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## Duality of (2,3,5)-distributions and Lagrangian cone structures

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**Abstract.** As was shown by a part of the authors, for a given (2,3,5)-distribution  $D$  on a 5-dimensional manifold  $Y$ , there is, locally, a Lagrangian cone structure  $C$  on another 5-dimensional manifold  $X$  which consists of abnormal or singular paths of  $(Y, D)$ . We give a characterization of the class of Lagrangian cone structures corresponding to (2,3,5)-distributions. Thus we complete the duality between (2,3,5)-distributions and Lagrangian cone structures via pseudo-product structures of type  $G_2$ . A local example of non-flat perturbations of the global model of flat Lagrangian cone structure which corresponds to (2,3,5)-distributions is given.

### §1. Introduction

A distribution  $D$  on a 5-dimensional manifold  $Y$  is called a (2,3,5)-distribution if there is a local section  $\eta_1, \eta_2$  of  $D$  such that

$$\eta_1, \eta_2, [\eta_1, \eta_2], [\eta_1, [\eta_1, \eta_2]], [\eta_2, [\eta_1, \eta_2]]$$

form a local frame of the tangent bundle to  $Y$ , in other words, if  $D$  has the weak growth (2,3,5), namely, if  $\text{rank}(\partial\mathcal{D}) = 3$  and  $\text{rank}(\partial^{(2)}\mathcal{D}) = 5$ , where  $\partial\mathcal{D} := [\mathcal{D}, \mathcal{D}] (= \mathcal{D} + [\mathcal{D}, \mathcal{D}])$ , the derived system, and  $\partial^{(2)}\mathcal{D} := [\mathcal{D}, \partial\mathcal{D}] (= \mathcal{D} + \partial\mathcal{D} + [\mathcal{D}, \partial\mathcal{D}])$  for the sheaf  $\mathcal{D}$  of section-germs to  $D$ .

The geometry and classification problem of (2,3,5)-distributions are studied after E. Cartan ([13]), related to the simple Lie group  $G_2$ , by many mathematicians ([5][8][19][26][27][28][29][30]). The (2,3,5)-distributions are related to many problems, for instance, to the problem of “rolling balls” ([1][11][7][6]), to indefinite conformal metrics ([21][19]), to non-linear differential equations ([22]), and so on.

In [17][16][18], we studied the global duality of  $G_2$ -homogeneous (flat) (2,3,5)-distribution and a Lagrangian cone structure from Cayley’s split Octonions and classified the related generic singularities. In [15], we associated locally with any given (2,3,5)-distribution  $D$  on a 5-dimensional manifold  $Y$ , a Lagrangian cone structure  $C$  on another 5-dimensional manifold  $X$ , which consists of abnormal or singular paths of  $(Y, D)$ , in the sense of sub-Riemannian geometry or geometric control theory (see [20][2]). Moreover it was shown in [15] that the original space  $Y$  turns to be the totality of singular paths of the “Lagrangian cone structure”  $(X, C)$ , when the cone field  $C$  is regarded as a control system on  $X$ .

In this paper, we give the characterization of the class of Lagrangian cone structures corresponding to (2,3,5)-distributions, and thus we complete the duality between (2,3,5)-distributions and Lagrangian cone structures (Theorem 3.1). The duality is actually understood via pseudo-product structure of  $G_2$ -type  $E = K \oplus L$  on a 6-dimensional manifold  $Z$  (§2), which is regarded both as the prolongation of  $(Y, D)$  and  $(X, C)$  in the sense of Bryant ([9][8]), via the double fibration

$$(Y, D) \xleftarrow{\pi_Y} (Z, E) \xrightarrow{\pi_X} (X, C).$$

We realize, for the characterization a class of Lagrangian cone structures, that the language of cone structures is actually lacking: We introduce, regarding the cone structures as control systems, the notions of linear approximations and osculating bundles of cone structures, as well as the exact definition of non-degenerate Lagrangian cone structures (Definition 2.3).

We remark that our correspondence is purely local in nature: It is “spatially” local for  $(Z, E)$  while “spatially and directionally” local for  $(Y, D)$  and for  $(X, C)$ . Moreover the “directional locality” for the distribution  $(Y, D)$  is resolved by taking linear hull, however it is not the case for the cone structure  $(X, C)$ . This fact makes our duality delicate.

It is clear that  $(2, 3, 5)$ -distributions form an open set, for Whitney  $C^\infty$ -topology, in the space of all distributions of rank 2 on a 5-dimensional manifold. In particular a  $(2, 3, 5)$ -distribution remains a  $(2, 3, 5)$ -distribution by sufficiently small perturbations with compact supports. However it is not clear such a stability for cone structures corresponding to  $(2, 3, 5)$ -distributions via the duality. We give a local example of non-flat perturbations of the global model of flat Lagrangian cone structure ([16]), which corresponds to  $(2, 3, 5)$ -distributions (Example 4.3). It is open the existence of non-flat *global* perturbations of Lagrangian cone structures which correspond to  $(2, 3, 5)$ -distributions. The classification of non-degenerate Lagrangian cone structures based on their symmetries is an interesting open problem, regarded our duality and the studies on  $G_2$ -contact structures ([12][19][26]). It should be desirable the direct study on symmetries of non-degenerate Lagrangian cone structures.

The cone structure was first given in [5] by a foliation on the space  $P((\partial D)^\perp) \subset P(T^*Y)$  for the derived system  $\partial D$  of a  $(2, 3, 5)$ -distribution  $D$ , which is an essentially same foliation in the space  $P(D) \subset P(TY)$  of [15]. See also [5][14]. In fact there exists the natural fiber-preserving diffeomorphism  $P(D) \rightarrow P((\partial D)^\perp)$  which preserves also the foliation induced from singular paths of  $D$ . Moreover the Lagrangian cone structure  $C \subset TX$ , which is contained in a contact structure  $D' \subset TX$  on  $X$ , has the essentially same information with the Jacobi curves introduced in [3][4]. In fact each cone  $C_x \subset D'_x, (x \in X)$  gives the (reduced) Jacobi curve associated to the singular path  $x$  of  $D$  in Lagrangian Grassmannian of  $D'_x$  by taking tangent planes to  $C_x$ .

