left\|\Gamma^{\leq 2}|u|^{p_S}\right\|_{L_t^1\mathcal{L}_r^2L_\omega^2} \\ &\leq C\left\|\|u\|^{p_S-2}_{\mathcal{L}_r^\infty L_\omega^\infty}\left(\left\|\Gamma^{\leq 2}u\right\|_{\mathcal{L}_r^{p_S}L_\omega^2}\|u\|_{\mathcal{L}_r^{p_S+1}L_\omega^\infty} + \left\|\Gamma^{\leq 1}u\right\|^2_{\mathcal{L}_r^4L_\omega^4}\right)\right\|_{L_t^1} \\ &\leq C\left\|\left\|\Gamma^{\leq 1}u\right\|^{p_S-2}_{L_x^{p_S}}\left\|\Gamma^{\leq 2}u\right\|^2_{\mathcal{L}_r^{p_S}L_\omega^2}\langle t\rangle^{(-1/p_S)(p_S-2)}\right\|_{L_t^1} \\ &\leq C\left\|\left\|\Gamma^{\leq 2}u\right\|^{p_S}_{\mathcal{L}_r^{p_S}L_\omega^2}\langle t\rangle^{p_S-4}\right\|_{L_t^1} \\ &\leq C\left\|\Gamma^{\leq 2}u\right\|^{p_S}_{L_t^{p_*p_S}\mathcal{L}_r^{p_S}L_\omega^2}\left\|\langle t\rangle^{p_S-4}\right\|_{L_t^{1+p_S}} \\ &\leq CC_M^{p_S}\varepsilon^{p_S}, \end{aligned}$$

and the same for the last term

$$\begin{aligned}\left\|\Gamma^{\le 2}|\partial u|^q\right\|_{L^2_x} &\le C\left\|\Gamma^{\le 2}\partial u\right\|_{L^2_x}\|\partial u\|^{q-1}_{L^\infty_x} + C\left\|\Gamma^{\le 1}\partial u\right\|^2_{L^4_x}\|\partial u\|^{q-2}_{L^\infty_x}\\ &\le C\left\|\Gamma^{\le 2}\partial u\right\|^q_{L^2_x}\left(\langle t\rangle^{-(1/2)(q-1)} + \langle t\rangle^{-(1/2-1/4)2-(1/2)(q-2)}\right)\\ &\le CC_M^q\varepsilon^q\langle t\rangle^{(1-q)/2}.\end{aligned}$$

By the definition of $\tilde{T}$ and ε_0, we know

$$B(T) \le C_M\varepsilon\big(1/4 + CC_M^{p_S}\varepsilon^{p_S-1} + CC_M^{q-1}\varepsilon^{q-1}\big) \le C_M\varepsilon\big(1/4 + CC_M^{p_S}\varepsilon_0^{p_S-1}\big).$$

Part 3: The boundness of $A(t), B(t)$.

Now, by choosing ε_0 and the constant c (in $\tilde{T}_\varepsilon$) small enough, we conclude $A(T) \le C_M\varepsilon^{1/p_S}/2$ and $B(T) \le C_M\varepsilon/2$, which completes the proof.

4.2.4 Proof of Claim 4.9

Before the discussion, set $h = \left\{\partial_x^{\le 1}f, g\right\}$, by equation (1.9) we see

$$\left|\left\{\partial_x^{\le 3}u, \partial_x^{\le 2}\partial_t u\right\}\right|_{t=0} \le C\varepsilon\left|\partial_x^{\le 2}h\right|,$$

$$\begin{aligned}\left|\partial_x^{\le 1}\partial_t^2 u\right|_{t=0} &= \left|\partial_x^{\le 1}(|\partial u|^q + |u|^{p_S} + \Delta u)\right|_{t=0}\\ &\le C\varepsilon\left|\left\{\partial_x^{\le 1}h|h|^{q-1}, |h|^{p_S}, \partial_x^{\le 2}h\right\}\right|,\end{aligned}$$

$$\begin{aligned}\left|\partial_t^3 u\right|_{t=0} &\le C\left|\left\{\partial\partial_t u|\partial u|^{q-1}, \partial_t u|u|^{p_S-1}, \partial_t\Delta u\right\}\right|_{t=0}\\ &\le C\left|\left\{\left(|\partial u|^q + |u|^{p_S} + \Delta u\right)|\partial u|^{q-1}, \partial_x\partial_t u|\partial u|^{q-1}, \partial_t u|u|^{p_S-1}, \partial_t\Delta u\right\}\right|\\ &\le C\varepsilon\left|\left\{|h|^{2q-1}, |h|^{p_S+q-1}, \partial_x^{\le 1}h|h|^{q-1}, |h|^{p_S}, \partial_x^{\le 2}h\right\}\right|.\end{aligned}$$

Set $M = \Lambda + \Lambda^q + \Lambda^{p_S} + \Lambda^{2q-1} + \Lambda^{p_S+q-1}$, we want to show that all of the initial norms can be controlled by $C\varepsilon M$, where C does not depend on ε and M.

