HOKKAIDO UNIVERSITY

Title	Pfaffian Systems of Confluent Hypergeometric Functions of Two Variables
Author(s)	向井, 重雄
Degree Grantor	北海道大学
Degree Name	博士(理学)
Dissertation Number	甲第14641号
Issue Date	2021-09-24
DOI	https://doi.org/10.14943/doctoral.k14641
Doc URL	https://hdl.handle.net/2115/83469
Type	doctoral thesis
File Information	Shigeo_Mukai.pdf

Hokkaido University Collection of Scholarly and Academic Papers : HUSCAP

Pfaffian Systems of Confluent Hypergeometric Functions of Two Variables

(２変数合流型超幾何関数のパフィアン系)

by

Shigeo MUKAI

Department of Mathematics, Graduate School of Science,
Hokkaido University

September, 2021

Abstract

We study Pfaffian systems of confluent hypergeometric functions of two variables with rank three and rank four. This paper is composed of two parts. In part I, we study Pfaffian systems of two variables with rank three by using rational twisted cohomology groups associated with Euler type integrals of these functions. We give bases of the cohomology groups, whose intersection matrices depend only on parameters. Each connection matrix of our Pfaffian systems admits a decomposition into five parts, each of which is the product of a constant matrix and a rational 1-form on the space of variables. In part II, we consider confluences of Euler type integrals expressing solutions to Appell's F_2 system of hypergeometric differential equations, and study systems of confluent hypergeometric differential equations of rank four of two variables. Our consideration is based on a confluence transforming the abelian group $(\mathbb{C}^\times)^2$ to the Jordan group of size two. For each system obtained by our study, we give its Pfaffian system with a connection matrix admitting a decomposition into four or five parts, each of which is the product of a matrix depending only on parameters and a rational 1-form in two variables. We classify these Pfaffian systems under an equivalence relation. Any system obtained by our study is equivalent to one of Humbert's Ψ_1 system, Humbert's Ξ_1 system, and the system satisfied by the product of two Kummer's confluent hypergeometric functions.

Contents

1 Introduction **2**

2 General Hypergeometoric Function **5**

3 Twisted Cohomology Group **7**

4 Intersection Matrices **8**

5 Pfaffian Systems of Gauss type, Kummer type and Hermite type **15**

6 Pfaffian Systems of Appell's F_1 and Humbert's Φ_1 **22**

7 List of Pfaffian Systems (of single variable type) **30**

8 List of Pfaffian Systems (of two variable type) **32**

9 Pfaffian system of Appell's F_2 **37**

10 Equivalence of Pfaffian systems **39**

11 Confluences of Appell's F_2 system **41**

12 Detailed Calculations of Confluences **49**

12.1 Confluence from Appell's F_2 to Humbert's Ξ_1 50

12.2 Confluence from Appell's F_2 to Humbert's Ψ_1 55

12.3 Confluence from Appell's F_2 to Hörn's H_2 58

12.4 Confluence from Appell's F_2 to the product of Kummer's functions 62

13 Classification of confluent Pfaffian systems **63**

14 Automorphism Group **68**

1 Introduction

Kummer's hypergeometric series is defined by

$$ {}_1F_1(a,c;x) = \sum_{n=0}^{\infty} \frac{(a,n)}{(c,n)(1,n)} x^n, \tag{1.1} $$

and it satisfies Kummer's hypergeometric differential equation

$$ \Big[x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (c-x)\frac{\mathrm{d}}{\mathrm{d}x} - a \Big] f(x) = 0. \tag{1.2} $$

It is well known that we can obtain them from Gauss' hypergeometric series ${}_2F_1(a,b,c;x)$ and differential equation by introducing an infinitesimal parameter ε, expressing the parameter b and the variable x as $b = 1/\varepsilon$ and εx, and taking the limit $\varepsilon \to 0$. By applying this confluence process to Appell's hypergeometric series of two variables, Humbert P. introduces seven confluent hypergeometric series of two variables in [Hu], and Hörn J. completes a list of such series in [Ho].

It is pointed out in [GRS] and [KK] that we can study confluent hypergeometric functions in a unified way by regarding integrands of their integral representations as characters $u_\lambda(\beta)$ of maximal abelian subgroups of the universal covering of $\mathrm{GL}(n,\mathbb{C})$. Here complex parameters in a confluent hypergeometric function correspond to β and a grade of confluence is expressed as a partition λ of a natural number n.

Part I of this paper consisting of Sections 2 through 8 has been published in Kyushu Journal of Mathematics as [Mu]. In part I, we study Pfaffian systems of confluent hypergeometric functions in terms of intersection forms on rational twisted de Rham cohomology groups $H^1(\Omega^\bullet(x), \nabla_\omega)$ associated with Euler type integrals of them (refer to Section 3 for definitions of this cohomology group and the intersection form). In the case of Appell's hypergeometric function F_1, the connection matrix of its Pfaffian system is expressed by the intersection form on $H^1(\Omega^\bullet(x), \nabla_\omega)$ in [Mt2]. In this expression, it is important to find a basis of the rational twisted de Rham cohomology group so that its intersection matrix depends only on parameters. In confluent cases, we are suggested to use rational 1-forms $\phi_{i;k}$ by the study in [KHT], which mentions that these forms are convenient to see processes of confluences. It is shown in [KTn] that intersection numbers of $\phi_{i;k}$ depend only on parameters. In part I, we also give formulas to evaluate intersection numbers of those forms, by which we can see the deformation of intersection numbers of these forms by confluences. By using our formulas, we choose frames of Pfaffian systems of confluent hypergeometric functions of one variable with rank two and those of two variables with rank three. It turns out that their connection matrices W admit decompositions, that is,

(1) $W = \mathrm{d}(f_1)W_1 + \mathrm{d}(f_2)W_2$ in one variable cases, and $W = \sum_{i=1}^{5} \mathrm{d}(f_i)W_i$ in two variable cases, where d is the exterior derivative on the space of variables;

(2) $\mathrm{d}(f_i)$ is a rational 1-form on the space of variables;

(3) W_i is a square matrix depending only on parameters; it is of size two in one variable cases and of size three in two variable cases.

We list frames and $\mathrm{d}(f_i)W_i$ in one variable cases in Section 7 and those in two variable cases in Section 8. The connection matrix W satisfies

$$ W\,C - C\,{}^tW = O, $$

where $2\pi\sqrt{-1}C$ is the intersection matrix of the basis of $H^1(\Omega^\bullet(x), \nabla_\omega)$ corresponding to the frame of our Pfaffian system. In the two variable case, we can easily show that the integrability condition reduces to $W \wedge W = O$ because of $\mathrm{d}W = O$.

We also study in Section 5 and Section 6 whether the decomposition of the connection matrix of our Pfaffian system is preserved under a limit process of confluence. For example, each part of the connection matrix of our Pfaffian system of Gauss's hypergeometric function converges to that of Kummer's one by a confluence. For a confluence from Kummer's one to Hermite's one, though the connection matrix of our Pfaffian system converges, its decomposition is not preserved; a part of the connection matrix diverges. We study how to avoid this phenomenon.

In part II, we consider confluences of Euler type integrals expressing solutions to Appell's F_2 system of hypergeometric differential equations, and study systems of confluent hypergeometric differential equations of rank four of two variables. Appell's F_2 system admits solutions expressed by Euler type integrals

$$\iint_\Delta t_1^{b_1-1}(1-t_1)^{c_1-b_1-1}t_2^{b_2-1}(1-t_2)^{c_2-b_2-1}(1-t_1x-t_2y)^{-a}\mathrm{d}t_1\mathrm{d}t_2, \tag{1.3}$$

where Δ are 2-chains with certain cycle conditions. By setting

$$\begin{array}{ccccccc} \mathbf{l}_0 = 1, & \mathbf{l}_1 = t_1, & \mathbf{l}_2 = 1-t_1, & \mathbf{l}_3 = t_2, & \mathbf{l}_4 = 1-t_2, & \mathbf{l}_5 = 1-xt_1-yt_2, \\ \alpha_0 = a-c_1-c_2, & \alpha_1 = b_1, & \alpha_2 = c_1-b_1, & \alpha_3 = b_2, & \alpha_4 = c_2-b_2, & \alpha_5 = -a, \end{array}$$

and regarding $\mathbf{l}_0, \mathbf{l}_1, \ldots, \mathbf{l}_5$ as independent variables of $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$, we can consider the integrand $u_{F_2} = \prod_{i=0}^5 \mathbf{l}_i^{\alpha_i}$ in (1.3) as a character of (the universal covering of) the abelian group of $(\mathbb{C}^\times)^6$ embedded diagonally in $\mathrm{GL}(6,\mathbb{C})$. We use confluences which transform a subgroup $(\mathbb{C}^\times)^2$ of $(\mathbb{C}^\times)^6$ to the Jordan group $J(2) = \left\{ \begin{bmatrix} h_1 & h_2 \\ 0 & h_1 \end{bmatrix} \mid h_1 \in \mathbb{C}^\times,\ h_2 \in \mathbb{C} \right\} \subset \mathrm{GL}(2,\mathbb{C})$. Here a confluence is an operator conjugating invertible diagonal matrices of size 2 by an element $q(\varepsilon) \in \mathrm{GL}(2,\mathbb{C}(\varepsilon))$ with an infinitesimal parameter ε, and taking the limit $\varepsilon \to 0$. By choosing different indices i and j $(0 \le i, j \le 5)$ and using this confluence, we have a confluence of a character, which transforms $\mathbf{l}_i^{\alpha_i}\mathbf{l}_j^{\alpha_j}$ on $(\mathbb{C}^\times)^2$ to $\mathbf{l}_i^{\alpha_i'}\exp(\beta_i'\mathbf{l}_j/\mathbf{l}_i)$ on $J(2)$. We apply this confluence of character to Euler type integrals (1.3). From now on, $\mathbf{l}_1, \ldots, \mathbf{l}_5$ are regarded as linear forms in t_1, t_2, and l_k $(k = 1, \ldots, 5)$ denotes the line $\mathbf{l}_k = 0$ in the complex projective plane $\mathbb{P}^2$ with the line at infinity l_0 regarded as $\mathbf{l}_0 = 0$. Note that the intersections $l_1 \cap l_2$ and $l_3 \cap l_4$ lie on the line l_0, and that any other three lines l_p, l_q, l_r $((p,q,r) \neq (0,1,2), (0,3,4))$ of the six are in generic position in $\mathbb{P}^2$. In this application to (1.3), parameters α_i, α_j and $\mathbf{l}_j$ vary as

$$\alpha_i(\varepsilon) = \alpha_i + (1-1/\varepsilon)\alpha_j,\ \alpha_j(\varepsilon) = \alpha_j/\varepsilon, \quad \mathbf{l}_j(\varepsilon) = \mathbf{l}_i + \varepsilon(\mathbf{l}_j - \mathbf{l}_i)$$

according to the change of ε, and the integrand u_{F_2} in (1.3) is changed into

$$\mathbf{l}_i^{\alpha_i(\varepsilon)}\mathbf{l}_j(\varepsilon)^{\alpha_j(\varepsilon)} \prod_{0\le k\le 5}^{k\neq i,j} \mathbf{l}_k^{\alpha_k}. \tag{1.4}$$

If the configuration of the six lines is kept invariant topologically under the variation of the line $l_j(\varepsilon) : \mathbf{l}_j(\varepsilon) = 0$ for $\varepsilon \neq 0$, i.e., each space $\mathbb{P}^2 - l_j(\varepsilon) - \cup_{0\le k\le 5}^{k\neq j} l_k$ for $\varepsilon \neq 0$ is homeomorphic to $\mathbb{P}^2 - \cup_{0\le k\le 5} l_k$, then there exist a projective transformation g of $\mathbb{P}^2$ and an action of $h \in (\mathbb{C}^\times)^6$

on the six linear forms which transform $\mathbf{l}_0, \cdots, \mathbf{l}_j(\varepsilon), \ldots, \mathbf{l}_5$ into $\mathbf{l}_0, \cdots, \mathbf{l}_4, \mathbf{l}'_5(\varepsilon)$, where

$$\mathbf{l}'_5(\varepsilon) = 1 + x't_1 + y't_2$$

and x', y' are rational functions of x, y, and ε. Since we can rewrite (1.4) in terms of $\mathbf{l}_0, \cdots, \mathbf{l}_4, \mathbf{l}'_5(\varepsilon)$ by using g and h, we realize this application to (1.3) as an operator $\mathcal{C}(i,j)$ on (1.3) substituting $(\alpha_i(\varepsilon), \alpha_j(\varepsilon))$ and (x', y') for (α_i, α_j) and (x, y), multiplying a factor arising from h to it, and taking the limit $\varepsilon \to 0$. In this way, we have transformations of parameters α_i, α_j and variables x, y with an infinitesimal parameter ε just like those used in the confluence from Gauss' hypergeometric series and differential equation to Kummer's ones. Note that the line l_0 cannot move and movements of $\ell_1, \ldots, \ell_4$ should be restricted under this condition for the variation of $\mathbf{l}_j(\varepsilon)$. Note also that if this condition for the variation of $\mathbf{l}_j(\varepsilon)$ is not satisfied, then we cannot transform (1.4) into integrands of (1.3) by any projective transformations of $\mathbb{P}^2$ and any actions of $(\mathbb{C}^\times)^6$. Hence we have thirteen confluences $\mathcal{C}(0,j)$ $(1 \leq j \leq 5)$, $\mathcal{C}(i,5)$ $(1 \leq i \leq 4)$, $\mathcal{C}(1,2)$, $\mathcal{C}(2,1)$, $\mathcal{C}(3,4)$ and $\mathcal{C}(4,3)$ in Definition 11.3.

In Section 12, we execute $\mathcal{C}(i,j)$, and we obtain four confluent systems of differential equations: Humbert's Ξ_1 system by $\mathcal{C}(0,j)$ $(1 \leq j \leq 4)$, Humbert's Ψ_1 system by $\mathcal{C}(1,2)$, $\mathcal{C}(2,1)$, $\mathcal{C}(3,4)$, $\mathcal{C}(4,3)$, Hörn's H_2 system by $\mathcal{C}(i,5)$ $(1 \leq i \leq 4)$, and the system satisfied by the product ${}_1F_1(a,c;x)\ {}_1F_1(a,c;y)$ of the Kummer's hypergeometric series by $\mathcal{C}(0,5)$. We give Pfaffian systems of the four confluent systems in Propositions 12.1–12.4. The connection matrix W of each Pfaffian system admits the following decomposition:

(1) $W = \sum_{i=1}^{m} \mathrm{d}(f_i) W_i$, where m is 4 or 5 and d is the exterior derivative on the space of the independent variables;

(2) $\mathrm{d}(f_i)$ is a rational 1-form on the space of variables;

(3) W_i is a square matrix of size four depending only on parameters.

We remark that there exists only one system of differential equations obtained by once application of this kind of confluences to Appell's F_1 system of hypergeometric differential equations as shown in Section 6.

We classify obtained Pfaffian systems by the equivalence relation given in Definition 10.1 between two Pfaffian systems

$$\mathrm{d}\mathbf{F}(x,y) = W\mathbf{F}(x,y), \quad \mathrm{d}\tilde{\mathbf{F}}(x,y) = \tilde{W}\tilde{\mathbf{F}}(x,y)$$

of same rank with regular locus X and $\tilde{X}$, respectively. Under this equivalence relation, we can show that a Pfaffian system of Appell's F_2 and that of Appell's F_3 are equivalent by a birational map $(x,y) \mapsto (1/x, 1/y)$, since two systems are equivalent if one is transformed into the other by a birational map $\psi(x,y)$ whose restriction to X is a biregular morphism between X and $\tilde{X}$. Theorems 13.1 and 13.2 state that the Pfaffian system of Humbert's Ξ_1 is equivalent to that of Hörn's H_2, and that of Humbert's Ψ_1 is not equivalent to them under some non-integral conditions on parameters, respectively. We can prove Theorem 13.1 by finding transformations which give the equivalence. To prove Theorem 13.2, we introduce the type of spectral partitions for our Pfaffian system with a connection matrix W satisfying (1),(2),(3) as the set of five partitions of the number 4 given by the multiplicities of the roots of the characteristic polynomial of W_i

$(i = 1, \ldots, 5)$ in W. The type of spectral partitions of the Pfaffian system of Humbert's Ψ_1 is different from that of Humbert's Ξ_1 under the previously stated conditions on parameters. Since the type of spectral partitions is not kept invariant under transformations by birational maps, only this difference is not enough to show Theorem 13.2. To complete our proof of Theorem 13.2, we need to determine the structure of the group of birational maps whose restrictions to the regular locus of Ψ_1 are biregular; it is studied in Section 14.

We conclude that any system of differential equations obtained by our confluences on Appell's F_2 system is equivalent to one of the following three: Humbert's Ψ_1 system, Humbert's Ξ_1 system, and the system satisfied by the product ${}_1F_1(a, c; x){}_1F_1(a', c; y)$.

2 General Hypergeometoric Function

By referring to [KK, Section 2.], we introduce general hypergeometric functions, and prepare some notations and terminologies associated with them.

Let $\lambda = (\lambda_0, \cdots, \lambda_{m-1})$ be a partition of n, which is a set of natural numbers $\lambda_0 \geq \lambda_1 \cdots \geq \lambda_{m-1}$ satisfying $|\lambda| = \lambda_0 + \cdots + \lambda_{m-1} = n$. For each λ_k, $J(\lambda_k)$ denotes the Jordan group

$$\left\{ \sum_{0 \leq j < \lambda_k} h_j \Lambda_{\lambda_k}^j | h_j \in \mathbb{C}, h_0 \neq 0 \right\}$$

of size λ_k, where Λ_{λ_k} is the shift matrix of size λ_k given by

$$\Lambda_{\lambda_k} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}. \tag{2.1}$$

Let H_λ be a maximal abelian subgroup

$$\begin{bmatrix} J(\lambda_0) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & J(\lambda_{m-1}) \end{bmatrix}$$

in $\mathrm{GL}(n) = \mathrm{GL}(n, \mathbb{C})$. We use also the notation $[h_0, \ldots, h_{\lambda_k - 1}]$ standing for the element $\sum_{0 \leq j < \lambda_k} h_j \Lambda_{\lambda_k}^j \in J(\lambda_k)$. Define an isomorphism

$$\iota : H_\lambda \to \prod_k (\mathbb{C}^\times \times \mathbb{C}^{\lambda_k - 1})$$

by corresponding $h = (h^{(0)}, \ldots, h^{(m-1)}), h^{(k)} = [h_{k;0}, \ldots, h_{k;\lambda_k - 1}]$ to

$$\iota(h) = (h_{0;0}, \ldots, h_{0;\lambda_0 - 1}, \ldots, h_{m-1;0}, \ldots, h_{m-1;\lambda_{m-1} - 1}).$$

Let $\tilde{H}_\lambda$ be the universal covering group of H_λ:

$$\tilde{H}_\lambda = \tilde{J}(\lambda_0) \times \cdots \times \tilde{J}(\lambda_{m-1}).$$

Then the map ι can be lifted to the map $\tilde{H}_\lambda \to \prod_k (\tilde{\mathbb{C}}^\times \times \mathbb{C}^{\lambda_k - 1})$, which is also denoted by ι. We give the explicit form of the characters of $\tilde{H}_\lambda$, namely, the complex analytic homomorphisms from $\tilde{H}_\lambda$ to the complex torus $\mathbb{C}^\times$. To define them, we introduce functions $\theta_j(v)\,(j \geq 0)$ of $v = (v_0, v_1, v_2, \ldots)$ from the generating function

$$\begin{aligned} \sum_{i=0}^{\infty} \theta_i(v) T^i &= \log(v_0 + v_1 T + v_2 T^2 + \cdots) \\ &= \log v_0 + \log\left(1 + \frac{v_1}{v_0} T + \frac{v_2}{v_0} T^2 + \cdots\right). \end{aligned} \tag{2.2}$$

Notice that $\theta_0(v) = \log v_0$, and that $\theta_i(v)\,(i \geq 1)$ is a weighted homogeneous polynomial of $v_1/v_0, \ldots, v_i/v_0$ of total weight i when the weight of v_j is assigned to j.

Proposition 2.1. ([GRS]).
(a) *Let $u_{\lambda_k} : \tilde{J}(\lambda_k) \to \mathbb{C}^\times$ be a character. Then there exist some complex constants $\beta^{(k)} = (\beta_{k;0}, \ldots, \beta_{k;\lambda_k - 1}) \in \mathbb{C}^{\lambda_k}$ such that*

$$u_{\lambda_k}(\beta^{(k)}; h^{(k)}) = \exp\left(\sum_{0 \leq j < \lambda_k} \beta_{k;j} \theta_j(\iota(h^{(k)}))\right) = h_{k;0}^{\beta_{k;0}} \exp\left(\sum_{1 \leq j < \lambda_k} \beta_{k;j} \theta_j(\iota(h^{(k)}))\right), \tag{2.3}$$

where $h^{(k)} = [h_{k;0}, \ldots, h_{k;\lambda_k - 1}] \in J(\lambda_k)$. This character will be denoted by $u_{\lambda_k}(\beta^{(k)}; \cdot)$ to indicate the dependence on $\beta^{(k)}$.
(b) *Let u_λ be a character of $\tilde{H}_\lambda$. Then there are complex constants $\beta = (\beta^{(0)}, \ldots, \beta^{(m-1)}) \in \mathbb{C}^n$, $\beta^{(k)} \in \mathbb{C}^{\lambda_k}$ such that*

$$u_\lambda(\beta; h) = \prod_{0 \leq k < m} u_{\lambda_k}(\beta^{(k)}; h^{(k)}), \tag{2.4}$$

where $h = (h^{(0)}, \ldots, h^{(m-1)}) \in \tilde{H}_\lambda$ and $h^{(k)} \in \tilde{J}(\lambda_k)$. The character u_λ with β is denoted by $u_\lambda(\beta; \cdot)$.

For the sake of simplicity, we write $\theta_j(h)$ instead of $\theta_j(\iota(h))$ by abuse of notation. To give the space on which the general hypergeometric function is defined, we introduce the following terminology.

Definition 2.1. Let $\lambda = (\lambda_0, \ldots, \lambda_{m-1})$ be a partition of n. A subdiagram of λ is a set of nonnegative integers $\mu = (\mu_0, \ldots, \mu_{m-1})$ such that

$$0 \leq \mu_i \leq \lambda_i, \quad 0 \leq i < m,$$

and is denoted by $\mu \subset \lambda$. The integer $|\mu| := \mu_0 + \cdots + \mu_{m-1}$ is called the weight of μ. Let $\mathrm{M}(r, n)$ be the set of $r \times n$ complex matrices. We restrict our consideration to the case $r = 2$. The set $\mathrm{M}(2, 2)$ will be denoted simply as $\mathrm{M}(2)$. For an element $z \in \mathrm{M}(2, n)$, we write

$$z = \left[z^{(0)} \ \cdots \ z^{(m-1)}\right], \quad z^{(k)} = (z_0^{(k)}, \ldots, z_{\lambda_k - 1}^{(k)}),$$

where $z_i^{(k)}$ are column vectors. For a subdiagram μ of λ of weight two, we put

$$z_\mu = (z_0^{(0)}, \ldots, z_{\mu_0 - 1}^{(0)}, \ldots, z_0^{(m-1)}, \ldots, z_{\mu_{m-1} - 1}^{(m-1)}) \in \mathrm{M}(2).$$

Definition 2.2. The set

$$Z_{2,n} = \{z \in \mathrm{M}(2,n) \mid \det z_\mu \neq 0 \text{ for any subdiagram } \mu \subset \lambda \text{ of weight two}\}$$

is called the *generic stratum* of $\mathrm{M}(2,n)$ with respect to H_λ.

Note that the generic stratum $Z_{2,n}$ is a Zariski open subset of $M(2,n)$. Put

$$E = \left\{(\dot{t}, z) \in \mathbb{P}^1 \times Z_{2,n} \mid h_{k;0} = \dot{t} z_0^{(k)} \neq 0 \, (0 \leq k < m)\right\},$$

where $\dot{t} = [t_0 \;\; t_1]$ are the homogeneous coordinates of $\mathbb{P}^1$ and $h_{k;0}$ denotes the homogeneous linear function of $\dot{t}$ defined by the column vector $z_0^{(k)}$. Let $\pi : E \to Z_{2,n}$ be the projection map defined by $\pi(\dot{t}, z) = z$. Then we know that E is a C^∞ fiber bundle over $Z_{2,n}$ with the fiber

$$X := \mathbb{P}^1 \backslash \bigcup_k H^{(k)}(z), \tag{2.5}$$

where $H^{(k)}(z)$ is the point in $\mathbb{P}^1$ satisfying the equation $h_{k;0} = 0$. Note that these points are in general position by virtue of the definition of $Z_{2,n}$.

Let a rational 1-form φ be holomorphic on X.

Definition 2.3. The function defined by the integral

$$F(\beta; z) = \int_{\Delta(z)} (u_\lambda)\varphi \tag{2.6}$$

is called a *general hypergeometric function of type* λ (**GHF** of type λ, for short), where $\Delta(z)$ is an element of a kind of homology groups defined in [MMT, Section 3.].

Remark 2.1. In our study, we consider $F(\beta; z)$ for some fixed twisted cycle Δ. We need not explicit forms of $\Delta(z)$ but the convergence of the integral (2.6).

Remark 2.2. When $\lambda = (1, \ldots, 1)$, GHF of type λ can be regarded as an integral representation of a solution to Lauricella's F_D systems of hypergeometric differential equations.

3 Twisted Cohomology Group

We take distinct m points of the complex projective line $\mathbb{P}^1$, and select its homogeneous coordinates $[t_0 \;\; t_1]$ so that these points are expressed as

$$x_0 = \infty, x_1 = 0, x_2, \ldots, x_{m-1}$$

by the associated affine coordinate $t = t_1/t_0 \, (s = t_0/t_1)$. Put

$$x = \{x_0, x_1, x_2, \ldots, x_{m-1}\}, \quad X = \mathbb{P}^1 \backslash x;$$

$X = \mathbb{C}$ for $m = 1$ and $X = \mathbb{C}\backslash\{0\}$ for $m = 2$. The character u_λ is written by

$$u_\lambda = \exp\Big\{\sum_{k=1}^{\lambda_1 - 1} \beta_{0;k}\theta_k(h^{(0)})\Big\} \times \prod_{i=1}^{m-1} {h_{i;0}}^{\beta_{i;0}} \exp\Big\{\sum_{k=1}^{\lambda_i - 1} \beta_{i;k}\theta_k(h^{(i)})\Big\}, \tag{3.1}$$

where $\beta_{i;k} \in \mathbb{C}$, $\beta_{i;\lambda_i-1} \neq 0$ and $\sum_{i=0}^{m-1} \beta_{i;0} = 0$. We assume that if $\lambda_i = 1$ then $\beta_{i;0} \notin \mathbb{Z}$. Linear forms h's are written by $h_{0;0} = s$ and $h_{i;0} = t - x_i$ $(i \geq 1)$. Let ω be a logarithmic 1-form defined by

$$\omega := \mathrm{d}_t(\log(u_\lambda)), \tag{3.2}$$

and ∇_ω be a connection $\mathrm{d}_t + \omega\wedge$ with respect to ω. Note that

$$\mathrm{d}_t(u_\lambda \varphi) = u_\lambda \nabla_\omega(\varphi).$$

We define complexes with differential ∇_ω :

$$\begin{aligned}(\Omega^\bullet(x), \nabla_\omega) &: \Omega^0(x) \xrightarrow{\nabla_\omega} \Omega^1(x) \xrightarrow{\nabla_\omega} 0 \to 0,\\ (S^\bullet(x), \nabla_\omega) &: S^0(x) \xrightarrow{\nabla_\omega} S^1(x) \xrightarrow{\nabla_\omega} S^2(x) \xrightarrow{\nabla_\omega} 0,\end{aligned}$$

where $\Omega^k(x)$ is the vector space of rational k-forms on $\mathbb{P}^1$ admitting poles only on x, and $S^k(x)$ is the vector space of smooth k-forms on $\mathbb{P}^1$ which rapidly decrease at x_i for any i. The k-th cohomology groups $H^k(\Omega^\bullet(x), \nabla_\omega)$ and $H^k(S^\bullet(x), \nabla_\omega)$ of above complexes are called rational and rapidly decreasing twisted de Rham cohomology groups with respect to ∇_ω, respectively. It is shown in [Km] that the dimension of $H^1(\Omega^\bullet(x), \nabla_\omega)$ is $n-2$, which is equal to the rank of the associated confluent hypergeometric system of differential equations. The next lemma is shown in several references (cf. [Mj1, Theorem 2], [Mj2, Proposition 3.1] and [Ml, p. 82 ii]).

Lemma 3.1. *We have a canonical isomorphism*

$$H^k(\Omega^\bullet(x), \nabla_\omega) \simeq H^k(S^\bullet(x), \nabla_\omega).$$

For $k \neq 1$, $H^k(\Omega^\bullet(x), \nabla_\omega)$ and $H^k(S^\bullet(x), \nabla_\omega)$ vanish.

By the isomorphism in Lemma 3.1,

$$\iota_\omega : H^1(\Omega^\bullet(x), \nabla_\omega) \quad \longrightarrow \quad H^1(S^\bullet(x), \nabla_\omega),$$

the intersection number $\left\langle \varphi, \psi \right\rangle_\omega$ of $\varphi \in H^1(\Omega^\bullet(x), \nabla_\omega)$ and $\psi \in H^1(\Omega^\bullet(x), \nabla_{-\omega})$ is defined by

$$\left\langle \varphi, \psi \right\rangle_\omega = \int_{\mathbb{P}}^1 \iota_\omega(\varphi) \wedge (\psi). \tag{3.3}$$

4 Intersection Matrices

There are some studies of intersection numbers of $H^1(\Omega^\bullet(x), \nabla_\omega)$ (cf. [Mt1], [MMT]). In general, intersection number that is expressed by parameters and variables. In this section, we show that each intersection number of 1-forms $\{\mathrm{d}_t(\theta_i(h))\}$ depends only on parameters. We set

$$\phi_{i;k} := \mathrm{d}_t(\theta_k(h^{(i)})). \tag{4.1}$$

To give its explicit form, we introduce rational functions $p_k(v)$ $(k \geq 0)$ of $v = (v_0, v_1, v_2, \ldots)$ from the generating function

$$\sum_{k=0}^{\infty} p_k(v) T^k = \frac{v_0}{v_0 + v_1 T + v_2 T^2 + \cdots}. \tag{4.2}$$

Using p_k, we have

$$\phi_{0;k} = p_k(\iota(h^{(0)}))\frac{\mathrm{d}s}{s}, \quad \phi_{i;k} = p_k(\iota(h^{(i)}))\frac{\mathrm{d}t}{t-x_i}\ (i \geq 1).$$

For the sake of simplicity, we write $p_k(h)$ instead of $p_k(\iota(h))$ by abuse of notation. Using these forms, we express ω as

$$\omega = \mathrm{d}_t(\log(u_\lambda(\beta;h))) = \Big\{\frac{1}{s}\sum_{k=1}^{\lambda_0-1}\beta_{0;k}p_k\big(h^{(0)}\big)\Big\}\mathrm{d}s + \sum_{i=1}^{m-1}\Big\{\frac{1}{t-x_i}\sum_{k=0}^{\lambda_i-1}\beta_{i;k}p_k\big(h^{(i)}\big)\Big\}\mathrm{d}t. \tag{4.3}$$

Let

$$\sum_{k=1}^{\lambda_0-1}\left(\frac{\alpha_{0;k}}{s^{k+1}}\right)\mathrm{d}s + \sum_{i=1}^{m-1}\sum_{k=0}^{\lambda_i-1}\left(\frac{\alpha_{i;k}}{(t-x_i)^{k+1}}\right)\mathrm{d}t \tag{4.4}$$

be the partial fraction expansion of ω. In order to express $\alpha_{i;k}$ by β's and h's, we introduce polynomials p_j^k:

$$p_j^k(v_1,\dots,v_j) := \sum_{m's}(-1)^{m_1+\cdots+m_j}\frac{(m_1+\cdots+m_j)!}{m_1!\cdots m_j!}v_1^{m_1}\cdots v_j^{m_j}. \tag{4.5}$$

Here, the summation is carried out by all combinations under the conditions: $m_1+2m_2+\cdots+jm_j = j$ and $m_1+m_2+\cdots+m_j = k$, for $(m_1,\cdots,m_j)\in\mathbb{N}_0^j$ ($\mathbb{N}_0 = \mathbb{N}\cup\{0\}$).

Lemma 4.1. *We have*

$$\alpha_{i;0} = \beta_{i;0}, \quad \alpha_{i;k} = \sum_{j=k}^{\lambda_i-1}\beta_{i;j}p_j^k(h_{i;1},\dots,h_{i;j}). \tag{4.6}$$

Proof. We substitute $x = (v_1/v_0)T+\cdots+(v_j/v_0)T^j$ into $1/(1+x) = \sum_{k=0}^{\infty}(-1)^k x^k$, and expand by T. Then we have

$$\frac{v_0}{v_0+v_1T+\cdots+v_jT^j} = \sum_{k=0}^{\infty}(-1)^k\sum_{m's}\frac{k!}{m_1!\cdots m_j!}\frac{v_1^{m_1}\cdots v_j^{m_j}}{v_0^k}T^J,$$

where $m_1+m_2+\cdots+m_j = k$ and $m_1+2m_2+\cdots+jm_j = J$. Compare this with (4.2), then $p_j(v)$ is expressed as

$$p_j(v) = \sum_{m's}(-1)^{m_1+\cdots+m_j}\frac{(m_1+\cdots+m_j)!}{m_1!\cdots m_j!}\frac{v_1^{m_1}\cdots v_j^{m_j}}{v_0^{m_1+\cdots+m_j}},$$

where the summation is carried out by all combinations under the condition: $m_1+2m_2+\cdots+jm_j = j$. Comparing this with (4.5), we have

$$p_j(v_0,v_1,\dots,v_j) = \sum_{k=0}^{j}\frac{p_j^k(v_1,\dots,v_j)}{v_0^k}. \tag{4.7}$$

From (4.3) and (4.4), the principal part of the Laurent expansion of ω at x_i is

$$\sum_{k=0}^{\lambda_i-1}\frac{\alpha_{i;k}}{(t-x_i)^{k+1}} = \frac{1}{t-x_i}\sum_{j=0}^{\lambda_i-1}\beta_{i;j}\sum_{k=0}^{j}\frac{p_j^k(h_{i;1},\dots,h_{i;j})}{(t-x_i)^k}.$$

Comparing coefficients of $1/(t-x_i)^{k+1}$, we have (4.6). □

The following is a key lemma to evaluate intersection numbers.

Lemma 4.2. *Let φ and ψ be elements of $H^1(\Omega^\bullet(x), \nabla_\omega)$ and $H^1(\Omega^\bullet(x), \nabla_{-\omega})$, and the pole orders at x_q of φ and ψ be bounded by that of ω:*

$$\underset{t=x_q}{ord}(\varphi) \leq \underset{t=x_q}{ord}(\omega),\ \underset{t=x_q}{ord}(\psi) \leq \underset{t=x_q}{ord}(\omega),\ (0 \leq q \leq m-1).$$

Let G_q be the rational function defined by the identity

$$G_q \cdot \omega = \varphi.$$

Then we have

$$\left\langle \varphi, \psi \right\rangle_\omega = 2\pi\sqrt{-1} \sum_{q=0}^{m-1} \operatorname*{Res}_{t=x_q}\{G_q\psi\}. \tag{4.8}$$

Proof. In [MMT, Theorem 4.1], the intersection number is evaluated as $2\pi\sqrt{-1}\sum_{q=1}^{m} \mathrm{Res}_{t=x_q}\{f_q\psi\}$, where f_q is the formal power series solution to $\nabla_\omega f_q = \varphi$ in a small neighborhood U_q of x_q. Since G_q is holomorphic in U_q,

$$\varphi - \nabla_\omega G_q = \varphi - (\mathrm{d}_t G_q + G_q \cdot \omega) = -\mathrm{d}_t G_q \tag{4.9}$$

is holomorphic in U_q. As a formal power series, $f_q - G_q$ satisfies $\nabla_\omega(f_q - G_q) = -\mathrm{d}_t G_q$. Let $d_p(t-x_q)^p$ be the leading term of $f_q - G_q$, then the leading term of $\nabla_\omega(f_q - G_q)$ becomes

$$(\alpha_{q;\lambda_q-1} + p\delta_{\lambda_q,1}) d_p (t-x_q)^{p-\lambda_q}, \tag{4.10}$$

where $\delta_{\lambda_q,1}$ is Kronecker's symbol. From Lemma 4.1, $\alpha_{q;\lambda_q-1} = \beta_{q;\lambda_q-1} p_{\lambda_q-1}^{\lambda_q-1}$ and $\alpha_{q;0} = \beta_{q;0}$. Since $\alpha_{q;\lambda_q-1} \neq 0$ and $\alpha_{q;0} \notin \mathbb{Z}$ if $\lambda_q = 1$, the coefficient of (4.10) is not zero. This means $p \geq \lambda_q$. Because the degree of ψ is not more than the order of ω, we have

$$\operatorname*{Res}_{t=x_q}\{f_q\psi\} = \operatorname*{Res}_{t=x_q}\{G_q\psi\}.$$

□

We consider linear maps

$$M_q : \mathcal{O}_{U_q} \ni f \quad \mapsto \quad f\omega \in \Omega_{U_q}(\lambda_q x_q) \quad (0 \leq q \leq m-1),$$

where $\mathcal{O}_{U_q}$ is the vector space of holomorphic functions on U_q and $\Omega_{U_q}(\lambda_q x_q)$ is that of meromorphic 1-forms on U_q admitting a pole of oder λ_q only on x_q. These descend to the maps between the quotient spaces

$$\tilde{M}_q : \mathcal{O}_{U_q}/(t-x_q)^{\lambda_q}\mathcal{O}_{U_q} \to \Omega_{U_q}(\lambda_q x_q)/\Omega_{U_q}, \tag{4.11}$$

where $\mathcal{O}_{U_q}/(t-x_q)^{\lambda_q}\mathcal{O}_{U_q}$ is regarded as $\mathcal{O}_{U_0}/s^{\lambda_0}\mathcal{O}_{U_0}$ for $q=0$. Let A_q be the representation matrix of $\tilde{M}_q$ with respect to bases

$${}^t\left[1\ (t-x_q)\ \cdots\ (t-x_q)^{\lambda_q-1}\right] \quad \text{and} \quad {}^t\left[\frac{\mathrm{d}t}{(t-x_q)^{\lambda_q}}\ \frac{\mathrm{d}t}{(t-x_q)^{\lambda_q-1}}\ \cdots\ \frac{\mathrm{d}t}{t-x_q}\right],$$

where they are regarded as

$$ {}^t\left[1\ s\ \cdots\ s^{\lambda_0-1}\right] \quad \text{and} \quad {}^t\left[\frac{\mathrm{d}s}{s^{\lambda_0}}\ \frac{\mathrm{d}s}{s^{\lambda_0-1}}\ \cdots\ \frac{\mathrm{d}s}{s}\right] \quad \text{for} \quad q=0. $$

Then A_q is expressed as

$$ A_q = \begin{bmatrix} \alpha_{q;\lambda_q-1} & \alpha_{q;\lambda_q-2} & \cdots & \alpha_{q;1} & \alpha_{q;0} \\ & \alpha_{q;\lambda_q-1} & \cdots & \alpha_{q;2} & \alpha_{q;1} \\ & & \ddots & \vdots & \vdots \\ & \mathbf{0} & & \alpha_{q;\lambda_q-1} & \alpha_{q,\lambda_q-2} \\ & & & & \alpha_{q,\lambda_q-1} \end{bmatrix} \tag{4.12} $$

by (4.4).

Lemma 4.3. *Let B be a Jordan matrix of size n given by*

$$ B = \sum_{0\le j<n} b_j \Lambda_n^j \quad (b_0 \neq 0). $$

The inverse of B becomes

$$ B^{-1} = \frac{1}{b_0} \sum_{0\le j<n} p_j(b) \Lambda_n^j. $$

Proof. Substitute $v_j = b_j\,(0 \le j < n)$, $v_k = 0\,(\forall k \ge n)$ in (4.2), and multiply $\sum_{0\le k<n} b_k T^k$ to its both sides. Then we have

$$ \sum_{k=0}^{n-1} b_k T^k \sum_{j=0}^{\infty} p_j(b) T^j = b_0. \tag{4.13} $$

Comparing coefficients of T^k of (4.13), we can check $BB^{-1} = I_n$. □

We define the *reverse order matrix* of a square matrix.

Definition 4.1. Let U be a square matrix of size n expressed by

$$ U = \begin{bmatrix} U_{(1,1)} & U_{(1,2)} & \cdots & U_{(1,n-1)} & U_{(1,n)} \\ U_{(2,1)} & U_{(2,2)} & \cdots & U_{(2,n-1)} & U_{(2,n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ U_{(n-1,1)} & U_{(n-1,2)} & \cdots & U_{(n-1,n-1)} & U_{(n-1,n)} \\ U_{(n,1)} & U_{(n,2)} & \cdots & U_{(n,n-1)} & U_{(n,n)} \end{bmatrix}. $$

Then a square matrix

$$ {}^R U := [U_{(n+1-j,n+1-i)}]_{(i,j)} = \begin{bmatrix} U_{(n,n)} & U_{(n-1,n)} & \cdots & U_{(2,n)} & U_{(1,n)} \\ U_{(n,n-1)} & U_{(n-1,n-1)} & \cdots & U_{(2,n-1)} & U_{(1,n-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ U_{(n,2)} & U_{(n-1,2)} & \cdots & U_{(2,2)} & U_{(1,2)} \\ U_{(n,1)} & U_{(n-1,1)} & \cdots & U_{(2,1)} & U_{(1,1)} \end{bmatrix} \tag{4.14} $$

is called the *reverse order matrix* of U.

We prepare the next lemma.

Lemma 4.4. *Suppose that a Jordan matrix $A \in J(n, \mathbb{C})$ is expressed as*

$$A = \sum_{k=0}^{n-1} a_k \Lambda_n^{n-k-1},$$

where Λ_n is the shift matrix of size n in (2.2), and $\Lambda_n^0 = I_n$ is the unit matrix. If $\{a_k\}_{0 \le k < n}$ are linear combinations

$$a_0 = b_0, \quad a_k = \sum_{j=k}^{n-1} b_j p_j^k(x_1, \dots, x_k),$$

of p_j^k given in (4.5), then the matrix A is represented by

$$A = {}^R U B U, \tag{4.15}$$

where

$$B = \sum_{k=0}^{n-1} b_k \Lambda_n^{n-k-1}, \quad U = \begin{bmatrix} p_{n-1}^{n-1} & p_{n-1}^{n-2} & \cdots & p_{n-1}^1 & 0 \\ & p_{n-2}^{n-2} & \cdots & p_{n-2}^1 & 0 \\ & & \ddots & \vdots & \vdots \\ & \mathbf{0} & & p_1^1 & 0 \\ & & & & p_0^0 \end{bmatrix},$$

and ${}^R U = [U_{(n+1-j,n+1-i)}]_{(i,j)}$ is the reverse order matrix of U.

