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Extrinsic flat surfaces along a curve on a surface in the unit three-sphere

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Abstract

In this paper we consider the curves on the surface in the unit 3-sphere. For a regular curve on a surface in the unit 3-sphere, we have a moving frame along the curve which is called a spherical Darboux frame. We induce two special vector fields along the curve with respect to the spherical Darboux frame and investigate the singularities of extrinsic flat great circular surfaces associated to these vector fields.

Keywords: the unit 3-sphere, singularity, great circular surfaces, spherical duality.

Mathematics Subject classification: 51H20, 57R45, 58C25.

1 Introduction

Touching or slicing a surface are two basic geometric methods to recognize the shape of a surface in the Euclidean 3-space. If we consider the normal slice of the surface with a normal plane at a point, we have a normal curvature of the surface at the point which induces the principal curvatures of the surface.

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The tangent plane of a surface at a point is determined by the unit normal vector at the point, so that it induces the notion of Gauss maps and the principal curvatures again. These are the methods of slicing and touching for defining the Gaussian curvature and the mean curvature of the surface at a point. Therefore, the curvatures at a point of a surface are obtained by using the tangent plane and the normal planes at the point. If we consider a curve on a surface, we have families of tangent planes and rectifying normal planes along the curve, respectively. As the envelopes of these two families of planes, the osculating developable surface and the normal developable surface of the surface along the curve were defined in [3, 4, 7]. In order to investigate the singularities of these developable surfaces, some new invariants of the curve on the surface were induced which have interesting geometric properties.

In this paper we consider the curve on the surface in the unit 3-sphere in the Euclidean 4-space and investigate the properties analogous to those of [3, 4, 7]. Associated to the curve on the surface in the unit 3-sphere, we have a natural moving frame along the curve which is called a spherical Darboux frame along the curve (cf. §2). It is the main tool to study the geometric properties of the curves on the surface in the unit 3-sphere. The spherical Darboux frame comprises the position vector of the curve, the unit tangent vector of the curve, the normal vector of the surface along the curve and the unit tangent vector of the surface which is normal to the curve. The last two vector fields offer important information on the geometric properties of curves on the surface in the unit 3-sphere. We have two special directions in the spherical Darboux frame at each point of the curve which are defined by vectors in the Darboux rectifying plane and the tangent plane of the surface, respectively. We respectively call such vector fields the rectifying and the osculating Darboux vector fields.

For the surface in the unit 3-sphere, we have known that there are two kinds of curvatures. One is called intrinsic Gaussian curvature denoted by K_I, which is just the Gaussian curvature defined by the induced Riemannian metric on the surface. The other is called the extrinsic Gaussian curvature denoted by K_e, which is defined by $K_e = K_I - 1$. In Euclidean space, surfaces with vanishing Gaussian curvature are developable surfaces which belong to a special class of ruled surfaces [5, 6]. In the unit 3-sphere, we call the surface with vanishing extrinsic Gaussian curvature the extrinsic flat surface and we can show that an extrinsic flat surface is (at least locally) parametrized as a great circular surface. Therefore, the notion of great circular surface is one of the analogous notions with ruled surface in Euclidean space and the notion of extrinsic flat surface is one of the analogous notions with the developable surface in Euclidean space. This is one of the motivations for us to investigate the extrinsic flat surfaces along the curve on the surface in the unit 3-sphere.

We show that the spherical dual surfaces (cf. [9]) associated with the last two unit vector fields of the spherical Darboux frame are the extrinsic flat great circular surfaces in [8]. These extrinsic flat great circular surfaces are generated by the rectifying and the osculating Darboux vector fields along the curve. We can show that they are tangent or normal to the surface along the curve in the unit 3-sphere. So we can study the geometric properties of the curve through these extrinsic flat surfaces. Then we obtain the invariants which characterize the singularities of these extrinsic flat great circular surfaces. As a consequence, we give a classification of the singularities of these extrinsic flat surfaces by using the invariants (cf. Theorem 5.4 and Theorem 5.5). We also consider the geometric meaning of these invariants.

A brief organization of this paper is given as follows: In §2, we give basic concepts, Frenet-Serret type formulae and the Darboux vector fields along the curve in the surface in the unit 3-sphere. We also review the notions and basic properties of great circular surfaces and extrinsic flat surfaces. In §3, we give the definitions of the osculating and normal extrinsic flat surfaces along the curve on the surface in the 3-sphere. They are the spherical dualities of normal vector and binormal vector respectively. In §4, we define the height functions, we also calculate the critical value sets of the extrinsic flat surfaces. In §5, we prove that the height functions are versal unfoldings and give the classifications of the singularities of the extrinsic flat surfaces. In §6, we study the geometric meanings of the invariants of the curve on the surface in the unit 3-sphere by the contact between the spheres and the spherical curves.

All maps considered here are of class C^∞ unless otherwise stated.

2 The basic concepts

In this section we give the basic concepts in this paper. Let $\mathbb{R}^4$ be a four-dimensional Euclidean space with scalar product defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$ for any two vectors $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$, $\boldsymbol{y} = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$. For any three vectors $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$, $\boldsymbol{y} = (y_1, y_2, y_3, y_4)$, $\boldsymbol{z} = (z_1, z_2, z_3, z_4)$, their vector product is defined by

$$\boldsymbol{x} \times \boldsymbol{y} \times \boldsymbol{z} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix},$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4$ is the canonical basis of $\mathbb{R}^4$.

Let $\boldsymbol{\gamma} : I \to \mathbb{R}^4$ be a regular curve in $\mathbb{R}^4$ (i.e., $\dot{\boldsymbol{\gamma}}(t) \neq \mathbf{0}$ for any $t \in I$), where I is an open interval. The arc-length of the curve $\boldsymbol{\gamma}$ measured from $\boldsymbol{\gamma}(t_0)(t_0 \in I)$ is $s(t) = \int_{t_0}^{t} \parallel \dot{\boldsymbol{\gamma}}(t) \parallel \mathrm{d}t$. The parameter s is determined such that $\parallel \boldsymbol{\gamma}'(s) \parallel = 1$ for the curve, where $\boldsymbol{\gamma}'(s) = \mathrm{d}\boldsymbol{\gamma}/\mathrm{d}s(s)$ is the unit tangent vector of $\boldsymbol{\gamma}$ at s.

We define the unit 3-sphere in $\mathbb{R}^4$ by

$$S^3 = \{\boldsymbol{x} \in \mathbb{R}^4 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}.$$

Let $\boldsymbol{x} : U \longrightarrow S^3$ be a regular surface in S^3, where $U \subseteq \mathbb{R}^2$ is an open subset. We also write $M = \boldsymbol{x}(U)$ and identify M and U through the embedding $\boldsymbol{x}$. Let $\overline{\boldsymbol{\gamma}} : I \longrightarrow U$ be a regular curve in the open subset $U \subseteq \mathbb{R}^2$. We have a curve embedded in the surface $M \subset S^3$ defined by $\boldsymbol{\gamma} = \boldsymbol{x} \circ \overline{\boldsymbol{\gamma}} : I \longrightarrow M \subset S^3 \subset \mathbb{R}^4$. Since $\boldsymbol{\gamma}$ is a regular curve in $\mathbb{R}^4$, we can reparameterize it by the arc-length s. So we have the unit tangent vector of $\boldsymbol{\gamma}(s)$ given by $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$. For the surface M, we define the unit normal vector of M by

$$\boldsymbol{n}(u) = \frac{\boldsymbol{x}(u) \times \boldsymbol{x}_{u_1}(u) \times \boldsymbol{x}_{u_2}(u)}{\|\boldsymbol{x}(u) \times \boldsymbol{x}_{u_1}(u) \times \boldsymbol{x}_{u_2}(u)\|}.$$

Then we have the unit normal vector field of M along $\boldsymbol{\gamma}$ defined by $\boldsymbol{n}_\gamma = \boldsymbol{n} \circ \overline{\boldsymbol{\gamma}} : I \longrightarrow S^3$. We also define a unit vector by $\boldsymbol{b}(s) = \boldsymbol{\gamma}(s) \times \boldsymbol{t}(s) \times \boldsymbol{n}_\gamma(s)$. Then we have a orthonormal frame field $\{\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{n}_\gamma(s), \boldsymbol{b}(s)\}$ of $\mathbb{R}^4$ along $\boldsymbol{\gamma}$. By standard arguments, we have the following *Frenet-Serret type formulae.*

$$\begin{cases} \boldsymbol{\gamma}'(s) = \boldsymbol{t}(s) \\ \boldsymbol{t}'(s) = -\boldsymbol{\gamma}(s) + \kappa_n(s)\boldsymbol{n}_\gamma(s) + \kappa_g(s)\boldsymbol{b}(s) \\ \boldsymbol{n}_\gamma'(s) = -\kappa_n(s)\boldsymbol{t}(s) - \tau_g(s)\boldsymbol{b}(s) \\ \boldsymbol{b}'(s) = -\kappa_g(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{n}_\gamma(s) \end{cases},$$

Here,

$$\begin{aligned} &\kappa_g(s) = \langle \boldsymbol{t}'(s), \boldsymbol{b}(s) \rangle = \det(\boldsymbol{\gamma}(s), \boldsymbol{\gamma}'(s), \boldsymbol{\gamma}''(s), \boldsymbol{n}_\gamma(s)), \\ &\kappa_n(s) = \langle \boldsymbol{t}'(s), \boldsymbol{n}_\gamma(s) \rangle = \langle \boldsymbol{\gamma}''(s), \boldsymbol{n}_\gamma(s) \rangle, \\ &\tau_g(s) = \langle \boldsymbol{b}'(s), \boldsymbol{n}_\gamma(s) \rangle = \det(\boldsymbol{\gamma}(s), \boldsymbol{\gamma}'(s), \boldsymbol{n}_\gamma(s), \boldsymbol{n}_\gamma'(s)). \end{aligned}$$

We call $\{\boldsymbol{\gamma}, \boldsymbol{t}, \boldsymbol{n}_\gamma, \boldsymbol{b}\}$ a *spherical Darboux frame* of $\boldsymbol{\gamma}$. We call κ_g the *geodesic curvature* of $\boldsymbol{\gamma}$, τ_g the *geodesic torsion* of $\boldsymbol{\gamma}$ and κ_n the *normal curvature* of $\boldsymbol{\gamma}$.