In [29], it was shown that the Cartan tensor of any  $(2, 3, 5)$ -distribution is given by the fundamental invariant of Jacobi curves of singular paths and, in particular, the Cartan tensor is determined by the projective equivalence classes of the point-wise curves  $P(C_x), x \in X$  of the corresponding Lagrangian cone structure  $(X, C)$ . We give a short proof (Proposition 4.1), related to the study on  $G_2$ -contact structures ([12][19]), that the  $(2, 3, 5)$ -distribution which corresponds to a cubic Lagrangian cone structure via our duality is necessarily flat, by using Zelenko’s theorem [29] (Proposition 4.1). Since the degrees of cone structures are invariant under isomorphisms of cone structures and by Theorem 3.1 of the present paper, we see that any cone structure which corresponds to a flat  $(2, 3, 5)$  structure must be cubic. Then we can say that, to check the flatness of a  $(2, 3, 5)$ -distribution is easier, if it is given by a corresponding Lagrangian cone structure. In fact the condition  $\partial(T_s C) \subset O_s^{(2)} C$  of Theorem 3.1 is checked by straightforward computations of differentials and then it is sufficient to see the degree of the cone is cubic or not. However it is a difficult task, given a  $(2, 3, 5)$ -distribution, to get the corresponding Lagrangian cone structure concretely.

In §2, we review the results given in the previous paper [15] with additional explanations. In particular we give the exact definition of (non-degenerate) Lagrangian cone structures (Definition 2.3).

In §3, we complete the duality between  $(2, 3, 5)$ -distributions and non-degenerate Lagrangian cone structures with an additional condition via pseudo-product structures of type  $G_2$ .

We conclude this paper by several remarks related to the duality in §4.

All manifolds and mappings are supposed to be of class  $C^\infty$  unless otherwise stated.

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## §2. Pseudo-product structures of $G_2$ -type

Let  $D$  be a  $(2, 3, 5)$ -distribution on a 5-dimensional manifold  $Y$ . Let  $Z := P(D) = (D - 0)/\mathbb{R}^\times$  be the space of tangential lines in  $D$ ,  $Z := \{(y, \ell) \mid y \in Y, \ell \subset D_y (\subset T_y Y), \dim(\ell) = 1\}$ . Then  $\dim(Z) = 6$  and the projection  $\pi_Y : Z \rightarrow Y$  is an  $\mathbb{R}P^1$ -bundle.

We define a subbundle  $E \subset TZ$  of rank 2, *Cartan prolongation* of  $D \subset TY$ , by setting for each  $(y, \ell) \in Z$ ,  $\ell \subset D_y$ ,  $E_{(y, \ell)} := \pi_{Y*}^{-1}(\ell) (\subset T_{(y, \ell)} Z)$ . Then  $E$  is a distribution with (weak) growth  $(2, 3, 4, 5, 6)$ :  $\text{rank}(E) = 2$ ,  $\text{rank}(\partial E) = 3$ ,  $\text{rank}(\partial^{(2)} E) = 4$ ,  $\text{rank}(\partial^{(3)} E) = 5$ ,  $\text{rank}(\partial^{(4)} E) = 6$ .

Then we see that there exists an *intrinsic* decomposition

$$E = K \oplus L$$

of  $E$  with  $L := \text{Ker}(\pi_{Y*}) \subset E$  and a complementary line subbundle  $K$  of  $E$ , a *pseudo-product structure* in the sense of N. Tanaka [24][25].

We will explain this in terms of “geometric control theory” ([2][20]).

A control system  $\mathbb{C} : \mathcal{U} \xrightarrow{F} TM \rightarrow M$  on a manifold  $M$  is given by a locally trivial fibration  $\pi_{\mathcal{U}} : \mathcal{U} \rightarrow M$  over  $M$  and a map  $F : \mathcal{U} \rightarrow TM$  such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{F} & TM \\ \pi_{\mathcal{U}} \searrow & & \swarrow \pi_{TM} \\ & M & \end{array}$$

Any section  $s : M \rightarrow \mathcal{U}$  defines a vector field  $F \circ s : M \rightarrow TM$  over  $M$ . Via a local triviality on  $M$ , a control system is given by a family of vector fields  $f_u(x) = F(x, u)$  over  $M$ ,  $(x, u) \in \mathcal{U}$ ,  $x \in M$ .

A distribution  $D \subset TM$  is regarded as a control system  $\mathbb{D} : D \hookrightarrow TM \rightarrow M$ , by the inclusion.

Two control systems  $\mathbb{C} : \mathcal{U} \xrightarrow{F} TM \xrightarrow{\pi_{TM}} M$  and  $\mathbb{C}' : \mathcal{U}' \xrightarrow{F'} TM' \xrightarrow{\pi_{TM'}} M'$  are called *isomorphic* if the diagram

$$\begin{array}{ccccccc} \mathcal{U} & \xrightarrow{F} & TM & \xrightarrow{\pi_{TM}} & M \\ \psi \downarrow & & \varphi_* \downarrow & & \downarrow \varphi \\ \mathcal{U}' & \xrightarrow{F'} & TM' & \xrightarrow{\pi_{TM'}} & M' \end{array}$$

commutes for some diffeomorphisms  $\psi$  and  $\varphi$ . Here  $\varphi_*$  is the differential of  $\varphi$ .

The pair  $(\psi, \varphi)$  of diffeomorphisms is called an *isomorphism* of the control systems  $\mathbb{C}$  and  $\mathbb{C}'$ .

Given a control system  $\mathbb{C} : \mathcal{U} \xrightarrow{F} TM \rightarrow M$ , an  $L^\infty$  (measurable, essentially bounded) map  $c : [a, b] \rightarrow \mathcal{U}$  is called an *admissible control* if the curve

$$\gamma := \pi_{\mathcal{U}} \circ c : [a, b] \rightarrow M$$

satisfies the differential equation

$$\dot{\gamma}(t) = F(c(t)) \quad (\text{a.e. } t \in [a, b]).$$

Then the Lipschitz curve  $\gamma$  is called a *trajectory*. If we write  $c(t) = (x(t), u(t))$ , then  $x(t) = \gamma(t)$  and

$$\dot{x}(t) = F(x(t), u(t)), \quad (\text{a.e. } t \in [a, b]).$$

We use the term “*path*” for a smooth ( $C^\infty$ ) immersive trajectory regarded up to parametrisation.

The totality  $\mathcal{C}$  of admissible controls  $c : [a, b] \rightarrow \mathcal{U}$  with a given initial point  $q_0 \in M$  is a Banach manifold. The *endpoint mapping*  $\text{End} : \mathcal{C} \rightarrow M$  is defined by

$$\text{End}(c) := \pi_{\mathcal{U}} \circ c(b).$$

An admissible control  $c : [a, b] \rightarrow \mathcal{U}$  with the initial point  $\pi_{\mathcal{U}}(c(a)) = q_0$  is called *singular* or *abnormal*, if  $c \in \mathcal{C}$  is a singular point of  $\text{End}$ , namely if the differential  $\text{End}_* : T_c \mathcal{C} \rightarrow T_{\text{End}(c)} M$  is not surjective. If  $c$  is a singular control, then the trajectory  $\gamma = \pi_{\mathcal{U}} \circ c$  is called a *singular trajectory* or an *abnormal extremal*.