Proof. Since each entry of A is regarded as a linear combination of $\{b_j\}$, it is enough to show $(\partial/\partial b_j) A_{k,l} = (\partial/\partial b_j)({}^R U B U)_{k,l}$. The partial derivatives of the left hand side of equation (4.15) are

$$\frac{\partial}{\partial b_0} A = \Lambda_n^{n-1}, \quad \frac{\partial}{\partial b_j} A_{k,l} = \frac{\partial a_{n+k-l-1}}{\partial b_j} = \begin{cases} p_j^{n+k-l-1}(x) & (k \le l, n+k-l-1 \le j), \\ 0 & (\text{otherwise}), \end{cases}$$

and those of the right hand side of (4.15) are

$$\begin{aligned} \frac{\partial}{\partial b_0}({}^R U B U) &= {}^R U \Lambda_n^{n-1} U = \Lambda_n^{n-1}, \\ \frac{\partial}{\partial b_j}({}^R U B U)_{k,l} &= ({}^R U \Lambda_n^{n-j-1} U)_{k,l} = \sum_{m=1}^{j+1} {}^R U_{k,m} U_{m+n-j-1,l} = \sum_{m=1}^{j+1} p_{m-1}^{k-1}(x) p_{j-m+1}^{n-l}(x) \\ &= p_j^{n+k-l-1}(x) \qquad (n+k-l-1 \le j). \end{aligned}$$

Thus they coincide. □

As a result, we can see the next formulation.

Proposition 4.1. *The intersection number of $\phi_{i;k}$ and $\phi_{j;l}$ becomes*

$$\left\langle \phi_{i;k}, \phi_{j;l} \right\rangle_\omega = 2\pi\sqrt{-1}\left(\delta_{ij} \frac{\psi_{k+l+1-\lambda_i}^{(i)}}{\beta_{i;\lambda_i-1}} - \epsilon_{ijl} \frac{\psi_{k+1-\lambda_i}^{(i)}}{\beta_{i;\lambda_i-1}} - \epsilon_{jik} \frac{\psi_{l+1-\lambda_j}^{(j)}}{\beta_{j;\lambda_j-1}} + \delta_{ijkl} \frac{\psi_{1-\lambda_{i'}}^{(i')}}{\beta_{i';\lambda_{i'}-1}}\right), \tag{4.16}$$

where δ_{ij} is Kronecker's symbol, ϵ_{ijl} is defined by

$$\epsilon_{ijl} = \begin{cases} 1 & (i=0, j\neq 0, l=0), \\ 1 & (i=1, j=0, l=0), \\ 0 & (\text{otherwise}), \end{cases}$$

$i' := 1-i$, δ_{ijkl} is defined by

$$\delta_{ijkl} = \begin{cases} 1 & (i=j=0, k=l=0), \\ 1 & (i=j=1, k=l=0), \\ 0 & (\text{otherwise}), \end{cases}$$

and $\psi_k^{(i)}$ is defined by

$$\psi_0^{(i)}(\beta) := 1, \quad \psi_k^{(i)}(\beta) := p_k(\beta_{i;\lambda_i-1}, \beta_{i;\lambda_i-2}, \ldots, \beta_{i;\lambda_i-k-1}). \tag{4.17}$$

We regard x, h as variables, and β as parameters. In particular, the intersection number $\langle \phi_{i;k}, \phi_{j;l} \rangle_\omega$ depends only on parameters.

Proof. We use (4.8) in Lemma 4.2. From (4.1), we express g_ω, G_q and g_l by

$$\omega =: g_\omega \mathrm{d}t, \quad G_q = \frac{p_k(h^{(i)})}{g_\omega(t-x_i)} =: \sum_{m=0}^{\infty} b_{q;m}(t-x_q)^m, \quad \phi_{j;l} = \frac{p_l(h^{(j)})}{(t-x_j)}\mathrm{d}t =: g_l \mathrm{d}t. \tag{4.18}$$

In case $i=j$, in the right hand side of (4.16), we have $\epsilon_{ijl} = \epsilon_{jik} = 0$. Using Lemma 4.1 and 4.4, we can express A_i by:

$$A_i = {}^R V B_i V,$$

where

$$B_i = \begin{bmatrix} \beta_{i;\lambda_i-1} & \beta_{i;\lambda_i-2} & \cdots & \beta_{i;1} & \beta_{i;0} \\ & \beta_{i;\lambda_i-1} & \cdots & \beta_{i;2} & \beta_{i;1} \\ & & \ddots & \vdots & \vdots \\ & \mathbf{0} & & \beta_{i;\lambda_i-1} & \beta_{i;\lambda_i-2} \\ & & & & \beta_{i;\lambda_i-1} \end{bmatrix},$$

$$V = \begin{bmatrix} p_{\lambda_i-1}^{\lambda_i-1} & p_{\lambda_i-1}^{\lambda_i-2} & \cdots & p_{\lambda_i-1}^1 & 0 \\ & p_{\lambda_i-2}^{\lambda_i-2} & \cdots & p_{\lambda_i-2}^1 & 0 \\ & & \ddots & \vdots & \vdots \\ & \mathbf{0} & & p_1^1 & 0 \\ & & & & p_0^0 \end{bmatrix},$$

and ${}^R V$ is the reverse order matrix of V. Note that A_i includes variables and parameters, and B_i depends only on parameters. Using (4.7), we have

$$\phi_{i;k} = p_k(h^{(i)})\frac{\mathrm{d}t}{t-x_i} = \sum_{j=0}^{k} p_k^j(h_{i;1}, \ldots, h_{i;j})\frac{\mathrm{d}t}{(t-x_i)^{j+1}},$$

$$\phi_{i;l} = p_l(h^{(i)})\frac{\mathrm{d}t}{t-x_i} = \sum_{j=0}^{l} p_l^j(h_{i;1}, \ldots, h_{i;j})\frac{\mathrm{d}t}{(t-x_i)^{j+1}},$$

where p_k^j and p_l^j include no parameters. Put row vectors

$$\mathbf{c} = [p_k^{\lambda_i - j}]_{1 \le j \le \lambda_i} = [\underbrace{0 \cdots 0}_{\lambda_i - k - 1} p_k^k \cdots p_k^0],$$
$$\mathbf{d} = [p_l^{j-1}]_{1 \le j \le \lambda_i} = [\underbrace{p_l^0 \cdots p_l^l}_{l+1} 0 \cdots 0],$$
$$\mathbf{e}_i = [\delta_{ij}]_{1 \le j \le \lambda_i} = [\underbrace{0 \cdots 0}_{i-1} 1 0 \cdots 0].$$

Since A_i is the representation matrix of $\tilde{M}_i$ defined in (4.12), the intersection number becomes

$$\begin{aligned}\left\langle \phi_{i;k}, \phi_{i:l} \right\rangle_\omega &= \int_{\mathbb{P}^1} \iota_\omega(\phi_{i;k}) \wedge (\phi_{i;l}) = 2\pi\sqrt{-1}\mathbf{c} A_i^{-1\,t}\mathbf{d} \\ &= 2\pi\sqrt{-1}\mathbf{c} V^{-1} B_i^{-1\,R} V^{-1\,t}\mathbf{d} = 2\pi\sqrt{-1}\mathbf{e}_{(\lambda_i - k)} B_i^{-1\,t}\mathbf{e}_{(l+1)} \\ &= 2\pi\sqrt{-1}(B_i^{-1})_{(\lambda_i - k, l+1)},\end{aligned}$$

where $(B^{-1})_{(p,q)}$ means (p,q)-entry of matrix B^{-1}. Using Lemma 4.3, we obtain the inverse matrix of B_i:

$$(B_i^{-1})_{(\lambda_i - k, l+1)} = \frac{\psi^{(i)}_{k+l+1-\lambda_i}}{\beta_{i;\lambda_i - 1}}, \tag{4.19}$$

where $\psi_k^{(i)}$ is given in (4.17). The value $2\pi\sqrt{-1}(B_i^{-1})_{(\lambda_i - k, l+1)}$ is equal to the first term of the right hand side of (4.16).

In case $i = j$ but $q \neq i$, if the order of pole of ω at x_q is more than one, the function $G_q g_l$ is holomorphic in U_q. If $i = j = 0$, $q = 1$ and $\lambda_q = 1$, then we have

$$\operatorname*{Res}_{t=x_q}\{G_q \phi_{j;l}\} = \operatorname*{Res}_{t=0}\{\frac{p_k(h^{(0)})}{(g_\omega)(-t)} p_l(h^{(0)}) \frac{\mathrm{d}s}{s}\} = \lim_{t \to 0}\{\frac{p_k(h^{(0)}) p_l(h^{(0)})}{t\, g_\omega}\} = \frac{p_k^0 p_l^0}{\alpha_{1;0}}.$$

If $i = j = 1$, $q = 0$ and $\lambda_q = 1$, then we have

$$\operatorname*{Res}_{t=x_q}\{G_q \phi_{j;l}\} = \operatorname*{Res}_{t=\infty}\{\frac{p_k(h^{(1)})}{(g_\omega)t} p_l(h^{(1)}) \frac{\mathrm{d}t}{t}\} = \lim_{t \to \infty}\{\frac{-p_k(h^{(1)}) p_l(h^{(1)})}{t\, g_\omega}\} = \frac{p_k^0 p_l^0}{\alpha_{0;0}}.$$

Those values are equivalent to the last term of the right hand side of (4.16).

In case $i \neq j$, $q \neq i$ and $q \neq j$, since the function $G_q g_l$ is holomorphic in U_q, we have

$$\operatorname*{Res}_{t=x_q}\{G_q \phi_{j;l}\} = 0.$$

In the right hand side of (4.16), $\delta_{ij} = \delta_{ijkl} = 0$.

In case $i \neq j$ and $q = i$, since the function G_i is holomorphic in U_i, the residue becomes

$$\operatorname*{Res}_{t=x_i}\{G_i \phi_{j;l}\} = -\{\delta_{i0} + \delta_{i1}\delta_{j0}\} b_{i;0} p_l^0.$$

It is easy to see $p_l^0 = \delta_{l0}$. By (4.19), $b_{i;0}$ is equal to $\psi^{(i)}_{k+1-\lambda_i}/\beta_{i;\lambda_i - 1}$.

In case $i \neq j$ and $q = j$, using $G' = g_l/g_\omega$, we obtain

$$\operatorname*{Res}_{t=x_j}\{G_j \phi_{j;l}\} = \operatorname*{Res}_{t=x_j}\{G' \phi_{i;k}\} = -\{\delta_{j0} + \delta_{j1}\delta_{i0}\} b_{j;0} p_k^0.$$

By (4.19), $b_{j;0}$ is equal to $\psi^{(j)}_{l+1-\lambda_j}/\beta_{j;\lambda_j - 1}$. □

5 Pfaffian Systems of Gauss type, Kummer type and Hermite type

We explain how to obtain Pfaffian systems of confluent hypergeometric functions examplifying **GHF** of one variable. We show that Gauss's hypergeometric series

$$ {}_2F_1(a,b,c;x) = \sum_{m_1=0}^{\infty} \frac{(a)_{m_1}(b)_{m_1}}{(c)_{m_1}(1)_{m_1}} x^{m_1} $$

can be regarded as a general hypergeometric function of type $\lambda = (1,1,1,1)$, where $(a)_k := a(a+1)\cdots(a+k-1) = \Gamma(a+k)/\Gamma(a)$. It has an integral representation

$$ {}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_\gamma t^a (1-t)^{c-a}(1-xt)^{-b}\varphi_{0,G} \quad (\Re(c) > \Re(a) > 0), \tag{5.1} $$

where $\gamma = (0,1)$ and $\varphi_{0,G} = \mathrm{d}t/(t(1-t))$. It is a solution to Gauss's hypergeometric differential equation:

$$ \left[x(1-x)(\frac{\mathrm{d}}{\mathrm{d}x})^2 + \{c-(a+b+1)x\}\frac{\mathrm{d}}{\mathrm{d}x} - ab \right] G(x) = 0. \tag{5.2} $$

Set $n = m = 4$, and h and β in **GHF** as follows:

$$ \begin{aligned} &h = (h^{(0)}, h^{(1)}, h^{(2)}, h^{(3)}), \quad h_{0;0} = s, \quad h_{1;0} = t, \quad h_{2;0} = 1-t, \quad h_{3;0} = 1-xt, \\ &\beta = (\beta^{(0)}, \beta^{(1)}, \beta^{(2)}, \beta^{(3)}), \quad \beta_{0;0} = b-c, \quad \beta_{1;0} = a, \quad \beta_{2;0} = c-a, \quad \beta_{3;0} = -b. \end{aligned} $$

Then we have

$$ \begin{aligned} u_{(1,1,1,1)} &= \prod_{i=1}^{3} h_{i;0}^{\beta_{i;0}} = t^a(1-t)^{c-a}(1-xt)^{-b}, \\ \phi_{i;0} &= \mathrm{d}_t(\varepsilon_0(h^{(i)})) \quad (0 \le i \le 3), \\ \omega_G &= \mathrm{d}_t \log(u_{(1,1,1,1)}) = \sum_{i=1}^{3} \beta_{i;0}\phi_{i;0}, \\ \varphi_{0,G} &= \phi_{1;0} - \phi_{2;0}. \end{aligned} $$

A solution to (5.2)

$$ G_0 = \int_\gamma u_{(1,1,1,1)}\varphi_{0,G} \tag{5.3} $$

is equal to ${}_2F_1$ up to a non-zero constant. We summarize variables, parameters and 1-forms in a table:

variables :	$x_0 = \infty$,	$x_3 = \frac{1}{x}$,	$x_1 = 0$,	$x_2 = 1$,
parameters :	$\beta_{0;0} = b-c$,	$\beta_{3;0} = -b$,	$\beta_{1;0} = a$,	$\beta_{2;0} = c-a$,
1-forms :	$\phi_{0;0} = \frac{\mathrm{d}s}{s}$,	$\phi_{3;0} = \frac{x\mathrm{d}t}{xt-1}$,	$\phi_{1;0} = \frac{\mathrm{d}t}{t}$,	$\phi_{2;0} = \frac{\mathrm{d}t}{t-1}$,

where $\{x_i\}_{0\le i\le 3}$ are poles of ω_G.

Kummer's confluent hypergeometric function ${}_1F_1(a,c;d_1x') = \sum_{m_1=0}^{\infty} \frac{(a)_{m_1}}{(c)_{m_1}(1)_{m_1}} {d_1}^{m_1}{x'}^{m_1}$ can be regarded as a **GHF** of type $\lambda = (2,1,1)$. It has an integral representation

$$ {}_1F_1(a,c;d_1x') = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_\gamma t^a(1-t)^{c-a}\exp(d_1x't)\varphi_{0,K} \quad (\Re(c) > \Re(a) > 0), \tag{5.4} $$

where $\gamma = (0,1)$, $\varphi_{0,K} = \frac{\mathrm{d}t}{t(1-t)}$ and $d_1 \neq 0$. It is a solution to Kummer's hypergeometric differential equation

$$\left[x'(\frac{\mathrm{d}}{\mathrm{d}x'})^2 + \{c - d_1 x'\}\frac{\mathrm{d}}{\mathrm{d}x'} - ad_1\right] K(x') = 0. \tag{5.5}$$

Set $n = 4, m = 3$, and h and β in **GHF** as follows:

$$\begin{aligned} &h = (h^{(0)}, h^{(1)}, h^{(2)}), \quad \iota(h^{(0)}) = (h_{0;0}, h_{0;1}) = (s, x'), \quad h_{1;0} = t, \quad h_{2;0} = 1 - t, \\ &\beta = (\beta'^{(0)}, \beta^{(1)}, \beta^{(3)}), \quad \beta'^{(0)} = (\beta'_{0;0}, \beta'_{0;1}) = (-c, d_1), \quad \beta_{1;0} = a, \quad \beta_{2;0} = c - a. \end{aligned}$$

Then we have

$$\begin{aligned} u_{(2,1,1)} &= \exp(\beta'_{0;1}\theta_1(h^{(0)})) \prod_{i=1}^{2} h_{i;0}^{\beta_{i;0}} = \exp(d_1 x' t) t^a (1-t)^{c-a}, \\ \phi_{i;0} &= \mathrm{d}_t(\theta_0(h^{(i)}))\,(0 \leq i \leq 2), \quad \phi_{1;1} = \mathrm{d}_t(\theta_1(h^{(1)})), \\ \omega_K &= \mathrm{d}_t \log(u_{(2,1,1)}) = \beta'_{0;1}\phi_{0;1} + \sum_{i=1}^{2} \beta_{i;0}\phi_{i;0}, \\ \varphi_{0,K} &= \phi_{1;0} - \phi_{2;0}. \end{aligned}$$

A solution to (5.5)

$$K_0 = \int_\gamma u_{(2,1,1)}\varphi_{0,K} \tag{5.6}$$

is ${}_1F_1$ up to a non-zero constant. We summarize variables, parameters and 1-forms in a table:

variables :	$x_0 = \infty$,	$h_{0;1} = x'$,	$x_1 = 0$,	$x_2 = 1$,
parameters :	$\beta'_{0;0} = -c$,	$\beta'_{0;1} = d_1$,	$\beta_{1;0} = a$,	$\beta_{2;0} = c - a$,
1-forms :	$\phi_{0;0} = \frac{\mathrm{d}s}{s}$,	$\phi_{0;1} = -\frac{x'\mathrm{d}s}{s^2}$,	$\phi_{1;0} = \frac{\mathrm{d}t}{t}$,	$\phi_{2;0} = \frac{\mathrm{d}t}{t-1}$,

where $\{x_i\}_{1 \leq i \leq 3}$ are poles of ω_K.

Using the exterior derivatives $\mathrm{d}_x(f) := \mathrm{d}x \wedge (\mathrm{d}f/\mathrm{d}x)$ and $\mathrm{d}_{x'}(f) := \mathrm{d}x' \wedge (\mathrm{d}f/\mathrm{d}x')$, we define logarithmic 1-forms:

$$\begin{aligned} \omega_{G,x} &:= \mathrm{d}_x \log(u_{(1,1,1,1)}) = \frac{\beta_{3;0} t \mathrm{d}x}{xt - 1}, \\ \omega_{K,x'} &:= \mathrm{d}'_x \log(u_{(2,1,1)}) = \beta'_{0;1} t \mathrm{d}x'. \end{aligned}$$

For connections $\nabla_x := \mathrm{d}_x + \omega_x\wedge$ and $\nabla_{x'} := \mathrm{d}_{x'} + \omega_{x'}\wedge$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_\gamma u(t,x)\varphi = \int_\gamma u(t,x)\frac{\nabla_x}{\mathrm{d}x}(\varphi), \quad \frac{\mathrm{d}}{\mathrm{d}x'}\int_\gamma u(t,x')\varphi = \int_\gamma u(t,x')\frac{\nabla_{x'}}{\mathrm{d}x'}(\varphi).$$

Using these operators, we have

$$\begin{aligned} \frac{\nabla_x}{\mathrm{d}x}(\varphi_{0,G}) &= \frac{\mathrm{d}t}{t(1-t)} \times \frac{\beta_{3;0} t}{xt - 1} = \frac{\beta_{3;0}}{x - 1}(\frac{x\mathrm{d}t}{xt - 1} - \frac{\mathrm{d}t}{t - 1}), \\ \frac{\nabla_{x'}}{\mathrm{d}x'}(\varphi_{0,K}) &= \frac{\mathrm{d}t}{t(1-t)} \times \beta'_{0;1} t = -\frac{\beta'_{0;1}\mathrm{d}t}{t - 1}. \end{aligned}$$

We select bases of $H^1_G(\Omega^\bullet(x),\nabla_\omega)$ and $H^1_K(\Omega^\bullet(x),\nabla_\omega)$ as follows:

$$\begin{cases} \varphi_{0,G} = \phi_{1;0} - \phi_{2;0} = \frac{\mathrm{d}t}{t} - \frac{\mathrm{d}t}{t-1}, \\ \varphi_{1,G} := \frac{x-1}{\beta_{3;0}} \frac{\nabla_x}{\mathrm{d}x}(\varphi_{0,G}) = \phi_{3;0} - \phi_{2;0} = \frac{x\mathrm{d}t}{xt-1} - \frac{\mathrm{d}t}{t-1}, \end{cases}$$

$$\begin{cases} \varphi_{0,K} = \phi_{1;0} - \phi_{2;0} = \frac{\mathrm{d}t}{t} - \frac{\mathrm{d}t}{t-1}, \\ \varphi_{1,K} := \frac{1}{\beta'_{0;1}} \frac{\nabla_{x'}}{\mathrm{d}x'}(\varphi_{0,K}) = -\phi_{2;0} = -\frac{\mathrm{d}t}{t-1}. \end{cases}$$

The intersection matrix of $(\varphi_{0,G},\varphi_{1,G})$ becomes $2\pi\sqrt{-1}C_G$, and that of $(\varphi_{0,K},\varphi_{1,K})$ becomes $2\pi\sqrt{-1}C_K$, where

$$C_G = \begin{bmatrix} (\frac{1}{\beta_{1;0}} + \frac{1}{\beta_{2;0}}) & \frac{1}{\beta_{2;0}} \\ \frac{1}{\beta_{2;0}} & (\frac{1}{\beta_{2;0}} + \frac{1}{\beta_{3;0}}) \end{bmatrix} = \begin{bmatrix} \frac{c}{a(c-a)} & \frac{1}{c-a} \\ \frac{1}{c-a} & \frac{a+b-c}{b(c-a)} \end{bmatrix}, \tag{5.7}$$

$$C_K = \begin{bmatrix} (\frac{1}{\beta_{1;0}} + \frac{1}{\beta_{2;0}}) & \frac{1}{\beta_{2;0}} \\ \frac{1}{\beta_{2;0}} & \frac{1}{\beta_{2;0}} \end{bmatrix} = \begin{bmatrix} \frac{c}{a(c-a)} & \frac{1}{c-a} \\ \frac{1}{c-a} & \frac{1}{c-a} \end{bmatrix}. \tag{5.8}$$

In order to get the intersection number of $(\nabla_x/\mathrm{d}x)(\varphi)$ and ψ, we express $(\nabla_x/\mathrm{d}x)(\varphi)$ as a linear combination of $\varphi_{0,G}$ and $\varphi_{1,G}$. Let us prepare partial fraction expansions of rational 1-forms $(\nabla_x/\mathrm{d}x)(\varphi)$,

$$\begin{aligned}
\frac{\nabla_x}{\mathrm{d}x}(\varphi_{0,G}) &= \frac{\beta_{3;0}t}{xt-1}\left(\frac{\mathrm{d}t}{t(1-t)}\right) = \frac{\beta_{3;0}}{x-1}\left(\frac{x\mathrm{d}t}{xt-1} - \frac{\mathrm{d}t}{t-1}\right) = \frac{\beta_{3;0}}{x-1}\varphi_{1,G}, \\
\frac{\nabla_x}{\mathrm{d}x}(\varphi_{1,G}) &\equiv \frac{\nabla_x}{\mathrm{d}x}\left(\phi_{3;0} - \phi_{2;0} - \frac{1}{\beta_{3;0}}\omega_G\right) = \frac{-t}{xt-1}\left\{\beta_{1;0}\phi_{1;0} + (\beta_{2;0}+\beta_{3;0})\phi_{2;0}\right\} \\
&= -\frac{\beta_{1;0}\mathrm{d}t}{xt-1} - \frac{\beta_{2;0}+\beta_{3;0}}{xt-1}\frac{t\mathrm{d}t}{t-1} = -\frac{\beta_{1;0}}{x}\phi_{3;0} - \frac{\beta_{2;0}+\beta_{3;0}}{x}\phi_{3;0} + \frac{\beta_{2;0}+\beta_{3;0}}{x-1}\big(\phi_{3;0}-\phi_{2;0}\big) \\
&\equiv \frac{1}{x}\Big(-\beta_{3;0}\phi_{3;0} + \omega_G\Big) - \frac{\beta_{1;0}+\beta_{2;0}}{x}\phi_{3;0} + \frac{\beta_{2;0}+\beta_{3;0}}{x-1}\big(\phi_{3;0}-\phi_{2;0}\big) \\
&= \frac{1}{x}\big(\beta_{1;0}\phi_{1;0} + \beta_{2;0}\phi_{2;0}\big) - \frac{\beta_{1;0}+\beta_{2;0}}{x}\phi_{3;0} + \frac{\beta_{2;0}+\beta_{3;0}}{x-1}\big(\phi_{3;0}-\phi_{2;0}\big) \\
&= \frac{1}{x}\big\{\beta_{1;0}\varphi_{0,G} - (\beta_{1;0}+\beta_{2;0})\,\varphi_{1,G}\big\} + \frac{\beta_{2;0}+\beta_{3;0}}{x-1}\varphi_{1,G},
\end{aligned}$$

where $\equiv$ means equivalent as elements of $H^1_G(\Omega^\bullet(x),\nabla_\omega)$. Here note that $\omega_G = \nabla_\omega(1)$ is zero in $H^1_G(\Omega^\bullet(x),\nabla_\omega)$. Using these expansions, we have

$$\frac{\nabla_x}{\mathrm{d}x}\begin{bmatrix}\varphi_{0,G}\\ \varphi_{1,G}\end{bmatrix} = \left\{\frac{1}{x}W_{1,G} + \frac{1}{x-1}W_{2,G}\right\}\begin{bmatrix}\varphi_{0,G}\\ \varphi_{1,G}\end{bmatrix}, \tag{5.9}$$

where matrices W's are given by

$$W_{1,G} = \begin{bmatrix} 0 & 0 \\ \beta_{1;0} & -\beta_{1;0}-\beta_{2;0}\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & -c\end{bmatrix}, \quad W_{2,G} = \begin{bmatrix} 0 & \beta_{3;0} \\ 0 & \beta_{2;0}+\beta_{3;0}\end{bmatrix} = \begin{bmatrix} 0 & -b \\ 0 & c-a-b\end{bmatrix}.$$

By using G_0, we construct a vector-valued function ${}^t[G_0 \;\; G_1]$, where

$$G_1 = \frac{x-1}{-b}\frac{\mathrm{d}G_0}{\mathrm{d}x} = \int_\gamma u_{(1,1,1,1)}\varphi_{1,G}.$$

This vector-valued function satisfies a Pfaffian equation of Gauss's function,

$$\mathrm{d}_x\begin{bmatrix}G_0\\ G_1\end{bmatrix} = \left\{\frac{\mathrm{d}x}{x}W_{1,G} + \frac{\mathrm{d}x}{x-1}W_{2,G}\right\}\begin{bmatrix}G_0\\ G_1\end{bmatrix}, \tag{5.10}$$

which is equivalent to equation (5.9). We can see that:

(1) the connection matrix is divided into two parts, $\mathrm{d}_x(\log(x))W_{1,G}$ and $\mathrm{d}_x(\log(x-1))W_{2,G}$;

(2) $\mathrm{d}_x \log(x) = \mathrm{d}x/x$ and $\mathrm{d}_x \log(x-1) = \mathrm{d}x/(x-1)$ are rational 1-forms of variable x; and

(3) $W_{1,G}$ and $W_{2,G}$ are 2×2 matrices depend only on parameters β.

To obtain a Pfaffian system of ${}_1F_1$, we prepare partial fraction expansions of rational 1-forms $(\nabla_{x'}/\mathrm{d}x')(\varphi)$,

$$\begin{aligned}
\frac{\nabla_{x'}}{\mathrm{d}x'}(\varphi_{0,K}) &= \beta'_{0;1} t \left(\frac{\mathrm{d}t}{t(1-t)} \right) = -\beta'_{0;1} \frac{\mathrm{d}t}{t-1} = \beta'_{0;1} \varphi_{1,K}, \\
\frac{\nabla_{x'}}{\mathrm{d}x'}(\varphi_{1,K}) &= \beta'_{0;1} t\big(-\phi_{2;0}\big) \equiv -\frac{\beta'_{0;1}}{x'} \phi_{0;1} - \beta'_{0;1} \phi_{2;0} + \frac{\omega_K}{x'} \\
&= \frac{1}{x'} \big(\beta_{1;0} \phi_{1;0} + \beta_{2;0} \phi_{2;0} \big) - \beta'_{0;1} \phi_{2;0} \\
&= \frac{1}{x'} \big\{ \beta_{1;0} \varphi_{0,K} - (\beta_{1;0} + \beta_{2;0}) \, \varphi_{1,K} \big\} + \beta'_{0;1} \varphi_{1,K},
\end{aligned}$$

where $\equiv$ means equivalent as elements of $H^1_K(\Omega^\bullet(x), \nabla_\omega)$. Using these expansions, we have

$$\frac{\nabla_{x'}}{\mathrm{d}x'} \begin{bmatrix} \varphi_{0,K} \\ \varphi_{1,K} \end{bmatrix} = \left\{ \frac{1}{x'} W_{1,K} + W_{2,K} \right\} \begin{bmatrix} \varphi_{0,K} \\ \varphi_{1,K} \end{bmatrix}, \tag{5.11}$$

where

$$W_{1,K} = \begin{bmatrix} 0 & 0 \\ \beta_{1;0} & -\beta_{1;0} - \beta_{2;0} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & -c \end{bmatrix}, \quad W_{2,K} = \begin{bmatrix} 0 & \beta'_{0;1} \\ 0 & \beta'_{0;1} \end{bmatrix} = \begin{bmatrix} 0 & d_1 \\ 0 & d_1 \end{bmatrix}.$$

By using K_0, we construct a vector-valued function ${}^t[K_0 \;\; K_1]$, where

$$K_1 = \frac{1}{d_1} \frac{\mathrm{d}K_0}{\mathrm{d}x'} = \int_\gamma u_{(2,1,1)} \varphi_{1,K}.$$

This vector-valued function satisfies a Pfaffian equation of Kummer function,

$$\mathrm{d}_{x'} \begin{bmatrix} K_0 \\ K_1 \end{bmatrix} = \left\{ \frac{\mathrm{d}x'}{x'} W_{1,K} + \mathrm{d}x' W_{2,K} \right\} \begin{bmatrix} K_0 \\ K_1 \end{bmatrix}, \tag{5.12}$$

which is equivalent to equation (5.11).

We study a transformation from our Pfaffian system of Gauss's hypergeometric function to that of Kummer's one. We relate variables and parameters of ${}_2F_1$ to those of ${}_1F_1$ as

$$\begin{cases} x = \varepsilon x', \\ b = \frac{1}{\varepsilon} d_1, \end{cases}$$

and take the limit $\varepsilon \to 0$ with keeping the relations. Under this limit so called a confluence, the 1-forms $\phi_{3;0}$ and ω_G converge to

$$\begin{aligned}
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \phi_{3;0} &= \lim_{\varepsilon \to 0} \frac{x' \mathrm{d}t}{\varepsilon x' - 1} = -x' \mathrm{d}t = -\phi_{0;1}, \\
\lim_{\varepsilon \to 0} \omega_G &= \beta_{1;0} \phi_{1;0} + \beta_{2;0} \phi_{2;0} + \beta'_{0;1} \phi_{0;1} = \omega_K.
\end{aligned}$$

The intersection matrix C_G converges to

$$\lim_{\varepsilon\to 0} C_G = \lim_{\varepsilon\to 0} \begin{bmatrix} (\frac{1}{a}+\frac{1}{c-a}) & \frac{1}{c-a} \\ \frac{1}{c-a} & (\frac{1}{c-a}-\frac{\varepsilon}{d_1}) \end{bmatrix} = C_K.$$

Substitute the relation $x = \varepsilon x'$ into equation (5.10):

$$\mathrm{d}x' \begin{bmatrix} G_0 \\ G_1 \end{bmatrix} = \left\{ \frac{\mathrm{d}x'}{x'} W_{1,G} + \frac{\varepsilon \mathrm{d}x'}{\varepsilon x' - 1} W_{2,G} \right\} \begin{bmatrix} G_0 \\ G_1 \end{bmatrix}, \tag{5.13}$$

and take the limit

$$\lim_{\varepsilon\to 0} \frac{\mathrm{d}x'}{x'} W_{1,G} = \frac{\mathrm{d}x'}{x'} \begin{bmatrix} 0 & 0 \\ a & -c \end{bmatrix} = \frac{\mathrm{d}x'}{x'} W_{1,K},$$
$$\lim_{\varepsilon\to 0} \frac{\varepsilon \mathrm{d}x'}{\varepsilon x' - 1} W_{2,G} = \lim_{\varepsilon\to 0} \frac{\varepsilon \mathrm{d}x'}{\varepsilon x' - 1} \begin{bmatrix} 0 & -\frac{d_1}{\varepsilon} \\ 0 & c - a - \frac{d_1}{\varepsilon} \end{bmatrix} = \mathrm{d}x' W_{2,K}.$$

It turns out that the Pfaffian equation (5.10) is transformed into (5.12) by the confluence. In this case, the decomposition of connection matrix W is preserved under confluence.

We give an example of confluence which does not preserve decompositions of connection matrices. Instead of equation (5.5), we introduce a parameter ε to Kummer's hypergeometric function in (5.4) by

$$K_0^{\varepsilon}(x) = \int_{\gamma'} t'^a (1-\varepsilon t')^{c-a} \exp(d_1 x t') \varphi_{0,K} \quad (\Re(a) > 0, \Re(d_1 x) < 0), \tag{5.14}$$

where $\gamma' = (0, \infty)$ and $\varphi_{0,K} = \mathrm{d}t'/(t'(1-\varepsilon t'))$. By substituting

$$\begin{cases} x' = \frac{x}{\varepsilon}, \\ t = \varepsilon t', \end{cases} \tag{5.15}$$

into (5.6), we have (5.14). It is a solution to the following differential equation:

$$\left[x(\frac{\mathrm{d}}{\mathrm{d}x})^2 + \left\{ c - \frac{d_1}{\varepsilon} x \right\} \frac{\mathrm{d}}{\mathrm{d}x} - a\frac{d_1}{\varepsilon} \right] K_0^{\varepsilon}(x) = 0. \tag{5.16}$$

By rewriting t' to t into (5.11), we have

$$\frac{\nabla_x}{\mathrm{d}x} \begin{bmatrix} \varphi_{0,K} \\ \varphi_{1,K} \end{bmatrix} = \left\{ \frac{\varepsilon}{x} W_{1,K} + W_{2,K} \right\} \begin{bmatrix} \varphi_{0,K} \\ \varphi_{1,K} \end{bmatrix}, \tag{5.17}$$

where

$$\varphi_{0,K} = \frac{\mathrm{d}t}{t(1-\varepsilon t)}, \qquad \varphi_{1,K} = \frac{\varphi \mathrm{d}t}{1-\varepsilon t},$$
$$W_{1,K} = \begin{bmatrix} 0 & 0 \\ a & -c \end{bmatrix}, \qquad W_{2,K} = \begin{bmatrix} 0 & d_1 \\ 0 & d_1 \end{bmatrix}.$$

We construct a vector-valued function ${}^t[K_0^{\varepsilon} \;\; \varepsilon K_1^{\varepsilon}]$, where

$$K_1^{\varepsilon} = \frac{1}{d_1} \frac{\mathrm{d}K_0^{\varepsilon}}{\mathrm{d}x} = \frac{1}{\varepsilon d_1} \frac{\mathrm{d}K_0^{\varepsilon}}{\mathrm{d}x'} = \frac{1}{\varepsilon} \int_{\gamma'} u_{(2,1,1)} \varphi_{1,K}.$$

This vector-valued function satisfies a Pfaffian equation of Kummer's hypergeometric function,

$$\frac{\mathrm{d}_x}{\varepsilon}\begin{bmatrix} K_0^\varepsilon \\ \varepsilon K_1^\varepsilon \end{bmatrix} = \left\{ \frac{\mathrm{d}x}{x} W_{1,K} + \frac{\mathrm{d}x}{\varepsilon} W_{2,K} \right\} \begin{bmatrix} K_0^\varepsilon \\ \varepsilon K_1^\varepsilon \end{bmatrix}, \tag{5.18}$$

which is equivalent to (5.17).

Hermite's hypergeometric function $H_0(x'')$ can be regarded as a **GHF** of type $\lambda = (3,1)$. It has the form

$$H_0(x'') = \int_{\gamma'} t^a \exp(d'x''t - \frac{d't^2}{2})\varphi_{0,H} \quad (\Re(a) > 0, \Re(d') > 0), \tag{5.19}$$

where $\gamma' = (0,\infty)$ and $\varphi_{0,H} = \mathrm{d}t/t$. It is a solution to the following differential equation:

$$\left[(\frac{\mathrm{d}}{\mathrm{d}x''})^2 - d'x'' \frac{\mathrm{d}}{\mathrm{d}x''} - ad' \right] H_0(x'') = 0. \tag{5.20}$$

Set $n = 4, m = 2$, and h and β in **GHF** as follows:

$$\begin{aligned} h &= (h'^{(0)}, h^{(1)}), \quad \iota(h'^{(0)}) = (h'_{0;0}, h'_{0;1}, h'_{0;2}) = (s, 1, x''), \quad h_{1;0} = t, \\ \beta &= (\beta'^{(0)}, \beta^{(1)}), \quad \beta'^{(0)} = (\beta'_{0;0}, \beta'_{0;1}, \beta'_{0;2}) = (-a, 0, d'), \quad \beta_{1;0} = a. \end{aligned}$$

Then we have

$$\begin{aligned} u_{(3,1)} &= h_{1;0}^{\beta_{1;0}} \prod_{k=1}^{2} \exp(\beta'_{0;k}\theta_k(h'^{(0)})) = t^a \exp\left\{ d'(x''t - \frac{t^2}{2}) \right\}, \\ \phi'_{0;k} &= \mathrm{d}_t(\theta_k(h'^{(0)}))\,(0 \le k \le 2), \quad \phi_{1;0} = \mathrm{d}_t(\theta_0(h^{(1)})), \\ \omega_H &= \mathrm{d}_t \log(u_{(3,1)}) = \beta'_{0;1}\phi'_{0;1} + \beta'_{0;2}\phi'_{0;2} + \beta_{1;0}\phi_{1;0}, \\ \varphi_{0,H} &= \phi_{1;0}. \end{aligned}$$

We summarize variables, parameters and 1-forms in a table:

variables :	$x_0 = \infty$,	$h_{0;1} = 1$,	$h_{0;2} = x''$,	$x_1 = 0$,
parameters :	$\beta'_{0;0} = -a$,	$\beta'_{0;1} = 0$,	$\beta'_{0;2} = d'$,	$\beta_{1;0} = a$,
1-forms :	$\phi'_{0;0} = \frac{\mathrm{d}s}{s}$,	$\phi'_{0;1} = \mathrm{d}t$,	$\phi'_{0;2} = (x'' - t)\mathrm{d}t$,	$\phi_{1;0} = \frac{\mathrm{d}t}{t}$.

We define a logarithmic 1-form:

$$\omega_{H,x''} := \mathrm{d}_{x''} \log(u_{(3,1)}) = \beta'_{1;2} t \mathrm{d}x'',$$

and we have

$$\frac{\nabla_{x''}}{\mathrm{d}x''}(\varphi_{0,H}) = \frac{\mathrm{d}t}{t} \times \beta'_{1;2} t = \beta'_{1;2}\mathrm{d}t.$$

Considering above relations, we select bases of $H^1_K(\Omega^\bullet(x), \nabla_\omega)$ and $H^1_H(\Omega^\bullet(x), \nabla_\omega)$ as:

$$\begin{cases} \varphi_{0,K} = \phi_{1;0} - \phi_{2;0} = \frac{\mathrm{d}t}{t} - \frac{\varepsilon \mathrm{d}t}{\varepsilon t - 1}, \\ \varphi'_{1,K} := \frac{1}{\varepsilon}\varphi_{1,K} = \frac{\mathrm{d}t}{1 - \varepsilon t}, \end{cases}$$

$$\begin{cases} \varphi_{0,H} = \phi_{1;0} = \frac{\mathrm{d}t}{t}, \\ \varphi_{1,H} := \phi'_{0;1} = \mathrm{d}t. \end{cases}$$

The intersection matrix of $(\varphi_{0,K}, \varphi'_{1,K})$ becomes $2\pi\sqrt{-1}C'_K$, and that of $(\varphi_{0,H}, \varphi_{1,H})$ becomes $2\pi\sqrt{-1}C_H$, where

$$C'_K = \begin{bmatrix} (\frac{1}{\beta_{2;0}} + \frac{1}{\beta_{3;0}}) & \frac{1}{\varepsilon\beta_{3;0}} \\ \frac{1}{\varepsilon\beta_{3;0}} & \frac{1}{\varepsilon^2\beta_{3;0}} \end{bmatrix} = \begin{bmatrix} \frac{c}{a(c-a)} & \frac{1}{\varepsilon(c-a)} \\ \frac{1}{\varepsilon(c-a)} & \frac{1}{\varepsilon^2(c-a)} \end{bmatrix}, \tag{5.21}$$

$$C_H = \begin{bmatrix} \frac{1}{\beta_{2;0}} & 0 \\ 0 & \frac{1}{\beta'_{1;2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d'} \end{bmatrix}. \tag{5.22}$$

Then we have

$$\mathrm{d}_x \begin{bmatrix} K_0^\varepsilon \\ K_1^\varepsilon \end{bmatrix} = \left\{ \frac{\mathrm{d}x}{x} W'_{1,K} + \frac{\mathrm{d}x}{\varepsilon} W'_{2,K} \right\} \begin{bmatrix} K_0^\varepsilon \\ K_1^\varepsilon \end{bmatrix} \tag{5.23}$$

where

$$W'_{1,K} = \begin{bmatrix} 0 & 0 \\ \frac{a}{\varepsilon} & -c \end{bmatrix}, \quad W'_{2,K} = \begin{bmatrix} 0 & \varepsilon d_1 \\ 0 & d_1 \end{bmatrix}.$$

To express $(\nabla_{x''}/\mathrm{d}x'')(\varphi)$ as a linear combination of $\varphi_{H,0}$ and $\varphi_{H,1}$, we prepare partial fraction expansions of rational 1-forms $(\nabla_{x''}/\mathrm{d}x'')(\varphi)$,

$$\begin{aligned}
\frac{\nabla_{x''}}{\mathrm{d}x''}(\varphi_{0,H}) &= \beta'_{0;2} t \left(\frac{\mathrm{d}t}{t} \right) = \beta'_{0;2} \mathrm{d}t = \beta'_{0;2} \varphi_{1,H}, \\
\frac{\nabla_{x''}}{\mathrm{d}x''}(\varphi_{1,H}) &= \beta'_{0;2} t\big(\mathrm{d}t\big) = \beta'_{0;2} \left(t - x''\right) \mathrm{d}t + \beta'_{0;2} x'' \mathrm{d}t = -\beta'_{0;2}\phi'_{0;2} + \beta'_{0;2} x'' \phi'_{0;1} \\
&\equiv \omega_H - \beta'_{0;2}\phi'_{0;2} + \beta'_{0;2} x'' \phi'_{0;1} = \beta_{1;0}\phi_{1;0} + \beta'_{0;2} x'' \phi'_{0;1} \\
&= \beta_{1;0}\varphi_{0,H} + \beta'_{0;2} x'' \varphi_{1,H},
\end{aligned}$$

where $\equiv$ means equivalent as elements of $H^1_H(\Omega^\bullet(x), \nabla_\omega)$. Using these results, we have

$$\frac{\nabla_{x''}}{\mathrm{d}x''} \begin{bmatrix} \varphi_{0,H} \\ \varphi_{1,H} \end{bmatrix} = \{ W_{1,H} + x'' W_{2,H} \} \begin{bmatrix} \varphi_{0,H} \\ \varphi_{1,H} \end{bmatrix}, \tag{5.24}$$

where

$$W_{1,H} = \begin{bmatrix} 0 & \beta'_{0;2} \\ \beta_{1;0} & 0 \end{bmatrix} = \begin{bmatrix} 0 & d' \\ a & 0 \end{bmatrix},$$
$$W_{2,H} = \begin{bmatrix} 0 & 0 \\ 0 & \beta'_{0;2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & d' \end{bmatrix}.$$

We construct a vector-valued function ${}^t[H_0 \;\; H_1]$, where

$$H_1 = \frac{1}{d'x''} \frac{\mathrm{d}H_0}{\mathrm{d}x''} = \int_{\gamma'} u_{(3,1)} \varphi_{1,H}.$$

This function satisfies a Pfaffian equation of Hermite's hypergeometric function,

$$\mathrm{d}_{x''} \begin{bmatrix} H_0 \\ H_1 \end{bmatrix} = \mathrm{d}x'' \{ W_{1,H} + x'' W_{2,H} \} \begin{bmatrix} H_0 \\ H_1 \end{bmatrix}. \tag{5.25}$$

This equation is equivalent to (5.24).