We define two vector fields $D_o(s), D_r(s)$ along $\boldsymbol{\gamma}$ by

$$\begin{aligned} D_o(s) &= \tau_g(s)\boldsymbol{t}(s) - \kappa_n(s)\boldsymbol{b}(s), \\ D_r(s) &= \tau_g(s)\boldsymbol{t}(s) + \kappa_g(s)\boldsymbol{n}_\gamma(s), \end{aligned}$$

which are respectively called an *osculating Darboux vector field* and a *rectifying Darboux vector field* along $\boldsymbol{\gamma}$, respectively.

We define

$$\begin{aligned} \overline{D}_o(s) &= \frac{\tau_g(s)\boldsymbol{t}(s) - \kappa_n(s)\boldsymbol{b}(s)}{\sqrt{\kappa_n^2(s) + \tau_g^2(s)}} \text{ if } \kappa_n^2(s) + \tau_g^2(s) \neq 0, \\ \overline{D}_r(s) &= \frac{\tau_g(s)\boldsymbol{t}(s) + \kappa_g(s)\boldsymbol{n}_\gamma(s)}{\sqrt{\kappa_g^2(s) + \tau_g^2(s)}} \text{ if } \kappa_g^2(s) + \tau_g^2(s) \neq 0, \end{aligned}$$

which are respectively called a *unit osculating Darboux vector field* and a *unit rectifying Darboux vector field* along $\boldsymbol{\gamma}$.

We now define basic invariants related to the above vector fields.

$$\begin{aligned} \delta_o(s) &= \kappa_n^2(s)\kappa_g(s) + \kappa_g(s)\tau_g^2(s) + \kappa_n(s)\tau_g'(s) - \kappa_n'(s)\tau_g(s), \\ \sigma_o(s) &= (2\kappa_n'(s) - \kappa_g(s)\tau_g(s))\delta_o(s) - \kappa_n(s)\delta_o'(s) - \kappa_n^2(s)\tau_g(s), \\ \delta_r(s) &= \kappa_n(s)\kappa_g^2(s) + \kappa_n(s)\tau_g^2(s) + \kappa_g'(s)\tau_g(s) - \kappa_g(s)\tau_g'(s), \\ \sigma_r(s) &= (\kappa_n(s)\tau_g(s) + 2\kappa_g'(s))\delta_r(s) - \kappa_g(s)\delta_r'(s) + \kappa_g^2(s)\tau_g(s). \end{aligned}$$

We will investigate the geometric meanings of the above invariants in §4, 5, 6.

On the other hand, we briefly review the notions and basic properties of great circular surfaces and extrinsic flat surfaces (E-flat surfaces) as following [8]. Let $\mathbf{a}_i : I \longrightarrow S^3$ $(i = 0, 1, 2, 3)$ be smooth maps from an open interval I with $\langle \mathbf{a}_i(t), \mathbf{a}_j(t) \rangle = \delta_{ij}$, so that we have an orthonormal frame $\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ of $\mathbb{R}^4$. We now define a mapping

$$F_A : I \times [0, 2\pi) \longrightarrow S^3$$

by

$$F_A(t, \theta) = \cos\theta \mathbf{a}_1(t) + \sin\theta \mathbf{a}_3(t),$$

where we assume that $A(t) = (\mathbf{a}_0(t), \mathbf{a}_1(t), \mathbf{a}_2(t), \mathbf{a}_3(t)) \in SO(4)$. We have a great circle $F_A(\theta, t_0)$ for any fixed $t = t_0$. We call F_A (or the image of it) a *great circular surface.* We also call $\mathbf{a}_1(t)$ a *base curve* and $\mathbf{a}_3(t)$ a *directrix.* Each great circle $F_A(\theta, t_0)$ is called a *generating great circle.* By using the above orthonormal frame, we define the following fundamental invariants:

$$
\begin{aligned}
c_1(t) &= \langle \mathbf{a}_0'(t), \mathbf{a}_1(t) \rangle = -\langle \mathbf{a}_0(t), \mathbf{a}_1'(t) \rangle, & c_4(t) &= \langle \mathbf{a}_1'(t), \mathbf{a}_2(t) \rangle = -\langle \mathbf{a}_1(t), \mathbf{a}_2'(t) \rangle, \\
c_2(t) &= \langle \mathbf{a}_0'(t), \mathbf{a}_2(t) \rangle = -\langle \mathbf{a}_0(t), \mathbf{a}_2'(t) \rangle, & c_5(t) &= \langle \mathbf{a}_1'(t), \mathbf{a}_3(t) \rangle = -\langle \mathbf{a}_1(t), \mathbf{a}_3'(t) \rangle, \\
c_3(t) &= \langle \mathbf{a}_0'(t), \mathbf{a}_3(t) \rangle = -\langle \mathbf{a}_0(t), \mathbf{a}_3'(t) \rangle, & c_6(t) &= \langle \mathbf{a}_2'(t), \mathbf{a}_3(t) \rangle = -\langle \mathbf{a}_2(t), \mathbf{a}_3'(t) \rangle.
\end{aligned}
$$

The following are the fundamental differential equations for the great circular surface:

$$
\begin{cases}
\mathbf{a}_0'(t) &= c_1(t)\mathbf{a}_1(t) + c_2(t)\mathbf{a}_2(t) + c_3(t)\mathbf{a}_3(t) \\
\mathbf{a}_1'(t) &= -c_1(t)\mathbf{a}_0(t) + c_4(t)\mathbf{a}_2(t) + c_5(t)\mathbf{a}_3(t) \\
\mathbf{a}_2'(t) &= -c_2(t)\mathbf{a}_0(t) - c_4(t)\mathbf{a}_1(t) + c_6(t)\mathbf{a}_3(t) \\
\mathbf{a}_3'(t) &= -c_3(t)\mathbf{a}_0(t) - c_5(t)\mathbf{a}_1(t) - c_6(t)\mathbf{a}_2(t).
\end{cases}
$$

It can be written in the following form:

$$
\begin{pmatrix} \mathbf{a}_0'(t) \\ \mathbf{a}_1'(t) \\ \mathbf{a}_2'(t) \\ \mathbf{a}_3'(t) \end{pmatrix} = \begin{pmatrix} 0 & c_1(t) & c_2(t) & c_3(t) \\ -c_1(t) & 0 & c_4(t) & c_5(t) \\ -c_2(t) & -c_4(t) & 0 & c_6(t) \\ -c_3(t) & -c_5(t) & -c_6(t) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{a}_0(t) \\ \mathbf{a}_1(t) \\ \mathbf{a}_2(t) \\ \mathbf{a}_3(t) \end{pmatrix}.
$$

We remark that

$$
C(t) = \begin{pmatrix} 0 & c_1(t) & c_2(t) & c_3(t) \\ -c_1(t) & 0 & c_4(t) & c_5(t) \\ -c_2(t) & -c_4(t) & 0 & c_6(t) \\ -c_3(t) & -c_5(t) & -c_6(t) & 0 \end{pmatrix} \in \mathfrak{so}(4),
$$

where $\mathfrak{so}(4)$ is the Lie algebra of the rotation group $SO(4)$. Great circular surfaces have singularities generally. The following proposition has been shown in [8].

Proposition 2.1. *A point (θ_0, t_0) is a singular point of $F_A(\theta, t)$ if and only if $c_1(t_0)c_6(t_0) + c_3(t_0)c_4(t_0) = 0$ and $(\cos\theta_0, \sin\theta_0)$ is a nontrivial solution of the following simultaneous linear equations:*

$$
\begin{cases}
c_1(t_0)x + c_3(t_0)y &= 0 \\
c_4(t_0)x - c_6(t_0)y &= 0.
\end{cases}
$$

The Gaussian curvature K_I is given by the following (cf. [8]):

Proposition 2.2. *Let (θ, t) be a regular point of F_A. Then the Gaussian curvature is*

$$K_I(\theta, t) = \frac{-(c_1(t)c_6(t) + c_3(t)c_4(t))^2}{((\cos\theta c_4(t) - \sin\theta c_6(t))^2 + (\cos\theta c_1(t) + \sin\theta c_3(t))^2)^2} + 1.$$

We now define $K_e = K_I - 1$, which is called an *extrinsic curvature* of the surface in S^3. A great circular surface $F_A(\theta, t)$ is said to be *extrinsic flat* (briefly, *E-flat*) if $K_e \equiv 0$. We write $c_\kappa(t) = c_1(t)c_6(t) + c_3(t)c_4(t)$. Then we have the following corollary.

Corollary 2.3. *A great circular surface $F_A(\theta, t)$ is E-flat if and only if $c_\kappa(t) = 0$ for any t.*

3 Extrinsic flat surfaces along the curve on the surface in the 3-sphere

We now define surfaces in S^3 associated with $\boldsymbol{\gamma}$ in S^3. Let $\boldsymbol{\gamma} : I \longrightarrow M \subset S^3$ be a unit speed curve.