We begin with $A(0) = \|\Gamma^{\le 2}u(0,\cdot)\|_{\mathcal{L}^{p_S}_r L^3_\omega}(n=3)$, here we get

$$\begin{aligned}\|\Gamma^{\le 2}u(0,\cdot)\|_{\mathcal{L}^{p_S}_r L^3_\omega} &\le C\|\Gamma^{\le 3}u(0,\cdot)\|_{L^{p_S}_x}\\ &\le C\left\|\langle x\rangle^5\partial^{\le 3}u(0,\cdot)\right\|_{L^\infty_x}\left\|\langle x\rangle^{-2}\right\|_{L^{p_S}_x}\\ &\le C\varepsilon M.\end{aligned}$$

Similarly, for $B(0) = \|\partial\Gamma^{\le 2}u(0,\cdot)\|_{L^2_x}(n=2,3)$, we have

$$\|\partial\Gamma^{\le 2}u(0,\cdot)\|_{L^2_x} \le C\left\|\langle x\rangle^5\partial^{\le 3}u(0,\cdot)\right\|_{L^\infty_x}\left\|\langle x\rangle^{-3}\right\|_{L^2_x} \le C\varepsilon M.$$

For $\|r^{1-2/p_S}\partial_\omega^{\le 1}\Gamma^{\le 2}u(0,\cdot)\|_{\mathcal{L}_r^{p_S}L_\omega^{3/2}}(n=3)$ which comes from right hand side (RHS) of equation (4.3), we see

$$\begin{aligned}\|r^{1-2/p_S}\partial_\omega^{\le 1}\Gamma^{\le 2}u(0,\cdot)\|_{\mathcal{L}_r^{p_S}L_\omega^{3/2}} &\le C\|\langle x\rangle^{4-2/p_S}\,\partial^{\le 3}f\|_{L_x^{p_S}}\\ &\le C\big\|\langle x\rangle^{5}\,\partial^{\le 3}u(0,\cdot)\big\|_{L_x^\infty}\left\|\langle x\rangle^{-1-2/p_S}\right\|_{L_x^{p_S}}\\ &\le C\varepsilon M.\end{aligned}$$

The other three terms from equation (4.3) can be controlled by similar arguments. At last, for the term $\left\|\Gamma^{\le 2}(u,\partial_t u)(0,\cdot)\right\|_{X^b}(n=2)$ which comes from equation (4.4), we have

$$\left\|\Gamma^{\le 2}(u,\partial_t u)(0,\cdot)\right\|_{X^b}\le\left\|\Gamma^{\le 2}(u,\partial_t u)(0,\cdot)\right\|_{(H^1,L^2)}.$$

Through the previous discussion, we finish the proof of the Claim 4.9.

Chapter 5

Long time behavior of coupled *Strauss-Glassey* system

5.1 Main result

In this section, we consider the equation (1.10) which is proposed in 1.5. Where, the existence result presented in this chapter was obtained by the author's joint work [5]. We will discuss its long time existence in $n = 3$, $q < 2$, and the blow-up in $n \geq 3$. In this chapter we set

$$\sigma = \sigma(p, q) := \frac{p+2}{pq-1} - \frac{1}{pq} - \frac{n-1}{2}.$$

Theorem 5.1. *When $n = 3$, $\sigma < 0$, $q < 2$ and $2 < p < 3$, assume $f, \tilde{f} \in C^2(\mathbb{R}^3)$, $g, \tilde{g} \in C^1(\mathbb{R}^3)$ are spherical symmetric and supported in $B_1(0)$. Then there exists a positive number ε_0 such that for $0 < \varepsilon < \varepsilon_0$, there exists a global solution (u, v) of (1.10) that satisfying*

$$u, v, \partial_t u \in C(\mathbb{R}^+ \times \mathbb{R}^3).$$

For the blow-up result, we consider equation (1.10) in $n \geq 3$ dimensions.

Theorem 5.2. *Consider $n \geq 3$, assume $f = \tilde{f} = g = 0$,* $\operatorname{supp} \tilde{g} \subset B(0, 1)$ *nonnegative and not vanishes. Then, when $\sigma > 0$, for any $\varepsilon > 0$, there exists no weak solution (u, v) of (1.10) which satisfies* $\operatorname{supp}(u, v) \subset B(0, t+1)$ *and $\partial_t u, v \in L^\infty_{loc}(\mathbb{R}_+ \times \mathbb{R}^n)$.*

Remark 5.1. *Comparing Theorem 5.1 and Theorem 5.2, we find that the curve deduced by (1.15) is the critical curve for $q < 2$ indeed.*

5.2 Proof of Theorem 5.1

Before the proof, we consider a simple coordinate transform with $(u, u_t)|_{t=2} = (f, g)$, $(v, v_t)|_{t=2} = (\tilde{f}, \tilde{g})$. Due to the property of symmetry, (u, v) solve the equivalent 1-D integral equations in $t \geq 2$

$$\begin{aligned} u(t,r) &= \varepsilon u_o(t,r) + L|v|^q(t,r), \\ v(t,r) &= \varepsilon v_o(t,r) + L|\partial_t u|^p(t,r), \end{aligned}$$

where

$$\begin{aligned} u_o(t+2,r) &= \frac{1}{2r}\bigg((r+t)f(r+t) + (r-t)f(r-t) + \int_{r-t}^{r+t} \rho g(\rho)\,\mathrm{d}\,\rho\bigg), \\ v_o(t+2,r) &= \frac{1}{2r}\bigg((r+t)\tilde{f}(r+t) + (r-t)\tilde{f}(r-t) + \int_{r-t}^{r+t} \rho \tilde{g}(\rho)\,\mathrm{d}\,\rho\bigg), \\ LF(t,r) &= \frac{1}{2r}\int_0^t \int_{r-t+s}^{r+t-s} \rho F(s,\rho)\,\mathrm{d}\,\rho\,\mathrm{d}\,s. \end{aligned}$$

for $t \geq 0$. Here for convenience we consider $u|_{t<2} = v|_{t<2} = 0$.