We relate variables and parameters of G_0^ε to those of H_0 as

$$\begin{cases} x = 1 + \varepsilon x'', \\ d_1 = \frac{1}{\varepsilon} d', \\ c = a + \frac{1}{\varepsilon^2} d', \end{cases}$$

and take the limit $\varepsilon \to 0$ with keeping the relations. The intersection matrix C'_K converges to

$$\lim_{\varepsilon \to 0} C'_K = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d'} \end{bmatrix} = C_H.$$

We substitute the relation $x = 1 + \varepsilon x''$ into (5.23):

$$\mathrm{d}_{x''} \begin{bmatrix} K_0^\varepsilon \\ K_1^\varepsilon \end{bmatrix} = \left\{ \frac{\varepsilon \mathrm{d}x''}{1 + \varepsilon x''} W'_{1,K} + \mathrm{d}x'' W'_{2,K} \right\} \begin{bmatrix} K_0^\varepsilon \\ K_1^\varepsilon \end{bmatrix}, \tag{5.26}$$

where

$$\varepsilon W'_{1,K} = \begin{bmatrix} 0 & 0 \\ a & -\varepsilon a - \frac{d'}{\varepsilon} \end{bmatrix}, \quad W'_{2,K} = \begin{bmatrix} 0 & d' \\ 0 & \frac{d'}{\varepsilon} \end{bmatrix}.$$

Under this limit $\varepsilon \to 0$, neither $\varepsilon W'_{1,K}$ nor $W'_{2;K}$ converges. The decomposition of our connection matrix W is not preserved under this confluence, we need a recomposition of W to make each part of W converge under this confluence. We define matrices

$$W''_{1;K} = \varepsilon W'_{1;K} + W'_{2;K}, \quad W''_{2;K} = \varepsilon W'_{2;K},$$

and rewrite (5.26) into

$$\begin{aligned} \mathrm{d}_{x''} \begin{bmatrix} K_0^\varepsilon \\ K_1^\varepsilon \end{bmatrix} &= \left\{ \frac{\mathrm{d}x''}{1 + \varepsilon x''} W''_{1,K} + \frac{x'' \mathrm{d}x''}{1 + \varepsilon x''} W''_{2,K} \right\} \begin{bmatrix} K_0^\varepsilon \\ K_1^\varepsilon \end{bmatrix} \\ &= \left\{ \frac{1}{\varepsilon} \mathrm{d}_{x''}(\log(1 + \varepsilon x'')) W''_{1,K} + \{ \frac{1}{\varepsilon^2} \mathrm{d}_{x''}(\varepsilon x'' - \log(1 + \varepsilon x'')) \} W''_{2,K} \right\} \begin{bmatrix} K_0^\varepsilon \\ K_1^\varepsilon \end{bmatrix}. \end{aligned} \tag{5.27}$$

Under this limit $\varepsilon \to 0$, both $W''_{1,K}$ and $W''_{2,k}$ converge to

$$\lim_{\varepsilon \to 0} W''_{1,K} = W_{1,H}, \quad \lim_{\varepsilon \to 0} W''_{2,K} = W_{2,H}.$$

In this limit, the equation (5.27) goes into (5.25).

6 Pfaffian Systems of Appell's F_1 and Humbert's Φ_1

In this section, we show some examples of Pfaffian systems of two variables. We explain that Appell's hypergeometric series

$$F_1(a, b_1, b_2, c; x, y) = \sum_{m_1, m_2}^{\infty} \frac{(a)_{m_1+m_2} (b_1)_{m_1} (b_2)_{m_2}}{(c)_{m_1+m_2} (1)_{m_1} (1)_{m_2}} x^{m_1} y^{m_2}$$

can be regarded as **GHF** of type $\lambda = (1,1,1,1,1)$. It admits an integral representation

$$F_1(a,b_1,b_2,c;x,y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \times \int_\gamma t^a(1-t)^{c-a}(1-xt)^{-b_1}(1-yt)^{-b_2}\varphi_{0,F} \quad (\Re(c) > \Re(a) > 0), \tag{6.1}$$

where $\gamma = (0,1)$ and $\varphi_{0,F} = \mathrm{d}t/(t(1-t))$. It is a solution to Appell's system of hypergeometric differential equations:

$$\begin{cases} \left[x(1-x)\frac{\partial^2}{\partial x^2} + y(1-x)\frac{\partial^2}{\partial x\partial y} + \{c-(a+b_1+1)x\}\frac{\partial}{\partial x} - b_1 y\frac{\partial}{\partial y} - ab_1\right] Q(x,y) = 0, \\ \left[y(1-y)\frac{\partial^2}{\partial y^2} + x(1-y)\frac{\partial^2}{\partial x\partial y} + \{c-(a+b_2+1)y\}\frac{\partial}{\partial y} - b_2 x\frac{\partial}{\partial x} - ab_2\right] Q(x,y) = 0, \\ \left[(x-y)\frac{\partial^2}{\partial x\partial y} - b_2\frac{\partial}{\partial x} + b_1\frac{\partial}{\partial y}\right] Q(x,y) = 0. \end{cases} \tag{6.2}$$

To regard F_1 as **GHF**, we set $n = m = 5$, and h and β as follows:

$$h = (h^{(0)}, h^{(1)}, h^{(2)}, h^{(3)}, h^{(4)}), h_{0;0} = s, h_{1;0} = t, h_{2;0} = 1-t, h_{3;0} = 1-xt, h_{4;0} = 1-yt,$$
$$\beta = (\beta^{(0)}, \beta^{(1)}, \beta^{(2)}, \beta^{(3)}, \beta^{(4)}), \beta_{0;0} = b_1+b_2-c, \beta_{1;0} = a, \beta_{2;0} = c-a, \beta_{3;0} = -b_1, \beta_{4;0} = -b_2.$$

Then we have

$$\begin{aligned} u_{(1,1,1,1,1)} &= \prod_{i=0}^{4} h_{i;0}^{\beta_{i;0}} = t^a(1-t)^{c-a}(1-xt)^{-b_1}(1-yt)^{-b_2}, \\ \phi_{i;0} &= \mathrm{d}_t(\theta_0(h^{(i)})) \quad (0 \le i \le 4), \\ \omega_F &= \mathrm{d}_t \log(u_{(1,1,1,1,1)}) = \sum_{i=1}^{4} \beta_{i;0}\phi_{i;0}, \\ \varphi_{0,F} &= \phi_{1;0} - \phi_{2;0}. \end{aligned}$$

A solution to (6.2)

$$Q_0 = \int_\gamma u_{(1,1,1,1,1)}\varphi_{0,F} \tag{6.3}$$

is F_1 up to a non-zero constant. We summarize variables, parameters and 1-forms in a table:

variables :	$x_0 = \infty$,	$x_4 = \frac{1}{y}$,	$x_1 = 0$,	$x_2 = 1$,	$x_3 = \frac{1}{x}$,
parameters :	$\beta_{0;0} = b_1+b_2-c$,	$\beta_{4;0} = -b_2$,	$\beta_{1;0} = a$,	$\beta_{2;0} = c-a$,	$\beta_{3;0} = -b_1$,
1-forms :	$\phi_{0;0} = \frac{\mathrm{d}s}{s}$,	$\phi_{4;0} = \frac{y\mathrm{d}t}{yt-1}$,	$\phi_{1;0} = \frac{\mathrm{d}t}{t}$,	$\phi_{2;0} = \frac{\mathrm{d}t}{t-1}$,	$\phi_{3;0} = \frac{x\mathrm{d}t}{xt-1}$,

where $\{x_i\}_{0\le i\le 4}$ are poles of ω_F.

Humbert's hypergeometric series

$$\Phi_1(a,b_1,c;x,-d_2y') = \sum_{m_1,m_2}^{\infty} \frac{(a)_{m_1+m_2}(b_1)_{m_1}}{(c)_{m_1+m_2}(1)_{m_1}(1)_{m_2}} x^{m_1}(-d_2y')^{m_2}$$

can be regarded as a **GHF** of type $\lambda = (2,1,1,1)$. It has the form

$$\Phi_1(a,b_1,c;x,-d_2y') = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_\gamma t^a(1-t)^{c-a}(1-xt)^{-b_1}\exp(-d_2y't)\varphi_{0,\Phi} \quad (\Re(c) > \Re(a) > 0), \tag{6.4}$$

where $\gamma = (0,1)$, $\varphi_{0,\Phi} = \mathrm{d}t/(t(1-t))$ and $d_2 \neq 0$. It is a solution to a system of differential equations:

$$\begin{cases} \left[x(1-x)\frac{\partial^2}{\partial x^2} + y'(1-x)\frac{\partial^2}{\partial x \partial y'} + \{c-(a+b_1+1)x\}\frac{\partial}{\partial x} - b_1 y'\frac{\partial}{\partial y'} - ab_1\right] P(x,y') = 0, \\ \left[y'\frac{\partial^2}{\partial y'^2} + x\frac{\partial^2}{\partial x \partial y'} + \{c+d_2 y'\}\frac{\partial}{\partial y'} + d_2 x\frac{\partial}{\partial x} + ad_2\right] P(x,y') = 0, \\ \left[x\frac{\partial^2}{\partial x \partial y'} + d_2\frac{\partial}{\partial x} + b_1\frac{\partial}{\partial y'}\right] P(x,y') = 0. \end{cases} \tag{6.5}$$

To regard Φ_1 as **GHF**, we set $n=5, n=4$, and h and β as follows:

$$\begin{array}{llllll} h = (h^{(0)}, h^{(1)}, h^{(2)}, h^{(3)}), & \iota(h^{(0)}) = (s, y'), & h_{1;0} = t, & h_{2;0} = 1-t, & h_{3;0} = 1-xt, \\ \beta = (\beta'^{(0)}, \beta^{(1)}, \beta^{(2)}, \beta^{(3)}), & \beta'^{(0)} = (b_1 - c, -d_2), & \beta_{1;0} = a, & \beta_{2;0} = c-a, & \beta_{3;0} = -b_1. \end{array}$$

Then we have

$$\begin{aligned} u_{(2,1,1,1)} &= \exp(\beta'_{0;1}\theta_1(h^{(0)}))\prod_{i=1}^{3} h_{i;0}^{\beta_{i;0}}, \\ \phi_{i;0} &= \mathrm{d}_t(\theta_0(h^{(i)}))\,(0 \leq i \leq 3), \quad \phi_{0;1} = \mathrm{d}_t(\theta_1(h^{(0)})), \\ \omega_\Phi &= \mathrm{d}_t \log(u_{(2,1,1,1)}) = \beta'_{0;1}\phi_{0;1} + \sum_{i=1}^{3}\beta_{i;0}\phi_{i;0}, \\ \varphi_{0,\Phi} &= \phi_{1;0} - \phi_{2;0}. \end{aligned}$$

A solution to (6.5)

$$P_0 = \int_\gamma u_{(2,1,1,1)}\varphi_{0,\Phi} \tag{6.6}$$

is Φ_1 up to a non-zero constant. We summarize variables, parameters and 1-forms in a table:

variables :	$x_0 = \infty$,	$h_{0;1} = y'$,	$x_1 = 0$,	$x_2 = 1$,	$x_3 = \frac{1}{x}$,
parameters :	$\beta'_{0;0} = b_1 - c$,	$\beta'_{0;1} = -d_2$,	$\beta_{1;0} = a$,	$\beta_{2;0} = c-a$,	$\beta_{3;0} = -b_1$,
1-forms :	$\phi_{0;0} = \frac{\mathrm{d}s}{s}$,	$\phi_{0;1} = -\frac{y'\mathrm{d}s}{s^2}$,	$\phi_{1;0} = \frac{\mathrm{d}t}{t}$,	$\phi_{2;0} = \frac{\mathrm{d}t}{t-1}$,	$\phi_{3;0} = \frac{x\mathrm{d}t}{xt-1}$,

where $\{x_i\}_{0\leq i\leq 3}$ are poles of ω_Φ.

We introduce operators $\partial_x(f) := \mathrm{d}x \wedge (\partial f/\partial x)$, $\partial_y(f) := \mathrm{d}y \wedge (\partial f/\partial y)$ and $\partial_{y'}(f) := \mathrm{d}y' \wedge (\partial f/\partial y')$. We define logarithmic 1-forms:

$$\begin{aligned} \omega_{F,x} &:= \partial_x \log(u_{(1,1,1,1,1)}) = \frac{\beta_{3;0}t\mathrm{d}x}{xt-1}, \\ \omega_{\Phi,x} &:= \partial_x \log(u_{(2,1,1,1)}) = \frac{\beta_{3;0}t\mathrm{d}x}{xt-1}, \\ \omega_{F,y} &:= \partial_y \log(u_{(1,1,1,1,1)}) = \frac{\beta_{4;0}t\mathrm{d}y}{yt-1}, \\ \omega_{\Phi,y'} &:= \partial_{y'} \log(u_{(2,1,1,1)}) = \beta'_{0;1}t\mathrm{d}y'. \end{aligned}$$

We introduce operators $\nabla_x := \partial_x + \omega_x\wedge$, $\nabla_y := \partial_y + \omega_y\wedge$ and $\nabla_{y'} := \partial_{y'} + \omega_{y'}\wedge$, then we have

$$\frac{\partial}{\partial x}\int_\gamma u(t,x,y)\varphi = \int_\gamma u(t,x,y)\frac{\nabla_x}{\mathrm{d}x}(\varphi),$$

and so on. We have

$$\begin{aligned}
\frac{\nabla_x}{\mathrm{d}x}(\varphi_{0,F}) &= \frac{\mathrm{d}t}{t(1-t)} \times \frac{\beta_{3;0}t}{xt-1} = \frac{\beta_{3;0}}{x-1}(\frac{x\mathrm{d}t}{xt-1} - \frac{\mathrm{d}t}{t-1}),\\
\frac{\nabla_x}{\mathrm{d}x}(\varphi_{0,\Phi}) &= \frac{\mathrm{d}t}{t(1-t)} \times \frac{\beta_{3;0}t}{xt-1} = \frac{\beta_{3;0}}{x-1}(\frac{x\mathrm{d}t}{xt-1} - \frac{\mathrm{d}t}{t-1}),\\
\frac{\nabla_y}{\mathrm{d}y}(\varphi_{0,F}) &= \frac{\mathrm{d}t}{t(1-t)} \times \frac{\beta_{3;0}t}{yt-1} = \frac{\beta_{4;0}}{y-1}(\frac{y\mathrm{d}t}{yt-1} - \frac{\mathrm{d}t}{t-1}),\\
\frac{\nabla_{y'}}{\mathrm{d}y'}(\varphi_{0,\Phi}) &= \frac{\mathrm{d}t}{t(1-t)} \times \beta'_{0;1}t = -\beta'_{0;1}\frac{\mathrm{d}t}{t-1}.
\end{aligned}$$

We choose bases of $H^1_F(\Omega^\bullet(x), \nabla_\omega)$ and $H^1_\Phi(\Omega^\bullet(x), \nabla_\omega)$ as following:

$$\begin{cases}
\varphi_{0,F} = \phi_{1;0} - \phi_{2;0} = \frac{\mathrm{d}t}{t} - \frac{\mathrm{d}t}{t-1},\\
\varphi_{1,F} := \phi_{3;0} - \phi_{2;0} = \frac{x\mathrm{d}t}{xt-1} - \frac{\mathrm{d}t}{t-1},\\
\varphi_{2,F} := \phi_{4;0} - \phi_{2;0} = \frac{y\mathrm{d}t}{yt-1} - \frac{\mathrm{d}t}{t-1},
\end{cases}$$

$$\begin{cases}
\varphi_{0,\Phi} = \phi_{1;0} - \phi_{2;0} = \frac{\mathrm{d}t}{t} - \frac{\mathrm{d}t}{t-1},\\
\varphi_{1,\Phi} := \phi_{3;0} - \phi_{2;0} = \frac{x\mathrm{d}t}{xt-1} - \frac{\mathrm{d}t}{t-1},\\
\varphi_{2,\Phi} := -\phi_{2;0} = -\frac{\mathrm{d}t}{t-1}.
\end{cases}$$

The intersection matrix of $(\varphi_{0,F}, \varphi_{1,F}, \varphi_{2,F})$ becomes $2\pi\sqrt{-1}C_F$, and that of $(\varphi_{0,\Phi}, \varphi_{1,\Phi}, \varphi_{2,\Phi})$ becomes $2\pi\sqrt{-1}C_\Phi$, where

$$C_F = \begin{bmatrix}
(\frac{1}{\beta_{1;0}} + \frac{1}{\beta_{2;0}}) & \frac{1}{\beta_{2;0}} & \frac{1}{\beta_{2;0}}\\
\frac{1}{\beta_{2;0}} & (\frac{1}{\beta_{2;0}} + \frac{1}{\beta_{3;0}}) & \frac{1}{\beta_{2;0}}\\
\frac{1}{\beta_{2;0}} & \frac{1}{\beta_{2;0}} & (\frac{1}{\beta_{2;0}} + \frac{1}{\beta_{4;0}})
\end{bmatrix} = \begin{bmatrix}
\frac{c}{a(c-a)} & \frac{1}{c-a} & \frac{1}{c-a}\\
\frac{1}{c-a} & \frac{a+b_1-c}{b_1(c-a)} & \frac{1}{c-a}\\
\frac{1}{c-a} & \frac{1}{c-a} & \frac{a+b_2-c}{b_2(c-a)}
\end{bmatrix}, \tag{6.7}$$

$$C_\Phi = \begin{bmatrix}
(\frac{1}{\beta_{1;0}} + \frac{1}{\beta_{2;0}}) & \frac{1}{\beta_{2;0}} & \frac{1}{\beta_{2;0}}\\
\frac{1}{\beta_{2;0}} & (\frac{1}{\beta_{2;0}} + \frac{1}{\beta_{3;0}}) & \frac{1}{\beta_{2;0}}\\
\frac{1}{\beta_{2;0}} & \frac{1}{\beta_{2;0}} & \frac{1}{\beta_{2;0}}
\end{bmatrix} = \begin{bmatrix}
\frac{c}{a(c-a)} & \frac{1}{c-a} & \frac{1}{c-a}\\
\frac{1}{c-a} & \frac{a+b_1-c}{b_1(c-a)} & \frac{1}{c-a}\\
\frac{1}{c-a} & \frac{1}{c-a} & \frac{1}{c-a}
\end{bmatrix}. \tag{6.8}$$

As is in [Mt2], we have

$$\frac{\nabla_x}{\mathrm{d}x}\begin{bmatrix}\varphi_{0,F}\\ \varphi_{1,F}\\ \varphi_{2,F}\end{bmatrix} = \left\{\frac{1}{x}W_{1,F} + \frac{1}{x-1}W_{2,F} + \frac{1}{x-y}W_{5,F}\right\}\begin{bmatrix}\varphi_{0,F}\\ \varphi_{1,F}\\ \varphi_{2,F}\end{bmatrix}, \tag{6.9}$$

$$\frac{\nabla_y}{\mathrm{d}y}\begin{bmatrix}\varphi_{0,F}\\ \varphi_{1,F}\\ \varphi_{2,F}\end{bmatrix} = \left\{\frac{1}{y}W_{3,F} + \frac{1}{y-1}W_{4,F} + \frac{1}{y-x}W_{5,F}\right\}\begin{bmatrix}\varphi_{0,F}\\ \varphi_{1,F}\\ \varphi_{2,F}\end{bmatrix}, \tag{6.10}$$

where

$$W_{1,F} = \begin{bmatrix}0 & 0 & 0\\ \beta_{1;0} & \beta_{0;0}+\beta_{3;0} & \beta_{4;0}\\ 0 & 0 & 0\end{bmatrix}, \quad W_{2,F} = \begin{bmatrix}0 & \beta_{3;0} & 0\\ 0 & \beta_{2;0}+\beta_{3;0} & 0\\ 0 & \beta_{3;0} & 0\end{bmatrix},$$

$$W_{3,F} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \beta_{1;0} & \beta_{3;0} & \beta_{0;0}+\beta_{4;0}\end{bmatrix}, \quad W_{4,F} = \begin{bmatrix}0 & 0 & \beta_{4;0}\\ 0 & 0 & \beta_{4;0}\\ 0 & 0 & \beta_{2;0}+\beta_{4;0}\end{bmatrix}, \quad W_{5,F} = \begin{bmatrix}0 & 0 & 0\\ 0 & \beta_{4;0} & -\beta_{4;0}\\ 0 & -\beta_{3;0} & \beta_{3;0}\end{bmatrix}.$$

By using Q_0, we construct a vector-valued function ${}^t[Q_0 \ \ Q_1 \ \ Q_2]$, where

$$Q_1 = \frac{x-1}{-b_1}\frac{\partial Q_0}{\partial x} = \int_\gamma u_{(1,1,1,1,1)}\varphi_{1,F}, \quad Q_2 = \frac{y-1}{-b_2}\frac{\partial Q_0}{\partial y} = \int_\gamma u_{(1,1,1,1,1)}\varphi_{2,F}.$$

This vector-valued function satisfies a Pfaffian system of Appell's F_1 function,

$$\mathrm{d}_x \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix} = \left\{ \frac{\mathrm{d}x}{x} W_{1,F} + \frac{\mathrm{d}x}{x-1} W_{2,F} + \frac{\mathrm{d}x}{x-y} W_{5,F} \right\} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix}, \tag{6.11}$$

$$\mathrm{d}_y \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix} = \left\{ \frac{\mathrm{d}y}{y} W_{3,F} + \frac{\mathrm{d}y}{y-1} W_{4,F} + \frac{\mathrm{d}y}{y-x} W_{5,F} \right\} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix}. \tag{6.12}$$

This system is equivalent to the system of (6.9) and (6.10).

We have

$$\begin{aligned}
\frac{\nabla_x}{\mathrm{d}x}(\varphi_{0,\Phi}) &= \frac{\beta_{3;0}}{x-1}(\phi_{3;0}-\phi_{2;0}) = \frac{\beta_{3;0}}{x-1}\varphi_{1,\Phi}, \\
\frac{\nabla_x}{\mathrm{d}x}(\varphi_{2,\Phi}) &= -\frac{\beta_{3;0}}{x}\phi_{3;0} + \frac{\beta_{3;0}}{x-1}(\phi_{3;0}-\phi_{2;0}) = \frac{\beta_{3;0}}{x}(\varphi_{2,\Phi}-\varphi_{1,\Phi}) + \frac{\beta_{3;0}}{x-1}\varphi_{1,\Phi}, \\
\frac{\nabla_x}{\mathrm{d}x}(\varphi_{1,\Phi}) &\equiv \frac{\nabla_x}{\mathrm{d}x}(\phi_{3;0}-\phi_{2;0}-\frac{1}{\beta_{3;0}}\omega_\Phi) = \frac{-t}{xt-1}\left\{\beta'_{0;1}\phi_{0;1}+\beta_{1;0}\phi_{1;0}+(\beta_{2;0}+\beta_{3;0})\phi_{2;0}\right\} \\
&= -\beta'_{0;1}(\frac{1}{x}\phi_{0;1}+\frac{y'}{x^2}\phi_{3;0}) - \frac{\beta_{1;0}}{x}\phi_{3;0} - \frac{\beta_{2;0}+\beta_{3;0}}{x}\phi_{3;0} + \frac{\beta_{2;0}+\beta_{3;0}}{x-1}(\phi_{3;0}-\phi_{2;0}) \\
&\equiv \frac{1}{x}(-\beta'_{0;1}\phi_{0;1}-\beta_{3;0}\phi_{3;0}+\omega_\Phi) - \beta'_{0;1}\frac{y'}{x^2}\phi_{3;0} - \frac{\beta_{1;0}+\beta_{2;0}}{x}\phi_{3;0} + \frac{\beta_{2;0}+\beta_{3;0}}{x-1}(\phi_{3;0}-\phi_{2;0}) \\
&= \frac{1}{x}(\beta_{1;0}\phi_{1;0}+\beta_{2;0}\phi_{2;0}) - \beta'_{0;1}\frac{y'}{x^2}\phi_{3;0} - \frac{\beta_{1;0}+\beta_{2;0}}{x}\phi_{3;0} + \frac{\beta_{2;0}+\beta_{3;0}}{x-1}(\phi_{3;0}-\phi_{2;0}) \\
&= \beta'_{0;1}\frac{y'}{x^2}(\varphi_{2,\Phi}-\varphi_{1,\Phi}) + \frac{1}{x}\left\{\beta_{1;0}\varphi_{0,\Phi} - (\beta_{1;0}+\beta_{2;0})\varphi_{1,\Phi}\right\} + \frac{\beta_{2;0}+\beta_{3;0}}{x-1}\varphi_{1,\Phi}, \\
\frac{\nabla_{y'}}{\mathrm{d}y'}(\varphi_{0,\Phi}) &= \beta'_{0;1}t(\phi_{1;0}-\phi_{2;0}) = \beta'_{0;1}\varphi_{2,\Phi}, \\
\frac{\nabla_{y'}}{\mathrm{d}y'}(\varphi_{1,\Phi}) &= \beta'_{0;1}t(\phi_{3;0}-\phi_{2;0}) = \beta'_{0;1}(\frac{\mathrm{d}t}{xt-1} - \frac{\mathrm{d}t}{t-1}) = \frac{\beta'_{0;1}}{x}(\varphi_{1,\Phi}-\varphi_{2,\Phi}) + \beta'_{0;1}\varphi_{2,\Phi}, \\
\frac{\nabla_{y'}}{\mathrm{d}y'}(\varphi_{2,\Phi}) &= \beta'_{0;1}t(-\phi_{2;0}) \equiv -\frac{\beta'_{0;1}}{y'}\phi_{0;1} - \beta'_{0;1}\phi_{2;0} + \frac{\omega_\Phi}{y'} = \frac{1}{y'}(\beta_{1;0}\phi_{1;0}+\beta_{2;0}\phi_{2;0}+\beta_{3;0}\phi_{3;0}) - \beta'_{0;1}\phi_{2;0} \\
&= \frac{1}{y'}(\beta_{1;0}\varphi_{0,\Phi}+\beta_{3;0}\varphi_{1,\Phi}+\beta'_{0;0}\varphi_{2;\Phi}) + \beta'_{0;1}\varphi_{2,\Phi},
\end{aligned}$$

where $\equiv$ means equivalent as elements of $H^1_\Phi(\Omega^\bullet(x), \nabla_\omega)$. Using these results, we can write

$$\frac{\nabla_x}{\mathrm{d}x} \begin{bmatrix} \varphi_{0,\Phi} \\ \varphi_{1,\Phi} \\ \varphi_{2,\Phi} \end{bmatrix} = \left\{ \frac{1}{x} W_{1,\Phi} + \frac{1}{x-1} W_{2,\Phi} - \frac{y'}{x^2} W_{5,\Phi} \right\} \begin{bmatrix} \varphi_{0,\Phi} \\ \varphi_{1,\Phi} \\ \varphi_{2,\Phi} \end{bmatrix}, \tag{6.13}$$

$$\frac{\nabla_{y'}}{\mathrm{d}y'} \begin{bmatrix} \varphi_{0,\Phi} \\ \varphi_{1,\Phi} \\ \varphi_{2,\Phi} \end{bmatrix} = \left\{ \frac{1}{y'} W_{3,\Phi} + W_{4,\Phi} + \frac{1}{x} W_{5,\Phi} \right\} \begin{bmatrix} \varphi_{0,\Phi} \\ \varphi_{1,\Phi} \\ \varphi_{2,\Phi} \end{bmatrix}, \tag{6.14}$$

where

$$W_{1,\Phi} = \begin{bmatrix} 0 & 0 & 0 \\ \beta_{1;0} & \beta'_{0;0} + \beta_{3;0} & 0 \\ 0 & -\beta_{3;0} & \beta_{3;0} \end{bmatrix}, \quad W_{2,\Phi} = \begin{bmatrix} 0 & \beta_{3;0} & 0 \\ 0 & \beta_{2;0} + \beta_{3;0} & 0 \\ 0 & \beta_{3;0} & 0 \end{bmatrix},$$

$$W_{3,\Phi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \beta_{1;0} & \beta_{3;0} & \beta'_{0;0} \end{bmatrix}, \quad W_{4,\Phi} = \begin{bmatrix} 0 & 0 & \beta'_{0;1} \\ 0 & 0 & \beta'_{0;1} \\ 0 & 0 & \beta'_{0;1} \end{bmatrix}, \quad W_{5,\Phi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \beta'_{0;1} & -\beta'_{0;1} \\ 0 & 0 & 0 \end{bmatrix}.$$

By using P_0, we construct a vector-valued function ${}^t[P_0 \ P_1 \ P_2]$, where

$$P_1 = \frac{x-1}{-b_1}\frac{\partial P_0}{\partial x} = \int_\gamma u_{(2,1,1,1)}\varphi_{1,\Phi}, \quad P_2 = \frac{1}{-d_2}\frac{\partial P_0}{\partial y'} = \int_\gamma u_{(2,1,1,1)}\varphi_{2,\Phi}.$$

This vector-valued function satisfies a Pfaffian system of Humbert's Φ_1 function,

$$\mathrm{d}x \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix} = \left\{ \frac{\mathrm{d}x}{x} W_{1,\Phi} + \frac{\mathrm{d}x}{x-1} W_{2,\Phi} - \frac{y'}{\mathrm{d}x} x^2 W_{5,\Phi} \right\} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}, \tag{6.15}$$

$$\mathrm{d}y' \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix} = \left\{ \frac{\mathrm{d}y'}{y'} W_{3,\Phi} + \mathrm{d}y' W_{4,\Phi} + \frac{\mathrm{d}y'}{x} W_{5,\Phi} \right\} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}. \tag{6.16}$$

This system is equivalent to the system of (6.13) and (6.14).

We study a confluence from our Pfaffian system of F_1 to that of Φ_1. We relate variables and parameters of F_1 to those of Φ_1 as

$$\begin{cases} y = -\varepsilon y', \\ b_2 = \frac{1}{\varepsilon} d_2, \end{cases}$$

and take the limit $\varepsilon \to 0$ with keeping the relations. Under this limit, the 1-forms $\phi_{4;0}$ and ω_F converge to

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \phi_{4;0} = \lim_{\varepsilon \to 0} \frac{-y' \mathrm{d}t}{-\varepsilon y' - 1} = y' \mathrm{d}t = \phi_{0;1},$$
$$\lim_{\varepsilon \to 0} \omega_F = \beta_{1;0}\phi_{1;0} + \beta_{2;0}\phi_{2;0} + \beta_{3;0}\phi_{3;0} + \beta'_{0;1}\phi_{0;1} = \omega_\Phi.$$

The intersection matrix C_F converges to

$$\lim_{\varepsilon \to 0} C_F = \begin{bmatrix} (\frac{1}{\beta_{1;0}} + \frac{1}{\beta_{2;0}}) & \frac{1}{\beta_{2;0}} & \frac{1}{\beta_{2;0}} \\ \frac{1}{\beta_{2;0}} & (\frac{1}{\beta_{2;0}} + \frac{1}{\beta_{3;0}}) & \frac{1}{\beta_{2;0}} \\ \frac{1}{\beta_{2;0}} & \frac{1}{\beta_{2;0}} & \frac{1}{\beta_{2;0}} \end{bmatrix} = C_\Phi.$$

Components of the connection matrix of F_1 converges to

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left\{\frac{W_{1,F}}{x}+\frac{W_{5,F}}{x-y}\right\} \qquad (6.17)\\
&=\lim_{\varepsilon\to 0}\left\{\frac{1}{x}\begin{bmatrix}0 & 0 & 0\\ \beta_{1;0} & (\beta'_{0;0}+\beta_{3;0}-\frac{\beta'_{0;1}}{\varepsilon}) & \frac{\beta'_{0;1}}{\varepsilon}\\ 0 & 0 & 0\end{bmatrix}+\frac{1}{x+\varepsilon y'}\begin{bmatrix}0 & 0 & 0\\ 0 & \frac{\beta'_{0;1}}{\varepsilon} & -\frac{\beta'_{0;1}}{\varepsilon}\\ 0 & -\beta_{3;0} & \beta_{3;0}\end{bmatrix}\right\}\\
&=\frac{1}{x}\begin{bmatrix}0 & 0 & 0\\ \beta_{1;0} & (\beta'_{0;0}+\beta_{3;0}) & 0\\ 0 & -\beta_{3;0} & \beta_{3;0}\end{bmatrix}-\lim_{\varepsilon\to 0}\frac{y'}{x(x+\varepsilon y')}\begin{bmatrix}0 & 0 & 0\\ 0 & \beta'_{0;1} & -\beta'_{0;1}\\ 0 & 0 & 0\end{bmatrix}=\left\{\frac{1}{x}W_{1,\Phi}-\frac{y'}{x^2}W_{5,\Phi}\right\},
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0}(-\varepsilon)\left\{\frac{W_{4,F}}{y-1}+\frac{W_{5,F}}{y-x}\right\} \qquad (6.18)\\
&=\lim_{\varepsilon\to 0}\left\{\frac{\varepsilon}{1+\varepsilon y'}\begin{bmatrix}0 & 0 & \frac{\beta'_{0;1}}{\varepsilon}\\ 0 & 0 & \frac{\beta'_{0;1}}{\varepsilon}\\ 0 & 0 & \beta_{2;0}+\frac{\beta'_{0;1}}{\varepsilon}\end{bmatrix}+\frac{\varepsilon}{x+\varepsilon y'}\begin{bmatrix}0 & 0 & 0\\ 0 & \frac{\beta'_{0;1}}{\varepsilon} & -\frac{\beta'_{0;1}}{\varepsilon}\\ 0 & -\beta_{3;0} & \beta_{3;0}\end{bmatrix}\right\}\\
&=\begin{bmatrix}0 & 0 & \beta'_{0;1}\\ 0 & 0 & \beta'_{0;1}\\ 0 & 0 & \beta'_{0;1}\end{bmatrix}+\frac{1}{x}\begin{bmatrix}0 & 0 & 0\\ 0 & \beta'_{0;1} & -\beta'_{0;1}\\ 0 & 0 & 0\end{bmatrix}=\left\{W_{4,\Phi}+\frac{1}{x}W_{5,\Phi}\right\}.
\end{aligned}$$

These relations mean that the Pfaffian system (6.11) and (6.12) is transformed into the system (6.15) and (6.16). Especially, the decomposition of our connection matrix W is not preserved under this confluence, we need a recomposition of W to make each part of W converge under this confluence.

By changing independent variables, we can make the decompositions of connection matrices preserved under the confluence. We select x and $z := y/x$ as independent variables instead of x and y in the equations (6.11) and (6.12). Then these equations become

$$\mathrm{d}_x\begin{bmatrix}Q_0\\ Q_1\\ Q_2\end{bmatrix}=\left\{\frac{\mathrm{d}x}{x}W'_{1,F}+\frac{\mathrm{d}x}{x-1}W_{2,F}+\frac{z\mathrm{d}x}{xz-1}W_{4,F}\right\}\begin{bmatrix}Q_0\\ Q_1\\ Q_2\end{bmatrix}, \qquad (6.19)$$

$$\mathrm{d}_z\begin{bmatrix}Q_0\\ Q_1\\ Q_2\end{bmatrix}=\left\{\frac{\mathrm{d}z}{z}W_{3,F}+\frac{\mathrm{d}z}{z-1}W_{5,F}+\frac{x\mathrm{d}z}{xz-1}W_{4,F}\right\}\begin{bmatrix}Q_0\\ Q_1\\ Q_2\end{bmatrix}, \qquad (6.20)$$

where matrices W's are given by

$$W'_{1,F} = \begin{bmatrix} 0 & 0 & 0 \\ \beta_{1;0} & \beta_{0;0}+\beta_{3;0}+\beta_{4;0} & 0 \\ \beta_{1;0} & 0 & \beta_{0;0}+\beta_{3;0}+\beta_{4;0} \end{bmatrix},$$

$$W_{2,F} = \begin{bmatrix} 0 & \beta_{3;0} & 0 \\ 0 & \beta_{2;0}+\beta_{3;0} & 0 \\ 0 & \beta_{3;0} & 0 \end{bmatrix}, \quad W_{3,F} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \beta_{1;0} & \beta_{3;0} & \beta_{0;0}+\beta_{4;0} \end{bmatrix},$$

$$W_{4,F} = \begin{bmatrix} 0 & 0 & \beta_{4;0} \\ 0 & 0 & \beta_{4;0} \\ 0 & 0 & \beta_{2;0}+\beta_{4;0} \end{bmatrix}, \quad W_{5,F} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \beta_{4;0} & -\beta_{4;0} \\ 0 & -\beta_{3;0} & \beta_{3;0} \end{bmatrix}.$$

We select x and $z' := y'/x$ as independent variables instead of x and y' in the equations (6.15) and (6.16). By this change of variables, these equations are transformed into

$$\mathrm{d}_x \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix} = \left\{ \frac{\mathrm{d}x}{x} W'_{1,\Phi} + \frac{\mathrm{d}x}{x-1} W_{2,\Phi} + z' \mathrm{d}x W_{4,\Phi} \right\} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}, \tag{6.21}$$

$$\mathrm{d}_{z'} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix} = \left\{ \frac{\mathrm{d}z'}{z'} W_{3,\Phi} + \mathrm{d}z' W_{5,\Phi} + x \mathrm{d}z' W_{4,\Phi} \right\} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}, \tag{6.22}$$

where

$$W'_{1,\Phi} = \begin{bmatrix} 0 & 0 & 0 \\ \beta_{1;0} & \beta'_{0;0}+\beta_{3;0} & 0 \\ \beta_{1;0} & 0 & \beta'_{0;0}+\beta_{3;0} \end{bmatrix}, \quad W_{2,\Phi} = \begin{bmatrix} 0 & \beta_{3;0} & 0 \\ 0 & \beta_{2;0}+\beta_{3;0} & 0 \\ 0 & \beta_{3;0} & 0 \end{bmatrix},$$

$$W_{3,\Phi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \beta_{1;0} & \beta_{3;0} & \beta'_{0;0} \end{bmatrix}, \quad W_{4,\Phi} = \begin{bmatrix} 0 & 0 & \beta'_{0;1} \\ 0 & 0 & \beta'_{0;1} \\ 0 & 0 & \beta'_{0;1} \end{bmatrix}, \quad W_{5,\Phi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \beta'_{0;1} & -\beta'_{0;1} \\ 0 & 0 & 0 \end{bmatrix}.$$

We relate variables and parameters of F_1 to those of Φ_1 as

$$\begin{cases} z = -\varepsilon z', \\ b_2 = \frac{1}{\varepsilon} d_2, \end{cases}$$

and take the limit $\varepsilon \to 0$ with keeping the relations. In this case, we have

$$W'_{1,F} = W'_{1,\Phi}, \quad W_{2,F} = W_{2,\Phi}, \quad W_{3,F} = W_{3,\Phi},$$
$$\lim_{\varepsilon\to 0} \{\varepsilon W_{4,F}\} = W_{4,\Phi}, \quad \lim_{\varepsilon\to 0} \{\varepsilon W_{5,F}\} = W_{5,\Phi}.$$

Consequently, equations (6.19) and (6.20) converge into (6.21) and (6.22) with preserving the decompositions of the connection matrices.

The integrability condition is satisfied from $\mathrm{dd} = 0$. In fact, a connection 1-form W is defined by

$$\mathrm{d} \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{bmatrix} = W \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{bmatrix}$$

and we have

$$0 = \mathrm{dd}\begin{bmatrix}\varphi_0\\ \varphi_1\\ \varphi_2\end{bmatrix} = \mathrm{d}\left(W\begin{bmatrix}\varphi_0\\ \varphi_1\\ \varphi_2\end{bmatrix}\right) = [\mathrm{d}W - W\wedge W]\begin{bmatrix}\varphi_0\\ \varphi_1\\ \varphi_2\end{bmatrix},$$

and the Frobenius condition $\mathrm{d}W = W\wedge W$.

In our case, we can easily check by matrix calculation. We define the bracket product,

$$[A,B] := AB - BA. \tag{6.23}$$

Pfaffian systems of F_1 and Φ_1 satisfy the integrability conditions:

$$\begin{aligned}
&[W_{1,F}, W_{4,F}] = [W_{2,F}, W_{3,F}] = O,\\
&[W_{1,F}, W_{3,F}] = [W_{3,F}, W_{5,F}] = [W_{5,F}, W_{1,F}],\\
&[W_{2,F}, W_{4,F}] = [W_{4,F}, W_{5,F}] = [W_{5,F}, W_{2,F}],\\
&[W_{1,\Phi}, W_{3,\Phi}] = [W_{2,\Phi}, W_{3,\Phi}] = [W_{4,\Phi}, W_{5,\Phi}] = O,\\
&[W_{2,\Phi}, W_{4,\Phi}] = [W_{2,\Phi}, W_{5,\Phi}] = [W_{4,\Phi}, W_{1,\Phi}],\\
&[W_{3,\Phi}, W_{5,\Phi}] = [W_{5,\Phi}, W_{1,\Phi}].
\end{aligned}$$

7 List of Pfaffian Systems (of single variable type)

In this section, we list Pfaffian systems of **GHF** of type λ on $Z_{2,4}$ in Table 7.1. Although equations of Gauss, Kummer and Hermite types are shown already, they are included so that they can be compared with the others.

Table 7.1: Setting of confluent hypergeometric functions in a variable.

Function name	λ	Normal form of z	Normal form of β
Gauss's ${}_2F_1$	(1,1,1,1)	$\begin{bmatrix}1 & 0 & 1 & 1\\ 0 & 1 & -1 & -x\end{bmatrix}$	$(\beta_{0;0}, \beta_{1;0}, \beta_{2;0}, \beta_{3;0})$
Kummer's ${}_1F_1$	(2,1,1)	$\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & x & 1 & -1\end{bmatrix}$	$(\beta_{0;0}, \beta_{0;1}, \beta_{1;0}, \beta_{2;0})$
Bessel's J_a	(2,2)	$\begin{bmatrix}1 & 0 & 0 & -x/2\\ 0 & x/2 & 1 & 0\end{bmatrix}$	$(\beta_{0;0}, \beta_{0;1}, \beta_{1;0}, \beta_{1;1})$
Hermite's H_e	(3,1)	$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & x & 1\end{bmatrix}$	$(\beta_{0;0}, 0, \beta_{0;2}, \beta_{1;0})$
Airy's A_i	(4)	$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & -x\end{bmatrix}$	$(0, 0, 0, \beta_{0;3})$

As a consequence, we have the following.

Theorem 7.1. *Every system satisfies the following.*

1. *Any connection matrix of our Pfaffian system admits a decomposition into two components.*
2. *Each component is the product of a rational 1-form of the variable and a 2×2 matrix whose*

entries are rational functions of parameters.

If basis $\{\varphi_0, \varphi\}$ is selected, the Pfaffian system is given by

$$\mathrm{d}_x \begin{bmatrix} \varphi_0 \\ \varphi_1 \end{bmatrix} = \left\{ \sum_{i=1}^{2} \mathrm{d}_x(f_i) W_i \right\} \begin{bmatrix} \varphi_0 \\ \varphi_1 \end{bmatrix}, \tag{7.1}$$

where d_x is the exterior derivative on the space of variable x, $\mathrm{d}_x(f_i)$ is a rational form of variable x, and entries of W_i depend only on parameters. We introduce a non-zero parameter d so that each entry of C and W_i is homogeneous of degree -1 and 1 in parameter, respectively. We may set $d = \pm 1$, but this kind of homogeneity in parameter is lost.