If $\kappa_n^2(s) + \tau_g^2(s) \neq 0$, we define a great circular surface $OE_{\boldsymbol{\gamma}} : I \times [0, 2\pi) \longrightarrow S^3$ by

$$OE_{\boldsymbol{\gamma}}(s, \theta) = \cos\theta\boldsymbol{\gamma}(s) + \sin\theta\overline{D}_o(s).$$

If $\kappa_g^2(s) + \tau_g^2(s) \neq 0$, we define a great circular surface $NE_{\boldsymbol{\gamma}} : I \times [0, 2\pi) \longrightarrow S^3$ by

$$NE_{\boldsymbol{\gamma}}(s, \theta) = \cos\theta\boldsymbol{\gamma}(s) + \sin\theta\overline{D}_r(s).$$

For $OE_{\boldsymbol{\gamma}}$, we define a vector field

$$T_{\boldsymbol{n}_{\boldsymbol{\gamma}}}(s) = \frac{-\kappa_n(s)\boldsymbol{t}(s) - \tau_g(s)\boldsymbol{b}(s)}{\sqrt{\kappa_n^2(s) + \tau_g^2(s)}}.$$

Then we have an orthonormal frame

$$\{\mathbf{a}_{o0}(s), \mathbf{a}_{o1}(s), \mathbf{a}_{o2}(s), \mathbf{a}_{o3}(s)\} = \{\boldsymbol{n}_{\boldsymbol{\gamma}}(s), \boldsymbol{\gamma}(s), T_{\boldsymbol{n}_{\boldsymbol{\gamma}}}(s), \overline{D}_o(s)\}.$$

So we have

$$
\begin{aligned}
c_{o1}(s) &= \langle \mathbf{a}'_{o0}(s), \mathbf{a}_{o1}(s) \rangle = -\langle \mathbf{a}_{o0}(s), \mathbf{a}'_{o1}(s) \rangle = 0, \\
c_{o2}(s) &= \langle \mathbf{a}'_{o0}(s), \mathbf{a}_{o2}(s) \rangle = -\langle \mathbf{a}_{o0}(s), \mathbf{a}'_{o2}(s) \rangle = \sqrt{\kappa_n^2(s) + \tau_g^2(s)}, \\
c_{o3}(s) &= \langle \mathbf{a}'_{o0}(s), \mathbf{a}_{o3}(s) \rangle = -\langle \mathbf{a}_{o0}(s), \mathbf{a}'_{o3}(s) \rangle = 0, \\
c_{o4}(s) &= \langle \mathbf{a}'_{o1}(s), \mathbf{a}_{o2}(s) \rangle = -\langle \mathbf{a}_{o1}(s), \mathbf{a}'_{o2}(s) \rangle = -\frac{\kappa_n(s)}{\sqrt{\kappa_n^2(s) + \tau_g^2(s)}}, \\
c_{o5}(s) &= \langle \mathbf{a}'_{o1}(s), \mathbf{a}_{o3}(s) \rangle = -\langle \mathbf{a}_{o1}(s), \mathbf{a}'_{o3}(s) \rangle = \frac{\tau_g(s)}{\sqrt{\kappa_n^2(s) + \tau_g^2(s)}}, \\
c_{o6}(s) &= \langle \mathbf{a}'_{o2}(s), \mathbf{a}_{o3}(s) \rangle = -\langle \mathbf{a}_{o2}(s), \mathbf{a}'_{o3}(s) \rangle = \frac{\delta_o(s)}{(\kappa_n^2(s) + \tau_g^2(s))}.
\end{aligned}
$$

Therefore, $c_{o\kappa}(s) \equiv 0$. This means that $OE_{\boldsymbol{\gamma}}$ is an E-flat surface.

For $NE_{\boldsymbol{\gamma}}$, we define a vector field

$$
T_{\boldsymbol{b}}(s) = \frac{-\kappa_g(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s)}{\sqrt{\kappa_g^2(s) + \tau_g^2(s)}}.
$$

Then we have an orthonormal frame

$$
\{\mathbf{a}_{r0}(s), \mathbf{a}_{r1}(s), \mathbf{a}_{r2}(s), \mathbf{a}_{r3}(s)\} = \{\boldsymbol{b}(s), \boldsymbol{\gamma}(s), T_{\boldsymbol{b}}(s), \overline{D}_r(s)\}.
$$

So we have

$$
\begin{aligned}
c_{r1}(s) &= \langle \mathbf{a}'_{r0}(s), \mathbf{a}_{r1}(s) \rangle = -\langle \mathbf{a}_{r0}(s), \mathbf{a}'_{r1}(s) \rangle = 0, \\
c_{r2}(s) &= \langle \mathbf{a}'_{r0}(s), \mathbf{a}_{r2}(s) \rangle = -\langle \mathbf{a}_{r0}(s), \mathbf{a}'_{r2}(s) \rangle = \sqrt{\kappa_g^2(s) + \tau_g^2(s)}, \\
c_{r3}(s) &= \langle \mathbf{a}'_{r0}(s), \mathbf{a}_{r3}(s) \rangle = -\langle \mathbf{a}_{r0}(s), \mathbf{a}'_{r3}(s) \rangle = 0, \\
c_{r4}(s) &= \langle \mathbf{a}'_{r1}(s), \mathbf{a}_{r2}(s) \rangle = -\langle \mathbf{a}_{r1}(s), \mathbf{a}'_{r2}(s) \rangle = -\frac{\kappa_g(s)}{\sqrt{\kappa_g^2(s) + \tau_g^2(s)}}, \\
c_{r5}(s) &= \langle \mathbf{a}'_{r1}(s), \mathbf{a}_{r3}(s) \rangle = -\langle \mathbf{a}_{r1}(s), \mathbf{a}'_{r3}(s) \rangle = \frac{\tau_g(s)}{\sqrt{\kappa_g^2(s) + \tau_g^2(s)}}, \\
c_{r6}(s) &= \langle \mathbf{a}'_{r2}(s), \mathbf{a}_{r3}(s) \rangle = -\langle \mathbf{a}_{r2}(s), \mathbf{a}'_{r3}(s) \rangle = -\frac{\delta_r(s)}{(\kappa_g^2(s) + \tau_g^2(s))}.
\end{aligned}
$$

Therefore, $c_{r\kappa}(s) \equiv 0$. This means that $NE_{\boldsymbol{\gamma}}$ is an E-flat surface. Then we have the following proposition.

Proposition 3.1. *Let $\boldsymbol{\gamma} : I \longrightarrow M \subset S^3$ be a unit speed curve. Then we have the following*:

(1) *If $\kappa_n^2(s) + \tau_g^2(s) \neq 0$, then $OE_{\boldsymbol{\gamma}}$ is tangent to M at any point of $\boldsymbol{\gamma}(I)$.*

(2) *If $\kappa_g^2(s) + \tau_g^2(s) \neq 0$, then $NE_{\boldsymbol{\gamma}}$ is normal to M at any point of $\boldsymbol{\gamma}(I)$.*

Proof. (1) We have the following calculations:

$$\begin{aligned} \frac{\partial OE_{\boldsymbol{\gamma}}}{\partial \theta}(\theta, s) &= -\sin\theta\boldsymbol{\gamma}(s) + \cos\theta\overline{D}_o(s), \\ \frac{\partial OE_{\boldsymbol{\gamma}}}{\partial s}(\theta, s) &= \cos\theta\boldsymbol{t}(s) + \sin\theta\left(-\frac{\delta_o(s)}{\kappa_n^2(s)+\tau_g^2(s)}T_{\boldsymbol{n}_{\boldsymbol{\gamma}}}(s) - \frac{\tau_g(s)}{\sqrt{\kappa_n^2(s)+\tau_g^2(s)}}\boldsymbol{\gamma}(s)\right). \end{aligned}$$

These vectors are equal to $\overline{D}_o(s), \boldsymbol{t}(s)$ at $\theta = 0$, respectively. Since $\boldsymbol{t}(s), \overline{D}_o(s)$ are linearly independent tangent vectors of M at $\boldsymbol{\gamma}(s)$, these vectors generate the tangent space of M at $\boldsymbol{\gamma}(s)$. This means that $OE_{\boldsymbol{\gamma}}$ is tangent to M at any point of $\boldsymbol{\gamma}(I)$.

(2) We also have the following calculations:

$$\begin{aligned} \frac{\partial NE_{\boldsymbol{\gamma}}}{\partial \theta}(\theta, s) &= -\sin\theta\boldsymbol{\gamma}(s) + \cos\theta\overline{D}_r(s), \\ \frac{\partial NE_{\boldsymbol{\gamma}}}{\partial s}(\theta, s) &= \cos\theta\boldsymbol{t}(s) + \sin\theta\left(\frac{\delta_r(s)}{\kappa_g^2(s)+\tau_g^2(s)}T_{\boldsymbol{b}}(s) - \frac{\tau_g(s)}{\sqrt{\kappa_g^2(s)+\tau_g^2(s)}}\boldsymbol{\gamma}(s)\right). \end{aligned}$$

These vectors are equal to $\overline{D}_r(s), \boldsymbol{t}(s)$ at $\theta = 0$, respectively. Since $\boldsymbol{t}(s), \overline{D}_r(s)$ generate a normal plane of M at $\boldsymbol{\gamma}(s)$, $NE_{\boldsymbol{\gamma}}$ is normal to M at any point of $\boldsymbol{\gamma}(I)$. $\square$

By the above proposition, we call $OE_{\boldsymbol{\gamma}}$ an *osculating E-flat surface* along $\boldsymbol{\gamma}$ and $NE_{\boldsymbol{\gamma}}$ a *normal E-flat surface* along $\boldsymbol{\gamma}$, respectively.

We now consider the spherical duality from the view point of contact geometry. We briefly review some properties of contact manifolds and Legendrian submanifolds [1, Part III]. Let W be a $(2n+1)$-dimensional smooth manifold and K be a tangent hyperplane field on W. Locally such a field is defined as the field of zeros of a 1-form α. If tangent hyperplane field K is non-degenerate, we say that (W, K) is a *contact manifold* . Here K is said to be *non-degenerate* if $\alpha \wedge (d\alpha)^n \neq 0$ at any point of W. In this case K is called a *contact structure* and α is a *contact form.* A submanifold $i : L \subset W$

of a contact manifold (W, K) is a *Legendrian submanifold* if $\dim L = n$ and $di_p(T_pL) \subset K_{i(p)}$ at any point $p \in L$. We consider a smooth fiber bundle $\pi : N \to A$. The fiber bundle $\pi : N \to A$ is called a *Legendrian fibration* if its total space W is furnished with a contact structure and its fibers are Legendrian submanifolds. Let $\pi : N \to A$ be a Legendrian fibration. For a Legendrian submanifold $i : L \subset N$, a map $\pi \circ i : L \to A$ is called a *Legendrian map* . The image of the Legendrian map $\pi \circ i$ is called a *wavefront set* of i which is denoted by $W(i)$.

