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Singular fibers of two colored differentiable maps and cobordism invariants

Takahiro Yamamoto

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Singular fibers of two colored differentiable maps and cobordism invariants

Thesis submitted for the Degree of Doctor of Philosophy
in Graduate School of Science
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Introduction

Following the pioneering work of Thom [**49**] on the cobordism of embedding of manifolds, Rimańy and Szűcs [**36**] introduced the notion of cobordisms for singular maps. In particular, they constructed the "universal singular maps" by a Pontrjagin-Thom construction from which any other singular maps can be pulled back. Here, the *codimension* of a map $f : M \to N$ between manifolds is defined to be $k = \dim N - \dim M$. On the other hand, Saeki [**40**] defined the cobordism group for maps of negative codimension having only definite fold singularities, and he showed that the cobordism group of such functions is isomorphic to the h-cobordism group of homotopy spheres. Recently, this result was generalized for fold maps by Sadykov [**37**]. We note that if we consider the target manifold as the Euclidean space, then cobordism invariants of maps induce cobordism invariants of the source manifolds.

Furthermore, Saeki [**43**] developed the theory of singular fibers of differentiable maps of negative codimension. Here, the terminology "singular fiber" for differentiable map $f : M \to N$ and a point $q \in N$ refers to a certain right-left equivalent class of map germs along the inverse image

$$f : (M, f^{-1}(q)) \to (N, q),$$

not just inverse image $f^{-1}(q)$. For positive codimension case, the fiber over a point in N is a discrete set of points, as long as the map is generic enough, and we can study the topology of such maps by using the well-developed theory of multi-jet spaces. However, in the negative codimension case, the fiber over a point is no longer a discrete set, and is a complex of positive dimension in general. This means that the theory of multi-jet spaces is not sufficient any more, and in [**43**] we have seen that the topology of singular fibers plays an essential role in negative codimension case. Moreover, in that theory, Saeki constructed the cochain complex of singular fibers and he showed that the cohomology classes represent the cobordism invariants of the differentiable maps. As an explicit and important example in his theory of singular fibers, stable maps of closed orientable 4-manifolds into 3-manifolds were studied and singular fibers were completely classified (for a precise definition of equivalence relation, see §2 in Chapter 1 of the present thesis). Furthermore, Saeki obtained the following: For any stable map of an orientable closed 4-manifold into a connected 3-manifold, the number of singular fibers of $\widetilde{\mathrm{III}}^{12}$ type as depicted in Figure 0.1 and the Euler number of the source 4-manifold have the same parity, where $\widetilde{\mathrm{III}}^{*}$ is the names of the singular fibers (In the book [**43**], the symbol "III^{8}" is used instead of "$\widetilde{\mathrm{III}}^{12}$"). Note that mod 2 Euler number of 4-manifolds corresponds to the generator of unoriented cobordism groups of 4-manifolds (for details, see [**30**]).

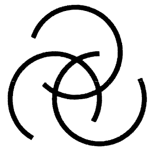

FIGURE 0.1. The singular fiber of $\widetilde{\mathrm{III}}^{12}$ type

Throughout this thesis, we study the topology of manifolds in terms of the singular fibers of differentiable maps and we obtain several formulas of Saeki's Euler number formula. This thesis consists of five Chapters.

In Chapter 1, we generalise Saeki's Euler number formula for possibly non-orientable 4-manifolds. We first classify the singular fibers where the source 4-manifolds may possibly be non-orientable (Theorem 2.4 in Chapter 1). Then we prove Theorem 4.7 in Chapter 1: Under certain homological conditions, for a stable map $f : M \to N$ of a closed 4-manifold M into a connected 3-manifold N, the total number of certain singular fibers and the Euler number $\chi(M)$ of the source 4-manifold M have the same parity,

$$\begin{aligned}\chi(M) \equiv\ & |\widetilde{\mathrm{III}}^{2,2,2}(f)| + |\widetilde{\mathrm{III}}^{2,7}(f)| + |\widetilde{\mathrm{III}}^{12}(f)| + |\widetilde{\mathrm{III}}_e^{13}(f)| \\ & + |\widetilde{\mathrm{III}}_B^{13}(f)| + |\widetilde{\mathrm{III}}^{25}(f)| + |\widetilde{\mathrm{III}}^{26}(f)| \pmod 2,\end{aligned}$$

where $|\mathcal{F}(f)|$ denotes the number of singular fibers of f of type $\mathcal{F}$. For the notation of singular fibers in the formula, see Figure 1.1.

Under certain homological conditions, which will be called the *two colorable condition*, for $f|_{S(f)} : S(f) \to N$ there exist non-empty disjoint open subsets R and B in N such that $R \cup B = N \setminus f(S(f))$ and $\overline{R} \cap \overline{B} = \partial R = \partial B = f(S(f))$ (for details, see §5). For points $q \in f(S(f))$, by combining the colouring (i.e. R or B) of the 3-dimensional open strata adjacent to q and the numbers of connected components of the fibers corresponding to these 3-dimensional strata (we note that the number is constant on each stratum) we can divide several C^∞ equivalence classes of singular fibers into two types A and B. In the formula of Theorem 4.7, $\widetilde{\mathrm{III}}_B^{13}$ denotes such a subclass of $\widetilde{\mathrm{III}}^{13}$. We note that $\widetilde{\mathrm{III}}_e^{13}$ is also a subclass of $\widetilde{\mathrm{III}}^{13}$, which consists of those singular fibers of type $\widetilde{\mathrm{III}}^{13}$ having an even number of connected components. In next Chapter, we show that without two colorable condition, we cannot expect Saeki's type Euler number formula. A part of the results of this Chapter has been obtained in author's master thesis [**51**] and the paper [**52**]

In Chapter 2, we introduce the notion of *two-colored map* and *the two-colored cobordism* among two-colored maps. Furthermore, we develop the theory of the singular fiber of two-colored maps. In this theory, we construct the cochain complex of the singular fibers of two-colored maps and we show that cohomology classes of cochain complex induce two-colored cobordism invariants of two-colored maps. From an actual calculation, we obtain several Euler number formulas of manifolds: Theorems 5.8, 5.9, 5.12 in Chapter 2. Finally, our Theorem 5.18 in Chapter 2 says that there is no Euler number formula of 4-manifolds in terms of the singular fibers of stable maps if we consider not two-colored map just only stable map into 3-manifold.

The contents of Chapter 3 and 4 are joint work with Saeki.

In Chapter 3, we consider the signature formula of 4-manifolds. We will give an "integral lift" of the Saeki's Euler number formula [**43**]. More precisely, we consider

C^∞ stable maps of *oriented* 4-manifolds into 3-manifolds, and we give a sign $+1$ or -1 to each of its III^8 type fiber, using the orientation of the source 4-manifold. Then we show that the algebraic number of III^8 type fibers coincides with the signature $\sigma(M)$ of the source oriented 4-manifold M (Theorem 5.5 in Chapter 3): namely

$$\sigma(M) = ||\mathrm{III}^8(f)|| \in \mathbb{Z},$$

where $||\mathrm{III}^8(f)||$ denote the algebraic number of III^8 type fibers of $f : M \to N$. The results of this Chapter has been obtained in [**44**].

In Chapter 4, we generalize the signature formula for stable maps of closed oriented 4-manifolds into $\mathbb{R}^3$ obtained in Chapter 3 for proper Thom-Boardman generic maps $f : M \to N$ of codimension -1. Here, a smooth map $f : M \to N$ between manifolds is said to be *Thom-Boardman generic* if its jet extension $j^r f : M \to J^r(M, N)$ is transverse to all the Thom-Boardman strata for all r (for details, see [**4**]), and f restricted to its Thom-Boardman singular sets is in general position (for details, see [**13**, Chapter VI, §5]). In order to work with integer coefficients, we will work with oriented maps. Recall that a differentiable map between manifolds of negative codimension is an oriented map if the regular parts of the fibers are consistently oriented (see also [**2**]). In this situation, we will show that the Poincaré dual to the homology class represented by the closure of the III^8-locus coincides with $f_! P_1(M)$ modulo torsion, (Theorem 2.5 in Chapter 4): namely

$$f_! P_1(M) \equiv [\overline{\mathrm{III}^8(f)}]^* \in H^3(N; \mathbb{Z}) \text{ modulo torsion},$$

where $f_!$ is the Gysin homomorphism induced by the map f and $P_1(M)$ is the first Pontrjagin class of M. The results of this Chapter is a part of the paper [**45**].

In Chapter 5, we state several conjectures for further developments of the theory of the singular fibers of differentiable maps.

Throughout this thesis, all manifolds and maps are of class C^∞ unless otherwise stated. For a finite set P, we denote by $|P|$ the number of its elements. For a topological space Y and a subset $X \subset Y$, $\overline{X}$ is the topological closure of X in Y, ∂X is the boundary of X which is defined to be $\overline{X} \setminus X$, the symbol "id_X" denotes the identity map of X. and the symbol $\chi(X)$ denote the Euler number of X. The symbol "$\cong$" denotes an appropriate isomorphism between algebraic objects. For a smooth manifold M, its tangent bundle is denoted by TM and its interior is denoted by $\mathrm{Int}M$. We call $p \in M$ the *singular point* of C^∞ map $f : M \to N$ if the rank of Jacobi matrix of f at p is strictly less than the dimension of the target manifold. In this sense, there is no regular point if the dimension of source manifold is strictly less than that of target manifold.

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Chapter 1

Classification of singular fibers of stable maps of 4-manifolds into 3-manifolds and its applications

CHAPTER 1

Classification of singular fibers of stable maps of 4-manifolds into 3-manifolds and its applications

1. Introduction

As pioneers, Kushner, Levine and Porto studied the singular fibres of stable maps of closed 3-manifolds into the plane in [**22**] and [**24**]. However, they did not state clearly the definitions of singular fibres and the equivalence relation among them. Recently, in the book [**43**], Saeki stated the precise definition of singular fibres, introduced an equivalence relation among them, and classified the singular fibres of stable maps of closed orientable 4-manifolds into 3-manifolds. Moreover, he proved: For any stable map of an orientable closed 4-manifold into a connected 3-manifold, the number of singular fibres of $\widetilde{\mathrm{III}}^{12}$ type as depicted in Figure 0.1 and the Euler number of the source 4-manifold are of the same parity, where $\widetilde{\mathrm{III}}^{*}$ mean the names of the singular fibers (In the book [**43**], the symbol "III^{8}" is used instead of "$\widetilde{\mathrm{III}}^{12}$")

Then it is natural to ask:

Is there similar formula if the source manifold is non-orientable?

In this paper we generalise Saeki's Euler number formula, giving an answer to the above question. We first classify the singular fibres in the general case where the source 4-manifold may possibly be non-orientable (see Theorem 2.4). Then we prove Theorem 4.7: Under certain homological conditions, for a stable map $f : M \to N$ of a closed 4-manifold M into a connected 3-manifold N, the total number of certain singular fibres and the Euler number $\chi(M)$ of the source 4-manifold M are of the same parity: namely,

$$\begin{aligned}\chi(M) \equiv\ & |\widetilde{\mathrm{III}}^{2,2,2}(f)| + |\widetilde{\mathrm{III}}^{2,7}(f)| + |\widetilde{\mathrm{III}}^{12}(f)| + |\widetilde{\mathrm{III}}_e^{13}(f)| \\ & + |\widetilde{\mathrm{III}}_B^{13}(f)| + |\widetilde{\mathrm{III}}^{25}(f)| + |\widetilde{\mathrm{III}}^{26}(f)| \pmod 2,\end{aligned}$$

where $|\mathcal{F}(f)|$ denotes the number of singular fibres of f of type $\mathcal{F}$. For the notation of singular fibres in the formula, see Figure 1.1, where "/" is used only for separating the figures.

We note that the C^∞ equivalence classes of fibres over points of N give a natural stratification of N. Then the set of regular values $N \setminus f(S(f))$ consists of 3-dimensional strata, while the set of singular values $f(S(f))$ consists of 2-, 1- and 0-dimensional strata, where $S(f)$ ($\subset M$) denotes the set of singular points of f. We assign to each 3-dimensional stratum the number of connected components of the fibre over a point in the stratum. Note that the number is constant on each stratum.

Under certain homological conditions, which will be called the *two colorable condition*, for $f|_{S(f)} : S(f) \to N$ there exist disjoint open subsets R and B of N such that $R \cup B = N \setminus f(S(f))$ and $\overline{R} \cap \overline{B} = \partial R = \partial B = f(S(f))$ (for details, see §5). For points $q \in f(S(f))$, by combining the colouring (i.e. R or B) of the 3-dimensional strata adjacent to q and the numbers of connected components of

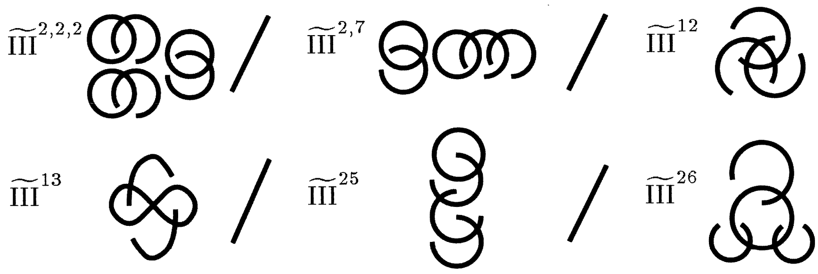

FIGURE 1.1. The singular fibers in the formula

the fibres corresponding to these 3-dimensional strata, we can divide several C^∞ equivalence classes of singular fibres into two types A and B. In the formula of Theorem 4.7, $\widetilde{\mathrm{III}}^{13}_B$ denotes such a subclass of $\widetilde{\mathrm{III}}^{13}$. We note that $\widetilde{\mathrm{III}}^{13}_e$ is also a subclass of $\widetilde{\mathrm{III}}^{13}$, which consists of those singular fibres of type $\widetilde{\mathrm{III}}^{13}$ with an even number of connected components.

If M is orientable, then $f|_{S(f)} : S(f) \to N$ always satisfies the two colorable condition (for details, see §5). In other words, the assumption of Theorem 4.7 is automatically satisfied. Furthermore, the singular fibres of types other than $\widetilde{\mathrm{III}}^{12}$ in the formula never appear. Thus Theorem 4.7 gives the Euler number formula obtained in [**43**] when the source 4-manifold is orientable.

In [**21**] Kobayashi constructed a stable map $g : \mathbb{C}P^2 \to \mathbb{R}^3$ such that $g(S(g))$ has exactly two triple points, which correspond to the singular fibres $\widetilde{\mathrm{III}}^{0,0,0}$ and $\widetilde{\mathrm{III}}^{12}$. Theorem 4.7 implies that $\chi(\mathbb{C}P^2)$ is odd. In fact, we have $\chi(\mathbb{C}P^2) = 3$. In [**38**, Example 3.7] Saeki constructed a non-orientable closed 4-manifold E as the total space of an $\mathbb{R}P^2$-bundle over $\mathbb{R}P^2$ together with a stable map $h : E \to \mathbb{R}^3$ such that $h(S(h))$ has 27 triple points. They consist of eight $\widetilde{\mathrm{III}}^{0,0,0}$ points, twelve $\widetilde{\mathrm{III}}^{0,0,2}$ points, six $\widetilde{\mathrm{III}}^{0,2,2}$ points and one $\widetilde{\mathrm{III}}^{2,2,2}$ point. Theorem 4.7 shows that $\chi(E)$ must be an odd number. Actually, we have $\chi(E) = 1$.

We have some direct consequences of Theorem 4.7 as follows.

COROLLARY 1.1. *Let M be a closed 4-manifold with odd Euler number and $f : M \to N$ a stable map of M into a connected 3-manifold N with $H_1(N; \mathbb{Z}_2) = 0$. Then $f|_{S(f)}$ has at least one triple point.*

COROLLARY 1.2. *Let f be a stable map as in Corollary 1.1. Then f has at least one singular fibre of $\widetilde{\mathrm{III}}^{12}$ type or $\widetilde{\mathrm{I}}^2$ type.*

Corollary 1.2 is proved as follows. The stable map f has at least one singular fibre as appearing in the formula of Theorem 4.7, since $\chi(M)$ is odd. If the source manifold of f is orientable, then f must have a singular fibre of type $\widetilde{\mathrm{III}}^{12}$. If the source manifold is non-orientable and f has no singular fibre of type $\widetilde{\mathrm{III}}^{12}$, then by using the description of local nearby fibres of singular fibres of types other than $\widetilde{\mathrm{III}}^{12}$ appearing in Theorem 4.7 (e.g. see Figure 1.7), we see that f has a singular fibre of type $\widetilde{\mathrm{I}}^2$.

We say that a stable map is *simple* if each connected component of every fibre has at most one singular point.

COROLLARY 1.3. *Let f be a stable map as in Corollary 1.1. If f is simple, then f has at least one singular fibre of type $\widetilde{\mathrm{III}}^{2,2,2}$.*

The hypothesis that the Euler number should be odd in Corollaries 1.1–1.3 is essential. In fact there exists a special generic map $f : S^3 \tilde{\times} S^1 \to \mathbb{R}^3$ such that $f|_{S(f)}$ is an embedding [**40**], where $S^3 \tilde{\times} S^1$ is the total space of a non-orientable S^3 bundle over S^1. A *special generic map* is a stable map which has only definite fold points as its singular points. In fact, all the singular fibres of f are of type $\widetilde{\mathrm{I}}^0$. We note that $\chi(S^3 \tilde{\times} S^1) = 0$.

As this example shows, we have the stable map of closed non-orientable 4-manifold with even Euler number into 3-manifold which has no singular fibers of types $\widetilde{\mathrm{III}}^{12}$ and $\widetilde{\mathrm{I}}^2$. We pose a following question.

Problem 1.4. Does there exist stable map from even Euler number 4-manifold into connected 3-manifold which has no $\widetilde{\mathrm{III}}^{12}$ and $\widetilde{\mathrm{I}}^2$ type fiber ?

This chapter is organized as follows. In §2, we give the definitions of equivalence relations among the fibres of stable maps. Furthermore, we classify the singular fibres up to C^∞ equivalence, and give a table of singular fibres for stable maps of closed possibly non-orientable 4-manifolds into 3-manifolds. In §3, we give some co-existence relations of singular fibres for stable maps of closed 4-manifolds into 3-manifolds. In §4, we show that the Euler number of the source manifold has the same parity as the total number of certain singular fibres (Theorem 4.7 and Proposition 4.5).

this paper has been obtained in the author's master's thesis [**51**] and [**52**].

2. Singular fibers of stable maps of 4-manifolds into 3-manifolds

In this section, we classify the singular fibers of stable maps of closed 4-manifolds into 3- manifolds.

Let us begin with some fundamental definitions.

DEFINITION 2.1. Let M_i be smooth manifolds and A_i subsets of M_i, $i = 0, 1$. A continuous map $g : A_0 \to A_1$ is said to be *smooth* if for every point $q \in A_0$, there exists a smooth map $\tilde{g} : V \to M_1$ defined on a neighbourhood V of q in M_0 such that $\tilde{g}|_{V \cap A_0} = g|_{V \cap A_0}$. A smooth map $g : A_0 \to A_1$ is a *diffeomorphism* if it is a homeomorphism and its inverse is also smooth. When there exists a diffeomorphism between A_0 and A_1, we say that they are *diffeomorphic*.

Let $f_i : M_i \to N_i$ be smooth maps, $i = 0, 1$. For $q_i \in N_i$, $i = 0, 1$, we say that the fibers over q_0 and q_1 are *diffeomorphic* if $f_0^{-1}(q_0) \subset M_0$ and $f_1^{-1}(q_1) \subset M_1$ are diffeomorphic in the above sense. Furthermore, we say that the fibers over q_0 and q_1 are C^∞ *equivalent* (or C^0 *equivalent*) if for some open neighbourhood U_i of q_i, there exist diffeomorphisms (resp. homeomorphisms) $\Phi : f_0^{-1}(U_0) \to f_1^{-1}(U_1)$ and $\varphi : U_0 \to U_1$ with $\varphi(q_0) = q_1$ which make the following diagram commutative:

$$\begin{array}{ccc} (f_0^{-1}(U_0), f_0^{-1}(q_0)) & \xrightarrow{\Phi} & (f_1^{-1}(U_1), f_1^{-1}(q_1)) \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ (U_0, q_0) & \xrightarrow{\varphi} & (U_1, q_1). \end{array}$$

If $q \in N$ is a regular value of a smooth map $f : M \to N$ between manifolds, then we call the map germ $f : (M, f^{-1}(q)) \to (N, q)$ along the set $f^{-1}(q)$ a *regular fiber*; otherwise, we call it a *singular fiber*.

Let us recall a characterization of stable maps of closed 4-manifolds into 3-manifolds. For smooth manifolds M and N, let us denote by $C^\infty(M, N)$ the space of all C^∞-maps $M \to N$, equipped with the Whitney C^∞-topology. In general, we say that $f \in C^\infty(M, N)$ is C^∞ *stable* (or *stable* for short) if the $\mathcal{A}$-orbit of f is open in $C^\infty(M, N)$. Here the $\mathcal{A}$-orbit of $f \in C^\infty(M, N)$ is defined as follows. Let Diff(N) denote the group of self-diffeomorphisms of N. Then the group Diff(M) $\times$ Diff(N) acts on $C^\infty(M, N)$ by $(\Phi, \Psi)f = \Psi \circ f \circ \Phi^{-1}$, where $(\Phi, \Psi) \in$ Diff(M) $\times$ Diff(N) and $f \in C^\infty(M, N)$. Then the $\mathcal{A}$-orbit of $f \in C^\infty(M, N)$ is the orbit through f with respect to this action.

PROPOSITION 2.2. *A smooth map $f : M \to N$ of a closed 4-manifold M into a 3-manifold N is C^∞ stable if and only if the following conditions are satisfied.*

(*i*) (*Local condition*) *For every $p \in M$, there exist local coordinates (x, y, z, w) and (X, Y, Z) around $p \in M$ and $f(p) \in N$, respectively, such that one of the following holds:*

$$(X \circ f, Y \circ f, Z \circ f)$$
$$= \begin{cases} (x, y, z), & p : \text{regular point}, \\ (x, y, z^2 + w^2), & p : \text{definite fold point}, \\ (x, y, z^2 - w^2), & p : \text{indefinite fold point}, \\ (x, y, z^3 + xz - w^2), & p : \text{cusp point}, \\ (x, y, z^4 + xz^2 + yz + w^2), & p : \text{definite swallow-tail point}, \\ (x, y, z^4 + xz^2 + yz - w^2), & p : \text{indefinite swallow-tail point}. \end{cases}$$

(*ii*) (*Global condition*) *Set $S(f) = \{p \in M \mid \operatorname{rank} df_p < 3\}$, which is a closed 2-dimensional submanifold of M under the above local condition. Then, for every $q \in f(S(f))$, $f^{-1}(q) \cap S(f)$ consists of at most three points and the multi-germ*

$$(f|_{S(f)}, f^{-1}(q) \cap S(f))$$

is right-left equivalent to one of the six multi-germs as described in Figure 1.2: (1) *corresponds to a single fold point,* (2) *and* (3) *represent normal crossings of two and three immersion germs, respectively, each of which corresponds to a fold point,* (4) *corresponds to a cusp point,* (5) *represents a transverse crossing of a cuspidal edge as in* (4) *and an immersion germ corresponding to a fold point as in* (1), *and* (6) *corresponds to a swallow-tail point.*

Proposition 2.2 can be proved by using the transversality theorem and the multi-transversality theorem, since the dimensions pair $(4, 3)$ is in the nice range in the sense of Mather **[26]** (for details, see **[13]**, **[25]** or **[14]**).

Let $f : M \to N$ be a stable map of a closed 4-manifold M into a 3-manifold N. For each regular point $x \in M$ of f, the fiber through x is a 1-dimensional submanifold near the point. For each singular point $p \in M$ of f, based on the local condition of Proposition 3.1 (i), it is easy to determine the diffeomorphism type of a neighbourhood of p in $f^{-1}(f(p))$ as follows.

LEMMA 2.3. *Every singular point p of a stable map $f : M \to N$ of a closed 4-manifold M into a 3-manifold N has one of the following neighbourhoods in its corresponding singular fiber $f^{-1}(f(p))$ (see Figure 1.3):*

(1) *isolated point diffeomorphic to $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 0\}$, if p is a definite fold point,*
(2) *union of two transverse arcs diffeomorphic to $\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = 0\}$, if p is an indefinite fold point,*
(3) *cuspidal arc diffeomorphic to $\{(x, y) \in \mathbb{R}^2 \mid x^3 - y^2 = 0\}$, if p is a cusp point,*
(4) *isolated point diffeomorphic to $\{(x, y) \in \mathbb{R}^2 \mid x^4 + y^2 = 0\}$, if p is a definite swallowtail point,*

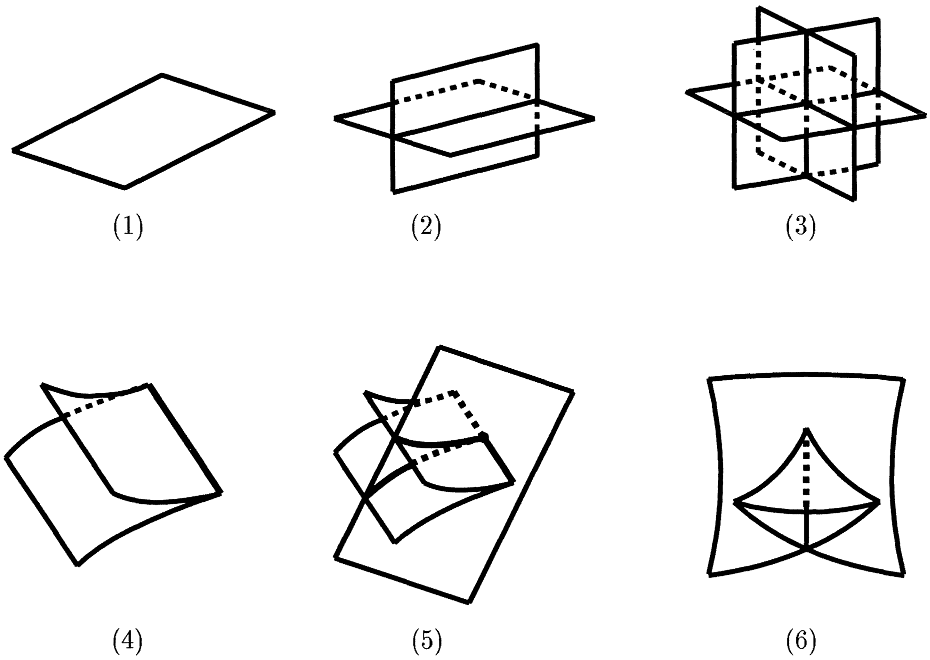

FIGURE 1.2. Multi-germs of $f|_{S(f)}$

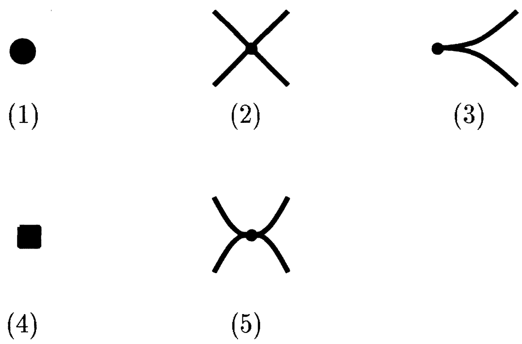

FIGURE 1.3. Neighbourhoods of singular points in singular fibers

(5) *union of two tangent arcs diffeomorphic to* $\{(x, y) \in \mathbb{R}^2 \mid x^4 - y^2 = 0\}$, *if* p *is an indefinite swallowtail point.*

We note that in Figure 1.3, both the black dot (1) and the black square (4) represent an isolated point, although the corresponding map germs are not C^∞ equivalent to each other; we use distinct symbols in order to distinguish them. We note that each singular point $p \in M$, except for a definite fold point and a definite swallow-tail point, is incident to some edges in its neighbourhood in $f^{-1}(f(p))$.

We note that a regular fiber of f is a closed 1-dimensional submanifold of M, namely a disjoint union of a finite number of circles. Thus, for a regular value q of f, the fiber of f over q is C^∞ equivalent to the disjoint union of a finite number of copies of a fiber of a trivial circle bundle. For the singular fibers of f, we have the following.

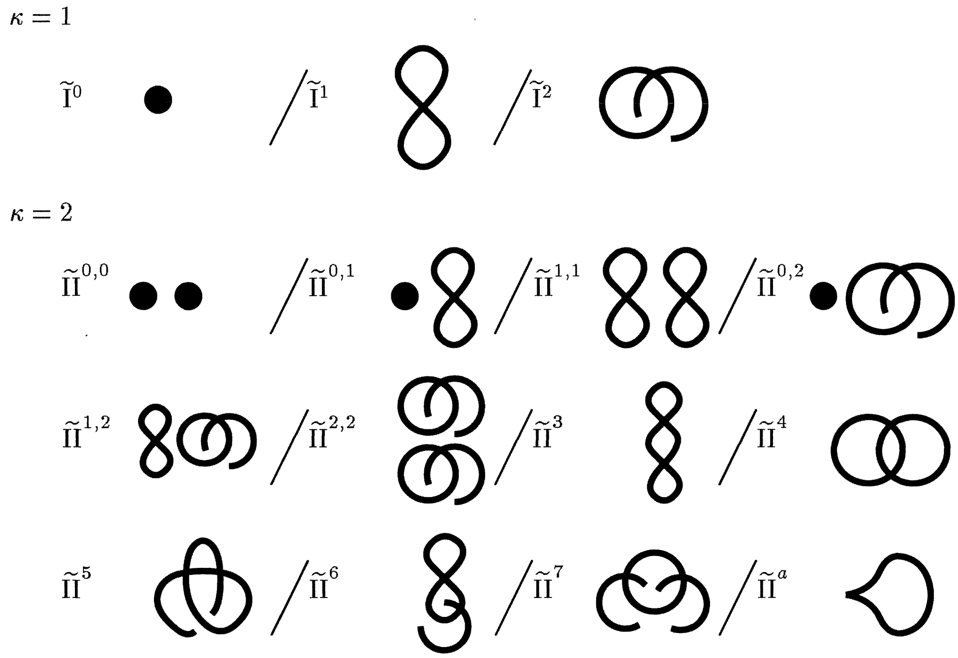

FIGURE 1.4. List of singular fibers; 1

THEOREM 2.4. *Let $f : M \to N$ be a stable map of a closed 4-manifold M into a 3-manifold N. Then, every singular fiber of f is C^∞ equivalent to the disjoint union of one of the fibers in the following list and a finite number of copies of a fiber of a trivial circle bundle:*

(1) *one of the fibers as depicted in Figure* 1.4,
(2) *a disconnected fiber* $\widetilde{\mathrm{III}}^{0,0,0}$, $\widetilde{\mathrm{III}}^{0,0,1}$, $\widetilde{\mathrm{III}}^{0,1,1}$, $\widetilde{\mathrm{III}}^{1,1,1}$, $\widetilde{\mathrm{III}}^{0,0,2}$, $\widetilde{\mathrm{III}}^{0,2,2}$, $\widetilde{\mathrm{III}}^{1,1,2}$, $\widetilde{\mathrm{III}}^{1,2,2}$, $\widetilde{\mathrm{III}}^{0,1,2}$, $\widetilde{\mathrm{III}}^{2,2,2}$, $\widetilde{\mathrm{III}}^{0,3}$, $\widetilde{\mathrm{III}}^{0,4}$, $\widetilde{\mathrm{III}}^{0,5}$, $\widetilde{\mathrm{III}}^{0,6}$, $\widetilde{\mathrm{III}}^{0,7}$, $\widetilde{\mathrm{III}}^{1,3}$, $\widetilde{\mathrm{III}}^{1,4}$, $\widetilde{\mathrm{III}}^{1,5}$, $\widetilde{\mathrm{III}}^{1,6}$, $\widetilde{\mathrm{III}}^{1,7}$, $\widetilde{\mathrm{III}}^{2,3}$, $\widetilde{\mathrm{III}}^{2,4}$, $\widetilde{\mathrm{III}}^{2,5}$, $\widetilde{\mathrm{III}}^{2,6}$, $\widetilde{\mathrm{III}}^{2,7}$, $\widetilde{\mathrm{III}}^{0,a}$, $\widetilde{\mathrm{III}}^{1,a}$ *or* $\widetilde{\mathrm{III}}^{2,a}$,
(3) *one of the connected fibers as depicted in Figure* 1.5.

The figure corresponding to each fiber listed in Theorem 2.4 (2) can be obtained by taking the disjoint union of the fibers in Figure 1.4 corresponding to the numbers or letters appearing in the superscript. For example, the figure of the fiber $\widetilde{\mathrm{III}}^{0,0,2}$ consists of two dots and a figure of $\widetilde{\mathrm{I}}^2$ type as shown in Figure 1.6.

In Figures 1.4 and 1.5, κ denotes the codimension of the set of points in N whose corresponding fibers are C^∞ equivalent to the relevant one (for details, see [**43**]). Furthermore, $\widetilde{\mathrm{I}}^*$, $\widetilde{\mathrm{II}}^*$, $\widetilde{\mathrm{III}}^*$ mean the names of the corresponding singular fibers, and "/" is used only for separating the figures.

We note that the list of singular fibers as in Figure 1.4 coincides with that appearing in the introduction of [**24**].

We note that the conclusion of Theorem 2.4 holds if f is proper even if M is not closed, where a continuous map is said to be *proper* if the inverse image of a compact set is always compact.

Theorem 2.4 can be proved in two steps. First we show that for a singular value q of f, the union of the components of $f^{-1}(q)$ containing singular points is diffeomorphic to one of the fibers listed in Theorem 2.4 in the sense of Definition 2.1. Second we show that if two singular fibers are diffeomorphic to each other, then

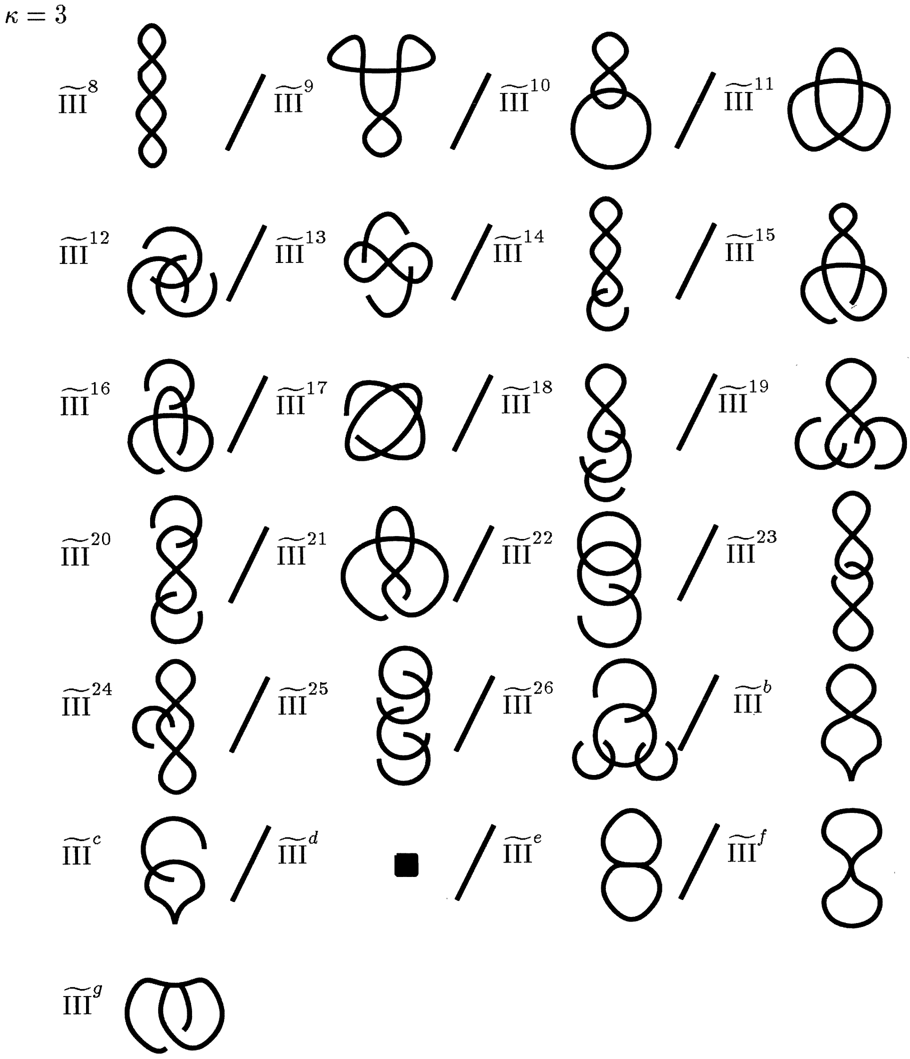

FIGURE 1.5. List of singular fibers; 2

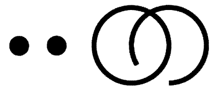

FIGURE 1.6. The singular fiber of type $\widetilde{\mathrm{III}}^{0,0,2}$

they are C^∞ equivalent in the sense of Definition 2.1, except for the two types of fibers $\widetilde{\mathrm{I}}^0$ and $\widetilde{\mathrm{III}}^d$. The proof is very similar to that of [**43**, Theorem 3.5], as we omit the proof here.

Remark 2.5. Each singular fiber described in Theorem 2.4 can be realized as a component (or as a union of some components) of a singular fiber of a stable map of a closed 4-manifold into $\mathbb{R}^3$. This can be seen as follows. Given a singular fiber, we can realize it as a singular fiber of a Morse function parameterized on D^2, $f_t : S \to [-1, 1], t \in D^2$, of a compact surface with boundary S into $[-1, 1]$, where D^2 denotes the unit disk in $\mathbb{R}^2$. We note that $F : S \times D^2 \to [-1,1] \times D^2$, defined by $F(x,t) = (f_t(x), t)$, is a smooth map and that F has the given singular fiber over $(0,0)$. We call S *a transverse surface* corresponding to the singular fiber (for details, see [**24**]). In this way we obtain a proper smooth map $F|_{\mathrm{Int}(S\times D^2)} : \mathrm{Int}(S \times D^2) \to \mathrm{Int}([-1,1] \times D^2)$. Then we can extend the map to a smooth map of a closed 4-manifold containing $\mathrm{Int}(S \times D^2)$ into $\mathbb{R}^3$. Perturbing the extended map slightly, we obtain a desired stable map.

If the source 4-manifold is orientable, then any transverse surface for any singular fiber is orientable. If the source 4-manifold is non-orientable, then there may exist a non-orientable transverse surface. The transverse surface which corresponds to the singular fiber of I^2 type is a punctured Möbius band. We note that there exists a stable map of a non-orientable 4-manifold into a 3-manifold such that the transverse surface is orientable for any fiber. (For instance, see the example just after Corollary 1.3 in §1.)

We note that for a stable map $f : M \to N$ of an orientable closed 4-manifold M into a 3-manifold N, the singular fibers of the following types never appear, since they have non-orientable transverse surfaces: $\widetilde{\mathrm{I}}^2$, $\widetilde{\mathrm{II}}^{0,2}$, $\widetilde{\mathrm{II}}^{1,2}$, $\widetilde{\mathrm{II}}^{2,2}$, $\widetilde{\mathrm{II}}^{5}$, $\widetilde{\mathrm{II}}^{6}$, $\widetilde{\mathrm{II}}^{7}$, $\widetilde{\mathrm{III}}^{0,0,2}$, $\widetilde{\mathrm{III}}^{0,2,2}$, $\widetilde{\mathrm{III}}^{1,1,2}$, $\widetilde{\mathrm{III}}^{1,2,2}$, $\widetilde{\mathrm{III}}^{0,1,2}$, $\widetilde{\mathrm{III}}^{2,2,2}$, $\widetilde{\mathrm{III}}^{0,5}$, $\widetilde{\mathrm{III}}^{0,6}$, $\widetilde{\mathrm{III}}^{0,7}$, $\widetilde{\mathrm{III}}^{1,5}$, $\widetilde{\mathrm{III}}^{1,6}$, $\widetilde{\mathrm{III}}^{1,7}$, $\widetilde{\mathrm{III}}^{2,3}$, $\widetilde{\mathrm{III}}^{2,4}$, $\widetilde{\mathrm{III}}^{2,5}$, $\widetilde{\mathrm{III}}^{2,6}$, $\widetilde{\mathrm{III}}^{2,7}$, $\widetilde{\mathrm{III}}^{2,a}$, $\widetilde{\mathrm{III}}^{13}$, $\widetilde{\mathrm{III}}^{14}$, $\widetilde{\mathrm{III}}^{15}$, $\widetilde{\mathrm{III}}^{16}$, $\widetilde{\mathrm{III}}^{17}$, $\widetilde{\mathrm{III}}^{18}$, $\widetilde{\mathrm{III}}^{19}$, $\widetilde{\mathrm{III}}^{20}$, $\widetilde{\mathrm{III}}^{21}$, $\widetilde{\mathrm{III}}^{22}$, $\widetilde{\mathrm{III}}^{23}$, $\widetilde{\mathrm{III}}^{24}$, $\widetilde{\mathrm{III}}^{25}$, $\widetilde{\mathrm{III}}^{26}$, $\widetilde{\mathrm{III}}^{c}$ and $\widetilde{\mathrm{III}}^{g}$.

Remark 2.6. For stable maps f of 4-manifolds into 3-manifolds, the triple points of $f|_{S(f)}$ correspond to the singular fiber of types $\widetilde{\mathrm{III}}^{0,0,0}$, $\widetilde{\mathrm{III}}^{0,0,1}$, $\widetilde{\mathrm{III}}^{0,1,1}$, $\widetilde{\mathrm{III}}^{1,1,1}$, $\widetilde{\mathrm{III}}^{0,0,2}$, $\widetilde{\mathrm{III}}^{0,2,2}$, $\widetilde{\mathrm{III}}^{1,1,2}$, $\widetilde{\mathrm{III}}^{1,2,2}$, $\widetilde{\mathrm{III}}^{0,1,2}$, $\widetilde{\mathrm{III}}^{2,2,2}$, $\widetilde{\mathrm{III}}^{0,3}$, $\widetilde{\mathrm{III}}^{0,4}$, $\widetilde{\mathrm{III}}^{0,5}$, $\widetilde{\mathrm{III}}^{0,6}$, $\widetilde{\mathrm{III}}^{0,7}$, $\widetilde{\mathrm{III}}^{1,3}$, $\widetilde{\mathrm{III}}^{1,4}$, $\widetilde{\mathrm{III}}^{1,5}$, $\widetilde{\mathrm{III}}^{1,6}$, $\widetilde{\mathrm{III}}^{1,7}$, $\widetilde{\mathrm{III}}^{2,3}$, $\widetilde{\mathrm{III}}^{2,4}$, $\widetilde{\mathrm{III}}^{2,5}$, $\widetilde{\mathrm{III}}^{2,6}$, $\widetilde{\mathrm{III}}^{2,7}$, $\widetilde{\mathrm{III}}^{8}$, $\widetilde{\mathrm{III}}^{9}$, $\widetilde{\mathrm{III}}^{10}$, $\widetilde{\mathrm{III}}^{11}$, $\widetilde{\mathrm{III}}^{12}$, $\widetilde{\mathrm{III}}^{13}$, $\widetilde{\mathrm{III}}^{14}$, $\widetilde{\mathrm{III}}^{15}$, $\widetilde{\mathrm{III}}^{16}$, $\widetilde{\mathrm{III}}^{17}$, $\widetilde{\mathrm{III}}^{18}$, $\widetilde{\mathrm{III}}^{19}$, $\widetilde{\mathrm{III}}^{20}$, $\widetilde{\mathrm{III}}^{21}$, $\widetilde{\mathrm{III}}^{22}$, $\widetilde{\mathrm{III}}^{21}$, $\widetilde{\mathrm{III}}^{22}$, $\widetilde{\mathrm{III}}^{23}$, $\widetilde{\mathrm{III}}^{24}$, $\widetilde{\mathrm{III}}^{25}$ and $\widetilde{\mathrm{III}}^{26}$ of f. Thus the number of triple points of $f|_{S(f)}$ coincides with the total number of singular fiber of types as above.

For stable maps of a closed 4-manifolds into 3-manifolds, if the source 4-manifolds are orientable, then the classification of the singular fibers with respect to the C^∞ equivalence coincides with respect to the C^0 equivalence (for details, see [**43**, Corollary 3.9]). We obtain similar result.

COROLLARY 2.7. *For two singular fibers of proper stable maps of (possibly non-orientable) 4-manifolds into 3-manifolds, the following two are equivalent.*

(1) *They are C^∞ equivalent.*
(2) *They are C^0 equivalent.*

3. Relations among the numbers of singular fibers

Let $f : M \to N$ be a stable map of a closed 4-manifold M into a 3-manifold N. In this section, we consider a natural stratification of N induced by the C^∞ equivalence classes of the fibers of f, and obtain some relations among the numbers of singular fibers of codimension three.

Let $f : M \to N$ be as above, and $\mathcal{F}$ be a C^∞ equivalence class of one of the singular fibers appearing in Theorem 2.4. We define $\mathcal{F}(f)$ to be the set of points $q \in N$ such that the fiber $f^{-1}(q)$ over q is C^∞ equivalent to the union of $\mathcal{F}$ and some regular fibers. Then we obtain a "stratification" of N which consists of the components of $\mathcal{F}(f)$ together with $N \setminus f(S(f))$, where $\mathcal{F}$ runs over all C^∞ equivalence classes of singular fibers [1].

[1]In this paper, each stratum of a stratification may not necessarily be connected.

We define $\mathcal{F}_o(f)$ (resp. $\mathcal{F}_e(f)$) to be the subset of $\mathcal{F}(f)$ consisting of the points $q \in \mathcal{F}(f)$ such that the number of connected components of $f^{-1}(q)$ is odd (resp. even). It is easy to see that the closures $\overline{\mathcal{F}(f)}$, $\overline{\mathcal{F}_o(f)}$ and $\overline{\mathcal{F}_e(f)}$ of $\mathcal{F}(f)$, $\mathcal{F}_o(f)$ and $\mathcal{F}_e(f)$, respectively, in N are $(3-\kappa)$-dimensional complexes in N, where κ is the codimension of $\mathcal{F}$. In particular, if the codimension κ is equal to two, then $\overline{\mathcal{F}_o(f)}$ and $\overline{\mathcal{F}_e(f)}$ are finite graphs embedded in N. Their vertices correspond to points over which f has a singular fiber of codimension three. For a C^∞ equivalence class $\mathcal{G}$ of singular fibers of codimension three, the degree of the vertex corresponding to $\mathcal{G}_o(f)$ (or $\mathcal{G}_e(f)$) in the graph $\overline{\mathcal{F}_o(f)}$ is given in Tables 1.1, 1.2 and 1.3. In the tables, only non-zero degrees are given: an empty column means that the corresponding degree is equal to zero. We note that the graphs $\overline{\widetilde{\mathrm{II}}^*_o(f)}$ or $\overline{\widetilde{\mathrm{II}}^*_e(f)}$, or the vertices $\widetilde{\mathrm{III}}^*_o(f)$ or $\widetilde{\mathrm{III}}^*_e(f)$ may possibly be empty depending on the stable map f.

These tables can be obtained by using the description of local nearby fibers as shown in Figure 1.7. We note that the degrees in the graph $\overline{\mathcal{F}_e(f)}$ can be obtained by interchanging $\mathcal{G}_o(f)$ with $\mathcal{G}_e(f)$ in the table corresponding to the graph $\overline{\mathcal{F}_o(f)}$.

The handshake lemma of the classical graph theory claims that for a finite graph, the sum of the degrees over all vertices is equal to the double of the number of edges and hence is always even. We apply this lemma to the graphs $\overline{\widetilde{\mathrm{II}}^{0,0}_o(f)}$, $\overline{\widetilde{\mathrm{II}}^{0,0}_e(f)}$, $\overline{\widetilde{\mathrm{II}}^{0,1}_o(f)}$, $\overline{\widetilde{\mathrm{II}}^{0,1}_e(f)}$, $\overline{\widetilde{\mathrm{II}}^{1,1}_o(f)}$, $\overline{\widetilde{\mathrm{II}}^{1,1}_e(f)}$, $\overline{\widetilde{\mathrm{II}}^{0,2}_o(f)}$, $\overline{\widetilde{\mathrm{II}}^{0,2}_e(f)}$, $\overline{\widetilde{\mathrm{II}}^{1,2}_o(f)}$, $\overline{\widetilde{\mathrm{II}}^{1,2}_e(f)}$, $\overline{\widetilde{\mathrm{II}}^{2,2}_o(f)}$, $\overline{\widetilde{\mathrm{II}}^{2,2}_e(f)}$, $\overline{\widetilde{\mathrm{II}}^{3}_o(f)}$, $\overline{\widetilde{\mathrm{II}}^{3}_e(f)}$, $\overline{\widetilde{\mathrm{II}}^{4}_o(f)}$, $\overline{\widetilde{\mathrm{II}}^{4}_e(f)}$, $\overline{\widetilde{\mathrm{II}}^{5}_o(f)}$, $\overline{\widetilde{\mathrm{II}}^{5}_e(f)}$, $\overline{\widetilde{\mathrm{II}}^{6}_o(f)}$, $\overline{\widetilde{\mathrm{II}}^{6}_e(f)}$, $\overline{\widetilde{\mathrm{II}}^{7}_o(f)}$, $\overline{\widetilde{\mathrm{II}}^{7}_e(f)}$, $\overline{\widetilde{\mathrm{II}}^{a}_o(f)}$ and $\overline{\widetilde{\mathrm{II}}^{a}_e(f)}$. Then we obtain 24 relations among the numbers of elements of $\mathcal{G}_o(f)$ and $\mathcal{G}_e(f)$ for C^∞ equivalence classes $\mathcal{G}$ of singular fibers of codimension three. We combine the relation obtained from $\overline{\mathrm{II}^*_o(f)}$ and that obtained from $\overline{\mathrm{II}^*_e(f)}$, we arrange these relations a certain matters. Then we obtain the following.

Proposition 3.1. *Let $f : M \to N$ be a stable map of a closed 4-manifold M into a 3-manifold N. Then the following numbers are always even:*

(1) $|\widetilde{\mathrm{III}}^{0,a}(f)| \equiv |\widetilde{\mathrm{III}}^{d}(f)| \pmod 2$,
(2) $|\widetilde{\mathrm{III}}^{1,a}(f)| \equiv |\widetilde{\mathrm{III}}^{8}(f)| \pmod 2$,
(3) $|\widetilde{\mathrm{III}}^{2,a}(f)| \equiv |\widetilde{\mathrm{III}}^{14}(f)| \equiv |\widetilde{\mathrm{III}}^{c}(f)| \pmod 2$,
(4) $|\widetilde{\mathrm{III}}^{13}(f)| \equiv |\widetilde{\mathrm{III}}^{20}(f)| \equiv 0 \pmod 2$,
(5) $|\widetilde{\mathrm{III}}^{d}(f)| + |\widetilde{\mathrm{III}}^{e}(f)| + |\widetilde{\mathrm{III}}^{f}(f)| + |\widetilde{\mathrm{III}}^{g}(f)| \equiv 0 \pmod 2$,
(6) $|\widetilde{\mathrm{III}}^{0,a}(f)| + |\widetilde{\mathrm{III}}^{1,a}(f)| + |\widetilde{\mathrm{III}}^{b}(f)| \equiv 0 \pmod 2$.

We note that the left hand side of the congruence (5) of Proposition 3.1 is nothing but the total number of definite and indefinite swallow-tail points of a stable map f. The congruences (3) and (6) of Proposition 3.1 imply that the total number of multi-germs which correspond to the transverse intersection of a fold sheet and a cuspidal edge in the target is always even for a stable map f.

	$*=0,0$	$0,1$	$1,1$	$0,2$	$1,2$	$2,2$	3	4	5	6	7	a
$\widetilde{\mathrm{III}}_o^{0,0,0}(f)$	3											
$\widetilde{\mathrm{III}}_e^{0,0,0}(f)$	3											
$\widetilde{\mathrm{III}}_o^{0,0,1}(f)$	1	2										
$\widetilde{\mathrm{III}}_e^{0,0,1}(f)$	1	2										
$\widetilde{\mathrm{III}}_o^{0,1,1}(f)$		2	1									
$\widetilde{\mathrm{III}}_e^{0,1,1}(f)$		2	1									
$\widetilde{\mathrm{III}}_o^{1,1,1}(f)$			3									
$\widetilde{\mathrm{III}}_e^{1,1,1}(f)$			3									
$\widetilde{\mathrm{III}}_o^{0,0,2}(f)$	2			2								
$\widetilde{\mathrm{III}}_e^{0,0,2}(f)$				2								
$\widetilde{\mathrm{III}}_o^{0,2,2}(f)$				4		1						
$\widetilde{\mathrm{III}}_e^{0,2,2}(f)$						1						
$\widetilde{\mathrm{III}}_o^{1,1,2}(f)$			2		2							
$\widetilde{\mathrm{III}}_e^{1,1,2}(f)$					2							
$\widetilde{\mathrm{III}}_o^{1,2,2}(f)$					4	1						
$\widetilde{\mathrm{III}}_e^{1,2,2}(f)$						1						
$\widetilde{\mathrm{III}}_o^{0,1,2}(f)$		2		1	1							
$\widetilde{\mathrm{III}}_e^{0,1,2}(f)$				1	1							
$\widetilde{\mathrm{III}}_o^{2,2,2}(f)$						6						
$\widetilde{\mathrm{III}}_e^{2,2,2}(f)$												
$\mathrm{III}_o^{0,3}(f)$		2					1					
$\widetilde{\mathrm{III}}_e^{0,3}(f)$		2					1					
$\widetilde{\mathrm{III}}_o^{0,4}(f)$		4						1				
$\widetilde{\mathrm{III}}_e^{0,4}(f)$								1				
$\widetilde{\mathrm{III}}_o^{0,5}(f)$		2		2					1			
$\widetilde{\mathrm{III}}_e^{0,5}(f)$									1			
$\widetilde{\mathrm{III}}_o^{0,6}(f)$		2		1						1		
$\widetilde{\mathrm{III}}_e^{0,6}(f)$				1						1		
$\widetilde{\mathrm{III}}_o^{0,7}(f)$				4							1	
$\widetilde{\mathrm{III}}_e^{0,7}(f)$											1	
$\widetilde{\mathrm{III}}_o^{1,3}(f)$			2				1					
$\widetilde{\mathrm{III}}_e^{1,3}(f)$			2				1					
$\widetilde{\mathrm{III}}_o^{1,4}(f)$			4					1				
$\widetilde{\mathrm{III}}_e^{1,4}(f)$								1				
$\widetilde{\mathrm{III}}_o^{1,5}(f)$			2		2				1			
$\widetilde{\mathrm{III}}_e^{1,5}(f)$									1			
$\widetilde{\mathrm{III}}_o^{1,6}(f)$			2		1					1		
$\widetilde{\mathrm{III}}_e^{1,6}(f)$					1					1		
$\widetilde{\mathrm{III}}_o^{1,7}(f)$					4						1	
$\widetilde{\mathrm{III}}_e^{1,7}(f)$											1	
$\widetilde{\mathrm{III}}_o^{2,3}(f)$					2		2					
$\widetilde{\mathrm{III}}_e^{2,3}(f)$					2							

TABLE 1.1. The degree of each vertex in the graphs $\overline{\mathrm{II}_o^*(f)}$

	$*=0,0$	$0,1$	$1,1$	$0,2$	$1,2$	$2,2$	3	4	5	6	7	a
$\widetilde{\mathrm{III}}^{2,4}_o(f)$					4			2				
$\widetilde{\mathrm{III}}^{2,4}_e(f)$												
$\widetilde{\mathrm{III}}^{2,5}_o(f)$					2	2			2			
$\widetilde{\mathrm{III}}^{2,5}_e(f)$												
$\widetilde{\mathrm{III}}^{2,6}_o(f)$					2	1				2		
$\widetilde{\mathrm{III}}^{2,6}_e(f)$						1						
$\widetilde{\mathrm{III}}^{2,7}_o(f)$						4					2	
$\widetilde{\mathrm{III}}^{2,7}_e(f)$												
$\widetilde{\mathrm{III}}^{0,a}_o(f)$		1										1
$\widetilde{\mathrm{III}}^{0,a}_e(f)$	1											1
$\widetilde{\mathrm{III}}^{1,a}_o(f)$			1									1
$\widetilde{\mathrm{III}}^{1,a}_e(f)$		1										1
$\widetilde{\mathrm{III}}^{2,a}_o(f)$					1							2
$\widetilde{\mathrm{III}}^{2,a}_e(f)$				1								
$\widetilde{\mathrm{III}}^{8}_o(f)$							3					
$\widetilde{\mathrm{III}}^{8}_e(f)$			1				2					
$\widetilde{\mathrm{III}}^{9}_o(f)$							3					
$\widetilde{\mathrm{III}}^{9}_e(f)$							3					
$\widetilde{\mathrm{III}}^{10}_o(f)$							4	1				
$\widetilde{\mathrm{III}}^{10}_e(f)$								1				
$\widetilde{\mathrm{III}}^{11}_o(f)$							3	3				
$\widetilde{\mathrm{III}}^{11}_e(f)$												
$\widetilde{\mathrm{III}}^{12}_o(f)$								6				
$\widetilde{\mathrm{III}}^{12}_e(f)$												
$\widetilde{\mathrm{III}}^{13}_o(f)$								1	4		1	
$\widetilde{\mathrm{III}}^{13}_e(f)$												
$\widetilde{\mathrm{III}}^{14}_o(f)$							2			2		
$\widetilde{\mathrm{III}}^{14}_e(f)$					1					1		
$\widetilde{\mathrm{III}}^{15}_o(f)$							2		1	2		
$\widetilde{\mathrm{III}}^{15}_e(f)$									1			
$\widetilde{\mathrm{III}}^{16}_o(f)$									2	2	2	
$\widetilde{\mathrm{III}}^{16}_e(f)$												
$\widetilde{\mathrm{III}}^{17}_o(f)$							3		3			
$\widetilde{\mathrm{III}}^{17}_e(f)$												
$\widetilde{\mathrm{III}}^{18}_o(f)$										4	1	
$\widetilde{\mathrm{III}}^{18}_e(f)$											1	
$\widetilde{\mathrm{III}}^{19}_o(f)$										4	1	
$\widetilde{\mathrm{III}}^{19}_e(f)$											1	
$\widetilde{\mathrm{III}}^{20}_o(f)$										4	1	
$\widetilde{\mathrm{III}}^{20}_e(f)$						1						
$\widetilde{\mathrm{III}}^{21}_o(f)$								1	3	2		
$\widetilde{\mathrm{III}}^{21}_e(f)$												

TABLE 1.2. The degree of each vertex in the graphs $\overline{\mathrm{II}^*_o(f)}$

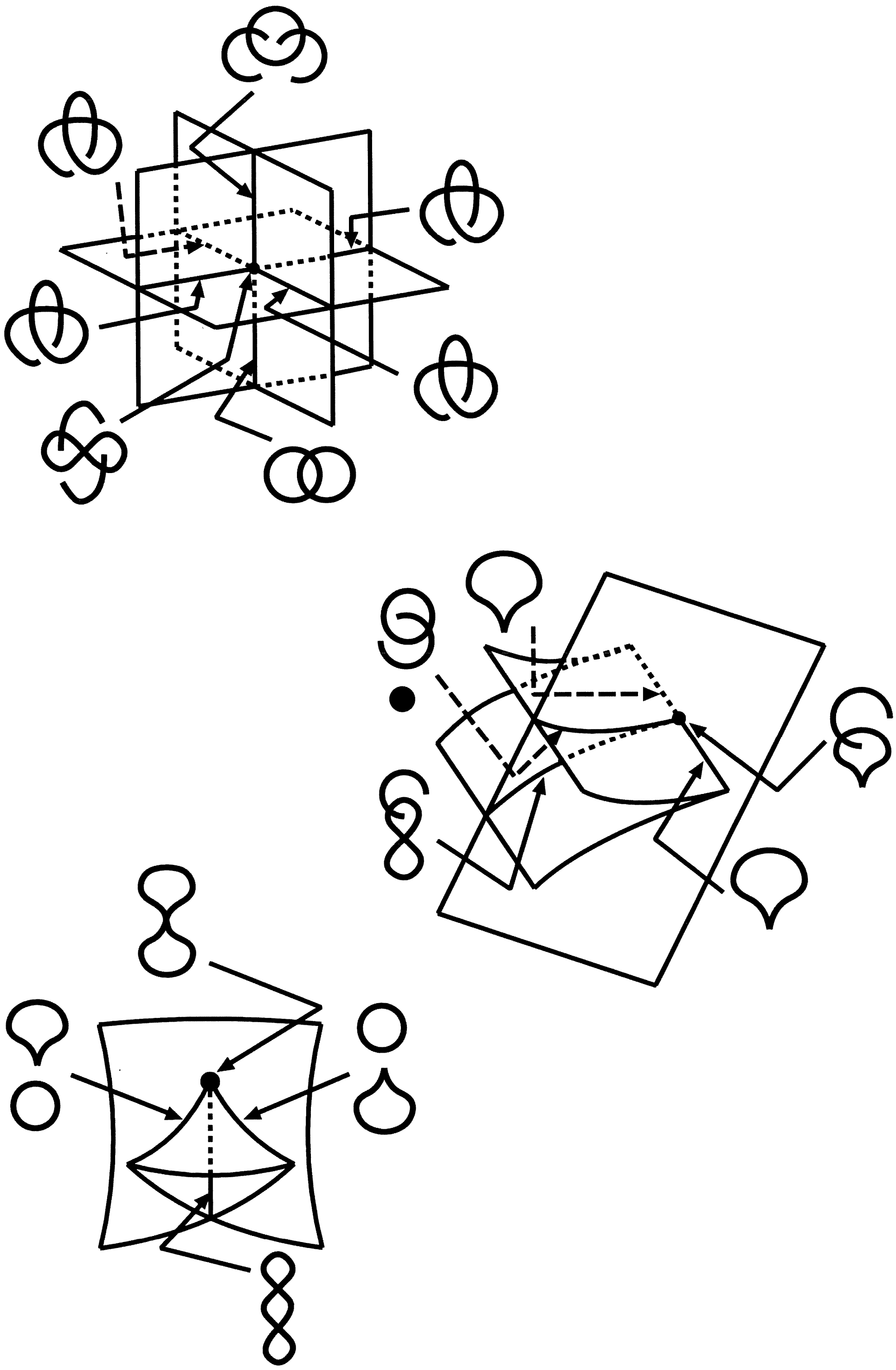

FIGURE 1.7. Descriptions of local nearby fibers of types $\widetilde{\mathrm{III}}^{13}$, $\widetilde{\mathrm{III}}^{c}$ and $\widetilde{\mathrm{III}}^{f}$

	$*=0,0$	$0,1$	$1,1$	$0,2$	$1,2$	$2,2$	3	4	5	6	7	a
$\widetilde{\mathrm{III}}^{22}_o(f)$								2		4		
$\widetilde{\mathrm{III}}^{22}_e(f)$												
$\widetilde{\mathrm{III}}^{23}_o(f)$							2			2		
$\widetilde{\mathrm{III}}^{23}_e(f)$										2		
$\widetilde{\mathrm{III}}^{24}_o(f)$							2			2		
$\widetilde{\mathrm{III}}^{24}_e(f)$										2		
$\widetilde{\mathrm{III}}^{25}_o(f)$											6	
$\widetilde{\mathrm{III}}^{25}_e(f)$												
$\widetilde{\mathrm{III}}^{26}_o(f)$											6	
$\widetilde{\mathrm{III}}^{26}_e(f)$												
$\widetilde{\mathrm{III}}^{b}_o(f)$							1					1
$\widetilde{\mathrm{III}}^{b}_e(f)$		1										1
$\widetilde{\mathrm{III}}^{c}_o(f)$										1		2
$\widetilde{\mathrm{III}}^{c}_e(f)$				1								
$\widetilde{\mathrm{III}}^{d}_o(f)$												2
$\widetilde{\mathrm{III}}^{d}_e(f)$	1											
$\widetilde{\mathrm{III}}^{e}_o(f)$								1				2
$\widetilde{\mathrm{III}}^{e}_e(f)$												
$\widetilde{\mathrm{III}}^{f}_o(f)$							1					
$\widetilde{\mathrm{III}}^{f}_e(f)$												2
$\widetilde{\mathrm{III}}^{g}_o(f)$									1			2
$\widetilde{\mathrm{III}}^{g}_e(f)$												

TABLE 1.3. The degree of each vertex in the graphs $\overline{\mathrm{II}^*_o(f)}$

By combining the above 24 co-existence relations in a certain manner, we obtain the following co-existence formula:

$$
\begin{aligned}
&|\widetilde{\mathrm{III}}^{0,0,0}(f)| + |\widetilde{\mathrm{III}}^{0,0,1}(f)| + |\widetilde{\mathrm{III}}^{0,1,1}(f)| + |\widetilde{\mathrm{III}}^{1,1,1}(f)| + |\widetilde{\mathrm{III}}^{0,2,2}(f)| + |\widetilde{\mathrm{III}}^{1,2,2}(f)| \\
&+ |\widetilde{\mathrm{III}}^{0,3}(f)| + |\widetilde{\mathrm{III}}^{0,4}(f)| + |\widetilde{\mathrm{III}}^{0,5}(f)| + |\widetilde{\mathrm{III}}^{0,7}(f)| + |\widetilde{\mathrm{III}}^{1,3}(f)| + |\widetilde{\mathrm{III}}^{1,4}(f)| \\
&+ |\widetilde{\mathrm{III}}^{1,5}(f)| + |\widetilde{\mathrm{III}}^{1,7}(f)| + |\widetilde{\mathrm{III}}^{2,6}(f)| + |\widetilde{\mathrm{III}}^{8}(f)| + |\widetilde{\mathrm{III}}^{9}(f)| + |\widetilde{\mathrm{III}}^{10}(f)| \\
&+ |\widetilde{\mathrm{III}}^{11}(f)| + |\widetilde{\mathrm{III}}^{13}(f)| + |\widetilde{\mathrm{III}}^{15}(f)| + |\widetilde{\mathrm{III}}^{18}(f)| + |\widetilde{\mathrm{III}}^{19}(f)| + |\widetilde{\mathrm{III}}^{20}(f)| \\
&+ |\widetilde{\mathrm{III}}^{21}(f)| + |\widetilde{\mathrm{III}}^{d}_e(f)| + |\widetilde{\mathrm{III}}^{e}_e(f)| + |\widetilde{\mathrm{III}}^{f}_o(f)| + |\widetilde{\mathrm{III}}^{g}_o(f)| \equiv 0 \pmod 2. \qquad (*)
\end{aligned}
$$

This formula $(*)$ will be used in the following section.

4. Parity of the Euler characteristic

In this section we study the relationship between the total number of certain singular fibers and the Euler number of the source 4-manifold, based on the co-existence results among singular fibers obtained in the previous section.

Let $f : M \to N$ be a stable map of a closed 4-manifold M into a 3-manifold N. Recall the stratification of N by the C^∞ equivalence classes of singular fibers obtained in the previous section. We further subdivide it in the following way. For $n = 0, 1, 2, \ldots$, we define $\mathcal{O}_n(f)$ to be the set of points in $N \setminus f(S(f))$ such that the number of connected components of the associated fiber is equal to n. Then we obtain another stratification of N, which consists of the components of $\mathcal{F}(f)$

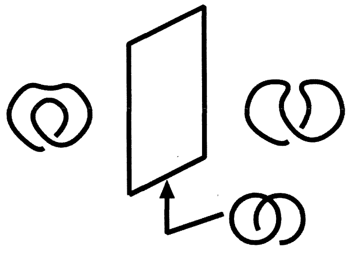

FIGURE 1.8. Deformation of the I^2 fiber

together with $\mathcal{O}_n(f)$, where $\mathcal{F}$ runs over all C^∞ equivalence classes of singular fibers appearing in Theorem 2.4 and $n = 0, 1, 2, \ldots$. Thus the set of regular values $N \setminus f(S(f))$ consists of 3-dimensional strata $\mathcal{O}_n(f)$ $(n = 0, 1, 2, \ldots)$, while the set of singular values $f(S(f))$ consists of 2-, 1- and 0-dimensional strata. Throughout this section, we consider this subdivided stratification and not that obtained in § 3. Then we assign to each 3-dimensional stratum $\mathcal{O}_n(f)$ the number n.

Let X be a closed subset of a manifold Y. We say that $Y \setminus X$ has a *two colour decomposition* if there exist non-empty disjoint open subsets R and B of Y such that $Y \setminus X = R \cup B$ and $\overline{R} \cap \overline{B} = \partial R = \partial B = X$. We call the pair (R, B) a *two colour decomposition* (or a *colouring*) for $Y \setminus X$. In the following, we study the condition under which $N \setminus f(S(f))$ has a colouring (R, B) for a stable map $f : M \to N$ as above.

Let $f : M \to N$ be as above, and we define $\triangle_o(f)$ (resp. $\triangle_e(f)$) as the set of points in $N \setminus f(S(f))$ such that the number of connected components of the associated fiber is odd (resp. even). Then $\triangle_o(f)$ and $\triangle_e(f)$ are unions of 3-dimensional strata of the above stratification. It is easy to see that they are disjoint open subsets of N. If M is orientable, then we have $\overline{\triangle_o(f)} \cap \overline{\triangle_e(f)} = \partial\triangle_o(f) = \partial\triangle_e(f) = f(S(f))$, since the difference between the numbers of connected components of the fibers associated with the two 3-dimensional strata adjacent to each 2-dimensional stratum is always equal to one. Therefore, if M is orientable, then the pair $(\triangle_o(f), \triangle_e(f))$ is a colouring for $N \setminus f(S(f))$. If $S \subset f(S(f))$ is a 2-dimensional stratum whose corresponding fiber is of I^2 type, then the numbers of connected components of the fibers associated with the two 3-dimensional strata adjacent to S are the same (see Figure 1.8). Hence, if f has a fiber of I^2 type, then we have $\overline{\triangle_o(f)} \cap \overline{\triangle_e(f)} \neq f(S(f))$. Thus if M is non-orientable, then the pair $(\triangle_o(f), \triangle_e(f))$ may not be a two colour decomposition for $N \setminus f(S(f))$.

The following lemma which claims that $N \setminus f(S(f))$ has a two colour decomposition under certain homological conditions is well known (see [**33**]).

LEMMA 4.1. *Let $g : X \to Y$ be a stable map of a closed surface X into a connected 3-manifold Y such that either $H_1(Y; \mathbb{Z}_2) = 0$ or $g_*[X] = 0 \in H_2(Y; \mathbb{Z}_2)$, where $[X] \in H_2(X; \mathbb{Z}_2)$ is the fundamental class of X. Then $Y \setminus g(X)$ has a two colour decomposition (R, B).*

We call the assumption that $g : X \to Y$ satisfies either $H_1(Y; \mathbb{Z}_2) = 0$ or $g_*[X] = 0 \in H_2(Y; \mathbb{Z}_2)$, the *two colorable condition*. We note that this is a sufficient condition for the existence of a two colour decomposition and that this is not a necessary condition in general.

Remark 4.2. There is a characterization of stable maps of closed surfaces into 3-manifolds similar to Proposition 2.2. We note that a stable map g of a closed surface into a 3-manifold has only Whitney umbrella points as its singular points

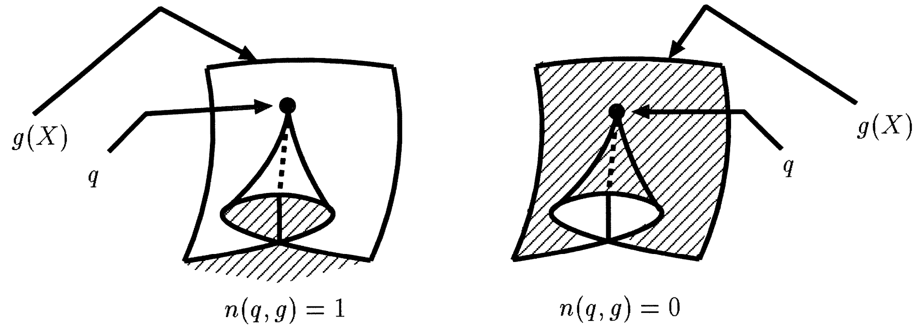

FIGURE 1.9. Index of a Whitney umbrella point q

and g is an immersion with normal crossings outside of the Whitney umbrella points.

We note that if $f : M \to N$ is a stable map of a closed 4-manifold into a 3-manifold, then the singular point set $S(f)$ is a smooth submanifold of M of codimension two and the map $f|_{S(f)} : S(f) \to N$ is a topologically stable singular surface. Here, a smooth map $g : X \to Y$ of a closed surface into a 3-manifold is a *topologically stable singular surface* if there exist a C^∞ stable map $\tilde{g} : \tilde{X} \to \tilde{Y}$ of a closed surface into a 3-manifold, and homeomorphisms $\phi : X \to \tilde{X}$ and $\psi : Y \to \tilde{Y}$ such that $\psi \circ g = \tilde{g} \circ \phi$. We note that Lemma 4.1 can be applied to topologically stable singular surfaces as well. Thus we can apply this lemma to $f|_{S(f)}$. For a stable map $f : M \to N$ as above, if we assume that the map $f|_{S(f)} : S(f) \to N$ satisfies the two colorable condition, then there exists a two colour decomposition (R, B) for $N \setminus f(S(f))$.

The following theorem relates the number of triple points of g in the target manifold and the topology of the source manifold.

THEOREM 4.3 (A. Szűcs [**48**], J. J. Nuño Ballesteros-O. Saeki [**33**]). *Let $g : X \to Y$ be a stable map of a closed surface X into a connected 3-manifold Y. Suppose g satisfies the two colorable condition. Then we have*

$$T(g) + \sum_{q:\text{Whitney umbrella}} n(q, g) \equiv \chi(X) \pmod 2,$$

where $T(g)$ is the total number of triple points of g in the target, and $n(q, g) \in \{0, 1\}$ is the index of a Whitney umbrella point q defined by using the two colour decomposition (R, B) for $Y \setminus g(X)$ as in Figure 1.9 in which the shadowed regions indicate R.

Each swallow-tail point of a stable map f of a 4-manifold into a 3-manifold corresponds to a Whitney umbrella point of the topologically stable singular surface $f|_{S(f)}$. We note that Theorem 4.3 can also be applied to topologically stable singular surfaces.

Let us combine the co-existence relation of singular fibers $(*)$ obtained in §3 and the relation between the number of singular fibers of f and the number of triple points of $f|_{S(f)}$ obtained in Remark 2.6. Then we obtain

$$\begin{aligned} T(f|_{S(f)}) \equiv\ & |\widetilde{\mathrm{III}}^{0,0,2}(f)| + |\widetilde{\mathrm{III}}^{1,1,2}(f)| + |\widetilde{\mathrm{III}}^{0,1,2}(f)| + |\widetilde{\mathrm{III}}^{2,2,2}(f)| + |\widetilde{\mathrm{III}}^{0,6}(f)| \\ & + |\widetilde{\mathrm{III}}^{1,6}(f)| + |\widetilde{\mathrm{III}}^{2,3}(f)| + |\widetilde{\mathrm{III}}^{2,4}(f)| + |\widetilde{\mathrm{III}}^{2,5}(f)| + |\widetilde{\mathrm{III}}^{2,7}(f)| \\ & + |\widetilde{\mathrm{III}}^{12}(f)| + |\widetilde{\mathrm{III}}^{14}(f)| + |\widetilde{\mathrm{III}}^{16}(f)| + |\widetilde{\mathrm{III}}^{17}(f)| + |\widetilde{\mathrm{III}}^{22}(f)| \\ & + |\widetilde{\mathrm{III}}^{23}(f)| + |\widetilde{\mathrm{III}}^{24}(f)| + |\widetilde{\mathrm{III}}^{25}(f)| + |\widetilde{\mathrm{III}}^{26}(f)| \pmod 2. \end{aligned}$$

If $f : M \to \mathbb{R}^3$ is a stable map of a closed 4-manifold M into $\mathbb{R}^3$ which has no swallow-tail points, then we have

$$T(f|_{S(f)}) \equiv \chi(S(f)) \pmod 2$$

by Theorem 4.3 or by the following result of Banchoff [**3**]: *For any self-transverse immersion $f : X \to \mathbb{R}^3$ of a closed surface X into $\mathbb{R}^3$, the number of triple points of f in $\mathbb{R}^3$ and the Euler number of the surface X have the same parity.* We note that if f has no swallow-tail points, then singular fibers of types $\widetilde{\mathrm{III}}^d$, $\widetilde{\mathrm{III}}^e$, $\widetilde{\mathrm{III}}^f$ and $\widetilde{\mathrm{III}}^g$ never appear.

Furthermore, we have the following.

THEOREM 4.4 (T. Fukuda [**11**], O. Saeki [**41**]). *Let $h : V \to N$ be a stable map of a closed n-dimensional manifold V $(n \geq 3)$ into a 3-manifold N. Then we have*

$$\chi(V) \equiv \chi(S(h)) \pmod 2.$$

Thus we obtain the following proposition.

PROPOSITION 4.5. *Let $f : M \to \mathbb{R}^3$ be a stable map of a closed 4-manifold into $\mathbb{R}^3$ which has no swallow-tail points. Then we have*

$$\begin{aligned} \chi(M) \equiv\ & |\widetilde{\mathrm{III}}^{0,0,2}(f)| + |\widetilde{\mathrm{III}}^{1,1,2}(f)| + |\widetilde{\mathrm{III}}^{0,1,2}(f)| + |\widetilde{\mathrm{III}}^{2,2,2}(f)| + |\widetilde{\mathrm{III}}^{0,6}(f)| \\ & + |\widetilde{\mathrm{III}}^{1,6}(f)| + |\widetilde{\mathrm{III}}^{2,3}(f)| + |\widetilde{\mathrm{III}}^{2,4}(f)| + |\widetilde{\mathrm{III}}^{2,5}(f)| + |\widetilde{\mathrm{III}}^{2,7}(f)| \\ & + |\widetilde{\mathrm{III}}^{12}(f)| + |\widetilde{\mathrm{III}}^{14}(f)| + |\widetilde{\mathrm{III}}^{16}(f)| + |\widetilde{\mathrm{III}}^{17}(f)| + |\widetilde{\mathrm{III}}^{22}(f)| \\ & + |\widetilde{\mathrm{III}}^{23}(f)| + |\widetilde{\mathrm{III}}^{24}(f)| + |\widetilde{\mathrm{III}}^{25}(f)| + |\widetilde{\mathrm{III}}^{26}(f)| \pmod 2. \end{aligned}$$

In the following, we consider a stable map $f : M \to N$ of a closed 4-manifold M into a connected 3-manifold N such that $f|_{S(f)}$ satisfies the two colorable condition. Then, based on a fixed colouring (R, B) for $N \setminus f(S(f))$, we divide several C^∞ equivalence classes of singular fibers into two types, A and B, as follows.

First, for any C^∞ equivalence class $\mathcal{E}$ of singular fibers of codimension one, the 2-dimensional stratum $\mathcal{E}(f)$ is locally adjacent to two 3-dimensional strata. If $\mathcal{E} = \widetilde{\mathrm{I}}^0$ or $\widetilde{\mathrm{I}}^1$, then the difference between the numbers of connected components of the fibers associated with the two 3-dimensional strata adjacent to $\mathcal{E}(f)$ is always equal to one. Let us take a point $y \in \mathcal{E}(f)$ with $\mathcal{E} = \widetilde{\mathrm{I}}^0$ or $\widetilde{\mathrm{I}}^1$. If the 3-dimensional stratum adjacent to y which has a larger associated number is contained in R, then we say that the fiber over y is of type $\mathcal{E}_A$, otherwise $\mathcal{E}_B$. In this way, we can divide the stratum $\mathcal{E}(f)$ into $\mathcal{E}_A(f)$ and $\mathcal{E}_B(f)$. If $\mathcal{E} = \mathrm{I}^2$, then the difference between the numbers of connected components of the fibers associated with the two 3-dimensional strata adjacent to $\widetilde{\mathrm{I}}^2(f)$ is always equal to zero, as shown in Figure 1.8. Thus we cannot divide singular fibers of I^2 type into two types by this method.

For any C^∞ equivalence class $\mathcal{F}$ of singular fibers of codimension two, except for $\widetilde{\mathrm{II}}^a$, $\mathcal{F}(f)$ is locally adjacent to four 3-dimensional strata. Now we combine the numbers of connected components of the fibers and the "colors" of these 3-dimensional strata. We divide $\mathcal{F}(f)$ into two types A and B in the following way.

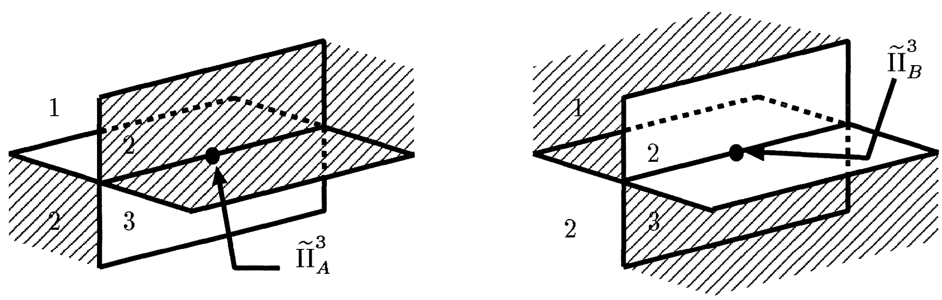

FIGURE 1.10. Types A and B for $\widetilde{\mathrm{II}}^3$

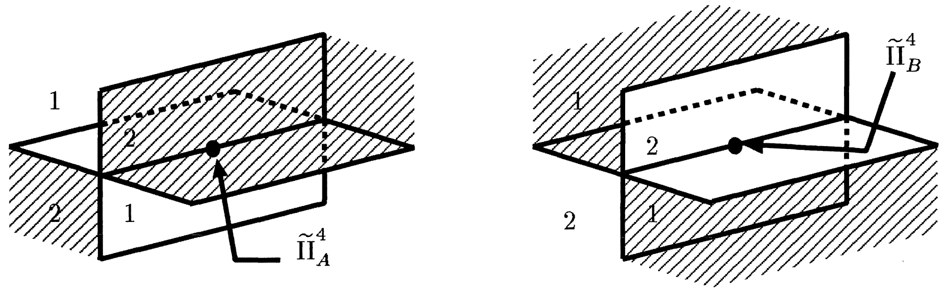

FIGURE 1.11. Types A and B for $\widetilde{\mathrm{II}}^4$

Let us take a point $y \in \mathcal{F}(f)$ with $\mathcal{F} = \widetilde{\mathrm{II}}^{0,0}, \widetilde{\mathrm{II}}^{0,1}, \widetilde{\mathrm{II}}^{1,1}, \widetilde{\mathrm{II}}^{3}$ or $\widetilde{\mathrm{II}}^{5}$. If the two 3-dimensional strata adjacent to y which are contained in R have the same associated number then we say that the fiber over y is of type $\mathcal{F}_A$, otherwise $\mathcal{F}_B$ (see Figure 1.10). Let us take a point $y \in \mathcal{F}(f)$ with $\mathcal{F} = \widetilde{\mathrm{II}}^{4}$. If the two 3-dimensional strata adjacent to y which have a larger associated number are contained in R, then we say that the fiber over y is of type $\mathcal{F}_A$, otherwise $\mathcal{F}_B$ (see Figure 1.11). In a way similar to that for I^0 and I^1, singular fibers of $\widetilde{\mathrm{II}}^{a}$ type are divided into two types A and B. In this way, we can divide the stratum $\mathcal{F}(f)$ into $\mathcal{F}_A(f)$ and $\mathcal{F}_B(f)$. However, if $\mathcal{F} = \widetilde{\mathrm{II}}^{0,2}, \widetilde{\mathrm{II}}^{1,2}, \widetilde{\mathrm{II}}^{2,2}, \widetilde{\mathrm{II}}^{6}$ and $\widetilde{\mathrm{II}}^{7}$, then we cannot divide into two types by these method (see Figure 1.12).

In Figures 1.10 and 1.12 the numbers attached to 3-dimensional strata are the numbers of connected components of the associated fibers when the singular value at the center has no regular component in its fiber. Furthermore, the shadowed regions indicate R.

For the equivalence classes of codimension three, we see that $\widetilde{\mathrm{III}}^{0,0,0}(f)$, $\widetilde{\mathrm{III}}^{0,0,1}(f)$, $\widetilde{\mathrm{III}}^{0,1,1}(f)$, $\widetilde{\mathrm{III}}^{1,1,1}(f)$, $\widetilde{\mathrm{III}}^{0,3}(f)$, $\widetilde{\mathrm{III}}^{0,4}(f)$, $\widetilde{\mathrm{III}}^{0,5}(f)$, $\widetilde{\mathrm{III}}^{1,3}(f)$, $\widetilde{\mathrm{III}}^{1,4}(f)$, $\widetilde{\mathrm{III}}^{1,5}(f)$, $\widetilde{\mathrm{III}}^{0,a}(f)$, $\widetilde{\mathrm{III}}^{1,a}(f)$, $\widetilde{\mathrm{III}}^{8}(f)$, $\widetilde{\mathrm{III}}^{9}(f)$, $\widetilde{\mathrm{III}}^{10}(f)$, $\widetilde{\mathrm{III}}^{11}(f)$, $\widetilde{\mathrm{III}}^{12}(f)$, $\widetilde{\mathrm{III}}^{13}(f)$, $\widetilde{\mathrm{III}}^{15}(f)$, $\widetilde{\mathrm{III}}^{17}(f)$, $\widetilde{\mathrm{III}}^{21}(f)$, $\widetilde{\mathrm{III}}^{b}(f)$, $\widetilde{\mathrm{III}}^{d}(f)$, $\widetilde{\mathrm{III}}^{e}(f)$, $\widetilde{\mathrm{III}}^{f}(f)$ and $\widetilde{\mathrm{III}}^{g}(f)$ can be divided into two types A and B, in a way similar to that for the codimension 2 case (see Figures 1.13, 1.14, 1.15, 1.16, 1.20, 1.21, 1.22, 1.24, 1.25, 1.26, 1.31, 1.32, 1.33, 1.34, 1.35, 1.36, 1.37, 1.38, 1.40, 1.41, 1.44, 1.47, 1.49, 1.50, 1.51 and 1.52). In Figures 1.13, 1.14 and 1.52 the numbers attached to 3-dimensional strata are chosen in the same

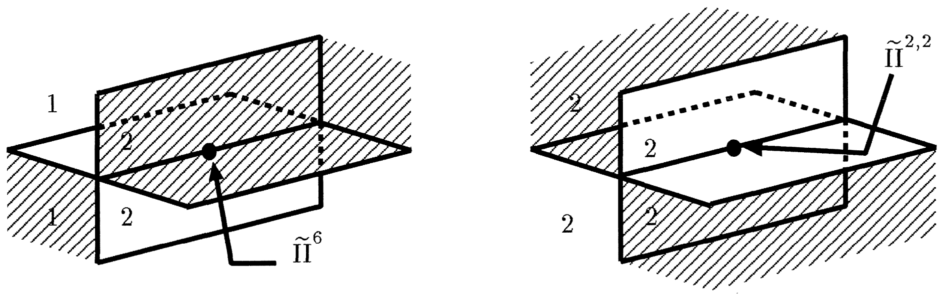

FIGURE 1.12. Codimension 2 strata which cannot be divided into two types

way as in Figures 1.10 and 1.12 and the shadowed regions indicate R. Singular fibers of codimension three of the other types cannot be divided into two types A and B.

Let $f : M \to N$ be as above and suppose that $f|_{S(f)}$ satisfies the two colorable condition. Let $\mathcal{F}$ be the C^∞ equivalence class of one of the singular fibers of codimension two appearing in Figure 1.4. If $\mathcal{F}$ can be divided into two types A and B, then we define $\mathcal{F}_A(f)$ (or $\mathcal{F}_B(f)$) to be the set of points $y \in \mathcal{F}(f)$ such that the fiber over y is of type A (resp. type B). If $\mathcal{F}$ cannot be divided into two types, then $\mathcal{F}_A(f)$ or $\mathcal{F}_B(f)$ is not defined and we just consider $\mathcal{F}(f)$. Note that then $\overline{\mathcal{F}_A(f)}$, $\overline{\mathcal{F}_B(f)}$ and $\overline{\mathcal{F}(f)}$ are finite graphs embedded in N. Their vertices correspond to points over which f has a singular fiber of codimension three.

We again apply the handshake lemma to the graphs $\overline{\widetilde{\mathrm{III}}_A^{0,0}(f)}$, $\overline{\widetilde{\mathrm{III}}_B^{0,0}(f)}$, $\overline{\widetilde{\mathrm{III}}_A^{0,1}(f)}$, $\overline{\widetilde{\mathrm{III}}_B^{0,1}(f)}$, $\overline{\widetilde{\mathrm{III}}_A^{1,1}(f)}$, $\overline{\widetilde{\mathrm{III}}_B^{1,1}(f)}$, $\overline{\widetilde{\mathrm{III}}^{0,2}(f)}$, $\overline{\widetilde{\mathrm{III}}^{1,2}(f)}$, $\overline{\widetilde{\mathrm{III}}^{2,2}(f)}$, $\overline{\widetilde{\mathrm{III}}_A^{3}(f)}$, $\overline{\widetilde{\mathrm{III}}_B^{3}(f)}$, $\overline{\widetilde{\mathrm{III}}_A^{4}(f)}$, $\overline{\widetilde{\mathrm{III}}_B^{4}(f)}$, $\overline{\widetilde{\mathrm{III}}_A^{5}(f)}$, $\overline{\widetilde{\mathrm{III}}_B^{5}(f)}$, $\overline{\widetilde{\mathrm{III}}^{6}(f)}$, $\overline{\widetilde{\mathrm{III}}^{7}(f)}$, $\overline{\widetilde{\mathrm{III}}_A^{a}(f)}$ and $\overline{\widetilde{\mathrm{III}}_B^{a}(f)}$. Then we obtain 19 co-existence relations among the numbers of singular fibers of codimension three. We note that the relation obtained from the graph $\overline{\widetilde{\mathrm{II}}_B^{*}(f)}$ can also be obtained from the graph $\overline{\widetilde{\mathrm{II}}_A^{*}(f)}$, by interchanging $\widetilde{\mathrm{III}}_A^{*}(f)$ and $\widetilde{\mathrm{III}}_B^{*}(f)$. In the following Proposition 4.6, we give the relations obtained from the graph $\overline{\widetilde{\mathrm{II}}_A^{*}(f)}$ and $\overline{\widetilde{\mathrm{II}}^{*}(f)}$.

PROPOSITION 4.6. *Let $f : M \to N$ be a stable map of a closed 4-manifold M into a connected 3-manifold N. Suppose that $f|_{S(f)}$ satisfies the two colorable condition. Then the following numbers are always even:*

(1) $|\widetilde{\mathrm{III}}^{0,0,0}(f)| + |\widetilde{\mathrm{III}}^{0,0,1}(f)| + |\widetilde{\mathrm{III}}^{0,0,2}(f)| + |\widetilde{\mathrm{III}}_A^{0,a}(f)| + |\widetilde{\mathrm{III}}_A^{d}(f)|$,
(2) $|\widetilde{\mathrm{III}}^{0,1,2}(f)| + |\widetilde{\mathrm{III}}^{0,6}(f)| + |\widetilde{\mathrm{III}}_A^{0,a}(f)| + |\widetilde{\mathrm{III}}_A^{1,a}(f)| + |\widetilde{\mathrm{III}}_A^{b}(f)|$,
(3) $|\widetilde{\mathrm{III}}^{0,1,1}(f)| + |\widetilde{\mathrm{III}}^{1,1,1}(f)| + |\widetilde{\mathrm{III}}^{1,1,2}(f)| + |\widetilde{\mathrm{III}}^{1,6}(f)| + |\widetilde{\mathrm{III}}_A^{1,a}(f)| + |\widetilde{\mathrm{III}}_B^{8}(f)|$,
(4) $|\widetilde{\mathrm{III}}^{2,a}(f)| + |\widetilde{\mathrm{III}}^{c}(f)|$,
(5) $|\widetilde{\mathrm{III}}^{2,a}(f)| + |\widetilde{\mathrm{III}}^{14}(f)|$,
(6) $|\widetilde{\mathrm{III}}^{20}(f)|$,
(7) $|\widetilde{\mathrm{III}}^{0,3}(f)| + |\widetilde{\mathrm{III}}^{1,3}(f)| + |\widetilde{\mathrm{III}}^{2,3}(f)| + |\widetilde{\mathrm{III}}_A^{8}(f)| + |\widetilde{\mathrm{III}}^{9}(f)| + |\widetilde{\mathrm{III}}_A^{11}(f)|$
$+ |\widetilde{\mathrm{III}}^{14}(f)| + |\widetilde{\mathrm{III}}_A^{17}(f)| + |\widetilde{\mathrm{III}}^{23}(f)| + |\widetilde{\mathrm{III}}^{24}(f)| + |\widetilde{\mathrm{III}}_A^{b}(f)| + |\widetilde{\mathrm{III}}_A^{f}(f)|$,
(8) $|\widetilde{\mathrm{III}}^{0,4}(f)| + |\widetilde{\mathrm{III}}^{1,4}(f)| + |\widetilde{\mathrm{III}}^{2,4}(f)| + |\widetilde{\mathrm{III}}^{10}(f)| + |\widetilde{\mathrm{III}}_A^{11}(f)| + |\widetilde{\mathrm{III}}_B^{13}(f)|$
$+ |\widetilde{\mathrm{III}}_A^{21}(f)| + |\widetilde{\mathrm{III}}^{22}(f)| + |\widetilde{\mathrm{III}}_B^{e}(f)|$,

(9) $|\widetilde{\mathrm{III}}^{0,5}(f)| + |\widetilde{\mathrm{III}}^{1,5}(f)| + |\widetilde{\mathrm{III}}^{2,5}(f)| + |\widetilde{\mathrm{III}}^{15}(f)| + |\widetilde{\mathrm{III}}^{16}(f)| + |\widetilde{\mathrm{III}}^{17}_A(f)|$
$+ |\widetilde{\mathrm{III}}^{21}_B(f)| + |\widetilde{\mathrm{III}}^{g}_B(f)|$,

(10) $|\widetilde{\mathrm{III}}^{14}(f)| + |\widetilde{\mathrm{III}}^{c}(f)|$,

(11) $|\widetilde{\mathrm{III}}^{13}(f)| + |\widetilde{\mathrm{III}}^{20}(f)|$,

(12) $|\widetilde{\mathrm{III}}^{0,a}(f)| + |\widetilde{\mathrm{III}}^{1,a}(f)| + |\widetilde{\mathrm{III}}^{2,a}(f)| + |\widetilde{\mathrm{III}}^{b}(f)| + |\widetilde{\mathrm{III}}^{c}(f)|$.

Items (1)–(11) of the above proposition correspond to the graphs $\overline{\widetilde{\mathrm{II}}^{0,0}_A(f)}$, $\overline{\widetilde{\mathrm{II}}^{0,1}_A(f)}$, $\overline{\widetilde{\mathrm{II}}^{1,1}_A(f)}$, $\overline{\widetilde{\mathrm{II}}^{0,2}(f)}$, $\overline{\widetilde{\mathrm{II}}^{1,2}(f)}$, $\overline{\widetilde{\mathrm{II}}^{2,2}(f)}$, $\overline{\widetilde{\mathrm{II}}^{3}_A(f)}$, $\overline{\widetilde{\mathrm{II}}^{4}_A(f)}$, $\overline{\widetilde{\mathrm{II}}^{5}_A(f)}$, $\overline{\widetilde{\mathrm{II}}^{6}(f)}$ and $\overline{\widetilde{\mathrm{II}}^{7}(f)}$ respectively. Item (12) corresponds to both $\overline{\widetilde{\mathrm{II}}^{a}_A(f)}$ and $\overline{\widetilde{\mathrm{II}}^{a}_B(f)}$. If we combine the relations obtained from $\overline{\widetilde{\mathrm{II}}^{*}_A(f)}$ and that obtained from $\overline{\widetilde{\mathrm{II}}^{*}_B(f)}$ and we arrange these relations a certain manner, then we obtain exactly the same co-existence relations as those in Proposition 3.1. We omit the table of the degrees of the vertices in these graphs.

For a stable map $f : M \to N$ of a closed 4-manifold into a 3-manifold, the symbol $\widetilde{\mathrm{III}}^{*}_o(f)$ (resp. $\widetilde{\mathrm{III}}^{*}_e(f)$) denotes the set of points $y \in N$ such that the fiber $f^{-1}(y)$ over y is C^∞ equivalent to the union of the $\widetilde{\mathrm{III}}^{*}$ type fiber and some copies of a fiber of the trivial circle bundle and that the total number of connected components of $f^{-1}(y)$ is odd (resp. even). Furthermore, the symbol $\widetilde{\mathrm{III}}^{*}_A(f)$ (resp. $\widetilde{\mathrm{III}}^{*}_B(f)$) denotes the set of points $y \in N$ such that the fiber $f^{-1}(y)$ over y is C^∞ equivalent to the union of the $\widetilde{\mathrm{III}}^{*}$ type fiber and some copies of a fiber of the trivial circle bundle and that the colouring of a neighbourhood of $y \in N$ is of A type (resp. B type).

Let us now state and prove the main theorem of this paper.

THEOREM 4.7. *Let $f : M \to N$ be a stable map of a closed 4-manifold M into a connected 3-manifold N such that $H_1(N;\mathbb{Z}_2) = 0$ or $f_*[S(f)] = 0 \in H_2(N;\mathbb{Z}_2)$, where $S(f) \subset M$ denotes the singular point set of f and $[S(f)] \in H_2(M;\mathbb{Z}_2)$ is the homology class represented by $S(f)$. Then we have*

$$\begin{aligned}\chi(M) \equiv\ & |\widetilde{\mathrm{III}}^{2,2,2}(f)| + |\widetilde{\mathrm{III}}^{2,7}(f)| + |\widetilde{\mathrm{III}}^{12}(f)| + |\widetilde{\mathrm{III}}^{13}_e(f)| \\ & + |\widetilde{\mathrm{III}}^{13}_B(f)| + |\widetilde{\mathrm{III}}^{25}(f)| + |\widetilde{\mathrm{III}}^{26}(f)| \pmod 2,\end{aligned}$$

where $\chi(M)$ denotes the Euler number of M.

By Remark 3.1 or Proposition 4.6, the number of singular fibers of type $\widetilde{\mathrm{III}}^{13}$ is even for any stable map $f : M \to N$ as in Theorem 4.7. Thus, $|\widetilde{\mathrm{III}}^{13}_A(f)|$ and $|\widetilde{\mathrm{III}}^{13}_B(f)|$ have the same parity. Therefore, we may replace $\widetilde{\mathrm{III}}^{13}_A(f)$ with $\widetilde{\mathrm{III}}^{13}_B(f)$ in Theorem 4.7. Similarly we may replace $\widetilde{\mathrm{III}}^{13}_o(f)$ with $\widetilde{\mathrm{III}}^{13}_e(f)$.

PROOF OF THEOREM 4.7. We add up those items which correspond to the graphs $\overline{\widetilde{\mathrm{II}}^{0,0}_A(f)}$, $\overline{\widetilde{\mathrm{II}}^{0,1}_A(f)}$, $\overline{\widetilde{\mathrm{II}}^{1,1}_A(f)}$, $\overline{\widetilde{\mathrm{II}}^{3}_A(f)}$, $\overline{\widetilde{\mathrm{II}}^{4}_B(f)}$ and $\overline{\widetilde{\mathrm{II}}^{5}_B(f)}$ of Proposition 4.6, and we further add to it the co-existence relations obtained by the graphs $\overline{\widetilde{\mathrm{II}}^{2,2}_o(f)}$ and $\overline{\widetilde{\mathrm{II}}^{7}_o(f)}$ in §4:

$$\overline{\widetilde{\mathrm{II}}^{2,2}_o(f)} : |\widetilde{\mathrm{III}}^{0,2,2}(f)| + |\widetilde{\mathrm{III}}^{1,2,2}(f)| + |\widetilde{\mathrm{III}}^{2,6}(f)| + |\widetilde{\mathrm{III}}^{20}_e(f)| \equiv 0 \pmod 2,$$

$$\begin{aligned}\overline{\widetilde{\mathrm{II}}^{7}_o(f)} :\ & |\widetilde{\mathrm{III}}^{0,7}(f)| + |\widetilde{\mathrm{III}}^{1,7}(f)| + |\widetilde{\mathrm{III}}^{13}_o(f)| + |\widetilde{\mathrm{III}}^{18}(f)| + |\widetilde{\mathrm{III}}^{19}(f)| \\ & + |\widetilde{\mathrm{III}}^{20}_o(f)| \equiv 0 \pmod 2.\end{aligned}$$

In this way we obtain

$$\begin{aligned}&|\widetilde{\mathrm{III}}^{0,0,0}(f)| + |\widetilde{\mathrm{III}}^{0,0,1}(f)| + |\widetilde{\mathrm{III}}^{0,1,1}(f)| + |\widetilde{\mathrm{III}}^{1,1,1}(f)| + |\widetilde{\mathrm{III}}^{0,0,2}(f)| + |\widetilde{\mathrm{III}}^{0,2,2}(f)| \\ &+ |\widetilde{\mathrm{III}}^{1,1,2}(f)| + |\widetilde{\mathrm{III}}^{1,2,2}(f)| + |\widetilde{\mathrm{III}}^{0,1,2}(f)| + |\widetilde{\mathrm{III}}^{0,3}(f)| + |\widetilde{\mathrm{III}}^{0,4}(f)| + |\widetilde{\mathrm{III}}^{0,5}(f)| \\ &+ |\widetilde{\mathrm{III}}^{0,6}(f)| + |\widetilde{\mathrm{III}}^{0,7}(f)| + |\widetilde{\mathrm{III}}^{1,3}(f)| + |\widetilde{\mathrm{III}}^{1,4}(f)| + |\widetilde{\mathrm{III}}^{1,5}(f)| + |\widetilde{\mathrm{III}}^{1,6}(f)| \\ &+ |\widetilde{\mathrm{III}}^{1,7}(f)| + |\widetilde{\mathrm{III}}^{2,3}(f)| + |\widetilde{\mathrm{III}}^{2,4}(f)| + |\widetilde{\mathrm{III}}^{2,5}(f)| + |\widetilde{\mathrm{III}}^{2,6}(f)| + |\widetilde{\mathrm{III}}^{8}(f)| \\ &+ |\widetilde{\mathrm{III}}^{9}(f)| + |\widetilde{\mathrm{III}}^{10}(f)| + |\widetilde{\mathrm{III}}^{11}(f)| + |\widetilde{\mathrm{III}}^{13}_{o}(f)| + |\widetilde{\mathrm{III}}^{13}_{A}(f)| + |\widetilde{\mathrm{III}}^{14}(f)| \\ &+ |\widetilde{\mathrm{III}}^{15}(f)| + |\widetilde{\mathrm{III}}^{16}(f)| + |\widetilde{\mathrm{III}}^{17}(f)| + |\widetilde{\mathrm{III}}^{18}(f)| + |\widetilde{\mathrm{III}}^{19}(f)| + |\widetilde{\mathrm{III}}^{20}(f)| \\ &+ |\widetilde{\mathrm{III}}^{21}(f)| + |\widetilde{\mathrm{III}}^{22}(f)| + |\widetilde{\mathrm{III}}^{23}(f)| + |\widetilde{\mathrm{III}}^{24}(f)| + |\widetilde{\mathrm{III}}^{d}_{A}(f)| + |\widetilde{\mathrm{III}}^{e}_{A}(f)| \\ &+ |\widetilde{\mathrm{III}}^{f}_{A}(f)| + |\widetilde{\mathrm{III}}^{g}_{A}(f)| \equiv 0 \pmod 2.\end{aligned}$$

Combining this formula with the relation between the number of singular fibers of f and the number of triple points of $f|_{S(f)}$ obtained in Remark 2.6, we obtain

$$\begin{aligned} T(f|_{S(f)}) &= |\widetilde{\mathrm{III}}^{2,2,2}(f)| + |\widetilde{\mathrm{III}}^{2,7}(f)| + |\widetilde{\mathrm{III}}^{12}(f)| + |\widetilde{\mathrm{III}}^{13}_{e}(f)| + |\widetilde{\mathrm{III}}^{13}_{B}(f)| \\ &\quad + |\widetilde{\mathrm{III}}^{25}(f)| + |\widetilde{\mathrm{III}}^{26}(f)| + |\widetilde{\mathrm{III}}^{d}_{A}(f)| + |\widetilde{\mathrm{III}}^{e}_{A}(f)| + |\widetilde{\mathrm{III}}^{f}_{A}(f)| \\ &\quad + |\widetilde{\mathrm{III}}^{g}_{A}(f)| \pmod 2.\end{aligned}$$

On the other hand, by Theorem 4.3, we have

$$\chi(S(f)) \equiv T(f|_{S(f)}) + \sum_{q:\ \text{swallow-tail point of } f} n(q, f|_{S(f)}) \pmod 2.$$

Then we obtain

$$\begin{aligned} \chi(S(f)) &\equiv |\widetilde{\mathrm{III}}^{2,2,2}(f)| + |\widetilde{\mathrm{III}}^{2,7}(f)| + |\widetilde{\mathrm{III}}^{12}(f)| + |\widetilde{\mathrm{III}}^{13}_{e}(f)| + |\widetilde{\mathrm{III}}^{13}_{B}(f)| \\ &\quad + |\widetilde{\mathrm{III}}^{25}(f)| + |\widetilde{\mathrm{III}}^{26}(f)| + |\widetilde{\mathrm{III}}^{d}_{A}(f)| + |\widetilde{\mathrm{III}}^{e}_{A}(f)| + |\widetilde{\mathrm{III}}^{f}_{A}(f)| \\ &\quad + |\widetilde{\mathrm{III}}^{g}_{A}(f)| + \sum_{q:\ \text{swallow-tail point of } f} n(q, f|_{S(f)}) \pmod 2.\end{aligned}$$

Furthermore, by the definitions of $n(q, f|_{S(f)})$ and type A (see Figures 1.9 and 1.52), we have

$$\sum_{q:\ \text{swallow-tail point of } f} n(q, f|_{S(f)}) = |\widetilde{\mathrm{III}}^{d}_{A}(f)| + |\widetilde{\mathrm{III}}^{e}_{A}(f)| + |\widetilde{\mathrm{III}}^{f}_{A}(f)| + |\widetilde{\mathrm{III}}^{g}_{A}(f)|.$$

Combining these equations with Theorem 4.4, we obtain Theorem 4.7. □

The homological hypothesis in Theorem 4.7 and in Corollaries 1.1 and 1.3 is essential. In fact, let $f : \mathbb{R}P^2 \times \mathbb{R}P^2 \to \mathbb{R}P^2 \times \mathbb{R}$ be defined by $f(x, y) = (x, \varphi(y))$, where φ is any stable Morse function on $\mathbb{R}P^2$. Then $f|_{S(f)}$ has no triple points, although $\chi(\mathbb{R}P^2 \times \mathbb{R}P^2) = 1$. We note that $H_1(\mathbb{R}P^2 \times \mathbb{R}; \mathbb{Z}_2) \neq 0$ and $f_*[S(f)] \neq 0$ in $H_2(\mathbb{R}P^2 \times \mathbb{R}; \mathbb{Z}_2)$.

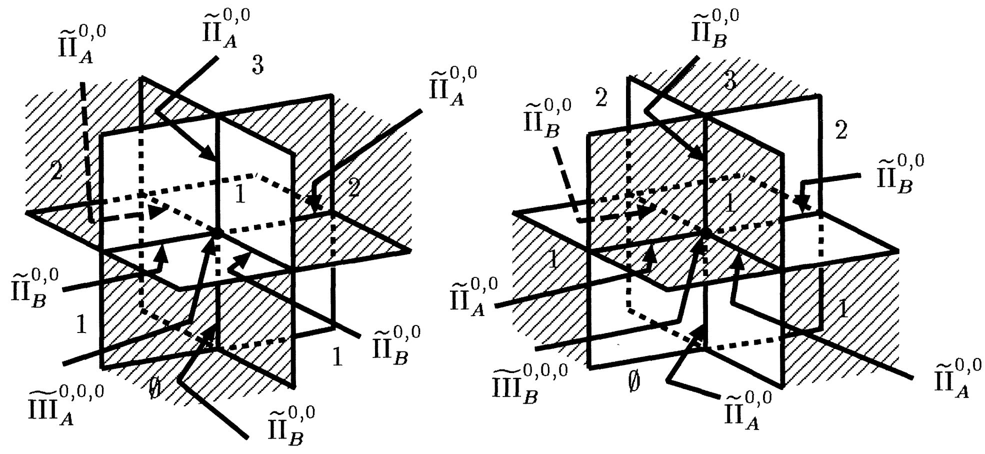

FIGURE 1.13. Types A and B for $\widetilde{\mathrm{III}}^{0,0,0}$

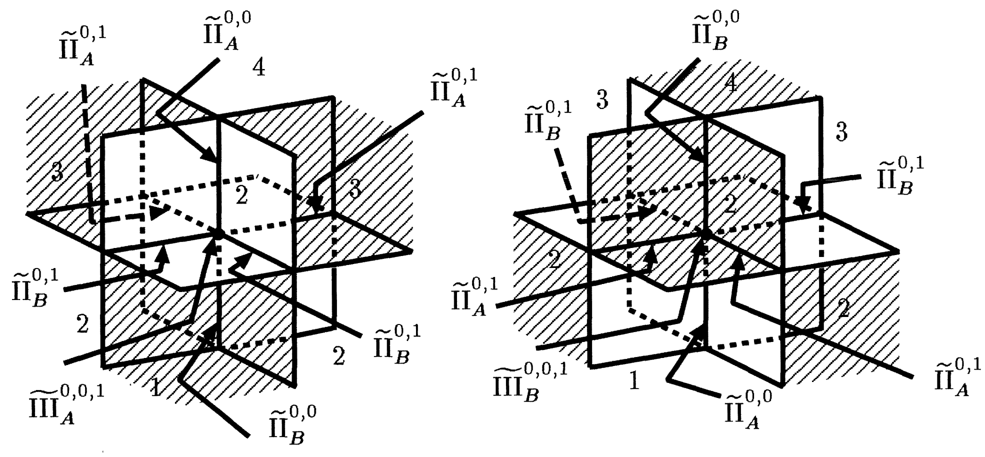

FIGURE 1.14. Types A and B for $\widetilde{\mathrm{III}}^{0,0,1}$

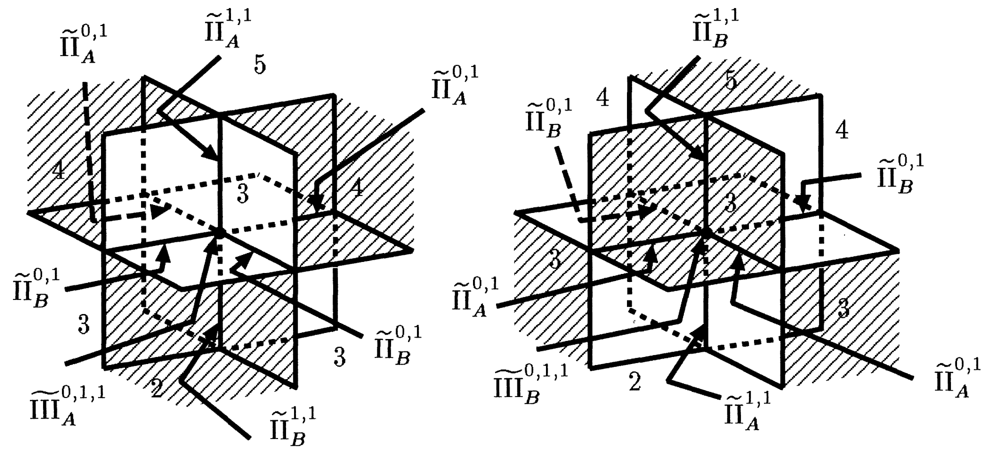

FIGURE 1.15. Types A and B for $\widetilde{\mathrm{III}}^{0,1,1}$

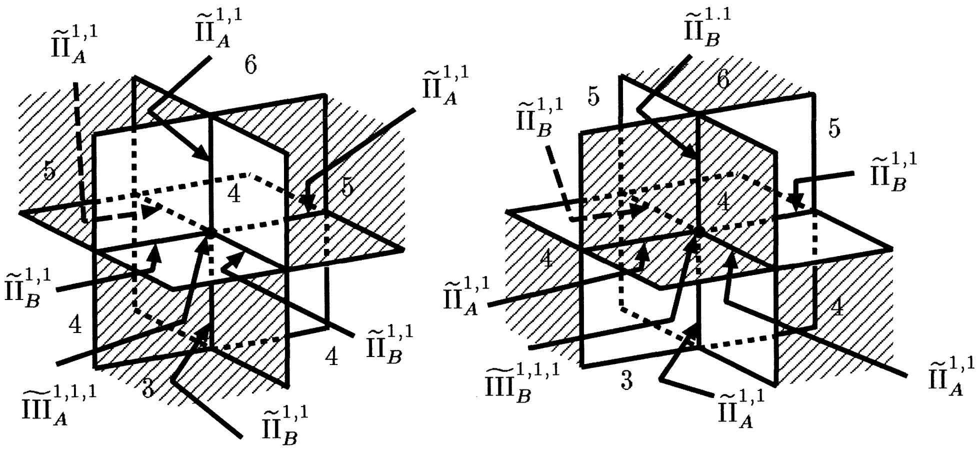

FIGURE 1.16. Types A and B for $\widetilde{III}^{1,1,1}$

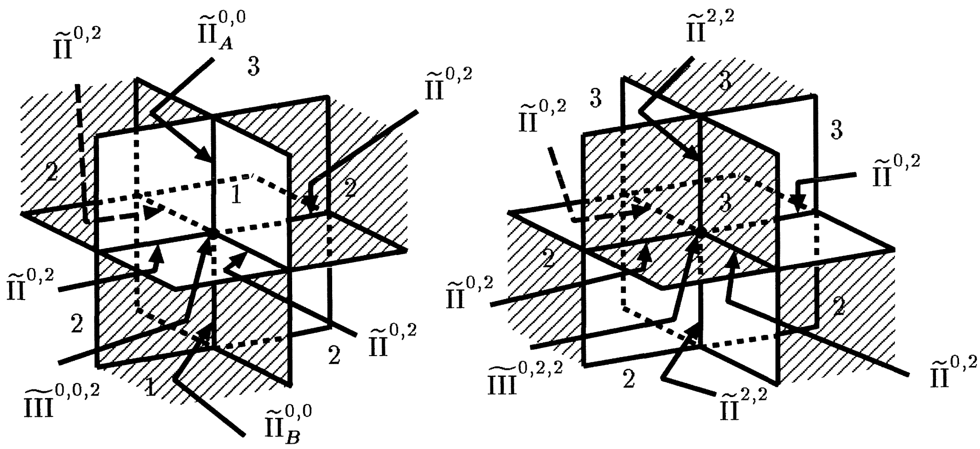

FIGURE 1.17. $\widetilde{III}^{0,0,2}$ and $\widetilde{III}^{0,2,2}$ can not be divided into two types

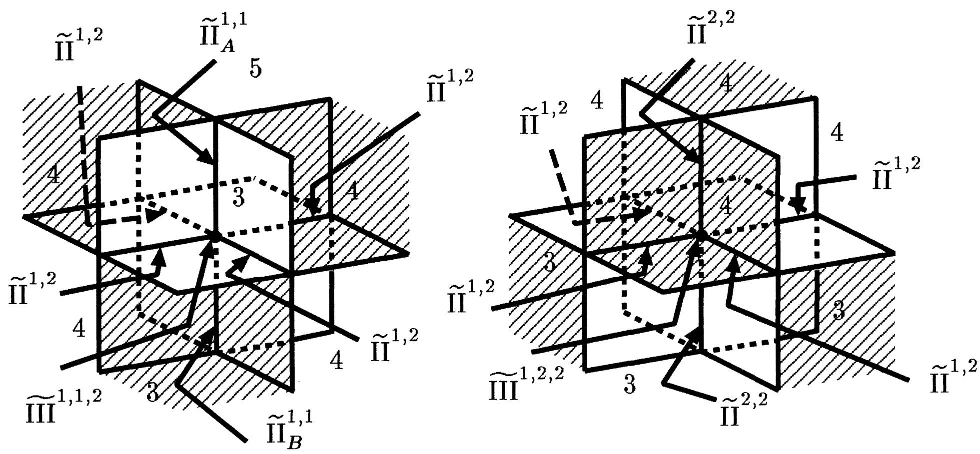

FIGURE 1.18. $\widetilde{III}^{1,1,2}$ and $\widetilde{III}^{1,2,2}$ can not be divided into two types

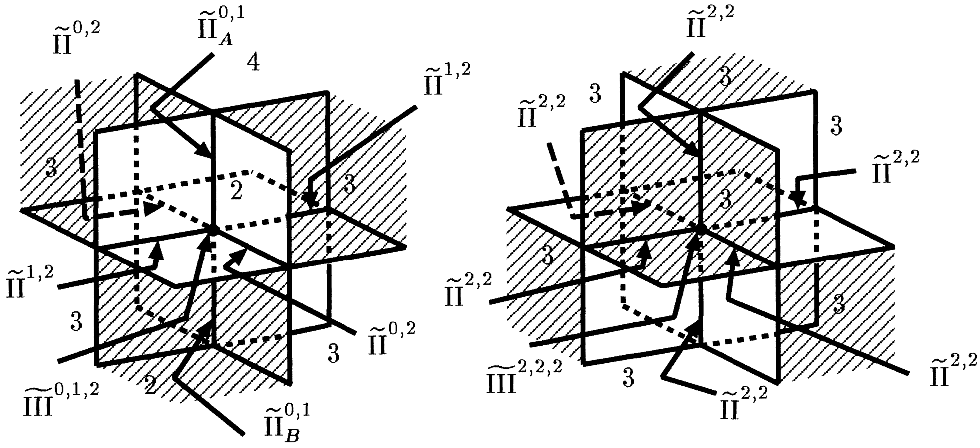

FIGURE 1.19. $\widetilde{\mathrm{III}}^{0,1,2}$ and $\widetilde{\mathrm{III}}^{2,2,2}$ can not be divided into two types

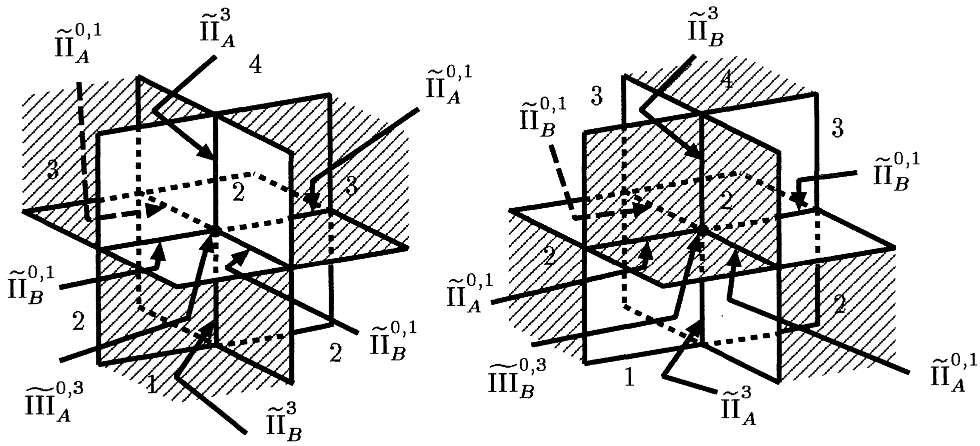

FIGURE 1.20. Types A and B for $\widetilde{\mathrm{III}}^{0,3}$

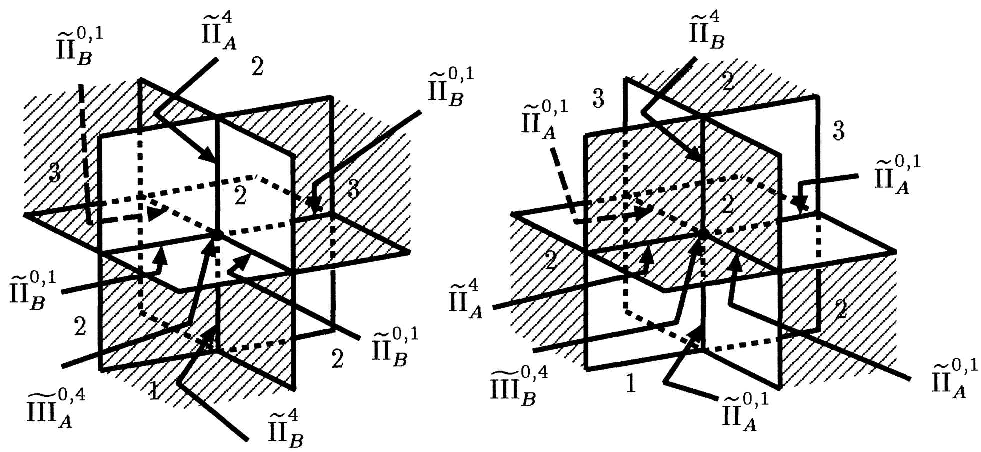

FIGURE 1.21. Types A and B for $\widetilde{\mathrm{III}}^{0,4}$

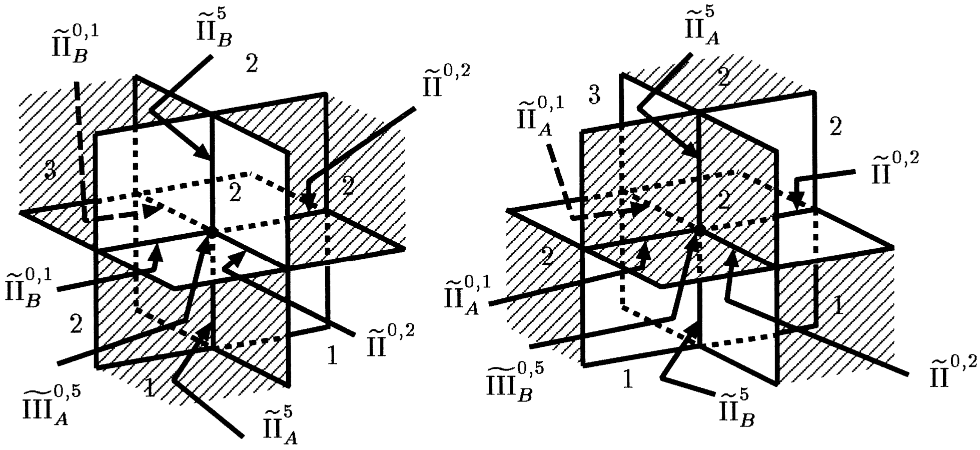

FIGURE 1.22. Types A and B for $\widetilde{\mathrm{III}}^{0,5}$

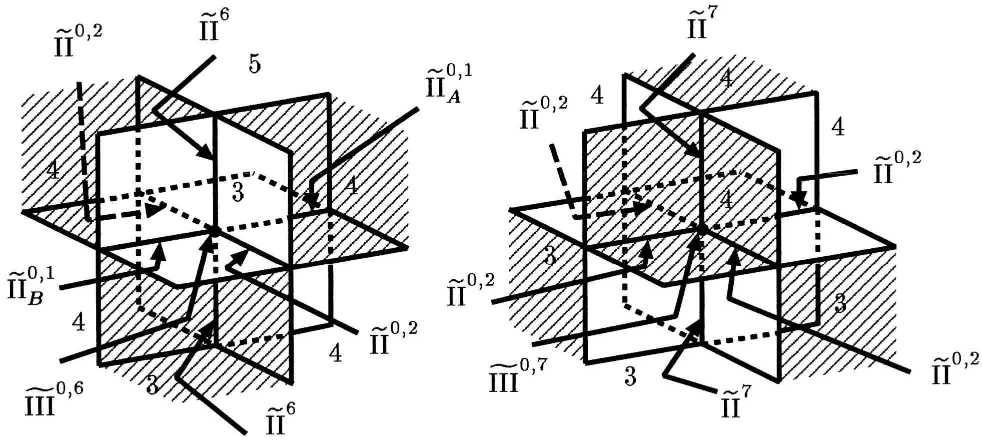

FIGURE 1.23. $\widetilde{\mathrm{III}}^{0,6}$ and $\widetilde{\mathrm{III}}^{0,7}$ can not be divided into two types

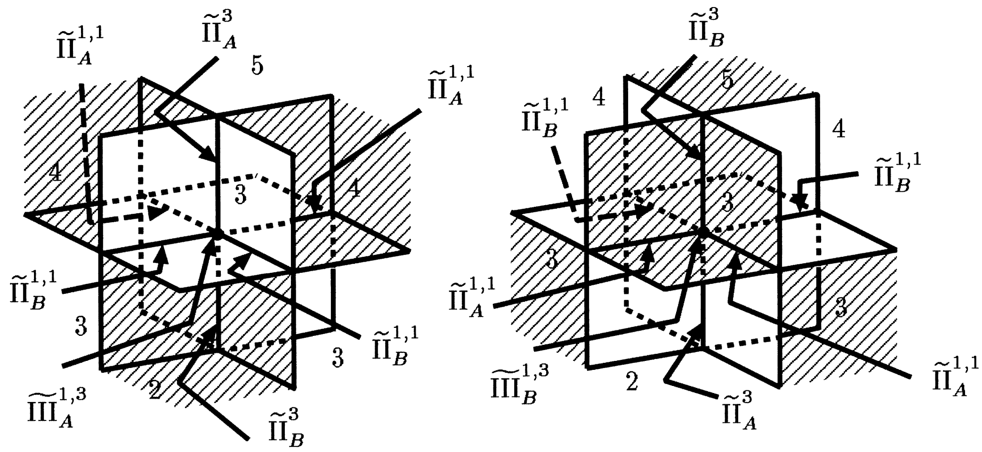

FIGURE 1.24. Types A and B for $\widetilde{\mathrm{III}}^{1,3}$

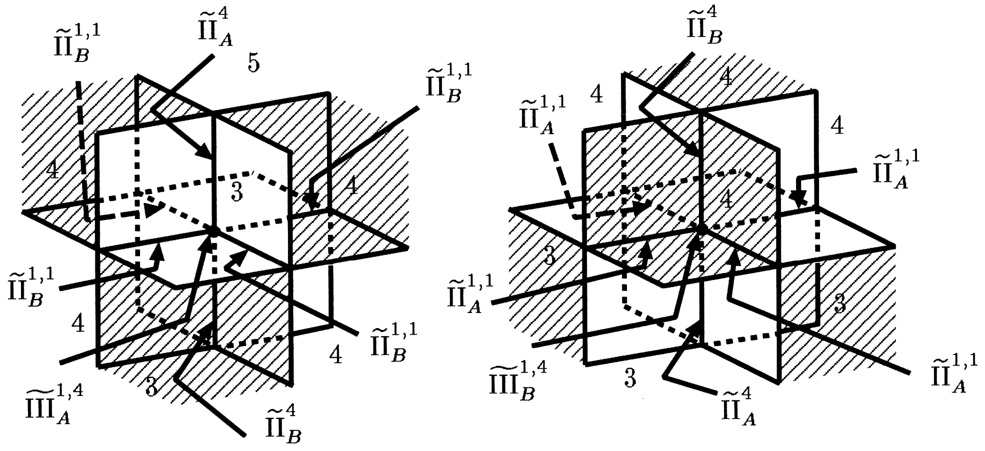

FIGURE 1.25. Types A and B for $\widetilde{\mathrm{III}}^{1,4}$

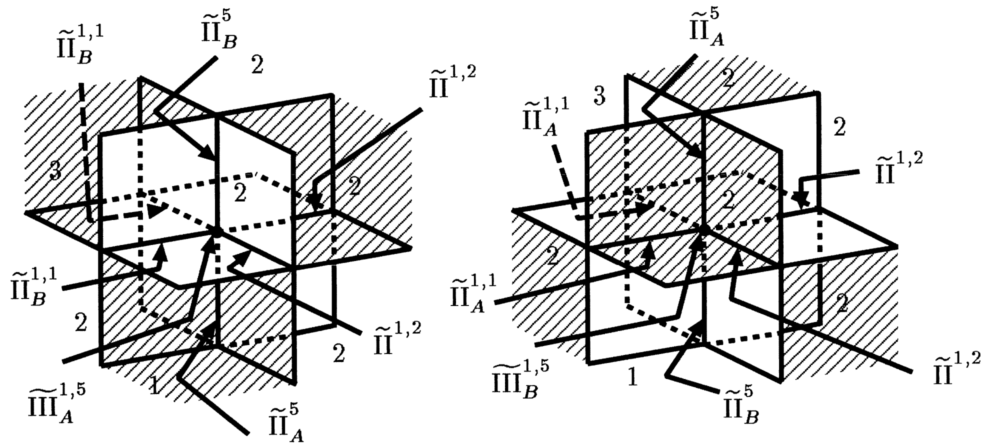

FIGURE 1.26. Types A and B for $\widetilde{\mathrm{III}}^{1,5}$

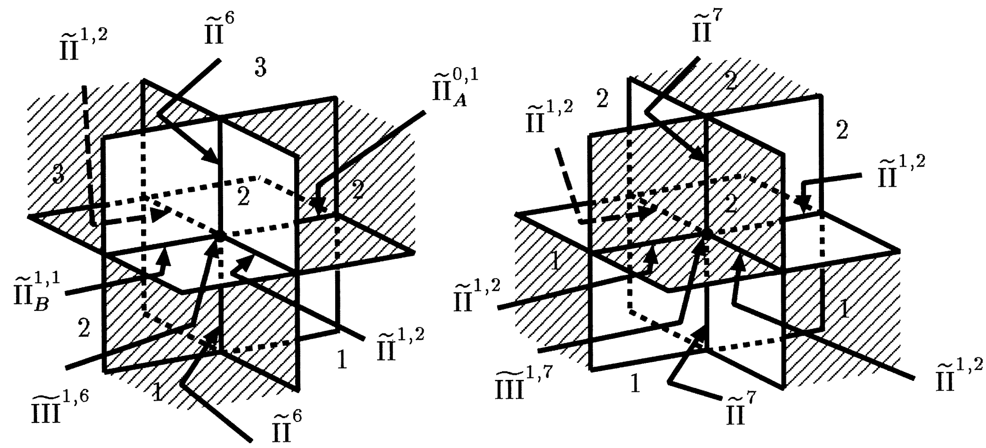

FIGURE 1.27. $\widetilde{\mathrm{III}}^{1,6}$ and $\widetilde{\mathrm{III}}^{1,7}$ can not be divided into two types

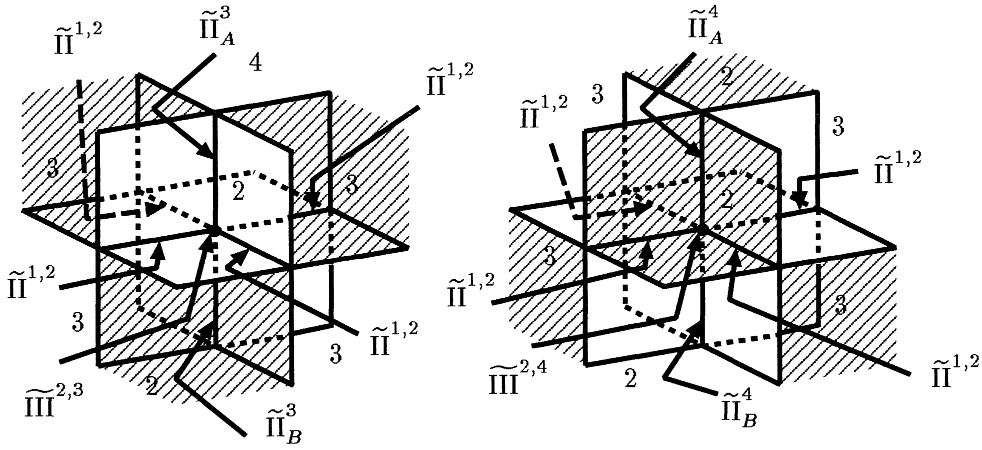

FIGURE 1.28. $\widetilde{III}^{2,3}$ and $\widetilde{III}^{2,4}$ can not be divided into two types

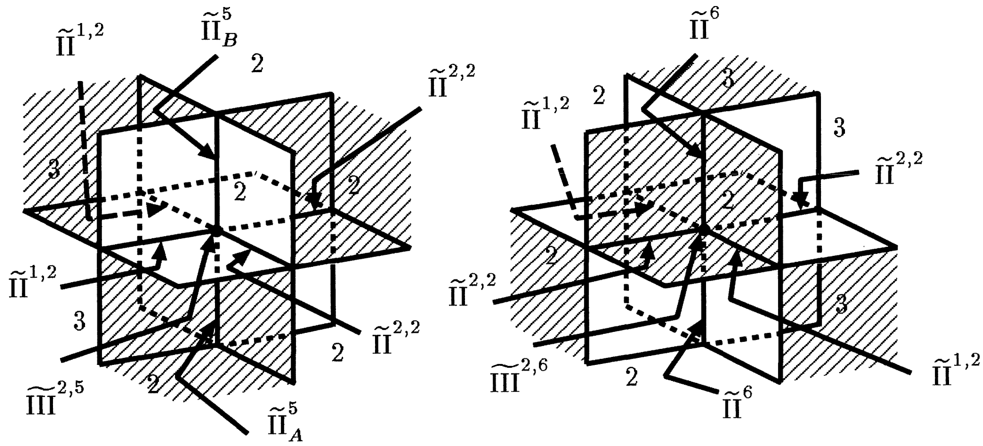

FIGURE 1.29. $\widetilde{III}^{2,5}$ and $\widetilde{III}^{2,6}$ can not be divided into two types

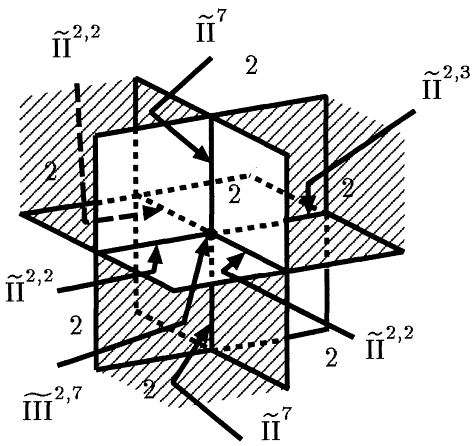

FIGURE 1.30. $\widetilde{III}^{2,7}$ can not be divided into two types

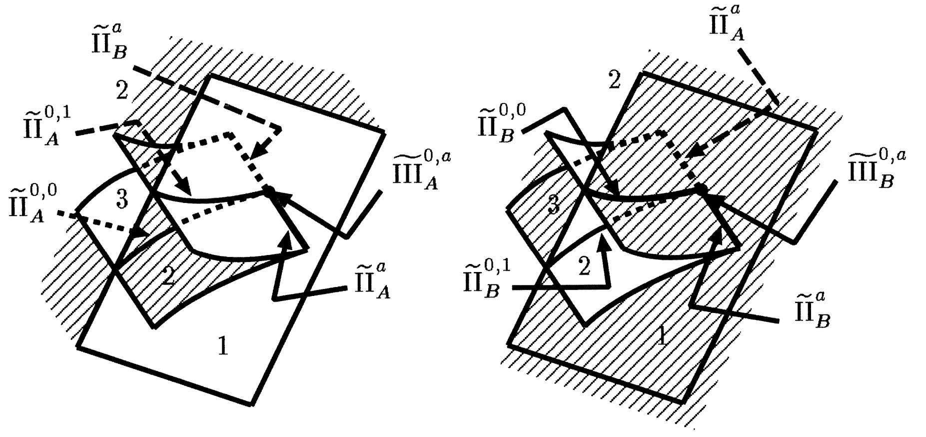

FIGURE 1.31. Types A and B for $\widetilde{\mathrm{III}}^{0,a}$

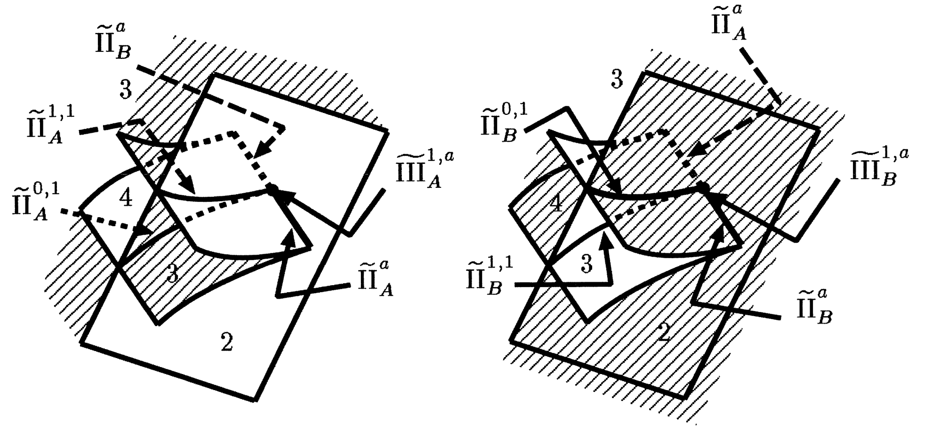

FIGURE 1.32. Types A and B for $\widetilde{\mathrm{III}}^{1,a}$

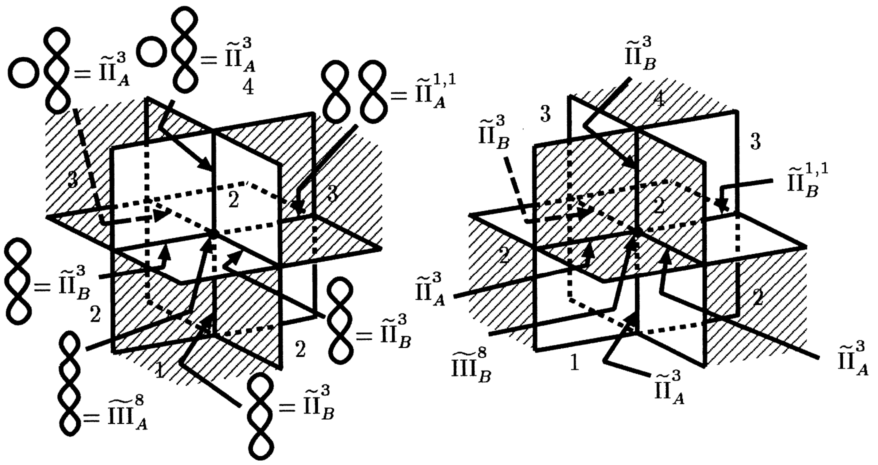

FIGURE 1.33. Types A and B for $\widetilde{\mathrm{III}}^8$

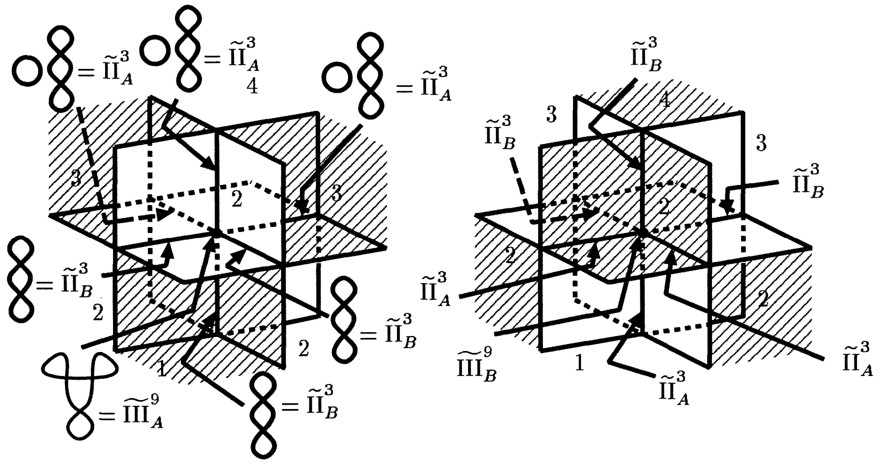

FIGURE 1.34. Types A and B for $\widetilde{\mathrm{III}}^9$

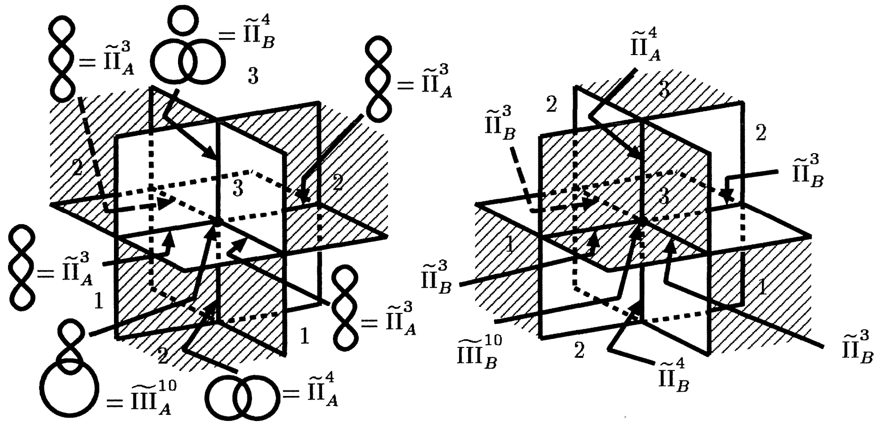

FIGURE 1.35. Types A and B for $\widetilde{\mathrm{III}}^{10}$

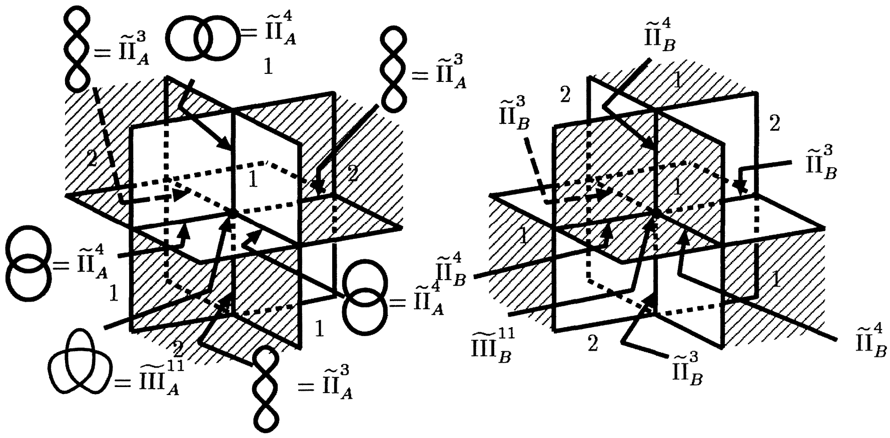

FIGURE 1.36. Types A and B for $\widetilde{\mathrm{III}}^{11}$

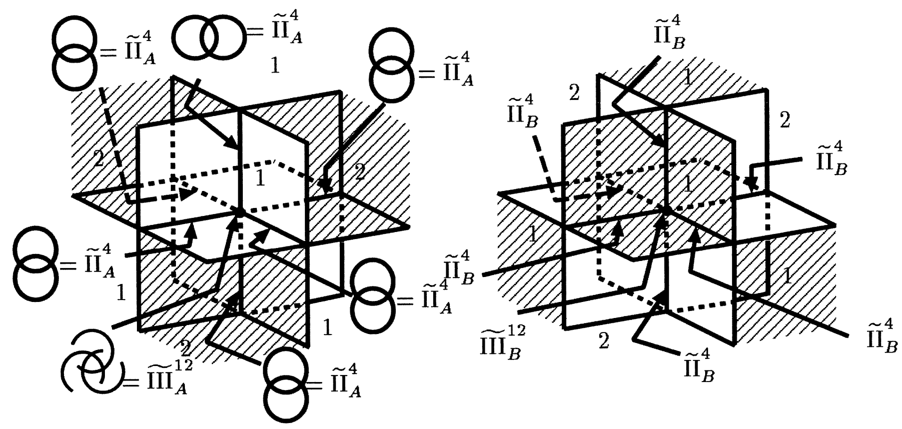

FIGURE 1.37. Types A and B for $\widetilde{III}^{12}$

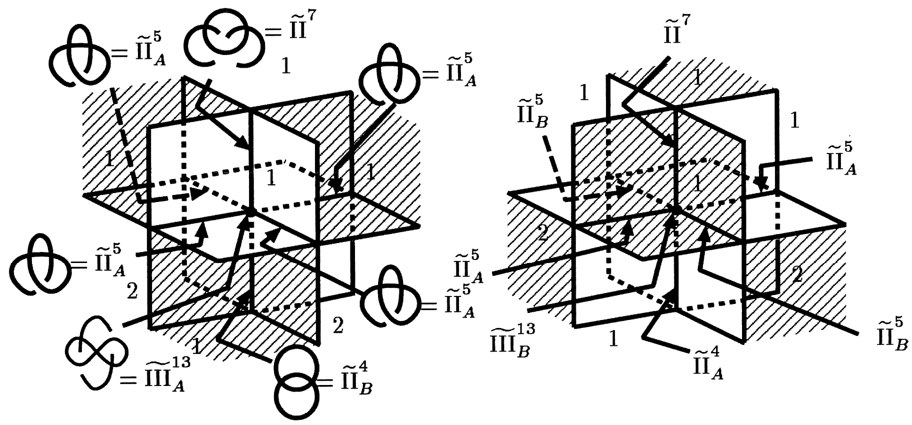

FIGURE 1.38. Types A and B for $\widetilde{III}^{13}$

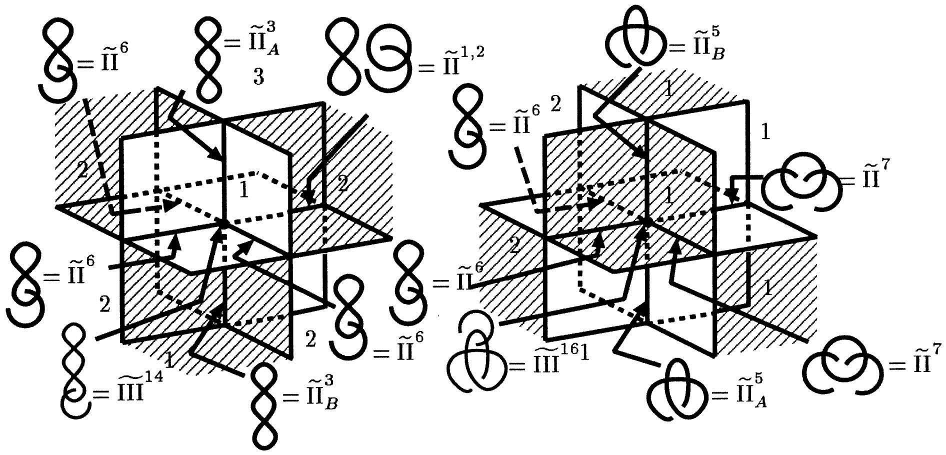

FIGURE 1.39. $\widetilde{III}^{14}$ and $\widetilde{III}^{16}$ can not be divided into two types