The exact forms are shown in the following subsections.

7.1. Gauss's ${}_2F_1$ $\lambda = (1,1,1,1)$

variables: $x_0 = \infty$, $x_1 = 0$, $x_2 = 1$, $x_3 = \dfrac{1}{x}$,

parameters: $\beta_{0;0} = b_1$, $\beta_{1;0} = b_2$, $\beta_{2;0} = b_3$, $\beta_{3;0} = b_4$,

1-forms: $\phi_{0;0} = \dfrac{\mathrm{d}s}{s}$, $\phi_{1;0} = \dfrac{\mathrm{d}t}{t}$, $\phi_{2;0} = \dfrac{\mathrm{d}t}{t-1}$, $\phi_{3;0} = \dfrac{x\mathrm{d}t}{xt-1}$,

conditions: $b_1 + b_2 + b_3 + b_4 = 0$, $b_1,\ b_2,\ b_3,\ b_4 \notin \mathbb{Z}$,

$$\begin{cases} \varphi_0 = \phi_{1;0} - \phi_{2;0}, \\ \varphi_1 = \phi_{3;0} - \phi_{2;0}, \end{cases} \qquad \begin{cases} F_0 = \int u_\lambda \varphi_0, \\ F_1 = \int u_\lambda \varphi_1 = \frac{x-1}{b_4} \frac{\mathrm{d}}{\mathrm{d}x} F_0, \end{cases} \qquad C = \begin{bmatrix} (\frac{1}{b_2} + \frac{1}{b_3}) & \frac{1}{b_3} \\ \frac{1}{b_3} & (\frac{1}{b_3} + \frac{1}{b_4}) \end{bmatrix},$$

$f_1 = \log(x)$, $f_2 = \log(x-1)$,

$$W_1 = \begin{bmatrix} 0 & 0 \\ b_2 & b_1 + b_4 \end{bmatrix}, \qquad W_2 = \begin{bmatrix} 0 & b_4 \\ 0 & b_3 + b_4 \end{bmatrix}.$$

7.2. Kummer's ${}_1F_1$ $\lambda = (2,1,1)$

variables: $x_0 = \infty$, $h_{0;1} = x$, $x_1 = 0$, $x_2 = 1$,

parameters: $\beta_{0;0} = b_1$, $\beta_{0;1} = d$, $\beta_{1;0} = b_2$, $\beta_{2;0} = b_3$,

1-forms: $\phi_{0;0} = \dfrac{\mathrm{d}s}{s}$, $\phi_{0;1} = -x\dfrac{\mathrm{d}s}{s^2}$, $\phi_{1;0} = \dfrac{\mathrm{d}t}{t}$, $\phi_{2;0} = \dfrac{\mathrm{d}t}{t-1}$,

conditions: $b_1 + b_2 + b_3 = 0$, $b_2,\ b_3 \notin \mathbb{Z}$, $d \neq 0$,

$$\begin{cases} \varphi_0 = \phi_{1;0} - \phi_{2;0}, \\ \varphi_1 = -\phi_{2;0}, \end{cases} \qquad \begin{cases} F_0 = \int u_\lambda \varphi_0, \\ F_1 = \int u_\lambda \varphi_1 = \frac{1}{d} \frac{\mathrm{d}}{\mathrm{d}x} F_0, \end{cases} \qquad C = \begin{bmatrix} (\frac{1}{b_2} + \frac{1}{b_3}) & \frac{1}{b_3} \\ \frac{1}{b_3} & \frac{1}{b_3} \end{bmatrix},$$

$f_1 = \log(x)$, $f_2 = x$,

$$W_1 = \begin{bmatrix} 0 & 0 \\ b_2 & b_1 \end{bmatrix}, \qquad W_2 = \begin{bmatrix} 0 & d \\ 0 & d \end{bmatrix}.$$

7.3. Bessel's J_a $\lambda = (2,2)$

variables: $x_0 = \infty$, $h_{0;1} = \dfrac{x}{2}$, $x_1 = 0$, $h_{1;1} = -\dfrac{x}{2}$,

parameters: $\beta_{0;0} = b_1$, $\beta_{0;1} = d$, $\beta_{1;0} = b_2$, $\beta_{1;1} = d$,

1-forms: $\phi_{0;0} = \dfrac{\mathrm{d}s}{s}$, $\phi_{0;1} = -\dfrac{x}{2}\dfrac{\mathrm{d}s}{s^2}$, $\phi_{1;0} = \dfrac{\mathrm{d}t}{t}$, $\phi_{1;1} = \dfrac{x}{2}\dfrac{\mathrm{d}t}{t^2}$,

conditions: $b_1 + b_2 = 0$, $d \neq 0$,

$$\begin{cases} \varphi_0 = \phi_{1;0}, \\ \varphi_1 = \phi_{0;1} - \phi_{1;1}, \end{cases} \quad \begin{cases} F_0 = \int u_\lambda \varphi_0, \\ F_1 = \int u_\lambda \varphi_1 = \frac{x}{d}\frac{\mathrm{d}}{\mathrm{d}x} F_0, \end{cases} \quad C = \begin{bmatrix} 0 & -\frac{2}{d} \\ -\frac{2}{d} & 0 \end{bmatrix},$$

$f_1 = \log(x)$, $f_2 = \dfrac{x^2}{2}$,

$$W_1 = \begin{bmatrix} 0 & d \\ \frac{b_2^2}{d} & 0 \end{bmatrix}, \qquad W_2 = \begin{bmatrix} 0 & 0 \\ -d & 0 \end{bmatrix}.$$

7.4. Hermite's H_e $\lambda = (3,1)$

variables: $x_0 = \infty$, $h_{0;1} = 1$, $h_{0;2} = x$, $x_1 = 0$,

parameters: $\beta_{0;0} = b_1$, $\beta_{0;1} = 0$, $\beta_{0;2} = d$, $\beta_{1;0} = b_2$,

1-forms: $\phi_{0;0} = \dfrac{\mathrm{d}s}{s}$, $\phi_{0;1} = -\dfrac{\mathrm{d}s}{s^2}$, $\phi_{0;2} = (\dfrac{1}{s^3} - \dfrac{x}{s^2})\mathrm{d}s$, $\phi_{1;0} = \dfrac{\mathrm{d}t}{t}$,

conditions: $b_1 + b_2 = 0$, $b_2 \notin \mathbb{Z}$, $d \neq 0$,

$$\begin{cases} \varphi_0 = \phi_{1;0}, \\ \varphi_1 = \phi_{0;1}, \end{cases} \quad \begin{cases} F_0 = \int u_\lambda \varphi_0, \\ F_1 = \int u_\lambda \varphi_1 = \frac{1}{d}\frac{\mathrm{d}}{\mathrm{d}x} F_0, \end{cases} \quad C = \begin{bmatrix} \frac{1}{b_2} & 0 \\ 0 & \frac{1}{d} \end{bmatrix},$$

$f_1 = x$, $f_2 = \dfrac{x^2}{2}$,

$$W_1 = \begin{bmatrix} 0 & d \\ b_2 & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}.$$

7.5. Airy's A_i $\lambda = (4)$

variables: $x_0 = \infty$, $h_{0;1} = 1$, $h_{0;2} = 0$, $h_{0;3} = -x$,

parameters: $\beta_{0;0} = 0$, $\beta_{0;1} = 0$, $\beta_{0;2} = 0$, $\beta_{0;3} = d$,

1-forms: $\phi_{0;0} = -\dfrac{\mathrm{d}t}{t}$, $\phi_{0;1} = \mathrm{d}t$, $\phi_{0;2} = -t\mathrm{d}t$, $\phi_{0;3} = (t^2 - x)\mathrm{d}t$,

condition: $d \neq 0$,

$$\begin{cases} \varphi_0 = \phi_{0;1}, \\ \varphi_1 = \phi_{0;2}, \end{cases} \quad \begin{cases} F_0 = \int u_\lambda \varphi_0, \\ F_1 = \int u_\lambda \varphi_1 = \frac{1}{d}\frac{\mathrm{d}}{\mathrm{d}x} F_0, \end{cases} \quad C = \begin{bmatrix} 0 & \frac{1}{d} \\ \frac{1}{d} & 0 \end{bmatrix},$$

$f_1 = x$, $f_2 = \dfrac{x^2}{2}$,

$$W_1 = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 & 0 \\ d & 0 \end{bmatrix}.$$

8 List of Pfaffian Systems (of two variable type)

In this section, we list Pfaffian systems of the general hypergeometric functions of type λ on $Z_{2,5}$ in Table 8.1. Although systems of F_1 and Φ_1 have been shown already, they are included so that they can be compared with the others.

As a consequence, we have the following.

Theorem 8.1. *Every system satisfies the following.*

Table 8.1: Setting of confluent hypergeometric functions in two variables.

Function name	λ	Normal form of z	Normal form of β
Appell's F_1	(1,1,1,1,1)	$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & -x & -y \end{bmatrix}$	$(\beta_{0;0}, \beta_{1;0}, \beta_{2;0}, \beta_{3;0}, \beta_{4;0})$
Hörn's G_2		$\begin{bmatrix} 1 & 0 & 1 & x & 1 \\ 0 & 1 & -1 & 1 & y \end{bmatrix}$	$(\beta_{0;0}, \beta_{1;0}, \beta_{2;0}, \beta_{3;0}, \beta_{4;0})$
Humbert's Φ_1	(2,1,1,1)	$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & y & 1 & -1 & -x \end{bmatrix}$	$(\beta_{0;0}, \beta_{0;1}, \beta_{1;0}, \beta_{2;0}, \beta_{3;0})$
Humbert's Φ_2		$\begin{bmatrix} 1 & 0 & 0 & x & y \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(\beta_{0;0}, \beta_{0;1}, \beta_{1;0}, \beta_{2;0}, \beta_{3;0})$
Hörn's Γ_1		$\begin{bmatrix} 1 & 0 & 0 & 1 & x \\ 0 & y & 1 & -1 & 1 \end{bmatrix}$	$(\beta_{0;0}, \beta_{0;1}, \beta_{1;0}, \beta_{2;0}, \beta_{3;0})$
Humbert's Φ_3	(2,2,1)	$\begin{bmatrix} 1 & 0 & 0 & y & x \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$	$(\beta_{0;0}, \beta_{0;1}, \beta_{1;0}, \beta_{1;1}, \beta_{2;0})$
Hörn's Γ_2		$\begin{bmatrix} 1 & 0 & 0 & x & 1 \\ 0 & y & 1 & 0 & -1 \end{bmatrix}$	$(\beta_{0;0}, \beta_{0;1}, \beta_{1;0}, \beta_{1;1}, \beta_{2;0})$
[KK] $\Phi_{(3,1,1)}$	(3,1,1)	$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & x & y & 1 & -1 \end{bmatrix}$	$(\beta_{0;0}, 0, \beta_{0;2}, \beta_{1;0}, \beta_{2;0})$
[KK] $\Phi_{(3,2)}$	(3,2)	$\begin{bmatrix} 1 & 0 & 0 & 0 & y \\ 0 & 1 & x & 1 & 0 \end{bmatrix}$	$(\beta_{0;0}, 0, \beta_{0;2}, \beta_{1;0}, \beta_{1;1})$
[KK] $\Phi_{(4,1)}$	(4,1)	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & x & y & 1 \end{bmatrix}$	$(\beta_{0;0}, 0, 0, \beta_{0;3}, \beta_{1;0})$
[KK] $\Phi_{(5)}$	(5)	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & x & y \end{bmatrix}$	$(0, 0, 0, 0, \beta_{0;4})$

1. *Any connection matrix of our Pfaffian system admits a decomposition into five components.*
2. *Each component is the product of a rational 1-form of variables and a 3×3 matrix whose entries depend only on parameters.*
3. *Any Pfaffian system satisfies the integrability condition.*

If basis $\{\varphi_0, \varphi_1, \varphi_2\}$ is selected, the Pfaffian system is express as

$$\mathrm{d} \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{bmatrix} = \left\{ \sum_{i=1}^{5} \mathrm{d}(f_i) W_i \right\} \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{bmatrix}, \tag{8.1}$$

where d *is the exterior derivative on the space of variables x and y, $\mathrm{d}(f_i)$ is a rational 1-form of x and y, and W_i is a 3×3 matrix, whose entries depend only on parameters. We introduce non-zero parameters d_j so that each entry of C and W_i is homogeneous of degree -1 and 1 in parameter, respectively. We may set $d_j = \pm 1$, but this kind of homogeneity in parameter is lost.*

The exact forms are shown in the following subsections.

8.1. Appell's F_1 $\lambda=(1,1,1,1,1)$

variables: $x_0=\infty,\quad x_1=0,\quad x_2=1,\quad x_3=\dfrac{1}{x},\quad x_4=\dfrac{1}{y},$

parameters: $\beta_{0;0}=b_1,\quad \beta_{1;0}=b_2,\quad \beta_{2;0}=b_3,\quad \beta_{3;0}=b_4,\quad \beta_{4;0}=b_5,$

1-forms: $\phi_{0;0}=\dfrac{\mathrm{d}s}{s},\quad \phi_{1;0}=\dfrac{\mathrm{d}t}{t},\quad \phi_{2;0}=\dfrac{\mathrm{d}t}{t-1},\quad \phi_{3;0}=\dfrac{x\mathrm{d}t}{xt-1},\quad \phi_{4;0}=\dfrac{y\mathrm{d}t}{yt-1},$

conditions: $b_1+b_2+b_3+b_4+b_5=0,\quad b_j\notin\mathbb{Z}\,(1\le j\le 5),$

$$\begin{cases}\varphi_0=\phi_{1;0}-\phi_{2;0},\\ \varphi_1=\phi_{3;0}-\phi_{2;0},\\ \varphi_2=\phi_{4;0}-\phi_{2;0},\end{cases}\quad \begin{cases}F_0=\int u_\lambda\varphi_0,\\ F_1=\int u_\lambda\varphi_1=\frac{x-1}{b_4}\frac{\partial}{\partial x}F_0,\\ F_2=\int u_\lambda\varphi_2=\frac{y-1}{b_5}\frac{\partial}{\partial y}F_0,\end{cases}\quad C=\begin{bmatrix}(\frac{1}{b_2}+\frac{1}{b_3}) & \frac{1}{b_3} & \frac{1}{b_3}\\ \frac{1}{b_3} & (\frac{1}{b_3}+\frac{1}{b_4}) & \frac{1}{b_3}\\ \frac{1}{b_3} & \frac{1}{b_3} & (\frac{1}{b_3}+\frac{1}{b_5})\end{bmatrix},$$

f_i:	$\log(x)$	$\log(x-1)$	$\log(y)$	$\log(y-1)$	$\log(x-y)$
W_i:	$\begin{bmatrix}0&0&0\\ b_2&b_1+b_4&b_5\\ 0&0&0\end{bmatrix}$	$\begin{bmatrix}0&b_4&0\\ 0&b_3+b_4&0\\ 0&b_4&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\ 0&0&0\\ b_2&b_4&b_1+b_5\end{bmatrix}$	$\begin{bmatrix}0&0&b_5\\ 0&0&b_5\\ 0&0&b_3+b_5\end{bmatrix}$	$\begin{bmatrix}0&0&0\\ 0&b_5&-b_5\\ 0&-b_4&b_4\end{bmatrix}$

8.2. Hörn's G_2 $\lambda=(1,1,1,1,1)$

variables: $x_0=\infty,\quad x_1=0,\quad x_2=1,\quad x_3=-x,\quad x_4=-\dfrac{1}{y},$

parameters: $\beta_{0;0}=b_1,\quad \beta_{1;0}=b_2,\quad \beta_{2;0}=b_3,\quad \beta_{3;0}=b_4,\quad \beta_{4;0}=b_5,$

1-forms: $\phi_{0;0}=\dfrac{\mathrm{d}s}{s},\quad \phi_{1;0}=\dfrac{\mathrm{d}t}{t},\quad \phi_{2;0}=\dfrac{\mathrm{d}t}{t-1},\quad \phi_{3;0}=\dfrac{\mathrm{d}t}{t+x},\quad \phi_{4;0}=\dfrac{y\mathrm{d}t}{yt+1},$

conditions: $b_1+b_2+b_3+b_4+b_5=0,\quad b_j\notin\mathbb{Z}\,(1\le j\le 5),$

$$\begin{cases}\varphi_0=\phi_{1;0},\\ \varphi_1=\phi_{1;0}-\phi_{3;0},\\ \varphi_2=\phi_{4;0},\end{cases}\quad \begin{cases}F_0=\int u_\lambda\varphi_0,\\ F_1=\int u_\lambda\varphi_1=\frac{x}{b_4}\frac{\partial}{\partial x}F_0,\\ F_2=\int u_\lambda\varphi_2=\frac{y}{b_5}\frac{\partial}{\partial y}F_0,\end{cases}\quad C=\begin{bmatrix}(\frac{1}{b_1}+\frac{1}{b_2}) & \frac{1}{b_2} & \frac{1}{b_1}\\ \frac{1}{b_2} & (\frac{1}{b_2}+\frac{1}{b_4}) & 0\\ \frac{1}{b_1} & 0 & (\frac{1}{b_1}+\frac{1}{b_5})\end{bmatrix},$$

f_i:	$\log(x)$	$\log(x+1)$	$\log(y)$	$\log(y+1)$	$\log(xy-1)$
	$\begin{bmatrix}0&b_4&0\\ 0&b_2+b_4&0\\ 0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\ b_1+b_5&b_3+b_4&-b_5\\ 0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&b_5\\ 0&0&0\\ 0&0&b_1+b_5\end{bmatrix}$	$\begin{bmatrix}0&0&0\\ 0&0&0\\ b_2+b_4&-b_4&b_3+b_5\end{bmatrix}$	$\begin{bmatrix}0&0&0\\ -b_5&b_5&b_5\\ -b_4&b_4&b_4\end{bmatrix}$

8.3. Humbert's Φ_1 $\lambda=(2,1,1,1)$

variables: $x_0=\infty,\quad h_{0;1}=y,\quad x_1=0,\quad x_2=1,\quad x_3=\dfrac{1}{x},$

parameters: $\beta_{0;0}=b_1,\quad \beta_{0;1}=d_5,\quad \beta_{1;0}=b_2,\quad \beta_{2;0}=b_3,\quad \beta_{3;0}=b_4,$

1-forms: $\phi_{0;0}=\dfrac{\mathrm{d}s}{s},\quad \phi_{0;1}=-y\dfrac{\mathrm{d}s}{s^2},\quad \phi_{1;0}=\dfrac{\mathrm{d}t}{t},\quad \phi_{2;0}=\dfrac{\mathrm{d}t}{t-1},\quad \phi_{3;0}=\dfrac{x\mathrm{d}t}{xt-1},$

conditions: $b_1+b_2+b_3+b_4=0,\quad b_j\notin\mathbb{Z}\,(2\le j\le 4),\quad d_5\ne 0,$

$$\begin{cases}\varphi_0=\phi_{1;0}-\phi_{2;0},\\ \varphi_1=\phi_{3;0}-\phi_{2;0},\\ \varphi_2=-\phi_{2;0},\end{cases}\quad \begin{cases}F_0=\int u_\lambda\varphi_0,\\ F_1=\int u_\lambda\varphi_1=\frac{x-1}{b_4}\frac{\partial}{\partial x}F_0,\\ F_2=\int u_\lambda\varphi_2=\frac{1}{d_5}\frac{\partial}{\partial y}F_0,\end{cases}\quad C=\begin{bmatrix}(\frac{1}{b_2}+\frac{1}{b_3}) & \frac{1}{b_3} & \frac{1}{b_3}\\ \frac{1}{b_3} & (\frac{1}{b_3}+\frac{1}{b_4}) & \frac{1}{b_3}\\ \frac{1}{b_3} & \frac{1}{b_3} & \frac{1}{b_3}\end{bmatrix},$$

f_i:	$\log(x)$	$\log(x-1)$	$\log(y)$	y	$\dfrac{y}{x}$
W_i:	$\begin{bmatrix}0&0&0\\ b_2&b_1+b_4&0\\ 0&-b_4&b_4\end{bmatrix}$	$\begin{bmatrix}0&b_4&0\\ 0&b_3+b_4&0\\ 0&b_4&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\ 0&0&0\\ b_2&b_4&b_1\end{bmatrix}$	$\begin{bmatrix}0&0&d_5\\ 0&0&d_5\\ 0&0&d_5\end{bmatrix}$	$\begin{bmatrix}0&0&0\\ 0&d_5&-d_5\\ 0&0&0\end{bmatrix}$

8.4. Humbert's Φ_2 $\lambda = (2,1,1,1)$

variables: $x_0 = \infty$, $h_{0;1} = 1$, $x_1 = 0$, $x_2 = -x$, $x_3 = -y$,

parameters: $\beta_{0;0} = b_2$, $\beta_{0;1} = d_3$, $\beta_{1;0} = b_1$, $\beta_{2;0} = b_4$, $\beta_{3;0} = b_5$,

1-forms: $\phi_{0;0} = \dfrac{\mathrm{d}s}{s}$, $\phi_{0;1} = -\dfrac{\mathrm{d}s}{s^2}$, $\phi_{1;0} = \dfrac{\mathrm{d}t}{t}$, $\phi_{2;0} = \dfrac{\mathrm{d}t}{t+x}$, $\phi_{3;0} = \dfrac{\mathrm{d}t}{t+y}$,

conditions: $b_1 + b_2 + b_4 + b_5 = 0$, $b_1, b_4, b_5 \notin \mathbb{Z}$, $d_3 \neq 0$,

$$\begin{cases} \varphi_0 = \phi_{0;1}, \\ \varphi_1 = \phi_{2;0}, \\ \varphi_2 = \phi_{3;0}, \end{cases} \quad \begin{cases} F_0 = \int u_\lambda \varphi_0, \\ F_1 = \int u_\lambda \varphi_1 = \frac{1}{b_4}\frac{\partial}{\partial x} F_0, \\ F_2 = \int u_\lambda \varphi_2 = \frac{1}{b_5}\frac{\partial}{\partial y} F_0, \end{cases} \quad C = \begin{bmatrix} -\frac{b_2}{b_3^2} & -\frac{1}{d_3} & -\frac{1}{d_3} \\ -\frac{1}{d_3} & \frac{1}{b_4} & 0 \\ -\frac{1}{d_3} & 0 & \frac{1}{b_5} \end{bmatrix},$$

f_i:	$\log(x)$	x	$\log(y)$	y	$\log(x-y)$
W_i:	$\begin{bmatrix} 0 & 0 & 0 \\ d_3 & b_1+b_4 & b_5 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & b_4 & 0 \\ 0 & -d_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_3 & b_4 & b_1+b_5 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & b_5 \\ 0 & 0 & 0 \\ 0 & 0 & -d_3 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & b_5 & -b_5 \\ 0 & -b_4 & b_4 \end{bmatrix}$

8.5. Hörn's Γ_1 $\lambda = (2,1,1,1)$

variables: $x_0 = \infty$, $h_{0;1} = y$, $x_1 = 0$, $x_2 = 1$, $x_3 = -x$,

parameters: $\beta_{0;0} = b_1$, $\beta_{0;1} = d_5$, $\beta_{1;0} = b_2$, $\beta_{2;0} = b_3$, $\beta_{3;0} = b_4$,

1-forms: $\phi_{0;0} = \dfrac{\mathrm{d}s}{s}$, $\phi_{0;1} = -y\dfrac{\mathrm{d}s}{s^2}$, $\phi_{1;0} = \dfrac{\mathrm{d}t}{t}$, $\phi_{2;0} = \dfrac{\mathrm{d}t}{t-1}$, $\phi_{3;0} = \dfrac{\mathrm{d}t}{t+x}$,

conditions: $b_1 + b_2 + b_3 + b_4 = 0$, $b_2, b_3, b_4 \notin \mathbb{Z}$, $d_5 \neq 0$,

$$\begin{cases} \varphi_0 = \phi_{1;0}, \\ \varphi_1 = \phi_{1;0} - \phi_{3;0}, \\ \varphi_2 = \phi_{0;1}, \end{cases} \quad \begin{cases} F_0 = \int u_\lambda \varphi_0, \\ F_1 = \int u_\lambda \varphi_1 = \frac{x}{b_4}\frac{\partial}{\partial x} F_0, \\ F_2 = \int u_\lambda \varphi_2 = \frac{y}{d_5}\frac{\partial}{\partial y} F_0, \end{cases} \quad C = \begin{bmatrix} \frac{1}{b_2} & \frac{1}{b_2} & -\frac{1}{d_5} \\ \frac{1}{b_2} & (\frac{1}{b_2} + \frac{1}{b_4}) & 0 \\ -\frac{1}{d_5} & 0 & -\frac{b_1}{b_5^2} \end{bmatrix},$$

f_i:	$\log(x)$	$\log(x+1)$	$\log(y)$	y	xy
W_i:	$\begin{bmatrix} 0 & b_4 & 0 \\ 0 & b_2+b_4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ b_1 & b_3+b_4 & -d_5 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & d_5 \\ 0 & 0 & 0 \\ 0 & 0 & b_1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_2+b_4 & -b_4 & d_5 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ d_5 & -d_5 & 0 \\ b_4 & -b_4 & 0 \end{bmatrix}$

8.6. Humbert's Φ_3 $\lambda = (2,2,1)$

variables: $x_0 = \infty$, $h_{0;1} = 1$, $x_1 = 0$, $h_{1;1} = y$, $x_2 = -x$,

parameters: $\beta_{0;0} = b_2$, $\beta_{0;1} = d_3$, $\beta_{1;0} = b_1$, $\beta_{1;1} = d_5$, $\beta_{2;0} = b_4$,

1-forms: $\phi_{0;0} = \dfrac{\mathrm{d}s}{s}$, $\phi_{0;1} = -\dfrac{\mathrm{d}s}{s^2}$, $\phi_{1;0} = \dfrac{\mathrm{d}t}{t}$, $\phi_{1;1} = -y\dfrac{\mathrm{d}t}{t^2}$, $\phi_{2;0} = \dfrac{\mathrm{d}t}{t+x}$,

conditions: $b_1 + b_2 + b_4 = 0$, $b_4 \notin \mathbb{Z}$, $d_3 \neq 0$, $d_5 \neq 0$,

$$\begin{cases} \varphi_0 = \phi_{0;1}, \\ \varphi_1 = \phi_{2;0}, \\ \varphi_2 = \phi_{1;0}, \end{cases} \quad \begin{cases} F_0 = \int u_\lambda \varphi_0, \\ F_1 = \int u_\lambda \varphi_1 = \frac{1}{b_4}\frac{\partial}{\partial x} F_0, \\ F_2 = \int u_\lambda \varphi_2 = \frac{1}{d_5}\frac{\partial}{\partial y} F_0, \end{cases} \quad C = \begin{bmatrix} -\frac{b_2}{d_3^2} & -\frac{1}{d_3} & -\frac{1}{d_3} \\ -\frac{1}{d_3} & \frac{1}{b_4} & 0 \\ -\frac{1}{d_3} & 0 & 0 \end{bmatrix},$$

f_i:	$\log(x)$	x	$\log(y)$	y	$\dfrac{y}{x}$
W_i:	$\begin{bmatrix} 0 & 0 & 0 \\ d_3 & b_1+b_4 & 0 \\ 0 & -b_4 & b_4 \end{bmatrix}$	$\begin{bmatrix} 0 & b_4 & 0 \\ 0 & -d_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_3 & b_4 & b_1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & d_5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -d_5 & d_5 \\ 0 & 0 & 0 \end{bmatrix}$

8.7. Hörn's Γ_2 $\lambda = (2,2,1)$

variables: $x_0 = \infty$, $h_{0;1} = y$, $x_1 = 0$, $h_{1;1} = x$, $x_2 = 1$,

parameters: $\beta_{0;0} = b_1$, $\beta_{0;1} = d_5$, $\beta_{1;0} = b_2$, $\beta_{1;1} = d_4$, $\beta_{2;0} = b_3$,

1-forms: $\phi_{0;0} = \dfrac{\mathrm{d}s}{s}$, $\phi_{0;1} = -y\dfrac{\mathrm{d}s}{s^2}$, $\phi_{1;0} = \dfrac{\mathrm{d}t}{t}$, $\phi_{1;1} = -x\dfrac{\mathrm{d}t}{t^2}$, $\phi_{2;0} = \dfrac{\mathrm{d}t}{t-1}$,

conditions: $b_1 + b_2 + b_3 = 0$, $b_3 \notin \mathbb{Z}$, $d_4 \neq 0$, $d_5 \neq 0$,

$$\begin{cases} \varphi_0 = \phi_{1;0}, \\ \varphi_1 = -\phi_{1;1}, \\ \varphi_2 = \phi_{0;1}, \end{cases} \quad \begin{cases} F_0 = \int u_\lambda \varphi_0, \\ F_1 = \int u_\lambda \varphi_1 = \frac{x}{d_4}\frac{\partial}{\partial x}F_0, \\ F_2 = \int u_\lambda \varphi_2 = \frac{y}{d_5}\frac{\partial}{\partial y}F_0, \end{cases} \quad C = \begin{bmatrix} 0 & -\frac{1}{d_4} & -\frac{1}{d_5} \\ -\frac{1}{d_4} & -\frac{b_2}{b_4^2} & 0 \\ -\frac{1}{d_5} & 0 & -\frac{b_1}{d_5^2} \end{bmatrix},$$

f_i:	$\log(x)$	x	$\log(y)$	y	xy
W_i:	$\begin{bmatrix} 0 & d_4 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ b_1 & d_4 & -d_5 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & d_5 \\ 0 & 0 & 0 \\ 0 & 0 & b_1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_2 & -d_4 & d_5 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ d_5 & 0 & 0 \\ d_4 & 0 & 0 \end{bmatrix}$

8.8. Kimura's $\Phi_{(3,1,1)}$ $\lambda = (3,1,1)$

variables: $x_0 = \infty$, $h_{0;1} = x$, $h_{0;2} = y$, $x_1 = 0$, $x_2 = 1$,

parameters: $\beta_{0;0} = b_1$, $\beta_{0;1} = 0$, $\beta_{0;2} = d_5$, $\beta_{1;0} = b_2$, $\beta_{2;0} = b_3$,

1-forms: $\phi_{0;0} = \dfrac{\mathrm{d}s}{s}$, $\phi_{0;1} = -x\dfrac{\mathrm{d}s}{s^2}$, $\phi_{0;2} = (\dfrac{x^2}{s^3} - \dfrac{y}{s^2})\mathrm{d}s$, $\phi_{1;0} = \dfrac{\mathrm{d}t}{t}$, $\phi_{2;0} = \dfrac{\mathrm{d}t}{t-1}$,

conditions: $b_1 + b_2 + b_3 = 0$, $b_2,\ b_3 \notin \mathbb{Z}$, $d_5 \neq 0$,

$$\begin{cases} \varphi_0 = \phi_{1;0}, \\ \varphi_1 = \phi_{1;0} - \phi_{2;0}, \\ \varphi_2 = \phi_{0;1}, \end{cases} \quad \begin{cases} F_0 = \int u_\lambda \varphi_0, \\ F_1 = \int u_\lambda \varphi_1 = (x\frac{\partial}{\partial x}F_0 + y\frac{\partial}{\partial y}F_0 - b_1 F_0)/b_3, \\ F_2 = \int u_\lambda \varphi_2 = \frac{x}{d_5}\frac{\partial}{\partial y}F_0, \end{cases} \quad C = \begin{bmatrix} \frac{1}{b_2} & \frac{1}{b_2} & 0 \\ \frac{1}{b_2} & (\frac{1}{b_2} + \frac{1}{b_3}) & 0 \\ 0 & 0 & \frac{1}{d_5} \end{bmatrix},$$

f_i:	$\log(x)$	x	$\dfrac{y}{x}$	$\dfrac{1}{2}\left(\dfrac{y}{x}\right)^2$	$(\dfrac{x^2}{2} - y)$
W_i:	$\begin{bmatrix} b_1 & b_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & d_5 \\ -b_3 & b_3 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & d_5 \\ 0 & 0 & 0 \\ -b_1 & -b_3 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d_5 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ d_5 & -d_5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

8.9. Kimura's $\Phi_{(3,2)}$ $\lambda = (3,2)$

variables: $x_0 = \infty$, $h_{0;1} = 1$, $h_{0;2} = x$, $x_1 = 0$, $h_{1;1} = y$,

parameters: $\beta_{0;0} = b_2$, $\beta_{0;1} = 0$, $\beta_{0;2} = d_4$, $\beta_{1;0} = b_1$, $\beta_{1;1} = d_5$,

1-forms: $\phi_{0;0} = \dfrac{\mathrm{d}s}{s}$, $\phi_{0;1} = -\dfrac{\mathrm{d}s}{s^2}$, $\phi_{0;2} = (\dfrac{1}{s^3} - \dfrac{x}{s^2})\mathrm{d}s$, $\phi_{1;0} = \dfrac{\mathrm{d}t}{t}$, $\phi_{1;1} = -y\dfrac{\mathrm{d}t}{t^2}$,

conditions: $b_1 + b_2 = 0$, $d_4 \neq 0$, $d_5 \neq 0$,

$$\begin{cases} \varphi_0 = \phi_{1;0}, \\ \varphi_1 = \phi_{0;1}, \\ \varphi_2 = -\phi_{1;1}, \end{cases} \quad \begin{cases} F_0 = \int u_\lambda \varphi_0, \\ F_1 = \int u_\lambda \varphi_1 = \frac{1}{d_4}\frac{\partial}{\partial x}F_0, \\ F_2 = \int u_\lambda \varphi_2 = \frac{y}{d_5}\frac{\partial}{\partial y}F_0, \end{cases} \quad C = \begin{bmatrix} 0 & 0 & -\frac{1}{d_5} \\ 0 & \frac{1}{d_4} & 0 \\ -\frac{1}{d_5} & 0 & -\frac{b_1}{d_5^2} \end{bmatrix},$$

f_i:	x	$\dfrac{x^2}{2}$	$\log(y)$	y	xy
W_i:	$\begin{bmatrix} 0 & d_4 & 0 \\ b_1 & 0 & -d_5 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & d_4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & d_5 \\ 0 & 0 & 0 \\ 0 & 0 & b_1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ d_5 & 0 & 0 \\ 0 & -d_4 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ d_4 & 0 & 0 \end{bmatrix}$

8.10. **Kimura's** $\Phi_{(4,1)}$ $\lambda=(4,1)$

variables: $x_0=\infty$, $h_{0;1}=1$, $h_{0;2}=x$, $h_{0;3}=y$, $x_1=0$,

parameters: $\beta_{0;0}=b_1$, $\beta_{0;1}=0$, $\beta_{0;2}=0$, $\beta_{0;3}=d_5$, $\beta_{1;0}=-b_1$,

1-forms: $\phi_{0;0}=-\dfrac{\mathrm{d}t}{t}$, $\phi_{0;1}=\mathrm{d}t$, $\phi_{0;2}=(x-t)\mathrm{d}t$, $\phi_{0;3}=(y-2xt+t^2)\mathrm{d}t$, $\phi_{1;0}=\dfrac{\mathrm{d}t}{t}$,

conditions: $b_1\notin\mathbb{Z}$, $d_5\neq 0$,

$$\begin{cases}\varphi_0=\phi_{1;0},\\ \varphi_1=\phi_{0;1},\\ \varphi_2=\phi_{0;2},\end{cases}\quad\begin{cases}F_0=\int u_\lambda\varphi_0,\\ F_1=\int u_\lambda\varphi_1=\frac{1}{d_5}\frac{\partial}{\partial y}F_0,\\ F_2=\int u_\lambda\varphi_2=(\frac{\partial}{\partial x}F_0+x\frac{\partial}{\partial y}F_0)/d_5,\end{cases}\quad C=\begin{bmatrix}-\frac{1}{b_1}&0&0\\0&0&\frac{1}{d_5}\\0&\frac{1}{d_5}&0\end{bmatrix},$$

f_i:	x	$\dfrac{x^2}{2}-y$	x^2-y	$xy-\dfrac{2}{3}x^3$	$\dfrac{1}{2}(x^2-y)^2$
W_i:	$\begin{bmatrix}0&0&d_5\\-b_1&0&0\\0&b_1&0\end{bmatrix}$	$\begin{bmatrix}0&-d_5&0\\0&0&0\\b_1&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&d_5\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&d_5&0\\0&0&d_5\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&d_5&0\end{bmatrix}$

8.11. **Kimura's** $\Phi_{(5)}$ $\lambda=(5)$

variables: $x_0=\infty$, $h_{0;1}=1$, $h_{0;2}=0$, $h_{0;3}=x$, $h_{0;4}=y$,

parameters: $\beta_{0;0}=0$, $\beta_{0;1}=0$, $\beta_{0;2}=0$, $\beta_{0;3}=0$, $\beta_{0;4}=d_5$,

1-forms: $\phi_{0;0}=-\dfrac{\mathrm{d}t}{t}$, $\phi_{0;1}=\mathrm{d}t$, $\phi_{0;2}=-t\mathrm{d}t$, $\phi_{0;3}=(x+t^2)\mathrm{d}t$, $\phi_{0;4}=(y-2xt-t^3)\mathrm{d}t$,

condition: $d_5\neq 0$,

$$\begin{cases}\varphi_0=\phi_{0;1},\\ \varphi_1=\phi_{0;2},\\ \varphi_2=\phi_{0;3},\end{cases}\quad\begin{cases}F_0=\int u_\lambda\varphi_0,\\ F_1=\int u_\lambda\varphi_1=-\frac{1}{d_5}\frac{\partial}{\partial y}F_0,\\ F_2=\int u_\lambda\varphi_2=xF_0-\frac{1}{d_5}\frac{\partial}{\partial x}F_0,\end{cases}\quad C=\frac{1}{d_5}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix},$$

f_i:	x	$\dfrac{x^2}{2}$	y	xy	$(\dfrac{y^2}{2}-\dfrac{x^3}{3})$
W_i:	$\begin{bmatrix}0&0&-d_5\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}d_5&0&0\\0&2d_5&0\\0&0&d_5\end{bmatrix}$	$\begin{bmatrix}0&-d_5&0\\0&0&-d_5\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\d_5&0&0\\0&d_5&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\d_5&0&0\end{bmatrix}$

9 Pfaffian system of Appell's F_2

Appell's F_2 hypergeometric series is defined by

$$F_2(a,b_1,b_2,c_1,c_2;x,y)=\sum_{m_1,m_2=0}^{\infty}\frac{(a)_{m_1+m_2}(b_1)_{m_1}(b_2)_{m_2}}{(c_1)_{m_1}(c_2)_{m_2}(1)_{m_1}(1)_{m_2}}x^{m_1}y^{m_2},\tag{9.1}$$

where x,y are complex variables, a,b_1,b_2,c_1,c_2 are complex parameters, $c_1,c_2\neq 0,-1,-2,\dots$, and $(a)_m=\Gamma(a+m)/\Gamma(a)$. This series converges absolutely in the domain $\{(x,y)\in\mathbb{C}^2\mid |x|+|y|<1\}$, and admits an Euler type integral

$$\begin{aligned}F_2(a,b_1,b_2,c_1,c_2;x,y)&=\frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(b_1)\Gamma(c_1-b_1)\Gamma(b_2)\Gamma(c_2-b_2)}\iint_{\Delta}u_{F_2}\varphi_{F_2},\\ u_{F_2}&=t_1^{b_1}(1-t_1)^{c_1-b_1}t_2^{b_2}(1-t_2)^{c_2-b_2}(1-xt_1-yt_2)^{-a},\\ \varphi_{F_2}&=\frac{\mathrm{d}t_1\wedge\mathrm{d}t_2}{t_1(t_1-1)t_2(t_2-1)},\quad \Delta=(0,1)\times(0,1),\end{aligned}\tag{9.2}$$

under the convergence condition $\Re(c_1) > \Re(b_1) > 0$, $\Re(c_2) > \Re(b_2) > 0$ for the integral.

This series satisfies Appell's F_2 system of hypergeometric differential equations:

$$\begin{cases} \left[x(1-x)\frac{\partial^2}{\partial x^2} - xy\frac{\partial^2}{\partial x \partial y} + \{c_1 - (a+b_1+1)x\}\frac{\partial}{\partial x} - b_1 y \frac{\partial}{\partial y} - ab_1\right] f(x,y) = 0, \\ \left[y(1-y)\frac{\partial^2}{\partial y^2} - xy\frac{\partial^2}{\partial x \partial y} + \{c_2 - (a+b_2+1)y\}\frac{\partial}{\partial y} - b_2 x \frac{\partial}{\partial x} - ab_2\right] f(x,y) = 0. \end{cases} \tag{9.3}$$

It is known that (9.3) is a regular singular system of rank four with singular locus

$$S_{F_2} = \left\{(x,y) \in \mathbb{C}^2 \,\middle|\, x(x-1)y(y-1)(x+y-1) = 0\right\} \cup L_\infty,$$

where L_∞ is the line at infinity in the projective plane $\mathbb{P}^2$ of the (x,y)-space. By changing Δ into suitable cycles in the (t_1,t_2)-space for the Euler type integral (9.2), we can give linearly independent solutions to (9.3) around any point in the complement of the singular locus S_{F_2}.

We set

$$\begin{cases} \varphi_{0,F} = \frac{\mathrm{d}T}{(1-xt_1-yt_2)t_1t_2}, \\ \varphi_{1,F} = \frac{(1-x)\mathrm{d}T}{(1-xt_1-yt_2)(t_1-1)t_2}, \\ \varphi_{2,F} = \frac{(1-y)\mathrm{d}T}{(1-xt_1-yt_2)t_1(t_2-1)}, \\ \varphi_{3,F} = \frac{(1-x-y)\mathrm{d}T}{(1-xt_1-yt_2)(t_1-1)(t_2-1)}, \end{cases}$$

where $\mathrm{d}T := \mathrm{d}t_1 \wedge \mathrm{d}t_2$, and define functions

$$f_i(x,y) := \iint_\Delta u_{F_2}\varphi_{i,F} \quad (0 \le i \le 3), \quad \Delta = (0,1)\times(0,1). \tag{9.4}$$

It is known that the series $F_2(a,b_1,b_2,c_1,c_2;x,y)$ spans the 1-dimensional space of single valued holomorphic solutions to (9.3) around $(0,0)$. The linear combination

$$f = f_0 - f_1 - f_2 + f_3, \tag{9.5}$$

belongs to this space, and it is a non-zero constant multiplication to $F_2(a,b_1,b_2,c_1,c_2;x,y)$. Its partial derivatives $(\partial/\partial x)f$, $(\partial/\partial y)f$ and $(\partial^2/\partial x\partial y)f$ are expressed by

$$\begin{aligned} &\frac{\partial}{\partial x}f = \frac{a}{x-1}f_1 - \frac{a}{x+y-1}f_3, \quad \frac{\partial}{\partial y}f = \frac{a}{y-1}f_2 - \frac{a}{x+y-1}f_3, \\ &\frac{\partial^2}{\partial x\partial y}f = \frac{a(a+b_1+b_2-c_1-c_2+1)}{(x+y-1)^2}f_3 - \frac{ab_2}{(x-1)(x+y-1)}f_1 - \frac{ab_1}{(y-1)(x+y-1)}f_2. \end{aligned}$$

Thus the vector-valued function $\mathbf{F}_{F_2} := {}^t\!\left[f_0 \quad f_1 \quad f_2 \quad f_3\right]$ satisfies a system of first order differential equations equivalent to (9.3):

$$\begin{aligned} \partial_x \mathbf{F}_{F_2} &= \left\{\tfrac{\mathrm{d}x}{x}W_{1,F} + \tfrac{\mathrm{d}x}{x-1}W_{2,F} + \tfrac{\mathrm{d}x}{x+y-1}W_{5,F}\right\}\mathbf{F}_{F_2}, \\ \partial_y \mathbf{F}_{F_2} &= \left\{\tfrac{\mathrm{d}y}{y}W_{3,F} + \tfrac{\mathrm{d}y}{y-1}W_{4,F} + \tfrac{\mathrm{d}y}{x+y-1}W_{5,F}\right\}\mathbf{F}_{F_2}, \end{aligned} \tag{9.6}$$

where $\partial_x \mathbf{F} = \mathrm{d}x(\partial/\partial x)\mathbf{F}$, $\partial_y \mathbf{F} = \mathrm{d}y(\partial/\partial y)\mathbf{F}$ and

$$W_{1,F} := \begin{bmatrix} -b_1 & b_1 - c_1 & 0 & 0 \\ -b_1 & b_1 - c_1 & 0 & 0 \\ 0 & 0 & -b_1 & b_1 - c_1 \\ 0 & 0 & -b_1 & b_1 - c_1 \end{bmatrix}, \quad W_{2,F} := \begin{bmatrix} 0 & c_1 - b_1 & 0 & 0 \\ 0 & c_1 - b_1 + b_2 - a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \end{bmatrix},$$

$$W_{3,F} := \begin{bmatrix} -b_2 & 0 & b_2 - c_2 & 0 \\ 0 & -b_2 & 0 & b_2 - c_2 \\ -b_2 & 0 & b_2 - c_2 & 0 \\ 0 & -b_2 & 0 & b_2 - c_2 \end{bmatrix}, \quad W_{4,F} := \begin{bmatrix} 0 & 0 & c_2 - b_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c_2 - b_2 + b_1 - a & 0 \\ 0 & 0 & b_1 & 0 \end{bmatrix},$$

$$W_{5,F} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2 - b_2 \\ 0 & 0 & 0 & c_1 - b_1 \\ 0 & 0 & 0 & c_1 + c_2 - b_1 - b_2 - a \end{bmatrix}.$$

The system (9.6) is called a Pfaffian system of Appell's F_2.