We now consider the following double fibrations over S^3:

$$
\begin{aligned}
&\Delta = \{(\boldsymbol{v}, \boldsymbol{w}) \in S^3 \times S^3 | \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0\},\\
&\pi_1 \ : \ \Delta \ni (\boldsymbol{v} \ \ \boldsymbol{w}) \ \longmapsto \boldsymbol{v} \in S^3, \quad \pi_2 \ : \ \Delta \ni (\boldsymbol{v} \ \ \boldsymbol{w}) \ \longmapsto \boldsymbol{w} \in S^3,\\
&\theta_1 = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle |_\Delta, \quad \theta_2 = \langle \boldsymbol{v}, d\boldsymbol{w} \rangle |_\Delta.
\end{aligned}
$$

Here, $\langle d\boldsymbol{v}, \boldsymbol{w} \rangle = \sum_{i=1}^4 w_i dv_i$ and $\langle \boldsymbol{v}, d\boldsymbol{w} \rangle = \sum_{i=1}^4 v_i dw_i$. Since $d(\langle \boldsymbol{v}, \boldsymbol{w} \rangle) = \langle d\boldsymbol{v}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, d\boldsymbol{w} \rangle$ and $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0$ on Δ, $\theta_1^{-1}(0)$ and $\theta_2^{-1}(0)$ define the same tangent hyperplane field over Δ which is denoted by K. The following proposition is well-known.

Proposition 3.2. *Under the above notations, (Δ, K) is a contact manifold and both of π_i $(i = 1, 2)$ are Legendrian fibrations.*

We say that a C^∞-mapping $\mathcal{L} : U \longrightarrow \Delta$ is an *isotropic mapping* if $\mathcal{L}^*\theta_i = 0$ ($i = 1$ or 2). We remark that the isotropic mapping is a Legendrian immersion if it is an immersion. If we have an isotropic mapping $\mathcal{L} : U \longrightarrow \Delta$, then we say that $\pi_1 \circ \mathcal{L}(U)$ and $\pi_2 \circ \mathcal{L}(U)$ are Δ-*dual* to each other.

Proposition 3.3. *Let $\boldsymbol{\gamma} : I \longrightarrow M \subset S^3$ be a unit speed curve. Then we have the following*:

(1) *If $\kappa_n^2(s) + \tau_g^2(s) \neq 0$, then $\boldsymbol{n}_{\boldsymbol{\gamma}}$ is the Δ-dual of $OE_{\boldsymbol{\gamma}}$.*
(2) *If $\kappa_g^2(s) + \tau_g^2(s) \neq 0$, then $\boldsymbol{b}$ is the Δ-dual of $NE_{\boldsymbol{\gamma}}$.*

Proof. (1) We define a mapping $\mathcal{L}_o : I \times [0, 2\pi) \longrightarrow \Delta$ by $\mathcal{L}_o(s, \theta) = (\boldsymbol{n}_{\boldsymbol{\gamma}}(s), OE_{\boldsymbol{\gamma}}(s, \theta))$. Then

$$\langle \boldsymbol{n}'_{\boldsymbol{\gamma}}(s), OE_{\boldsymbol{\gamma}}(s, \theta) \rangle = \langle -\kappa_n(s)\boldsymbol{t}(s) - \tau_g(s)\boldsymbol{b}(s), OE_{\boldsymbol{\gamma}}(s, \theta) \rangle = 0.$$

This means that $\mathcal{L}_o$ is an isotropic mapping, so that $\boldsymbol{n}_{\boldsymbol{\gamma}}$ is the Δ-dual of $OE_{\boldsymbol{\gamma}}$.

(2) We also define a mapping $\mathcal{L}_r : I \times [0, 2\pi) \longrightarrow \Delta$ by $\mathcal{L}_r(s,\theta) = (\boldsymbol{b}(s), NE_{\boldsymbol{\gamma}}(s,\theta))$. Then

$$\langle \boldsymbol{b}'(s), NE_{\boldsymbol{\gamma}}(s,\theta)\rangle = \langle -\kappa_g(s)\boldsymbol{t}(s) + \tau_g(s)\boldsymbol{n}_{\boldsymbol{\gamma}}(s), NE_{\boldsymbol{\gamma}}(s,\theta)\rangle = 0.$$

This means that $\mathcal{L}_r$ is an isotropic mapping, so that $\boldsymbol{b}$ is the Δ-dual of $NE_{\boldsymbol{\gamma}}$. □

4 Height functions

In order to study the singularities of the osculating and the normal E-flat surfaces along $\boldsymbol{\gamma}$, respectively, we introduce two families of functions and apply the theory of unfoldings. Let $\boldsymbol{\gamma} : I \longrightarrow M \subset S^3$ be a unit speed curve, then we define two families of functions as follows:

$$\begin{array}{ll} H_o : I \times S^3 \longrightarrow \mathbb{R}, & H_o(s,\boldsymbol{v}) = \langle \boldsymbol{n}_\gamma(s), \boldsymbol{v}\rangle, \\ H_r : I \times S^3 \longrightarrow \mathbb{R}, & H_r(s,\boldsymbol{v}) = \langle \boldsymbol{b}(s), \boldsymbol{v}\rangle. \end{array}$$

For any fixed $\boldsymbol{v} \in S^3$, we denote $h_{o\boldsymbol{v}}(s) = H_o(s,\boldsymbol{v})$, $h_{r\boldsymbol{v}}(s) = H_r(s,\boldsymbol{v})$. Then we have the following two propositions on $h_{o\boldsymbol{v}}$ and $h_{r\boldsymbol{v}}$.

For simplification, we denote the following symbols:

$$\begin{aligned} \rho_o(s) &= \sqrt{\delta_o^2(s) + \kappa_n^2(s)\tau_g^2(s) + \kappa_n^4(s)}, \\ \rho_r(s) &= \sqrt{\delta_r^2(s) + \kappa_g^2(s)\tau_g^2(s) + \kappa_g^4(s)}. \end{aligned}$$

Proposition 4.1. *Let $\boldsymbol{\gamma} : I \longrightarrow M \subset S^3$ be a unit speed curve with $\kappa_n^2(s) + \tau_g^2(s) \neq 0$, then we have the following:*
(1) $h_{o\boldsymbol{v}}(s) = 0$ if and only if there exist $\lambda, \eta, \xi \in \mathbb{R}$ with $\lambda^2 + \eta^2 + \xi^2 = 1$ such that $\boldsymbol{v} = \mu\boldsymbol{\gamma}(s) + \eta\boldsymbol{t}(s) + \xi\boldsymbol{b}(s)$.
(2) $h_{o\boldsymbol{v}}(s) = h'_{o\boldsymbol{v}}(s) = 0$ if and only if there exist $\theta \in \mathbb{R}$ such that

$$\boldsymbol{v} = \cos\theta\boldsymbol{\gamma}(s) + \sin\theta\overline{D}_o(s).$$

(3) $h_{o\boldsymbol{v}}(s) = h'_{o\boldsymbol{v}}(s) = h''_{o\boldsymbol{v}}(s) = 0$ if and only if $(\delta_o(s), \kappa_n(s)) \neq (0,0)$ and

$$\boldsymbol{v} = \pm\frac{1}{\rho_o(s)}\left(\delta_o(s)\boldsymbol{\gamma}(s) - \kappa_n(s)\sqrt{\kappa_n^2(s) + \tau_g^2(s)}\overline{D}_o(s)\right)$$

or $(\delta_o(s), \kappa_n(s)) = (0,0)$ and $\boldsymbol{v} = \cos\theta\boldsymbol{\gamma}(s) + \sin\theta\boldsymbol{t}(s)$.

(4) $h_{\boldsymbol{ov}}(s)=h'_{\boldsymbol{ov}}(s)=h''_{\boldsymbol{ov}}(s)=h'''_{\boldsymbol{ov}}(s)=0$ *if and only if* $(\delta_o(s),\kappa_n(s))\neq(0,0)$, $\sigma_o(s)=0$ *and*

$$\boldsymbol{v}=\pm\frac{1}{\rho_o(s)}\left(\delta_o(s)\boldsymbol{\gamma}(s)-\kappa_n(s)\sqrt{\kappa_n^2(s)+\tau_g^2(s)}\overline{D}_o(s)\right)$$

or $(\delta_o(s),\kappa_n(s))=(0,0)$ *and* $\boldsymbol{v}=\cos\theta\boldsymbol{\gamma}(s)+\sin\theta\boldsymbol{t}(s)$.

(5) $h_{\boldsymbol{ov}}(s)=h'_{\boldsymbol{ov}}(s)=h''_{\boldsymbol{ov}}(s)=h'''_{\boldsymbol{ov}}(s)=h^{(4)}_{\boldsymbol{ov}}(s)=0$ *if and only if* $(\delta_o(s),\kappa_n(s))\neq(0,0)$, $\sigma_o(s)=\sigma'_o(s)=0$ *and*

$$\boldsymbol{v}=\pm\frac{1}{\rho_o(s)}\left(\delta_o(s)\boldsymbol{\gamma}(s)-\kappa_n(s)\sqrt{\kappa_n^2(s)+\tau_g^2(s)}\overline{D}_o(s)\right)$$

or $(\delta_o(s),\kappa_n(s))=(0,0)$ *and* $\boldsymbol{v}=\cos\theta\boldsymbol{\gamma}(s)+\sin\theta\boldsymbol{t}(s)$.