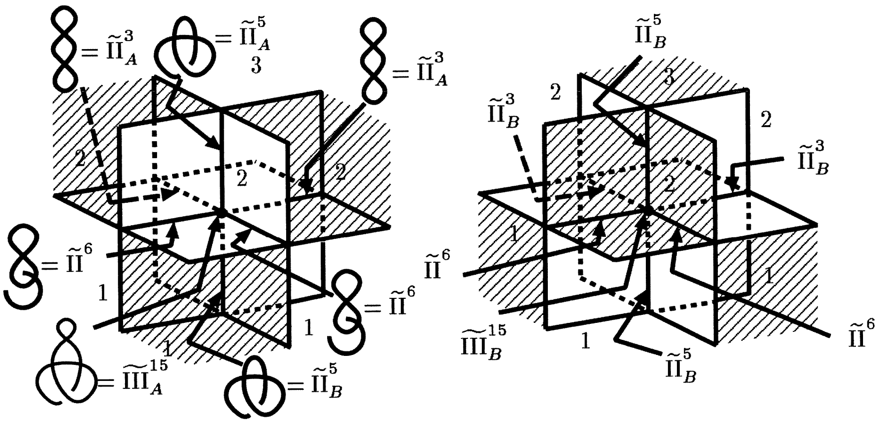

FIGURE 1.40. Types A and B for $\widetilde{\mathrm{III}}^{15}$

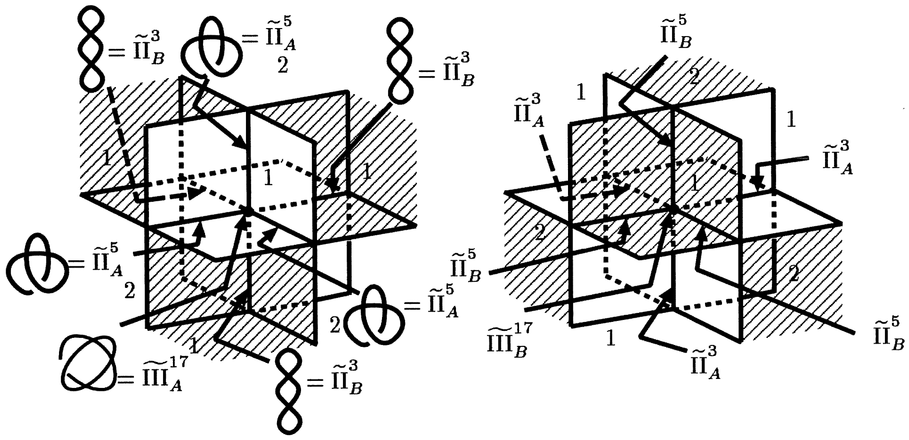

FIGURE 1.41. Types A and B for $\widetilde{\mathrm{III}}^{17}$

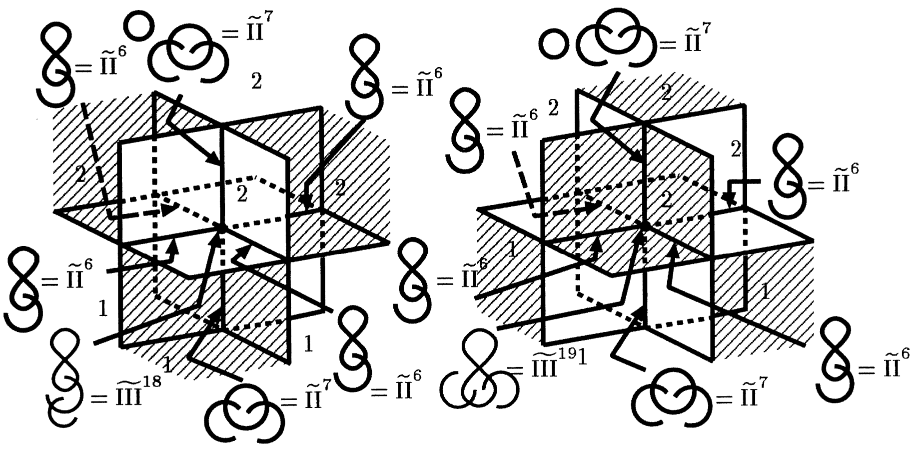

FIGURE 1.42. $\widetilde{\mathrm{III}}^{18}$ and $\widetilde{\mathrm{III}}^{19}$ can not be divided into two types

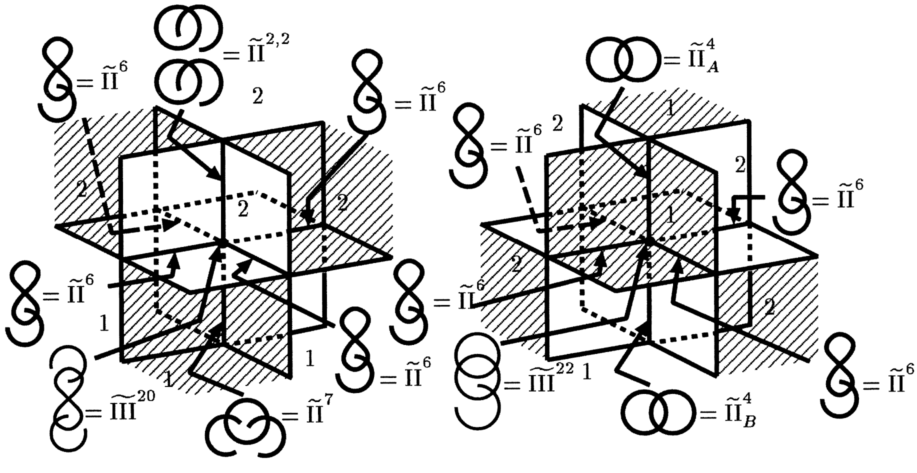

FIGURE 1.43. $\widetilde{\mathrm{III}}^{20}$ and $\widetilde{\mathrm{III}}^{22}$ can not be divided into two types

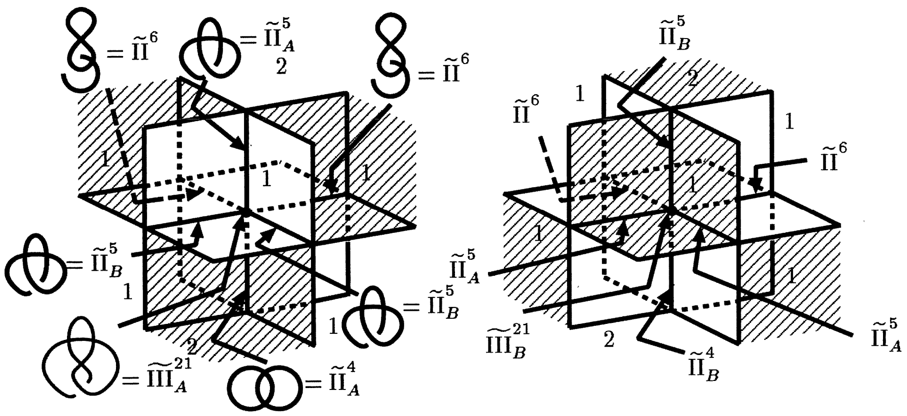

FIGURE 1.44. Types A and B for $\widetilde{\mathrm{III}}^{21}$

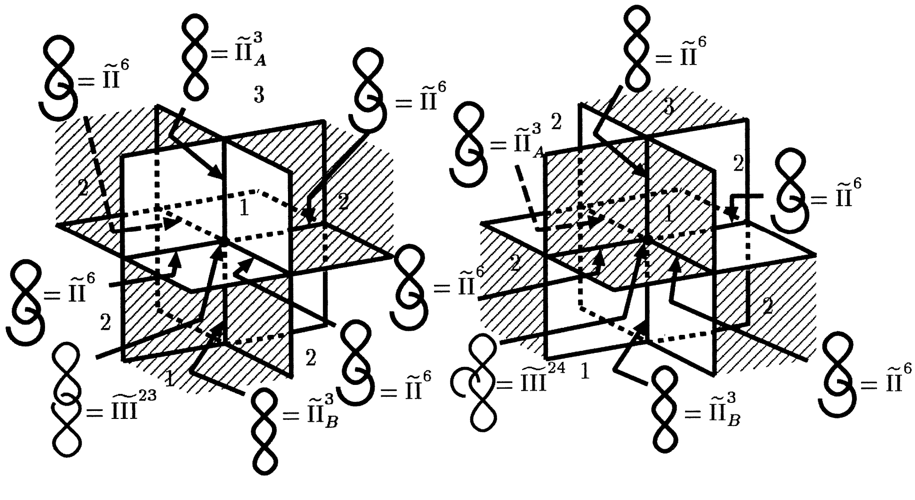

FIGURE 1.45. $\widetilde{\mathrm{III}}^{23}$ and $\widetilde{\mathrm{III}}^{24}$ can not be divided into two types

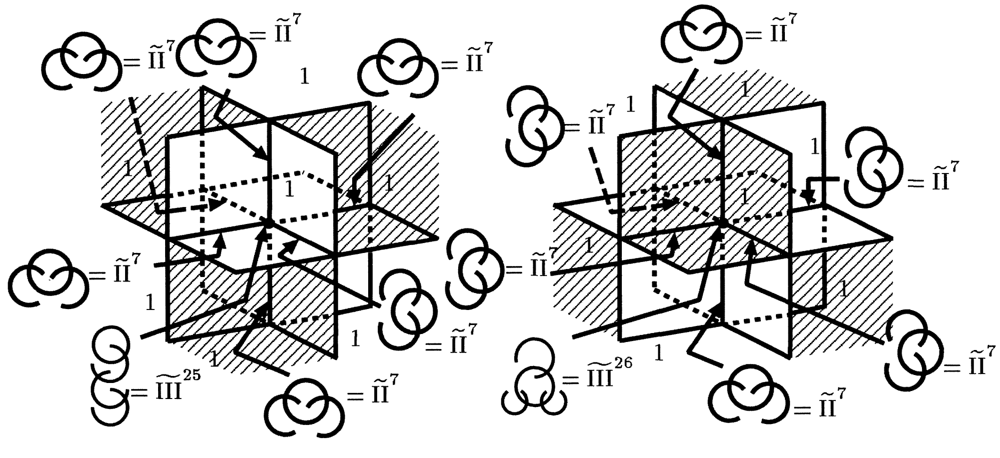

FIGURE 1.46. $\widetilde{\mathrm{III}}^{25}$ and $\widetilde{\mathrm{III}}^{26}$ can not be divided into two types

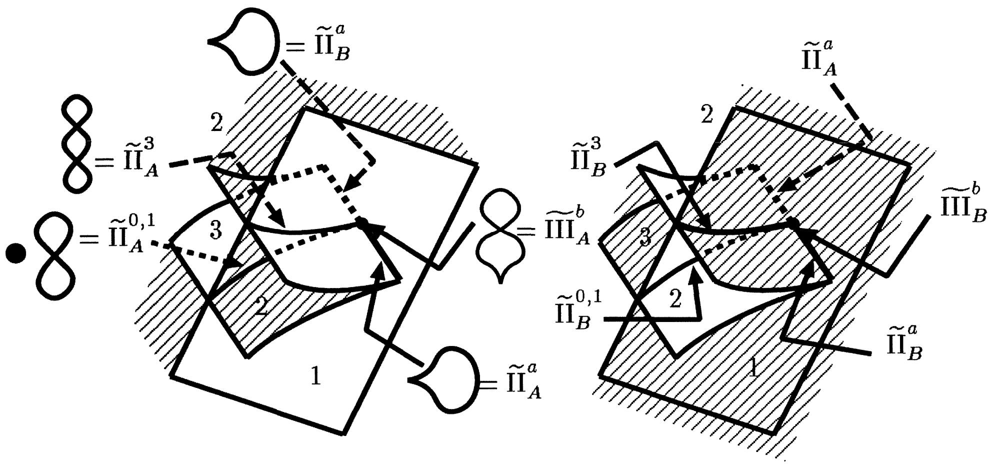

FIGURE 1.47. Types A and B for $\widetilde{\mathrm{III}}^b$

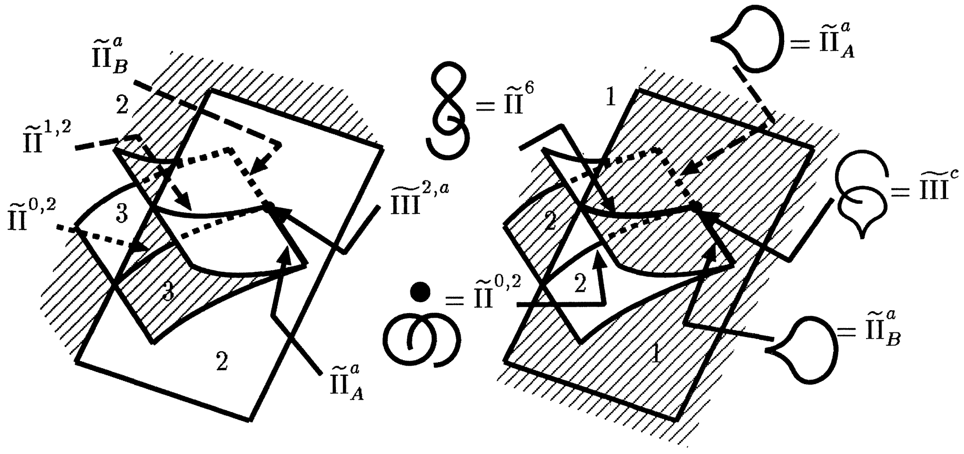

FIGURE 1.48. $\widetilde{\mathrm{III}}^{2,a}$ and $\widetilde{\mathrm{III}}^c$ can not be divided into two types

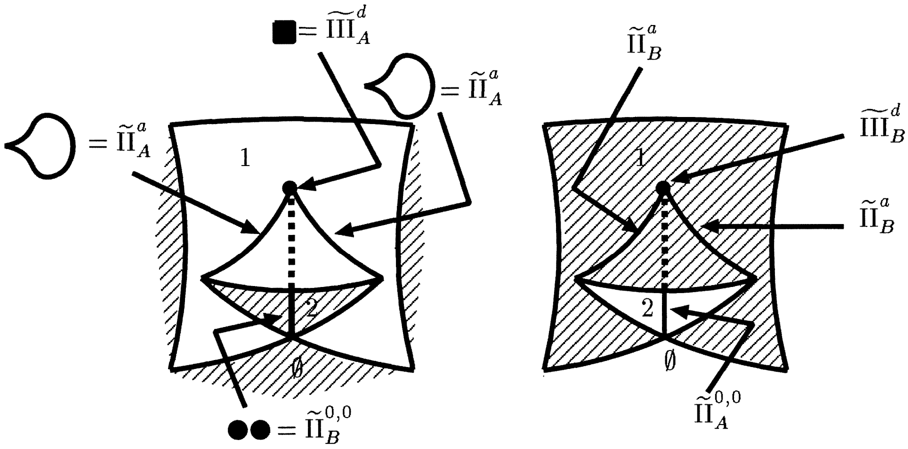

FIGURE 1.49. Types A and B for $\widetilde{\text{III}}^d$

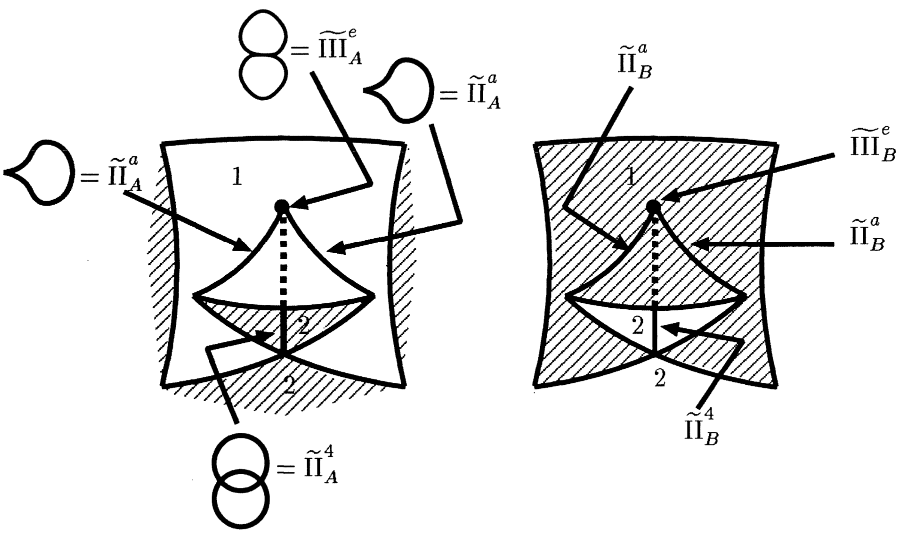

FIGURE 1.50. Types A and B for $\widetilde{\text{III}}^e$

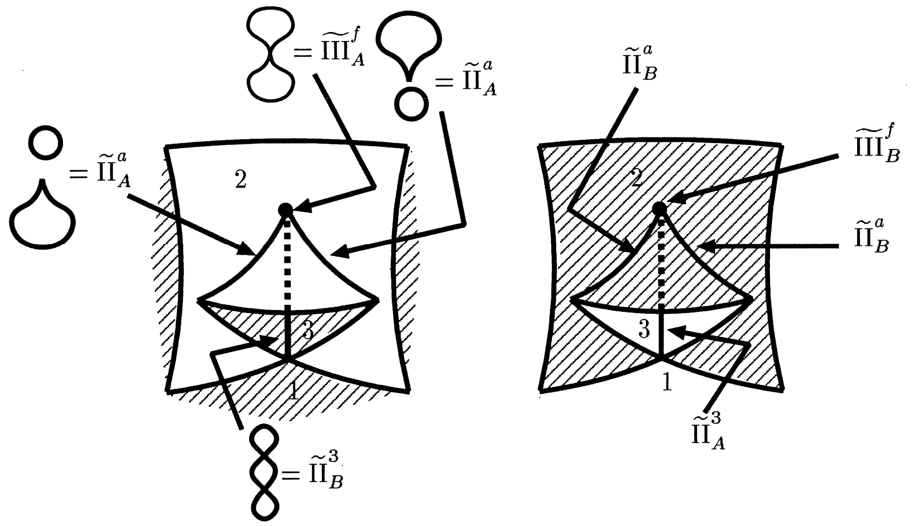

FIGURE 1.51. Types A and B for $\widetilde{\mathrm{III}}^f$

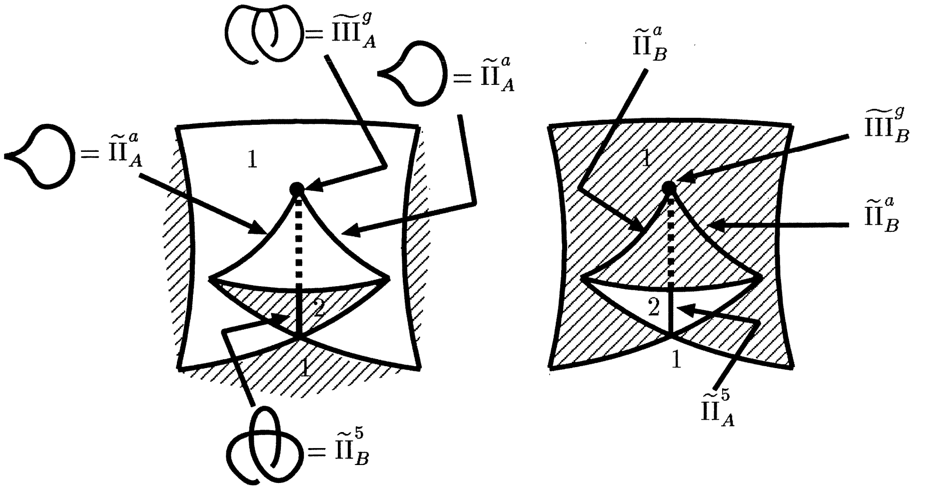

FIGURE 1.52. Types A and B for $\widetilde{\mathrm{III}}^g$

Chapter 2

Universal complex of singular fibers of two-colored maps

CHAPTER 2

Universal complex of singular fibers of two-colored maps

Dedicate to Katsunori Nakajima's marriage.

1. Introduction

As a pioneer, Vassilyev [**50**] constructed the universal complex of multi-singularities. The theory of Vassilyev was further developed by Kazarian [**15**]. Secondary, Saeki [**43**] constructed the universal complex of the singular fibers of differentiable maps and he showed that the cohomology classes of these universal complex induce cobordism invariant of the differentiable maps. If we consider the target manifold as the Euclidean space, then the cobordism invariants of maps induces cobordism invariants of the source manifolds. In this way, for the study of the topology of manifolds, the universal complex is very powerful tools.

In this Chapter, we consider the singular fibers of differentiable maps from global point of view. More precisely, we introduce the notion of *two-colored maps* (roughly speaking, the two-colored map is the map equipped with two color decomposition of target manifold complement to discriminant set, for details see §2) and the *two-colored cobordism* among two-colored maps. Furthermore, we develop the theory of singular fiber of two-colored maps. In this theory, we construct the cochain complex of the singular fibers of two-colored maps and we show that cohomology classes of the cochain complex induce two-colored cobordism invariants among two-colored maps. Note that if we consider the Euclidean spaces as the target manifold, then the two-colored cobordism invariants induce cobordism invariants of the source manifolds. From an actual calculation, we obtain several Euler number formulas of manifolds: Theorems 5.8, 5.9, 5.12 and interesting result Theorem 5.18: There is no Euler number formula of 4-manifolds in terms of the singular fibers of stable maps if we consider not two-colored stable maps but stable maps.

This Chapter is organized as follows. In §2, we give some fundamental definitions concerning the singular fibers of two-colored maps. In §3, we construct cochain complex of the singular fibers of two-colored maps. In §4, we present the classification of the singular fiber of two-colored stable maps of possibly non-orientable 5-manifolds into 4-manifolds. Furthermore, we give the cohomology groups of the several cochain complex which we restrict the singularities of the stable maps of 5-manifolds into 4-manifolds, the cohomology groups of the cochain complex of the stable maps of 5-manifolds into 6-manifolds and the cohomology groups of the cochain complex of the stable maps of 3-manifolds into 3-manifolds. In §5, we consider geometrical meaning of the cohomology classes of the cochain complex of the singular fibers of two-colored maps. From the cohomology group obtained in §4, we obtain several mod 2 Euler number formulas of manifolds.

2. Preparation

In this section, we prepare basic definitions of this Chapter.

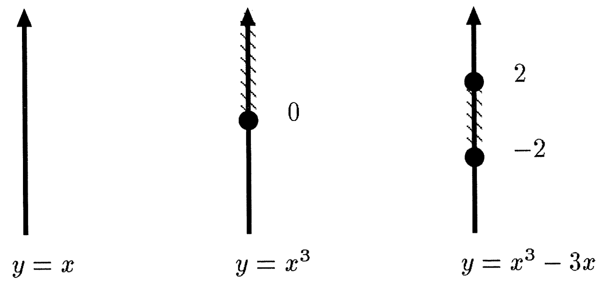

FIGURE 2.1. Examples of two-colored maps

DEFINITION 2.1. Let $f : M^n \to N^p$ be a differentiable map of a n-manifold into a p-manifold such that $n+1 \geq p$. We say that f is *two colorable*, if there exists disjoint non-empty open subset $R, B \subset N \setminus f(S(f))$ which satisfy the following condition;

$$N \setminus f(S(f)) = R \cup B \text{ and } \overline{R} \cap \overline{B} = \partial R = \partial B = f(S(f)), \tag{2.1}$$

where $S(f)$ denote the singular point set of f. If f has no critical values, then we also say that f is two colorable. In this case, we consider if $N = R$ (or $N = B$), then $B = \emptyset$ (resp. $R = \emptyset$). We call the pair (R, B), *two color decomposition* or *color* of f.

The examples of two-colored maps are given in Figure 2.1, where the shadowed regions indicate R.

The condition of two colorable is characterized by the following.

PROPOSITION 2.2. *Let $f : M^n \to N^p$ be a differentiable map such that $n+1 \geq p$. Then f is two colorable if and only if $f_*[S(f)] = 0 \in H^c_{p-1}(N; \mathbb{Z}_2) \cong H^1(N; \mathbb{Z}_2)$, where $S(f)$ denote the set of singular points of f and $[S(f)]$ denote the homology class represented by $S(f)$.*

PROOF. Let $x_0 \in N \setminus f(S(f))$ be a fixed point. Then for every $x \in N \setminus f(S(f))$, we take a regular curve γ connecting x_0 and x such that γ intersects $f(S(f))$ transversely at a finite points. We prove that the parity of this number, $\#(\gamma \cap f(S(f)))$, does not depend on the chosen curve γ. In fact, let γ' be another curve as above. Then

$$\#(\gamma \cap f(S(f))) + \#(\gamma') \cap f(S(f)) \equiv [\gamma \cup \gamma'] \cdot [f(S(f))] \equiv 0 \mod 2$$

where $[\gamma \cup \gamma'] \cdot [f(S(f))]$ is the modulo 2 intersection number of the homology class represented by the closed curve $\gamma \cup \gamma'$ and that represented by $f(S(f))$. Thus we can define

$$R = \{x \in N \setminus f(S(f)) \mid \#(\gamma \cap f(S(f))) \text{ is even}\}$$
$$B = \{x \in N \setminus f(S(f)) \mid \#(\gamma \cap f(S(f))) \text{ is odd }\}$$

It is obvious that R and B are disjoint nonempty open subset. For the second part of this proof, we omit here. □

We call the pair $(f, (R, B))$ of a two-colorable map f and disjoint open subsets R, B which satisfy the above condition *two-colored map*. We note that for a two-colored map $(f, (R, B))$ we obtain a new two-colored map $(f, (\tilde{R}, \tilde{B}))$ where $\tilde{R} = B$, and $\tilde{B} = R$. We call the two-colored map $(f, (\tilde{R}, \tilde{B}))$ the *two-colored conjugate of* $(f, (R, B))$. We consider they are different two-colored map.

We introduce the equivalence relation among the singular fibers of two-colored maps.

DEFINITION 2.3. Let M_i be smooth manifolds and A_i subsets of M_i, $(i = 0, 1)$. A continuous map $g : A_0 \to A_1$ is said to be *smooth* if for every point $q \in A_0$, there exists a smooth map $\tilde{g} : V \to M_1$ defined on a neighbourhood V of q in M_0 such that $\tilde{g}|_{V \cap A_0} = g|_{V \cap A_0}$. A smooth map $g : A_0 \to A_1$ is a *diffeomorphism* if it is a homeomorphism and its inverse is also smooth. When there exists a diffeomorphism between A_0 and A_1, we say that they are *diffeomorphic*.

Let $(f_i, (R_i, B_i)) : M_i \to N_i$ be two-colored maps and $q_i \in N$, $i = 0, 1$. For $q_i \in N_i$, we call that the fibers over q_0 and q_1 are C^∞ *equivalent* (or C^0 *equivalent*) if there exists open neighborhood U_i of $q_i \in N_i$, $i = 0, 1$, and diffeomorphisms (resp. homeomorphisms) $\Phi : (f^{-1}(U_0), f^{-1}(q_0)) \to (f^{-1}(U_1), f^{-1}(q_1))$ and $\varphi : (U_0, q_0) \to (U_1, q_1)$ which makes following diagram commutative,

$$\begin{array}{ccc} (f_0{}^{-1}(U_0), f_0{}^{-1}(q_0)) & \xrightarrow{\Phi} & (f_1{}^{-1}(U_1), f_1{}^{-1}(q_1)) \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ (U_0, q_0) & \xrightarrow{\varphi} & (U_1, q_1). \end{array} \tag{2.2}$$

Furthermore, we call that the fibers over q_0 and q_1 are *two-colored* C^∞ *equivalent* (or two-colored C^0 *equivalent*) if there exists open neighborhoods U_i of $q_i \in N_i$ $(i = 0, 1)$ diffeomorphism (resp. homeomorphism) Φ as above and diffeomorphism (resp. homeomorphism) $\varphi : (U_0, q_0) \to (U_1, q_1)$ with

$$\varphi(U_0 \cap R_0) = U_1 \cap R_1 \tag{2.3}$$

which makes diagram (2.2) commutative.

Let $(f, (R, B)) : M \to N$ be a two-colored map. We note that there exists a point $q \in N$ such that the fiber of $(f, (R, B))$ over q is not two-colored C^∞ equivalent to the fiber of two-colored conjugate of $(f, (R, B))$ over q. We call the latter fiber is the *two-colored conjugate* of the original fiber.

3. Cochain complex of singular fibers of two-colored maps

In this section, we construct the theory of the singular fibers of two-colored maps. For Propositions and Corollaries of this sections, the proofs are very similar to that of [**43**, §7 and 8], as we omit the proofs here.

For a pair of nonnegative integers (n, p) such that $n + 1 \geq p$, we denote by $\mathcal{CT}_{pr}(n, p)$ (or by $\mathcal{T}_{pr}(n, p)$) the set of all proper two-colored Thom maps (resp. proper Thom maps) of n-manifolds into p-manifolds, which is a pair of stratified map with respect to Whitney regular stratification of M and N such that it is a submersion on each stratum and satisfies a certain regularity conditions (for more details, refer to [**13**]) and nonempty disjoint open subsets R, B in $N \backslash \cup_{\lambda \in \Lambda} \mathcal{N}_\lambda$, where $\mathcal{N}_\lambda$ is the stratum of N of positive codimension such that $R \cup B = N \setminus \cup_{\lambda \in \Lambda} \mathcal{N}_\lambda$ and $\overline{R} \cap \overline{B} = \partial R = \partial B = \cup_{\lambda \in \Lambda} \mathcal{N}_\lambda$. Furthermore, we denote by $\mathcal{CS}^\infty_{pr}(n, p)$ (resp. $\mathcal{S}^\infty_{pr}(n, p)$) the set of the proper two-colored stable maps (resp. proper stable maps) of n-manifolds into p-manifolds and we denote by $\mathcal{CS}^0_{pr}(n, p)$ (or by $\mathcal{S}^0_{pr}(n, p)$) the set of all two-colored C^0 stable maps (resp. all C^0 stable maps) which are elements of $\mathcal{CT}_{pr}(n, p)$ (resp. $\mathcal{T}_{pr}(n, p)$), where C^0 stable maps are defined as follows. We say that $f \in C^\infty(M, N)$ is C^0 stable if the C^0-$\mathcal{A}$-orbit of f is open in $C^\infty(M, N)$, where M and N are smooth manifolds. Here the C^0-$\mathcal{A}$-orbit is defined as follows. Let Homeo(N) denote the group of self-homeomorphisms of N. Then the group Homeo(M) $\times$ Homeo(N) acts on $C^\infty(M, N)$ by $(\Phi, \Psi)f = \Psi \circ f \circ \Phi^{-1}$, where $(\Phi, \Psi) \in$ Homeo(M) $\times$ Homeo(N) and $f \in C^\infty(M, N)$. Then C^0-$\mathcal{A}$-orbit of $f \in C^\infty(M, N)$ is the orbit through f with respect to this action.

We note that $\mathcal{CS}^{\infty}_{pr}(n,p) \subset \mathcal{CT}_{pr}(n,p)$ (resp. $\mathcal{S}^{\infty}_{pr}(n,p) \subset \mathcal{T}_{pr}(n,p)$). We note also that $\mathcal{CS}^{0}_{pr}(n,p) = \mathcal{CS}^{\infty}_{pr}(n,p)$ (resp. $\mathcal{S}^{0}_{pr}(n,p) = \mathcal{S}^{\infty}_{pr}(n,p)$) for nice dimension pair (n,p) in the sense of Mather [**26**] by [**8**].

LEMMA 3.1. *Let $\mathfrak{F}$ be a two-colored C^0 equivalence class and $(f,(R,B)) : M \to N$ be in $\mathcal{CT}_{pr}(n,p)$. Set $\mathfrak{F}(f)$ be the set of point in N whose corresponding fiber is two-colored C^0 equivalent to $\mathfrak{F}$. Then $\mathfrak{F}(f)$ is C^0 submanifold of N of constant codimension if it is nonempty. Furthermore, this codimension does not depend on a particular choice of $(f,(R,B)) : M \to N$.*

The first assertion of this lemma can by proved by Thom's second isotopy lemma (for example, see [**13**, Chapter II, §5]).

By virtue of the above lemma, we can define the codimension of the two-colored C^0 equivalence class $\mathfrak{F}$ by the codimension of $\mathfrak{F}(f)$ in N : cod $\mathfrak{F} :=$ cod $\mathfrak{F}(f)$. We note that this definition is well-defined.

Let us introduce the following notion which will play essential role throughout of this Chapter.

DEFINITION 3.2. Suppose that an equivalence relation $c\rho = c\rho_{n,p}$ (or $\rho = \rho_{n,p}$) among the fibers of proper two-colored Thom maps (resp. proper Thom maps) from n-manifold into p-manifold is given. We say that the relation $c\rho$ (resp. ρ) is *two-colored admissible* (resp. *admissible*) if the following conditions are satisfied.

(1) If two fiber are two-colored C^0 equivalent (resp. C^0 equivalent), then they are also equivalent with respect to $c\rho$ (resp. ρ).
(2) For any two proper two-colored Thom maps $(f_i(R_i,B_i)) : M_i^n \to N_i^p$ and for any points $q_i \in N_i, i = 0,1$, such that the fibers over q_i are equivalent to each other with respect to $c\rho$ (resp. ρ), there exist neighborhoods U_i of q_i in N_i, $(i = 0,1)$, and a homeomorphism $\varphi : U_0 \to U_1$ such that $\varphi(q_0) = q_1$ and $\varphi(U_0 \cap \mathfrak{F}(f_0)) = U_1 \cap \mathfrak{F}(f_1)$ for every equivalence class $\mathfrak{F}$ of fibers with respect to $c\rho$ (resp. ρ), where $\mathfrak{F}(f_i)$ is the set of points in N_i which has a fiber of f_i of type $\mathfrak{F}$.

For example, the two-colored C^0 equivalence is clearly two-colored admissible in the above sense. We denote the two-colored C^0 equivalence relation among the fibers of elements of $\mathcal{CT}_{pr}(n,p)$ by $c\rho^0_{n,p}$. We note that the C^0 equivalence is not two-colored admissible. We denote the C^0 equivalence relation among the fibers of elements of $\mathcal{T}_{pr}(n,p)$ by $\rho^0_{n,p}$.

In the following argument, we fix a two-colored admissible equivalence relation $c\rho = c\rho_{n,p}$ as in Definition 3.2.

Then we obtain the following.

LEMMA 3.3. *For every equivalence class $\mathfrak{F}$ with respect to a two-colored admissible equivalence relation $c\rho$, and for every proper two-colored Thom map $(f,(R,B))$: $M \to N$, the subset $\mathfrak{F}(f)$ of N is a union of strata and is a C^0 submanifold of N of constant codimension if it is nonempty. Furthermore, this codimension does not depend on a particular choice of $(f,(R,B))\colon M \to N$ in $\mathcal{CT}_{pr}(n,p)$.*

From this Lemma, the codimension of the equivalence class with respect to $c\rho$ makes sense.

For an equivalence class $\widetilde{\mathfrak{F}}$ of fibers with respect to $c\rho$ with $\kappa = \kappa(\widetilde{\mathfrak{F}})$, let $\partial\widetilde{\mathfrak{F}}$ be the set of equivalence classes $\widetilde{\mathfrak{G}}$ of fibers with respect to $c\rho$ of codimension $\kappa+1$ such that $\widetilde{\mathfrak{G}}(f) \subset \overline{\widetilde{\mathfrak{F}}(f)} \setminus \widetilde{\mathfrak{F}}(f)$ for every $f \in \mathcal{CT}_{pr}(n,p)$.

Let $c\rho$ be a two-colored admissible equivalence relation as in Definition 3.2 for the fibers of elements of $\mathcal{CT}_{pr}(n,p)$. For $\kappa \in \mathbb{Z}$, let $C^{\kappa}(\mathcal{CT}_{pr}(n,p), c\rho)$ be the formal $\mathbb{Z}_2$-vector space spanned by the all equivalence class of the fiber of elements of $\mathcal{CT}_{pr}(n,p)$ of codimension κ with respect to the equivalence relation $c\rho$, which may possibly be infinitely dimension vector space. If there is no such equivalence class (for example, if $\kappa > p$ or $\kappa < 0$), then we simply put $C^\kappa(\mathcal{CT}_{pr}(n,p), c\rho) = 0$.

Then we define $\mathbb{Z}_2$-linear map $\delta_\kappa : C^\kappa(\mathcal{CT}_{pr}(n,p), c\rho) \to C^{\kappa+1}(\mathcal{CT}_{pr}(n,p), c\rho)$ by

$$\delta_\kappa(\mathfrak{F}) = \sum_{\kappa(\mathfrak{G})=\kappa+1} n_{\mathfrak{F}}(\mathfrak{G})\mathfrak{G}$$

where $\mathfrak{F}$ is the equivalence class of fibers of elements of $\mathcal{CT}_{pr}(n,p)$ and $n_{\mathfrak{F}}(\mathfrak{G}) \in \mathbb{Z}_2$ is the number modulo 2 of the components $\mathfrak{F}(f)$ which locally adjacent to the component $\mathfrak{G}(f)$ (For examples, see Figure 1.7). We note that the definition of the coefficient $n_{\mathfrak{F}}(\mathfrak{G}) \in \mathbb{Z}_2$ and the map δ_κ is well-defined by virtue of the above lemma 3.3 and the definition of an admissible equivalence relation.

Furthermore, we can prove that $\delta_{\kappa+1} \circ \delta_\kappa = 0$. We call the resulting cochain complex

$$\mathcal{C}(\mathcal{CT}_{pr}(n,p), c\rho) = (C^\kappa(\mathcal{CT}_{pr}(n,p), c\rho), \delta_\kappa)_\kappa$$

the *universal complex of singular fibers for proper two-colored Thom maps from n-manifold into p-manifold with respect to the admissible equivalence relation* $c\rho$, and we denote its cohomology group of dimension κ by $H^\kappa(\mathcal{CT}_{pr}(n,p), c\rho)$.

Then we obtain the geometric characterization of the coboundary of the cochain complex of the singular fibers of two-colored map.

PROPOSITION 3.4. *For every equivalence class $\mathfrak{F}$ of fibers with respect to a two-colored admissible equivalence relation $c\rho$, and for every $(f,(R,B)) : M \to N$ in $\mathcal{CT}_{pr}(n,p)$, the $\mathbb{Z}_2$-chain*

$$\sum_{\mathfrak{G}\in\partial\mathfrak{F}} n_{\mathfrak{F}}(\mathfrak{G})\overline{\mathfrak{G}(f)}$$

(of closed support) is a cycle in N and represents the zero homology class in the homology $H^c_{p-\kappa-1}(N;\mathbb{Z}_2)$ of closed support, where κ denotes the codimension of $\mathfrak{F}$.

3.1. Homomorphism induced by changing of color. Let us consider the two-colored admissible equivalence relation $c\rho_{n,p}$. Then we obtain homomorphism $\gamma_\kappa : C^\kappa(\mathcal{CT}_{pr}(n,p), c\rho_{n,p}) \to C^\kappa(\mathcal{CT}_{pr}(n,p), c\rho_{n,p})$ defined by

$$\gamma_\kappa(\mathfrak{F}) = \widetilde{\mathfrak{F}},$$

where $\widetilde{\mathfrak{F}}$ is the class represented by the singular fiber of two-colored conjugate of $\mathfrak{F}$. We note that γ_κ is the involution ($\gamma_\kappa \circ \gamma_\kappa = id_{C^\kappa(\mathcal{CT}_{pr}(n,p),c\rho_{n,p})}$) and $\{\gamma_\kappa\}_\kappa$ is cochain maps. We note that the quotient of this involution induce natural monomorphism.

$$C^\kappa(\mathcal{CT}_{pr}(n,p), c\rho_{n,p})/\gamma_\kappa \to C^\kappa(\mathcal{T}_{pr}(n,p), \rho_{n,p}),$$

where $\rho_{n,p}$ is a admissible equivalence relation induced by losing the condition with respect to the color of $c\rho_{n,p}$ and $C^\kappa(\mathcal{T}_{pr}(n,p), \rho_{n,p})$ is the formal $\mathbb{Z}_2$ vector space spanned by the equivalence classes with respect to $\rho_{n,p}$ obtained in the book [**43**]. In particular, this quotient is not isomorphism because there are Thom maps which are not two colorable.

3.2. Homomorphism induced by suspension. A singular fiber of codimension k two-colored map into a p-manifold can naturally be identified with a singular fiber of a codimension k two-colored map into a $(p+1)$-manifold. In general we can identify with a singular fiber of codimension k two-colored map into a $(p+l)$-manifold.

DEFINITION 3.5. Let $(f,(R,B)) : M \to N$ be in $\mathcal{CT}_{pr}(n,p)$ with $k = p - n$. For a positive integer l, we call the map

$$(f \times id_{\mathbb{R}^l}, (R \times \mathbb{R}^l, B \times \mathbb{R}^l)) : M \times \mathbb{R}^l \to N \times \mathbb{R}^l$$

the *l-suspension of f*. We note that $S(f \times id_{\mathbb{R}^l}) = S(f) \times \mathbb{R}^l$ and $(f \times id_{\mathbb{R}^l})(S(f) \times \mathbb{R}^l) = f(S(f)) \times \mathbb{R}^l$. Then we obtain two-colored map $(f \times id_{\mathbb{R}^l}, (R \times \mathbb{R}^l, B \times \mathbb{R}^l))$. Thus, the l-th suspension of a proper two-colored Thom map is again a proper two-colored Thom map. Furthermore, to the fiber of $(f, (R, B))$ over a point $q \in N$, we associate the fiber of $f \times id_{\mathbb{R}^l}$ over $(y, 0)$. We call that the latter fiber is *obtained from the original fiber by the l-th suspension.*

DEFINITION 3.6. Let fix an integer k. Suppose that for each dimension pair (n, p) with $p - n = k$ and $\min(n, p) \geq 0$, we are given a two-colored admissible equivalence relation $c\rho_{n,p}$ for the fibers of elements of $\mathcal{CT}_{pr}(n, p)$. Such a system of equivalence relations

$$\mathcal{CR}_k = \{c\rho_{n,p} \mid p - n = k, \min(n, p) \geq 0\}$$

is said to be *stable* if the following condition is satisfied: if two fiber of proper two-colored Thom maps from n-manifolds into p-manifolds are equivalent with respect to $c\rho_{n,p}$, then their l-th suspensions are also equivalent with respect to $c\rho_{n+l,p+l}$ for all $l > 0$.

Suppose that a stable system of admissible equivalence relations $\mathcal{CR}_k$ is given for the fibers of proper two-colored Thom maps of codimension k. Then, for every pair (n, p) with $p - n = k$ and a positive integer l, the suspension induces a natural map

$$s_\kappa : C^\kappa(\mathcal{CT}_{pr}(n + l, p + l), c\rho_{n+l,p+l}) \to C^\kappa(\mathcal{CT}_{pr}(n, p), c\rho_{n,p})$$

for $\kappa \in \mathbb{Z}$. More precisely, when $0 \leq \kappa \leq p$, for an equivalence class $\widetilde{\mathfrak{F}} \in C^\kappa(\mathcal{CT}_{pr}(n + l, p + l), c\rho_{n+l,p+l})$ of fibers with respect to $c\rho_{n+l,p+l}$, we define $s_\kappa(\widetilde{\mathfrak{F}})$ to be the sum of all those equivalence classes of fibers of codimension κ with respect to $c\rho_{n,p}$ whose l-suspensions are contained in $\widetilde{\mathfrak{F}}$. For $\kappa > p$ or $\kappa < 0$, we simply put $s_\kappa = 0$. We note that $s_{\dot{\kappa}}$ is well-defined $\mathbb{Z}_2$-linear map by virtue of Definition 3.2.

LEMMA 3.7. *The $\mathbb{Z}_2$-linear map s_κ is a monomorphism for every $\kappa < p$.*

LEMMA 3.8. *The system of $\mathbb{Z}_2$-linear maps $\{s_\kappa\}_\kappa$ defines a cochain map*

$$\mathcal{C}(\mathcal{CT}_{pr}(n + l, p + l), c\rho_{n+l,p+l}) \to \mathcal{C}(\mathcal{CT}_{pr}(n, p), c\rho_{n,p}).$$

In other words, we have $\delta_\kappa \circ s_\kappa = s_{\kappa+1} \circ \delta_\kappa$ for all $\kappa \in \mathbb{Z}$.

It follows easily from the definition of s_κ that the composition of

$$s_\kappa : C^\kappa(\mathcal{CT}_{pr}(n+l+l', p+l+l'), c\rho_{n+l+l',p+l+l'}) \to C^\kappa(\mathcal{CT}_{pr}(n+l, p+l), c\rho_{n+l,p+l})$$

and

$$s_\kappa : C^\kappa(\mathcal{CT}_{pr}(n + l, p + l), c\rho_{n+l,p+l}) \to C^\kappa(\mathcal{CT}_{pr}(n, p), c\rho_{n,p})$$

coincides with

$$s_\kappa : C^\kappa(\mathcal{CT}_{pr}(n + l + l', p + l + l'), c\rho_{n+l+l',p+l+l'}) \to C^\kappa(\mathcal{CT}_{pr}(n, p), c\rho_{n,p}).$$

By this observation together with Lemma 3.8, for a fixed integer k, the projective limit

$$\mathcal{C}(\widetilde{\mathcal{CT}}_{pr}(k), \mathcal{CR}_k) = \lim_{\leftarrow p} \mathcal{C}(\mathcal{CT}_{pr}(p - k, p), c\rho_{p-k,p})$$

is well-defined as a cochain complex, where $\widetilde{\mathcal{CT}}_{pr}(k)$ is defined by

$$\widetilde{\mathcal{CT}}_{pr}(k) = \cup_{p-n=k}\mathcal{CT}_{pr}(n, p).$$

We call $\mathcal{C}(\mathcal{CT}_{pr}(k), \mathcal{CR}_k)$ the *universal complex of singular fibers for codimension k proper two-colored Thom maps with respect to the stable system of two-colored admissible equivalence relation* $\mathcal{CR}_k$. We write its cohomology group of dimension κ by $H^\kappa(\widetilde{\mathcal{CT}}_{pr}(k), \mathcal{CR}_k)$.

LEMMA 3.9. *The natural map*

$$\Phi^{\kappa}_{n,p} : C^{\kappa}(\widetilde{\mathcal{CT}}_{pr}(k), \mathcal{CR}_k) \to C^{\kappa}(\widetilde{\mathcal{CT}}_{pr}(n,p), c\rho_{n,p})$$

induced by the projection is a monomorphism if $\kappa \leq p$. *Furthermore, the system of* $\mathbb{Z}_2$*-linear maps* $\{\Phi^{\kappa}_{n,p}\}_{\kappa}$ *defines a cochain map*

$$\mathcal{C}(\widetilde{\mathcal{CT}}_{pr}(k), \mathcal{CR}_k) \to \mathcal{C}(\widetilde{\mathcal{CT}}_{pr}(n,p), c\rho_{n,p}).$$

3.3. Restriction of the class of singular fibers.

DEFINITION 3.10. A two-colored C^0 equivalence class $\mathfrak{F}$ of fiber of elements of $\mathcal{CT}_{pr}(n,p)$ is said to be *under* another two-colored C^0 equivalence class $\mathfrak{G}$ of fibers if for some representative $(f,(R,B)) : (M, f^{-1}(q)) \to (N,q)$ of $\mathfrak{F}$, there is a point q' arbitrary close to q which has a fiber of type $\mathfrak{G}$. In this case, we also say that $\mathfrak{G}$ is *over* $\mathfrak{F}$.

Let $\Delta = \Delta_{n,p}$ be the set of two-colored C^0 equivalence classes of the fiber of elements of $\mathcal{CT}_{pr}(n,p)$. We call Δ *ascending set* if for an arbitrary equivalence class in the set Δ, every class over it also belongs to the set Δ. On the other hand, we call Δ *descending set* if every class under it also belongs to the set. We say a proper two-colored Thom map $f : M \to N$ of n-manifold into p-manifold a Δ*-map* if its fibers all lie in Δ.

Let $\Delta = \Delta_{n,p}$ be as above and let $c\rho^{\Delta} = c\rho^{\Delta}_{n,p}$ be a two-colored equivalence relation among the fibers of Δ-maps which is two-colored admissible in the sense as in Definition 3.2. Then, we can naturally define the universal complex $\mathcal{C}(\Delta_{n,p}, c\rho^{\Delta})$ of singular fibers for Δ-maps with respect to the two-colored admissible equivalence relation $c\rho^{\Delta}$. We write the corresponding cohomology group of dimension κ by $H^{\kappa}(\Delta_{n,p}, c\rho^{\Delta})$.

Now let us vary the dimension pair (n,p) keeping the codimension $p - n = k$ fixed. Let

$$\widetilde{\Delta} = \widetilde{\Delta}_k = \cup_{p-n=k}\Delta_{n,p}$$

be a set of two-colored C^0 equivalence classes of fibers of proper two-colored Thom maps of codimension k such that each $\Delta_{n,p}$ is an ascending set of two-colored C^0 equivalence classes of fibers of elements of $\mathcal{CT}_{pr}(n,p)$, and that $\widetilde{\Delta}$ is closed under suspension in the sense of Definition 3.5. For example, the set of all C^0 equivalence classes of elements of $\widetilde{\mathcal{CS}}^0_{pr}(k)$ is such a set, where $\widetilde{\mathcal{CS}}^0_{pr}(k)$ is defined by

$$\widetilde{\mathcal{CS}}^0_{pr}(k) = \cup_{p-n=k}\mathcal{CS}^0_{pr}(n.p)$$

We say that a proper two-colored Thom map of codimension k is a $\widetilde{\Delta}_k$-map if its fibers all lies in $\widetilde{\Delta}_k$. We use the same notation $\widetilde{\Delta} = \widetilde{\Delta}_k$ for the set of all $\widetilde{\Delta}_k$-maps.

Let $\mathcal{CR}^{\Delta}_k = \{c\rho^{\Delta_{p-k,p}}_{p-k,p}\}_p$ be a system of equivalence relations, where each $c\rho^{\Delta_{p-k,p}}_{p-k,p}$ is a two-colored admissible equivalence relation among the fibers of $\Delta_{p-k,p}$-maps. Furthermore, we assume that the system $\mathcal{CR}^{\widetilde{\Delta}}_k$ of two-colored admissible equivalence relation is stable in the sense of Definition 3.6.

Then, we can naturally define the universal complex of singular fibers

$$\mathcal{C}(\widetilde{\Delta}_k, \mathcal{CR}^{\widetilde{\Delta}}_k)$$

for $\widetilde{\Delta}_k$-maps with respect to the stable system of two-colored admissible equivalence relation $\mathcal{CR}^{\widetilde{\Delta}}_k$. We write its cohomology group of dimension κ by $H^{\kappa}(\widetilde{\Delta}_k, \mathcal{CR}^{\widetilde{\Delta}}_k)$.

When a class of proper two-colored Thom map is given, let us consider the following definitions.

DEFINITION 3.11. (1). Let $\Gamma_{n,p} = \Gamma$ be a subset of $\mathcal{CT}_{pr}(n,p)$. We denote by $\Delta(\Gamma_{n,p}) = \Delta(\Gamma)$ the set of all C^0 equivalence classes of fibers of elements $\Gamma_{n,p}$. Then, it is clear that $\Delta(\Gamma_{n,p})$ is an ascending set and the set of all $\Delta(\Gamma_{n,p})$-maps contain the original set $\Gamma_{n,p}$ of maps. For two-colored admissible equivalence relation $c\rho^\Gamma$ among the elements of $\Delta(\Gamma_{n,p})$, we define the universal complex of singular fibers for $\Gamma_{n,p}$ with respect to $c\rho^\Gamma$ by

$$\mathcal{C}(\Gamma_{n,p}, c\rho^\Gamma) := \mathcal{C}(\Delta(\Gamma_{n,p}), c\rho^\Gamma).$$

Furthermore, we denote the corresponding cohomology group of dimension κ by $H^\kappa(\Gamma_{n,p}, c\rho^\Gamma)$.
(2). Let $\widetilde{\Gamma}_k = \widetilde{\Gamma}$ be a subset of $\mathcal{CT}_{pr}(k)$. We denote by $\Delta(\widetilde{\Gamma}_k) = \Delta(\widetilde{\Gamma})$ the set of all C^0 equivalence classes of fibers of elements of $\widetilde{\Gamma}_k$ and their suspensions. Then, we have

$$\Delta(\widetilde{\Gamma}_k) = \cup_{p-n=k}\Delta(\Gamma_{n,p})$$

where $\Delta(\Gamma_{n,p})$ is the set of C^0 equivalence classes in $\Delta(\widetilde{\Gamma}_k)$ of fibers of maps of n-manifolds into p-manifolds and each $\Delta(\Gamma_{n,p})$ is an ascending set. Furthermore, $\Delta(\widetilde{\Gamma}_k)$ is closed under suspension. Then, it is clear that the set of all $\Delta(\widetilde{\Gamma}_k)$-maps contain the original set $\widetilde{\Gamma}_k$ of maps. For a stable system of admissible equivalence relations $\mathcal{R}^{\widetilde{\Gamma}_k}$ among the elements of $\Delta(\widetilde{\Gamma}_k)$, we define the universal complex of singular fibers for $\widetilde{\Gamma}_k$ with respect to $\mathcal{R}_k^{\widetilde{\Gamma}}$ by

$$\mathcal{C}(\widetilde{\Gamma}_k, \mathcal{R}_k^{\widetilde{\Gamma}}) = \mathcal{C}(\Delta(\widetilde{\Gamma}_k), \mathcal{R}_k^{\widetilde{\Gamma}}).$$

Furthermore, we denote the corresponding cohomology group of codimension κ by $H^\kappa(\widetilde{\Gamma}_k, \mathcal{R}_k^{\widetilde{\Gamma}})$

4. Stable maps of n-manifolds into p-manifolds, $p = n-1, n, n+1$

Now let us consider a more specific situation, the case of proper two-colored C^∞ stable maps of n-manifolds into p-manifolds.

4.1. Stable Maps of 5-Manifolds into 4-Manifolds.

Since $(5,4)$ is a nice dimension pair in the sense of Mather [**26**], if dim$M = 5$ and dim$N = 4$, then the set of all C^∞ stable maps is open and dense in $C^\infty(M,N)$ with respect to Whitney C^∞ topology as long as M is compact.

The following characterization of proper C^∞ stable maps from 5-manifolds into 4-manifolds is well-known.

PROPOSITION 4.1. *A proper smooth map $f : M \to N$ from a 5-manifold into a 4-manifold is C^∞ stable if and only if the following conditions are satisfied.*

> (*i*) (*Local condition*) *For every $p \in M$, there exist local coordinates (a,b,c,x,y) and (X,Y,Z,W) about $p \in M$ and $f(p) \in N$ respectively such that one of the following holds:*

$$(X \circ f, Y \circ f, Z \circ f, W \circ f)$$

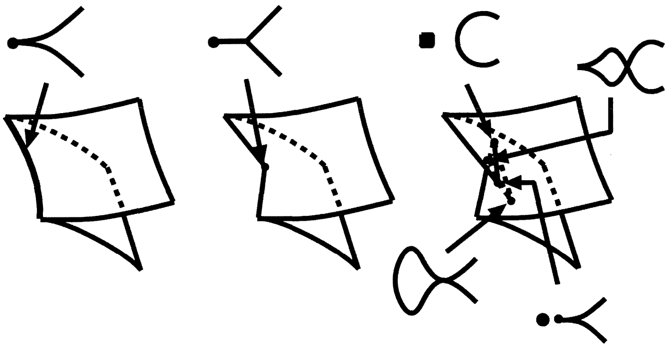

FIGURE 2.2. Map germ corresponding to a butterfly point

$$= \begin{cases} (a,b,c,x) & p: \textit{regular point}, \\ (a,b,c,x^2+y^2) & p: \textit{definite fold}, \\ (a,b,c,x^2-y^2) & p: \textit{indefinite fold}, \\ (a,b,c,x^3+ax-y^2) & p: \textit{cusp}, \\ (a,b,c,x^4+ax^2+bx+y^2) & p: \textit{definite swallow-tail}, \\ (a,b,c,x^4+ax^2+bx-y^2) & p: \textit{indefinite swallow-tail}, \\ (a,b,c,x^5+ax^4+bx^3+cx^2-y^2) & p: \textit{butterfly}, \\ (a,b,c,3x^2y+y^3+a(x^2+y^2)+bx+cy) & p: \textit{definite } D_4, \\ (a,b,c,3x^2y-y^3+a(x^2+y^2)+bx+cy) & p: \textit{indefinite } D_4, \end{cases}$$

(ii) (Global condition) Set $S(f) = \{p \in M \mid \text{rank}df_p < 4\}$*, which is a regular closed 3-dimension submanifold of M under the above condition 1. Then for every* $q \in f(S(f))$, $f^{-1}(q) \cap S(f)$ *consists at most four points and the multi-germ*

$$(f|_{S(f)}, f^{-1}(q) \cap S(f))$$

is smoothly right-left equivalent to one of the thirteen multi-germs as follows:

(1) *single immersion germ which corresponding to a fold point,*
(2) *normal crossing of two immersion germs, each of which corresponds to a fold point,*
(3) *normal crossing of three immersion germs, each of which corresponds to a fold point,*
(4) *map germ corresponding to a cusp point,*
(5) *transverse crossing of a cusp germ and an immersion germ corresponding to a fold point,*
(6) *map germ corresponding to a swallow-tail point,*
(7) *normal crossing of four immersion germs, each of which corresponds to a fold point,*
(8) *transverse crossing of cusp germ and a normal crossing of two immersion germs which corresponding to a fold point,*
(9) *transverse crossing of a swallow-tail germ and an immersion germ corresponding to a fold point,*
(10) *normal crossing of two cusp germs,*
(11) *map germ corresponds to a butterfly point,*
(12) *map germ corresponds to a definite* D_4 *point,*
(13) *map germ corresponds to an indefinite* D_4 *point.*

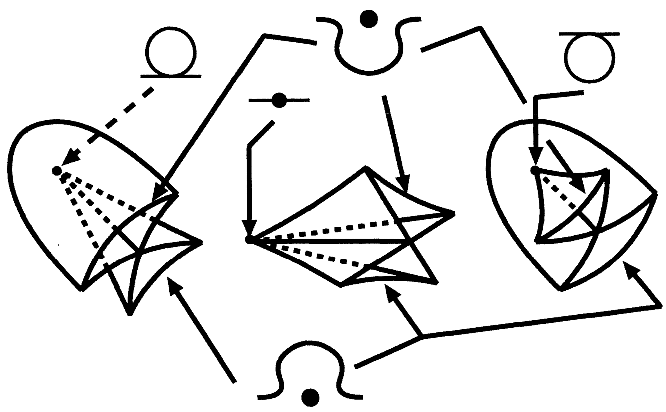

FIGURE 2.3. Map germ corresponding to a definite D_4 point

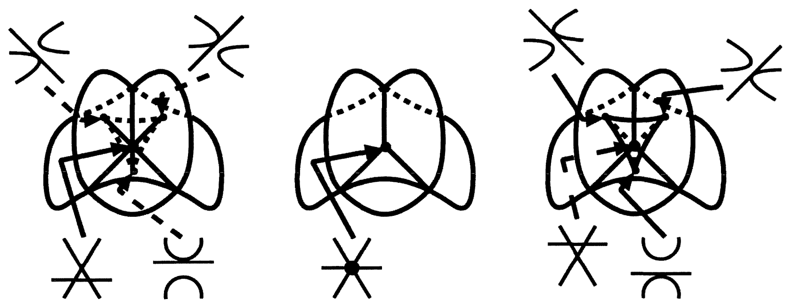

FIGURE 2.4. Map germ corresponding to an indefinite D_4 point

We note that the map germs (1) – (6) in Proposition 4.1 correspond to the suspension of the map germs in Figure 1.2. The map germs (11)–(13) are described in Figures 2.2, 2.3 and 2.4 respectively, in order to draw 3-dimensional objects in a dimensional space, we have depicted three "sections" by 3-dimensional spaces for each object. In Figures 2.2, 2.3 and 2.4, around 1-dimensional complex are the neighborhood of corresponding singular point in singular fibers.

Proposition 4.1 can be proved by using the transversality theorem and the multi-transversality theorem, since the dimensions pair (5, 4) is in the nice range in the sense of Mather **[26]** (for details, see **[13]**, **[25]** or **[14]**).

We call a D_4 point a $\Sigma^{2,2,0}$ point as well. The following remark of D_4 is obtained in **[44]**

Remark 4.2. The normal forms for D_4 points are slightly different from the usual ones (see, for example, **[1]**). We have chosen them so that at an indefinite D_4 point, f can be represented as

$$(a, \eta, \zeta) \mapsto (a, \eta, \Im(\zeta^3) + \Re(\bar{\eta}\zeta) + a|\zeta|^2)$$

by using complex numbers, where $i = \sqrt{-1}$, $\eta = b + ic$, $\zeta = x + iy$, $\Im$ means the imaginary part, and $\Re$ means the real part.

Set $\tau = \exp(2\pi i/3)$. Then with respect to the chosen coordinates, we have

$$f \circ \tilde{\varphi}_\tau = \varphi_\tau \circ f,$$

where $\tilde{\varphi}_\tau$ and φ_τ are orientation preserving diffeomorphisms defined by

$$\begin{aligned} \tilde{\varphi}_\tau(a,\eta,\zeta) &= (a,\tau\eta,\tau\zeta), \quad \text{and} \\ \varphi_\tau(X,Y+iZ,W) &= (X,\tau(Y+iZ),W) \end{aligned}$$

respectively. This shows that an indefinite D_4 point (or a local fiber through an indefinite D_4 point) has a (orientation preserving) symmetry of order 3.

Set $\tau' = \exp(\pi i/3)$ so that we have ${\tau'}^2 = \tau$. Then we have

$$f \circ \tilde{\varphi}_{\tau'} = \varphi_{\tau'} \circ f,$$

where $\tilde{\varphi}_{\tau'}$ and $\varphi_{\tau'}$ are diffeomorphisms defined by

$$\begin{aligned} \tilde{\varphi}_{\tau'}(a,\eta,\zeta) &= (-a,-\tau'\eta,\tau'\zeta), \quad \text{and} \\ \varphi_\tau(X,Y+iZ,W) &= (-X,-\tau'(Y+iZ),-W) \end{aligned}$$

respectively. Note that $\tilde{\varphi}_{\tau'}$ is orientation reversing while $\varphi_{\tau'}$ is orientation preserving. This shows that an indefinite D_4 point (or a local fiber through an indefinite D_4 point) has a symmetry of order 6 and that the generator reverses the "local orientation" of the fiber. In fact, we have $\tilde{\varphi}_\tau = \tilde{\varphi}_{\tau'}^2$ and $\varphi_\tau = \varphi_{\tau'}^2$.

Remark 4.3. According to du Plessis and Wall [**8**], if (n,p) is in the nice range in the sense of Mather [**26**], a smooth map between manifolds of dimension n and p is C^∞ stable if and only if it is C^0 stable. Hence, the above proposition gives a characterization of C^0 stable maps of 5-manifolds into 4-manifolds as well, since $(5,4)$ is in the nice range.

Let $f : M \to N$ be a stable map of a closed 5-manifold M into a 4-manifold N. For each regular point $x \in M$ of f, the fiber through x is a 1-dimensional submanifold near the point (see Figure 2.5 (0)). For each singular point $p \in M$ of f, based on the local condition of Proposition 3.1 (i), it is easy to determine the diffeomorphism type of a neighbourhood of p in $f^{-1}(f(p))$ as follows.

LEMMA 4.4. *Every point p of a stable map $f : M \to N$ of a closed 5-manifold M into a 4-manifold N has one of the following neighborhood in its corresponding singular fiber $f^{-1}(f(p))$ (see Figure 2.5):*

(1) *isolated point diffeomorphic to $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 0\}$, if p is a definite fold point,*
(2) *union of two transverse arcs diffeomorphic to $\{(x,y) \in \mathbb{R}^2 \mid x^2-y^2 = 0\}$, if p is an indefinite fold point,*
(3) *$(2,3)$-cuspidal arc diffeomorphic to $\{(x,y) \in \mathbb{R}^2 \mid x^3-y^2 = 0\}$, if p is a cusp point,*
(4) *isolated point diffeomorphic to $\{(x,y) \in \mathbb{R}^2 \mid x^4+y^2 = 0\}$, if p is a definite swallowtail,*
(5) *union of two tangent arcs diffeomorphic to $\{(x,y) \in \mathbb{R}^2 \mid x^4-y^2 = 0\}$, if p is an indefinite swallowtail,*
(6) *$(2,5)$-cuspidal arc diffeomorphic to $\{(x,y) \in \mathbb{R}^2 \mid x^5-y^2 = 0\}$, if p is a butterfly point,*
(7) *union of an arc and a point diffeomorhic to $\{(x,y) \in \mathbb{R}^2 \mid 3x^2y+y^3 = 0\}$, if p is a definite $\Sigma^{2,2,0}$ point,*
(8) *union of three arcs meeting at a point diffeomorphic to $\{(x,y) \in \mathbb{R}^2 \mid 3x^2-y^3 = 0\}$, if p is an indefinite $\Sigma^{2,2,0}$ point. point.*

In Figure 2.5, both the black dot (1) and the black square (4) represent an isolated point, although the corresponding map germs are not C^∞ equivalent to each other; we use distinct symbols in order to distinguish them. We note that each singular point $p \in M$, except for a definite fold point and a definite swallowtail point, is incident to some edges in its neighbourhood in $f^{-1}(f(p))$.

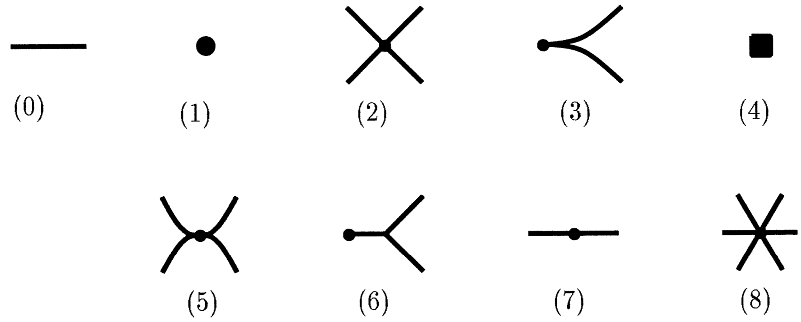

FIGURE 2.5. Neighborhoods of a singular point in a singular fibers of proper stable maps of 5-manifolds into 4-manifolds

We also note that a regular fiber of f is a closed 1-dimensional submanifold of M, namely, a disjoint union of finite number of circles. Thus, for a regular value q of f, the fiber of f over q is C^∞ equivalent to the disjoint union of a finite number of copies of a fiber of a trivial circle bundle. For the singular fibers of f, we have the following.

THEOREM 4.5. *Let $f : M \to N$ be a stable map of a closed 4-manifold M into a 3-manifold N. Then, every singular fiber of f is C^∞ equivalent to the disjoint union of one of the fibers in the following list and a finite number of copies of a fiber of a trivial circle bundle:*

(1) *one of the fibers as depicted in Figure* 1.4,

(2) *a disconnected fiber* $\widetilde{\mathrm{III}}^{0,0,0}$, $\widetilde{\mathrm{III}}^{0,0,1}$, $\widetilde{\mathrm{III}}^{0,1,1}$, $\widetilde{\mathrm{III}}^{1,1,1}$, $\widetilde{\mathrm{III}}^{0,0,2}$, $\widetilde{\mathrm{III}}^{0,2,2}$, $\widetilde{\mathrm{III}}^{1,1,2}$, $\widetilde{\mathrm{III}}^{1,2,2}$, $\widetilde{\mathrm{III}}^{0,1,2}$, $\widetilde{\mathrm{III}}^{2,2,2}$, $\widetilde{\mathrm{III}}^{0,3}$, $\widetilde{\mathrm{III}}^{0,4}$, $\widetilde{\mathrm{III}}^{0,5}$, $\widetilde{\mathrm{III}}^{0,6}$, $\widetilde{\mathrm{III}}^{0,7}$, $\widetilde{\mathrm{III}}^{1,3}$, $\widetilde{\mathrm{III}}^{1,4}$, $\widetilde{\mathrm{III}}^{1,5}$, $\widetilde{\mathrm{III}}^{1,6}$, $\widetilde{\mathrm{III}}^{1,7}$, $\widetilde{\mathrm{III}}^{2,3}$, $\widetilde{\mathrm{III}}^{2,4}$, $\widetilde{\mathrm{III}}^{2,5}$, $\widetilde{\mathrm{III}}^{2,6}$, $\widetilde{\mathrm{III}}^{2,7}$, $\widetilde{\mathrm{III}}^{0,a}$, $\widetilde{\mathrm{III}}^{1,a}$, $\widetilde{\mathrm{III}}^{2,a}$, 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$\widetilde{\mathrm{IV}}^{2,e}$, $\widetilde{\mathrm{IV}}^{2,f}$, $\widetilde{\mathrm{IV}}^{2,g}$ *and* $\widetilde{\mathrm{IV}}^{a,a}$

(3) *one of the connected fibers as depicted in Figures* 1.5, 2.6, 2.7 *and* 2.8.

The figure corresponding to each fiber listed in Theorem 4.5 (2) can be obtained by taking the disjoint union of the fibers in Figure 1.4 or 1.5 corresponding to the

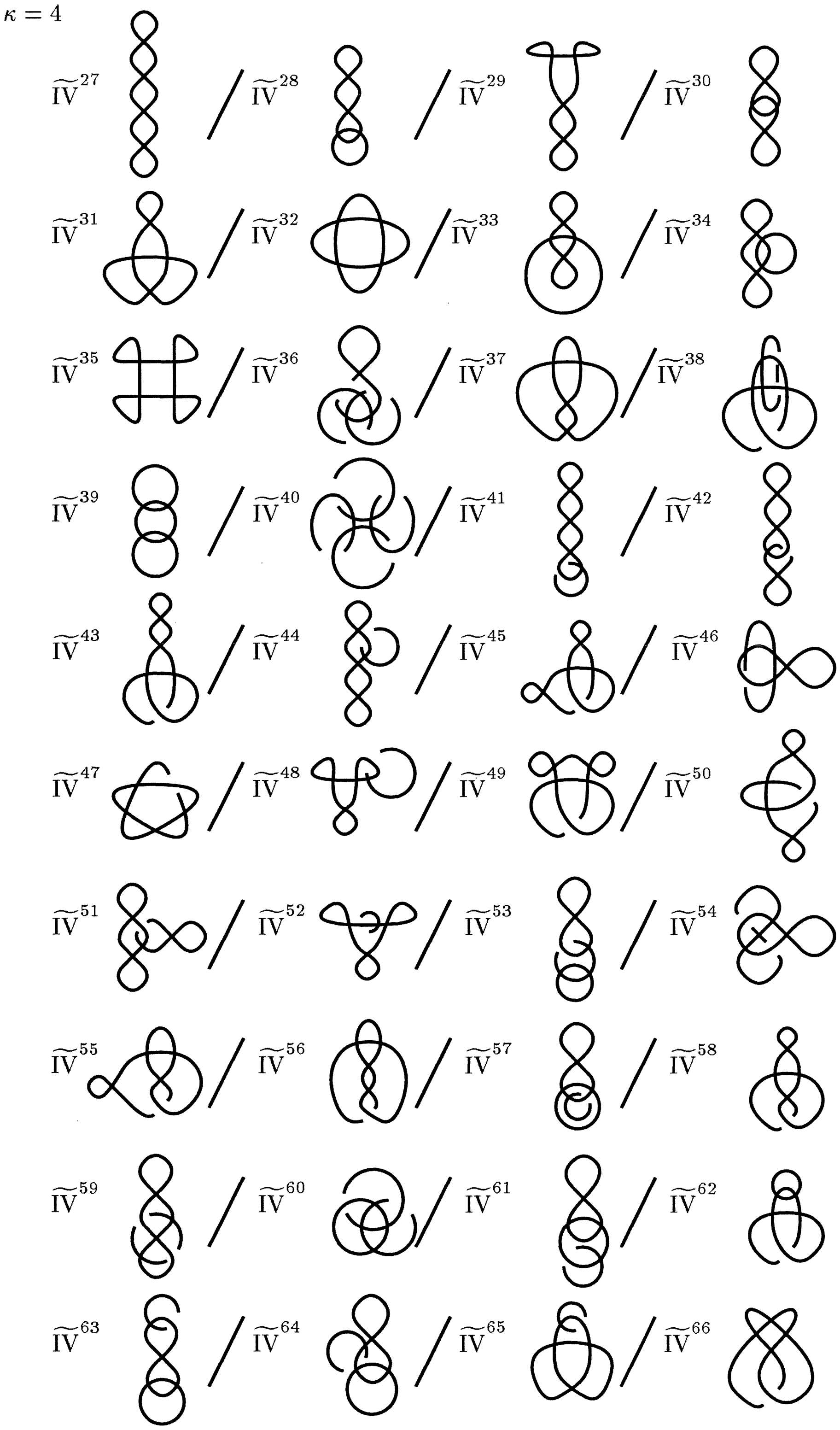

FIGURE 2.6. List of $\kappa = 4$ singular fibers; 1

$\kappa = 4$

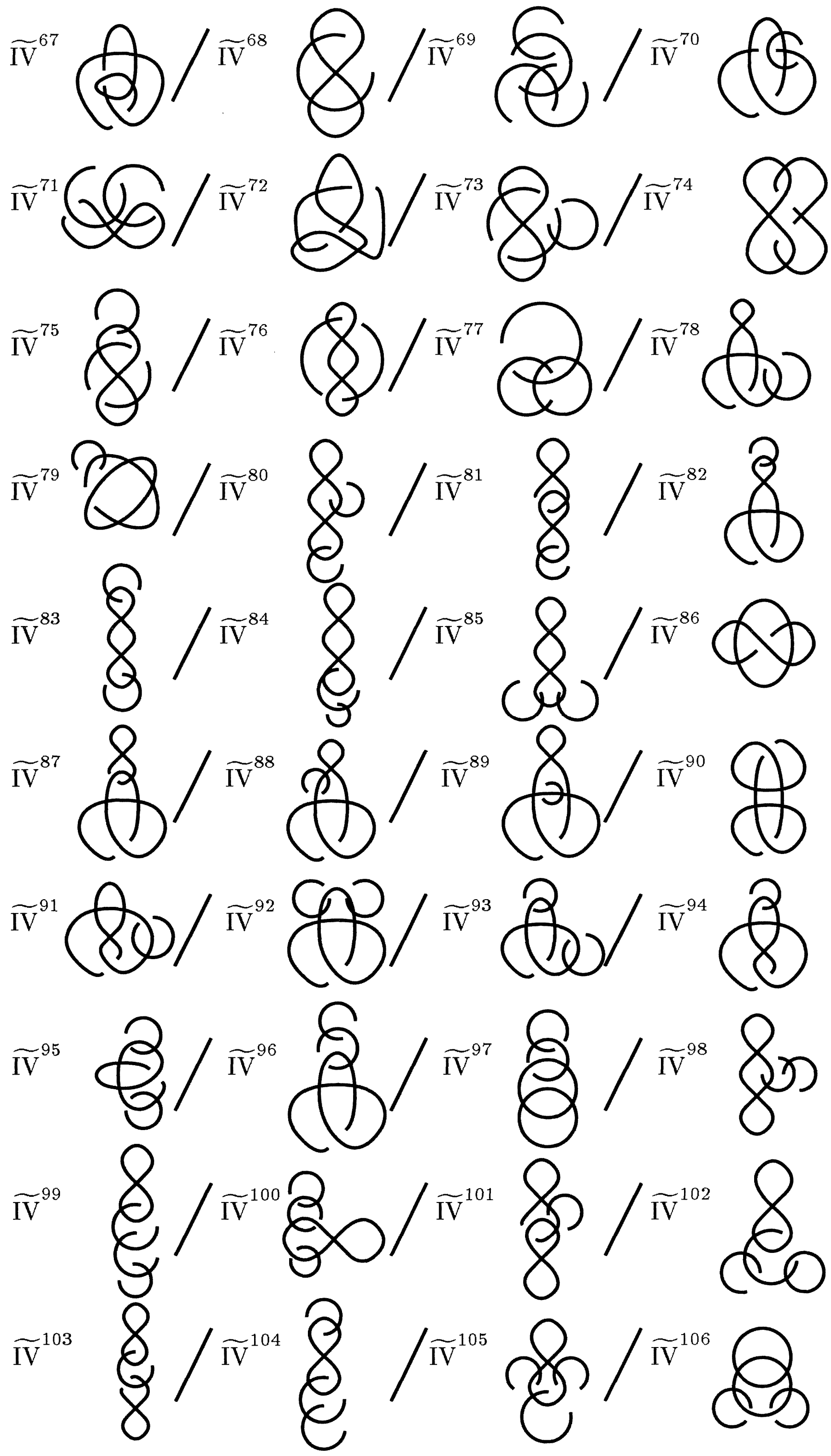

FIGURE 2.7. List of $\kappa = 4$ singular fibers; 2

$\kappa = 4$

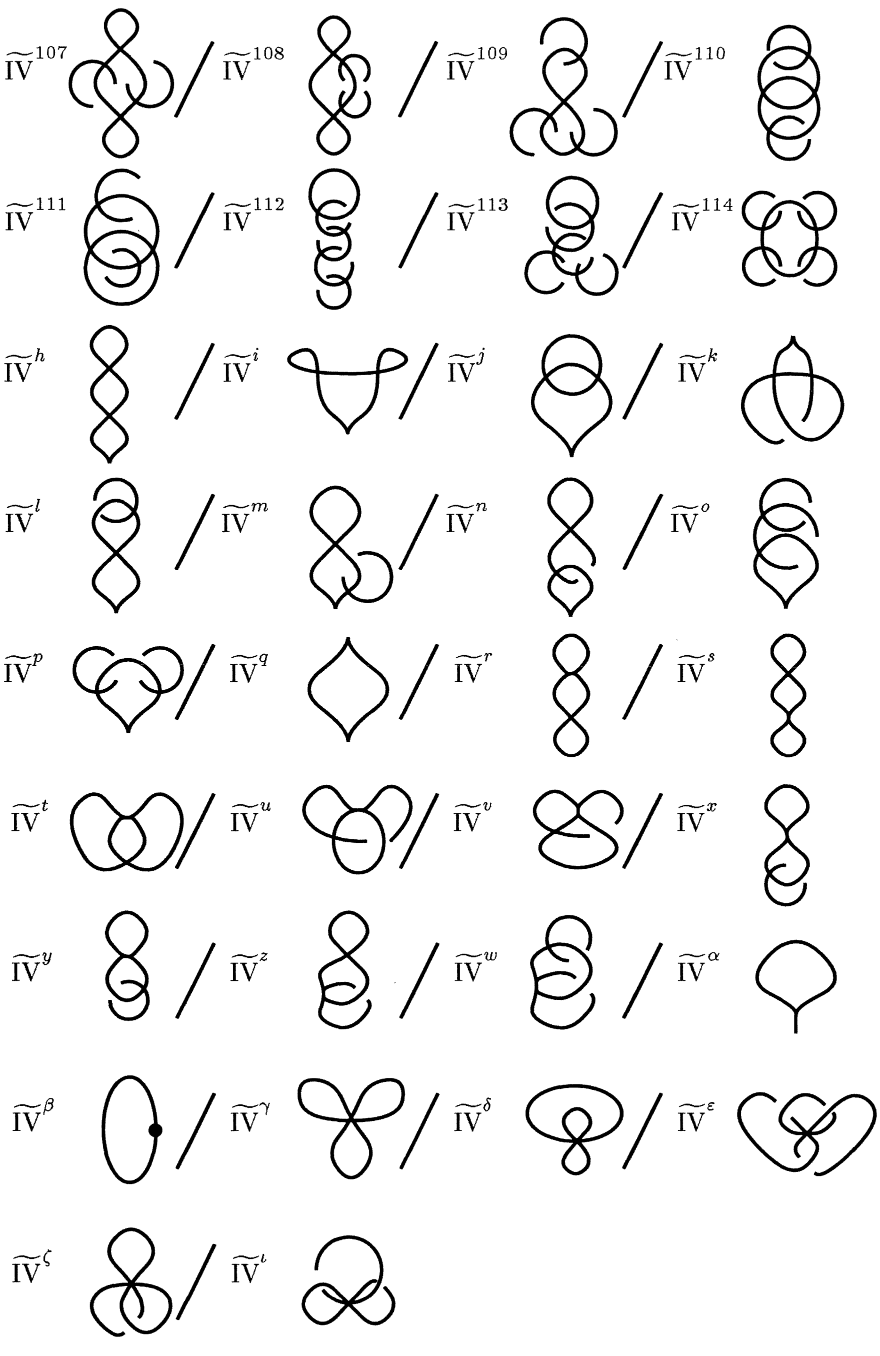

FIGURE 2.8. List of $\kappa = 4$ singular fibers; 3

numbers or letters appearing in the superscript. For example, the figure of the fiber $\widetilde{\mathrm{III}}^{0,0,2}$ consists of two dots and a figure of $\widetilde{\mathrm{I}}^2$ type.

In Figures 1.4, 1.5, 2.6, 2.7 and 2.8 κ denotes the codimension of the set of points in N whose corresponding fibers are C^∞ equivalent to the relevant one.

Furthermore, $\widetilde{\mathrm{I}}^{*}$, $\widetilde{\mathrm{II}}^{*}$, $\widetilde{\mathrm{III}}^{*}$ and $\widetilde{\mathrm{IV}}^{*}$ mean the names of the corresponding singular fibers, and "/" is used only for separating the figures.

In Figures 2.6, 2.7 and 2.8 the singular fibers $\widetilde{\mathrm{IV}}^{*}$ ($* = 27, 28, \dots, 113$ and 114) types are over a point of $f|_{S(f)}$ of (7) in Proposition 4.1 (*ii*), the singular fibers $\widetilde{\mathrm{IV}}^{h}$, $\widetilde{\mathrm{IV}}^{i}$, $\widetilde{\mathrm{IV}}^{j}$, $\widetilde{\mathrm{IV}}^{k}$, $\widetilde{\mathrm{IV}}^{l}$, $\widetilde{\mathrm{IV}}^{m}$, $\widetilde{\mathrm{IV}}^{n}$, $\widetilde{\mathrm{IV}}^{o}$ and $\widetilde{\mathrm{IV}}^{p}$ types are over a point of $f|_{S(f)}$ of (8) in Proposition 4.1 (*ii*), the singular fiber $\widetilde{\mathrm{IV}}^{q}$ is over a point of $f|_{S(f)}$ of (10) in Proposition 4.1 (*ii*), the singular fiber of $\widetilde{\mathrm{IV}}^{r}$, $\widetilde{\mathrm{IV}}^{s}$, $\widetilde{\mathrm{IV}}^{t}$, $\widetilde{\mathrm{IV}}^{t}$, $\widetilde{\mathrm{IV}}^{u}$, $\widetilde{\mathrm{IV}}^{v}$, $\widetilde{\mathrm{IV}}^{x}$, $\widetilde{\mathrm{IV}}^{y}$, $\widetilde{\mathrm{IV}}^{z}$, $\widetilde{\mathrm{IV}}^{w}$ types are over a point of $f|_{S(f)}$ of (9) in Proposition 4.1 (*ii*), the singular fiber $\widetilde{\mathrm{IV}}^{\alpha}$ is over a point of $f|_{S(f)}$ of (11) in Proposition 4.1 (*ii*), the singular fiber of $\widetilde{\mathrm{IV}}^{\beta}$ type is over a point of $f|_{S(f)}$ of (12) in Proposition 4.1 (*ii*), the singular fiber of $\widetilde{\mathrm{IV}}^{\gamma}$, $\widetilde{\mathrm{IV}}^{\delta}$, $\widetilde{\mathrm{IV}}^{\varepsilon}$, $\widetilde{\mathrm{IV}}^{\zeta}$ and $\widetilde{\mathrm{IV}}^{\iota}$ types are over a point of $f|_{S(f)}$ of (13) in Proposition 4.1 (*ii*),

We note that the conclusion of Theorem 4.5 holds if f is proper even if M is not closed, where a continuous map is said to be *proper* if the inverse image of a compact set is always compact.

Theorem 4.5 can be proved in two steps. First we show that for a singular value q of f, the union of the components of $f^{-1}(q)$ containing singular points is diffeomorphic to one of the fibers listed in Theorem 4.5 in the sense of Definition 2.3. Second we show that if two singular fibers are diffeomorphic to each other, then they are C^{∞} equivalent in the sense of Definition 2.3, except for the two types of fibers I^{0} and $\widetilde{\mathrm{III}}^{d}$. The proof is very similar to that of [**43**, Theorem 3.5], as we omit the proof here.

Remark 4.6. Each singular fiber described in Theorem 4.5 can be realized as a component (or as a union of some components) of a singular fiber of a stable map of a closed 5-manifold into $\mathbb{R}^4$. This can be seen as follows. Given a singular fiber, we can realize it as a singular fiber of a Morse function parameterized on D^3, $f_t : S \to [-1,1], t \in D^3$, of a compact surface with boundary S into $[-1,1]$, where D^3 denotes the unit disk in $\mathbb{R}^3$. We note that $F : S \times D^3 \to [-1,1] \times D^3$, defined by $F(x,t) = (f_t(x), t)$, is a smooth map and that F has the given singular fiber over $(0,0)$. We call S *a transverse surface* corresponding to the singular fiber (for details, see [**24**]). In this way we obtain a proper smooth map $F|_{\mathrm{Int}S \times \mathrm{Int}D^3} : \mathrm{Int}S \times \mathrm{Int}D^3 \to (-1,1) \times \mathrm{Int}D^3$. Then we can extend the map to a smooth map of a closed 5-manifold containing $\mathrm{Int}S\times$ $\mathrm{Int}D^3$ into $\mathbb{R}^4$. Perturbing the extended map slightly, we obtain a desired stable map.

If the source 5-manifold is orientable, then any transverse surface for any singular fiber is orientable. If the source 5-manifold is non-orientable, then there may exist a non-orientable transverse surface. The transverse surface which corresponds to the singular fiber of I^2 type is a punctured Möbius band. We note that there exists a stable map of a non-orientable 4-manifold into a 3-manifold such that the transverse surface is orientable for any fiber. (For instance, see the example just after Corollary 1.3 in Chapter 1.)

We note that for a stable map $f : M \to N$ of an orientable closed 5-manifold M into a 4-manifold N, the singular fibers of the following types never appear, since they have non-orientable transverse surfaces: $\widetilde{\mathrm{I}}^{2}$, $\widetilde{\mathrm{II}}^{0,2}$, $\widetilde{\mathrm{II}}^{1,2}$, $\widetilde{\mathrm{II}}^{2,2}$, $\widetilde{\mathrm{II}}^{5}$, $\widetilde{\mathrm{II}}^{6}$, $\widetilde{\mathrm{II}}^{7}$, $\widetilde{\mathrm{III}}^{0,0,2}$, $\widetilde{\mathrm{III}}^{0,2,2}$, $\widetilde{\mathrm{III}}^{1,1,2}$, $\widetilde{\mathrm{III}}^{1,2,2}$, $\widetilde{\mathrm{III}}^{0,1,2}$, $\widetilde{\mathrm{III}}^{2,2,2}$, $\widetilde{\mathrm{III}}^{0,5}$, $\widetilde{\mathrm{III}}^{0,6}$, $\widetilde{\mathrm{III}}^{0,7}$, $\widetilde{\mathrm{III}}^{1,5}$, $\widetilde{\mathrm{III}}^{1,6}$, $\widetilde{\mathrm{III}}^{1,7}$, $\widetilde{\mathrm{III}}^{2,3}$, $\widetilde{\mathrm{III}}^{2,4}$, $\widetilde{\mathrm{III}}^{2,5}$, $\widetilde{\mathrm{III}}^{2,6}$, $\widetilde{\mathrm{III}}^{2,7}$, $\widetilde{\mathrm{III}}^{2,a}$, $\widetilde{\mathrm{III}}^{13}$, $\widetilde{\mathrm{III}}^{14}$, $\widetilde{\mathrm{III}}^{15}$, $\widetilde{\mathrm{III}}^{16}$, $\widetilde{\mathrm{III}}^{17}$, $\widetilde{\mathrm{III}}^{18}$, $\widetilde{\mathrm{III}}^{19}$, $\widetilde{\mathrm{III}}^{20}$, $\widetilde{\mathrm{III}}^{21}$, $\widetilde{\mathrm{III}}^{22}$, $\widetilde{\mathrm{III}}^{23}$, $\widetilde{\mathrm{III}}^{24}$, $\widetilde{\mathrm{III}}^{25}$, $\widetilde{\mathrm{III}}^{26}$, $\widetilde{\mathrm{III}}^{c}$, $\widetilde{\mathrm{III}}^{g}$, $\widetilde{\mathrm{IV}}^{0,0,0,2}$, $\widetilde{\mathrm{IV}}^{0,0,2,2}$, $\widetilde{\mathrm{IV}}^{0,2,2,2}$, $\widetilde{\mathrm{IV}}^{1,1,1,2}$, $\widetilde{\mathrm{IV}}^{1,1,2,2}$, $\widetilde{\mathrm{IV}}^{1,2,2,2}$, $\widetilde{\mathrm{IV}}^{0,0,1,2}$, $\widetilde{\mathrm{IV}}^{0,1,1,2}$, $\widetilde{\mathrm{IV}}^{0,1,2,2}$, $\widetilde{\mathrm{IV}}^{2,2,2,2}$, $\widetilde{\mathrm{IV}}^{0,0,5}$,

$\widetilde{\mathrm{IV}}^{0,0,6}$, $\widetilde{\mathrm{IV}}^{0,0,7}$, $\widetilde{\mathrm{IV}}^{0,1,5}$, $\widetilde{\mathrm{IV}}^{0,1,6}$, $\widetilde{\mathrm{IV}}^{0,1,7}$, $\widetilde{\mathrm{IV}}^{1,1,5}$, $\widetilde{\mathrm{IV}}^{1,1,6}$, $\widetilde{\mathrm{IV}}^{1,1,7}$, $\widetilde{\mathrm{IV}}^{0,2,3}$, $\widetilde{\mathrm{IV}}^{0,2,4}$, $\widetilde{\mathrm{IV}}^{0,2,5}$, $\widetilde{\mathrm{IV}}^{0,2,6}$, $\widetilde{\mathrm{IV}}^{0,2,7}$, $\widetilde{\mathrm{IV}}^{1,2,3}$, $\widetilde{\mathrm{IV}}^{1,2,4}$, $\widetilde{\mathrm{IV}}^{1,2,5}$, $\widetilde{\mathrm{IV}}^{1,2,6}$, $\widetilde{\mathrm{IV}}^{1,2,7}$, $\widetilde{\mathrm{IV}}^{2,2,3}$, $\widetilde{\mathrm{IV}}^{2,2,4}$, $\widetilde{\mathrm{IV}}^{2,2,5}$, $\widetilde{\mathrm{IV}}^{2,2,6}$, $\widetilde{\mathrm{IV}}^{2,2,7}$, $\widetilde{\mathrm{IV}}^{3,5}$, $\widetilde{\mathrm{IV}}^{3,6}$, $\widetilde{\mathrm{IV}}^{3,7}$, $\widetilde{\mathrm{IV}}^{4,5}$, $\widetilde{\mathrm{IV}}^{4,6}$, $\widetilde{\mathrm{IV}}^{4,7}$, $\widetilde{\mathrm{IV}}^{5,5}$, $\widetilde{\mathrm{IV}}^{5,6}$, $\widetilde{\mathrm{IV}}^{5,7}$, $\widetilde{\mathrm{IV}}^{6,6}$, $\widetilde{\mathrm{IV}}^{6,7}$, $\widetilde{\mathrm{IV}}^{7,7}$, $\widetilde{\mathrm{IV}}^{5,a}$, $\widetilde{\mathrm{IV}}^{6,a}$, $\widetilde{\mathrm{IV}}^{7,a}$, $\widetilde{\mathrm{IV}}^{0,13}$, $\widetilde{\mathrm{IV}}^{0,14}$, $\widetilde{\mathrm{IV}}^{0,15}$, $\widetilde{\mathrm{IV}}^{0,16}$, $\widetilde{\mathrm{IV}}^{0,17}$, $\widetilde{\mathrm{IV}}^{0,18}$, $\widetilde{\mathrm{IV}}^{0,19}$, $\widetilde{\mathrm{IV}}^{0,20}$, $\widetilde{\mathrm{IV}}^{0,21}$, $\widetilde{\mathrm{IV}}^{0,22}$, $\widetilde{\mathrm{IV}}^{0,23}$, $\widetilde{\mathrm{IV}}^{0,24}$, $\widetilde{\mathrm{IV}}^{0,25}$, $\widetilde{\mathrm{IV}}^{0,26}$, $\widetilde{\mathrm{IV}}^{1,13}$, $\widetilde{\mathrm{IV}}^{1,14}$, $\widetilde{\mathrm{IV}}^{1,15}$, $\widetilde{\mathrm{IV}}^{1,16}$, $\widetilde{\mathrm{IV}}^{1,17}$, $\widetilde{\mathrm{IV}}^{1,18}$, $\widetilde{\mathrm{IV}}^{1,19}$, $\widetilde{\mathrm{IV}}^{1,20}$, $\widetilde{\mathrm{IV}}^{1,21}$, $\widetilde{\mathrm{IV}}^{1,22}$, $\widetilde{\mathrm{IV}}^{1,23}$, $\widetilde{\mathrm{IV}}^{1,24}$, $\widetilde{\mathrm{IV}}^{1,25}$, $\widetilde{\mathrm{IV}}^{1,26}$, $\widetilde{\mathrm{IV}}^{2,8}$, $\widetilde{\mathrm{IV}}^{2,9}$, $\widetilde{\mathrm{IV}}^{2,10}$, $\widetilde{\mathrm{IV}}^{2,11}$, $\widetilde{\mathrm{IV}}^{2,12}$, $\widetilde{\mathrm{IV}}^{2,13}$, $\widetilde{\mathrm{IV}}^{2,14}$, $\widetilde{\mathrm{IV}}^{2,15}$, $\widetilde{\mathrm{IV}}^{2,16}$, $\widetilde{\mathrm{IV}}^{2,17}$, $\widetilde{\mathrm{IV}}^{2,18}$, $\widetilde{\mathrm{IV}}^{2,19}$, $\widetilde{\mathrm{IV}}^{2,20}$, $\widetilde{\mathrm{IV}}^{2,21}$, $\widetilde{\mathrm{IV}}^{2,22}$, $\widetilde{\mathrm{IV}}^{2,23}$, $\widetilde{\mathrm{IV}}^{2,24}$, $\widetilde{\mathrm{IV}}^{2,25}$, $\widetilde{\mathrm{IV}}^{2,26}$, $\widetilde{\mathrm{IV}}^{2,b}$, $\widetilde{\mathrm{IV}}^{0,c}$, $\widetilde{\mathrm{IV}}^{1,c}$, $\widetilde{\mathrm{IV}}^{2,c}$, $\widetilde{\mathrm{IV}}^{41}$, $\widetilde{\mathrm{IV}}^{42}$, $\widetilde{\mathrm{IV}}^{43}$, $\widetilde{\mathrm{IV}}^{44}$, $\widetilde{\mathrm{IV}}^{45}$, $\widetilde{\mathrm{IV}}^{46}$, $\widetilde{\mathrm{IV}}^{47}$, $\widetilde{\mathrm{IV}}^{48}$, $\widetilde{\mathrm{IV}}^{49}$, $\widetilde{\mathrm{IV}}^{50}$, $\widetilde{\mathrm{IV}}^{51}$, $\widetilde{\mathrm{IV}}^{52}$, $\widetilde{\mathrm{IV}}^{53}$, $\widetilde{\mathrm{IV}}^{54}$, $\widetilde{\mathrm{IV}}^{55}$, $\widetilde{\mathrm{IV}}^{56}$, $\widetilde{\mathrm{IV}}^{57}$, $\widetilde{\mathrm{IV}}^{58}$, $\widetilde{\mathrm{IV}}^{59}$, $\widetilde{\mathrm{IV}}^{60}$, $\widetilde{\mathrm{IV}}^{61}$, $\widetilde{\mathrm{IV}}^{62}$, $\widetilde{\mathrm{IV}}^{63}$, $\widetilde{\mathrm{IV}}^{64}$, $\widetilde{\mathrm{IV}}^{65}$, $\widetilde{\mathrm{IV}}^{66}$, $\widetilde{\mathrm{IV}}^{67}$, $\widetilde{\mathrm{IV}}^{68}$, $\widetilde{\mathrm{IV}}^{69}$, $\widetilde{\mathrm{IV}}^{70}$, $\widetilde{\mathrm{IV}}^{71}$, $\widetilde{\mathrm{IV}}^{72}$, $\widetilde{\mathrm{IV}}^{73}$, $\widetilde{\mathrm{IV}}^{74}$, $\widetilde{\mathrm{IV}}^{75}$, $\widetilde{\mathrm{IV}}^{76}$, $\widetilde{\mathrm{IV}}^{77}$, $\widetilde{\mathrm{IV}}^{78}$, $\widetilde{\mathrm{IV}}^{79}$, $\widetilde{\mathrm{IV}}^{80}$, $\widetilde{\mathrm{IV}}^{81}$, $\widetilde{\mathrm{IV}}^{82}$, $\widetilde{\mathrm{IV}}^{83}$, $\widetilde{\mathrm{IV}}^{84}$, $\widetilde{\mathrm{IV}}^{85}$, $\widetilde{\mathrm{IV}}^{86}$, $\widetilde{\mathrm{IV}}^{87}$, $\widetilde{\mathrm{IV}}^{88}$, $\widetilde{\mathrm{IV}}^{89}$, $\widetilde{\mathrm{IV}}^{90}$, $\widetilde{\mathrm{IV}}^{91}$, $\widetilde{\mathrm{IV}}^{92}$, $\widetilde{\mathrm{IV}}^{93}$, $\widetilde{\mathrm{IV}}^{94}$, $\widetilde{\mathrm{IV}}^{95}$, $\widetilde{\mathrm{IV}}^{96}$, $\widetilde{\mathrm{IV}}^{97}$, $\widetilde{\mathrm{IV}}^{98}$, $\widetilde{\mathrm{IV}}^{99}$, $\widetilde{\mathrm{IV}}^{100}$, $\widetilde{\mathrm{IV}}^{101}$, $\widetilde{\mathrm{IV}}^{102}$, $\widetilde{\mathrm{IV}}^{103}$, $\widetilde{\mathrm{IV}}^{104}$, $\widetilde{\mathrm{IV}}^{105}$, $\widetilde{\mathrm{IV}}^{106}$, $\widetilde{\mathrm{IV}}^{107}$, $\widetilde{\mathrm{IV}}^{108}$, $\widetilde{\mathrm{IV}}^{109}$, $\widetilde{\mathrm{IV}}^{110}$, $\widetilde{\mathrm{IV}}^{111}$, $\widetilde{\mathrm{IV}}^{112}$, $\widetilde{\mathrm{IV}}^{113}$, $\widetilde{\mathrm{IV}}^{114}$, $\widetilde{\mathrm{IV}}^{k}$, $\widetilde{\mathrm{IV}}^{l}$, $\widetilde{\mathrm{IV}}^{m}$, $\widetilde{\mathrm{IV}}^{n}$, $\widetilde{\mathrm{IV}}^{o}$, $\widetilde{\mathrm{IV}}^{p}$, $\widetilde{\mathrm{IV}}^{u}$, $\widetilde{\mathrm{IV}}^{v}$, $\widetilde{\mathrm{IV}}^{x}$, $\widetilde{\mathrm{IV}}^{y}$, $\widetilde{\mathrm{IV}}^{z}$, $\widetilde{\mathrm{IV}}^{w}$, $\widetilde{\mathrm{IV}}^{\zeta}$ and $\widetilde{\mathrm{IV}}^{\iota}$.

We note that the list of the singular fibers of stable maps of orientable 5-manifold into 4-manifolds is also obtained in [**20**], [**44**] and in Chapter 3.

For stable maps of closed (possibly non-orientable) 4-manifolds into 3-manifolds, C^∞ equivalent and C^0 equivalent are coincide, refer to Corollary 2.7. Similarly, we obtain the following.

COROLLARY 4.7. *For two singular fibers of proper stable maps of closed (possibly non-orientable)* 5*-manifolds into* 4*-manifolds, the following two are equivalent.*

(1) *They are C^∞ equivalent.*
(2) *They are C^0 equivalent.*

4.2. Classification of singular fibers of stable maps of 5-manifolds into 4-manifolds with respect to the two-colored C^0 equivalence. If we consider a two-colored stable map $(f,(R,B)) : M \to N$ from a closed 5-manifold into a connected 4-manifold then we can classify the equivalence class with respect to two-colored C^0 equivalent relation as following. As the two-colored C^0 equivalence is more sensitive than C^0 equivalence, we can classify a equivalence class with respect to C^0 equivalence into 2 types A, B with respect to two-colored C^0 equivalence. We assign to each 4-dimension region the number of circle components in the fiber over a point in the region. Note that the number is constant on each regions. In the following, we combine this number and the color R or B.

Let $\mathcal{O}$ be any class of codimension 0 and $(f,(R,B)) : M \to N$ be two-colored map and the fiber of $(f,(R,B))$ over q is C^0 equivalent to $\mathcal{O}$. If q is contained in R (resp. B), then we denote the equivalence class with respect to C^0 equivalence of $f^{-1}(q)$ by $\mathcal{O}_A$ (resp. $\mathcal{O}_B$).

Let $\mathcal{E}$ be any class of codimension 1 with respect to C^0 equivalence relation, $(f,(R,B)) : M \to N$ be two-colored map and the fiber of $(f,(R,B))$ over q is C^0 equivalent to $\mathcal{E}$. Then $\mathcal{E}(f)$ is adjacent to two 4-dimension regions. If $\mathcal{E}$ is I^0 or I^1 then the difference between the number of connected components of the fibers associated with the two 4-dimension regions adjacent to $\mathcal{E}(f)$ is equal to one. Let us take a point $y \in \mathcal{E}(f)$ with $\mathcal{E} = \widetilde{\mathrm{I}}^0, \widetilde{\mathrm{I}}^1$. If the 4-dimension region adjacent to y which has larger associated number is contained in R, then we denote the equivalence class of the fiber over y with respect to two-colored C^0 equivalence is of type $\mathcal{E}_A$, otherwise $\mathcal{E}_B$. If $\mathcal{E} = \widetilde{\mathrm{I}}^2$, then the difference between the numbers of connected components of the fibers associated with the two 4-dimension regions adjacent to $\mathrm{I}^2(f)$ is always equal to zero, as in Figure 1.8. Thus we cannot divide the equivalence class $\mathcal{E}$ with respect to C^0 equivalence into two type with respect to two-colored C^0 equivalence.

Let $\mathcal{F}$ be any class of codimension 2 with respect to C^0 equivalence relation, $(f,(R,B)) : M \to N$ be two-colored map and the fiber of $(f,(R,B))$ over q is C^0 equivalent to $\mathcal{F}$. We note that $\mathcal{F}(f)$ is locally adjacent to four 4-dimension regions. We divide $\mathcal{F}$ into two types in the following way. Let us take a point $y \in \mathcal{F}(f)$ with $\mathcal{F} = \widetilde{\mathrm{II}}^{0,0}, \widetilde{\mathrm{II}}^{0,1}, \widetilde{\mathrm{II}}^{1,1}, \widetilde{\mathrm{II}}^{3}$ or $\widetilde{\mathrm{II}}^{5}$. If the two 4-dimension regions adjacent to y which are contained in R have the same associated number then we say that the equivalence class of fiber over y with respect to two-colored C^0 equivalence is of type $\mathcal{F}_A$, otherwise $\mathcal{F}_B$ (see Figure 1.10). Let us take a point $y \in \mathcal{F}(f)$ with $\mathcal{F} = \widetilde{\mathrm{II}}^{4}$. If the two 4 dimension regions adjacent to y which have a larger associated number are contained in R, then we say that the equivalence class of the fiber over y with respect to two-colored C^0 equivalence is of type $\mathcal{F}_A$, otherwise $\mathcal{F}_B$ (see Figure 1.11). In a way similar to that for $\widetilde{\mathrm{I}}^{0}$ and $\widetilde{\mathrm{I}}^{1}$, singular fibers of $\widetilde{\mathrm{II}}^{a}$ type are divided into two types. In this way, we can divide the equivalence class $\mathcal{F} = \mathrm{II}^{0,0}, \mathrm{II}^{0,1}, \mathrm{II}^{1,1}, \mathrm{II}^3, \mathrm{II}^4$ and II^a with respect to C^0 equivalence into $\mathcal{F}_A$ and $\mathcal{F}_B$ with respect to two-colored C^0 equivalence. However, if $\mathcal{F} = \widetilde{\mathrm{II}}^{0,2}, \widetilde{\mathrm{II}}^{1,2}, \widetilde{\mathrm{II}}^{2,2}$, $\widetilde{\mathrm{II}}^{6}$ and $\widetilde{\mathrm{II}}^{7}$, then we cannot be divided into two types by these method (see Figure 1.12).

In case of $\kappa = 3$ we see $\widetilde{\mathrm{III}}^{0,0,0}, \widetilde{\mathrm{III}}^{0,0,1}, \widetilde{\mathrm{III}}^{0,1,1}, \widetilde{\mathrm{III}}^{1,1,1}, \widetilde{\mathrm{III}}^{0,3}, \widetilde{\mathrm{III}}^{0,4}, \widetilde{\mathrm{III}}^{0,5}$, $\widetilde{\mathrm{III}}^{1,3}, \widetilde{\mathrm{III}}^{1,4}, \widetilde{\mathrm{III}}^{1,5}, \widetilde{\mathrm{III}}^{8}, \widetilde{\mathrm{III}}^{9}, \widetilde{\mathrm{III}}^{10}, \widetilde{\mathrm{III}}^{11}, \widetilde{\mathrm{III}}^{12}, \widetilde{\mathrm{III}}^{13}, \widetilde{\mathrm{III}}^{15}, \widetilde{\mathrm{III}}^{17}, \widetilde{\mathrm{III}}^{21}, \widetilde{\mathrm{III}}^{0,a}, \widetilde{\mathrm{III}}^{1,a}$, $\widetilde{\mathrm{III}}^{b}, \widetilde{\mathrm{III}}^{d}, \widetilde{\mathrm{III}}^{e}, \widetilde{\mathrm{III}}^{f}$ and $\widetilde{\mathrm{III}}^{g}$ have type A, B, by the similarly way as Chapter 1. The C^∞ equivalence class of the other types do not have A and B types with respect to two-colored C^∞ equivalence relation.

In case of $\kappa = 4$ we see $\widetilde{\mathrm{IV}}^{0,0,0,0}, \widetilde{\mathrm{IV}}^{0,0,0,1}, \widetilde{\mathrm{IV}}^{0,0,1,1}, \widetilde{\mathrm{IV}}^{0,1,1,1}, \widetilde{\mathrm{IV}}^{1,1,1,1}, \widetilde{\mathrm{IV}}^{0,0,3}$, $\widetilde{\mathrm{IV}}^{0,0,4}, \widetilde{\mathrm{IV}}^{0,0,5}, \widetilde{\mathrm{IV}}^{0,1,3}, \widetilde{\mathrm{IV}}^{0,1,4}, \widetilde{\mathrm{IV}}^{0,1,5}, \widetilde{\mathrm{IV}}^{1,1,3}, \widetilde{\mathrm{IV}}^{1,1,4}, \widetilde{\mathrm{IV}}^{1,1,5}, \widetilde{\mathrm{IV}}^{0,0,a}, \widetilde{\mathrm{IV}}^{0,1,a}$, $\widetilde{\mathrm{IV}}^{1,1,a}, \widetilde{\mathrm{IV}}^{3,3}, \widetilde{\mathrm{IV}}^{3,4}, \widetilde{\mathrm{IV}}^{3,5}, \widetilde{\mathrm{IV}}^{4,4}, \widetilde{\mathrm{IV}}^{4,5}, \widetilde{\mathrm{IV}}^{5,5}, \widetilde{\mathrm{IV}}^{3,a}, \widetilde{\mathrm{IV}}^{4,a}, \widetilde{\mathrm{IV}}^{5,a}, \widetilde{\mathrm{IV}}^{0,8}, \widetilde{\mathrm{IV}}^{0,9}$, $\widetilde{\mathrm{IV}}^{0,10}, \widetilde{\mathrm{IV}}^{0,11}, \widetilde{\mathrm{IV}}^{0,12}, \widetilde{\mathrm{IV}}^{0,13}, \widetilde{\mathrm{IV}}^{0,15}, \widetilde{\mathrm{IV}}^{0,17}, \widetilde{\mathrm{IV}}^{0,21}, \widetilde{\mathrm{IV}}^{1,8}, \widetilde{\mathrm{IV}}^{1,9}, \widetilde{\mathrm{IV}}^{1,10}, \widetilde{\mathrm{IV}}^{1,11}$, $\widetilde{\mathrm{IV}}^{1,12}, \widetilde{\mathrm{IV}}^{1,13}, \widetilde{\mathrm{IV}}^{1,15}, \widetilde{\mathrm{IV}}^{1,17}, \widetilde{\mathrm{IV}}^{1,21}, \widetilde{\mathrm{IV}}^{0,b}, \widetilde{\mathrm{IV}}^{1,b}, \widetilde{\mathrm{IV}}^{0,d}, \widetilde{\mathrm{IV}}^{1,d}, \widetilde{\mathrm{IV}}^{0,e}, \widetilde{\mathrm{IV}}^{1,e}, \widetilde{\mathrm{IV}}^{0,f}$, $\widetilde{\mathrm{IV}}^{1,f}, \widetilde{\mathrm{IV}}^{0,g}, \widetilde{\mathrm{IV}}^{1,g}, \widetilde{\mathrm{IV}}^{a,a}, \widetilde{\mathrm{IV}}^{27}, \widetilde{\mathrm{IV}}^{28}, \widetilde{\mathrm{IV}}^{29}, \widetilde{\mathrm{IV}}^{30}, \widetilde{\mathrm{IV}}^{31}, \widetilde{\mathrm{IV}}^{32}, \widetilde{\mathrm{IV}}^{33}, \widetilde{\mathrm{IV}}^{34}, \widetilde{\mathrm{IV}}^{35}$, $\widetilde{\mathrm{IV}}^{36}, \widetilde{\mathrm{IV}}^{37}, \widetilde{\mathrm{IV}}^{38}, \widetilde{\mathrm{IV}}^{39}, \widetilde{\mathrm{IV}}^{40}, \widetilde{\mathrm{IV}}^{43}, \widetilde{\mathrm{IV}}^{45}, \widetilde{\mathrm{IV}}^{46}, \widetilde{\mathrm{IV}}^{47}, \widetilde{\mathrm{IV}}^{49}, \widetilde{\mathrm{IV}}^{50}, \widetilde{\mathrm{IV}}^{54}, \widetilde{\mathrm{IV}}^{55}, \widetilde{\mathrm{IV}}^{56}$, $\widetilde{\mathrm{IV}}^{58}, \widetilde{\mathrm{IV}}^{59}, \widetilde{\mathrm{IV}}^{60}, \widetilde{\mathrm{IV}}^{62}, \widetilde{\mathrm{IV}}^{66}, \widetilde{\mathrm{IV}}^{67}, \widetilde{\mathrm{IV}}^{68}, \widetilde{\mathrm{IV}}^{70}, \widetilde{\mathrm{IV}}^{71}, \widetilde{\mathrm{IV}}^{72}, \widetilde{\mathrm{IV}}^{74}, \widetilde{\mathrm{IV}}^{76}, \widetilde{\mathrm{IV}}^{77}, \widetilde{\mathrm{IV}}^{86}$, $\widetilde{\mathrm{IV}}^{90}, \widetilde{\mathrm{IV}}^{h}, \widetilde{\mathrm{IV}}^{i}, \widetilde{\mathrm{IV}}^{j}, \widetilde{\mathrm{IV}}^{k}, \widetilde{\mathrm{IV}}^{q}, \widetilde{\mathrm{IV}}^{r}, \widetilde{\mathrm{IV}}^{s}, \widetilde{\mathrm{IV}}^{t}, \widetilde{\mathrm{IV}}^{u}, \widetilde{\mathrm{IV}}^{v}, \widetilde{\mathrm{IV}}^{z}, \widetilde{\mathrm{IV}}^{\alpha}, \widetilde{\mathrm{IV}}^{\beta}, \widetilde{\mathrm{IV}}^{\gamma}, \widetilde{\mathrm{IV}}^{\delta}$, $\widetilde{\mathrm{IV}}^{\varepsilon}, \widetilde{\mathrm{IV}}^{\zeta}$ and $\widetilde{\mathrm{IV}}^{\eta}$ have A and B types, for details see Figures 2.10, 2.11 ... 2.177 and 2.178. The C^∞ equivalence class of the other types do not have A and B types with respect to two-colored C^∞ equivalence relation.

4.3. Universal complex of singular fibers of two-colored maps of closed 5-manifold into 4-manifold. The equivalence relation $c\rho^0_{5,4}$ among the fiber of the element of $\mathcal{CT}_{pr}(5,4)$ is so strong, we may choice weaker equivalence relation than the above. We note that for smooth map of $(n+1)$-manifold into n-manifold, the regular fiber is the disjoint union of the trivial circle bundles.

DEFINITION 4.8. Let $c\rho_{p+1,p}$ be an admissible equivalence relation among the fiber of the elements of $\mathcal{CT}_{pr}(p+1,p)$. We say that two fibers of proper two-colored Thom maps from $p+1$-manifolds into p-manifolds, $p \geq 0$, are *equivalent with respect to* $c\rho_{p+1,p}$ *modulo* m *circle components* if one of them is equivalent with respect to $c\rho_{p+1,p}$ to the disjoint union of the other one and lm copies of a fiber of the trivial circle bundle for some nonnegative integer l. We denote this equivalence relation by $c\rho_{p+1,p}(m)$. Given a subset $\Gamma_{p+1,p}$ of $\mathcal{CT}_{pr}(p+1,p)$, we shall use the same notation $c\rho_{p+1,p}(m)$ for the equivalence relation for $\Delta(\Gamma_{p+1,p})$.

Remark 4.9. We can also show that the system of admissible equivalence relation $\mathcal{CR}^0_{-1}(2) = \{c\rho^0_{p+1,p}(2)\}_{p\geq 0}$ for the fibers of elements of $\mathcal{CT}_{pr}(-1)$ is stable in the sense of Definition 3.6.

Then by the construction in subsection 3 of §3, we obtain the cochain complex of $\mathbb{Z}_2$-coefficients

$$\mathcal{C}(\mathcal{CS}^0_{pr}(n,p), c\rho^0_{n,p}(2)) = (C^\kappa(\mathcal{CS}^0_{pr}(n,p), c\rho^0_{n,p}(2)), \delta_\kappa)_\kappa.$$

In the following of subsections, we restrict our interest to the stable maps of 5-manifolds into 4-manifolds and subclasses of this maps. Then, we calculate several cohomology groups of universal complex of singular fibers of stable maps of certain classes.

PROPOSITION 4.10. *The cohomology groups of the universal complex of singular fibers for proper two-colored C^0 stable maps of* 5*-manifolds into* 4*-manifolds with respect to the two-colored C^0 equivalence modulo two connected components*

$$(C^\kappa(\mathcal{CS}^0_{pr}(5,4), c\rho^0_{5,4}(2)), \delta_\kappa)_\kappa$$

are given as follows;

$$\begin{aligned}
&H^0 \cong \mathbb{Z}_2 \ \textit{generated by } [0]\\
&H^1 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \ \textit{generated by } [\widetilde{\mathrm{I}}^0_o + \widetilde{\mathrm{I}}^1_e] \equiv [\widetilde{\mathrm{I}}^0_e + \widetilde{\mathrm{I}}^o]\\
&\qquad \textit{and } [\widetilde{\mathrm{I}}^0_A + \widetilde{\mathrm{I}}^1_A + \widetilde{\mathrm{I}}^2_o] \equiv [\widetilde{\mathrm{I}}^0_A + \widetilde{\mathrm{I}}^1_A + \widetilde{\mathrm{I}}^2_e] \equiv [\widetilde{\mathrm{I}}^0_B + \widetilde{\mathrm{I}}^1_B + \widetilde{\mathrm{I}}^2_o] \equiv [\widetilde{\mathrm{I}}^0_B + \widetilde{\mathrm{I}}^1_B + \widetilde{\mathrm{I}}^2_e]\\
&H^2 = \{0\}\\
&H^3 \cong \mathbb{Z}_2 \ \textit{generated by } [\widetilde{\mathrm{III}}^{2,2,2} + \widetilde{\mathrm{III}}^{2,7} + \widetilde{\mathrm{III}}^{12} + \widetilde{\mathrm{III}}^{13}_{o,A} + \widetilde{\mathrm{III}}^{13}_{e,B} + \widetilde{\mathrm{III}}^{25} + \widetilde{\mathrm{III}}^{26}]\\
&\qquad \equiv [\widetilde{\mathrm{III}}^{2,2,2} + \widetilde{\mathrm{III}}^{2,7} + \widetilde{\mathrm{III}}^{12} + \widetilde{\mathrm{III}}^{13}_{e,A} + \widetilde{\mathrm{III}}^{13}_{o,B} + \widetilde{\mathrm{III}}^{25} + \widetilde{\mathrm{III}}^{26}]
\end{aligned}$$

where 0 *denotes* $0_{0,A} + 0_{o,B} + 0_{e,A} + 0_{e,B}$, $\widetilde{\mathrm{I}}^0_A$ *denotes* $\widetilde{\mathrm{I}}^0_{o,A} + \widetilde{\mathrm{I}}^0_{e,A}$, *others denote similarly ways and* [$*$] *denote the cohomology class represented by the cocycle* $*$.

For the incidence number $[\mathrm{III}^*_{o,A}, \mathcal{F}]$ of the cochain maps δ_i $(i = 1,2,3)$, refer to tables, 2.4, 2.5, $\cdots$, 2.42 and 2.43. In the tables, only non-zero degrees are given; an empty column means that the corresponding degree is equal to zero. For a C^0 equivalence class $\mathcal{F}$ of the singular fibers of codimension 4 with respect to the two-colored C^0 -equivalence mod two circle components, the incidence number of $\mathcal{F}$ with $\widetilde{\mathrm{III}}^*_{o,A}$ or $\widetilde{\mathrm{III}}^{**}_o$ is given in tables. These tables can be obtained by using the description of local nearby fibers as shown in Figure 1.7, 2.2, 2.3 and 2.4. We note that the incidence number $[\widetilde{\mathrm{III}}^*_{o,B}, \mathcal{F}]$ can be obtained by interchanging $\mathcal{F}_A$ with $\mathcal{F}_B$ in the tables, $[\widetilde{\mathrm{III}}^*_{e,A}, \mathcal{F}]$ can be obtained by interchanging $\mathcal{F}_o$ with $\mathcal{F}_e$ in the tables and $[\widetilde{\mathrm{III}}^*_{e,B}, \mathcal{F}]$ can be also obtained by similarly ways.

In the following cochain complex, we consider the equivalence relation of C^0 equivalence not two-colored C^0 equivalence.

We denote by $\mathcal{M}_{pr}(n,p)$ the set of proper C^∞ Morin maps of n-manifolds into p-manifolds which are elements of $\mathcal{T}_{pr}(n,p)$, where *Morin map* is the map whose singular points are only A_k-types (for details, see [**7**]).

PROPOSITION 4.11. *The cohomology groups of the universal complex of singular fibers for proper C^∞ Morin maps of proper* 5*-manifolds into* 4*-manifolds with respect to the C^0 equivalence modulo two circle components*

$$(C^\kappa(\mathcal{M}_{pr}(5,4), \rho^0_{5,4}(2)), \delta_\kappa)_\kappa$$

are given as follows;

$$\begin{array}{ll} H^0 \cong \mathbb{Z}_2 & \text{generated by } [\widetilde{0}_o + \widetilde{0}_e] \\ H^1 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{generated by } [\widetilde{\mathrm{I}}^0_o + \widetilde{\mathrm{I}}^1_e] \equiv [\widetilde{\mathrm{I}}^0_e + \widetilde{\mathrm{I}}^1_o],\ [\widetilde{\mathrm{I}}^2_o + \widetilde{\mathrm{I}}^2_e] \\ H^2 \cong \mathbb{Z}_2 & \text{generated by } [\widetilde{\mathrm{II}}^{0,2}_o + \widetilde{\mathrm{II}}^{1,2}_e + \widetilde{\mathrm{II}}^{6}_e] \equiv [\widetilde{\mathrm{II}}^{0,2}_e + \widetilde{\mathrm{II}}^{1,2}_o + \widetilde{\mathrm{II}}^{6}_o] \\ H^3 = \{0\} & \end{array}$$

where $\mathcal{F}_o$ (resp. $\mathcal{F}_e$) is the equivalence class of $\mathcal{F}$ which has odd (resp. even) circle components.

We denote by $\mathcal{SW}_{pr}(n,p)$ the set of proper C^∞ swallow-tail maps of n-manifolds into p-manifolds which are elements of $\mathcal{T}_{pr}(n,p)$, where *Swallow-tail map* is the map whose singular points are only swallow-tail point, cusp point and fold point.

PROPOSITION 4.12. *The cohomology groups of the universal complex of singular fibers for proper C^∞ Swallow-tail maps of 5-manifolds into 4-manifolds with respect to the C^0 equivalence modulo two circle components*

$$(C^\kappa(\mathcal{SW}_{pr}(5,4), c\rho^0_{5,4}(2)), \delta_\kappa)_\kappa$$

are given as follows;

$$\begin{array}{ll} H^0 \cong \mathbb{Z}_2 & \text{generated by } [\widetilde{0}_o + \widetilde{0}_e] \\ H^1 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{generated by } [\widetilde{\mathrm{I}}^0_o + \widetilde{\mathrm{I}}^1_e] \equiv [\widetilde{\mathrm{I}}^0_e + \widetilde{\mathrm{I}}^1_o],\ [\widetilde{\mathrm{I}}^2_0 + \widetilde{\mathrm{I}}^2_e] \\ H^2 \cong \mathbb{Z}_2 & \text{generated by } [\widetilde{\mathrm{II}}^{0,2}_o + \widetilde{\mathrm{II}}^{1,2}_e + \widetilde{\mathrm{II}}^{6}_e] \equiv [\widetilde{\mathrm{II}}^{0,2}_e + \widetilde{\mathrm{II}}^{1,2}_o + \widetilde{\mathrm{II}}^{6}_o] \\ H^3 \cong \mathbb{Z}_2 & \text{generated by } [\widetilde{\mathrm{III}}^{0,a}] \end{array}$$

where $\mathcal{F}$ denote $\mathcal{F}_o + \mathcal{F}_e$ and the other symbols $\mathcal{F}_*$ ($* = o$ or e) are as above.

We denote by $\mathcal{C}_{pr}(n,p)$ the set of proper C^∞ cusp maps of n-manifolds into p-manifolds which are elements of $\mathcal{T}_{pr}(n,p)$, where *Cusp map* is the map whose singular points are only cusp point and fold point.

PROPOSITION 4.13. *The cohomology groups of the universal complex of singular fibers for proper C^∞ cusp maps from 5-manifolds into 4-manifolds with respect to the C^0 equivalence modulo two circle components*

$$(C^\kappa(\mathcal{C}_{pr}(5,4), \rho^0_{5,4}(2)), \delta_\kappa)_\kappa$$

are given as follows;

$$\begin{array}{ll} H^0 \cong \mathbb{Z}_2 & \text{generated by } [\widetilde{0}_o + \widetilde{0}_e] \\ H^1 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{generated by } [\widetilde{\mathrm{I}}^0_o + \widetilde{\mathrm{I}}^1_e] \equiv [\widetilde{\mathrm{I}}^0_e + \widetilde{\mathrm{I}}^1_o],\ [\widetilde{\mathrm{I}}^2_0 + \widetilde{\mathrm{I}}^2_e] \\ H^2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{generated by } [\widetilde{\mathrm{II}}^{0,2}_o + \widetilde{\mathrm{II}}^{1,2}_e + \widetilde{\mathrm{II}}^{6}_e] \equiv [\widetilde{\mathrm{II}}^{0,2}_e + \widetilde{\mathrm{II}}^{1,2}_o + \widetilde{\mathrm{II}}^{6}_o], \\ & \qquad [\widetilde{\mathrm{II}}^{0,0} + \widetilde{\mathrm{II}}^{0,1} + \widetilde{\mathrm{II}}^{1,1} + \widetilde{\mathrm{II}}^{2,2} + \widetilde{\mathrm{II}}^{3} + \widetilde{\mathrm{II}}^{4} + \widetilde{\mathrm{II}}^{5} + \widetilde{\mathrm{II}}^{7}] \\ H^3 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{generated by } [\widetilde{\mathrm{III}}^{0,0,2} + \widetilde{\mathrm{III}}^{1,1,2} + \widetilde{\mathrm{III}}^{0,1,2} + \widetilde{\mathrm{III}}^{2,2,2} + \widetilde{\mathrm{III}}^{0,6} + \widetilde{\mathrm{III}}^{1,6} \end{array}$$

$$+ \widetilde{\mathrm{III}}^{2,3} + \widetilde{\mathrm{III}}^{2,4} + \widetilde{\mathrm{III}}^{2,5} + \widetilde{\mathrm{III}}^{2,7} + \widetilde{\mathrm{III}}^{12} + \widetilde{\mathrm{III}}^{16} + \widetilde{\mathrm{III}}^{21} + \widetilde{\mathrm{III}}^{22} + \widetilde{\mathrm{III}}^{23}$$

$$+ \widetilde{\mathrm{III}}^{24} + \widetilde{\mathrm{III}}^{25} + \widetilde{\mathrm{III}}^{26} + \widetilde{\mathrm{III}}^{c}]$$

$$\text{and } [\widetilde{\mathrm{III}}^{0,0,0}_o + \widetilde{\mathrm{III}}^{0,0,1}_e + \widetilde{\mathrm{III}}^{0,1,1}_o + \widetilde{\mathrm{III}}^{1,1,1}_e + \widetilde{\mathrm{III}}^{0,2,2}_o + \widetilde{\mathrm{III}}^{1,2,2}_e + \widetilde{\mathrm{III}}^{0,3}_o + \widetilde{\mathrm{III}}^{1,3}_e$$

$$+ \widetilde{\mathrm{III}}^{0,4}_o + \widetilde{\mathrm{III}}^{1,4}_e + \widetilde{\mathrm{III}}^{0,5}_o + \widetilde{\mathrm{III}}^{1,5}_e + \widetilde{\mathrm{III}}^{0,7}_o + \widetilde{\mathrm{III}}^{1,7}_e + \widetilde{\mathrm{III}}^{2,6}_o + \widetilde{\mathrm{III}}^{9}_o + \widetilde{\mathrm{III}}^{10}_o + \widetilde{\mathrm{III}}^{15}_o$$

$$+ \widetilde{\mathrm{III}}^{18}_o + \widetilde{\mathrm{III}}^{19}_o + \widetilde{\mathrm{III}}^{b}_e]$$

where the symbols $\mathcal{F}$ and $\mathcal{F}_*$ ($* = o$ or e) are as above.

We denote by $\mathcal{F}_{pr}(n,p)$ the set of proper C^∞ fold maps of n-manifolds into p-manifolds which are elements of $\mathcal{T}_{pr}(n,p)$, where *Fold map* is the map whose singular points are only fold point.

PROPOSITION 4.14. *The cohomology groups of the universal complex of singular fibers for proper C^∞ fold maps from 5-manifolds into 4-manifolds with respect to the C^0 equivalence modulo two circle components*

$$(C^\kappa(\mathcal{F}_{pr}(5,4), \rho^0_{5,4}(2)), \delta_\kappa)_\kappa$$

are given as follows;

$H^0 \cong \mathbb{Z}_2$ *generated by* $[\widetilde{0}_o + \widetilde{0}_e]$

$H^1 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *generated by* $[\widetilde{\mathrm{I}}^0_e + \widetilde{\mathrm{I}}^0_o]$, $[\widetilde{\mathrm{I}}^1_o + \widetilde{\mathrm{I}}^0_o]$ *and* $[\widetilde{\mathrm{II}}^2_o + \widetilde{\mathrm{II}}^2_e]$

$H^2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *generated by* $[\widetilde{\mathrm{II}}^{0,0}]$, $[\widetilde{\mathrm{II}}^{0,1}_o] \equiv [\widetilde{\mathrm{II}}^{0,1}_e]$,
$[\widetilde{\mathrm{II}}^{1,1} + \widetilde{\mathrm{II}}^{2,2} + \widetilde{\mathrm{II}}^{3} + \widetilde{\mathrm{II}}^{4} + \widetilde{\mathrm{II}}^{5} + \widetilde{\mathrm{II}}^{7}]$, $[\widetilde{\mathrm{II}}^{0,2}]$ *and* $[\widetilde{\mathrm{II}}^{0,2}_o + \widetilde{\mathrm{II}}^{1,2}_o + \widetilde{\mathrm{II}}^{6}_o]$

$H^3 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *generated by* $[\widetilde{\mathrm{III}}^{0,0,2}]$,
$[\widetilde{\mathrm{III}}^{0,1,2}_e + \widetilde{\mathrm{III}}^{0,6}_o] \equiv [\widetilde{\mathrm{III}}^{0,1,2}_o + \widetilde{\mathrm{III}}^{0,6}_e]$,
$[\widetilde{\mathrm{III}}^{1,1,2} + \widetilde{\mathrm{III}}^{2,2,2} + \widetilde{\mathrm{III}}^{1,6} + \widetilde{\mathrm{III}}^{2,3} + \widetilde{\mathrm{III}}^{2,4} + \widetilde{\mathrm{III}}^{2,5} + \widetilde{\mathrm{III}}^{2,7} + \widetilde{\mathrm{III}}^{12} + \widetilde{\mathrm{III}}^{16}$
$+ \widetilde{\mathrm{III}}^{21} + \widetilde{\mathrm{III}}^{22} + \widetilde{\mathrm{III}}^{23} + \widetilde{\mathrm{III}}^{24} + \widetilde{\mathrm{III}}^{25} + \widetilde{\mathrm{III}}^{26}]$,
$[\widetilde{\mathrm{III}}^{0,1,1} + \widetilde{\mathrm{III}}^{0,2,2} + \widetilde{\mathrm{III}}^{0,3} + \widetilde{\mathrm{III}}^{0,4} + \widetilde{\mathrm{III}}^{0,5} + \widetilde{\mathrm{III}}^{0,7}]$
$\equiv [\widetilde{\mathrm{III}}^{1,1,1} + \widetilde{\mathrm{III}}^{1,2,2} + \widetilde{\mathrm{III}}^{1,3} + \widetilde{\mathrm{III}}^{1,4} + \widetilde{\mathrm{III}}^{1,5} + \widetilde{\mathrm{III}}^{1,7} + \widetilde{\mathrm{III}}^{2,6} + \widetilde{\mathrm{III}}^{9} + \widetilde{\mathrm{III}}^{10}$
$+ \widetilde{\mathrm{III}}^{15} + \widetilde{\mathrm{III}}^{18} + \widetilde{\mathrm{III}}^{19}]$

where the symbols $\mathcal{F}$ *and* $\mathcal{F}_*$ $(* = o$ *or* $e)$ *are as above.*

Let us consider the cochain complex

$$(C^\kappa(\mathcal{S}^\infty_{pr}(5,4), \rho^\infty_{5,4}(2)), \delta_\kappa)_\kappa.$$

Then the generators of coboundary of this cochain complex are in the tables 2.1, 2.2 and 2.3. By straightforward calculation, we obtain the following. In tables 2.1, 2.2 and 2.3 can be obtained by using the description of local nearby fibers as shown in Figure 1.7. In the tables 2.1,2.2 and 2.3, the symbols $\mathcal{F}$ represent $\mathcal{F}_o + \mathcal{F}_e$ and the item $\delta_\kappa(\mathcal{F}_e)$ can be obtained by interchanging $\mathcal{G}_o$ with $\mathcal{G}_e$ in the item corresponding to $\mathcal{F}_o$.

PROPOSITION 4.15. *The cohomology groups of the universal complex of singular fibers for proper* C^∞ *stable maps from* 5*-manifolds into* 4*-manifolds with respect to the* C^0 *equivalence modulo two connected components*

$$(C^\kappa(\mathcal{S}^0_{pr}(5,4), \rho^0_{5,4}(2)), \delta_\kappa)_\kappa$$

are given as follows;

$H^0 \cong \mathbb{Z}_2$ *generated by* $[\widetilde{0}_o + \widetilde{0}_e]$
$H^1 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *generated by* $[\widetilde{\mathrm{I}}^0_o + \widetilde{\mathrm{I}}^1_e] \equiv [\widetilde{\mathrm{I}}^0_e + \widetilde{\mathrm{I}}^1_o]$, $[\widetilde{\mathrm{I}}^2_o + \widetilde{\mathrm{I}}^2_e]$
$H^2 \cong \mathbb{Z}_2$ *generated by* $[\widetilde{\mathrm{II}}^{0,2}_o + \widetilde{\mathrm{II}}^{1,2}_e + \widetilde{\mathrm{II}}^{6}_e] \equiv [\widetilde{\mathrm{II}}^{0,2}_e + \widetilde{\mathrm{II}}^{1,2}_o + \widetilde{\mathrm{II}}^{6}_o]$
$H^3 = \{0\}$

where the symbols $\mathcal{F}$ *and* $\mathcal{F}_*$ $(* = o$ *or* $e)$ *are as above.*

Furthermore if we restrict our attentions to the case of orientable manifolds, then we obtain the following. We denote by $\mathcal{S}^\infty_{pr}(n,p)^{ori}$ the set of proper stable maps of orientable n-manifolds into p-manifolds which are elements of $\mathcal{T}_{pr}(n,p)$.