Let  $D \subset TY$  be a  $(2, 3, 5)$ -distribution. Then, it can be shown that for any point  $y$  of  $Y$  and for any direction  $\ell \subset D_y$ , there exists uniquely a *singular  $D$ -path* (an immersed abnormal extremal for  $D$ ) through  $y$  with the given direction  $\ell$ . Thus the singular  $D$ -paths form another five dimensional manifold  $X$ .

Let  $Z = P(D) = (D - 0)/\mathbb{R}^\times$  be the space of tangential lines in  $D$ ,  $\dim(Z) = 6$ . Then  $Z$  is naturally foliated by the liftings of singular  $D$ -paths, and we have locally double fibrations:

$$Y \xleftarrow{\pi_Y} Z \xrightarrow{\pi_X} X.$$

If we put  $L = \text{Ker}(\pi_{Y*})$ ,  $K = \text{Ker}(\pi_{X*})$ , then we have a decomposition  $E = K \oplus L$  by sub-bundles of rank 1.

We denote, for any distribution  $E$ , by  $\mathcal{E}$  the sheaf of local sections to  $E$ . We set

$$\partial \mathcal{E} := [\mathcal{E}, \mathcal{E}] = \mathcal{E} + [\mathcal{E}, \mathcal{E}], \quad \partial^{(2)} \mathcal{E} := [\mathcal{E}, \partial \mathcal{E}] = \mathcal{E} + \partial \mathcal{E} + [\mathcal{E}, \partial \mathcal{E}]$$

and so on. If  $\partial \mathcal{E}$  is generated by a local sections of a distribution, then we denote it by  $\partial E$ .

**DEFINITION 2.1.** A distribution  $(Z, E)$  of rank 2 on a 6-dimensional manifold  $Z$  with a decomposition  $E = K \oplus L$  by subbundles  $K, L$  of rank 1 is called a pseudo-product structures of  $G_2$ -type if  $E$  has small growth  $(2, 3, 4, 5, 6)$  and moreover satisfies that

$$\begin{aligned} [\mathcal{K}, \mathcal{L}] &= \partial \mathcal{E}, \quad [\mathcal{K}, \partial \mathcal{E}] = \partial^{(2)} \mathcal{E}, \quad [\mathcal{L}, \partial \mathcal{E}] = \partial \mathcal{E}, \\ [\mathcal{K}, \partial^{(2)} \mathcal{E}] &= \partial^{(3)} \mathcal{E}, \quad [\mathcal{L}, \partial^{(2)} \mathcal{E}] = \partial^{(2)} \mathcal{E}, \quad [\mathcal{K}, \partial^{(3)} \mathcal{E}] = \partial^{(3)} \mathcal{E}, \quad [\mathcal{L}, \partial^{(3)} \mathcal{E}] = \partial^{(4)} \mathcal{E}. \end{aligned}$$

Then, by taking the gradation of the filtration

$$\mathcal{E} \subset \partial \mathcal{E} \subset \partial^{(2)} \mathcal{E} \subset \partial^{(3)} \mathcal{E} \subset \partial^{(4)} \mathcal{E},$$

we have, at each point  $z \in Z$ , the symbol algebra:

$$\begin{aligned} \mathfrak{m} &= \mathfrak{g}_{-5} \oplus \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} = \langle e_6 \rangle \oplus \langle e_5 \rangle \oplus \langle e_4 \rangle \oplus \langle e_3 \rangle \oplus \langle e_1, e_2 \rangle, \\ [e_1, e_2] &= e_3, \quad [e_1, e_3] = e_4, \quad [e_2, e_3] = 0, \quad [e_1, e_4] = e_5, \quad [e_2, e_4] = 0, \quad [e_1, e_5] = 0, \quad [e_2, e_5] = e_6, \end{aligned}$$

with the decomposition  $\mathfrak{g}_{-1} = \mathfrak{k} \oplus \mathfrak{l} = \langle e_1 \rangle \oplus \langle e_2 \rangle$ .

Then we have

**THEOREM 2.2.** *There exists a natural bijective correspondence of local isomorphism classes between (2,3,5)-distributions and pseudo-product structures of  $G_2$ -type.*

*Proof:* First let us make sure that the prolongation  $E$  of a (2,3,5)-distribution  $D$  on a 5-dimensional manifold  $Y$  has small growth (2,3,4,5,6).

Let  $\eta_1, \eta_2$  be a local frame of  $D$ . Then, setting

$$\eta_3 := [\eta_1, \eta_2], \quad \eta_4 := [\eta_1, \eta_3], \quad \eta_5 := [\eta_2, \eta_3],$$

we have a local frame  $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$  of  $TY$ . For each  $y \in Y$ , directions in  $D_y$  are, locally, parametrized via  $\eta_1(y) + t\eta_2(y)$  ( $t \in \mathbb{R}$ ). Then, for any system of local coordinates  $y = (y_1, y_2, y_3, y_4, y_5)$  of  $Y$  centered at base point of  $Y$ ,  $(y, t)$  form a system of local coordinates of  $Z$  such that  $\pi_Y$  is expressed by  $(y, t) \mapsto y$ . We regard  $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$  as vector-fields over  $Z$ . Then

$$\zeta_1 := \eta_1 + t\eta_2 \quad \zeta_2 := \frac{\partial}{\partial t},$$

form a local frame of  $E$ , and  $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \zeta_2$  of  $TZ$ .

Since  $[\zeta_1, \zeta_2] = [\eta_1 + t\eta_2, \zeta_2] = -\eta_2$ , we have

$$\partial \mathcal{E} = \langle \zeta_1, \zeta_2, \eta_2 \rangle = \langle \eta_1, \eta_2, \zeta_2 \rangle,$$

which is of rank 3. Here  $\langle \zeta_1, \zeta_2, \eta_2 \rangle$  means the distribution generated by  $\zeta_1, \zeta_2, \eta_2$ . Since  $[\zeta_1, \eta_2] = [\eta_1 + t\eta_2, \eta_2] = \eta_3$  and  $[\zeta_2, \eta_2] = 0$ , we have

$$\partial^{(2)} \mathcal{E} = \langle \eta_1, \eta_2, \eta_3, \zeta_2 \rangle,$$

which is of rank 4. Since  $[\zeta_1, \eta_3] = [\eta_1 + t\eta_2, \eta_3] = \eta_4 + t\eta_5$  and  $[\zeta_2, \eta_3] = 0$ , we have

$$\partial^{(3)} \mathcal{E} = \langle \eta_1, \eta_2, \eta_3, \eta_4 + t\eta_5, \zeta_2 \rangle,$$

that is of rank 5. Since  $[\zeta_2, \eta_4 + t\eta_5] = \eta_5$ , we have  $\partial^{(4)} \mathcal{E} = TZ$ . Therefore  $E$  has small growth (2,3,4,5,6).