Proof. (1) Since $\boldsymbol{v}\in S^3$, there exist $\mu,\eta,\lambda,\xi\in\mathbb{R}$ with $\mu^2+\eta^2+\lambda^2+\xi^2=1$ such that $\boldsymbol{v}=\mu\boldsymbol{\gamma}(s)+\eta\boldsymbol{t}(s)+\lambda\boldsymbol{n}_\gamma(s)+\xi\boldsymbol{b}(s)$. From $h_{\boldsymbol{ov}}(s)=\langle\boldsymbol{n}_\gamma(s),\boldsymbol{v}\rangle=0$, we have $\lambda=0$. So $\boldsymbol{v}=\mu\boldsymbol{\gamma}(s)+\eta\boldsymbol{t}(s)+\xi\boldsymbol{b}(s)$ and $\mu^2+\eta^2+\xi^2=1$. The converse direction also holds.

(2) $h_{\boldsymbol{ov}}(s)=h'_{\boldsymbol{ov}}(s)=0$ if and only if (1) holds and $h'_{\boldsymbol{ov}}(s)=\langle-\kappa_n(s)\boldsymbol{t}(s)-\tau_g(s)\boldsymbol{b}(s),\boldsymbol{v}\rangle=\langle-\kappa_n(s)\boldsymbol{t}(s)-\tau_g(s)\boldsymbol{b}(s),\mu\boldsymbol{\gamma}(s)+\eta\boldsymbol{t}(s)+\xi\boldsymbol{b}(s)\rangle=-\kappa_n(s)\eta-\tau_g(s)\xi=0$. If $\tau_g(s)\neq0$, we have $\xi=-\kappa_n(s)\eta/\tau_g(s)$, and

$$\boldsymbol{v}=\mu\boldsymbol{\gamma}(s)+\eta(\boldsymbol{t}(s)-\frac{\kappa_n(s)}{\tau_g(s)}\boldsymbol{b}(s))=\mu\boldsymbol{\gamma}(s)+\frac{\eta}{\tau_g(s)}D_o(s)$$

$$=\mu\boldsymbol{\gamma}(s)+\eta\frac{\sqrt{\kappa_n^2(s)+\tau_g^2(s)}}{\tau_g(s)}\overline{D}_o(s).$$

Since

$$\mu^2+\eta^2\frac{\kappa_n^2(s)+\tau_g^2(s)}{\tau_g^2(s)}=1,$$

for some $\theta\in\mathbb{R}$, we can put

$$\begin{cases}\mu=\cos\theta\\ \eta\dfrac{\sqrt{\kappa_n^2(s)+\tau_g^2(s)}}{\tau_g(s)}=\sin\theta\end{cases},$$

Then we have

$$\boldsymbol{v}=\cos\theta\boldsymbol{\gamma}(s)+\sin\theta\overline{D}_o(s).$$

If $\tau_g(s)=0, \kappa_n(s)\neq 0$, we have $\eta=0$. In this case, $D_o(s)=-\kappa_n(s)\boldsymbol{b}(s)$ and $\overline{D}_o(s)=\pm\boldsymbol{b}(s)$. So $\boldsymbol{v}=\mu\boldsymbol{\gamma}(s)+\xi\boldsymbol{b}(s)=\mu\boldsymbol{\gamma}(s)\pm\xi\overline{D}_o(s)$. Then we can put $\mu=\cos\theta, \pm\xi=\sin\theta$, so $\boldsymbol{v}=\cos\theta\boldsymbol{\gamma}(s)+\sin\theta\overline{D}_o(s)$. Anyway, we have

$$\boldsymbol{v}=\cos\theta\boldsymbol{\gamma}(s)+\sin\theta\overline{D}_o(s).$$

(3) $h_{o\boldsymbol{v}}(s)=h'_{o\boldsymbol{v}}(s)=h''_{o\boldsymbol{v}}(s)=0$ if and only if (2) holds and

$$\begin{aligned} h''_{o\boldsymbol{v}}(s) &= \langle \kappa_n(s)\boldsymbol{\gamma}(s)+(-\kappa'_n(s)+\kappa_g(s)\tau_g(s))\boldsymbol{t}(s)-(\kappa_n^2(s)+\tau_g^2(s))\boldsymbol{n}_\gamma(s) \\ &\quad -(\kappa_n(s)\kappa_g(s)+\tau'_g(s))\boldsymbol{b}(s), \cos\theta\boldsymbol{\gamma}(s)+\sin\theta\overline{D}_o(s)\rangle \\ &= \kappa_n(s)\cos\theta+\delta_o(s)\sin\theta/\sqrt{\kappa_n^2(s)+\tau_g^2(s)}=0. \end{aligned}$$

That is $\kappa_n(s)\sqrt{\kappa_n^2(s)+\tau_g^2(s)}\cos\theta+\delta_o(s)\sin\theta=0$.

When $(\delta_o(s),\kappa_n(s))\neq(0,0)$, we have

$$\boldsymbol{v}=\pm\frac{1}{\rho_o(s)}\left(\delta_o(s)\boldsymbol{\gamma}(s)-\kappa_n(s)\sqrt{\kappa_n^2(s)+\tau_g^2(s)}\overline{D}_o(s)\right).$$

When $\delta_o(s)=\kappa_n(s)=0$, we have $\overline{D}_o(s)=\boldsymbol{t}(s)$. In this case, we have $\boldsymbol{v}=\cos\theta\boldsymbol{\gamma}(s)+\sin\theta\boldsymbol{t}(s)$.

(4) $h_{o\boldsymbol{v}}(s)=h'_{o\boldsymbol{v}}(s)=h''_{o\boldsymbol{v}}(s)=h'''_{o\boldsymbol{v}}(s)=0$ if and only if (3) holds and

$$\begin{aligned} h'''_{o\boldsymbol{v}}(s) &= \pm((2\kappa'_n(s)-\kappa_g(s)\tau_g(s))\delta_o(s)-\kappa_n(s)\delta'_o(s)-\kappa_n^2(s)\tau_g(s))/\rho_o(s) \\ &= \pm\sigma_o(s)/\rho_o(s)=0. \end{aligned}$$

When $(\delta_o(s),\kappa_n(s))\neq(0,0)$, we have $\sigma_o(s)=0$ and

$$\boldsymbol{v}=\pm\frac{1}{\rho_o(s)}\left(\delta_o(s)\boldsymbol{\gamma}(s)-\kappa_n(s)\sqrt{\kappa_n^2(s)+\tau_g^2(s)}\overline{D}_o(s)\right).$$

When $\delta_o(s)=\kappa_n(s)=0$, we have $\sigma_o(s)=0$ and $\boldsymbol{v}=\cos\theta\boldsymbol{\gamma}(s)+\sin\theta\boldsymbol{t}(s)$.

(5)$h_{o\boldsymbol{v}}(s)=h'_{o\boldsymbol{v}}(s)=h''_{o\boldsymbol{v}}(s)=h'''_{o\boldsymbol{v}}(s)=h^{(4)}_{o\boldsymbol{v}}(s)=0$ if and only if (4) holds and

$$\begin{aligned} h^{(4)}_{o\boldsymbol{v}}(s) &= \pm((2\kappa'_n(s)-\kappa_g(s)\tau_g(s))\delta_o(s)-\kappa_n(s)\delta'_o(s)-\kappa_n^2(s)\tau_g(s))'/\rho_o(s) \\ &= \pm\sigma'_o(s)/\rho_o(s)=0. \end{aligned}$$

When $(\delta_o(s),\kappa_n(s))\neq(0,0)$, we have $\sigma_o(s)=\sigma'_o(s)=0$ and

$$\boldsymbol{v}=\pm\frac{1}{\rho_o(s)}\left(\delta_o(s)\boldsymbol{\gamma}(s)-\kappa_n(s)\sqrt{\kappa_n^2(s)+\tau_g^2(s)}\overline{D}_o(s)\right).$$

When $\delta_o(s) = \kappa_n(s) = 0$, we have $\sigma_o(s) = \sigma_o'(s) = 0$ and $\boldsymbol{v} = \cos\theta\boldsymbol{\gamma}(s) + \sin\theta\boldsymbol{t}(s)$.

$\square$

Proposition 4.2. *Let $\boldsymbol{\gamma} : I \longrightarrow M$ be a unit speed curve with $\kappa_g^2(s)+\tau_g^2(s) \neq 0$, then we have the following:*
(1) $h_{r\boldsymbol{v}}(s) = 0$ if and only if there exist $\mu, \eta, \xi \in \mathbb{R}$ with $\mu^2+\eta^2+\xi^2 = 1$ such that $\boldsymbol{v} = \mu\boldsymbol{\gamma}(s) + \eta\boldsymbol{t}(s) + \xi\boldsymbol{n}_\gamma(s)$.
(2) $h_{r\boldsymbol{v}}(s) = h'_{r\boldsymbol{v}}(s) = 0$ if and only if there exist $\theta \in \mathbb{R}$ such that

$$\boldsymbol{v} = \cos\theta\boldsymbol{\gamma}(s) + \sin\theta\overline{D}_r(s).$$