PROPOSITION 4.16. *The cohomology groups of the universal complex of singular fibers for proper* C^∞ *stable maps from orientable* 5*-manifolds into* 4*-manifolds with respect to the* C^0 *equivalence modulo two connected components*

$$(C^\kappa(\mathcal{S}^0_{pr}(5,4)^{ori}, \rho^0_{5,4}(2)), \delta_\kappa)_\kappa$$

are given as follows;

$H^0 \cong \mathbb{Z}_2$ *generated by* $[\widetilde{0}_o + \widetilde{0}_e]$
$H^1 \cong \mathbb{Z}_2$ *generated by* $[\widetilde{\mathrm{I}}^0_o + \widetilde{\mathrm{I}}^1_e] \equiv [\widetilde{\mathrm{I}}^0_e + \widetilde{\mathrm{I}}^1_o]$
$H^2 = \{0\}$
$H^3 \cong \mathbb{Z}_2$ *generated by* $[\widetilde{\mathrm{III}}^{12}_o + \widetilde{\mathrm{III}}^{12}_e]$

where the symbols $\mathcal{F}$ *and* $\mathcal{F}_*$ $(* = o$ *or* $e)$ *are as above.*

κ	generator(s)
1	$\delta_0(\mathcal{O}_o) = \widetilde{\mathrm{I}}_o^0 + \widetilde{\mathrm{I}}_e^1 + \widetilde{\mathrm{I}}_o^0 + \widetilde{\mathrm{I}}_e^1$
2	$\delta_1(\widetilde{\mathrm{I}}_o^0) = \widetilde{\mathrm{II}}_o^{0,1} + \widetilde{\mathrm{II}}_e^{0,1} + \widetilde{\mathrm{II}}_e^{a},$ $\delta_1(\widetilde{\mathrm{I}}_e^0) = \widetilde{\mathrm{II}}_o^{0,1} + \widetilde{\mathrm{II}}_e^{0,1} + \widetilde{\mathrm{II}}_o^{a},$ $\delta_1(\widetilde{\mathrm{I}}_o^2) = \widetilde{\mathrm{II}}_o^{0,2} + \widetilde{\mathrm{II}}_e^{0,2} + \widetilde{\mathrm{II}}_o^{1,2} + \widetilde{\mathrm{II}}_e^{1,2} + \widetilde{\mathrm{II}}_o^{6} + \widetilde{\mathrm{II}}_e^{6}$
3	$\delta_2(\widetilde{\mathrm{II}}_o^{0,0}) = \widetilde{\mathrm{III}}^{0,0,0} + \widetilde{\mathrm{III}}^{0,0,1} + \widetilde{\mathrm{III}}_e^{0,a} + \widetilde{\mathrm{III}}_o^{d},$ $\delta_2(\widetilde{\mathrm{II}}_o^{0,1}) = \widetilde{\mathrm{III}}_o^{0,a} + \widetilde{\mathrm{III}}_e^{1,a} + \widetilde{\mathrm{III}}_o^{b},$ $\delta_2(\widetilde{\mathrm{II}}_o^{1,1}) = \widetilde{\mathrm{III}}^{0,1,1} + \widetilde{\mathrm{III}}^{1,1,1} + \widetilde{\mathrm{III}}_o^{8} + \widetilde{\mathrm{III}}_o^{1,a},$ $\delta_2(\widetilde{\mathrm{II}}_o^{0,2}) = \widetilde{\mathrm{III}}^{0,1,2} + \widetilde{\mathrm{III}}^{0,6} + \widetilde{\mathrm{III}}_e^{2,a} + \widetilde{\mathrm{III}}_o^{c},$ $\delta_2(\widetilde{\mathrm{II}}_o^{1,2}) = \widetilde{\mathrm{III}}^{0,1,2} + \widetilde{\mathrm{III}}^{1,6} + \widetilde{\mathrm{III}}_o^{14} + \widetilde{\mathrm{III}}_o^{2,a},$ $\delta_2(\widetilde{\mathrm{II}}_o^{2,2}) = \widetilde{\mathrm{III}}^{0,2,2} + \widetilde{\mathrm{III}}^{1,2,2} + \widetilde{\mathrm{III}}^{2,6} + \widetilde{\mathrm{III}}_o^{20},$ $\delta_2(\widetilde{\mathrm{II}}_o^{3}) = \widetilde{\mathrm{III}}^{0,3} + \widetilde{\mathrm{III}}^{1,3} + \widetilde{\mathrm{III}}_o^{8} + \widetilde{\mathrm{III}}^{9} + \widetilde{\mathrm{III}}_o^{11} + \widetilde{\mathrm{III}}_o^{17} + \widetilde{\mathrm{III}}_o^{b} + \widetilde{\mathrm{III}}_o^{f},$ $\delta_2(\widetilde{\mathrm{II}}_o^{4}) = \widetilde{\mathrm{III}}^{0,4} + \widetilde{\mathrm{III}}^{1,4} + \widetilde{\mathrm{III}}^{10} + \widetilde{\mathrm{III}}_o^{11} + \widetilde{\mathrm{III}}_o^{e} + \widetilde{\mathrm{III}}_o^{13} + \widetilde{\mathrm{III}}_o^{21},$ $\delta_2(\widetilde{\mathrm{II}}_o^{5}) = \widetilde{\mathrm{III}}^{0,5} + \widetilde{\mathrm{III}}^{1,5} + \widetilde{\mathrm{III}}^{15} + \widetilde{\mathrm{III}}_o^{17} + \widetilde{\mathrm{III}}_o^{21} + \widetilde{\mathrm{III}}_o^{g},$ $\delta_2(\widetilde{\mathrm{II}}_o^{6}) = \widetilde{\mathrm{III}}^{0,6} + \widetilde{\mathrm{III}}^{1,6} + \widetilde{\mathrm{III}}_e^{14} + \widetilde{\mathrm{III}}_o^{c},$ $\delta_2(\widetilde{\mathrm{II}}_o^{7}) = \widetilde{\mathrm{III}}^{0,7} + \widetilde{\mathrm{III}}^{1,7} + \widetilde{\mathrm{III}}_o^{13} + \widetilde{\mathrm{III}}^{18} + \widetilde{\mathrm{III}}^{19} + \widetilde{\mathrm{III}}_o^{20},$ $\delta_2(\widetilde{\mathrm{II}}_o^{a}) = \widetilde{\mathrm{III}}^{0,a} + \widetilde{\mathrm{III}}^{1,a} + \widetilde{\mathrm{III}}^{b}$
4	$\delta_3(\widetilde{\mathrm{III}}_o^{0,0,0}) = \widetilde{\mathrm{IV}}^{0,0,0,1} + \widetilde{\mathrm{IV}}_e^{0,0,a} + \widetilde{\mathrm{IV}}_o^{0,d},$ $\delta_3(\widetilde{\mathrm{III}}_o^{0,0,1}) = \widetilde{\mathrm{IV}}^{0,0,0,1} + \widetilde{\mathrm{IV}}_o^{0,0,a} + \widetilde{\mathrm{IV}}_e^{0,1,a} + \widetilde{\mathrm{IV}}_o^{0,b} + \widetilde{\mathrm{IV}}_o^{1,d},$ $\delta_3(\widetilde{\mathrm{III}}_o^{0,1,1}) = \widetilde{\mathrm{IV}}^{0,1,1,1} + \widetilde{\mathrm{IV}}_o^{0,8} + \widetilde{\mathrm{IV}}_o^{0,1,a} + \widetilde{\mathrm{IV}}_e^{1,1,a} + \widetilde{\mathrm{IV}}_o^{1,b},$ $\delta_3(\widetilde{\mathrm{III}}_o^{1,1,1}) = \widetilde{\mathrm{IV}}^{0,1,1,1} + \widetilde{\mathrm{IV}}_o^{1,8} + \widetilde{\mathrm{IV}}_o^{1,1,a}$ $\delta_3(\widetilde{\mathrm{III}}_o^{0,3}) = \widetilde{\mathrm{IV}}^{0,1,3} + \widetilde{\mathrm{IV}}_o^{0,8} + \widetilde{\mathrm{IV}}^{0,9} + \widetilde{\mathrm{IV}}_o^{0,11} + \widetilde{\mathrm{IV}}_o^{0,b} + \widetilde{\mathrm{IV}}_o^{h} + \widetilde{\mathrm{IV}}_o^{i}$ $\quad + \widetilde{\mathrm{IV}}_o^{0,f} + \widetilde{\mathrm{IV}}_e^{3,a} + \widetilde{\mathrm{IV}}_o^{0,17},$ $\delta_3(\widetilde{\mathrm{III}}_o^{1,3}) = \widetilde{\mathrm{IV}}^{0,1,3} + \widetilde{\mathrm{IV}}_o^{1,8} + \widetilde{\mathrm{IV}}^{1,9} + \widetilde{\mathrm{IV}}_o^{1,11} + \widetilde{\mathrm{IV}}_o^{29} + \widetilde{\mathrm{IV}}_o^{1,b} + \widetilde{\mathrm{IV}}_o^{1,f}$ $\quad + \widetilde{\mathrm{IV}}_o^{3,a} + \widetilde{\mathrm{IV}}_o^{1,17},$ $\delta_3(\widetilde{\mathrm{III}}_o^{0,4}) = \widetilde{\mathrm{IV}}^{0,1,4} + \widetilde{\mathrm{IV}}^{0,10} + \widetilde{\mathrm{IV}}_o^{0,11} + \widetilde{\mathrm{IV}}_o^{j} + \widetilde{\mathrm{IV}}_o^{0,e} + \widetilde{\mathrm{IV}}_e^{4,a} + \widetilde{\mathrm{IV}}_o^{0,13}$ $\quad + \widetilde{\mathrm{IV}}_o^{0,21},$ $\delta_3(\widetilde{\mathrm{III}}_o^{1,4}) = \widetilde{\mathrm{IV}}^{0,1.4} + \widetilde{\mathrm{IV}}^{1,10} + \widetilde{\mathrm{IV}}_o^{1,11} + \widetilde{\mathrm{IV}}_o^{28} + \widetilde{\mathrm{IV}}_o^{1,e} + \widetilde{\mathrm{IV}}_o^{4,a} + \widetilde{\mathrm{IV}}_o^{1,13}$ $\quad + \widetilde{\mathrm{IV}}_o^{1,21},$ $\delta_3(\widetilde{\mathrm{III}}_o^{8}) = \widetilde{\mathrm{IV}}^{0,8} + \widetilde{\mathrm{IV}}^{1,8} + \widetilde{\mathrm{IV}}_o^{h} + \widetilde{\mathrm{IV}}_o^{s},$ $\delta_3(\widetilde{\mathrm{III}}_o^{9}) = \widetilde{\mathrm{IV}}^{0,9} + \widetilde{\mathrm{IV}}^{1,9} + \widetilde{\mathrm{IV}}_e^{29} + \widetilde{\mathrm{IV}}_o^{31} + \widetilde{\mathrm{IV}}_o^{46} + \widetilde{\mathrm{IV}}_o^{i} + \widetilde{\mathrm{IV}}_o^{\gamma},$ $\delta_3(\widetilde{\mathrm{III}}_o^{10}) = \widetilde{\mathrm{IV}}^{0,10} + \widetilde{\mathrm{IV}}^{1,10} + \widetilde{\mathrm{IV}}_e^{28} + \widetilde{\mathrm{IV}}_o^{31} + \widetilde{\mathrm{IV}}_o^{j} + \widetilde{\mathrm{IV}}_o^{r} + \widetilde{\mathrm{IV}}_o^{54} + \widetilde{\mathrm{IV}}_o^{55}$ $\quad + \widetilde{\mathrm{IV}}_o^{58} + \widetilde{\mathrm{IV}}_o^{59},$ $\delta_3(\widetilde{\mathrm{III}}_o^{11}) = \widetilde{\mathrm{IV}}^{0,11} + \widetilde{\mathrm{IV}}^{1,11} + \widetilde{\mathrm{IV}}^{31} + \widetilde{\mathrm{IV}}_o^{66} + \widetilde{\mathrm{IV}}_o^{67} + \widetilde{\mathrm{IV}}_o^{68} + \widetilde{\mathrm{IV}}_o^{t} + \widetilde{\mathrm{IV}}_o^{\gamma},$ $\delta_3(\widetilde{\mathrm{III}}_o^{12}) = \widetilde{\mathrm{IV}}^{0,12} + \widetilde{\mathrm{IV}}^{1,12} + \widetilde{\mathrm{IV}}^{36} + \widetilde{\mathrm{IV}}_o^{70} + \widetilde{\mathrm{IV}}_o^{71},$ $\delta_3(\widetilde{\mathrm{III}}_o^{0,a}) = \widetilde{\mathrm{IV}}^{0,1,a} + \widetilde{\mathrm{IV}}^{0,b} + \widetilde{\mathrm{IV}}_o^{\alpha},$ $\delta_3(\widetilde{\mathrm{III}}_o^{1,a}) = \widetilde{\mathrm{IV}}^{0,1,a} + \widetilde{\mathrm{IV}}^{1,b} + \widetilde{\mathrm{IV}}_o^{h} + \widetilde{\mathrm{IV}}_o^{s},$ $\delta_3(\widetilde{\mathrm{III}}_o^{b}) = \widetilde{\mathrm{IV}}^{0,b} + \widetilde{\mathrm{IV}}^{1,b} + \widetilde{\mathrm{IV}}_e^{h} + \widetilde{\mathrm{IV}}_e^{s} + \widetilde{\mathrm{IV}}_o^{\alpha},$ $\delta_3(\widetilde{\mathrm{III}}_o^{d}) = \widetilde{\mathrm{IV}}^{0,d} + \widetilde{\mathrm{IV}}^{1,d} + \widetilde{\mathrm{IV}}_e^{\alpha},$

TABLE 2.1. Generators for the coboundary groups of $\mathcal{C}(\mathcal{S}_{pr}^{\infty}(5,4), \rho_{5,4}^{\infty}(2))$

κ	generator(s)
4	$\delta_3(\widetilde{\mathrm{III}}_o^{e}) = \widetilde{\mathrm{IV}}^{0,e} + \widetilde{\mathrm{IV}}^{1,e} + \widetilde{\mathrm{IV}}^{r} + \widetilde{\mathrm{IV}}_o^{v} + \widetilde{\mathrm{IV}}_o^{\gamma} + \widetilde{\mathrm{IV}}_o^{t} + \widetilde{\mathrm{IV}}^{\zeta}$
	$\delta_3(\widetilde{\mathrm{III}}_o^{f}) = \widetilde{\mathrm{IV}}^{0,f} + \widetilde{\mathrm{IV}}^{1,f} + \widetilde{\mathrm{IV}}^{s} + \widetilde{\mathrm{IV}}_o^{t} + \widetilde{\mathrm{IV}}_o^{\alpha} + \widetilde{\mathrm{IV}}_e^{\gamma} + \widetilde{\mathrm{IV}}_o^{u},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{0,0,2}) = \widetilde{\mathrm{IV}}^{0,0,0,2} + \widetilde{\mathrm{IV}}^{0,0,1,2} + \widetilde{\mathrm{IV}}^{0,0,6} + \widetilde{\mathrm{IV}}_e^{0,2,a} + \widetilde{\mathrm{IV}}_o^{0,c} + \widetilde{\mathrm{IV}}_o^{2,d},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{0,2,2}) = \widetilde{\mathrm{IV}}^{0,1,2,2} + \widetilde{\mathrm{IV}}^{0,2,6} + \widetilde{\mathrm{IV}}_0^{0,20} + \widetilde{\mathrm{IV}}_e^{2,2,a} + \widetilde{\mathrm{IV}}_o^{2,c},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{1,1,2}) = \widetilde{\mathrm{IV}}^{1,1,1,2} + \widetilde{\mathrm{IV}}^{0,1,1,2} + \widetilde{\mathrm{IV}}_o^{1,14} + \widetilde{\mathrm{IV}}_o^{2,8} + \widetilde{\mathrm{IV}}^{1,1,6} + \widetilde{\mathrm{IV}}_o^{1,2,a},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{1,2,2}) = \widetilde{\mathrm{IV}}^{0,1,2,2} + \widetilde{\mathrm{IV}}^{1,2,6} + \widetilde{\mathrm{IV}}_o^{1,20} + \widetilde{\mathrm{IV}}_o^{2,14} + \widetilde{\mathrm{IV}}_o^{2,2,a},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{0,1,2}) = \widetilde{\mathrm{IV}}^{0,1,6} + \widetilde{\mathrm{IV}}_o^{0,14} + \widetilde{\mathrm{IV}}_o^{0,2,a} + \widetilde{\mathrm{IV}}_e^{1,2,a} + \widetilde{\mathrm{IV}}_o^{2,b} + \widetilde{\mathrm{IV}}_o^{1,c},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{2,2,2}) = \widetilde{\mathrm{IV}}^{0,2,2,2} + \widetilde{\mathrm{IV}}^{1,2,2,2} + \widetilde{\mathrm{IV}}^{2,2,6} + \widetilde{\mathrm{IV}}_o^{2,20},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{0,5}) = \widetilde{\mathrm{IV}}^{0,1,5} + \widetilde{\mathrm{IV}}^{0,15} + \widetilde{\mathrm{IV}}_o^{0,17} + \widetilde{\mathrm{IV}}_o^{0,21} + \widetilde{\mathrm{IV}}_e^{5,a} + \widetilde{\mathrm{IV}}_o^{k} + \widetilde{\mathrm{IV}}_o^{0,g},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{0,6}) = \widetilde{\mathrm{IV}}^{0,1,6} + \widetilde{\mathrm{IV}}_e^{0,14} + \widetilde{\mathrm{IV}}_o^{0,c} + \widetilde{\mathrm{IV}}_e^{6,a} + \widetilde{\mathrm{IV}}_o^{l} + \widetilde{\mathrm{IV}}_o^{m} + \widetilde{\mathrm{IV}}_o^{n},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{0,7}) = \widetilde{\mathrm{IV}}^{0,1,7} + \widetilde{\mathrm{IV}}_o^{0,13} + \widetilde{\mathrm{IV}}^{0,18} + \widetilde{\mathrm{IV}}^{0,19} + \widetilde{\mathrm{IV}}_o^{0,20} + \widetilde{\mathrm{IV}}_e^{7,a} + \widetilde{\mathrm{IV}}_o^{o}$ $+ \widetilde{\mathrm{IV}}_o^{p},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{1,5}) = \widetilde{\mathrm{IV}}^{0,1,5} + \widetilde{\mathrm{IV}}^{1,15} + \widetilde{\mathrm{IV}}_o^{1,17} + \widetilde{\mathrm{IV}}_o^{1,21} + \widetilde{\mathrm{IV}}_o^{43} + \widetilde{\mathrm{IV}}_o^{5,a} + \widetilde{\mathrm{IV}}_o^{1,g},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{1,6}) = \widetilde{\mathrm{IV}}^{0,1,6} + \widetilde{\mathrm{IV}}_e^{1,14} + \widetilde{\mathrm{IV}}_o^{41} + \widetilde{\mathrm{IV}}_o^{42} + \widetilde{\mathrm{IV}}_o^{44} + \widetilde{\mathrm{IV}}_o^{1,c} + \widetilde{\mathrm{IV}}_o^{6,a},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{1,7}) = \widetilde{\mathrm{IV}}^{0,1,7} + \widetilde{\mathrm{IV}}_o^{1,13} + \widetilde{\mathrm{IV}}^{1,18} + \widetilde{\mathrm{IV}}^{1,19} + \widetilde{\mathrm{IV}}_o^{1,20} + \widetilde{\mathrm{IV}}_o^{7,a} + \widetilde{\mathrm{IV}}_o^{84}$ $+ \widetilde{\mathrm{IV}}_o^{85},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{2,3}) = \widetilde{\mathrm{IV}}^{0,2,3} + \widetilde{\mathrm{IV}}^{1,2,3} + \widetilde{\mathrm{IV}}^{3,6} + \widetilde{\mathrm{IV}}_o^{2,8} + \widetilde{\mathrm{IV}}^{2,9} + \widetilde{\mathrm{IV}}_o^{2,11} + \widetilde{\mathrm{IV}}_o^{2,17}$ $+ \widetilde{\mathrm{IV}}_o^{41} + \widetilde{\mathrm{IV}}_o^{48} + \widetilde{\mathrm{IV}}_o^{2,b} + \widetilde{\mathrm{IV}}_o^{2,f},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{2,4}) = \widetilde{\mathrm{IV}}^{0,2,4} + \widetilde{\mathrm{IV}}^{1,2,4} + \widetilde{\mathrm{IV}}^{4,6} + \widetilde{\mathrm{IV}}^{2,10} + \widetilde{\mathrm{IV}}_o^{2,11} + \widetilde{\mathrm{IV}}_o^{2,13} + \widetilde{\mathrm{IV}}_o^{2,21}$ $+ \widetilde{\mathrm{IV}}_o^{63} + \widetilde{\mathrm{IV}}_o^{2,e},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{2,5}) = \widetilde{\mathrm{IV}}^{0,2,5} + \widetilde{\mathrm{IV}}^{1,2,5} + \widetilde{\mathrm{IV}}^{5,6} + \widetilde{\mathrm{IV}}^{2,15} + \widetilde{\mathrm{IV}}_o^{2,17} + \widetilde{\mathrm{IV}}_o^{2,21} + \widetilde{\mathrm{IV}}_o^{82}$ $+ \widetilde{\mathrm{IV}}_o^{2,g},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{2,6}) = \widetilde{\mathrm{IV}}^{0,2,6} + \widetilde{\mathrm{IV}}^{1,2,6} + \widetilde{\mathrm{IV}}_e^{2,14} + \widetilde{\mathrm{IV}}_o^{80} + \widetilde{\mathrm{IV}}_o^{81} + \widetilde{\mathrm{IV}}_o^{2,c},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{2,7}) = \widetilde{\mathrm{IV}}^{0,2,7} + \widetilde{\mathrm{IV}}^{1,2,7} + \widetilde{\mathrm{IV}}^{6,7} + \widetilde{\mathrm{IV}}_o^{2,13} + \widetilde{\mathrm{IV}}^{2,18} + \widetilde{\mathrm{IV}}^{2,19} + \widetilde{\mathrm{IV}}_o^{2,20}$ $+ \widetilde{\mathrm{IV}}_o^{104} + \widetilde{\mathrm{IV}}_o^{109},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{13}) = \widetilde{\mathrm{IV}}^{0,13} + \widetilde{\mathrm{IV}}^{1,13} + \widetilde{\mathrm{IV}}^{54} + \widetilde{\mathrm{IV}}^{59},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{14}) = \widetilde{\mathrm{IV}}^{0,14} + \widetilde{\mathrm{IV}}^{1,14} + \widetilde{\mathrm{IV}}^{41} + \widetilde{\mathrm{IV}}^{42} + \widetilde{\mathrm{IV}}^{44} + \widetilde{\mathrm{IV}}_o^{l} + \widetilde{\mathrm{IV}}_o^{x},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{15}) = \widetilde{\mathrm{IV}}^{0,15} + \widetilde{\mathrm{IV}}^{1,15} + \widetilde{\mathrm{IV}}_e^{43} + \widetilde{\mathrm{IV}}_o^{46} + \widetilde{\mathrm{IV}}_o^{55} + \widetilde{\mathrm{IV}}_o^{58} + \widetilde{\mathrm{IV}}_o^{k} + \widetilde{\mathrm{IV}}_o^{z}$ $+ \widetilde{\mathrm{IV}}_o^{\zeta},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{16}) = \widetilde{\mathrm{IV}}^{0,16} + \widetilde{\mathrm{IV}}^{1,16} + \widetilde{\mathrm{IV}}_o^{67} + \widetilde{\mathrm{IV}}^{78} + \widetilde{\mathrm{IV}}_o^{79} + \widetilde{\mathrm{IV}}_o^{82} + \widetilde{\mathrm{IV}}^{87} + \widetilde{\mathrm{IV}}^{88}$ $+ \widetilde{\mathrm{IV}}^{89} + \widetilde{\mathrm{IV}}_o^{91} + \widetilde{\mathrm{IV}}_o^{94} + \widetilde{\mathrm{IV}}_o^{w},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{17}) = \widetilde{\mathrm{IV}}^{0,17} + \widetilde{\mathrm{IV}}^{1,17} + \widetilde{\mathrm{IV}}^{46} + \widetilde{\mathrm{IV}}_o^{66} + \widetilde{\mathrm{IV}}_o^{67} + \widetilde{\mathrm{IV}}_o^{68} + \widetilde{\mathrm{IV}}_o^{u},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{18}) = \widetilde{\mathrm{IV}}^{0,18} + \widetilde{\mathrm{IV}}^{1,18} + \widetilde{\mathrm{IV}}_o^{59} + \widetilde{\mathrm{IV}}_o^{78} + \widetilde{\mathrm{IV}}_o^{81} + \widetilde{\mathrm{IV}}_e^{84} + \widetilde{\mathrm{IV}}_o^{91} + \widetilde{\mathrm{IV}}^{101}$ $+ \widetilde{\mathrm{IV}}_o^{o},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{19}) = \widetilde{\mathrm{IV}}^{0,19} + \widetilde{\mathrm{IV}}^{1,19} + \widetilde{\mathrm{IV}}_o^{54} + \widetilde{\mathrm{IV}}_o^{78} + \widetilde{\mathrm{IV}}_o^{80} + \widetilde{\mathrm{IV}}_e^{85} + \widetilde{\mathrm{IV}}_o^{91} + \widetilde{\mathrm{IV}}^{101}$ $+ \widetilde{\mathrm{IV}}_o^{p},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{20}) = \widetilde{\mathrm{IV}}^{0,20} + \widetilde{\mathrm{IV}}^{1,20} + \widetilde{\mathrm{IV}}^{80} + \widetilde{\mathrm{IV}}^{81},$
	$\delta_3(\widetilde{\mathrm{III}}_o^{21}) = \widetilde{\mathrm{IV}}^{0,21} + \widetilde{\mathrm{IV}}^{1,21} + \widetilde{\mathrm{IV}}^{55} + \widetilde{\mathrm{IV}}^{58} + \widetilde{\mathrm{IV}}_o^{66} + \widetilde{\mathrm{IV}}_o^{67} + \widetilde{\mathrm{IV}}_o^{68} + \widetilde{\mathrm{IV}}_o^{v}$ $+ \widetilde{\mathrm{IV}}_o^{\zeta},$

TABLE 2.2. Generators for the coboundary groups of $\mathcal{C}(\mathcal{S}^{\infty}_{pr}(5,4), \rho^{\infty}_{5,4}(2))$

κ	generator(s)
4	$\delta_3(\widetilde{\mathrm{III}}^{22}_{o}) = \widetilde{\mathrm{IV}}^{0,22} + \widetilde{\mathrm{IV}}^{1,22} + \widetilde{\mathrm{IV}}^{53} + \widetilde{\mathrm{IV}}^{57} + \widetilde{\mathrm{IV}}^{61} + \widetilde{\mathrm{IV}}^{63}_{o} + \widetilde{\mathrm{IV}}^{64} + \widetilde{\mathrm{IV}}^{65}_{o}$ $+ \widetilde{\mathrm{IV}}^{68}_{o} + \widetilde{\mathrm{IV}}^{70}_{o} + \widetilde{\mathrm{IV}}^{73}_{o} + \widetilde{\mathrm{IV}}^{75}_{o} + \widetilde{\mathrm{IV}}^{91}_{o} + \widetilde{\mathrm{IV}}^{94}_{o} + \widetilde{\mathrm{IV}}^{y}_{o}$,
	$\delta_3(\widetilde{\mathrm{III}}^{23}_{o}) = \widetilde{\mathrm{IV}}^{0,23} + \widetilde{\mathrm{IV}}^{1,23} + \widetilde{\mathrm{IV}}^{42}_{e} + \widetilde{\mathrm{IV}}^{45}_{o} + \widetilde{\mathrm{IV}}^{55}_{o} + \widetilde{\mathrm{IV}}^{66}_{o} + \widetilde{\mathrm{IV}}^{n}_{o}$
	$\delta_3(\widetilde{\mathrm{III}}^{24}_{o}) = \widetilde{\mathrm{IV}}^{0,24} + \widetilde{\mathrm{IV}}^{1,24} + \widetilde{\mathrm{IV}}^{44}_{e} + \widetilde{\mathrm{IV}}^{45}_{o} + \widetilde{\mathrm{IV}}^{48}_{o} + \widetilde{\mathrm{IV}}^{51} + \widetilde{\mathrm{IV}}^{52} + \widetilde{\mathrm{IV}}^{55}_{o}$ $+ \widetilde{\mathrm{IV}}^{65}_{o} + \widetilde{\mathrm{IV}}^{79}_{o} + \widetilde{\mathrm{IV}}^{m}_{o}$,
	$\delta_3(\widetilde{\mathrm{III}}^{25}_{o}) = \widetilde{\mathrm{IV}}^{0,25} + \widetilde{\mathrm{IV}}^{1,25} + \widetilde{\mathrm{IV}}^{71}_{o} + \widetilde{\mathrm{IV}}^{75}_{o} + \widetilde{\mathrm{IV}}^{93}_{o} + \widetilde{\mathrm{IV}}^{99} + \widetilde{\mathrm{IV}}^{100} + \widetilde{\mathrm{IV}}^{104}_{o}$
	$\delta_3(\widetilde{\mathrm{III}}^{26}_{o}) = \widetilde{\mathrm{IV}}^{0,26} + \widetilde{\mathrm{IV}}^{1,26} + \widetilde{\mathrm{IV}}^{73}_{o} + \widetilde{\mathrm{IV}}^{93}_{o} + \widetilde{\mathrm{IV}}^{102} + \widetilde{\mathrm{IV}}^{105} + \widetilde{\mathrm{IV}}^{109}_{o}$,
	$\delta_3(\widetilde{\mathrm{III}}^{2,a}_{o}) = \widetilde{\mathrm{IV}}^{0,2,a} + \widetilde{\mathrm{IV}}^{1,2,a} + \widetilde{\mathrm{IV}}^{2,b} + \widetilde{\mathrm{IV}}^{6,a} + \widetilde{\mathrm{IV}}^{l}_{o} + \widetilde{\mathrm{IV}}^{x}_{o}$,
	$\delta_3(\widetilde{\mathrm{III}}^{c}_{o}) = \widetilde{\mathrm{IV}}^{0,c} + \widetilde{\mathrm{IV}}^{1,c} + \widetilde{\mathrm{IV}}^{l}_{o} + \widetilde{\mathrm{IV}}^{m} + \widetilde{\mathrm{IV}}^{n} + \widetilde{\mathrm{IV}}^{x}_{e}$,
	$\delta_3(\widetilde{\mathrm{III}}^{g}_{o}) = \widetilde{\mathrm{IV}}^{0,g} + \widetilde{\mathrm{IV}}^{1,g} + \widetilde{\mathrm{IV}}^{u}_{o} + \widetilde{\mathrm{IV}}^{v}_{o} + \widetilde{\mathrm{IV}}^{z} + \widetilde{\mathrm{IV}}^{\zeta}_{e}$

TABLE 2.3. Generators for the coboundary groups of $\mathcal{C}(\mathcal{S}^{\infty}_{pr}(5,4), \rho^{\infty}_{5,4}(2))$

Remark 4.17. We can apply Proposition 3.4 as follows. The $\mathbb{Z}_2$-homology class (of closed support) in the target 4-manifold represented by a cycle corresponding to a coboundary of the universal complex of singular fibers modulo 2 circle components always vanishes.

COROLLARY 4.18. *Let $f : M \to N$ be in $\mathcal{CS}^0_{pr}(5,4)$. The following numbers are always even.*

(1) $|\widetilde{\mathrm{III}}^{0,a}(f)| + |\widetilde{\mathrm{III}}^{d}(f)|$
(2) $|\widetilde{\mathrm{III}}^{0,a}(f)| + |\widetilde{\mathrm{III}}^{1,a}(f)| + |\widetilde{\mathrm{III}}^{b}(f)|$
(3) $|\widetilde{\mathrm{III}}^{8}(f)| + |\widetilde{\mathrm{III}}^{1,a}(f)|$
(4) $|\widetilde{\mathrm{III}}^{2,a}(f)| + |\widetilde{\mathrm{III}}^{c}(f)|$
(5) $|\widetilde{\mathrm{III}}^{2,a}(f)| + |\widetilde{\mathrm{III}}^{14}(f)|$
(6) $|\widetilde{\mathrm{III}}^{20}(f)|$
(7) $|\widetilde{\mathrm{III}}^{8}(f)| + |\widetilde{\mathrm{III}}^{11}(f)| + |\widetilde{\mathrm{III}}^{17}(f)| + |\widetilde{\mathrm{III}}^{b}(f)| + |\widetilde{\mathrm{III}}^{f}(f)|$
(8) $|\widetilde{\mathrm{III}}^{11}(f)| + |\widetilde{\mathrm{III}}^{13}(f)| + |\widetilde{\mathrm{III}}^{21}(f)| + |\widetilde{\mathrm{III}}^{e}(f)|$
(9) $|\widetilde{\mathrm{III}}^{17}(f)| + |\widetilde{\mathrm{III}}^{21}(f)| + |\widetilde{\mathrm{III}}^{g}(f)|$
(10) $|\widetilde{\mathrm{III}}^{14}(f)| + |\widetilde{\mathrm{III}}^{c}(f)|$
(11) $|\widetilde{\mathrm{III}}^{13}(f)| + |\widetilde{\mathrm{III}}^{20}(f)|$

Remark 4.19. It is easy to see that the eleven numbers appearing in Corollary 4.18 are all even if and only if the following arguments fold.

(1) $|\widetilde{\mathrm{III}}^{0,a}(f)| \equiv |\widetilde{\mathrm{III}}^{d}(f)|$
(2) $|\widetilde{\mathrm{III}}^{1,a}(f)| \equiv |\widetilde{\mathrm{III}}^{8}(f)|$
(3) $|\widetilde{\mathrm{III}}^{2,a}(f)| \equiv |\widetilde{\mathrm{III}}^{c}(f)| \equiv |\widetilde{\mathrm{III}}^{14}(f)|$
(4) $|\widetilde{\mathrm{III}}^{13}(f)| \equiv |\widetilde{\mathrm{III}}^{20}(f)| \equiv 0 \pmod 2$
(5) $|\widetilde{\mathrm{III}}^{d}(f)| + |\widetilde{\mathrm{III}}^{e}(f)| + |\widetilde{\mathrm{III}}^{f}(f)| + |\widetilde{\mathrm{III}}^{g}(f)| \equiv 0 \pmod 2$
(6) $|\widetilde{\mathrm{III}}^{0,a}(f)| + |\widetilde{\mathrm{III}}^{1,a}(f)| + |\widetilde{\mathrm{III}}^{b}(f)| \equiv 0 \pmod 2$

4.4. Stable Maps of 5-Manifolds into 6-Manifolds. Since $(5, 6)$ is a nice dimension pair in the sense of Mather [**26**], if $\dim M = 5$ and $\dim N = 6$, then the set of all C^∞ stable maps is open and dense in $C^\infty(M, N)$ with respect to Whitney C^∞ topology as long as M is compact.

The following characterization of proper C^∞ stable maps from 5-manifolds into 6-manifolds is well-known.

PROPOSITION 4.20. *A proper smooth map $f : M \to N$ from a 5-manifold into a 6-manifold is C^∞ stable if and only if the following conditions are satisfied.*

(i) (Local condition) For every $p \in M$, there exist local coordinates (a, b, c, d, x) and (X, Y, Z, W, U, V) about $p \in M$ and $f(p) \in N$ respectively such that one of the following holds:

$$(X \circ f,\ Y \circ f,\ Z \circ f,\ W \circ f,\ U \circ f,\ V \circ f)$$

$$= \begin{cases} (a, b, c, d, x, 0) & p : \textit{immersion point} \\ (a, b, c, d, ax, x^2) & p : \textit{Whitney umbrella point} \\ (a, b, c, d, ax + bx^2, cx + x^3) & p : \Sigma^{1,1,0} \textit{ point} \end{cases}$$

(ii) (Global condition) Set $S(f) = \{p \in M \mid \mathrm{rank} df_p < 6\}$, is coincide with source manifold M Then for every $q \in f(M)$, $f^{-1}(q) \cap M$ consists at most six points and the multi-germ

$$(f, f^{-1}(q))$$

is smoothly right-left equivalent to one of the thirteen multi-germs as follows:

(1) *single immersion,*
(2) *normal crossing of two immersion germs,*
(3) *normal crossing of three immersion germs,*
(4) *map germ corresponding to a Whitney umbrella point,*
(5) *normal crossing of four immersion germs,*
(6) *transverse crossing a Whitney umbrella germ and an immersion germ,*
(7) *normal crossing of five immersion germs,*
(8) *transverse crossing of a Whitney umbrella germ and a normal crossing two immersion germs,*
(9) *map germ corresponding to a $\Sigma^{1,1,0}$ point,*
(10) *normalcrossing of six immersion germs,*
(11) *transverse crossing of a Whitney umbrella germ and a normal crossing three immersion germs,*
(12) *transverse crossing of two Whitney umbrella germs,*
(13) *transverse crossing of a $\Sigma^{1,1,0}$ germ and an immersion germ.*

Remark 4.21. Since $(5, 6)$ is in the nice range in the sense of Mather [**26**], the above proposition gives a characterization of C^0 stable maps of 5-manifolds into 6-manifolds, for details see [**8**] or Remark 4.3 in this Chapter.

By a straightforward calculation, we obtain the following

PROPOSITION 4.22. *The cohomology groups of the universal complex of singular fibers for proper two-colored C^0 stable maps from 5-manifolds into 6-manifolds with respect to two-colored C^0 equivalence modulo one connected component*

$$((C^\kappa(\mathcal{CS}^0_{pr}(5, 6), c\rho^0_{5,6}(1))), \delta_\kappa)_\kappa$$

are given as follows;

$H^0 \cong \mathbb{Z}_2$ *generated by* $[\emptyset_A + \emptyset_B]$
$H^1 = \{0\}$
$H^2 = \{0\}$
$H^3 \cong \mathbb{Z}_2$ *generated by* $[WU_A + immer \times 3] \equiv [WU_B + immer \times 3]$
$H^4 = \{0\}$

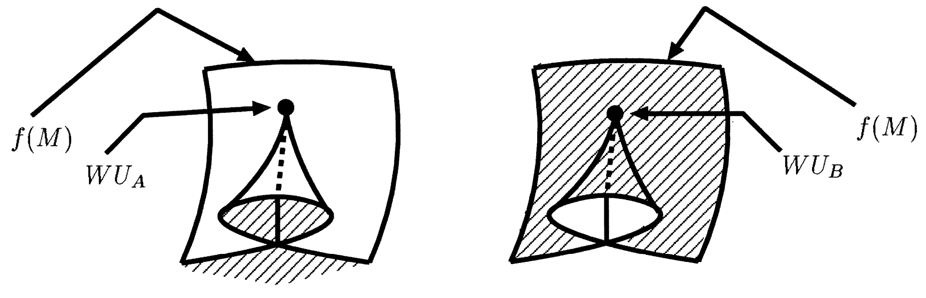

FIGURE 2.9. Equivalence classes of Whitney umbrella point

$H^5 \cong \mathbb{Z}_2$ *generated by* $[\Sigma^{1,1,0}]$,

where $\emptyset_*$ $(* = A, B)$ *denote the equivalence classes of emptysets,* WU_*, $(* = A, B)$ *denotes the equivalence classes of the Whitney umbrella point (for details, see Figure 2.9),* $\Sigma^{1,1,0}$ *denote the equivalence class of the* $\Sigma^{1,1,0}$ *point.*

4.5. Stable Maps of 3-Manifolds into 3-Manifolds. Since $(3,3)$ is a nice dimension pair in the sense of Mather [**26**], if $\dim M = 3$ and $\dim N = 3$, then the set of all C^∞ stable maps is open and dense in $C^\infty(M, N)$ with respect to Whitney C^∞ topology as long as M is compact.

The following characterization of proper C^∞ stable maps from 3-manifolds into 3-manifolds is well-known.

PROPOSITION 4.23. *A proper smooth map* $f : M \to N$ *from a 3-manifold into a 3-manifold is* C^∞ *stable if and only if the following conditions are satisfied.*

(ii) (Local condition) For every $p \in M$, *there exist local coordinates* (a, b, x) *and* (X, Y, Z) *about* $p \in M$ *and* $f(p) \in N$ *respectively such that one of the following holds:*

$$(X \circ f, Y \circ f, Z \circ f)$$

$$= \begin{cases} (a, b, x) & p : \textit{Regular point} \\ (a, b, x^2) & p : \textit{Fold point} \\ (a, b, x^3 + ax) & p : \textit{Cusp point} \\ (a, b, x^4 + ax^2 + bx) & p : \textit{Swallow-tail point} \end{cases}$$

(ii) (Global condition) Set $S(f) = \{p \in M \mid \mathrm{rank} df_p < 3\}$, *is closed 2-dimension submanifold of* M. *Then for every* $q \in f(S(f))$, $f^{-1}(q) \cap S(f)$ *consists of at most three points and the multi-germ*

$$(f, f^{-1}(q) \cap S(f))$$

is smoothly right-left equivalent to one of the thirteen multi-germs as follows:

(1) *single immersion germ which corresponds to a fold point,*
(2) *normal crossing of two immersion germs, each of which corresponds to a fold,*
(3) *map germ corresponding to a cusp point,*
(4) *normal crossing of three immersion germs,*
(5) *transverse crossing of a cusp germ and an immersion germ which corresponds to a fold germ,*
(6) *map germ corresponding to a swallow-tail point.*

Remark 4.24. Since $(3, 3)$ is in the nice range in the sense of Mather [**26**], the above proposition gives a characterization of C^0 stable maps of 3-manifolds into 3-manifolds, for details see [**8**] or Remark 4.3 in this Chapter.

By a straightforward calculation, we obtain the following

PROPOSITION 4.25. *The cohomology groups of the universal complex of singular fibers for proper two-colored C^0 stable maps from 3-manifolds into 3-manifolds with respect to C^0 equivalence modulo 2 connected components of fibers*

$$((C^\kappa(\mathcal{CS}^0_{pr}(3,3), c\rho^0_{3,3}(2))), \delta_\kappa)_\kappa$$

are given as follows;

$H^0 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *generated by* $[0_o]$ *and* $[0_e]$
$H^1 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *generated by* $[F_{o,A}] \equiv [F_{o,B}]$ *and* $[F_{e,A}] \equiv [F_{e,B}]$
$H^2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *generated by* $[C_o]$ *and* $[C_e]$

where 0 denote the point of regular and F denote the fold point and C denote the cusp point.

5. Geometrical meaning of cohomology classes

In this section, we show that one can obtain a lot of geometrical information from the cohomology classes of the universal complexes of singular fibers by using concrete examples of two-colored Thom maps.

First, we continue the construction of the theory of the singular fibers of two-colored maps §3.

Let $\Gamma = \Gamma_{n,p}$ be a subset of $\mathcal{CT}_{pr}(n,p)$ and $c\rho^\Gamma = c\rho^{\Gamma_{n,p}}_{n,p}$ a two-colored admissible equivalence relation among the fibers of elements of $\Gamma_{n,p}$.

DEFINITION 5.1. Let

$$c = \sum_{\kappa(\widetilde{\mathfrak{F}})=\kappa} n_{\widetilde{\mathfrak{F}}} \widetilde{\mathfrak{F}}$$

be a κ-dimensional cochain of the complex $\mathcal{C}(\Gamma_{n,p}, \rho_{n,p})$, where $n_{\widetilde{\mathfrak{F}}} \in \mathbb{Z}_2$. For a Thom map $f : M \to N$ which is an element of $\Gamma = \Gamma_{n,p}$, we define $c(f)$ to be the closure of the set of points $q \in N$ such that the fiber over q belongs to some $\widetilde{\mathfrak{F}}$ with $n_{\widetilde{\mathfrak{F}}} \neq 0$. If c is a cocycle, then $c(f)$ is a $\mathbb{Z}_2$-cycle of closed support of codimension κ of the target manifold N. In addition, if M is closed and $\kappa > 0$, then $c(f)$ is a $\mathbb{Z}_2$-cycle in the usual sense.

LEMMA 5.2. *Suppose that c and c' are κ-dimensional cocycle of the complex $\mathcal{C}(\Gamma_{n,p}, \rho_{n,p})$ which are cohomologus. Then $c(f)$ and $c'(f)$ are homologus in N for every $f \in \Delta(\Gamma_{n,p})$.*

PROOF. There exists a $(\kappa - 1)$ dimension cochain d of the complex such that $c - c' = \delta_{\kappa-1} d$. Then we see that $c(f) - c'(f) = \partial d(f)$. Hence the result follows. □

DEFINITION 5.3. Let $[c]$ be a κ-dimensional cohomology class of the complex $\mathcal{C}(\Gamma_{n,p}, c\rho^{\Gamma_{n,p}}_{n,p})$. For a proper two-colored Thom map $f : M \to N$ which is an element of $\Delta(\Gamma_{n,p})$ we define $[c(f)] \in H^c_{p-\kappa}(N; \mathbb{Z}_2)$ to be the homology class represented by the cycle $c(f)$ of closed support, where c is a cocycle representing $[c]$ and p=dim$N$. By Lemma 5.2, this is well-defined. When M is closed and $\kappa > 0$, we can also regard $[c(f)]$ as an element of $H_{p-\kappa}(N; \mathbb{Z}_2)$.

Then we can define the map

$$\varphi_f : H^\kappa(\Gamma_{n,p}, \rho^{\Gamma_{n,p}}_{n,p}) \to H^\kappa(N; \mathbb{Z}_2)$$

by $\varphi_f([c]) = [c(f)]^*$, where $[c(f)]^* \in H^\kappa(N; \mathbb{Z}_2)$ is the Poincaré dual to $[c(f)] \in H^c_{p-\kappa}(N; \mathbb{Z}_2)$. This is clearly a homomorphism induced by two-colored Thom map

f. When M is closed and $\kappa > 0$, we can also regard φ_f as a homomorphism into the cohomology group $H^\kappa_c(N;\mathbb{Z}_2)$ of compact support.

In the following, we define cobordism of two-colored maps with a given set of singular fibers and show that the homomorphism φ_f induced by a two-colored Thom map f is a cobordism invariant of f when restricted to a certain subgroup.

Let

$$\widetilde{\Delta} = \widetilde{\Delta}_k = \cup_{p-n=k}\Delta_{n,p}$$

be a set of two-colored C^0 equivalence classes of the fibers of proper two-colored Thom maps of codimension k such that each $\Delta_{n,p}$ is an ascending set of two-colored C^0 equivalence classes of fibers of elements of $\mathcal{CT}_{pr}(n,p)$, and that $\widetilde{\Delta}$ is closed under suspension in the sense of Definition 3.5. Recall that a proper two-colored Thom map $(f,(R,B)) : M \to N$ of codimension k is a $\widetilde{\Delta}_k$-maps if its fibers all are in $\widetilde{\Delta}_k$. If M is a manifold with boundary, then we also suppose that $f(\partial M) \subset \partial N$ and for collar neighborhoods $C = \partial M \times [0,1)$ and $C' = \partial N \times [0,1)$ of ∂M and ∂N, we have $f|_C = (f|_{\partial M}) \times \mathrm{id}_{[0,1)}$.

Definition 5.4. For a smooth manifold N, two-colored $\widetilde{\Delta}_k$-maps $(f_0,(R_0,B_0) : M_0 \to N$ and $(f_1,(R_1,B_1)) : M_1 \to N$ of closed manifolds M_0 and M_1 are said to be two-colored $\widetilde{\Delta}_k$-cobordant if there exist a compact manifold W with boundary the disjoint union of M_0 and M_1 and a $\widetilde{\Delta}_k$-map $(F,(R,B)) : W \to N \times [0,1]$ such that $f_i = F|_{M_i} : M_i \to N \times \{i\}$, $i = 0,1$ and $N \times \{i\} \cap R = R_i$. We call F a two-colored $\widetilde{\Delta}_k$-cobordism between f_0 and f_1.

Remark 5.5. In Definition 5.4, when the dimensions of the source manifolds M_0 and M_1 are equal to n, we have only to give $\Delta_{n,p}$ and $\Delta_{n+1,p+1}$ instead of the whole $\widetilde{\Delta}_k$ in order to define the notion of $\widetilde{\Delta}_k$-cobordisms. We will sometimes talk about $\widetilde{\Delta}_k$-cobordisms even when only $\Delta_{n,p}$ and $\Delta_{n+1,p+1}$ are given.

Let

$$s_{\kappa *} : H^\kappa(\Delta_{n+1,p+1}, c\rho^{\Delta_{n+1,p+1}}_{n+1,p+1}) \to H^\kappa(\Delta_{n,p}, c\rho^{\Delta_{n,p}}_{n,p})$$

be the homomorphism induced by the suspension, where $\mathcal{R}^{\widetilde{\Delta}}_k = \{c\rho^{\Delta_{p-k,p}}_{p-k,p}\}_p$ is a stable system of two-colored admissible equivalence relation for $\widetilde{\Delta}$.

Lemma 5.6. *Let $(f_i,(R_i,B_i)) : M_i \to N$, $i = 0,1$, be two-colored Thom maps which are $\widetilde{\Delta}_k$-maps, where we assume that M_i are closed. If they are $\widetilde{\Delta}_k$-cobordant, then for every κ we have*

$$\varphi_{f_0}|_{\mathrm{Im} s_{\kappa *}} = \varphi_{f_1}|_{\mathrm{Im} s_{\kappa *}} : \mathrm{Im} s_{\kappa *} \to H^\kappa(N;\mathbb{Z}_2).$$

Proof. Let $(F,(R,B)) : W \to N \times [0,1]$ be a Δ_k-cobordism between $(f_0,(R_0,B_o))$ and $(f_1,(R_1,B_1))$. Let c be any κ-dimensional cocycle of the complex $\mathcal{C}(\Delta_{n+1,p+1}, c\rho^{\Delta_{n+1,p+1}}_{n+1,p+1})$ and we put $\bar{c} = s_\kappa(c) \in C^\kappa(\Delta_{n,p}, c\rho^{\Delta_{n,p}}_{n,p})$. Then we see that $0 = (\delta c)((F,(R,B))) = \partial c((F,(R,B))) = \bar{c}((f_1,(R_1,B_1)))\times\{1\} - \bar{c}((f_0,(R_0,B_0)))\times\{0\}$, since c is a cocycle. Then the result follows immediately. □

We note that the above Lemma show that the κ dimension cohomology classes induce the *bordism invariant* of two-colored Thom map from closed $\kappa+1$-manifold into κ-manifold. The smooth maps $f_0 : M_0 \to N$ and $f_1 : M_1 \to N$ of n-manifolds M_0 and M_1 are said to be *bordant* if there exist a compact manifold W with boundary the disjoint union M_0 and M_1, and a smooth map $F : W \to N \times [0,1]$ such that $f_i = F|_{M_i} : M_i \to N \times \{i\}$, $i = 0,1$ (for detail, see [**43**]). In particular, if N is contractible, it induce a cobordant of the source manifolds.

We prepare the following definition for subsection 5.4.

DEFINITION 5.7. The pair $\{\Gamma_{n+1,p+1}, c\rho_{n+1,p+1}^{\Gamma_{n+1,p+1}}\}$ and $\{\Gamma_{n,p}, c\rho_{n,p}^{\Gamma_{n,p}}\}$ are said to be *compatible* at dimension κ if the homomorphism

$$s_{\kappa *} : H^{\kappa}(\Gamma_{n+1,p+1}, c\rho_{n+1,p+1}^{\Gamma_{n+1,p+1}}) \to H^{\kappa}(\Gamma_{n,p}, c\rho_{n,p}^{\Gamma_{n,p}})$$

is surjective.

From Lemma 5.6 and the result of section 4.2, we obtain the following Euler number formulas.

5.1. Stable maps of closed 5-manifolds into 4-manifolds.

THEOREM 5.8. *Let $f : M \to N$ be a colorable map from closed 4-manifold into 3-manifold. Then we have following congruence relation;*

$$\chi(M) \equiv |\widetilde{\mathrm{III}}^{2,2,2}(f)| + |\widetilde{\mathrm{III}}^{2,7}(f)| + |\widetilde{\mathrm{III}}^{12}(f)| + |\widetilde{\mathrm{III}}^{13}_{o,A}(f)| \\ + |\widetilde{\mathrm{III}}^{13}_{e,B}(f)| + |\widetilde{\mathrm{III}}^{25}(f)| + |\widetilde{\mathrm{III}}^{26}(f)| \pmod 2.$$

PROOF. If $N^p = \mathbb{R} \times N'$, then the set of all bordism class of closed l-manifolds into N, denoted by $n_l(N)$, forms an abelian group. We note that the dimension pair $(4,3)$ is in the nice range in the sense of Mather, we can choice the representation of the bordism group as stable map. Furthermore, we can choice representation of $n_4(N^1 \times \mathbb{R}^2)$ as two-colored map $(f, (R, B))$ if we determine color (R, B) of maps as non-compact subset in $N \times \mathbb{R}^2 \setminus f(S(f))$ is R. On the other hand, we have

$$n_4(\mathbb{R}^3) \cong n_4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 = < w_4 >_{\mathbb{Z}_2} \oplus < w_1^4 >_{\mathbb{Z}_2},$$

where n_4 is unoriented cobordism group of 4-manifolds and its generators are w_4 and w_1^4, where w_i is i-th Stiefel-Whitney classes (for details, see [**7**]). For $[c] \in H^3(\mathcal{CS}^{\infty}_{pr}(5,4), c\rho^{\infty}_{5,4}(2))$, we can define homomorphism $[c] : n_4(\mathbb{R}^3) \to H^c_3(\mathbb{R}^3, \mathbb{Z}_2) \cong \mathbb{Z}_2$ by $[c]([f]) = [c(f)]$. We note that this definition is well-defined by virtue of above Proposition 5.6. We have the generator, $S = [S : \mathbb{R}P^2 \tilde{\times} \mathbb{R}P^2 \to \mathbb{R}^3] = (1,1) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong n_4(\mathbb{R}^3)$ and $[c](S) = 1$, this map obtained by Saeki in [**38**, Example 3.7] which has 27-th triple points and one of them correspond to $\mathrm{III}^{2,2,2}$ type fiber. We have the generator $K = [K : \mathbb{C}P^2 \to \mathbb{R}^3] = (1,0) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong n_4(\mathbb{R}^3)$ and $[c](K) = 1$, this map obtained by Kobayashi in [**21**] which has two triple points and one of them correspond to III^{12} type fiber. So we have the homomorphism induced by $[c]$ is just a projection to the first component. □

From Theorem 5.8, we can also obtain a pretty interesting Corollaries 1.1, 1.2 and 1.3 in Chapter 1.

5.2. Stable maps of closed 5-manifolds into 6-manifolds. By the similarly argument of the proof of Theorem 5.8, the generator of $H^3(\mathcal{CS}^{\infty}_{pr}(5,6), \rho_{5,6})$ induces the following formula. Actually, we take the generator in $n_2(\mathbb{R}^3)$ as boys surface $b : \mathbb{R}P^2 \to \mathbb{R}^3$. We note that $n_2(\mathbb{R}^3) \cong n_2 = < w_2 >_{\mathbb{Z}_2}$.

THEOREM 5.9 (A.Szűcs). *Let $f : M^2 \to N^3$ be colorable map from closed 2-manifold into connected 3-manifold. Then we have*

$$\chi(M^2) \equiv T(f) + \sum_{q:\,Whitney\ Umbrella} ind(q; f) \pmod 2,$$

where $T(f)$ is the number of triple points and $ind(q; f) \in \mathbb{Z}_2$ is the index of Whitney Umbrella points.

On the other hand, we have the non-trivial cohomology class in 5-dimension cohomology group. We have following conjecture and problem.

Conjecture 5.10. Let $f : M^4 \to N^5$ be colorable map from closed 4-manifold into connected 5-manifold. Then we have

$$\chi(M^4) \equiv \Sigma^{1,1,0}(f) \pmod 2,$$

where $\Sigma^{1,1,0}(f)$ is the number of $\Sigma^{1,1,0}$ points.

Problem 5.11. Let us construct the stable maps from odd Euler number closed 4-manifold into $\mathbb{R}^5$ which has odd number $\Sigma^{1,1,0}$ points.

5.3. Stable maps of closed 3-manifolds into 3-manifolds. By the similarly argument of the proof of Theorem 5.8, the generator of $H^2(\mathcal{CS}^\infty_{pr}(3,3), c\rho_{3,3}(2))$ induces the following formula. Actually, we take the generator in $n_2(\mathbb{R}^2)$ as $\mathbb{R}P^2 \to \mathbb{R}^2$ obtained by M.Kobayashi **[21]** which has three cusp points. We note that $n_2(\mathbb{R}^2) \cong n_2 =< w_2 >_{\mathbb{Z}_2}$, and the map $[C_0] : n_2(\mathbb{R}^2) \to H^c_2(\mathbb{R}^2; \mathbb{Z}_2)$ is 0-map.

THEOREM 5.12. *Let $f : M^2 \to N^2$ be colorable map from closed 2-manifold into connected 2-manifold. Then we have*

$$\chi(M^2) \equiv C(f) \pmod 2,$$

where $C(f)$ denote the number of cusp points of f.

5.4. Characterization of the cocycle. In this section, we shall give a necessary and sufficient condition for a certain cochain of the universal complex to be a cocycle in terms of the homomorphism induced by two-colored Thom maps. For Lemmas of this subsection, the proofs are very similar to that of **[43**, § 12], as we omit the proofs here.

DEFINITION 5.13. Let $(f,(R,B)) : M \to N$ be a proper two-colored Thom map and $(g,(R,B)) : V \to N$ be a two-colored smooth map which is transverse to f and to all the strata of N. Put

$$\widetilde{V} = \{(x,y) \in M \times N \mid f(x) = g(y)\} \subset M \times V$$

and consider the following commutative diagram;

$$\begin{array}{ccc} \widetilde{V} & \xrightarrow{\widetilde{g}} & M \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{g} & N, \end{array}$$

where $\widetilde{g}$ and $\widetilde{f}$ are the restrictions of the projections to the first and second factors respectively. We note that $\widetilde{V}$ is a smooth manifold of dimension $\dim V + \dim M - \dim N$ and $\widetilde{f}$ is a proper Thom map. We call $\widetilde{f}$ the pull-back of by g and say that $\widetilde{f}$ is obtained by pulling back f by g.

DEFINITION 5.14. Suppose that

$$\Gamma_{n,p} \subset \mathcal{CT}_{pr}(n,p) \text{ and } \Gamma_{n+l,p+l} \subset \mathcal{CT}_{pr}(n+l,p+l)$$

are given with $l > 0$ such that the l-suspension of an element of $\Gamma_{n,p}$ always belong to $\Gamma_{n+l,p+l}$. Let $(f,(R,B)) : M \to N$ be an element of $\Gamma_{n+l,p+l}$ and $(g,(R,B)) : \mathrm{Int}D^p \to N$ an smooth two-colored map which is transverse to f and to all the strata of N. We note that the pull-back $\widetilde{f}$ of f by g is then an element of $\mathcal{CT}_{pr}(n,p)$. If the fiber of $\widetilde{f}$ always belong to $\Delta(\Gamma_{n,p})$, then we say that $\Gamma_{n,p}$ is *transversely complete with respect to* $\Gamma_{n+l,p+l}$.

Furthermore, we say that

$$\widetilde{\Gamma}_k = \cup_{p-n=k}\Gamma_{n,p} \subset \widetilde{\mathcal{CT}}_{pr}(k)$$

is *transversely complete* if it is closed under suspension and if $\Gamma_{n,p}$ is transversely complete with respect to $\Gamma_{n+l,p+l}$ or all n, p and l.

We note that $\mathcal{CT}_{pr}(k)$ is clearly transversely complete.

LEMMA 5.15. *If* $\Gamma_{n,p}$ *is transversely complete with respect to* $\Gamma_{n+l,p+l}$, *then the natural* $\mathbb{Z}_2$*-linear map*

$$s_\kappa : C^\kappa(\Gamma_{n+l,p+l}, c\rho_{n+l,p+l}^{\Gamma_{n+l,p+l}}) \to C^\kappa(\Gamma_{n,p}, c\rho_{n,p}^{\Gamma_{n,p}})$$

induced by the suspension is a injective for any $\kappa \leq p$ *where* $c\rho_{n+l,p+l}^{\Gamma_{n+l,p+l}}$ *and* $c\rho_{n,p}^{\Gamma_{n,p}}$ *are admissible equivalence relations for the fibers of elements of* $\Gamma_{n+l,p+l}$ *and* $\Gamma_{n,p}$, *respectively, which are stable in the sense of Definition 3.6.*

Let $\widetilde{\Delta} = \widetilde{\Delta}_k$ be as in the previous, and let $\mathcal{R}_k^{\widetilde{\Delta}} = \{c\rho_{p-k,p}^{\Delta_{p-k,p}}\}_p$ be a stable system of two-colored admissible equivalence relation for $\widetilde{\Delta}_k$.

Let c be an any cochain in $C^\kappa(\Delta_{n,p}, c\rho_{n,p}^{\Delta_{n,p}})$ with $0 < \kappa < p$. Set $\lambda = \kappa - k$. Since we always have $C^{\kappa+1}(\Delta_{\lambda,\kappa}, c\rho_{\lambda,\kappa}^{\Delta_{\lambda,\kappa}}) = 0$,

$$\delta_\kappa : C^\kappa(\Delta_{\lambda,\kappa}, c\rho_{\lambda,\kappa}^{\Delta_{\lambda,\kappa}}) \to C^{\kappa+1}(\Delta_{\lambda,\kappa}, c\rho_{\lambda,\kappa}^{\Delta_{\lambda,\kappa}})$$

is zero map, and hence $s_\kappa c \in C^\kappa(\Delta_{\lambda,\kappa}, c\rho_{\lambda,\kappa}^{\Delta_{\lambda,\kappa}})$ is a cocycle of the complex $\mathcal{C}(\Delta_{\lambda,\kappa}, c\rho_{\lambda,\kappa}^{\Delta_{\lambda,\kappa}})$, where

$$s_\kappa : C^\kappa(\Delta_{n,p}, c\rho_{n,p}^{\Delta_{n,p}}) \to C^\kappa(\Delta_{\lambda,\kappa}, c\rho_{\lambda,\kappa}^{\Delta_{\lambda,\kappa}})$$

is the homomorphism induced by the $(p-\kappa)$-th suspension. Therefore, for a $\Delta(\Gamma_{\lambda,\kappa})$-map $f : M \to N$, the homology class $[s_\kappa c(f)] \in H_0^c(N;\mathbb{Z}_2)$ represented by $s_\kappa c(f)$ is well-defined.

LEMMA 5.16. *Suppose that* $\Gamma_{\lambda,k}$ *is transversely complete with respect to* $\Gamma_{n,p}$, *where* $0 < \kappa < p$ *and* $p - n = \kappa - \lambda = k$. *Then a cochain* c *in* $C^\kappa(\Gamma_{n,p}, c\rho_{n,p}^{\Gamma_{n,p}})$ *is a cocycle of the complex* $\mathcal{C}(\Gamma_{n,p}, c\rho_{n,p}^{\Gamma_{n,p}})$ *if and only if* $[s_\kappa c(f)] = 0 \in H_0(N;\mathbb{Z}_2)$ *for every* $\Delta(\Gamma_{\lambda,\kappa})$*-map* $(f,(R,B)) : M \to N$ *such that both* M *and* N *are closed and that* f *is* $\widetilde{\Delta}_k$*-cobordant to a nonsingular map.*

COROLLARY 5.17. *If there exist cochain* $c \in C^3(\mathcal{S}_{pr}^0(5,4), \rho_{5,4}^0(2))$ *such that* $[c(f)] \equiv \chi(M)$ *for every stable map* $f : M \to N$ *from closed* 4*-manifold into* 3*-manifold then* $[c] \in H^3(\mathcal{C}(\mathcal{S}_{pr}^0(5,4), \rho_{5,4}^0(2))$.

Saeki's Euler number formula (for details, see Introduction of this thesis) imply that the equivalence class of $\widetilde{\mathrm{III}}^{12}$ is in $H^3(\mathcal{C}(\mathcal{S}_{pr}^0(5,4), \rho_{5,4}^0(2))$. Actually $\widetilde{\mathrm{III}}^{12}$ is in $H^3(\mathcal{C}(\mathcal{S}_{pr}^0(5,4), \rho_{5,4}^0(2))$ (Proposition 4.16).

Combing Corollary 5.17 and the direct calculation of the third cohomology of the cochain complex Proposition 4.16, we obtain the following Theorem.

THEOREM 5.18. *There is no formula of same type as Theorem 5.8 if we consider not two-colored stable maps but the stable maps from closed* 4*-manifold into* 3*-manifold.*