Note that  $\mathcal{L}$  is generated by  $\zeta_2$ . Moreover there exists a generator of  $\mathcal{H}$  of form  $\zeta_1 + e(y, t)\zeta_2$ . In fact the function  $e(y, t)$  is uniquely determined by the condition  $[\mathcal{H}, \partial^{(3)} \mathcal{E}] = \partial^{(3)} \mathcal{E}$ , which is equivalent to the condition

$$e\eta_5 + [\eta_1, \eta_4] + t[\eta_1, \eta_5] + t[\eta_2, \eta_4] + t^2[\eta_1, \eta_5] \equiv 0, \quad \text{mod. } \partial^{(3)} \mathcal{E}.$$

Then other remaining conditions that  $E = K \oplus L$  is a pseudo-product structure of type  $G_2$  follow.

Conversely suppose  $E = K \oplus L$  is a pseudo-product structure of type  $G_2$ . Then  $L$  is the Cauchy characteristic of  $\partial E$  (see [10]). Let  $Y$  be the leaf space of  $L$ , which is locally defined 5 dimensional manifold. Moreover  $Z$  has a system of local coordinates  $(y, t)$  centered at the base point such that  $\pi_Y$  is given by  $(y, t) \mapsto y$ . Let  $D$  be the reduction of  $\partial E$  by  $L$ . Take a local frame  $\eta_1, \eta_2$  of  $D$  such that, regarded as vector fields over  $Z$ ,  $\eta_1$  generates the quotient bundle  $(\partial E)/E$ . Moreover  $\zeta_1 = \eta_1 + \varphi(y, t)\eta_2$  and  $\zeta_2 = \partial/\partial t$  generates  $K$  and  $L$  respectively for some function  $\varphi(y, t)$  with  $\varphi(0, 0) = 0$ . Since

$$[\zeta_1, \zeta_2] = [\eta_1 + \varphi\eta_2, \zeta_2] = -(\partial\varphi/\partial t)\eta_2,$$

we have that  $\partial\varphi/\partial t \neq 0$ . Set  $\zeta_3 := \eta_2$ . Then

$$[\zeta_1, \zeta_3] = [\eta_1, \eta_2] + \eta_2(\varphi)\eta_2 \equiv [\eta_1, \eta_2] \quad \text{mod. } \partial \mathcal{E}.$$

Therefore  $\eta_1, \eta_2, [\eta_1, \eta_2]$  are linearly independent point-wise on  $Y$ . We set  $\zeta_4 := \eta_3 = [\eta_1, \eta_2]$  as a vector field over  $Z$ . Then

$$[\zeta_1, \zeta_4] = [\eta_1, \eta_3] + \varphi[\eta_2, \eta_3] - \eta_3(\varphi)\eta_2 \equiv [\eta_1, \eta_3] + \varphi[\eta_2, \eta_3] \pmod{\partial^{(2)}\mathcal{E}},$$

and  $[\zeta_2, \zeta_4] = [\partial/\partial t, \eta_3] = 0$ . Set  $\eta_4 = [\eta_1, \eta_3], \eta_5 = [\eta_2, \eta_3]$  and  $\zeta_5 = \eta_4 + \varphi\eta_5$ . Then  $\eta_4(0) \in (\partial^{(3)}E)_0 \setminus (\partial^{(2)}E)_0$ . Then we have that  $[\zeta_2, \zeta_5](0) \notin (\partial^{(3)}E)_0$ , while  $[\zeta_2, \zeta_5] = (\partial\varphi/\partial t)\eta_5(0)$ . Therefore  $\eta_5(0) \notin (\partial^{(3)}E)_0$ . Therefore  $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$  are linearly independent point-wise. Thus we see that  $D$  is a  $(2, 3, 5)$ -distribution.

These correspondences induce the bijection between local isomorphism classes of  $(2, 3, 5)$ -distributions and pseudo-product structures of  $G_2$ -type on a 5-manifold.  $\square$

Note that the original  $(2, 3, 5)$ -distribution  $D$  is obtained as the linear hull of the cone field (“bowtie”) induced from  $K$ :

$$D_y = \text{linear hull} \left( \bigcup_{z \in \pi_Y^{-1}(y)} \pi_{Y*}(K_z) \subset T_y Y \right).$$

Also, the  $(2, 3, 5)$ -distribution  $D$  is obtained as the reduction of  $\partial E$  by Cauchy characteristic  $L = \text{Ker}(\pi_{Y*})$ .

On the other hand we obtain a cone field  $C \subset TX$  on  $X$  by setting, for each  $x \in X$ ,

$$C_x := \bigcup_{z \in \pi_X^{-1}(x)} \pi_{X*}(L_z) \subset T_x X. \quad (\Delta)$$

Now, to make sure, we formulate exactly the notion of Lagrangian cone structures (see [12]):

**DEFINITION 2.3.** (1) Let  $X$  be a manifold of dimension  $m$ . A subset  $C \subset TX$  is called a *cone structure* if there is an  $\mathbb{R}^\times$ -invariant subset  $\underline{C} \subset \mathbb{R}^m$ , a model cone, such that, for any  $x \in X$ , there exist an open neighborhood  $U$  of  $x$  and a local triviality  $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^m$  of  $\pi : TX \rightarrow X$  over  $U$  satisfying  $\Phi(\pi^{-1}(U) \cap C) = U \times \underline{C}$ .

(2) Suppose that the model cone  $\underline{C}$  is non-singular away from the origin in  $\mathbb{R}^m$ . Then  $P(C)$  is a submanifold of  $P(TX)$ . For each section  $s : X \rightarrow P(C)$  for the projection  $P(C) \rightarrow X$ , we have the subbundle  $T_s C \subset TX$  by taking tangent planes of  $C_x$  along the direction  $s(x)$  at every point  $x \in X$ . We call the distribution  $T_s C$  the *linear approximation* of  $C$  along  $s$ .

(3) A cone structure  $C \subset TX$  is called a *Lagrangian cone structure* if there exists a contact structure  $D'$  on  $X$  such that  $C \subset D'$  and, for any section  $s : X \rightarrow P(C)$ ,  $T_s C$  is a Lagrangian subbundle of  $D'$ . The last condition is equivalent to that, for any  $x \in X$ ,  $C_x \setminus \{0\}$  is a Lagrangian submanifold of the linear symplectic manifold  $D'_x$ , or equivalently,  $P(C_x)$  is a Legendrian submanifold of the contact manifold  $P(D'_x)$  induced from the conformal symplectic vector space  $D'_x$ .