Here we denote $f(|x|) = f(-|x|) = f(x)$ and the rest is similar. Then the lower limits of the integrals (to ρ) may be replaced by $|r-t|$ or $|r-(t-s)|$ because of the symmetric assumption. To control the iteration procedure, we need to estimate some derivatives. With the notation $w = \partial_t u$, we find that

$$w(t,r) = \varepsilon \partial_t u_o(t,r) + r^{-1} K_+ |v|^q(t,r), \tag{5.1}$$

$$v(t,r) = \varepsilon v_o(t,r) + L|w|^p(t,r), \tag{5.2}$$

$$\partial_r\{rv(t,r)\} = \varepsilon \partial_r\big(r v_o(t,r)\big) + K_-|w|^p(t,r), \tag{5.3}$$

where

$$K_\pm F(t,r) = \frac{1}{2}\int_0^t (r+t-s)F(s,r+t-s) \pm (r-t+s)F(s,r-t+s)\,\mathrm{d}\,s.$$

To control the norm of (w, v), we set

$$\|(w,v)\| = \|\omega_1 w\|_{L^\infty_{t,r}} + \|\omega_2 v\|_{L^\infty_{t,r}} + \|\omega_3 r\partial_r v\|_{L^\infty_{t,r}}$$

where weight functions $\omega_1, \omega_2, \omega_3$ are defined by

$$\omega_1(t,r) = \begin{cases} \langle r\rangle \langle t-r\rangle^{\mu/p+q-2} & (r < t/2), \\ \langle t-r\rangle^{\mu/p} \langle t+r\rangle^{q-1} & (r \geq t/2); \end{cases}$$

$$\omega_2(t,r) = \begin{cases} \langle r\rangle^{p-2} \langle t+r\rangle^{3-p+\mu/pq} & (r < t/2), \\ \langle t-r\rangle^{\mu/pq} \langle t+r\rangle & (r \geq t/2); \end{cases}$$

$$\omega_3(t,r) = \langle t-r\rangle^{\mu+pq-2p}$$

for $t \geq 0$ and $r \geq 0$, with a fixed $\mu < 1$ which satisfies

$$-\mu - pq + 2p \leq p - 3 - \mu/pq. \tag{5.4}$$

Now we consider the system of integral equations (5.1)-(5.3) in the closed subset of complete metric space

$$X_\varepsilon = \Big\{(w,v) : w, v, r\partial_r v \in C\big([2,\infty)\times\mathbb{R}\big), \|(w,v)\| \leq C_1\varepsilon, \\ \text{supp}(w,v) \subset \{t-r \geq 1, t \geq 2\}\Big\},$$

where C_1 will be determined later.

Lemma 5.3. *Suppose that (1.15) is satisfied,* $(w,v) \in X_\varepsilon$. *Then we have*

$$\|\omega_2 L|w|^p\|_{L^\infty_{t,r}} \leq C\|\omega_1 w\|^p_{L^\infty_{t,r}}, \tag{5.5}$$

$$\|\omega_1 r^{-1} K_+|v|^q\|_{L^\infty_{t,r}} \leq C\|\omega_2 v\|^q_{L^\infty_{t,r}} + C\|\omega_2 v\|^{q-1}_{L^\infty_{t,r}}\|\omega_3 r\partial_r v\|_{L^\infty_{t,r}}, \tag{5.6}$$

$$\|\omega_3 K_-|w|^p\|_{L^\infty_{t,r}} \leq C\|\omega_1 w\|^p_{L^\infty_{t,r}}. \tag{5.7}$$

Lemma 5.4. *Suppose that (1.15) is satisfied,* $(w,v), (\bar w, \bar v) \in X_\varepsilon$. *Set*

$$\tilde\omega_1 = \begin{cases} r\langle t-r\rangle^{\mu/p+q-2} & (r < t/2), \\ \langle t-r\rangle^{\mu/p} \langle t+r\rangle^{q-1} & (r \geq t/2); \end{cases}$$

$$\tilde\omega_2 = \begin{cases} r^{p-2}\langle t+r\rangle^{3-p+\mu/pq} & (r < t/2), \\ \langle t-r\rangle^{\mu/pq} \langle t+r\rangle & (r \geq t/2). \end{cases}$$

Then we have

$$\|\tilde\omega_2 L(|w|^p - |\bar w|^p)\|_{L^\infty_{t,r}} \leq C\|\tilde\omega_1(w,\bar w)\|^{p-1}_{L^\infty_{t,r}}\|\tilde\omega_1(w-\bar w)\|_{L^\infty_{t,r}}, \tag{5.8}$$

$$\|\tilde\omega_1 r^{-1} K_+(|v|^q - |\bar v|^q)\|_{L^\infty_{t,r}} \leq C\|\tilde\omega_2(v,\bar v)\|^{q-1}_{L^\infty_{t,r}}\|\tilde\omega_2(v-\bar v)\|_{L^\infty_{t,r}}. \tag{5.9}$$

Here, we apply the fixed point theorem with mapping

$$P : (w, v) \mapsto (Pw, Pv) := (\varepsilon \partial_t u_o + r^{-1} K_+ |v|^q, \varepsilon v_o + L|w|^p).$$

Firstly we check that P is well defined in $X_\varepsilon \to X_\varepsilon$. By expression (5.1)-(5.3), it is obvious that supp$(Pw, Pv) \subset \{t - r \geq 1, t \geq 2\}$ and $Pw, Pv, r\partial_r Pv \in C(\mathbb{R}^+ \times \mathbb{R})$. To estimate $\|(Pw, Pv)\|$, we begin with

$$\|\omega_1 Pw\|_{L^\infty_{t,r}} \leq \varepsilon \|\omega_1 \partial_t u_o\|_{L^\infty_{t,r}} + \|\omega_1 r^{-1} K_+ |v|^q\|_{L^\infty_{t,r}}.$$