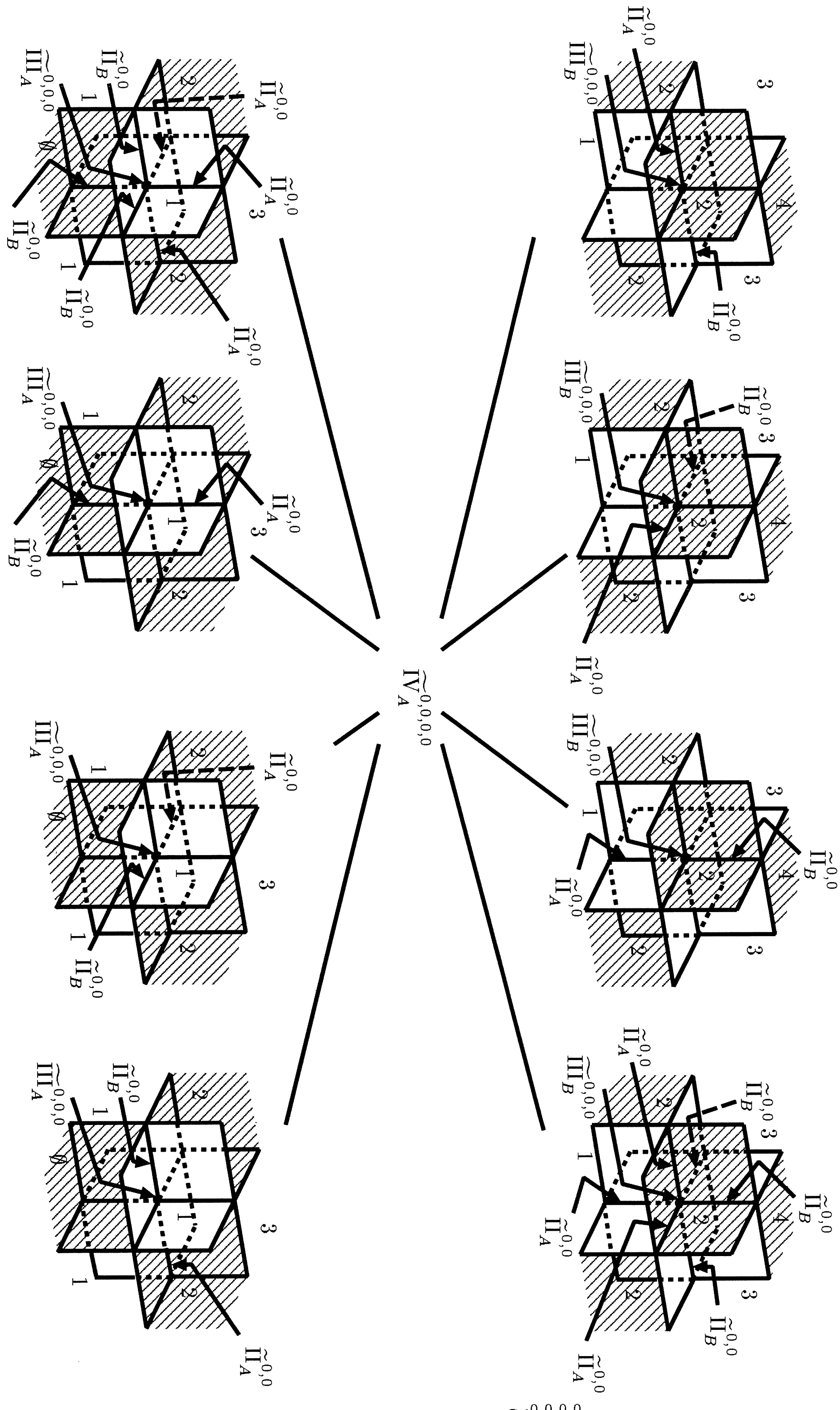

FIGURE 2.10. Type A for $\widetilde{\mathrm{IV}}^{0,0,0,0}$

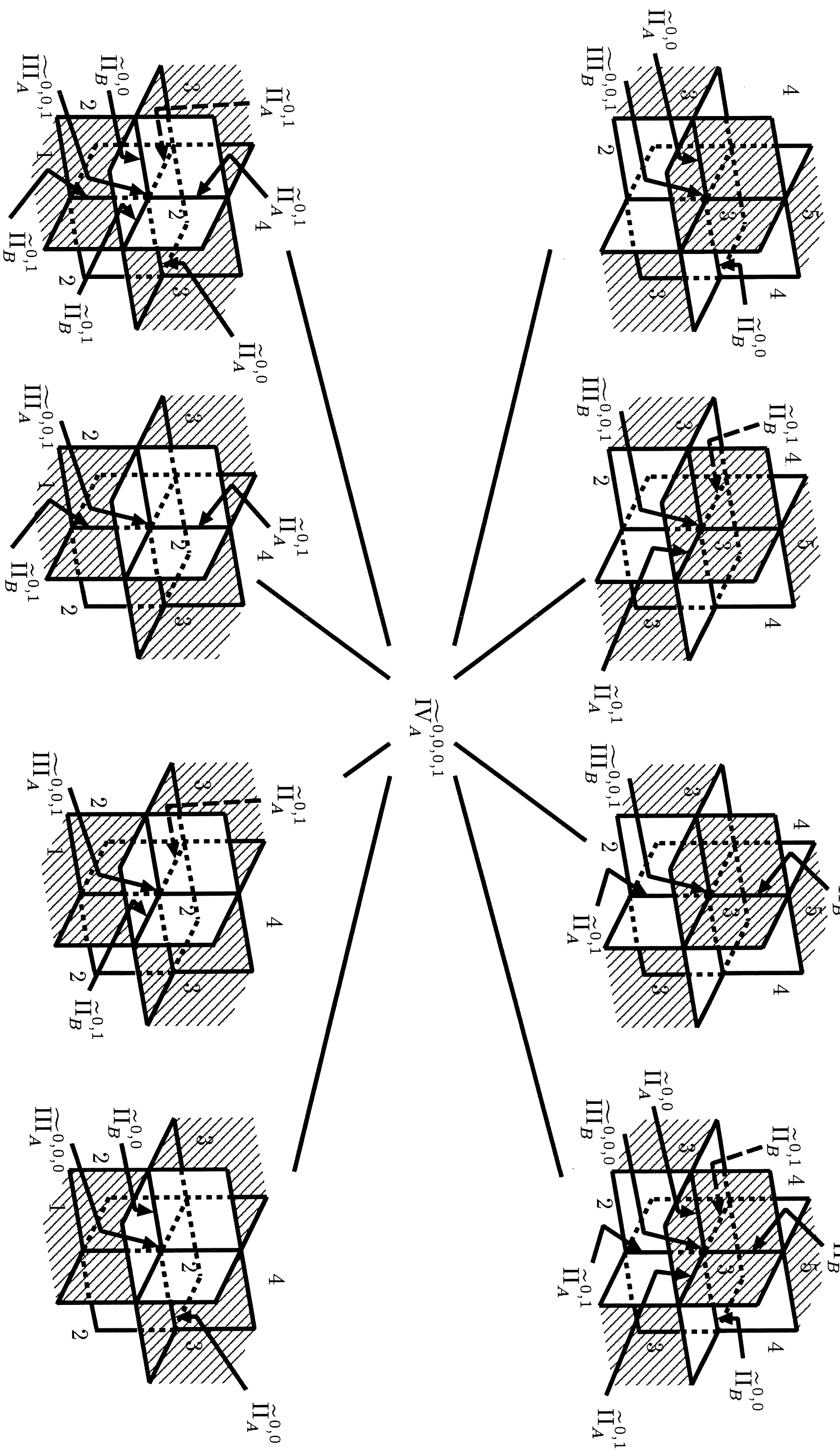

FIGURE 2.11. Type A for $\widetilde{\mathrm{IV}}^{0,0,0,1}$

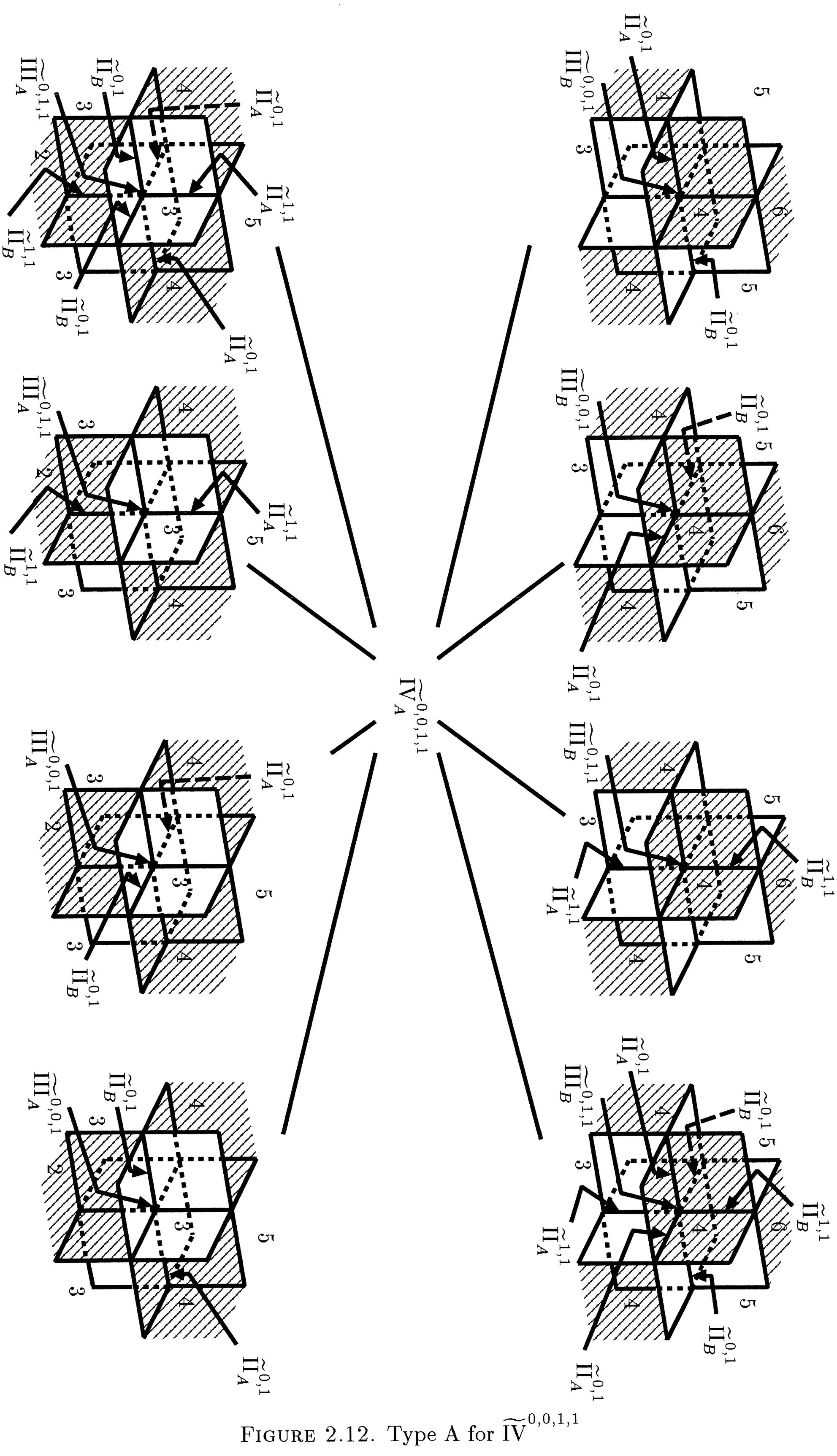

FIGURE 2.12. Type A for $\widetilde{\mathrm{IV}}^{0,0,1,1}$

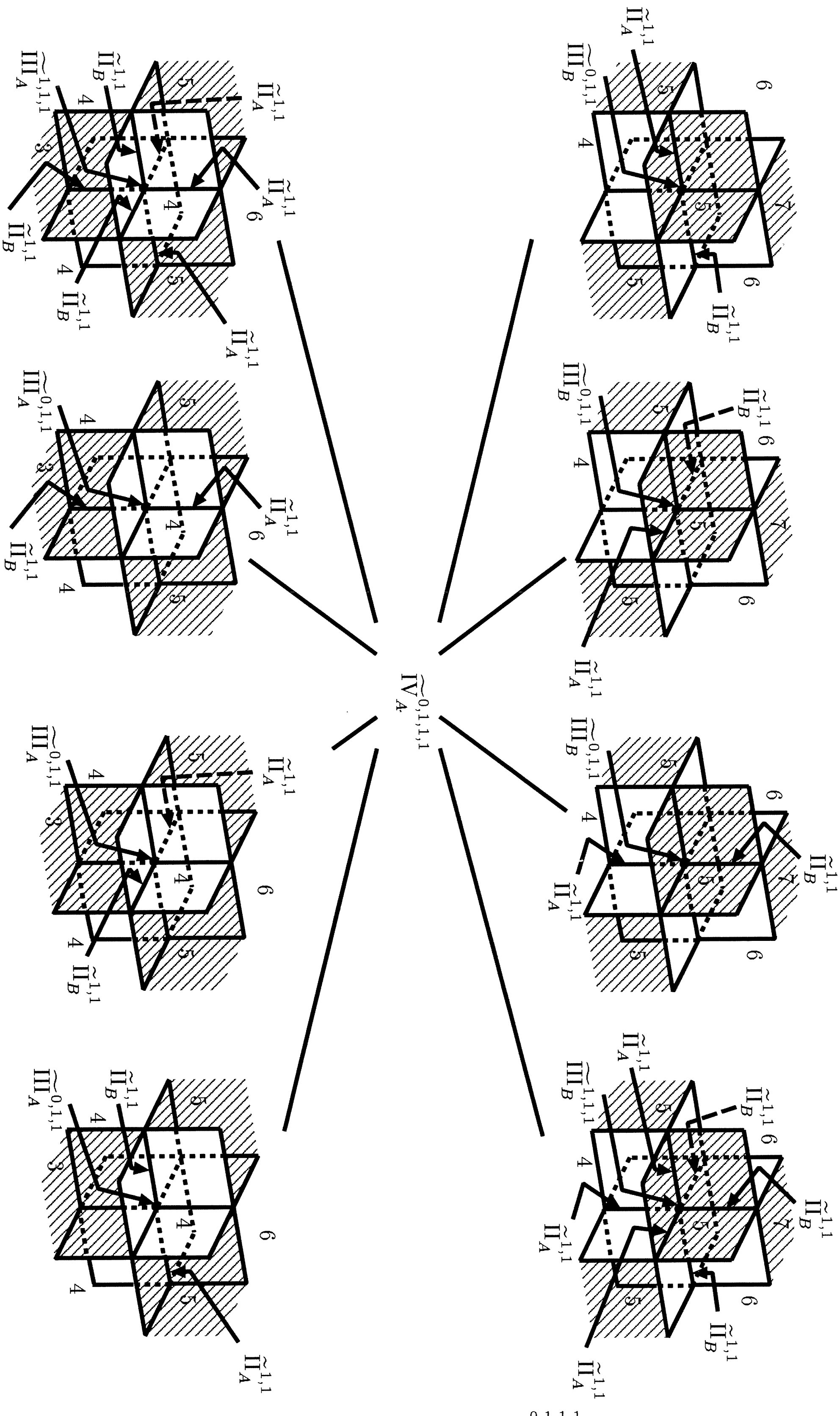

FIGURE 2.13. Type A for $\widetilde{\mathrm{IV}}^{0,1,1,1}$

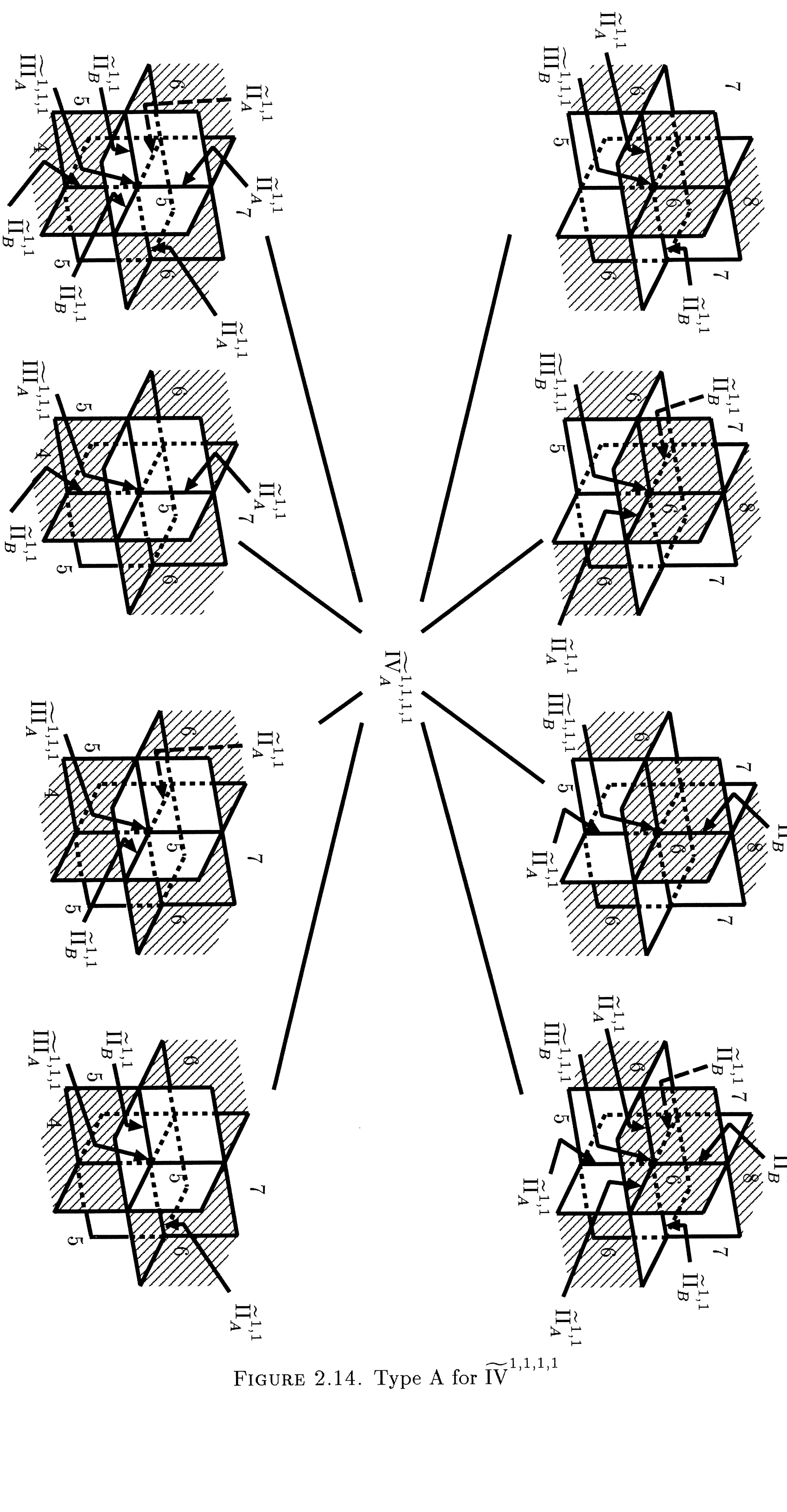

FIGURE 2.14. Type A for $\widetilde{\mathrm{IV}}^{1,1,1,1}$

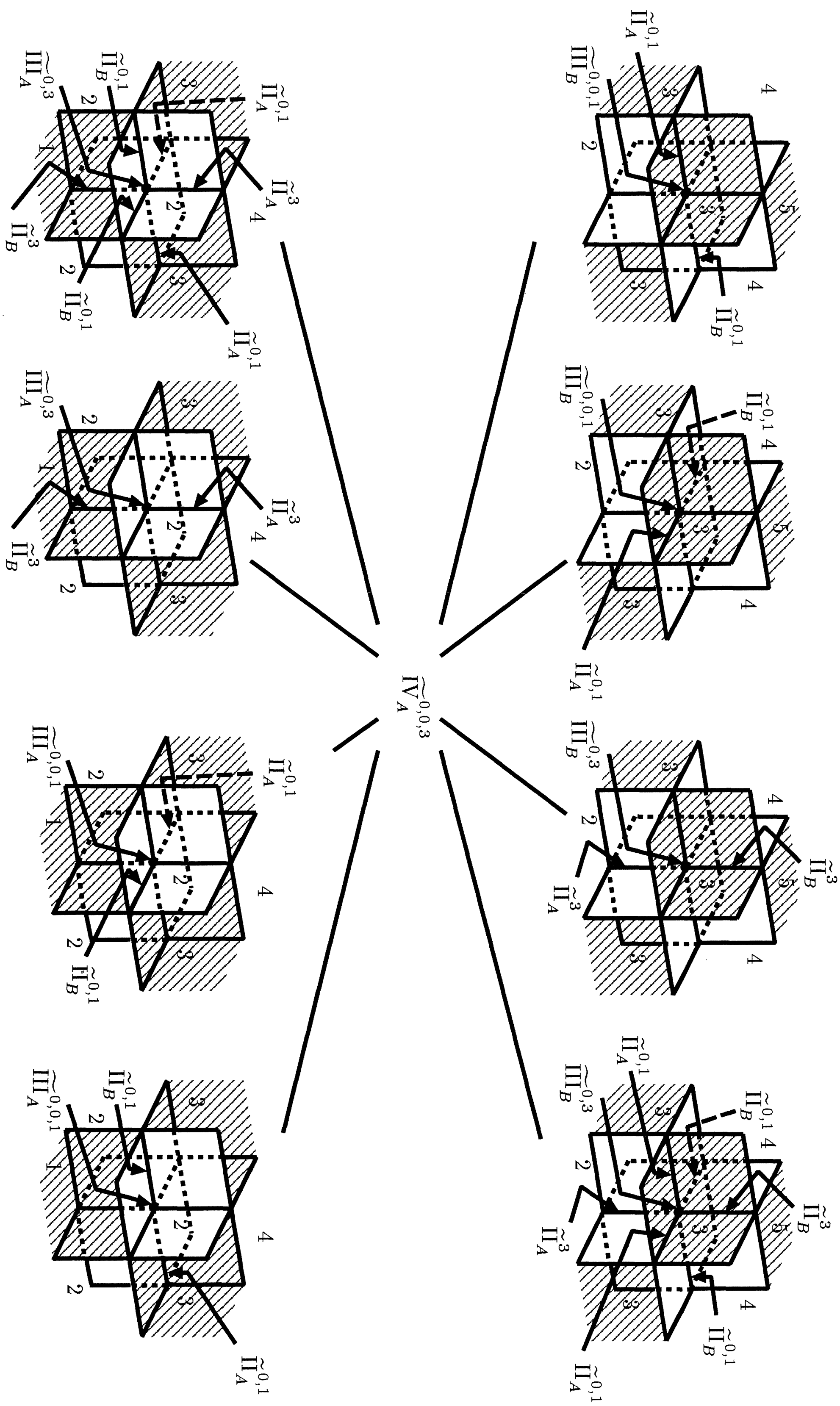

FIGURE 2.15. Type A for $\widetilde{IV}^{0,0,3}$

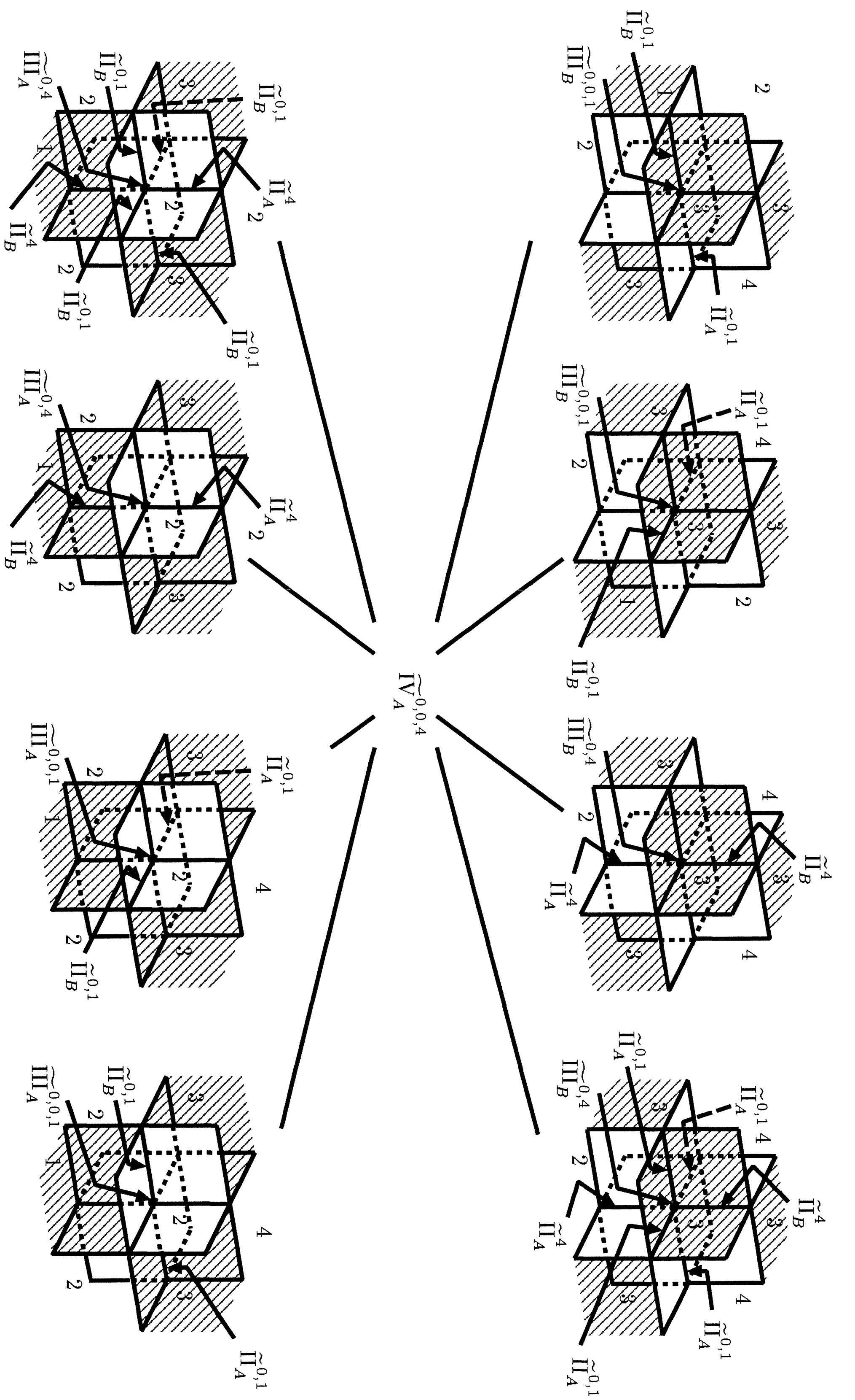

FIGURE 2.16. Type A for $\widetilde{\mathrm{IV}}^{0,0,4}$

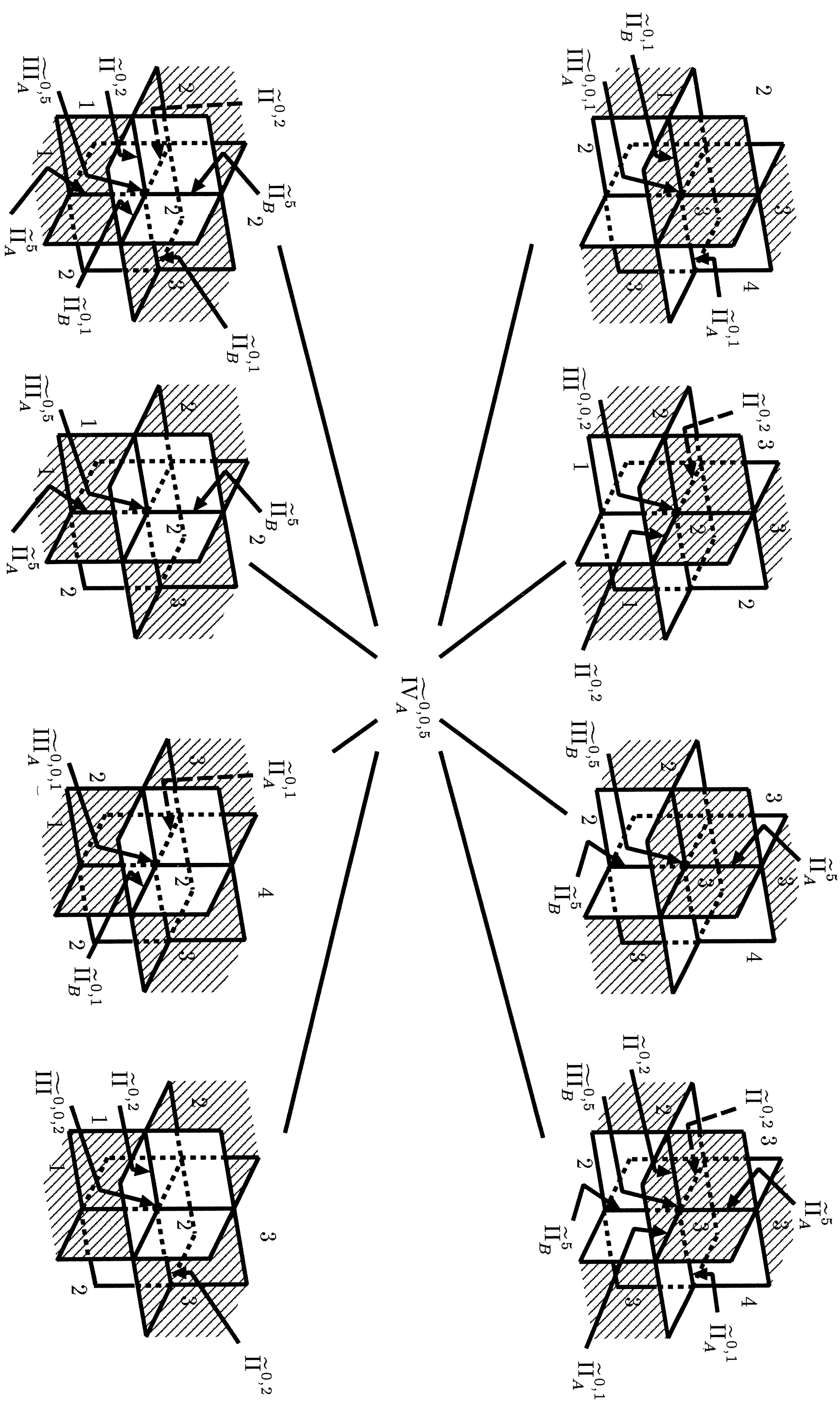

FIGURE 2.17. Type A for $\widetilde{\mathrm{IV}}^{0,0,5}$

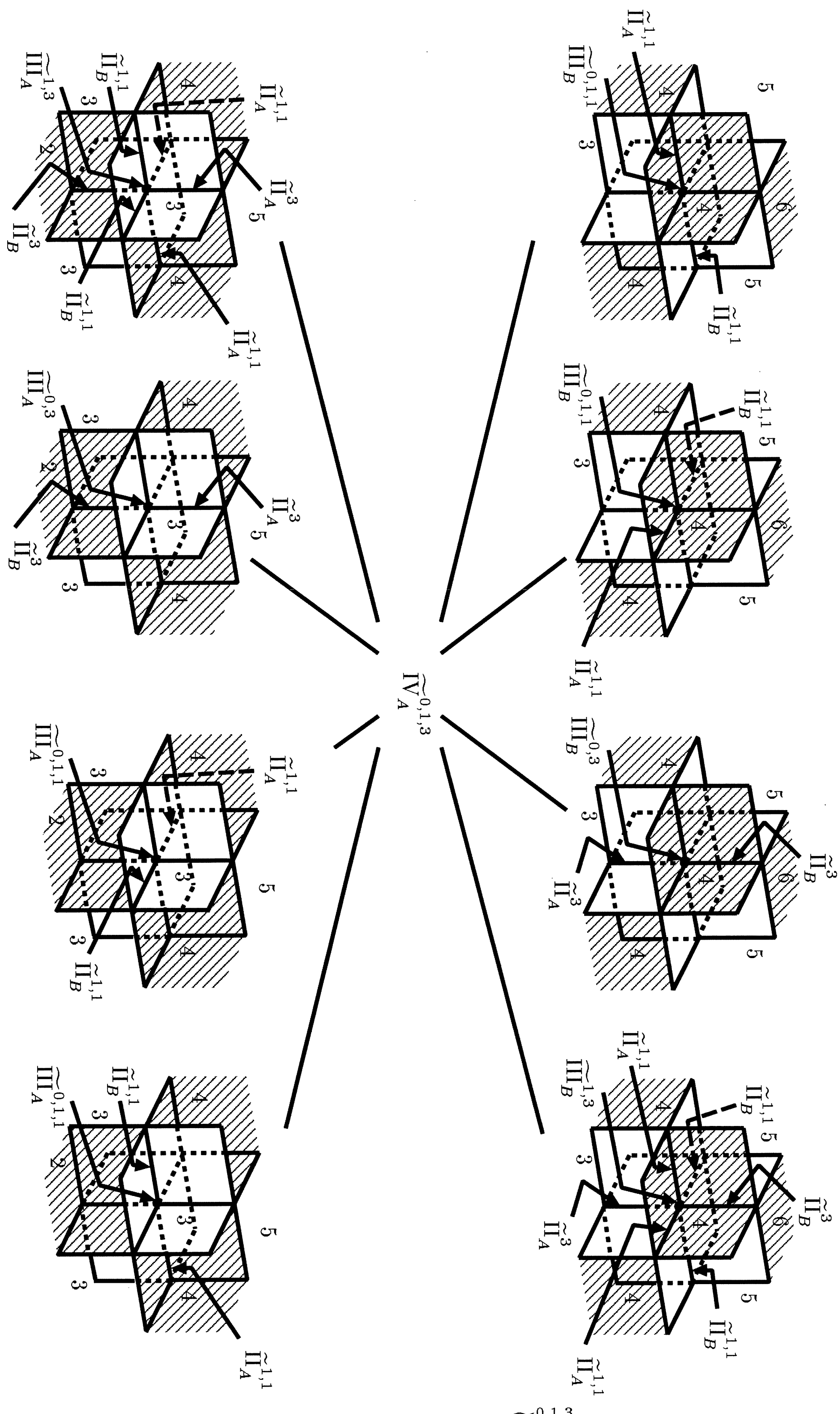

FIGURE 2.18. Type A for $\widetilde{\mathrm{IV}}^{0,1,3}$

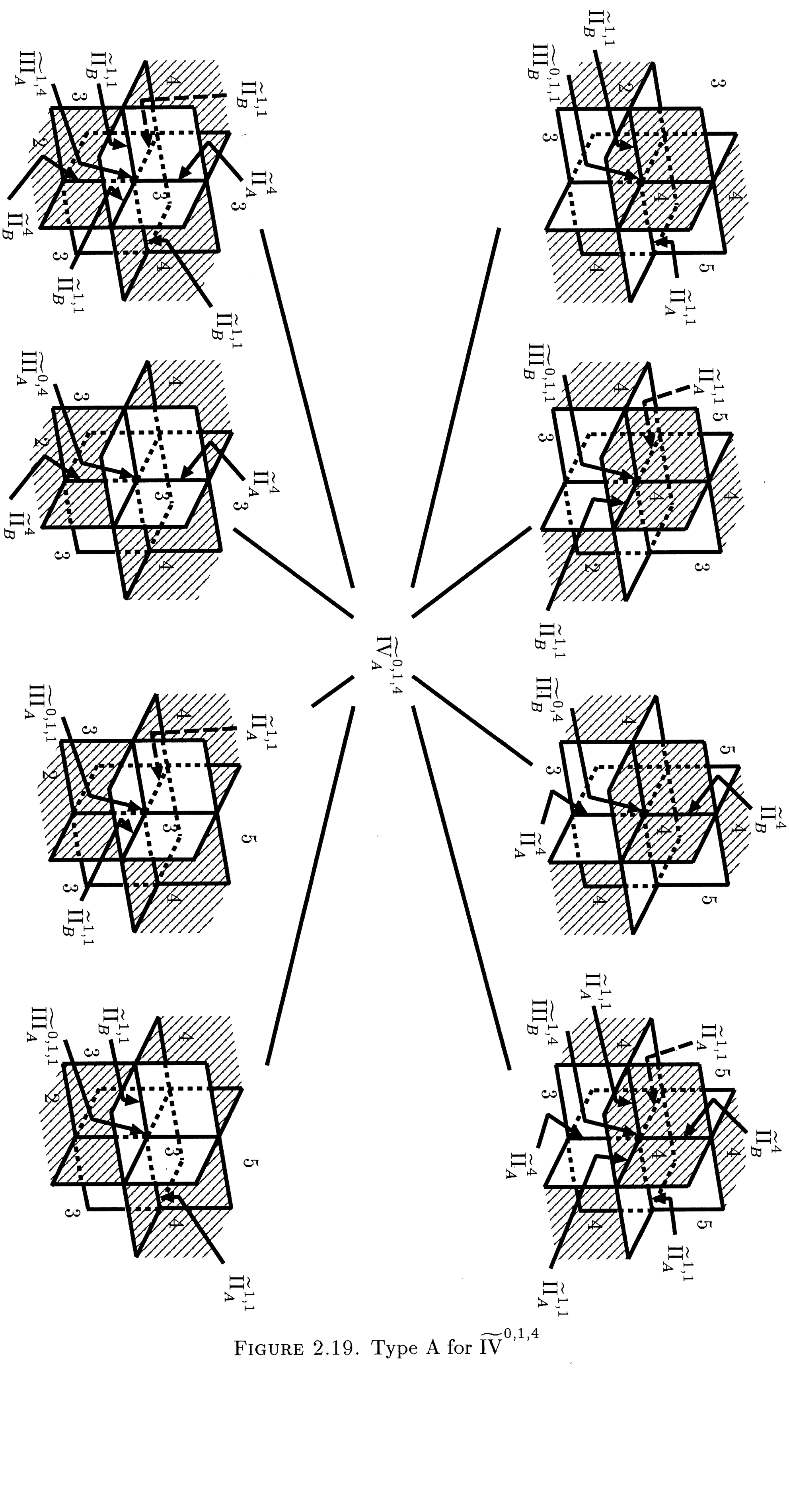

FIGURE 2.19. Type A for $\widetilde{IV}^{0,1,4}$

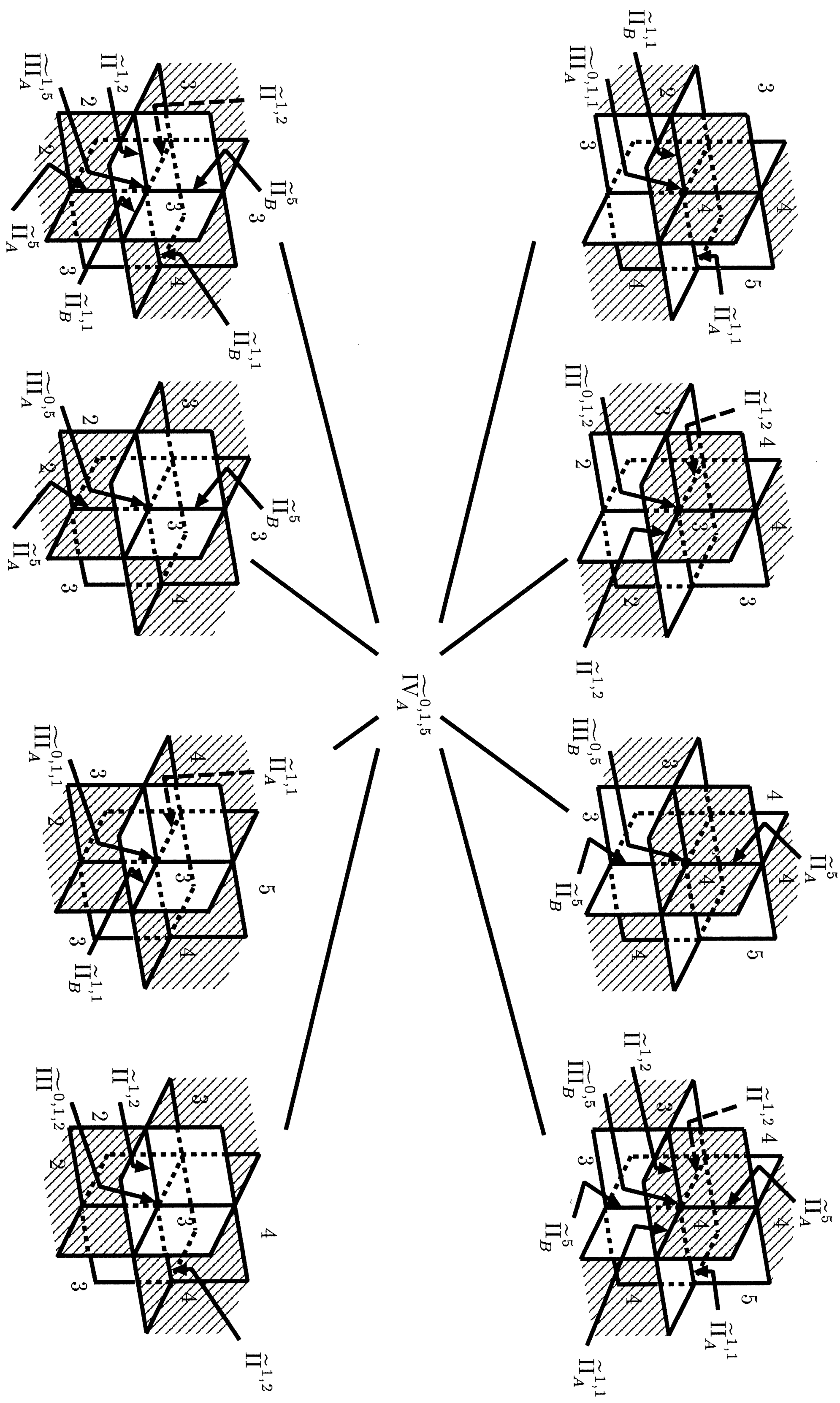

FIGURE 2.20. Type A for $\widetilde{\mathrm{IV}}^{0,1,5}$

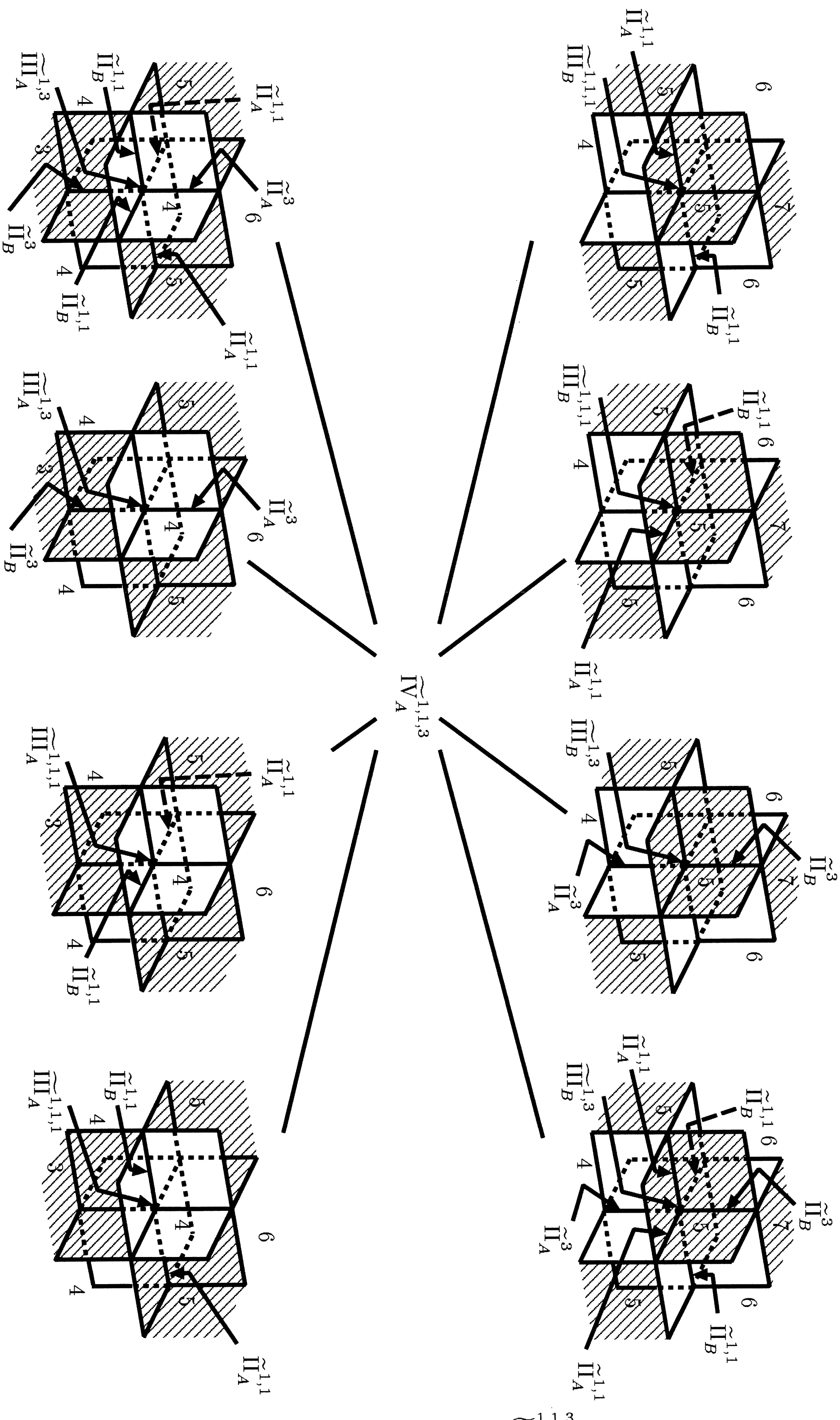

FIGURE 2.21. Type A for $\widetilde{\mathrm{IV}}^{1,1,3}$

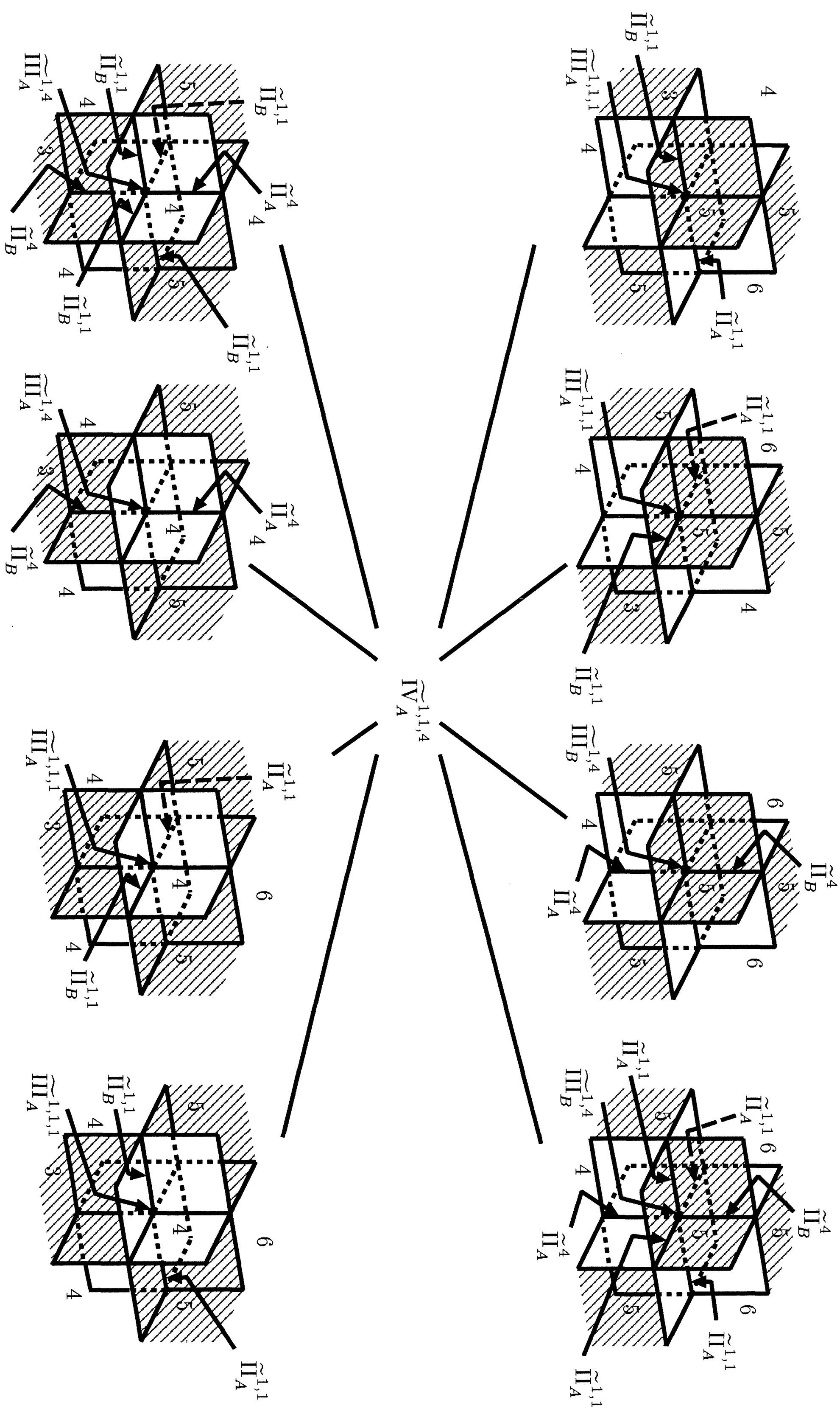

FIGURE 2.22. Type A for $\widetilde{\mathrm{IV}}^{1,1,4}$

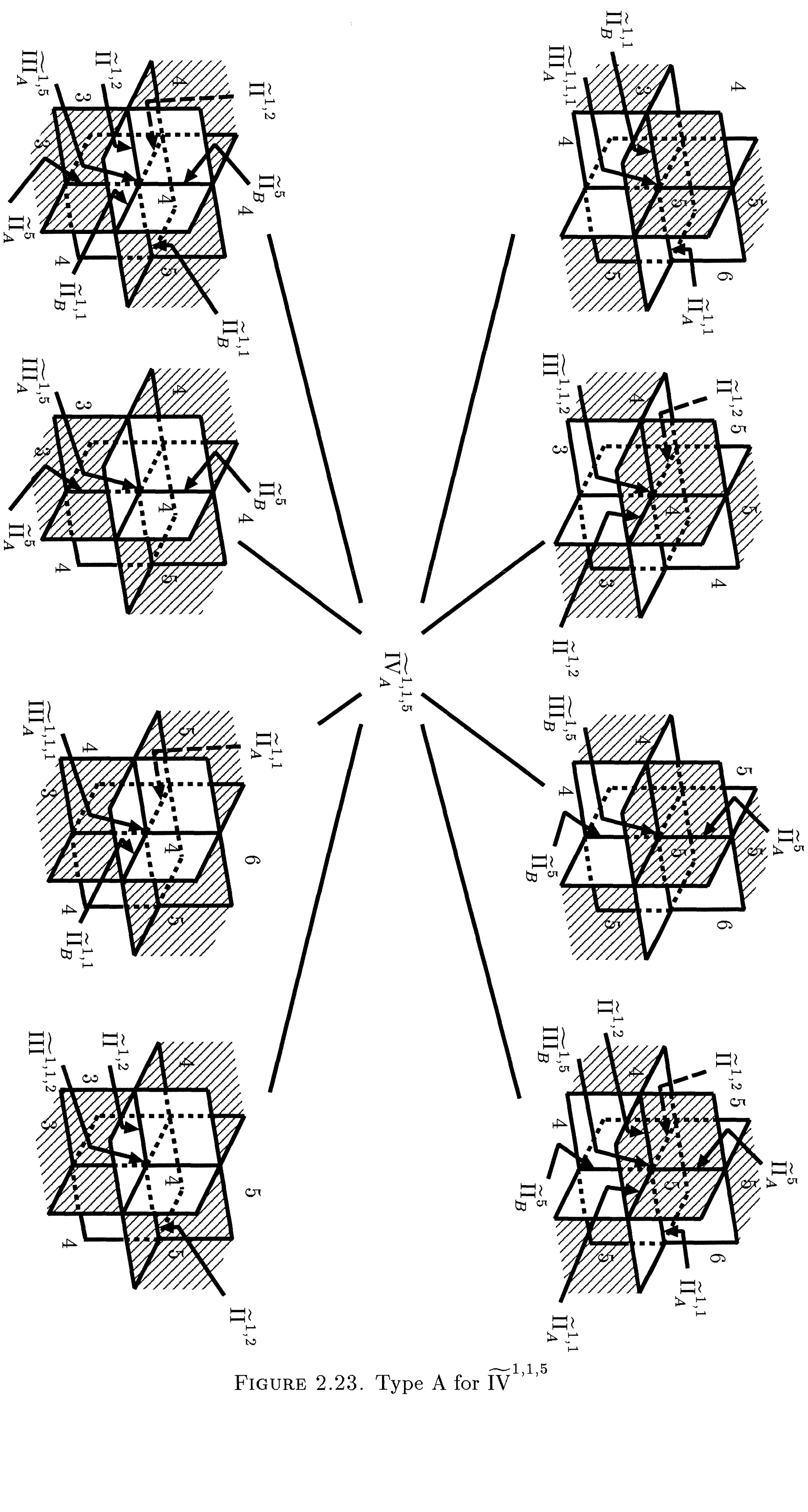

FIGURE 2.23. Type A for $\widetilde{\mathrm{IV}}^{1,1,5}$

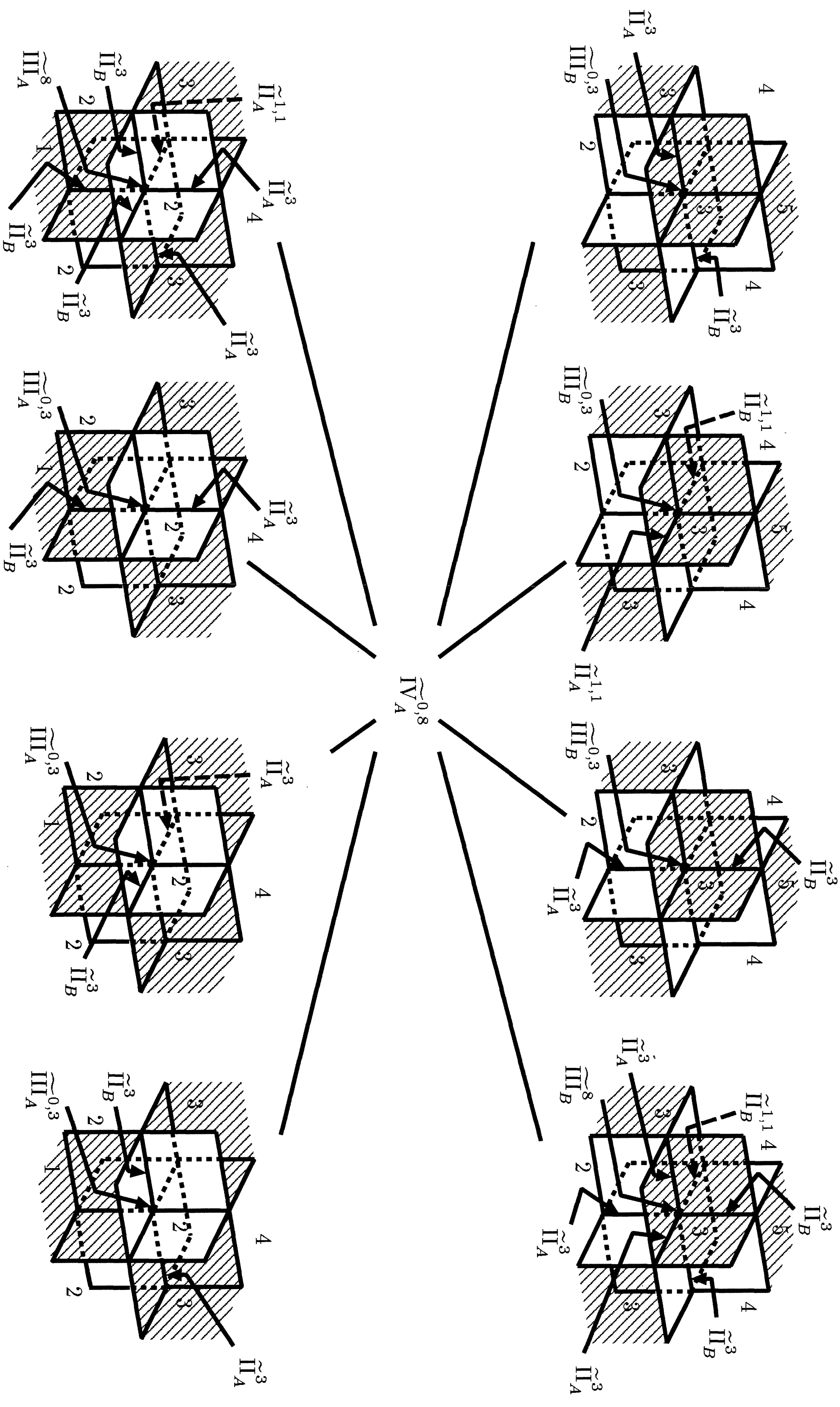

FIGURE 2.24. Type A for $\widetilde{\mathrm{IV}}^{0,8}$

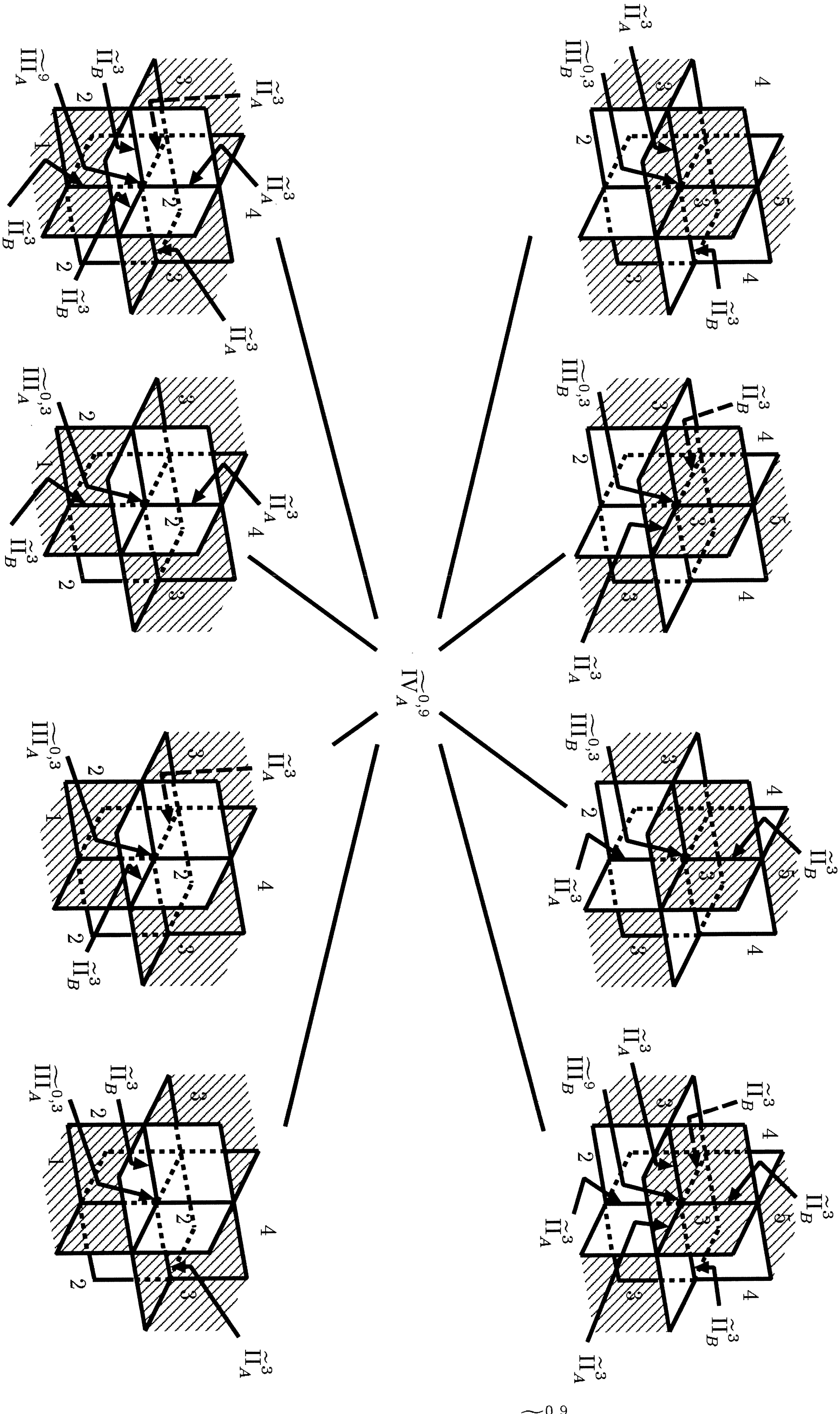

FIGURE 2.25. Type A for $\widetilde{\mathrm{IV}}^{0,9}$

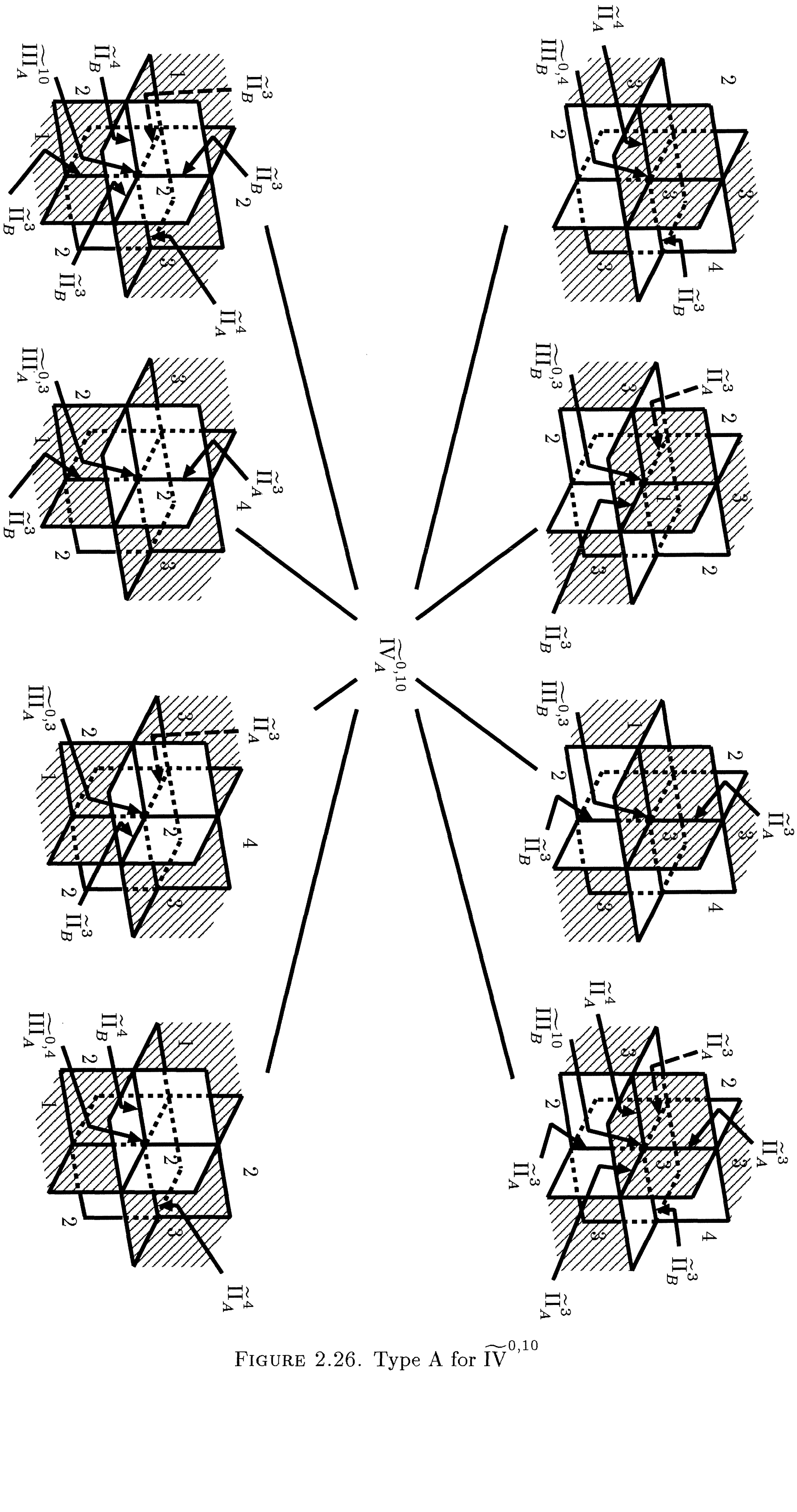

FIGURE 2.26. Type A for $\widetilde{\mathrm{IV}}^{0,10}$

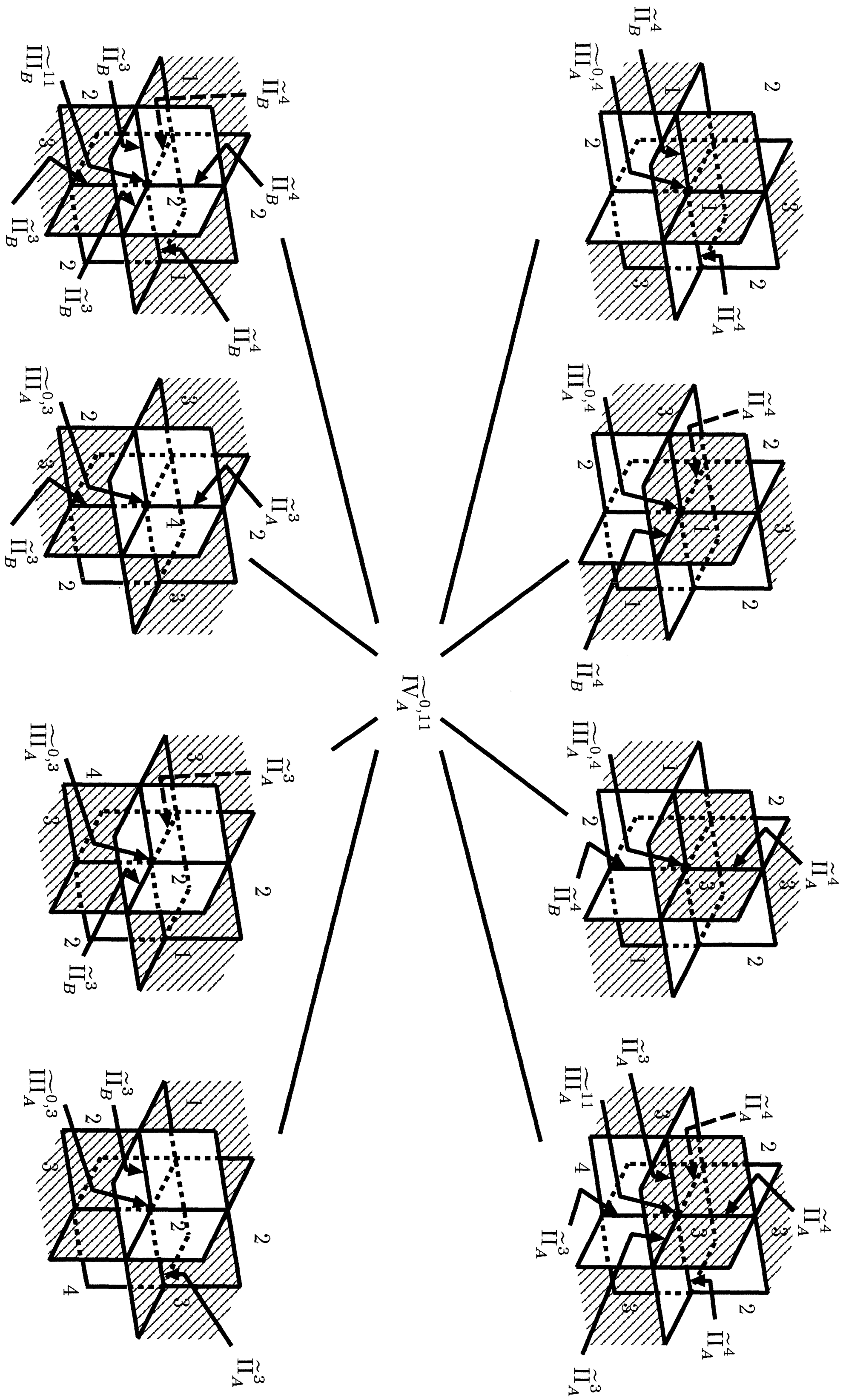

FIGURE 2.27. Type A for $\widetilde{\mathrm{IV}}^{0,11}$

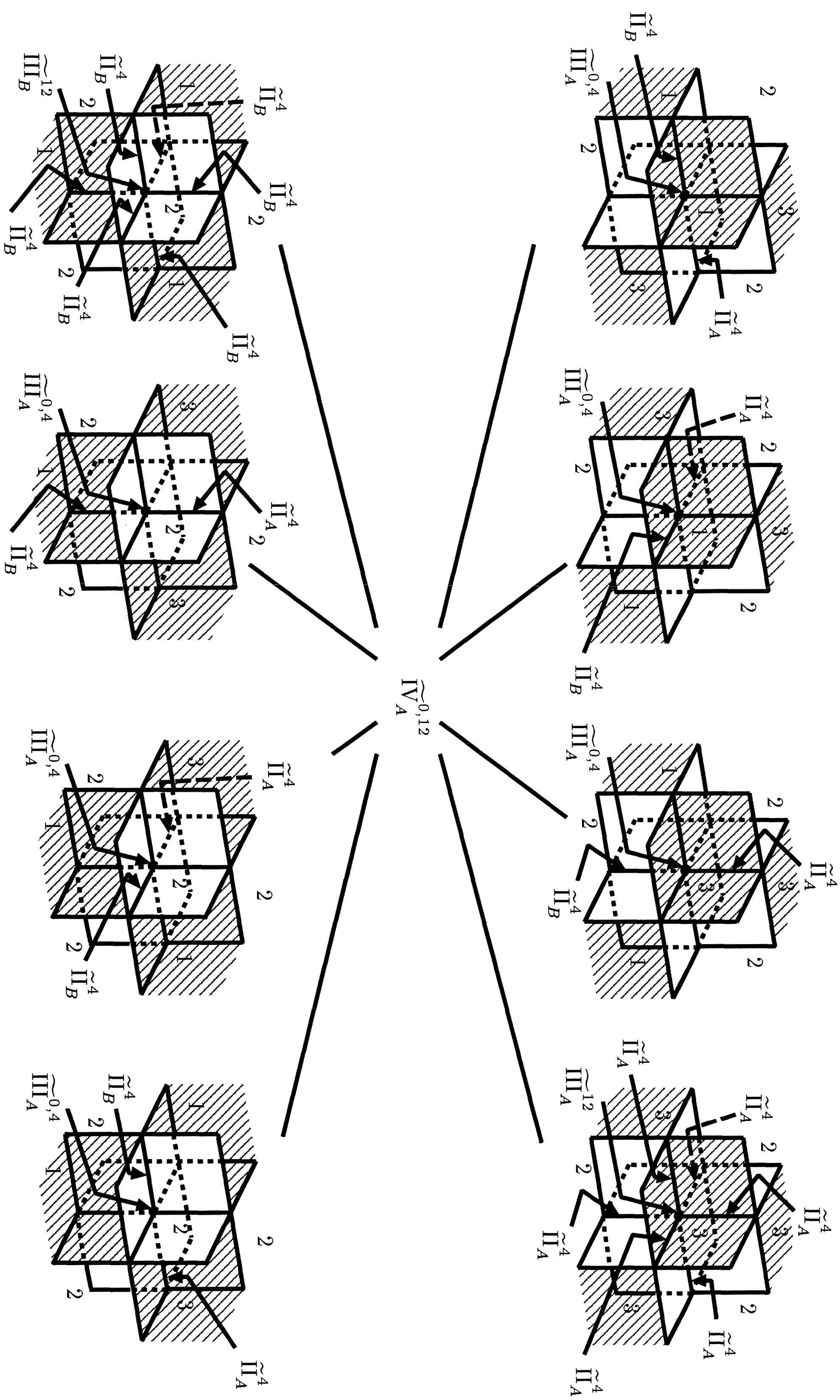

FIGURE 2.28. Type A for $\widetilde{\mathrm{IV}}^{0,12}$

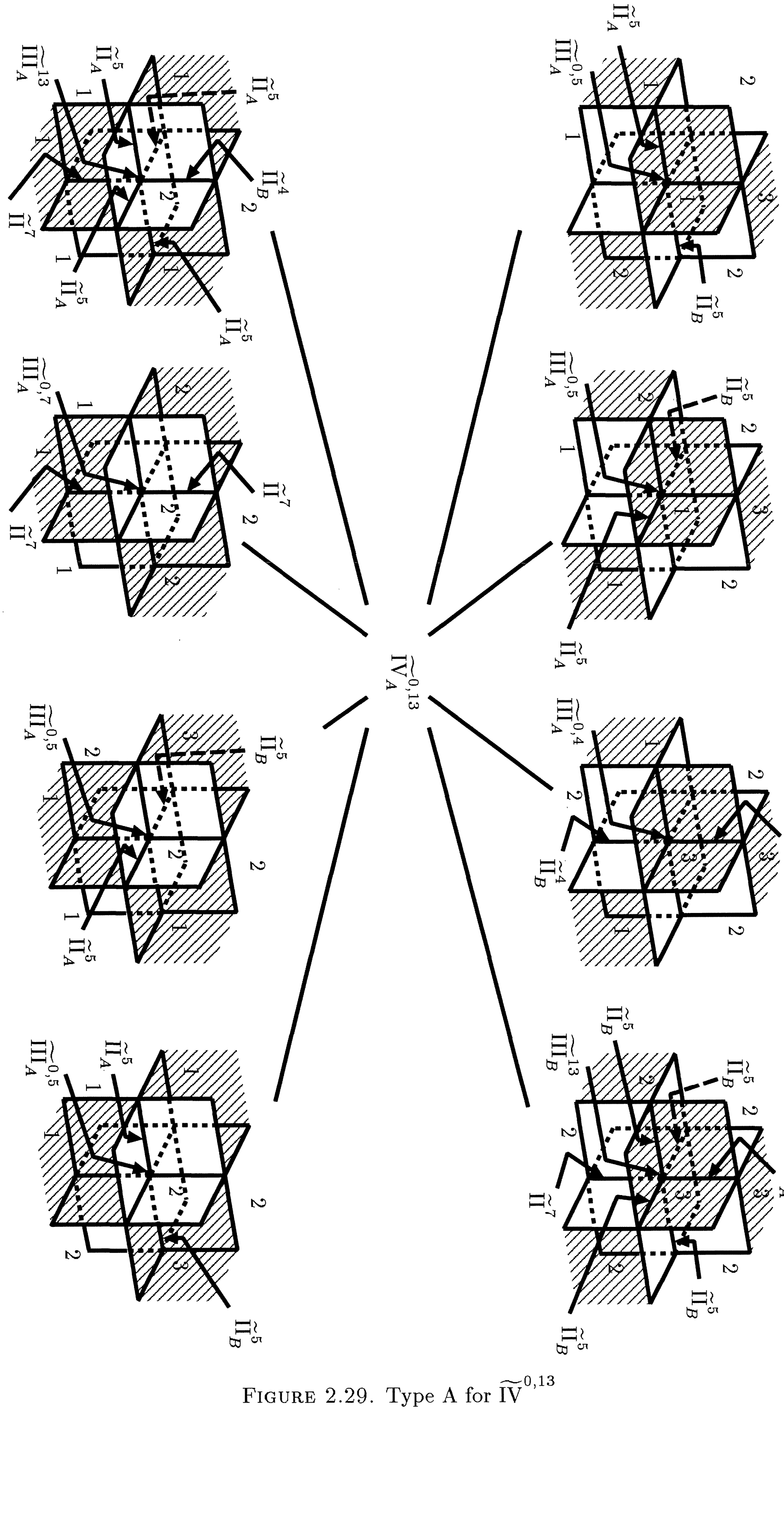

FIGURE 2.29. Type A for $\widetilde{\mathrm{IV}}^{0,13}$

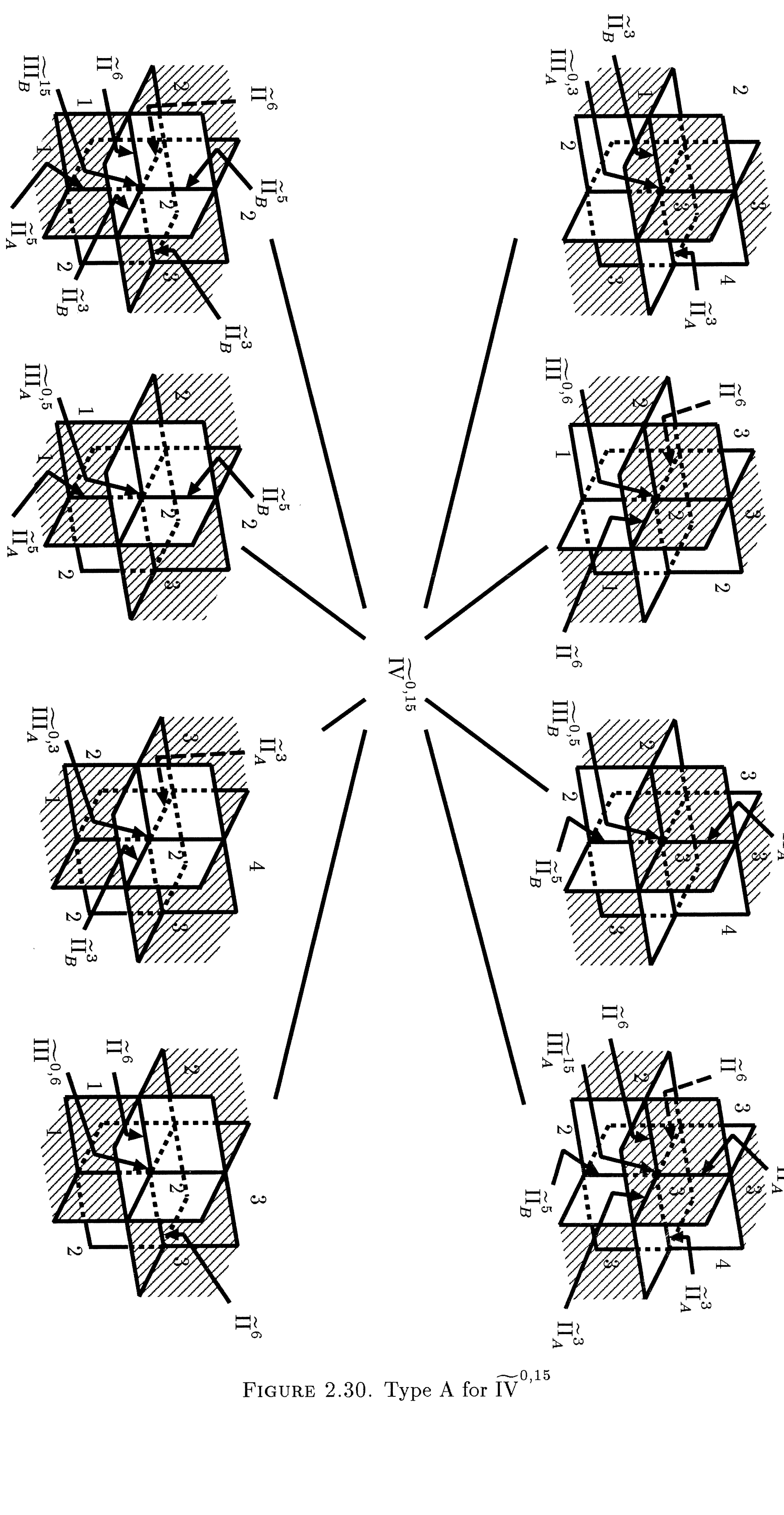

FIGURE 2.30. Type A for $\widetilde{\mathrm{IV}}^{0,15}$

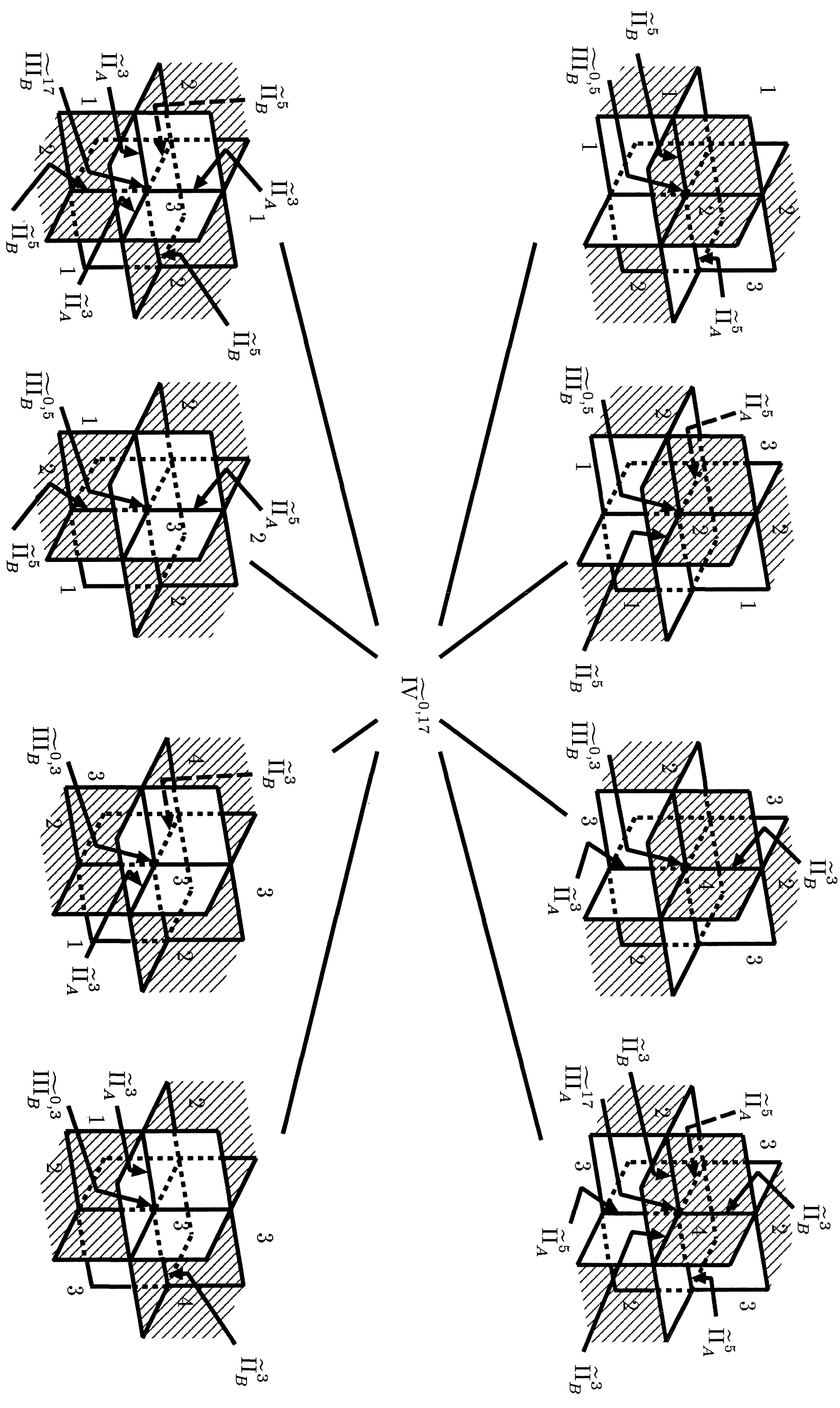

FIGURE 2.31. Type A for $\widetilde{\mathrm{IV}}^{0,17}$

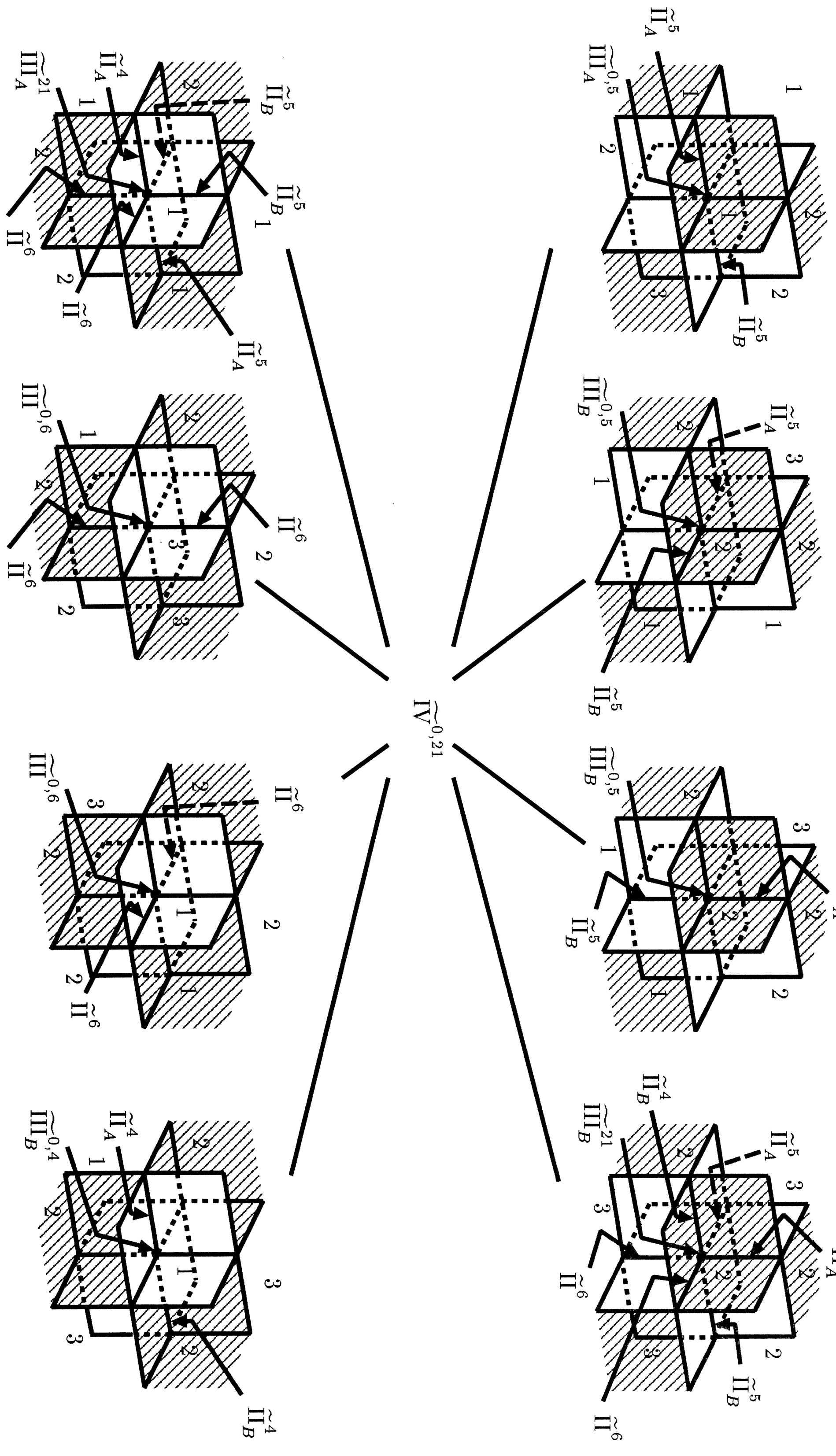

FIGURE 2.32. Type A for $\widetilde{\mathrm{IV}}^{0,21}$

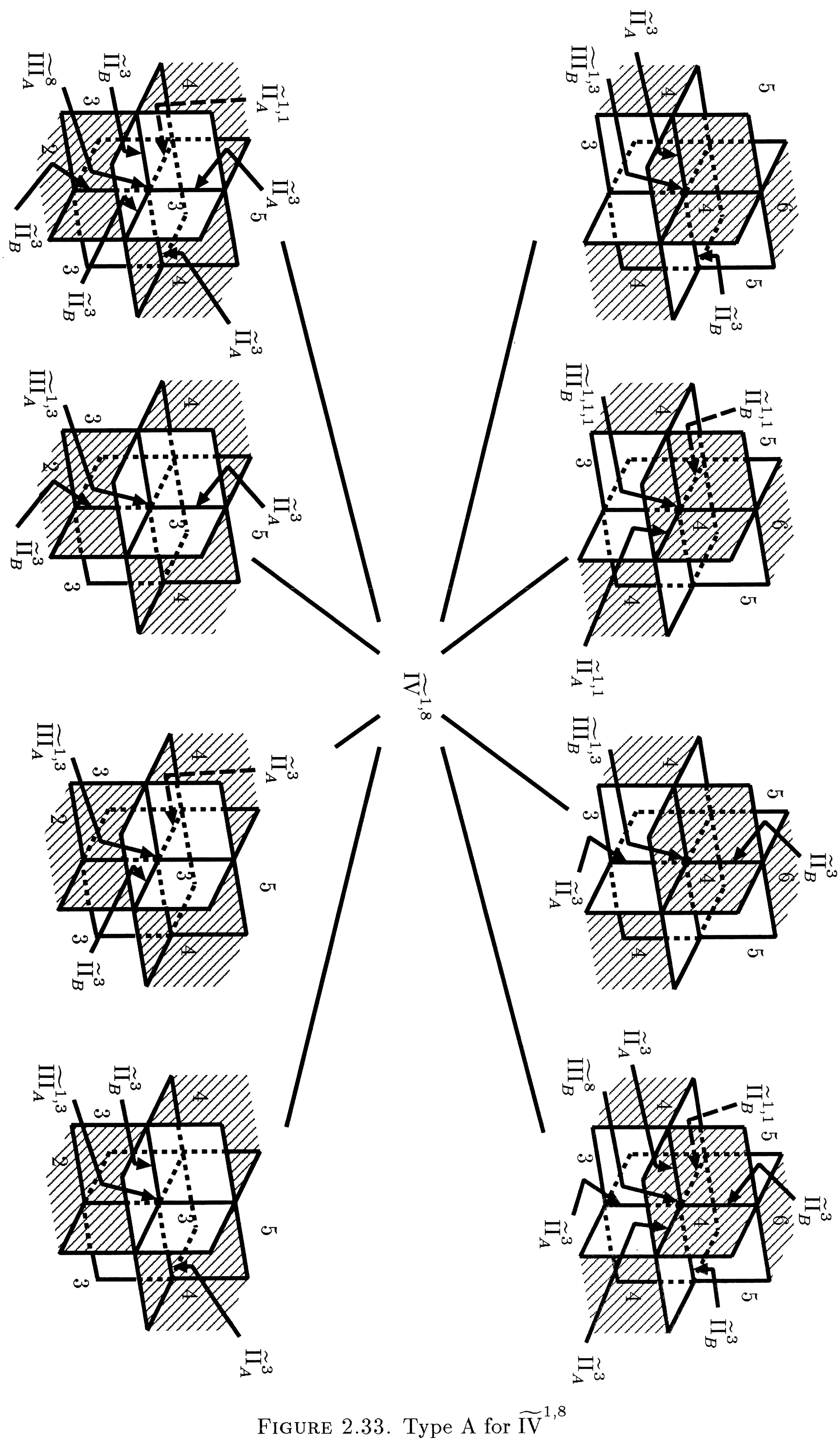

FIGURE 2.33. Type A for $\widetilde{\mathrm{IV}}^{1,8}$

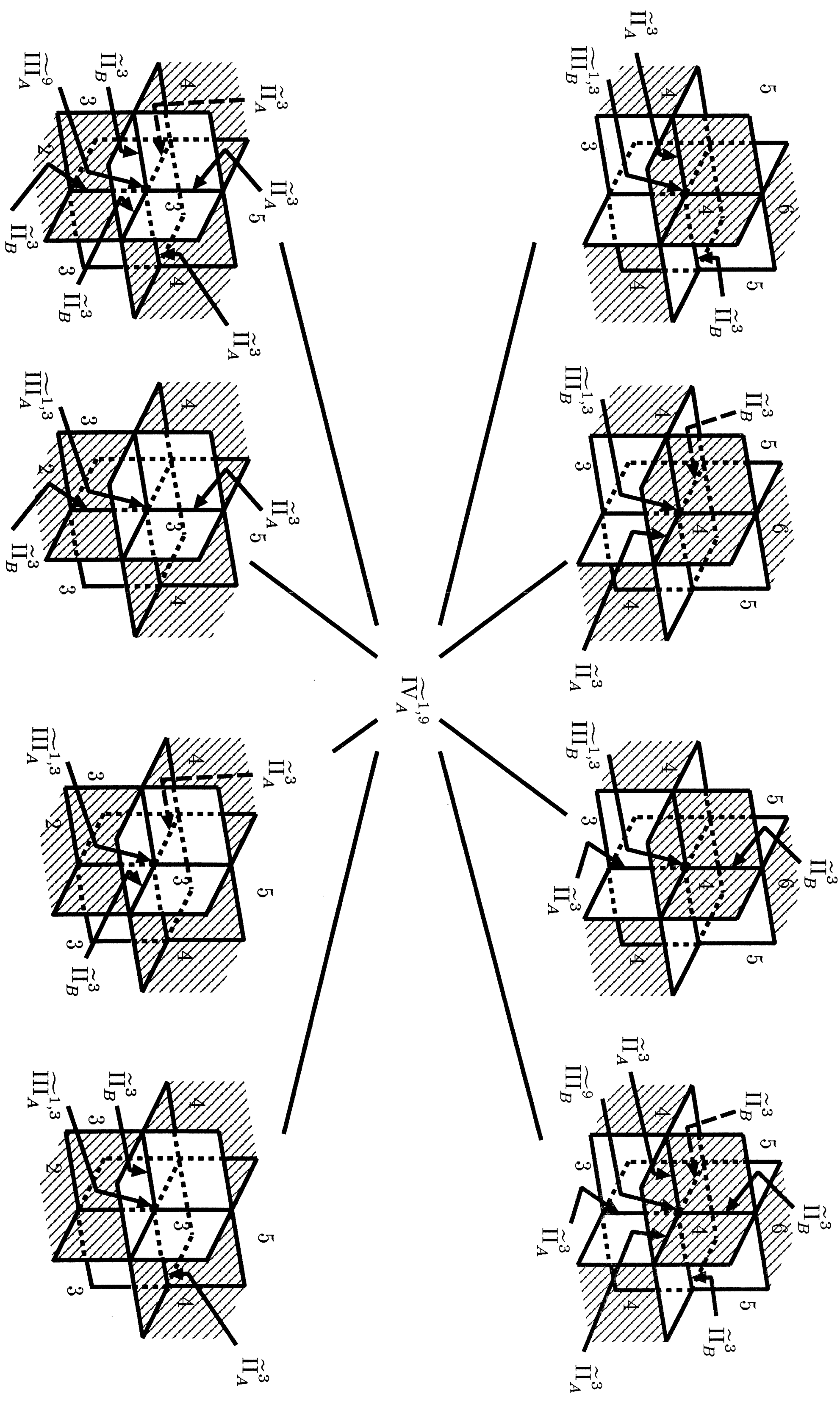

FIGURE 2.34. Type A for $\widetilde{\mathrm{IV}}^{1,9}$

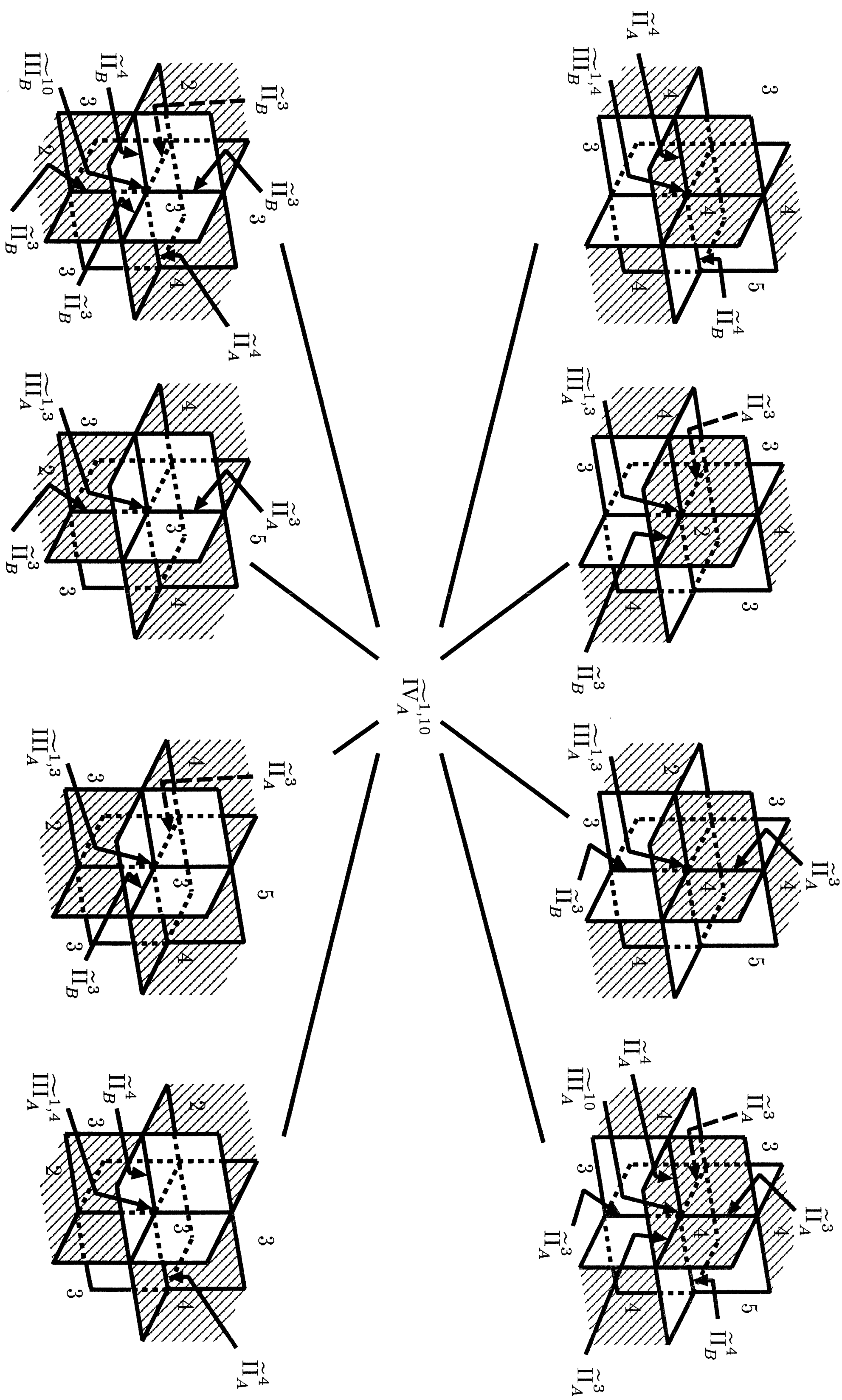

FIGURE 2.35. Type A for $\widetilde{\mathrm{IV}}^{1,10}$

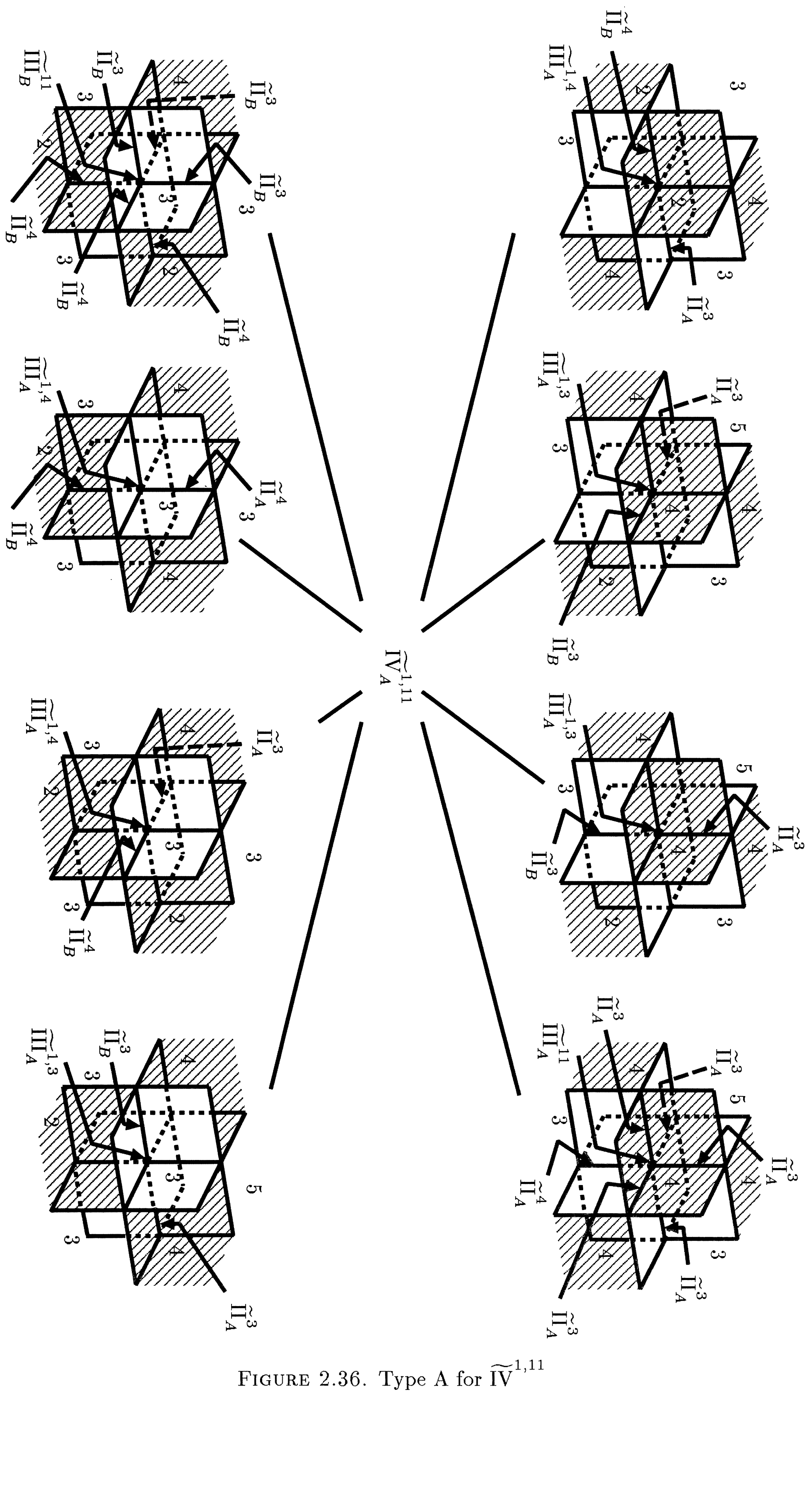

FIGURE 2.36. Type A for $\widetilde{\mathrm{IV}}^{1,11}$

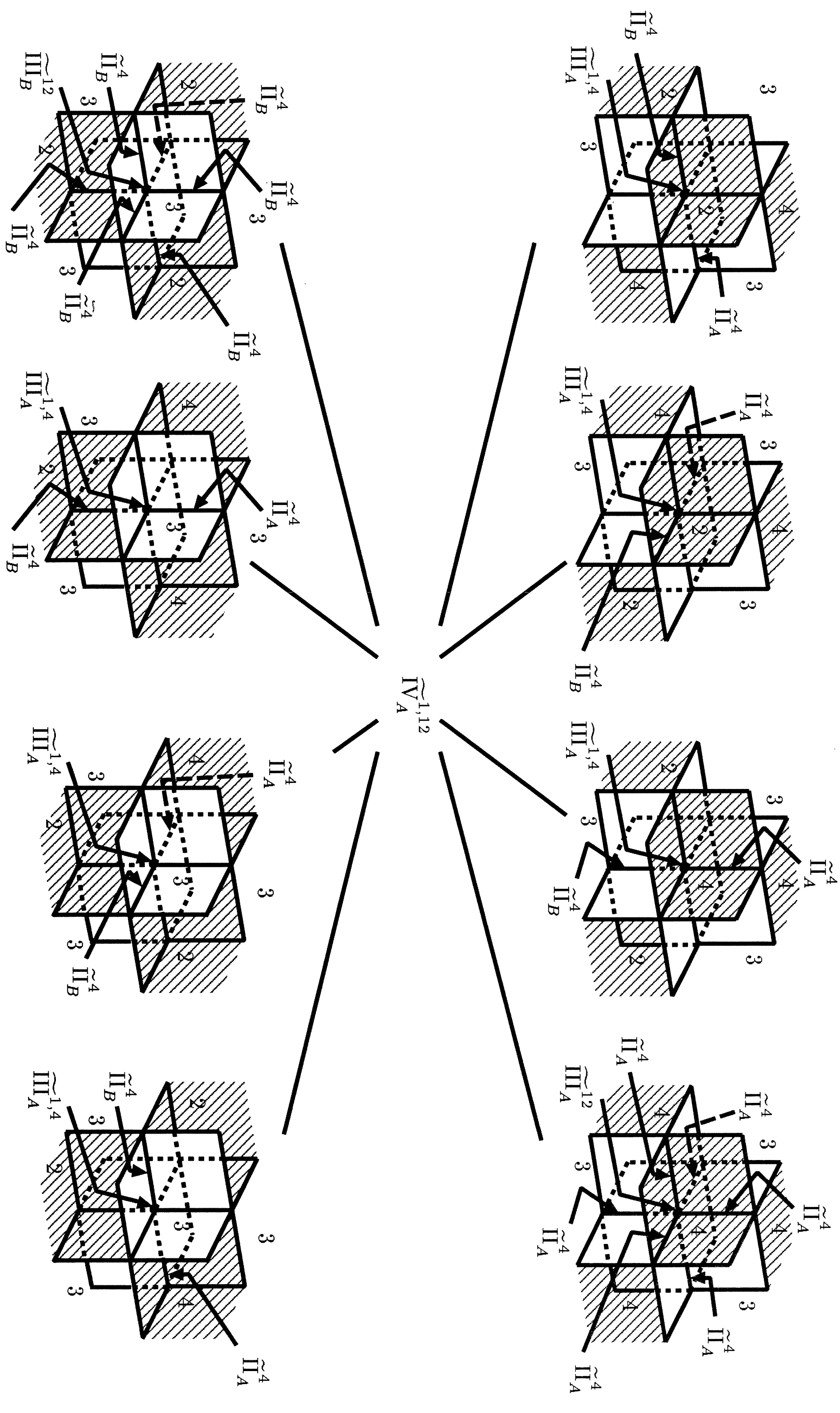

FIGURE 2.37. Type A for $\widetilde{\mathrm{IV}}^{1,12}$

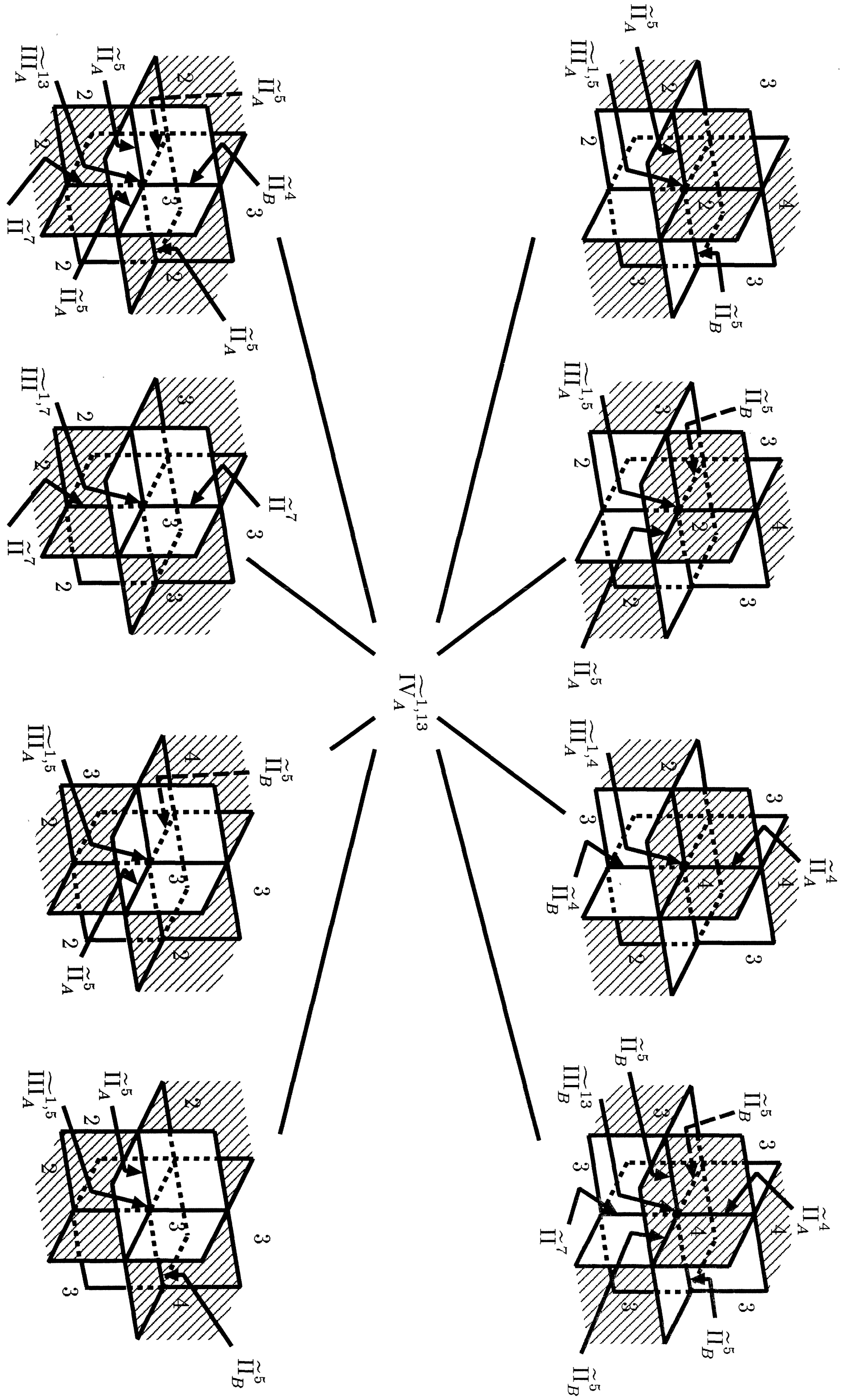

FIGURE 2.38. Type A for $\widetilde{\mathrm{IV}}^{1,13}$

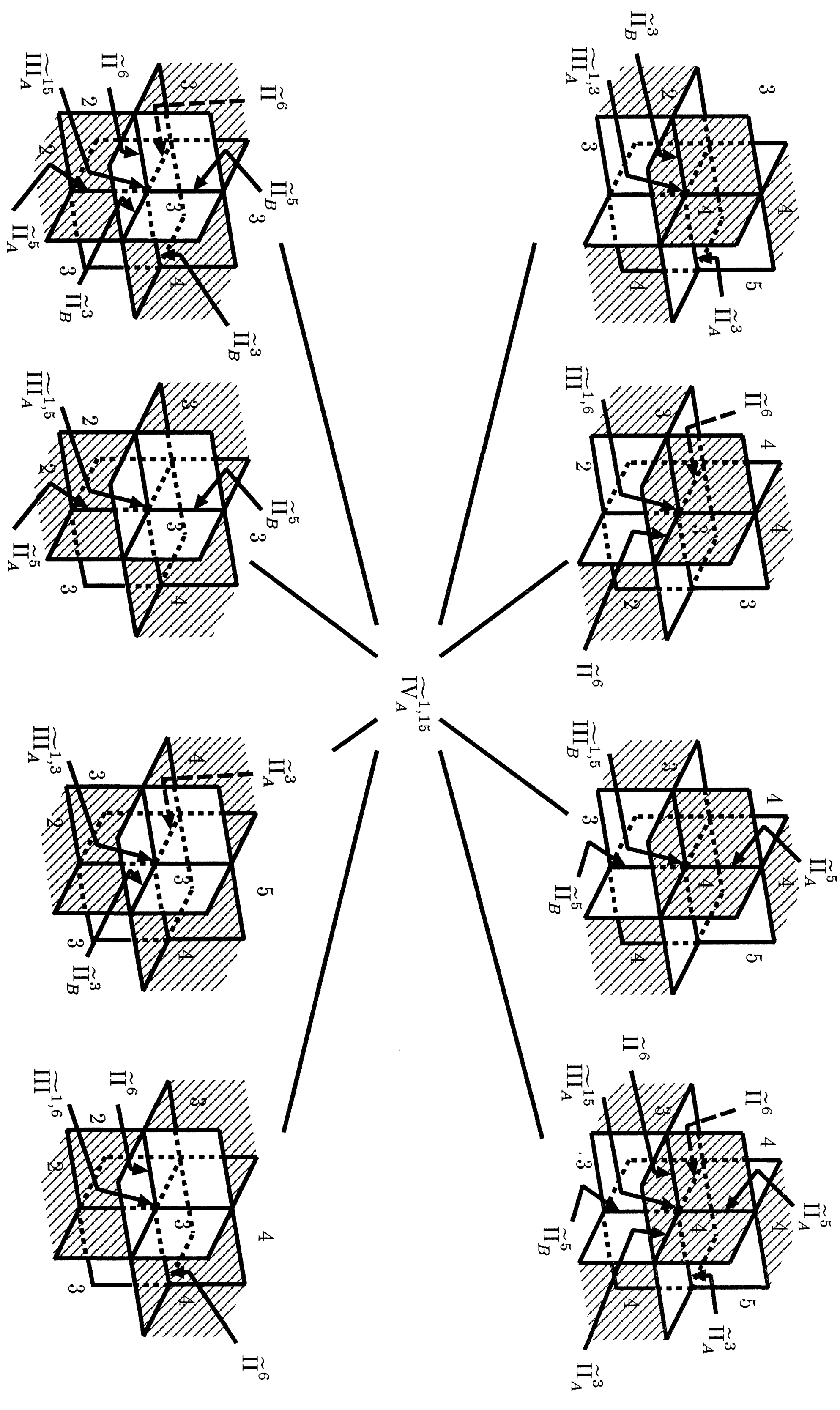

FIGURE 2.39. Type A for $\widetilde{\mathrm{IV}}^{1,15}$

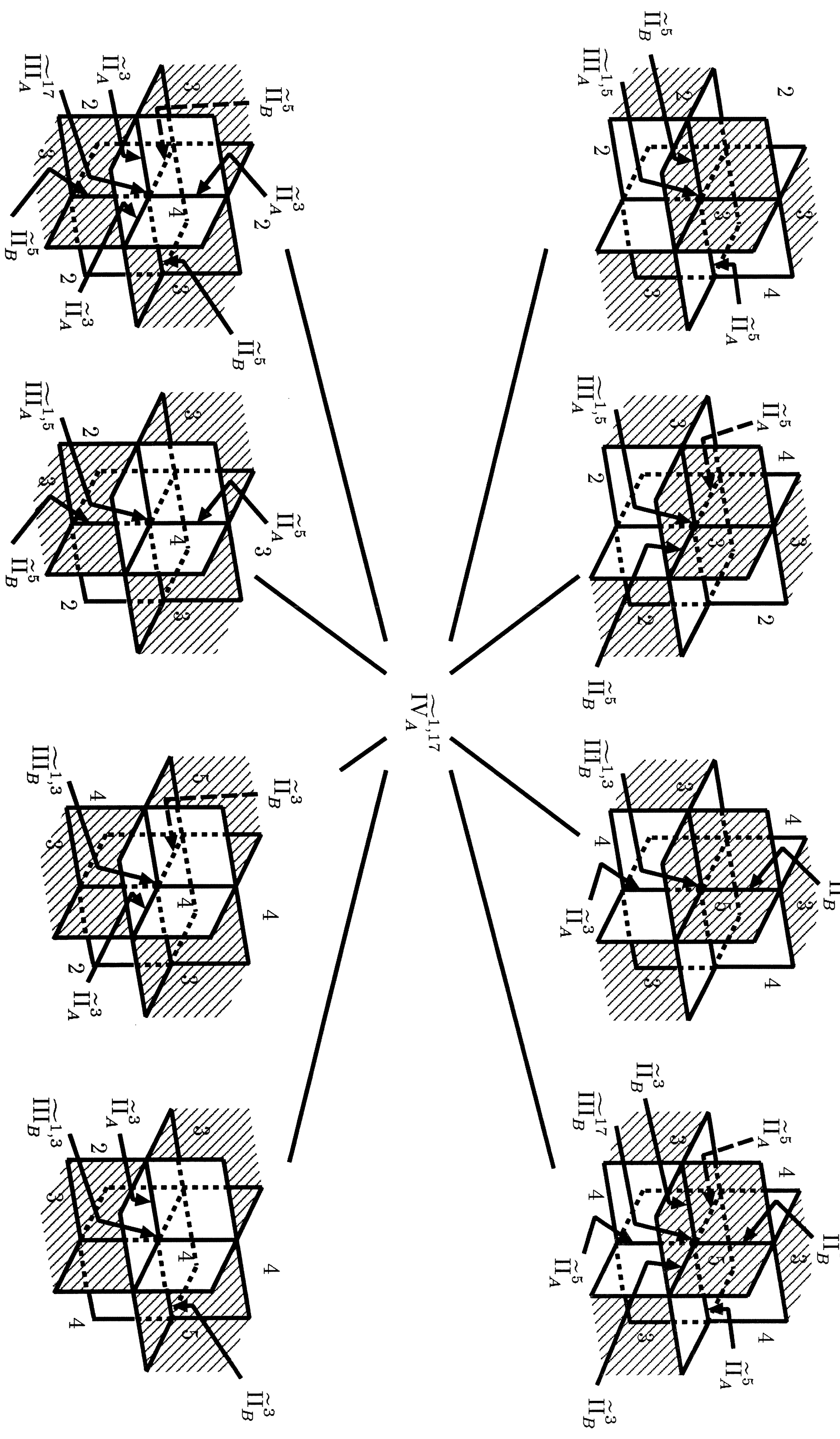

FIGURE 2.40. Type A for $\widetilde{\mathrm{IV}}^{1,17}$

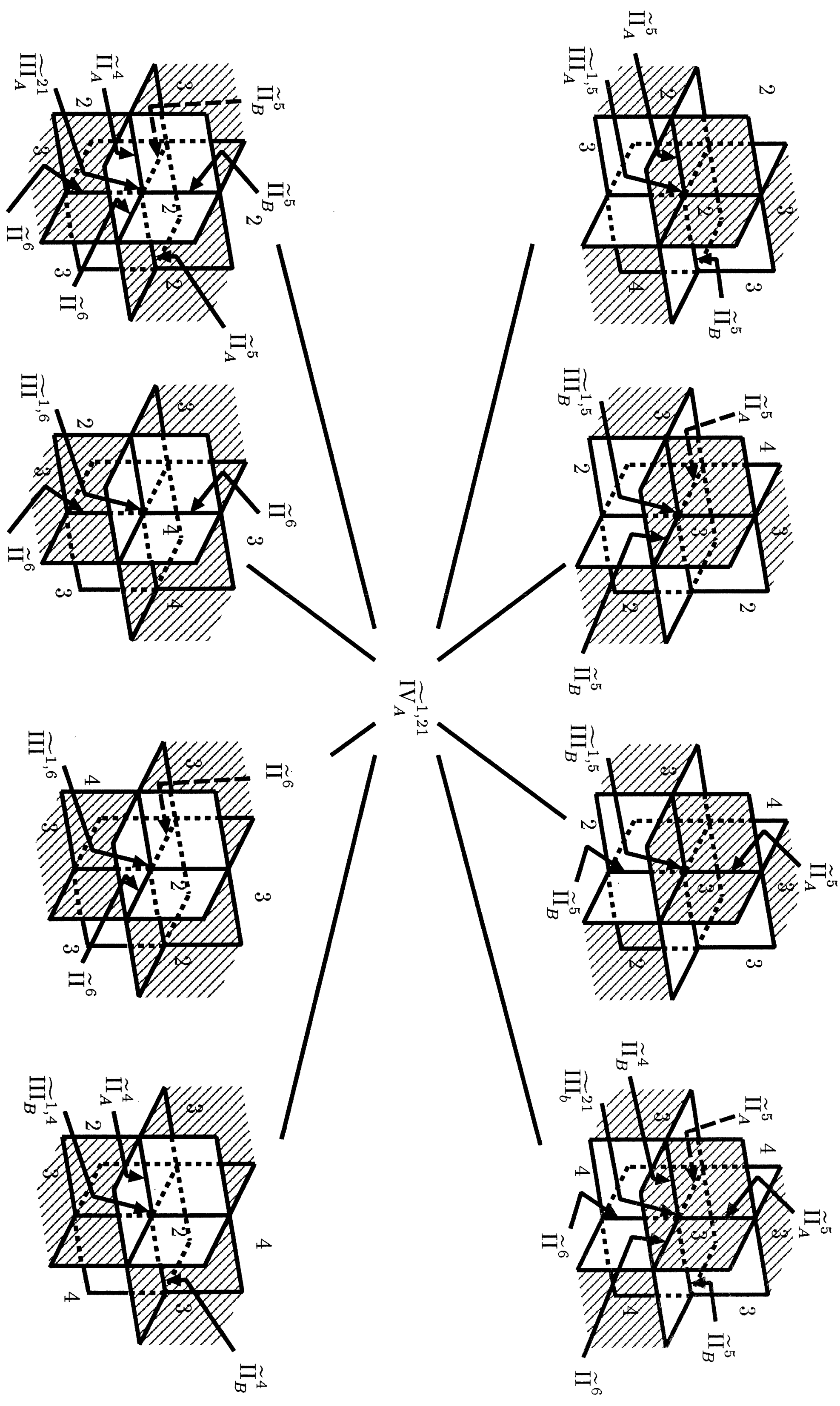

FIGURE 2.41. Type A for $\widetilde{\mathrm{IV}}^{1,21}$

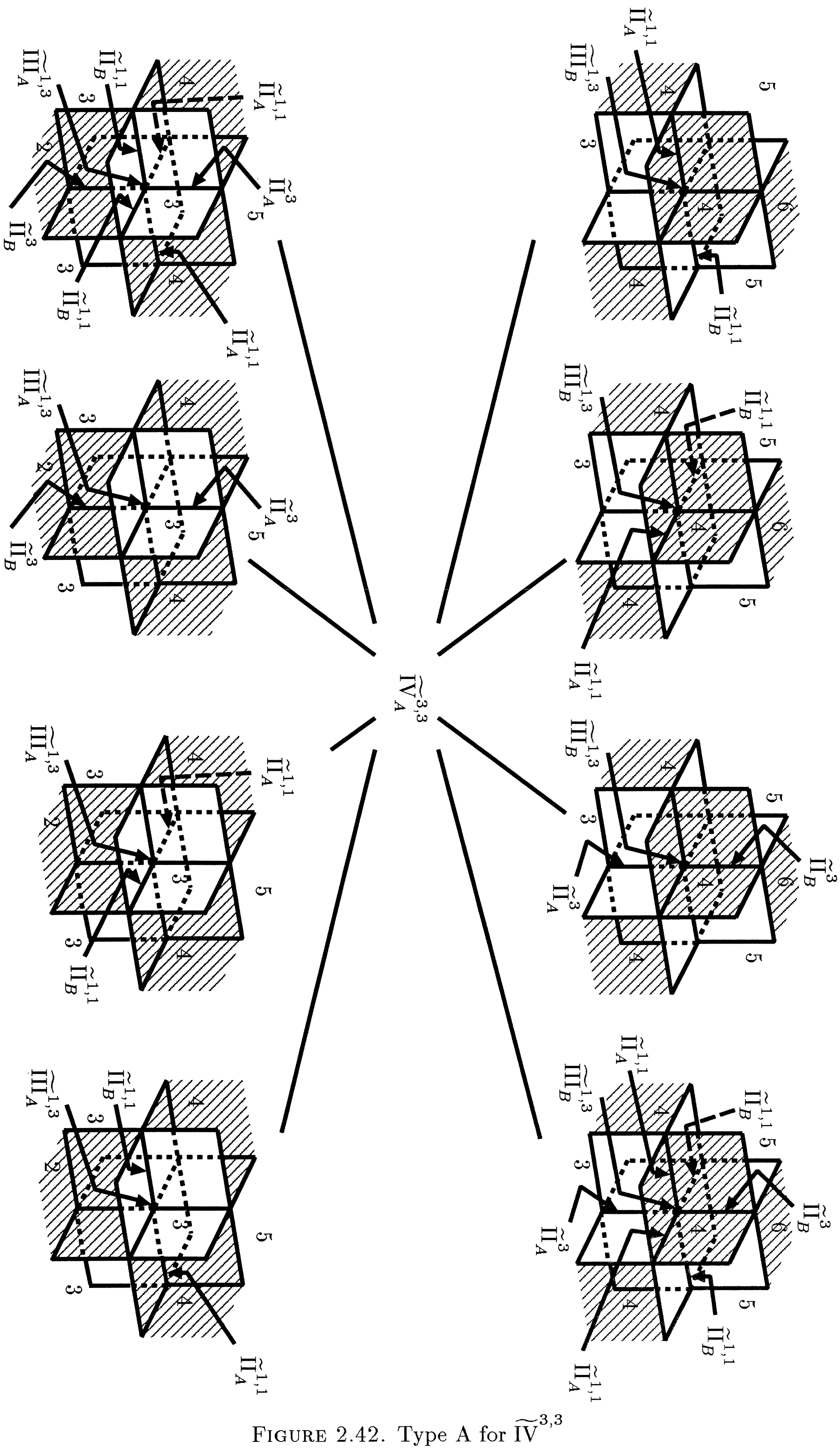

FIGURE 2.42. Type A for $\widetilde{IV}^{3,3}$

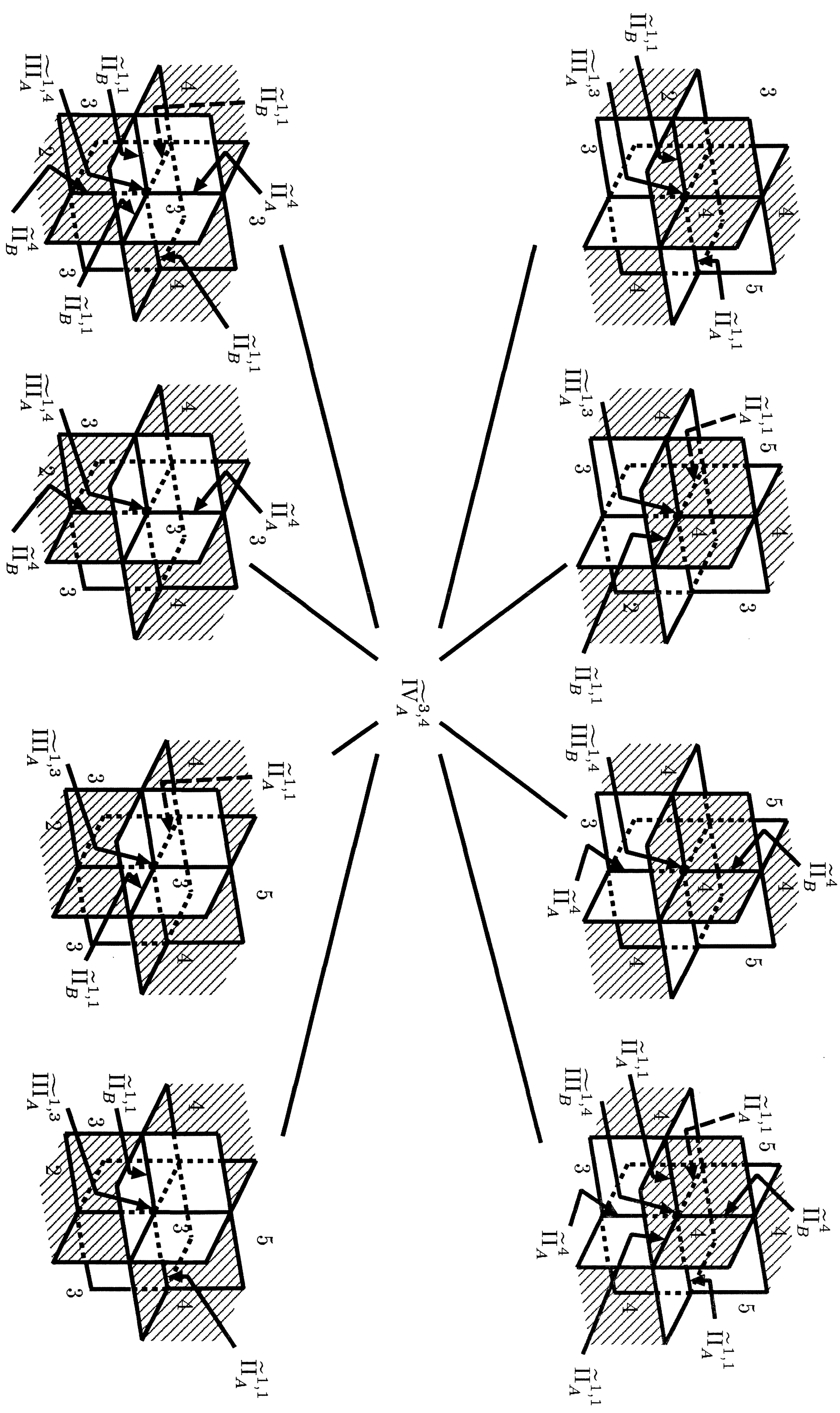

FIGURE 2.43. Type A for $\widetilde{\mathrm{IV}}^{3,4}$

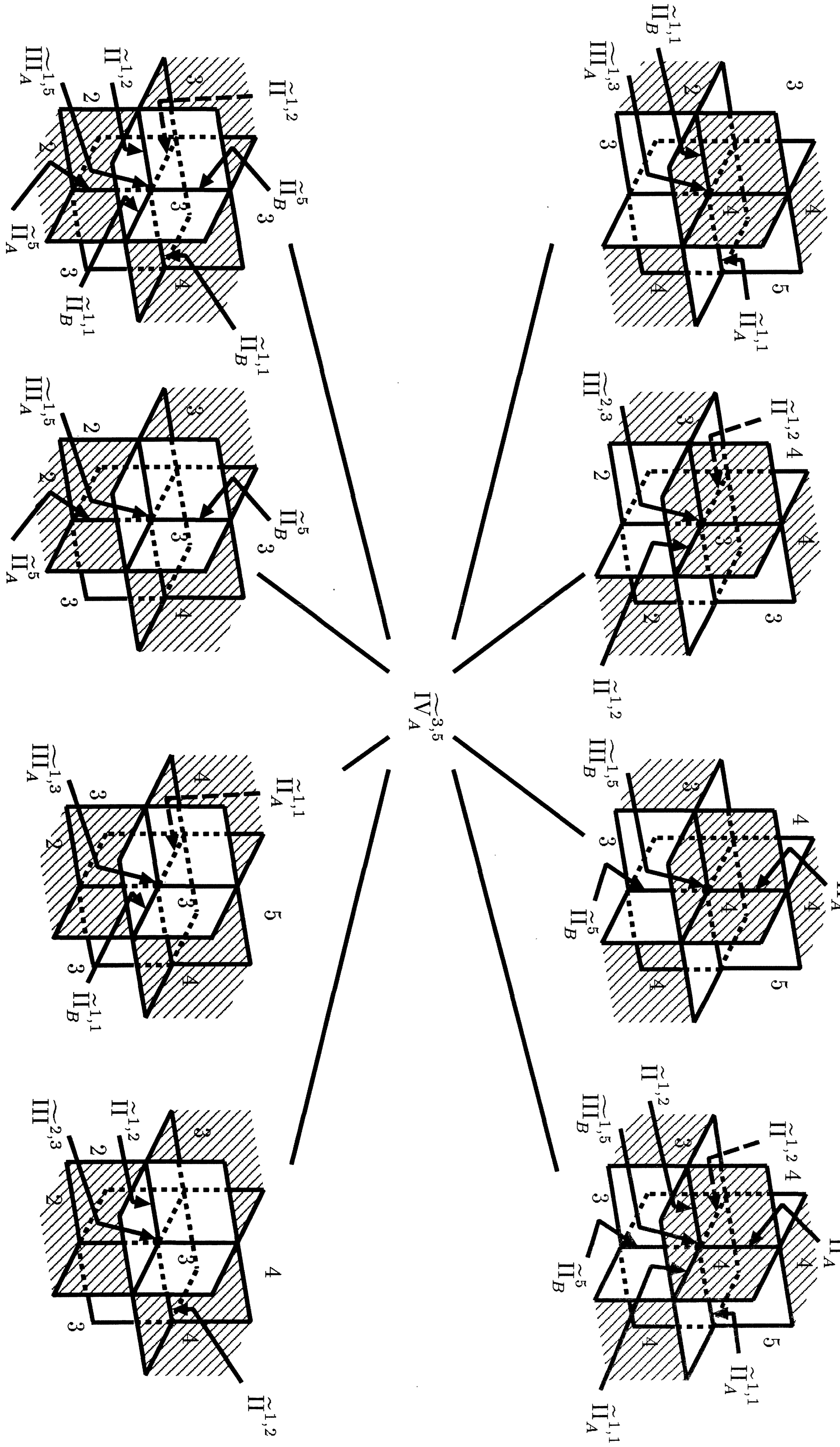

FIGURE 2.44. Type A for $\widetilde{\mathrm{IV}}^{3,5}$

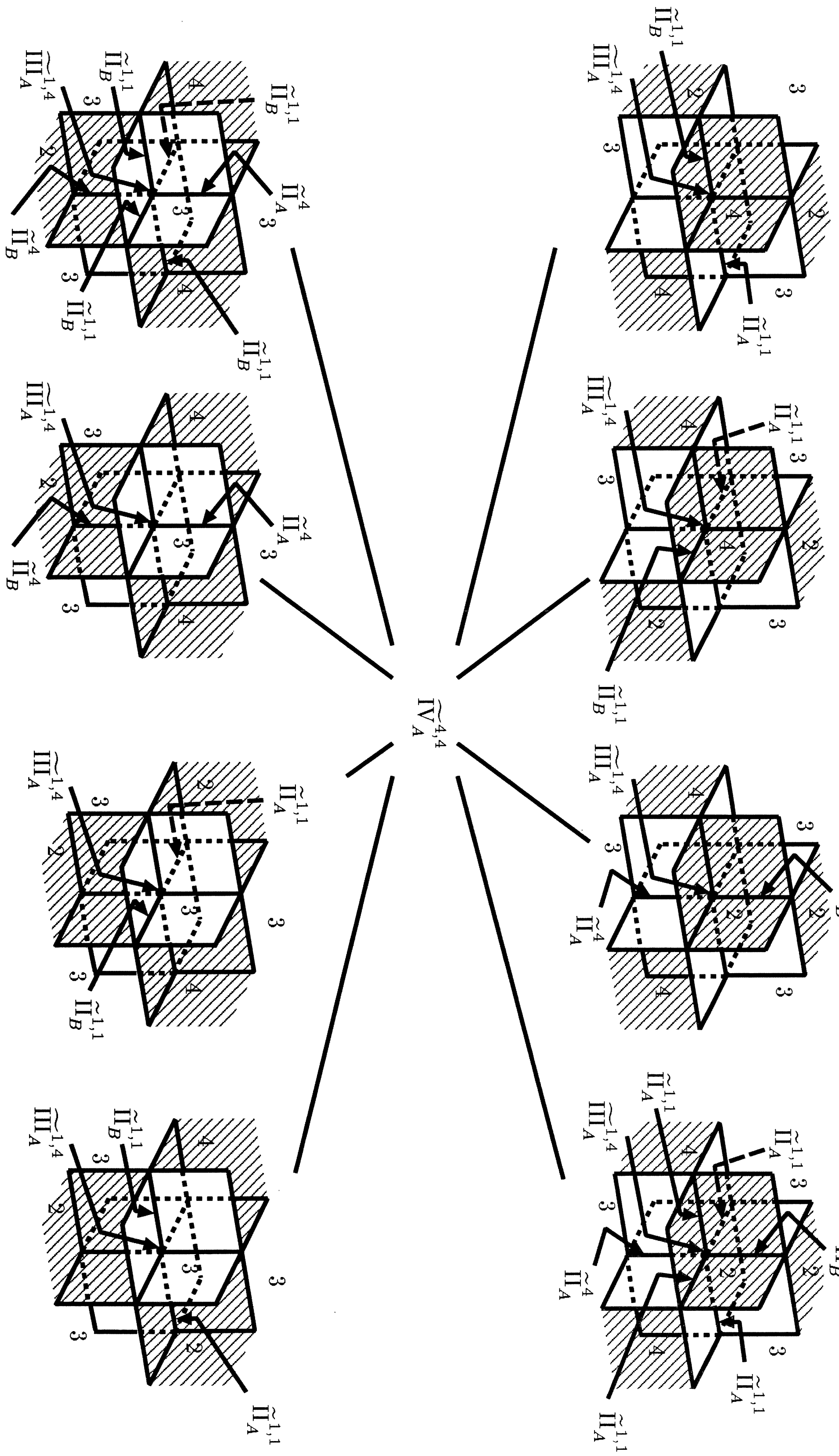

FIGURE 2.45. Type A for $\widetilde{\mathrm{IV}}^{4,4}$

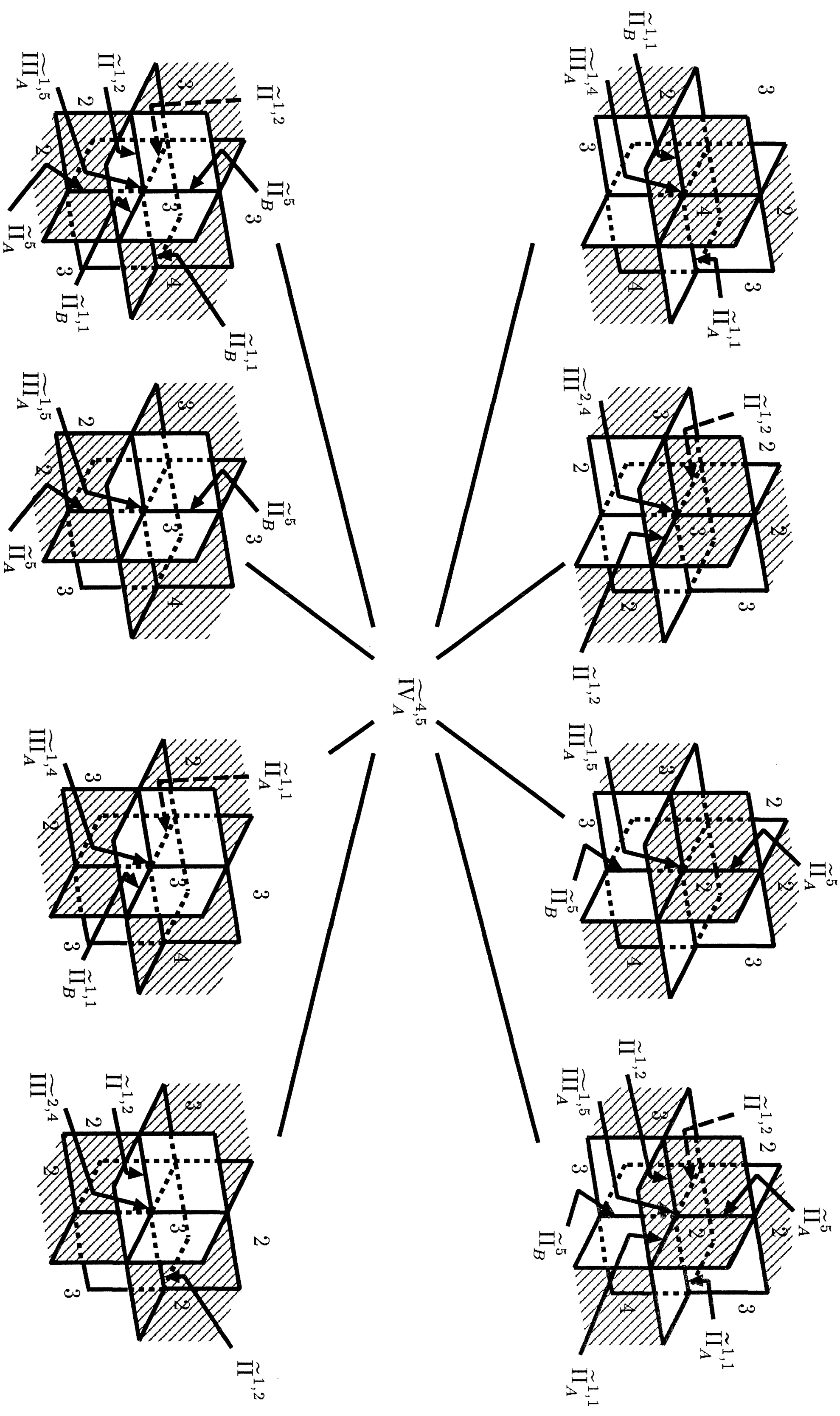

FIGURE 2.46. Type A for $\widetilde{\mathrm{IV}}^{4,5}$

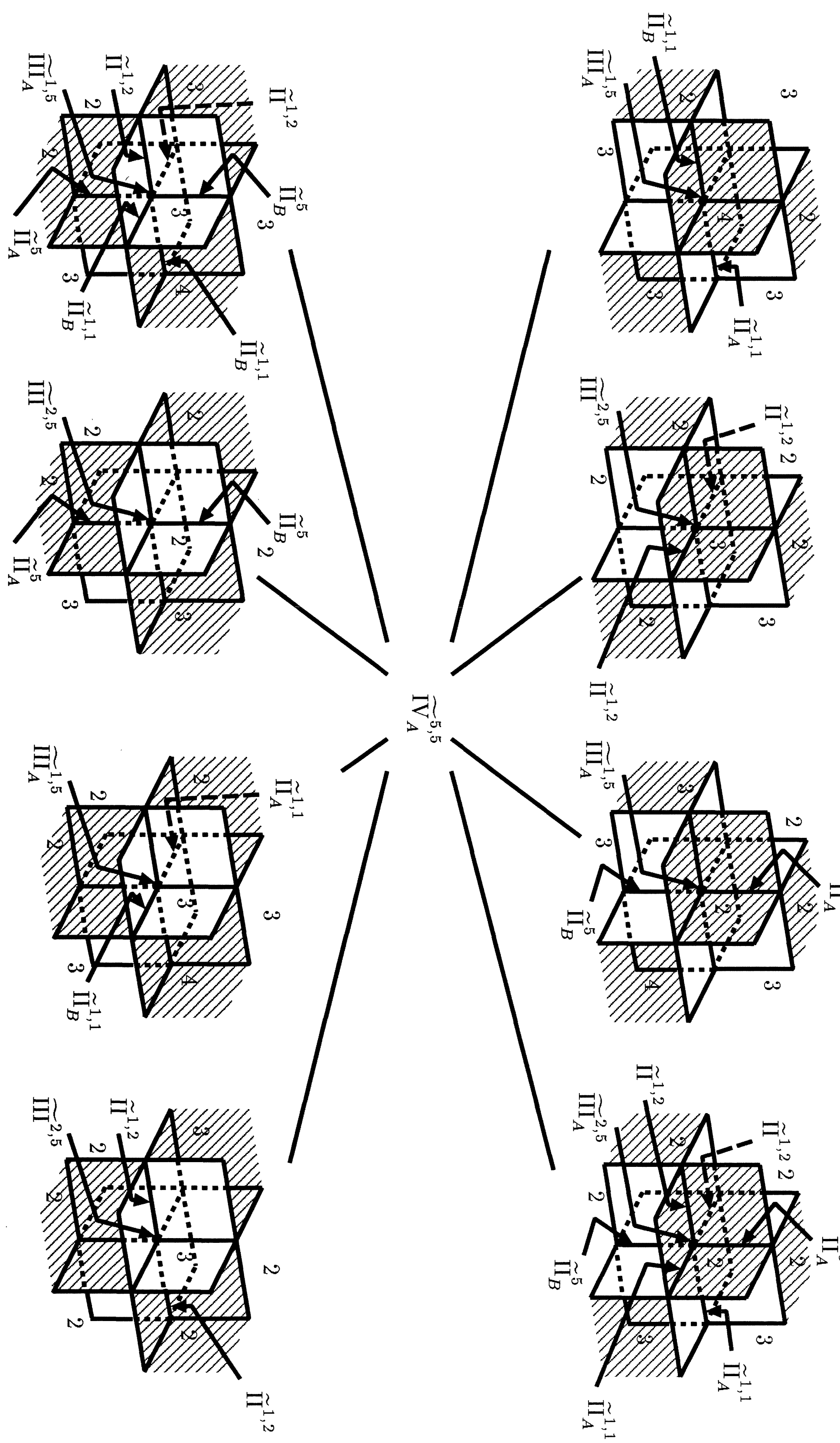

FIGURE 2.47. Type A for $\widetilde{\mathrm{IV}}^{5,5}$

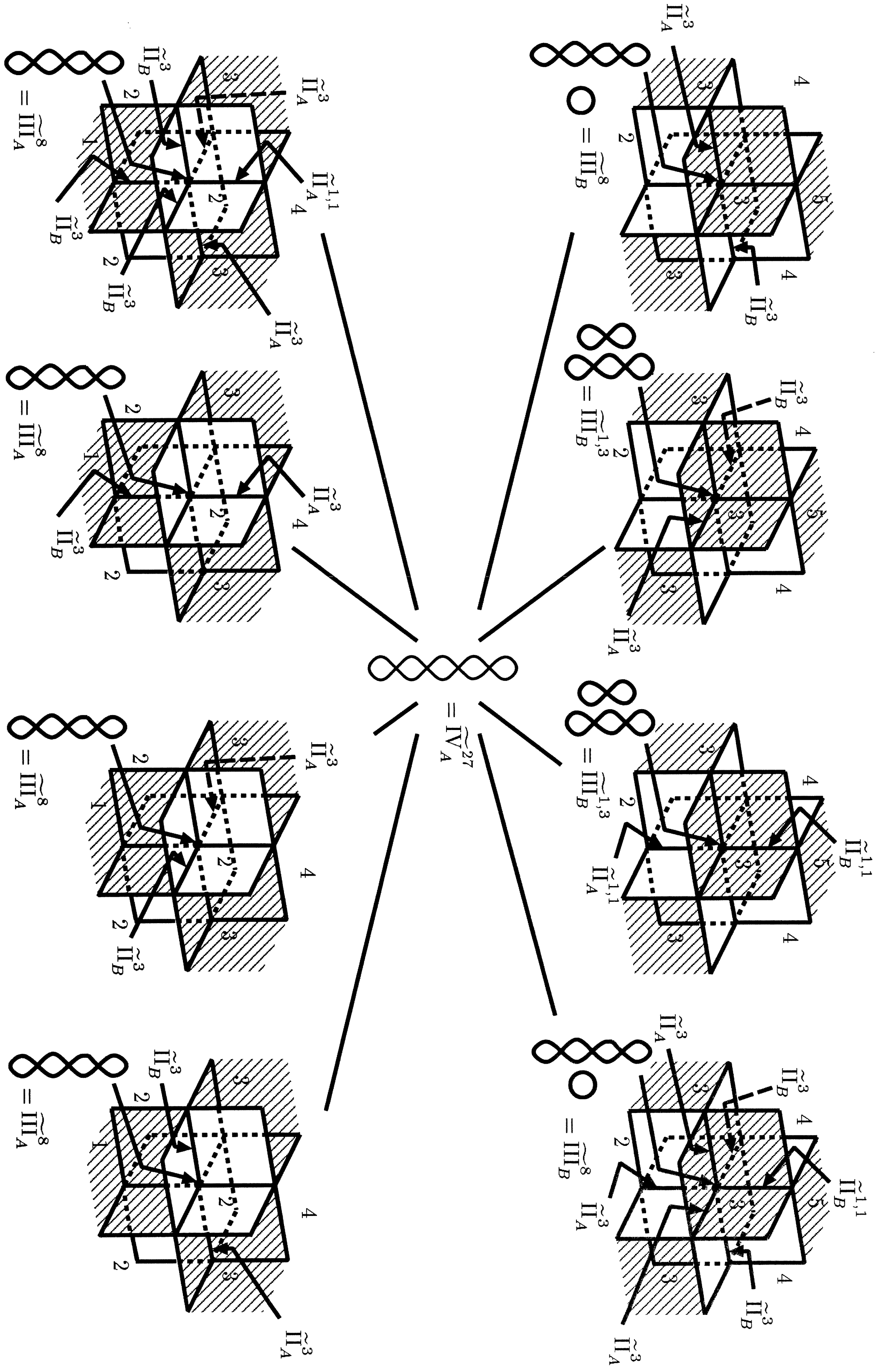

FIGURE 2.48. Type A for $\widetilde{\mathrm{IV}}^{27}$

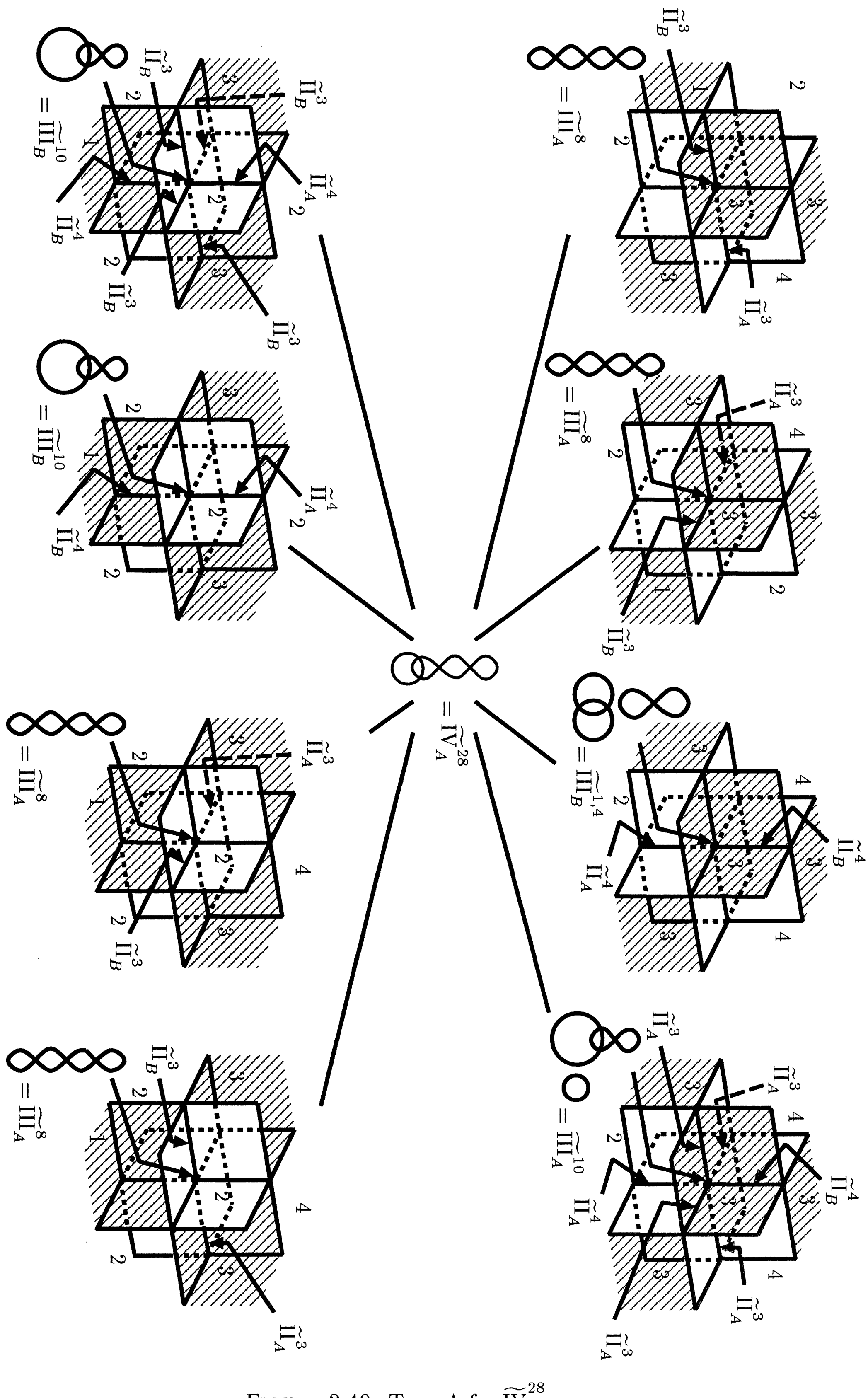

FIGURE 2.49. Type A for $\widetilde{\text{IV}}^{28}$

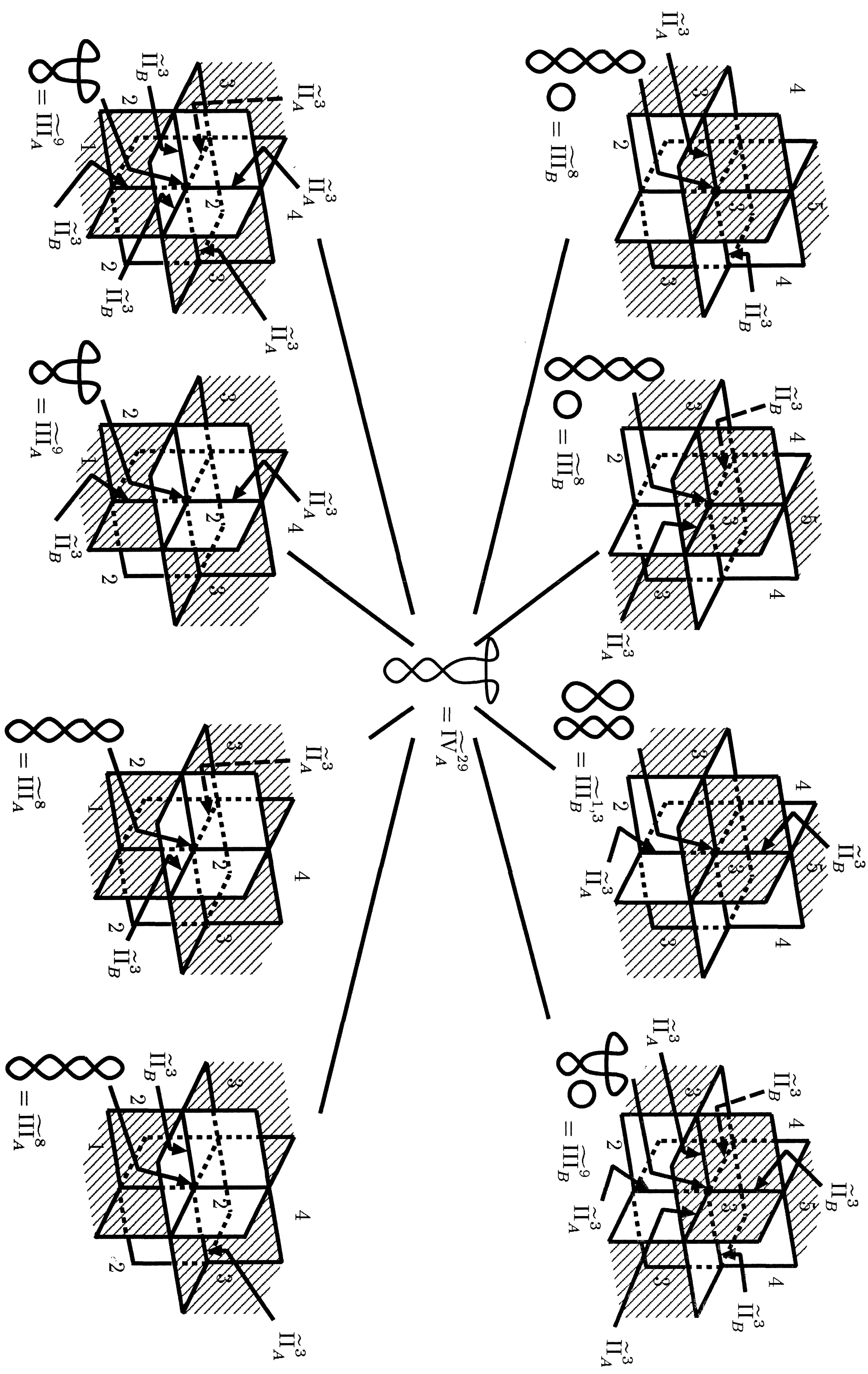

FIGURE 2.50. Type A for $\widetilde{\mathrm{IV}}^{29}$

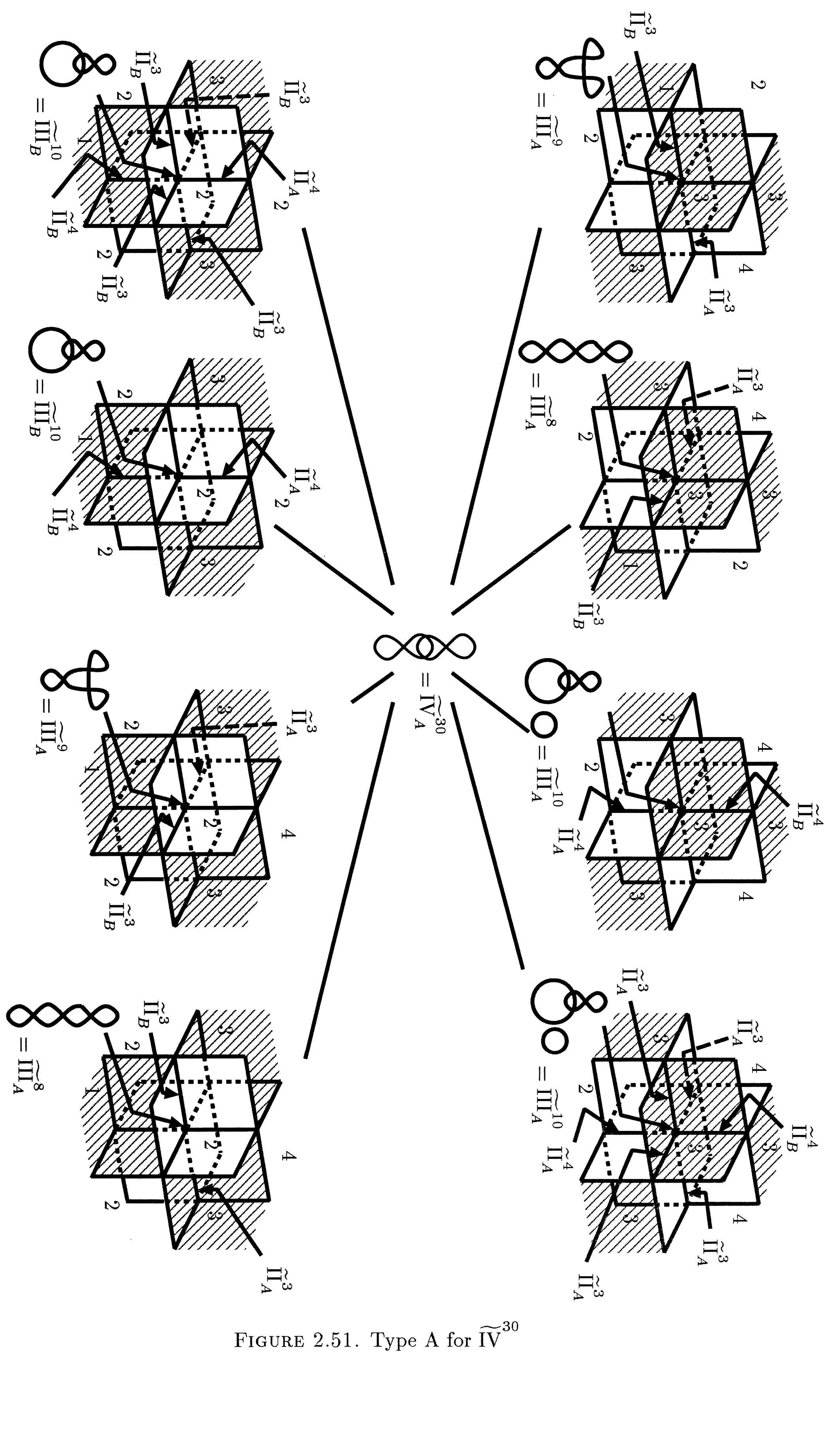

FIGURE 2.51. Type A for $\widetilde{IV}^{30}$

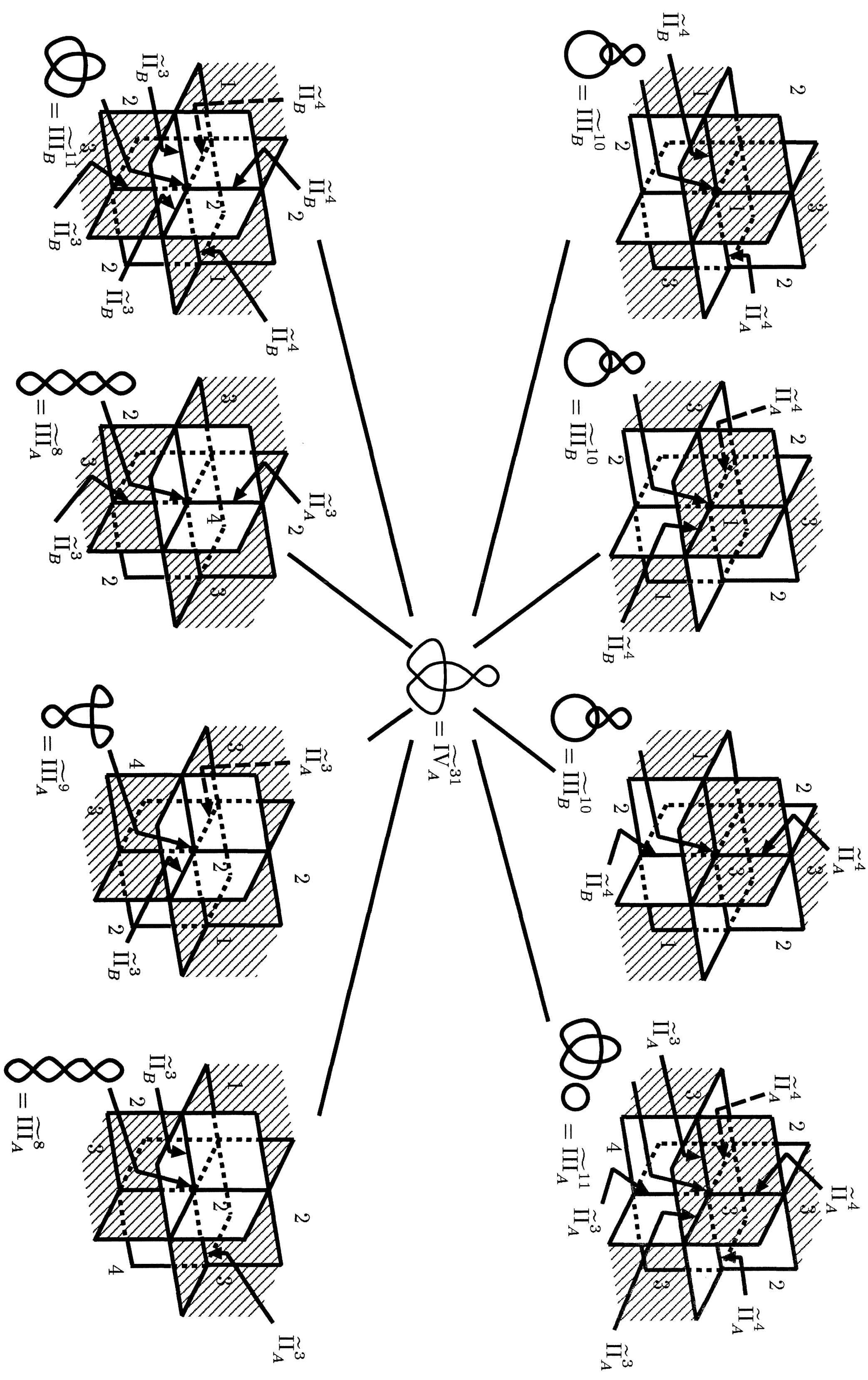

FIGURE 2.52. Type A for $\widetilde{\mathrm{IV}}^{31}$

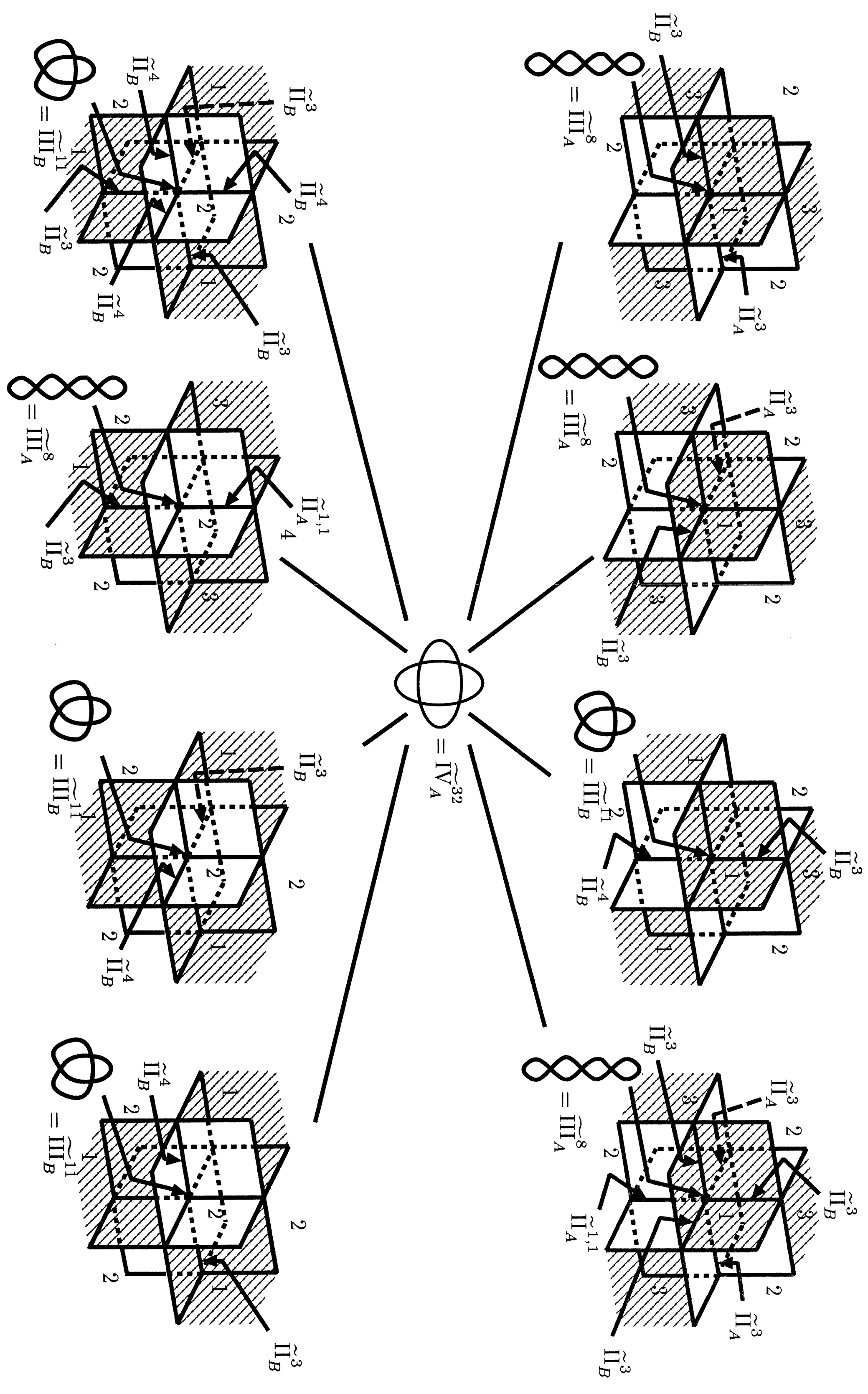

FIGURE 2.53. Type A for $\widetilde{IV}^{32}$

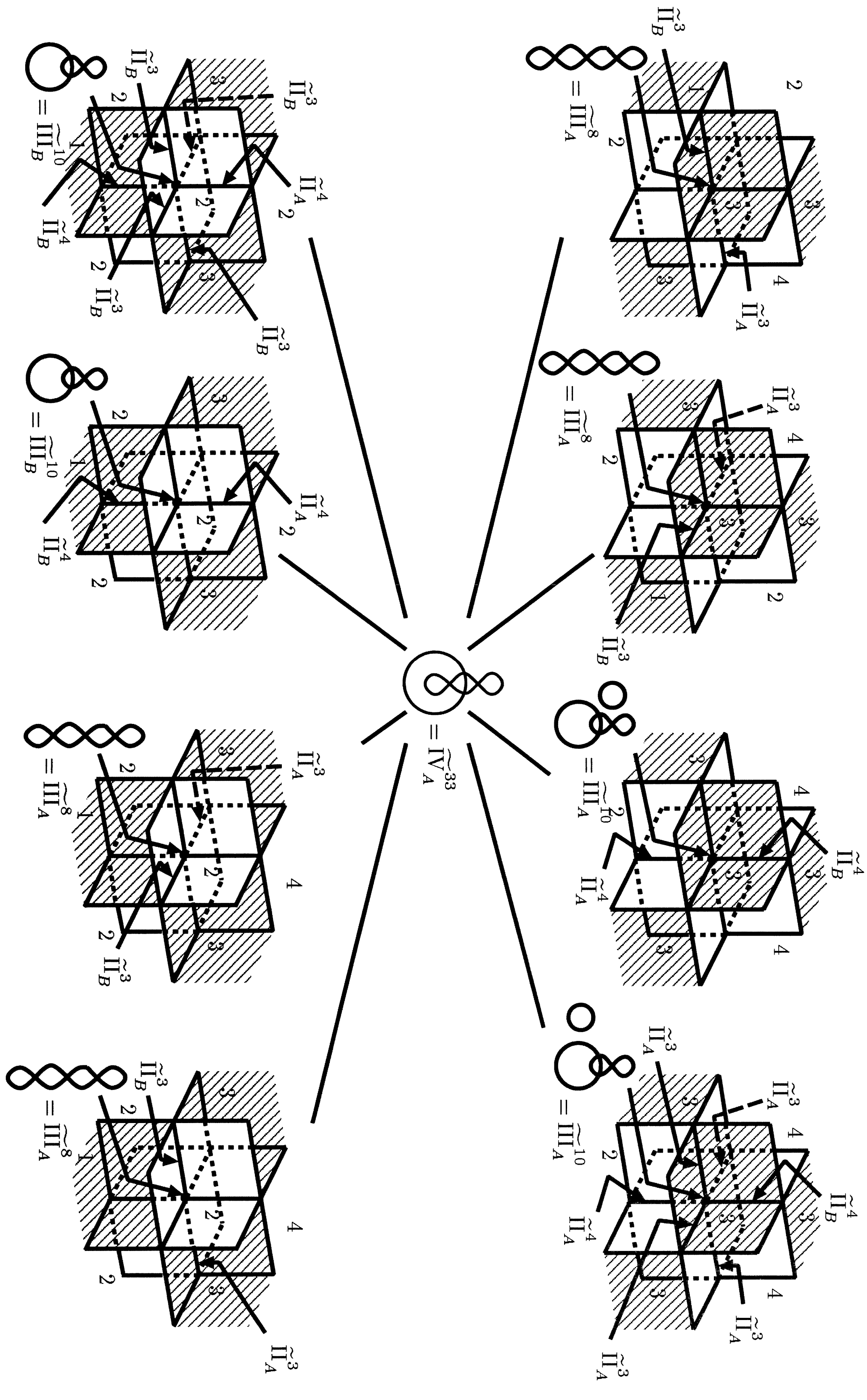

FIGURE 2.54. Type A for $\widetilde{\mathrm{IV}}^{33}$

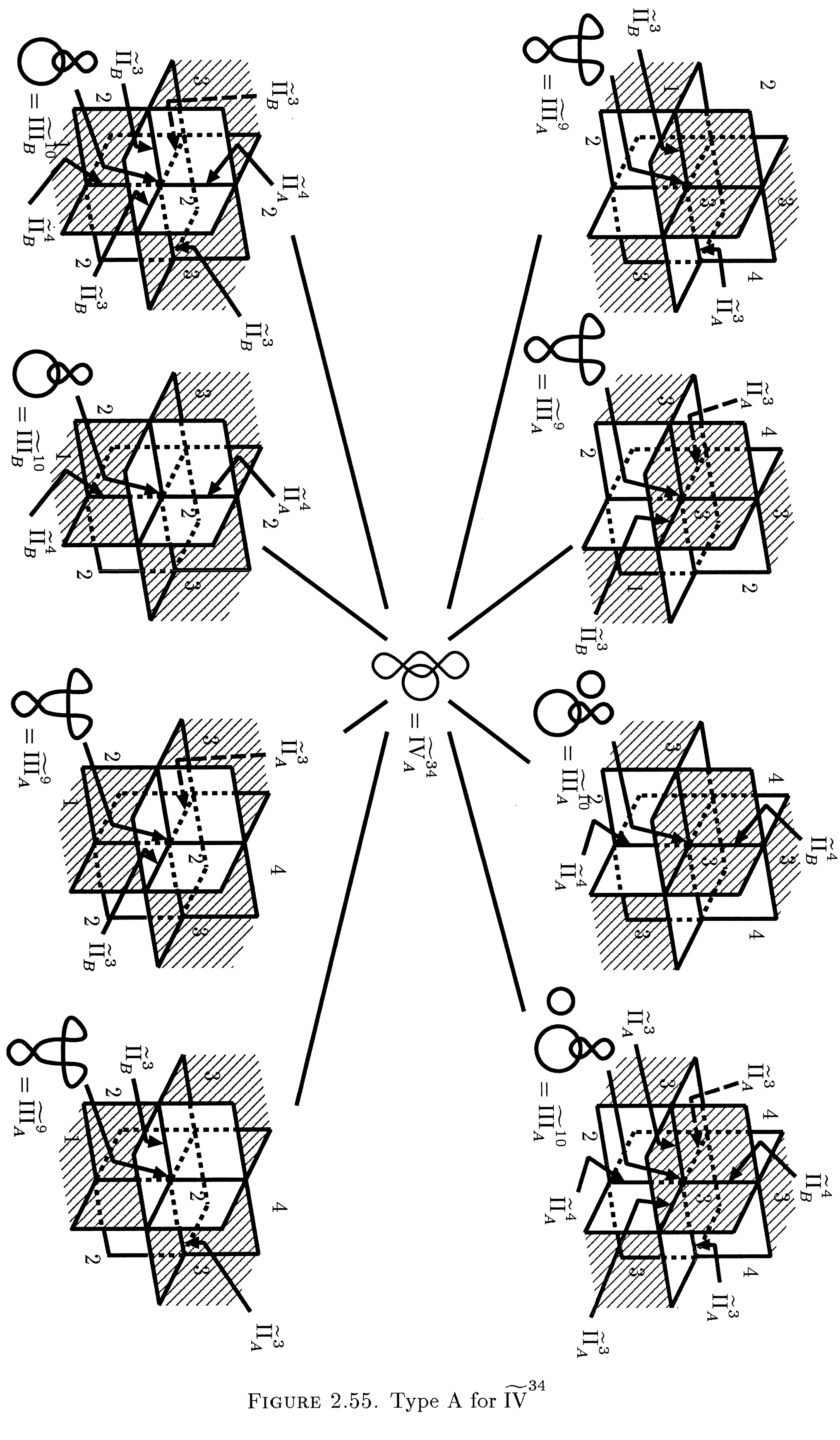

FIGURE 2.55. Type A for $\widetilde{\mathrm{IV}}^{34}$

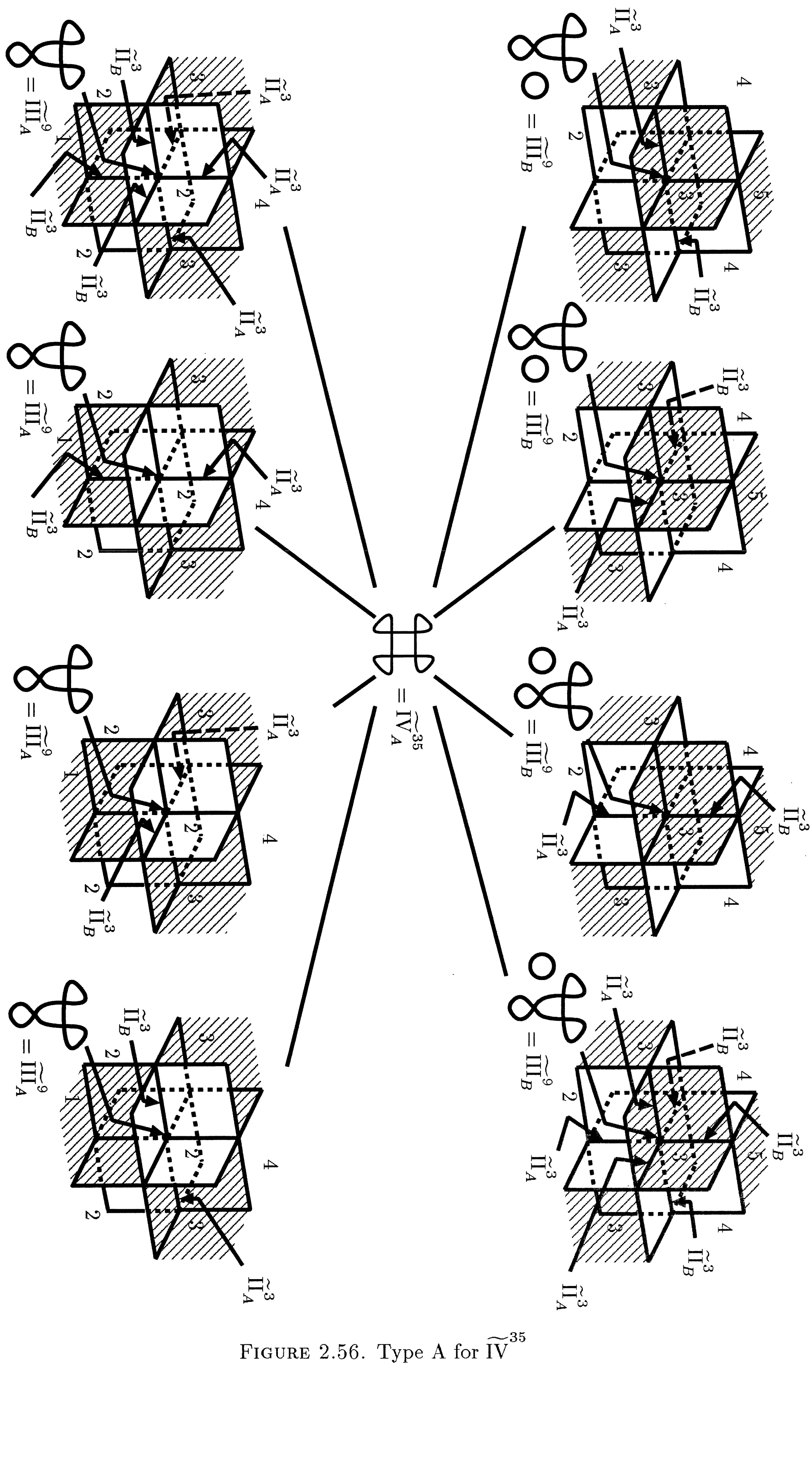

Figure 2.56. Type A for $\widetilde{\mathrm{IV}}^{35}$

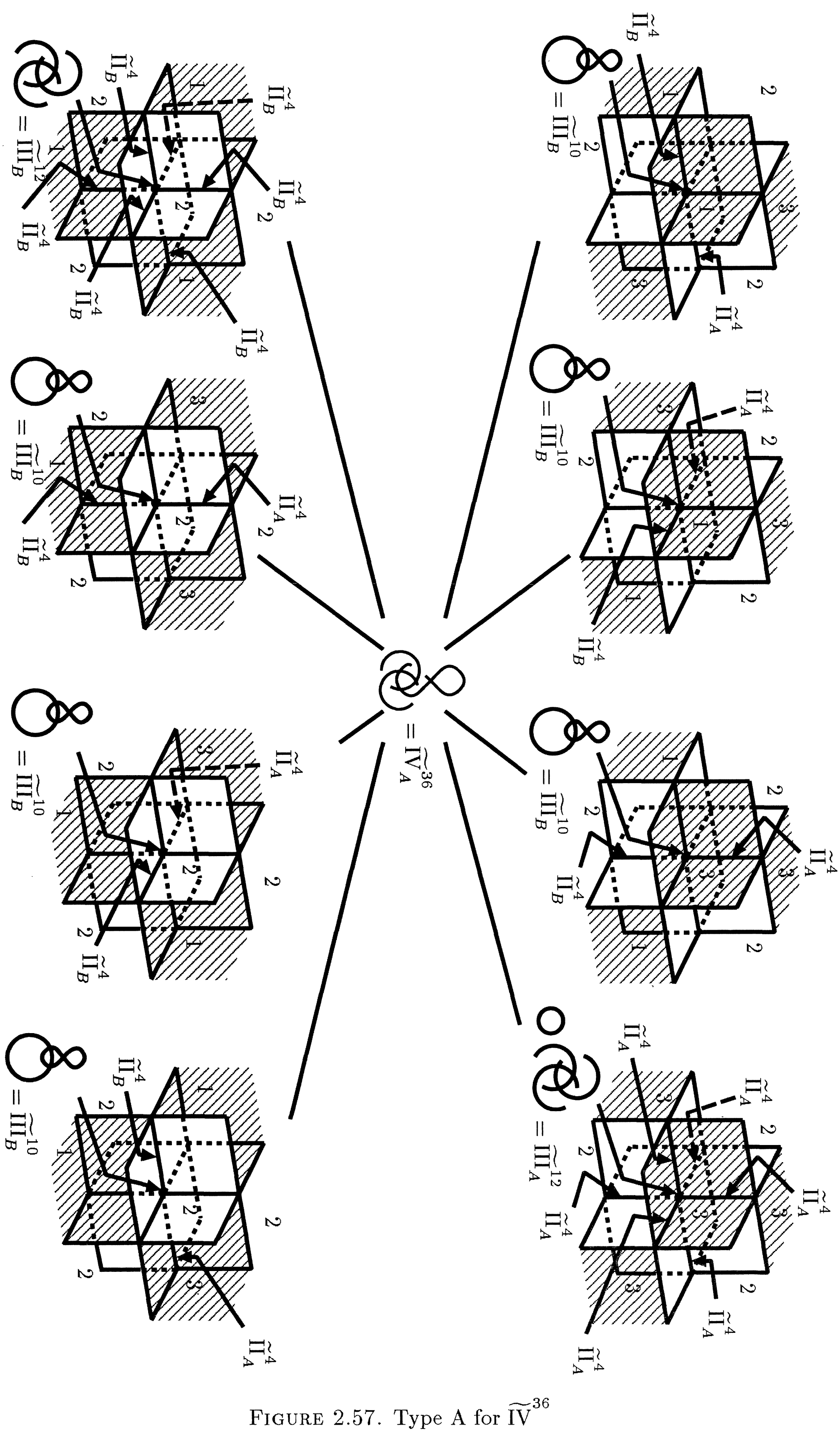

FIGURE 2.57. Type A for $\widetilde{\mathrm{IV}}^{36}$

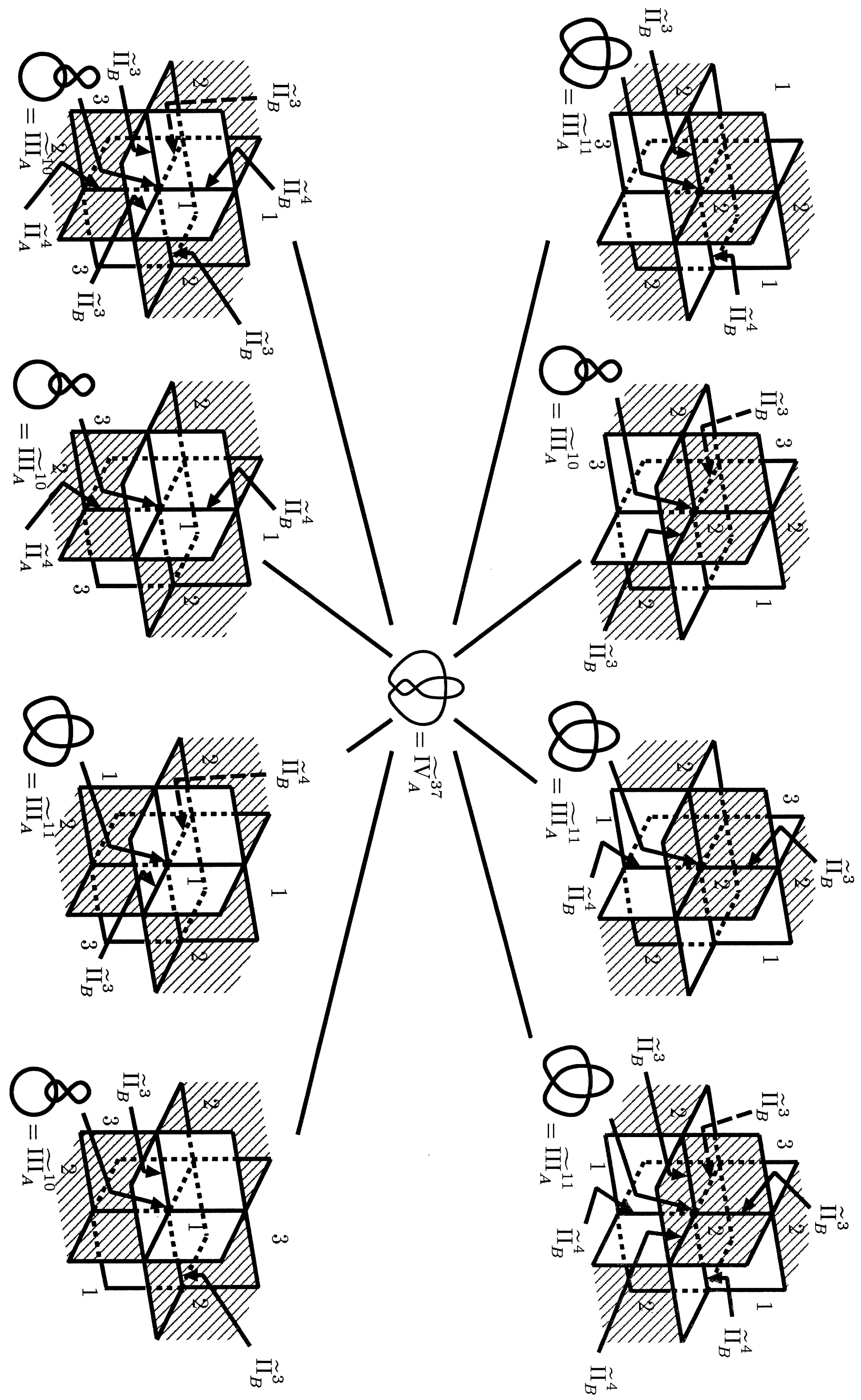

FIGURE 2.58. Type A for $\widetilde{\mathrm{IV}}^{37}$

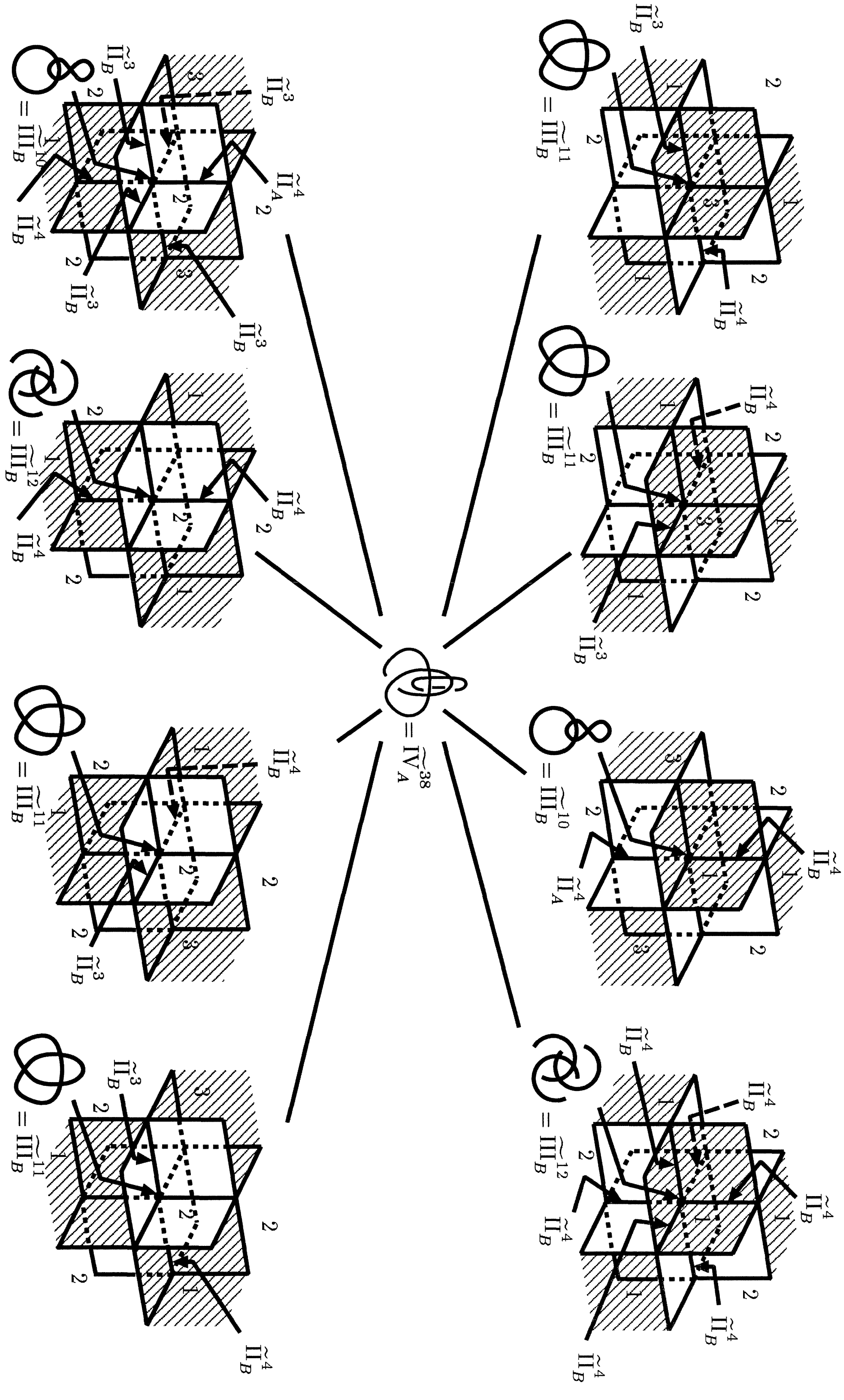

FIGURE 2.59. Type A for $\widetilde{\mathrm{IV}}^{38}$

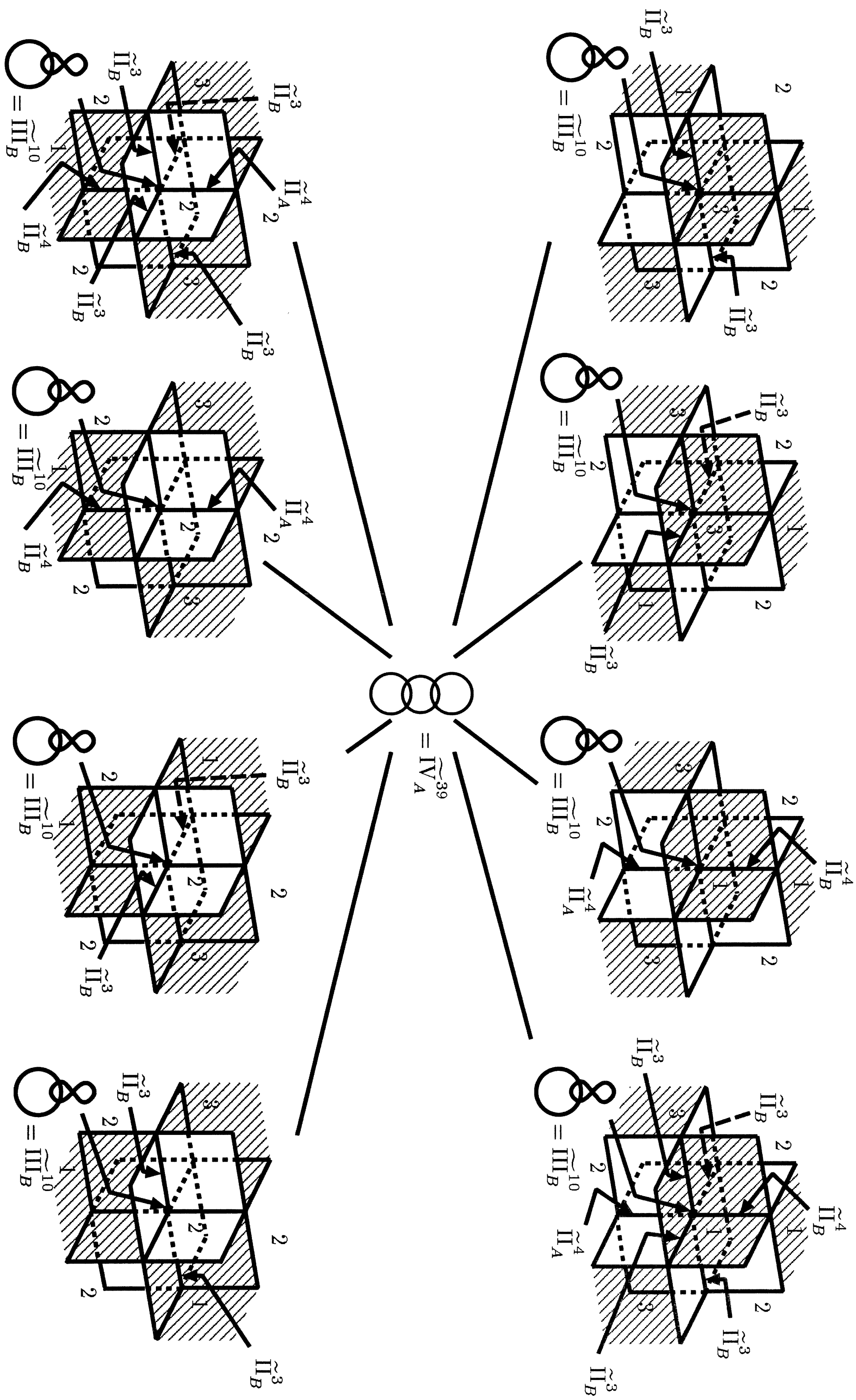

FIGURE 2.60. Type A for $\widetilde{IV}^{39}$

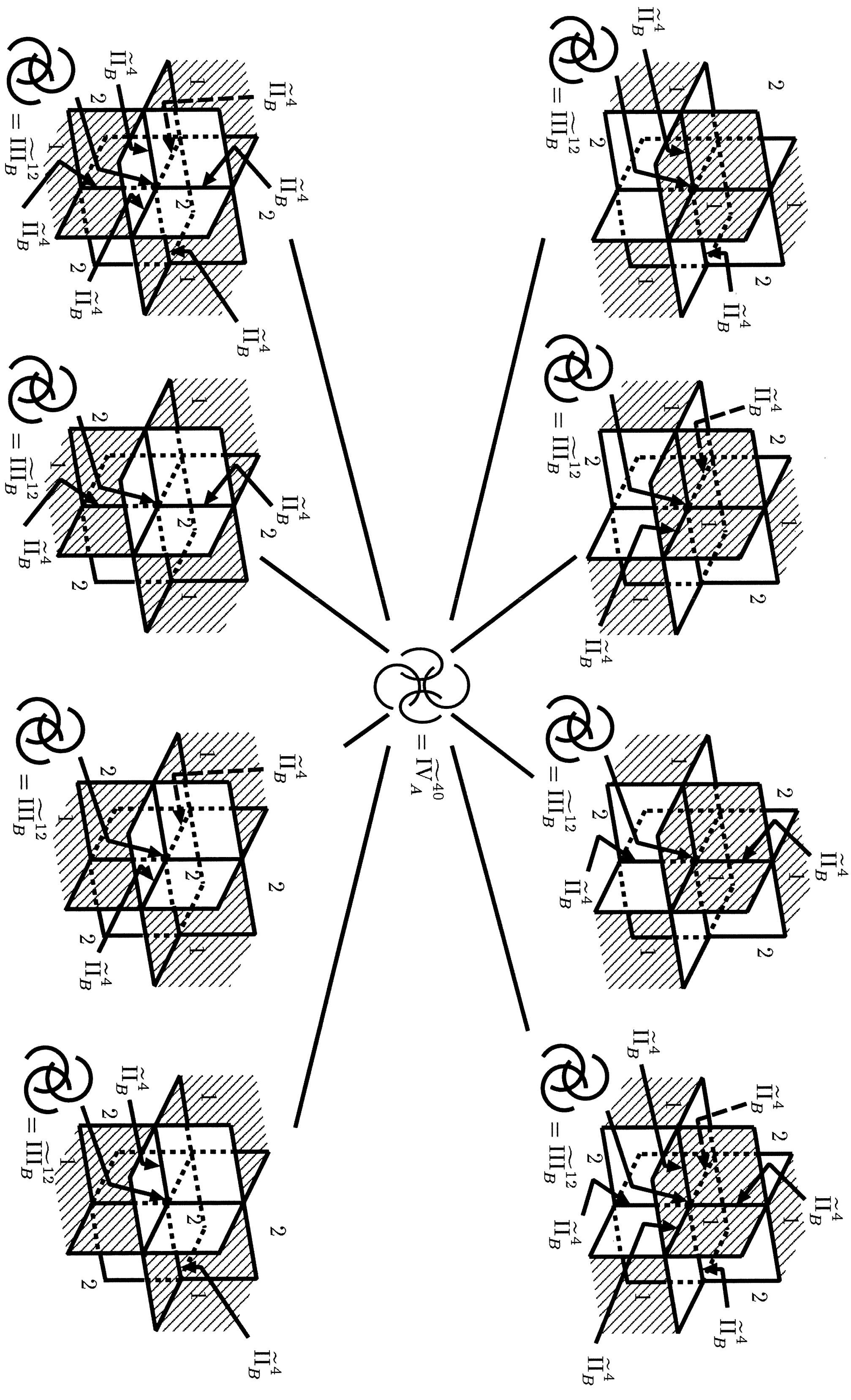

FIGURE 2.61. Type A for $\widetilde{\mathrm{IV}}^{40}$

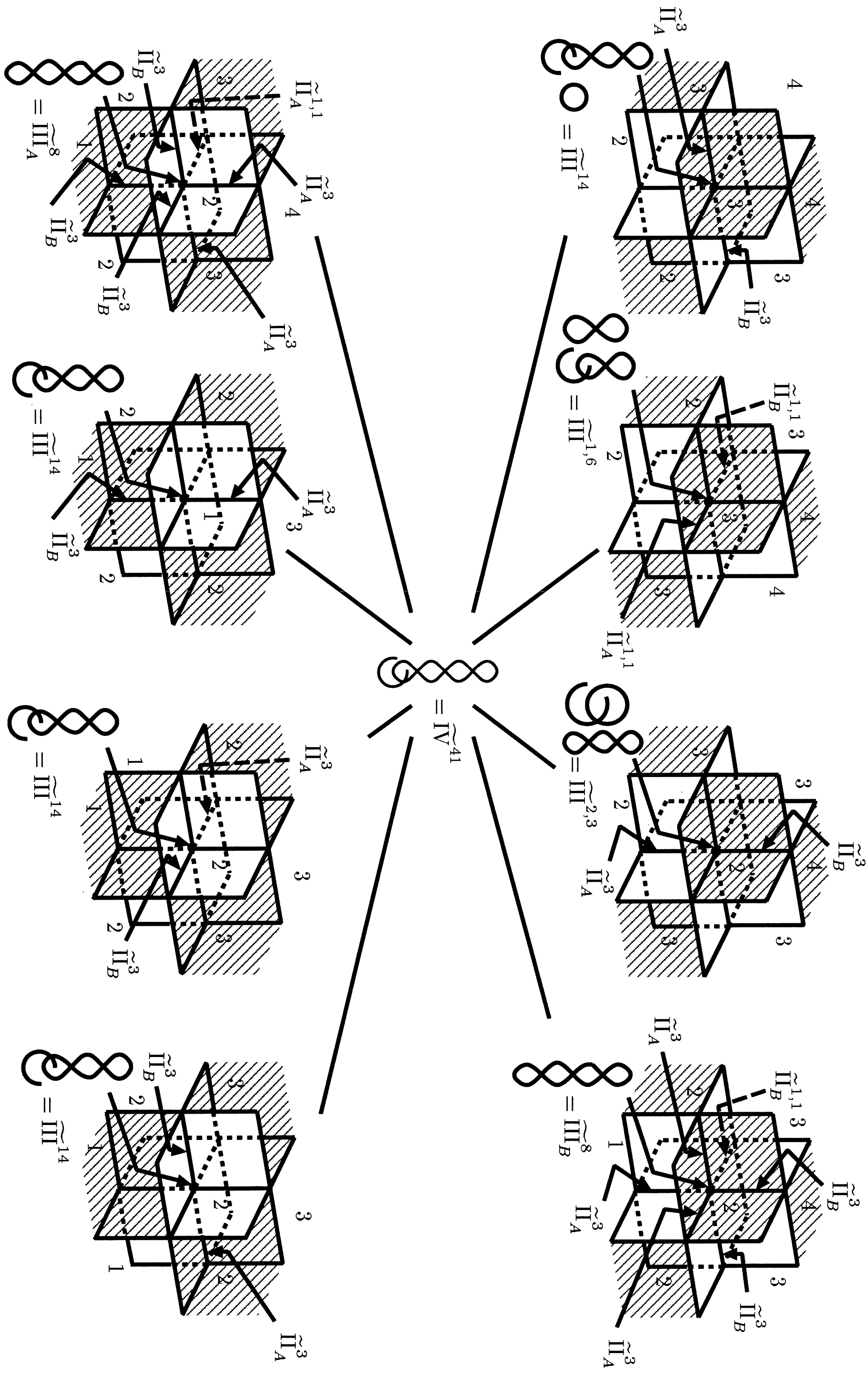

FIGURE 2.62. $\widetilde{\mathrm{IV}}^{41}$ can not divide into two types

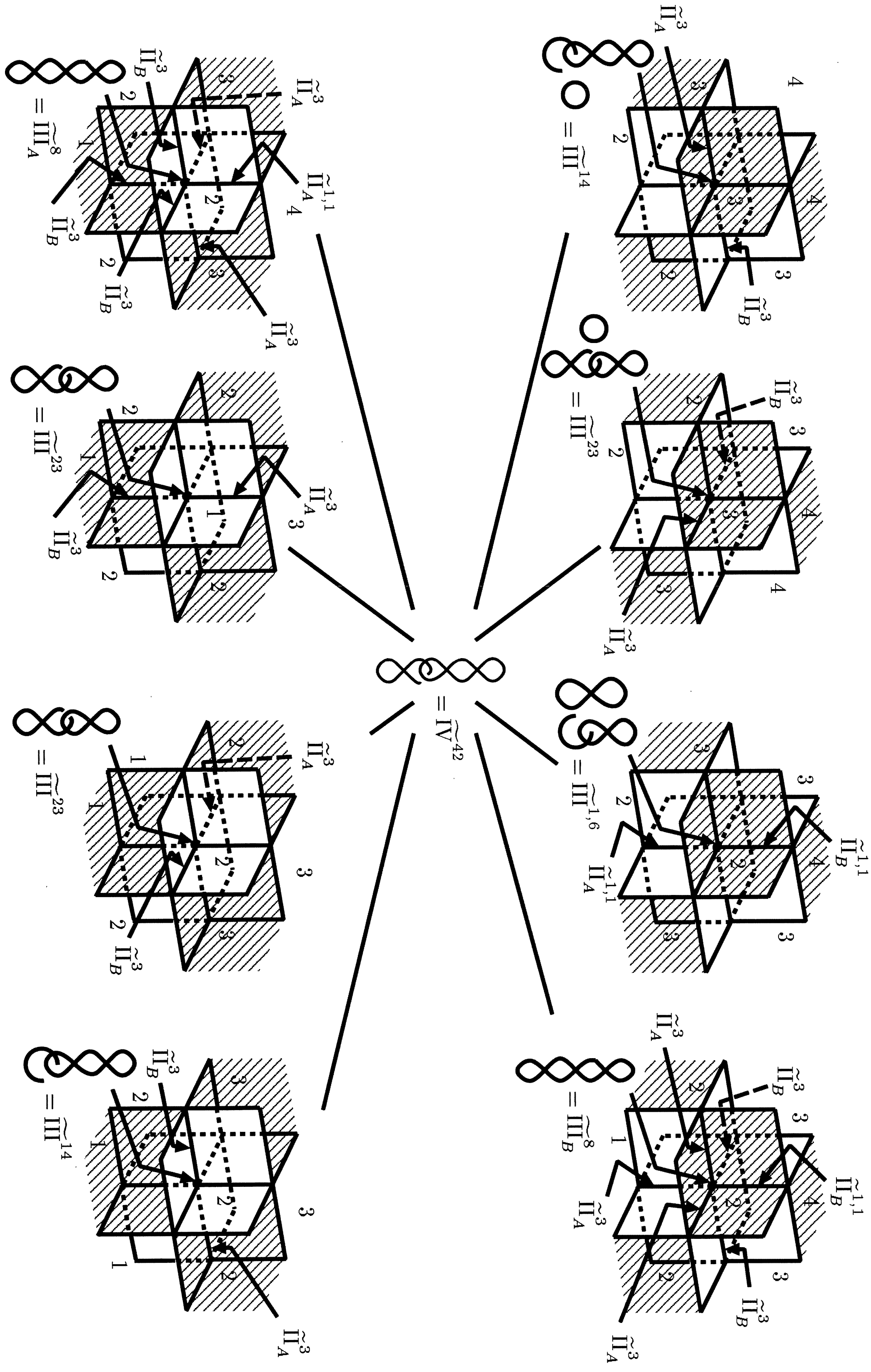

FIGURE 2.63. $\widetilde{\mathrm{IV}}^{42}$ can not divide into two types

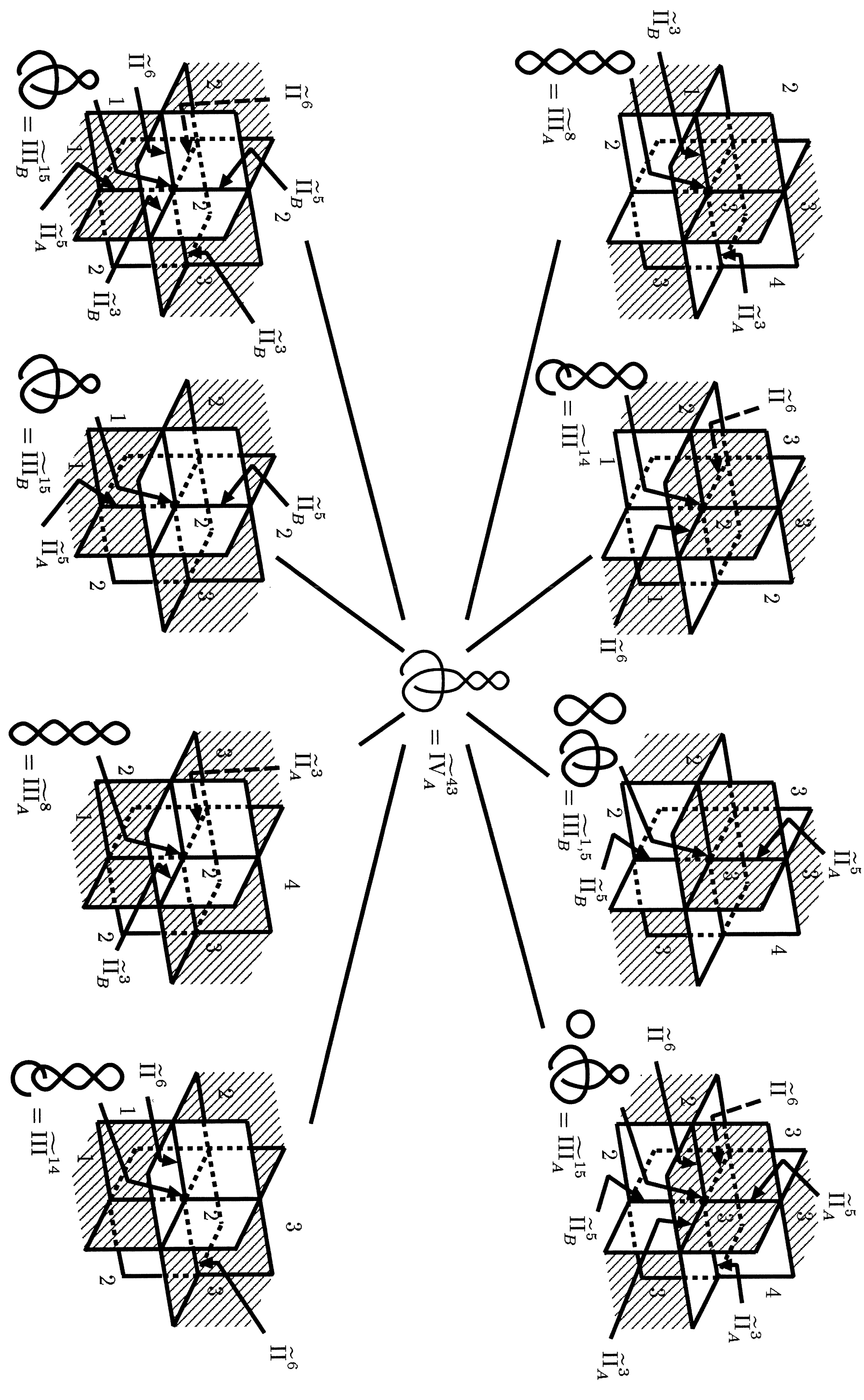

FIGURE 2.64. Type A for $\widetilde{\mathrm{IV}}^{43}$

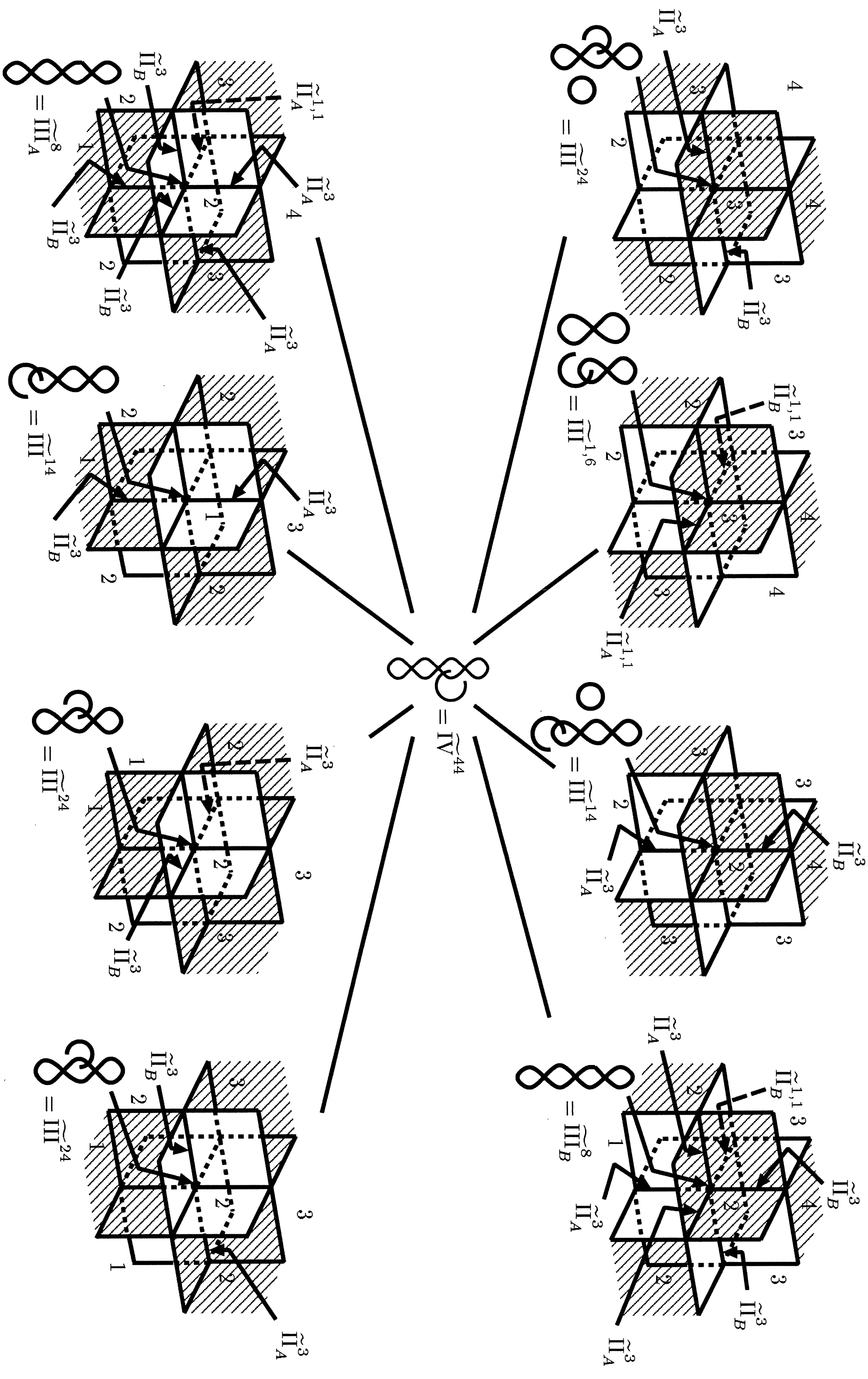

FIGURE 2.65. $\widetilde{IV}^{44}$ can not divide into two types

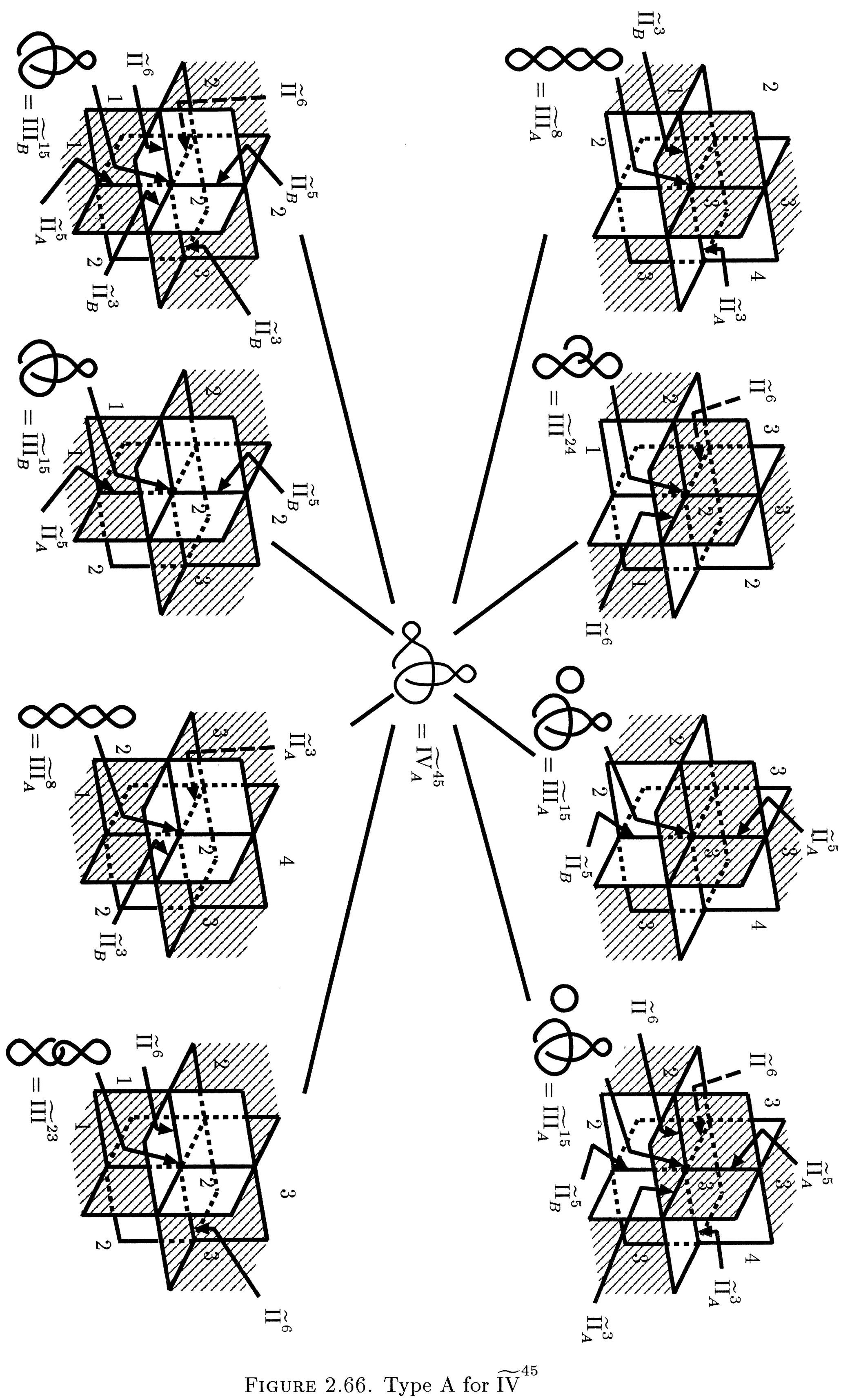

FIGURE 2.66. Type A for $\widetilde{\mathrm{IV}}^{45}$

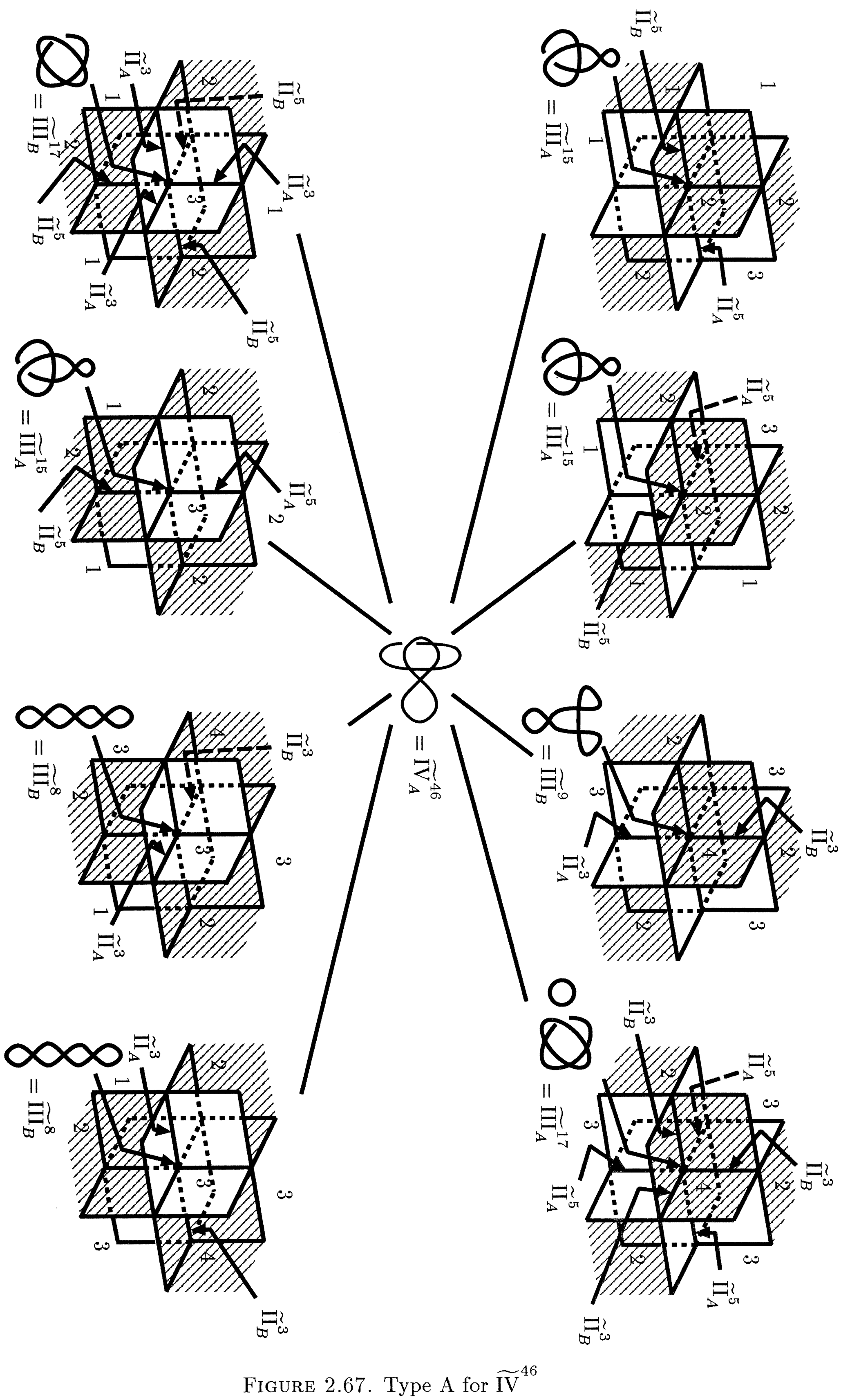

FIGURE 2.67. Type A for $\widetilde{\mathrm{IV}}^{46}$

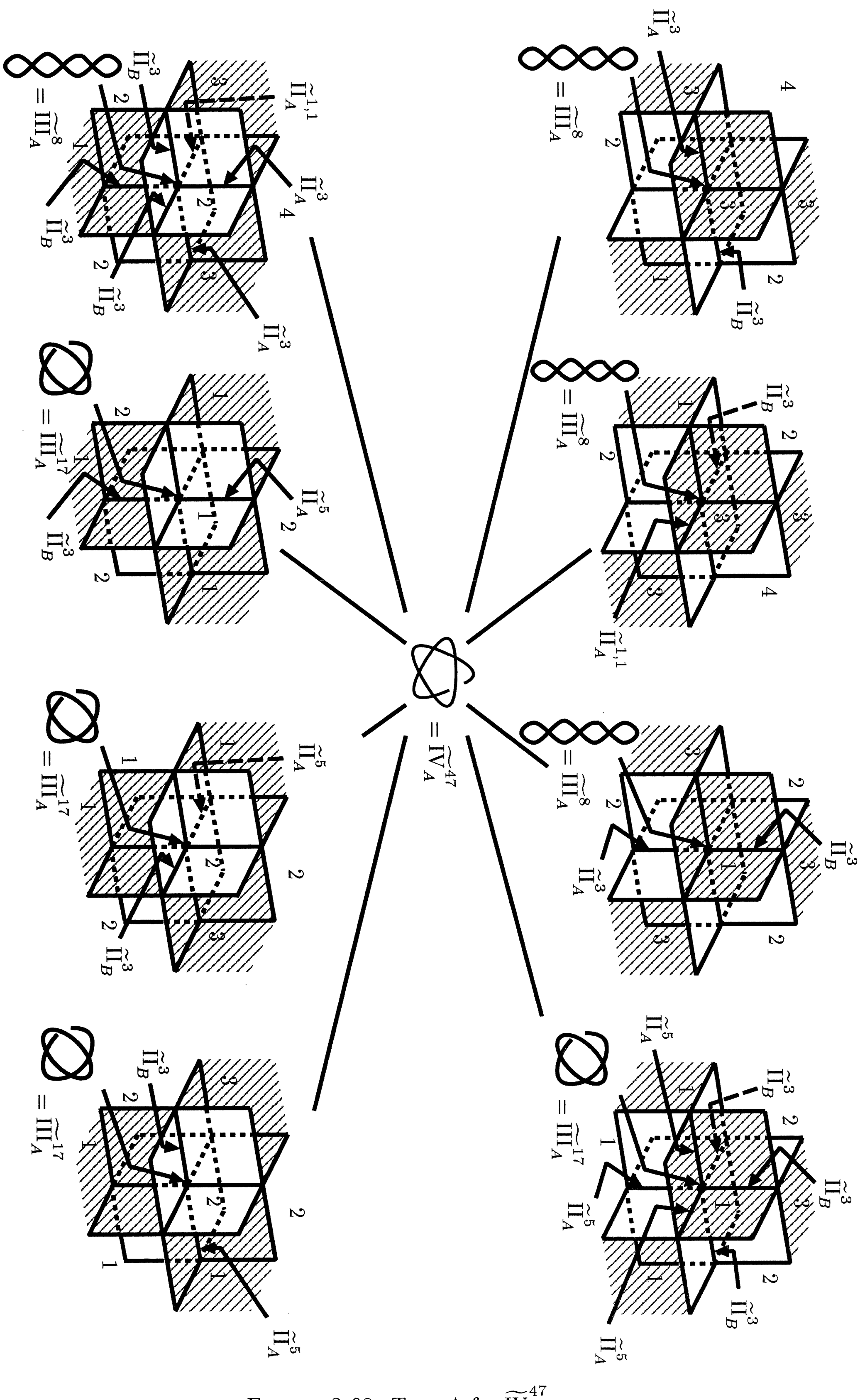

FIGURE 2.68. Type A for $\widetilde{\mathrm{IV}}^{47}$

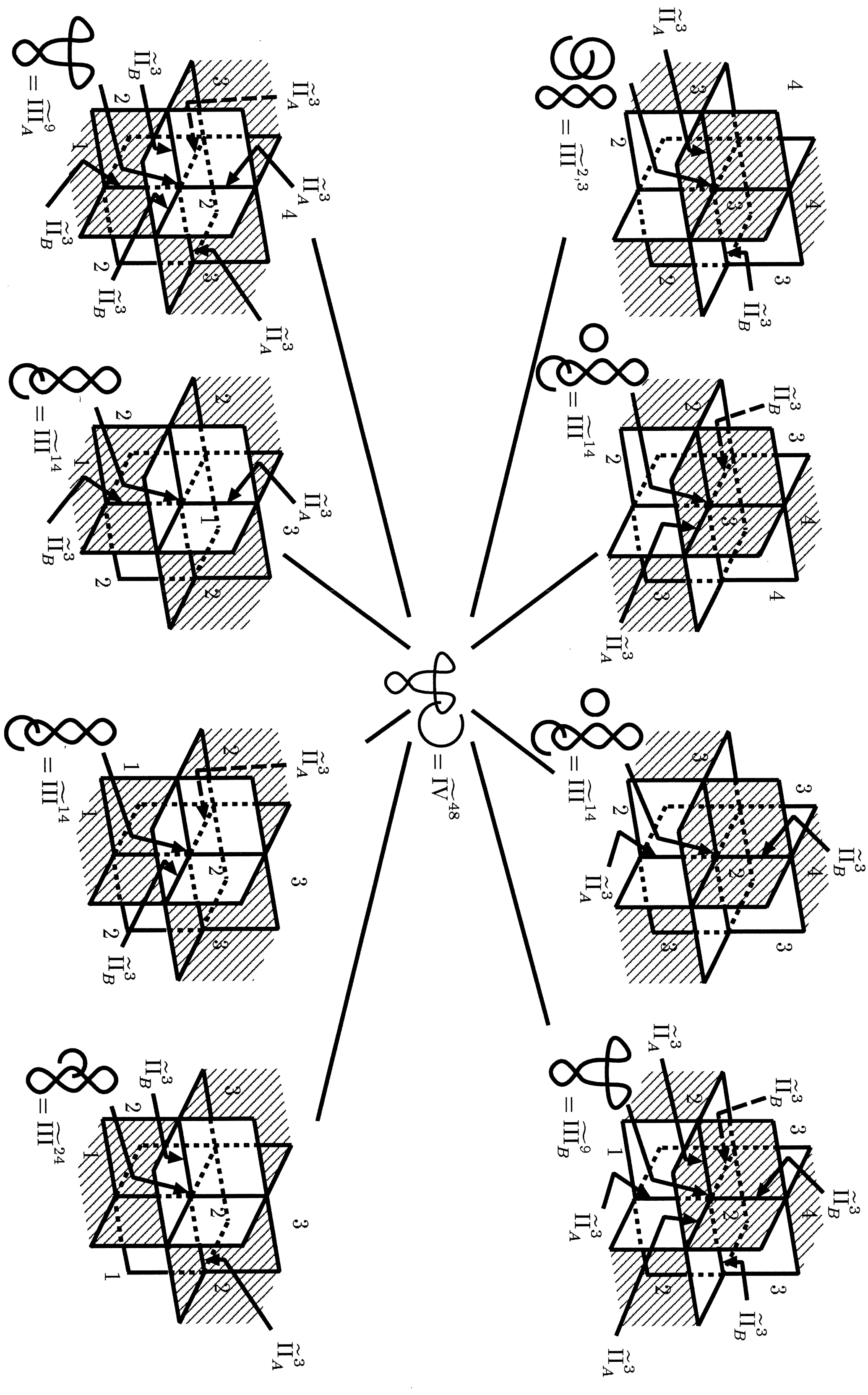

FIGURE 2.69. $\widetilde{\mathrm{IV}}^{48}$ can not divide into two types

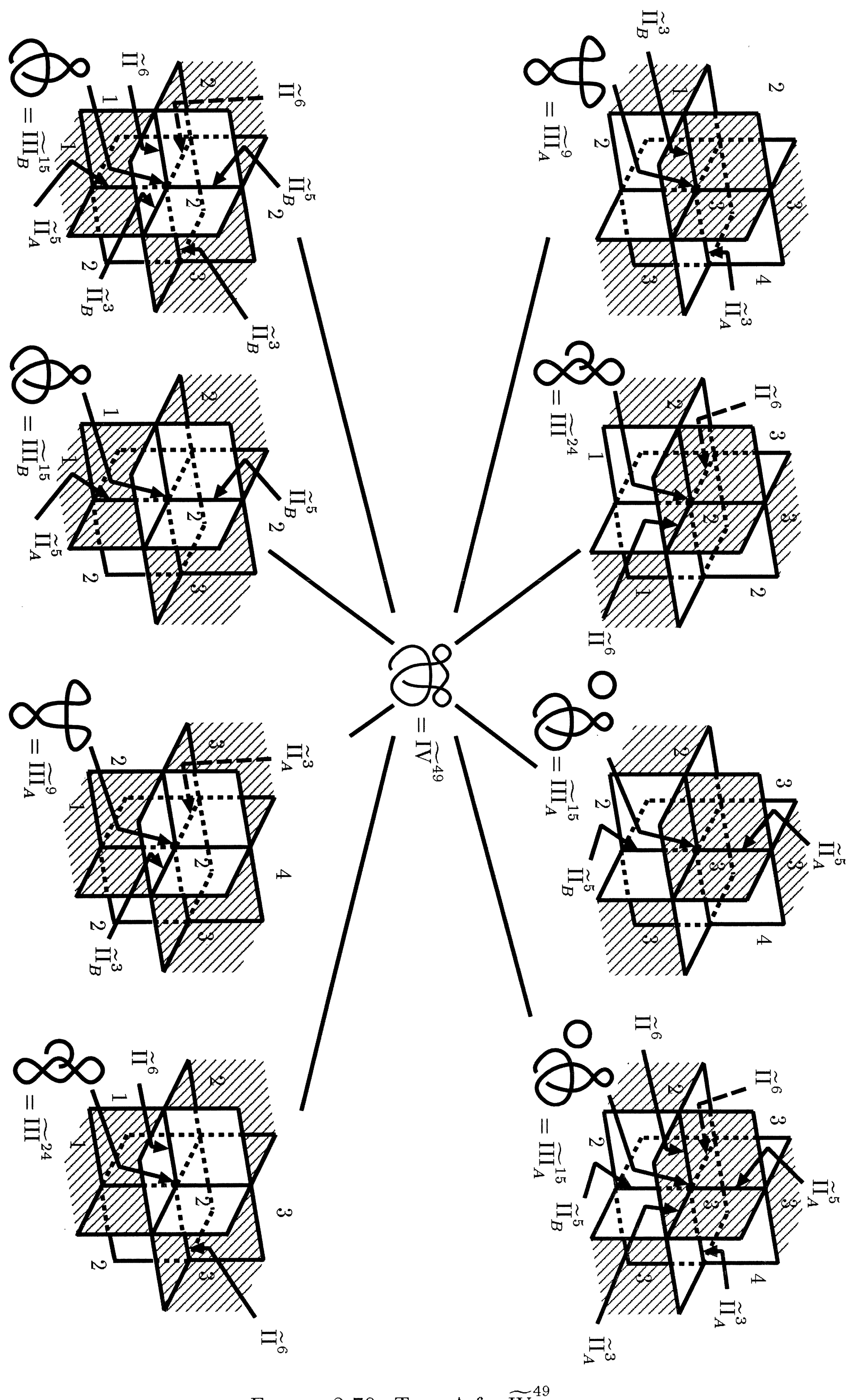

FIGURE 2.70. Type A for $\widetilde{\mathrm{IV}}^{49}$

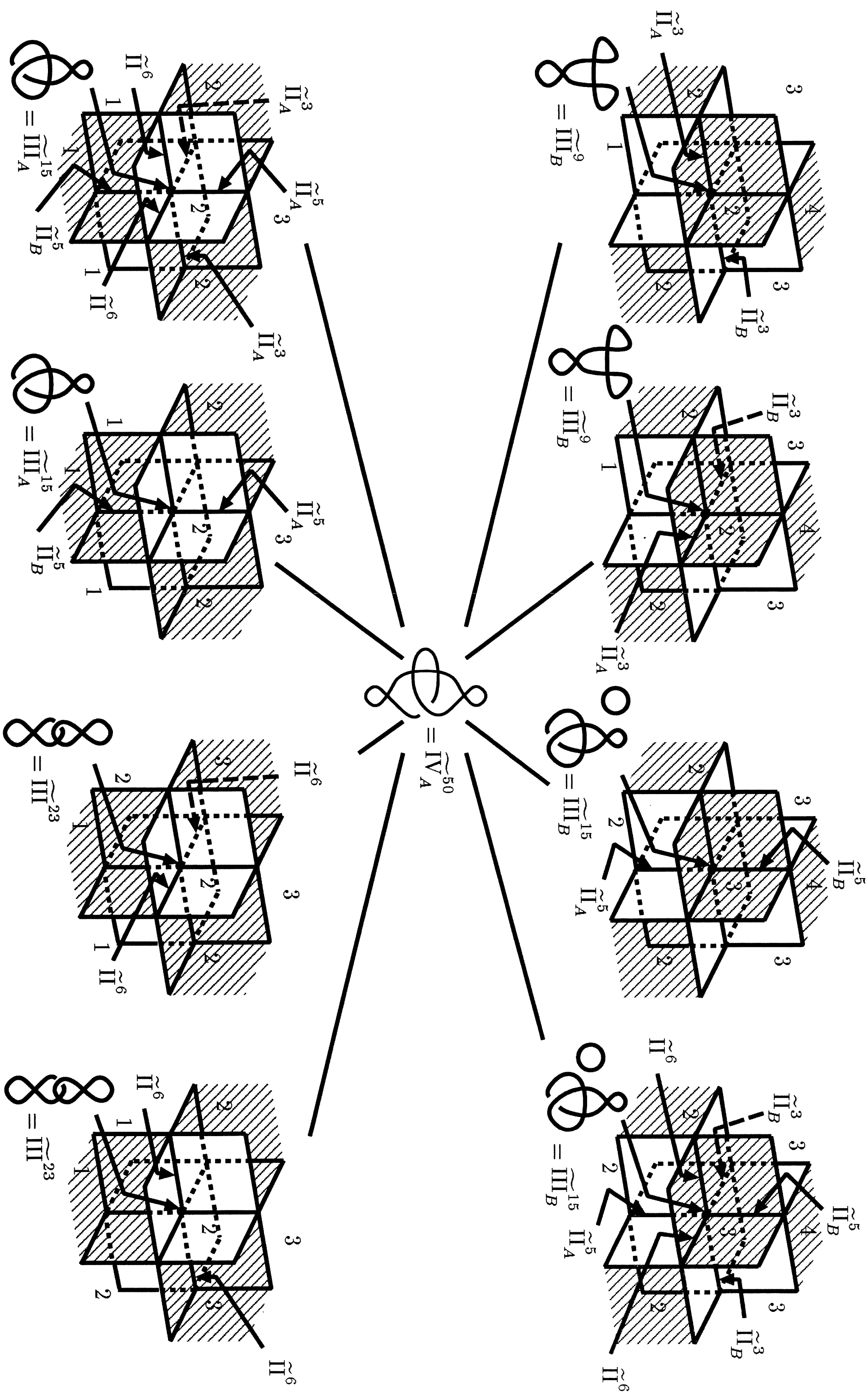

FIGURE 2.71. Type A for $\widetilde{\mathrm{IV}}^{50}$

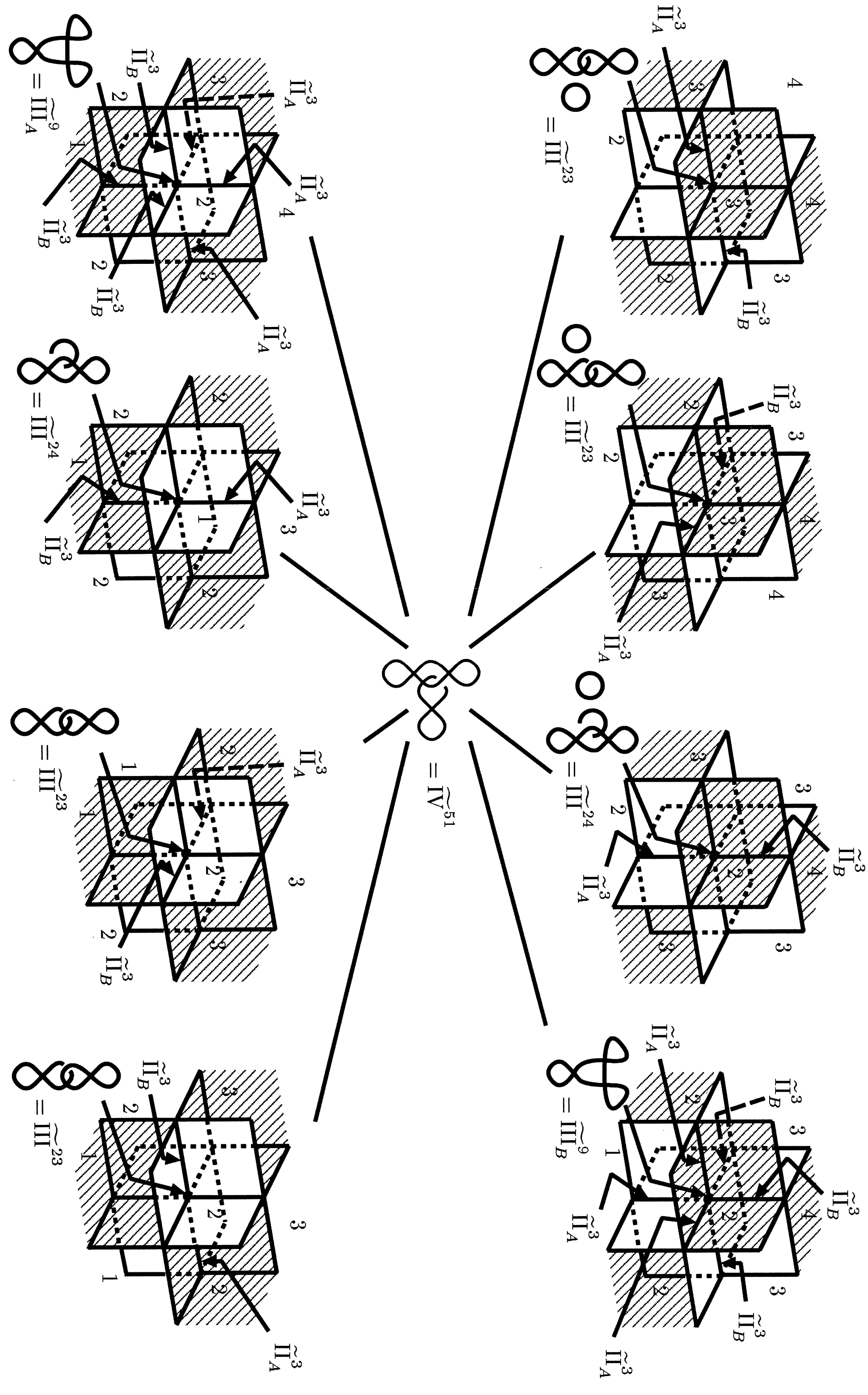

FIGURE 2.72. $\widetilde{\mathrm{IV}}^{51}$ can not divide into two types

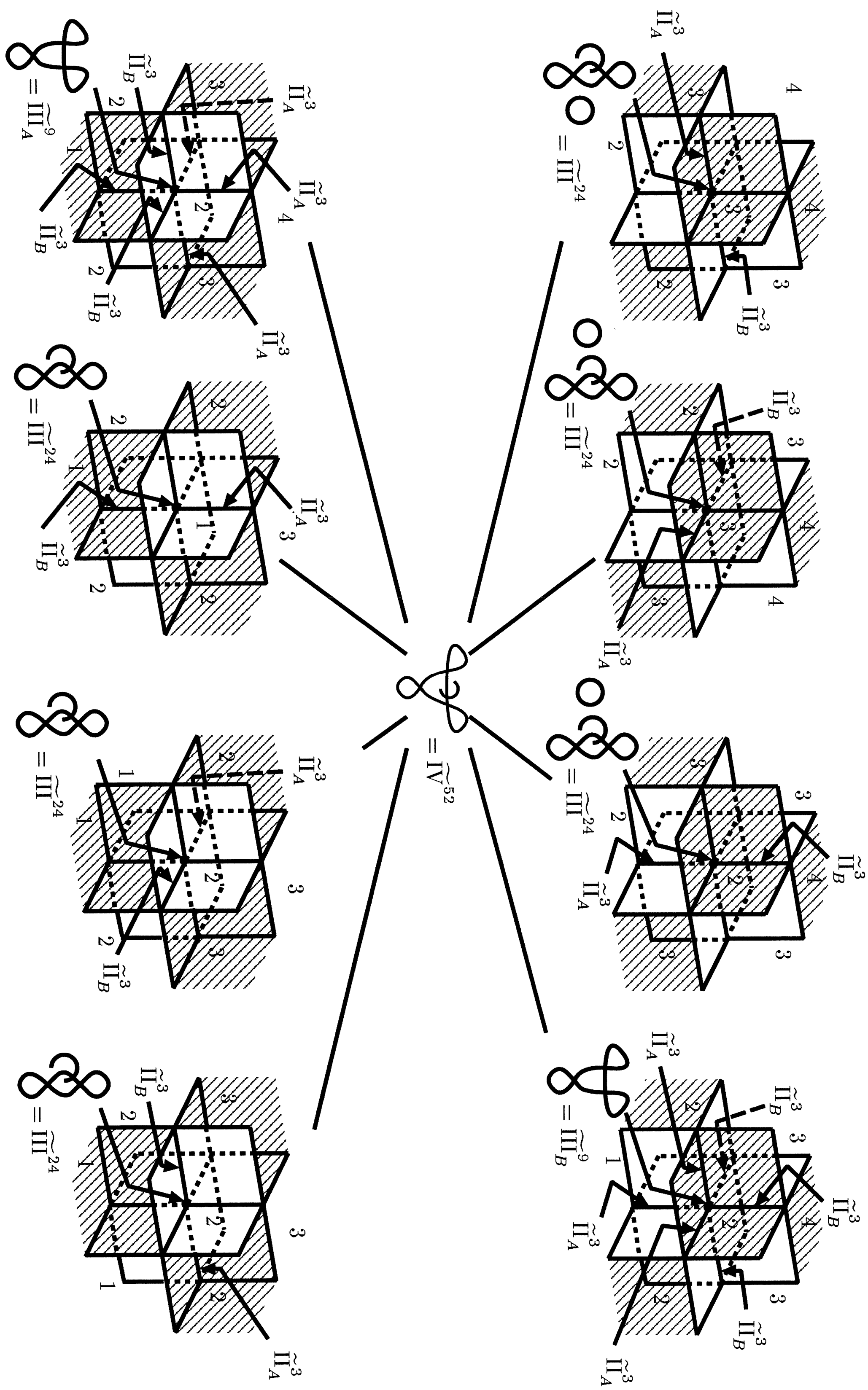

FIGURE 2.73. $\widetilde{\mathrm{IV}}^{52}$ can not divide into two types

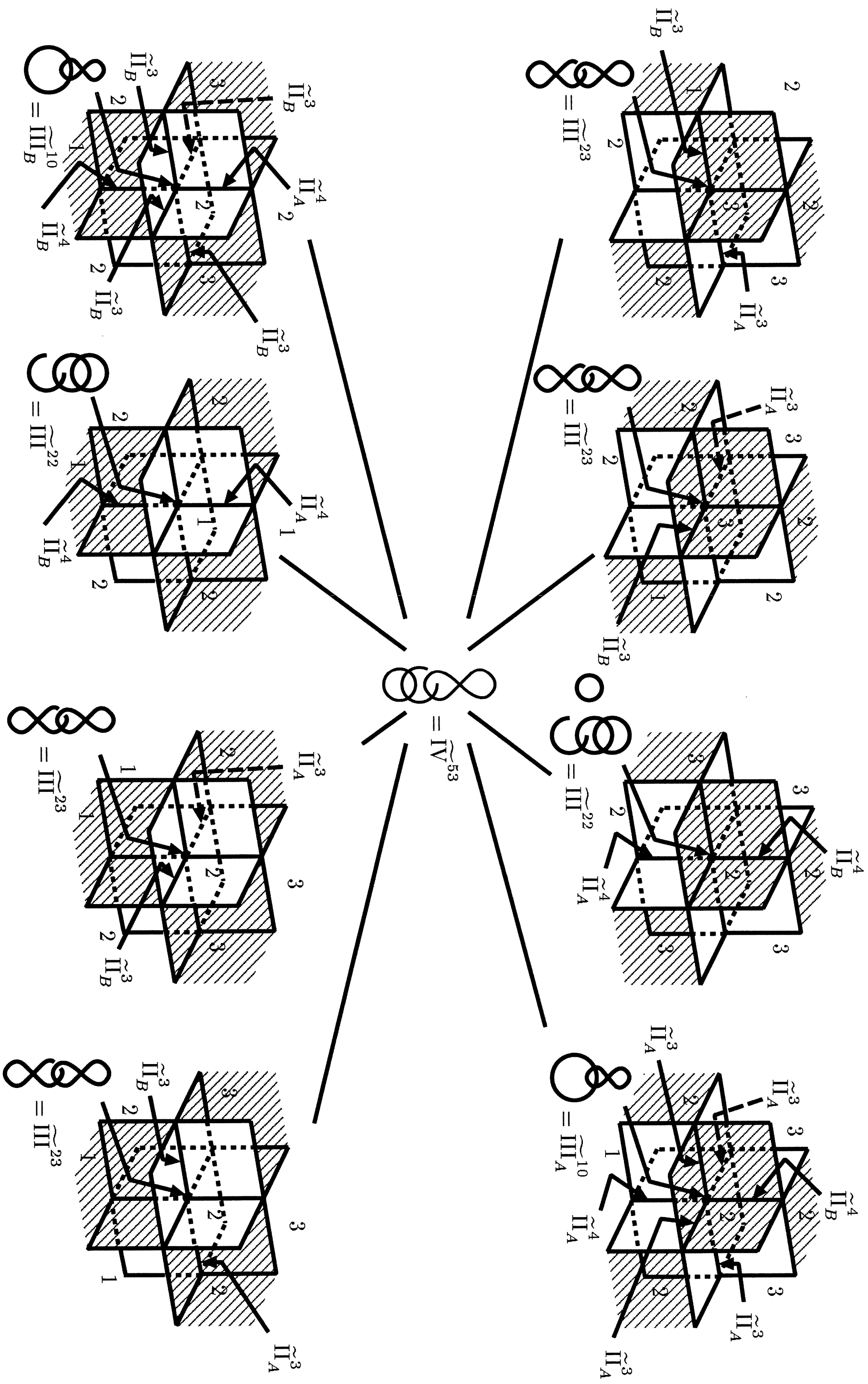

FIGURE 2.74. $\widetilde{\mathrm{IV}}^{53}$ can not divide into two types

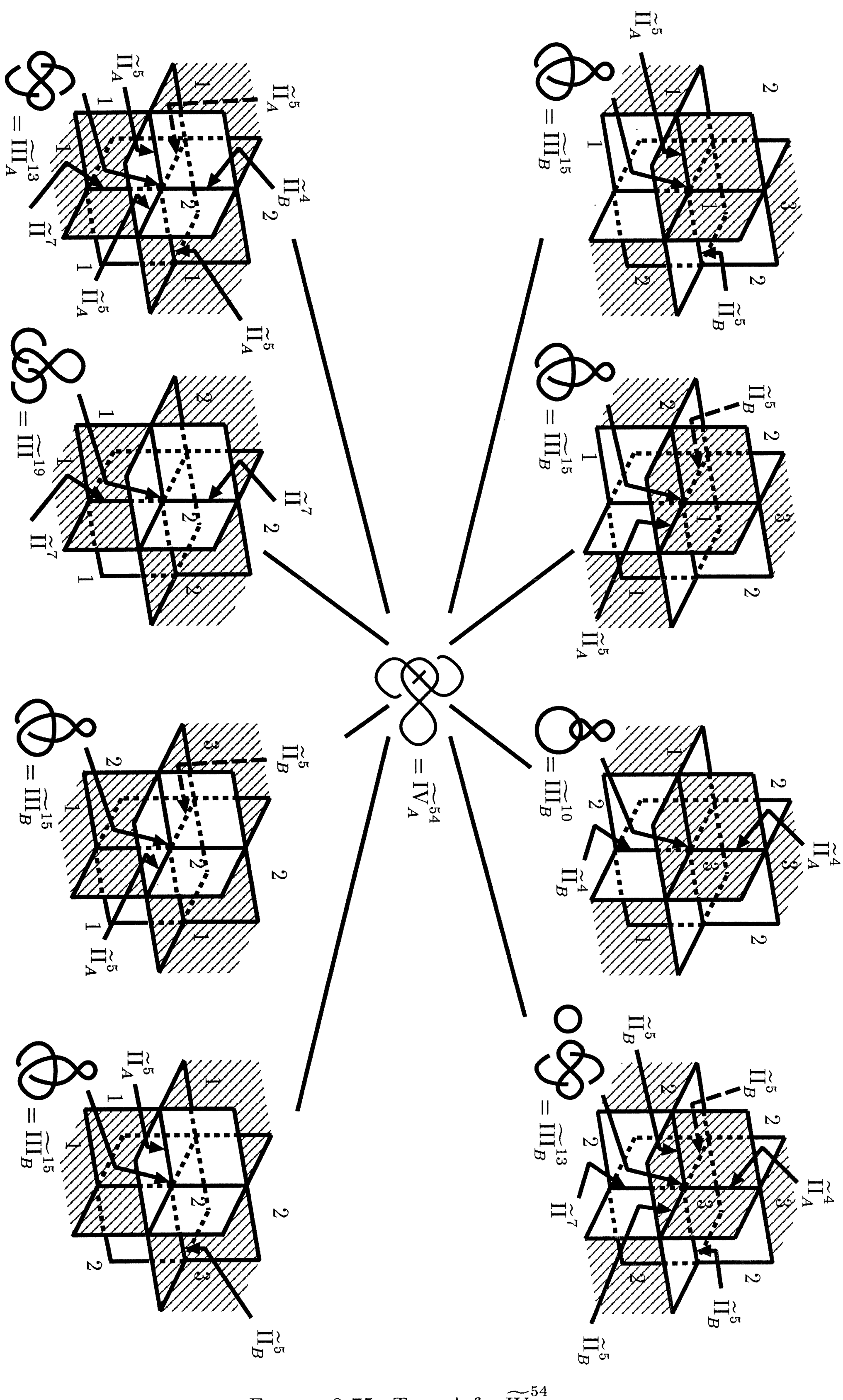

FIGURE 2.75. Type A for $\widetilde{IV}^{54}$

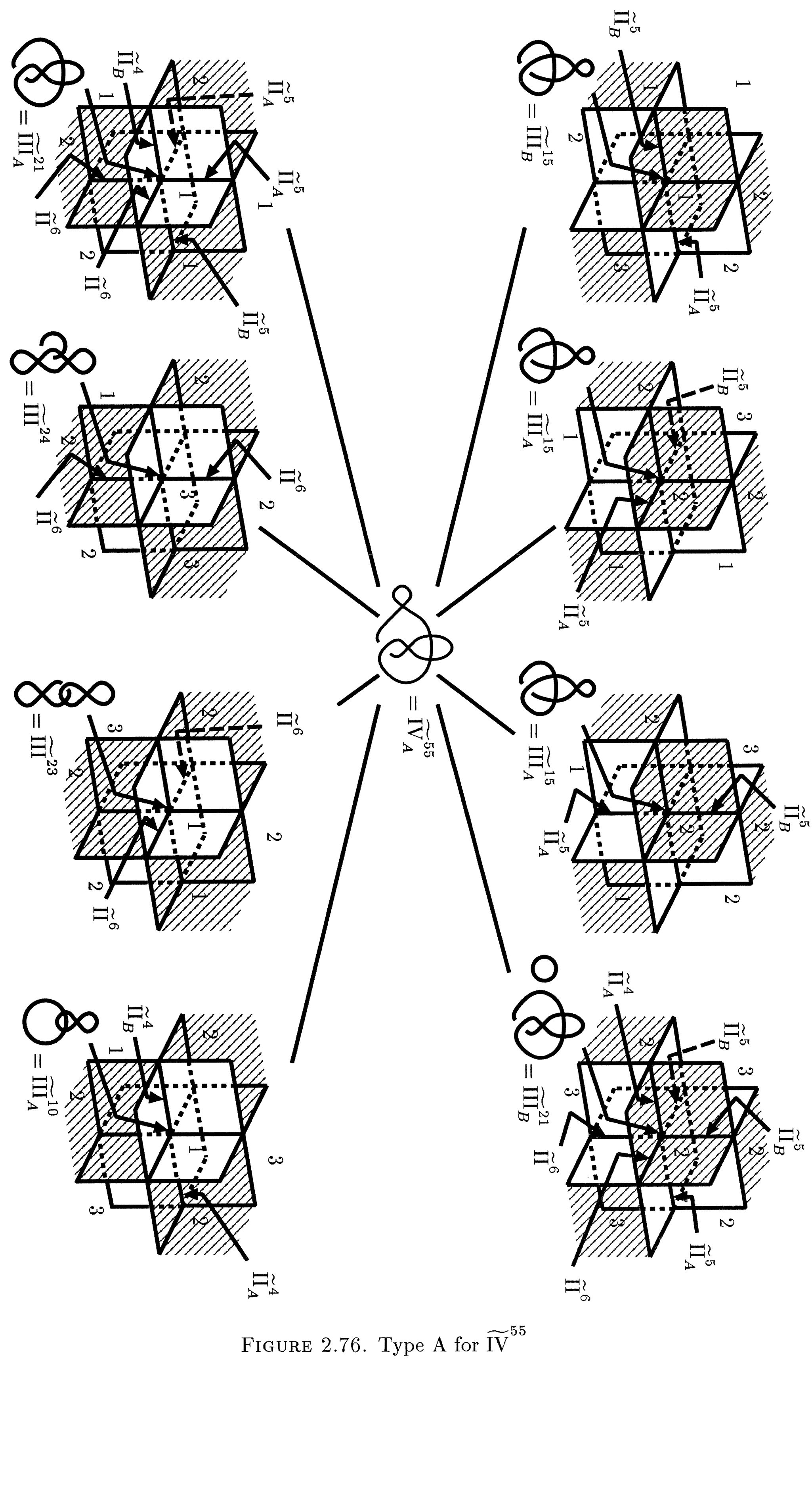

FIGURE 2.76. Type A for $\widetilde{\mathrm{IV}}^{55}$

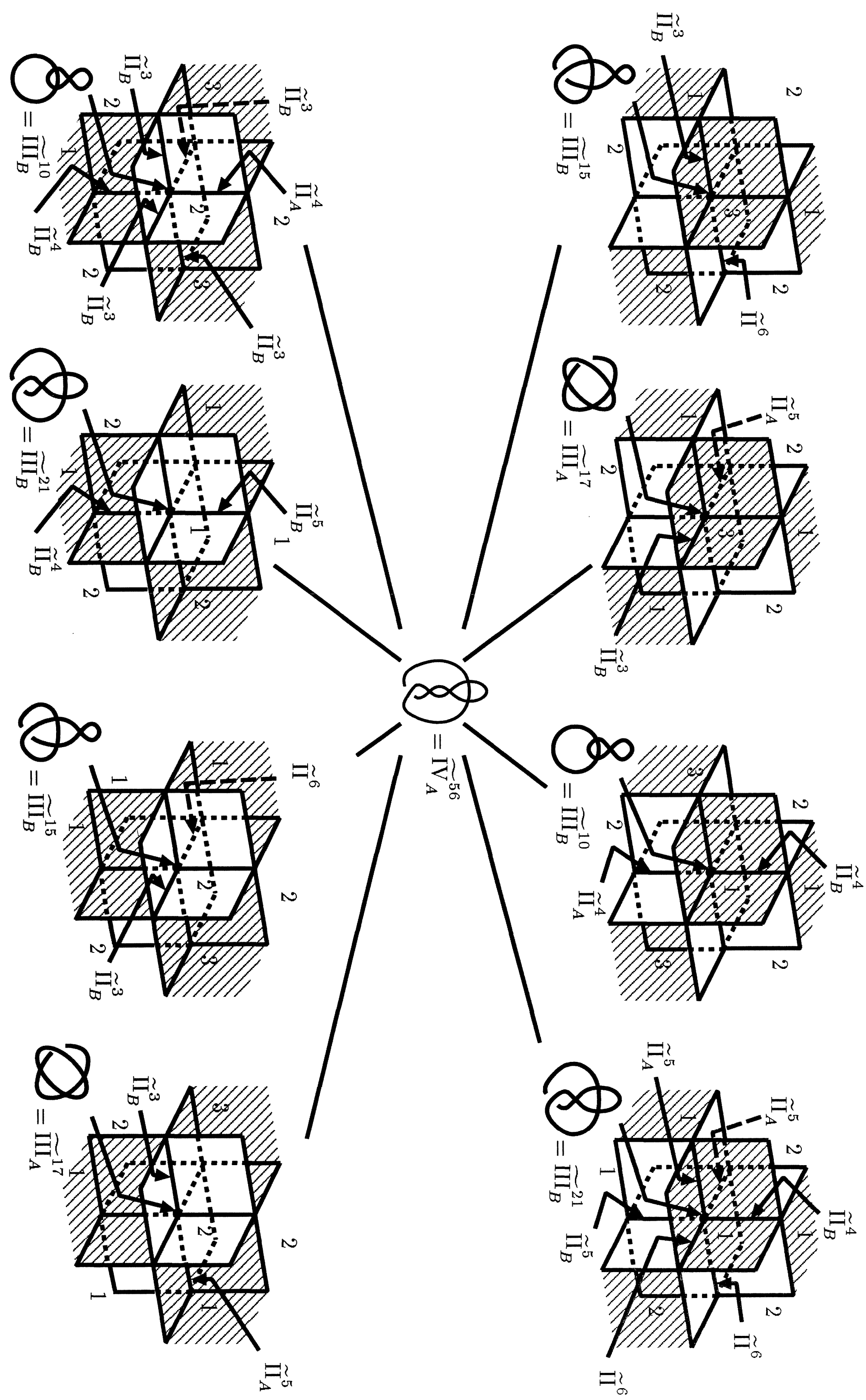

FIGURE 2.77. Type A for $\widetilde{\mathrm{IV}}^{56}$

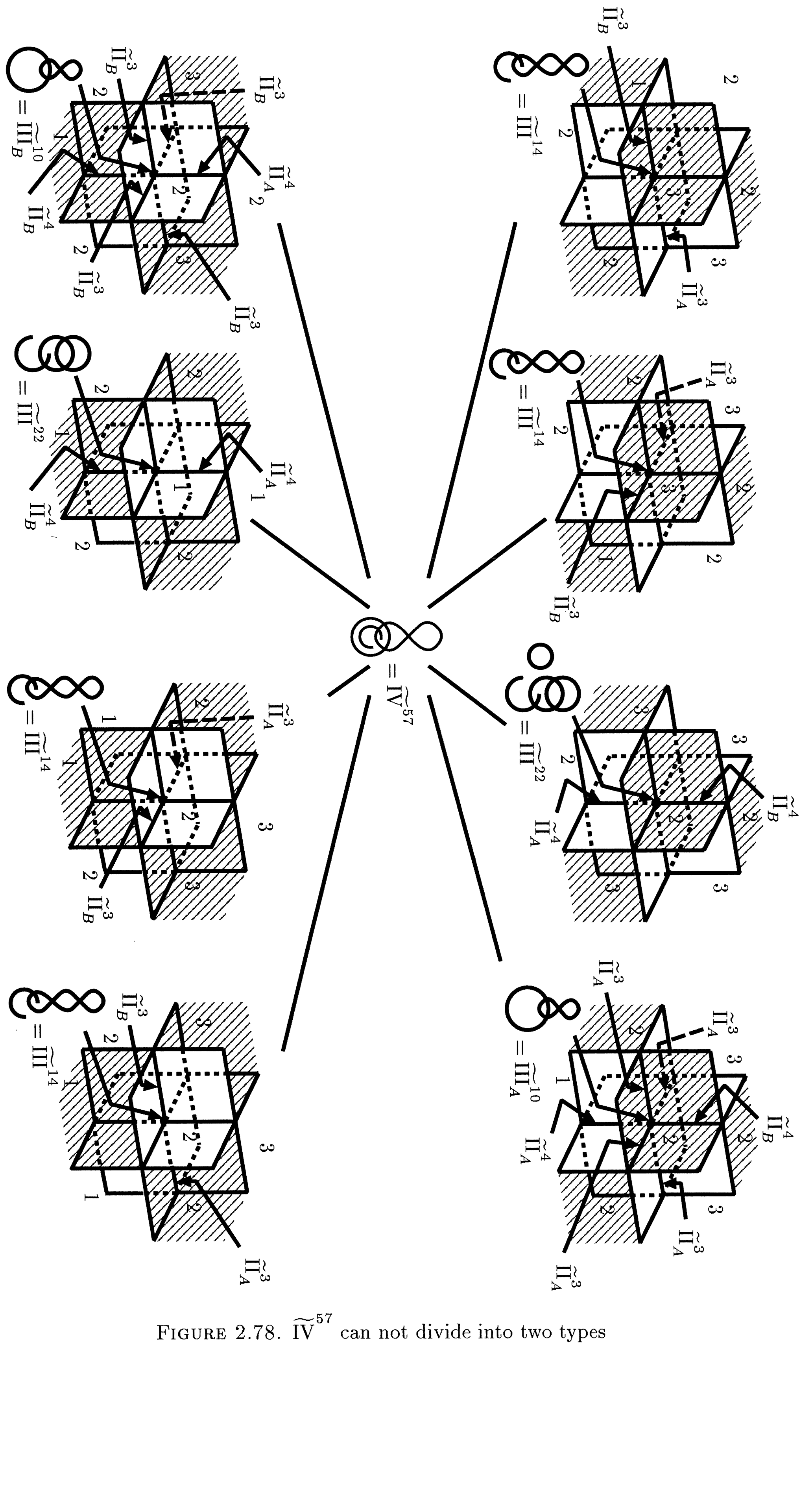

FIGURE 2.78. $\widetilde{\mathrm{IV}}^{57}$ can not divide into two types

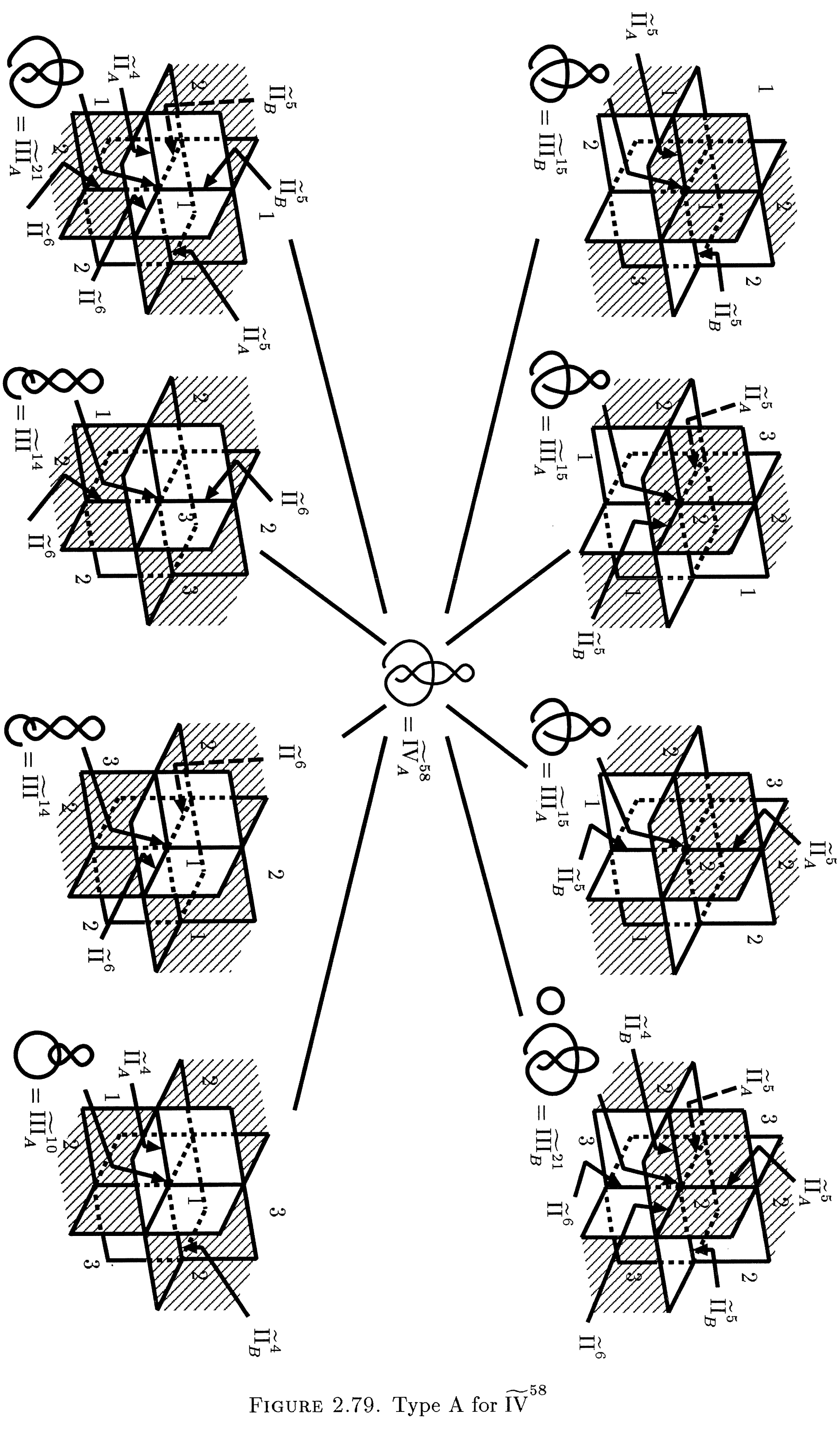

FIGURE 2.79. Type A for $\widetilde{\mathrm{IV}}^{58}$

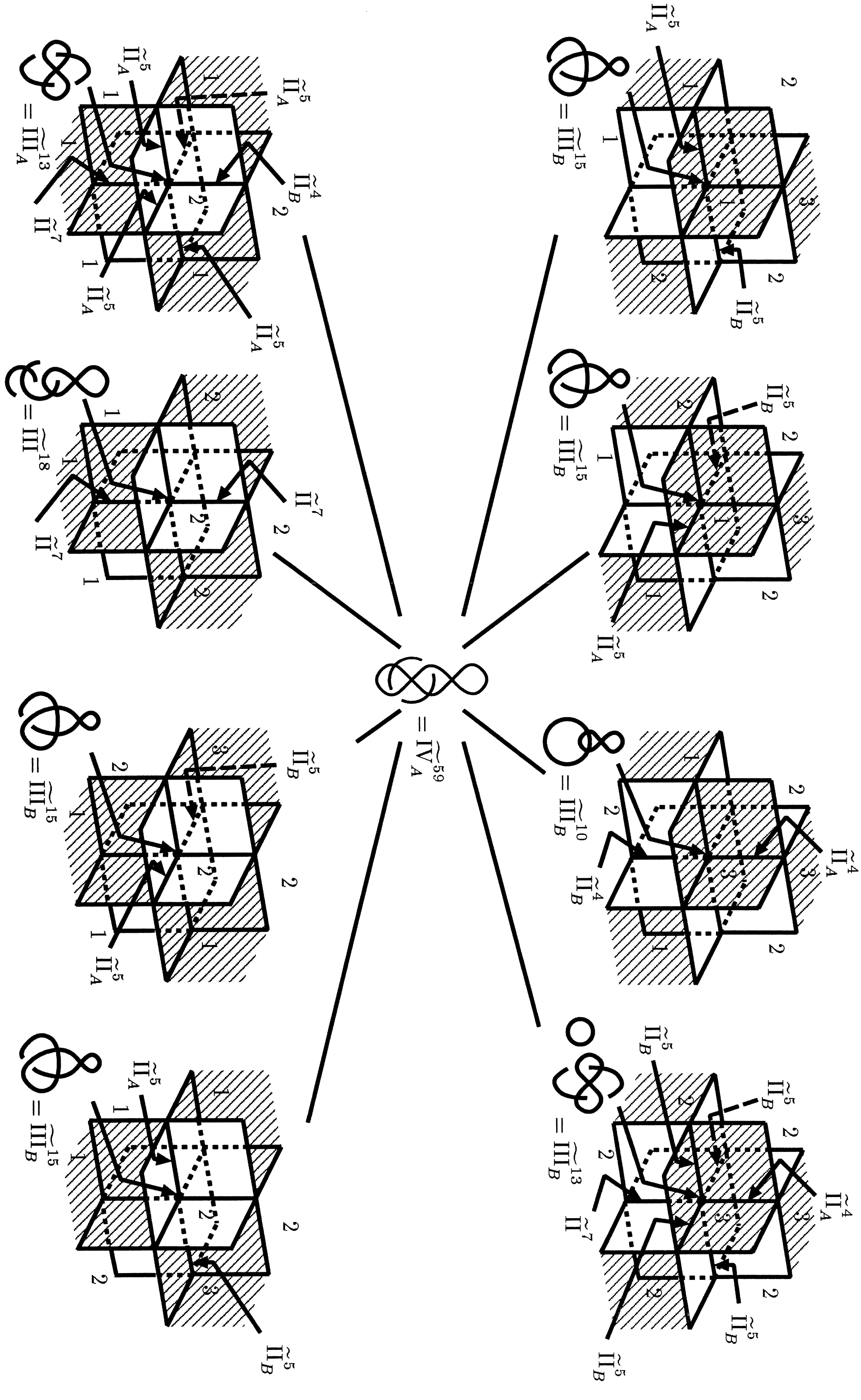

FIGURE 2.80. Type A for $\widetilde{\mathrm{IV}}^{59}$

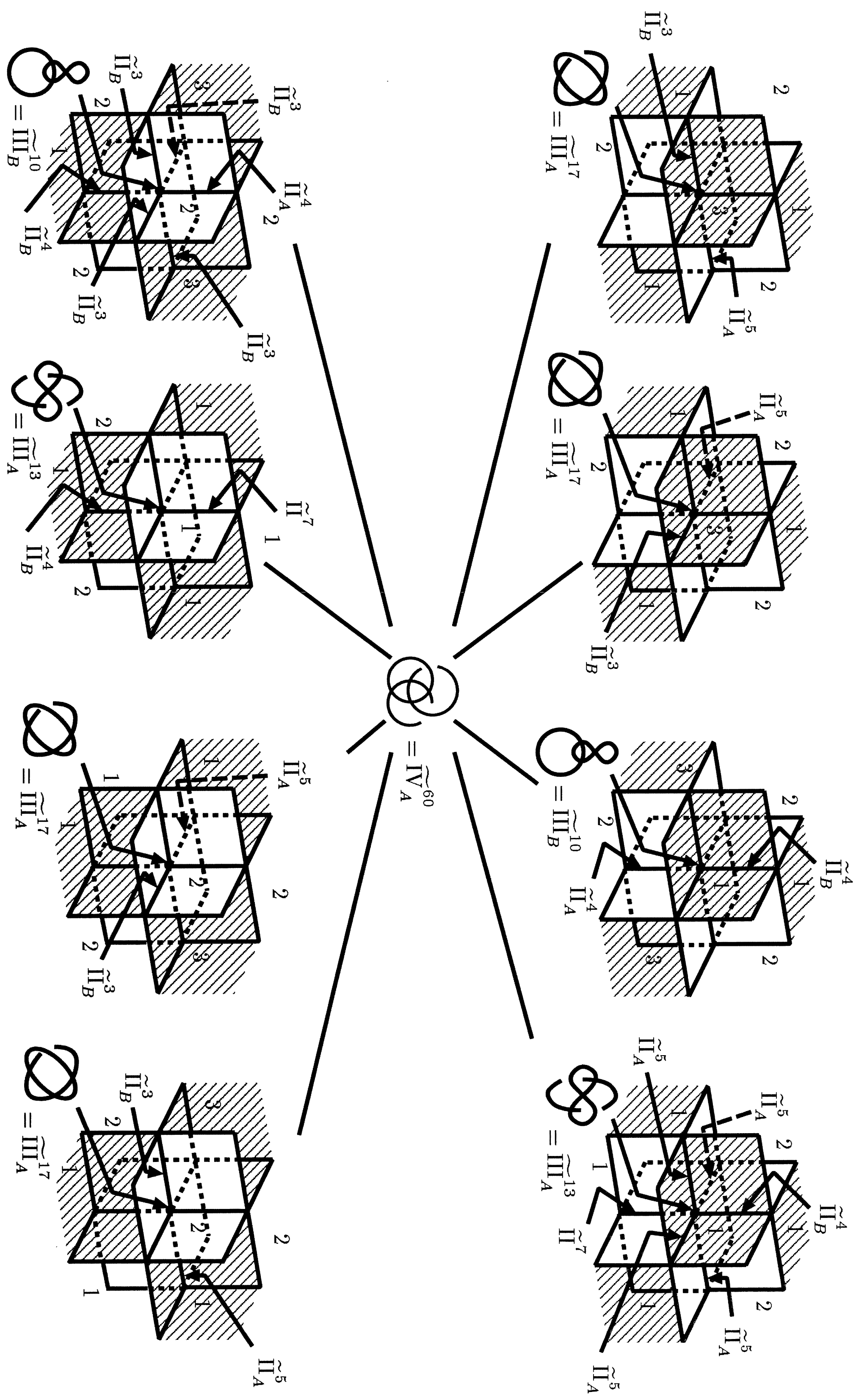

FIGURE 2.81. Type A for $\widetilde{\mathrm{IV}}^{60}$

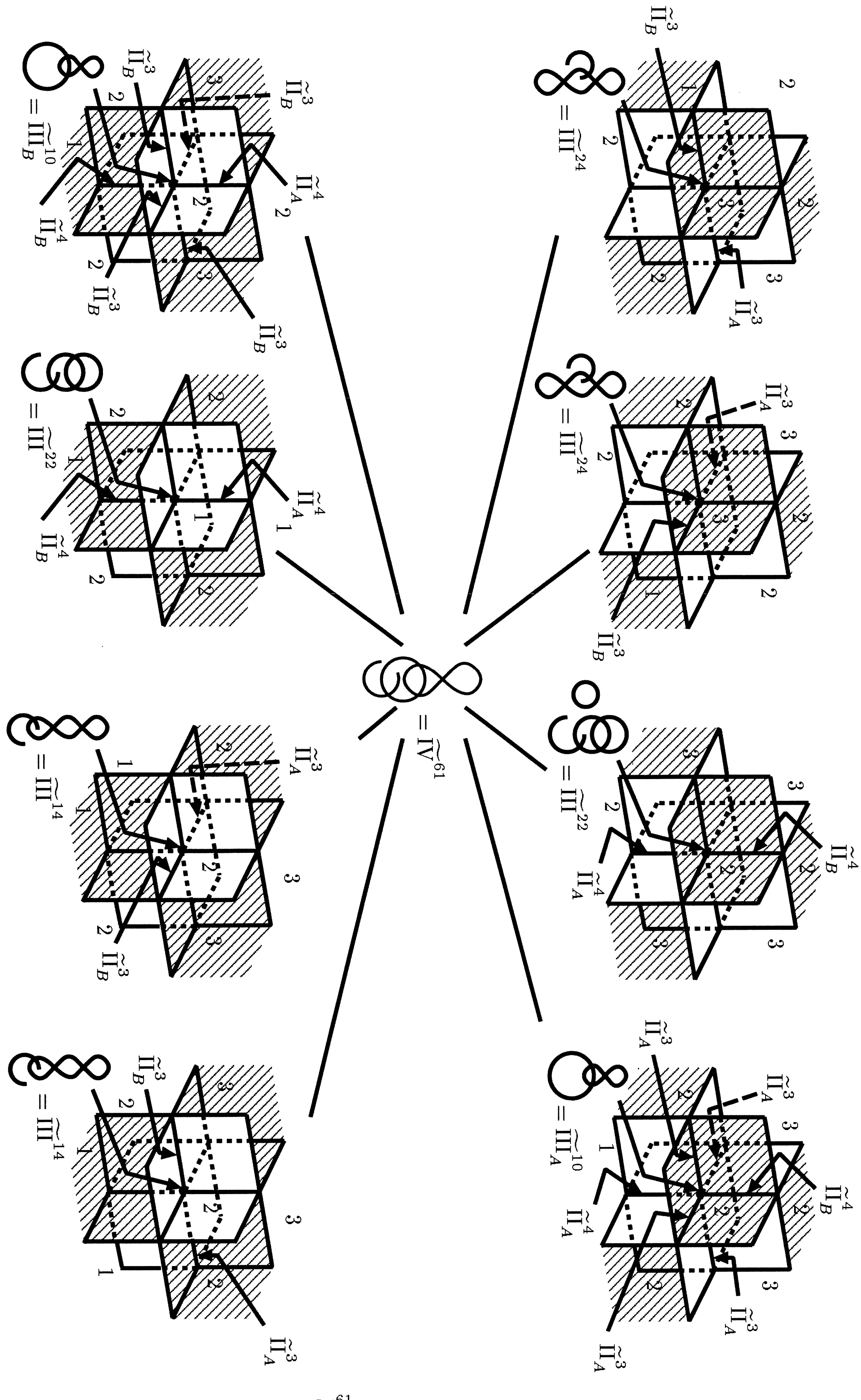

FIGURE 2.82. $\widetilde{\mathrm{IV}}^{61}$ can not divide into two types

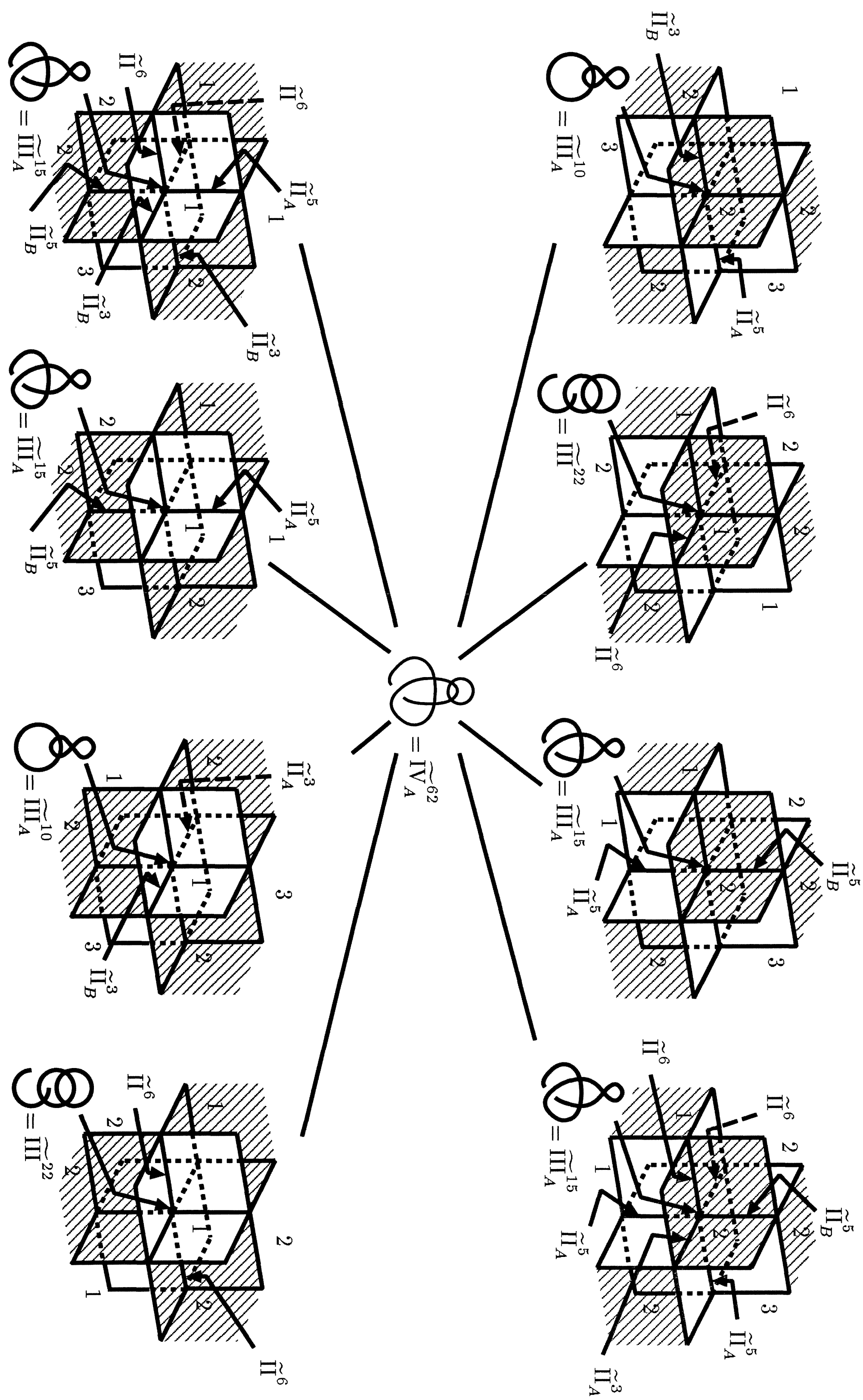

FIGURE 2.83. Type A for $\widetilde{\mathrm{IV}}^{62}$

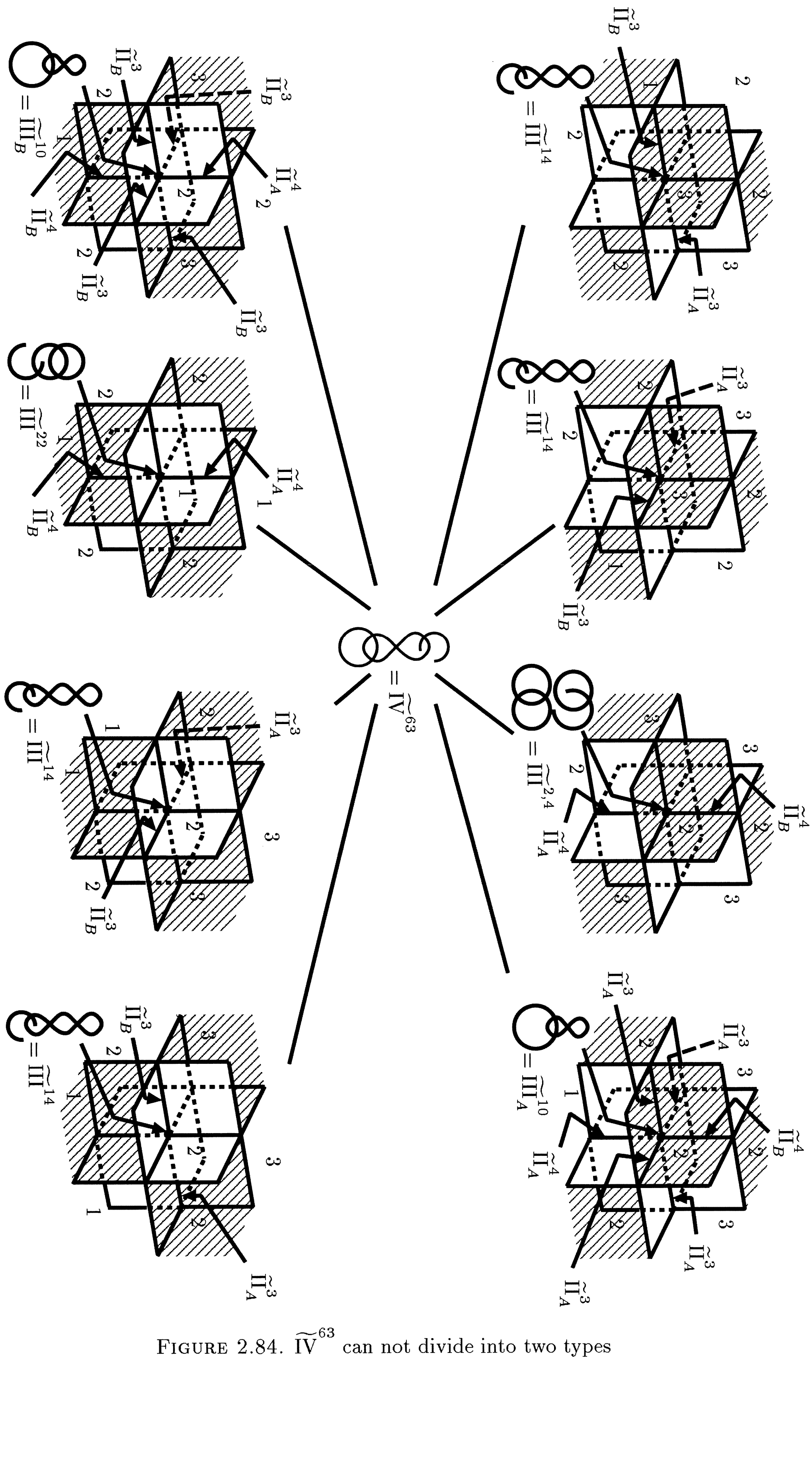

FIGURE 2.84. $\widetilde{\mathrm{IV}}^{63}$ can not divide into two types

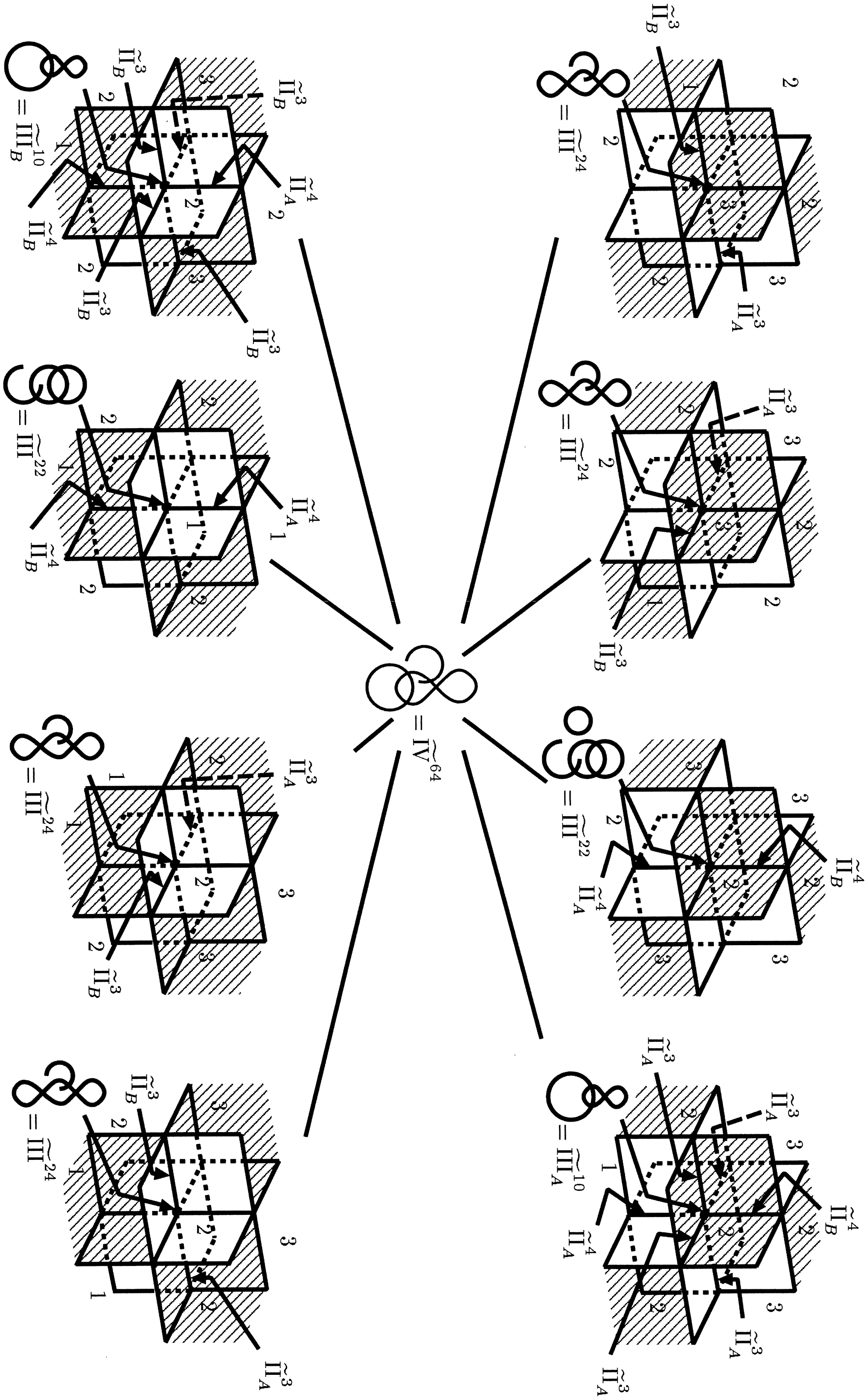

FIGURE 2.85. $\widetilde{\mathrm{IV}}^{64}$ can not divide into two types

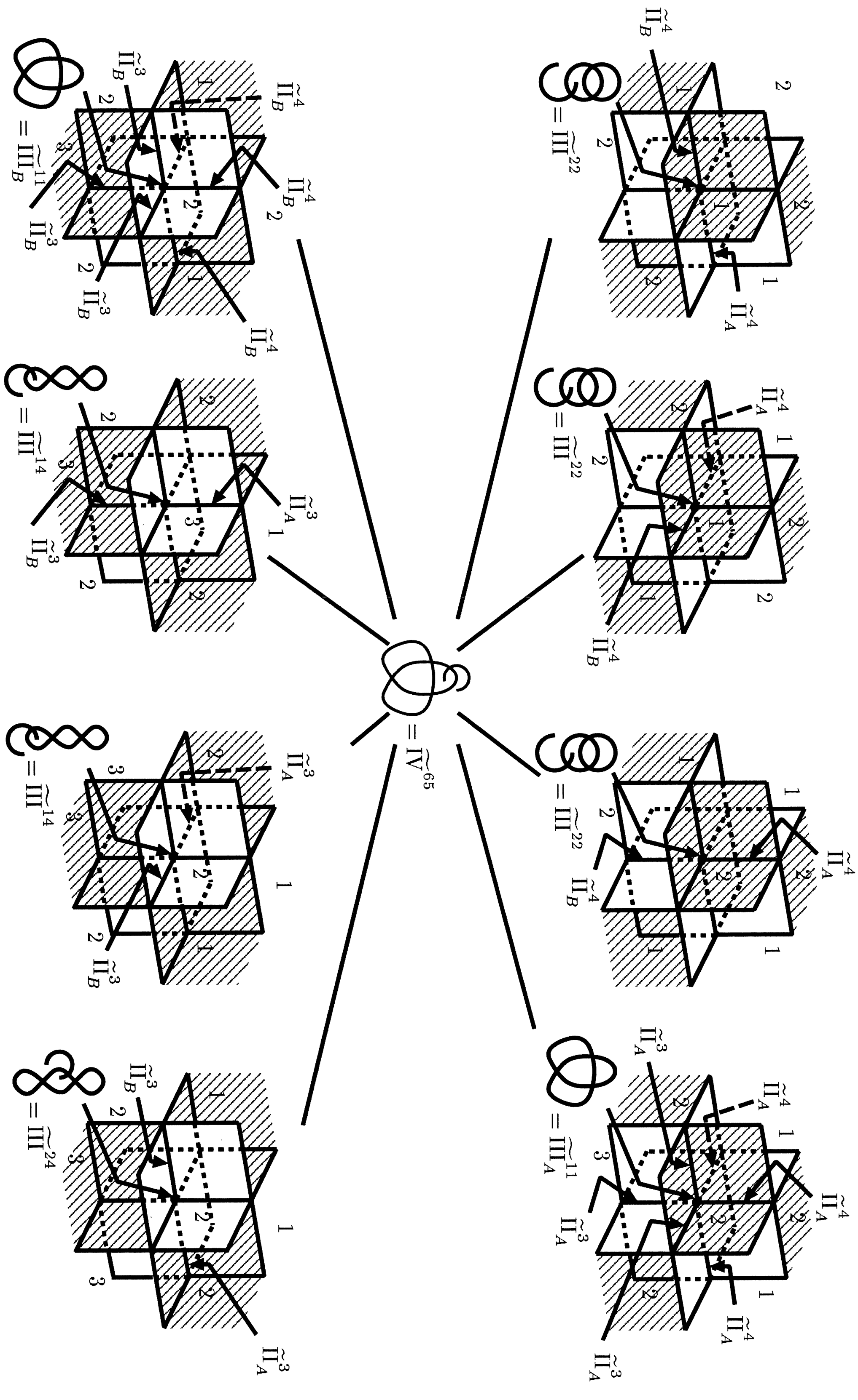

FIGURE 2.86. $\widetilde{\mathrm{IV}}^{65}$ can not divide into two types

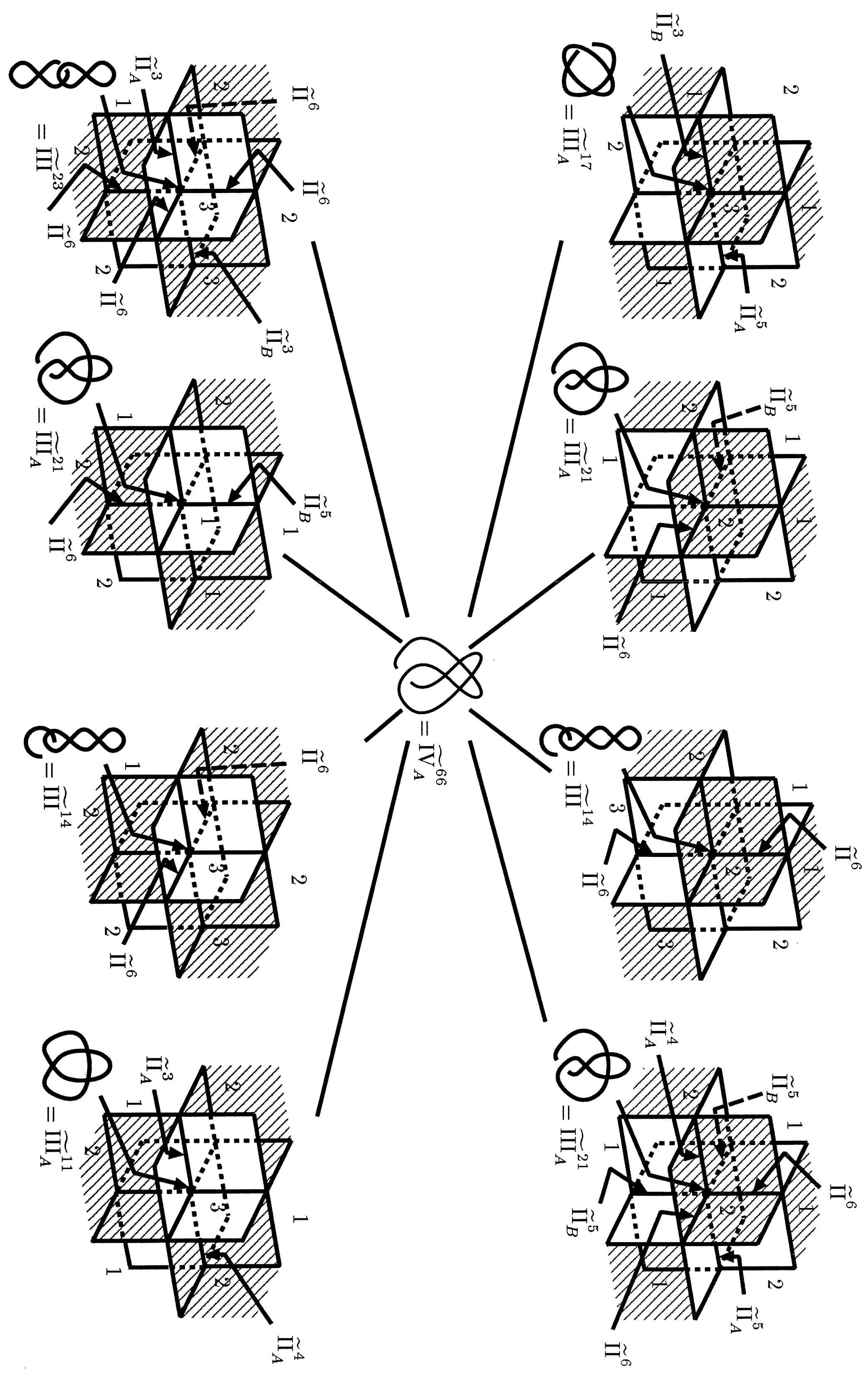

FIGURE 2.87. Type A for $\widetilde{\mathrm{IV}}^{66}$

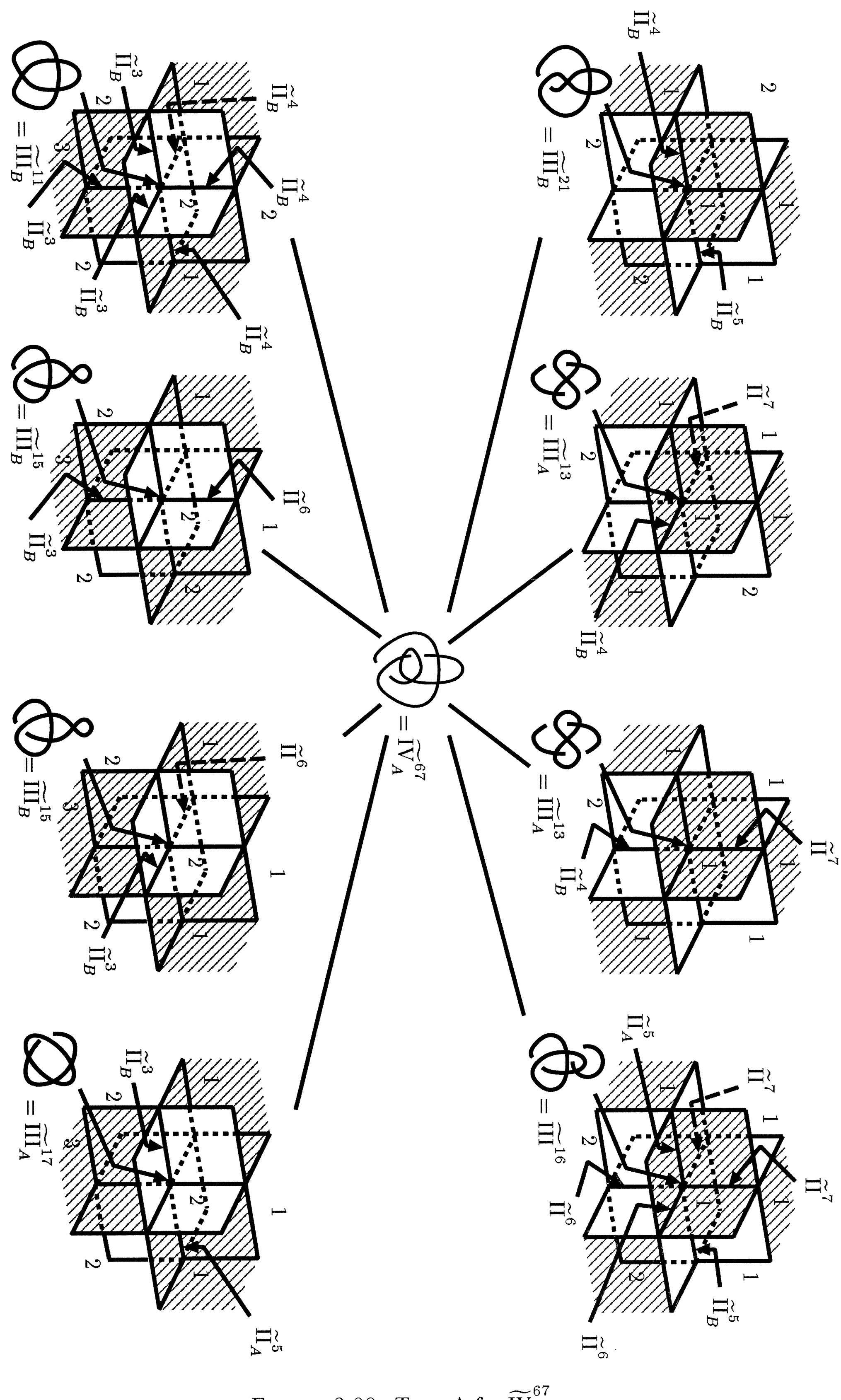