(3) $h_{r\boldsymbol{v}}(s) = h'_{r\boldsymbol{v}}(s) = h''_{r\boldsymbol{v}}(s) = 0$ if and only if $(\delta_r(s), \kappa_g(s)) \neq (0,0)$ and

$$\boldsymbol{v} = \pm\frac{1}{\rho_r(s)}\left(\delta_r(s)\boldsymbol{\gamma}(s) + \kappa_g(s)\sqrt{\kappa_g^2(s)+\tau_g^2(s)}\overline{D}_r(s)\right)$$

or $(\delta_r(s), \kappa_g(s)) = (0,0)$ and $\boldsymbol{v} = \cos\theta\boldsymbol{\gamma}(s) + \sin\theta\boldsymbol{t}(s)$.
(4) $h_{r\boldsymbol{v}}(s) = h'_{r\boldsymbol{v}}(s) = h''_{r\boldsymbol{v}}(s) = h'''_{r\boldsymbol{v}}(s) = 0$ if and only if $(\delta_r(s), \kappa_g(s)) \neq (0,0)$, $\sigma_r(s) = 0$ and

$$\boldsymbol{v} = \pm\frac{1}{\rho_r(s)}\left(\delta_r(s)\boldsymbol{\gamma}(s) + \kappa_g(s)\sqrt{\kappa_g^2(s)+\tau_g^2(s)}\overline{D}_r(s)\right)$$

or $(\delta_r(s), \kappa_g(s)) = (0,0)$ and $\boldsymbol{v} = \cos\theta\boldsymbol{\gamma}(s) + \sin\theta\boldsymbol{t}(s)$.
(5) $h_{r\boldsymbol{v}}(s) = h'_{r\boldsymbol{v}}(s) = h''_{r\boldsymbol{v}}(s) = h'''_{r\boldsymbol{v}}(s) = h^{(4)}_{r\boldsymbol{v}}(s) = 0$ if and only if $(\delta_r(s), \kappa_g(s)) \neq (0,0)$ $\sigma_r(s) = \sigma_r'(s) = 0$ and

$$\boldsymbol{v} = \pm\frac{1}{\rho_r(s)}\left(\delta_r(s)\boldsymbol{\gamma}(s) + \kappa_g(s)\sqrt{\kappa_g^2(s)+\tau_g^2(s)}\overline{D}_r(s)\right)$$

or $(\delta_r(s), \kappa_g(s)) = (0,0)$ and $\boldsymbol{v} = \cos\theta\boldsymbol{\gamma}(s) + \sin\theta\boldsymbol{t}(s)$.

We can show Propositions 4.2 by exactly the same arguments as those for Proposition 4.1. So we omit here.

5 Singularities of E-flat surfaces along the curve on the surface in the unit 3-sphere

In this section we classify the singularities of $OE_{\boldsymbol{\gamma}}$ and $NE_{\boldsymbol{\gamma}}$ as an application of the unfolding theory of functions. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \longrightarrow \mathbb{R}$ be a function germ, we call F an *r-parameter unfolding* of f, where $f(s) = F_{\boldsymbol{x}_0}(s, \boldsymbol{x}_0)$. If $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$ and $f^{(k+1)}(s_0) \neq 0$, then we say that f has *A_k-singularity* at s_0. Let F be an r-parameter unfolding of f and f has A_k-singularity ($k \geq 1$) at s_0. We denote the $(k-1)$-jet of the partial derivative $\partial F/\partial x_i$ at s_0 as

$$j^{(k-1)}\left(\frac{\partial F}{\partial x_i}(s, \boldsymbol{x}_0)\right)(s_0) = \sum_{j=0}^{k-1} \alpha_{ji}(s - s_0)^j, \quad (i = 1, \cdots, r).$$

If the rank of $k \times r$ matrix $(\alpha_{ji})_{j=0,\dots,k-1; i=1,\dots,r}$ is k ($k \leq r$), then F is called a *versal unfolding* of f. The *discriminant set* of F is defined by

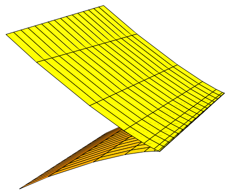
$$D_F = \left\{ \boldsymbol{x} \in \mathbb{R}^r \mid \exists s \in \mathbb{R},\ F(s, \boldsymbol{x}) = \frac{\partial F}{\partial s}(s, \boldsymbol{x}) = 0 \right\}.$$

For the discriminant set of F, we have the following theorem [2].

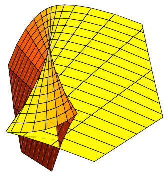
Theorem 5.1. *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \longrightarrow \mathbb{R}$ be an r-parameter unfolding of f which has A_k-singularity at s_0. Suppose F is a versal unfolding of f, then we have the following assertions.*

(a) *If $k = 1$, then D_F is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.*
(b) *If $k = 2$, then D_F is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.*
(c) *If $k = 3$, then D_F is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$.*

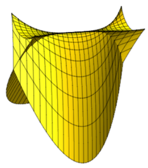
Here, C is the *ordinary cusp* defined by $\{(x_1, x_2) \mid x_1 = u^3, x_2 = u^2\}$ and $C \times \mathbb{R} = \{(x_1, x_2, x_3) \mid x_1 = u^3, x_2 = u^2, x_3 = v\}$ is called a *cuspidal edge* (Fig.1). Moreover, $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is called a *swallow tail* (Fig.2). Here we also give some other types of singularities. The *cuspidal beaks* is given by $CBK = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = v, x_2 = -2u^3 + uv^2, x_3 = 3u^4 - u^2 v^2)\}$(Fig.3). The *cuspidal lips* is given by $CL = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 3u^4 + 2u^2 v^2, x_2 = u^3 + uv^2, x_3 = v)\}$(Fig.4). The *cuspidal cross cap* is given by $CCR = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = u, x_2 = uv^3, v^2\}$(Fig.5).



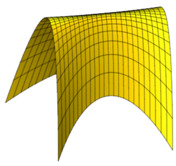
The cuspidal edge
Figure 1



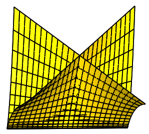
The swallow tail
Figure 2



The cuspidal beaks
Figure 3



The cuspidal lips
Figure 4



The cuspidal cross cap
Figure 5

By Propositions 4.1, 4.2, the discriminant sets of H_o and H_r are as follows:

$$D_{H_o} = Im\ OE_{\boldsymbol{\gamma}} = \{\cos\theta\boldsymbol{\gamma}(s) + \sin\theta\overline{D}_o(s) \mid s \in I, \theta \in \mathbb{R}\},$$
$$D_{H_r} = Im\ NE_{\boldsymbol{\gamma}} = \{\cos\theta\boldsymbol{\gamma}(s) + \sin\theta\overline{D}_r(s) \mid s \in I, \theta \in \mathbb{R}\}.$$

These are the spherical dual surfaces of $\boldsymbol{n}_\gamma$ and $\boldsymbol{b}$ respectively. By Propositions 4.1 and 4.2, we know that there are no $A_k(k \leq 3)$ singularities for the height functions $h_{o\boldsymbol{v}}$ if $(\delta_o(s), \kappa_n(s)) = (0, 0)$ and $h_{r\boldsymbol{v}}$ if $(\delta_r(s), \kappa_g(s)) = (0, 0)$. We have the following key propositions on H_o when $(\delta_o(s), \kappa_n(s)) \neq (0, 0)$ and H_r when $(\delta_r(s), \kappa_g(s)) \neq (0, 0)$.

Proposition 5.2. *If h_{ov_0} has A_k-singularity $(k = 1, 2, 3)$ at s_0, then H_o is a versal unfolding of h_{ov_0}.*

Proof. For $\boldsymbol{v} \in S^3$, we have $\boldsymbol{v} = (v_1, v_2, v_3, v_4) = (\pm(1-v_2^2-v_3^2-v_4^2)^{1/2}, v_2, v_3, v_4)$, without the loss of the generality, we consider $\boldsymbol{v} = ((1-v_2^2-v_3^2-v_4^2)^{1/2}, v_2, v_3, v_4)$ and $\boldsymbol{n}_\gamma(s) = (\boldsymbol{n}_{\gamma 1}(s), \boldsymbol{n}_{\gamma 2}(s), \boldsymbol{n}_{\gamma 3}(s), \boldsymbol{n}_{\gamma 4}(s))$, then

$$H_o(s, \boldsymbol{v}) = \langle \boldsymbol{n}_\gamma(s), \boldsymbol{v}\rangle = (1-v_2^2-v_3^2-v_4^2)^{1/2}\boldsymbol{n}_{\gamma 1}(s) + \boldsymbol{n}_{\gamma 2}(s)v_2 + \boldsymbol{n}_{\gamma 3}(s)v_3 + \boldsymbol{n}_{\gamma 4}(s)v_4.$$

Thus we have

$$\frac{\partial H}{\partial v_2}(s,\boldsymbol{v}) = -\frac{v_2}{v_1}\boldsymbol{n}_{\gamma 1}(s) + \boldsymbol{n}_{\gamma 2}(s), \frac{\partial H}{\partial v_3}(s,\boldsymbol{v}) = -\frac{v_3}{v_1}\boldsymbol{n}_{\gamma 1}(s) + \boldsymbol{n}_{\gamma 3}(s),$$
$$\frac{\partial H}{\partial v_4}(s,\boldsymbol{v}) = -\frac{v_4}{v_1}\boldsymbol{n}_{\gamma 1}(s) + \boldsymbol{n}_{\gamma 4}(s);$$
$$\frac{\partial^2 H}{\partial s\partial v_2}(s,\boldsymbol{v}) = -\frac{v_2}{v_1}\boldsymbol{n}'_{\gamma 1}(s) + \boldsymbol{n}'_{\gamma 2}(s),\ \frac{\partial^2 H}{\partial s\partial v_3}(s,\boldsymbol{v}) = -\frac{v_3}{v_1}\boldsymbol{n}'_{\gamma 1}(s) + \boldsymbol{n}'_{\gamma 3}(s),$$
$$\frac{\partial^2 H}{\partial s\partial v_4}(s,\boldsymbol{v}) = -\frac{v_4}{v_1}\boldsymbol{n}'_{\gamma 1}(s) + \boldsymbol{n}'_{\gamma 4}(s);$$
$$\frac{\partial^3 H}{\partial s^2\partial v_2}(s,\boldsymbol{v}) = -\frac{v_2}{v_1}\boldsymbol{n}''_{\gamma 1}(s) + \boldsymbol{n}''_{\gamma 2}(s),\ \frac{\partial^3 H}{\partial s^2\partial v_3}(s,\boldsymbol{v}) = -\frac{v_3}{v_1}\boldsymbol{n}''_{\gamma 1}(s) + \boldsymbol{n}''_{\gamma 3}(s),$$
$$\frac{\partial^3 H}{\partial s^2\partial v_4}(s,\boldsymbol{v}) = -\frac{v_4}{v_1}\boldsymbol{n}''_{\gamma 1}(s) + \boldsymbol{n}''_{\gamma 4}(s).$$