Since $f \in C^2$, $g \in C^1$, and supp $u_o \subset \{(t, r) : 3 \geq t - r \geq 1\}$ where $\omega_1 \lesssim \langle t \rangle$, we see

$$\|\omega_1 \partial_t u_o\|_{L^\infty_{t,r}} \leq C_{f,g}.$$

Moreover, by (5.6), and $(w, v) \in X_\varepsilon$, we conclude

$$\|\omega_1 Pw\|_{L^\infty_{t,r}} \leq C_{f,g}\varepsilon + C\varepsilon^q.$$

The estimates for the remaining terms are similar, noticing $r\partial_r Pv = \partial_r(rPv) - Pv$ and $\omega_3 \lesssim \omega_2$, by (5.5) and (5.7) we finally have

$$\|(w, v)\| \leq C_{f,g,\tilde{f},\tilde{g}}\varepsilon + C\varepsilon^p + C\varepsilon^q \leq C_1\varepsilon$$

for $C_1 \geq 2C_{f,g,\tilde{f},\tilde{g}}$ and ε small enough.

Similarly, by (5.8) and (5.9) we have P is contraction in a weaker sense. However it is enough to obtain the fixed point (u, v) which solves (5.1)-(5.3). So we complete the proof.

5.2.1 Proof of (5.5) and (5.8)

First we prove (5.5). Let $r > 0$. Considering $D = \{(s, \rho) : t - r \leq s + \rho \leq t + r, 1 \leq s - \rho \leq t - r\}$ is the influence domain of (t, r) intersect with $\{(s, \rho) : s - \rho \geq 1\}$. Set $D_1 = D \cap \{(s, \rho) : \rho < s/2\}$, $D_2 = D \cap \{(s, \rho) : \rho \geq s/2\}$, then we find

$$\begin{aligned} |\omega_2 L|w|^p| &\leq \omega_2 \frac{C}{r} \int_D \rho\, \omega_1^{-p}(s, \rho) \|\omega_1 w\|^p_{L^\infty_{t,r}} \,\mathrm{d}\,\rho \,\mathrm{d}\, s \\ &= C\omega_2 \|\omega_1 w\|^p_{L^\infty_{t,r}} \frac{1}{r} \left(\int_{D_1} + \int_{D_2} \right) \rho\, \omega_1^{-p}(s, \rho) \,\mathrm{d}\,\rho \,\mathrm{d}\, s. \end{aligned} \tag{5.10}$$

Part 1: $(s, \rho) \in D_1$.

Here $\langle s\rangle \approx \langle s-\rho\rangle \approx \langle s+\rho\rangle$, take $\tau = s+\rho$, $\sigma = s-\rho$, by (5.4) we obtain

$$\begin{aligned}\frac{1}{r}\int_{D_1} \rho\,\omega_1^{-p}(s,\rho)\,\mathrm{d}\,\rho\,\mathrm{d}\,s &= \frac{1}{r}\int_{D_1} \rho\,\langle\rho\rangle^{-p}\,\langle s-\rho\rangle^{-\mu-pq+2p}\,\mathrm{d}\,\rho\,\mathrm{d}\,s\\ &\le \frac{C}{r}\int_{t-r}^{t+r}\int_{(t-r)/3}^{t-r} \langle\tau-\sigma\rangle^{1-p}\,\langle\tau\rangle^{-\mu-pq+2p}\,\mathrm{d}\,\sigma\,\mathrm{d}\,\tau \qquad (5.11)\\ &\le \frac{C}{r}\int_{t-r}^{t+r} \langle\tau-t+r\rangle^{2-p}\,\langle\tau\rangle^{p-3-\mu/pq}\,\mathrm{d}\,\tau.\end{aligned}$$

Part 1.1: $r < t/2$.

Here $t-r \approx t \approx t+r$, we conclude

$$\begin{aligned}RHS\ of\ (5.11) &\le C\langle t+r\rangle^{p-3-\mu/pq}\,r^{-1}\big((r+1)^{3-p}-1^{3-p}\big)\\ &\le C\langle t+r\rangle^{p-3-\mu/pq}\,\langle r\rangle^{2-p} \qquad (5.12)\\ &= C\omega_2^{-1}.\end{aligned}$$

Part 1.2: $r \ge t/2$.

Here $r \approx t \approx t+r$, similarly we have

$$\begin{aligned}RHS\ of\ (5.11) &\le C\langle t\rangle^{-1}\left(\langle t-r\rangle^{p-3-\mu/pq}\int_{t-r}^{2(t-r)}\langle\tau-t+r\rangle^{2-p}\,\mathrm{d}\,\tau\right.\\ &\qquad\left.+\int_{2(t-r)}^{t+r}\langle\tau\rangle^{-1-\mu/pq}\,\mathrm{d}\,\tau\right) \qquad (5.13)\\ &\le C\langle t\rangle^{-1}\,\langle t-r\rangle^{-\mu/pq}\\ &\le C\omega_2^{-1}.\end{aligned}$$

Part 2: $(s,\rho)\in D_2$.