FIGURE 2.88. Type A for $\widetilde{\mathrm{IV}}^{67}$

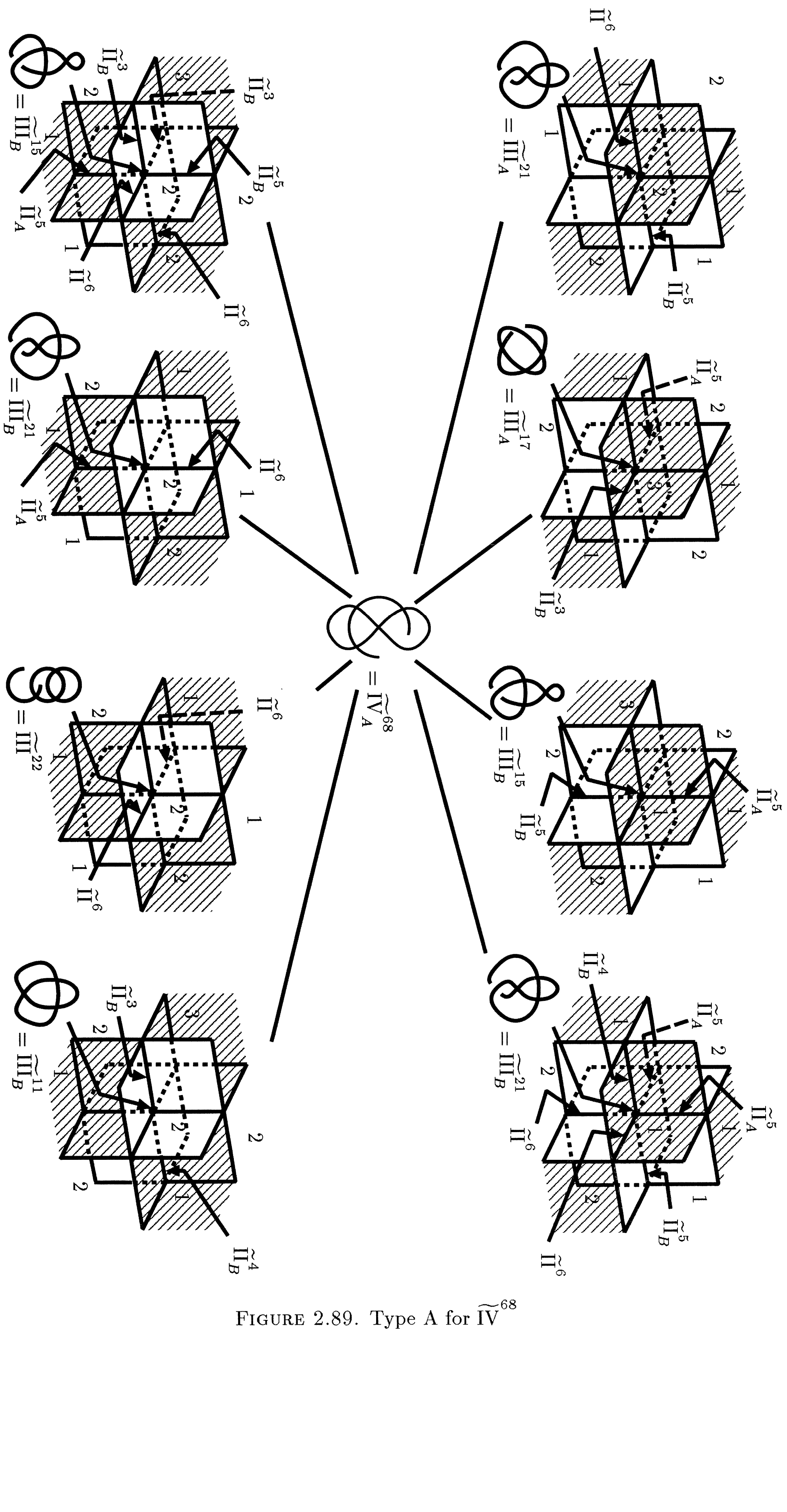

FIGURE 2.89. Type A for $\widetilde{\mathrm{IV}}^{68}$

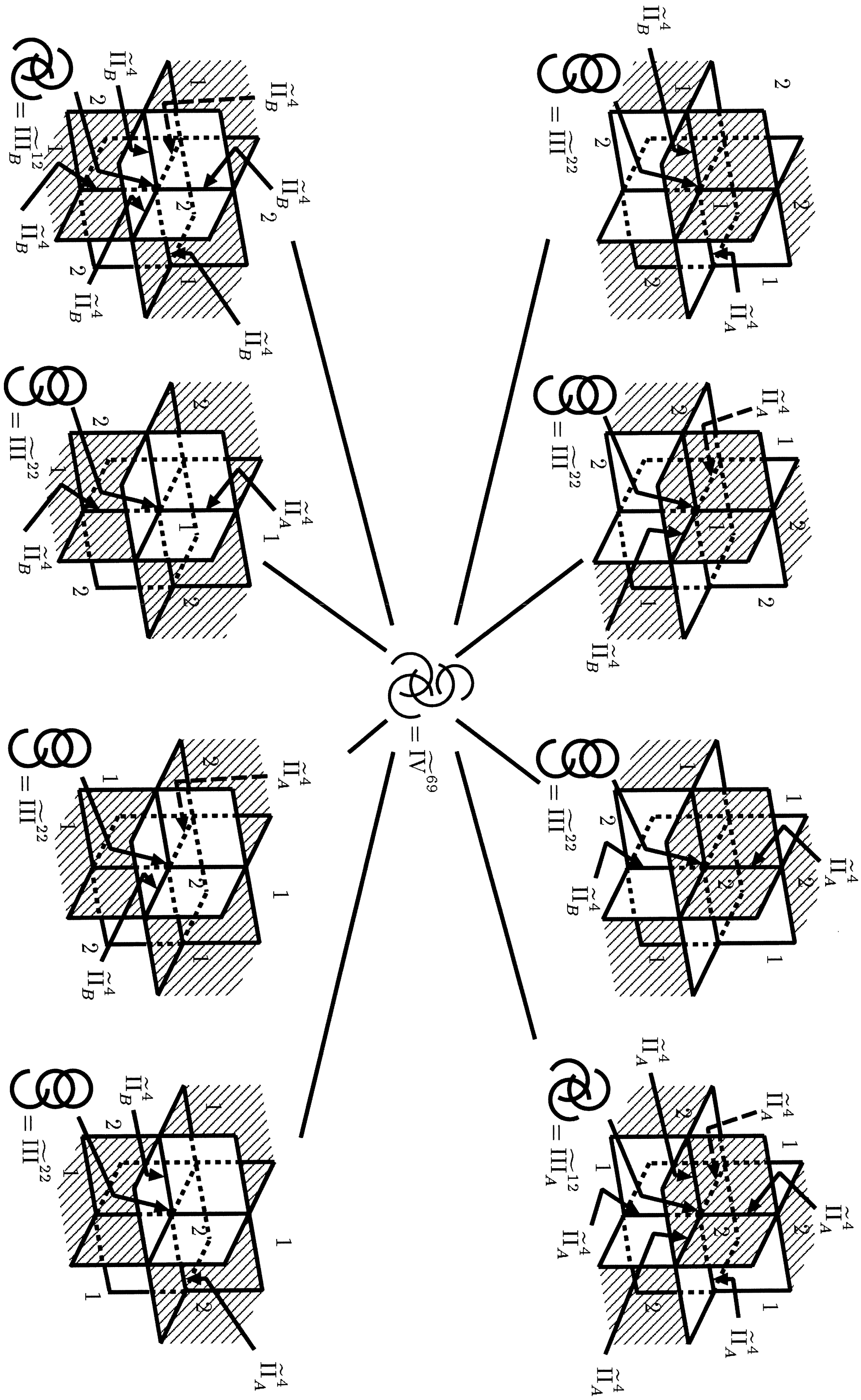

FIGURE 2.90. $\widetilde{\mathrm{IV}}^{69}$ can not divide into two types

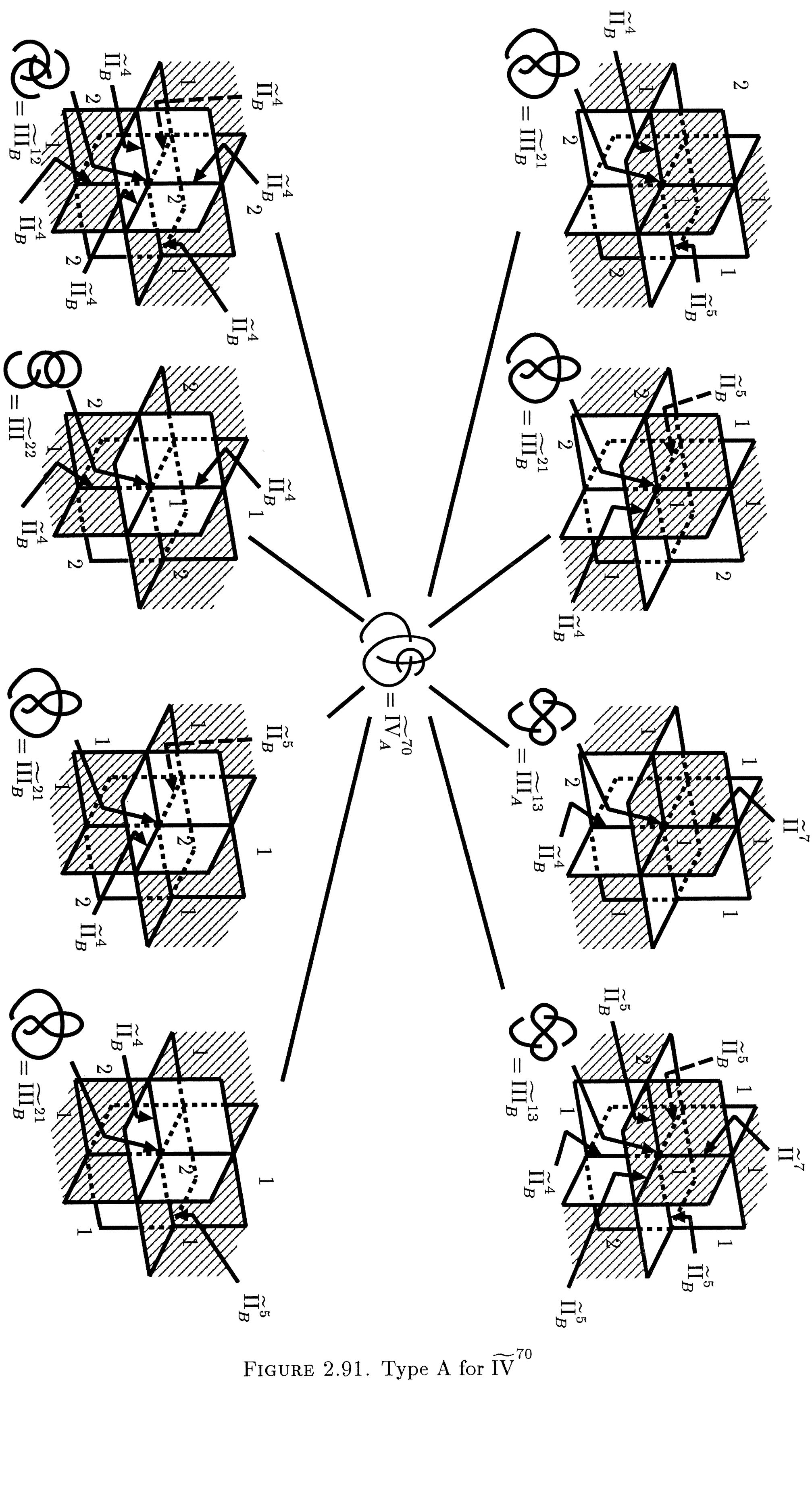

FIGURE 2.91. Type A for $\widetilde{IV}^{70}$

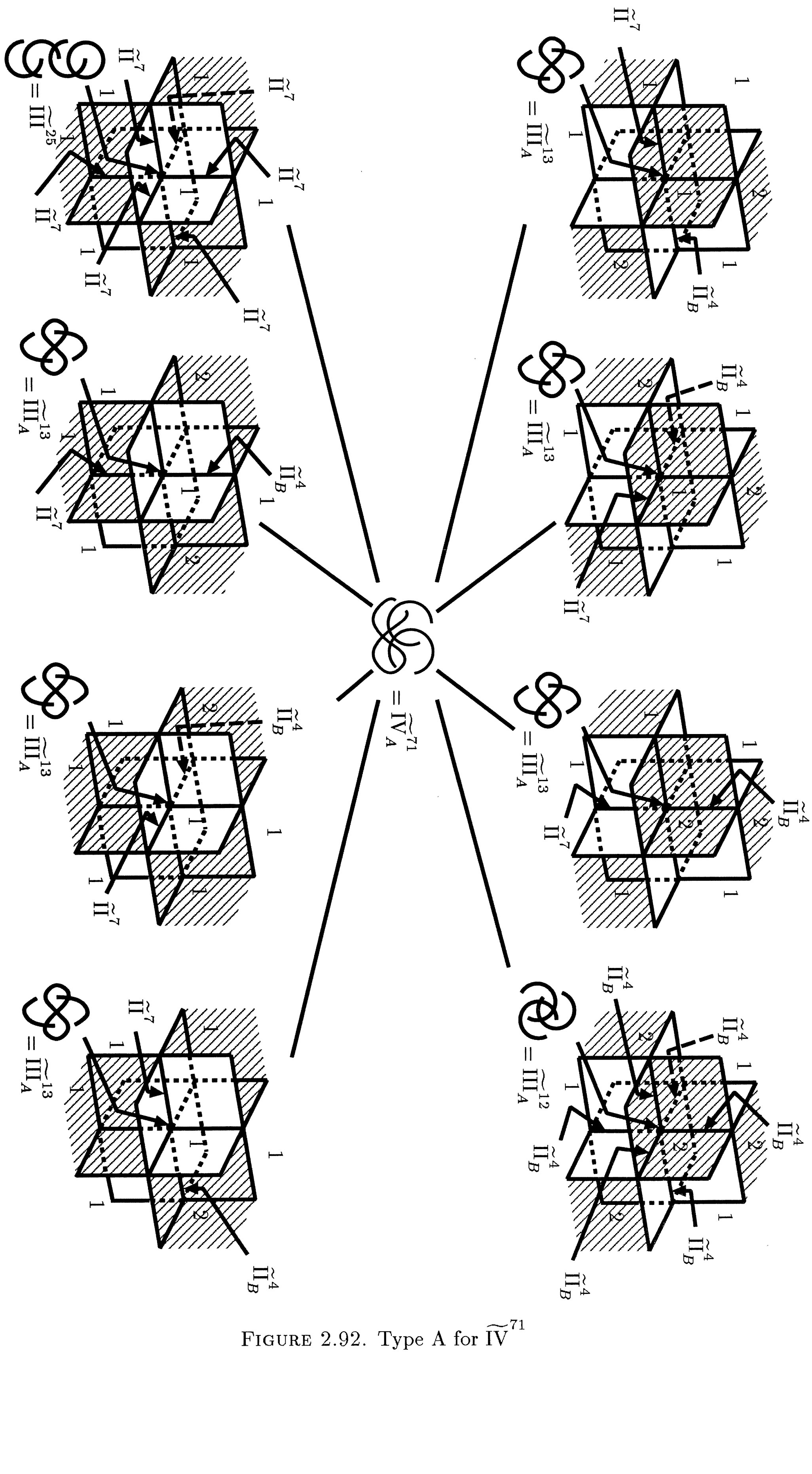

FIGURE 2.92. Type A for $\widetilde{\mathrm{IV}}^{71}$

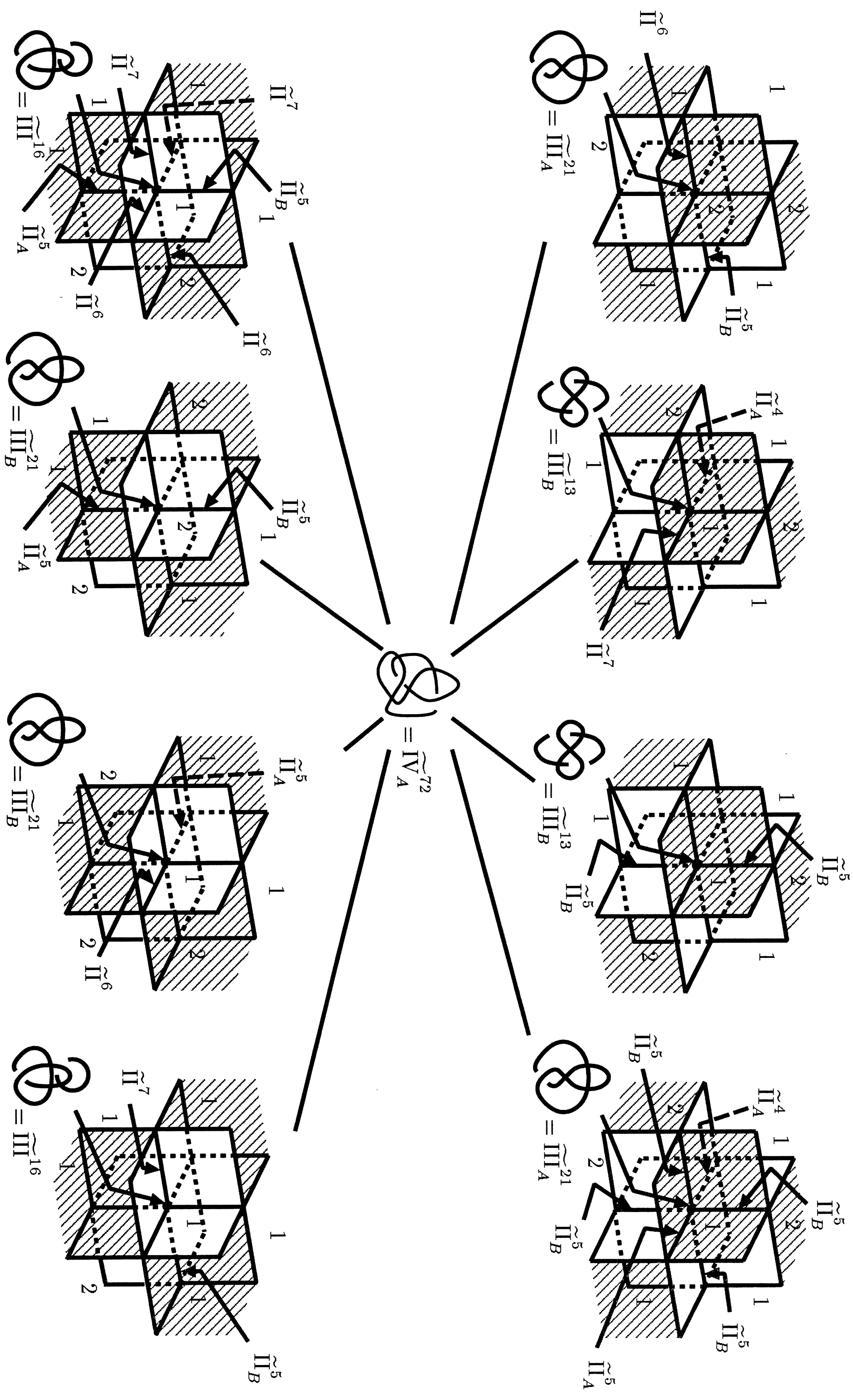

FIGURE 2.93. Type A for $\widetilde{\mathrm{IV}}^{72}$

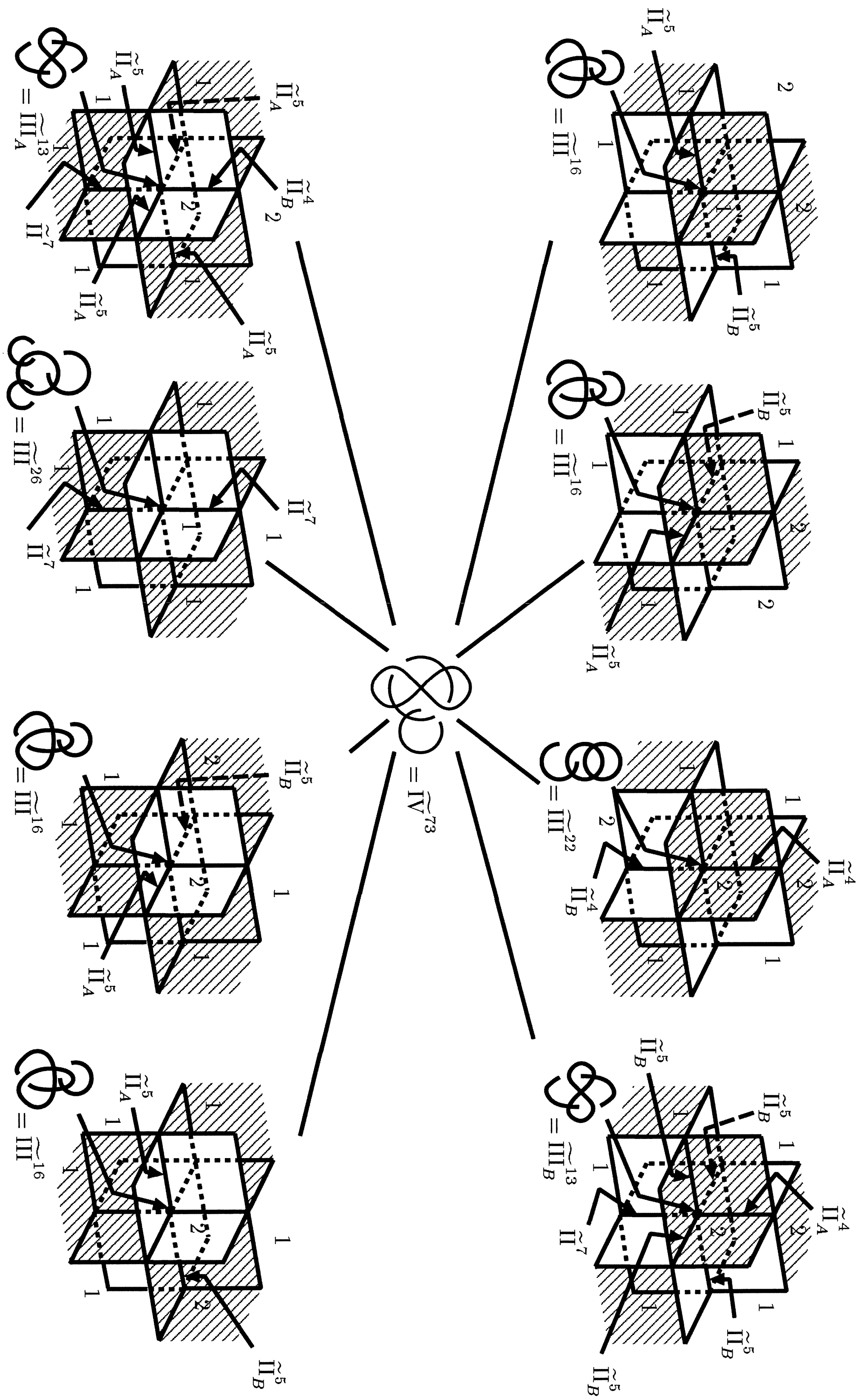

FIGURE 2.94. $\widetilde{\mathrm{IV}}^{73}$ can not divide into two types

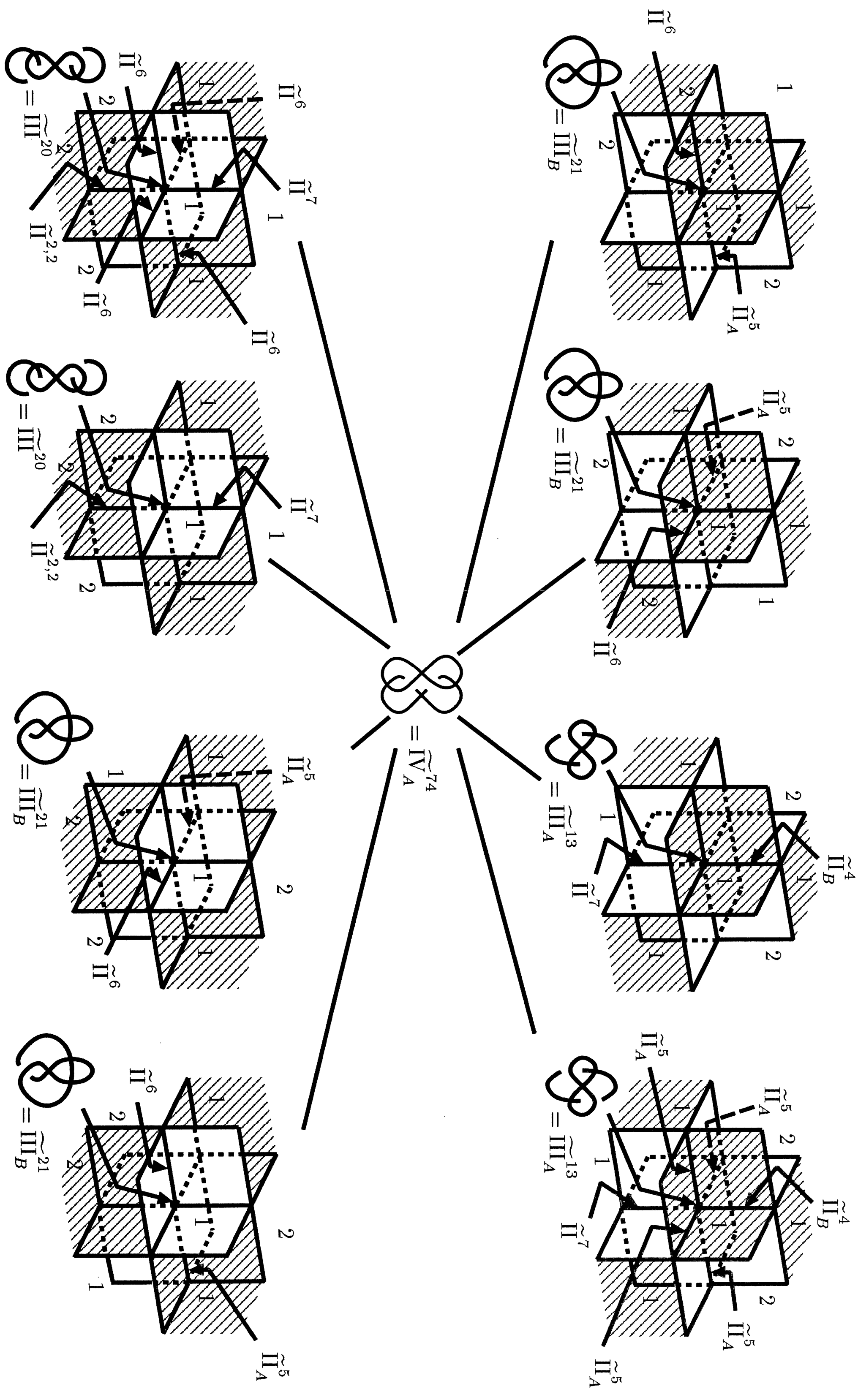

FIGURE 2.95. Type A for $\widetilde{\mathrm{IV}}^{74}$

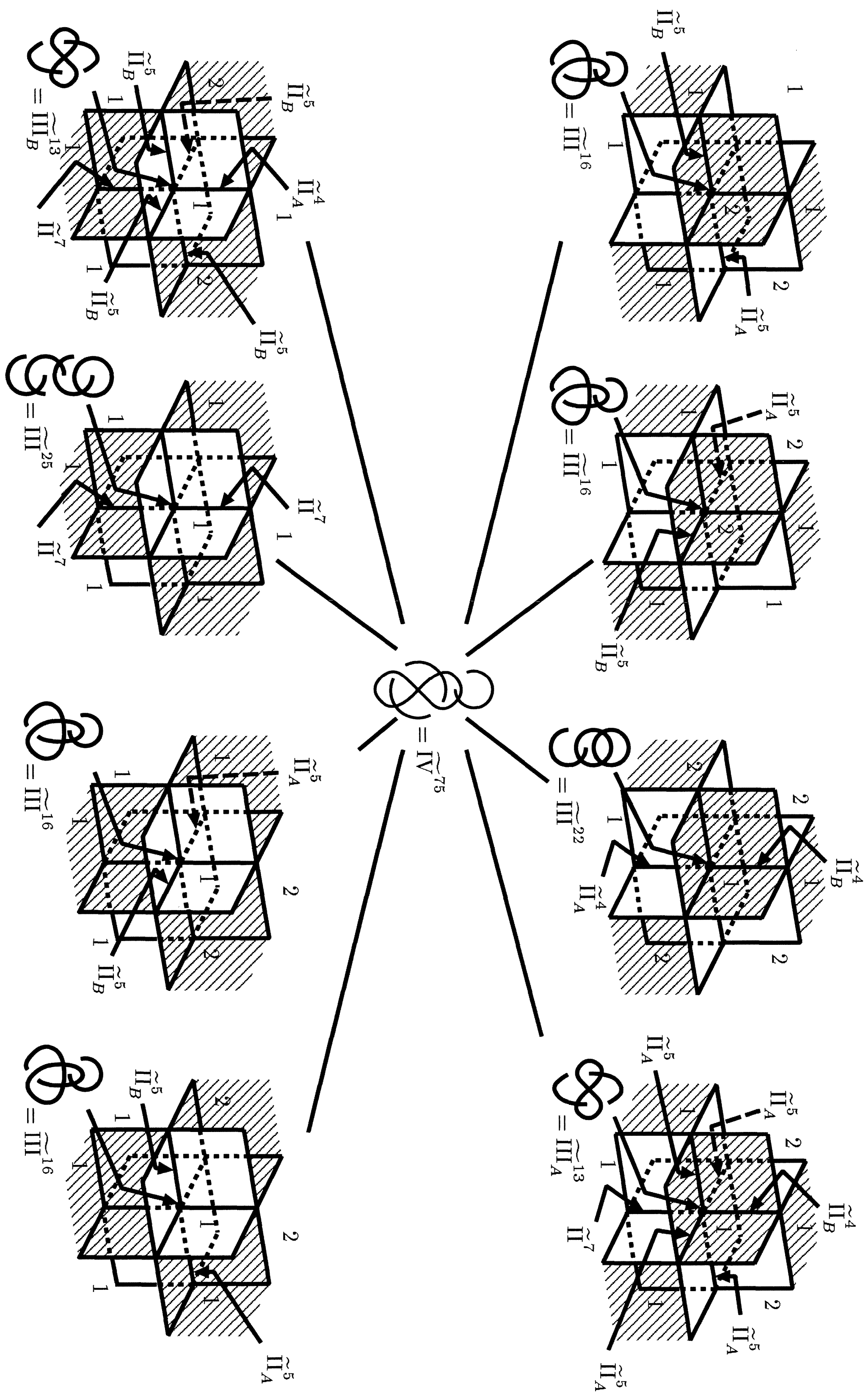

FIGURE 2.96. $\widetilde{\mathrm{IV}}^{75}$ can not divide into two types

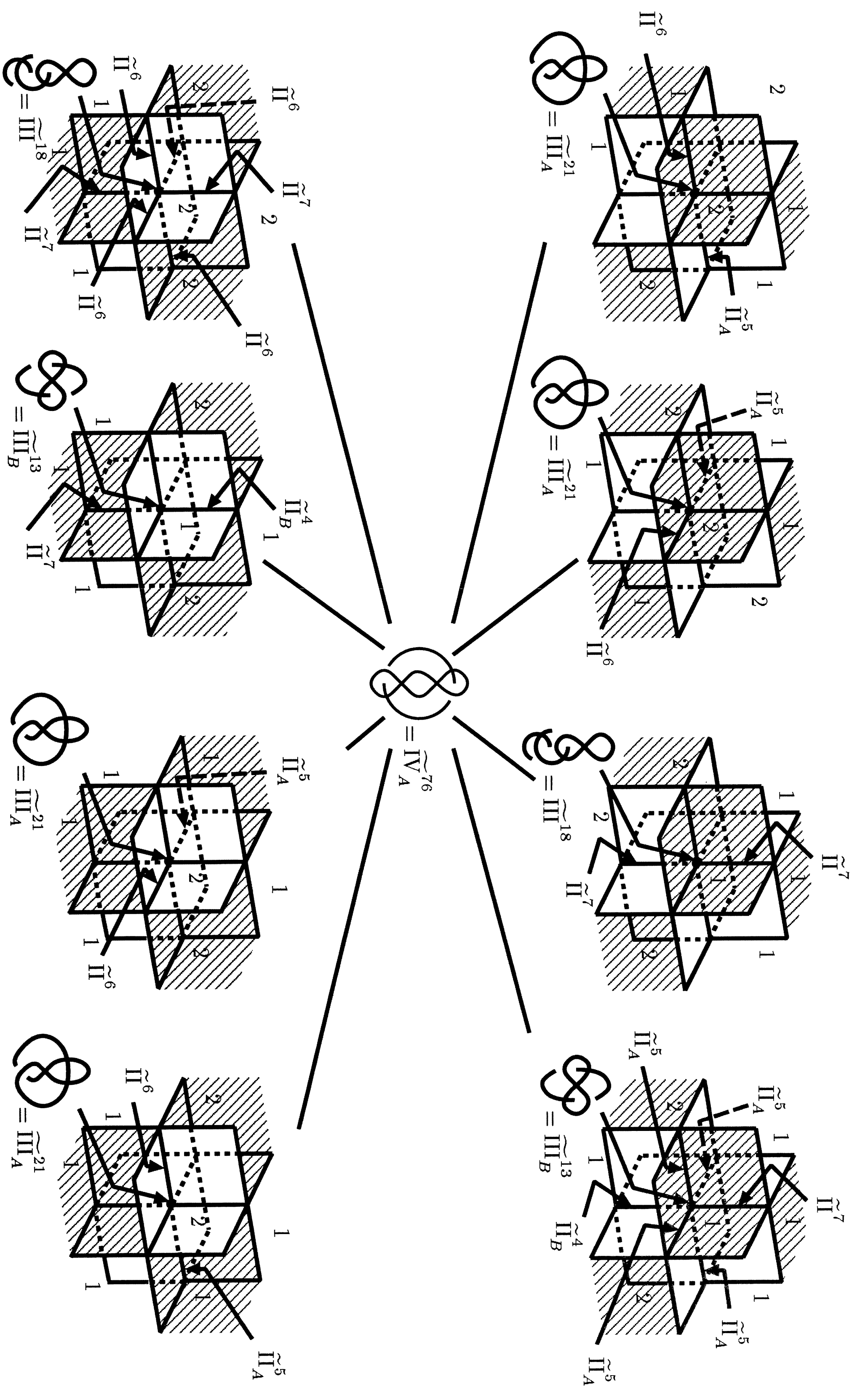

FIGURE 2.97. Type A for $\widetilde{\mathrm{IV}}^{76}$

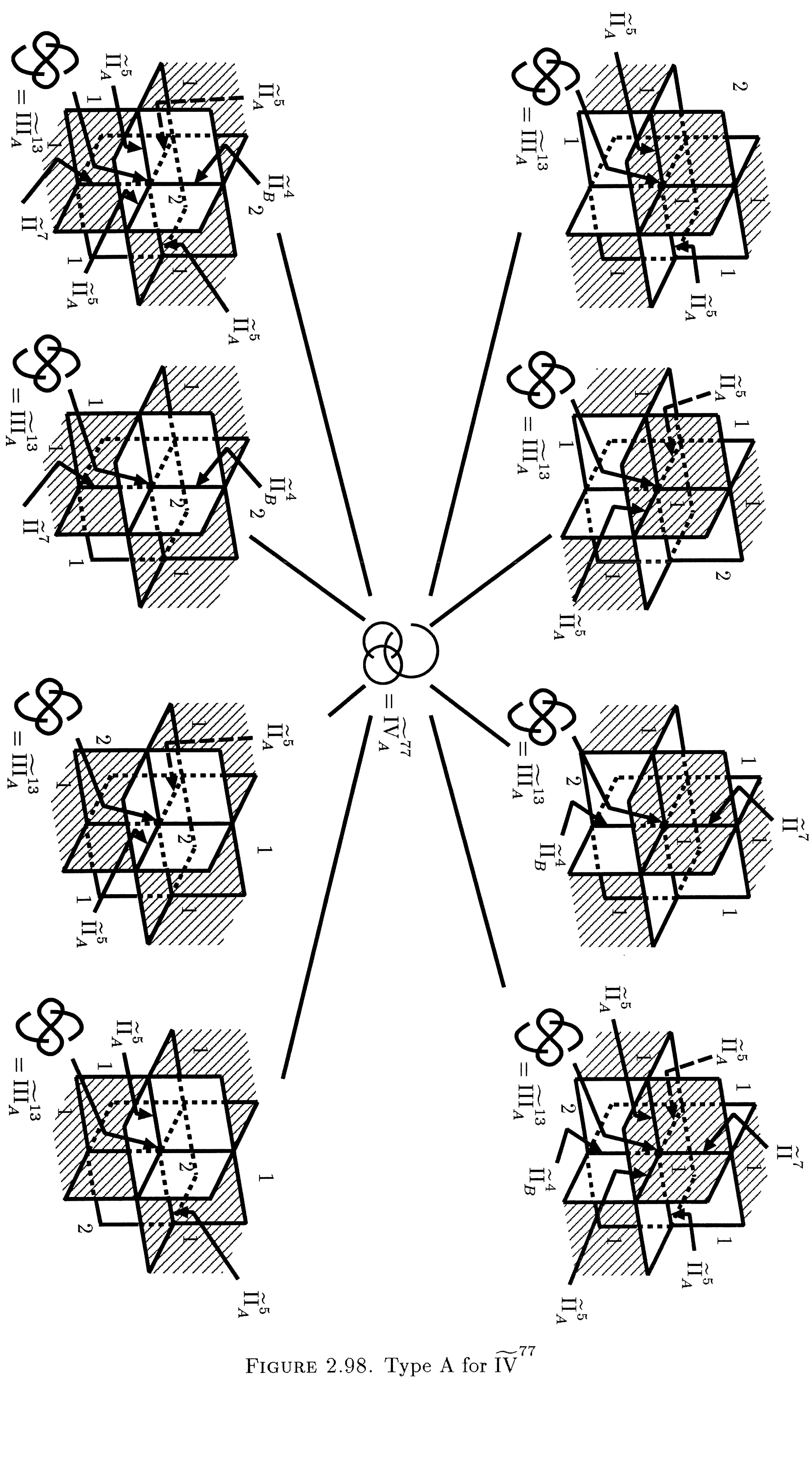

FIGURE 2.98. Type A for $\widetilde{\mathrm{IV}}^{77}$

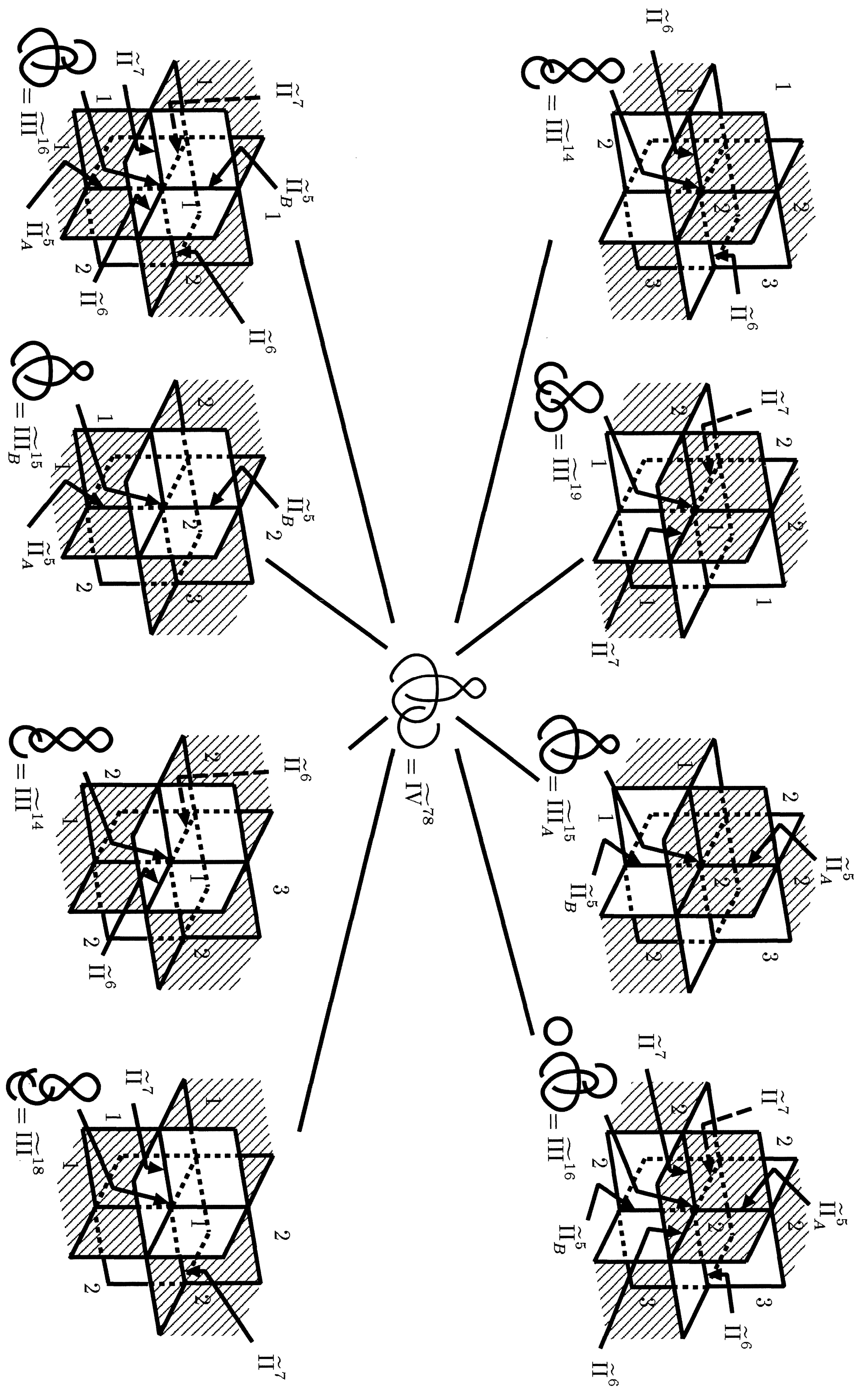

FIGURE 2.99. $\widetilde{\mathrm{IV}}^{78}$ can not divide into two types

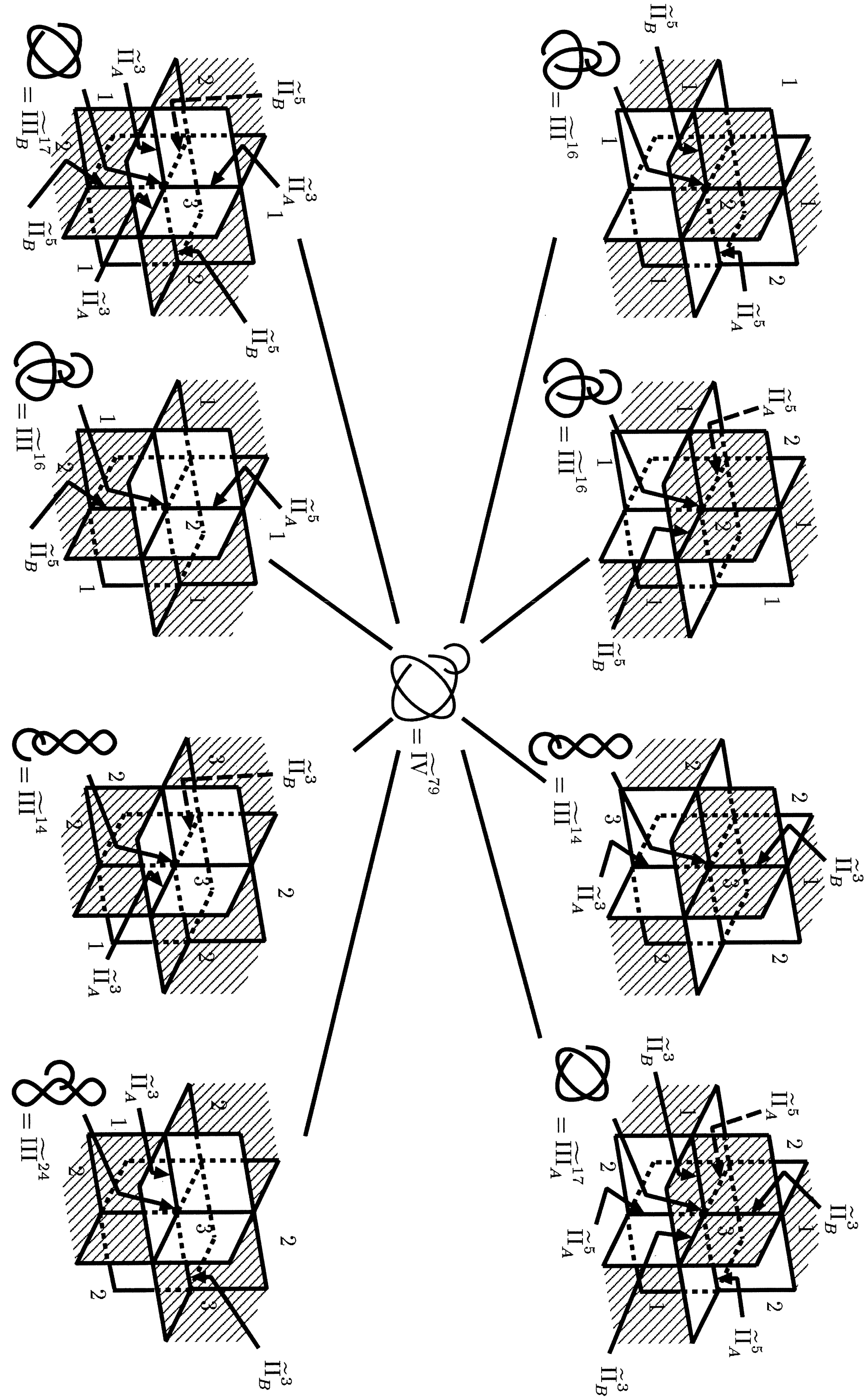

FIGURE 2.100. $\widetilde{\mathrm{IV}}^{79}$ can not divide into two types

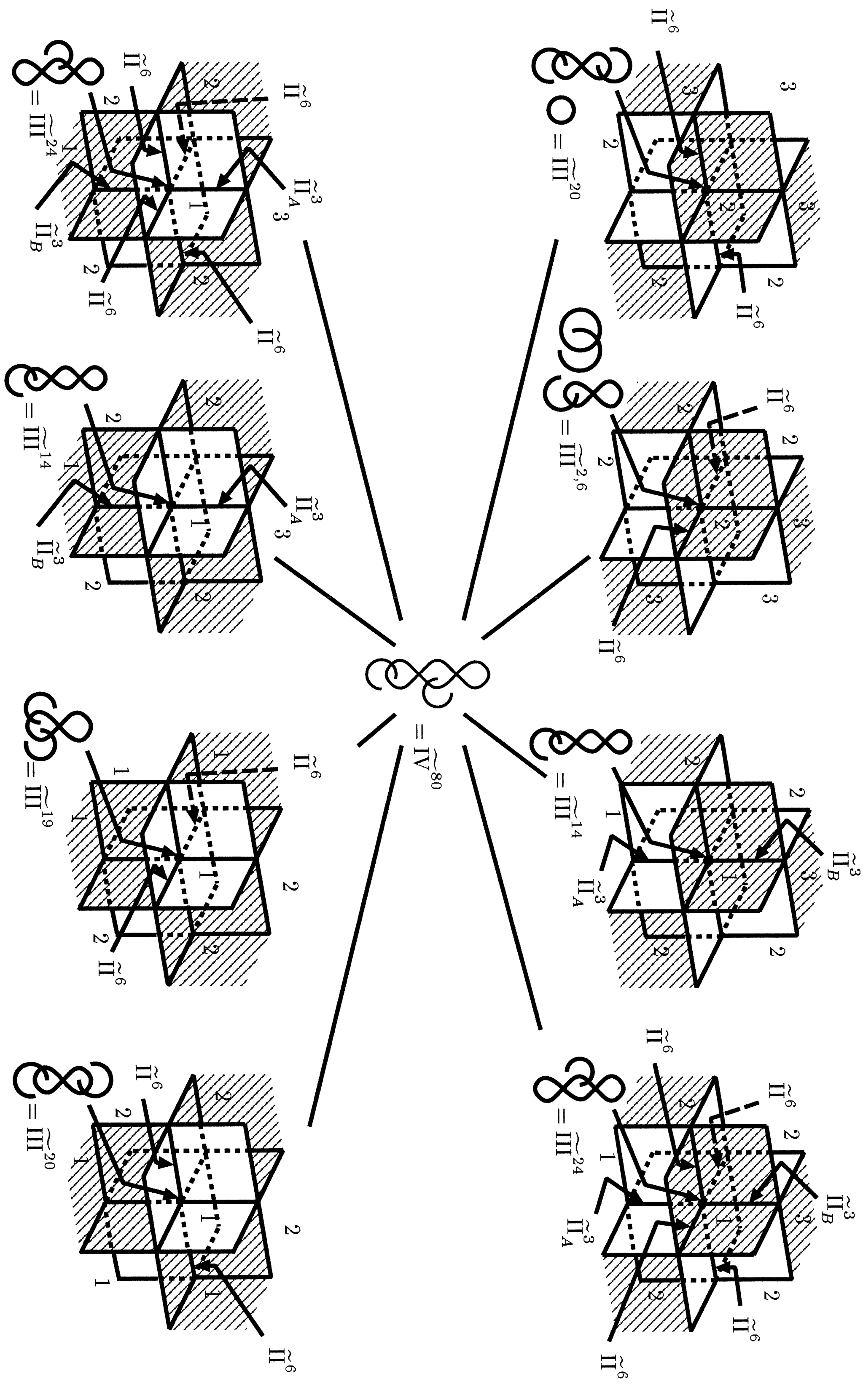

FIGURE 2.101. $\widetilde{\mathrm{IV}}^{80}$ can not divide into two types

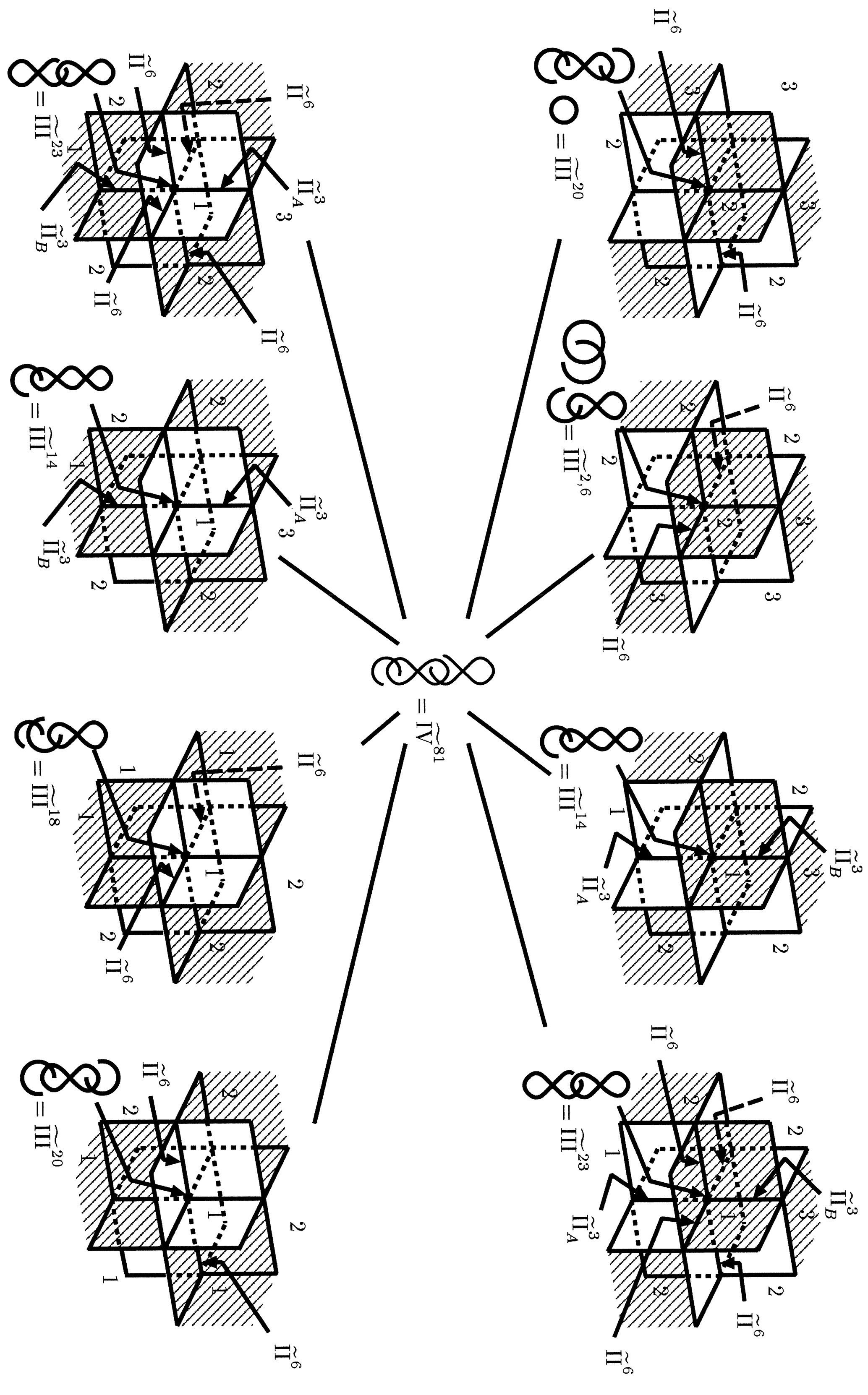

FIGURE 2.102. $\widetilde{\mathrm{IV}}^{81}$ can not divide into two types

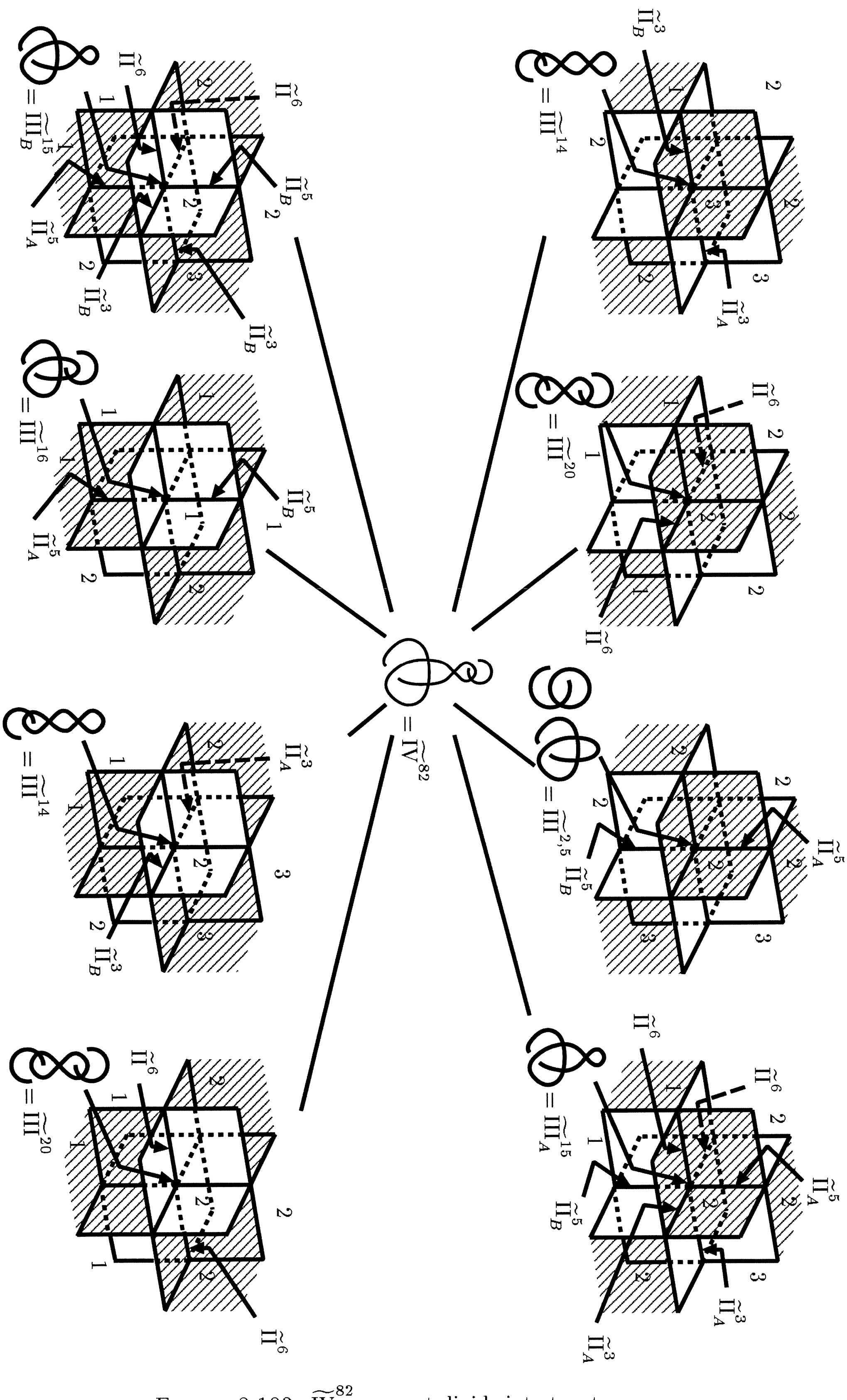

FIGURE 2.103. $\widetilde{\mathrm{IV}}^{82}$ can not divide into two types

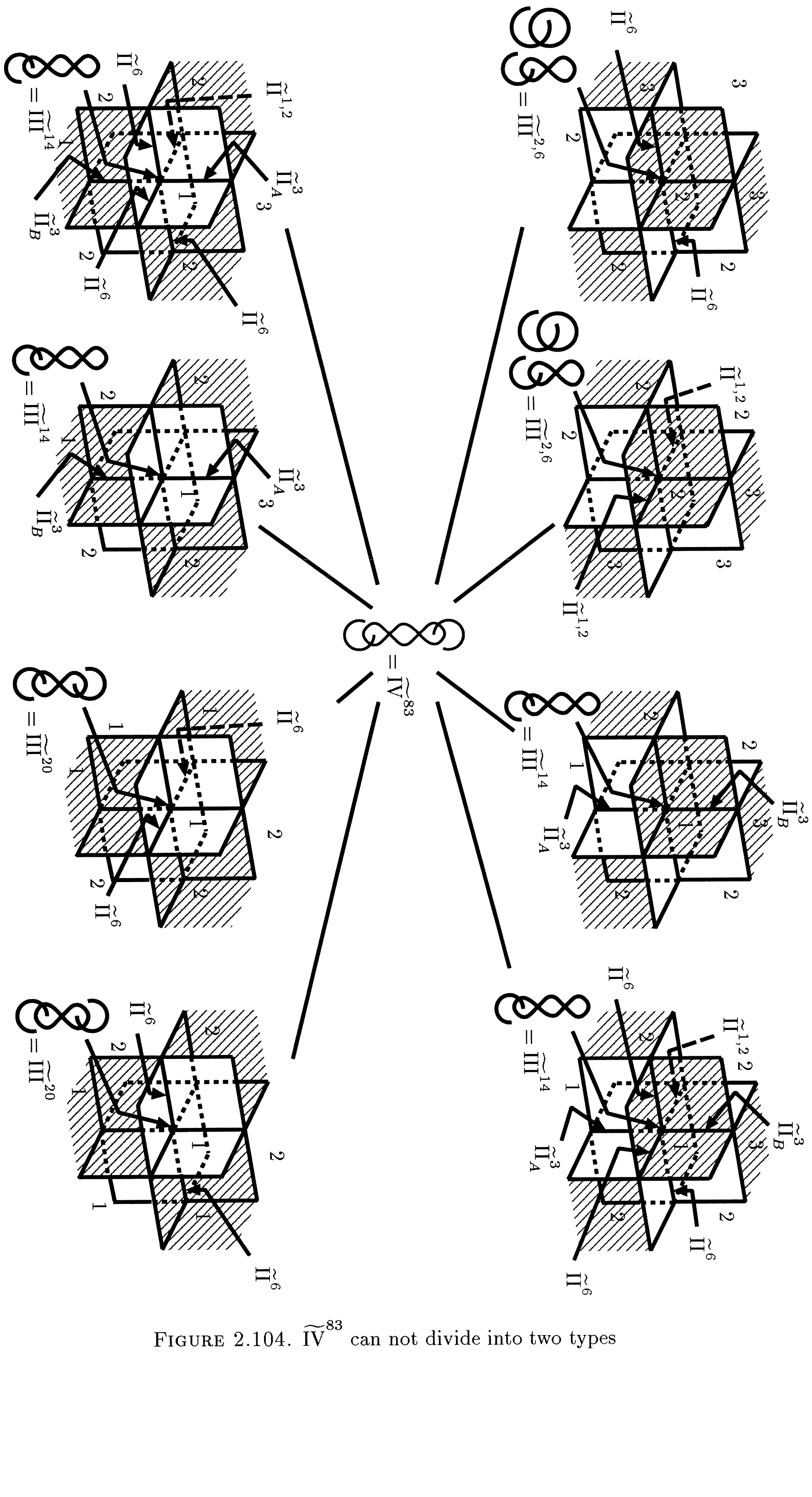

FIGURE 2.104. $\widetilde{\mathrm{IV}}^{83}$ can not divide into two types

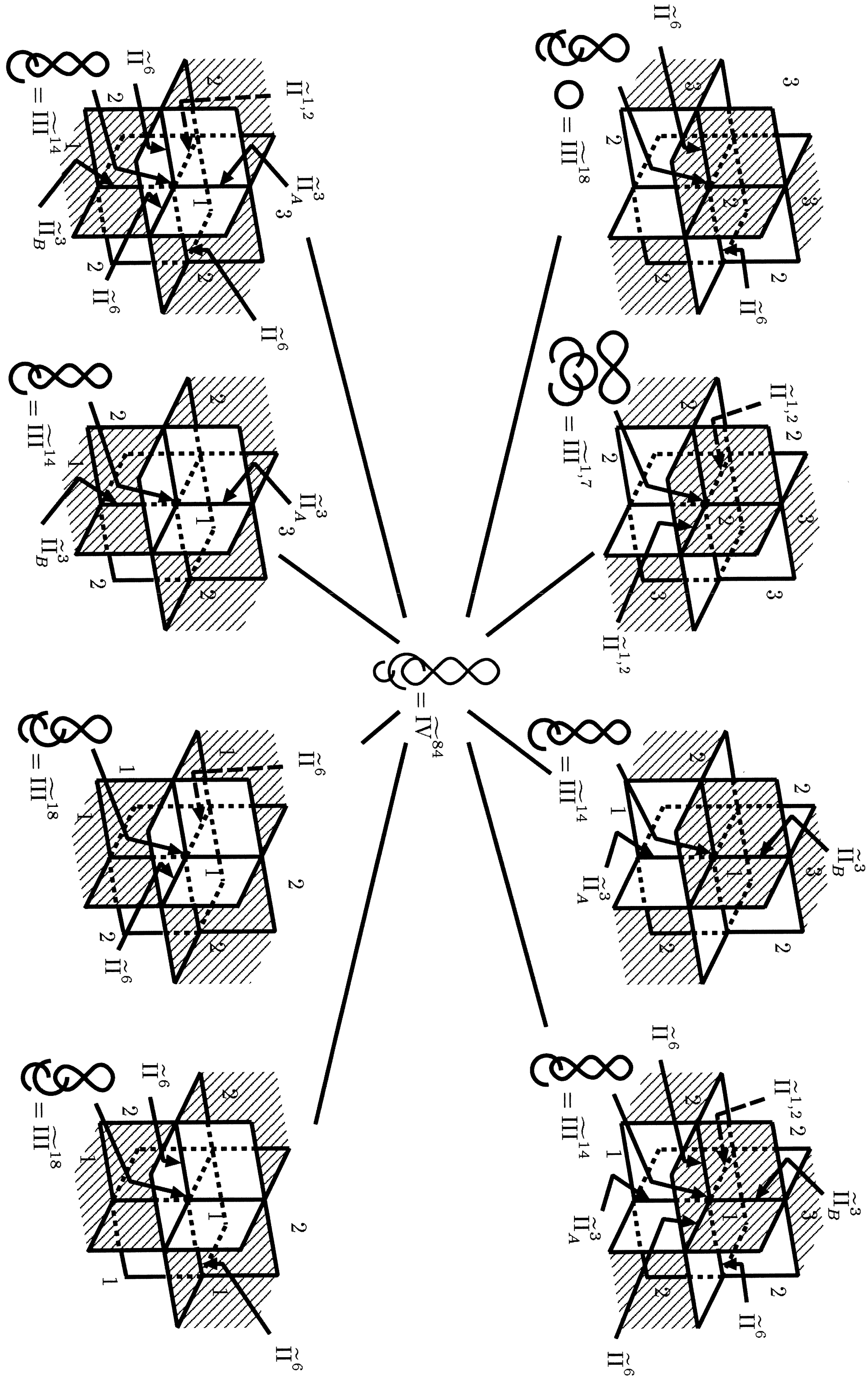

FIGURE 2.105. $\widetilde{IV}^{84}$ can not divide into two types

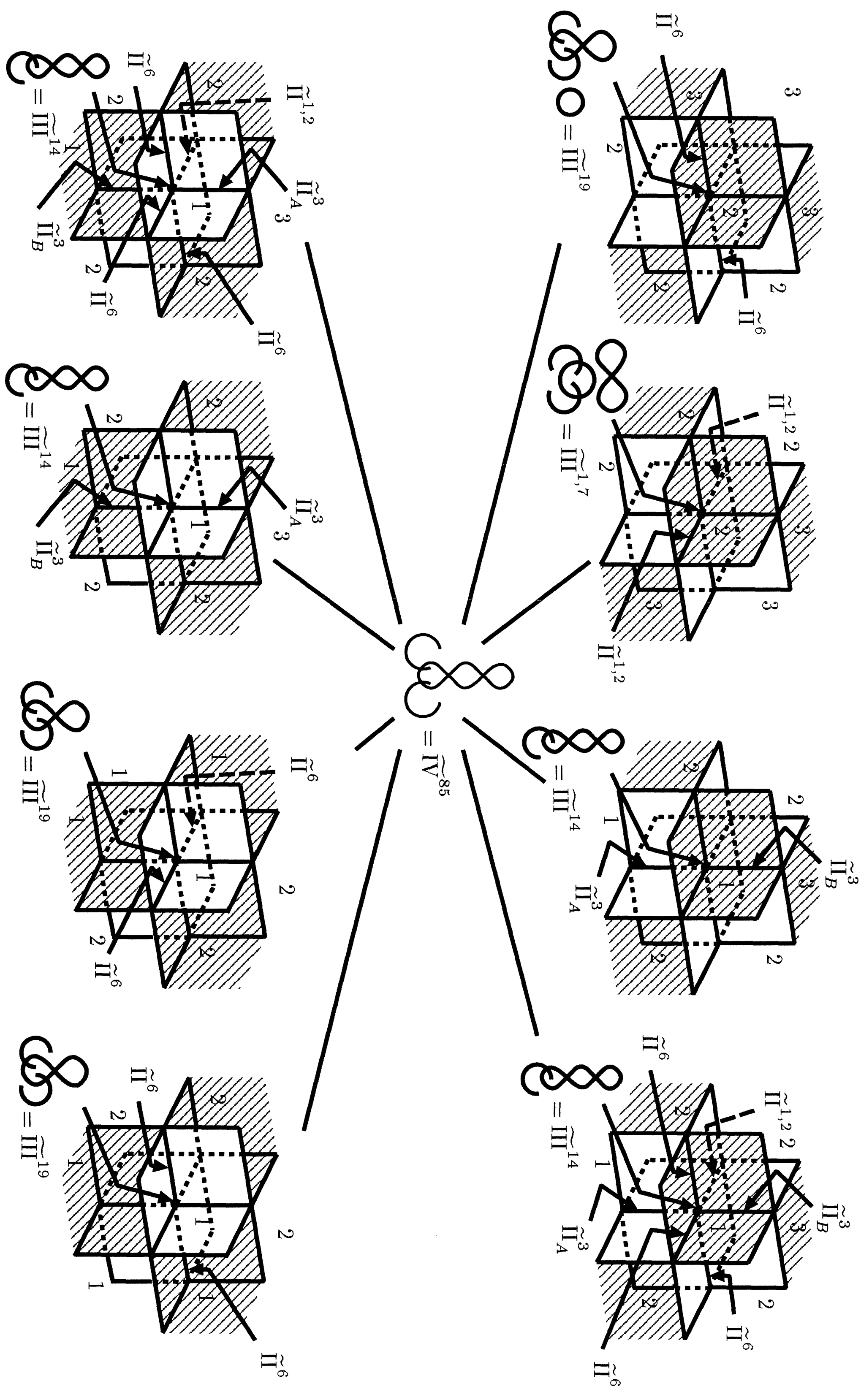

FIGURE 2.106. $\widetilde{\mathrm{IV}}^{85}$ can not divide into two types

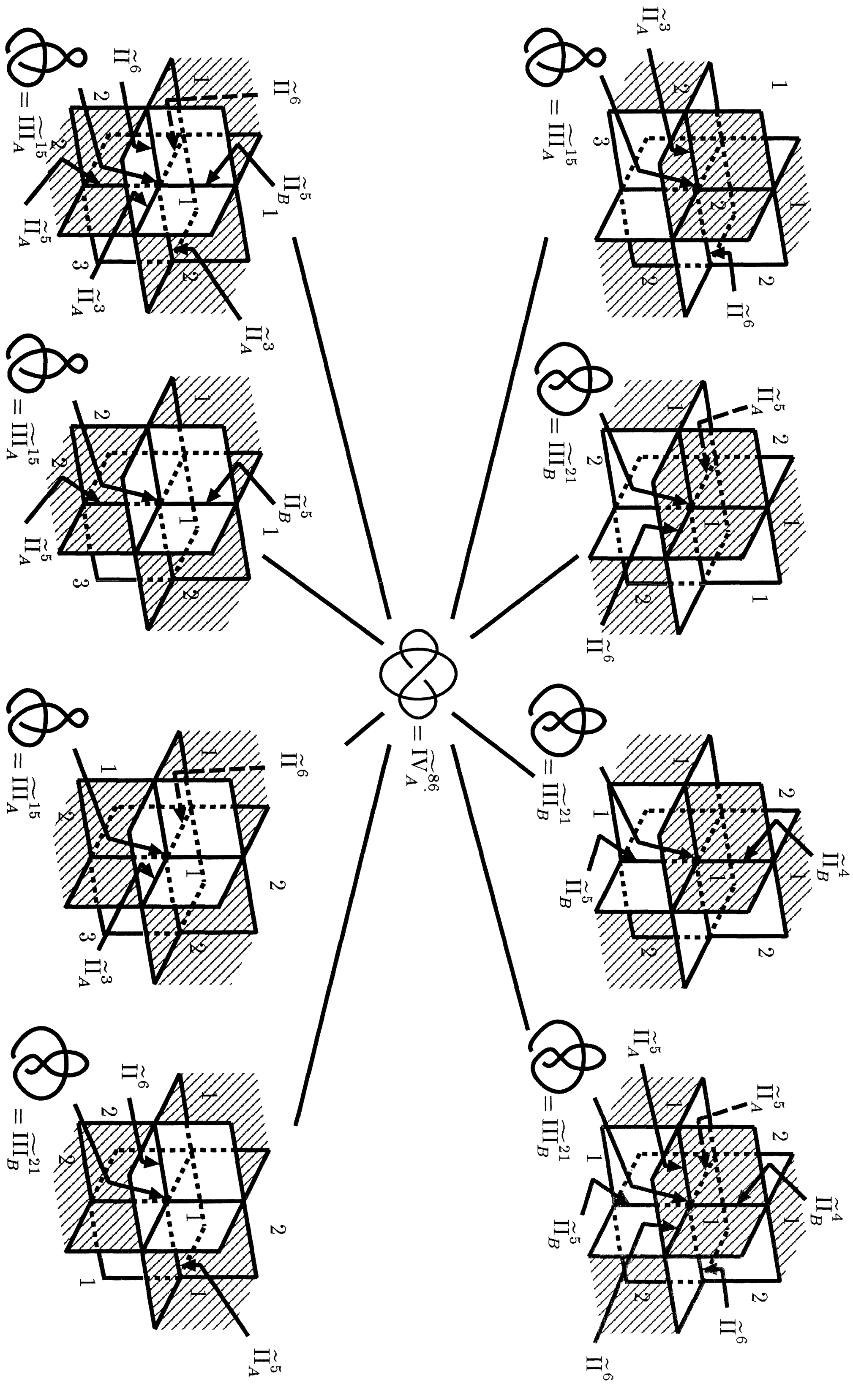

FIGURE 2.107. Type A for $\widetilde{\mathrm{IV}}^{86}$

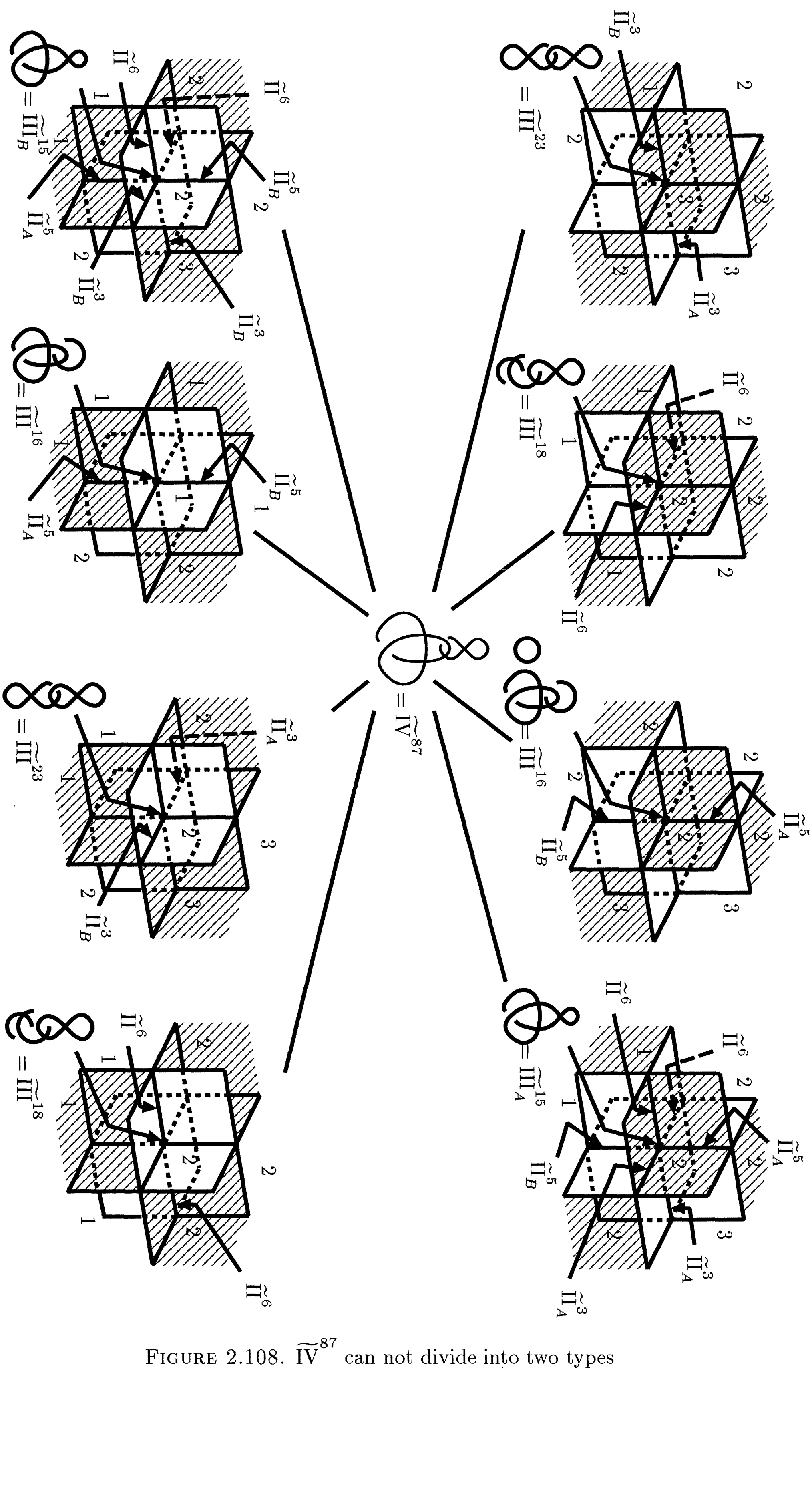

FIGURE 2.108. $\widetilde{\mathrm{IV}}^{87}$ can not divide into two types

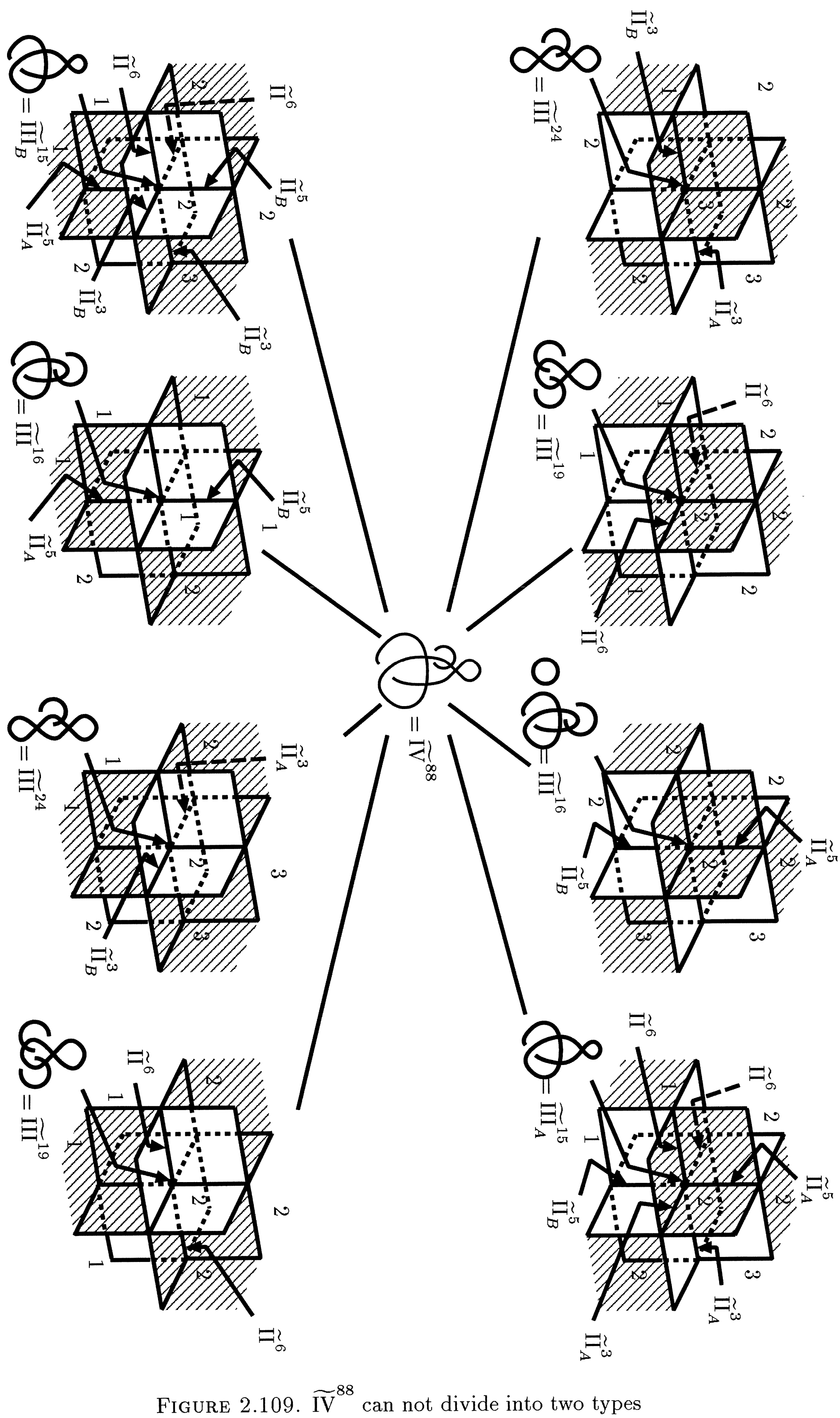

FIGURE 2.109. $\widetilde{\mathrm{IV}}^{88}$ can not divide into two types

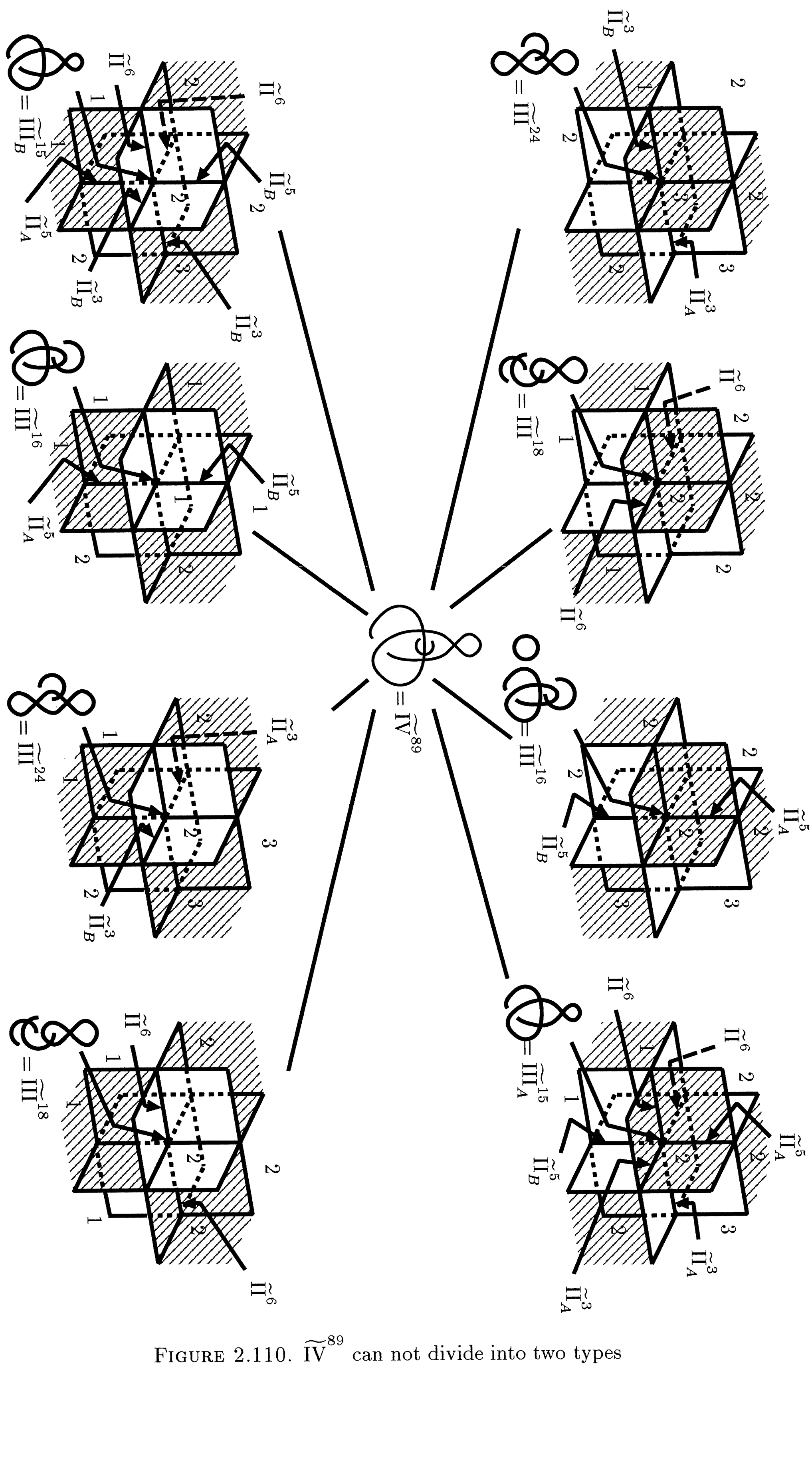

FIGURE 2.110. $\widetilde{\mathrm{IV}}^{89}$ can not divide into two types

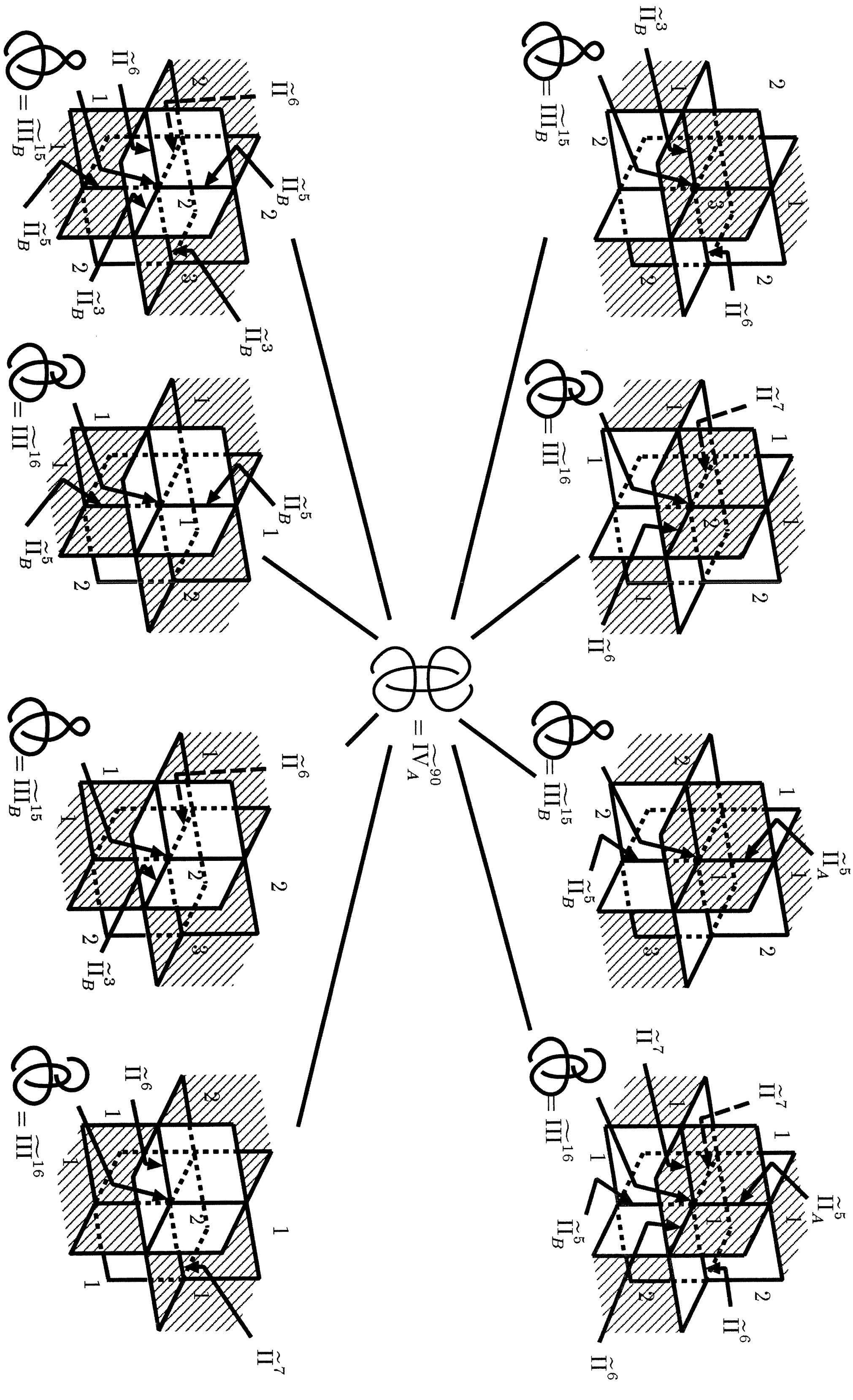

FIGURE 2.111. Type A for $\widetilde{\mathrm{IV}}^{90}$

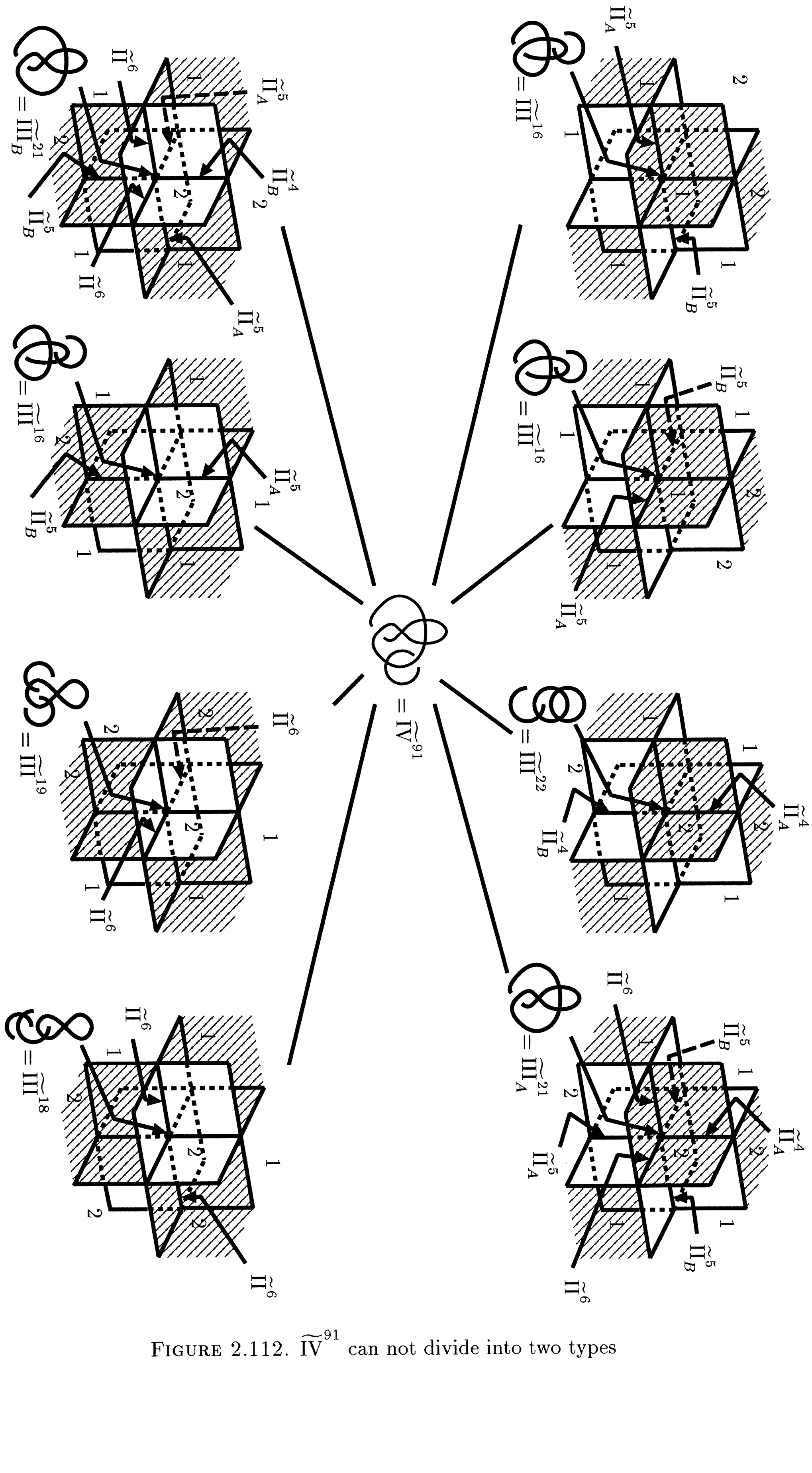

FIGURE 2.112. $\widetilde{\mathrm{IV}}^{91}$ can not divide into two types

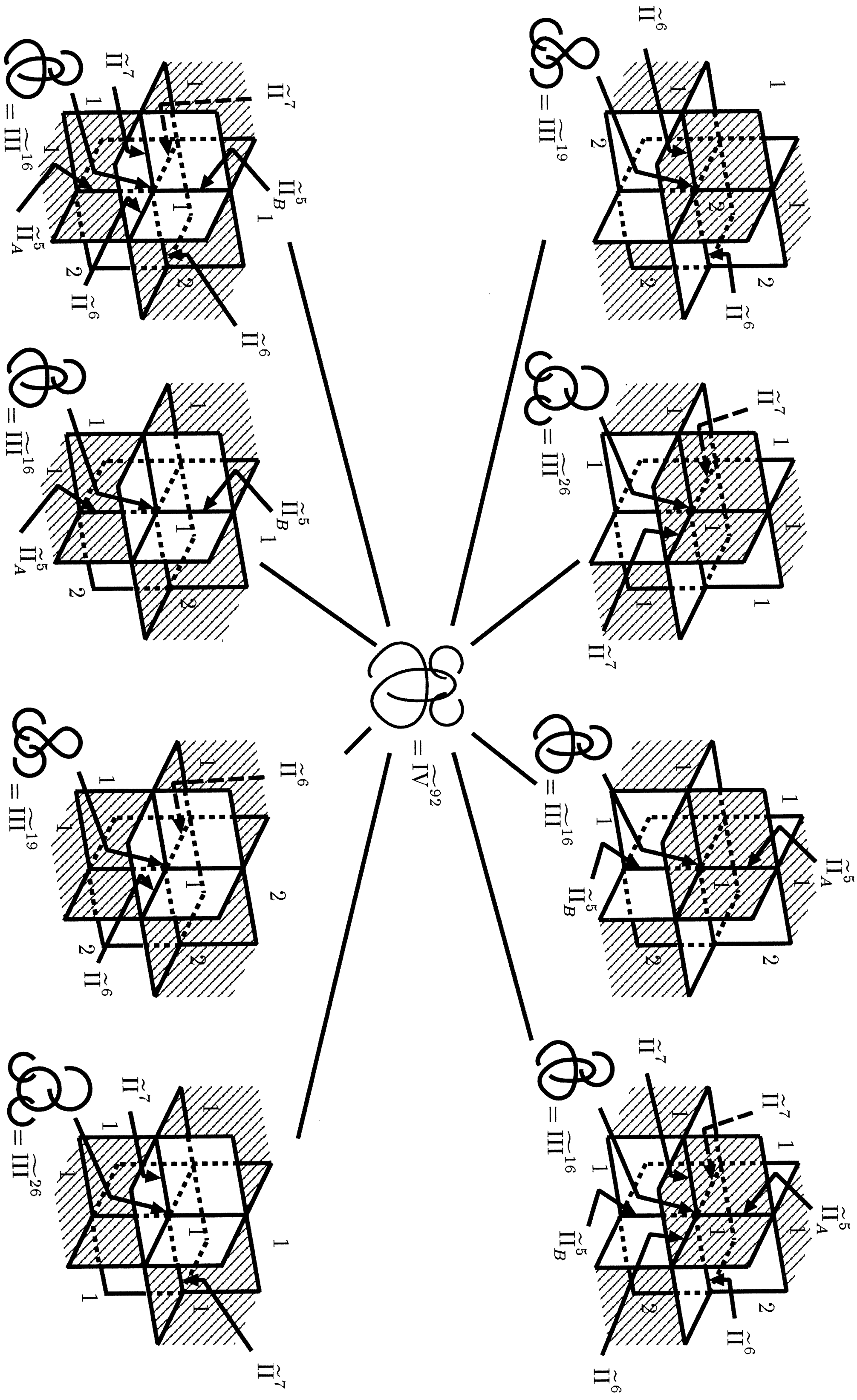

FIGURE 2.113. $\widetilde{\mathrm{IV}}^{92}$ can not divide into two types

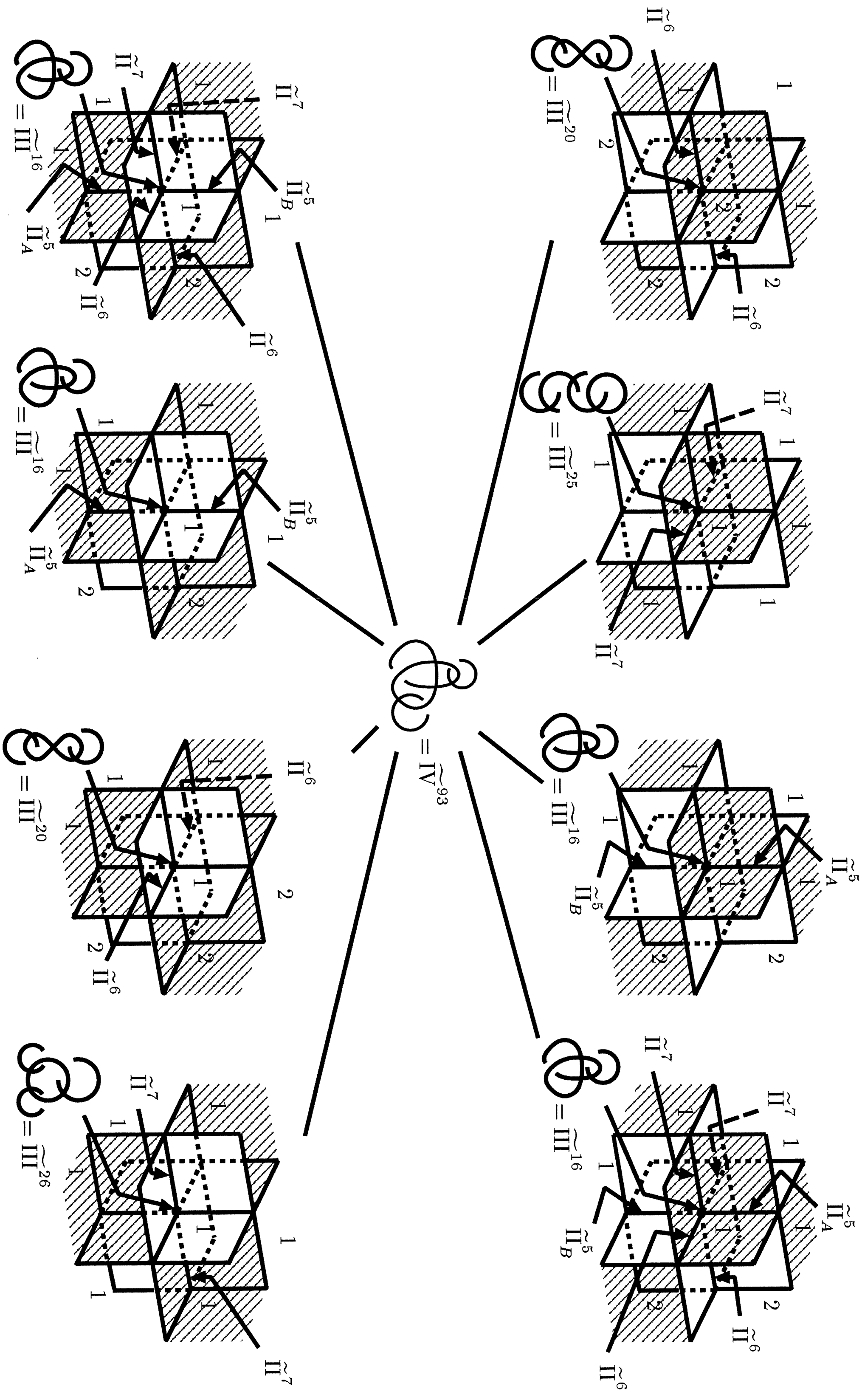

FIGURE 2.114. $\widetilde{\mathrm{IV}}^{93}$ can not divide into two types

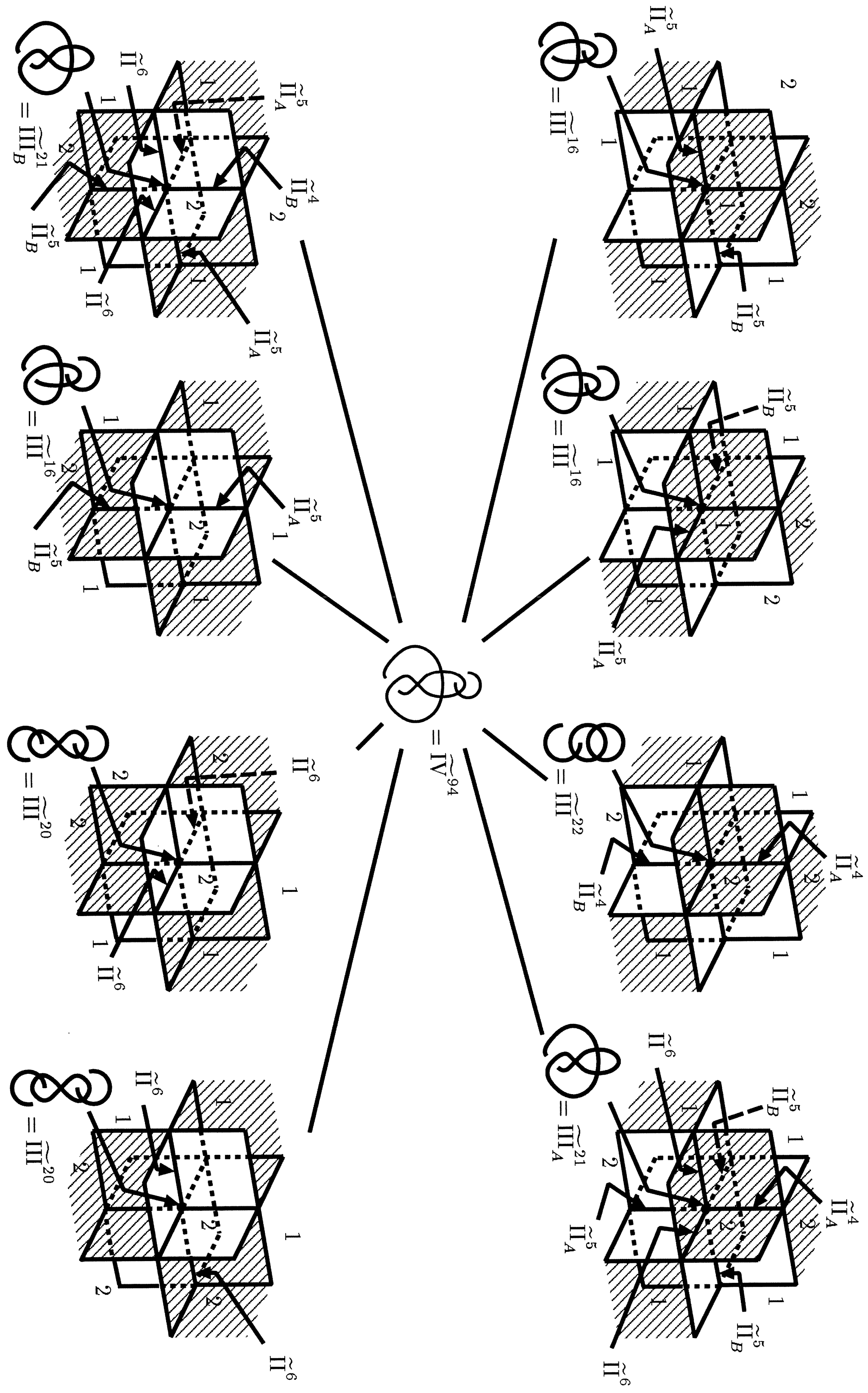

FIGURE 2.115. $\widetilde{\mathrm{IV}}^{94}$ can not divide into two types

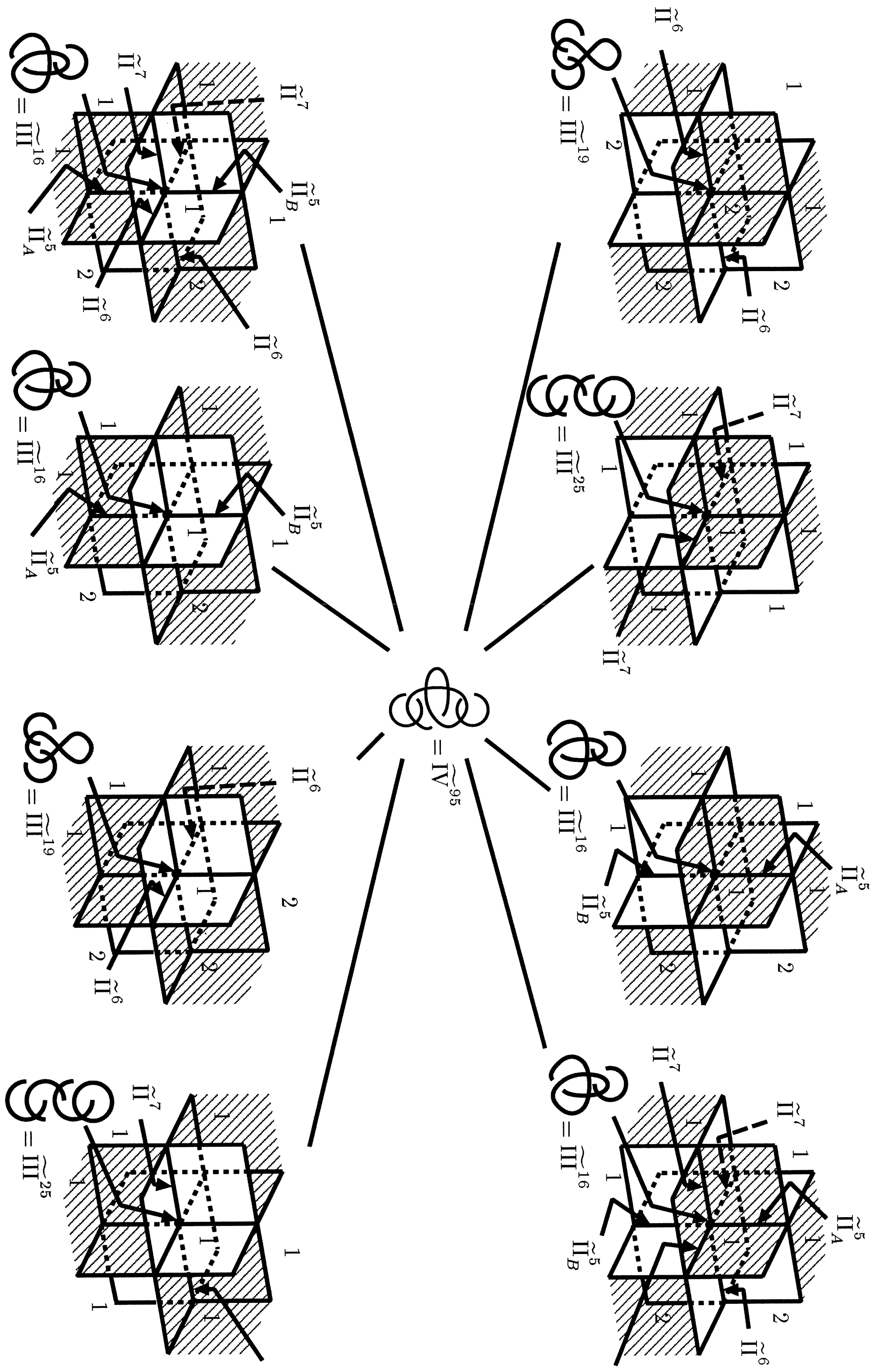

FIGURE 2.116. $\widetilde{\mathrm{IV}}^{95}$ can not divide into two types

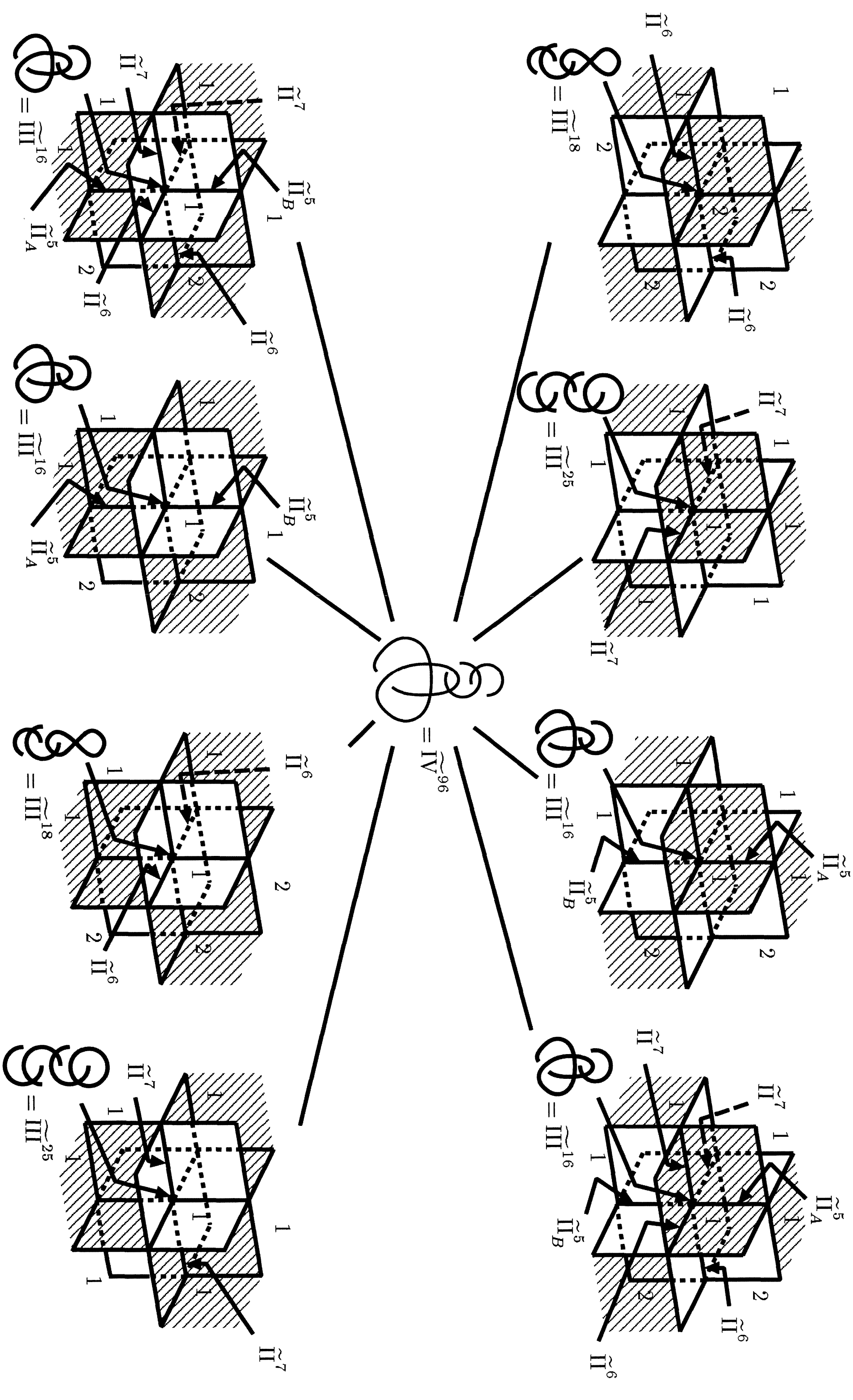

FIGURE 2.117. $\widetilde{\mathrm{IV}}^{96}$ can not divide into two types

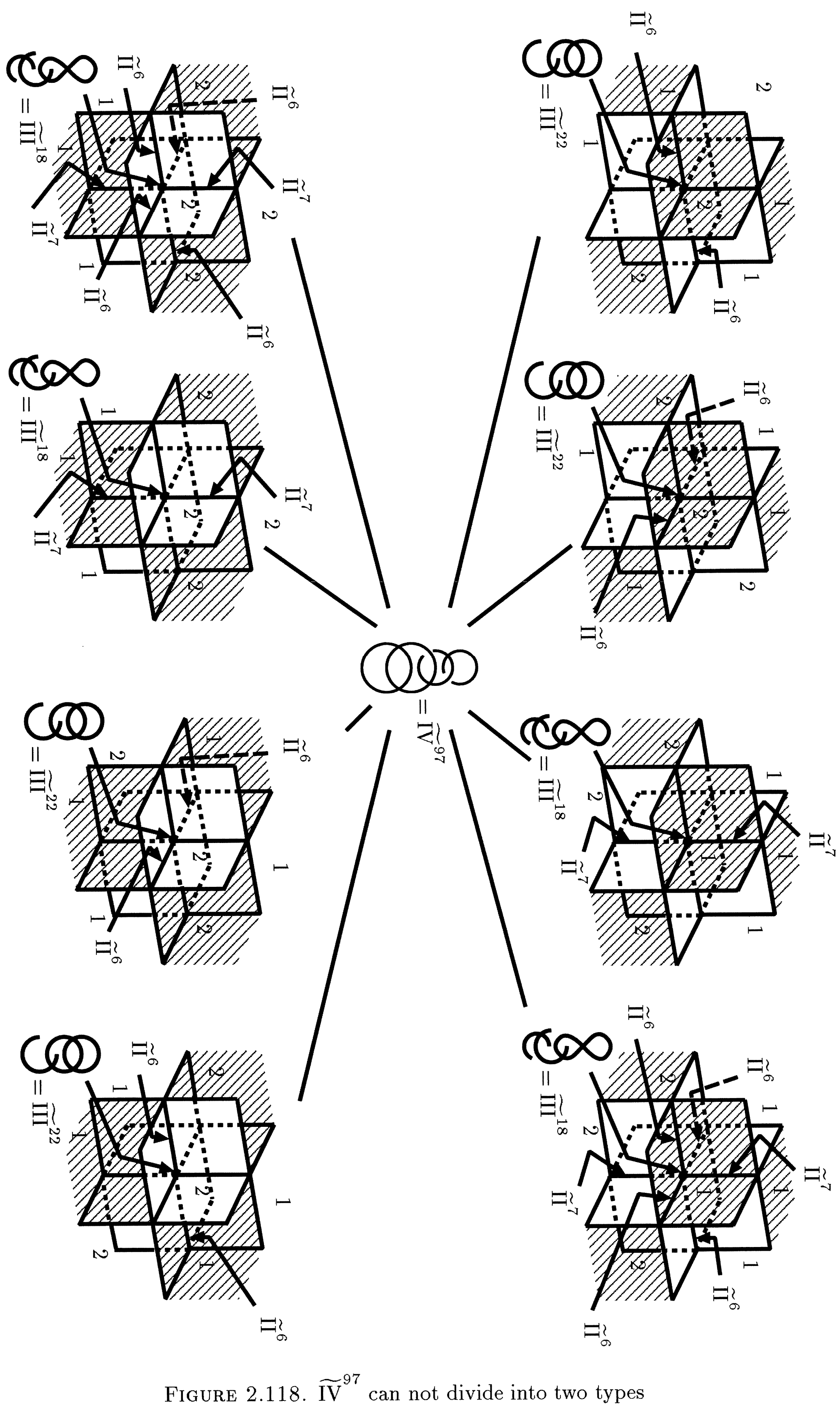

FIGURE 2.118. $\widetilde{\mathrm{IV}}^{97}$ can not divide into two types

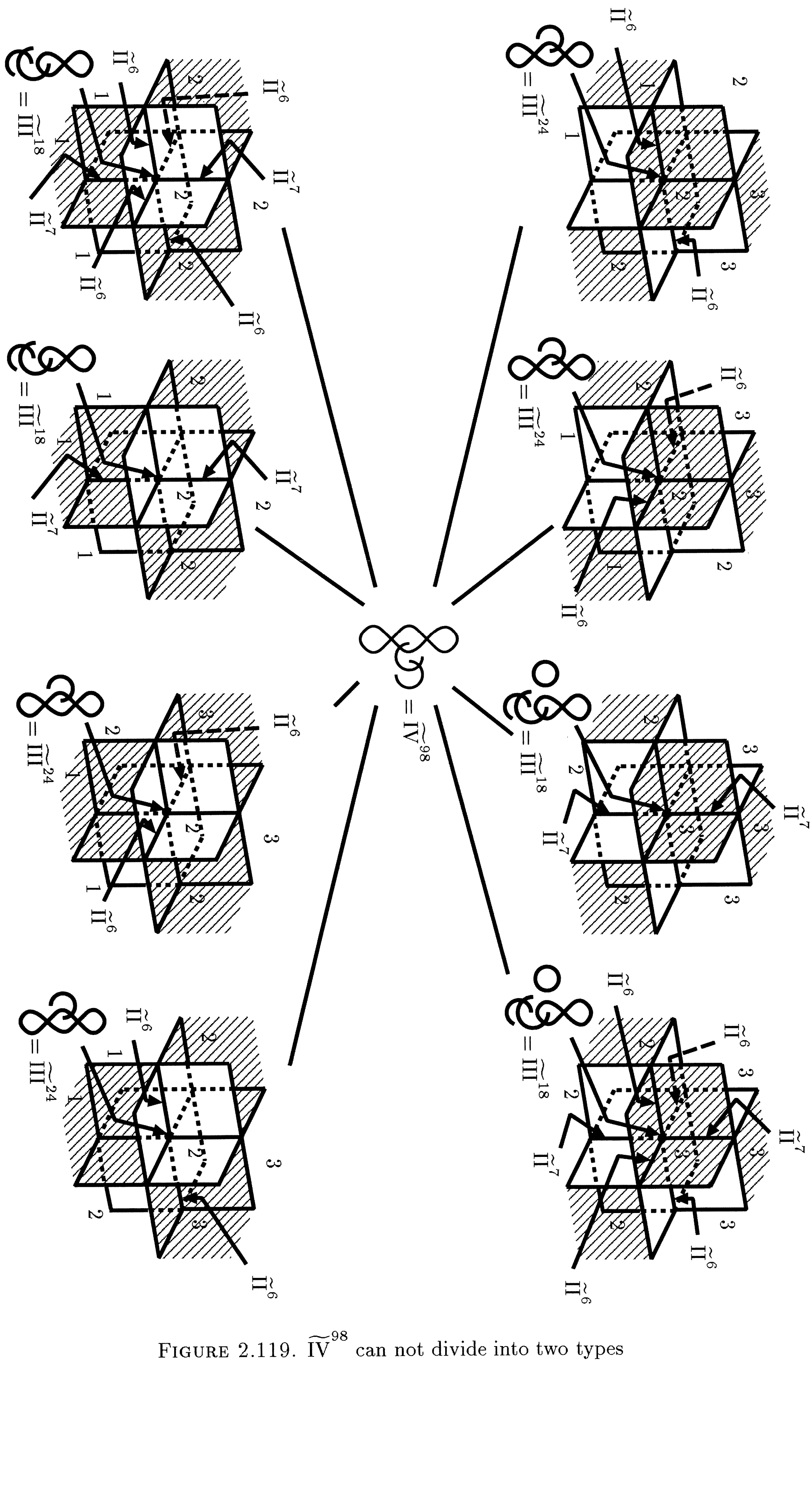

FIGURE 2.119. $\widetilde{\mathrm{IV}}^{98}$ can not divide into two types

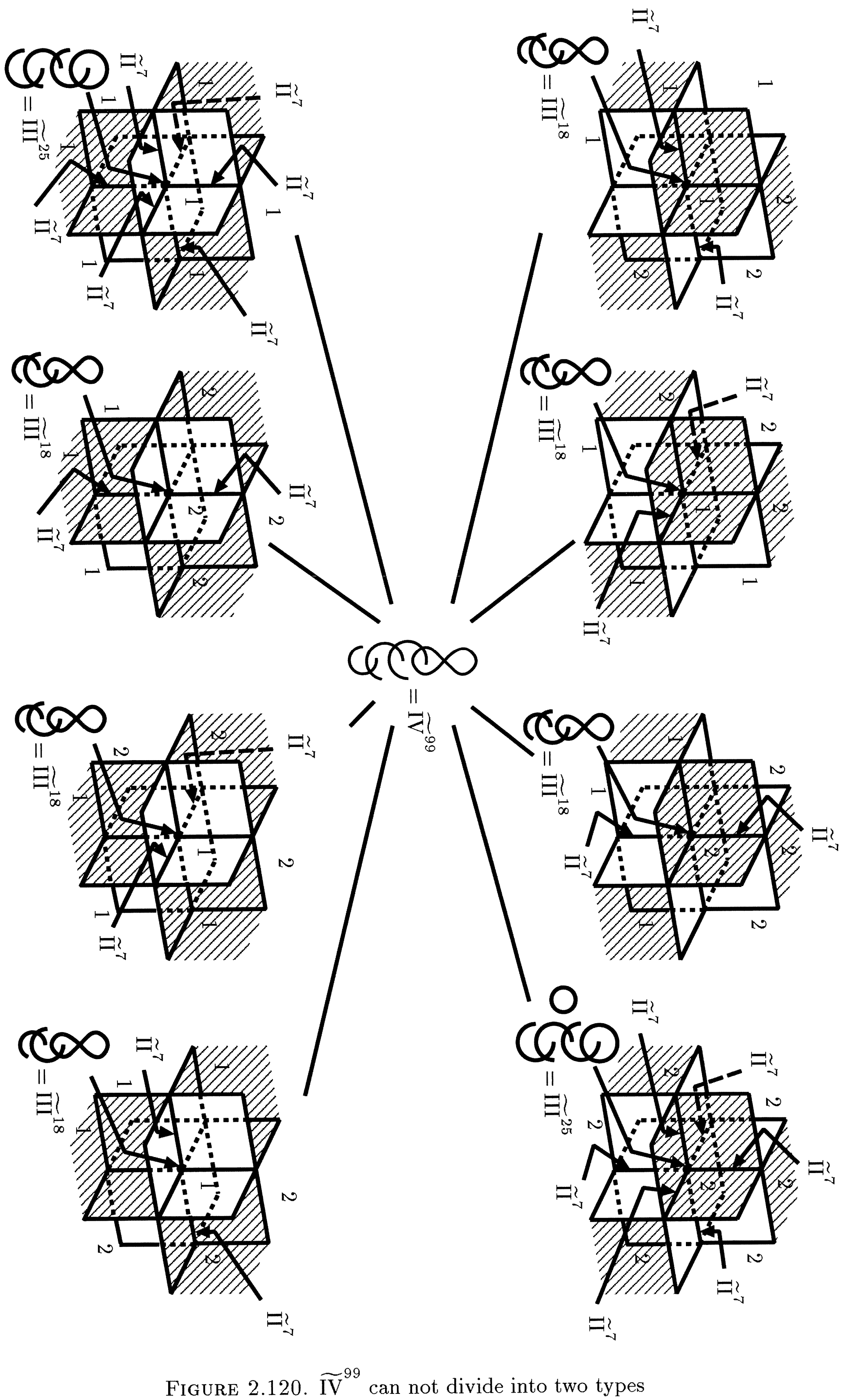

FIGURE 2.120. $\widetilde{\mathrm{IV}}^{99}$ can not divide into two types

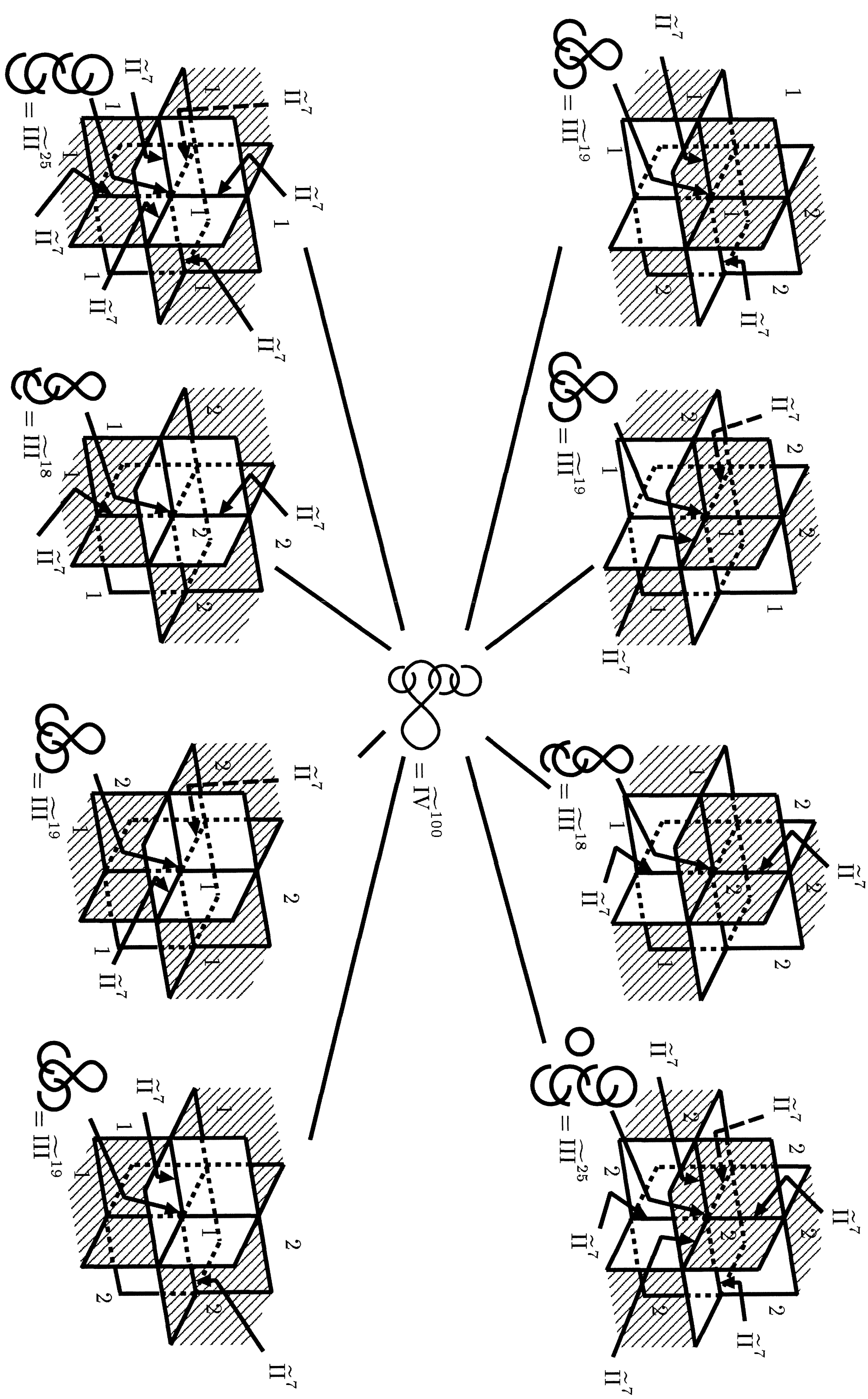

FIGURE 2.121. $\widetilde{\text{IV}}^{100}$ can not divide into two types

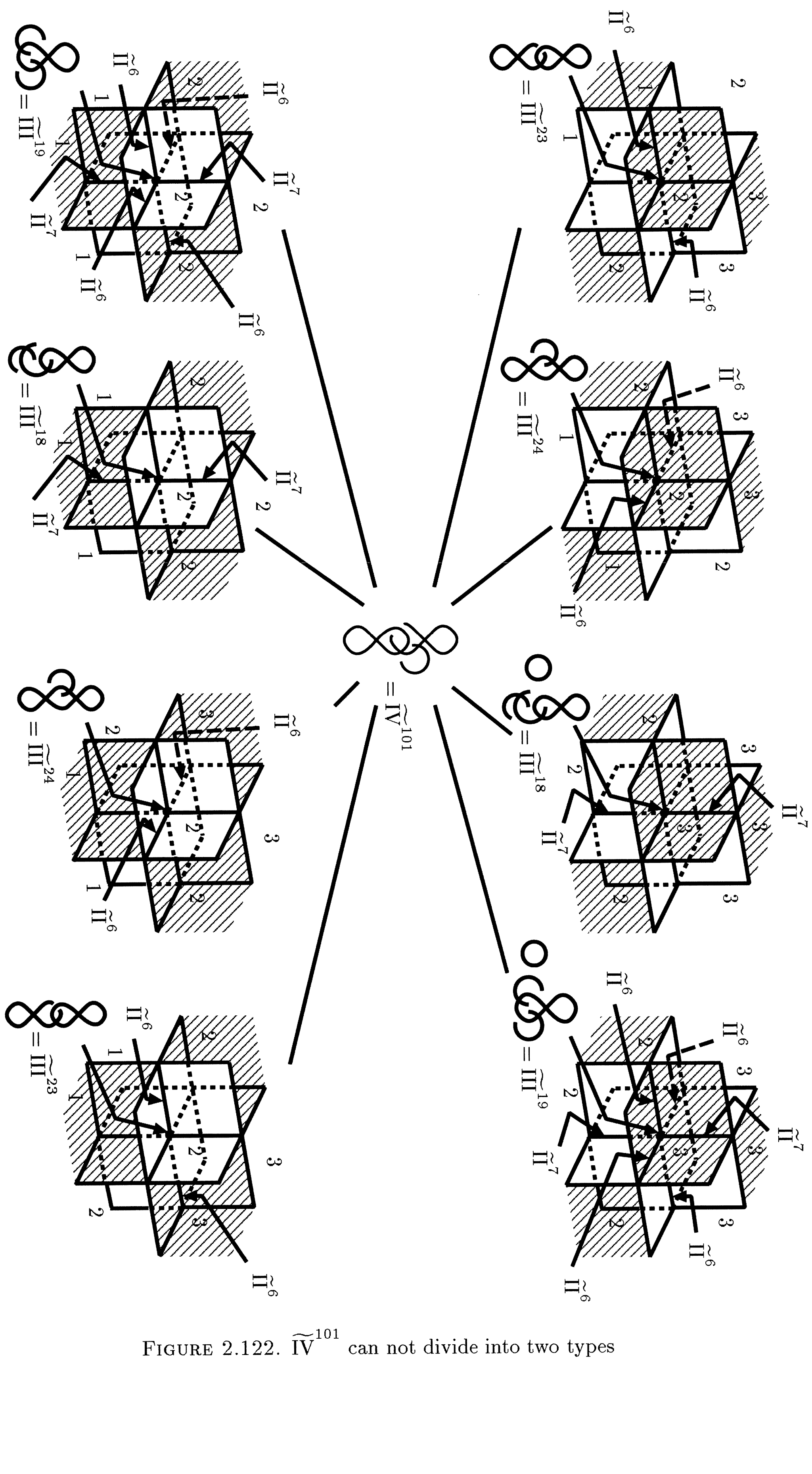

FIGURE 2.122. $\widetilde{\mathrm{IV}}^{101}$ can not divide into two types

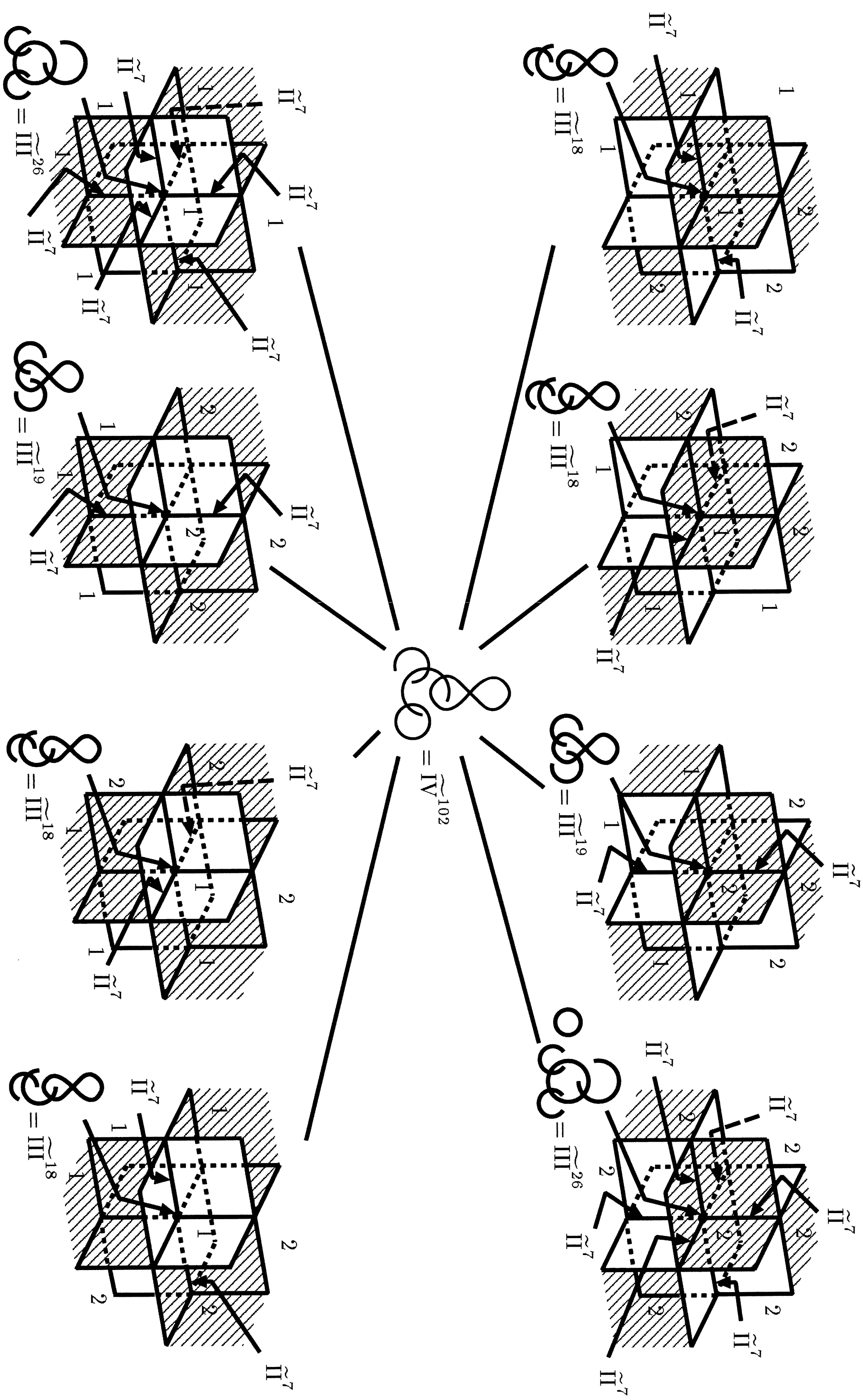

FIGURE 2.123. $\widetilde{IV}^{102}$ can not divide into two types

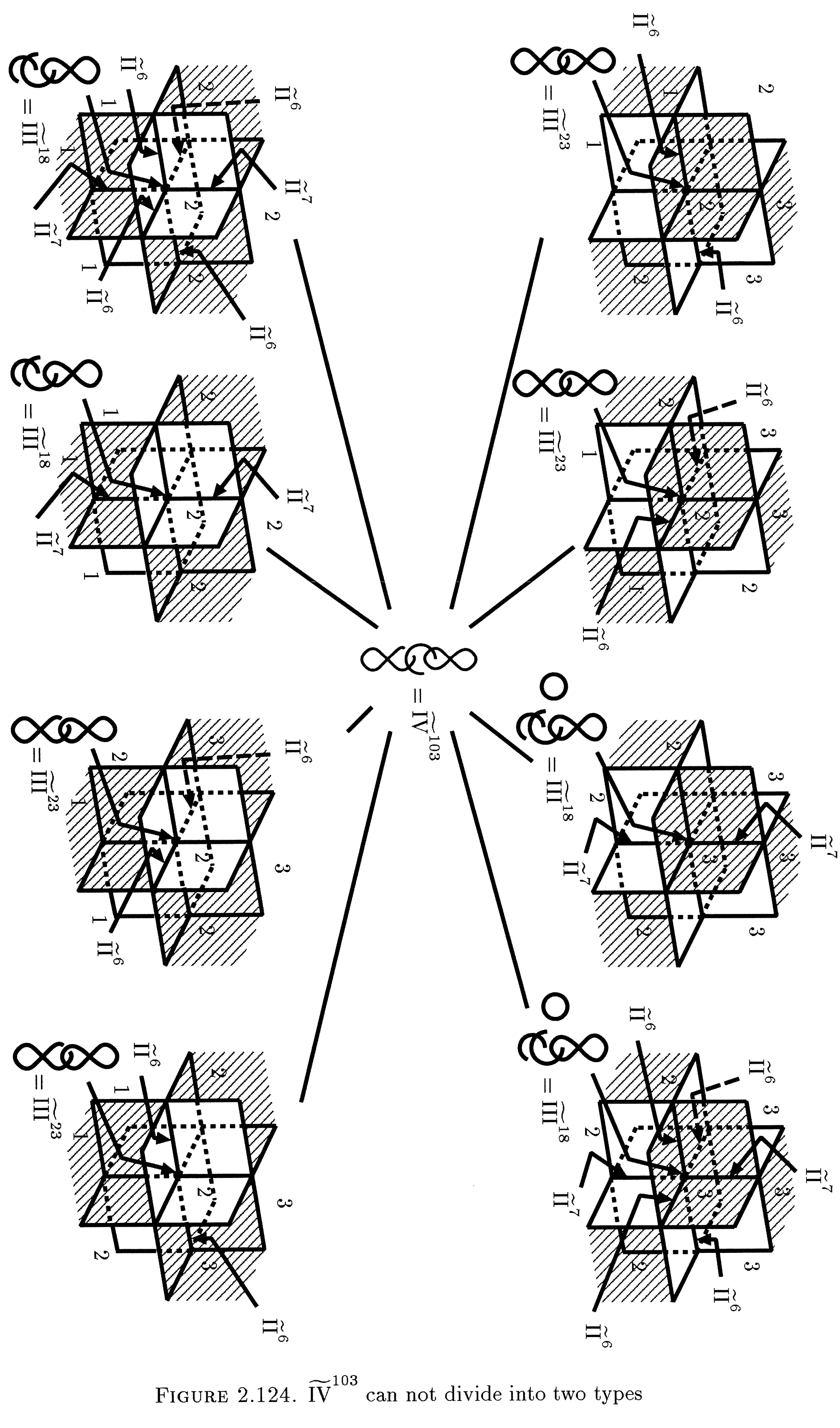

FIGURE 2.124. $\widetilde{\mathrm{IV}}^{103}$ can not divide into two types

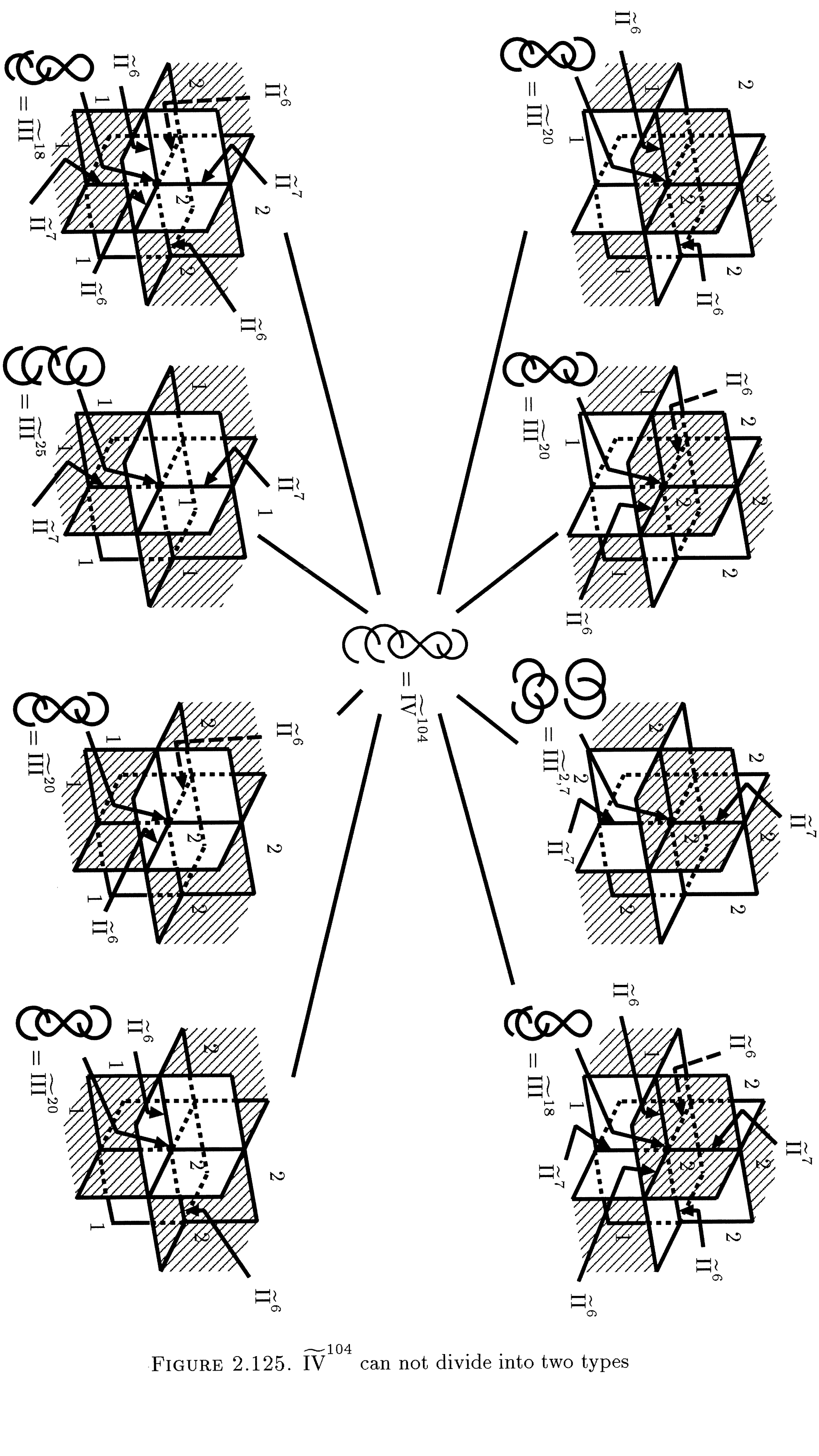

FIGURE 2.125. $\widetilde{\mathrm{IV}}^{104}$ can not divide into two types

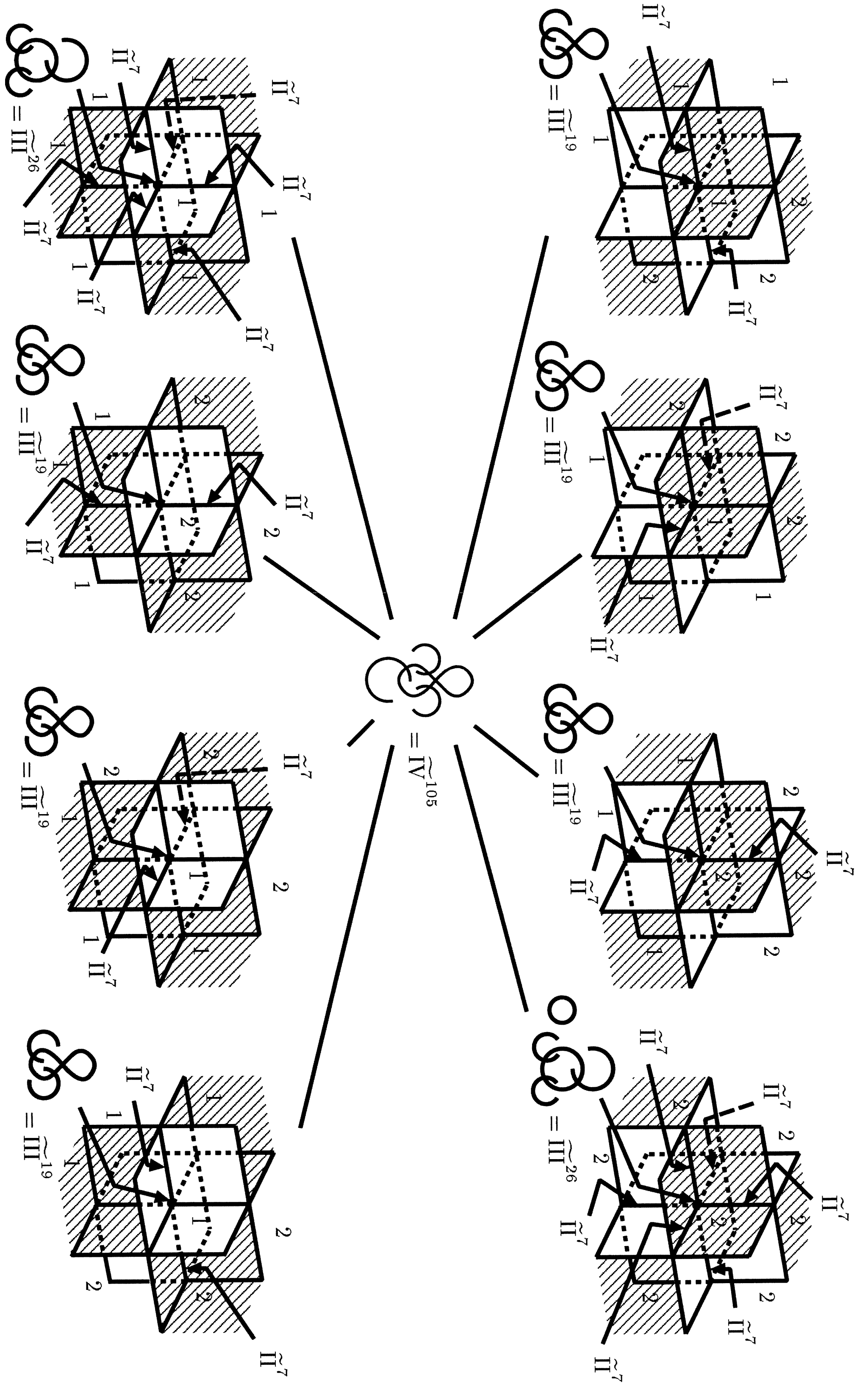

FIGURE 2.126. $\widetilde{\mathrm{IV}}^{105}$ can not divide into two types

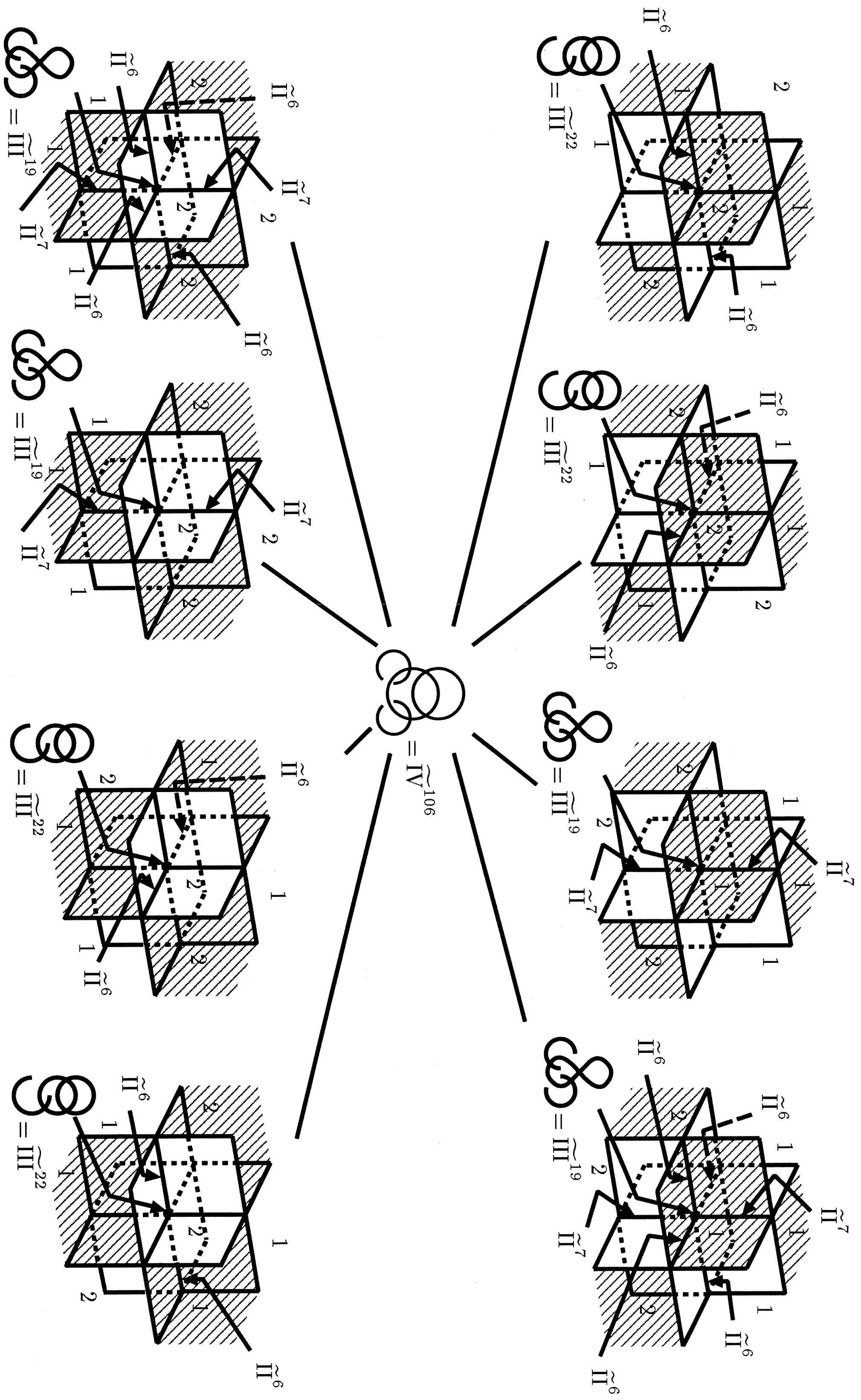

FIGURE 2.127. $\widetilde{\mathrm{IV}}^{106}$ can not divide into two types

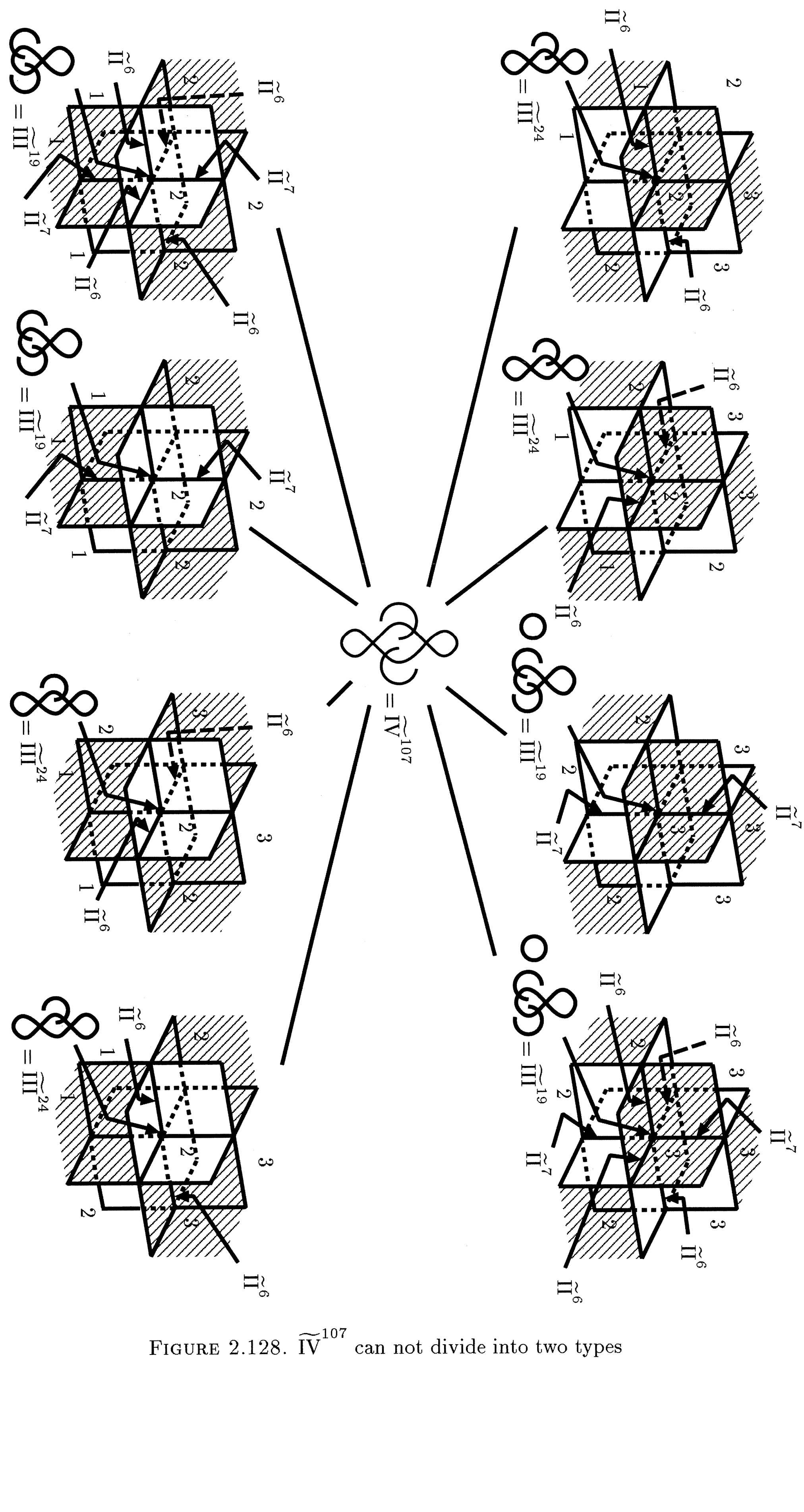

FIGURE 2.128. $\widetilde{\mathrm{IV}}^{107}$ can not divide into two types

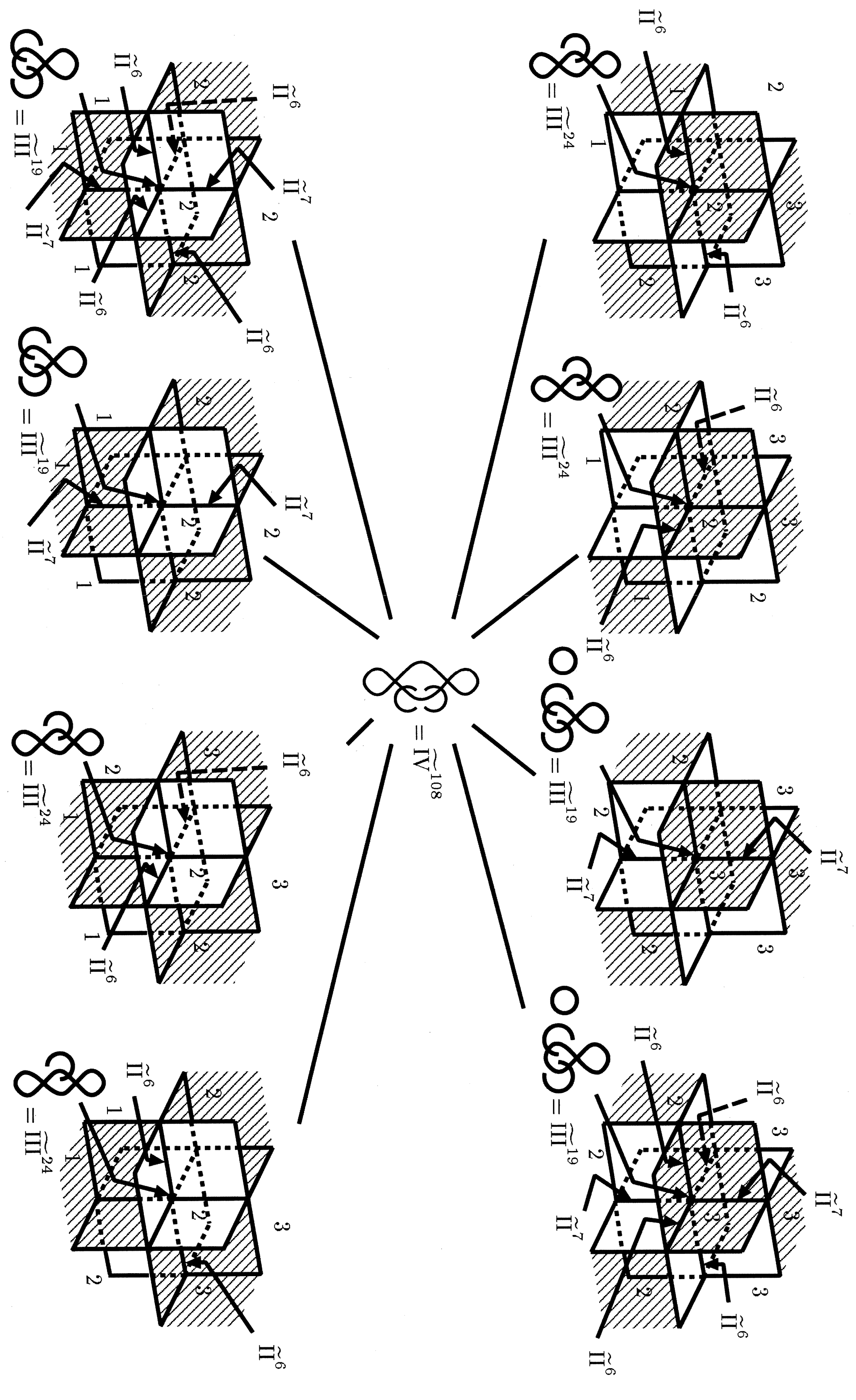

FIGURE 2.129. $\widetilde{\mathrm{IV}}^{108}$ can not divide into two types

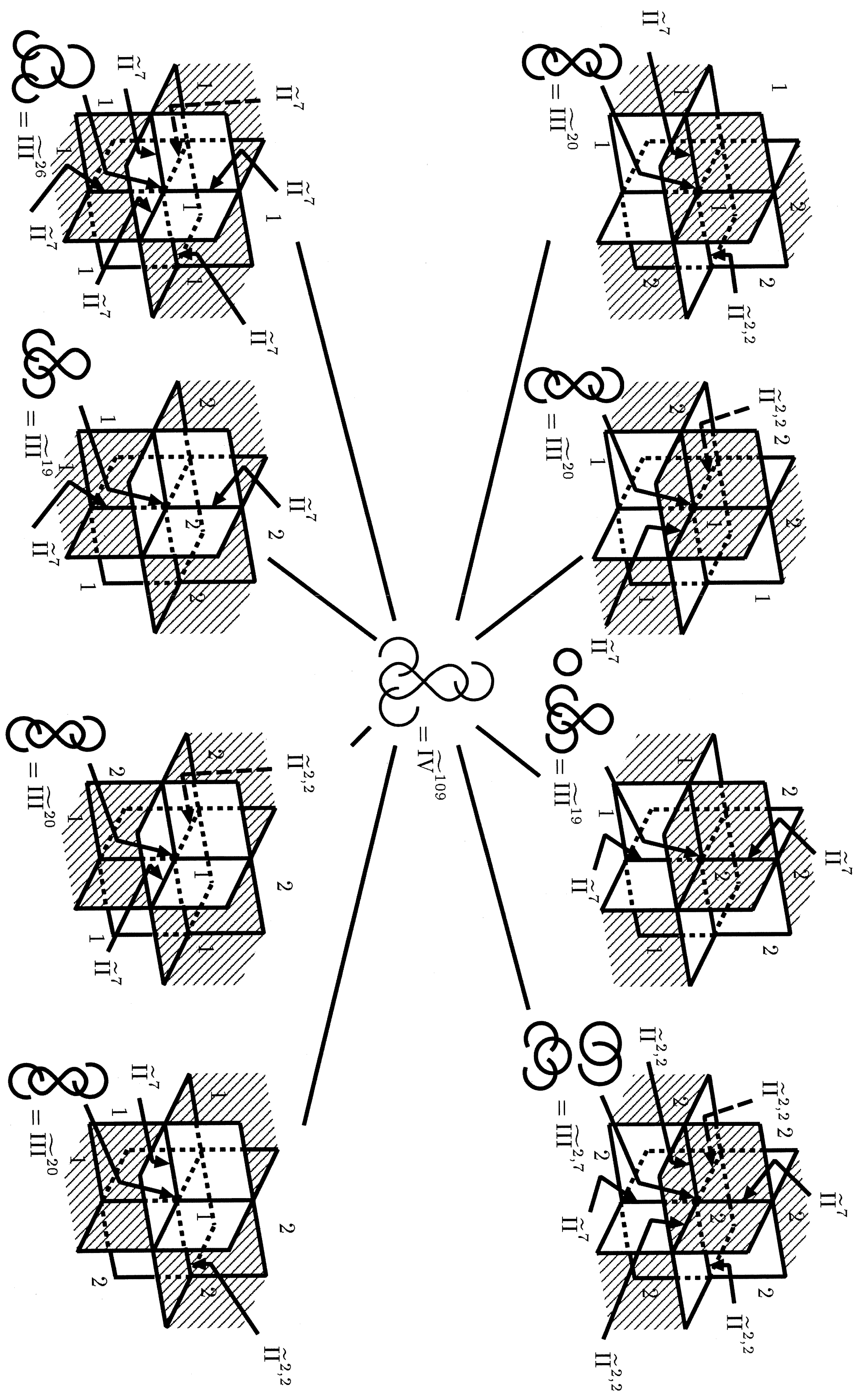

FIGURE 2.130. $\widetilde{\mathrm{IV}}^{109}$ can not divide into two types

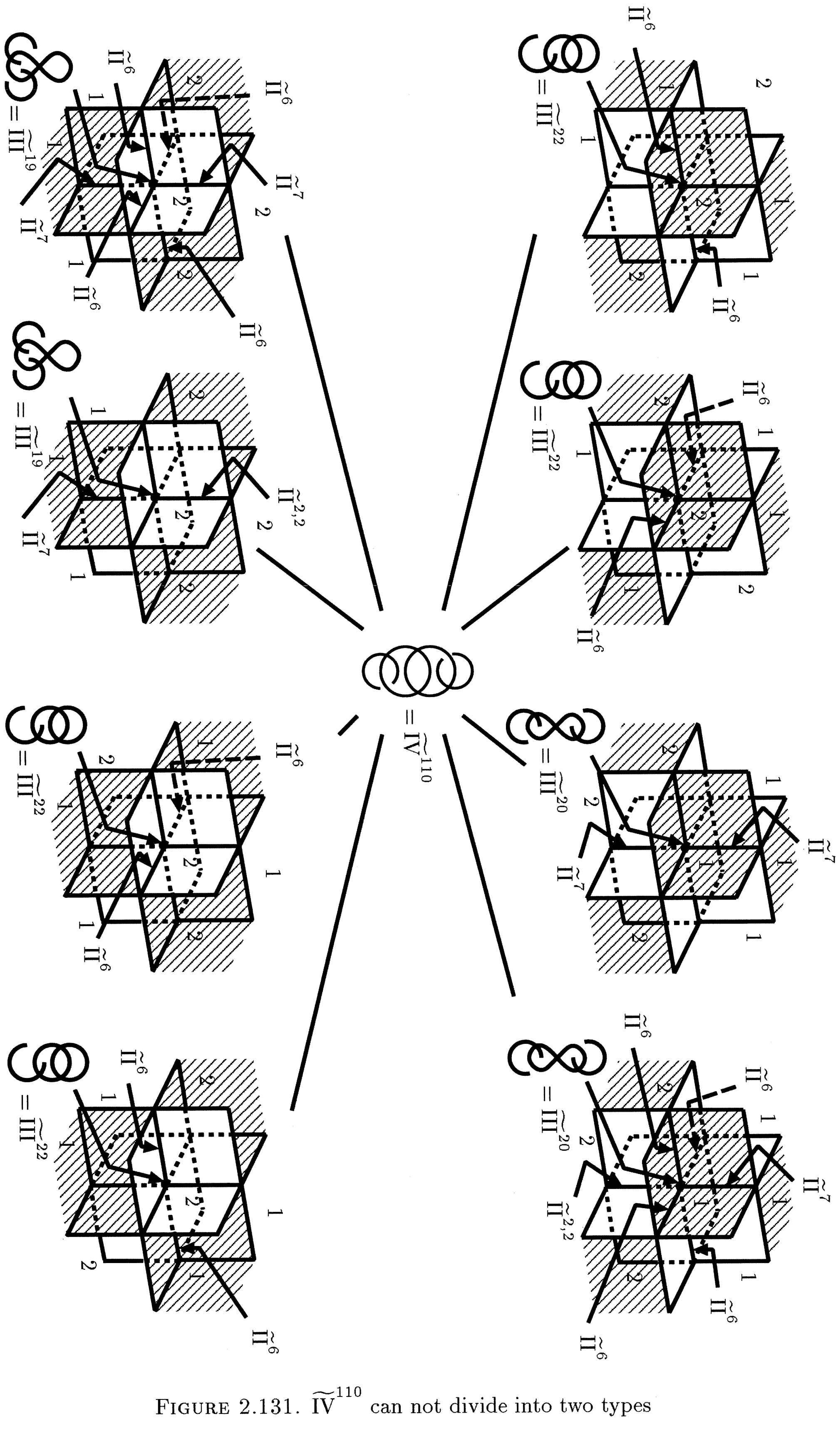

FIGURE 2.131. $\widetilde{\mathrm{IV}}^{110}$ can not divide into two types

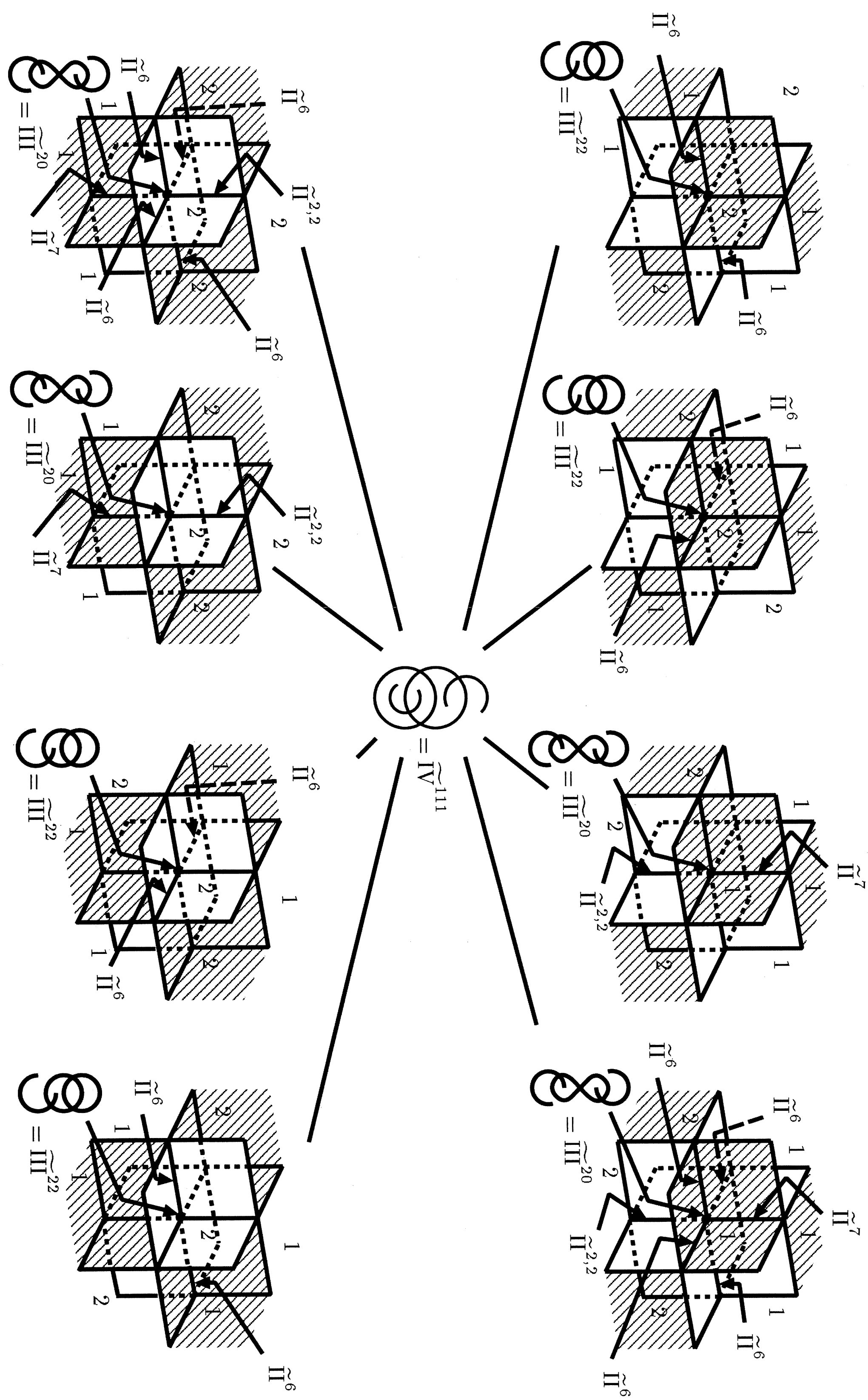

FIGURE 2.132. $\widetilde{\mathrm{IV}}^{111}$ can not divide into two types

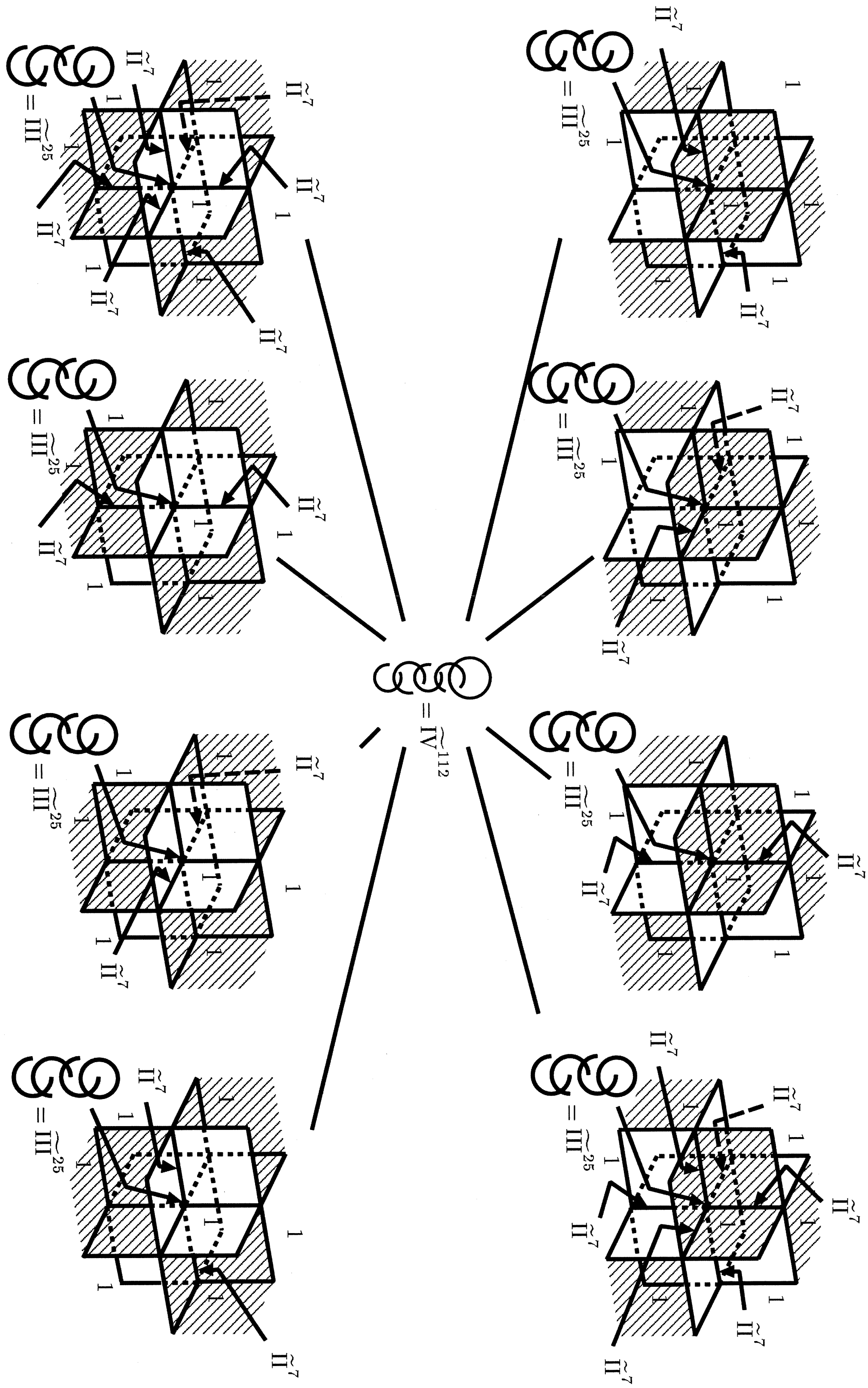

FIGURE 2.133. $\widetilde{\mathrm{IV}}^{112}$ can not divide into two types

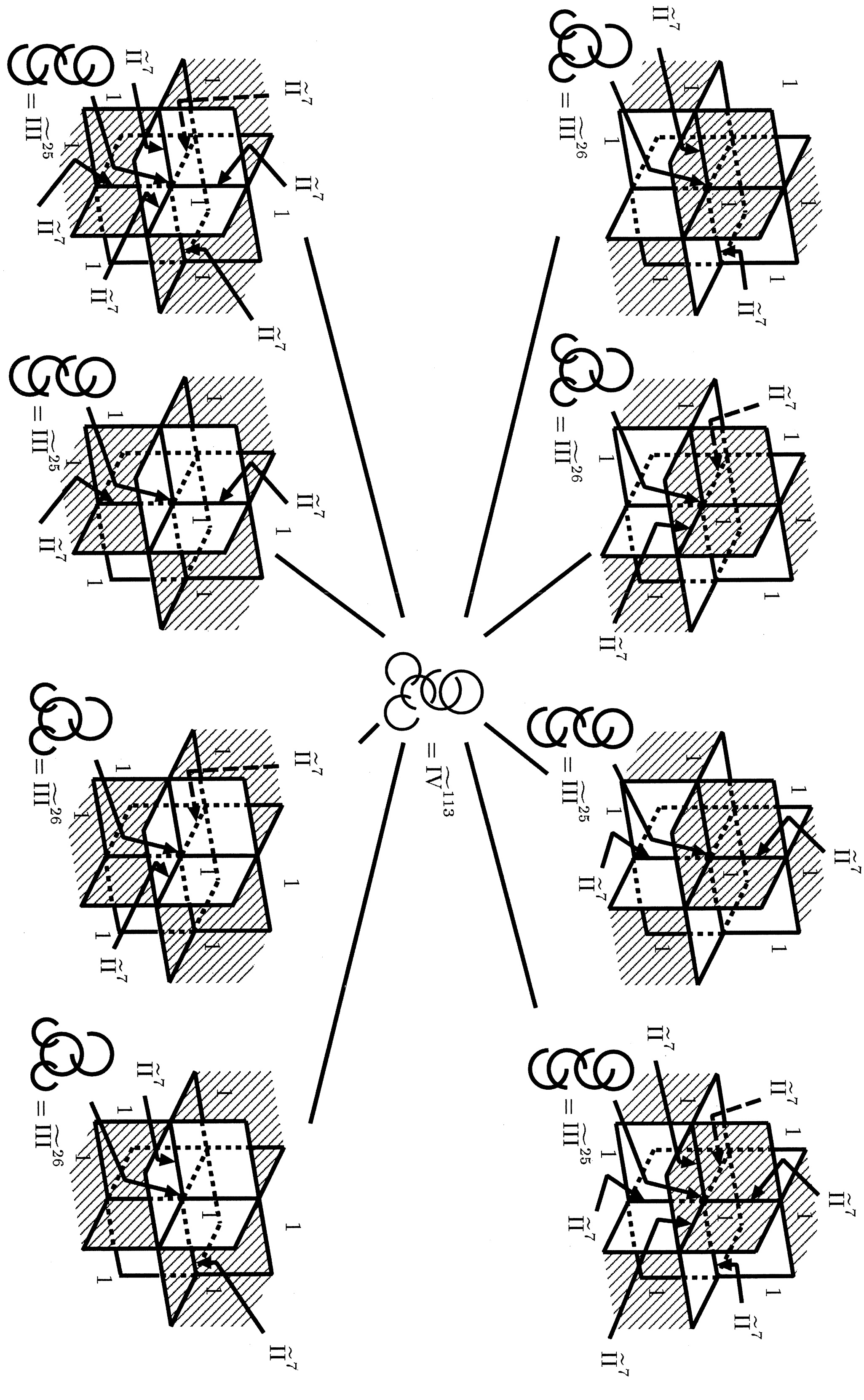

FIGURE 2.134. $\widetilde{IV}^{113}$ can not divide into two types

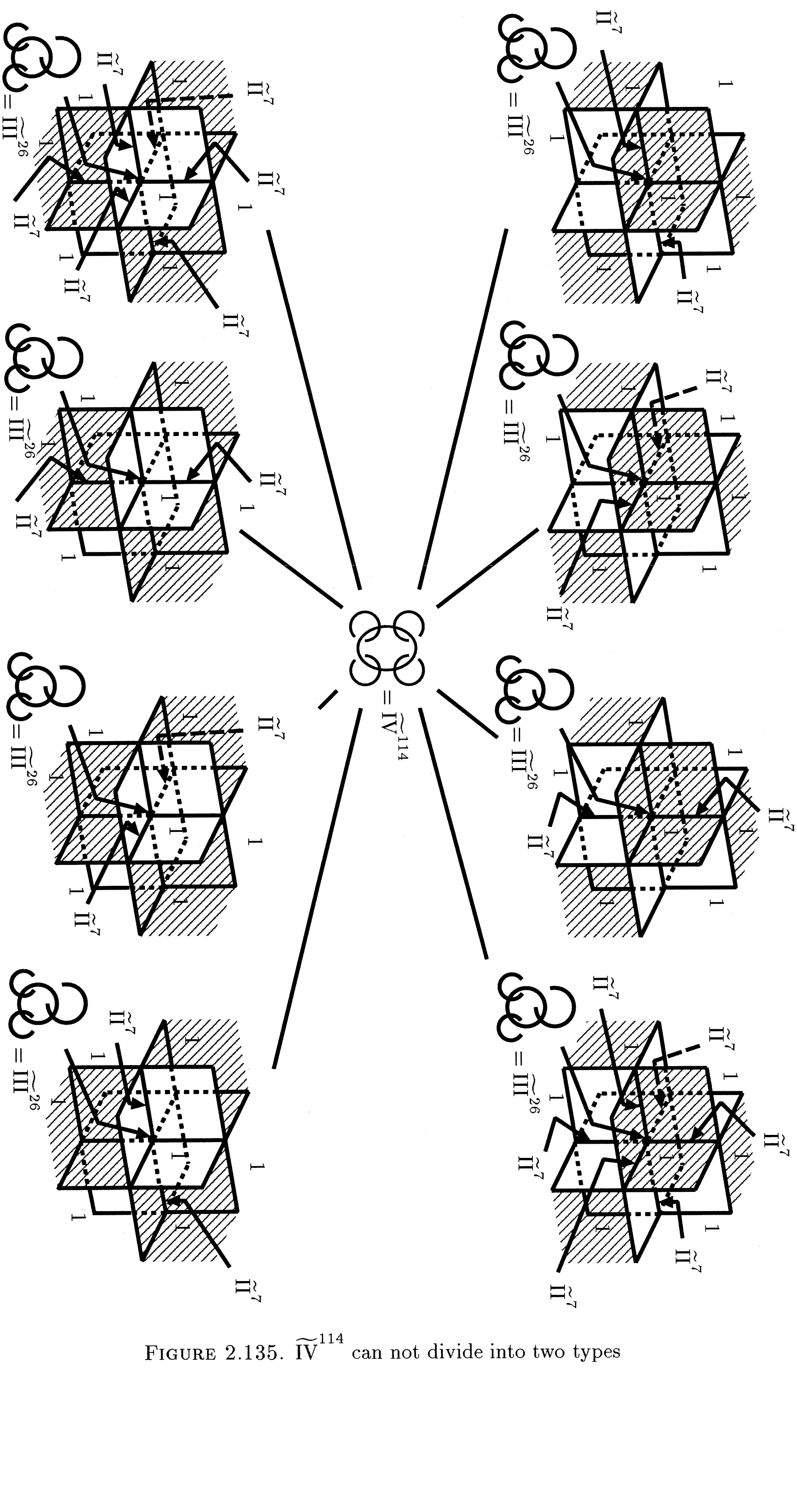

FIGURE 2.135. $\widetilde{\mathrm{IV}}^{114}$ can not divide into two types

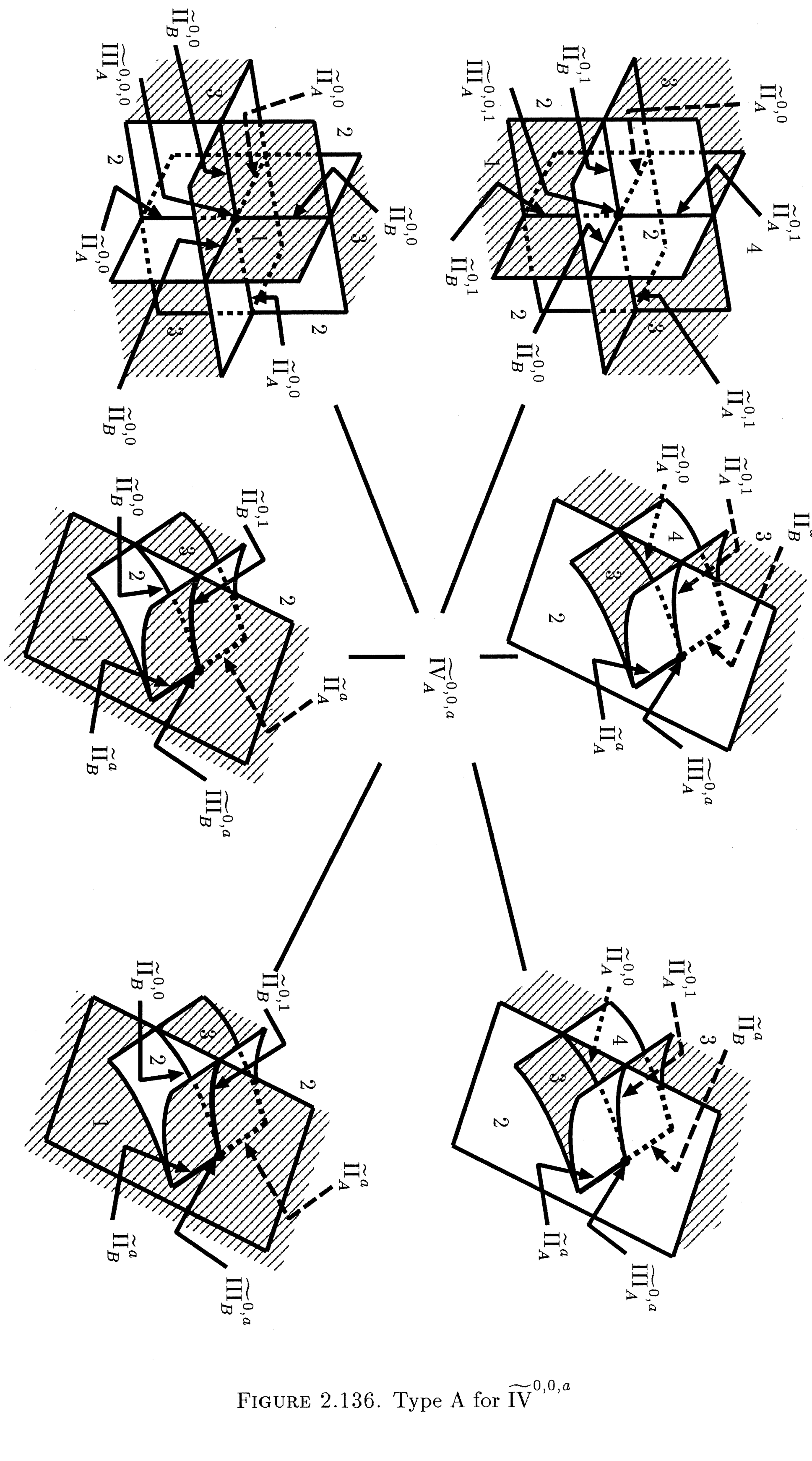

FIGURE 2.136. Type A for $\widetilde{\mathrm{IV}}^{0,0,a}$

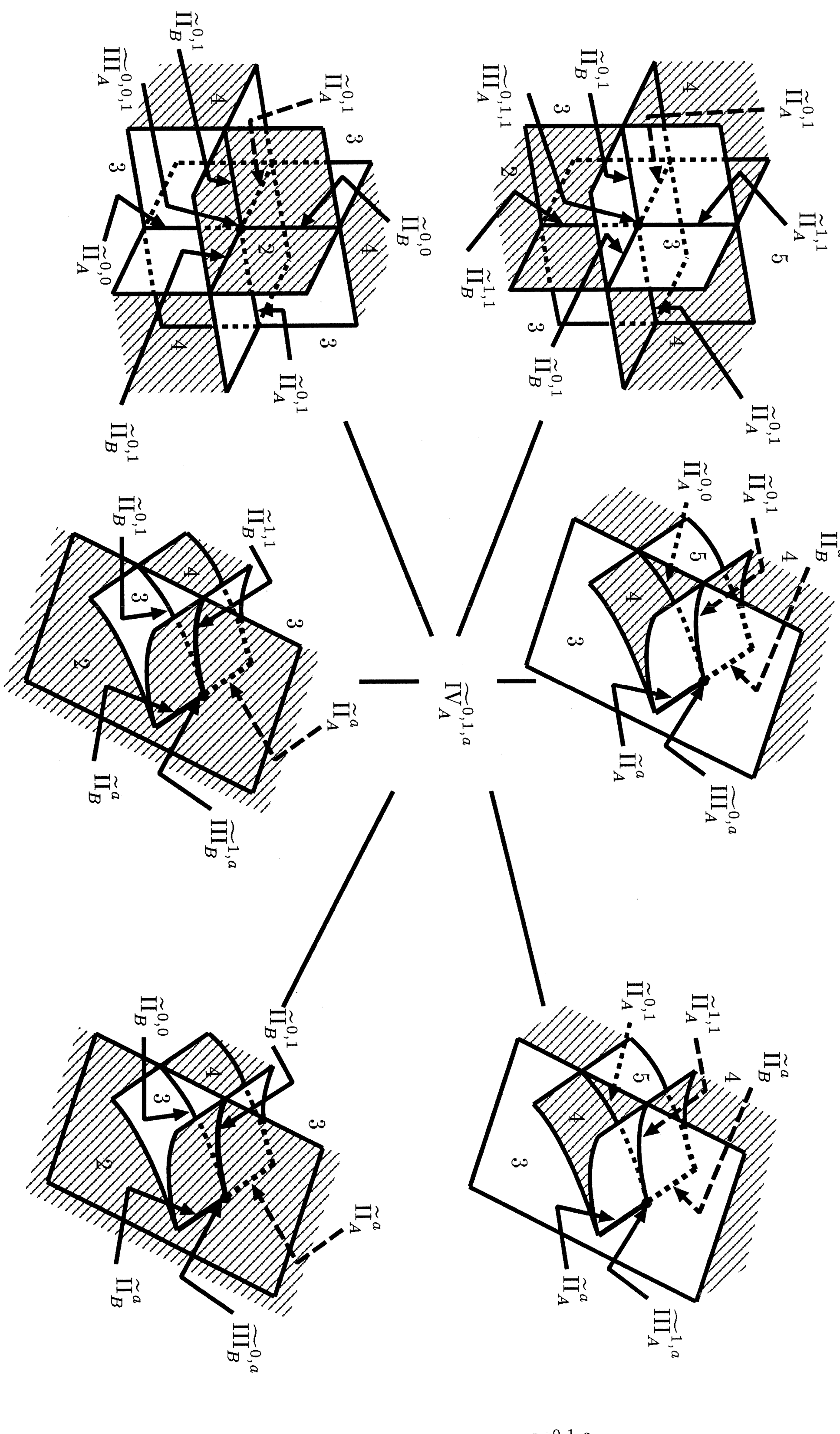

FIGURE 2.137. Type A for $\widetilde{\mathrm{IV}}^{0,1,a}$

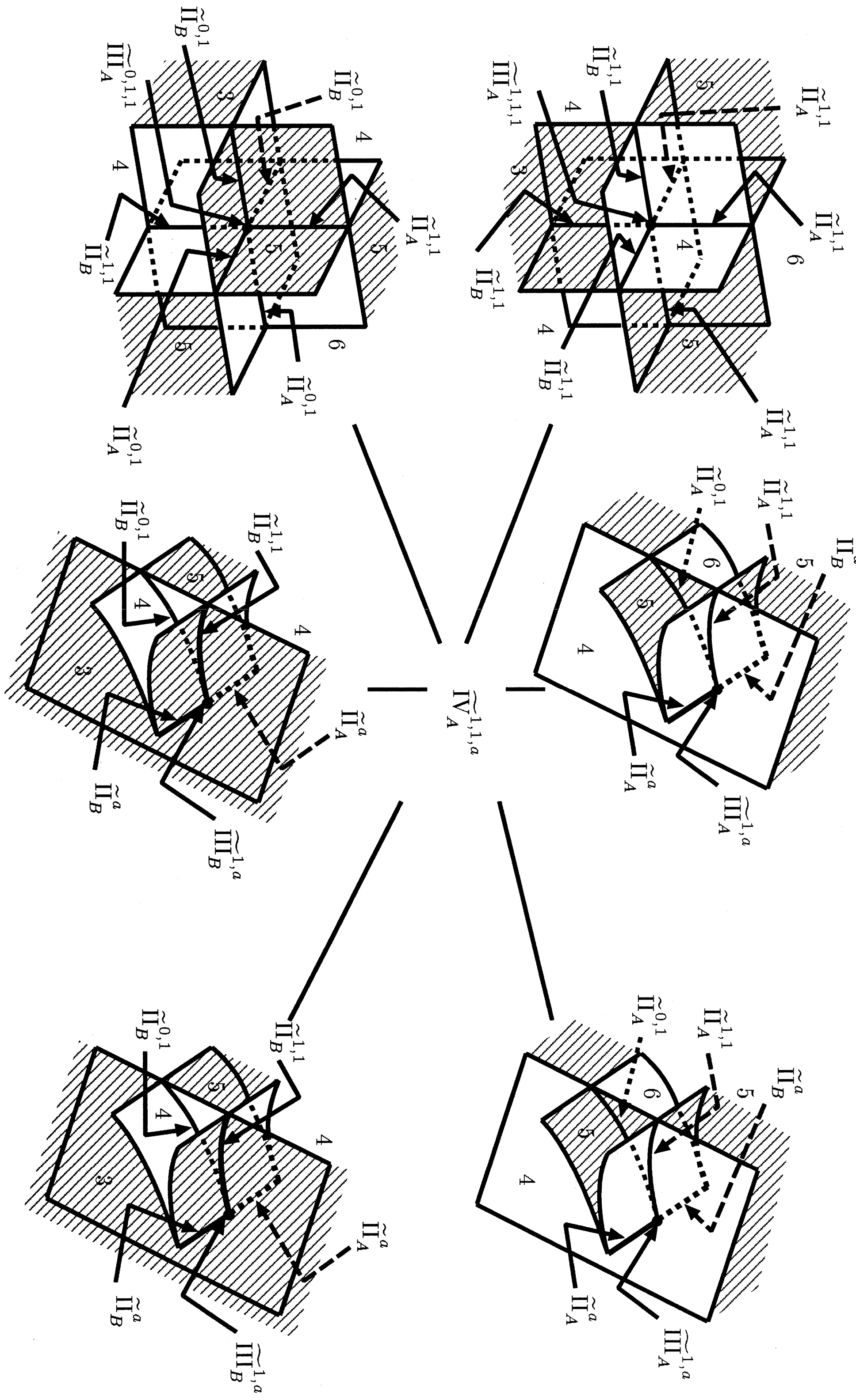

FIGURE 2.138. Type A for $\widetilde{\mathrm{IV}}^{1,1,a}$

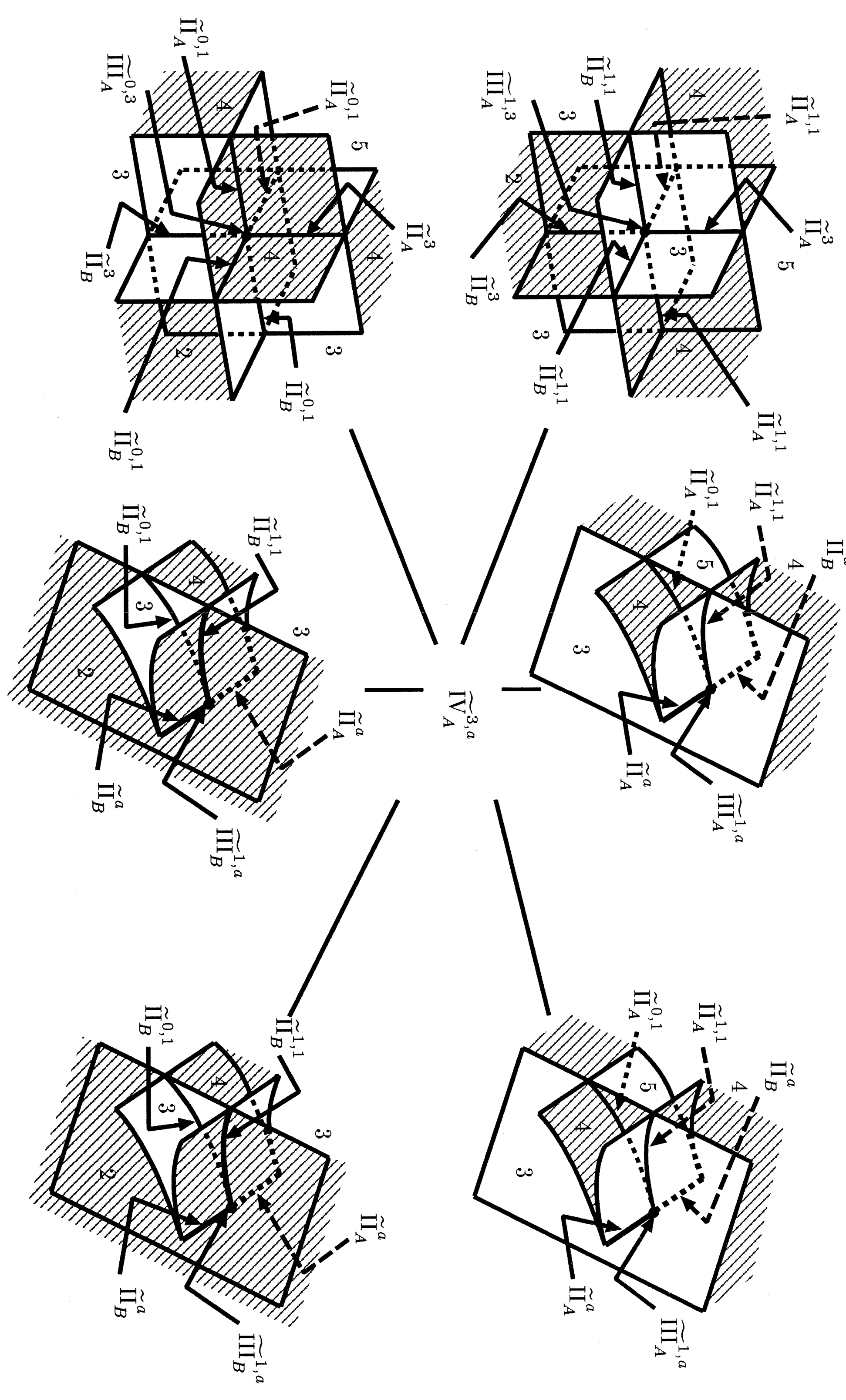

FIGURE 2.139. Type A for $\widetilde{\mathrm{IV}}^{3,a}$

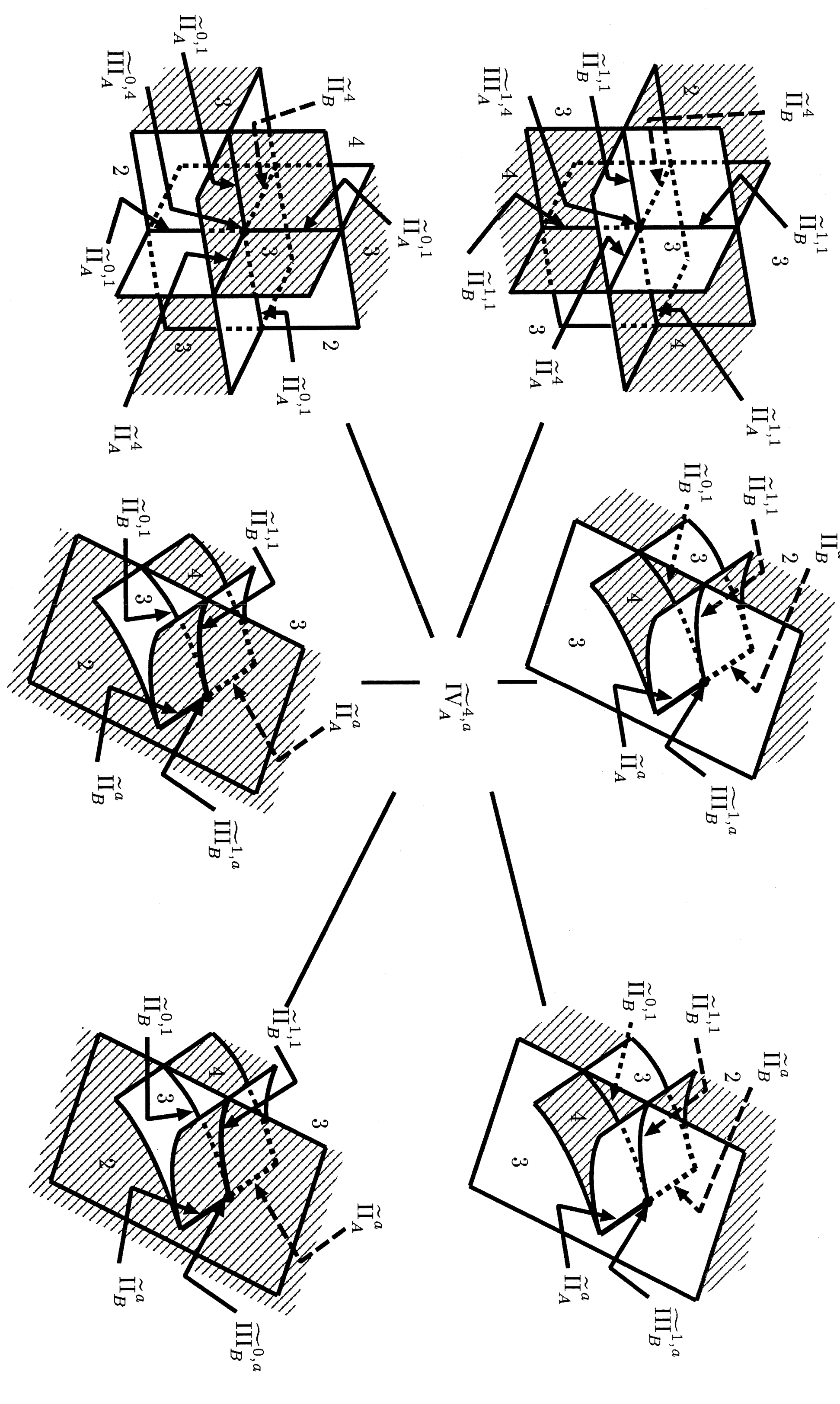

FIGURE 2.140. Type A for $\widetilde{\mathrm{IV}}^{4,a}$

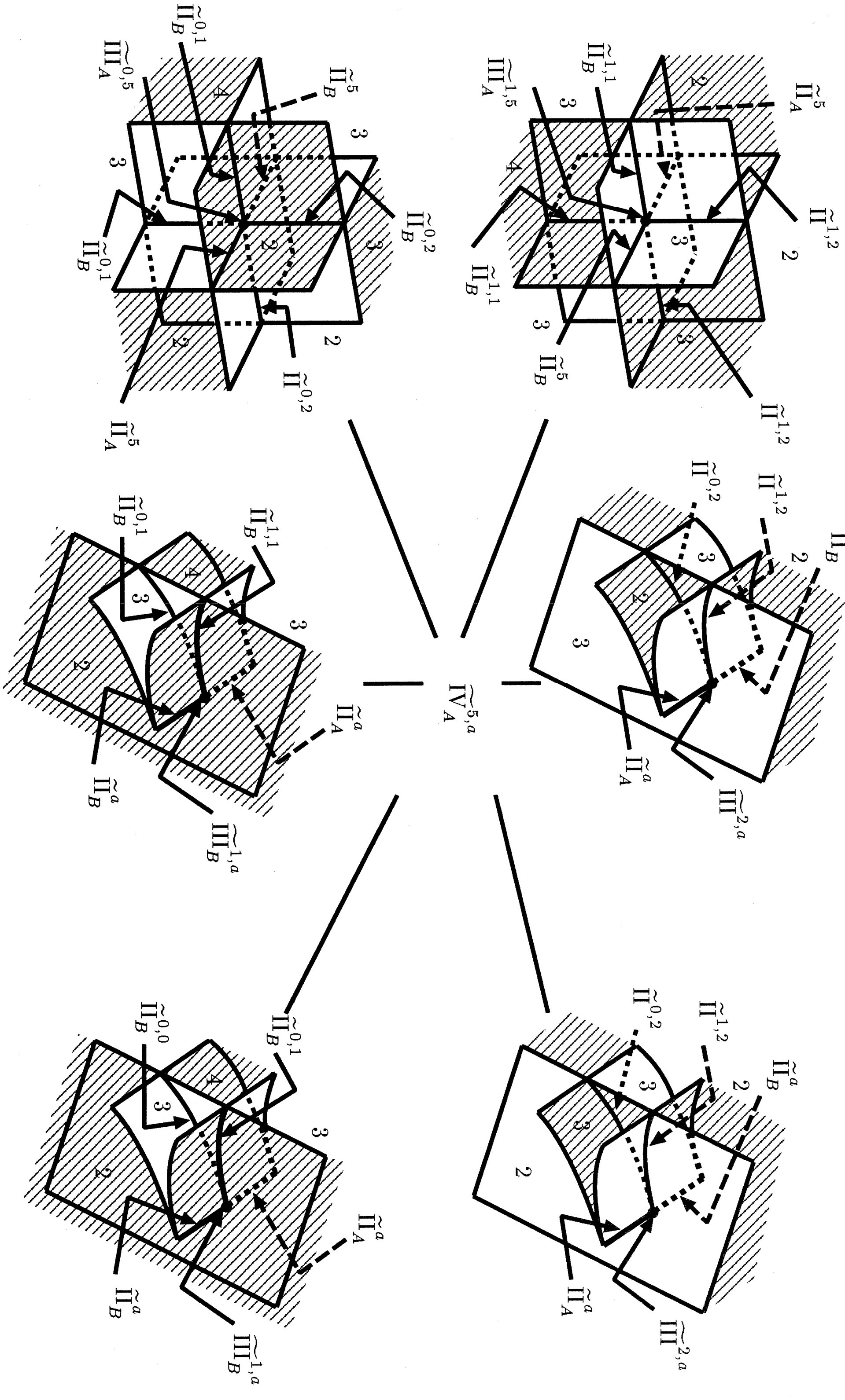

FIGURE 2.141. Type A for $\widetilde{IV}^{5,a}$

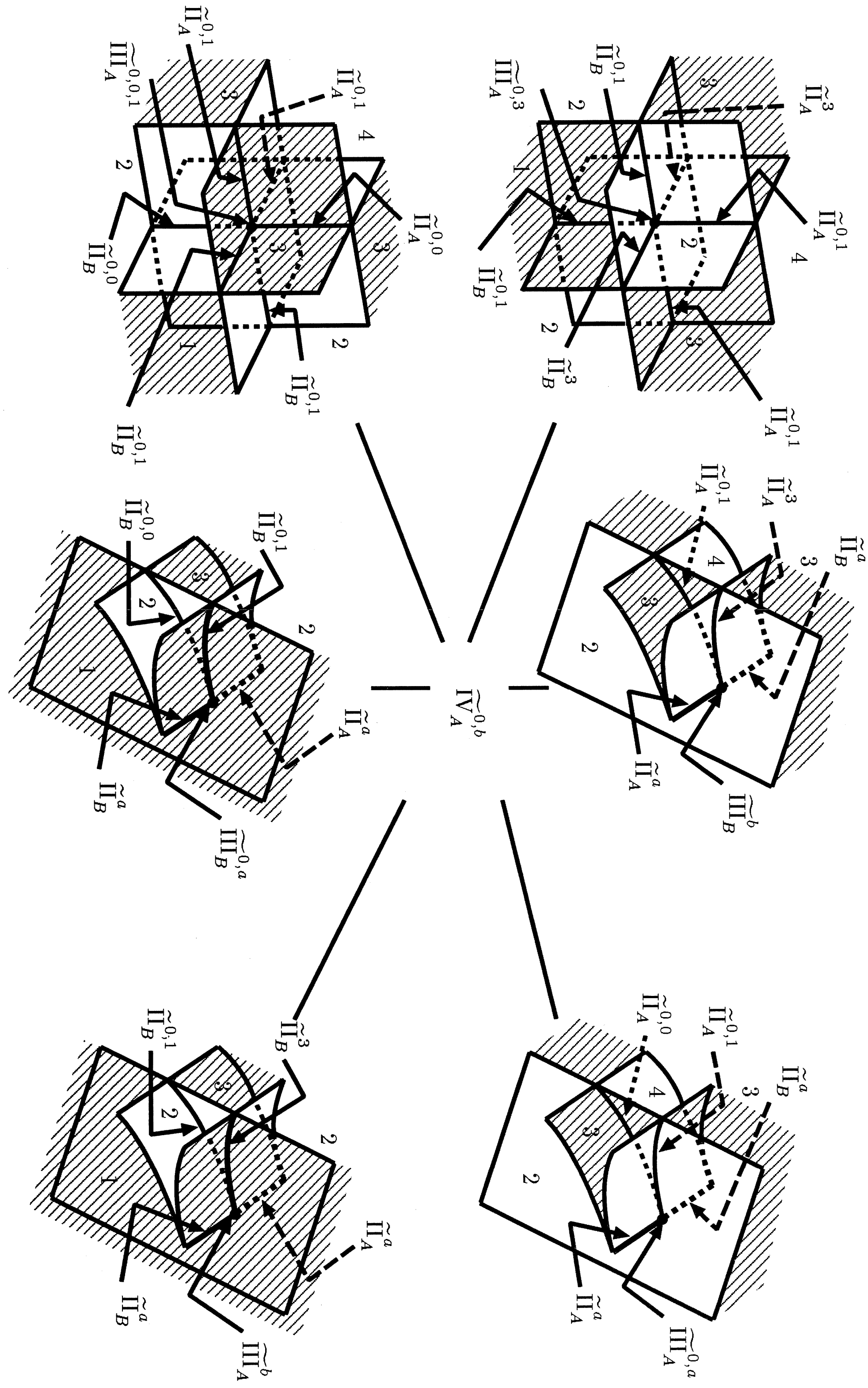

FIGURE 2.142. Type A for $\widetilde{\mathrm{IV}}^{0,b}$

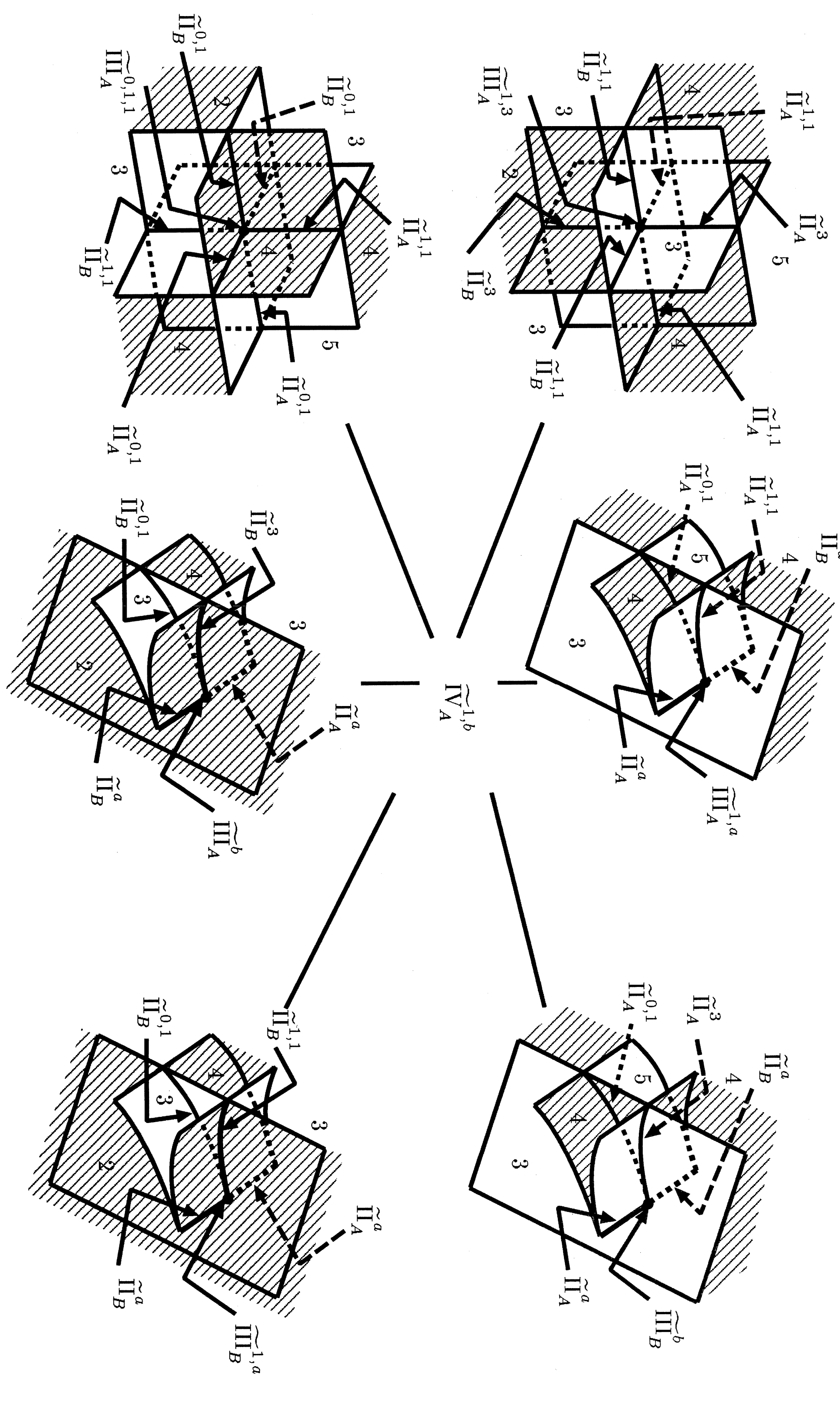

FIGURE 2.143. Type A for $\widetilde{\mathrm{IV}}^{1,b}$

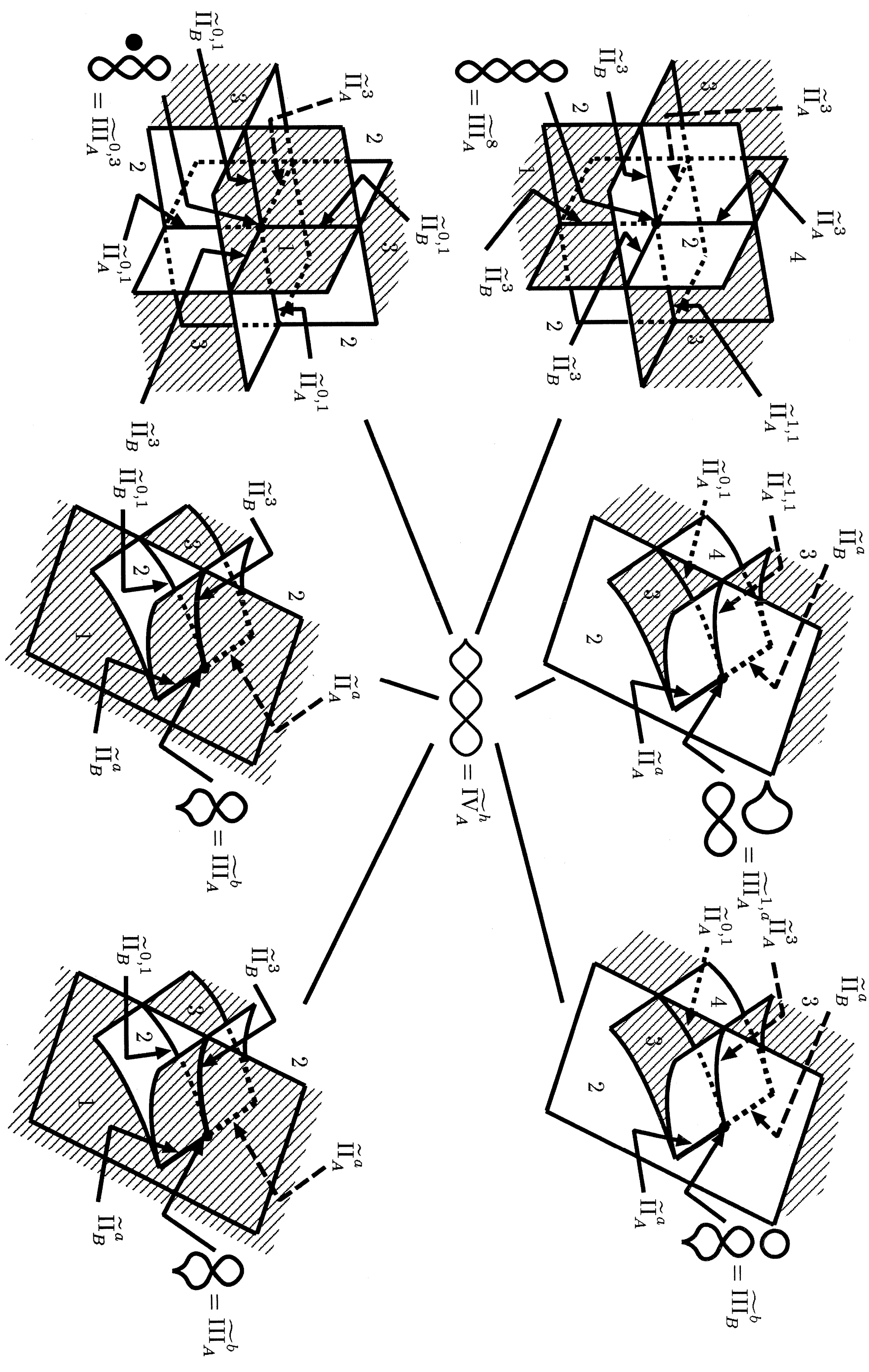

FIGURE 2.144. Type A for $\widetilde{\mathrm{IV}}^h$

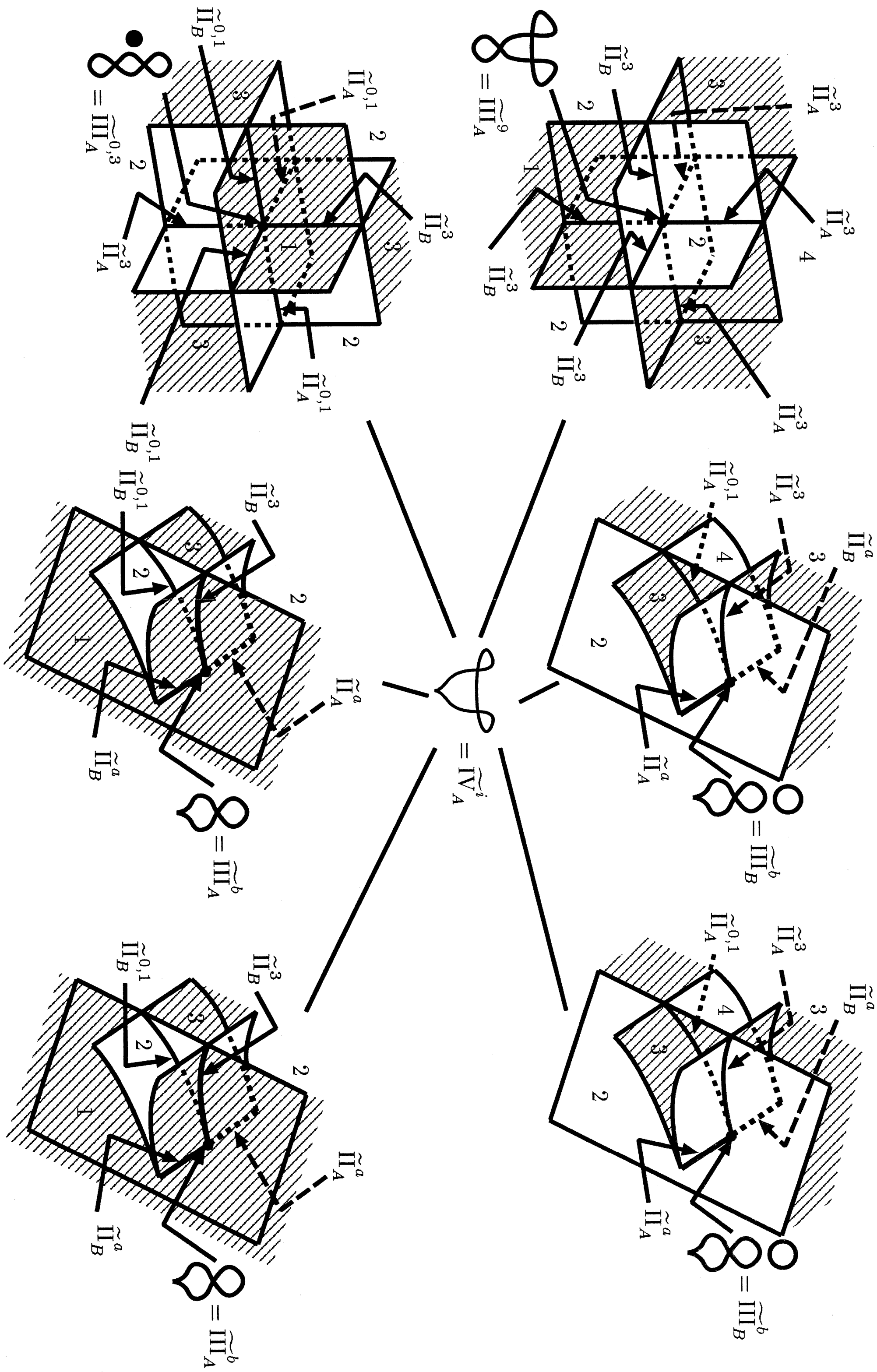

FIGURE 2.145. Type A for $\widetilde{\mathrm{IV}}^i$

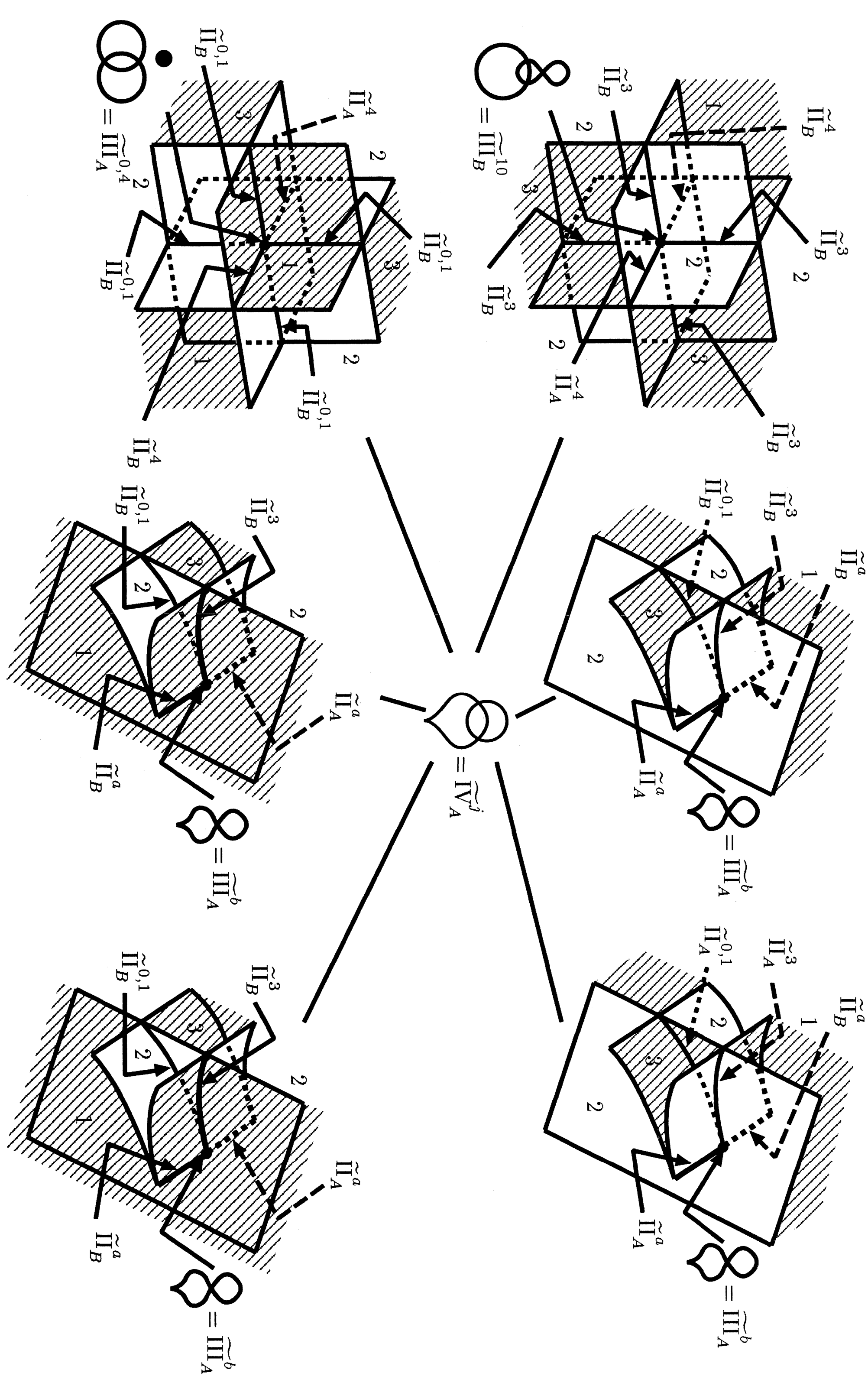

FIGURE 2.146. Type A for $\widetilde{\mathrm{IV}}^j$

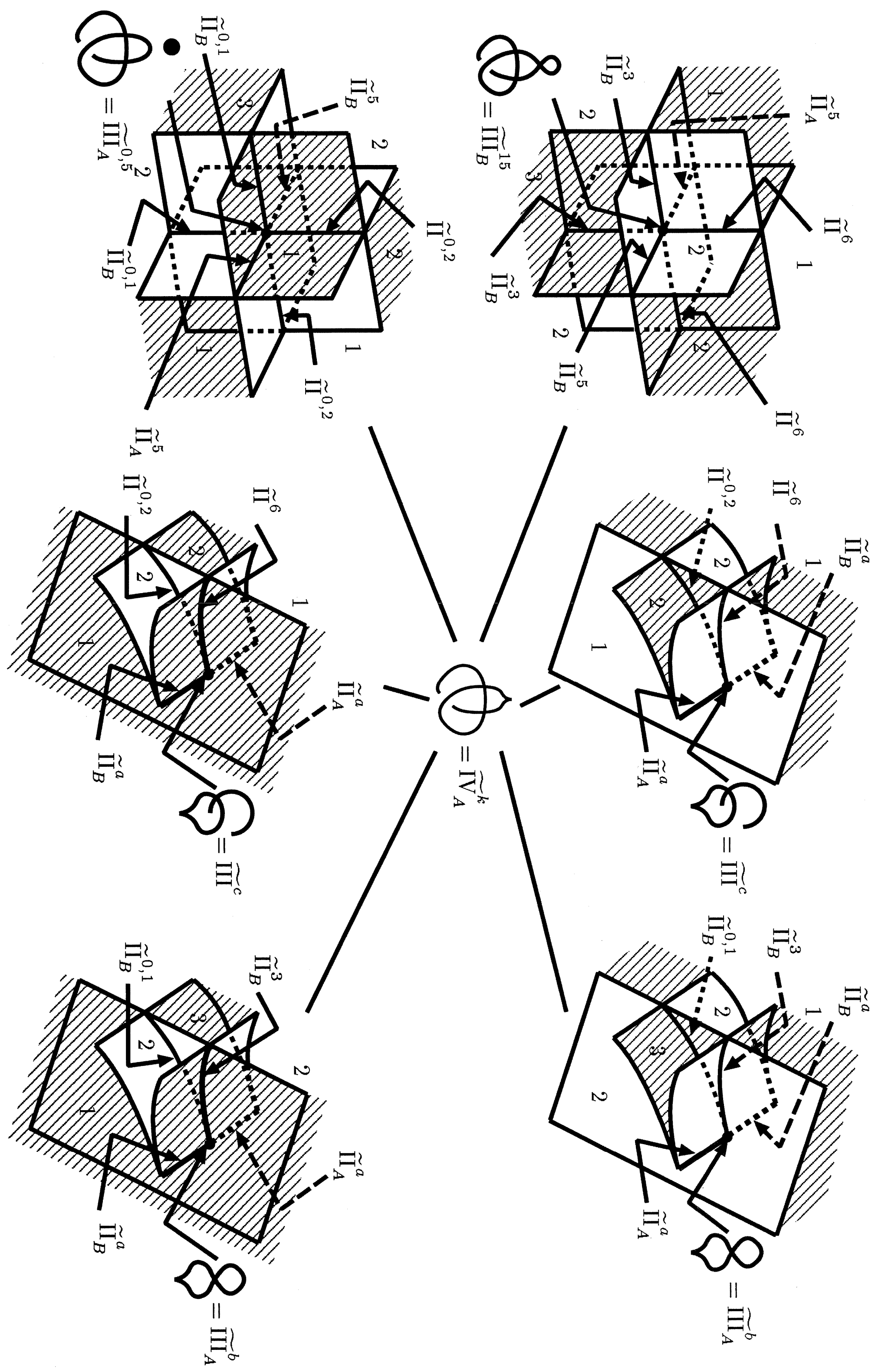

FIGURE 2.147. Type A for $\widetilde{\mathrm{IV}}^k$

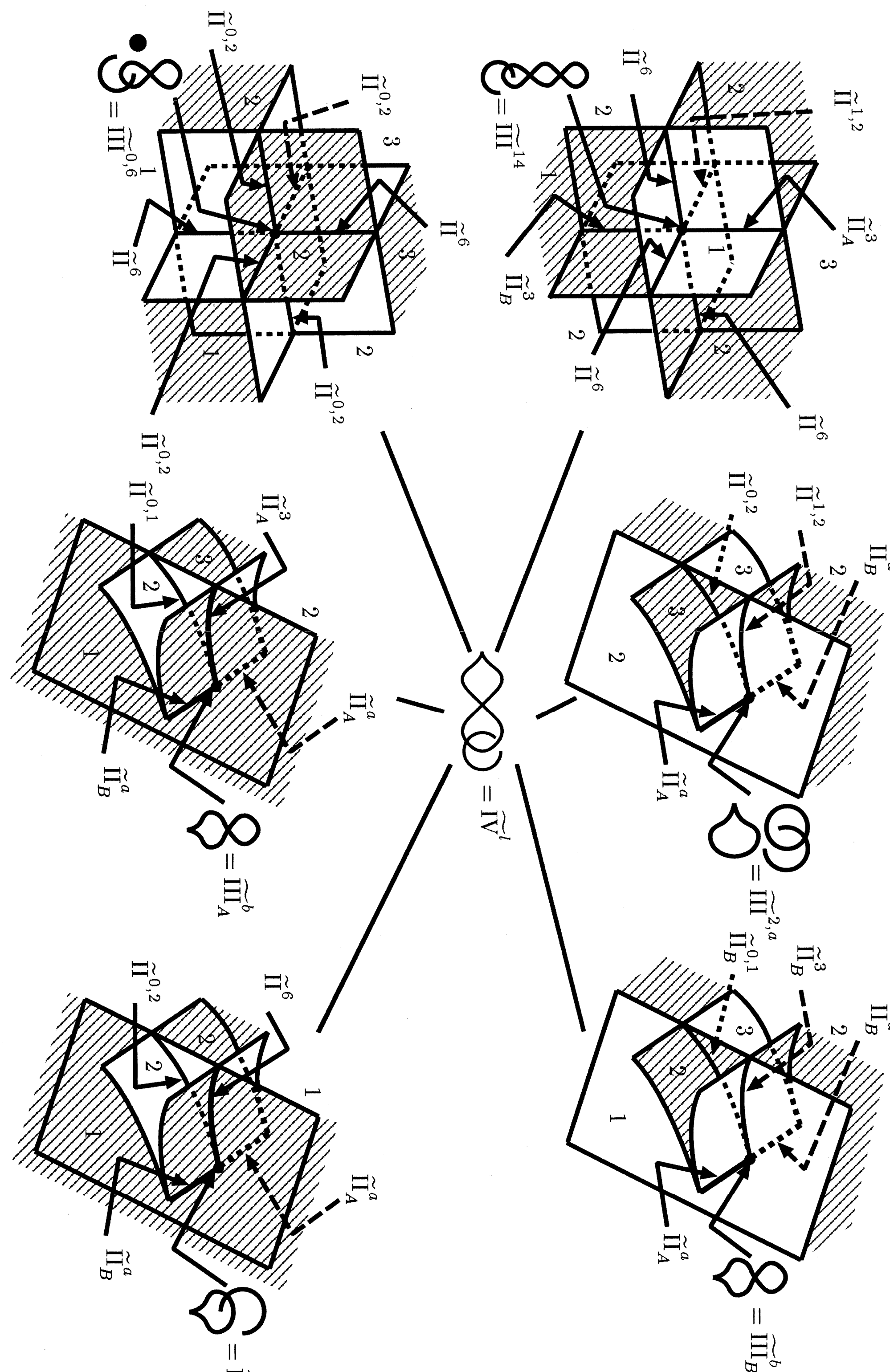

FIGURE 2.148. $\widetilde{\mathrm{IV}}^{l}$ can not divide into two types

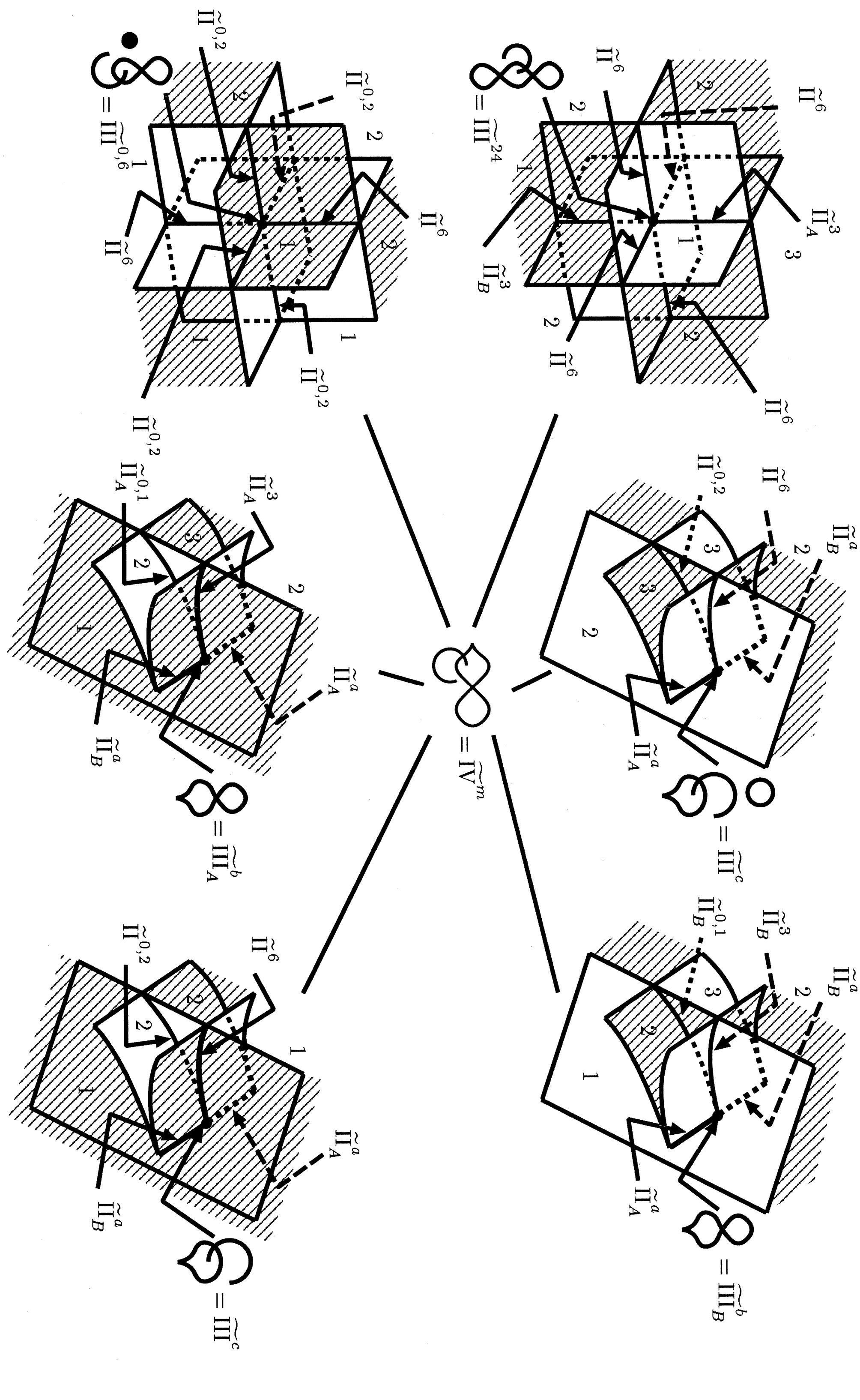

FIGURE 2.149. $\widetilde{\mathrm{IV}}^m$ can not divide into two types

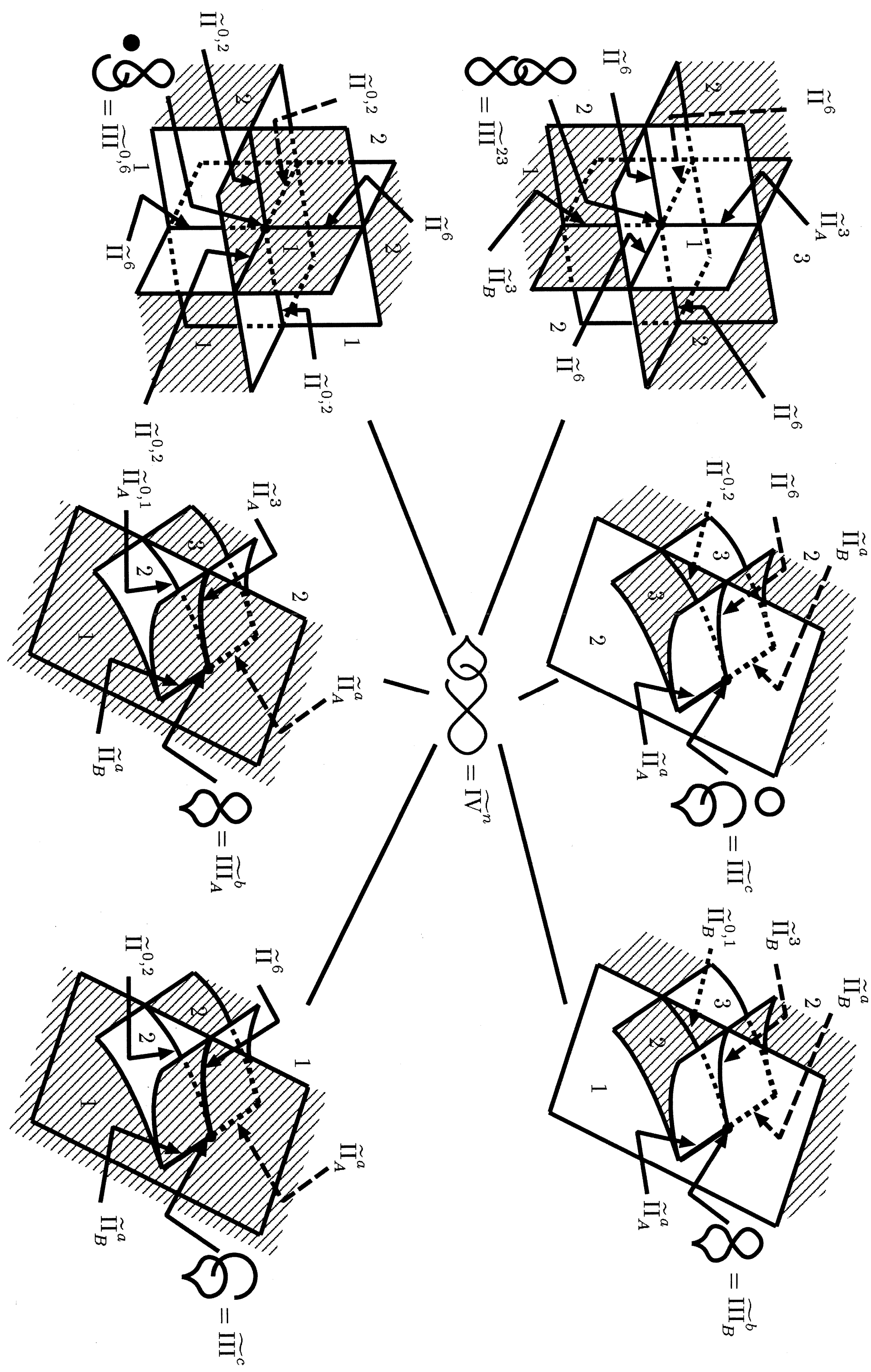

FIGURE 2.150. $\widetilde{IV}^n$ can not divide into two types

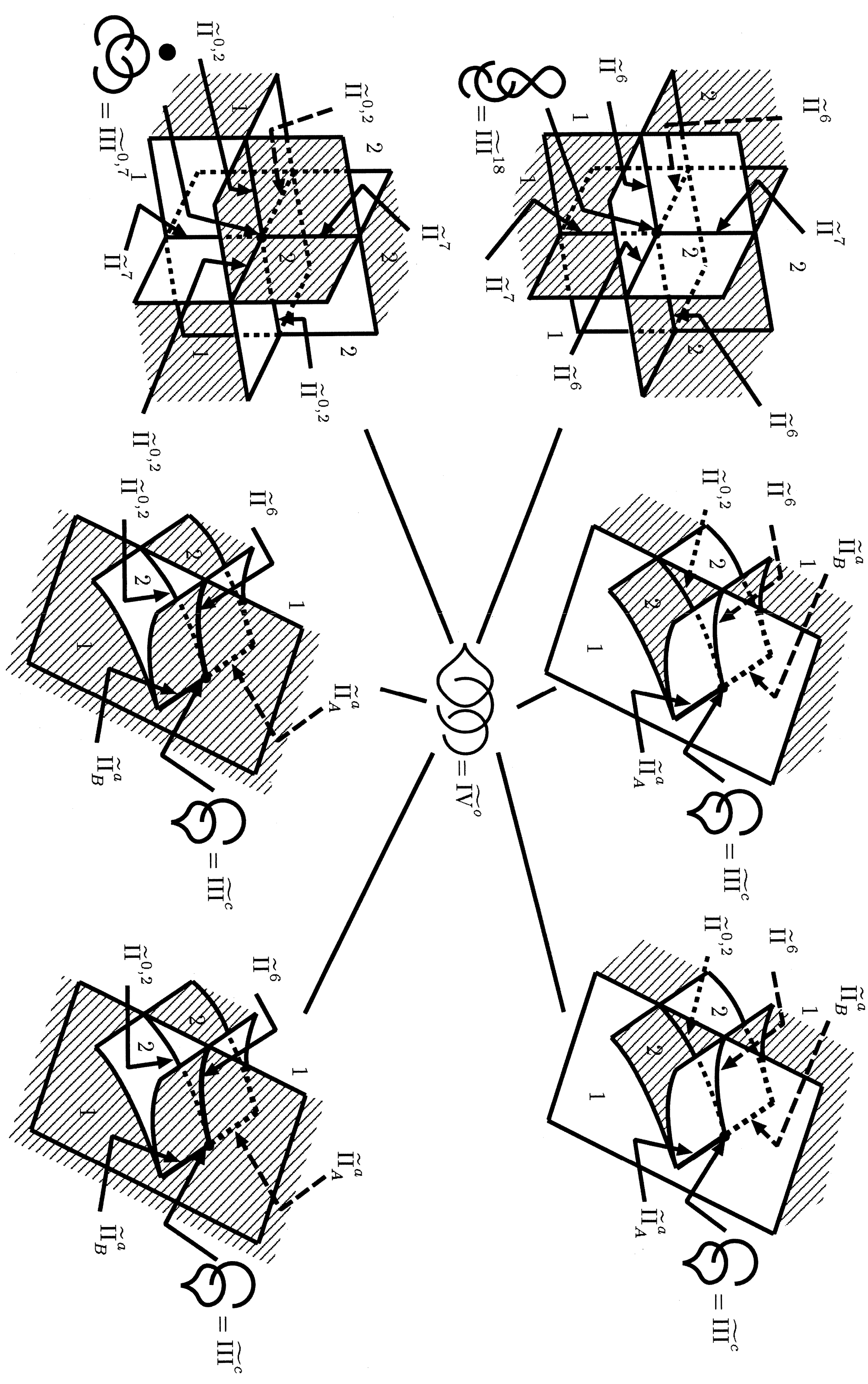

FIGURE 2.151. $\widetilde{\mathrm{IV}}^{o}$ can not divide into two types

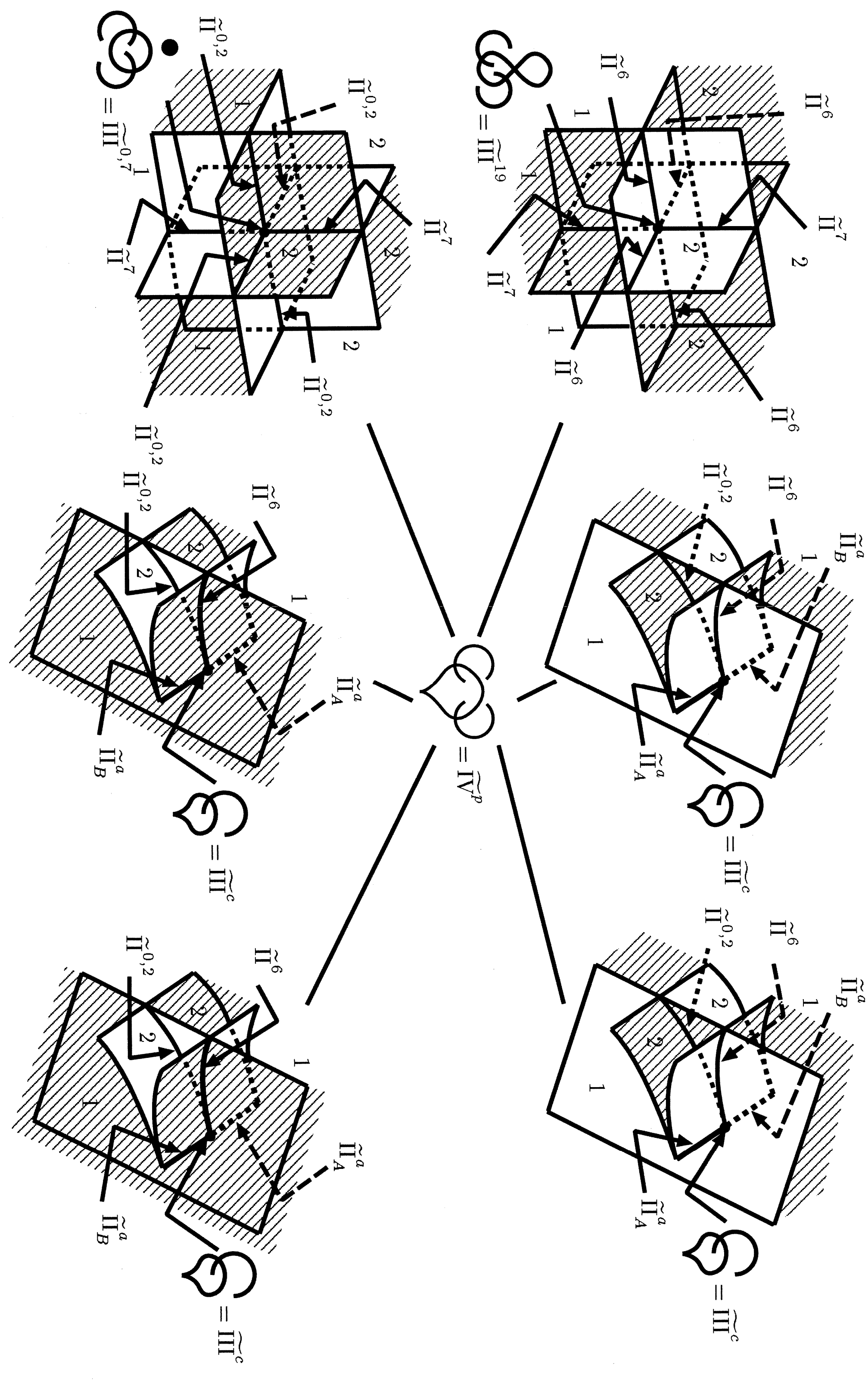

FIGURE 2.152. $\widetilde{\mathrm{IV}}^{p}$ can not divide into two types

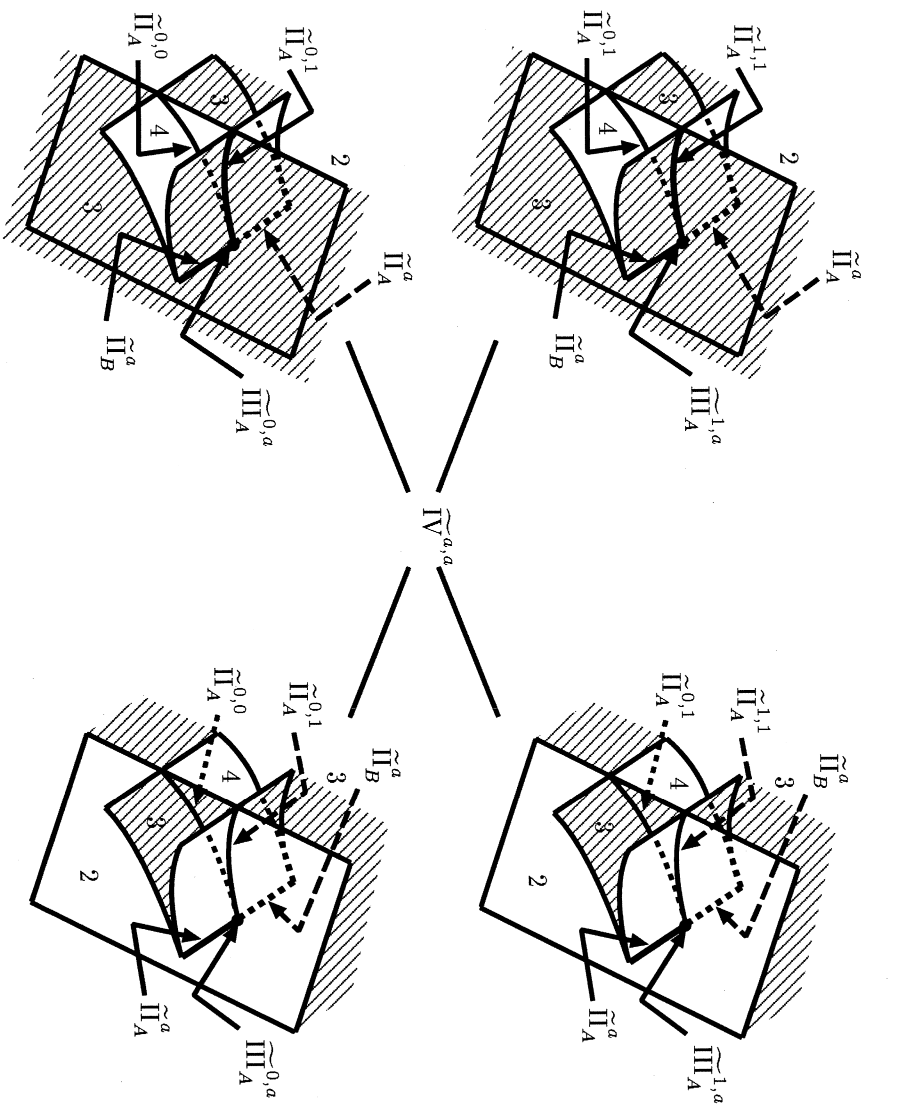

FIGURE 2.153. Type A for $\widetilde{\mathrm{IV}}^{a,a}$

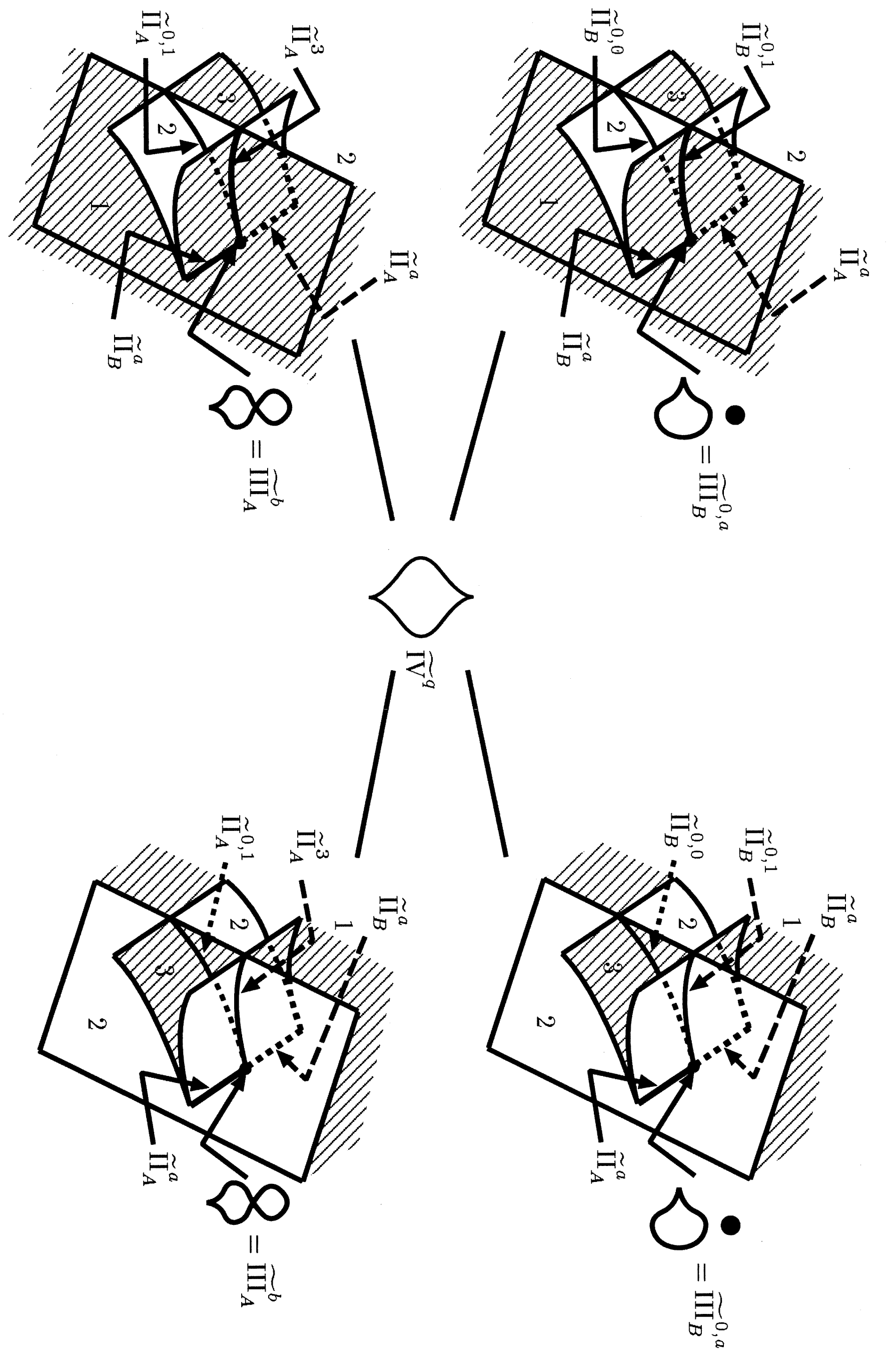

FIGURE 2.154. Type A for $\widetilde{\mathrm{IV}}^q$

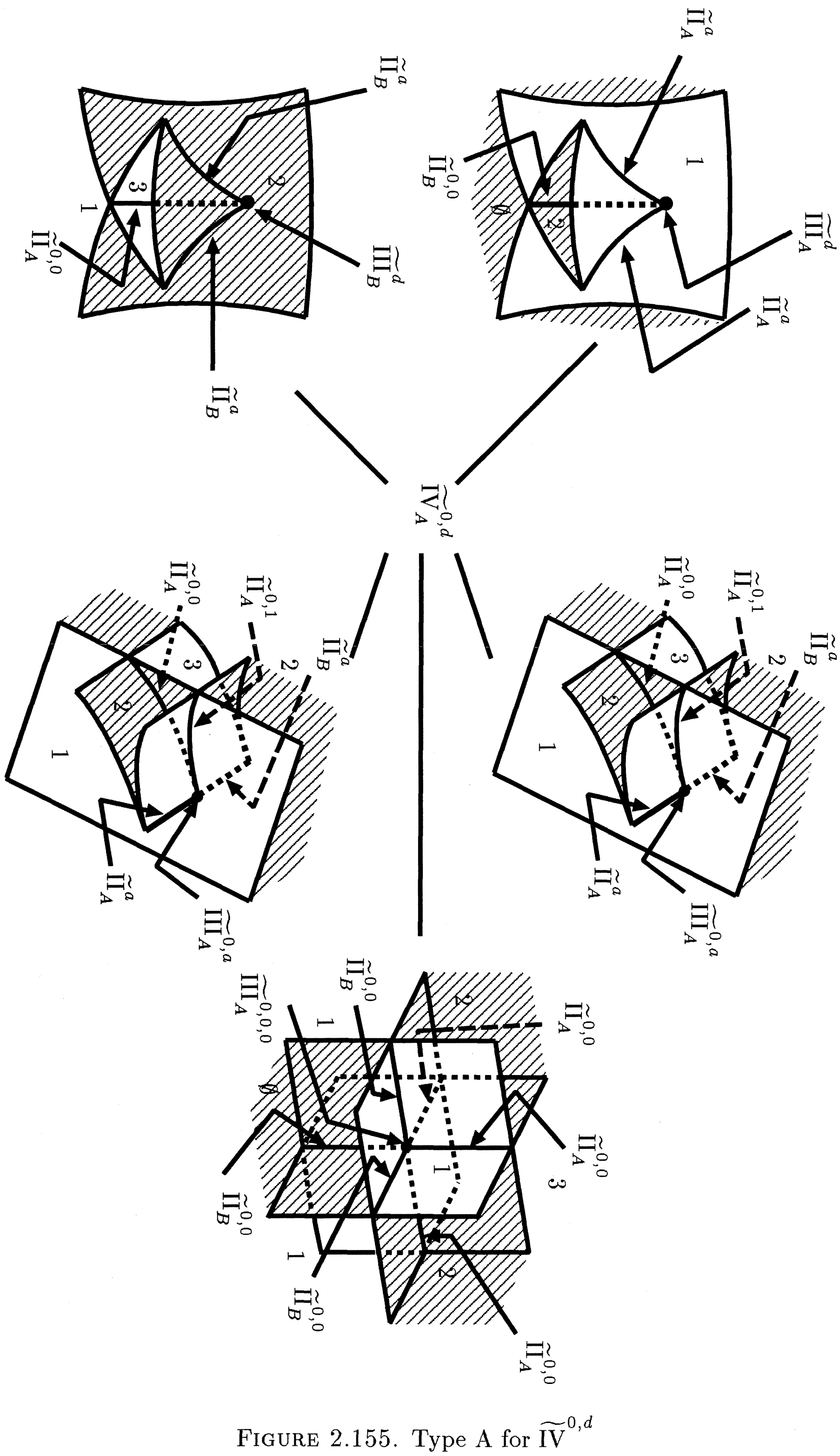

FIGURE 2.155. Type A for $\widetilde{\mathrm{IV}}^{0,d}$

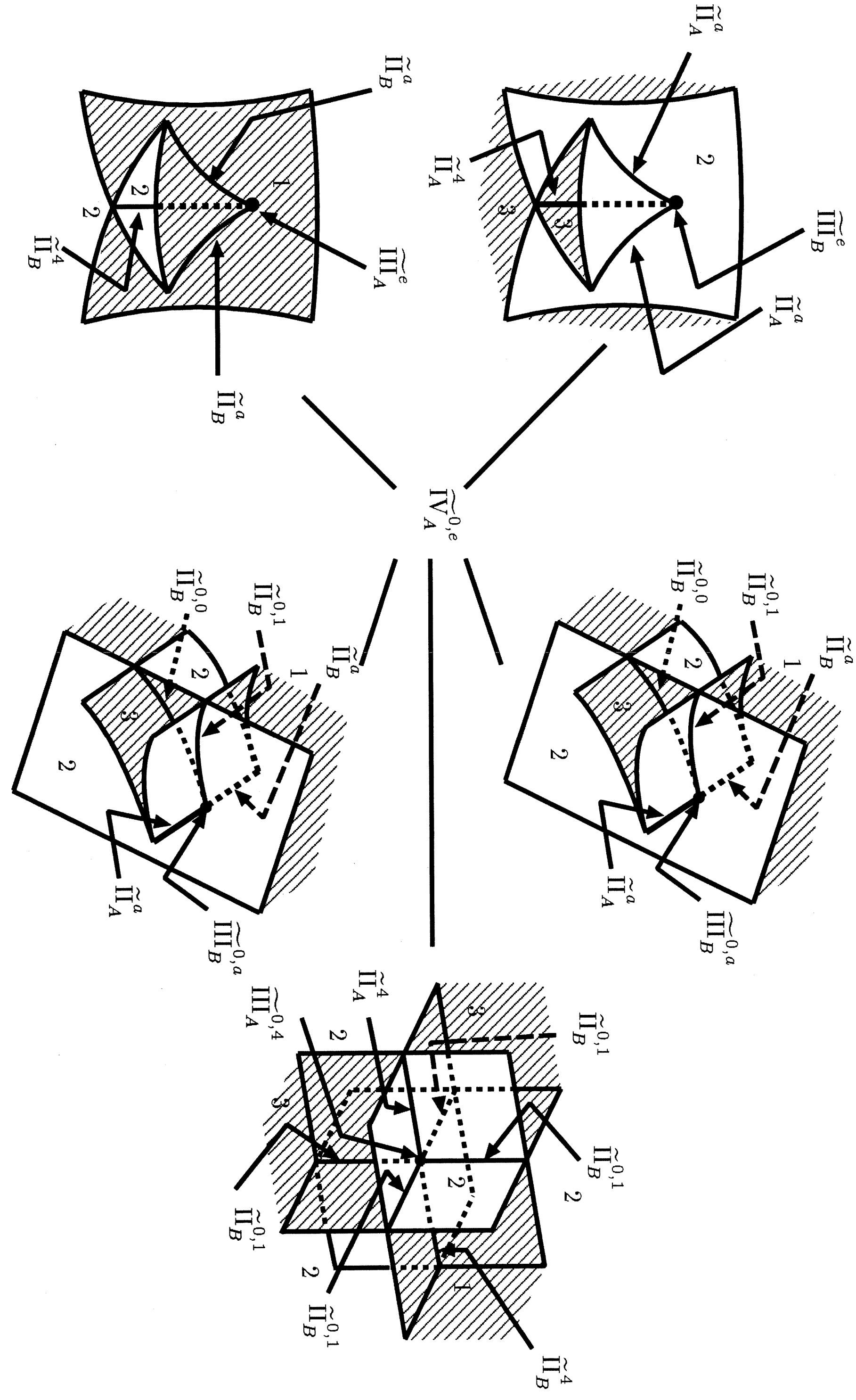

FIGURE 2.156. Type A for $\widetilde{\mathrm{IV}}^{0,e}$

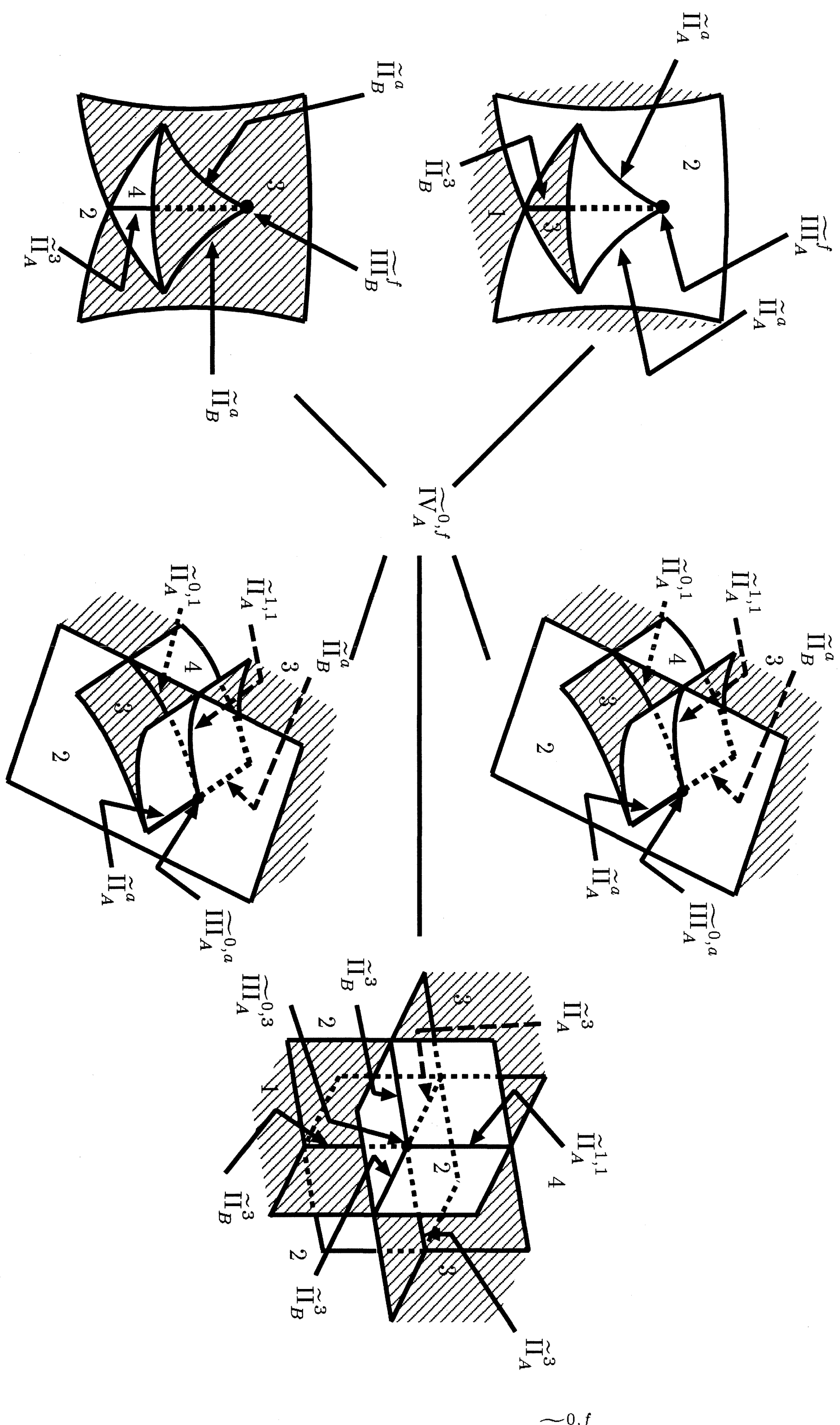

FIGURE 2.157. Type A for $\widetilde{\mathrm{IV}}^{0,f}$

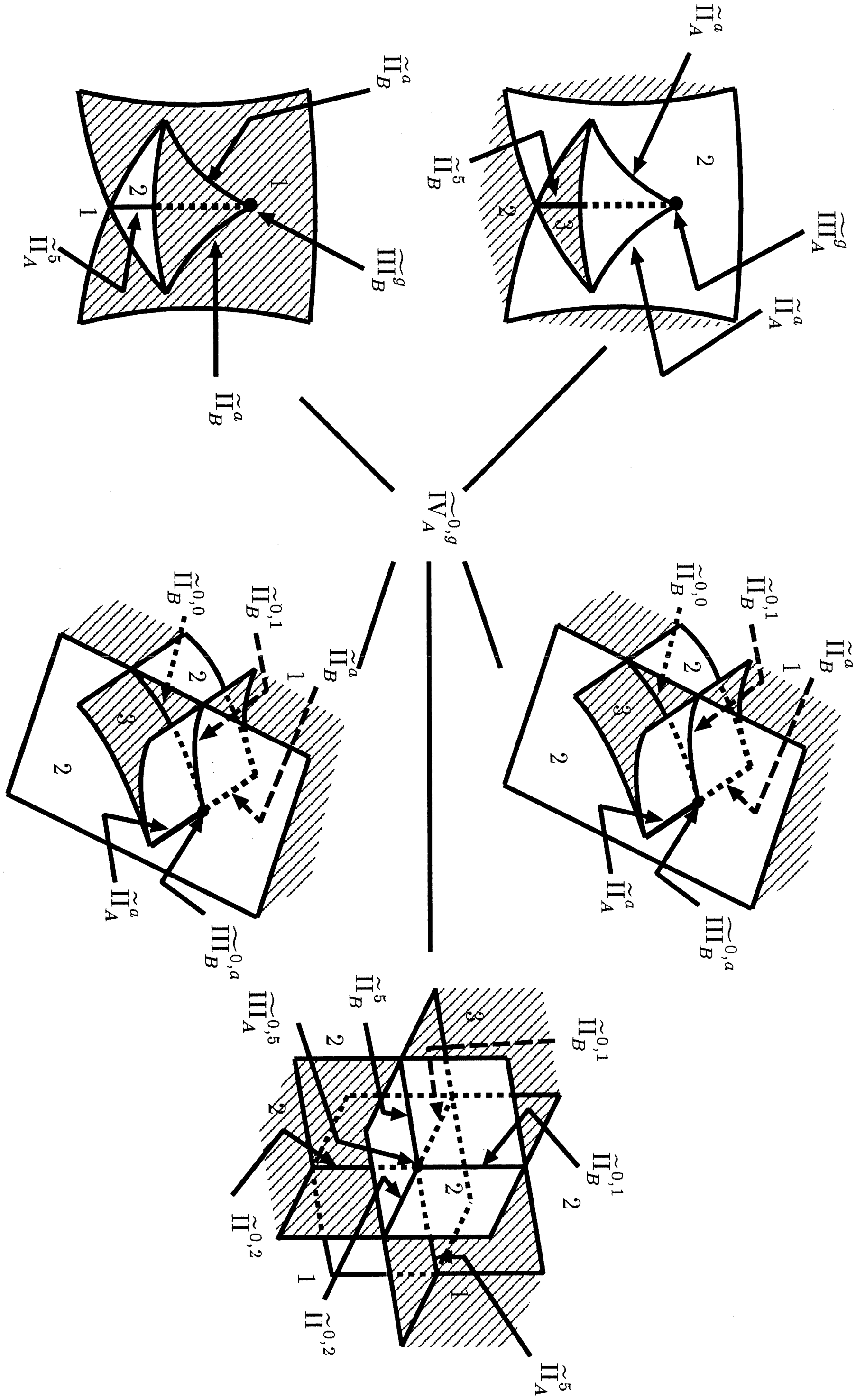

FIGURE 2.158. Type A for $\widetilde{\mathrm{IV}}^{0,g}$

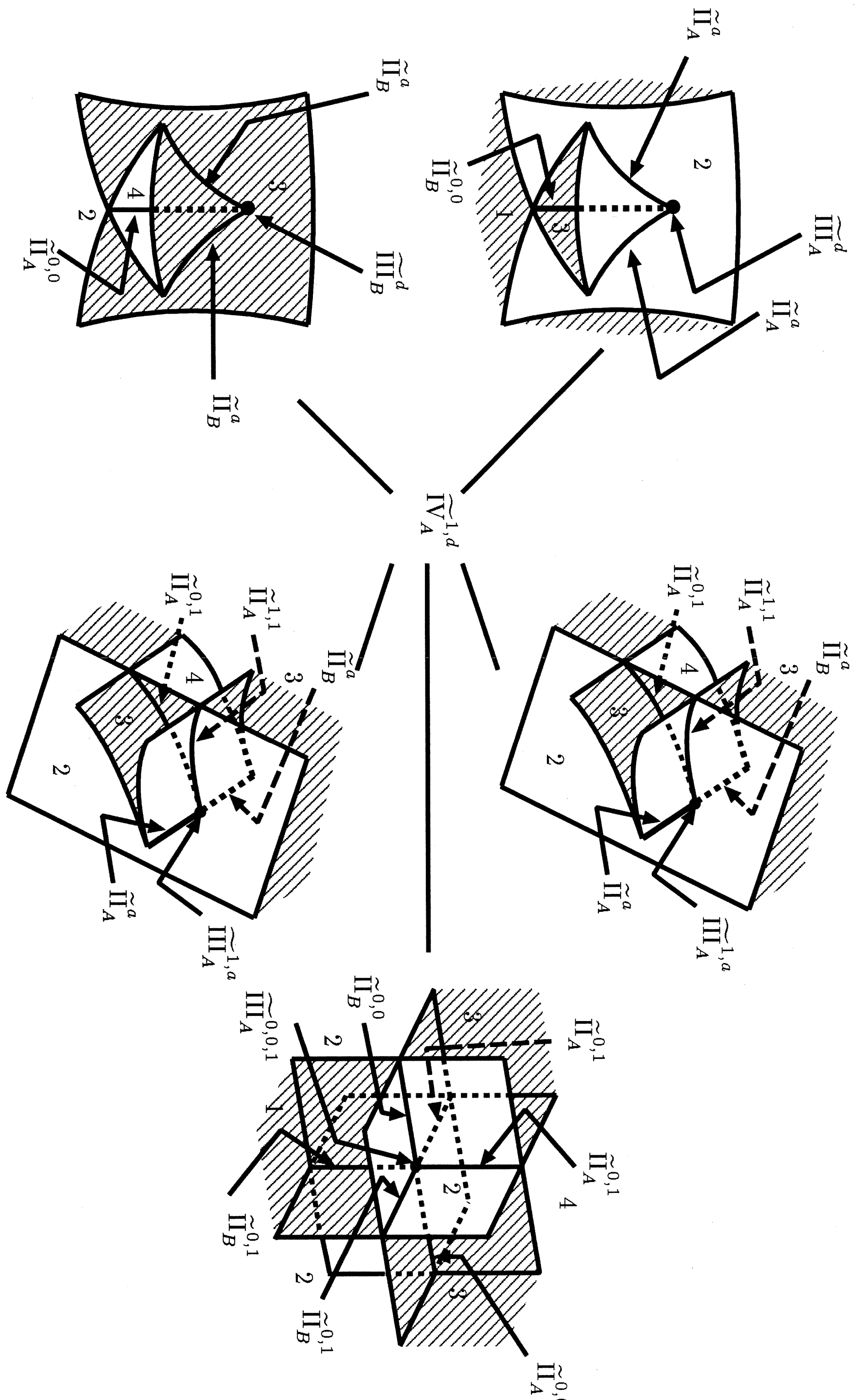

FIGURE 2.159. Type A for $\widetilde{\mathrm{IV}}^{1,d}$

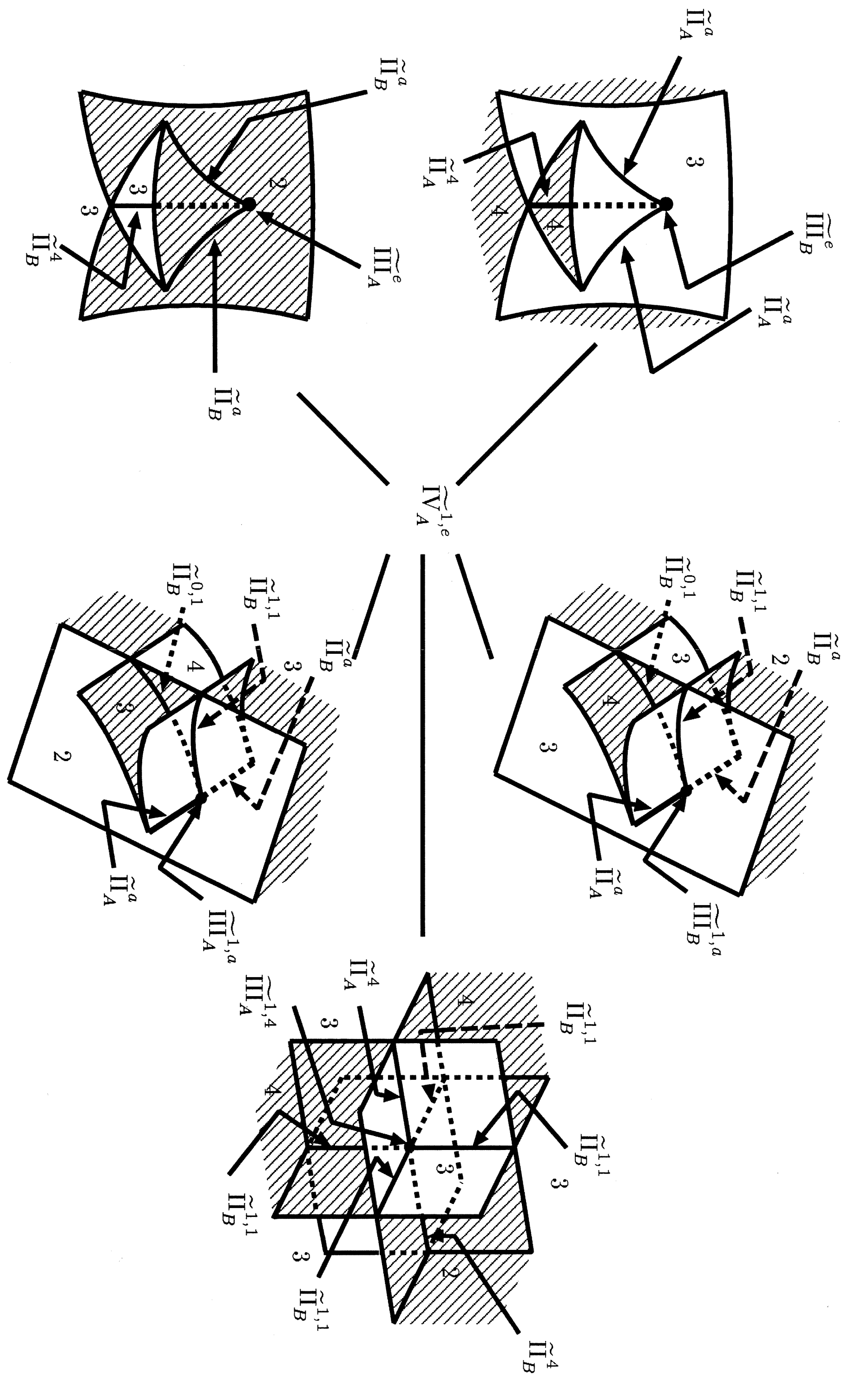