(4) Let  $\dim(X) = 5$ . A Lagrangian cone structure  $C \subset TX$  for a contact structure  $D' \subset TX$  is called *non-degenerate* if the spatial projective curve segment  $P(C_x) \subset P(D'_x) \cong P^3$  is non-degenerate, i.e. the first, second and third derivatives of a parametrization of  $P(C_x)$  are linearly independent.

From the condition (4), for each direction field  $s$  of  $C$ , we define *osculating bundles*  $O_s^{(2)} C \subset TX$  of rank 3 and  $O_s^{(3)} C \subset TX$  of rank 4, generated by osculating planes  $O_2$  and 3-dimensional osculating spaces  $O_3$  to  $P(C_x)$  with direction  $s$ . Then the contact structure  $D'$  coincides with  $O_s^{(3)} C$  which is independent of  $s$ .

Because distributions are regarded as cone structures of special type, the notion of Lagrangian cone structures is a natural generalization for that of Lagrangian subbundle of the tangent bundle over a contact manifold.

LEMMA 2.4. *In our case, the above  $C \subset TX$  defined as  $(\Delta)$  corresponding to a (2,3,5)-distribution  $D \subset TY$  is a non-degenerate Lagrangian cone structure in the sense of Definition 2.3.*

*Proof:* By the condition  $[\mathcal{H}, \partial \mathcal{E}] = \partial^{(2)} \mathcal{E}$ ,  $C$  satisfies the conditions (1)(2) of Definition 2.3. By the condition  $[\mathcal{K}, \partial^{(3)} \mathcal{E}] = \partial^{(3)} \mathcal{E}$ ,  $K$  is the Cauchy characteristic of  $\partial^{(3)} \mathcal{E}$ . Then the distribution  $D' \subset TX$  induced from  $\partial^{(3)} \mathcal{E}$  is a contact structure by the condition  $[\mathcal{L}, \partial^{(3)} \mathcal{E}] = \partial^{(4)} \mathcal{E}$ . Moreover  $\partial^{(2)} \mathcal{E}$  projects to tangent spaces to  $C_x$  along  $\pi_x^{-1}(x)$ . For any section  $s : X \rightarrow L$ ,  $s(x) \neq 0$ , we have that the linear approximation  $T_s C$  is a Lagrangian subbundle of  $D'$  by the condition  $[\mathcal{L}, \partial^{(3)} \mathcal{E}] = \partial^{(3)} \mathcal{E}$ . Therefore  $C$  satisfies also the condition (3) of Definition 2.3. Thus  $(X, C)$  is a Lagrangian cone structure. Moreover by the condition  $[\mathcal{H}, \partial^{(2)} \mathcal{E}] = \partial^{(3)} \mathcal{E}$ , the condition (4) of Definition 2.3 is satisfied. Therefore  $(X, C)$  is a non-degenerate Lagrangian cone structure.  $\square$

Now, we regard the cone field  $C \subset TX$  as a control system over  $X$ :

$$\mathbb{C} : L \xrightarrow{\pi_{X*}|_L} TX \rightarrow X,$$

for the subbundle  $L$  of  $TZ$ . Then we have shown in [15] the following theorem:

THEOREM 2.5. (Duality Theorem [15]) *Singular paths of the control system*

$$\mathbb{C} : L \xrightarrow{\pi_{X*}|_L} TX \rightarrow X$$

are given by  $\pi_X$ -images of  $\pi_Y$ -fibers.

Therefore, for any  $x \in X$  and for any direction  $\ell \in C_x$ , there exists uniquely a singular  $\mathbb{C}$ -paths passing through  $x$  with the direction  $\ell$  at  $x$ .

Thus the original space  $Y$  is identified with the space of singular paths for  $(X, C)$ , while  $X$  is the space of singular paths for  $(Y, D)$ .

We recall the local characterization of singular controls.

For a control system  $\mathbb{C} : \mathcal{U} \xrightarrow{F} TM \rightarrow M$  on a manifold  $M$ , we consider the fibre-product  $\mathcal{U} \times_M T^*M$ , and define the *Hamiltonian function*  $H : \mathcal{U} \times_M T^*M \rightarrow \mathbb{R}$  of the control system  $F : \mathcal{U} \rightarrow TM$  by

$$H(x, p, u) := \langle p, F(x, u) \rangle, \quad ((x, u), (x, p)) \in \mathcal{U} \times_M T^*M.$$

A singular control  $(x(t), u(t))$  is characterized by the liftability to an *abnormal bi-extremal*  $(x(t), p(t), u(t))$  satisfying the constrained Hamiltonian equation

$$\begin{cases} \dot{x}_i(t) = \frac{\partial H}{\partial p_i}(x(t), p(t), u(t)), & (1 \leq i \leq m) \\ \dot{p}_i(t) = -\frac{\partial H}{\partial x_i}(x(t), p(t), u(t)), & (1 \leq i \leq m) \\ \frac{\partial H}{\partial u_j}(x(t), p(t), u(t)) = 0, & (1 \leq j \leq r), \quad p(t) \neq 0. \end{cases}$$

Let  $E \subset TZ$  be a distribution on a manifold  $Z$  regarded as a control system. A singular path  $x(t)$  for  $E \subset TZ$  is called *regular singular* if it is associated with an abnormal bi-extremal  $(x(t), p(t), u(t))$  such that  $p(t) \in (\partial E)^\perp \setminus (\partial^{(2)}E)^\perp \subset T^*Z$ . A singular path  $x(t)$  for  $E \subset TZ$  is called *totally irregular singular* if any associated abnormal bi-extremals  $(x(t), p(t), u(t))$  satisfies that  $p(t) \in (\partial^{(2)}E)^\perp \subset T^*Z$ .