Here $\langle\rho\rangle \approx \langle s\rangle \approx \langle s+\rho\rangle$, take $\tau = s+\rho$, $\sigma = s-\rho$, then we conclude

$$\begin{aligned}\frac{1}{r}\int_{D_2} \rho\,\omega_1^{-p}(s,\rho)\,\mathrm{d}\,\rho\,\mathrm{d}\,s &= \frac{1}{r}\int_{D_2} \rho\,\langle s-\rho\rangle^{-\mu}\,\langle s+\rho\rangle^{-p(q-1)}\,\mathrm{d}\,\rho\,\mathrm{d}\,s\\ &\le \frac{C}{r}\int_{t-r}^{t+r}\langle\tau\rangle^{1-p(q-1)}\,\mathrm{d}\,\tau\int_{1}^{t-r}\langle\sigma\rangle^{-\mu}\,\mathrm{d}\,\sigma. \qquad (5.14)\end{aligned}$$

Part 2.1: $r < t/2$.

Here we have

$$RHS\ of\ (5.14) \le C\langle t-r\rangle^{1-p(q-1)}\,\langle t-r\rangle^{1-\mu} \le C\omega_2^{-1}. \qquad (5.15)$$

Part 2.2: $r \ge t/2$.

Here we have

$$RHS\ of\ (5.14) \le C\langle t\rangle^{-1}\,\langle t-r\rangle^{2-p(q-1)}\,\langle t-r\rangle^{1-\mu} \le C\omega_2^{-1}. \qquad (5.16)$$

Thus (5.10)-(5.16) give $|\omega_2 L|w|^p| \le C\|\omega_1 w\|^p_{L^\infty_{t,r}}$ which completes the proof of (5.5). The proof of (5.8) is similar, since that $\big||a|^q-|b|^q\big| \lesssim (|a|^{q-1}+|b|^{q-1})\,|a-b|$ for $q>1$.

5.2.2 Proof of (5.6) and (5.9)

For (5.6), we divide the proof into two main cases: $r \geq 1/4$ and $r < 1/4$.

Part 1: $r \geq 1/4$.

For this situation, we see $r \approx \langle r \rangle$. Similar to the proof of (5.5) we have

$$\begin{aligned} \left|\omega_1 r^{-1} K_+ |v|^q\right| &\leq \|\omega_2 v\|_{L^\infty_{t,r}}^q \omega_1 r^{-1} \int_0^t \frac{(r+t-s)}{\omega_2(s, r+t-s)^q} + \frac{|r-t+s|}{\omega_2(s,|r-t+s|)^q} \,\mathrm{d}\, s \\ &\equiv \|\omega_2 v\|_{L^\infty_{t,r}}^q \omega_1 r^{-1} \int_0^t I + II \,\mathrm{d}\, s \end{aligned} \tag{5.17}$$

Part 1.1: $r < t/2$. Estimates about I.

In this part, we obtain $\omega_1 r^{-1} \approx \langle t-r \rangle^{\mu/p+q-2}$.

Part 1.1.1: $0 \leq s < 2(r+t)/3$.

This means $r+t-s > s/2$, then we conclude

$$\begin{aligned} \int_0^{2(t+r)/3} I \,\mathrm{d}\, s &= \int_0^{2(t+r)/3} (r+t-s) \langle 2s-r-t \rangle^{-\mu/p} \langle r+t \rangle^{-q} \,\mathrm{d}\, s \\ &\leq C \langle t+r \rangle^{1-q} \int_0^{2(t+r)/3} \langle 2s-r-t \rangle^{-\mu/p} \,\mathrm{d}\, s \\ &\leq C \langle t-r \rangle^{2-q-\mu/p} \\ &\leq C r \omega_1^{-1}. \end{aligned} \tag{5.18}$$

Part 1.1.2: $2(r+t)/3 \leq s \leq t$.

This means $r+t-s \leq s/2$, then similarly we have

$$\begin{aligned} \int_{2(t+r)/3}^t I \,\mathrm{d}\, s &= \int_{2(t+r)/3}^t (r+t-s) \langle r+t-s \rangle^{-q(p-2)} \langle r+t \rangle^{-3q+pq-\mu/p} \,\mathrm{d}\, s \\ &\leq C \langle t+r \rangle^{2-q(p-2)} \langle r+t \rangle^{-3q+pq-\mu/p} \\ &\leq C r \omega_1^{-1}. \end{aligned} \tag{5.19}$$

Part 1.2: $r \geq t/2$. Estimates about I.

Here we always have $r+t-s \geq s/2$. Then it is similar to (5.18).

Part 1.3: $r \geq t/2$. Estimates about II.

Part 1.3.1: $0 \leq s < t-r$.

Here $|r-t+s| = t-s-r$. It is similar to (5.18)-(5.19).

Part 1.3.2: $t-r \leq s < 2(t-r)$.

Here $r-t+s<s/2$, then we have

$$\begin{aligned}\int_{t-r}^{2(t-r)} II\,\mathrm{d}\,s &= \int_{t-r}^{2(t-r)} (r-t+s)\langle r-t+s\rangle^{-q(p-2)}\,\langle r-t+2s\rangle^{-3q+pq-\mu/p}\,\mathrm{d}\,s\\ &\le C\langle t-r\rangle^{2-q(p-2)}\,\langle t-r\rangle^{-3q+pq-\mu/p}\\ &\le Cr\omega_1^{-1}.\end{aligned}\tag{5.20}$$

Part 1.3.3: $2(t-r)\le s<t$.

Here $r-t+s\ge s/2$, then we also have

$$\begin{aligned}\int_{2(t-r)}^{t} II\,\mathrm{d}\,s &= \int_{2(t-r)}^{t} (r-t+s)\langle t-r\rangle^{-\mu/p}\,\langle r-t+2s\rangle^{-q}\,\mathrm{d}\,s\\ &\le C\langle t-r\rangle^{-\mu/p}\int_{2(t-r)}^{t}\langle r-t+2s\rangle^{1-q}\,\mathrm{d}\,s\\ &\le C\langle t-r\rangle^{-\mu/p}\,\langle t+r\rangle^{2-q}\\ &\le Cr\omega_1^{-1}.\end{aligned}\tag{5.21}$$

Part 1.4: $r<t/2$. Estimates about II.