For a fixed $\boldsymbol{v}_0 = (v_{01}, v_{02}, v_{03}, v_{04})$ with $v_{01} \neq 0$, the 2-jet of $\frac{\partial H}{\partial v_i}(s,\boldsymbol{v}_0)(i = 2,3,4)$ at s_0 is

$$j^{(2)}\frac{\partial H}{\partial v_i}(s,\boldsymbol{v}_0)(s_0) = -\frac{v_{0i}}{v_{01}}\boldsymbol{n}_{\gamma 1}(s_0) + \boldsymbol{n}_{\gamma i}(s_0) + (-\frac{v_{0i}}{v_{01}}\boldsymbol{n}'_{\gamma 1}(s_0) + \boldsymbol{n}'_{\gamma i}(s_0))(s-s_0)$$
$$+\frac{1}{2}(-\frac{v_{0i}}{v_{01}}\boldsymbol{n}''_{\gamma 1}(s_0) + \boldsymbol{n}''_{\gamma i}(s_0))(s-s_0)^2.$$

It is enough to show that the rank of the matrix A is three, where

$$A = -\begin{pmatrix} \frac{v_{02}}{v_{01}}\boldsymbol{n}_{\gamma 1}(s_0) - \boldsymbol{n}_{\gamma 2}(s_0) & \frac{v_{03}}{v_{01}}\boldsymbol{n}_{\gamma 1}(s_0) - \boldsymbol{n}_{\gamma 3}(s_0) & \frac{v_{04}}{v_{01}}\boldsymbol{n}_{\gamma 1}(s_0) - \boldsymbol{n}_{\gamma 4}(s_0) \\ \frac{v_{02}}{v_{01}}\boldsymbol{n}'_{\gamma 1}(s_0) - \boldsymbol{n}'_{\gamma 2}(s_0) & \frac{v_{03}}{v_{01}}\boldsymbol{n}'_{\gamma 1}(s_0) - \boldsymbol{n}'_{\gamma 3}(s_0) & \frac{v_{04}}{v_{01}}\boldsymbol{n}'_{\gamma 1}(s_0) - \boldsymbol{n}'_{\gamma 4}(s_0) \\ \frac{v_{02}}{v_{01}}\boldsymbol{n}''_{\gamma 1}(s_0) - \boldsymbol{n}''_{\gamma 2}(s_0) & \frac{v_{03}}{v_{01}}\boldsymbol{n}''_{\gamma 1}(s_0) - \boldsymbol{n}''_{\gamma 3}(s_0) & \frac{v_{04}}{v_{01}}\boldsymbol{n}''_{\gamma 1}(s_0) - \boldsymbol{n}''_{\gamma 4}(s_0) \end{pmatrix}.$$

By straight calculation, we have

$$\begin{aligned}\det A &= \langle \boldsymbol{n}_\gamma(s_0) \times \boldsymbol{n}'_\gamma(s_0) \times \boldsymbol{n}''_\gamma(s_0), \boldsymbol{v}_0\rangle / v_{01} \\ &= \langle \delta_o(s_0)\boldsymbol{\gamma}(s_0) - \kappa_n(s_0)\sqrt{\kappa_n^2(s_0) + \tau_g^2(s_0)}\overline{D}_o(s_0), \boldsymbol{v}_0\rangle / v_{01}.\end{aligned}$$

Since $\boldsymbol{v}_0 \in D_{H_o}$ is a singular point, we have

$$\boldsymbol{v}_0 = \pm\frac{1}{\rho_o(s_0)}\left(\delta_o(s_0)\boldsymbol{\gamma}(s_0) - \kappa_n(s_0)\sqrt{\kappa_n^2(s_0) + \tau_g^2(s_0)}\overline{D}_o(s_0)\right),$$

and $\rho_o(s_0) \neq 0$. Therefore we have

$$\det A = \pm \rho_o(s_0)/v_{01} \neq 0.$$

So the rank of A is three, this completes the proof. $\square$

Proposition 5.3. *If $h_{rv_0}(s)$ has A_k-singularity $(k = 1, 2, 3)$ at s_0, then H_r is a versal unfolding of h_{rv_0}.*

The proof of Proposition 5.3 is similar to the proof of Proposition 5.2, so we omit it. We have the following theorem:

Theorem 5.4. *Let $\boldsymbol{\gamma} : I \longrightarrow M \subset S^3$ be a unit speed curve.*
(A) *For osculating E-flat surface OE_γ along $\boldsymbol{\gamma}$, suppose $\kappa_n^2(s_0) + \tau_g^2(s_0) \neq 0$, $(\delta_o(s_0), \kappa_n(s_0)) \neq 0$, we have the following assertions:*

(a) *OE_γ is locally diffeomorphic to the cuspidal edge at (s_0, θ_0) if and only if $\sigma_o(s_0) \neq 0$ and $\kappa_n(s_0)\sqrt{\kappa_n^2(s_0) + \tau_g^2(s_0)} \cos\theta_0 + \delta_o(s_0)\sin\theta_0 = 0$.*

(b) *OE_γ is locally diffeomorphic to the swallowtail at (s_0, θ_0) if and only if $\sigma_o(s_0) = 0$, $\sigma_o'(s_0) \neq 0$ and $\kappa_n(s_0)\sqrt{\kappa_n^2(s_0) + \tau_g^2(s_0)} \cos\theta_0 + \delta_o(s_0)\sin\theta_0 = 0$.*

(B) *For normal E-flat surface NE_γ along $\boldsymbol{\gamma}$, suppose $\kappa_g^2(s_0) + \tau_g^2(s_0) \neq 0$ and $(\delta_r(s_0), \kappa_g(s_0)) \neq 0$, we have the following assertions:*

(a) *NE_γ is locally diffeomorphic to the cuspidal edge at (s_0, θ_0) if and only if $\sigma_r(s_0) \neq 0$ and $\kappa_g(s_0)\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)} \cos\theta_0 - \delta_r(s_0)\sin\theta_0 = 0$.*

(b) *NE_γ is locally diffeomorphic to the swallowtail at (s_0, θ_0) if and only if $\sigma_r(s_0) = 0$, $\sigma_r'(s_0) \neq 0$ and $\kappa_g(s_0)\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)} \cos\theta_0 - \delta_r(s_0)\sin\theta_0 = 0$.*

Proof. By Propositions 4.1 and 4.2, the discriminant sets of H_o and H_r are OE_γ and NE_γ respectively. We also have h_{ov_0}, h_{rv_0} have A_k singularities ($k = 1, 2, 3$) if and only if the above conditions hold respectively. By Propositions 5.2 and 5.3, H_o and H_r are versal unfoldings of h_{ov_0} and h_{rv_0} at any singular point $s_0 \in I$ respectively. By applying Theorem 5.1, we can get the above assertions. $\square$

In [8], the classifications of singularities are given for the extrinsic flat great circular surface by using criteria for singularities of fronts. For the detailed descriptions of fronts, see [1].

Theorem 5.4 in our paper can be considered an a special case of Theorem 5.1 in [8]. But in our case, $c_{o2}(s) \neq 0$ and $c_{r2}(s) \neq 0$, so we don't have cuspidal cross caps on OE_γ and NE_γ.

For the special case when $\kappa_n(s) \equiv 0$, we have $\mathbf{a}_{o2}(s) = -\boldsymbol{b}(s)$, $\mathbf{a}_{o3}(s) = \overline{D}_o(s) = \boldsymbol{t}(s)$. In this case, $c_{o1}(s) = 0$, $c_{o2}(s) = \tau_g(s)$, $c_{o3}(s) = 0$, $c_{o4}(s) = 0$, $c_{o5}(s) = 1$ and $c_{o6}(s) = \kappa_g(s)$.

For the special case when $\kappa_g(s) \equiv 0$, we have $\mathbf{a}_{r2}(s) = \boldsymbol{n}_\gamma(s)$, $\mathbf{a}_{r3}(s) = \overline{D}_r(s) = \boldsymbol{t}(s)$. In this case, $c_{r1}(s) = 0$, $c_{r2}(s) = \tau_g(s)$, $c_{r3}(s) = 0$, $c_{r4}(s) = 0$, $c_{r5}(s) = 1$ and $c_{r6}(s) = -\kappa_n(s)$.

By Theorem 5.2 in [8], we have the following theorem for the E-flat surfaces along $\boldsymbol{\gamma}$.