10 Equivalence of Pfaffian systems

In this section, we define an equivalence relation between two Pfaffian systems.
Let

$$(W, \mathbf{F}) : \mathrm{d}\mathbf{F} = W\mathbf{F} \tag{10.1}$$

be an integrable Pfaffian system on $\mathbb{P}^2$ for an unknown vector-valued holomorphic function $\mathbf{F}$ of size n with a rational connection matrix $W = W(x, y)$. Note that W is an $n \times n$ matrix whose entries are rational 1-forms of x, y and satisfies the integrability condition

$$\mathrm{d}W = W \wedge W.$$

Let X be the regular locus of a system of differential equations equivalent to (10.1), $\mathcal{O}(X)$ be the ring of holomorphic functions on X, $\mathcal{R}(X)$ be the intersection of the field $\mathbb{C}(x, y)$ of rational functions in x, y, and $\mathcal{O}(X)$, and $\mathcal{R}^\times(X)$ be its unit group. We introduce the following actions on a local solution $\mathbf{F}(x, y)$ to this Pfaffian system:

(a) a multiplication of $h(x, y)^\alpha$ ($h(x, y) \in \mathcal{R}^\times(X)$, $\alpha \in \mathbb{C}$) to $\mathbf{F}(x, y)$, i.e.,

$$\mathbf{F}(x, y) \mapsto h(x, y)^\alpha \mathbf{F}(x, y),$$

(b) a left multiplication of $g(x, y) \in \mathrm{GL}(4, \mathcal{R}(X))$ to $\mathbf{F}(x, y)$, i.e.,

$$\mathbf{F}(x, y) \mapsto g(x, y)\mathbf{F}(x, y),$$

(c) a change of the independent variables by a birational map $\psi(x, y)$ whose restriction to X is a biregular morphism, i.e.,

$$\mathbf{F}(x, y) \mapsto \mathbf{F}(\psi^{-1}(x, y)).$$

The transformed function $\tilde{\mathbf{F}}(x,y)$ under the action in (a), (b) or (c) satisfies

$$(\tilde{W},\tilde{\mathbf{F}}) : \mathrm{d}\tilde{\mathbf{F}} = \tilde{W}\tilde{\mathbf{F}}, \tag{10.2}$$

where connection matrix $\tilde{W}(x,y)$ is given as follows:

(a)
$$\tilde{W}(x,y) = W(x,y) + \alpha\frac{\mathrm{d}h(x,y)}{h(x,y)},$$

(b)
$$\tilde{W}(x,y) = g(x,y)W(x,y)g(x,y)^{-1} + \mathrm{d}g(x,y)g(x,y)^{-1},$$

(c)
$$\tilde{W}(x,y) = W(\psi^{-1}(x,y)).$$

Definition 10.1. If a Pfaffian system $(W,\mathbf{F})$ changes into $(\tilde{W},\tilde{\mathbf{F}})$ by a series of these actions, then $(W,\mathbf{F})$ and $(\tilde{W},\tilde{\mathbf{F}})$ are equivalent, and this equivalence is denoted by

$$(W,\mathbf{F}) \sim (\tilde{W},\tilde{\mathbf{F}}). \tag{10.3}$$

Under this definition, we have the following.

Proposition 10.1. *Pfaffian systems of Appell's F_2 and F_3 are equivalent under Definition 10.1. Here refer to [Er, Section 5.9] for Appell's F_3 system of hypergeometric differential equations.*

Proof. For the Pfaffian system (9.6) of Appell's F_2, multiply the local solution $\mathbf{F}_{F_2}$ by $x^{b_1}y^{b_2}$, then we have

$$\mathbf{F}_{F_3} = {}^t\Big[f_0'' \quad f_1'' \quad f_2'' \quad f_3''\Big] := x^{b_1}y^{b_2}\mathbf{F}_{F_2}.$$

Apply a birational map $x = 1/x''$, $y = 1/y''$ to it, then the Pfaffian system is transformed into

$$\begin{aligned}
\partial_{x''}\mathbf{F}_{F_3} &= \left\{\tfrac{\mathrm{d}x''}{x''}W''_{1,F} + \tfrac{\mathrm{d}x''}{x''-1}W_{2,F} + \tfrac{(y''-1)\mathrm{d}x''}{x''y''-x''-y''}W_{5,F}\right\}\mathbf{F}_{F_3},\\
\partial_{y''}\mathbf{F}_{F_3} &= \left\{\tfrac{\mathrm{d}y''}{y''}W''_{3,F} + \tfrac{\mathrm{d}y''}{y''-1}W_{4,F} + \tfrac{(x''-1)\mathrm{d}y''}{x''y''-x''-y''}W_{5,F}\right\}\mathbf{F}_{F_3},
\end{aligned} \tag{10.4}$$

where $W''_{1,F} = -(b_1I_4 + W_{1,F} + W_{2,F} + W_{5,F})$, $W''_{3,F} = -(b_2I_4 + W_{3,F} + W_{4,F} + W_{5,F})$ and I_4 is the unit matrix.

A linear combination

$$f'' := f_0'' - f_1'' - f_2'' + f_3''$$

is one of solutions to

$$\begin{cases}
\Big[x''(1-x'')\frac{\partial^2}{\partial x''^2} + y''\frac{\partial^2}{\partial x''\partial y''} + \{B_0 - (b_1+B_1+1)x''\}\frac{\partial}{\partial x''} - b_1B_1\Big]f''(x'',y'') = 0,\\
\Big[y''(1-y'')\frac{\partial^2}{\partial y''^2} + x''\frac{\partial^2}{\partial x''\partial y''} + \{B_0 - (b_2+B_2+1)y''\}\frac{\partial}{\partial y''} - b_2B_2\Big]f''(x'',y'') = 0,
\end{cases} \tag{10.5}$$

where $B_0 = b_1 + b_2 - a + 1$ and $B_1 = b_1 - c_1 + 1$, $B_2 = b_2 - c_2 + 1$. This is Appell's F_3 system of hypergeometric differential equations [Er]. Thus its Pfaffian system (10.4) is equivalent to the Pfaffian system (9.6) of Appell's F_2. □

11 Confluences of Appell's F_2 system

In this section, we study confluences of the Pfaffian system of Appell's F_2. We express the integrand u_{F_2} as

$$u_{F_2} = \prod_{i=1}^{5} \mathbf{l}_i^{\alpha_i}, \quad \begin{matrix} \mathbf{l}_1 = t_1, & \mathbf{l}_2 = 1 - t_1, & \mathbf{l}_3 = t_2, & \mathbf{l}_4 = 1 - t_2, & \mathbf{l}_5 = 1 - xt_1 - yt_2, \\ \alpha_1 = b_1, & \alpha_2 = c_1 - b_1, & \alpha_3 = b_2, & \alpha_4 = c_2 - b_2, & \alpha_5 = -a. \end{matrix}$$

Let ℓ_0 be the line at infinity in $\mathbb{P}^2$ and let ℓ_i $(1 \leq i \leq 5)$ be the lines given by $\mathbf{l}_i = 0$. Here $\mathbf{l}_0$ is regarded as $1 = 1 + 0 \cdot t_1 + 0 \cdot t_2$. Note that the intersection of ℓ_1 and ℓ_2 and that of ℓ_3 and ℓ_4 are in the line ℓ_0. By regarding lines in $\mathbb{P}^2$ as points in the dual space of $\mathbb{P}^2$, we express the configuration of the six lines as in Figure 11.1.

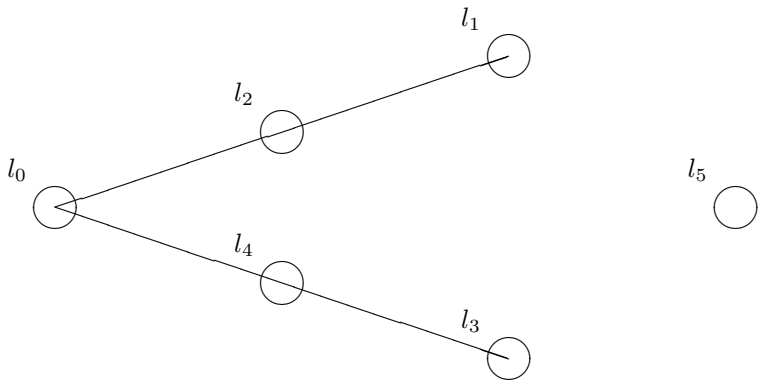


Figure 11.1: Configuration of six lines

Figure 11.1 means that if circles ◯ are in a line in the dual space, then the lines corresponding to the circles intersect at a point in the original space. The linear forms are expressed as $\begin{bmatrix} \mathbf{l}_0 & \cdots & \mathbf{l}_5 \end{bmatrix} = \begin{bmatrix} 1 & t_1 & t_2 \end{bmatrix} z$ by a 3×6 matrix

$$z = [z_{ij}]_{\substack{0 \leq i \leq 2 \\ 0 \leq j \leq 5}} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & -x \\ 0 & 0 & 0 & 1 & -1 & -y \end{bmatrix}, \tag{11.1}$$

which satisfies

$$\begin{aligned} &\operatorname{rank}(0,1,2;z) = \operatorname{rank}(0,3,4;z) = 2, \\ &\operatorname{rank}(p,q,r;z) = 3 \quad \text{for } 0 \leq p < q < r \leq 5,\ (p,q,r) \neq (0,1,2), (0,3,4), \end{aligned} \tag{11.2}$$

where $\operatorname{rank}(p,q,r;z)$ denotes the rank of the minor matrix consisting of p,q,r-th column vectors of the 3×6 matrix z. Let Z_0 be the set of 3×6 matrices satisfying (11.2). Note that if a 3×6 matrix z belongs to Z_0 then the configuration of the six lines given by $\begin{bmatrix} 1 & t_1 & t_2 \end{bmatrix} z = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}$ is same as that in Figure 11.1 topologically. Moreover, we can easily show the following.

Proposition 11.1. *For any element $z \in Z_0$, there exist $g \in \mathrm{GL}(3,\mathbb{C})$, $h \in D_6$ and $(x,y) \in \mathbb{P}^2\backslash S_{F_2} = \{(x,y) \in \mathbb{C}^2 \mid xy(x-1)(y-1)(x+y-1) \neq 0\}$ such that*

$$g \cdot z \cdot h = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & -x \\ 0 & 0 & 0 & 1 & -1 & -y \end{bmatrix}, \tag{11.3}$$

where D_6 is the subgroup of $\mathrm{GL}(6,\mathbb{C})$ consisting of diagonal matrices; it is isomorphic to $(\mathbb{C}^\times)^6$. The element $(x,y) \in \mathbb{P}^2\backslash S_{F_2}$ is uniquely determined, and $g \in \mathrm{GL}(3,\mathbb{C})$ and $h \in D_6$ are unique up to non-zero constant multiples.

We call the right hand side of (11.3) *the standard form of $z \in Z_0$, and x, y in the standard form the normalized variables of z, which are denoted by $x(z)$, $y(z)$ in case of expressing their dependence on z.*

We extend the integrals defining $f_0(x,y),\dots,f_3(x,y)$ in (9.4) to functions on the set Z_0, which are invariant under the left $\mathrm{GL}(3,\mathbb{C})$ action on Z_0, and covariant under the right D_6 action on Z_0. Let $\mathfrak{Z}_0$ be a subset in $\mathbb{P}^2 \times Z_0$ given by

$$\left\{(t,z) \in \mathbb{P}^2 \times Z_0 \mid t \cdot z = \begin{bmatrix} \mathbf{l}_0(z) & \cdots & \mathbf{l}_5(z) \end{bmatrix} \neq \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}\right\},$$

where the linear form $\mathbf{l}_i(z)$ is the homogenization of $z_{0i} + z_{1i}t_1 + z_{2i}t_2$ expressed by affine coordinates (t_1,t_2). We have a natural projection from $\mathfrak{Z}_0$ to Z_0:

$$pr : \mathfrak{Z}_0 \ni (t,z) \mapsto z \in Z_0,$$

and a map from $\mathfrak{Z}_0$ to D_6:

$$\lambda : \mathfrak{Z}_0 \ni (t,z) \mapsto \mathrm{diag}(1, \mathbf{l}_1(z)/\mathbf{l}_0(z), \dots, \mathbf{l}_5(z)/\mathbf{l}_0(z)) \in D_6. \tag{11.4}$$

Suppose that $\alpha = {}^t\begin{bmatrix} \alpha_0 & \dots & \alpha_5 \end{bmatrix} \in (\mathbb{C}-\mathbb{Z})^6$ satisfies

$$\alpha_0 + \cdots + \alpha_5 = 0.$$

By using the map λ, we pull back the character $\chi_0(\alpha;\cdot) : D_6 \ni h = \mathrm{diag}(h_0,\dots,h_5) \mapsto h_0^{\alpha_0}\cdots h_5^{\alpha_5} \in \mathbb{C}^\times$ to a (multi-valued) map

$$u_0^\alpha : \mathfrak{Z}_0 \ni (t,z) \mapsto u_0^\alpha(t,z) = \mathbf{l}_0(z)^{\alpha_0}\cdots\mathbf{l}_5(z)^{\alpha_5} = \prod_{i=0}^{5}(z_{0i} + z_{1i}t_1 + z_{2i}t_2)^{\alpha_i} \in \mathbb{C}^\times.$$

We set

$$\begin{aligned} &\varphi_0 = \mathrm{d}_t \log(\mathbf{l}_1(z)/\mathbf{l}_5(z)) \wedge \mathrm{d}_t \log(\mathbf{l}_3(z)/\mathbf{l}_5(z)), \quad \varphi_1 = \mathrm{d}_t \log(\mathbf{l}_2(z)/\mathbf{l}_5(z)) \wedge \mathrm{d}_t \log(\mathbf{l}_3(z)/\mathbf{l}_5(z)), \\ &\varphi_2 = \mathrm{d}_t \log(\mathbf{l}_1(z)/\mathbf{l}_5(z)) \wedge \mathrm{d}_t \log(\mathbf{l}_4(z)/\mathbf{l}_5(z)), \quad \varphi_3 = \mathrm{d}_t \log(\mathbf{l}_2(z)/\mathbf{l}_5(z)) \wedge \mathrm{d}_t \log(\mathbf{l}_4(z)/\mathbf{l}_5(z)), \\ &\varphi = \varphi_0 - \varphi_1 - \varphi_2 + \varphi_3, \end{aligned}$$

where $\mathrm{d}_t = \mathrm{d}t_1(\partial/\partial t_1) + \mathrm{d}t_2(\partial/\partial t_2)$ is the holomorphic exterior derivative on $\mathbb{P}^2$. For any $z \in Z_0$, we take a 2-chain Δ with some cycle conditions in the fiber $pr^{-1}(z) \subset \mathbb{P}^2$. Note that we have the continuation of Δ in the fiber over any point in a small neighborhood of $z \in Z_0$ by the local triviality.

Definition 11.1. We define general hypergeometric functions $F_i(\alpha; z)$ $(i = 0, \dots, 3)$ and $F(\alpha; z)$ $(i = \emptyset)$ on Z_0 as the analytic continuations of

$$F_i(\alpha; z) := \iint_\Delta u_0^\alpha(t, z)\varphi_i, \qquad F(\alpha; z) := \iint_\Delta u_0^\alpha(t, z)\varphi.$$

They are single-valued holomorphic functions in any small neighborhood of $z \in Z_0$.

Remark 11.1. In the definition of a general hypergeometric function in [KHT], α is supposed to be

$$\alpha_0 + \cdots + \alpha_5 = -3$$

and the 2-form in the integral is multiplied $\mathbf{l}_p(z)\mathbf{l}_q(z)\mathbf{l}_5(z)/\det(z_{ij})_{\substack{0\le i\le 2\\ j=p,q,5}}$ for some $1 \le p < q \le 4$ to ours.

By [KHT, Proposition 1.2] and this remark, we see that $F_i(\alpha; z)$ admits the invariance for $\mathrm{GL}(3, \mathbb{C})$ and the covariance for D_6 as the following.

Proposition 11.2. *The general hypergeometric functions $F_i(\alpha; z)$ $(i = 0, \dots, 3, \emptyset)$ satisfy*

$$F_i(\alpha; g \cdot z) = F_i(\alpha; z), \quad F_i(\alpha; z \cdot h) = \Big(\prod_{i=0}^{5} h_i^{\alpha_i}\Big) F_i(\alpha; z),$$

for $g \in \mathrm{GL}(3, \mathbb{C})$ and $h \in D_6$ in small neighborhoods of their identities.

Remark 11.2. Let z be an element of Z_0, g and h be elements of $\mathrm{GL}(3, \mathbb{C})$ and D_6 changing z into the standard form as in (11.3), and $x(z)$, $y(z)$ be the normalized variables of z. By Proposition 11.2, $F_i(\alpha; z)$ $(i = 0, \dots, 3, \emptyset)$ can be expressed in terms of Euler type integrals expressing solutions to Appell's F_2 system of differential equations with substitution of $x(z)$, $y(z)$ for the independent variables x, y of the system and the factor $\prod_{i=0}^{5} h_i^{\alpha_i}$, where $h_0, \dots, h_5 \in \mathbb{C}^\times$ are the entries of h. In particular, if the 2-chain Δ is the image of $(0, 1) \times (0, 1)$ under g regarded as a projective transformation, then $F_i(\alpha, z)$ is expressed as

$$F_i(\alpha; z) = \Big(\prod_{i=0}^{5} h_i^{\alpha_i}\Big)^{-1} f_i(x(z), y(z)),$$

where $f_i(x(z), y(z))$ are given in (9.4) and (9.5).

We study confluences of solutions to Appell's F_2 system by using general hypergeometric functions. Though the definition of the general hypergeometric function is slightly different from that in [KHT], we can apply confluences in [KHT] to ours, since the confluences are essentially defined by some kinds of limits for maximal abelian groups in $\mathrm{GL}(n, \mathbb{C})$ and for their characters. Let us explain the simplest case which will be used in our study. We can regard the group $D_2 = \left\{ \begin{bmatrix} h_i & 0 \\ 0 & h_j \end{bmatrix} \;\middle|\; h_i, h_j \in \mathbb{C}^\times \right\}$ isomorphic to $(\mathbb{C}^\times)^2$ as a subgroup of D_6. A character of this group is expressed as

$$D_2 \ni \begin{bmatrix} h_i & 0 \\ 0 & h_j \end{bmatrix} \mapsto h_i^{\alpha_i} \cdot h_j^{\alpha_j} \in \mathbb{C}^\times.$$

A character of the Jordan group is expressed as

$$J(2) \ni \begin{bmatrix} h_i & h_k \\ 0 & h_i \end{bmatrix} \mapsto h_i^{\alpha_i+\alpha_j} \cdot \exp(\alpha_k \frac{h_k}{h_i}) \in \mathbb{C}^\times.$$

We set a subgroup B_2 of $\mathrm{GL}(2,\mathbb{C})$ and a 3-dimensional vector space $\bar{B}_2$ as

$$B_2 := \Big\{ \begin{bmatrix} x & z \\ 0 & y \end{bmatrix} \in \mathrm{GL}(2,\mathbb{C}) \Big\}, \quad \bar{B}_2 := \Big\{ \begin{bmatrix} x & z \\ 0 & y \end{bmatrix} \in \mathrm{M}(2,\mathbb{C}) \Big\}.$$

We identify $\bar{B}_2$ with the vector space $\mathbb{C}^3$ via a map

$$\iota : \bar{B}_2 \ni \begin{bmatrix} x & z \\ 0 & y \end{bmatrix} \mapsto \begin{bmatrix} x & y & z \end{bmatrix} \in \mathbb{C}^3. \tag{11.5}$$

The logarithmic map from a neighborhood U_{I_2} of the identity I_2 to $\bar{B}_2$ is given by

$$\log(H) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (H - I_2)^n, \quad (H \in U_{I_2}). \tag{11.6}$$

It admits an expression

$$\log(H) = \begin{bmatrix} \log(h_i) & x_k \\ 0 & \log(h_j) \end{bmatrix}, \quad x_k := \begin{cases} h_k/h_i & (h_i = h_j), \\ \frac{h_k}{h_j - h_i} \log(h_j/h_i) & (h_i \neq h_j), \end{cases}$$

where $h_i - 1$, $h_j - 1$ and h_k are sufficiently close to 0.

For a fixed column vector $\alpha = {}^t\begin{bmatrix} \alpha_i & \alpha_j & \alpha_k \end{bmatrix} \in \mathbb{C}^3$, we define a function $\chi_1(\alpha; H)$ on U_{I_2} by

$$\chi_1(\alpha; H) := \exp\{\iota \circ \log(H) \cdot \alpha\} \quad (H \in U_{I_2}). \tag{11.7}$$

This function can be extended to a (multi-valued) function on B_2 by the analytic continuation.

Remark 11.3. Since B_2 is not commutative, $\chi_1(\alpha; H)$ is not a homomorphism. However, the both of restrictions $\chi_1(\alpha; H)$ to D_2 and to $J(2)$ are characters. In fact, we have

$$\chi_1(\alpha; \begin{bmatrix} h_i & 0 \\ 0 & h_j \end{bmatrix}) = h_i^{\alpha_i} h_j^{\alpha_j}, \quad \chi_1(\alpha; \begin{bmatrix} h_i & h_k \\ 0 & h_i \end{bmatrix}) = h_i^{\alpha_i+\alpha_j} \exp(\alpha_k \frac{h_k}{h_i})$$

by (11.7).

By a matrix $q(\varepsilon') = \begin{bmatrix} 1 & 1-\varepsilon' \\ 0 & \varepsilon' \end{bmatrix}$ instead of $g(\varepsilon) = \begin{bmatrix} 1 & 1 \\ 0 & \varepsilon \end{bmatrix}$ used in the study of confluences of characters in [KTn, (4.1) in Lemma 4.1], we set a linear transformation $L_{\varepsilon'}$ of $\bar{B}_2$ as

$$L_{\varepsilon'} : H = \begin{bmatrix} h_i & h_k \\ 0 & h_j \end{bmatrix} \mapsto q(\varepsilon') H q(\varepsilon')^{-1}, \tag{11.8}$$

which is represented by

$$\mathbb{C}^3 \ni \begin{bmatrix} h_i & h_j & h_k \end{bmatrix} \mapsto \begin{bmatrix} h_i & h_j & h_k \end{bmatrix} L(\varepsilon') \in \mathbb{C}^3, \quad L(\varepsilon') := \begin{bmatrix} 1 & 0 & 1 - 1/\varepsilon' \\ 0 & 1 & 1/\varepsilon' - 1 \\ 0 & 0 & 1/\varepsilon' \end{bmatrix},$$

via the map $\iota(H) = \begin{bmatrix} h_i & h_j & h_k \end{bmatrix}$. By using this representation matrix $L(\varepsilon')$, we set a linear transformation of $\alpha = {}^t\begin{bmatrix} \alpha_i & \alpha_j & \alpha_k \end{bmatrix} \in \mathbb{C}^3$ by

$$ {}^tL_{\varepsilon'} : \begin{bmatrix} \alpha_i \\ \alpha_j \\ \alpha_k \end{bmatrix} \mapsto L(\varepsilon') \begin{bmatrix} \alpha_i \\ \alpha_j \\ \alpha_k \end{bmatrix}. \tag{11.9} $$

Proposition 11.3. *We have*

$$ \chi_1({}^tL_{\varepsilon'}(\alpha); H) = \chi_1(\alpha; L_{\varepsilon'}(H)). \tag{11.10} $$

Proof. We have

$$ \begin{aligned} \chi_1({}^tL_{\varepsilon'}(\alpha); H) &= \exp(\iota \circ \log(H) \cdot {}^tL_{\varepsilon'}(\alpha)) = \exp(\iota(\log(H)) \cdot L(\varepsilon') \cdot \alpha) \\ &= \exp(\iota \circ L_{\varepsilon'}(\log(H)) \cdot \alpha) = \exp(\iota \circ \log(L_{\varepsilon'}(H)) \cdot \alpha) = \chi_1(\alpha; L_{\varepsilon'}(H)), \end{aligned} $$

since $L_{\varepsilon'}(\log(H)) = q(\varepsilon')\log(H)q(\varepsilon')^{-1} = \log(q(\varepsilon')Hq(\varepsilon')^{-1}) = \log(L_{\varepsilon'}(H))$. □

By extending the 2×2 matrix $g(\varepsilon) = \begin{bmatrix} 1 & 1 \\ 0 & \varepsilon \end{bmatrix}$ studied in [KTn, (4.3) in Lemma 4.1] to a 3×3 matrix

$$ R(\varepsilon'') := \begin{bmatrix} 1 & 1-\varepsilon'' & 0 \\ 0 & \varepsilon'' & 0 \\ 0 & 0 & \varepsilon'' \end{bmatrix} \quad (\varepsilon'' \in \mathbb{C}^\times), $$

and using the map $\iota(H) = \begin{bmatrix} h_i & h_j & h_k \end{bmatrix}$, we define a linear transformation $R_{\varepsilon''}$ of $\bar{B}_2$ as

$$ R_{\varepsilon''} : H \mapsto R_{\varepsilon''}(H) := \iota^{-1} \circ (\iota(H) \cdot R(\varepsilon'')), \tag{11.11} $$

which admits an expression

$$ R_{\varepsilon''} : H = \begin{bmatrix} h_i & h_k \\ 0 & h_j \end{bmatrix} \mapsto \begin{bmatrix} h_i & 0 \\ 0 & h_i \end{bmatrix} + \varepsilon'' \begin{bmatrix} 0 & h_k \\ 0 & h_j - h_i \end{bmatrix}. $$

By a straightforward calculation, we can see that $R_{\varepsilon''}$ and $L_{\varepsilon'}$ are commutative.

By using the linear transformations $L_{\varepsilon'}$ and $R_{\varepsilon''}$, and introducing an infinitesimal parameter ε, we define a deformation of the group B_2 as a map $\psi_\varepsilon : B_2 \to \bar{B}_2$ given by

$$ \psi_\varepsilon(H) := L_\varepsilon \circ R_\varepsilon(H), \tag{11.12} $$

which admits an expression

$$ \psi_\varepsilon : H = \begin{bmatrix} h_i & h_k \\ 0 & h_j \end{bmatrix} \mapsto \psi_\varepsilon(H) = \begin{bmatrix} h_i & (1-\varepsilon)(h_j - h_i) + h_k \\ 0 & h_i + \varepsilon(h_j - h_i) \end{bmatrix}. $$

Definition 11.2 (A deformation of $\chi_1(\alpha; H)$)**.** We define a deformation of $\chi_1(\alpha; H)$ as its pull back under the map $\psi_\varepsilon : B_2 \to \bar{B}_2$:

$$ \psi_\varepsilon^* \chi_1(\alpha; H) := \chi_1(\alpha; \psi_\varepsilon(H)) = \chi_1({}^tL_\varepsilon(\alpha); R_\varepsilon(H)). \tag{11.13} $$

Proposition 11.4 (Confluences of D_2 and a character on D_2)**.** *We set*

$$H = \begin{bmatrix} h_i & 0 \\ 0 & h_j \end{bmatrix} \in D_2 \subset B_2, \quad \alpha = {}^t\begin{bmatrix} \alpha_i & \alpha_j & \alpha_k \end{bmatrix}.$$

(1) *The group $J(2)$ is given as the set of limits of $\psi_\varepsilon(H)$ for elements $H \in D_2$ as $\varepsilon \to 0$, i.e.*

$$J(2) = \{\lim_{\varepsilon\to 0} \psi_\varepsilon(H) \mid H \in D_2\}.$$

(2) *A character $h_i^{\alpha_i+\alpha_j} \exp(\alpha_k(h_j - h_i)/h_i)$ on $J(2)$ is given as the limit of the character $\psi_\varepsilon^* \chi_1(\alpha; H)$ on D_2 as $\varepsilon \to 0$, i.e.,*

$$h_i^{\alpha_i+\alpha_j} \exp(\alpha_k \frac{h_j - h_i}{h_i}) = \lim_{\varepsilon\to 0} \psi_\varepsilon^* \chi_1(\alpha; H).$$

Proof. (1) As $\varepsilon \to 0$, $\psi_\varepsilon(H)$ converges to

$$\lim_{\varepsilon\to 0} \psi_\varepsilon(H) = \begin{bmatrix} h_i & h_j - h_i \\ 0 & h_i \end{bmatrix} \in J(2).$$

(2) As in Remark 11.3, we have $\chi_1(\alpha; H) = h_i^{\alpha_i} h_j^{\alpha_j}$. Its deformation is written by

$$\psi_\varepsilon^* \chi_1(\alpha; H) = h_i^{\alpha_i+(1-1/\varepsilon)\alpha_k} \big\{h_i + \varepsilon(h_j - h_i)\big\}^{\alpha_j-(1-1/\varepsilon)\alpha_k}.$$

By taking the limit $\varepsilon \to 0$, we have

$$\lim_{\varepsilon\to 0} \psi_\varepsilon^* \chi_1(\alpha; H) = \chi_1(\alpha; \begin{bmatrix} h_i & h_j - h_i \\ 0 & h_i \end{bmatrix}) = h_i^{\alpha_i+\alpha_j} \exp(\alpha_k \frac{h_j - h_i}{h_i}).$$

□

In (11.13), ψ_ε^* is considered as an operator acting on a character of D_2. We extend it to an operator acting on a character of D_6 as follows. We take indices i, j $(0 \le i, j \le 5,\ i \ne j)$ and fix them. For the indices (i, j), we extend R_ε to a map $R_{ij,\varepsilon} : D_6 \to \bar{D}_6$ by sending the i, j-th entries h_i, h_j in D_6 by R_ε and by keeping the other entries invariant, i.e.,

$$R_{ij,\varepsilon}(\mathrm{diag}(h_0, \dots, h_i, \dots, h_j, \dots, h_5)) = \mathrm{diag}(h_0, \dots, h_i, \dots, h_i + \varepsilon(h_j - h_i), \dots, h_5),$$

where $^-$ denotes the closures. We identify $\bar{D}_6$ with $\mathbb{C}^6$ by the natural map

$$\iota_6 : \bar{D}_6 \ni h = \mathrm{diag}(h_0, \dots, h_5) \mapsto \begin{bmatrix} h_0 & \cdots & h_5 \end{bmatrix} \in \mathbb{C}^6.$$

Under this identification, the map $R_{ij,\varepsilon}$ is represented by the right multiplication of the matrix

$$q_{ij}(\varepsilon) := I_6 + (1 - \varepsilon)(E_{ij} - E_{jj}) \in \mathrm{M}(6, \mathbb{C}), \tag{11.14}$$

to $\iota_6(h)$ for $h \in \bar{D}_6$, where E_{ij} is the (i, j) matrix unit in $\mathrm{M}(6, \mathbb{C})$. This means that

$$\iota_6 \circ R_{ij,\varepsilon}(h) = \iota_6(h) \cdot q_{ij}(\varepsilon) \quad (h \in D_6).$$

Similarly, we extend ${}^tL_\varepsilon$ to a linear transformation ${}^tL_{ij,\varepsilon} : \mathbb{C}^6 \to \mathbb{C}^6$ by

$${}^tL_{ij,\varepsilon}\big({}^t\begin{bmatrix} \alpha_0 & \dots & \alpha_i & \dots & \alpha_j & \dots & \alpha_5 \end{bmatrix}\big) = {}^t\begin{bmatrix} \alpha_0 & \dots & \alpha_i + (1-1/\varepsilon)\alpha_j & \dots & \alpha_j/\varepsilon & \dots & \alpha_5 \end{bmatrix},$$

which is represented by the left multiplication of $q_{ij}^{-1}(\varepsilon)$ to $\alpha = {}^t\begin{bmatrix}\alpha_0 & \cdots & \alpha_5\end{bmatrix}$:

$$ {}^tL_{ij,\varepsilon}(\alpha) = q_{ij}^{-1}(\varepsilon)\cdot\alpha. $$

Using $R_{ij,\varepsilon}$ and ${}^tL_{ij,\varepsilon}$, we define an operator $\psi^*_{ij,\varepsilon}$ acting on the character $\chi_0(\alpha;h)$ of D_6 as

$$ \psi^*_{ij,\varepsilon}\chi_0(\alpha;h) := \chi_0({}^tL_{ij,\varepsilon}(\alpha);R_{ij,\varepsilon}(h)). \tag{11.15} $$

We operate $\psi^*_{ij,\varepsilon}$ on $f_i(x,y)$ $(i=0,\dots,3,\emptyset)$ in (9.4) and (9.5) by regarding them as general hypergeometric functions $F_i(\alpha;z)$. Since $u_0^\alpha(t,z)$ is the pull back of the character $\chi_0(\alpha;h)$ under the map $\lambda:\mathfrak{Z}_0\to D_6$, we set

$$ \psi^*_{ij,\varepsilon}f_i(x,y) = \psi^*_{ij,\varepsilon}F_i(\alpha;z) := \int_\Delta \psi^*_{ij,\varepsilon}\chi_0(\alpha,\lambda(t,z))\varphi_i \quad (i=0,\dots,3,\emptyset). \tag{11.16} $$

Since

$$ \psi^*_{ij,\varepsilon}\chi_0(\alpha,\lambda(t,z)) = \chi_0({}^tL_{ij,\varepsilon}(\alpha),R_{ij,\varepsilon}(\lambda(t,z))), $$
$$ {}^tL_{ij,\varepsilon}(\alpha) = q_{ij}^{-1}(\varepsilon)\cdot\alpha, \quad R_{ij,\varepsilon}(\lambda(t,z)) = \lambda(t,z)\cdot q_{ij}(\varepsilon) = \lambda(t,z\cdot q_{ij}(\varepsilon)), $$

we have

$$ \int_\Delta \psi^*_{ij,\varepsilon}\chi_0(\alpha,\lambda(t,z))\varphi_i = \int_\Delta \chi_0(q_{ij}^{-1}(\varepsilon)\cdot\alpha,\lambda(t,z\cdot q_{ij}(\varepsilon)))\varphi_i. $$

Thus the operator $\psi^*_{ij,\varepsilon}$ on $f_i(x,y)$ in (11.16) is represented by transformations of z, α into $z(\varepsilon)$, $\alpha(\varepsilon)$ in the general hypergeometric function:

$$ \psi^*_{ij,\varepsilon}F_i(\alpha,z) = F_i(\alpha(\varepsilon);z(\varepsilon)), \tag{11.17} $$

where $z(\varepsilon)$ and $\alpha(\varepsilon)$ are given by matrix multiplications

$$ z(\varepsilon) := z\cdot q_{ij}(\varepsilon), \quad \alpha(\varepsilon) := q_{ij}^{-1}(\varepsilon)\cdot\alpha. \tag{11.18} $$

Note that $z(\varepsilon)$ does not satisfy the rank conditions (11.2) defining the set Z_0 for any i,j $(0\le i,j\le 5, i\ne j)$. We temporally assume that the 3×6 matrix $z(\varepsilon)$ belongs to Z_0. Then by Remark 11.2, $F(\alpha(\varepsilon);z(\varepsilon))$ in (11.17) can be expressed by the normalized variables $x(z(\varepsilon)),y(z(\varepsilon))$ and parameters $\alpha(\varepsilon)$ as

$$ F_i(\alpha(\varepsilon);z(\varepsilon)) = (\prod_{k=0}^5 h_k(\varepsilon)^{\alpha_k(\varepsilon)})^{-1} f_i(\alpha(\varepsilon);x(z(\varepsilon)),y(z(\varepsilon))), $$

where $h(\varepsilon)=\mathrm{diag}(h_0(\varepsilon),\dots,h_5(\varepsilon))\in D_6$ is used to change $z(\varepsilon)$ into its standard form as in (11.3). Hence we have

$$ \psi^*_{ij,\varepsilon}f_i(\alpha;x,y) = \psi^*_{ij,\varepsilon}F_i(\alpha;z) = F_i(\alpha(\varepsilon);z(\varepsilon)) = (\prod_{k=0}^5 h_k(\varepsilon)^{\alpha_k(\varepsilon)})^{-1} f_i(\alpha(\varepsilon);x(z(\varepsilon)),y(z(\varepsilon))), \tag{11.19} $$

that is, the operator $\psi^*_{ij,\varepsilon}$ on $f_i(\alpha;x,y)$ is realized by the transformations of x, y, α into $x(z(\varepsilon))$, $y(z(\varepsilon))$, $\alpha(\varepsilon)$ for $f_i(x,y)$, and the multiplication of the factor $(\prod_{k=0}^5 h_k(\varepsilon)^{\alpha_k(\varepsilon)})^{-1}$ to it. Recall that we define the operator $\psi^*_{ij,\varepsilon}$ by the confluence of a character on D_6. We will calculate the limit $\lim_{\varepsilon\to 0}\psi^*_{ij,\varepsilon}f_i(\alpha;x,y)$ by using the right hand side of (11.19).

Here we list (i,j) such that the operator $\psi^*_{ij,\varepsilon}$ on $f_i(\alpha;x,y)$ can be defined.

Proposition 11.5. *Suppose that ε is a non-zero complex number with sufficiently small absolute value. There are thirteen pairs of indices (i,j) $(0 \leq i,j \leq 5,\ i \neq j)$ such that the set Z_0 contains $z(\varepsilon) = z \cdot q_{ij}(\varepsilon)$ for any $z \in Z_0$. We list them as follows:*

$$\begin{array}{llll} (0,1), & (0,2), & (0,3), & (0,4), \\ (1,2), & (2,1), & (3,4), & (4,3), \\ (1,5), & (2,5), & (3,5), & (4,5), \\ (0,5). & & & \end{array} \tag{11.20}$$

For any (i,j) $(0 \leq i,j \leq 5,\ i \neq j)$ not listed in (11.20), and any $z \in Z_0$, the set Z_0 contains neither $z(\varepsilon)$ nor any matrix obtained by a permutation of its columns.

Proof. We fix any element $(x,y) \in \mathbb{P}^2 \backslash S_{F_2}$. Let V_{pq} be the linear span of the column vectors $\mathbf{z}_p$ and $\mathbf{z}_q$.

Firstly, we consider $(i,0)$ $(1 \leq i \leq 5)$. In this case, the column vector $\mathbf{z}_0(\varepsilon)$ is given as

$$\mathbf{z}_0(\varepsilon) = \mathbf{z}_i + \varepsilon(\mathbf{z}_0 - \mathbf{z}_i).$$

In the case of $i = 1, 2$, $\mathbf{z}_i$ does not belong to V_{34} by $\mathrm{rank}(i,3,4;z(1)) = 3$. Thus $\mathbf{z}_0(\varepsilon), \mathbf{z}_3, \mathbf{z}_4$ are linearly independent, which contradicts the condition $\mathrm{rank}(0,3,4;z(\varepsilon)) = 2$ $(\varepsilon \neq 0)$. In the case of $i = 3, 4, 5$, $\mathbf{z}_i$ does not belong to V_{12} by the condition $\mathrm{rank}(1,2,i;z(1)) = 3$, which contradicts the condition $\mathrm{rank}(0,1,2;z(\varepsilon)) = 2$ $(\varepsilon \neq 0)$. Hence, we have $z(\varepsilon) \notin Z_0$ for $(i,0)$ $(1 \leq i \leq 5)$.

Secondly, we consider the case $(i,1)$ $(i = 0, 2, \ldots, 5)$. We have $\dim(V_{0j}) = 2$ for $j = 1, 2$ by the condition $\mathrm{rank}(0,j,3;z(1)) = 3$. The column vector $\mathbf{z}_1$ belongs to V_{02} by the condition $\mathrm{rank}(0,1,2;z(1)) = 2$. We consider the column vector

$$\mathbf{z}_1(\varepsilon) = \mathbf{z}_i + \varepsilon(\mathbf{z}_1 - \mathbf{z}_i).$$

In the case $i = 0, 2$, $\mathbf{z}_1(\varepsilon)$ satisfies that $\mathbf{z}_1(\varepsilon) \neq \mathbf{z}_0, \mathbf{z}_2$, and $\mathbf{z}_1(\varepsilon) \in V_{02}$. Since we can see

$$\begin{aligned} \mathrm{rank}(0,1,2;z(\varepsilon)) = 2, \quad \mathrm{rank}(0,1,r;z(\varepsilon)) = \mathrm{rank}(0,2,r;z(1)) = 3, \\ \mathrm{rank}(1,2,r;z(\varepsilon)) = \mathrm{rank}(0,2,r;z(1)) = 3 \quad (3 \leq r \leq 5), \end{aligned}$$

we have $z(\varepsilon) \in Z_0$ for $(0,1)$ and $(2,1)$.
In the case $i = 3, 4, 5$, note that

$$\mathrm{rank}(0,1,2;z(\varepsilon)) = \mathrm{rank}(0,i,2;z(1)) = 3 \quad (\varepsilon \neq 1).$$

Hence we have $z(\varepsilon) \notin Z_0$ for $(i,1)$ $(3 \leq i \leq 5)$.

Thirdly, we consider the case (i,j) for $j = 2, 3, 4$ and $0 \leq i \leq 5$, $i \neq j$. By a similar argument to the previous, we can show that $z(\varepsilon) \in Z_0$ for $(0,2)$ and $(1,2)$, and $z(\varepsilon) \notin Z_0$ for $(i,2)$ $(3 \leq i \leq 5)$. By exchanging the roles of $\mathbf{z}_1$, $\mathbf{z}_2$ and $\mathbf{z}_3$, $\mathbf{z}_4$ in previous results, $z(\varepsilon) \in Z_0$ for $(0,3)$, $(0,4)$, $(4,3)$ and $(3,4)$, and $z(\varepsilon) \notin Z_0$ for $(i,3)$ and $(i,4)$ $(i = 1, 2, 5)$.