Part 1.4.1: $0\le s<t-r$.

It is similar to (5.18)-(5.19).

Part 1.4.2: $t-r\le s\le t$.

Here we have

$$\begin{aligned}\int_{t-r}^{t} II\,\mathrm{d}\,s &= \int_{t-r}^{t} (r-t+s)\langle r-t+s\rangle^{-q(p-2)}\,\langle r-t+2s\rangle^{-3q+pq-\mu/p}\,\mathrm{d}\,s\\ &\le C\langle r\rangle^{2-q(p-2)}\,\langle t+r\rangle^{-3q+pq-\mu/p}\\ &\le Cr\omega_1^{-1}.\end{aligned}\tag{5.22}$$

Part 2: $r<1/4$.

In this part $\langle r\rangle\approx 1$. For the convenience of proof, we set $r^+=r+t-s, r^-=r-t+s$. Then we get

$$|\omega_1 r^{-1}K_+|v|^q|\le C\omega_1 r^{-1}\int_0^t\left|r^+|v|^q(s,r^+)+r^-|v|^q(s,|r^-|)\right|\mathrm{d}\,s\tag{5.23}$$

where

$$\begin{aligned}\left|r^+|v|^q(s,r^+)+r^-|v|^q(s,|r^-|)\right| \le&\left|r^+|v|^q(s,r^+)+r^-|v|^q(s,r^+)\right|\\ &+\left|r^-|v|^q(s,r^+)-r^-|v|^q(s,|r^-|)\right|\end{aligned}$$

then

$$RHS\ of\ (5.23) \le C\omega_1\bigg(\int_0^t |v|^q(s,r^+)\,\mathrm{d}\,s + r^{-1}\int_0^t |r^-|\Big||v|^q(s,r^+) - |v|^q(s,|r^-|)\Big|\,\mathrm{d}\,s\bigg)$$
$$\equiv C\omega_1 I + C\omega_1 II.$$

Part 2.1: Estimates about I.

It is similar to (5.18)-(5.19).

Part 2.2: Estimates about II.

For this part, we should consider $\partial_r v$. Since $r^+ - |r^-| \le 2r$, we obtain

$$r^{-1}\Big||v|^q(s,r^+) - |v|^q(s,|r^-|)\Big| \le C|\partial_r v(s,r^*)||v|^{q-1}(s,r^*)$$

for $|r^-| < r^*(s) < r^+$. Then we have

$$\begin{aligned} II &\le C\int_0^t |r^-||\partial_r v(s,r^*)||v|^{q-1}(s,r^*)\,\mathrm{d}\,s \\ &\le C\|\omega_2 v\|_{L^\infty_{t,r}}^{q-1}\|\omega_3 r\partial_r v\|_{L^\infty_{t,r}}\int_0^t \left(\omega_2^{-(q-1)}\omega_3^{-1}\right)(s,r^*)\,\mathrm{d}\,s \\ &\equiv C\|\omega_2 v\|_{L^\infty_{t,r}}^{q-1}\|\omega_3 r\partial_r v\|_{L^\infty_{t,r}}\int_0^t III\,\mathrm{d}\,s \end{aligned}$$

Here we notice $\langle t\rangle \lesssim \langle |r^-| + s + r\rangle \lesssim \langle r^* + s\rangle$.

Part 2.2.1: $r^* < s/2$.

Here $\langle s \pm r^*\rangle \gtrsim \langle r^*\rangle \gtrsim \langle t - s\rangle$, by (5.4), we see

$$\begin{aligned} \omega_1\int_{2r^*<s<t} III\,\mathrm{d}\,s &\le C\int_{2r^*<s<t} \langle r^*\rangle^{(1-q)(p-2)} \\ &\qquad \times \langle s - r^*\rangle^{(1/p-1)(\mu+pq-2p)+(1-q)(3-p+\mu/pq)}\,\mathrm{d}\,s \\ &\le C\int_{2r^*<s<t} \langle r^*\rangle^{-1-\mu-pq+2p+\mu/pq}\,\mathrm{d}\,s \\ &\le C\int_{2r^*<s<t} \langle t-s\rangle^{p-4}\,\mathrm{d}\,s \\ &\le C. \end{aligned}$$

Part 2.2.2: $r^* \ge s/2$.

Here $\langle s + r^*\rangle \gtrsim \langle s - r^*\rangle \gtrsim \langle 2s - t\rangle$, similarly we have

$$\omega_1\int_{s<2r^*\wedge t} III\,\mathrm{d}\,s \le C\int_{s<2r^*\wedge t}\langle 2s-t\rangle^{p-4}\,\mathrm{d}\,s \le C.$$

In summary, we complete the proof. To verify (5.9), without distinguishing whether $r > 1/4$ or not we have a proof just like (5.10)-(5.22).

5.2.3 Proof of (5.7)

We follow the same process as before

$$|\omega_3 K_-|w|^p| \le \|\omega_1 v\|^p_{L^\infty_{t,r}} \omega_3 \int_0^t \frac{r+t-s}{\omega_1(s, r+t-s)^p} + \frac{|r-t+s|}{\omega_1(s,|r-t+s|)^p}\,\mathrm{d}\,s.$$

Both part in the integration is similar to the last proof. Here we only show the proof of $r \ge t/2$, $(t-r) \le s$ for $|r-t+s|$ part.

Part 1: $t-r \le s < 2(t-r)$.