FIGURE 2.160. Type A for $\widetilde{\mathrm{IV}}^{1,e}$

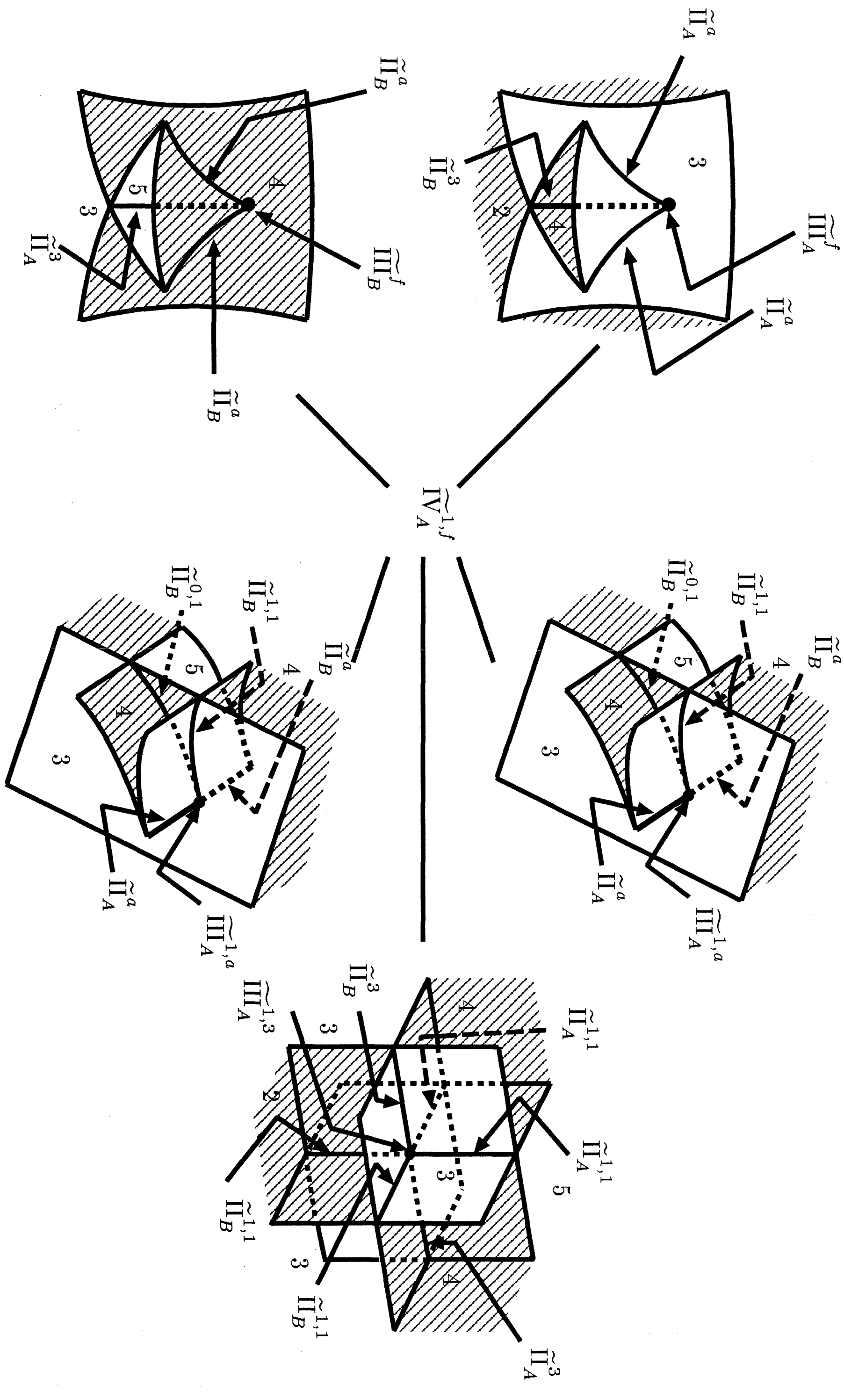

FIGURE 2.161. Type A for $\widetilde{\mathrm{IV}}^{1,f}$

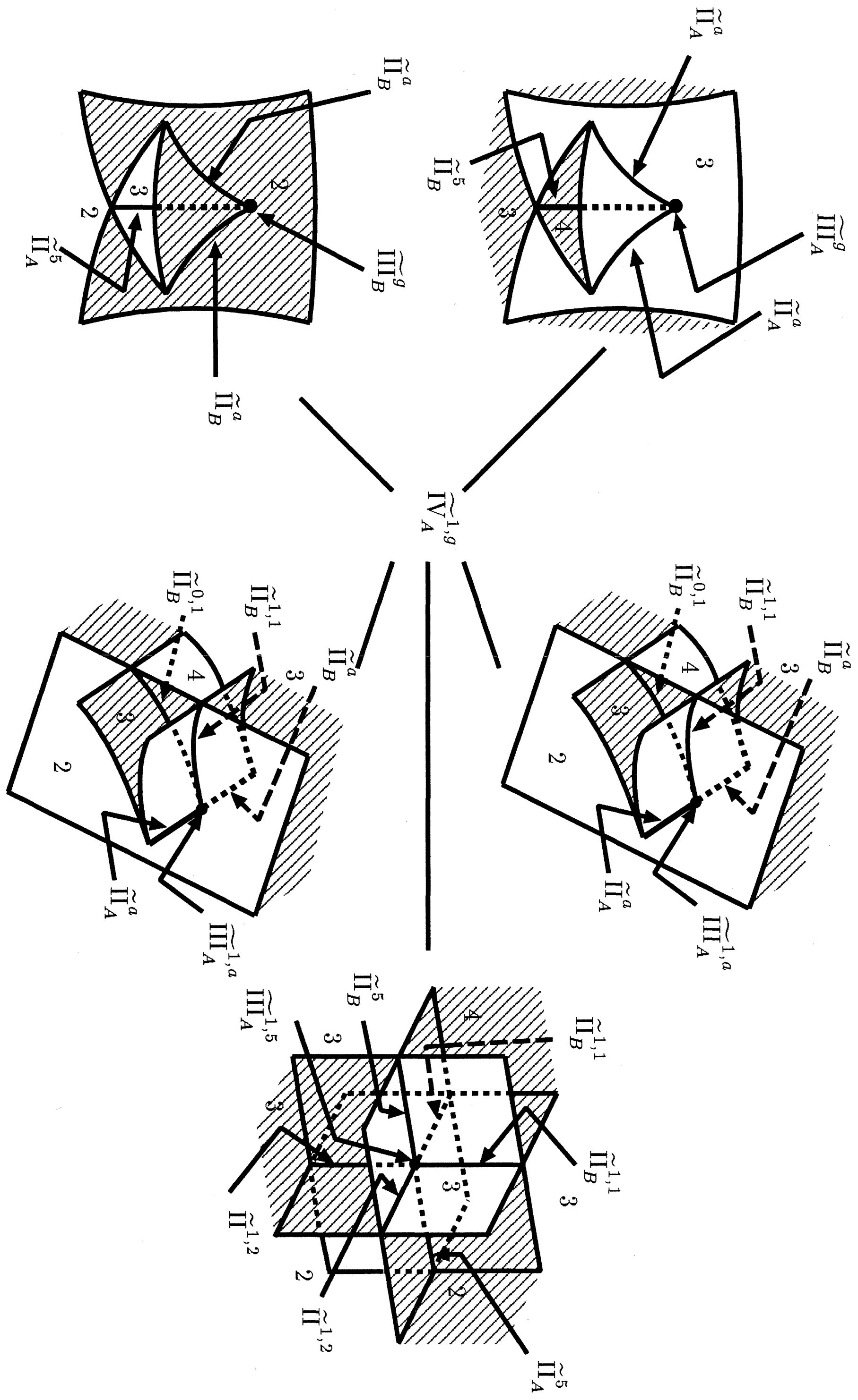

FIGURE 2.162. Type A for $\widetilde{\mathrm{IV}}^{1,g}$

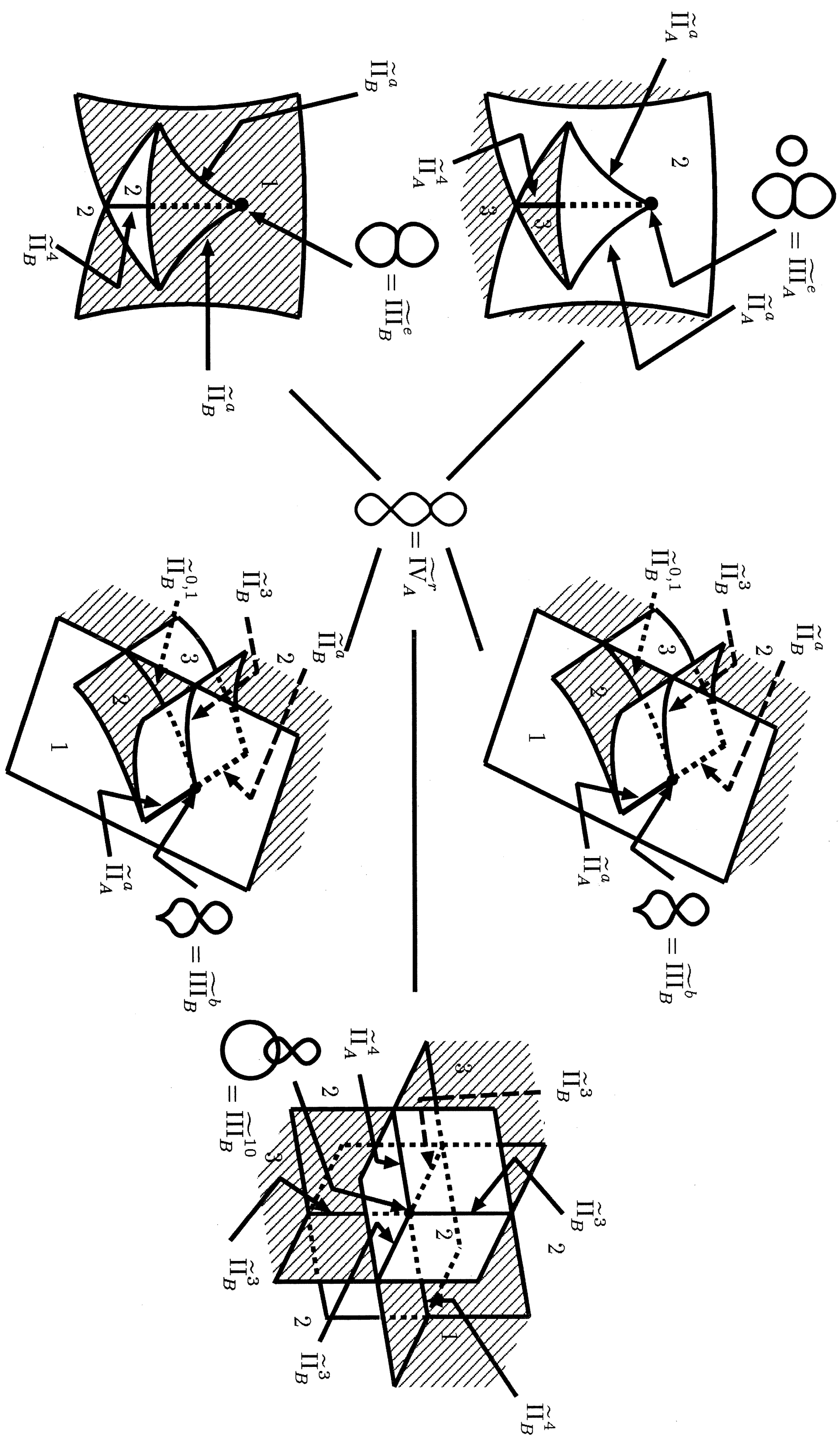

FIGURE 2.163. Type A for $\widetilde{\mathrm{IV}}^r$

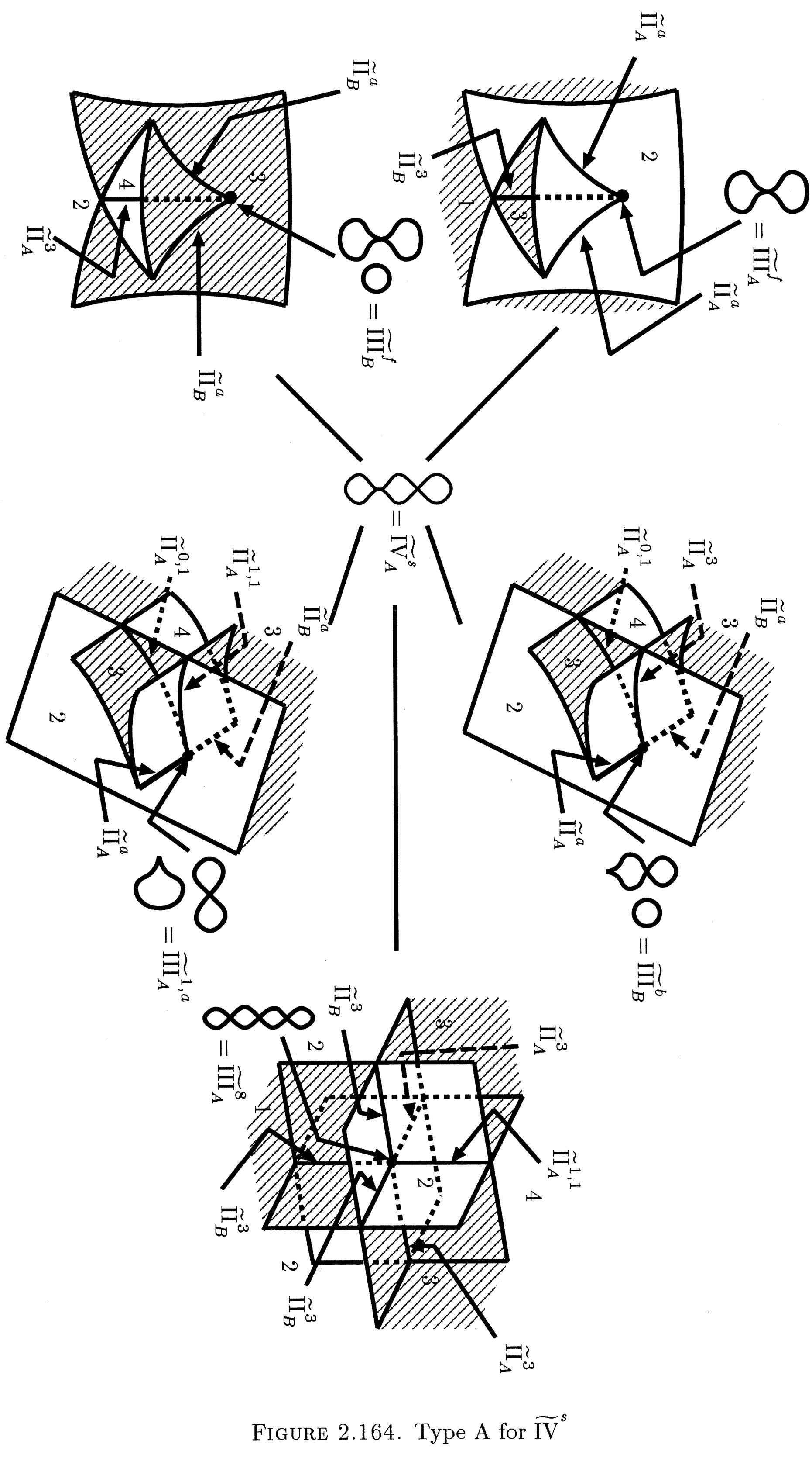

FIGURE 2.164. Type A for $\widetilde{\mathrm{IV}}^s$

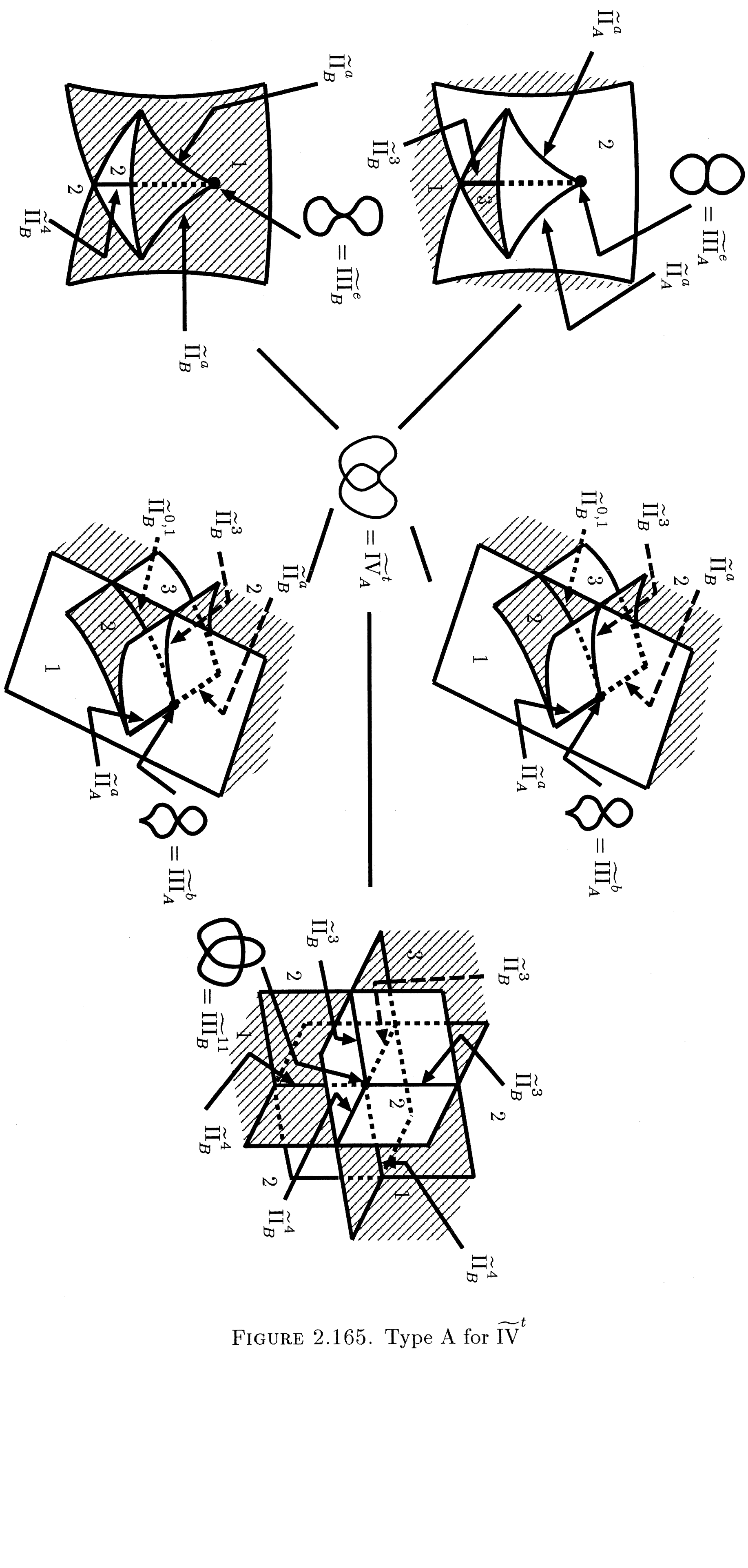

FIGURE 2.165. Type A for $\widetilde{\mathrm{IV}}^t$

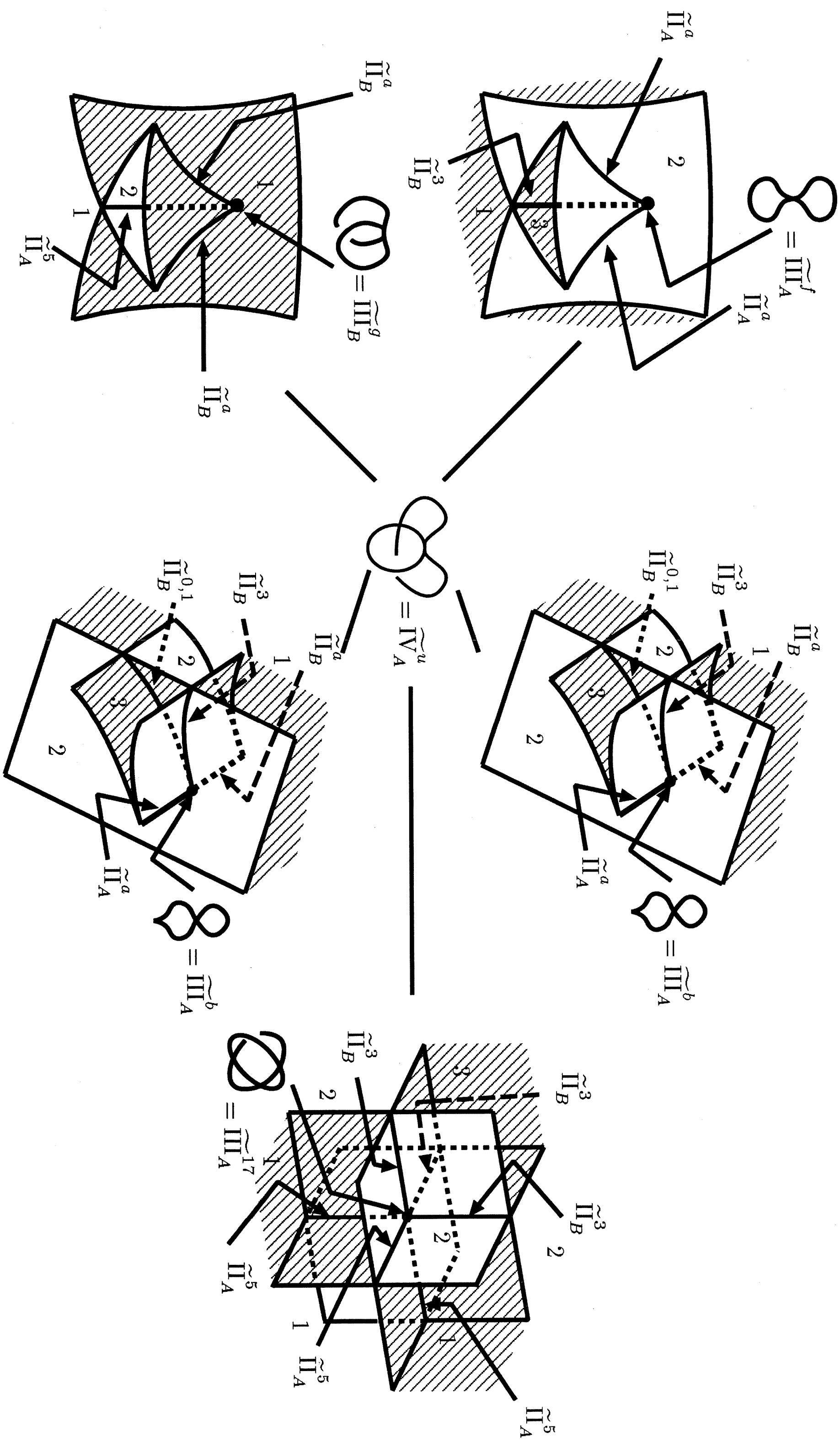

FIGURE 2.166. Type A for $\widetilde{\mathrm{IV}}^u$

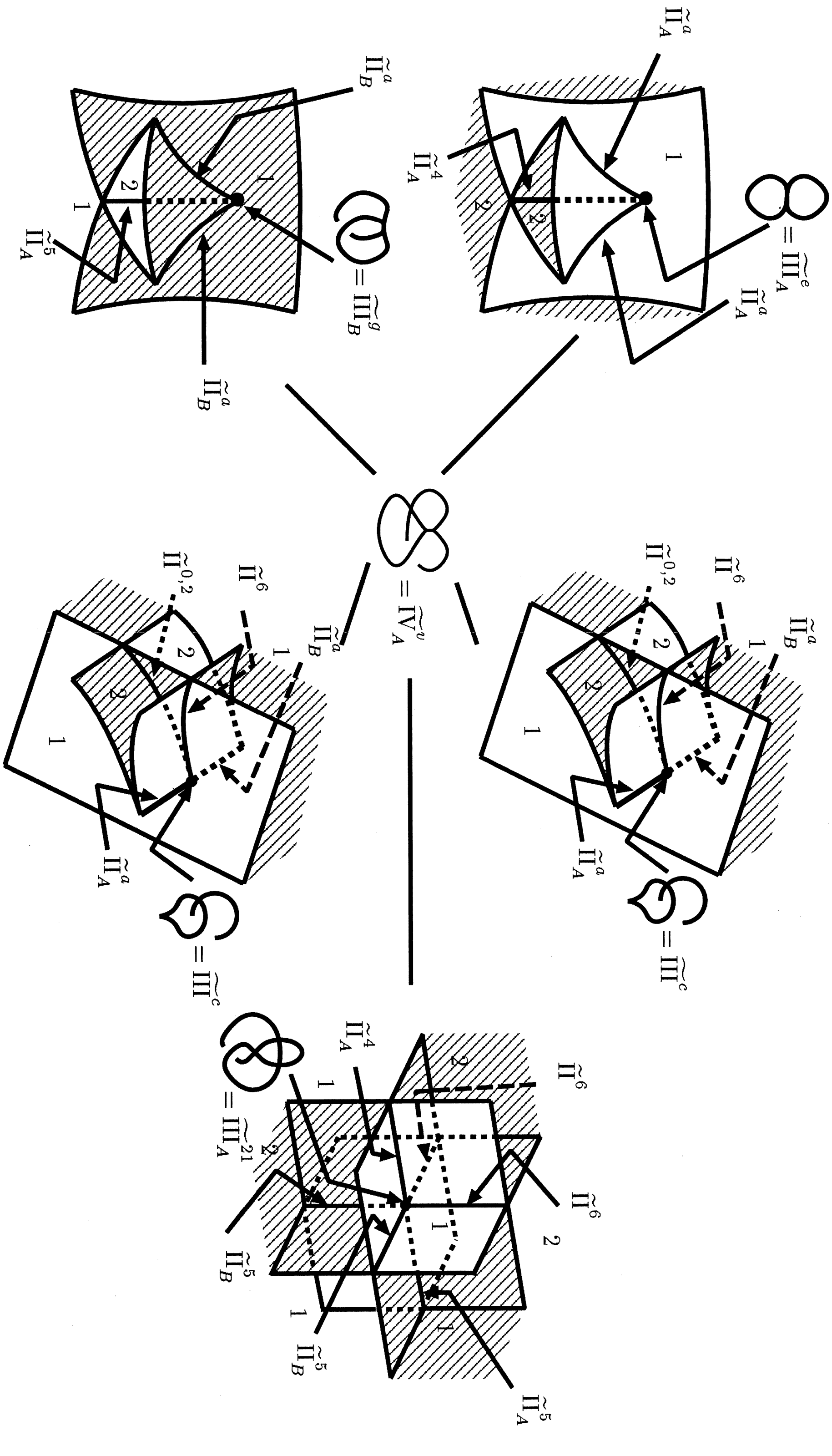

FIGURE 2.167. Type A for $\widetilde{\mathrm{IV}}^{v}$

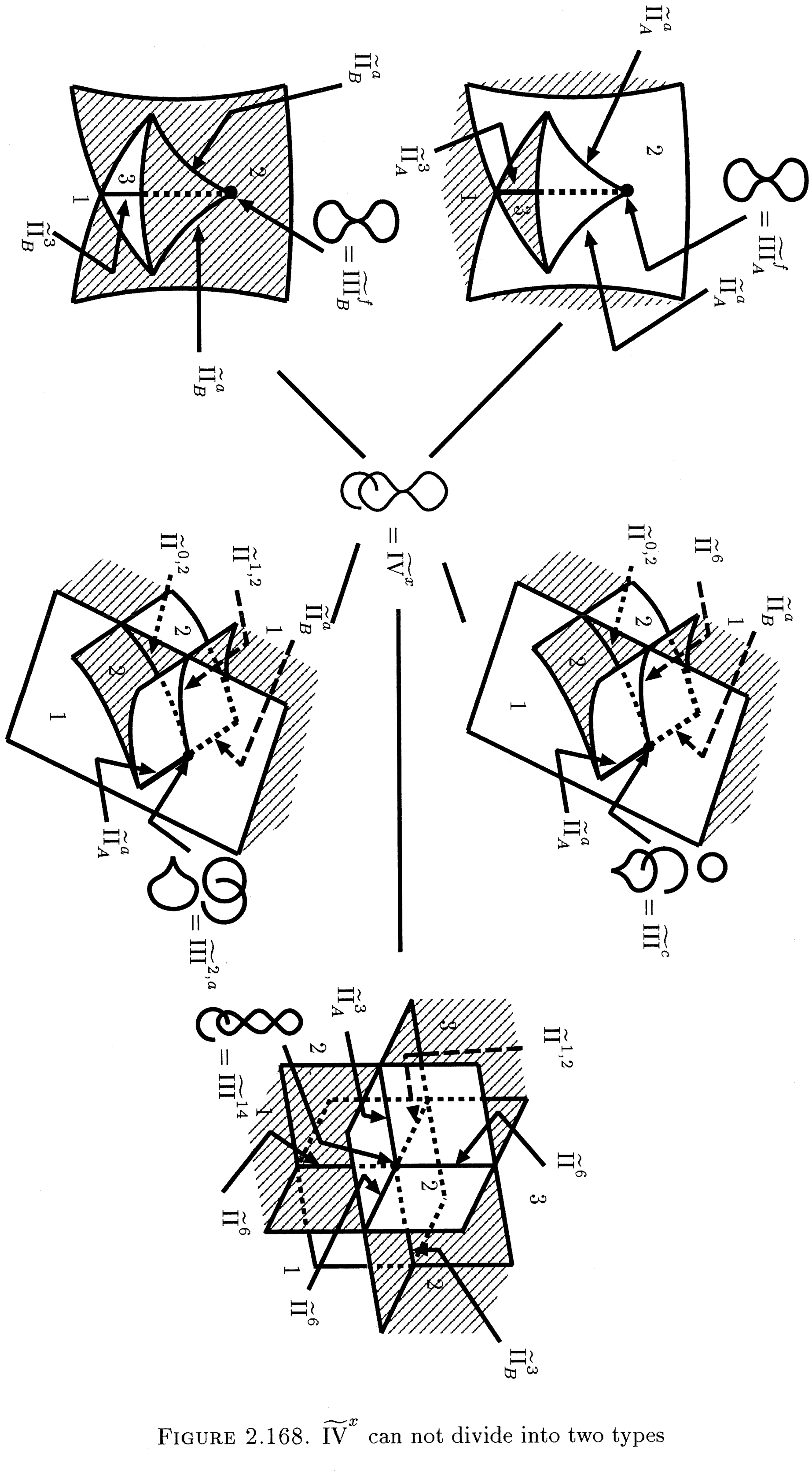

FIGURE 2.168. $\widetilde{\mathrm{IV}}^x$ can not divide into two types

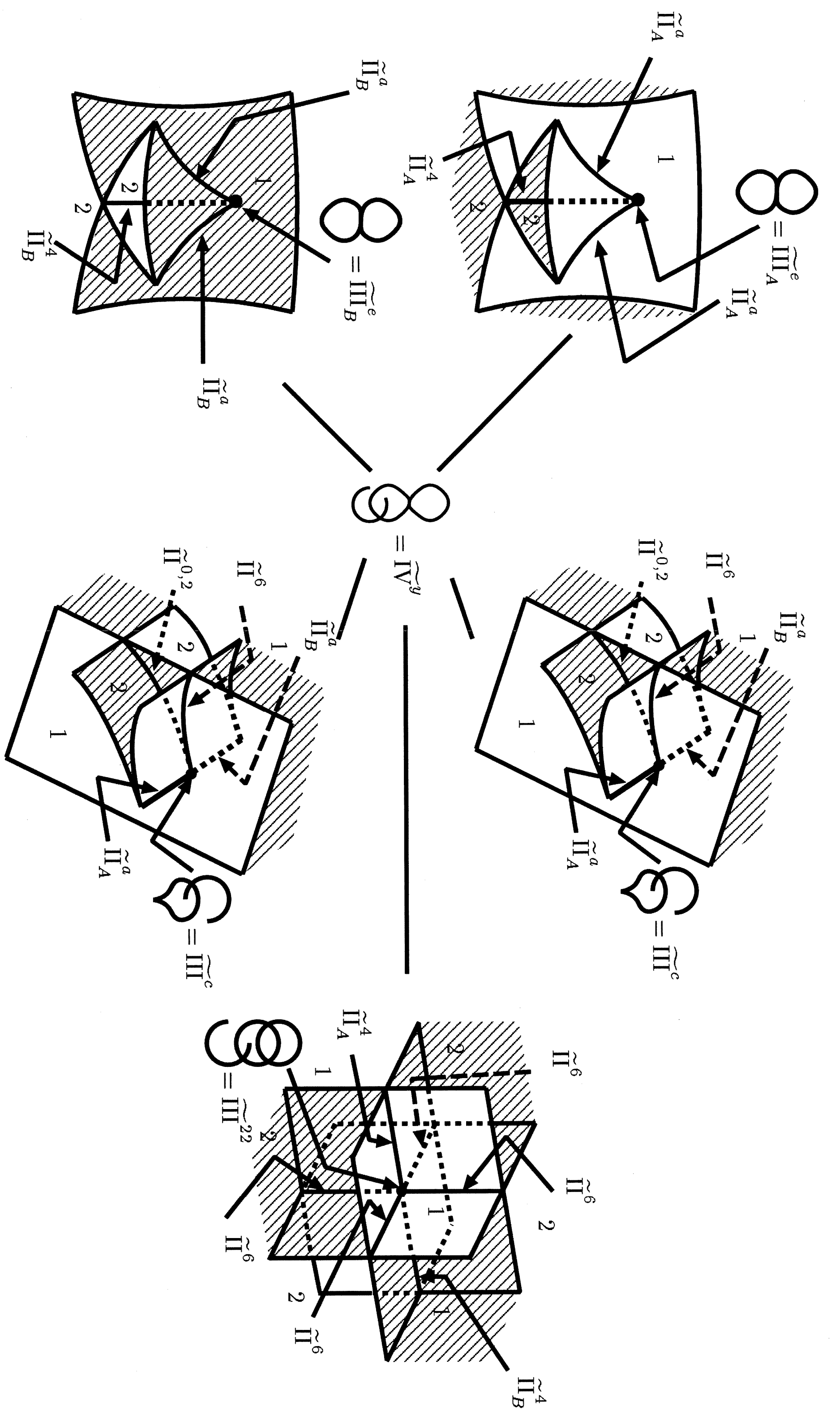

FIGURE 2.169. $\widetilde{\mathrm{IV}}^y$ can not divide into two types

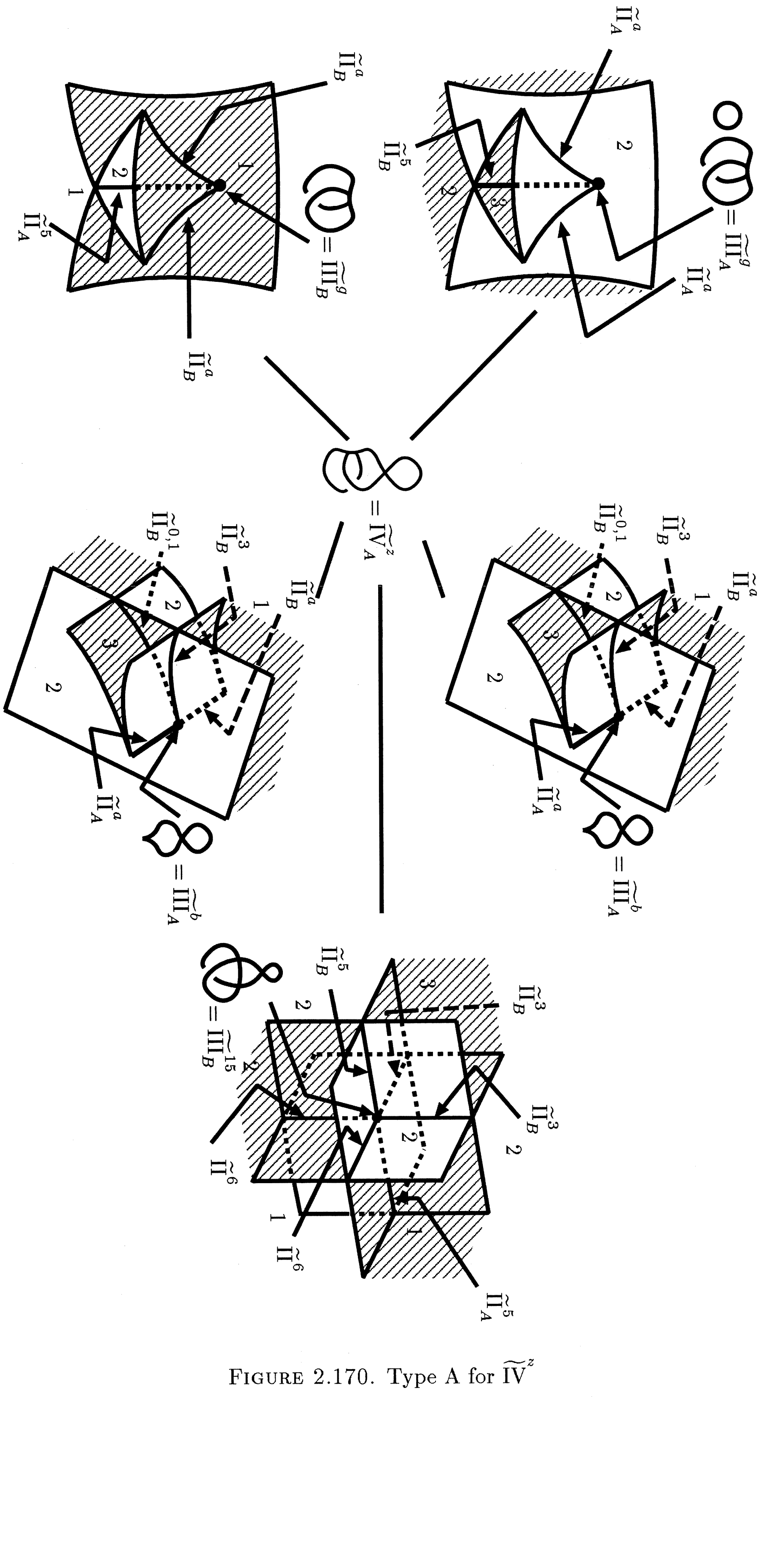

FIGURE 2.170. Type A for $\widetilde{\mathrm{IV}}^z$

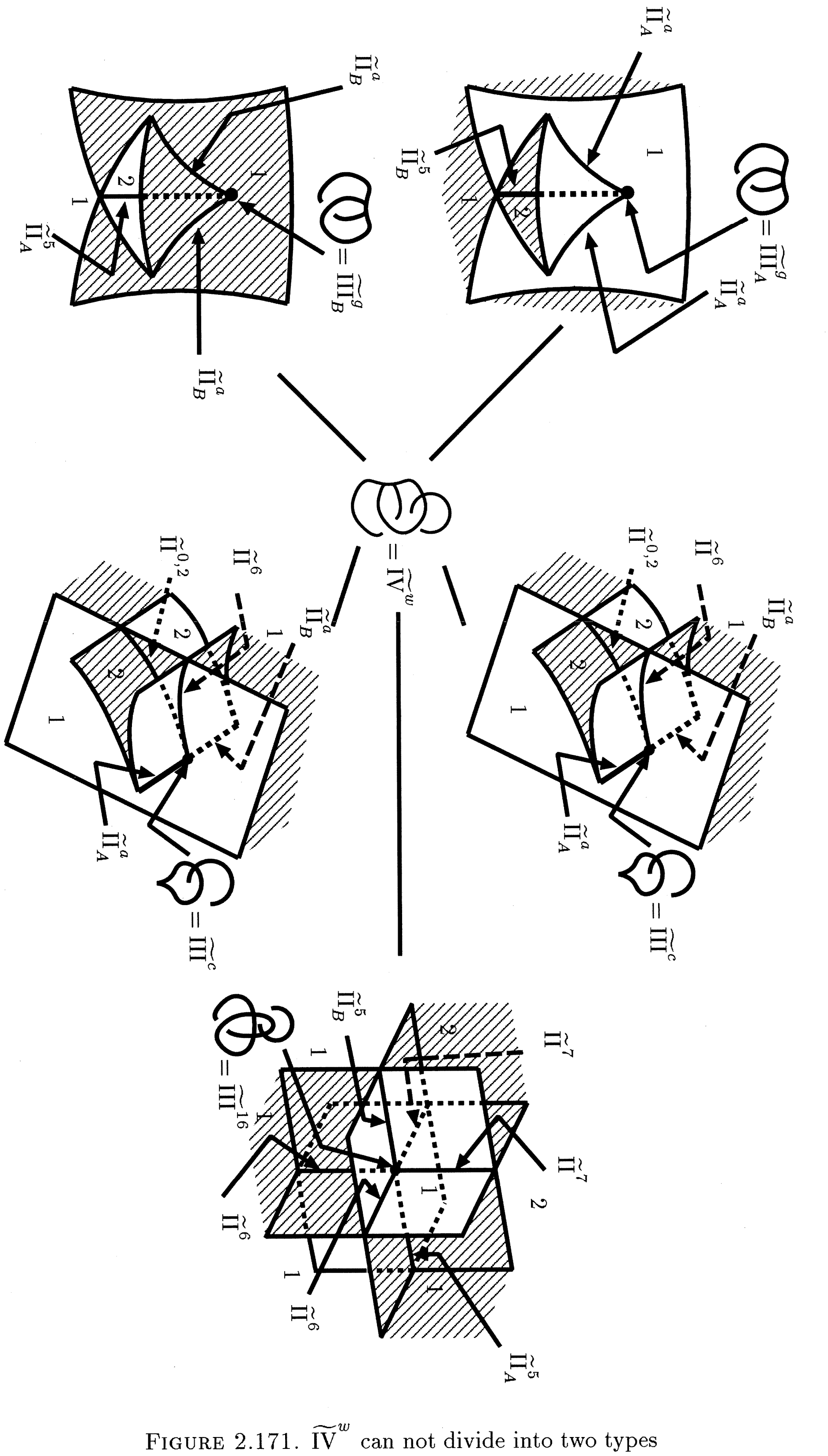

FIGURE 2.171. $\widetilde{\mathrm{IV}}^{w}$ can not divide into two types

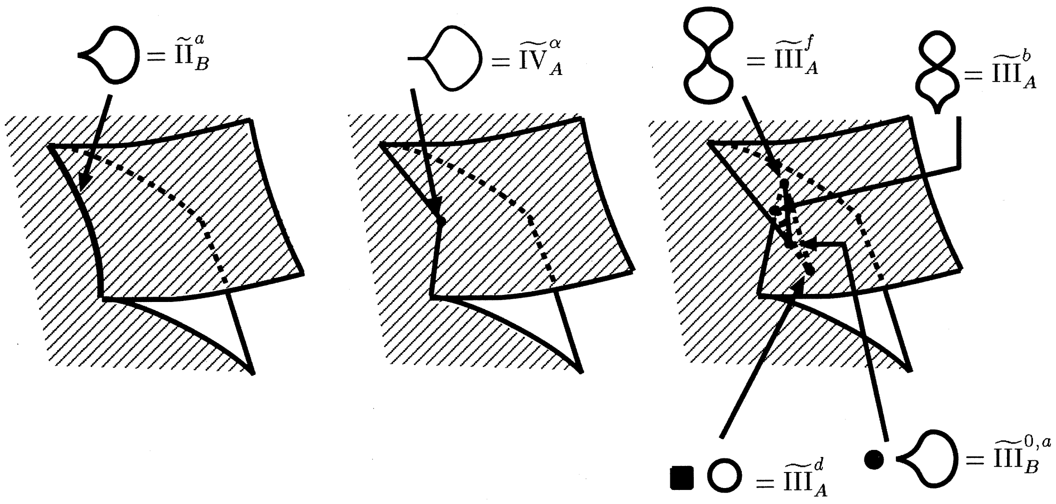

FIGURE 2.172. Type A for $\widetilde{\mathrm{IV}}^\alpha$

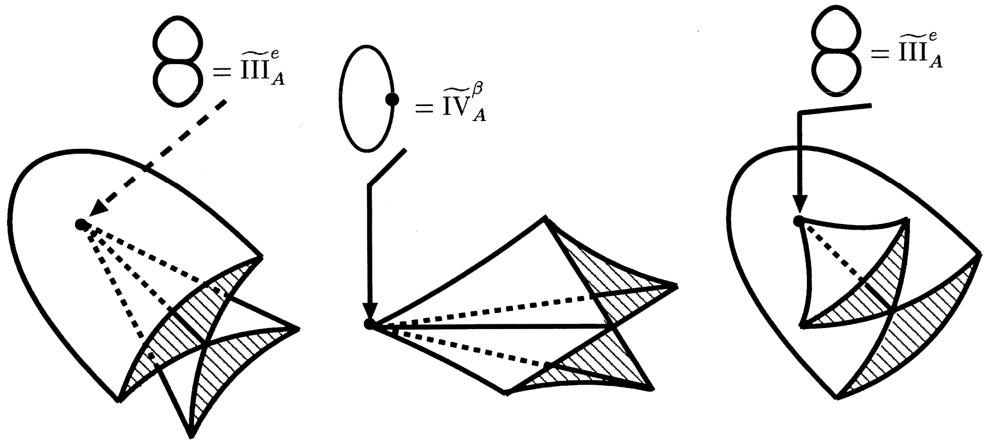

FIGURE 2.173. Type A for $\widetilde{\mathrm{IV}}^\beta$

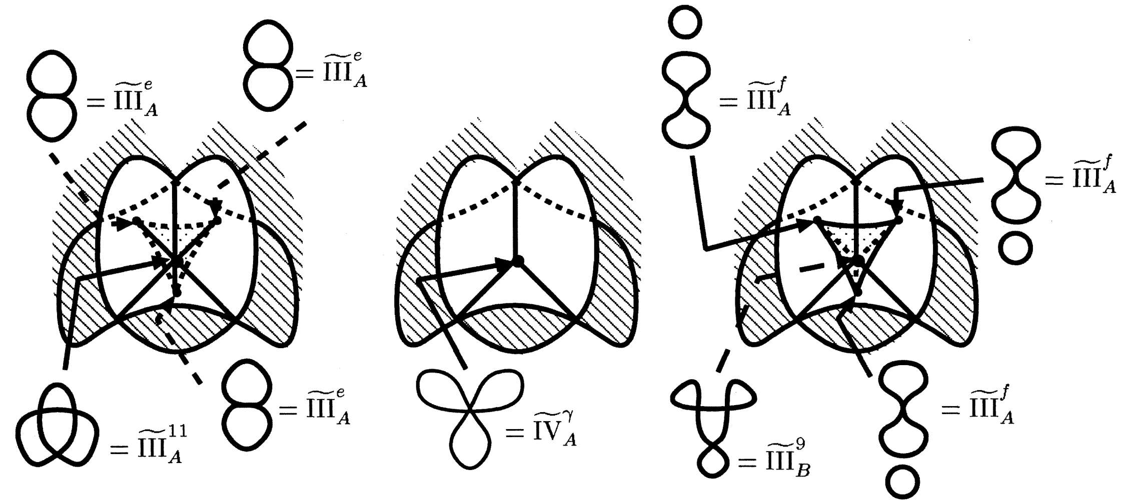

FIGURE 2.174. Type A for $\widetilde{\mathrm{IV}}^\gamma$

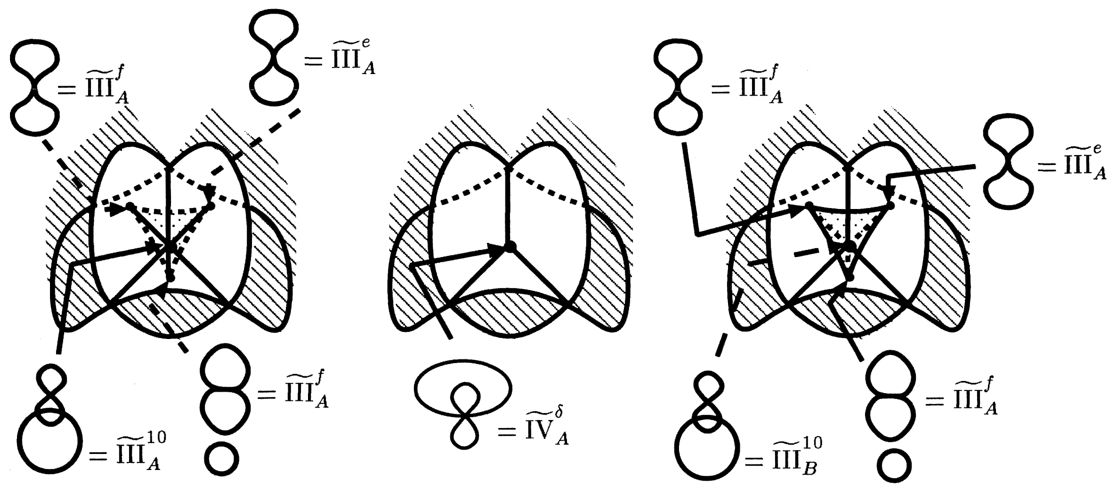

FIGURE 2.175. Type A for $\widetilde{\mathrm{IV}}^\delta$

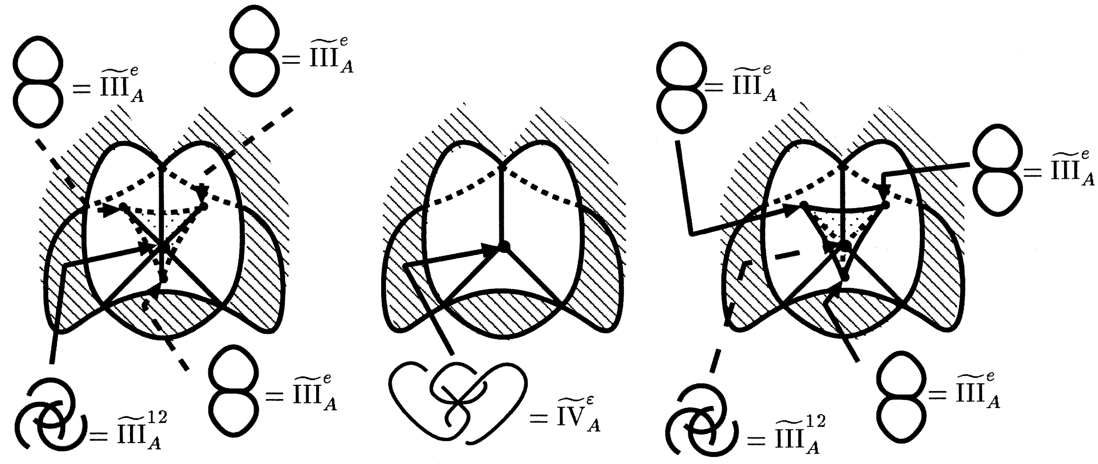

FIGURE 2.176. Type A for $\widetilde{\mathrm{IV}}^{\varepsilon}$

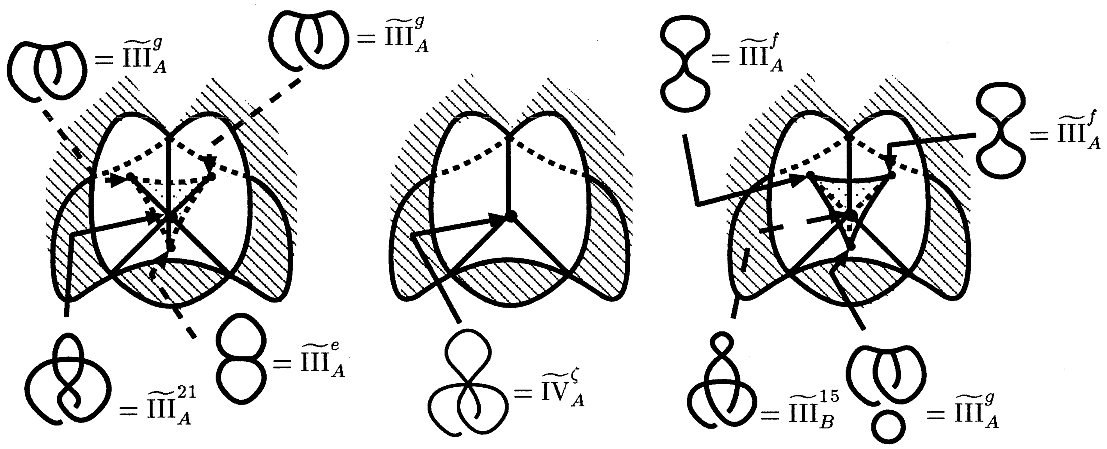

FIGURE 2.177. Type A for $\widetilde{\mathrm{IV}}^{\zeta}$

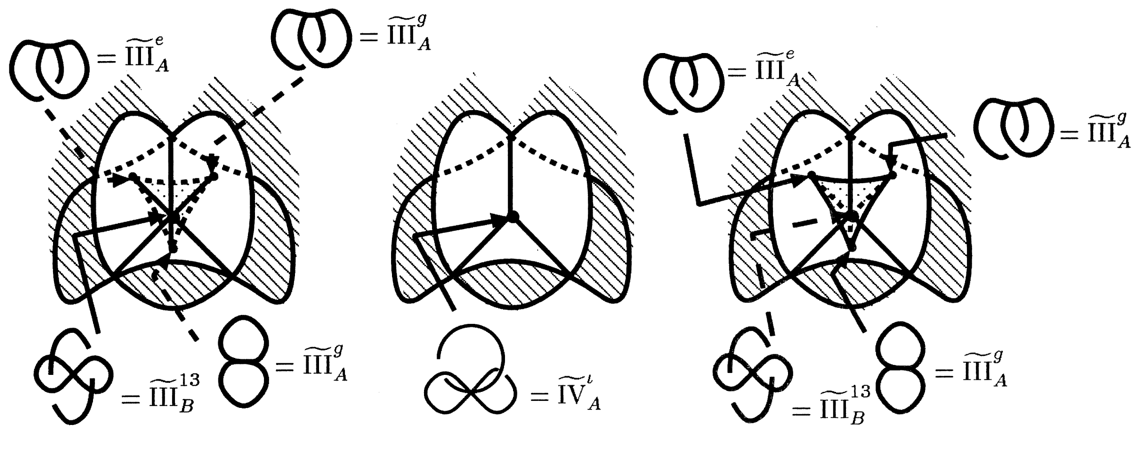

FIGURE 2.178. Type A for $\widetilde{\mathrm{IV}}^{\iota}$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,a	1,a	b	d	e	f	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$\mathrm{IV}^{0,0,0,0}_{o,A}(f)$	4																								
$\mathrm{IV}^{0,0,0,0}_{o,B}(f)$																									
$\mathrm{IV}^{0,0,0,0}_{e,A}(f)$																									
$\mathrm{IV}^{0,0,0,0}_{e,B}(f)$	4																								
$\mathrm{IV}^{0,0,0,1}_{o,A}(f)$		3																							
$\mathrm{IV}^{0,0,0,1}_{o,B}(f)$	1																								
$\mathrm{IV}^{0,0,0,1}_{e,A}(f)$	1																								
$\mathrm{IV}^{0,0,0,1}_{e,B}(f)$		3																							
$\mathrm{IV}^{0,0,1,1}_{o,A}(f)$			2																						
$\mathrm{IV}^{0,0,1,1}_{o,B}(f)$		2																							
$\mathrm{IV}^{0,0,1,1}_{e,A}(f)$		2																							
$\mathrm{IV}^{0,0,1,1}_{e,B}(f)$			2																						
$\mathrm{IV}^{0,1,1,1}_{o,A}(f)$				1																					
$\mathrm{IV}^{0,1,1,1}_{o,B}(f)$			3																						
$\mathrm{IV}^{0,1,1,1}_{e,A}(f)$			3																						
$\mathrm{IV}^{0,1,1,1}_{e,B}(f)$				1																					
$\mathrm{IV}^{1,1,1,1}_{o,A}(f)$																									
$\mathrm{IV}^{1,1,1,1}_{o,B}(f)$				4																					
$\mathrm{IV}^{1,1,1,1}_{e,A}(f)$				4																					
$\mathrm{IV}^{1,1,1,1}_{e,B}(f)$																									
$\mathrm{IV}^{0,0,0,2}_{o}(f)$																				3					
$\mathrm{IV}^{0,0,0,2}_{e}(f)$	1																			3					
$\mathrm{IV}^{0,0,2,2}_{o}(f)$																					2				
$\mathrm{IV}^{0,0,2,2}_{e}(f)$																				4	2				
$\mathrm{IV}^{0,2,2,2}_{o}(f)$																					6				1
$\mathrm{IV}^{0,2,2,2}_{e}(f)$																									1
$\mathrm{IV}^{1,1,1,2}_{o}(f)$																						3			
$\mathrm{IV}^{1,1,1,2}_{e}(f)$				1																		3			
$\mathrm{IV}^{1,1,2,2}_{o}(f)$																							2		
$\mathrm{IV}^{1,1,2,2}_{e}(f)$																						4	2		
$\mathrm{IV}^{1,2,2,2}_{o}(f)$																							6		1
$\mathrm{IV}^{1,2,2,2}_{e}(f)$																									1
$\mathrm{IV}^{0,0,1,2}_{o}(f)$																				1				2	
$\mathrm{IV}^{0,0,1,2}_{e}(f)$		1																		1				2	
$\mathrm{IV}^{0,1,1,2}_{o}(f)$																						1		2	
$\mathrm{IV}^{0,1,1,2}_{e}(f)$			1																			1		2	
$\mathrm{IV}^{0,1,2,2}_{o}(f)$																					1		1		
$\mathrm{IV}^{0,1,2,2}_{e}(f)$																					1		1	4	
$\mathrm{IV}^{2,2,2,2}_{o}(f)$																									
$\mathrm{IV}^{2,2,2,2}_{e}(f)$																									8

TABLE 2.4. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_{A} : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,a	c	g
$\mathrm{IV}^{0,0,0,0}_{\mathrm{o},A}(f)$																												
$\mathrm{IV}^{0,0,0,0}_{\mathrm{o},B}(f)$																												
$\mathrm{IV}^{0,0,0,0}_{\mathrm{e},A}(f)$																												
$\mathrm{IV}^{0,0,0,0}_{\mathrm{e},B}(f)$																												
$\mathrm{IV}^{0,0,0,1}_{\mathrm{o},A}(f)$																												
$\mathrm{IV}^{0,0,0,1}_{\mathrm{o},B}(f)$																												
$\mathrm{IV}^{0,0,0,1}_{\mathrm{e},A}(f)$																												
$\mathrm{IV}^{0,0,0,1}_{\mathrm{e},B}(f)$																												
$\mathrm{IV}^{0,0,1,1}_{\mathrm{o},A}(f)$																												
$\mathrm{IV}^{0,0,1,1}_{\mathrm{o},B}(f)$																												
$\mathrm{IV}^{0,0,1,1}_{\mathrm{e},A}(f)$																												
$\mathrm{IV}^{0,0,1,1}_{\mathrm{e},B}(f)$																												
$\mathrm{IV}^{0,1,1,1}_{\mathrm{o},A}(f)$																												
$\mathrm{IV}^{0,1,1,1}_{\mathrm{o},B}(f)$																												
$\mathrm{IV}^{0,1,1,1}_{\mathrm{e},A}(f)$																												
$\mathrm{IV}^{0,1,1,1}_{\mathrm{e},B}(f)$																												
$\mathrm{IV}^{1,1,1,1}_{\mathrm{o},A}(f)$																												
$\mathrm{IV}^{1,1,1,1}_{\mathrm{o},B}(f)$																												
$\mathrm{IV}^{1,1,1,1}_{\mathrm{e},A}(f)$																												
$\mathrm{IV}^{1,1,1,1}_{\mathrm{e},B}(f)$																												
$\mathrm{IV}^{0,0,0,2}_{\mathrm{o}}(f)$																												
$\mathrm{IV}^{0,0,0,2}_{\mathrm{e}}(f)$																												
$\mathrm{IV}^{0,0,2,2}_{\mathrm{o}}(f)$																												
$\mathrm{IV}^{0,0,2,2}_{\mathrm{e}}(f)$																												
$\mathrm{IV}^{0,2,2,2}_{\mathrm{o}}(f)$																												
$\mathrm{IV}^{0,2,2,2}_{\mathrm{e}}(f)$																												
$\mathrm{IV}^{1,1,1,2}_{\mathrm{o}}(f)$																												
$\mathrm{IV}^{1,1,1,2}_{\mathrm{e}}(f)$																												
$\mathrm{IV}^{1,1,2,2}_{\mathrm{o}}(f)$																												
$\mathrm{IV}^{1,1,2,2}_{\mathrm{e}}(f)$																												
$\mathrm{IV}^{1,2,2,2}_{\mathrm{o}}(f)$																												
$\mathrm{IV}^{1,2,2,2}_{\mathrm{e}}(f)$																												
$\mathrm{IV}^{0,0,1,2}_{\mathrm{o}}(f)$																												
$\mathrm{IV}^{0,0,1,2}_{\mathrm{e}}(f)$																												
$\mathrm{IV}^{0,1,1,2}_{\mathrm{o}}(f)$																												
$\mathrm{IV}^{0,1,1,2}_{\mathrm{e}}(f)$																												
$\mathrm{IV}^{0,1,2,2}_{\mathrm{o}}(f)$																												
$\mathrm{IV}^{0,1,2,2}_{\mathrm{e}}(f)$																												
$\mathrm{IV}^{2,2,2,2}_{\mathrm{o}}(f)$																												
$\mathrm{IV}^{2,2,2,2}_{\mathrm{e}}(f)$																												

TABLE 2.5. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,a	1,a	b	d	e	f	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$\mathrm{IV}^{0,0,3}_{o,A}(f)$		2			2																				
$\mathrm{IV}^{0,0,3}_{o,B}(f)$																									
$\mathrm{IV}^{0,0,3}_{e,A}(f)$																									
$\mathrm{IV}^{0,0,3}_{e,B}(f)$		2			2																				
$\mathrm{IV}^{0,0,4}_{o,A}(f)$		4					2																		
$\mathrm{IV}^{0,0,4}_{o,B}(f)$																									
$\mathrm{IV}^{0,0,4}_{e,A}(f)$																									
$\mathrm{IV}^{0,0,4}_{e,B}(f)$							2																		
$\mathrm{IV}^{0,0,5}_{o,A}(f)$		2																		2					
$\mathrm{IV}^{0,0,5}_{o,B}(f)$																				2					