Fourthly, we consider the case $(i,5)$ $(0 \leq i \leq 5)$. In this case, the column vector $\mathbf{z}_5(\varepsilon)$ is given as

$$\mathbf{z}_5(\varepsilon) = \mathbf{z}_i + \varepsilon(\mathbf{z}_5 - \mathbf{z}_i).$$

We check the conditions $\mathrm{rank}(p,q,5;z(\varepsilon)) = 3 \quad (0 \le p < q \le 4)$. In the case of $\mathbf{z}_i \in V_{pq}$ $(0 \le p < q \le 4)$, we have $\mathrm{rank}(p,q,5;z(\varepsilon)) = \mathrm{rank}(p,q,5;z(1)) = 3$ for the ε. In the case of $\mathbf{z}_i \notin V_{pq}$, we have

$$\det\begin{bmatrix}\mathbf{z}_p & \mathbf{z}_q & \mathbf{z}_5(\varepsilon)\end{bmatrix} = (1-\varepsilon)\det\begin{bmatrix}\mathbf{z}_p & \mathbf{z}_q & \mathbf{z}_i\end{bmatrix} + \varepsilon\det\begin{bmatrix}\mathbf{z}_p & \mathbf{z}_q & \mathbf{z}_5\end{bmatrix}.$$

Because $|\varepsilon|$ is sufficiently small, $\det\begin{bmatrix}\mathbf{z}_p & \mathbf{z}_q & \mathbf{z}_5(\varepsilon)\end{bmatrix}$ is a value close to $\det\begin{bmatrix}\mathbf{z}_p & \mathbf{z}_q & \mathbf{z}_i\end{bmatrix} \ne 0$. This means $\mathrm{rank}(p,q,5;z(\varepsilon)) = 3$. Hence, we have $z(\varepsilon) \notin Z_0$ for $(i,5) \quad (0 \le i \le 4)$.

Finally, in each case for (i,j) not listed in (11.20), there do not exist two triples (p,q,r) $(0 \le p < q < r \le 5)$ such that $\mathrm{rank}(p,q,r;z(\varepsilon)) = 2$ for $\varepsilon \ne 0$ with small absolute value. Hence Z_0 does not contain any matrix obtained by a permutation of columns of $z(\varepsilon)$. □

We take a pair of indices (i,j) listed in (11.20) of Proposition 11.5, and fix it. By using (11.19), we define our confluence $\mathcal{C}(i,j)$ on Appell's F_2 system (9.3) of hypergeometric differential equations, its solutions $f(x,y)$ in (9.5), and its Pfaffian system (9.6). To distinguish variables and parameters used in $\mathcal{C}(i,j)$ from those of the original Appell's F_2 system, we prepare

$$z' = \begin{bmatrix}\mathbf{z}'_0 & \cdots & \mathbf{z}'_5\end{bmatrix} = \begin{bmatrix}1 & 0 & 1 & 0 & 1 & 1\\ 0 & 1 & -1 & 0 & 0 & -x'\\ 0 & 0 & 0 & 1 & -1 & -y'\end{bmatrix}$$

with independent variables $(x',y') \in \mathbb{P}^2 \backslash S_{F_2}$, parameters $\alpha' = {}^t\begin{bmatrix}\alpha'_0 & \cdots & \alpha'_5\end{bmatrix}$, and coordinates $\begin{bmatrix}1 & t'_1 & t'_2\end{bmatrix}$ of $\mathbb{P}^2$ for our confluence $\mathcal{C}(i,j)$. We set

$$z'(\varepsilon) = \begin{bmatrix}\mathbf{z}'_0 & \cdots & \mathbf{z}'_j(\varepsilon) & \cdots & \mathbf{z}'_5\end{bmatrix}, \quad \mathbf{z}'_j(\varepsilon) = (1-\varepsilon)\mathbf{z}'_i + \varepsilon\mathbf{z}'_j.$$

Let $g(\varepsilon)$ and $h(\varepsilon)$ be elements of $\mathrm{GL}(3,\mathbb{C})$ and D_6 changing $z'(\varepsilon)$ into the standard form as in (11.3), and $x(z'(\varepsilon))$, $y(z'(\varepsilon))$ be the normalized variables of $z'(\varepsilon)$. By regarding $x(z'(\varepsilon))$, $y(z'(\varepsilon))$ as functions of x', y' and ε, we express them by

$$x(z'(\varepsilon)) = x(x',y',\varepsilon), \quad y(z'(\varepsilon)) = y(x',y',\varepsilon).$$

Definition 11.3 (Confluence $\mathcal{C}(i,j)$)**.** For a pair of indices (i,j) listed in (11.20) of Proposition 11.5, we set transformations

$$\begin{cases} x = x(x',y',\varepsilon), \\ y = y(x',y',\varepsilon), \end{cases} \qquad \begin{cases} \alpha_i = \alpha'_i + (1-1/\varepsilon)\alpha'_j, \\ \alpha_j = \alpha'_j/\varepsilon, \\ \alpha_k = \alpha'_k \quad (k \ne i,j), \end{cases} \tag{11.21}$$

$$\begin{bmatrix}1 & t_1 & t_2\end{bmatrix} = \begin{bmatrix}1 & t'_1 & t'_2\end{bmatrix} g(\varepsilon)^{-1}.$$

We define the confluence $\mathcal{C}(i,j)$ on Appell's F_2 system (9.3) and its solutions by applying the transformations (11.21) to (9.3) and its solutions, and taking limit $\varepsilon \to 0$ for the transformed ones with the factor $(\prod_{k=0}^{5} h_k(\varepsilon)^{\alpha_k(\varepsilon)})^{-1}$ in (11.19). We also define the confluence $\mathcal{C}(i,j)$ on the Pfaffian (9.6) by applying the transformations (11.21) to the connection matrix and local solutions of (9.6), and taking limit $\varepsilon \to 0$ for the transformed ones with the factor.

Remark 11.4. In the confluence $\mathcal{C}(i,j)$ on (9.6), there are cases where we need a transformation of the unknown function $\mathbf{F}_{F_2}(x,y)$ by a 4×4 matrix with ε; for details refer to subsections 12.2 and 12.3.

12 Detailed Calculations of Confluences

In this section, we execute our confluences. We study $\mathcal{C}(0,1)$, $\mathcal{C}(0,2)$, $\mathcal{C}(0,3)$ and $\mathcal{C}(0,4)$ in subsection 12.1, $\mathcal{C}(1,2)$, $\mathcal{C}(2,1)$, $\mathcal{C}(3,4)$ and $\mathcal{C}(4,3)$ in subsection 12.2, $\mathcal{C}(1,5)$, $\mathcal{C}(2,5)$, $\mathcal{C}(3,5)$ and $\mathcal{C}(4,5)$ in subsection 12.3, and $\mathcal{C}(0,5)$ in subsection 12.4.

12.1 Confluence from Appell's F_2 to Humbert's Ξ_1

To study the confluences $\mathcal{C}(0,1),\dots,\mathcal{C}(0,4)$, we introduce Humbert's hypergeometric series

$$\Xi_1(\alpha_1,\alpha_2,\beta,\gamma;x,y)=\sum_{m_1,m_2=0}^{\infty}\frac{(\alpha_1)_{m_1}(\alpha_2)_{m_2}(\beta)_{m_1}}{(\gamma)_{m_1+m_2}(1)_{m_1}(1)_{m_2}}x^{m_1}y^{m_2}, \tag{12.1}$$

where $\gamma\notin\mathbb{Z}_{\leq 0}$. This function satisfies Humbert's Ξ_1 system of differential equations:

$$\begin{cases}\left[x(1-x)\frac{\partial^2}{\partial x^2}+y\frac{\partial^2}{\partial x\partial y}+\{\gamma-(\alpha_1+\beta+1)x\}\frac{\partial}{\partial x}-\alpha_1\beta\right]\Xi_1(x,y)=0,\\ \left[y\frac{\partial^2}{\partial y^2}+x\frac{\partial^2}{\partial x\partial y}+\{\gamma-y\}\frac{\partial}{\partial y}-\alpha_2\right]\Xi_1(x,y)=0.\end{cases} \tag{12.2}$$

In this subsection, we show that the Pfaffian system (9.6) of Appell's F_2 converges to a Pfaffian system equivalent to Humbert's Ξ_1 system of differential equations by the confluences $\mathcal{C}(0,1),\dots,\mathcal{C}(0,4)$. At first, we consider the confluence $\mathcal{C}(0,4)$ given by the deformation

$$\mathbf{z}_4'(\varepsilon)=(1-\varepsilon)\mathbf{z}_0'+\varepsilon\mathbf{z}_4', \tag{12.3}$$

by an infinitesimal parameter ε. The deformed matrix is

$$z'(\varepsilon):=\begin{bmatrix}1&0&1&0&1&1\\0&1&-1&0&0&-x'\\0&0&0&1&-\varepsilon&-y'\end{bmatrix},$$

and its standard form is

$$g\cdot z'(\varepsilon)\cdot h=\begin{bmatrix}1&0&1&0&1&1\\0&1&-1&0&0&-x'\\0&0&0&1&-1&-y'/\varepsilon\end{bmatrix},$$

where $g=\mathrm{diag}(1,1,1/\varepsilon)$ and $h=\mathrm{diag}(1,1,1,\varepsilon,1,1)$. The variables of integral in (9.4) are transformed by

$$\begin{bmatrix}1&t_1'&t_2'\end{bmatrix}g^{-1}=\begin{bmatrix}1&t_1&t_2\end{bmatrix}.$$

Hence, we have transformations

$$\begin{cases}t_1=t_1',\\ t_2=\varepsilon t_2',\end{cases}\qquad\begin{cases}x=x',\\ y=y'/\varepsilon.\end{cases} \tag{12.4}$$

The parameters α_0, α_4 are transformed by

$$\begin{bmatrix}\alpha_0\\ \alpha_4\end{bmatrix} = q(\varepsilon)^{-1}\begin{bmatrix}\alpha_0'\\ \alpha_4'\end{bmatrix}. \tag{12.5}$$

We set $\alpha_4' := -d$, and

$$\alpha_4 = (c_2 - b_2) = -d/\varepsilon, \quad \alpha_0 = (a - c_1 - b_2) + d/\varepsilon. \tag{12.6}$$

Here, the non-zero parameter d is introduced in order to give Pfaffian systems the homogeneity of parameter as in Theorems 7.1, 8.1. As $\varepsilon \to 0$, the integrand u_{F_2} converges to

$$\lim_{\varepsilon\to 0}(h_3^{-\alpha_3}u_{F_2}) = \lim_{\varepsilon\to 0}(\varepsilon^{-b_2}u_{F_2}) = {t'}_1^{b_1}(1-t_1')^{c_1-b_1}{t'}_2^{b_2}\exp(dt_2')(1-x't_1'-y't_2')^{-a}.$$

Thus we have Euler type integrals expressing solutions to some system of differential equations by the confluence $\mathcal{C}(0,4)$ on those to Appell's F_2 system (9.3). To change this system into Humbert's Ξ_1 system in (12.2), we consider further transformations

$$\begin{cases} t_1 = t_1' = x''t_1'', \\ t_2 = \varepsilon t_2' = \varepsilon y'' t_2'', \end{cases} \quad \begin{cases} x = x' = 1/x'', \\ y = y'/\varepsilon = 1/(\varepsilon y''). \end{cases} \tag{12.7}$$

As $\varepsilon \to 0$, the integrand u_{F_2} converges to

$$\lim_{\varepsilon\to 0}((x'')^{-b_1}(\varepsilon y'')^{-b_2}u_{F_2}) = {t''}_1^{b_1}(1-x''t_1'')^{c_1-b_1}{t''}_2^{b_2}\exp(dy''t_2'')(1-t_1''-t_2'')^{-a},$$

which is denoted by u_{Ξ_1}. Applying the replacements (12.7) and (12.6) to (9.3), we have a system

$$\begin{cases} \Big[x''(1-x'')\frac{\partial^2}{\partial {x''}^2} + y''\frac{\partial^2}{\partial x''\partial y''} + \{(b_1+b_2-a+1) - (2b_1-c_1+2)x''\}\frac{\partial}{\partial x''} \\ \qquad\qquad -b_1(b_1-c_1+1)\Big]\,(x''^{-b_1}(\varepsilon y'')^{-b_2}f(\frac{1}{x''},\frac{1}{\varepsilon y''})) = 0, \\ \Big[y''(1-\varepsilon y'')\frac{\partial^2}{\partial {y''}^2} + x''\frac{\partial^2}{\partial x''\partial y''} + \{(b_1+b_2-a+1) - (d+\varepsilon b_2+2\varepsilon)y''\}\frac{\partial}{\partial y''} \\ \qquad\qquad -b_2(d+\varepsilon)\Big]\,(x''^{-b_1}(\varepsilon y'')^{-b_2}f(\frac{1}{x''},\frac{1}{\varepsilon y''})) = 0. \end{cases}$$

By the confluence, this system converges to

$$\begin{cases} \Big[x''(1-x'')\frac{\partial^2}{\partial {x''}^2} + y''\frac{\partial^2}{\partial x''\partial y''} + \{(b_1+b_2-a+1) - (2b_1-c_1+2)x''\}\frac{\partial}{\partial x''} \\ \qquad\qquad -b_1(b_1-c_1+1)\Big]\,g(x'',y'') = 0, \\ \Big[y''\frac{\partial^2}{\partial {y''}^2} + x''\frac{\partial^2}{\partial x''\partial y''} + \{(b_1+b_2-a+1) - dy''\}\frac{\partial}{\partial y''} - b_2 d\Big]\,g(x'',y'') = 0, \end{cases} \tag{12.8}$$

which is Humbert's Ξ_1 system of differential equations. If $\Re(dy') < 0$, then the solution (9.2) to the system (9.3) converges to

$$\lim_{\varepsilon\to 0} x''^{-b_1}(\varepsilon y'')^{-b_2}f(\frac{1}{x''},\frac{1}{\varepsilon y''}) = \iint_{\Delta''} u_{\Xi_1}\varphi_{\Xi_1},$$

where $\Delta'' = (0, 1/x'') \times (0,\infty)$ and $\varphi_{\Xi_1} = \mathrm{d}t_1'' \wedge \mathrm{d}t_2''/(t_1''t_2''(1-x''t_1''))$. We define a function

$$g(x'',y'') := \iint_{\Delta''} u_{\Xi_1}\varphi_{\Xi_1}, \tag{12.9}$$

which is a solution to (12.8). By the confluence, f_i converges to

$$g_i(x'', y'') := \lim_{\varepsilon \to 0} x''^{-b_1} (\varepsilon y'')^{-b_2} f_i(\frac{1}{x''}, \frac{1}{\varepsilon y''}).$$

The solution g is expressed by a linear combination

$$g = g_0 - g_1 - g_2 + g_3. \tag{12.10}$$

Applying the replacements (12.4) and (12.6) to (9.6), we have the system

$$\begin{aligned}
\partial_{x''}(x^{b_1}y^{b_2})\mathbf{F}_{F_2} &= \left\{ \frac{-\mathrm{d}x''}{x''}(b_1 I_4 + W_{1,F} + W_{2,F}) + \frac{\mathrm{d}x''}{x''-1} W_{2,F} - \frac{\varepsilon y'' \mathrm{d}x''}{x''(x''+\varepsilon y'' - \varepsilon x'' y'')} W_{5,F} \right\} (x^{b_1}y^{b_2})\mathbf{F}_{F_2}, \\
\partial_{y''}(x^{b_1}y^{b_2})\mathbf{F}_{F_2} &= \left\{ \frac{-\mathrm{d}y''}{y''}(b_2 I_4 + W_{3,F} + W_{4,F} + W_{5,F}) + \frac{\varepsilon \mathrm{d}y''}{\varepsilon y'' - 1} W_{4,F} \right. \\
&\qquad \left. + \frac{\varepsilon(1-x'')\mathrm{d}y'}{x''+\varepsilon y'' - \varepsilon x'' y''} W_{5,F} \right\} (x^{b_1}y^{b_2})\mathbf{F}_{F_2},
\end{aligned}$$

and as $\varepsilon \to 0$, this system converges to

$$\begin{aligned}
\partial_{x''}\mathbf{F}_{\Xi_1} &= \left\{ \tfrac{\mathrm{d}x''}{x''} W_{1,\Xi} + \tfrac{\mathrm{d}x''}{x''-1} W_{2,\Xi} - \tfrac{y''\mathrm{d}x''}{x''^2} W_{5,\Xi} \right\} \mathbf{F}_{\Xi_1}, \\
\partial_{y''}\mathbf{F}_{\Xi_1} &= \left\{ \tfrac{\mathrm{d}y''}{y''} W_{3,\Xi} + \mathrm{d}y'' W_{4,\Xi} + \tfrac{\mathrm{d}y''}{x''} W_{5,\Xi} \right\} \mathbf{F}_{\Xi_1},
\end{aligned} \tag{12.11}$$

where $W_{2,\Xi} := W_{2,F}$, $W_{1,\Xi} := -(b_1 I_4 + W_{1,F} + W_{2,F})$, $W_{3,\Xi} := -(b_2 I_4 + W_{3,F} + W_{4,F} + W_{5,F})$, $W_{4,\Xi} := \lim_{\varepsilon \to 0}(-\varepsilon)(W_{4,F} + W_{5,F})$, $W_{5,\Xi} := \lim_{\varepsilon \to 0}(\varepsilon W_{5,F})$ and $\mathbf{F}_{\Xi_1} := {}^t\begin{bmatrix} g_0 & g_1 & g_2 & g_3 \end{bmatrix}$. They are

$$W_{1,\Xi} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ b_1 & a-b_1-b_2 & 0 & 0 \\ 0 & 0 & 0 & c_1-b_1 \\ 0 & -b_2 & b_1 & c_1-2b_1 \end{bmatrix}, \quad W_{2,\Xi} = \begin{bmatrix} 0 & c_1-b_1 & 0 & 0 \\ 0 & c_1-b_1+b_2-a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \end{bmatrix},$$

$$W_{3,\Xi} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_2 & 0 & a-b_1-b_2 & b_1-c_1 \\ 0 & b_2 & -b_1 & a+b_1-c_1-b_2 \end{bmatrix}, \; W_{4,\Xi} = \begin{bmatrix} 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \end{bmatrix}, \; W_{5,\Xi} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d \end{bmatrix}.$$

Using (12.10) and (12.11), we have partial derivatives $(\partial/\partial x'')g$, $(\partial/\partial y'')g$ and $(\partial^2/\partial x''\partial y'')g$ of g, which are expressed by

$$\begin{aligned}
\frac{\partial}{\partial x''} g &= \frac{-b_1}{x''}(g_0 - g_1 - g_2 + g_3) + \frac{a}{x''(x''-1)} g_1, \\
\frac{\partial}{\partial y''} g &= \frac{-b_2}{y''}(g_0 - g_1 - g_2 + g_3) + \frac{a}{y''}(g_3 - g_2), \\
\frac{\partial^2}{\partial x''\partial y''} g &= \frac{b_1 b_2}{x'' y''}(g_0 - g_1 - g_2 + g_3) - \frac{a b_1}{x'' y''}(g_3 - g_2) + \frac{ad}{x''^2} g_3.
\end{aligned}$$

Thus we have the following.

Proposition 12.1. *By the confluence $\mathcal{C}(0,4)$, the Pfaffian system (9.6) converges to the Pfaffian system (12.11), which is equivalent to Humbert's Ξ_1 system (12.2) of differential equations and the matrices $W_{1,\Xi}, \ldots, W_{5,\Xi}$ depend only on the parameters. This system is called a Pfaffian system of Humbert's Ξ_1.*

Next, we consider the confluence $\mathcal{C}(0,3)$. Before introducing an infinitesimal parameter ε, we give a standard form of

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & -x \\ 0 & 0 & 0 & -1 & 1 & -y \end{bmatrix}.$$

It is

$$g \cdot \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & -\tilde{x} \\ 0 & 0 & 0 & 1 & -1 & -\tilde{y} \end{bmatrix} \cdot h = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & -\tilde{x}/(1-\tilde{y}) \\ 0 & 0 & 0 & -1 & 1 & \tilde{y}/(1-\tilde{y}) \end{bmatrix},$$

where

$$g = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad h = \mathrm{diag}(1,1,1,1,1,\frac{1}{1-\tilde{y}}).$$

Now, we use the following transformation

$$t_2 = 1 - \tilde{t}_2, \quad \begin{cases} x = \frac{\tilde{x}}{1-\tilde{y}}, \\ y = \frac{-\tilde{y}}{1-\tilde{y}}, \end{cases} \tag{12.12}$$

and exchange the role of $\mathbf{z}_3$ and $\mathbf{z}_4$ in the study of $\mathcal{C}(0,4)$, and define a new integrand

$$\tilde{u}_{F_2} := (\frac{1}{1-\tilde{y}})^a u_{F_2} = t_1^{b_1}(1-t_1)^{c_1-b_1}\tilde{t}_2^{c_2-b_2}(1-\tilde{t}_2)^{b_2}(1-\tilde{x}t_1-\tilde{y}\tilde{t}_2)^{-a}.$$

Applying (12.12) to (9.3), we have

$$\begin{cases} \left[\tilde{x}(1-\tilde{x})\frac{\partial^2}{\partial\tilde{x}^2} - \tilde{x}\tilde{y}\frac{\partial^2}{\partial\tilde{x}\partial\tilde{y}} + \{c_1 - (a+b_1+1)\tilde{x}\}\frac{\partial}{\partial\tilde{x}} - b_1\tilde{y}\frac{\partial}{\partial\tilde{y}} - ab_1\right]\tilde{f}(\tilde{x},\tilde{y}) = 0, \\ \Big[\tilde{y}(1-\tilde{y})\frac{\partial^2}{\partial\tilde{y}^2} - \tilde{x}\tilde{y}\frac{\partial^2}{\partial\tilde{x}\partial\tilde{y}} + \{c_2 - (a+c_2-b_2+1)\tilde{y}\}\frac{\partial}{\partial\tilde{y}} \\ \qquad\qquad -(c_2-b_2)\tilde{x}\frac{\partial}{\partial\tilde{x}} - a(c_2-b_2)\Big]\tilde{f}(\tilde{x},\tilde{y}) = 0, \end{cases} \tag{12.13}$$

where $\tilde{f}(\tilde{x},\tilde{y}) := (1-\tilde{y})^{-a} f(\tilde{x}/(1-\tilde{y}), -\tilde{y}/(1-\tilde{y}))$. There is a solution $\tilde{f}(\tilde{x},\tilde{y})$ to (12.13) expressed by the integral

$$\tilde{f}(\tilde{x},\tilde{y}) = \iint_{\tilde{\Delta}} \tilde{u}_{F_2}\tilde{\varphi}_{F_2},$$

where $\tilde{\Delta} = (0,1)\times(0,1)$ and $\tilde{\varphi}_{F_2} = \mathrm{d}t_1 \wedge \mathrm{d}\tilde{t}_2/(t_1(t_1-1)\tilde{t}_2(\tilde{t}_2-1))$. Apply (12.12) to

$$\tilde{f}_i(\tilde{x},\tilde{y}) := \iint_{\tilde{\Delta}} \tilde{u}_{F_2}\varphi_{i,F} \quad (0 \le i \le 3)$$

and (9.6), then we have a Pfaffian system of Appell's F_2:

$$\begin{aligned} \partial_{\tilde{x}}\tilde{\mathbf{F}}_{F_2} &= \left\{\tfrac{\mathrm{d}\tilde{x}}{\tilde{x}}W_{1,F} + \tfrac{\mathrm{d}\tilde{x}}{\tilde{x}-1}W_{5,F} + \tfrac{\mathrm{d}\tilde{x}}{\tilde{x}+\tilde{y}-1}W_{2,F}\right\}\tilde{\mathbf{F}}_{F_2}, \\ \partial_{\tilde{y}}\tilde{\mathbf{F}}_{F_2} &= \left\{\tfrac{\mathrm{d}\tilde{y}}{\tilde{y}}W_{3,F} + \tfrac{\mathrm{d}\tilde{y}}{\tilde{y}-1}W_{0,F} + \tfrac{\mathrm{d}\tilde{y}}{\tilde{x}+\tilde{y}-1}W_{2,F}\right\}\tilde{\mathbf{F}}_{F_2}, \end{aligned} \tag{12.14}$$

where

$$W_{0,F} := -(aI_4 + \sum_{i=1}^{5} W_{i,F}) = \begin{bmatrix} b_1+b_2-a & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 \\ b_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{F}}_{F_2} := \begin{bmatrix} \tilde{f}_0 \\ \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \end{bmatrix}.$$

The confluence $\mathcal{C}(0,3)$ consists of the transformation

$$\begin{cases} t_1 = x' t'_1, \\ \tilde{t}_2 = \varepsilon y' t'_2, \end{cases} \quad \begin{cases} \tilde{x} = \frac{1}{x'}, \\ \tilde{y} = \frac{1}{\varepsilon y'}, \end{cases} \quad \begin{cases} b_2 = -d/\varepsilon, \\ c_2 = c' - d/\varepsilon. \end{cases} \tag{12.15}$$

and the limit $\varepsilon \to 0$, and the integrand $\tilde{u}_{F_2}$ converges to

$$\lim_{\varepsilon\to 0}(\tilde{x}^{b_1}\tilde{y}^{c'}\tilde{u}_{F_2}) = {t'}_1^{b_1}(1-x't'_1)^{c_1-b_1}{t'}_2^{c'}\exp(dy't'_2)(1-t'_1-t'_2)^{-a},$$

which is denoted by $\tilde{u}_{\Xi_1}$. By the confluence, the system (12.13) converges to

$$\begin{cases} \Big[x'(1-x')\frac{\partial^2}{\partial x'^2} + y'\frac{\partial^2}{\partial x'\partial y'} + \{(b_1+c'-a+1) - (2b_1-c_1+2)x'\}\frac{\partial}{\partial x'} \\ \qquad\qquad -b_1(b_1-c_1+1)\Big]\,\tilde{g}(x',y') = 0, \\ \Big[y'\frac{\partial^2}{\partial y'^2} + x'\frac{\partial^2}{\partial x'\partial y'} + \{(b_1+c'-a+1) - dy'\}\frac{\partial}{\partial y'} - c'd\Big]\,\tilde{g}(x',y') = 0, \end{cases} \tag{12.16}$$

which is Humbert's Ξ_1 system of differential equations, too. If $\Re(dy') < 0$, then the solution $\tilde{f}$ converges to

$$\tilde{g}(x',y') := \lim_{\varepsilon\to 0} {x'}^{-b_1}(\varepsilon y')^{-c'}\tilde{f}(\frac{1}{x'},\frac{1}{\varepsilon y'}).$$

By the confluence, the Pfaffian system (12.14) converges to

$$\begin{aligned} \partial_{x'}\tilde{F}_{\Xi_1} &= \left\{\tfrac{\mathrm{d}x'}{x'}\tilde{W}_{1,\Xi} + \tfrac{\mathrm{d}x'}{x'-1}\tilde{W}_{2,\Xi} - \tfrac{y'\mathrm{d}x'}{x'^2}\tilde{W}_{5,\Xi}\right\}\tilde{F}_{\Xi_1}, \\ \partial_{y'}\tilde{F}_{\Xi_1} &= \left\{\tfrac{\mathrm{d}y'}{y'}\tilde{W}_{3,\Xi} + \mathrm{d}y'\tilde{W}_{4,\Xi} + \tfrac{\mathrm{d}y'}{x'}\tilde{W}_{5,\Xi}\right\}\tilde{F}_{\Xi_1}, \end{aligned} \tag{12.17}$$

where $\tilde{W}_{2,\Xi} := W_{5,F}$, $\tilde{W}_{1,\Xi} := -(b_1 I_4 + W_{1,F} + W_{5,F})$, $\tilde{W}_{3,\Xi} := -(c' I_4 + W_{3,F} + W_{0,F} + W_{2,F})$, $\tilde{W}_{4,\Xi} := \lim_{\varepsilon\to 0}(-\varepsilon)(W_{0,F}+W_{2,F})$, $\tilde{W}_{5,\Xi} := \lim_{\varepsilon\to 0}(\varepsilon W_{2,F})$, $\tilde{F}_{\Xi_1} := {}^t\Big[\tilde{g}_0 \quad \tilde{g}_1 \quad \tilde{g}_2 \quad \tilde{g}_3\Big]$ and $\tilde{g}_i(x',y') := \lim_{\varepsilon\to 0} {x'}^{-b_1}(\varepsilon y')^{-c'}\tilde{f}_i(1/x', 1/(\varepsilon y'))$ $(0 \le i \le 3)$. They are

$$\tilde{W}_{1,\Xi} = \begin{bmatrix} 0 & c_1-b_1 & 0 & 0 \\ b_1 & c_1-2b_1 & 0 & -c' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_1 & a-b_1-c' \end{bmatrix}, \quad \tilde{W}_{2,\Xi} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2-b_2 \\ 0 & 0 & 0 & c_1-b_1 \\ 0 & 0 & 0 & c_1+c_2-b_1-b_2-a \end{bmatrix},$$

$$\tilde{W}_{3,\Xi} = \begin{bmatrix} a-b_1-c' & b_1-c_1 & c' & 0 \\ -b_1 & a+b_1-c_1-c' & 0 & c' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{W}_{4,\Xi} = \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{bmatrix}, \quad \tilde{W}_{5,\Xi} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -d & 0 & 0 \end{bmatrix}.$$

Proposition 12.2. *The Pfaffian system (12.14) of Appell's F_2 converges to the Pfaffian system (12.17) by the confluence $\mathcal{C}(0,3)$. This system is equivalent to the Pfaffian system (12.11) of Humbert's Ξ_1 under Definition 10.1.*

Proof. By replacing $c' = b_2$ in (12.17) and using a matrix

$$\tilde{U} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

we have

$$\tilde{U}\tilde{W}_{i,\Xi}\tilde{U}^{-1} = W_{i,\Xi} \quad (1 \leq i \leq 5),$$

which shows the equivalence. □

To study the confluences $\mathcal{C}(0,1)$ and $\mathcal{C}(0,2)$, replace the role of x and y and that of (b_1, c_1) and (b_2, c_2) in the previous consideration. Then we have the following.

Corollary 12.1. *By each of the confluences $\mathcal{C}$(0,1) and $\mathcal{C}$(0,2), the Pfaffian system (9.6) of Appell's F_2 converges to a Pfaffian system equivalent to the Pfaffian system (12.11) of Humbert's Ξ_1 under Definition 10.1.*

12.2 Confluence from Appell's F_2 to Humbert's Ψ_1

In order to study the confluences $\mathcal{C}(1,2)$, $\mathcal{C}(2,1)$, $\mathcal{C}(3,4)$ and $\mathcal{C}(4,3)$, we introduce Humbert's hypergeometric series

$$\Psi_1(\alpha, \beta, \gamma_1, \gamma_2; x, y) = \sum_{m_1, m_2 = 0}^{\infty} \frac{(\alpha)_{m_1+m_2}(\beta)_{m_1}}{(\gamma_1)_{m_1}(\gamma_2)_{m_2}(1)_{m_1}(1)_{m_2}} x^{m_1} y^{m_2}, \tag{12.18}$$

where $\gamma_1, \gamma_2 \notin \mathbb{Z}_{\leq 0}$. This function satisfies Humbert's Ψ_1 system of differential equations:

$$\begin{cases} \left[x(1-x)\frac{\partial^2}{\partial x^2} - xy\frac{\partial^2}{\partial x \partial y} + \{\gamma_1 - (\alpha+\beta+1)x\}\frac{\partial}{\partial x} - \beta y \frac{\partial}{\partial y} - \alpha\beta \right] \Psi_1(x,y) = 0, \\ \left[y\frac{\partial^2}{\partial y^2} + \{\gamma_2 - y\}\frac{\partial}{\partial y} - x\frac{\partial}{\partial x} - \alpha \right] \Psi_1(x,y) = 0. \end{cases} \tag{12.19}$$

In this subsection, we show that a Pfaffian system equivalent to (9.6) of Appell's F_2 converges to a Pfaffian system equivalent to Humbert's Ψ_1 system of differential equations by the confluences $\mathcal{C}(1,2)$, $\mathcal{C}(2,1)$, $\mathcal{C}(3,4)$ and $\mathcal{C}(4,3)$. At first, we consider the confluence $\mathcal{C}(3,4)$. We multiply the integrand u_{F_2} by $(-1)^{\alpha_4}$, and give the deformation

$$\mathbf{z}'_4(\varepsilon) = (1-\varepsilon)\mathbf{z}'_3 + \varepsilon \mathbf{z}'_4.$$

The deformed matrix becomes

$$z'(\varepsilon) = \begin{bmatrix} 1 & 0 & 1 & 0 & -\varepsilon & 1 \\ 0 & 1 & -1 & 0 & 0 & -x' \\ 0 & 0 & 0 & 1 & 1 & -y' \end{bmatrix}.$$

Its standard form is

$$g \cdot z'(\varepsilon) \cdot h = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & -x' \\ 0 & 0 & 0 & 1 & -1 & -\varepsilon y' \end{bmatrix},$$

where $g = \mathrm{diag}(1, 1, \varepsilon)$ and $h = \mathrm{diag}(1, 1, 1, 1/\varepsilon, -1/\varepsilon, 1)$. Hence, we use replacements

$$t_2 = t_2'/\varepsilon, \quad y = \varepsilon y', \quad b_2 = c_2 + d/\varepsilon. \tag{12.20}$$

As $\varepsilon \to 0$, $\mathbf{z}_4'(\varepsilon) = {}^t\begin{bmatrix} -\varepsilon & 0 & 1 \end{bmatrix}$ approaches to $\mathbf{z}_3' = {}^t\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. In this case, the integrand u_{F_2} multiplied by $(-1)^{\alpha_4}(\varepsilon)^{\alpha_3+\alpha_4}$ converges to

$$\lim_{\varepsilon\to 0}((-1)^{c_2-b_2}\varepsilon^{c_2}u_{F_2}) = t_1^{b_1}(1-t_1)^{c_1-b_1}t_2'^{c_2}\exp(\frac{d}{t_2'})(1-xt_1-y't_2')^{-a}. \tag{12.21}$$

To obtain Euler type integrals expressing solutions to Humbert's Ψ_1 system, we multiply this function by y'^{c_2-1}, which is denoted by u_{Ψ_1}. By this confluence, the system (9.3) converges to

$$\begin{cases} \Big[x(1-x)\frac{\partial^2}{\partial x^2} - xy'\frac{\partial^2}{\partial x\partial y'} + \{c_1 - (a+b_1-c_2+2)x\}\frac{\partial}{\partial x} \\ \qquad\qquad -b_1y'\frac{\partial}{\partial y'} - (a-c_2+1)b_1\Big]p(x,y') = 0, \\ \Big[y'\frac{\partial^2}{\partial y'^2} + \{(2-c_2) - dy'\}\frac{\partial}{\partial y'} - dx\frac{\partial}{\partial x} - (a-c_2+1)d'\Big]p(x,y') = 0, \end{cases} \tag{12.22}$$

which is Humbert's Ψ_1 system of differential equation systems. If $\Re(d) < 0$, then the solution (9.2) to the system (9.3) converges to

$$\lim_{\varepsilon\to 0}\varepsilon^{c_2-1}y'^{c_2-1}f(x,\varepsilon y') = p(x,y').$$

In this case, we can see

$$\lim_{\varepsilon\to 0}\varphi_{0,F} = \lim_{\varepsilon\to 0}\varphi_{2,F} = \frac{\mathrm{d}t_1\wedge\mathrm{d}t_2'}{(1-xt_1-y't_2')t_1t_2'}, \quad \lim_{\varepsilon\to 0}\varphi_{1,F} = \lim_{\varepsilon\to 0}\varphi_{3,F} = \frac{(1-x)\mathrm{d}t_1\wedge\mathrm{d}t_2'}{(1-xt_1-y't_2')(t_1-1)t_2'},$$

which mean that the independence of $\varphi_{0,F}$ and $\varphi_{2,F}$ and that of $\varphi_{1,F}$ and $\varphi_{3,F}$ are not kept under this limit. We transform $\mathbf{F}_{F_2}$ into

$$\dot{\mathbf{F}}_{F_2} = \begin{bmatrix} \dot{f}_0 \\ \dot{f}_1 \\ \dot{f}_2 \\ \dot{f}_3 \end{bmatrix} := \dot{U}\mathbf{F}_{F_2}, \quad \dot{U} := \begin{bmatrix} 1/\varepsilon & 0 & -1/\varepsilon & 0 \\ 0 & 1/\varepsilon & 0 & -1/\varepsilon \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $\dot{\mathbf{F}}_{F_2}$ satisfies a Pfaffian system equivalent to (9.6):

$$\begin{aligned} \partial_x\dot{\mathbf{F}}_{F_2} &= \Big\{\tfrac{\mathrm{d}x}{x}\dot{W}_{1,F} + \tfrac{\mathrm{d}x}{x-1}\dot{W}_{2,F} + \tfrac{\mathrm{d}x}{x+y-1}\dot{W}_{5,F}\Big\}\dot{\mathbf{F}}_{F_2}, \\ \partial_y\dot{\mathbf{F}}_{F_2} &= \Big\{\tfrac{\mathrm{d}y}{y}\dot{W}_{3,F} + \tfrac{\mathrm{d}y}{y-1}\dot{W}_{4,F} + \tfrac{\mathrm{d}y}{x+y-1}\dot{W}_{5,F}\Big\}\dot{\mathbf{F}}_{F_2}, \end{aligned} \tag{12.23}$$

where

$$\dot{W}_{i,F} := \dot{U}W_{i,F}\dot{U}^{-1} \quad (1 \le i \le 5).$$

Applying the deformation (12.20) to (12.23), we have the system

$$\begin{aligned} \partial_x(\varepsilon y^{c_2-1})\dot{\mathbf{F}}_{F_2} &= \Big\{\tfrac{\mathrm{d}x}{x}\dot{W}_{1,F} + \tfrac{\mathrm{d}x}{x-1}(\dot{W}_{2,F}+\dot{W}_{5,F}) - \tfrac{\varepsilon y'\mathrm{d}x}{(x-1)(x+\varepsilon y'-1)}\dot{W}_{5,F}\Big\}(\varepsilon y^{c_2-1})\dot{\mathbf{F}}_{F_2}, \\ \partial_{y'}(\varepsilon y^{c_2-1})\dot{\mathbf{F}}_{F_2} &= \Big\{\tfrac{\mathrm{d}y'}{y'}((c_2-1)I_4+\dot{W}_{3,F}) + \tfrac{\varepsilon\mathrm{d}y'}{\varepsilon y'-1}\dot{W}_{4,F} + \tfrac{\varepsilon\mathrm{d}y'}{x+\varepsilon y'-1}\dot{W}_{5,F}\Big\}(\varepsilon y^{c_2-1})\dot{\mathbf{F}}_{F_2}, \end{aligned}$$

and as $\varepsilon \to 0$, this system converges to

$$\begin{aligned} \partial_x \mathbf{F}_{\Psi_1} &= \left\{ \tfrac{\mathrm{d}x}{x} W_{1,\Psi} + \tfrac{\mathrm{d}x}{x-1} W_{2,\Psi} - \tfrac{y'\mathrm{d}x}{(x-1)^2} W_{5,\Psi} \right\} \mathbf{F}_{\Psi_1}, \\ \partial_{y'} \mathbf{F}_{\Psi_1} &= \left\{ \tfrac{\mathrm{d}y'}{y'} W_{3,\Psi} + \mathrm{d}y' W_{4,\Psi} + \tfrac{\mathrm{d}y'}{x-1} W_{5,\Psi} \right\} \mathbf{F}_{\Psi_1}, \end{aligned} \tag{12.24}$$

where $W_{1,\Psi} := W_{1,F}$, $W_{2,\Psi} := \lim_{\varepsilon\to 0}(\dot{W}_{2,F} + \dot{W}_{5,F})$, $W_{3,\Psi} := \lim_{\varepsilon\to 0}\big((c_2-1)I_4 + \dot{W}_{3,F}\big)$, $W_{4,\Psi} := \lim_{\varepsilon\to 0}(-\varepsilon \dot{W}_{4,F})$, $W_{5,\Psi} := \lim_{\varepsilon\to 0}(\varepsilon \dot{W}_{5,F})$, $\mathbf{F}_{\Psi_1} := {}^t\Big[p_0 \quad p_1 \quad p_2 \quad p_3\Big]$ and $p_i(x,y') := \lim_{\varepsilon\to 0} \varepsilon^{c_2} (y')^{c_2-1} \dot{f}_i(x, \varepsilon y')$ $(0 \le i \le 3)$. They are

$$W_{1,\Psi} = \begin{bmatrix} -b_1 & b_1-c_1 & 0 & 0 \\ -b_1 & b_1-c_1 & 0 & 0 \\ 0 & 0 & -b_1 & b_1-c_1 \\ 0 & 0 & -b_1 & b_1-c_1 \end{bmatrix}, \quad W_{2,\Psi} = \begin{bmatrix} 0 & c_1-b_1 & 0 & 0 \\ 0 & c_1-b_1-a & 0 & 0 \\ 0 & 0 & 0 & c_1-b_1 \\ 0 & d & 0 & c_1+c_2-b_1-a \end{bmatrix},$$

$$W_{3,\Psi} = \begin{bmatrix} c_2-1 & 0 & 0 & 0 \\ 0 & c_2-1 & 0 & 0 \\ -d & 0 & -1 & 0 \\ 0 & -d & 0 & -1 \end{bmatrix}, \; W_{4,\Psi} = \begin{bmatrix} 0 & 0 & b_1-a & 0 \\ 0 & 0 & b_1 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; W_{5,\Psi} = \begin{bmatrix} 0 & 0 & 0 & b_1-c_1 \\ 0 & 0 & 0 & a+b_1-c_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d \end{bmatrix}.$$

The solution $p(x,y')$ to (12.22) satisfies

$$p = p_0 - p_1, \quad \frac{\partial}{\partial x} p = \frac{a}{x-1} p_1 + \frac{ay'}{(x-1)^2} p_3, \quad \frac{\partial}{\partial y'} p = \frac{c_2-1}{y'} p_0 - \frac{c_2-1}{y'} p_1 - a p_2 - \frac{a}{x-1} p_3,$$
$$\frac{\partial^2}{\partial x \partial y'} p = \left\{ \frac{a(a+b_1-c_1)}{(x-1)^2} - \frac{ady'}{(x-1)^3} \right\} p_3 + \left\{ \frac{a(c_2-1)}{(x-1)y'} - \frac{ad}{(x-1)^2} \right\} p_1 + \frac{ab_1}{(x-1)} p_2.$$

Thus we have the following.