From the pseudo-product structure on  $E \subset TZ$ , we have

**THEOREM 2.6.** (Asymmetry Theorem [15]) *A singular path for  $E \hookrightarrow TZ \rightarrow Z$  is either a  $\pi_Y$ -fibre or a  $\pi_X$ -fibre. Each  $\pi_Y$ -fibre is regular singular, while each  $\pi_X$ -fibre is totally irregular singular.*

### §3. Complete duality

The description of the duality on  $(2, 3, 5)$ -distributions  $(Y, D)$  and non-degenerate Lagrangian cone structures  $(X, C)$  via  $(Z, E)$  which is given in §2 should be completed by answering the question: *What kinds of non-degenerate Lagrangian cone structures do they correspond to  $(2, 3, 5)$ -distributions?*

Then we have

**THEOREM 3.1.** *There exist natural bijective correspondences of isomorphism classes:*

$$\{(2, 3, 5)\text{-distributions } (Y, D)\} / \cong \longleftrightarrow \left\{ \begin{array}{l} \text{pseudo-product structures of } G_2\text{-type } (Z, E): \\ (2, 3, 4, 5, 6)\text{-distributions } E \text{ with a decomposition} \\ E = K \oplus L, \text{ rank}(K) = \text{rank}(L) = 1, \\ [\mathcal{H}, \mathcal{L}] = \partial \mathcal{E} \text{ } (:= [\mathcal{E}, \mathcal{E}] = \mathcal{E} + [\mathcal{E}, \mathcal{E}]), \\ [\mathcal{H}, \partial \mathcal{E}] = \partial^{(2)} \mathcal{E}, \text{ } [\mathcal{L}, \partial \mathcal{E}] = \partial \mathcal{E}, \\ [\mathcal{H}, \partial^{(2)} \mathcal{E}] = \partial^{(3)} \mathcal{E}, \text{ } [\mathcal{L}, \partial^{(2)} \mathcal{E}] = \partial^{(2)} \mathcal{E}, \\ [\mathcal{H}, \partial^{(3)} \mathcal{E}] = \partial^{(3)} \mathcal{E}, \text{ } [\mathcal{L}, \partial^{(3)} \mathcal{E}] = \partial^{(4)} \mathcal{E}. \end{array} \right\} / \cong$$

$$\longleftrightarrow \left\{ \begin{array}{l} \text{non-degenerate Lagrangian cone structures } (X, C) \\ \text{on 5-dimensional manifolds } X \text{ with the condition} \\ \partial(T_s C) \subset \mathcal{O}_s^{(2)} C, \text{ for any direction field } s \text{ of } C. \end{array} \right\} / \cong$$

*Proof of Theorem 3.1.*

Let  $X$  be a 5-dimensional manifold and  $C \subset TX$  a non-degenerate Lagrangian cone structure (Definition 2.3). Then  $Z = P(C) := (C \setminus (\text{zero-section})) / \mathbb{R}^\times$  is a 6-dimensional manifold and that  $\pi_X : Z \rightarrow X$  is a  $C^\infty$ -fibration with projective curves  $P(C_x) \subset P(T_x X) \cong P^4$  as fibers.

By the non-degeneracy condition, we have that the first, second and third derivatives are linearly independent everywhere on  $P(C_x)$ , for any  $x \in X$ .

Then we define a subbundle  $E \subset TZ$  of rank 2 by setting

$$E_{(x, \ell)} := (\pi_X)_*^{-1}(\ell),$$

for each  $(x, \ell) \in Z$  as the prolongation of the cone structure  $C \in TX$ . We set  $K = \text{Ker}((\pi_X)_*)$ .

Let  $x = (x_1, x_2, x_3, x_4, x_5)$  be a system of local coordinates of  $X$  and  $x, \theta$  that of  $Z$  such that  $\pi_X : Z \rightarrow X$  is given by  $(x, \theta) \mapsto x$  and  $E$  is generated by  $\zeta_1 = \frac{\partial}{\partial \theta}$  and a vector field  $\zeta_2(x, \theta)$  of form

$$\zeta_2(x, \theta) = \frac{\partial}{\partial x_1} + A \frac{\partial}{\partial x_2} + B \frac{\partial}{\partial x_3} + S \frac{\partial}{\partial x_4} + T \frac{\partial}{\partial x_5},$$

where  $A, B, S, T$  are function-germs of  $x, \theta$ . The projective curve  $C_x \subset P(T_x X)$  is given by

$$\theta \mapsto [1 : A(x, \theta) : B(x, \theta) : S(x, \theta) : T(x, \theta)]$$

in homogeneous coordinates, for each  $x \in X$ .

We have, on  $Z$ ,

$$[\zeta_1, \zeta_2](x, \theta) = \frac{\partial \zeta_2}{\partial \theta}(x, \theta) =: \zeta_3,$$

and

$$[\zeta_1, \zeta_3](x, \theta) = \frac{\partial^2 \zeta_2}{\partial \theta^2}(x, \theta) =: \zeta_4.$$

In local coordinates,

$$\zeta_3 = A_\theta \frac{\partial}{\partial x_2} + B_\theta \frac{\partial}{\partial x_3} + S_\theta \frac{\partial}{\partial x_4} + T_\theta \frac{\partial}{\partial x_5}, \quad \zeta_4 = A_{\theta\theta} \frac{\partial}{\partial x_2} + B_{\theta\theta} \frac{\partial}{\partial x_3} + S_{\theta\theta} \frac{\partial}{\partial x_4} + T_{\theta\theta} \frac{\partial}{\partial x_5},$$

and

$$\zeta_5 = A_{\theta\theta\theta} \frac{\partial}{\partial x_2} + B_{\theta\theta\theta} \frac{\partial}{\partial x_3} + S_{\theta\theta\theta} \frac{\partial}{\partial x_4} + T_{\theta\theta\theta} \frac{\partial}{\partial x_5}.$$

Any direction field  $s$  of  $C$  is given by  $x \mapsto (x, \theta(x))$  for some functions  $\theta(x)$  of  $x$  and the linear approximation  $T_s C$  of  $C$  along the direction field  $s$  is generated by  $\zeta_2(x, \theta(x)), \frac{\partial \zeta_2}{\partial \theta}(x, \theta(x))$ .

Moreover the osculating bundles  $O_s^{(2)} C$  and  $O_s^{(3)} C$  are generated by  $\zeta_2(x, \theta(x)), \frac{\partial \zeta_2}{\partial \theta}(x, \theta(x)), \frac{\partial^2 \zeta_2}{\partial \theta^2}(x, \theta(x))$  and by  $\zeta_2(x, \theta(x)), \frac{\partial \zeta_2}{\partial \theta}(x, \theta(x)), \frac{\partial^2 \zeta_2}{\partial \theta^2}(x, \theta(x)), \frac{\partial^3 \zeta_2}{\partial \theta^3}(x, \theta(x))$  respectively.

By the condition  $\partial(T_s C) \subset O_s^{(2)} C$ , we have that  $[\zeta_2, \zeta_3] \equiv 0, \text{ mod. } \langle \zeta_1, \zeta_2, \zeta_3, \zeta_4 \rangle$ . Then there exists uniquely a function  $U(x, \theta)$  such that  $\tilde{\zeta}_2 = \zeta_2 + U \zeta_1$  is the Cauchy characteristic vector field of  $\partial E$ , so that  $[\tilde{\zeta}_2, \zeta_3] \equiv 0, \text{ mod. } \langle \zeta_1, \zeta_2, \zeta_3 \rangle$ .