$\mathrm{IV}^{0,0,5}_{e,A}(f)$																									
$\mathrm{IV}^{0,0,5}_{e,B}(f)$																									
$\mathrm{IV}^{0,0,6}_{o}(f)$		1																		1					
$\mathrm{IV}^{0,0,6}_{e}(f)$																				1					
$\mathrm{IV}^{0,0,7}_{o}(f)$																				4					
$\mathrm{IV}^{0,0,7}_{e}(f)$																									
$\mathrm{IV}^{0,1,3}_{o,A}(f)$			2			1																			
$\mathrm{IV}^{0,1,3}_{o,B}(f)$					1																				
$\mathrm{IV}^{0,1,3}_{e,A}(f)$					1																				
$\mathrm{IV}^{0,1,3}_{e,B}(f)$			2			1																			
$\mathrm{IV}^{0,1,4}_{o,A}(f)$			4					1																	
$\mathrm{IV}^{0,1,4}_{o,B}(f)$							1																		
$\mathrm{IV}^{0,1,4}_{e,A}(f)$							1																		
$\mathrm{IV}^{0,1,4}_{e,B}(f)$								1																	
$\mathrm{IV}^{0,1,5}_{o,A}(f)$			2																					2	
$\mathrm{IV}^{0,1,5}_{o,B}(f)$																								2	
$\mathrm{IV}^{0,1,5}_{e,A}(f)$																									
$\mathrm{IV}^{0,1,5}_{e,B}(f)$																									
$\mathrm{IV}^{0,1,6}_{o}(f)$			1																					1	
$\mathrm{IV}^{0,1,6}_{e}(f)$																								1	
$\mathrm{IV}^{0,1,7}_{o}(f)$																								4	
$\mathrm{IV}^{0,1,7}_{e}(f)$																									
$\mathrm{IV}^{1,1,3}_{o,A}(f)$				2																					
$\mathrm{IV}^{1,1,3}_{o,B}(f)$						2																			
$\mathrm{IV}^{1,1,3}_{e,A}(f)$						2																			
$\mathrm{IV}^{1,1,3}_{e,B}(f)$				2																					

TABLE 2.6. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,a	c	g
$\mathrm{IV}^{0,0,3}_{o,A}(f)$																												
$\mathrm{IV}^{0,0,3}_{o,B}(f)$																												
$\mathrm{IV}^{0,0,3}_{e,A}(f)$																												
$\mathrm{IV}^{0,0,3}_{e,B}(f)$																												
$\mathrm{IV}^{0,0,4}_{o,A}(f)$																												
$\mathrm{IV}^{0,0,4}_{o,B}(f)$																												
$\mathrm{IV}^{0,0,4}_{e,A}(f)$																												
$\mathrm{IV}^{0,0,4}_{e,B}(f)$																												
$\mathrm{IV}^{0,0,5}_{o,A}(f)$	2																											
$\mathrm{IV}^{0,0,5}_{o,B}(f)$																												
$\mathrm{IV}^{0,0,5}_{e,A}(f)$																												
$\mathrm{IV}^{0,0,5}_{e,B}(f)$	2																											
$\mathrm{IV}^{0,0,6}_{o}(f)$		2																										
$\mathrm{IV}^{0,0,6}_{e}(f)$		2																										
$\mathrm{IV}^{0,0,7}_{o}(f)$			2																									
$\mathrm{IV}^{0,0,7}_{e}(f)$			2																									
$\mathrm{IV}^{0,1,3}_{o,A}(f)$																												
$\mathrm{IV}^{0,1,3}_{o,B}(f)$																												
$\mathrm{IV}^{0,1,3}_{e,A}(f)$																												
$\mathrm{IV}^{0,1,3}_{e,B}(f)$																												
$\mathrm{IV}^{0,1,4}_{o,A}(f)$																												
$\mathrm{IV}^{0,1,4}_{o,B}(f)$																												
$\mathrm{IV}^{0,1,4}_{e,A}(f)$																												
$\mathrm{IV}^{0,1,4}_{e,B}(f)$																												
$\mathrm{IV}^{0,1,5}_{o,A}(f)$				1																								
$\mathrm{IV}^{0,1,5}_{o,B}(f)$	1																											
$\mathrm{IV}^{0,1,5}_{e,A}(f)$	1																											
$\mathrm{IV}^{0,1,5}_{e,B}(f)$				1																								
$\mathrm{IV}^{0,1,6}_{o}(f)$		1			1																							
$\mathrm{IV}^{0,1,6}_{e}(f)$		1			1																							
$\mathrm{IV}^{0,1,7}_{o}(f)$			1			1																						
$\mathrm{IV}^{0,1,7}_{e}(f)$			1			1																						
$\mathrm{IV}^{1,1,3}_{o,A}(f)$																												
$\mathrm{IV}^{1,1,3}_{o,B}(f)$																												
$\mathrm{IV}^{1,1,3}_{e,A}(f)$																												
$\mathrm{IV}^{1,1,3}_{e,B}(f)$																												

TABLE 2.7. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,a	1,a	b	d	e	f	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$IV^{1,1,4}_{o,A}(f)$				4																					
$IV^{1,1,4}_{o,B}(f)$								2																	
$IV^{1,1,4}_{e,A}(f)$								2																	
$IV^{1,1,4}_{e,B}(f)$																									
$IV^{1,1,5}_{o,A}(f)$				2																		2			
$IV^{1,1,5}_{o,B}(f)$																						2			
$IV^{1,1,5}_{e,A}(f)$																									
$IV^{1,1,5}_{e,B}(f)$																									
$IV^{1,1,6}_{o}(f)$				1																		1			
$IV^{1,1,6}_{e}(f)$																						1			
$IV^{1,1,7}_{o}(f)$																						4			
$IV^{1,1,7}_{e}(f)$																									
$IV^{0,2,3}_{o}(f)$																								2	
$IV^{0,2,3}_{e}(f)$					1																			2	
$IV^{0,2,4}_{o}(f)$																								4	
$IV^{0,2,4}_{e}(f)$							1																		
$IV^{0,2,5}_{o}(f)$																					2			2	
$IV^{0,2,5}_{e}(f)$																									
$IV^{0,2,6}_{o}(f)$																					1			2	
$IV^{0,2,6}_{e}(f)$																					1				
$IV^{0,2,7}_{o}(f)$																					4				
$IV^{0,2,7}_{e}(f)$																									
$IV^{1,2,3}_{o}(f)$																						2			
$IV^{1,2,3}_{e}(f)$						1																2			
$IV^{1,2,4}_{o}(f)$																						4			
$IV^{1,2,4}_{e}(f)$								1																	
$IV^{1,2,5}_{o}(f)$																						2	2		
$IV^{1,2,5}_{e}(f)$																									
$IV^{1,2,6}_{o}(f)$																						2	1		
$IV^{1,2,6}_{e}(f)$																							1		
$IV^{1,2,7}_{o}(f)$																							4		
$IV^{1,2,7}_{e}(f)$																									
$IV^{2,2,3}_{o}(f)$																							2		
$IV^{2,2,3}_{e}(f)$																							2		
$IV^{2,2,4}_{o}(f)$																							4		
$IV^{2,2,4}_{e}(f)$																									
$IV^{2,2,5}_{o}(f)$																							2		2
$IV^{2,2,5}_{e}(f)$																									

TABLE 2.8. Table of incident numbers of $[III^*_{0,A} : \mathcal{F}]$ or $[III^*_A : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,a	c	g
$\mathrm{IV}^{1,1,4}_{o,A}(f)$																												
$\mathrm{IV}^{1,1,4}_{o,B}(f)$																												
$\mathrm{IV}^{1,1,4}_{e,A}(f)$																												
$\mathrm{IV}^{1,1,4}_{e,B}(f)$																												
$\mathrm{IV}^{1,1,5}_{o,A}(f)$																												
$\mathrm{IV}^{1,1,5}_{o,B}(f)$				2																								
$\mathrm{IV}^{1,1,5}_{e,A}(f)$				2																								
$\mathrm{IV}^{1,1,5}_{e,B}(f)$																												
$\mathrm{IV}^{1,1,6}_{o}(f)$					2																							
$\mathrm{IV}^{1,1,6}_{e}(f)$					2																							
$\mathrm{IV}^{1,1,7}_{o}(f)$						2																						
$\mathrm{IV}^{1,1,7}_{e}(f)$						2																						
$\mathrm{IV}^{0,2,3}_{o}(f)$							1																					
$\mathrm{IV}^{0,2,3}_{e}(f)$							1																					
$\mathrm{IV}^{0,2,4}_{o}(f)$								1																				
$\mathrm{IV}^{0,2,4}_{e}(f)$								1																				
$\mathrm{IV}^{0,2,5}_{o}(f)$									1																			
$\mathrm{IV}^{0,2,5}_{e}(f)$	1								1																			
$\mathrm{IV}^{0,2,6}_{o}(f)$										1																		
$\mathrm{IV}^{0,2,6}_{e}(f)$		2								1																		
$\mathrm{IV}^{0,2,7}_{o}(f)$											1																	
$\mathrm{IV}^{0,2,7}_{e}(f)$			2								1																	
$\mathrm{IV}^{1,2,3}_{o}(f)$							1																					
$\mathrm{IV}^{1,2,3}_{e}(f)$							1																					
$\mathrm{IV}^{1,2,4}_{o}(f)$								1																				
$\mathrm{IV}^{1,2,4}_{e}(f)$								1																				
$\mathrm{IV}^{1,2,5}_{o}(f)$									1																			
$\mathrm{IV}^{1,2,5}_{e}(f)$				1					1																			
$\mathrm{IV}^{1,2,6}_{o}(f)$										1																		
$\mathrm{IV}^{1,2,6}_{e}(f)$					2					1																		
$\mathrm{IV}^{1,2,7}_{o}(f)$											1																	
$\mathrm{IV}^{1,2,7}_{e}(f)$						2					1																	
$\mathrm{IV}^{2,2,3}_{o}(f)$							2																					
$\mathrm{IV}^{2,2,3}_{e}(f)$							2																					
$\mathrm{IV}^{2,2,4}_{o}(f)$																												
$\mathrm{IV}^{2,2,4}_{e}(f)$								4																				
$\mathrm{IV}^{2,2,5}_{o}(f)$																												
$\mathrm{IV}^{2,2,5}_{e}(f)$									4																			