Proposition 12.3. *By the confluence $\mathcal{C}(3,4)$, the Pfaffian system (12.23) of Appell's F_2 converges to the Pfaffian system (12.24), which is equivalent to Humbert's Ψ_1 system (12.19) of differential equations and the matrices $W_{1,\Psi}, \ldots, W_{5,\Psi}$ depend only on the parameters. This system is called a Pfaffian system of Humbert's Ψ_1.*

Next, we consider the confluence $\mathcal{C}(4,3)$. Before introducing an infinitesimal parameter ε, we exchange parameters b_2 and $(c_2 - b_2)$ and use the transformation (12.12). The confluence $\mathcal{C}(4,3)$ consists of the transformation

$$\tilde{t}_2 = t_2'/\varepsilon, \quad \tilde{y} = \varepsilon y', \quad b_2 = -d/\varepsilon. \tag{12.25}$$

and the limit $\varepsilon \to 0$, and the integrand $\tilde{u}_{F_2}$ converges to

$$\lim_{\varepsilon\to 0}((-1)^{b_2} \varepsilon \tilde{y}^{c_2-1} \tilde{u}_{F_2}) = y'^{c_2-1} t_1^{b_1} (1-t_1)^{c_1-b_1} t'^{c_2}_2 \exp(\frac{d}{t_2'})(1 - \tilde{x}t_1 - y't_2')^{-a}.$$

By this confluence, the system (12.13) converges to

$$\begin{cases} \Big[\tilde{x}(1-\tilde{x})\frac{\partial^2}{\partial \tilde{x}^2} - \tilde{x}y'\frac{\partial^2}{\partial \tilde{x}\partial y'} + \{c_1 - (a+b_1-c_2+2)\tilde{x}\}\frac{\partial}{\partial \tilde{x}} \\ \qquad\qquad -b_1 y' \frac{\partial}{\partial y'} - (a-c_2+1)b_1\Big]\tilde{p}(\tilde{x},y') = 0, \\ \Big[y'\frac{\partial^2}{\partial y'^2} + \{(2-c_2) - dy'\}\frac{\partial}{\partial y'} - dx\frac{\partial}{\partial \tilde{x}} - (a-c_2+1)d\Big]\tilde{p}(\tilde{x},y') = 0, \end{cases} \tag{12.26}$$

where

$$\tilde{p}(\tilde{x}, y') := \lim_{\varepsilon \to 0} \varepsilon^{c_2 - 1} y'^{c_2 - 1} \tilde{f}(\tilde{x}, \varepsilon y').$$

This is Humbert's Ψ_1 system of differential equations, too. In this case, we can see

$$\lim_{\varepsilon \to 0} \varphi_{0,F} = \lim_{\varepsilon \to 0} \varphi_{2,F} = \frac{\mathrm{d}t_1 \wedge \mathrm{d}t_2'}{(1 - \tilde{x}t_1 - y't_2')t_1 t_2'}, \quad \lim_{\varepsilon \to 0} \varphi_{1,F} = \lim_{\varepsilon \to 0} \varphi_{3,F} = \frac{(1 - \tilde{x})\mathrm{d}t_1 \wedge \mathrm{d}t_2'}{(1 - \tilde{x}t_1 - y't_2')(t_1 - 1)t_2'},$$

which mean that the independence of $\varphi_{0,F}$ and $\varphi_{2,F}$ and that of $\varphi_{1,F}$ and $\varphi_{3,F}$ are not kept under this limit. We transform $\tilde{\mathbf{F}}_{F_2}$ into

$$\ddot{\mathbf{F}}_{F_2} = \begin{bmatrix} \ddot{f}_0 \\ \ddot{f}_1 \\ \ddot{f}_2 \\ \ddot{f}_3 \end{bmatrix} := \ddot{U}\tilde{\mathbf{F}}_{F_2}, \quad \ddot{U} = \begin{bmatrix} -1/\varepsilon & 0 & 1/\varepsilon & 0 \\ 0 & -1/\varepsilon & 0 & 1/\varepsilon \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $\ddot{\mathbf{F}}_{F_2}$ satisfies a Pfaffian system equivalent to (12.14):

$$\begin{aligned} \partial_{\tilde{x}} \ddot{\mathbf{F}}_{F_2} &= \left\{ \tfrac{\mathrm{d}\tilde{x}}{\tilde{x}} \ddot{W}_{1,F} + \tfrac{\mathrm{d}\tilde{x}}{\tilde{x}-1} \ddot{W}_{5,F} + \tfrac{\mathrm{d}\tilde{x}}{\tilde{x}+\tilde{y}-1} \ddot{W}_{2,F} \right\} \ddot{\mathbf{F}}_{F_2}, \\ \partial_{\tilde{y}} \ddot{\mathbf{F}}_{F_2} &= \left\{ \tfrac{\mathrm{d}\tilde{y}}{\tilde{y}} \ddot{W}_{3,F} + \tfrac{\mathrm{d}\tilde{y}}{\tilde{y}-1} \ddot{W}_{0,F} + \tfrac{\mathrm{d}\tilde{y}}{\tilde{x}+\tilde{y}-1} \ddot{W}_{2,F} \right\} \ddot{\mathbf{F}}_{F_2}, \end{aligned} \tag{12.27}$$

where

$$\ddot{W}_{i,F} = \ddot{U} W_{i,F} \ddot{U}^{-1} \quad (0 \leq i \leq 5).$$

Applying the deformation (12.25) to (12.27), and taking the limit $\varepsilon \to 0$, we have

$$\begin{aligned} \partial_{\tilde{x}} \ddot{\mathbf{F}}_{\Psi_1} &= \left\{ \tfrac{\mathrm{d}\tilde{x}}{\tilde{x}} \ddot{W}_{1,\Psi} + \tfrac{\mathrm{d}\tilde{x}}{\tilde{x}-1} \ddot{W}_{2,\Psi} - \tfrac{y'\mathrm{d}\tilde{x}}{(\tilde{x}-1)^2} \ddot{W}_{5,\Psi} \right\} \ddot{\mathbf{F}}_{\Psi_1}, \\ \partial_{y'} \ddot{\mathbf{F}}_{\Psi_1} &= \left\{ \tfrac{\mathrm{d}y'}{y'} \ddot{W}_{3,\Psi} + \mathrm{d}y' \ddot{W}_{4,\Psi} + \tfrac{\mathrm{d}y'}{\tilde{x}-1} \ddot{W}_{5,\Psi} \right\} \ddot{\mathbf{F}}_{\Psi_1}, \end{aligned} \tag{12.28}$$

where $\ddot{W}_{1,\Psi} := W_{1,F}$, $\ddot{W}_{2,\Psi} := \lim_{\varepsilon \to 0}(\ddot{W}_{2,F} + \ddot{W}_{5,F}) = W_{2,\Psi}$, $\ddot{W}_{3,\Psi} := \lim_{\varepsilon \to 0}\{(c_2 - 1)I_4 + \ddot{W}_{3,F}\} = W_{3,\Psi}$, $\ddot{W}_{4,\Psi} := \lim_{\varepsilon \to 0}(-\varepsilon \ddot{W}_{0,F}) = W_{4,\Psi}$, $\ddot{W}_{5,\Psi} := \lim_{\varepsilon \to 0}(\varepsilon \ddot{W}_{2,F}) = W_{5,\Psi}$, $\ddot{\mathbf{F}}_{\Psi_1} := {}^t\begin{bmatrix} \ddot{p}_0 & \ddot{p}_1 & \ddot{p}_2 & \ddot{p}_3 \end{bmatrix}$ and $\ddot{p}_i(\tilde{x}, y') := \lim_{\varepsilon \to 0} \varepsilon^{c_2} (y')^{c_2 - 1} \ddot{f}_i(\tilde{x}, \varepsilon y')$ $(0 \leq i \leq 3)$. Thus we have the following.

Proposition 12.4. *The Pfaffian system (12.27) of Appell's F_2 converges to the Pfaffian system (12.28) by the confluence $\mathcal{C}$(4,3). This system is equivalent to the Pfaffian system (12.24) of Humbert's Ψ_1 under Definition 10.1.*

Proof. We have only to note that $\ddot{W}_{i,\Psi} = W_{i,\Psi}$ $(1 \leq i \leq 5)$. □

To study the confluences $\mathcal{C}(1,2)$ and $\mathcal{C}(2,1)$, replace the role of x and y and that of (b_1, c_1) and (b_2, c_2) in the previous consideration. Then we have the following.

Corollary 12.2. *By each of the confluences $\mathcal{C}$(1,2) and $\mathcal{C}$(2,1), a Pfaffian system equivalent to (9.6) of Appell's F_2 converges to a Pfaffian system equivalent to (12.24) of Humbert's Ψ_1 under Definition 10.1.*

12.3 Confluence from Appell's F_2 to Hörn's H_2

In order to study the confluences $\mathcal{C}(1,5),\dots,\mathcal{C}(4,5)$, we introduce Hörn's hypergeometric series

$$\mathrm{H}_2(\alpha,\beta,\gamma,\delta;x,y)=\sum_{m_1,m_2=0}^{\infty}\frac{(\alpha)_{m_1-m_2}(\beta)_{m_1}(\gamma)_{m_2}}{(\delta)_{m_1}(1)_{m_1}(1)_{m_2}}x^{m_1}y^{m_2}, \tag{12.29}$$

where $\delta\notin\mathbb{Z}_{\leq 0}$. This function satisfies Hörn's H_2 system of differential equations:

$$\begin{cases}\left[x(1-x)\frac{\partial^2}{\partial x^2}+xy\frac{\partial^2}{\partial x\partial y}+\{\delta-(\alpha+\beta+1)x\}\frac{\partial}{\partial x}+\beta y\frac{\partial}{\partial y}-\alpha\beta\right]\mathrm{H}_2(x,y)=0,\\ \left[y\frac{\partial^2}{\partial y^2}-x\frac{\partial^2}{\partial x\partial y}+\{1-\alpha+y\}\frac{\partial}{\partial y}+\gamma\right]\mathrm{H}_2(x,y)=0.\end{cases} \tag{12.30}$$

In this subsection, we show that a Pfaffian system equivalent to (9.6) of Appell's F_2 converges to a Pfaffian system equivalent to Hörn's H_2 system of differential equations by the confluences $\mathcal{C}(1,5),\dots,\mathcal{C}(4,5)$. At first, we consider the confluence $\mathcal{C}(3,5)$. We multiply the integrand u_{F_2} by $(-y')^{\alpha_3}$, and give the deformation

$$\mathbf{z}'_5(\varepsilon)=(1-\varepsilon)\mathbf{z}'_3+\varepsilon\mathbf{z}'_5.$$

The deformed matrix becomes

$$z'(\varepsilon)=\begin{bmatrix}1&0&1&0&1&\varepsilon\\0&1&-1&0&0&-\varepsilon x'\\0&0&0&-y'&-1&-y'\end{bmatrix}.$$

Its standard form is

$$g\cdot z'(\varepsilon)\cdot h=\begin{bmatrix}1&0&1&0&1&1\\0&1&-1&0&0&-x'\\0&0&0&1&-1&-y'/\varepsilon\end{bmatrix},$$

where $g=\mathrm{diag}(1,1,1)$ and $h=\mathrm{diag}(1,1,1,-1/y',1,1/\varepsilon)$. Hence, we have

$$x=x',\quad y=y'/\varepsilon.$$

Note that the parameters α_3, α_5 are transformed into

$$\alpha_3+\alpha_5=\alpha'_3+\alpha'_5,\quad \alpha_5=\alpha'_5/\varepsilon.$$

We put $\alpha_3+\alpha_5=b_2-a:=b'$, $\alpha_4=c_2-b_2:=c'-b'$ and $\alpha'_5:=d$. By the confluence $\mathcal{C}(3,5)$ on Appell's F_2 system, we have a system of differential equations. To change this system into Hörn's H_2 system in (12.30), and we consider further transformations

$$t_2=y''t'_2,\quad y=\frac{1}{\varepsilon y''},\quad \begin{cases}a=d/\varepsilon,\\ b_2=b'+d/\varepsilon,\\ c_2=c'+d/\varepsilon.\end{cases} \tag{12.31}$$

As $\varepsilon\to 0$, $\mathbf{z}'_5(\varepsilon)={}^t\begin{bmatrix}\varepsilon & -\varepsilon x & -t'_2\end{bmatrix}$ approaches to $\mathbf{z}'_3={}^t\begin{bmatrix}0 & 0 & -t'_2\end{bmatrix}$, and the integrand u_{F_2} multiplied by $(1/y'')^{\alpha_3}(-\varepsilon)^{\alpha_5}$ converges to

$$\lim_{\varepsilon\to 0}(-\varepsilon)^{-a}(\varepsilon y)^{b_2}u_{F_2}=t_1^{b_1}(1-t_1)^{c_1-b_1}{t'_2}^{b'}(1-y''t'_2)^{c'-b'}\exp(d\frac{1-xt_1}{t'_2}),$$

which is denoted by u_{H_2}. By the confluence, the system (9.3) converges to

$$\begin{cases} \left[x(1-x)\frac{\partial^2}{\partial x^2} + xy''\frac{\partial^2}{\partial x\partial y''} + \{c_1 - (a'+b_1+1)x\}\frac{\partial}{\partial x} + b_1y''\frac{\partial}{\partial y''} - a'b_1\right]h(x,y'') = 0, \\ \left[y''\frac{\partial^2}{\partial y''^2} - x\frac{\partial^2}{\partial x\partial y''} + \{1-a'-dy''\}\frac{\partial}{\partial y''} + (c'-1)d\right]h(x,y'') = 0, \end{cases} \tag{12.32}$$

where

$$h(x,y'') := \lim_{\varepsilon\to 0}\varepsilon^{-a}y''^{-b_2}f(x,\frac{1}{\varepsilon y''}).$$

This is Hörn's H_2 system of differential equations. In this case, we can see

$$\lim_{\varepsilon\to 0}(\varphi_{0,F}/\varepsilon) = \frac{-\mathrm{d}t_1\wedge\mathrm{d}t_2'}{t_1t_2'^2}, \quad \lim_{\varepsilon\to 0}(\varphi_{1,F}/\varepsilon) = \frac{(x-1)y'\mathrm{d}t_1\wedge\mathrm{d}t_2'}{(t_1-1)t_2'^2},$$

which mean that 2-forms $\varphi_{0,F}$ and $\varphi_{1,F}$ vanish under this limit. We transform $\mathbf{F}_{F_2}$ into

$$\breve{\mathbf{F}}_{F_2} = \begin{bmatrix}\breve{f}_0\\ \breve{f}_1\\ \breve{f}_2\\ \breve{f}_3\end{bmatrix} := \breve{U}\mathbf{F}_{F_2}, \quad \breve{U} := \begin{bmatrix}1/\varepsilon & 0 & 0 & 0\\ 0 & 1/\varepsilon & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

Then $\breve{\mathbf{F}}_{F_2}$ satisfies a Pfaffian system equivalent to (9.6):

$$\begin{aligned} \partial_x\breve{\mathbf{F}}_{F_2} &= \left\{\tfrac{\mathrm{d}x}{x}\breve{W}_{1,F} + \tfrac{\mathrm{d}x}{x-1}\breve{W}_{2,F} + \tfrac{\mathrm{d}x}{x+y-1}\breve{W}_{5,F}\right\}\breve{\mathbf{F}}_{F_2}, \\ \partial_y\breve{\mathbf{F}}_{F_2} &= \left\{\tfrac{\mathrm{d}y}{y}\breve{W}_{3,F} + \tfrac{\mathrm{d}y}{y-1}\breve{W}_{4,F} + \tfrac{\mathrm{d}y}{x+y-1}\breve{W}_{5,F}\right\}\breve{\mathbf{F}}_{F_2}, \end{aligned} \tag{12.33}$$

where

$$\breve{W}_{i,F} := \breve{U}W_{i,F}\breve{U}^{-1} \quad (1\le i\le 5).$$

Applying the deformation (12.31) to (12.33), we have the system

$$\begin{aligned} \partial_x\breve{\mathbf{F}}'_{F_2} &= \left\{\frac{\mathrm{d}x}{x}\breve{W}_{1,F} + \frac{\mathrm{d}x}{x-1}\breve{W}_{2,F} + \frac{\varepsilon y''\mathrm{d}x}{1+\varepsilon(x-1)y''}\breve{W}_{5,F}\right\}\breve{\mathbf{F}}'_{F_2}, \\ \partial_{y''}\breve{\mathbf{F}}'_{F_2} &= \left\{-\frac{\mathrm{d}y''}{y''}(b_2I_4 + \breve{W}_{3,F} + \breve{W}_{4,F} + \breve{W}_{5,F}) + \frac{\varepsilon\mathrm{d}y''}{\varepsilon y''-1}\breve{W}_{4,F} + \frac{\varepsilon(x-1)\mathrm{d}y''}{1+\varepsilon(x-1)y''}\breve{W}_{5,F}\right\}\breve{\mathbf{F}}'_{F_2}, \end{aligned}$$

where $\breve{\mathbf{F}}'_{F_2} := (\varepsilon^{-a'}y^{b_2})\breve{\mathbf{F}}_{F_2}$ and, as $\varepsilon\to 0$, this system converges to

$$\begin{aligned} \partial_x\mathbf{F}_{\mathrm{H}_2} &= \left\{\tfrac{\mathrm{d}x}{x}W_{1,\mathrm{H}} + \tfrac{\mathrm{d}x}{x-1}W_{2,\mathrm{H}} + y''\mathrm{d}xW_{5,\mathrm{H}}\right\}\mathbf{F}_{\mathrm{H}_2}, \\ \partial_{y''}\mathbf{F}_{\mathrm{H}_2} &= \left\{\tfrac{\mathrm{d}y''}{y''}W_{3,\mathrm{H}} + \mathrm{d}y''W_{4,\mathrm{H}} + x\mathrm{d}y''W_{5,\mathrm{H}}\right\}\mathbf{F}_{\mathrm{H}_2}, \end{aligned} \tag{12.34}$$

where $W_{1,\mathrm{H}} := W_{1,F}$, $W_{2,\mathrm{H}} := \lim_{\varepsilon\to 0}\breve{W}_{2,F}$, $W_{3,\mathrm{H}} := \lim_{\varepsilon\to 0}(-b_2I_4 - \breve{W}_{3,F} - \breve{W}_{4,F} - \breve{W}_{5,F})$, $W_{4,\mathrm{H}} := \lim_{\varepsilon\to 0}(-\varepsilon)(\breve{W}_{4,F} + \breve{W}_{5,F})$, $W_{5,\mathrm{H}} := \lim_{\varepsilon\to 0}(\varepsilon\breve{W}_{5,F})$, $\mathbf{F}_{\mathrm{H}_2} := {}^t\begin{bmatrix}h_0 & h_1 & h_2 & h_3\end{bmatrix}$ and $h_i(x,y'') :=$

$\lim_{\varepsilon\to 0}\varepsilon^{-a}y''^{-b_2}\breve{f}_i(x,\frac{1}{\varepsilon y''})$ $(0\le i\le 3)$. They are

$$W_{1,\mathrm{H}}=\begin{bmatrix}-b_1 & b_1-c_1 & 0 & 0\\ -b_1 & b_1-c_1 & 0 & 0\\ 0 & 0 & -b_1 & b_1-c_1\\ 0 & 0 & -b_1 & b_1-c_1\end{bmatrix},\quad W_{2,\mathrm{H}}=\begin{bmatrix}0 & c_1-b_1 & 0 & 0\\ 0 & c_1-b_1+b' & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & d & 0 & 0\end{bmatrix},$$

$$W_{3,\mathrm{H}}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ d & 0 & -b_1-b' & b_1-c_1\\ 0 & d & -b_1 & b_1-c_1-b'\end{bmatrix},\ W_{4,\mathrm{H}}=\begin{bmatrix}0 & 0 & b'-c' & 0\\ 0 & 0 & 0 & b'-c'\\ 0 & 0 & d & 0\\ 0 & 0 & 0 & d\end{bmatrix},\ W_{5,\mathrm{H}}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & c'-b'\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -d\end{bmatrix}.$$

The solution $h(x,y'')$ to (12.32) satisfies

$$\begin{aligned} h&=h_3-h_2,\quad \frac{\partial}{\partial x}h=\frac{d}{x-1}h_1-dy''h_3,\\ \frac{\partial}{\partial y''}h&=\frac{d}{y''}(h_1-h_0)+\frac{b'}{y''}(h_2-h_3)-dh_2-d(x-1)h_3,\\ \frac{\partial^2}{\partial x\partial y''}h&=\left\{d(c_1-b_1+c'-1)+d^2(x-1)y''\right\}h_3-d^2h_1+b_1dh_2.\end{aligned}$$

Proposition 12.5. *By the confluence $\mathcal{C}(3,5)$, the Pfaffian system (12.33) of Appell's F_2 converges to the Pfaffian system (12.34), which is equivalent to Hörn's* H_2 *system (12.30) of differential equations, and the matrices $W_{1,\mathrm{H}},\dots,W_{5,\mathrm{H}}$ depend only on the parameters. This system is called a Pfaffian system of Hörn's* H_2.

Next, we consider the confluence $\mathcal{C}(4,5)$. Before introducing an infinitesimal parameter ε, we exchange parameters b_2 and (c_2-b_2) and use the transformation (12.12). The confluence $\mathcal{C}(4,5)$ consists of the transformation

$$\tilde{t}_2=y't'_2,\quad \tilde{y}=\frac{1}{\varepsilon y'},\quad \begin{cases}a=a'+d/\varepsilon,\\ c_2=b_2+d/\varepsilon.\end{cases} \tag{12.35}$$

and the limit $\varepsilon\to 0$. Then the integrand $\tilde{u}_{F_2}$ converges to

$$\lim_{\varepsilon\to 0}(-1)^{-a}\varepsilon^{c_2-b_2-a}\tilde{y}^{c_2-b_2}\tilde{u}_{F_2}=t_1^{b_1}(1-t_1)^{c_1-b_1}t'^{-a'}_2(1-y't'_2)^{b_2}\exp(d\frac{1-\tilde{x}t_1}{t'_2}).$$

By the confluence, the system (12.13) converges to

$$\begin{cases}\Big[\tilde{x}(1-\tilde{x})\frac{\partial^2}{\partial\tilde{x}^2}+\tilde{x}y'\frac{\partial^2}{\partial\tilde{x}\partial y'}+\{c_1-(b_1+b_2-c'+1)\tilde{x}\}\frac{\partial}{\partial\tilde{x}}\\ \qquad\qquad +b_1y'\frac{\partial}{\partial y'}+(c'-b_2)b_1\Big]\tilde{h}(\tilde{x},y')=0,\\ \Big[y'\frac{\partial^2}{\partial y'^2}-\tilde{x}\frac{\partial^2}{\partial\tilde{x}\partial y'}+\{1+(c'-b_2)+dy'\}\frac{\partial}{\partial y'}+(b_2-1)d\Big]\tilde{h}(\tilde{x},y')=0,\end{cases} \tag{12.36}$$

where

$$\tilde{h}(\tilde{x},y'):=\lim_{\varepsilon\to 0}\varepsilon^{-a}y'^{b_2-c_2}\tilde{f}(\tilde{x},\frac{1}{\varepsilon y'}).$$

This is Hörn's H_2 system of differential equations, too. In this case, we can see

$$\lim_{\varepsilon\to 0}(\varphi_{2,F}/\varepsilon)=\frac{-\mathrm{d}\tilde{t}_1\wedge\mathrm{d}t'_2}{\tilde{t}_1t'^2_2},\quad \lim_{\varepsilon\to 0}(\varphi_{3,F}/\varepsilon)=\frac{(\tilde{x}-1)y'\mathrm{d}\tilde{t}_1\wedge\mathrm{d}t'_2}{(\tilde{t}_1-1)t'^2_2},$$

which mean that 2-forms $\varphi_{2,F}$ and $\varphi_{3,F}$ vanish under this limit. We transform $\tilde{\mathbf{F}}_{F_2}$ into

$$\bar{\mathbf{F}}_{F_2}=\begin{bmatrix}\bar{f}_0\\ \bar{f}_1\\ \bar{f}_2\\ \bar{f}_3\end{bmatrix}:=\bar{U}\tilde{\mathbf{F}}_{F_2},\quad \bar{U}:=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1/\varepsilon&0\\0&0&0&1/\varepsilon\end{bmatrix}.$$

Then $\bar{\mathbf{F}}_{F_2}$ satisfies a Pfaffian system

$$\begin{aligned}\partial_{\tilde{x}}\bar{\mathbf{F}}_{F_2}&=\left\{\tfrac{\mathrm{d}\tilde{x}}{\tilde{x}}\bar{W}_{1,F}+\tfrac{\mathrm{d}\tilde{x}}{\tilde{x}-1}\bar{W}_{5,F}+\tfrac{\mathrm{d}\tilde{x}}{\tilde{x}+\tilde{y}-1}\bar{W}_{2,F}\right\}\bar{\mathbf{F}}_{F_2},\\ \partial_{\tilde{y}}\bar{\mathbf{F}}_{F_2}&=\left\{\tfrac{\mathrm{d}\tilde{y}}{\tilde{y}}\bar{W}_{3,F}+\tfrac{\mathrm{d}\tilde{y}}{\tilde{y}-1}\bar{W}_{0,F}+\tfrac{\mathrm{d}\tilde{y}}{\tilde{x}+\tilde{y}-1}\bar{W}_{2,F}\right\}\bar{\mathbf{F}}_{F_2},\end{aligned}\tag{12.37}$$

where

$$\bar{W}_{i,F}:=\bar{U}W_{i,F}\bar{U}^{-1}\quad(0\le i\le 5).$$

Applying the deformation (12.35) to (12.37), and taking the limit $\varepsilon\to 0$, we have

$$\begin{aligned}\partial_{\tilde{x}}\bar{\mathbf{F}}_{\mathrm{H}_2}&=\left\{\tfrac{\mathrm{d}\tilde{x}}{\tilde{x}}\bar{W}_{1,\mathrm{H}}+\tfrac{\mathrm{d}\tilde{x}}{\tilde{x}-1}\bar{W}_{2,\mathrm{H}}+y'\mathrm{d}\tilde{x}\bar{W}_{5,\mathrm{H}}\right\}\bar{\mathbf{F}}_{\mathrm{H}_2},\\ \partial_{y'}\bar{\mathbf{F}}_{\mathrm{H}_2}&=\left\{\tfrac{\mathrm{d}y'}{y'}\bar{W}_{3,\mathrm{H}}+\mathrm{d}y'\bar{W}_{4,\mathrm{H}}+\mathrm{d}y'\tilde{x}\bar{W}_{5,\mathrm{H}}\right\}\bar{\mathbf{F}}_{\mathrm{H}_2},\end{aligned}\tag{12.38}$$

where $\bar{W}_{1,\mathrm{H}}:=\bar{W}_{1,F}$, $\bar{W}_{2,\mathrm{H}}:=\lim_{\varepsilon\to 0}\bar{W}_{5,F}$, $\bar{W}_{3,\mathrm{H}}:=\lim_{\varepsilon\to 0}\big((b_2-c_2)I_4-\bar{W}_{3,F}-\bar{W}_{0,F}-\bar{W}_{2,F}\big)$, $\bar{W}_{4,\mathrm{H}}:=\lim_{\varepsilon\to 0}(-\varepsilon)(\bar{W}_{0,F}+\bar{W}_{2,F})$, $\bar{W}_{5,\mathrm{H}}:=\lim_{\varepsilon\to 0}(\varepsilon\bar{W}_{2,F})$, $\bar{\mathbf{F}}_{\mathrm{H}_2}:={}^t\begin{bmatrix}\bar{h}_0&\bar{h}_1&\bar{h}_2&\bar{h}_3\end{bmatrix}$ and $\bar{h}_i(\tilde{x},y'):=\lim_{\varepsilon\to 0}\varepsilon^{-a}y'^{b_2-c_2}\ \bar{f}_i(\tilde{x},1/(\varepsilon y'))$ $(0\le i\le 3)$. They are

$$\bar{W}_{1,\mathrm{H}}=\begin{bmatrix}-b_1&b_1-c_1&0&0\\-b_1&b_1-c_1&0&0\\0&0&-b_1&b_1-c_1\\0&0&-b_1&b_1-c_1\end{bmatrix},\quad \bar{W}_{2,\mathrm{H}}=\begin{bmatrix}0&0&0&0\\0&0&0&d\\0&0&0&c_1-b_1\\0&0&0&c_1-b_1+c'-b_2\end{bmatrix},$$

$$\bar{W}_{3,\mathrm{H}}=\begin{bmatrix}a'-b_1&b_1-c_1&d&0\\-b_1&a'+b_1-c_1&0&d\\0&0&0&0\\0&0&0&0\end{bmatrix},\ \bar{W}_{4,\mathrm{H}}=\begin{bmatrix}d&0&0&0\\0&d&0&0\\-b_2&0&0&0\\0&-b_2&0&0\end{bmatrix},\ \bar{W}_{5,\mathrm{H}}=\begin{bmatrix}0&0&0&0\\0&-d&0&0\\0&0&0&0\\0&b_2&0&0\end{bmatrix}.$$

Proposition 12.6. *The Pfaffian system (12.37) of Appell's F_2 converges to the Pfaffian system (12.38) by the confluence $\mathcal{C}(4,5)$. This system is equivalent to the Pfaffian system (12.34) of Hörn's* H_2 *under Definition 10.1.*

To study the confluences $\mathcal{C}(1,5)$ and $\mathcal{C}(2,5)$, replace the role of x and y and that of (b_1,c_1) and (b_2,c_2) in the previous consideration. Then we have the following.

Corollary 12.3. *By each of the confluences $\mathcal{C}(1,5)$ and $\mathcal{C}(2,5)$, a Pfaffian system equivalent to (9.6) of Appell's F_2 converges to a Pfaffian system equivalent to (12.34) of Hörn's* H_2 *under Definition 10.1.*

12.4 Confluence from Appell's F_2 to the product of Kummer's functions

In this subsection, we study the system of differential equations satisfied by the product of Kummer's hypergeometric functions ${}_1F_1(x')$ and ${}_1F_1(y')$. The confluence $\mathcal{C}(0,5)$ consists of the transformation

$$\begin{cases} x = \varepsilon dx', \\ y = \varepsilon dy', \end{cases} \qquad a = \frac{1}{\varepsilon}. \tag{12.39}$$

and the limit $\varepsilon \to 0$, and the integrand u_{F_2} converges to

$$\lim_{\varepsilon \to 0} u_{F_2} = t_1^{b_1}(1-t_1)^{c_1-b_1}\exp(dx't_1)t_2^{b_2}(1-t_2)^{c_2-b_2}\exp(dy't_2),$$

which is denoted by u_K. If $\Re(dx') < 0$ and $\Re(dy') < 0$, then the solution (9.2) to the system (9.3) converges as $\varepsilon \to 0$, and we define a function

$$k(x',y') := \lim_{\varepsilon \to 0} f(\varepsilon x', \varepsilon y') = \iint_{\Delta} u_K \varphi_{F_2}. \tag{12.40}$$

By this consequence, the system (9.3) converges to

$$\begin{cases} \left[x' \frac{\partial^2}{\partial x'^2} + \{c_1 - dx'\}\frac{\partial}{\partial x'} - b_1 d \right] k(x',y') = 0, \\ \left[y' \frac{\partial^2}{\partial y'^2} + \{c_2 - dy'\}\frac{\partial}{\partial y'} - b_2 d \right] k(x',y') = 0, \end{cases} \tag{12.41}$$

and $k(x',y')$ is a solution to this system. Since each of (12.41) can be regarded as a confluent hypergeometric differential equation of Kummer's ${}_1F_1$, the system (12.41) is denoted by ${}_1F_1^{\oplus 2}$.

By this confluence, f_i $(0 \leq i \leq 3)$ converge to

$$k_i(x',y') := \lim_{\varepsilon \to 0} f_i(\varepsilon x', \varepsilon y').$$

The solution k is expressed by a linear combination

$$k = k_0 - k_1 - k_2 + k_3.$$

Applying the deformation (12.39) to (9.6), we have a system

$$\begin{aligned} \partial_{x'}\mathbf{F}_{F_2} &= \left\{ \frac{\mathrm{d}x'}{x'}W_{1,F} - \frac{\varepsilon d\mathrm{d}x'}{1-\varepsilon dx'}W_{2,F} - \frac{\varepsilon d\mathrm{d}x'}{1-\varepsilon dx' - \varepsilon dy'}W_{5,F} \right\}\mathbf{F}_{F_2}, \\ \partial_{y'}\mathbf{F}_{F_2} &= \left\{ \frac{\mathrm{d}y'}{y'}W_{3,F} - \frac{\varepsilon d\mathrm{d}y'}{1-\varepsilon dy'}W_{4,F} - \frac{\varepsilon d\mathrm{d}y'}{1-\varepsilon dx' - \varepsilon dy'}W_{5,F} \right\}\mathbf{F}_{F_2}, \end{aligned}$$

and as $\varepsilon \to 0$, this system converges to

$$\begin{aligned} \partial_{x'}\mathbf{F}_K &= \left\{ \tfrac{\mathrm{d}x'}{x'}W_{1,K} + \mathrm{d}x'W_{2,K} \right\}\mathbf{F}_K, \\ \partial_{y'}\mathbf{F}_K &= \left\{ \tfrac{\mathrm{d}y'}{y'}W_{3,K} + \mathrm{d}y'W_{4,K} \right\}\mathbf{F}_K, \end{aligned} \tag{12.42}$$

where

$$W_{1,K} := \begin{bmatrix} -b_1 & b_1-c_1 & 0 & 0 \\ -b_1 & b_1-c_1 & 0 & 0 \\ 0 & 0 & -b_1 & b_1-c_1 \\ 0 & 0 & -b_1 & b_1-c_1 \end{bmatrix}, \quad W_{2,K} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d \end{bmatrix},$$

$$W_{3,K} := \begin{bmatrix} -b_2 & 0 & b_2-c_2 & 0 \\ 0 & -b_2 & 0 & b_2-c_2 \\ -b_2 & 0 & b_2-c_2 & 0 \\ 0 & -b_2 & 0 & b_2-c_2 \end{bmatrix}, \quad W_{4,K} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \end{bmatrix},$$

$$\mathbf{F}_K := {}^t\begin{bmatrix} k_0 & k_1 & k_2 & k_3 \end{bmatrix}.$$

Proposition 12.7. *By the confluence $\mathcal{C}(0,5)$, the Pfaffian system (9.6) of Appell's F_2 converges to the Pfaffian system (12.42), which is equivalent to the system (12.41) of differential equations and the matrices $W_{1,K}, \ldots, W_{4,K}$ depend only on the parameters. This system is called a Pfaffian system of Kummer's ${}_1F_1^{\oplus 2}$.*

13 Classification of confluent Pfaffian systems

As we have seen in the previous section, a Pfaffian system equivalent to Appell's F_2 converges to the Pfaffian system of Humbert's Φ_1, Ξ_1, Hörn's H_2 or the product ${}_1F_1^{\oplus 2}$ of Kummer's functions by confluences $\mathcal{C}(i,j)$. In this section, we study their equivalence under Definition 10.1.

In case of ${}_1F_1^{\oplus 2}$, the regular locus of this system is $\mathbb{C}^\times \times \mathbb{C}^\times$, which is different from those of the other systems. Hence, this system can not be equivalent to the other systems.

Theorem 13.1. *Pfaffian systems (12.11) of Humbert's Ξ_1 and (12.34) of Hörn's H_2 are equivalent.*

Proof. In the case of $b_2 \neq 0$, replace parameters of the Pfaffian system (12.34) of Hörn's H_2 as

$$b' = b_2 - a, \quad c' = -a,$$

and transform $\mathbf{F}_{\mathrm{H}_2}$ into

$$\mathbf{F}'_{\mathrm{H}_2} = \begin{bmatrix} h'_0 \\ h'_1 \\ h'_2 \\ h'_3 \end{bmatrix} := x^{b_1} U' \mathbf{F}_{\mathrm{H}_2}, \quad U' := \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & b_2 & 0 \\ 0 & 0 & 0 & b_2 \end{bmatrix},$$

and apply a birational map $x = 1/x''$. Then we have

$$\begin{aligned} \partial_{x''} \mathbf{F}'_{\mathrm{H}_2} &= \left\{ \tfrac{\mathrm{d}x''}{x''} W'_{1,\mathrm{H}} + \tfrac{\mathrm{d}x''}{x''-1} W'_{2,\mathrm{H}} - \tfrac{y''\mathrm{d}x''}{x''^2} W'_{5,\mathrm{H}} \right\} \mathbf{F}'_{\mathrm{H}_2}, \\ \partial_{y''} \mathbf{F}'_{\mathrm{H}_2} &= \left\{ \tfrac{\mathrm{d}y''}{y''} W'_{3,\mathrm{H}} + \mathrm{d}y'' W'_{4,\mathrm{H}} + \tfrac{\mathrm{d}y''}{x''} W'_{5,\mathrm{H}} \right\} \mathbf{F}'_{\mathrm{H}_2}, \end{aligned} \tag{13.1}$$

where $W'_{1,\mathrm{H}} = -(b_1 I_4 + U' W_{1,\mathrm{H}} U'^{-1} + U' W_{2,\mathrm{H}} U'^{-1})$ and $W'_{i,\mathrm{H}} = U' W_{i,\mathrm{H}} U'^{-1}$ $(2 \leq i \leq 5)$.

We can see

$$W_{i,\Xi} = W'_{i,\mathrm{H}} \quad (1 \le i \le 5), \tag{13.2}$$

where $W_{i,\Xi}$ are matrices in the Pfaffian system (12.11) of Humbert's Ξ.

In the case of $b_2 = 0$, use the unit matrix I_4 instead of U' in (13.1), then we have the same result as (13.2). □

We consider whether Humbert's Ξ_1 and Ψ_1 systems are equivalent, or not. By the action of a birational of transformation

$$x'' = \frac{Y}{X}, \quad y'' = Y,$$

on the Pfaffian system (12.11) of Humbert's Ξ_1, it is transformed into

$$\begin{aligned} \partial_X \mathbf{F}'_{\Xi_1} &= \left\{ \tfrac{\mathrm{d}X}{X} W'_{1,\Xi} + \tfrac{\mathrm{d}X}{X-Y} W'_{2,\Xi} + \mathrm{d}X W'_{5,\Xi} \right\} \mathbf{F}'_{\Xi_1}, \\ \partial_Y \mathbf{F}'_{\Xi_1} &= \left\{ \tfrac{\mathrm{d}Y}{Y} W'_{3,\Xi} + \tfrac{\mathrm{d}Y}{Y-X} W'_{2,\Xi} + \mathrm{d}Y W'_{4,\Xi} \right\} \mathbf{F}'_{\Xi_1}, \end{aligned} \tag{13.3}$$

where $W'_{1,\Xi} = -(b_1 I_4 + W_{1,\Xi} + W_{2,\Xi})$, $W'_{2,\Xi} = W_{2,\Xi}$, $W'_{3,\Xi} = (W_{1,\Xi} + W_{3,\Xi})$, $W'_{4,\Xi} = W_{4,\Xi}$, $W'_{5,\Xi} = W_{5,\Xi}$, $\mathbf{F}'_{\Xi_1} = (x'')^{b_1}(y'')^{b_2-a}\mathbf{F}_{\Xi_1}$. They are

$$W'_{1,\Xi} = \begin{bmatrix} -b_1 & b_1 - c_1 & 0 & 0 \\ -b_1 & b_1 - c_1 & 0 & 0 \\ 0 & 0 & -b_1 & b_1 - c_1 \\ 0 & 0 & -b_1 & b_1 - c_1 \end{bmatrix}, \quad W'_{2,\Xi} = \begin{bmatrix} 0 & c_1 - b_1 & 0 & 0 \\ 0 & c_1 - b_1 + b_2 - a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \end{bmatrix},$$

$$W'_{3,\Xi} = \begin{bmatrix} b_1 + b_2 - a & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 \\ b_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad W'_{4,\Xi} = \begin{bmatrix} 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \end{bmatrix}, \quad W'_{5,\Xi} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d \end{bmatrix}.$$

By the action of a birational transformation

$$x = \frac{X}{X - Y}, \quad y' = Y,$$

on the Pfaffian system (12.24) of Humbert's Ψ_1, it is transformed into

$$\begin{aligned} \partial_X \mathbf{F}'_{\Psi_1} &= \left\{ \tfrac{\mathrm{d}X}{X} W'_{1,\Psi} + \tfrac{\mathrm{d}X}{X-Y} W'_{2,\Psi} + \mathrm{d}X W'_{5,\Psi} \right\} \mathbf{F}'_{\Psi_1}, \\ \partial_Y \mathbf{F}'_{\Psi_1} &= \left\{ \tfrac{\mathrm{d}Y}{Y} W'_{3,\Psi} + \tfrac{\mathrm{d}Y}{Y-X} W'_{2,\Psi} + \mathrm{d}Y W'_{4,\Psi} \right\} \mathbf{F}'_{\Psi_1}, \end{aligned} \tag{13.4}$$

where $W'_{1,\Psi} = W_{1,\Psi}$, $W'_{2,\Psi} = -(b_1 I_4 + W_{1,\Psi} + W_{2,\Psi})$, $W'_{3,\Psi} = ((1 - c_2) I_4 + W_{2,\Psi} + W_{3,\Psi})$,

$W'_{4,\Psi} = (W_{4,\Psi} - W_{5,\Psi})$, $W'_{5,\Psi} = W_{5,\Psi}$, $\mathbf{F}'_{\Psi_1} = (x-1)^{b_1}(y')^{1-b_1-c_2}\mathbf{F}_{\Psi_1}$. They are

$$W'_{1,\Psi} = \begin{bmatrix} -b_1 & b_1-c_1 & 0 & 0 \\ -b_1 & b_1-c_1 & 0 & 0 \\ 0 & 0 & -b_1 & b_1-c_1 \\ 0 & 0 & -b_1 & b_1-c_1 \end{bmatrix}, \quad W'_{2,\Psi} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ b_1 & a-b_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -d & b_1 & a-b_1-c_2 \end{bmatrix},$$

$$W'_{3,\Psi} = \begin{bmatrix} 0 & c_1-b_1 & 0 & 0 \\ 0 & c_1-b_1-a & 0 & 0 \\ -d & 0 & -c_2 & c_1-b_1 \\ 0 & 0 & 0 & c_1-b_1-a \end{bmatrix}, \quad W'_{4,\Psi} = \begin{bmatrix} 0 & 0 & b_1-a & c_1-b_1 \\ 0 & 0 & b_1 & c_1-b_1-a \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \end{bmatrix},$$

$$W'_{5,\Psi} = \begin{bmatrix} 0 & 0 & 0 & b_1-c_1 \\ 0 & 0 & 0 & a+b_1-c_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d \end{bmatrix}.$$

The set of poles of the connection matrix in (13.3) is same as that in (13.4), and their coefficient matrices are different from each other.

Definition 13.1. For the connection matrix W of the Pfaffian system (13.3) (resp. (13.4)), we define the spectral partition of W'_i $(1 \leq i \leq 5)$ by the partition of the number 4 by the multiplicities of the roots of the characteristic polynomial of W'_i. We define the total type of spectral partitions of W by the pentad of spectral partitions of W'_i. The subset consisting of spectral partitions of W'_1, W'_2, W'_3 is called the first type of the spectral partitions of W, and its complement is called the second type of the spectral partitions of W.

We list the first and second types of spectral partitions of the connection matrices of (13.3) for Ξ_1 and (13.4) for Ψ_1 in Table 13.1 and 13.2. Here, we have $d \neq 0$, and we assume the other parameters in (13.3) and (13.4) are generic.

Table 13.1: First type of spectral partitions

Component of singular locus	Order of pole	Matrix	Ξ_1	Ψ_1
$X = 0$	1	W'_1	(2,2)	(2,2)
$X = Y$	1	W'_2	(3,1)	(2,1,1)
$Y = 0$	1	W'_3	(3,1)	(2,1,1)

Table 13.2: Second type of spectral partitions

Component of singular locus	Order of pole	Matrix	Ξ_1	Ψ_1
$\frac{1}{X} = 0$	2	W'_5	(3,1)	(3,1)
$\frac{1}{Y} = 0$	2	W'_4	(2,2)	(2,2)

The regular locus of each of (13.3) and (13.4) is

$$Q := \left\{ [1 : X : Y] \in \mathbb{P}^2 | XY(X-Y) \neq 0 \right\}. \tag{13.5}$$

Around each regular singular locus of $X=0$, $Y=0$ and $X-Y=0$ of the Pfaffian system (13.3) or (13.4), it has local solutions of the form $\mathbf{x}^{\lambda}v(X,Y)$ by [Ka][Theorem 2.1], where $\mathbf{x}$ is X, Y or $X-Y$, λ is an eigenvalue of the coefficient matrix of $\mathrm{d}\mathbf{x}/\mathbf{x}$ and $v(X,Y)$ is a holomorphic function. Note that the dimension of space of the local solutions of the form $\mathbf{x}^{\lambda}v(X,Y)$ is the dimension of the eigenspace of W_i' of eigenvalue λ, if W_i' is diagonalizable. The difference of the type of spectral partitions of (13.3) and that of (13.4) is not enough to prove their non-equivalence, since the type of spectral partitions is not kept invariant under birational transformations.