Taking the subbundle  $L \subset E$  generated by  $\tilde{\zeta}_2$ , we have a pseudo-product structure  $E = K \oplus L$  on  $Z$  satisfying the conditions

$$[\mathcal{H}, \mathcal{L}] = \partial \mathcal{E}, \quad [\mathcal{H}, \partial \mathcal{E}] = \partial^{(2)} \mathcal{E}, \quad [\mathcal{L}, \partial \mathcal{E}] = \partial \mathcal{E}, \quad [\mathcal{H}, \partial^{(2)} \mathcal{E}] = \partial^{(3)} \mathcal{E}.$$

By Jacobi identity,  $[\tilde{\zeta}_2, [\zeta_1, \zeta_3]] + [\zeta_1, [\zeta_3, \tilde{\zeta}_2]] + [\zeta_3, [\tilde{\zeta}_2, \zeta_1]] = 0$ , we have that

$$[\tilde{\zeta}_2, \zeta_4] \equiv [\zeta_1, [\tilde{\zeta}_2, \zeta_3]] \equiv 0, \text{ mod. } \langle \zeta_1, \zeta_2, \zeta_3, \zeta_4 \rangle.$$

Therefore the condition  $[\mathcal{L}, \partial^{(2)} \mathcal{E}] = \partial^{(2)} \mathcal{E}$  is satisfied. Since  $O_s^{(3)} C \subset TX$  is independent of  $s$  and is a contact structure on  $X$ , we have that  $[\mathcal{H}, \partial^{(3)} \mathcal{E}] = \partial^{(3)} \mathcal{E}$  and that  $[\mathcal{L}, \partial^{(3)} \mathcal{E}]$  generates the total tangent bundle  $TZ$ . Thus the last condition  $[\mathcal{L}, \partial^{(3)} \mathcal{E}] = \partial^{(4)} \mathcal{E}$  holds.

Consequently, if  $C$  is a non-degenerate Lagrangian cone structure with the condition that  $\partial(T_s C) \subset O_s^{(2)} C$  for any direction field  $s$  of  $C$ , then  $E = K \oplus L$  is a pseudo-product structure of  $G_2$ -type.

This completes the proof of Theorem 3.1.  $\square$

**REMARK 3.2.** The cone structure  $C \subset TX$  is regarded as the control system over  $X$ ,

$$\mathbb{C} : L \rightarrow TX \rightarrow X, \quad L \ni ((x, \ell), v) \mapsto (x, v) \mapsto x,$$

with 2-control parameters. In local coordinates, the control system  $\mathbb{C}$  is given by

$$F(x; r, \theta) := r \left( \frac{\partial}{\partial x_1} + A(x, \theta) \frac{\partial}{\partial x_2} + B(x, \theta) \frac{\partial}{\partial x_3} + S(x, \theta) \frac{\partial}{\partial x_4} + T(x, \theta) \frac{\partial}{\partial x_5} \right),$$

with the control parameters  $r, \theta$ .

#### §4. (2, 3, 5)-distributions and cubic Lagrangian cone structures

Let us denote by  $G'_2$  the automorphism group of the split octonion algebra  $\mathbb{O}'$ . Then for a Borel group subgroup  $B$  and parabolic subgroups  $P_1, P_2$  containing  $B$  of  $G'_2$ , we have a double fibration

$$Y = G'_2/P_1 \xleftarrow{\pi_Y} Z = G'_2/B \xrightarrow{\pi_X} X = G'_2/P_2,$$

a (2, 3, 5)-distribution  $D \subset TY$  on  $Y$ , a pseudo-product structure of type  $G_2$  as  $E = K \oplus L \subset TZ$  on  $Z$  and a non-degenerate Lagrangian cubic cone structure  $C \subset TX$  (see [16]). It is known also that  $Y$  is diffeomorphic to  $S^3 \times S^2$  (resp.  $Z$  to  $S^3 \times S^3$ ,  $X$  to  $S^2 \times S^3$ ). On each of three places, there exists Cartan's parabolic geometry as a natural non-flat geometry modeled on the homogeneous space. On  $Y$  it is the geometry of (2, 3, 5)-distributions. On  $Z$  it is the geometry of pseudo-product structures of type  $G_2$ . On  $X$  it is  $G_2$ -contact structures ([12][19]). Moreover any  $G_2$ -contact structure is accompanied with and is recovered from a non-degenerate Lagrangian cubic cone structure.

Hajime Sato [23] has suggested to the first author that any  $G_2$ -contact structure corresponding to a (2, 3, 5)-distribution should be flat, from the exact comparison of curvatures for associated Cartan connections on pseudo-product  $G_2$ -structure and on  $G_2$ -contact structures ([24][27]). Here we would like to provide alternative proof for the fact. In fact we have:

**PROPOSITION 4.1.** *Any (2, 3, 5)-distribution  $(Y, D)$  which corresponds to a cubic cone structure  $(X, C)$  must be flat. Any Lagrangian cone structure which corresponds to a flat (2, 3, 5)-distributions must be cubic.*

*Proof of Proposition 4.1.* For each  $x \in X$ , the cone  $C_x \subset D'_x(\subset T_x X)$  gives the (reduced) ‘‘Jacobi curve’’ in the sense of Agrachev and Zelenko [3][4][29][5]. Then, in [29], it is proved that ‘‘Cartan tensor’’ of  $D$  is recovered by a projective invariant, the fundamental invariant, a kind of cross ratio, of  $P(C_x)$  point-wise. In fact, for the cone  $C_x \subset D_x \cong \mathbb{R}^4$ , there is associated a curve  $P(C_x)$  in Grassmannian  $\text{Gr}(2, \mathbb{R}^4)$ , and the fundamental invariants is calculated from  $P(C_x)$  in projective invariant way.

Suppose a (2, 3, 5)-distribution  $D$  corresponds to a cubic cone structure  $C \subset D' \subset TX$ . Then the cone structure is non-degenerate. Since all non-degenerate cubic cones are projectively equivalent point-wise, the Cartan tensor of  $D$  coincides with the flat (2, 3, 5)-distribution. Therefore  $D$  must be flat.