Theorem 5.5. *Let $\boldsymbol{\gamma} : I \longrightarrow M \subset S^3$ be a unit speed curve.*
(A) *For osculating E-flat surface OE_γ along $\boldsymbol{\gamma}$, suppose $\tau_g(s) \neq 0$ and $\kappa_n(s) \equiv 0$, we have the following assertions*:
(1) *If $\kappa_g(s_0) \neq 0$, then $\theta_0 = 0$ or π and the following assertions hold*:

(a) *OE_γ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at (s_0, θ_0) if and only if $\tau_g(s_0) \neq 0$.*

(b) *OE_γ is never locally diffeomorphic to the swallowtail and the cuspidal cross cap.*

(2) *If $\kappa_g(s_0) = 0$, the following assertions hold*:

(c) *OE_γ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at (s_0, θ_0) if and only if $\theta_0 \neq 0, \pi$ and $\tau_g(s_0)\kappa_g'(s_0) \neq 0$.*

(d) *OE_γ is locally diffeomorphic to the cuspidal beaks at (s_0, θ_0) if and only if $\theta_0 = 0$ or π and $\tau_g(s_0)\kappa_g'(s_0) \neq 0$.*

(e) *OE_γ is never locally diffeomorphic to the swallowtail, the cuspidal lips and the cuspidal cross cap.*

(B) *For normal E-flat surface NE_γ along $\boldsymbol{\gamma}$, suppose $\tau_g(s) \neq 0$ and $\kappa_g(s) \equiv 0$, we have the following assertions*:
(1) *If $\kappa_n(s_0) \neq 0$, then $\theta_0 = 0$ or π and the following assertions hold*:

(a) *NE_γ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at (s_0, θ_0) if and only if $\tau_g(s_0) \neq 0$.*

(b) *NE_γ is never locally diffeomorphic to the swallowtail or cuspidal cross cap.*

(2) *If $\kappa_n(s_0) = 0$, the following assertions hold*:

(c) *NE_γ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at (s_0, θ_0) if and only if $\theta_0 \neq 0, \pi$ and $\tau_g(s_0)\kappa_n'(s_0) \neq 0$.*

(d) *NE_γ is locally diffeomorphic to the cuspidal beaks at (s_0, θ_0) if and only if $\theta_0 = 0$ or π and $\tau_g(s_0)\kappa_n'(s_0) \neq 0$.*

(e) *NE_γ is never locally diffeomorphic to the swallowtail, the cuspidal lips and the cuspidal cross cap.*

6 Invariants of curves on surfaces in the unit 3-sphere

In this section, we consider geometric meanings of the invariants σ_o and σ_r. Let $\Gamma : I \longrightarrow S^3$ be a regular curve and $F : S^3 \longrightarrow \mathbb{R}$ a submersion. We say that Γ and $F^{-1}(0)$ have contact of at least order k for $t = t_0$ if the function $f(t) = F \circ \Gamma(t)$ satisfies $f(t_0) = f'(t_0) = \cdots = f^{(k)}(t_0) = 0$. If Γ and $F^{-1}(0)$ have contact of at least order k for $t = t_0$ and satisfies the condition that $f^{(k+1)}(t_0) \neq 0$, then we say that Γ and $F^{-1}(0)$ have contact of order k for $t = t_0$. We define a function $\mathfrak{g} : S^3 \times S^3 \longrightarrow \mathbb{R}$ by $\mathfrak{g}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle$. For any fixed $\boldsymbol{v}_0 \in S^3$, we write $\mathfrak{g}_{\boldsymbol{v}_0} = \langle \boldsymbol{u}, \boldsymbol{v}_0 \rangle$. Then we have

$$\mathfrak{g}_{\boldsymbol{v}_0}^{-1}(0) = \{\boldsymbol{u} \in S^3 \mid \langle \boldsymbol{u}, \boldsymbol{v}_0 \rangle = 0\},$$

which is a great 2-sphere in S^3. We denote that $S^2(\boldsymbol{v}_0) = \mathfrak{g}_{\boldsymbol{v}_0}^{-1}(0)$. For a spherical curve $\boldsymbol{\sigma} : I \longrightarrow S^3$, we consider a function $g(s) = \langle \boldsymbol{\sigma}(s), \boldsymbol{v}_0 \rangle = \mathfrak{g}_{\boldsymbol{v}_0} \circ \boldsymbol{\sigma}(s)$. Then $\boldsymbol{\sigma}$ is tangent to $\mathfrak{g}_{\boldsymbol{v}_0}^{-1}(0)$ at $\boldsymbol{\sigma}(s_0)$ if and only if $g(s_0) = g'(s_0) = 0$. In this case $S^2(\boldsymbol{v}_0)$ is called a *tangent unit* 2*-sphere* of $\boldsymbol{\sigma}$ at $\boldsymbol{\sigma}(s_0)$ in S^3, which is written by $TS^2_{\boldsymbol{\sigma}}(\boldsymbol{v}_0)$. We remark that there are infinitely many tangent 2-spheres of $\boldsymbol{\sigma}$ at $\boldsymbol{\sigma}(s_0)$ in S^3.

Proposition 6.1. *Let $\boldsymbol{\gamma} : I \longrightarrow M \subset S^3$ be a unit speed curve with $\kappa_n^2(s) + \tau_g^2(s) \neq 0$ and $(\delta_o(s), \kappa_n(s)) \neq (0, 0)$. For $\boldsymbol{v}_0 = OE_{\boldsymbol{\gamma}}(s_0, \theta_0)$, we have the following*:
(1) The order of contact of $\boldsymbol{n}_\gamma(s)$ with $TS^2_{\boldsymbol{n}_\gamma}(\boldsymbol{v}_0)$ at $s = s_0$ is two if and only if

$$\delta_o(s_0) \sin\theta_0 + \kappa_n(s_0)\sqrt{\kappa_n^2(s_0) + \tau_g^2(s_0)} \cos\theta_0 = 0 \qquad (*)$$

and $\sigma_o(s_0) \neq 0$.
(2) The order of contact of $\boldsymbol{n}_\gamma(s)$ with $TS^2_{\boldsymbol{n}_\gamma}(\boldsymbol{v}_0)$ at $s = s_0$ is three if and only if $(*)$, $\sigma_o(s_0) = 0$ and $\sigma_o'(s_0) \neq 0$.

Proof. By the assertions (3), (4) of Proposition 4.1, the conditions $h_{o\boldsymbol{v}}(s_0) = h'_{o\boldsymbol{v}}(s_0) = h''_{o\boldsymbol{v}}(s_0) = 0$ and $h'''_{o\boldsymbol{v}}(s_0) \neq 0$ if and only if $(*)$ and $\sigma_o(s_0) \neq 0$. Since $\mathfrak{g}_{\boldsymbol{v}_0} \circ \boldsymbol{n}_\gamma(s) = h_{o\boldsymbol{v}}(s)$, the above condition means that $\boldsymbol{n}_\gamma(s)$ and $TS^2_{\boldsymbol{n}_\gamma}(\boldsymbol{v}_0)$ have contact of order two at $s = s_0$. For the proof of the assertion (2), we use the assertions (4), (5) of Proposition 4.1 exactly the same way as the above case. $\square$

By the same arguments of Proposition 6.1, we have the following proposition.

Proposition 6.2. *Let $\boldsymbol{\gamma} : I \longrightarrow M \subset S^3$ be a unit speed curve with $\kappa_g^2(s) + \tau_g^2(s) \neq 0$ and $(\delta_r(s), \kappa_g(s)) \neq (0,0)$. For $\boldsymbol{v}_0 = NE_{\boldsymbol{\gamma}}(s_0, \theta_0)$, we have the following*:
(1) The order of contact of $\boldsymbol{b}(s)$ with $TS^2_{\boldsymbol{b}}(\boldsymbol{v}_0)$ at $s = s_0$ is two if and only if

$$\delta_r(s_0)\sin\theta_0 - \kappa_g(s_0)\sqrt{\kappa_g^2(s_0) + \tau_g^2(s_0)}\cos\theta_0 = 0 \qquad (**)$$

and $\sigma_r(s_0) \neq 0$.
(2) The order of contact of $\boldsymbol{b}(s)$ with $TS^2_{\boldsymbol{b}}(\boldsymbol{v}_0)$ at $s = s_0$ is three if and only if $(**)$, $\sigma_r(s_0) = 0$ and $\sigma_r'(s_0) \neq 0$.

Therefore the geometric meaning of the classification results in Theorem 5.4 are given as follows.

Theorem 6.3. *Let $\boldsymbol{\gamma} : I \longrightarrow M \subset S^3$ be a unit speed curve.*
(A) *For the osculating E-flat surface $OE_{\boldsymbol{\gamma}}$ along $\boldsymbol{\gamma}$, suppose $\kappa_n^2(s_0) + \tau_g^2(s_0) \neq 0$, $(\delta_o(s_0), \kappa_n(s_0)) \neq (0,0)$ and $\boldsymbol{v}_0 = OE_{\boldsymbol{\gamma}}(s_0, \theta_0)$. We have the following assertions*:
(1) *$OE_{\boldsymbol{\gamma}}$ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at (s_0, θ_0) if and only if the order of contact of $\boldsymbol{n}_{\boldsymbol{\gamma}}(s)$ with $TS^2_{\boldsymbol{n}_{\boldsymbol{\gamma}}}(\boldsymbol{v}_0)$ at $s = s_0$ is two.*
(2) *$OE_{\boldsymbol{\gamma}}$ is locally diffeomorphic to the swallowtail at (s_0, θ_0) if and only if the order of contact of $\boldsymbol{n}_{\boldsymbol{\gamma}}(s)$ with $TS^2_{\boldsymbol{n}_{\boldsymbol{\gamma}}}(\boldsymbol{v}_0)$ at $s = s_0$ is three.*
(B) *For the normal E-flat surface $NE_{\boldsymbol{\gamma}}$ along $\boldsymbol{\gamma}$, suppose $\kappa_g^2(s_0) + \tau_g^2(s_0) \neq 0$, $(\delta_r(s_0), \kappa_g(s_0)) \neq (0,0)$ and $\boldsymbol{v}_0 = NE_{\boldsymbol{\gamma}}(s_0, \theta_0)$. We have the following assertions*:
(1) *$NE_{\boldsymbol{\gamma}}$ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at (s_0, θ_0) if and only if the order of contact of $\boldsymbol{b}(s)$ with $TS^2_{\boldsymbol{b}}(\boldsymbol{v}_0)$ at $s = s_0$ is two.*
(2) *$NE_{\boldsymbol{\gamma}}$ is locally diffeomorphic to the swallowtail at (s_0, θ_0) if and only if the order of contact of $\boldsymbol{b}(s)$ with $TS^2_{\boldsymbol{b}}(\boldsymbol{v}_0)$ at $s = s_0$ is three.*

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