Here $r-t+s < s/2$, then we see

$$\begin{aligned}\int_{t-r}^{2(t-r)} \frac{r-t+s}{\omega_1(s,r-t+s)^p}\,\mathrm{d}\,s &= \int_{t-r}^{2(t-r)} (r-t+s)\langle r-t+s\rangle^{-p}\,\langle r-t+2s\rangle^{-\mu-pq+2p}\,\mathrm{d}\,s \\ &\le C\langle t-r\rangle^{-\mu-pq+2p} \\ &\le C\omega_3^{-1}.\end{aligned}$$

Part 2: $2(t-r) \le s \le t$.

Here $r-t+s \ge s/2$, then we know

$$\begin{aligned}\int_{2(t-r)}^{t} \frac{r-t+s}{\omega_1(s,r-t+s)^p}\,\mathrm{d}\,s &= \int_{2(t-r)}^{t} (r-t+s)\langle t-r\rangle^{-\mu}\,\langle r-t+2s\rangle^{-pq+p}\,\mathrm{d}\,s \\ &\le C\langle t-r\rangle^{-\mu}\int_{2(t-r)}^{t}\langle r-t+2s\rangle^{1-pq+p}\,\mathrm{d}\,s \\ &\le C\langle t-r\rangle^{-\mu}\,\langle t-r\rangle^{2-pq+p} \\ &\le C\omega_3^{-1},\end{aligned}$$

which complete the proof.

5.3 Proof of Theorem 5.2

In this theorem we consider the coupled system

$$\begin{cases}\Box u = |v|^q, \qquad \Box v = |\partial_t u|^p, \\ (u,\partial_t u)|_{t=0} = \varepsilon\big(0,0\big),\ (v,\partial_t v)|_{t=0} = \varepsilon\big(0,\tilde{g}(x)\big),\end{cases} \tag{5.24}$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

We adopt the η_T defined in Section 3.3.1. We also define

$$\Psi_\beta(t,x;\lambda) := (t+r+\lambda)^{-\beta} F\left(\frac{n-1}{2}, \beta, n-1; \frac{2r}{t+r+\lambda}\right),$$

with β fixed later and F be the hypergeometric function. From the previous discussion, we know that for such Ψ_β we have Lemma 3.7, and $\Box\Psi_\beta = 0$ in the region $Q_\lambda := \{(t,x) : t+\lambda > |x|\}$.

Now, choose $a \gg 1$, and $\beta, \beta_1, \tilde{\beta}, \tilde{\beta}_1$ fixed later. We multiply $\partial_t(-\eta_T^a \Psi_{\beta_1})$ on both sides of $\Box u = |v|^q$. Noticing $\eta_T(T) = \partial_t\eta_T(T) = \partial_t\eta_T(0) = 0$ and $\eta_T(0) = 1$, we have

$$\int_0^T \int_{\mathbb{R}^n} |v|^q \partial_t(-\eta_T^a \Psi_{\beta_1}) \,\mathrm{d}\, x \,\mathrm{d}\, t = \int_0^T \int_{\mathbb{R}^n} \partial_t u \Box(\eta_T^a \Psi_{\beta_1}) \,\mathrm{d}\, x \,\mathrm{d}\, t.$$

Next, we multiply $\eta_T^a \Psi_{\tilde{\beta}_1}$ on both sides of $\Box v = |\partial_t u|^q$, similarly we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \varepsilon \tilde{g} \Psi_{\tilde{\beta}_1}(0) \,\mathrm{d}\, x + \int_0^T \int_{\mathbb{R}^n} |\partial_t u|^p \eta_T^a \Psi_{\tilde{\beta}_1} \,\mathrm{d}\, x \,\mathrm{d}\, t \\ =& \int_0^T \int_{\mathbb{R}^n} v \Box\big(\eta_T^a \Psi_{\tilde{\beta}_1}\big) \,\mathrm{d}\, x \,\mathrm{d}\, t. \end{aligned}$$

Similar to the previous proof, using chain law and *Hölder* inequality, noticing $\partial_t(-\eta_T^a) \geq 0$, we have

$$\beta_1 \int_0^T \tilde{G}_{\beta_1+1}(t;T) \,\mathrm{d}\, t \lesssim \left(\int_{T/2}^T G_\beta(t;T) \,\mathrm{d}\, t\right)^{\frac{1}{p}} S(T;\beta_1,\beta,p)\,, \tag{5.25}$$

$$\varepsilon \int_{\mathbb{R}^n} \tilde{g} \Psi_{\tilde{\beta}}(0) \,\mathrm{d}\, x + \int_0^T G_{\tilde{\beta}_1}(t;T) \,\mathrm{d}\, t \lesssim \left(\int_{T/2}^T \tilde{G}_{\tilde{\beta}}(t;T) \,\mathrm{d}\, t\right)^{\frac{1}{q}} S\Big(T;\tilde{\beta}_1,\tilde{\beta},q\Big)\,. \tag{5.26}$$

Since there is no damping term, we find

$$G_\beta(t,T) := \int_{\mathbb{R}^n} |\partial_t u|^p \eta_T^a \Psi_\beta \,\mathrm{d}\, x, \qquad \tilde{G}_\beta(t,T) := \int_{\mathbb{R}^n} |v|^q \eta_T^a \Psi_\beta \,\mathrm{d}\, x,$$

$$S(T;\beta_1,\beta,p) := T^{-2}\left(\int_{T/2}^T \int_{B(0,1+t)} \Psi_{\beta_1}^{p'} \Psi_\beta^{-\frac{p'}{p}} \,\mathrm{d}\, x \,\mathrm{d}\, t\right)^{\frac{1}{p'}} + T^{-1}\left(\int_{T/2}^T \int_{B(0,1+t)} |\partial_t \Psi_{\beta_1}|^{p'} \Psi_\beta^{-\frac{p'}{p}} \,\mathrm{d}\, x \,\mathrm{d}\, t\right)^{\frac{1}{p'}}.$$

Set$X_\beta(T) := \int_0^T G_\beta(t;T) \,\mathrm{d}\, t$, $\tilde{X}_\beta(T) := \int_0^T \tilde{G}_\beta(t;T) \,\mathrm{d}\, t$, we hope to construct the relationship between them and give the blow up result. Before that, similar to Lemma 3.10, we first give the some estimates of $S(T;\beta_1,\beta,p)$.