To solve this problem, we restrict the action (c) in Definition 10.1 to the group $\mathrm{Aut}(Q)$ of birational transformations of $\mathbb{P}^2$ whose restriction to Q are biregular morphisms from Q to Q. The ring $\mathcal{R}(Q)$ consists of rational functions admitting poles only on X, Y and $X-Y$. Note that $\mathcal{R}^{\times}(Q)$ is generated by $\mathbb{C}$, X, Y and $X-Y$. Let A_a be $\langle h^{\alpha}I_4 \,|\, h\in\mathcal{R}^{\times}(Q),\ \alpha\in\mathbb{C}\rangle$, and A be the group generated by A_a, $\mathrm{GL}(4,\mathcal{R}(Q))$ and $\mathrm{Aut}(Q)$ acting on the space of vector valued local holomorphic functions of size four on Q. Then we can see the next lemma.

Lemma 13.1. *An action of any element a of A on $\mathbf{F}(X,Y)$ can be expressed by*

$$a(\mathbf{F}(X,Y)) = q'(X,Y)p'(X,Y)(g')^*(\mathbf{F}(X,Y)), \tag{13.6}$$

where $q'\in A_a$, $p'\in\mathrm{GL}(4,\mathcal{R}(Q))$ and $g'\in\mathrm{Aut}(Q)$.

Proof. Note that any element of A_a and any element of $\mathrm{GL}(4,\mathcal{R}(Q))$ are commutative. For any $h(X,Y)\in\mathcal{R}^{\times}(Q)$, $\alpha\in\mathbb{C}$ and $g\in\mathrm{Aut}(Q)$,

$$g^*(h(X,Y)^{\alpha}) = \{g^*(h(X,Y))\}^{\alpha},$$

where $\alpha'\in\mathbb{C}$ and $g^*(h(X,Y))\in\mathcal{R}^{\times}(Q)$. For any $p(X,Y)\in\mathrm{GL}(4,\mathcal{R}(Q))$ and for any $g\in\mathrm{Aut}(Q)$,

$$\det(g^*(p(X,Y))) = g^*(\det(p(X,Y)))\in\mathcal{R}^{\times}(Q)\quad(\forall(X,Y)\in Q).$$

This means $g^*(p(X,Y))\in\mathrm{GL}(4,\mathcal{R}(Q))$. Therefore, the action of any element a of A can be written by the form (13.6). □

Though the structure of $\mathrm{Aut}(Q)$ is not simple as shown in Section 14, we may restrict $\mathrm{Aut}(Q)$ to a simple subgroup for our equivalence problem by the following lemma.

Lemma 13.2. *Let W be the connection matrix of Pfaffian systems (13.3) (resp. (13.4)), and $g:(X',Y')\to(X,Y)$ be an element of $\mathrm{Aut}(Q)$. If the pull back $g^*(W)$ of W under g does not admit higher poles along $X'=0$, $Y'=0$ or $X'-Y'$, then $g(X',Y')$ is one of (rX',rY'), $(rX',r(X'-Y'))$, (rY',rX'), $(rY',-r(X'-Y'))$, $(r(X'-Y'),rX')$, $(r(X'-Y'),-rY')$, where $r\in\mathbb{C}^{\times}$.*

Proof. Note that the connection matrix W of (13.3) (resp. (13.4)) has the terms $\mathrm{d}XW_5'$ and $\mathrm{d}YW_4'$, and that the parameter d in W is not zero. The structure of $\mathrm{Aut}(Q)$ is given in Lemma 14.1, which states that any element g of $\mathrm{Aut}(Q)$ takes the form (14.3). If $(m,n)\neq(0,0)$ in (14.3) for g, then one of factors X', Y' and $X'-Y'$ appears in the denominator of the expression of $g(X',Y')$. Thus the pull backs $g^*(\mathrm{d}XW_5')$ and $g^*(\mathrm{d}XW_4')$ of under g admit higher poles along one of the lines $X'=0$, $Y'=0$ and $X'-Y'=0$. If $(m,n)=(0,0)$ and $t=-1$ in (14.3) for

g, then the pull backs $g^*(\mathrm{d}XW_5')$ and $g^*(\mathrm{d}XW_4')$ of under g have higher poles along one of the lines X', Y' and $X'-Y'$ by the same reason as the case $(m,n)\neq(0,0)$. Since W_4' and W_5' are linearly independent, these higher poles cannot cancel each other. Hence (m,n,t) for g should be $(0,0,1)$ under the assumption of this lemma, $g(X',Y')$ should be one of the listed maps by Lemma 14.1. □

Because the lines $X=0$, $Y=0$ and $X-Y=0$ are not normal crossing at the origin, we blow up the origin by

$$X=x_1,\quad Y=x_1y_1.$$

Using this transformation, we transform the equation (13.3) for Ξ_1 or (13.4) for Ψ_1 into

$$\begin{aligned}\partial_{x_1}\mathbf{F}&=\left\{\tfrac{\mathrm{d}x_1}{x_1}W_{123}'+y_1\mathrm{d}x_1W_4'+\mathrm{d}x_1W_5'\right\}\mathbf{F},\\ \partial_{y_1}\mathbf{F}&=\left\{\tfrac{\mathrm{d}y_1}{y_1}W_3'+\tfrac{\mathrm{d}y_1}{y_1-1}W_2'+x_1\mathrm{d}y_1W_4'\right\}\mathbf{F},\end{aligned}\tag{13.7}$$

where $W_{123}':=W_1'+W_2'+W_3'$. The spectral partition of W_{123}' of (13.7) for Ξ_1 and that of Ψ_1 are in Table 13.3.

Table 13.3: Extra spectral partition

Order of pole	Matrix	Ξ_1	Ψ_1
1	W_{123}'	(2,1,1)	(2,2)

Theorem 13.2. *Suppose that Humbert's Ψ_1 system (12.19) of differential equations or Humbert's Ξ_1 system (12.2) of differential equations satisfies one of the following conditions:*

(1) γ_2, $\alpha-\beta$, $\alpha-\beta-\gamma_2\notin\mathbb{Z}$, for parameters α, β, γ_2 of Humbert Ψ_1 system (12.19);

(2) γ_2, $\alpha+\beta-\gamma_1$, $\alpha+\beta-\gamma_1-\gamma_2\notin\mathbb{Z}$, for parameters α, β, γ_1, γ_2 of Humbert Ψ_1 system (12.19);

(3) $\alpha_1-\beta$, $\alpha_1-\gamma$, $\beta-\gamma\notin\mathbb{Z}$, for parameters α_1, β, γ of Humbert Ξ_1 system (12.2).

Then the Pfaffian system system of Humbert's Ξ_1 in (13.3) which is equivalent to (12.2) is not equivalent to that of Humbert's Ψ_1 in (13.4) which is equivalent to (12.19).

Proof. Recall that we relate parameters in the system (12.8) equivalent to (13.3) and the system (12.2) as

$$b_1+b_2-a+1=\gamma,\quad b_1=\alpha_1,\quad b_1-c_1+1=\beta,\quad b_2=\alpha_2,\quad d=1;$$

and relate parameters in the system (12.22) equivalent to (13.4) and the system (12.19) as

$$c_1=\gamma_1,\quad a-c_2+1=\alpha,\quad b_1=\beta,\quad 2-c_2=\gamma_2,\quad d=1.$$

By considering the orders of pole divisors of the connection matrices of the Pfaffian systems, we may restrict $\mathrm{Aut}(Q)$ to the transformations in Lemma 13.2 for the problem of the equivalence. Since these transformations yield only the transpositions of three divisors $X=0$, $Y=0$ and $X-Y=0$ of simple pole of W, the first type of spectral partitions of the connection matrices of

(13.3) and (13.4) are kept invariant under these transformations. If the condition (1) for (13.4) is satisfied, then the space of solutions to (13.4) around $X - Y = 0$ is spanned by

$$(X-Y)^{\nu_i}\tilde{f}_i(X,Y) \quad (1 \leq i \leq 4), \tag{13.8}$$

where $\tilde{f}_i$ are holomorphic functions and exponents ν_i are equal to the eigenvalues of W_2'. In this case, the spectral partition of W_2' is $(2,1,1)$. Assume $\nu_1 = \nu_2$, then each of $\nu_1 - \nu_3 = \beta + \gamma_2 - \alpha - 1$, $\nu_1 - \nu_4 = \beta - \alpha + 1$ and $\nu_3 - \nu_4 = 2 - \gamma_2$ is not an integer under the condition (1). By the action of an element in $\mathrm{Aut}(Q)$ of the form in Lemma 13.2, the first type of spectral partitions is kept. By the action of an element in $\mathrm{GL}(4, \mathcal{R}(Q))$, the solutions are multiplied by X, Y and $X - Y$, and the exponents of the factor $X - Y$ shift only to integers. By the action of an elements in A_a, the solutions are multiplied by X^α, Y^α or $(X-Y)^\alpha$ ($\alpha \in \mathbb{C}$), and the difference $\nu_i - \nu_j$ ($i \neq j$) does not change. The first type of spectral partitions of Pfaffian system (13.3) is $\{(2,2),(3,1),(3,1)\}$. If the condition (1) for (13.4) is satisfied, then each of the differences between different eigenvalues of W_2' is not integer, and the set $\{\nu_i | 1 \leq i \leq 4\}$ changes to neither type $(2,2)$ nor $(3,1)$ mod $\mathbb{Z}$, by these actions. In this case, the Pfaffian system of Humbert's Ψ_1 can not be transformed into a Pfaffian system of Humbert's Ξ_1 by the action of any element in A. If the condition (2) or (3) is satisfied, then we can show non-equivalence by similar ways. □

14 Automorphism Group

In this section, we consider a group of birational transformations of $\mathbb{P}^2$.

Definition 14.1. Let Q be the open set in $\mathbb{P}^2$ given by

$$Q := \left\{ [z:x:y] \in \mathbb{P}^2 \,|\, zxy(x-y) \neq 0 \right\},$$

and $\mathrm{Aut}(Q)$ be the group of birational transformations of $\mathbb{P}^2$ whose restrictions to Q are biregular morphisms from Q to Q.

To study the structure of $\mathrm{Aut}(Q)$, we prepare the following group.

Definition 14.2. Let G be the set

$$G := \left\{ (r,m,n,s,t) \,|\, r \in \mathbb{C}^\times,\, m,n \in \mathbb{Z},\, s \in S,\, t \in \{\pm 1\} \right\}, \tag{14.1}$$

where S is the subgroup of $\mathrm{GL}(2,\mathbb{Z})$ generated by $\sigma_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ and $\sigma_2 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$; it is isomorphic to the symmetric group S_3. We introduce a group structure into G by

$$g_a g_b = (r_a' r_b, m_a' + m_b, n_a' + n_b, s_a' s_b, t_a t_b),$$

where $g_a = (r_a, m_a, n_a, s_a, t_a)$, $g_b = (r_b, m_b, n_b, s_b, t_b) \in G$,

$$r'_a := (r_a)^{t_b}, \quad \begin{bmatrix} m'_a \\ n'_a \end{bmatrix} := s_b^{-1} \begin{bmatrix} m''_a \\ n''_a \end{bmatrix}, \quad s'_a := \begin{cases} s_a & (t_b = 1), \\ \sigma_1 s_a \sigma_1 & (t_b = -1), \end{cases} \tag{14.2}$$

$$m''_a := \begin{cases} m_a & (t_b = 1), \\ n_a + \theta_m(s_a) & (t_b = -1), \end{cases} \quad n''_a := \begin{cases} n_a & (t_b = 1), \\ m_a + \theta_n(s_a) & (t_b = -1), \end{cases}$$

$$\theta_m(s) := \begin{cases} 1 & (s \in \{\sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\}), \\ 0 & (\text{others}), \end{cases} \quad \theta_n(s) := \begin{cases} -1 & (s \in \{\sigma_2, \sigma_1\sigma_2\}), \\ 0 & (\text{others}). \end{cases}$$

The group G is isomorphic to

$$\big((\mathbb{C}^\times \times \mathbb{Z}^2) \rtimes S_3\big) \rtimes \{\pm 1\}.$$

Lemma 14.1. *The group G acts on Q by*

$$g \cdot [1 : X : Y] = [1 : X' : Y'],$$

where $g = (r, m, n, s, t) \in G$, $(X, Y) = (x/z, y/z)$ and $(X', Y') = (x'/z', y'/z')$ are affine coordinates, and

$$\begin{cases} X' = (X_2)^t, \\ Y' = (Y_2)^t, \end{cases} \quad \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} := s \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}, \quad \begin{cases} X_1 := rX(\frac{-X}{X-Y})^m(\frac{X-Y}{Y})^n, \\ Y_1 := rY(\frac{-X}{X-Y})^m(\frac{X-Y}{Y})^n. \end{cases} \tag{14.3}$$

This action induces an isomorphism $\psi : G \ni g \mapsto \psi_g \in \mathrm{Aut}(Q)$.

Proof. At first, we characterize an element of $\mathrm{Aut}(Q)$. Let $\phi : Q \ni [z : x : y] \mapsto [z' : x' : y'] \in Q$ be an element of $\mathrm{Aut}(Q)$, then ϕ is expressed by homogeneous polynomials

$$\begin{aligned} &\begin{cases} z' = x^{m_{01}} y^{m_{02}} (x-y)^{m_{03}} (z^{m_{00}} f_0(\frac{x}{z}, \frac{y}{z})), \\ x' = x^{m_{11}} y^{m_{12}} (x-y)^{m_{13}} (z^{m_{10}} f_1(\frac{x}{z}, \frac{y}{z})), \\ y' = x^{m_{21}} y^{m_{22}} (x-y)^{m_{23}} (z^{m_{20}} f_2(\frac{x}{z}, \frac{y}{z})), \end{cases} \\ &m_{i0} + m_{i1} + m_{i_2} + m_{i3} = d \quad (0 \le i \le 2), \\ &m_{ij} \in \mathbb{Z}_{\ge 0} \quad (0 \le i \le 2,\ 0 \le j \le 3), \end{aligned} \tag{14.4}$$

where $z^{m_{i0}} f_i$ are polynomials of x, y, z of degree m_{i0}, and they are supposed to be not divided by x, y or $x - y$.

Let us show that f_0, f_1 and f_2 are constants. Factorize f_i as

$$f_i(X, Y) = \prod_{j=1}^{N_i} g_{ij}(X, Y)$$

in $\mathbb{C}[X, Y]$, where g_{ij} are irreducible polynomials. Because $\phi(Q) \subset Q$, if $[z : x : y]$ satisfies $f_1(x/z, y/z) = 0$, then it also satisfies $f_0(x/z, y/z) = 0$. Using Hirbert's Nullstellensatz, we can see that $f_0(X, Y)$ is divided by $g_{1j}(X, Y)$. Similarly, $f_1(X, Y)$ is divided by $g_{0j}(X, Y)$. Consequently, we have

$$f_1/f_0 = c_1 \in \mathbb{C}^\times. \tag{14.5}$$

In the same way, we have

$$f_2/f_0 = c_2 \in \mathbb{C}^\times. \tag{14.6}$$

Using (14.5) and (14.6), we rewrite (14.4) as

$$\begin{cases} z' = z^{m_{00}} x^{m_{01}} y^{m_{02}} (x-y)^{m_{03}}, \\ x' = c_1 z^{m_{10}} x^{m_{11}} y^{m_{12}} (x-y)^{m_{13}}, \\ y' = c_2 z^{m_{20}} x^{m_{21}} y^{m_{22}} (x-y)^{m_{23}}. \end{cases} \tag{14.7}$$

By the previous consideration, we can see that $(x'-y')/z' \neq 0$ and $z'/(x'-y') \neq 0$. Thus $x'-y'$ is factorized as

$$\begin{aligned} &x' - y' = c_3 z^{m_{30}} x^{m_{31}} y^{m_{32}} (x-y)^{m_{33}}, \\ &c_3 \in \mathbb{C}^\times, \quad m_{30} + m_{31} + m_{32} + m_{33} = d. \end{aligned} \tag{14.8}$$

Regard x' and y' as monomials of the variable z. Then $x'-y'$ is also a monomial of z, and we have

$$m_{10} = m_{20} = m_{30}. \tag{14.9}$$

Note that x'/z', y'/z' and $(x'-y')/z'$ have same degree as rational functions of X and Y. This degree d' is represented by

$$d' = n_{i1} + n_{i2} + n_{i3} \quad (n_{ij} \in \mathbb{Z}),$$

where $n_{ij} := m_{ij} - m_{0j}$ $(1 \leq i, j \leq 3)$. We set

$$q(i,j) := \max(i,j) - j \quad (i,j \in \mathbb{Z}).$$

Using (14.7), we take factors x/z, y/z and $(x-y)/z$ from $(x'-y')/z'$ as

$$\begin{aligned} \frac{x'-y'}{z'} = \frac{x'}{z'} - \frac{y'}{z'} = \left(\frac{x}{z}\right)^{\min(n_{11},n_{21})} \left(\frac{y}{z}\right)^{\min(n_{12},n_{22})} \left(\frac{x-y}{z}\right)^{\min(n_{13},n_{23})} \times g\left(\frac{x}{z}, \frac{y}{z}\right), \\ g(X,Y) := c_1 X^{q(n_{11},n_{21})} Y^{q(n_{12},n_{22})} (X-Y)^{q(n_{13},n_{23})} - c_2 X^{q(n_{21},n_{11})} Y^{q(n_{22},n_{12})} (X-Y)^{q(n_{23},n_{13})}, \end{aligned} \tag{14.10}$$

where $g(X,Y)$ is a non-constant polynomial of X and Y. Because $g(X,Y)$ does not have non-constant factors different from X, Y and $X-Y$, we can see that

$$\begin{cases} |n_{1j} - n_{2j}| \leq 1, \\ |n_{11} - n_{21}| + |n_{12} - n_{22}| + |n_{13} - n_{23}| = 2, \\ c_1 = (-1)^{n_{11} - n_{21} - 1} c_2. \end{cases}$$

This means that among three equalities $n_{11} = n_{21}, n_{12} = n_{22}$ and $n_{13} = n_{23}$, only one holds and the rest do not. For example, if $n_{11} = n_{21}$, then x'/z' and y'/z' are divided by $(x/z)^{n_{11}}$, and $(x'-y')/z'$ is also divided. In this case, we can see

$$n_{31} \geq n_{11} \quad (n_{11} = n_{21}).$$

For example, if $n_{12} > n_{22}$, then $(x'-y')/z'$ is divided by $(y/z)^{n_{22}}$ but not divided by $(y/z)^{n_{12}}$. Therefore, we can see

$$n_{32} = \min(n_{12}, n_{22}) \quad (n_{12} \neq n_{22}).$$

Let us give conditions satisfying n_{1j} and n_{3j}. Replace the role of $(x'-y')/z'$ in (14.10) by y'/z' as

$$\frac{y'}{z'} = \frac{x'}{z'} - \frac{x'-y'}{z'} = \left(\frac{x}{z}\right)^{\min(n_{11},n_{31})} \left(\frac{y}{z}\right)^{\min(n_{12},n_{32})} \left(\frac{x-y}{z}\right)^{\min(n_{13},n_{33})} \times g'\left(\frac{x}{z}, \frac{y}{z}\right),$$
$$g'(X,Y) := c_1 X^{q(n_{11},n_{31})} Y^{q(n_{12},n_{32})} (X-Y)^{q(n_{13},n_{33})} - c_3 X^{q(n_{31},n_{11})} Y^{q(n_{32},n_{12})} (X-Y)^{q(n_{33},n_{13})}.$$

Then we have

$$\begin{cases} |n_{1j} - n_{3j}| \leq 1, \\ |n_{11} - n_{31}| + |n_{12} - n_{32}| + |n_{13} - n_{33}| = 2, \\ c_1 = (-1)^{n_{11}-n_{31}-1} c_3 \end{cases}$$

from (14.7) and (14.8). We can see

$$\begin{cases} n_{2j} \geq n_{1j} \quad (n_{1j} = n_{3j}), \\ n_{2j} = \min(n_{1j}, n_{3j}) \quad (n_{1j} \neq n_{3j}). \end{cases}$$

Let us give conditions satisfying n_{2j} and n_{3j}. Replace the role of $(x'-y')/z'$ in (14.10) by x'/z', then we have

$$\begin{cases} |n_{3j} - n_{2j}| \leq 1, \\ |n_{31} - n_{21}| + |n_{32} - n_{22}| + |n_{33} - n_{23}| = 2, \\ c_3 = (-1)^{n_{31}-n_{21}} c_2, \end{cases}$$
$$\begin{cases} n_{1j} \geq n_{3j} \quad (n_{3j} = n_{2j}), \\ n_{1j} = \min(n_{3j}, n_{2j}) \quad (n_{3j} \neq n_{2j}). \end{cases}$$

We get the equalities

$$c := (-1)^{n_{11}-1} c_1 = (-1)^{n_{21}} c_2 = (-1)^{n_{31}} c_3.$$

We rewrite (14.7) and (14.8) as

$$\begin{cases} \frac{x'}{z'} = (-1)^{n_{11}-1} c (\frac{x}{z})^{n_{11}} (\frac{y}{z})^{n_{12}} (\frac{x-y}{z})^{n_{13}}, \\ \frac{y'}{z'} = (-1)^{n_{21}} c (\frac{x}{z})^{n_{21}} (\frac{y}{z})^{n_{22}} (\frac{x-y}{z})^{n_{23}}, \\ \frac{x'-y'}{z'} = (-1)^{n_{31}} c (\frac{x}{z})^{n_{31}} (\frac{y}{z})^{n_{32}} (\frac{x-y}{z})^{n_{33}}, \\ n_{i1} + n_{i2} + n_{i3} = d', \\ |n_{ij} - n_{kj}| \leq 1 \quad (1 \leq i < k \leq 3, 1 \leq j \leq 3), \\ n_{1j} + n_{2j} + n_{3j} = 3\min(n_{1j}, n_{2j}, n_{3j}) + 1 \quad (1 \leq j \leq 3). \end{cases} \tag{14.11}$$

We show that

$$d' = \pm 1.$$

Let $[1 : a' : b']$ be the image of a point $[1 : a : ba]$ in the line $y = bx\,(b \neq 0, 1)$ under ϕ. Then a', b' are written by

$$\begin{cases} a' = (-1)^{n_{11}-1} c a^{n_{11}} (ba)^{n_{12}} (a - ba)^{n_{13}} = (-1)^{n_{11}-1} c a^{d'} b^{n_{12}} (1-b)^{n_{13}}, \\ b' = (-1)^{n_{21}} c a^{n_{21}} (ba)^{n_{22}} (a - ba)^{n_{23}} = (-1)^{n_{21}} c a^{d'} b^{n_{22}} (1-b)^{n_{23}}. \end{cases}$$

If the inverse image of $[1 : a' : b']$ under ϕ is unique, then it is necessary that $|d'| = 1$, because

$$\phi(1 : \varepsilon_{|d'|}^k a : \varepsilon_{|d'|}^k ba]) = [1 : a' : b'] \quad (0 \leq k \leq |d'| - 1),$$

where $\varepsilon_{|d'|}$ denotes the $|d'|$-th root $e^{2\pi\sqrt{-1}/|d'|}$ of unity.

Let us consider the uniqueness of the inverse image. We divide our consideration into two cases whether the three images x', y' and $x' - y'$ in (14.7) and (14.8) have a common component or not.

Case 1: x', y' and $x' - y'$ have no common component. We define three curves

$$\begin{aligned} L &:= \left\{ [z : x : y] \in \mathbb{P}^2 \mid x' = a' z' \right\}, \\ M &:= \left\{ [z : x : y] \in \mathbb{P}^2 \mid y' = b' z' \right\}, \\ N &:= \left\{ [z : x : y] \in \mathbb{P}^2 \mid b' x' = a' y' \right\}, \end{aligned}$$

in $\mathbb{P}^2$. If the map $\varphi : Q \to Q$ is isomorphic, then each of intersections $L \cap M, L \cap N, M \cap N$ consists of one point. From Bézout's Theorem, if the map $\varphi : Q \to \mathbb{P}^2$ is isomorphic, then all of L, M, N are lines. From the conditions (14.7) and (14.8), if $[1 : a : b] \in Q$, then $[1 : a' : b'] \in Q$. It means that the map $\varphi : Q \to Q$ is isomorphic. As a result, we have

$$q(m_{i1}, m_{j1}) + q(m_{i2}, m_{j2}) + q(m_{i3}, m_{j3}) + q(m_{i0}, m_{j0}) = 1 \quad (i \neq j,\ 0 \leq i, j \leq 2).$$

From the condition $m_{10} = m_{20}$ in (14.9), we can see that

$$\begin{cases} |m_{11} - m_{01}| + |m_{12} - m_{02}| + |m_{13} - m_{03}| = 1, \\ |m_{21} - m_{01}| + |m_{22} - m_{02}| + |m_{23} - m_{03}| = 1, \\ |m_{21} - m_{11}| + |m_{22} - m_{12}| + |m_{23} - m_{13}| = 2, \\ m_{00} - m_{10} = d', \\ m_{20} = m_{10}. \end{cases} \tag{14.12}$$

In case of $d' = 1$, an isomorphism $\phi_1 \in \mathrm{Aut}(Q)$ satisfying both (14.11) and (14.12) should be

$$\begin{aligned} &[z' : x' : y'] = \phi_1([z : x : y]), \\ &\begin{bmatrix} x'/z' \\ y'/z' \end{bmatrix} = c A_3 \begin{bmatrix} x/z \\ y/z \end{bmatrix}, \\ &c \in \mathbb{C}^\times, \quad A_3 \in S. \end{aligned} \tag{14.13}$$

In case of $d' = -1$, an isomorphism $\phi_{-1} \in \mathrm{Aut}(Q)$ satisfying both (14.11) and (14.12) should be

$$\begin{aligned} &[z' : x' : y'] = \phi_{-1}([z : x : y]), \\ &\begin{bmatrix} x'/z' \\ y'/z' \end{bmatrix} = cB_3 \begin{bmatrix} z/x \\ z/y \end{bmatrix}, \\ &c \in \mathbb{C}^\times, \quad B_3 \in S. \end{aligned} \tag{14.14}$$

Hence, we conclude that φ takes the form of (14.13) or (14.14) in this case.

Case 2: x', y' and $x'-y'$ may have common components. We will show that any transformation satisfying (14.11) is expressed by a composite of transformations of the forms of (14.13) and (14.14).

In case of $d' = 1$, let a transformation

$$\varphi_1([z : x : y]) = [z' : x' : y'],$$

satisfy the conditions (14.11). A transformation of the form of (14.13) can be regarded as a permutation of $-x$, y and $x - y$. There is a transformation $\phi_1^{(1)} \in \mathrm{Aut}(Q)$

$$\phi_1^{(1)}([z' : x' : y']) = [z'' : x'' : y''],$$

such that the composite transformation $\phi_1^{(1)} \circ \varphi_1$ becomes

$$\begin{aligned} &\begin{cases} \frac{x''}{z''} = -c(\frac{-x}{z})^{n_1+1}(\frac{y}{z})^{n_2}(\frac{x-y}{z})^{n_3}, \\ \frac{y''}{z''} = c(\frac{-x}{z})^{n_1}(\frac{y}{z})^{n_2+1}(\frac{x-y}{z})^{n_3}, \\ \frac{x''-y''}{z''} = c(\frac{-x}{z})^{n_1}(\frac{y}{z})^{n_2}(\frac{x-y}{z})^{n_3+1}, \end{cases} \\ &n_i := \min(n_{i1}, n_{i2}, n_{i3}) \quad (1 \le i \le 3), \\ &n_1 + n_2 + n_3 = d' - 1 = 0. \end{aligned} \tag{14.15}$$

In order to show that $\phi_1^{(1)} \circ \varphi_1$ is a composite of transformations in Case 1, we choose transformations

$$\phi_{-1}^{(1)}([z_0 : x_0 : y_0]) = [z_1 : x_1 : y_1], \quad \begin{cases} \frac{x_1}{z_1} = \frac{z_0}{x_0}, \\ \frac{y_1}{z_1} = \frac{z_0}{x_0} - \frac{z_0}{y_0}, \end{cases}$$

$$\phi_{-1}^{(2)}([z_0 : x_0 : y_0]) = [z_2 : x_2 : y_2], \quad \begin{cases} \frac{x_2}{z_2} = \frac{z_0}{y_0} - \frac{z_0}{x_0}, \\ \frac{y_2}{z_2} = \frac{z_0}{y_0}, \end{cases}$$

in $\mathrm{Aut}(Q)$ of the form of (14.14). We define transformations $\varphi_{m\pm}, \varphi_{n\pm}$ by

$$\begin{aligned} \varphi_{m+} &:= \phi_{-1}^{(2)} \circ \phi_{-1}^{(1)}, \quad \varphi_{m-} := \phi_{-1}^{(-1)} \circ \phi_{-1}^{(-2)}, \\ \varphi_{n-} &:= \phi_{-1}^{(1)} \circ \phi_{-1}^{(2)}, \quad \varphi_{n+} := \phi_{-1}^{(-2)} \circ \phi_{-1}^{(-1)}, \end{aligned}$$

where $\phi_{-1}^{(-1)}$ and $\phi_{-1}^{(-2)}$ are the inverses of $\phi_{-1}^{(1)}$ and $\phi_{-1}^{(2)}$, respectively. The transformations $\varphi_{m\pm}, \varphi_{n\pm}$ are mutually commutative, and they satisfy

$$c(\varphi_m)^{n_1} \circ (\varphi_n)^{-n_2} = \phi_1^{(1)} \circ \varphi_1, \tag{14.16}$$

where $(\varphi_m)^n$ denotes $\underbrace{\varphi_{m+} \circ \cdots \circ \varphi_{m+}}_{n\text{times}} (n \geq 0)$ or $\underbrace{\varphi_{m-} \circ \cdots \circ \varphi_{m-}}_{|n|\text{times}} (n < 0)$. Because $\varphi_{m\pm}, \varphi_{n\pm}$ and the inverse $\phi_1^{(-1)}$ of $\phi_1^{(1)}$ are in $\mathrm{Aut}(Q)$, φ_1 is in $\mathrm{Aut}(Q)$.

In case of $d' = -1$, let a transformation

$$\varphi_{-1}([z : x : y]) = [z' : x' : y'],$$

satisfy the conditions (14.11). We use the Cremona transformation $\phi_{\mathrm{cre}} \in \mathrm{Aut}(Q)$

$$\phi_{\mathrm{cre}}([z_0 : x_0 : y_0]) = [x_0 y_0 : y_0 z_0 : z_0 x_0],$$

which can be written in the form of (14.14). The composite transformation $\varphi_{-1} \circ \phi_{\mathrm{cre}}$ is expressed by

$$\begin{cases} \frac{x'}{z'} = (-1)^{n_{12}} c (\frac{x_0}{z_0})^{n_{12}+1} (\frac{y_0}{z_0})^{n_{11}+1} (\frac{x_0 - y_0}{z_0})^{n_{13}}, \\ \frac{y'}{z'} = (-1)^{n_{22}+1} c (\frac{x_0}{z_0})^{n_{22}+1} (\frac{y_0}{z_0})^{n_{21}+1} (\frac{x_0 - y_0}{z_0})^{n_{23}}, \\ \frac{x'-y'}{z'} = (-1)^{n_{32}+1} c (\frac{x_0}{z_0})^{n_{32}+1} (\frac{y_0}{z_0})^{n_{31}+1} (\frac{x_0 - y_0}{z_0})^{n_{33}}, \\ (n_{i2}+1) + (n_{i1}+1) + n_{i3} = 1 \quad (1 \leq i \leq 3). \end{cases} \tag{14.17}$$

This is one of transformations in (14.11) with $d' = 1$. Thus φ_{-1} is in $\mathrm{Aut}(Q)$. Moreover, any transformation in (14.11) is in $\mathrm{Aut}(Q)$, and any transformation in $\mathrm{Aut}(Q)$ is written as (14.11). This transformation is represented by a composition of the forms of (14.13) and (14.14).

Next, we show that the map $\psi : G \ni g \mapsto \psi_g \in \mathrm{Aut}(Q)$ defined by (14.3) is a group homomorphism. There are subgroups $G_1 := \{(r, m, n, I_2, 1) \,|\, r \in \mathbb{C}^\times, m, n \in \mathbb{Z}\}$, $G_2 := \{(1, 0, 0, s, 1) \,|\, s \in S\}$ and $G_3 := \{(1, 0, 0, I_2, t) \,|\, t \in \{\pm 1\}\}$ of G. These groups are isomorphic to $\mathbb{C}^\times \times \mathbb{Z}^2$, S_3 and $\{\pm 1\}$, respectively. Note that for any element $g = (r_0, m_0, n_0, s_0, t_0) \in G$, there uniquely exist elements $g_1 = (r_0, m_0, n_0, I_2, 1) \in G_1$, $g_2 = (1, 0, 0, s_0, 1) \in G_2$ and $g_3 = (1, 0, 0, I_2, t_0) \in G_3$, such that $g = g_3 g_2 g_1$.

Let us consider the images $\psi(G_i)\,(1 \leq i \leq 3)$. From (14.3), the image $\psi(g_1)$ becomes $r(\varphi_m)^{m_0} \circ (\varphi_n)^{n_0}$. Thus $\psi(G_1)$ is the group

$$H_1 := \left\{ r(\varphi_m)^{m_0} \circ (\varphi_n)^{n_0} \,|\, r \in \mathbb{C}^\times, m_0, n_0 \in \mathbb{Z} \right\},$$

and the restriction $\psi|_{G_1}$ of the map ψ to G_1 is a group isomorphism. The images $\varphi_{\sigma_1} = \psi(1, 0, 0, \sigma_1, 1)$ and $\varphi_{\sigma_2} = \psi(1, 0, 0, \sigma_2, 1)$ are

$$\varphi_{\sigma_1}([1 : X : Y]) = [1 : X_1 : Y_1], \begin{cases} X_1 = -Y, \\ Y_1 = -X, \end{cases} \quad \varphi_{\sigma_2}([1 : X : Y]) = [1 : X_2 : Y_2], \begin{cases} X_2 = Y - X, \\ Y_2 = Y, \end{cases}$$

respectively. Thus $\psi(G_2)$ is the group $H_2 := \langle \varphi_{\sigma_1}, \varphi_{\sigma_2} \rangle$ generated by φ_{σ_1} and φ_{σ_2} and the restriction $\psi|_{G_2}$ is a group isomorphism. Let H_3 be a subgroup of $\mathrm{Aut}(Q)$ generated by the Cremona transformation. The element $(1, 0, 0, I_2, -1) \in G_3$ is mapped to ϕ_{cre} by ψ, then the restriction $\psi|_{G_3} : G_3 \to H_3$ is a group isomorphism.

The map ψ satisfies the relation

$$\psi(g) = \psi(g_3) \circ \psi(g_2) \circ \psi(g_1). \tag{14.18}$$

In order to extend the group homomorphisms $\psi|_{G_1}$ and $\psi|_{G_2}$, we define groups

$$G_{12} := \langle g_1, g_2 \,|\, g_1 \in G_1, g_2 \in G_2 \rangle\,, \quad H_{12} := \langle h_1, h_2 \,|\, h_1 \in H_1, h_2 \in H_2 \rangle\,,$$

which are generated by G_1 and G_2 and by H_1 and H_2, respectively. Any transformation in (14.11) with $d' = 1$ is in H_{12} by (14.16). Note that H_1 is a normal subgroup of H_{12} by relations,

$$\begin{aligned}
\varphi_{\sigma_1} \circ \varphi_{m+} \circ \varphi_{\sigma_1} &= \varphi_{n-}, & \varphi_{\sigma_1} \circ \varphi_{m-} \circ \varphi_{\sigma_1} &= \varphi_{n+},\\
\varphi_{\sigma_1} \circ \varphi_{n+} \circ \varphi_{\sigma_1} &= \varphi_{m-}, & \varphi_{\sigma_1} \circ \varphi_{n-} \circ \varphi_{\sigma_1} &= \varphi_{m+},\\
\varphi_{\sigma_2} \circ \varphi_{m+} \circ \varphi_{\sigma_2} &= \varphi_{m-}, & \varphi_{\sigma_2} \circ \varphi_{m-} \circ \varphi_{\sigma_2} &= \varphi_{m+},\\
\varphi_{\sigma_2} \circ \varphi_{n+} \circ \varphi_{\sigma_2} &= \varphi_{m+} \circ \varphi_{n+}, & \varphi_{\sigma_2} \circ \varphi_{n-} \circ \varphi_{\sigma_2} &= \varphi_{m-} \circ \varphi_{n-},
\end{aligned}$$

where φ_{σ_1} and φ_{σ_2} are of order 2. Using these relations, we have

$$\psi_1 \circ \psi_2 = r_1 r_2 \varphi_{s_1 s_2} \circ (\varphi_m)^{\dot{m}_1 + m_2} \circ (\varphi_n)^{\dot{n}_1 + n_2}, \quad \begin{bmatrix} \dot{m}_1 \\ \dot{n}_1 \end{bmatrix} := s_2^{-1} \begin{bmatrix} m_1 \\ n_1 \end{bmatrix}, \tag{14.19}$$

where $\psi_1 = r_1 \varphi_{s_1} \circ (\varphi_m)^{m_1} \circ (\varphi_n)^{n_1}$ and $\psi_2 = r_2 \varphi_{s_2} \circ (\varphi_m)^{m_2} \circ (\varphi_n)^{n_2}$. This composition is compatible with (14.2) in $t_b = 1$ case. Thus the restriction $\psi|_{G_{12}}$ is a group isomorphism G_{12} to H_{12}. Note that H_{12} is a normal subgroup of $\mathrm{Aut}(Q)$ by relations,

$$\begin{aligned}
\phi_{cre} \circ \varphi_{m+} \circ \phi_{cre} &= \varphi_{n+}, & \phi_{cre} \circ \varphi_{m-} \circ \phi_{cre} &= \varphi_{n-},\\
\phi_{cre} \circ \varphi_{n+} \circ \phi_{cre} &= \varphi_{m+}, & \phi_{cre} \circ \varphi_{n-} \circ \phi_{cre} &= \varphi_{m-},\\
\phi_{cre} \circ \varphi_{\sigma_1} \circ \phi_{cre} &= \varphi_{\sigma_1}, & \phi_{cre} \circ \varphi_{\sigma_2} \circ \phi_{cre} &= \varphi_{\sigma 1 \sigma_2 ß_1} \circ \varphi_{n-}.
\end{aligned}$$

Using these relations, we have

$$\psi_3 \circ \psi_4 = (\frac{r_3}{r_4}) \phi_{cre} \circ \varphi_{s'_3 s_4} \circ (\varphi_m)^{m'_3 + m_4} \circ (\varphi_n)^{n'_3 + n_4}, \quad \begin{bmatrix} m'_3 \\ n'_3 \end{bmatrix} := s_4^{-1} \begin{bmatrix} m''_3 \\ n''_3 \end{bmatrix}, \tag{14.20}$$

where $s'_3 = \sigma_1 s_3 \sigma_1$, $m''_3 = n_3 + \theta_m(s_3)$, $n''_3 = m_3 + \theta_n(s_3)$, $\psi_3 = r_3 \varphi_{s_3} \circ (\varphi_m)^{m_3} \circ (\varphi_n)^{n_3}$ and $\psi_4 = \phi_{cre} \circ r_4 \varphi_{s_4} \circ (\varphi_m)^{m_4} \circ (\varphi_n)^{n_4}$. This composition is compatible with (14.2) in $t_b = -1$ case.

Using (14.19) and (14.20), we can see

$$\psi(g_a g_b) = \psi(g_a) \circ \psi(g_b), \quad g_a, g_b \in \mathrm{Aut}(Q).$$

Hence ψ is a group homomorphism. Moreover, we have a series of normal subgroups

$$H_1 \lhd H_{12} \lhd \mathrm{Aut}(Q).$$

Because $\psi|_{G_1}$, $\psi|_{G_2}$ and $\psi|_{G_3}$ are isomorphic, the map $\psi : G \to \mathrm{Aut}(Q)$ is a group isomorphism. □

References

[Er] Erdélyi A.(editor), Higher transcendental functions, Vol. 1, McGraw-Hill, (1953).

[GRS] Gelfand I.M., Retakh V.S. and Serganova V.V., Generalized Airy functions, Schubert cells, and Jordan groups, Dokl. Akad. Nauk SSSR, **298** (1988), 17–21; Engl. transl. in Sov. Math. Dokl., **37** (1988), 8–12.

[Ha] Haraoka Y., On confluences of hypergeometric integrals, Kumamoto J. Math., **17** (2004), 1–8.

[Ho] Hörn J., Hypergeometrische Funktionen zweier Veranderlichen, Math. Ann., **105** (1931), 381–407; **111** (1935), 638–677; **113** (1937), 242–291.

[Hu] Humbert P., Sur les fonctions hypercylindriques, C.R. Acad. Sci. Paris, **171** (1920), 490–492.

[Ka] Kato M., Connection formulas and irreducibility conditions for Appell's F_2, Kyushu J. Math., **66** (2012), 325–363.

[KHT] Kimura H., Haraoka Y., Takano K., On confluences of the general hypergeometric systems, Proc. Japan Acad. Ser. A Math. Sci., **69** (1993), 99–104.

[KK] Kimura H. and Koitabashi T., Normalizer of maximal abelian subgroups of $\mathrm{GL}(n)$ and general hypergeometric functions, Kumamoto J. Math., **9** (1996), 13–43.

[KTk] Kimura H., Takano K., On confluences of general hypergeometric systems, Tohoku Math. J., **58** (2006), 1–31.

[KTn] Kimura, H. and Taneda, M., Analogue of flat bases and cohomological intersection numbers for general hypergeometric functions, J. Math. Sci. Univ. Tokyo, **6** (1999), 415-436.

[Mj1] Majima, H., ∇-Poincaré's Lemma and ∇-de Rham cohomology for an integrable connection with irregular singular points, Proc. Japan Acad. Ser. A Math. Sci. **59** (1983), no.4, 150–153.

[Mj2] Majima, H., Asymptotic analysis for integrable connections with irregular singular points, Lecture Notes in Math. **1075**, Springer-Verlag, Berlin-New York, (1984).

[Ml] Malgrange, B., Remarques sur les équations différentielles á points singuliers irréguliers, Équations différentielles et systèmes de Pfaff dans le champ complexe, Lecture Notes in Math., **712**, Springer-Verlag, Berlin, (1979), 77–86.

[Km] Kimura, H., On rational de Rham cohomology associated with the generalized confluent hypergeometric functions I, $\mathbb{P}^1$ case, Proc. Roy. Soc. Edinburgh, **127A** (1997), 145–155

[KMM] Kachi, N., Matsumoto, K., and Mihara, M., The perfectness of the intersection pairings for twisted cohomology and homology groups with respect to rational 1-forms, Kyushu J. Math., **53** (1999), 163–188.

[MMT] Majima, H., Matsumoto, K. and Takayama, N., Quadratic relations for confluent hypergeometric functions, Tohoku Math. J., **52** (2000), 489-513.

[Mt1] Matsumoto, K., Intersection numbers for 1-forms associated with confluent hypergeometric Functions, Funkcialaj Ekvacioj, **41** (1998), 291–308.

[Mt2] Matsumoto, K., Monodromy and Pfaffian of Lauricella's F_D in terms of the intersection forms of twisted (co)homology groups, Kyushu J. Math., **67** (2013), 367-387.

[Mu] Mukai S., Pfaffian systems of confluent hypergeometric functions of two variables, Kyushu J. Math., **74** (2020), 63–104.