Suppose a Lagrangian cone structure  $(X, C)$  corresponds to a flat (2, 3, 5)-distribution  $(Y, D)$ . The flat model  $(Y_0, D_0)$  has the standard cubic dual  $(X_0, C_0)$  as in [16]. Since  $(Y, D) \equiv (Y_0, D_0)$ , we see  $(X, C) \equiv (X_0, C_0)$  by Theorem 3.1. Then  $C$  is cubic, because the degree is invariant under the isomorphism of cone structures.  $\square$

**EXAMPLE 4.2.** (Cubic Lagrangian cone structures not corresponding to (2, 3, 5)-distributions.) Consider a cubic cone structure  $C$  on  $(\mathbb{R}^5, 0)$  around the direction  $\theta = 0$ ,

$$F(x; r, \theta) = r \left( \frac{\partial}{\partial x_1} + \theta \frac{\partial}{\partial x_2} + (\theta^2 + a) \frac{\partial}{\partial x_3} + (\theta^3 - 3\theta a) \frac{\partial}{\partial x_4} + \{x_3 \theta - 2x_2(\theta^2 + a) + x_1(\theta^3 - 3\theta a)\} \frac{\partial}{\partial x_5} \right),$$

defined by a  $C^\infty$  function  $a(x_1)$  with  $a(0) = 0$ .

Then  $C$  is a non-degenerate Lagrangian cone structure for the contact structure  $D' : dx_5 - x_3 dx_2 + 2x_2 dx_3 - x_1 dx_4 = 0$ . Moreover  $C$  satisfies the condition  $\partial(T_s C) \subset \mathcal{O}_s^{(2)} C$  for any  $s : X \rightarrow L \setminus \{0\}$ , to correspond to a (2, 3, 5)-distribution, if and only if  $a \neq 0$ . The case  $a \equiv 0$  corresponds to the  $G_2$ -homogeneous flat case ([16]).

The following gives examples of non-degenerate Lagrangian non-cubic cone structures which correspond to (2,3,5)-distributions and shows the necessity of the additional condition  $\partial(T_s C) \subset O_s^{(2)} C$  of Theorem 3.1.

EXAMPLE 4.3. (Non-cubic Lagrangian cone structures corresponding to (2,3,5)-distributions.) Consider a cone structure on  $(\mathbb{R}^5, 0)$  around the direction  $\theta = 0$ ,

$$F(x; r, \theta) = r \left( \frac{\partial}{\partial x_1} + \theta \frac{\partial}{\partial x_2} + (\theta^2 + b) \frac{\partial}{\partial x_3} + (\theta^3 + c) \frac{\partial}{\partial x_4} + \{x_3 \theta - 2x_2(\theta^2 + b) + x_1(\theta^3 + c)\} \frac{\partial}{\partial x_5} \right),$$

where  $b = b(\theta), c = c(\theta)$ , with  $\text{ord}_0 b(\theta) \geq 3$ ,  $\text{ord}_0 c(\theta) \geq 4$ .

Then  $F$  is a non-degenerate Lagrangian cone structure, for the contact structure  $D' : dx_5 - x_3 dx_2 + 2x_2 dx_3 - x_1 dx_4 = 0$ . Moreover  $F$  satisfies the condition  $\partial(T_s C) \subset O_s^{(2)} C$ , for any direction field  $s$ , to correspond to a (2,3,5)-distribution, if and only if  $c_\theta = 3\theta b_\theta - 3b$ .

If  $b_{\theta\theta\theta} \neq 0$ , for example, if  $b = \theta^4, c = \frac{9}{5}\theta^5$ , then  $C$  is not cubic. Therefore the corresponding (2,3,5)-distribution is never flat.

Here we present the computation of the prolongation  $(Z, E)$  from the above example of cone structures. The bundle  $E$  is generated by

$$\begin{cases} \zeta_1 &= \frac{\partial}{\partial \theta}, \\ \zeta_2 &= \frac{\partial}{\partial x_1} + \theta \frac{\partial}{\partial x_2} + (\theta^2 + b) \frac{\partial}{\partial x_3} + (\theta^3 + c) \frac{\partial}{\partial x_4} + \{x_3 \theta - 2x_2(\theta^2 + b) + x_1(\theta^3 + c)\} \frac{\partial}{\partial x_5}. \end{cases}$$

on the space  $Z$  with coordinates  $\theta, x_1, x_2, x_3, x_4, x_5$ . Then we have over  $Z$ ,

$$\begin{aligned} \zeta_3 &:= [\zeta_1, \zeta_2] = \frac{\partial}{\partial x_2} + (2\theta + b_\theta) \frac{\partial}{\partial x_3} + (3\theta^2 + c_\theta) \frac{\partial}{\partial x_4} + \{x_3 - 2x_2(2\theta + b_\theta) + x_1(3\theta^2 + c_\theta)\} \frac{\partial}{\partial x_5}, \\ \zeta_4 &:= [\zeta_1, \zeta_3] = (2 + b_{\theta\theta}) \frac{\partial}{\partial x_3} + (6\theta + c_{\theta\theta}) \frac{\partial}{\partial x_4} + \{-2x_2(2 + b_{\theta\theta}) + x_1(6\theta + c_{\theta\theta})\} \frac{\partial}{\partial x_5}, \\ [\zeta_2, \zeta_3] &= (c_\theta - 3\theta b_\theta + 3b) \frac{\partial}{\partial x_5} = 0, \\ \zeta_5 &:= [\zeta_1, \zeta_4] = b_{\theta\theta\theta} \frac{\partial}{\partial x_3} + (6 + c_{\theta\theta\theta}) \frac{\partial}{\partial x_4} + \{-2x_2 b_{\theta\theta\theta} + x_1(6 + c_{\theta\theta\theta})\} \frac{\partial}{\partial x_5}, \\ [\zeta_2, \zeta_4] &= (c_{\theta\theta} - 3\theta b_{\theta\theta}) \frac{\partial}{\partial x_5} = 0, \\ [\zeta_1, \zeta_5] &= b_{\theta\theta\theta\theta} \frac{\partial}{\partial x_3} + c_{\theta\theta\theta\theta} \frac{\partial}{\partial x_4} + (-2x_2 b_{\theta\theta\theta\theta} + x_1 c_{\theta\theta\theta\theta}) \frac{\partial}{\partial x_5}, \\ \zeta_6 &:= [\zeta_2, \zeta_5] = 3(2 + a_{\theta\theta}) \frac{\partial}{\partial x_5}. \end{aligned}$$

We have that  $\partial \mathcal{E} = \langle \zeta_1, \zeta_2, \zeta_3 \rangle, \partial^{(2)} \mathcal{E} = \langle \zeta_1, \zeta_2, \zeta_3, \zeta_4 \rangle, \partial^{(3)} \mathcal{E} = \langle \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5 \rangle$ , and  $\partial^{(4)} \mathcal{E} = \langle \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6 \rangle$ . Then  $E$  has the pseudo-product structure of  $G_2$ -type given by  $\mathcal{H} = \langle \zeta_1 \rangle, \mathcal{L} = \langle \zeta_2 \rangle$  and it descends to a non-flat (2,3,5)-distribution.

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