Lemma 5.5. *Considering $n \geq 3$, assume $T \geq 2$, $\lambda > 1$, we have*

$$S(T;\beta_1,\beta,p) \lesssim \begin{cases} T^{\frac{n+1}{2p'}-1}, & \beta > \frac{n-1}{2},\ \beta_1 \gg \frac{n-1}{2}; \\ T^{\frac{n+1}{p'}-\beta_1-2+\frac{\beta}{p}}, & \beta_1 < \frac{n-1}{2} - \frac{1}{p} \end{cases}$$

Proof. The proof of this lemma is almost same with Lemma 3.7, we take the second term of $S(T;\beta_1,\beta,p)$ under $\beta < \frac{n-1}{2}$, $\frac{n-1}{2} - 1 < \beta_1 < \frac{n-1}{2} - \frac{1}{p}$ as an example. Here, using Lemma 3.7 we get

$$|\partial_t \Psi_{\beta_1}| \approx (\lambda+t)^{-\beta_1-1}\left(1 - \frac{r}{t+\lambda}\right)^{\frac{n-1}{2}-\beta_1-1}, \qquad \Psi_\beta \approx (\lambda+t)^{-\beta}.$$

Noticing $(\frac{n-1}{2}-\beta_1-1)p'>-1$, we find

$$\begin{aligned}\int_{B(0,1+t)}|\partial_t\Psi_{\beta_1}|^{p'}\,\Psi_\beta^{-\frac{p'}{p}}\,\mathrm{d}\,x\lesssim&(t+1)^{n-1+(-\beta_1-1)p'-(-\beta)\frac{p'}{p}}\int_0^{t+1}\left(1-\frac{r}{t+\lambda}\right)^{(\frac{n-1}{2}-\beta_1-1)p'}\mathrm{d}\,r\\ \lesssim&(t+1)^{n+(-\beta_1-1)p'+\beta\frac{p'}{p}}.\end{aligned}$$

Substituted it to the second term of $S(T;\beta_1,\beta,p)$, we finally have

$$T^{-1}\left(\int_{T/2}^T\int_{B(0,1+t)}|\partial_t\Psi_{\beta_1}|^{p'}\,\Psi_\beta^{-\frac{p'}{p}}\,\mathrm{d}\,x\,\mathrm{d}\,t\right)^{\frac{1}{p'}}\lesssim T^{\frac{n+1}{p'}-\beta_1-2+\frac{\beta}{p}},$$

which is the result of this lemma. □

Now, we began to prove the blow-up. Fix $\beta<\frac{n-1}{2}-\frac{1}{\min(p,q)}$ and $\tilde\beta<\frac{n-1}{2}+\frac{1}{p'}$. First we consider the initial data part, take $\tilde\beta_1\gg\frac{n-1}{2}$. Using Lemma 5.5, by (5.26) we find

$$\varepsilon\lesssim\tilde X_{\tilde\beta}^{\frac{1}{q}}T^{\frac{n-1}{2}-\frac{n+1}{2q}}$$

Then, take $\beta_1=\tilde\beta-1$ and $\tilde\beta_1=\beta$. For $S(T;\beta_1,\beta,p)$ and $S(T;\tilde\beta_1,\tilde\beta,q)$ we use Lemma 5.5 separately. By (5.25) and (5.26) we have

$$\tilde X_{\tilde\beta}\lesssim X_\beta^{\frac{1}{p}}T^{\frac{n+1}{p'}-\tilde\beta-1+\frac{\beta}{p}},\qquad X_\beta\lesssim\ \tilde X_{\tilde\beta}^{\frac{1}{q}}T^{\frac{n+1}{q'}-\beta-2+\frac{\tilde\beta}{q}}.$$

Combining these equation, we have

$$\begin{aligned}\varepsilon\tilde X_{\tilde\beta}^{\frac{p}{pq-1}}X_{\tilde\beta}^{\frac{1}{pq-1}}\lesssim&\left(\tilde X_\beta^{\frac{1}{q}}T^{\frac{n-1}{2}-\frac{n+1}{2q}}\right)\left(X_\beta^{\frac{1}{p}}T^{\frac{n+1}{p'}-\tilde\beta-1+\frac{\beta}{p}}\right)^{\frac{p}{pq-1}}\left(\tilde X_{\tilde\beta}^{\frac{1}{q}}T^{\frac{n+1}{q'}-\beta-2+\frac{\tilde\beta}{q}}\right)^{\frac{1}{pq-1}}\\ \Leftrightarrow\varepsilon\lesssim&T^{-\sigma+\frac{(n-1)/2+1/p'-\tilde\beta}{q}}.\end{aligned}$$

Then, when $\sigma>0$, we can take $\tilde\beta<(n-1)/2+1/p'$ such that $-\sigma+\frac{(n-1)/2+1/p'-\tilde\beta}{q}<0$, which means T can not be infinity. Now we finish the proof.

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