TABLE 2.9. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,a	1,a	b	d	e	f	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$IV_{o}^{2,2,6}(f)$																							2		1
$IV_{e}^{2,2,6}(f)$																									1
$IV_{o}^{2,2,7}(f)$																									4
$IV_{e}^{2,2,7}(f)$																									
$IV_{o,A}^{0,0,a}(f)$		1																							
$IV_{o,B}^{0,0,a}(f)$														2											
$IV_{e,A}^{0,0,a}(f)$	1													2											
$IV_{e,B}^{0,0,a}(f)$																									
$IV_{o,A}^{0,1,a}(f)$			1											1											
$IV_{o,B}^{0,1,a}(f)$															1										
$IV_{e,A}^{0,1,a}(f)$		1													1										
$IV_{e,B}^{0,1,a}(f)$														1											
$IV_{o,A}^{1,1,a}(f)$				1											2										
$IV_{o,B}^{1,1,a}(f)$																									
$IV_{e,A}^{1,1,a}(f)$			1																						
$IV_{e,B}^{1,1,a}(f)$															2										
$IV_{o}^{0,2,a}(f)$																								1	
$IV_{e}^{0,2,a}(f)$														1						1					
$IV_{o}^{1,2,a}(f)$																						1			
$IV_{e}^{1,2,a}(f)$															1									1	
$IV_{o}^{2,2,a}(f)$																							1		
$IV_{e}^{2,2,a}(f)$																					1				
$IV_{o,A}^{0,8}(f)$					3				1																
$IV_{o,B}^{0,8}(f)$																									
$IV_{e,A}^{0,8}(f)$																									
$IV_{e,B}^{0,8}(f)$			1		2				1																
$IV_{o,A}^{0,9}(f)$					3					1															
$IV_{o,B}^{0,9}(f)$																									
$IV_{e,A}^{0,9}(f)$																									
$IV_{e,B}^{0,9}(f)$					3					1															
$IV_{o,A}^{0,10}(f)$					4		1																		
$IV_{o,B}^{0,10}(f)$											1														
$IV_{e,A}^{0,10}(f)$											1														
$IV_{e,B}^{0,10}(f)$							1																		
$IV_{o,A}^{0,11}(f)$					3		3																		
$IV_{o,B}^{0,11}(f)$												1													
$IV_{e,A}^{0,11}(f)$												1													
$IV_{e,B}^{0,11}(f)$																									

TABLE 2.10. Table of incident numbers of $[III_{0,A}^{*} : \mathcal{F}]$ or $[III_{A}^{*} : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,a	c	g
$\mathrm{IV}^{2,2,6}_{o}(f)$																												
$\mathrm{IV}^{2,2,6}_{e}(f)$										4																		
$\mathrm{IV}^{2,2,7}_{o}(f)$																												
$\mathrm{IV}^{2,2,7}_{e}(f)$											4																	
$\mathrm{IV}^{0,0,a}_{o,A}(f)$																												
$\mathrm{IV}^{0,0,a}_{o,B}(f)$																												
$\mathrm{IV}^{0,0,a}_{e,A}(f)$																												
$\mathrm{IV}^{0,0,a}_{e,B}(f)$																												
$\mathrm{IV}^{0,1,a}_{o,A}(f)$																												
$\mathrm{IV}^{0,1,a}_{o,B}(f)$																												
$\mathrm{IV}^{0,1,a}_{e,A}(f)$																												
$\mathrm{IV}^{0,1,a}_{e,B}(f)$																												
$\mathrm{IV}^{1,1,a}_{o,A}(f)$																												
$\mathrm{IV}^{1,1,a}_{o,B}(f)$																												
$\mathrm{IV}^{1,1,a}_{e,A}(f)$																												
$\mathrm{IV}^{1,1,a}_{e,B}(f)$																												
$\mathrm{IV}^{0,2,a}_{o}(f)$																										1		
$\mathrm{IV}^{0,2,a}_{e}(f)$																										1		
$\mathrm{IV}^{1,2,a}_{o}(f)$																										1		
$\mathrm{IV}^{1,2,a}_{e}(f)$																										1		
$\mathrm{IV}^{2,2,a}(f)$																												
$\mathrm{IV}^{2,2,a}(f)$																										4		
$\mathrm{IV}^{0,8}_{o,A}(f)$																												
$\mathrm{IV}^{0,8}_{o,B}(f)$																												
$\mathrm{IV}^{0,8}_{e,A}(f)$																												
$\mathrm{IV}^{0,8}_{e,B}(f)$																												
$\mathrm{IV}^{0,9}_{o,A}(f)$																												
$\mathrm{IV}^{0,9}_{o,B}(f)$																												
$\mathrm{IV}^{0,9}_{e,A}(f)$																												
$\mathrm{IV}^{0,9}_{e,B}(f)$																												
$\mathrm{IV}^{0,10}_{o,A}(f)$																												
$\mathrm{IV}^{0,10}_{o,B}(f)$																												
$\mathrm{IV}^{0,10}_{e,A}(f)$																												
$\mathrm{IV}^{0,10}_{e,B}(f)$																												
$\mathrm{IV}^{0,11}_{o,A}(f)$																												
$\mathrm{IV}^{0,11}_{o,B}(f)$																												
$\mathrm{IV}^{0,11}_{e,A}(f)$																												
$\mathrm{IV}^{0,11}_{e,B}(f)$																												

TABLE 2.11. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,a	1,a	b	d	e	f	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$IV^{0,12}_{o,A}(f)$							6																		
$IV^{0,12}_{o,B}(f)$													1												
$IV^{0,12}_{e,A}(f)$													1												
$IV^{0,12}_{e,B}(f)$																									
$IV^{0,13}_{o,A}(f)$							1																		
$IV^{0,13}_{o,B}(f)$																									
$IV^{0,13}_{e,A}(f)$																									
$IV^{0,13}_{e,B}(f)$																									
$IV^{0,14}_{o}(f)$					1																			1	
$IV^{0,14}_{e}(f)$																									
$IV^{0,15}_{o,A}(f)$					2																				
$IV^{0,15}_{o,B}(f)$																									
$IV^{0,15}_{e,A}(f)$																									
$IV^{0,15}_{e,B}(f)$																									
$IV^{0,16}_{o}(f)$																									
$IV^{0,16}_{e}(f)$																									
$IV^{0,17}_{o,A}(f)$																									
$IV^{0,17}_{o,B}(f)$					3																				
$IV^{0,17}_{e,A}(f)$																									
$IV^{0,17}_{e,B}(f)$																									
$IV^{0,18}_{o}(f)$																									
$IV^{0,18}_{e}(f)$																									
$IV^{0,19}_{o}(f)$																									
$IV^{0,19}_{e}(f)$																									
$IV^{0,20}_{o}(f)$																					1				
$IV^{0,20}_{e}(f)$																									
$IV^{0,21}_{o,A}(f)$																									
$IV^{0,21}_{o,B}(f)$							1																		
$IV^{0,21}_{e,A}(f)$																									
$IV^{0,21}_{e,B}(f)$																									
$IV^{0,22}_{o}(f)$							1																		
$IV^{0,22}_{e}(f)$																									
$IV^{0,23}_{o}(f)$					1																				
$IV^{0,23}_{e}(f)$																									
$IV^{0,24}_{o}(f)$					1																				
$IV^{0,24}_{e}(f)$																									
$IV^{0,25}_{o}(f)$																									
$IV^{0,25}_{e}(f)$																									
$IV^{0,26}_{o}(f)$																									
$IV^{0,26}_{e}(f)$																									

TABLE 2.12. Table of incident numbers of $[III^*_{0,A} : \mathcal{F}]$ or $[III^*_A : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,*a*	*c*	*g*
$\mathrm{IV}^{0,12}_{o,A}(f)$																												
$\mathrm{IV}^{0,12}_{o,B}(f)$																												
$\mathrm{IV}^{0,12}_{e,A}(f)$																												
$\mathrm{IV}^{0,12}_{e,B}(f)$																												
$\mathrm{IV}^{0,13}_{o,A}(f)$	4		1									1																
$\mathrm{IV}^{0,13}_{o,B}(f)$			1																									
$\mathrm{IV}^{0,13}_{e,A}(f)$																												
$\mathrm{IV}^{0,13}_{e,B}(f)$												1																
$\mathrm{IV}^{0,14}_{o}(f)$		2											1															
$\mathrm{IV}^{0,14}_{e}(f)$		1											1															
$\mathrm{IV}^{0,15}_{o,A}(f)$	1	2																										
$\mathrm{IV}^{0,15}_{o,B}(f)$		2												1														
$\mathrm{IV}^{0,15}_{e,A}(f)$														1														
$\mathrm{IV}^{0,15}_{e,B}(f)$	1																											
$\mathrm{IV}^{0,16}_{o}(f)$	1	2	2												1													
$\mathrm{IV}^{0,16}_{e}(f)$															1													
$\mathrm{IV}^{0,17}_{o,A}(f)$																1												
$\mathrm{IV}^{0,17}_{o,B}(f)$	3																											
$\mathrm{IV}^{0,17}_{e,A}(f)$																												
$\mathrm{IV}^{0,17}_{e,B}(f)$																1												
$\mathrm{IV}^{0,18}_{o}(f)$		4	1														1											
$\mathrm{IV}^{0,18}_{e}(f)$			1														1											
$\mathrm{IV}^{0,19}_{o}(f)$		4	1															1										
$\mathrm{IV}^{0,19}_{e}(f)$			1															1										
$\mathrm{IV}^{0,20}_{o}(f)$		4	1																1									
$\mathrm{IV}^{0,20}_{e}(f)$																			1									
$\mathrm{IV}^{0,21}_{o,A}(f)$	1	2																		1								
$\mathrm{IV}^{0,21}_{o,B}(f)$	2																											
$\mathrm{IV}^{0,21}_{e,A}(f)$																												
$\mathrm{IV}^{0,21}_{e,B}(f)$																				1								
$\mathrm{IV}^{0,22}_{o}(f)$		4																			1							
$\mathrm{IV}^{0,22}_{e}(f)$																					1							
$\mathrm{IV}^{0,23}_{o}(f)$		2																				1						
$\mathrm{IV}^{0,23}_{e}(f)$		2																				1						
$\mathrm{IV}^{0,24}_{o}(f)$		2																					1					
$\mathrm{IV}^{0,24}_{e}(f)$		2																					1					
$\mathrm{IV}^{0,25}(f)$			6																					1				
$\mathrm{IV}^{0,25}(f)$																								1				
$\mathrm{IV}^{0,26}(f)$			6																						1			
$\mathrm{IV}^{0,26}(f)$																									1			

TABLE 2.13. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,a	1,a	b	d	e	f	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$\mathrm{IV}^{1,8}_{o,A}(f)$						3																			
$\mathrm{IV}^{1,8}_{o,B}(f)$				1					1																
$\mathrm{IV}^{1,8}_{e,A}(f)$									1																
$\mathrm{IV}^{1,8}_{e,B}(f)$						2																			
$\mathrm{IV}^{1,9}_{o,A}(f)$						3																			
$\mathrm{IV}^{1,9}_{o,B}(f)$										1															
$\mathrm{IV}^{1,9}_{e,A}(f)$										1															
$\mathrm{IV}^{1,9}_{e,B}(f)$						3																			
$\mathrm{IV}^{1,10}_{o,A}(f)$						4		1			1														
$\mathrm{IV}^{1,10}_{o,B}(f)$																									
$\mathrm{IV}^{1,10}_{e,A}(f)$																									
$\mathrm{IV}^{1,10}_{e,B}(f)$								1			1														
$\mathrm{IV}^{1,11}_{o,A}(f)$						3		3				1													
$\mathrm{IV}^{1,11}_{o,B}(f)$																									
$\mathrm{IV}^{1,11}_{e,A}(f)$																									
$\mathrm{IV}^{1,11}_{e,B}(f)$												1													
$\mathrm{IV}^{1,12}_{o,A}(f)$								6					1												
$\mathrm{IV}^{1,12}_{o,B}(f)$																									
$\mathrm{IV}^{1,12}_{e,A}(f)$																									
$\mathrm{IV}^{1,12}_{e,B}(f)$													1												
$\mathrm{IV}^{1,13}_{o,A}(f)$								1																	
$\mathrm{IV}^{1,13}_{o,B}(f)$																									
$\mathrm{IV}^{1,13}_{e,A}(f)$																									
$\mathrm{IV}^{1,13}_{e,B}(f)$																									
$\mathrm{IV}^{1,14}_{o}(f)$						1																1			
$\mathrm{IV}^{1,14}_{e}(f)$																									
$\mathrm{IV}^{1,15}_{o,A}(f)$						2																			
$\mathrm{IV}^{1,15}_{o,B}(f)$																									
$\mathrm{IV}^{1,15}_{e,A}(f)$																									
$\mathrm{IV}^{1,15}_{e,B}(f)$																									
$\mathrm{IV}^{1,16}_{o}(f)$																									
$\mathrm{IV}^{1,16}_{e}(f)$																									
$\mathrm{IV}^{1,17}_{o,A}(f)$						3																			
$\mathrm{IV}^{1,17}_{o,B}(f)$																									
$\mathrm{IV}^{1,17}_{e,A}(f)$																									
$\mathrm{IV}^{1,17}_{e,B}(f)$																									
$\mathrm{IV}^{1,18}_{o}(f)$																									
$\mathrm{IV}^{1,18}_{e}(f)$																									

TABLE 2.14. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,a	c	g
$\mathrm{IV}^{1,8}_{o,A}(f)$																												
$\mathrm{IV}^{1,8}_{o,B}(f)$																												
$\mathrm{IV}^{1,8}_{e,A}(f)$																												
$\mathrm{IV}^{1,8}_{e,B}(f)$																												
$\mathrm{IV}^{1,9}_{o,A}(f)$																												
$\mathrm{IV}^{1,9}_{o,B}(f)$																												
$\mathrm{IV}^{1,9}_{e,A}(f)$																												
$\mathrm{IV}^{1,9}_{e,B}(f)$																												
$\mathrm{IV}^{1,10}_{o,A}(f)$																												
$\mathrm{IV}^{1,10}_{o,B}(f)$																												
$\mathrm{IV}^{1,10}_{e,A}(f)$																												
$\mathrm{IV}^{1,10}_{e,B}(f)$																												
$\mathrm{IV}^{1,11}_{o,A}(f)$																												
$\mathrm{IV}^{1,11}_{o,B}(f)$																												
$\mathrm{IV}^{1,11}_{e,A}(f)$																												
$\mathrm{IV}^{1,11}_{e,B}(f)$																												
$\mathrm{IV}^{1,12}_{o,A}(f)$																												
$\mathrm{IV}^{1,12}_{o,B}(f)$																												
$\mathrm{IV}^{1,12}_{e,A}(f)$																												
$\mathrm{IV}^{1,12}_{e,B}(f)$																												
$\mathrm{IV}^{1,13}_{o,A}(f)$				4		1																						
$\mathrm{IV}^{1,13}_{o,B}(f)$						1						1																
$\mathrm{IV}^{1,13}_{e,A}(f)$												1																
$\mathrm{IV}^{1,13}_{e,B}(f)$																												
$\mathrm{IV}^{1,14}_{o}(f)$					2								1															
$\mathrm{IV}^{1,14}_{e}(f)$					1								1															
$\mathrm{IV}^{1,15}_{o,A}(f)$				1	2									1														
$\mathrm{IV}^{1,15}_{o,B}(f)$					2																							
$\mathrm{IV}^{1,15}_{e,A}(f)$																												
$\mathrm{IV}^{1,15}_{e,B}(f)$				1										1														
$\mathrm{IV}^{1,16}_{o}(f)$				1	2	2									1													
$\mathrm{IV}^{1,16}_{e}(f)$															1													
$\mathrm{IV}^{1,17}_{o,A}(f)$				3																								
$\mathrm{IV}^{1,17}_{o,B}(f)$																1												
$\mathrm{IV}^{1,17}_{e,A}(f)$																1												
$\mathrm{IV}^{1,17}_{e,B}(f)$																												
$\mathrm{IV}^{1,18}_{o}(f)$					4	1											1											
$\mathrm{IV}^{1,18}_{e}(f)$						1											1											

TABLE 2.15. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,a	1,a	b	d	e	f	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$IV^{1,19}_o(f)$																									
$IV^{1,19}_e(f)$																									
$IV^{1,20}_o(f)$																						1			
$IV^{1,20}_e(f)$																									
$IV^{1,21}_{o,A}(f)$																									
$IV^{1,21}_{o,B}(f)$								1																	
$IV^{1,21}_{e,A}(f)$																									
$IV^{1,21}_{e,B}(f)$																									
$IV^{1,22}_o(f)$								1																	
$IV^{1,22}_e(f)$																									
$IV^{1,23}_o(f)$						1																			
$IV^{1,23}_e(f)$																									
$IV^{1,24}_o(f)$						1																			
$IV^{1,24}_e(f)$																									
$IV^{1,25}_o(f)$																									
$IV^{1,25}_e(f)$																									
$IV^{1,26}_o(f)$																									
$IV^{1,26}_e(f)$																									
$IV^{2,8}_o(f)$																						1			
$IV^{2,8}_e(f)$									1																
$IV^{2,9}_o(f)$																									
$IV^{2,9}_e(f)$										1															
$IV^{2,10}_o(f)$																									
$IV^{2,10}_e(f)$											1														
$IV^{2,11}_o(f)$																									
$IV^{2,11}_e(f)$												1													
$IV^{2,12}_o(f)$																									
$IV^{2,12}_e(f)$													1												
$IV^{2,13}_o(f)$																									
$IV^{2,13}_e(f)$																									
$IV^{2,14}_o(f)$																							1		
$IV^{2,14}_e(f)$																									
$IV^{2,15}_o(f)$																									
$IV^{2,15}_e(f)$																									
$IV^{2,16}_o(f)$																									
$IV^{2,16}_e(f)$																									

TABLE 2.16. Table of incident numbers of $[III^*_{0,A} : \mathcal{F}]$ or $[III^*_A : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,a	c	g
$\mathrm{IV}^{1,19}_{o}(f)$					4	1												1										
$\mathrm{IV}^{1,19}_{e}(f)$						1												1										
$\mathrm{IV}^{1,20}_{o}(f)$					4	1													1									
$\mathrm{IV}^{1,20}_{e}(f)$																			1									
$\mathrm{IV}^{1,21}_{o,A}(f)$				1	2																							
$\mathrm{IV}^{1,21}_{o,B}(f)$				2	2															1								
$\mathrm{IV}^{1,21}_{e,A}(f)$																				1								
$\mathrm{IV}^{1,21}_{e,B}(f)$																												
$\mathrm{IV}^{1,22}_{o}(f)$					4																1							
$\mathrm{IV}^{1,22}_{e}(f)$																					1							
$\mathrm{IV}^{1,23}_{o}(f)$					2																	1						
$\mathrm{IV}^{1,23}_{e}(f)$					2																	1						
$\mathrm{IV}^{1,24}_{o}(f)$					2																		1					
$\mathrm{IV}^{1,24}_{e}(f)$					2																		1					
$\mathrm{IV}^{1,25}_{o}(f)$						6																		1				
$\mathrm{IV}^{1,25}_{e}(f)$																								1				
$\mathrm{IV}^{1,26}_{o}(f)$						6																			1			
$\mathrm{IV}^{1,26}_{e}(f)$																									1			
$\mathrm{IV}^{2,8}_{o}(f)$							3																					
$\mathrm{IV}^{2,8}_{e}(f)$							2																					
$\mathrm{IV}^{2,9}_{o}(f)$							3																					
$\mathrm{IV}^{2,9}_{e}(f)$							3																					
$\mathrm{IV}^{2,10}_{o}(f)$							4	1																				
$\mathrm{IV}^{2,10}_{e}(f)$								1																				
$\mathrm{IV}^{2,11}_{o}(f)$							3	3																				
$\mathrm{IV}^{2,11}_{e}(f)$																												
$\mathrm{IV}^{2,12}_{o}(f)$								6																				
$\mathrm{IV}^{2,12}_{e}(f)$																												
$\mathrm{IV}^{2,13}_{o}(f)$								1	4		1																	
$\mathrm{IV}^{2,13}_{e}(f)$												1																
$\mathrm{IV}^{2,14}_{o}(f)$							2			2																		
$\mathrm{IV}^{2,14}_{e}(f)$										1			2															
$\mathrm{IV}^{2,15}_{o}(f)$							2		1	2																		
$\mathrm{IV}^{2,15}_{e}(f)$									1					1														
$\mathrm{IV}^{2,16}_{o}(f)$									2	2	2																	
$\mathrm{IV}^{2,16}_{e}(f)$															2													

TABLE 2.17. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_{A} : \mathcal{F}]$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,a	1,a	b	d	e	f	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$\mathrm{IV}_o^{2,17}(f)$																									
$\mathrm{IV}_e^{2,17}(f)$																									
$\mathrm{IV}_o^{2,18}(f)$																									
$\mathrm{IV}_e^{2,18}(f)$																									
$\mathrm{IV}_o^{2,19}(f)$																									
$\mathrm{IV}_e^{2,19}(f)$																									
$\mathrm{IV}_o^{2,20}(f)$																									1
$\mathrm{IV}_e^{2,20}(f)$																									
$\mathrm{IV}_o^{2,21}(f)$																									
$\mathrm{IV}_e^{2,21}(f)$																									
$\mathrm{IV}_o^{2,22}(f)$																									
$\mathrm{IV}_e^{2,22}(f)$																									
$\mathrm{IV}_o^{2,23}(f)$																									
$\mathrm{IV}_e^{2,23}(f)$																									
$\mathrm{IV}_o^{2,24}(f)$																									
$\mathrm{IV}_e^{2,24}(f)$																									
$\mathrm{IV}_o^{2,25}(f)$																									
$\mathrm{IV}_e^{2,25}(f)$																									
$\mathrm{IV}_o^{2,26}(f)$																									
$\mathrm{IV}_e^{2,26}(f)$																									
$\mathrm{IV}_{o,A}^{3,3}(f)$						4																			
$\mathrm{IV}_{o,B}^{3,3}(f)$																									
$\mathrm{IV}_{e,A}^{3,3}(f)$																									
$\mathrm{IV}_{e,B}^{3,3}(f)$						4																			
$\mathrm{IV}_{o,A}^{3,4}(f)$						4		2																	
$\mathrm{IV}_{o,B}^{3,4}(f)$																									
$\mathrm{IV}_{e,A}^{3,4}(f)$																									
$\mathrm{IV}_{e,B}^{3,4}(f)$								2																	
$\mathrm{IV}_{o,A}^{3,5}(f)$						2																			
$\mathrm{IV}_{o,B}^{3,5}(f)$																									
$\mathrm{IV}_{e,A}^{3,5}(f)$																									
$\mathrm{IV}_{e,B}^{3,5}(f)$																									
$\mathrm{IV}_o^{3,6}(f)$						1																			
$\mathrm{IV}_e^{3,6}(f)$																									
$\mathrm{IV}_o^{3,7}(f)$																									
$\mathrm{IV}_e^{3,7}(f)$																									
$\mathrm{IV}_{o,A}^{4,4}(f)$								8																	
$\mathrm{IV}_{o,B}^{4,4}(f)$																									
$\mathrm{IV}_{e,A}^{4,4}(f)$																									
$\mathrm{IV}_{e,B}^{4,4}(f)$																									

TABLE 2.18. Table of incident numbers of $[\mathrm{III}_{0,A}^* : \mathcal{F}]$ or $[\mathrm{III}_A^* : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,a	c	g
$\mathrm{IV}^{2,17}_{o}(f)$							3		3																			
$\mathrm{IV}^{2,17}_{e}(f)$																1												
$\mathrm{IV}^{2,18}_{o}(f)$										4	1																	
$\mathrm{IV}^{2,18}_{e}(f)$											1						2											
$\mathrm{IV}^{2,19}_{o}(f)$										4	1																	
$\mathrm{IV}^{2,19}_{e}(f)$											1							2										
$\mathrm{IV}^{2,20}_{o}(f)$										4	1																	
$\mathrm{IV}^{2,20}_{e}(f)$																			2									
$\mathrm{IV}^{2,21}_{o}(f)$								1	3	2																		
$\mathrm{IV}^{2,21}_{e}(f)$																				1								
$\mathrm{IV}^{2,22}_{o}(f)$								2		4																		
$\mathrm{IV}^{2,22}_{e}(f)$																					2							
$\mathrm{IV}^{2,23}_{o}(f)$							2			2																		
$\mathrm{IV}^{2,23}_{e}(f)$										2												2						
$\mathrm{IV}^{2,24}_{o}(f)$							2			2																		
$\mathrm{IV}^{2,24}_{e}(f)$										2													2					
$\mathrm{IV}^{2,25}_{o}(f)$											6																	
$\mathrm{IV}^{2,25}_{e}(f)$																								2				
$\mathrm{IV}^{2,26}(f)$											6																	
$\mathrm{IV}^{2,26}(f)$																									2			
$\mathrm{IV}^{3,3}_{o,A}(f)$																												
$\mathrm{IV}^{3,3}_{o,B}(f)$																												
$\mathrm{IV}^{3,3}_{e,A}(f)$																												
$\mathrm{IV}^{3,3}_{e,B}(f)$																												
$\mathrm{IV}^{3,4}_{o,A}(f)$																												
$\mathrm{IV}^{3,4}_{o,B}(f)$																												
$\mathrm{IV}^{3,4}_{e,A}(f)$																												
$\mathrm{IV}^{3,4}_{e,B}(f)$																												
$\mathrm{IV}^{3,5}_{o,A}(f)$				2			2																					
$\mathrm{IV}^{3,5}_{o,B}(f)$							2																					
$\mathrm{IV}^{3,5}_{e,A}(f)$																												
$\mathrm{IV}^{3,5}_{e,B}(f)$				2																								
$\mathrm{IV}^{3,6}_{o}(f)$					2		1																					
$\mathrm{IV}^{3,6}_{e}(f)$					2		1																					
$\mathrm{IV}^{3,7}_{o}(f)$						2	4																					
$\mathrm{IV}^{3,7}_{e}(f)$						2																						
$\mathrm{IV}^{4,4}_{o,A}(f)$																												
$\mathrm{IV}^{4,4}_{o,B}(f)$																												
$\mathrm{IV}^{4,4}_{e,A}(f)$																												
$\mathrm{IV}^{4,4}_{e,B}(f)$																												

TABLE 2.19. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_{A} : \mathcal{F}]$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,a	1,a	b	d	e	f	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$\mathrm{IV}^{4,5}_{o,A}(f)$								2																	
$\mathrm{IV}^{4,5}_{o,B}(f)$																									
$\mathrm{IV}^{4,5}_{e,A}(f)$																									
$\mathrm{IV}^{4,5}_{e,B}(f)$																									
$\mathrm{IV}^{4,6}_{o}(f)$								1																	
$\mathrm{IV}^{4,6}_{e}(f)$																									
$\mathrm{IV}^{4,7}_{o}(f)$																									
$\mathrm{IV}^{4,7}_{e}(f)$																									
$\mathrm{IV}^{5,5}_{o,A}(f)$																									
$\mathrm{IV}^{5,5}_{o,B}(f)$																									
$\mathrm{IV}^{5,5}_{e,A}(f)$																									
$\mathrm{IV}^{5,5}_{e,B}(f)$																									
$\mathrm{IV}^{5,6}_{o}(f)$																									
$\mathrm{IV}^{5,6}_{e}(f)$																									
$\mathrm{IV}^{5,7}_{o}(f)$																									
$\mathrm{IV}^{5,7}_{e}(f)$																									
$\mathrm{IV}^{6,6}_{o}(f)$																									
$\mathrm{IV}^{6,6}_{e}(f)$																									
$\mathrm{IV}^{6,7}_{o}(f)$																									
$\mathrm{IV}^{6,7}_{e}(f)$																									
$\mathrm{IV}^{7,7}_{o}(f)$																									
$\mathrm{IV}^{7,7}_{e}(f)$																									
$\mathrm{IV}^{3,a}_{o,A}(f)$						1																			
$\mathrm{IV}^{3,a}_{o,B}(f)$															2										
$\mathrm{IV}^{3,a}_{e,A}(f)$					1										2										
$\mathrm{IV}^{3,a}_{e,B}(f)$																									
$\mathrm{IV}^{4,a}_{o,A}(f)$								1																	
$\mathrm{IV}^{4,a}_{o,B}(f)$															4										
$\mathrm{IV}^{4,a}_{e,A}(f)$							1																		
$\mathrm{IV}^{4,a}_{e,B}(f)$																									
$\mathrm{IV}^{5,a}_{o,A}(f)$																									
$\mathrm{IV}^{5,a}_{o,B}(f)$															2										
$\mathrm{IV}^{5,a}_{e,A}(f)$																									
$\mathrm{IV}^{5,a}_{e,B}(f)$																									
$\mathrm{IV}^{6,a}_{o}(f)$															1										
$\mathrm{IV}^{6,a}_{e}(f)$																									
$\mathrm{IV}^{7,a}_{o}(f)$																									
$\mathrm{IV}^{7,a}_{e}(f)$																									

TABLE 2.20. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_{A} : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,a	c	g
$\mathrm{IV}^{4,5}_{o,A}(f)$				4				2																				
$\mathrm{IV}^{4,5}_{o,B}(f)$								2																				
$\mathrm{IV}^{4,5}_{e,A}(f)$																												
$\mathrm{IV}^{4,5}_{e,B}(f)$																												
$\mathrm{IV}^{4,6}_{o}(f)$					4			1																				
$\mathrm{IV}^{4,6}_{e}(f)$								1																				
$\mathrm{IV}^{4,7}_{o}(f)$						4		4																				
$\mathrm{IV}^{4,7}_{e}(f)$																												
$\mathrm{IV}^{5,5}_{o,A}(f)$				4					4																			
$\mathrm{IV}^{5,5}_{o,B}(f)$									4																			
$\mathrm{IV}^{5,5}_{e,A}(f)$																												
$\mathrm{IV}^{5,5}_{e,B}(f)$																												
$\mathrm{IV}^{5,6}_{o}(f)$				1	2				1	2																		
$\mathrm{IV}^{5,6}_{e}(f)$									1																			
$\mathrm{IV}^{5,7}_{o}(f)$						2			4		2																	
$\mathrm{IV}^{5,7}_{e}(f)$																												
$\mathrm{IV}^{6,6}_{o}(f)$					4					2																		
$\mathrm{IV}^{6,6}_{e}(f)$										2																		
$\mathrm{IV}^{6,7}_{o}(f)$						2				4	1																	
$\mathrm{IV}^{6,7}_{e}(f)$											1																	
$\mathrm{IV}^{7,7}_{o}(f)$											8																	
$\mathrm{IV}^{7,7}_{e}(f)$																												
$\mathrm{IV}^{3,a}_{o,A}(f)$																												
$\mathrm{IV}^{3,a}_{o,B}(f)$																												
$\mathrm{IV}^{3,a}_{e,A}(f)$																												
$\mathrm{IV}^{3,a}_{e,B}(f)$																												
$\mathrm{IV}^{4,a}_{o,A}(f)$																												
$\mathrm{IV}^{4,a}_{o,B}(f)$																												
$\mathrm{IV}^{4,a}_{e,A}(f)$																												
$\mathrm{IV}^{4,a}_{e,B}(f)$																												
$\mathrm{IV}^{5,a}_{o,A}(f)$				1																						2		
$\mathrm{IV}^{5,a}_{o,B}(f)$																										2		
$\mathrm{IV}^{5,a}_{e,A}(f)$	1																											
$\mathrm{IV}^{5,a}_{e,B}(f)$																												
$\mathrm{IV}^{6,a}_{o}(f)$					1																					1		
$\mathrm{IV}^{6,a}_{e}(f)$		1																								1		
$\mathrm{IV}^{7,a}_{o}(f)$						1																				4		
$\mathrm{IV}^{7,a}_{e}(f)$			1																									

TABLE 2.21. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_{A} : \mathcal{F}]$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,*a*	1,*a*	*b*	*d*	*e*	*f*	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$\mathrm{IV}^{0,b}_{o,A}(f)$		1			1									1											
$\mathrm{IV}^{0,b}_{o,B}(f)$																1									
$\mathrm{IV}^{0,b}_{e,A}(f)$																1									
$\mathrm{IV}^{0,b}_{e,B}(f)$														1											
$\mathrm{IV}^{1,b}_{o,A}(f)$			1			1																			
$\mathrm{IV}^{1,b}_{o,B}(f)$															1	1									
$\mathrm{IV}^{1,b}_{e,A}(f)$															1	1									
$\mathrm{IV}^{1,b}_{e,B}(f)$																									
$\mathrm{IV}^{2,b}_{o}(f)$																								1	
$\mathrm{IV}^{2,b}_{e}(f)$																1									
$\mathrm{IV}^{0,c}_{o}(f)$														1						1					
$\mathrm{IV}^{0,c}_{e}(f)$																									
$\mathrm{IV}^{1,c}_{o}(f)$															1									1	
$\mathrm{IV}^{1,c}_{e}(f)$																									
$\mathrm{IV}^{2,c}_{o}(f)$																					1				
$\mathrm{IV}^{2,c}_{e}(f)$																									
$\mathrm{IV}^{0,d}_{o,A}(f)$	1													2			1								
$\mathrm{IV}^{0,d}_{o,B}(f)$																									
$\mathrm{IV}^{0,d}_{e,A}(f)$																									
$\mathrm{IV}^{0,d}_{e,B}(f)$																	1								
$\mathrm{IV}^{0,e}_{o,A}(f)$							1																		
$\mathrm{IV}^{0,e}_{o,B}(f)$														2				1							
$\mathrm{IV}^{0,e}_{e,A}(f)$																		1							
$\mathrm{IV}^{0,e}_{e,B}(f)$																									
$\mathrm{IV}^{0,f}_{o,A}(f)$					1									2					1						
$\mathrm{IV}^{0,f}_{o,B}(f)$																									
$\mathrm{IV}^{0,f}_{e,A}(f)$																									
$\mathrm{IV}^{0,f}_{e,B}(f)$																			1						
$\mathrm{IV}^{0,g}_{o,A}(f)$																									
$\mathrm{IV}^{0,g}_{o,B}(f)$														2											
$\mathrm{IV}^{0,g}_{e,A}(f)$																									
$\mathrm{IV}^{0,g}_{e,B}(f)$																									

TABLE 2.22. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,a	c	g
$\mathrm{IV}^{0,b}_{o,A}(f)$																												
$\mathrm{IV}^{0,b}_{o,B}(f)$																												
$\mathrm{IV}^{0,b}_{e,A}(f)$																												
$\mathrm{IV}^{0,b}_{e,B}(f)$																												
$\mathrm{IV}^{1,b}_{o,A}(f)$																												
$\mathrm{IV}^{1,b}_{o,B}(f)$																												
$\mathrm{IV}^{1,b}_{e,A}(f)$																												
$\mathrm{IV}^{1,b}_{e,B}(f)$																												
$\mathrm{IV}^{2,b}_{o}(f)$																										1		
$\mathrm{IV}^{2,b}_{e}(f)$							1																			1		
$\mathrm{IV}^{0,c}_{o}(f)$		1																									1	
$\mathrm{IV}^{0,c}_{e}(f)$																											1	
$\mathrm{IV}^{1,c}_{o}(f)$					1																						1	
$\mathrm{IV}^{1,c}_{e}(f)$																											1	
$\mathrm{IV}^{2,c}_{o}(f)$										1																2		
$\mathrm{IV}^{2,c}_{e}(f)$																											2	
$\mathrm{IV}^{0,d}_{o,A}(f)$																												
$\mathrm{IV}^{0,d}_{o,B}(f)$																												
$\mathrm{IV}^{0,d}_{e,A}(f)$																												
$\mathrm{IV}^{0,d}_{e,B}(f)$																												
$\mathrm{IV}^{0,e}_{o,A}(f)$																												
$\mathrm{IV}^{0,e}_{o,B}(f)$																												
$\mathrm{IV}^{0,e}_{e,A}(f)$																												
$\mathrm{IV}^{0,e}_{e,B}(f)$																												
$\mathrm{IV}^{0,f}_{o,A}(f)$																												
$\mathrm{IV}^{0,f}_{o,B}(f)$																												
$\mathrm{IV}^{0,f}_{e,A}(f)$																												
$\mathrm{IV}^{0,f}_{e,B}(f)$																												
$\mathrm{IV}^{0,g}_{o,A}(f)$	1																											
$\mathrm{IV}^{0,g}_{o,B}(f)$																												1
$\mathrm{IV}^{0,g}_{e,A}(f)$																												1
$\mathrm{IV}^{0,g}_{e,B}(f)$																												

TABLE 2.23. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,*a*	1,*a*	*b*	*d*	*e*	*f*	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$\mathrm{IV}^{1,d}_{o,A}(f)$		1													2										
$\mathrm{IV}^{1,d}_{o,B}(f)$																	1								
$\mathrm{IV}^{1,d}_{e,A}(f)$																	1								
$\mathrm{IV}^{1,d}_{e,B}(f)$																									
$\mathrm{IV}^{1,e}_{o,A}(f)$								1										1							
$\mathrm{IV}^{1,e}_{o,B}(f)$															2										
$\mathrm{IV}^{1,e}_{e,A}(f)$																									
$\mathrm{IV}^{1,e}_{e,B}(f)$																		1							
$\mathrm{IV}^{1,f}_{o,A}(f)$						1																			
$\mathrm{IV}^{1,f}_{o,B}(f)$																			1						
$\mathrm{IV}^{1,f}_{e,A}(f)$															2				1						
$\mathrm{IV}^{1,f}_{e,B}(f)$																									
$\mathrm{IV}^{1,g}_{o,A}(f)$																									
$\mathrm{IV}^{1,g}_{o,B}(f)$															2										
$\mathrm{IV}^{1,g}_{e,A}(f)$																									
$\mathrm{IV}^{1,g}_{e,B}(f)$																									
$\mathrm{IV}^{2,d}_{o}(f)$																				1					
$\mathrm{IV}^{2,d}_{e}(f)$																	1								
$\mathrm{IV}^{2,e}_{o}(f)$																									
$\mathrm{IV}^{2,e}_{e}(f)$																		1							
$\mathrm{IV}^{2,f}_{o}(f)$																									
$\mathrm{IV}^{2,f}_{e}(f)$																			1						
$\mathrm{IV}^{2,g}_{o}(f)$																									
$\mathrm{IV}^{2,g}_{e}(f)$																									
$\mathrm{IV}^{a,a}_{o,A}(f)$															2										
$\mathrm{IV}^{a,a}_{o,B}(f)$																									
$\mathrm{IV}^{a,a}_{e,A}(f)$														2											
$\mathrm{IV}^{a,a}_{e,B}(f)$																									
$\mathrm{IV}^{27}_{o,A}(f)$									4																
$\mathrm{IV}^{27}_{o,B}(f)$																									
$\mathrm{IV}^{27}_{e,A}(f)$																									
$\mathrm{IV}^{27}_{e,B}(f)$						2			2																
$\mathrm{IV}^{28}_{o,A}(f)$									4																
$\mathrm{IV}^{28}_{o,B}(f)$								1			2														
$\mathrm{IV}^{28}_{e,A}(f)$											1														
$\mathrm{IV}^{28}_{e,B}(f)$																									

TABLE 2.24. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,a	c	g
$\mathrm{IV}^{1,d}_{o,A}(f)$																												
$\mathrm{IV}^{1,d}_{o,B}(f)$																												
$\mathrm{IV}^{1,d}_{e,A}(f)$																												
$\mathrm{IV}^{1,d}_{e,B}(f)$																												
$\mathrm{IV}^{1,e}_{o,A}(f)$																												
$\mathrm{IV}^{1,e}_{o,B}(f)$																												
$\mathrm{IV}^{1,e}_{e,A}(f)$																												
$\mathrm{IV}^{1,e}_{e,B}(f)$																												
$\mathrm{IV}^{1,f}_{o,A}(f)$																												
$\mathrm{IV}^{1,f}_{o,B}(f)$																												
$\mathrm{IV}^{1,f}_{e,A}(f)$																												
$\mathrm{IV}^{1,f}_{e,B}(f)$																												
$\mathrm{IV}^{1,g}_{o,A}(f)$				1																								1
$\mathrm{IV}^{1,g}_{o,B}(f)$																												
$\mathrm{IV}^{1,g}_{e,A}(f)$																												
$\mathrm{IV}^{1,g}_{e,B}(f)$																												1
$\mathrm{IV}^{2,d}_{o}(f)$																										2		
$\mathrm{IV}^{2,d}_{e}(f)$																												
$\mathrm{IV}^{2,e}_{o}(f)$								1																		2		
$\mathrm{IV}^{2,e}_{e}(f)$																												
$\mathrm{IV}^{2,f}_{o}(f)$							1																					
$\mathrm{IV}^{2,f}_{e}(f)$																										2		
$\mathrm{IV}^{2,g}_{o}(f)$									1																	2		
$\mathrm{IV}^{2,g}_{e}(f)$																												1
$\mathrm{IV}^{a,a}_{o,A}(f)$															2													
$\mathrm{IV}^{a,a}_{o,B}(f)$																												
$\mathrm{IV}^{a,a}_{e,A}(f)$														2														
$\mathrm{IV}^{a,a}_{e,B}(f)$																												
$\mathrm{IV}^{27}_{o,A}(f)$																												
$\mathrm{IV}^{27}_{o,B}(f)$																												
$\mathrm{IV}^{27}_{e,A}(f)$																												
$\mathrm{IV}^{27}_{e,B}(f)$																												
$\mathrm{IV}^{28}_{o,A}(f)$																												
$\mathrm{IV}^{28}_{o,B}(f)$																												
$\mathrm{IV}^{28}_{e,A}(f)$																												
$\mathrm{IV}^{28}_{e,B}(f)$																												

TABLE 2.25. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_{A} : \mathcal{F}]$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,a	1,a	b	d	e	f	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$\mathrm{IV}^{29}_{o,A}(f)$									2	2															
$\mathrm{IV}^{29}_{o,B}(f)$						1																			
$\mathrm{IV}^{29}_{e,A}(f)$																									
$\mathrm{IV}^{29}_{e,B}(f)$									2	1															
$\mathrm{IV}^{30}_{o,A}(f)$									2	2															
$\mathrm{IV}^{30}_{o,B}(f)$											2														
$\mathrm{IV}^{30}_{e,A}(f)$											2														
$\mathrm{IV}^{30}_{e,B}(f)$																									
$\mathrm{IV}^{31}_{o,A}(f)$									2	1															
$\mathrm{IV}^{31}_{o,B}(f)$											3	1													
$\mathrm{IV}^{31}_{e,A}(f)$												1													
$\mathrm{IV}^{31}_{e,B}(f)$																									
$\mathrm{IV}^{32}_{o,A}(f)$									4																
$\mathrm{IV}^{32}_{o,B}(f)$												4													
$\mathrm{IV}^{32}_{e,A}(f)$																									
$\mathrm{IV}^{32}_{e,B}(f)$																									
$\mathrm{IV}^{33}_{o,A}(f)$									4																
$\mathrm{IV}^{33}_{o,B}(f)$											2														
$\mathrm{IV}^{33}_{e,A}(f)$											2														
$\mathrm{IV}^{33}_{e,B}(f)$																									
$\mathrm{IV}^{34}_{o,A}(f)$										4															
$\mathrm{IV}^{34}_{o,B}(f)$											2														
$\mathrm{IV}^{34}_{e,A}(f)$											2														
$\mathrm{IV}^{34}_{e,B}(f)$																									
$\mathrm{IV}^{35}_{o,A}(f)$										4															
$\mathrm{IV}^{35}_{o,B}(f)$																									
$\mathrm{IV}^{35}_{e,A}(f)$																									
$\mathrm{IV}^{35}_{e,B}(f)$										4															
$\mathrm{IV}^{36}_{o,A}(f)$																									
$\mathrm{IV}^{36}_{o,B}(f)$											6		1												
$\mathrm{IV}^{36}_{e,A}(f)$													1												
$\mathrm{IV}^{36}_{e,B}(f)$																									
$\mathrm{IV}^{37}_{o,A}(f)$											4	4													
$\mathrm{IV}^{37}_{o,B}(f)$																									
$\mathrm{IV}^{37}_{e,A}(f)$																									
$\mathrm{IV}^{37}_{e,B}(f)$																									
$\mathrm{IV}^{38}_{o,A}(f)$																									
$\mathrm{IV}^{38}_{o,B}(f)$											2	4	2												
$\mathrm{IV}^{38}_{e,A}(f)$																									
$\mathrm{IV}^{38}_{e,B}(f)$																									

TABLE 2.26. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,a	c	g
$\mathrm{IV}^{29}_{o,A}(f)$																												
$\mathrm{IV}^{29}_{o,B}(f)$																												
$\mathrm{IV}^{29}_{e,A}(f)$																												
$\mathrm{IV}^{29}_{e,B}(f)$																												
$\mathrm{IV}^{30}_{o,A}(f)$																												
$\mathrm{IV}^{30}_{o,B}(f)$																												
$\mathrm{IV}^{30}_{e,A}(f)$																												
$\mathrm{IV}^{30}_{e,B}(f)$																												
$\mathrm{IV}^{31}_{o,A}(f)$																												
$\mathrm{IV}^{31}_{o,B}(f)$																												
$\mathrm{IV}^{31}_{e,A}(f)$																												
$\mathrm{IV}^{31}_{e,B}(f)$																												
$\mathrm{IV}^{32}_{o,A}(f)$																												
$\mathrm{IV}^{32}_{o,B}(f)$																												
$\mathrm{IV}^{32}_{e,A}(f)$																												
$\mathrm{IV}^{32}_{e,B}(f)$																												
$\mathrm{IV}^{33}_{o,A}(f)$																												
$\mathrm{IV}^{33}_{o,B}(f)$																												
$\mathrm{IV}^{33}_{e,A}(f)$																												
$\mathrm{IV}^{33}_{e,B}(f)$																												
$\mathrm{IV}^{34}_{o,A}(f)$																												
$\mathrm{IV}^{34}_{o,B}(f)$																												
$\mathrm{IV}^{34}_{e,A}(f)$																												
$\mathrm{IV}^{34}_{e,B}(f)$																												
$\mathrm{IV}^{35}_{o,A}(f)$																												
$\mathrm{IV}^{35}_{o,B}(f)$																												
$\mathrm{IV}^{35}_{e,A}(f)$																												
$\mathrm{IV}^{35}_{e,B}(f)$																												
$\mathrm{IV}^{36}_{o,A}(f)$																												
$\mathrm{IV}^{36}_{o,B}(f)$																												
$\mathrm{IV}^{36}_{e,A}(f)$																												
$\mathrm{IV}^{36}_{e,B}(f)$																												
$\mathrm{IV}^{37}_{o,A}(f)$																												
$\mathrm{IV}^{37}_{o,B}(f)$																												
$\mathrm{IV}^{37}_{e,A}(f)$																												
$\mathrm{IV}^{37}_{e,B}(f)$																												
$\mathrm{IV}^{38}_{o,A}(f)$																												
$\mathrm{IV}^{38}_{o,B}(f)$																												
$\mathrm{IV}^{38}_{e,A}(f)$																												
$\mathrm{IV}^{38}_{e,B}(f)$																												

TABLE 2.27. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,a	1,a	b	d	e	f	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$IV^{39}_{o,A}(f)$																									
$IV^{39}_{o,B}(f)$											8														
$IV^{39}_{e,A}(f)$																									
$IV^{39}_{e,B}(f)$																									
$IV^{40}_{o,A}(f)$																									
$IV^{40}_{o,B}(f)$													8												
$IV^{40}_{e,A}(f)$																									
$IV^{40}_{e,B}(f)$																									
$IV^{41}_{o}(f)$									1																
$IV^{41}_{e}(f)$																									
$IV^{42}_{o}(f)$									1																
$IV^{42}_{e}(f)$																									
$IV^{43}_{o,A}(f)$									2																
$IV^{43}_{o,B}(f)$																									
$IV^{43}_{e,A}(f)$																									
$IV^{43}_{e,B}(f)$																									
$IV^{44}_{o}(f)$									1																
$IV^{44}_{e}(f)$																									
$IV^{45}_{o,A}(f)$									2																
$IV^{45}_{o,B}(f)$																									
$IV^{45}_{e,A}(f)$																									
$IV^{45}_{e,B}(f)$																									
$IV^{46}_{o,A}(f)$																									
$IV^{46}_{o,B}(f)$									2	1															
$IV^{46}_{e,A}(f)$																									
$IV^{46}_{e,B}(f)$																									
$IV^{47}_{o,A}(f)$									4																
$IV^{47}_{o,B}(f)$																									
$IV^{47}_{e,A}(f)$																									
$IV^{47}_{e,B}(f)$																									
$IV^{48}_{o}(f)$										1															
$IV^{48}_{e}(f)$																									
$IV^{49}_{o,A}(f)$										2															
$IV^{49}_{o,B}(f)$																									
$IV^{49}_{e,A}(f)$																									
$IV^{49}_{e,B}(f)$																									

TABLE 2.28. Table of incident numbers of $[III^*_{0,A} : \mathcal{F}]$ or $[III^*_A : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,a	c	g
$\mathrm{IV}^{39}_{o,A}(f)$																												
$\mathrm{IV}^{39}_{o,B}(f)$																												
$\mathrm{IV}^{39}_{e,A}(f)$																												
$\mathrm{IV}^{39}_{e,B}(f)$																												
$\mathrm{IV}^{40}_{o,A}(f)$																												
$\mathrm{IV}^{40}_{o,B}(f)$																												
$\mathrm{IV}^{40}_{e,A}(f)$																												
$\mathrm{IV}^{40}_{e,B}(f)$																												
$\mathrm{IV}^{41}_{o}(f)$					1		1						3															
$\mathrm{IV}^{41}_{e}(f)$													1															
$\mathrm{IV}^{42}_{o}(f)$													1									2						
$\mathrm{IV}^{42}_{e}(f)$					1								1									1						
$\mathrm{IV}^{43}_{o,A}(f)$													2															
$\mathrm{IV}^{43}_{o,B}(f)$				1									2	2														
$\mathrm{IV}^{43}_{e,A}(f)$														1														
$\mathrm{IV}^{43}_{e,B}(f)$																												
$\mathrm{IV}^{44}_{o}(f)$					1								1										2					
$\mathrm{IV}^{44}_{e}(f)$													1										1					
$\mathrm{IV}^{45}_{o,A}(f)$																						1	1					
$\mathrm{IV}^{45}_{o,B}(f)$														2								1	1					
$\mathrm{IV}^{45}_{e,A}(f)$														2														
$\mathrm{IV}^{45}_{e,B}(f)$																												
$\mathrm{IV}^{46}_{o,A}(f)$														3														
$\mathrm{IV}^{46}_{o,B}(f)$																1												
$\mathrm{IV}^{46}_{e,A}(f)$																1												
$\mathrm{IV}^{46}_{e,B}(f)$																												
$\mathrm{IV}^{47}_{o,A}(f)$																4												
$\mathrm{IV}^{47}_{o,B}(f)$																												
$\mathrm{IV}^{47}_{e,A}(f)$																												
$\mathrm{IV}^{47}_{e,B}(f)$																												
$\mathrm{IV}^{48}_{o}(f)$							1						2										1					
$\mathrm{IV}^{48}_{e}(f)$													2															
$\mathrm{IV}^{49}_{o,A}(f)$																							2					
$\mathrm{IV}^{49}_{o,B}(f)$														2									2					
$\mathrm{IV}^{49}_{e,A}(f)$														2														
$\mathrm{IV}^{49}_{e,B}(f)$																												

TABLE 2.29. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,a	1,a	b	d	e	f	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$\mathrm{IV}^{50}_{o,A}(f)$																									
$\mathrm{IV}^{50}_{o,B}(f)$										2															
$\mathrm{IV}^{50}_{e,A}(f)$																									
$\mathrm{IV}^{50}_{e,B}(f)$																									
$\mathrm{IV}^{51}_{o}(f)$										1															
$\mathrm{IV}^{51}_{e}(f)$																									
$\mathrm{IV}^{52}_{o}(f)$										1															
$\mathrm{IV}^{52}_{e}(f)$																									
$\mathrm{IV}^{53}_{o}(f)$											1														
$\mathrm{IV}^{53}_{e}(f)$																									
$\mathrm{IV}^{54}_{o,A}(f)$																									
$\mathrm{IV}^{54}_{o,B}(f)$											1														
$\mathrm{IV}^{54}_{e,A}(f)$																									
$\mathrm{IV}^{54}_{e,B}(f)$																									
$\mathrm{IV}^{55}_{o,A}(f)$											1														
$\mathrm{IV}^{55}_{o,B}(f)$																									
$\mathrm{IV}^{55}_{e,A}(f)$																									
$\mathrm{IV}^{55}_{e,B}(f)$																									
$\mathrm{IV}^{56}_{o,A}(f)$																									
$\mathrm{IV}^{56}_{o,B}(f)$											2														
$\mathrm{IV}^{56}_{e,B}(f)$																									
$\mathrm{IV}^{56}_{e,B}(f)$																									
$\mathrm{IV}^{57}_{o}(f)$											1														
$\mathrm{IV}^{57}_{e}(f)$																									
$\mathrm{IV}^{58}_{o,A}(f)$											1														
$\mathrm{IV}^{58}_{o,B}(f)$																									
$\mathrm{IV}^{58}_{e,A}(f)$																									
$\mathrm{IV}^{58}_{e,B}(f)$																									
$\mathrm{IV}^{59}_{o,A}(f)$																									
$\mathrm{IV}^{59}_{o,B}(f)$											1														
$\mathrm{IV}^{59}_{e,A}(f)$																									
$\mathrm{IV}^{59}_{e,B}(f)$																									
$\mathrm{IV}^{60}_{o,A}(f)$																									
$\mathrm{IV}^{60}_{o,B}(f)$											2														
$\mathrm{IV}^{60}_{e,A}(f)$																									
$\mathrm{IV}^{60}_{e,B}(f)$																									

TABLE 2.30. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_{A} : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,a	c	g
$\mathrm{IV}^{50}_{o,A}(f)$														2								2						
$\mathrm{IV}^{50}_{o,B}(f)$																						2						
$\mathrm{IV}^{50}_{e,A}(f)$																												
$\mathrm{IV}^{50}_{e,B}(f)$														2														
$\mathrm{IV}^{51}_{o}(f)$																						2	1					
$\mathrm{IV}^{51}_{e}(f)$																						2	1					
$\mathrm{IV}^{52}_{o}(f)$																							3					
$\mathrm{IV}^{52}_{e}(f)$																							3					
$\mathrm{IV}^{53}_{o}(f)$																					1	4						
$\mathrm{IV}^{53}_{e}(f)$																					1							
$\mathrm{IV}^{54}_{o,A}(f)$												1						1										
$\mathrm{IV}^{54}_{o,B}(f)$														4				1										
$\mathrm{IV}^{54}_{e,A}(f)$																												
$\mathrm{IV}^{54}_{e,B}(f)$												1																
$\mathrm{IV}^{55}_{o,A}(f)$														2						1		1	1					
$\mathrm{IV}^{55}_{o,B}(f)$														1								1	1					
$\mathrm{IV}^{55}_{e,A}(f)$																												
$\mathrm{IV}^{55}_{e,A}(f)$																				1								
$\mathrm{IV}^{56}_{o,A}(f)$																2												
$\mathrm{IV}^{56}_{o,B}(f)$														2							2							
$\mathrm{IV}^{56}_{e,A}(f)$																												
$\mathrm{IV}^{56}_{e,B}(f)$																												
$\mathrm{IV}^{57}_{o}(f)$													4								1							
$\mathrm{IV}^{57}_{e}(f)$																					1							
$\mathrm{IV}^{58}_{o,A}(f)$													2	2						1								
$\mathrm{IV}^{58}_{o,B}(f)$													2	1														
$\mathrm{IV}^{58}_{e,A}(f)$																												
$\mathrm{IV}^{58}_{e,B}(f)$																				1								
$\mathrm{IV}^{59}_{o,A}(f)$												1					1											
$\mathrm{IV}^{59}_{o,B}(f)$														4			1											
$\mathrm{IV}^{59}_{e,A}(f)$																												
$\mathrm{IV}^{59}_{e,B}(f)$												1																
$\mathrm{IV}^{60}_{o,A}(f)$												2				4												
$\mathrm{IV}^{60}_{o,B}(f)$																												
$\mathrm{IV}^{60}_{e,A}(f)$																												
$\mathrm{IV}^{60}_{e,B}(f)$																												

TABLE 2.31. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,a	1,a	b	d	e	f	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$IV^{61}_{o}(f)$											1														
$IV^{61}_{e}(f)$																									
$IV^{62}_{o,A}(f)$											2														
$IV^{62}_{o,B}(f)$																									
$IV^{62}_{e,A}(f)$																									
$IV^{62}_{e,B}(f)$																									
$IV^{63}_{o}(f)$											1														
$IV^{63}_{e}(f)$																									
$IV^{64}_{o}(f)$											1														
$IV^{64}_{e}(f)$																									
$IV^{65}_{o}(f)$												1													
$IV^{65}_{e}(f)$																									
$IV^{66}_{o,A}(f)$												1													
$IV^{66}_{o,B}(f)$																									
$IV^{66}_{e,A}(f)$																									
$IV^{66}_{e,B}(f)$																									
$IV^{67}_{o,A}(f)$																									
$IV^{67}_{o,B}(f)$												1													
$IV^{67}_{e,A}(f)$																									
$IV^{67}_{e,B}(f)$																									
$IV^{68}_{o,A}(f)$																									
$IV^{68}_{o,B}(f)$												1													
$IV^{68}_{e,A}(f)$																									
$IV^{68}_{e,B}(f)$																									
$IV^{69}_{o}(f)$													1												
$IV^{69}_{e}(f)$																									
$IV^{70}_{o,A}(f)$																									
$IV^{70}_{o,B}(f)$													1												
$IV^{70}_{e,A}(f)$																									
$IV^{70}_{e,B}(f)$																									
$IV^{71}_{o,A}(f)$													1												
$IV^{71}_{o,B}(f)$																									
$IV^{71}_{e,A}(f)$																									
$IV^{71}_{e,B}(f)$																									
$IV^{72}_{o,A}(f)$																									
$IV^{72}_{o,B}(f)$																									
$IV^{72}_{e,A}(f)$																									
$IV^{72}_{e,B}(f)$																									

TABLE 2.32. Table of incident numbers of $[III^*_{0,A} : \mathcal{F}]$ or $[III^*_A : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,a	c	g
$IV^{61}_{o}(f)$														2							1		2					
$IV^{61}_{e}(f)$																					1							
$IV^{62}_{o,A}(f)$														4							2							
$IV^{62}_{o,B}(f)$																					2							
$IV^{62}_{e,A}(f)$																												
$IV^{62}_{e,B}(f)$																												
$IV^{63}_{o}(f)$								1					4								1							
$IV^{63}_{e}(f)$																												
$IV^{64}_{o}(f)$																					1		4					
$IV^{64}_{e}(f)$																					1							
$IV^{65}_{o}(f)$													2								3		1					
$IV^{65}_{e}(f)$																												
$IV^{66}_{o,A}(f)$													2			1				3		1						
$IV^{66}_{o,B}(f)$													2									1						
$IV^{66}_{e,A}(f)$																												
$IV^{66}_{e,B}(f)$																												
$IV^{67}_{o,A}(f)$												2			1	1												
$IV^{67}_{o,B}(f)$														2	1					1								
$IV^{67}_{e,A}(f)$																												
$IV^{67}_{e,B}(f)$																												
$IV^{68}_{o,A}(f)$																1				1	1							
$IV^{68}_{o,B}(f)$														2						2	1							
$IV^{68}_{e,A}(f)$																												
$IV^{68}_{e,B}(f)$																												
$IV^{69}_{o}(f)$																					6							
$IV^{69}_{e}(f)$																												
$IV^{70}_{o,A}(f)$												1									1							
$IV^{70}_{o,B}(f)$												1								4	1							
$IV^{70}_{e,A}(f)$																												
$IV^{70}_{e,B}(f)$																												
$IV^{71}_{o,A}(f)$												6												1				
$IV^{71}_{o,B}(f)$																								1				
$IV^{71}_{e,A}(f)$																												
$IV^{71}_{e,B}(f)$																												
$IV^{72}_{o,A}(f)$															2					2								
$IV^{72}_{o,B}(f)$												2			2					2								
$IV^{72}_{e,A}(f)$																												
$IV^{72}_{e,B}(f)$																												

TABLE 2.33. Table of incident numbers of $[III^*_{0,A} : \mathcal{F}]$ or $[III^*_A : \mathcal{F}]$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,a	1,a	b	d	e	f	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$\mathrm{IV}^{73}_{o}(f)$																									
$\mathrm{IV}^{73}_{e}(f)$																									
$\mathrm{IV}^{74}_{o,A}(f)$																									
$\mathrm{IV}^{74}_{o,B}(f)$																									
$\mathrm{IV}^{74}_{e,A}(f)$																									
$\mathrm{IV}^{74}_{e,B}(f)$																									
$\mathrm{IV}^{75}_{o}(f)$																									
$\mathrm{IV}^{75}_{e}(f)$																									
$\mathrm{IV}^{76}_{o,A}(f)$																									
$\mathrm{IV}^{76}_{o,B}(f)$																									
$\mathrm{IV}^{76}_{e,A}(f)$																									
$\mathrm{IV}^{76}_{e,B}(f)$																									
$\mathrm{IV}^{77}_{o,A}(f)$																									
$\mathrm{IV}^{77}_{o,B}(f)$																									
$\mathrm{IV}^{77}_{e,A}(f)$																									
$\mathrm{IV}^{77}_{e,B}(f)$																									
$\mathrm{IV}^{78}_{o}(f)$																									
$\mathrm{IV}^{78}_{e}(f)$																									
$\mathrm{IV}^{79}_{o}(f)$																									
$\mathrm{IV}^{79}_{e}(f)$																									
$\mathrm{IV}^{80}_{o}(f)$																									
$\mathrm{IV}^{80}_{e}(f)$																									
$\mathrm{IV}^{81}_{o}(f)$																									
$\mathrm{IV}^{81}_{e}(f)$																									
$\mathrm{IV}^{82}_{o}(f)$																									
$\mathrm{IV}^{82}_{e}(f)$																									
$\mathrm{IV}^{83}_{o}(f)$																									
$\mathrm{IV}^{83}_{e}(f)$																									
$\mathrm{IV}^{84}_{o}(f)$																									
$\mathrm{IV}^{84}_{e}(f)$																									
$\mathrm{IV}^{85}_{o}(f)$																									
$\mathrm{IV}^{85}_{e}(f)$																									
$\mathrm{IV}^{86}_{o,A}(f)$																									
$\mathrm{IV}^{86}_{o,B}(f)$																									
$\mathrm{IV}^{86}_{e,A}(f)$																									
$\mathrm{IV}^{86}_{e,B}(f)$																									

TABLE 2.34. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,a	c	g
$\mathrm{IV}^{73}_{o}(f)$												1			4						1				1			
$\mathrm{IV}^{73}_{e}(f)$																												
$\mathrm{IV}^{74}_{o,A}(f)$												2							2									
$\mathrm{IV}^{74}_{o,B}(f)$																			2	4								
$\mathrm{IV}^{74}_{e,A}(f)$																												
$\mathrm{IV}^{74}_{e,B}(f)$																												
$\mathrm{IV}^{75}_{o}(f)$												1			4						1			1				
$\mathrm{IV}^{75}_{e}(f)$																												
$\mathrm{IV}^{76}_{o,A}(f)$																	2			4								
$\mathrm{IV}^{76}_{o,B}(f)$												2					2											
$\mathrm{IV}^{76}_{e,A}(f)$																												
$\mathrm{IV}^{76}_{e,B}(f)$																												
$\mathrm{IV}^{77}_{o,A}(f)$												8																
$\mathrm{IV}^{77}_{o,B}(f)$																												
$\mathrm{IV}^{77}_{e,A}(f)$																												
$\mathrm{IV}^{77}_{e,B}(f)$																												
$\mathrm{IV}^{78}_{o}(f)$													2	1	1		1	1										
$\mathrm{IV}^{78}_{e}(f)$															1													
$\mathrm{IV}^{79}_{o}(f)$													2		3	1							1					
$\mathrm{IV}^{79}_{e}(f)$																												
$\mathrm{IV}^{80}_{o}(f)$										1			2					1	1				2					
$\mathrm{IV}^{80}_{e}(f)$																			1									
$\mathrm{IV}^{81}_{o}(f)$										1			2				1		1			2						
$\mathrm{IV}^{81}_{e}(f)$																			1									
$\mathrm{IV}^{82}_{o}(f)$									1				2	1	1				2									
$\mathrm{IV}^{82}_{e}(f)$																												
$\mathrm{IV}^{83}_{o}(f)$										2			4						2									
$\mathrm{IV}^{83}_{e}(f)$																												
$\mathrm{IV}^{84}_{o}(f)$						1							4				2											
$\mathrm{IV}^{84}_{e}(f)$																	1											
$\mathrm{IV}^{85}_{o}(f)$						1							4					2										
$\mathrm{IV}^{85}_{e}(f)$																		1										
$\mathrm{IV}^{86}_{o,A}(f)$														4														
$\mathrm{IV}^{86}_{o,B}(f)$																				4								
$\mathrm{IV}^{86}_{e,A}(f)$																												
$\mathrm{IV}^{86}_{e,B}(f)$																												

TABLE 2.35. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,a	1,a	b	d	e	f	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$IV^{87}_{o}(f)$																									
$IV^{87}_{e}(f)$																									
$IV^{88}_{o}(f)$																									
$IV^{88}_{e}(f)$																									
$IV^{89}_{o}(f)$																									
$IV^{89}_{e}(f)$																									
$IV^{90}_{o,A}(f)$																									
$IV^{90}_{o,B}(f)$																									
$IV^{90}_{e,A}(f)$																									
$IV^{90}_{e,B}(f)$																									
$IV^{91}_{o}(f)$																									
$IV^{91}_{e}(f)$																									
$IV^{92}_{o}(f)$																									
$IV^{92}_{e}(f)$																									
$IV^{93}_{o}(f)$																									
$IV^{93}_{e}(f)$																									
$IV^{94}_{o}(f)$																									
$IV^{94}_{e}(f)$																									
$IV^{95}_{o}(f)$																									
$IV^{95}_{e}(f)$																									
$IV^{96}_{o}(f)$																									
$IV^{96}_{e}(f)$																									
$IV^{97}_{o}(f)$																									
$IV^{97}_{e}(f)$																									
$IV^{98}_{o}(f)$																									
$IV^{98}_{e}(f)$																									
$IV^{99}_{o}(f)$																									
$IV^{99}_{e}(f)$																									
$IV^{100}_{o}(f)$																									
$IV^{100}_{e}(f)$																									
$IV^{101}_{o}(f)$																									
$IV^{101}_{e}(f)$																									
$IV^{102}_{o}(f)$																									
$IV^{102}_{e}(f)$																									
$IV^{103}_{o}(f)$																									
$IV^{103}_{e}(f)$																									
$IV^{104}_{o}(f)$																									
$IV^{104}_{e}(f)$																									
$IV^{105}_{o}(f)$																									
$IV^{105}_{e}(f)$																									

TABLE 2.36. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,a	c	g
$\mathrm{IV}^{87}_{o}(f)$														1	1		2					2						
$\mathrm{IV}^{87}_{e}(f)$															1													
$\mathrm{IV}^{88}_{o}(f)$														1	1			2					2					
$\mathrm{IV}^{88}_{e}(f)$															1													
$\mathrm{IV}^{89}_{o}(f)$														1	1		2						2					
$\mathrm{IV}^{89}_{e}(f)$															1													
$\mathrm{IV}^{90}_{o,A}(f)$															4													
$\mathrm{IV}^{90}_{o,B}(f)$														4	4													
$\mathrm{IV}^{90}_{e,A}(f)$																												
$\mathrm{IV}^{90}_{e,A}(f)$																												
$\mathrm{IV}^{91}_{o}(f)$															3		1	1		1	1							
$\mathrm{IV}^{91}_{e}(f)$																												
$\mathrm{IV}^{92}_{o}(f)$															4			2							2			
$\mathrm{IV}^{92}_{e}(f)$																												
$\mathrm{IV}^{93}_{o}(f)$															4				2					1	1			
$\mathrm{IV}^{93}_{e}(f)$																												
$\mathrm{IV}^{94}_{o}(f)$															3				2	1	1							
$\mathrm{IV}^{94}_{e}(f)$																												
$\mathrm{IV}^{95}_{o}(f)$															4			2						2				
$\mathrm{IV}^{95}_{e}(f)$																												
$\mathrm{IV}^{96}_{o}(f)$															4		2							2				
$\mathrm{IV}^{96}_{e}(f)$																												
$\mathrm{IV}^{97}_{o}(f)$																	4				4							
$\mathrm{IV}^{97}_{e}(f)$																												
$\mathrm{IV}^{98}_{o}(f)$																	2						4					
$\mathrm{IV}^{98}_{e}(f)$																	2											
$\mathrm{IV}^{99}_{o}(f)$																	6							1				
$\mathrm{IV}^{99}_{e}(f)$																								1				
$\mathrm{IV}^{100}_{o}(f)$																	2	4						1				
$\mathrm{IV}^{100}_{e}(f)$																								1				
$\mathrm{IV}^{101}_{o}(f)$																	1	1				2	2					
$\mathrm{IV}^{101}_{e}(f)$																	1	1										
$\mathrm{IV}^{102}_{o}(f)$																	4	2							1			
$\mathrm{IV}^{102}_{e}(f)$																									1			
$\mathrm{IV}^{103}_{o}(f)$																	2					4						
$\mathrm{IV}^{103}_{e}(f)$																	2											
$\mathrm{IV}^{104}_{o}(f)$											1						2		4					1				
$\mathrm{IV}^{104}_{e}(f)$																												
$\mathrm{IV}^{105}_{o}(f)$																		6							1			
$\mathrm{IV}^{105}_{e}(f)$																									1			

TABLE 2.37. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,a	1,a	b	d	e	f	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$\mathrm{IV}_o^{106}(f)$																									
$\mathrm{IV}_e^{106}(f)$																									
$\mathrm{IV}_o^{107}(f)$																									
$\mathrm{IV}_e^{107}(f)$																									
$\mathrm{IV}_o^{108}(f)$																									
$\mathrm{IV}_e^{108}(f)$																									
$\mathrm{IV}_o^{109}(f)$																									
$\mathrm{IV}_e^{109}(f)$																									
$\mathrm{IV}_o^{110}(f)$																									
$\mathrm{IV}_e^{110}(f)$																									
$\mathrm{IV}_o^{111}(f)$																									
$\mathrm{IV}_e^{111}(f)$																									
$\mathrm{IV}_o^{112}(f)$																									
$\mathrm{IV}_e^{112}(f)$																									
$\mathrm{IV}_o^{113}(f)$																									
$\mathrm{IV}_e^{113}(f)$																									
$\mathrm{IV}_o^{114}(f)$																									
$\mathrm{IV}_e^{114}(f)$																									
$\mathrm{IV}_{o,A}^{h}(f)$					1				1						1	2									
$\mathrm{IV}_{o,B}^{h}(f)$																									
$\mathrm{IV}_{e,A}^{h}(f)$																									
$\mathrm{IV}_{e,B}^{h}(f)$																1									
$\mathrm{IV}_{o,A}^{i}(f)$					1					1						2									
$\mathrm{IV}_{o,B}^{i}(f)$																									
$\mathrm{IV}_{e,A}^{i}(f)$																									
$\mathrm{IV}_{e,B}^{i}(f)$																2									
$\mathrm{IV}_{o,A}^{j}(f)$							1									4									
$\mathrm{IV}_{o,B}^{j}(f)$											1														
$\mathrm{IV}_{e,A}^{j}(f)$																									
$\mathrm{IV}_{e,B}^{j}(f)$																									
$\mathrm{IV}_{o,A}^{k}(f)$																2									
$\mathrm{IV}_{o,B}^{k}(f)$																									
$\mathrm{IV}_{e,A}^{k}(f)$																									
$\mathrm{IV}_{e,B}^{k}(f)$																									
$\mathrm{IV}_o^{l}(f)$																1									
$\mathrm{IV}_e^{l}(f)$																									

TABLE 2.38. Table of incident numbers of $[\mathrm{III}_{0,A}^* : \mathcal{F}]$ or $[\mathrm{III}_A^* : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,a	c	g
$\mathrm{IV}^{106}_o(f)$																		4			4							
$\mathrm{IV}^{106}_e(f)$																												
$\mathrm{IV}^{107}_o(f)$																		2					4					
$\mathrm{IV}^{107}_e(f)$																		2										
$\mathrm{IV}^{108}_o(f)$																		2					4					
$\mathrm{IV}^{108}_e(f)$																		2										
$\mathrm{IV}^{109}_o(f)$											1							2	4						1			
$\mathrm{IV}^{109}_e(f)$																												
$\mathrm{IV}^{110}_o(f)$																		2	2		4							
$\mathrm{IV}^{110}_e(f)$																												
$\mathrm{IV}^{111}_o(f)$																			4		4							
$\mathrm{IV}^{111}_e(f)$																												
$\mathrm{IV}^{112}_o(f)$																								8				
$\mathrm{IV}^{112}_e(f)$																												
$\mathrm{IV}^{113}_o(f)$																								4	4			
$\mathrm{IV}^{113}_e(f)$																												
$\mathrm{IV}^{114}_o(f)$																									8			
$\mathrm{IV}^{114}_e(f)$																												
$\mathrm{IV}^{h}_{o,A}(f)$																												
$\mathrm{IV}^{h}_{o,B}(f)$																												
$\mathrm{IV}^{h}_{e,A}(f)$																												
$\mathrm{IV}^{h}_{e,B}(f)$																												
$\mathrm{IV}^{i}_{o,A}(f)$																												
$\mathrm{IV}^{i}_{o,B}(f)$																												
$\mathrm{IV}^{i}_{e,A}(f)$																												
$\mathrm{IV}^{i}_{e,B}(f)$																												
$\mathrm{IV}^{j}_{o,A}(f)$																												
$\mathrm{IV}^{j}_{o,B}(f)$																												
$\mathrm{IV}^{j}_{e,A}(f)$																												
$\mathrm{IV}^{j}_{e,B}(f)$																												
$\mathrm{IV}^{k}_{o,A}(f)$	1																										2	
$\mathrm{IV}^{k}_{o,B}(f)$														1													2	
$\mathrm{IV}^{k}_{e,A}(f)$																												
$\mathrm{IV}^{k}_{e,B}(f)$																												
$\mathrm{IV}^{l}_{o}(f)$		1											1													1	1	
$\mathrm{IV}^{l}_{e}(f)$																												

TABLE 2.39. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,*a*	1,*a*	*b*	*d*	*e*	*f*	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$\mathrm{IV}^m_o(f)$																1									
$\mathrm{IV}^m_e(f)$																									
$\mathrm{IV}^n_o(f)$																1									
$\mathrm{IV}^n_e(f)$																									
$\mathrm{IV}^o_o(f)$																									
$\mathrm{IV}^o_e(f)$																									
$\mathrm{IV}^p_o(f)$																									
$\mathrm{IV}^p_e(f)$																									
$\mathrm{IV}^q_{o,A}(f)$																2									
$\mathrm{IV}^q_{o,B}(f)$														2											
$\mathrm{IV}^q_{e,A}(f)$																									
$\mathrm{IV}^q_{e,B}(f)$																									
$\mathrm{IV}^r_{o,A}(f)$																									
$\mathrm{IV}^r_{o,B}(f)$											1					2		1							
$\mathrm{IV}^r_{e,A}(f)$																		1							
$\mathrm{IV}^r_{e,B}(f)$																									
$\mathrm{IV}^s_{o,A}(f)$									1						1				1						
$\mathrm{IV}^s_{o,B}(f)$																									
$\mathrm{IV}^s_{e,A}(f)$																									
$\mathrm{IV}^s_{e,B}(f)$																1			1						
$\mathrm{IV}^t_{o,A}(f)$																2		1							
$\mathrm{IV}^t_{e,B}(f)$												1							1						
$\mathrm{IV}^t_{e,A}(f)$																									
$\mathrm{IV}^t_{e,B}(f)$																									
$\mathrm{IV}^u_{o,A}(f)$																2			1						
$\mathrm{IV}^u_{o,B}(f)$																									
$\mathrm{IV}^u_{e,A}(f)$																									
$\mathrm{IV}^u_{e,B}(f)$																									
$\mathrm{IV}^v_{o,A}(f)$																		1							
$\mathrm{IV}^v_{o,B}(f)$																									
$\mathrm{IV}^v_{e,A}(f)$																									
$\mathrm{IV}^v_{e,B}(f)$																									
$\mathrm{IV}^x_o(f)$																			1						
$\mathrm{IV}^x_e(f)$																									
$\mathrm{IV}^y_o(f)$																		1							
$\mathrm{IV}^y_e(f)$																									

TABLE 2.40. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,a	c	g
$\mathrm{IV}^m_o(f)$		1																					1				1	
$\mathrm{IV}^m_e(f)$																											1	
$\mathrm{IV}^n_o(f)$		1																				1					1	
$\mathrm{IV}^n_e(f)$																											1	
$\mathrm{IV}^o_o(f)$			1														1										4	
$\mathrm{IV}^o_e(f)$																												
$\mathrm{IV}^p_o(f)$			1															1									4	
$\mathrm{IV}^p_e(f)$																												
$\mathrm{IV}^q_{o,A}(f)$																												
$\mathrm{IV}^q_{o,B}(f)$																												
$\mathrm{IV}^q_{e,A}(f)$																												
$\mathrm{IV}^q_{e,B}(f)$																												
$\mathrm{IV}^r_{o,A}(f)$																												
$\mathrm{IV}^r_{o,B}(f)$																												
$\mathrm{IV}^r_{e,A}(f)$																												
$\mathrm{IV}^r_{e,B}(f)$																												
$\mathrm{IV}^s_{o,A}(f)$																												
$\mathrm{IV}^s_{o,B}(f)$																												
$\mathrm{IV}^s_{e,A}(f)$																												
$\mathrm{IV}^s_{e,B}(f)$																												
$\mathrm{IV}^t_{o,A}(f)$																												
$\mathrm{IV}^t_{o,B}(f)$																												
$\mathrm{IV}^t_{e,A}(f)$																												
$\mathrm{IV}^t_{e,B}(f)$																												
$\mathrm{IV}^u_{o,A}(f)$																1												
$\mathrm{IV}^u_{o,B}(f)$																												1
$\mathrm{IV}^u_{e,A}(f)$																												
$\mathrm{IV}^u_{e,B}(f)$																												
$\mathrm{IV}^v_{o,A}(f)$																				1							2	
$\mathrm{IV}^v_{o,B}(f)$																											2	1
$\mathrm{IV}^v_{e,A}(f)$																												
$\mathrm{IV}^v_{e,B}(f)$																												
$\mathrm{IV}^x_o(f)$													1													1		
$\mathrm{IV}^x_e(f)$																											1	
$\mathrm{IV}^y_o(f)$																					1						2	
$\mathrm{IV}^y_e(f)$																												

TABLE 2.41. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,0,0	0,0,1	0,1,1	1,1,1	0,3	1,3	0,4	1,4	8	9	10	11	12	0,a	1,a	b	d	e	f	0,0,2	0,2,2	1,1,2	1,2,2	0,1,2	2,2,2
$\mathrm{IV}^z_{o,A}(f)$																2									
$\mathrm{IV}^z_{o,B}(f)$																									
$\mathrm{IV}^z_{e,A}(f)$																									
$\mathrm{IV}^z_{e,B}(f)$																									
$\mathrm{IV}^w_o(f)$																									
$\mathrm{IV}^w_e(f)$																									
$\mathrm{IV}^\alpha_{o,A}(f)$																1			1						
$\mathrm{IV}^\alpha_{o,B}(f)$														1											
$\mathrm{IV}^\alpha_{e,A}(f)$																	1								
$\mathrm{IV}^\alpha_{e,B}(f)$																									
$\mathrm{IV}^\beta_{o,A}(f)$																		2							
$\mathrm{IV}^\beta_{o,B}(f)$																									
$\mathrm{IV}^\beta_{e,A}(f)$																									
$\mathrm{IV}^\beta_{e,B}(f)$																									
$\mathrm{IV}^\gamma_{o,A}(f)$												1						3							
$\mathrm{IV}^\gamma_{o,B}(f)$										1															
$\mathrm{IV}^\gamma_{e,A}(f)$																			3						
$\mathrm{IV}^\gamma_{e,B}(f)$																									
$\mathrm{IV}^\delta_{o,A}(f)$											2								4						
$\mathrm{IV}^\delta_{o,B}(f)$																									
$\mathrm{IV}^\delta_{e,A}(f)$																		2							
$\mathrm{IV}^\delta_{e,B}(f)$																									
$\mathrm{IV}^\varepsilon_{o,A}(f)$													2					6							
$\mathrm{IV}^\varepsilon_{o,B}(f)$																									
$\mathrm{IV}^\varepsilon_{e,A}(f)$																									
$\mathrm{IV}^\varepsilon_{e,B}(f)$																									
$\mathrm{IV}^\zeta_{o,A}(f)$																		1	2						
$\mathrm{IV}^\zeta_{o,B}(f)$																									
$\mathrm{IV}^\zeta_{e,A}(f)$																									
$\mathrm{IV}^\zeta_{e,B}(f)$																									
$\mathrm{IV}^\iota_{o,A}(f)$																		2							
$\mathrm{IV}^\iota_{o,B}(f)$																									
$\mathrm{IV}^\iota_{e,A}(f)$																									
$\mathrm{IV}^\iota_{e,B}(f)$																									

TABLE 2.42. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

	0,5	0,6	0,7	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7	13	14	15	16	17	18	19	20	21	22	23	24	25	26	2,a	c	g
$\mathrm{IV}^{z}_{o,A}(f)$																												
$\mathrm{IV}^{z}_{o,B}(f)$														1														1
$\mathrm{IV}^{z}_{e,A}(f)$																												1
$\mathrm{IV}^{z}_{e,B}(f)$																												
$\mathrm{IV}^{w}_{o}(f)$															1												2	1
$\mathrm{IV}^{w}_{e}(f)$																												
$\mathrm{IV}^{\alpha}_{o,A}(f)$																												
$\mathrm{IV}^{\alpha}_{o,B}(f)$																												
$\mathrm{IV}^{\alpha}_{e,A}(f)$																												
$\mathrm{IV}^{\alpha}_{e,B}(f)$																												
$\mathrm{IV}^{\beta}_{o,A}(f)$																												
$\mathrm{IV}^{\beta}_{o,B}(f)$																												
$\mathrm{IV}^{\beta}_{e,A}(f)$																												
$\mathrm{IV}^{\beta}_{e,B}(f)$																												
$\mathrm{IV}^{\gamma}_{o,A}(f)$																												
$\mathrm{IV}^{\gamma}_{o,B}(f)$																												
$\mathrm{IV}^{\gamma}_{e,A}(f)$																												
$\mathrm{IV}^{\gamma}_{e,B}(f)$																												
$\mathrm{IV}^{\delta}_{o,A}(f)$																												
$\mathrm{IV}^{\delta}_{o,B}(f)$																												
$\mathrm{IV}^{\delta}_{e,A}(f)$																												
$\mathrm{IV}^{\delta}_{e,B}(f)$																												
$\mathrm{IV}^{\varepsilon}_{o,A}(f)$																												
$\mathrm{IV}^{\varepsilon}_{o,B}(f)$																												
$\mathrm{IV}^{\varepsilon}_{e,A}(f)$																												
$\mathrm{IV}^{\varepsilon}_{e,B}(f)$																												
$\mathrm{IV}^{\zeta}_{o,A}(f)$																				1								2
$\mathrm{IV}^{\zeta}_{o,B}(f)$														1														
$\mathrm{IV}^{\zeta}_{e,A}(f)$																												1
$\mathrm{IV}^{\zeta}_{e,B}(f)$																												
$\mathrm{IV}^{\iota}_{o,A}(f)$																												4
$\mathrm{IV}^{\iota}_{o,B}(f)$												2																
$\mathrm{IV}^{\iota}_{e,A}(f)$																												
$\mathrm{IV}^{\iota}_{e,B}(f)$																												

TABLE 2.43. Table of incident numbers of $[\mathrm{III}^*_{0,A} : \mathcal{F}]$ or $[\mathrm{III}^*_A : \mathcal{F}]$

Chapter 3

Singular fibers of stable maps and signatures of 4-manifolds

CHAPTER 3

Singular fibers of stable maps and signatures of 4-manifolds

(joint work with O.Saeki)

1. Introduction

In [**43**] Saeki developed the theory of singular fibers of generic differentiable maps between manifolds of negative codimension. Here, the *codimension* of a map $f : M \to N$ between manifolds is defined to be $k = \dim N - \dim M$. For $k \geq 0$, the fiber over a point in N is a discrete set of points, as long as the map is generic enough, and we can study the topology of such maps by using the well-developed theory of multi-jet spaces. However, in the case where $k < 0$, the fiber over a point is no longer a discrete set, and is a complex of positive dimension $-k$ in general. This means that the theory of multi-jet spaces is not sufficient any more, and in [**43**] we have seen that the topology of singular fibers plays an essential role in such a study.

In [**43**], as an explicit and important example of the theory of singular fibers, C^∞ stable maps of closed orientable 4-manifolds into 3-manifolds were studied and their singular fibers were completely classified up to the natural equivalence relation, called the C^∞ (or C^0) right-left equivalence (for a precise definition, see §2 of the present paper). Furthermore, it was proved that the number of singular fibers of a specific type (in the terminology of [**43**], singular fibers of type III^8) of such a map is congruent modulo two to the Euler characteristic of the source 4-manifold (see [**43**, Theorem 5.1] and also Corollary 5.6 of the present paper).

In this Chapter, we will give an "integral lift" of this modulo two Euler characteristic formula. More precisely, we consider C^∞ stable maps of *oriented* 4-manifolds into 3-manifolds, and we give a sign $+1$ or -1 to each of its III^8 type fiber, using the orientation of the source 4-manifold. Then we show that the algebraic number of III^8 type fibers coincides with the signature of the source oriented 4-manifold (Theorem 5.5).

For certain Lefschetz fibrations, similar signature formulas have already been proved by Matsumoto [**27, 28**], Endo [**10**], etc. Our formula can be regarded as their analogues from the viewpoint of singularity theory of generic differentiable maps. The most important difference between Lefschetz fibrations and generic differentiable maps is that not all manifolds can admit a Lefschetz fibration, while every manifold admits a generic differentiable map. (In fact, a single manifold admits a lot of generic differentiable maps.) Furthermore, it is known that similar signature formulas do not hold for arbitrary Lefschetz fibrations, since there exist oriented surface bundles over oriented surfaces with nonzero signatures (see [**29**]). In this sense, our formula is more general (see Remark 7.7). Our proof of the formula is based on the abundance of such generic maps in some sense.

More precisely, our proof of the formula goes as follows. We first define the notion of a chiral singular fiber (for a precise definition, see §2). Roughly speaking, if a fiber can be transformed to its "orientation reversal" by an orientation preserving homeomorphism of the source manifold, then we call it an achiral fiber; otherwise, a chiral fiber. On the other hand, we classify singular fibers of proper C^∞ stable maps of orientable 5-manifolds into 4-manifolds by using methods developed in [**43**]. Then, for proper C^∞ stable maps of oriented 4-manifolds into 3-manifolds, and those of oriented 5-manifolds into 4-manifolds, we determine those singular fibers in the classification lists which are chiral. Furthermore, for each chiral singular fiber that appears discretely, we define its sign ($= \pm 1$).

Let us consider two C^∞ stable maps of 4-manifolds into a 3-manifold which are oriented bordant. Then by using a generic bordism between them, which is a generic differentiable map of a 5-manifold into a 4-manifold, and by looking at the III^8-fiber locus in the target 4-manifold, we show the oriented bordism invariance of the algebraic numbers of III^8 type fibers of the original stable maps of 4-manifolds. Finally, we verify our formula for an explicit example of a stable map of an oriented 4-manifold with signature $+1$. (In fact, this final step is not so easy and needs a careful analysis.) Combining all these, we will prove our formula.

This Chapter is organized as follows. In §2 we give some fundamental definitions concerning singular fibers of generic differentiable maps, among which is the notion of a chiral singular fiber. In §3 we recall the classification of singular fibers of proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds obtained in [**43**]. In §4 we present the classification of singular fibers of proper C^∞ stable maps of orientable 5-manifolds into 4-manifolds. In §5 we determine those singular fibers in the classification lists which are chiral. Furthermore, for each chiral singular fiber that appears discretely, we define its sign by using the orientation of the source manifold. In §6 we prove the oriented bordism invariance of the algebraic number of III^8 type fibers. This is proved by looking at the adjacencies of the chiral singular fiber loci in the target manifold. In §7 we investigate the explicit example of a C^∞ stable map of an oriented 4-manifold into a 3-manifold constructed in [**43**]. In order to calculate the signature of the source 4-manifold, we will compute the self-intersection number of the surface of definite fold points by using normal sections coming from the surface of indefinite fold points. This procedure needs some technical details so that this section will be rather long. By combining the result of §6 with the computation of the example, we prove our main theorem. Finally in §8, we define the universal complex of chiral singular fibers for proper C^∞ stable maps of 5-manifolds into 4-manifolds and compute its third cohomology group. This will give an interpretation of our formula from the viewpoint of the theory of singular fibers of generic differentiable maps as developed in [**43**].

The authors would like to express their thanks to András Szűcs for drawing their attention to the work of Conner–Floyd. They would also like to thank Goo Ishikawa for his invaluable comments and encouragement.

2. Preliminaries

Let us begin by some fundamental definitions. For some of the definitions of the singular fibers, the equivalence relations among them refer to Chapter 2, 3 and [**43**].

DEFINITION 2.1. Let $\mathfrak{F}$ be a C^0 equivalence class of a fiber of a proper smooth map in the sense of Definition 2.1 in Chapter 1. For a proper smooth map $f : M \to N$ between smooth manifolds, we denote by $\mathfrak{F}(f)$ the set consisting of those points of N over which lies a fiber of type $\mathfrak{F}$. It is known that if the smooth map f is generic enough (for example if f is a Thom map [**13**]), then $\mathfrak{F}(f)$ is a union of strata of N and is a C^0 submanifold of N of constant codimension (for details, see [**43**, Chap. 7]). Furthermore, this codimension $\kappa = \kappa(\mathfrak{F})$ does not depend on the choice of f and we call it the *codimension* of $\mathfrak{F}$. We also say that a fiber belonging to $\mathfrak{F}$ is a *codimension* κ *fiber.*

Let us introduce the following weaker relation for (singular) fibers.

DEFINITION 2.2. Let $f_i : (M_i, (f_i)^{-1}(y_i)) \to (N_i, y_i)$ be proper smooth map germs along fibers with $n = \dim M_i$ and $p = \dim N_i$, $i = 0, 1$, with $n \geq p$. We may assume that N_i is the p-dimensional open disk Int D^p and that y_i is its center 0, $i = 0, 1$. We say that the two fibers are C^0 (or C^∞) *equivalent modulo regular fibers* if there exist $(n-p)$-dimensional closed manifolds F_i, $i = 0, 1$, such that the disjoint union of f_0 and the map germ $\pi_0 : (F_0 \times \mathrm{Int}D^p, F_0 \times \{0\}) \to (\mathrm{Int}D^p, 0)$ defined by the projection to the second factor is C^0 (resp. C^∞) equivalent to the disjoint union of f_1 and the map germ $\pi_1 : (F_1 \times \mathrm{Int}D^p, F_1 \times \{0\}) \to (\mathrm{Int}D^p, 0)$ defined by the projection to the second factor.

Note that by the very definition, any two regular fibers are C^∞ equivalent modulo regular fibers to each other as long as their dimensions of the source and the target are the same.

For the C^0 equivalence modulo regular fibers, we use the same notation as in Definition 2.1. Then all the assertions in Definition 2.1 hold for C^0 equivalence classes modulo regular fibers as well.

The following definition is not so important in this paper. However, in order to compare it with Definition 2.4, we recall it. For details, refer to [**43**].

DEFINITION 2.3. Let $\mathfrak{F}$ be a C^0 equivalence class of a fiber of a proper Thom map. Let us consider arbitrary homeomorphisms $\tilde{\varphi}$ and φ which make the diagram

$$\begin{array}{ccc} (f^{-1}(U_0), f^{-1}(y)) & \xrightarrow{\tilde{\varphi}} & (f^{-1}(U_1), f^{-1}(y)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ (U_0, y) & \xrightarrow{\varphi} & (U_1, y) \end{array}$$

commutative, where f is a proper Thom map such that the fiber over y belongs to $\mathfrak{F}$, and U_i are open neighborhoods of y. Note that then we have $\varphi(\mathfrak{F}(f) \cap U_0) = \mathfrak{F}(f) \cap U_1$. We say that $\mathfrak{F}$ is *co-orientable* if φ always preserves the local orientation of the normal bundle of $\mathfrak{F}(f)$ at y.

We also call any fiber belonging to a co-orientable C^0 equivalence class a *co-orientable fiber.*

In particular, if the codimension of $\mathfrak{F}$ coincides with the dimension of the target of f, then φ above should preserve the local orientation of the target at y.

Note that if $\mathfrak{F}$ is co-orientable, then $\mathfrak{F}(f)$ has orientable normal bundle for every proper Thom map f.

The following definition plays an essential role in this paper. Compare this with Definition 2.3.

DEFINITION 2.4. Let $\mathfrak{F}$ be a C^0 equivalence class of a fiber of a proper Thom map of an *oriented* manifold. We say that $\mathfrak{F}$ is *achiral* if there exist homeomorphisms $\tilde{\varphi}$ and φ which make the diagram

$$\begin{array}{ccc} (f^{-1}(U_0), f^{-1}(y)) & \xrightarrow{\tilde{\varphi}} & (f^{-1}(U_1), f^{-1}(y)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ (U_0, y) & \xrightarrow{\varphi} & (U_1, y) \end{array} \tag{2.1}$$

commutative such that the homeomorphism $\tilde{\varphi}$ reverses the orientation and that the homeomorphism

$$\varphi|_{\mathfrak{F}(f)\cap U_0} : \mathfrak{F}(f) \cap U_0 \to \mathfrak{F}(f) \cap U_1 \tag{2.2}$$

preserves the local orientation of $\mathfrak{F}(f)$ at y, where f is a proper Thom map such that the fiber over y belongs to $\mathfrak{F}$, and U_i are open neighborhoods of y.

Note that if the codimension of $\mathfrak{F}$ coincides with the dimension of the target of f, then the condition about the homeomorphism (2.2) is redundant. Note also that the above definition does not depend on the choice of f or y.

Moreover, we say that $\mathfrak{F}$ is *chiral* if it is not achiral.

We also call any fiber belonging to a chiral (resp. achiral) C^0 equivalence class a *chiral fiber* (resp. *achiral fiber*).

Furthermore, we have the following.

LEMMA 2.5. *Suppose that the codimension of $\mathfrak{F}$ is strictly smaller than the dimension of the target. Then $\mathfrak{F}$ is achiral if and only if there exist homeomorphisms $\tilde{\varphi}$ and φ making the diagram* (2.1) *commutative such that the homeomorphism $\tilde{\varphi}$ preserves the orientation and that the homeomorphism* (2.2) *reverses the orientation.*

PROOF. Let $f : (M, f^{-1}(y)) \to (N, y)$ be a representative of $\mathfrak{F}$. Let us consider the Whitney stratifications $\mathcal{M}$ and $\mathcal{N}$ of M and N respectively with respect to which f satisfies certain regularity conditions [**12**]. We may assume that y belongs to a top dimensional stratum of $\mathcal{N}$. By our hypothesis, the dimension k of this stratum is strictly positive. Let Δ be a small open disk of codimension k centered at y in N which intersects with the stratum transversely at y. Set $M' = f^{-1}(\Delta)$. Note that $f' = f|_{M'} : M' \to \Delta$ is a proper Thom map.

By the second isotopy lemma, we see that the map germ $f : (M, f^{-1}(y)) \to (N, y)$ is C^0 equivalent to the map germ $f' \times \mathrm{id}_{\mathbb{R}^k} : (M' \times \mathbb{R}^k, f'^{-1}(y) \times 0) \to (\Delta \times \mathbb{R}^k, y \times 0)$. Since k is positive, it is now easy to construct orientation reversing homeomorphisms $\tilde{\varphi}$ and φ making the diagram (2.1) commutative such that the homeomorphism (2.2) reverses the orientation. Then the lemma follows immediately. This completes the proof. □

We warn the reader that even if a fiber is chiral, homeomorphisms $\tilde{\varphi}$ and φ making the diagram (2.1) commutative may not satisfy any of the following.

(1) The homeomorphism $\tilde{\varphi}$ preserves the orientation and the homeomorphism (2.2) preserves the orientation.
(2) The homeomorphism $\tilde{\varphi}$ reverses the orientation and the homeomorphism (2.2) reverses the orientation.

This is because $f^{-1}(U_i)$ may not be connected.

For example, a regular fiber is achiral if and only if the fiber manifold admits an orientation reversing homeomorphism. The disjoint union of an achiral fiber and an achiral regular fiber is clearly achiral. The disjoint union of a chiral fiber and an achiral regular fiber is always chiral.

In what follows, we consider only those maps of codimension -1 so that a regular fiber is always of dimension 1. Note that every compact 1-dimensional manifold admits an orientation reversing homeomorphism. Therefore, for two fibers which are C^0 equivalent modulo regular fibers, one is chiral if and only if so is the other. Therefore, we can speak of a chiral (or achiral) C^0 equivalence class modulo regular fibers as well.

3. Singular fibers of stable maps of 4-manifolds into 3-manifolds

In this section, we consider proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds and recall the classification of their singular fibers obtained in [**43**].

The characterization of proper C^∞ stable maps of 4-manifolds into 3-manifolds is obtained in Chapter 1, Proposition 2.2.

In the following, we assume that the 4-manifold M is orientable. Using Proposition 2.2 in Chapter 1, Saeki obtained the following classification of singular fibers [**43**].

THEOREM 3.1. *Let $f : M \to N$ be a proper C^∞ stable map of an orientable 4-manifold M into a 3-manifold N. Then, every singular fiber of f is C^∞ (and hence C^0) equivalent modulo regular fibers to one of the fibers as in Figure 3.1. Furthermore, no two fibers appearing in the list are C^0 equivalent modulo regular fibers.*

Remark 3.2. In Figure 3.1, κ denotes the codimension of the relevant singular fiber in the sense of Definition 2.1. Furthermore, $\mathrm{I}^*, \mathrm{II}^*$ and III^* mean the names of the corresponding singular fibers, and "/" is used only for separating the figures. Note that we have named the fibers so that each connected fiber has its own digit or letter, and a disconnected fiber has the name consisting of the digits or letters of its connected components. Hence, the number of digits or letters in the superscript coincides with the number of connected components that contain singular points.

Remark 3.3. For proper C^∞ stable maps of 3-manifolds into the plane, a similar classification of singular fibers was obtained by Kushner, Levine and Porto [**22**], although they did not mention explicitly the equivalence relation for their classification. Their classification was in fact based on the "diffeomorphism modulo regular fibers".

4. Singular fibers of stable maps of 5-manifolds into 4-manifolds

In this section we present the classification of singular fibers of orientable 5-manifolds into 4-manifolds.

We note that we alreadly have the characterization of proper stable maps of 5-manifolds into 4-manifolds: Proposition 4.1 in Chapter 2, and the diffeomorphism types of a neighborhood of a point in its corresponding fiber: Lemma 4.4 in Chapter 2.

Then by an argument similar to that in [**43**, Chap. 3], we can prove the following, whose proof is left to the reader.

THEOREM 4.1. *Let $f : M \to N$ be a proper C^∞ stable map of an orientable 5-manifold M into a 4-manifold N. Then, every singular fiber of f is C^0 equivalent modulo regular fibers to one of the fibers as follows:*

(1) *the suspension of a fiber appearing in Theorem 3.1, or*
(2) *a disconnected fiber* $\mathrm{IV}^{0,0,0,0}$, $\mathrm{IV}^{0,0,0,1}$, $\mathrm{IV}^{0,0,1,1}$, $\mathrm{IV}^{0,1,1,1}$, $\mathrm{IV}^{1,1,1,1}$, $\mathrm{IV}^{0,0,2}$, $\mathrm{IV}^{0,1,2}$, $\mathrm{IV}^{1,1,2}$, $\mathrm{IV}^{0,0,3}$, $\mathrm{IV}^{0,1,3}$, $\mathrm{IV}^{1,1,3}$, $\mathrm{IV}^{0,4}$, $\mathrm{IV}^{0,5}$, $\mathrm{IV}^{0,6}$, $\mathrm{IV}^{0,7}$, $\mathrm{IV}^{0,8}$, $\mathrm{IV}^{1,4}$, $\mathrm{IV}^{1,5}$, $\mathrm{IV}^{1,6}$, $\mathrm{IV}^{1,7}$, $\mathrm{IV}^{1,8}$, $\mathrm{IV}^{2,2}$, $\mathrm{IV}^{2,3}$, $\mathrm{IV}^{3,3}$, $\mathrm{IV}^{0,0,a}$, $\mathrm{IV}^{0,1,a}$, $\mathrm{IV}^{1,1,a}$, $\mathrm{IV}^{0,b}$, $\mathrm{IV}^{1,b}$, $\mathrm{IV}^{2,a}$, $\mathrm{IV}^{3,a}$, $\mathrm{IV}^{a,a}$, $\mathrm{IV}^{0,c}$, $\mathrm{IV}^{0,d}$, $\mathrm{IV}^{0,e}$, $\mathrm{IV}^{1,c}$, $\mathrm{IV}^{1,d}$, *or* $\mathrm{IV}^{1,e}$, *or*
(3) *one of the connected fibers as depicted in Figure* 3.2.

Furthermore, no two fibers appearing in the list (1)–(3) *above are C^0 equivalent modulo regular fibers.*

For the fibers as in Theorem 4.1 (1), we use the same names as those of the corresponding fibers in Theorem 3.1. Note that the names of the fibers are consistent with the convention mentioned in Remark 3.2. Therefore, the figure corresponding to each fiber listed in Theorem 4.1 (2) can be obtained by taking the disjoint union of the fibers in Figure 3.1 corresponding to the digits or letters appearing in the superscript. For example, the figure for the fiber $\mathrm{IV}^{0,0,0,1}$ consists of three dots and a "figure 8".

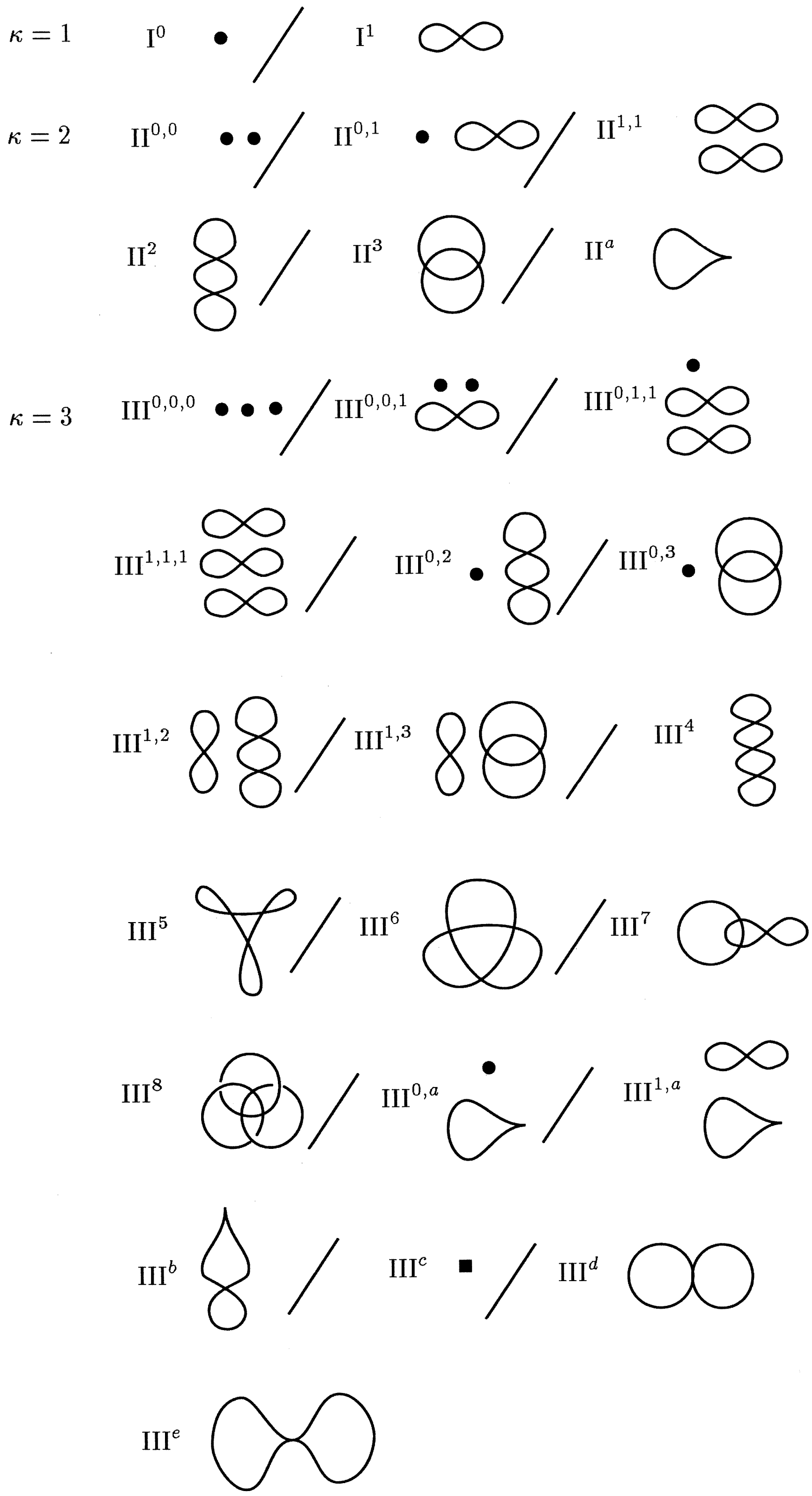

FIGURE 3.1. List of singular fibers of proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds

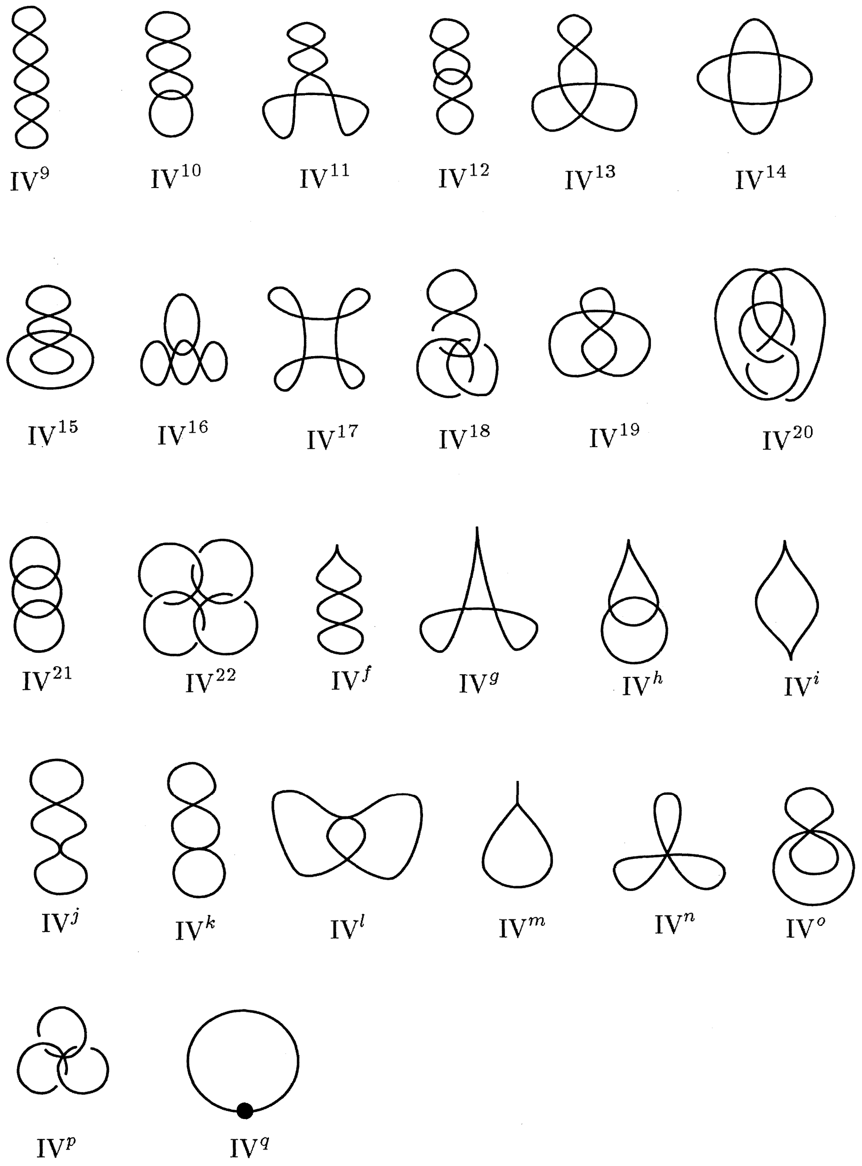

FIGURE 3.2. List of codimension 4 connected singular fibers of proper C^∞ stable maps of orientable 5-manifolds into 4-manifolds

In Figure 3.2, we did not use "/" as in Figure 3.1, since the depicted fibers are all connected and are easy to recognize.

Note also that the codimensions of the fibers in Theorem 4.1 (1) coincide with those of the corresponding fibers in Theorem 3.1. Furthermore, the fibers in Theorem 4.1 (2) and (3) all have codimension 4.

Remark 4.2. The result of Theorem 4.1 holds for the classification up to C^∞ equivalence as well. As a consequence, we see that two fibers are C^0 equivalent if and only if they are C^∞ equivalent (for related results, refer to [**43**, Chap. 3]). This should be compared with a result of Damon [**9**] about stable map germs in nice dimensions.

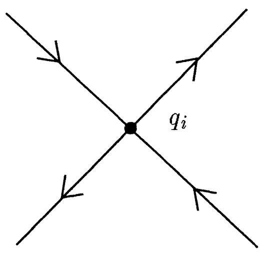

FIGURE 3.3. Orientations of the four arcs incident to a singular point

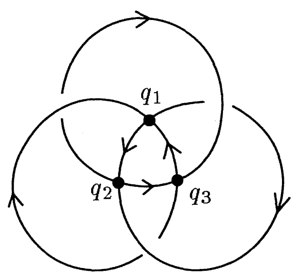

FIGURE 3.4. Cyclic order of the three singular points

5. Chiral singular fibers and their signs

In this section we determine those singular fibers of proper stable maps of oriented 4-manifolds into 3-manifolds which are chiral. We also define a sign ($= \pm 1$) for each chiral singular fiber of codimension 3.

Let us first consider a fiber of type III^8. Let $f : (M, f^{-1}(y)) \to (N, y)$ be a map germ representing the fiber of type III^8 with $f^{-1}(y)$ being connected, where M is an orientable 4-manifold and N is a 3-manifold. We assume that M is oriented. Let us denote the three singular points of f contained in $f^{-1}(y)$ by q_1, q_2 and q_3.

Let us fix an orientation of a neighborhood of y in N. Then for every regular point $q \in f^{-1}(y)$, we can define the local orientation of the fiber near q by the "fiber first" convention; i.e., we give the orientation to the fiber at q so that the ordered 4-tuple $\langle v, v_1, v_2, v_3 \rangle$ of tangent vectors at q gives the orientation of M, where v is a tangent vector of the fiber at q which corresponds to its orientation, and v_1, v_2 and v_3 are tangent vectors of M at q such that the ordered 3-tuple $\langle df_q(v_1), df_q(v_2), df_q(v_3) \rangle$ corresponds to the local orientation of N at y. Note that the set of regular points in $f^{-1}(y)$ consists of six open arcs and each of them gets its orientation by the above rule.

Each singular point q_i is incident to four open arcs. We see easily that their orientations should be as depicted in Figure 3.3 by considering the orientations induced on the nearby fibers.

For each pair (q_i, q_j), $i \neq j$, of singular points, we have exactly two open arcs of $f^{-1}(y)$ which connect q_i and q_j. Furthermore, the orientations of the two open arcs coincide with each other in the sense that one of the two arcs goes from q_i to q_j if and only if so does the other one. Then we see that the orientations on the six open arcs define a cyclic order of the three singular points q_1, q_2 and q_3 (see Figure 3.4). By renaming the three singular points if necessary, we may assume that this cyclic order is given by $\langle q_1, q_2, q_3 \rangle$.

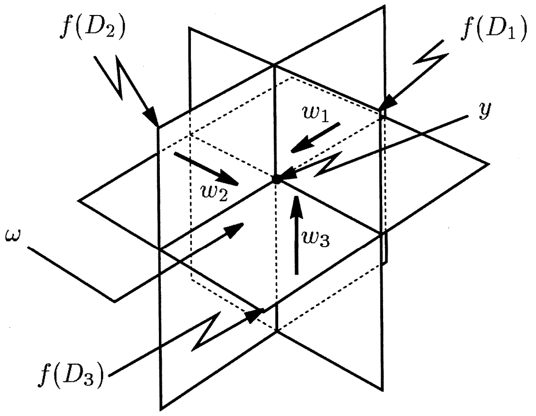

FIGURE 3.5. Vectors w_i normal to $f(D_i)$ pointing toward ω

Let D_i be a sufficiently small open disk neighborhood of q_i in $S(f)$. Since the multi-germ $(f|_{S(f)}, f^{-1}(y) \cap S(f))$ corresponds to the triple point as depicted in Figure 1.2 (4) in Chapter 1, the images $f(D_1), f(D_2)$ and $f(D_3)$ are open 2-disks in N in general position forming a triple point at y. They divide a neighborhood of y in N into eight octants. For each octant ω, take a point in it and count the number of connected components of the regular fiber over the point. It should be equal either to 1 or to 2 and it does not depend on the choice of the point (for details, see [**43**, Figure 3.6]). When it is equal to k $(= 1, 2)$, we call ω a *k-octant.*

Choose a 1-octant ω. Let w_i be a normal vector to $f(D_i)$ in N pointing toward ω at a point incident to that octant, $i = 1, 2, 3$ (see Figure 3.5).

We may identify a neighborhood of y in N with $\mathbb{R}^3$. Then the local orientation at y corresponding to the ordered 3-tuple of vectors $\langle w_1, w_2, w_3 \rangle$ depends only on the cyclic order of the three open disks $f(D_1)$, $f(D_2)$ and $f(D_3)$ and is well-defined, once a 1-octant is chosen. Then we say that the fiber $f^{-1}(y)$ is *positive* if the orientation corresponding to $\langle w_1, w_2, w_3 \rangle$ coincides with the local orientation of N at y which we chose at the beginning; otherwise, *negative*. We define the *sign* of the fiber to be $+1$ (or -1) if it is positive (resp. negative).

LEMMA 5.1. *The above definition does not depend on the choices of the following data, and the sign of a* III^8 *type fiber is well-defined as long as the source* 4*-manifold is oriented:*

(1) *the* 1*-octant* ω*,*
(2) *the local orientation of* N *at* y*.*

PROOF. (1) It is easy to see that any two adjacent octants have distinct numbers of connected components of their associated regular fibers; i.e., a 1-octant is adjacent to 2-octants, but never to another 1-octant, and vice versa. Therefore, in order to move from the chosen 1-octant to another 1-octant, one has to cross the open disks $f(D_1), f(D_2)$ and $f(D_3)$ even number of times. Every time one crosses an open disk, the associated normal vector corresponding to that open disk changes the direction, while the other two vectors remain parallel. Therefore, after crossing the open disks even number of times, we get the same orientation determined by the associated ordered normal vectors.

(2) If we reverse the local orientation of N at y, then the regular parts of fibers get opposite orientations. Therefore, in the above definition, the cyclic order of the three singular points is reversed. Hence, the resulting local orientation at y

determined by the three normal vectors is also reversed. Thus, the sign of the fiber is well-defined. □

For a fiber which is a disjoint union of a III^8 type fiber and a finite number of copies of a fiber of the trivial circle bundle (i.e., for a fiber equivalent to a III^8 type fiber modulo regular fibers), we say that it is positive (resp. negative) if the III^8-fiber component is positive (resp. negative). We define the *sign* of such a fiber to be $+1$ (or -1) if it is positive (resp. negative).

Remark 5.2. It should be noted that if we reverse the orientation of the source 4-manifold, then the sign of a III^8 type fiber necessarily changes.

COROLLARY 5.3. *A fiber equivalent to a* III^8 *type fiber modulo regular fibers is always chiral.*

PROOF. If it is achiral, then a representative of a III^8 type singular fiber and its copy with the orientation of the source 4-manifold being reversed are C^0 equivalent with respect to an orientation preserving homeomorphism between the sources (i.e., with respect to a homeomorphism $\tilde{\varphi}$ as in the diagram (2.1)). Let us take local orientations at the target points so that the homeomorphism between the target manifolds (i.e., the homeomorphism φ in the diagram (2.1)) preserves the orientation. Then by our definition of the sign, we see that the two III^8 type fibers should have the same sign, which is a contradiction in view of Remark 5.2. Therefore, the desired conclusion follows. This completes the proof. □

Let us now consider the other singular fibers appearing in Theorem 3.1. By using similar arguments, we can determine the chiral singular fibers among the list. More precisely, we have the following.

PROPOSITION 5.4. *A singular fiber of a proper* C^∞ *stable map of an oriented 4-manifold into a 3-manifold is chiral if and only if it contains a fiber of type* III^5, III^7 *or* III^8.

PROOF. For fibers of types III^5 and III^7, we can define their signs as for a III^8 type fiber. Therefore, they are chiral. Details are left to the reader.

For the other fibers, we can find homeomorphisms $\tilde{\varphi}$ and φ as in Definition 2.4. For example, let us consider a II^2 type fiber. Let $f : (M, f^{-1}(y)) \to (N, y)$ be a proper smooth map germ representing a fiber of type II^2, and let q_1 and q_2 be the two singular points contained in $f^{-1}(y)$, both of which are indefinite fold points. We fix orientations of M and N near $f^{-1}(y)$ and y respectively. Then the regular part of $f^{-1}(y)$ is naturally oriented by the "fiber first" convention.

It is easy to show that the involution of $f^{-1}(y)$ as in Figure 3.6 reverses the orientation of the regular part of $f^{-1}(y)$. Note that this involution fixes the two singular points q_1 and q_2 pointwise.

By §3, there exist coordinates (x_i, y_i, z_i, w_i) and (X, Y, Z) around q_i, $i = 1, 2$, and $f(q_i) = y$ respectively such that f is given by

$$(x_1, y_1, z_1, w_1) \mapsto (x_1, y_1, z_1^2 - w_1^2)$$

around q_1, and by

$$(x_2, y_2, z_2, w_2) \mapsto (x_2, z_2^2 - w_2^2, y_2)$$

around q_2 with respect to these coordinates. Then we may assume that the above involution is consistent with the involutions defined by

$$(x_1, y_1, z_1, w_1) \mapsto (x_1, y_1, -z_1, w_1)$$

around q_1 and by

$$(x_2, y_2, z_2, w_2) \mapsto (x_2, y_2, -z_2, z_2)$$

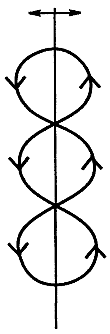

FIGURE 3.6. Orientation reversing involution of a II^2 type fiber

around q_2. Then we can extend this involution of a neighborhood of $\{q_1, q_2\}$ to a self-diffeomorphism $\tilde{\varphi}$ of $f^{-1}(U)$ for a sufficiently small open disk neighborhood U of y in N so that the diagram

$$\begin{array}{ccc} (f^{-1}(U), f^{-1}(y)) & \xrightarrow{\tilde{\varphi}} & (f^{-1}(U), f^{-1}(y)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ (U, y) & \xrightarrow{\mathrm{id}_U} & (U, y) \end{array}$$

is commutative, by using the relative version of Ehresmann's fibration theorem (see [**6**], [**23**, §3], [**5**, §8.12], [**43**, §1]), where id_U denotes the identity map of U. Note that the diffeomorphism $\tilde{\varphi}$ thus constructed is orientation reversing. Hence, the fiber $f^{-1}(y)$ is achiral according to Definition 2.4.

We can use similar arguments for the other fibers to show that they are achiral. Details are left to the reader. □

Let us now state the main theorem of this paper. For a closed oriented 4-manifold, we denote by $\sigma(M)$ the signature of M. Furthermore, for a C^∞ stable map $f : M \to N$ into a 3-manifold N, we denote by $||\mathrm{III}^8(f)||$ the algebraic number of III^8 type fibers of f; i.e., it is the sum of the signs over all fibers of f equivalent to a III^8 type fiber modulo regular fibers.

THEOREM 5.5. *Let M be a closed oriented 4-manifold and N a 3-manifold. Then, for any C^∞ stable map $f : M \to N$, we have*

$$\sigma(M) = ||\mathrm{III}^8(f)|| \in \mathbb{Z}.$$

The proof of Theorem 5.5 will be given in §7.

Since for an oriented 4-manifold, the signature and the Euler characteristic have the same parity, we immediately obtain the following, which was obtained in [**43**].

COROLLARY 5.6. *Let M be a closed orientable 4-manifold and N a 3-manifold. Then, for any C^∞ stable map $f : M \to N$, the number of fibers of f equivalent to a III^8 type fiber modulo regular fibers has the same parity as the Euler characteristic of M.*

Note that in the proof of our main theorem, we do not use the above corollary. In other words, our proof gives a new proof for the above modulo two Euler characteristic formula.

6. Cobordism invariance of the algebraic number of III^8 type fibers

In order to prove Theorem 5.5, let us first show that the algebraic number of III^8 type fibers is an oriented cobordism invariant of the source 4-manifold.

Let us begin by a list of chiral singular fibers of proper C^∞ stable maps of 5-manifolds into 4-manifolds. We can prove the following proposition by an argument similar to that in the previous section.

PROPOSITION 6.1. *A singular fiber of a proper C^∞ stable map of an oriented 5-manifold into a 4-manifold is chiral if and only if it is C^0 equivalent modulo regular fibers to a fiber of type* III^5, III^7, III^8, $\mathrm{IV}^{0,5}$, $\mathrm{IV}^{0,7}$, $\mathrm{IV}^{0,8}$, $\mathrm{IV}^{1,5}$, $\mathrm{IV}^{1,7}$, $\mathrm{IV}^{1,8}$, IV^{10}, IV^{11}, IV^{12}, IV^{13}, IV^{18}, IV^{g}, IV^{h}, *or* IV^{k}.

For example, in order to show that the fibers of types IV^o, IV^p and IV^q are achiral, we can use the symmetry of order 6 of an indefinite D_4 point as in Remark 4.2 in Chapter 2. The proof of Proposition 6.1 is left to the reader.

Note that for each chiral singular fiber of codimension 4, we can define its sign $(= \pm 1)$, as long as the source 5-manifold is oriented. In what follows, we fix such a definition of a sign for each chiral singular fiber of codimension 4 once and for all, although we do not mention it explicitly.

Let $\mathfrak{F}$ be a C^0 equivalence class modulo regular fibers. For a proper C^∞ stable map $f : M \to N$ of an oriented 5-manifold M into a 4-manifold N, we denote by $\mathfrak{F}(f)$ the set of all $y \in N$ over which lies a fiber of type $\mathfrak{F}$. Note that $\mathfrak{F}(f)$ is a regular C^∞ submanifold of N of codimension $\kappa(\mathfrak{F})$, where $\kappa(\mathfrak{F})$ denotes the codimension of the C^0 equivalence class modulo regular fibers $\mathfrak{F}$.

In general, if $\mathfrak{F}$ is chiral, then $\mathfrak{F}(f)$ is orientable. For $\mathfrak{F} = \mathrm{III}^8$, we introduce the orientation on $\mathrm{III}^8(f)$ as follows.

Take a point $y \in \mathrm{III}^8(f)$. Note that the singular value set $f(S(f))$ near y consists of three codimension 1 "sheets" meeting along $\mathrm{III}^8(f)$ in general position. Let D_y be a small open 3-disk centered at y in N which intersects $\mathrm{III}^8(f)$ transversely exactly at y and is transverse to the three sheets of $f(S(f))$. Put $M' = f^{-1}(D_y)$, which is a smooth 4-dimensional submanifold of M with trivial normal bundle and hence is orientable. Let us consider the proper smooth map

$$h = f|_{f^{-1}(D_y)} : M' \to D_y,$$

which is a C^∞ stable map by virtue of Proposition 2.2 in Chapter 1. Note that the fiber of h over y is of type III^8. Let M'' be the component of M' containing the III^8 type fiber.

Let $\mathcal{O}_{M''}$ be the orientation of M'' with respect to which the III^8 type fiber is positive. Then let $\mathcal{O}_\nu$ be the orientation of the normal bundle ν to M'' in M such that $\mathcal{O}_\nu \oplus \mathcal{O}_{M''}$ is consistent with the orientation of the 5-manifold M. By the differential $df : TM \to TN$ at a point in M'', $\mathcal{O}_\nu$ corresponds to a normal direction to D_y in N at y. Now we orient $\mathrm{III}^8(f)$ at y so that this direction is consistent with the orientation of $\mathrm{III}^8(f)$.

It is easy to see that this orientation varies continuously with respect to $y \in \mathrm{III}^8(f)$ and hence defines an orientation on $\mathrm{III}^8(f)$.

Now let $\mathfrak{F}$ be the C^0 equivalence class modulo regular fibers of one of the codimension 4 fibers appearing in Proposition 6.1; i.e., $\mathfrak{F}$ is a chiral singular fiber of codimension 4. Note that $\mathfrak{F}(f)$ is a discrete set in N. Take a point $y \in \mathfrak{F}(f)$ and a sufficiently small open disk neighborhood Δ_y of y in N. We orient the source 5-manifold so that the fiber over y gets the sign $+1$. Then $\Delta_y \cap \mathrm{III}^8(f)$ consists of several oriented arcs which have a common end point at y. Let us define the *incidence coefficient* $[\mathrm{III}^8 : \mathfrak{F}] \in \mathbb{Z}$ to be the number of arcs coming into y minus the number of arcs going out of y. Note that this does not depend on the point y nor on the map f.

Remark 6.2. Let $\mathfrak{F}$ be the C^0 equivalence class modulo regular fibers of a codimension 4 *achiral* singular fiber. Then we can define the incidence coefficient $[\mathrm{III}^8 : \mathfrak{F}] \in \mathbb{Z}$ in exactly the same manner as above. However, this should always vanish, since the homeomorphism φ as in (2.1) reverses the orientation of $\Delta_y \cap \mathrm{III}^8(f)$.

LEMMA 6.3. *The incidence coefficient* $[\mathrm{III}^8 : \mathfrak{F}]$ *vanishes for every* C^0 *equivalence class modulo regular fibers* $\mathfrak{F}$ *of codimension* 4 *that is chiral.*

PROOF. It is not difficult to see that for $y \in \mathfrak{F}(f)$, $\Delta_y \cap \mathrm{III}^8(f) \neq \emptyset$ if and only if $\mathfrak{F} = \mathrm{IV}^{0,8}$, $\mathrm{IV}^{1,8}$ or IV^{18}. Furthermore, for each of these three cases, the number of arcs of $\Delta_y \cap \mathrm{III}^8(f)$ is equal to 2 and exactly one of them is coming into y. Thus the result follows. □

Remark 6.4. The above lemma shows that the closure of $\mathrm{III}^8(f)$ is a regular oriented 1-dimensional submanifold of N near the points over which lies a chiral singular fiber of codimension 4. However, the closure of $\mathrm{III}^8(f)$, as a whole, is not even a topological manifold in general. For example, suppose that f admits a IV^{22} type fiber. Then the closure of $\mathrm{III}^8(f)$ forms a graph (i.e. a 1-dimensional complex) and each point of $\mathrm{IV}^{22}(f)$ is a vertex of degree 8, i.e. it has 8 incident edges. Furthermore, four of them are incoming edges and the other four are outgoing edges.

Let us recall the following definition (for details, refer to [**7**].)

DEFINITION 6.5. Let N be a manifold and $f_i : M_i \to N$ a continuous map of a closed oriented n-dimensional manifold M_i into N, $i = 0, 1$. We say that f_0 and f_1 are *oriented bordant* if there exist a compact oriented $(n+1)$-dimensional manifold W and a continuous map $F : W \to N \times [0,1]$ with the following properties:

(1) ∂W is identified with the disjoint union of $-M_0$ and M_1, where $-M_0$ denotes the manifold M_0 with the reversed orientation, and
(2) $F|_{M_i} : M_i \to N \times \{i\}$ is identified with f_i, $i = 0, 1$.

We call the map $F : W \to N \times [0,1]$ an *oriented bordism* between f_0 and f_1.

Note that if $M_0 = M_1$, and f_0 and f_1 are homotopic, then they are oriented bordant. Furthermore, if the target manifold N is contractible, then f_0 and f_1 are oriented bordant if and only if their source manifolds M_0 and M_1 are oriented cobordant as oriented manifolds.

For a given manifold N and a nonnegative integer n, the set of all oriented bordism classes of maps of closed oriented n-dimensional manifolds into N forms an additive group under the disjoint union. We call it the *n-dimensional oriented bordism group of N*.

Note that in the usual definition, an oriented bordism is a map into N and not into $N \times [0,1]$. However, it is easy to see that the above definition is equivalent to the usual one.

As a consequence of Lemma 6.3, we get the following.

LEMMA 6.6. *Let N be a 3-manifold and $f_i : M_i \to N$ a C^∞ stable map of a closed oriented 4-manifold M_i into N, $i = 0, 1$. If f_0 and f_1 are oriented bordant, then we have*

$$||\mathrm{III}^8(f_0)|| = ||\mathrm{III}^8(f_1)||.$$

PROOF. Let $F : W \to N \times [0,1]$ be an oriented bordism between f_0 and f_1. Take sufficiently small collar neighborhoods $C_0 = M_0 \times [0,1)$ and $C_1 = M_1 \times (0,1]$ of M_0 and M_1 in W respectively. We may assume that

$$\begin{aligned} F|_{M_0 \times [0,\varepsilon)} &= f_0 \times \mathrm{id}_{[0,\varepsilon)}, \quad \text{and} \\ F|_{M_1 \times (1-\varepsilon,1]} &= f_1 \times \mathrm{id}_{(1-\varepsilon,1]} \end{aligned}$$

for a sufficiently small $\varepsilon > 0$. Furthermore, we may assume that F is a smooth map with $F^{-1}(N \times (0,1)) = \mathrm{Int}W$. Then by a standard argument, we can approximate F by a generic map F' such that $F'|_{C_0 \cup C_1} = F|_{C_0 \cup C_1}$ and that $F'|_{\mathrm{Int}W} : \mathrm{Int}W \to N \times (0,1)$ is a proper C^∞ stable map. In the following, let us denote F' again by F.

By Lemma 6.3, we see that the closure of $\mathrm{III}^8(F)$ is a finite graph each of whose edge is oriented. Furthermore, for each vertex lying in $N \times (0,1)$, the number of incoming edges is equal to that of outgoing edges. Furthermore, its vertices lying in $N \times \{0,1\}$ have degree one and they coincide exactly with the union of

$$\begin{aligned} \mathrm{III}^8(F) \cap (N \times \{0\}) &= \mathrm{III}^8(f_0) \quad \text{and} \\ \mathrm{III}^8(F) \cap (N \times \{1\}) &= \mathrm{III}^8(f_1). \end{aligned}$$

Therefore, by virtue of Remark 5.2 we have

$$-||\mathrm{III}^8(f_0)|| + ||\mathrm{III}^8(f_1)|| = 0,$$

since $\partial W = (-M_0) \cup M_1$. Hence the result follows. □

By combining Lemma 6.6 with a work of Conner–Floyd [**7**], we get the following.

PROPOSITION 6.7. *Let N be a 3-manifold and $f_i : M_i \to N$ a C^∞ stable map of a closed oriented 4-manifold M_i into N, $i = 0, 1$. If M_0 and M_1 are oriented cobordant as oriented 4-manifolds, then we have*

$$||\mathrm{III}^8(f_0)|| = ||\mathrm{III}^8(f_1)||.$$

PROOF. Recall that the oriented cobordism groups Ω_n of n-dimensional manifolds for $0 \le n \le 4$ satisfy the following:

$$\Omega_n \cong \begin{cases} 0, & n = 1, 2, 3, \\ \mathbb{Z}, & n = 0, 4. \end{cases}$$

Furthermore, the 4-dimensional oriented bordism group of N is isomorphic to

$$\sum_{p+q=4} H_p(N; \Omega_q)$$

modulo (odd) torsion [**7**, §15]. Therefore, if the 4-dimensional manifolds M_0 and M_1 are oriented cobordant, then mf_0 and mf_1 are oriented bordant for some odd integer m, where mf_i denotes the map of the disjoint union of m copies of M_i into N such that on each copy it is given by f_i, $i = 0, 1$.

Thus by Lemma 6.6, we have

$$m||\mathrm{III}^8(f_0)|| = m||\mathrm{III}^8(f_1)||,$$

which implies the desired equality. This completes the proof. □

7. An explicit example

In this section, we study an explicit example of a C^∞ stable map of a closed oriented 4-manifold with nonzero signature into $\mathbb{R}^3$. Combining this with Proposition 6.7, we will prove our main theorem of this paper.

In [**43**, Chap. 6], Saeki constructed an explicit example of a C^∞ stable map $f : M \to \mathbb{R}^3$ of a closed 4-manifold M with exactly one III^8 type fiber such that f has only fold points as its singularities. Recall that M is diffeomorphic to $\mathbb{C}P^2 \sharp 2\overline{\mathbb{C}P^2}$ if we ignore the orientation. We would like to orient M and determine the sign of this III^8 type fiber.

In fact, by Proposition 6.7, we already know that there exists a constant c such that the algebraic number of III^8 type fibers is c times the signature of the source oriented 4-manifold (for details, see the argument in the proof of Theorem 5.5 below). In the above-mentioned example, the algebraic number of III^8 type fibers is equal to ± 1, and the signature of the source 4-manifold is equal to ± 1. Therefore, this constant c should be equal to ± 1. Thus, for the proof of our main theorem, it suffices to determine the sign of the constant c.

This procedure might seem easy, but in fact it is not. As a matter of fact, the construction of the above example in [**43**] was already very complicated, although the example itself seems to be a natural one. Therefore, in this section, we carefully study the example and determine the sign of the constant c. We will describe the argument in details, since the technique in this section can be very useful in determining the self-intersection number of the surface of singular point set in general situations. At the end of this section, we give a new proof of the self-intersection number formula based on our study.

In the construction given in [**43**], the orientation of the source 4-manifold M was not given explicitly. Here, we first orient the source 4-manifold so that the III^8 type fiber gets the positive sign, and then determine the signature of the source 4-manifold with respect to the chosen orientation.

LEMMA 7.1. *If we orient the source* 4*-manifold* M *of the above example so that the* III^8 *type fiber is positive, then the signature of* M *is equal to* $+1$.

PROOF. In general, let $f : M \to \mathbb{R}^3$ be a C^∞ stable map of a closed oriented 4-manifold into the 3-dimensional Euclidean space which has only fold points as its singularities. In view of a result in [**42**] (see also [**35**] or [**37**]), we have

$$3\sigma(M) = S_0(f) \cdot S_0(f), \tag{7.1}$$

where $S_0(f)$ denotes the surface of definite fold singularities of f, and $S_0(f) \cdot S_0(f)$ denotes its self-intersection number (or equivalently, its normal Euler number) in M. Therefore, in order to determine the signature of the source 4-manifold of the explicit example mentioned above, we have only to compute the self-intersection number $S_0(f) \cdot S_0(f)$.

Let us first consider the III^8 type fiber and denote the three singular points in it by q_1, q_2 and q_3 as in Figure 3.4. Furthermore, we orient the regular part of the III^8 type fiber so that it corresponds to the cyclic order $\langle q_1, q_2, q_3 \rangle$ of the three singular points.

The image $f(S(f))$ of the singular point set of f around the point y corresponding to the III^8 type fiber consists of three sheets $f(D_1)$, $f(D_2)$ and $f(D_3)$, where D_i is a small disk neighborhood of q_i in $S(f)$, $i = 1, 2, 3$. We may assume that the three sheets $f(D_1)$, $f(D_2)$ and $f(D_3)$ are situated as in Figure 3.5 and that the "front octant" ω in the figure is a 1-octant; i.e., the fiber over a point in the octant is connected.

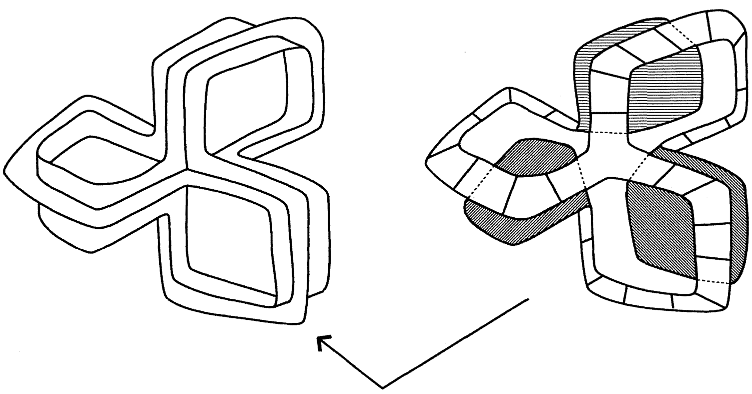

FIGURE 3.7. Boy surface P

Let w_i be a normal vector to the i-th sheet of $f(S(f))$ in N pointing toward ω at a point incident to that octant, $i = 1, 2, 3$. We orient $\mathbb{R}^3$ so that the ordered 3-tuple of vectors $\langle w_1, w_2, w_3 \rangle$ is consistent with the orientation, i.e., $\mathbb{R}^3$ is endowed with the "right-handed orientation" in the usual sense.

Now we orient the source 4-manifold of f so that the regular part of the III^8 type fiber gets the orientation as indicated in Figure 3.4 in the sense of §5 by the "fiber first" convention. Then the sign of the III^8 type fiber is equal to $+1$.

Recall that $S_0(f)$ consists of three 2-spheres $\tilde{S}_0$, $\tilde{S}_1$ and $\tilde{S}_2$, and that the surface $S_1(f)$ of indefinite fold points is a real projective plane whose image $P = f(S_1(f))$ is the Boy surface in $\mathbb{R}^3$ (see Figure 3.7). Furthermore, the embedded 2-sphere $S_0 = f(\tilde{S}_0)$ surrounds the Boy surface, and the disjoint 2-spheres $S_1 = f(\tilde{S}_1)$ and $S_2 = f(\tilde{S}_2)$ are contained in the bounded region of $\mathbb{R}^3 \setminus P$ so that S_1 surrounds S_2 (for details, see [**43**, Chap. 6]).

We have obviously a continuous map $h_0 : S_0 \times [0,1] \to \mathbb{R}^3$ with the following properties:

(1) $h_0|_{S_0 \times \{0\}} = \mathrm{id}_{S_0} : S_0 \times \{0\} \to S_0$.
(2) $h_0|_{S_0 \times (0,1)}$ is a diffeomorphism onto the region bounded by S_0 and P.
(3) $h_0(S_0 \times \{1\}) = P$.
(4) $h_0|_{S_0 \times \{1\}}$ is a homeomorphism outside of a 1-dimensional subcomplex C_0 of S_0 as depicted in Figure 3.8. The image $h_0((S_0 \setminus C_0) \times \{1\})$ coincides with the complement of the multiple point set of the Boy surface P in P.

Set $B = h_0(S_0 \times [0, 1/2])$. Then $N(\tilde{S}_0) = f^{-1}(B)$ can be identified with a normal disk bundle ν_0 to $\tilde{S}_0$ in M. In order to calculate the self-intersection number $\tilde{S}_0 \cdot \tilde{S}_0$ in M, let us consider the disk bundle $\tilde{\nu}_0$ over $\tilde{S}_0$ which is obtained from ν_0 by identifying the antipodal points on each disk fiber. In other words, the S^1-bundle $\partial\tilde{\nu}_0$ associated with the disk bundle $\tilde{\nu}_0$ corresponds to the $\mathbb{R}P^1$-bundle associated with the real 2-plane bundle afforded by ν_0. Note that the self-intersection number of the zero section of ν_0 is equal to one half of the self-intersection number of that of $\tilde{\nu}_0$.

Let us construct a section of the S^1-bundle $\partial\tilde{\nu}_0$ associated with $\tilde{\nu}_0$ over a certain subset $\tilde{X}_0$ of $\tilde{S}_0$. More precisely, for each point $\tilde{x}$ of $\tilde{X}_0$, we will choose a pair of antipodal points on the circle fiber of $\partial\nu_0$ over $\tilde{x}$ continuously with respect to $\tilde{x}$ so that the projection restricted to the set of all these points is a double covering map onto $\tilde{X}_0$, where $\partial\nu_0$ is the S^1-bundle associated with the disk bundle ν_0.

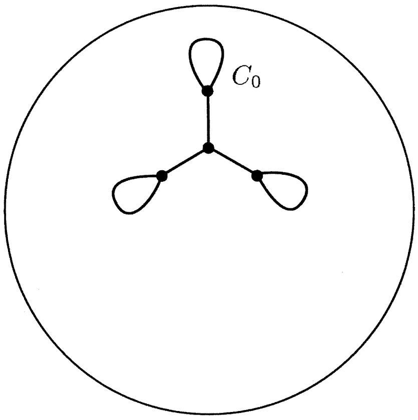

FIGURE 3.8. 1-Dimensional subcomplex C_0 on the 2-sphere S_0

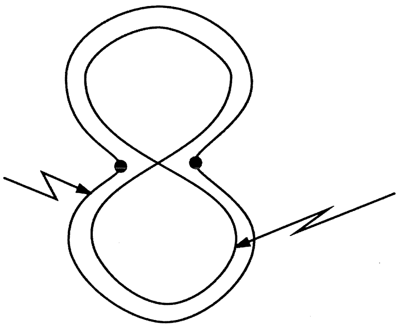

FIGURE 3.9. Two points on a regular fiber near a singular fiber of type I^1

Let $N(C_0)$ be a regular neighborhood of C_0 in S_0. Set $\tilde{C}_0 = f^{-1}(C_0)$ and $N(\tilde{C}_0) = f^{-1}(N(C_0))$. Note that $\tilde{C}_0$ is a 1-dimensional subcomplex of $\tilde{S}_0$ and that $N(\tilde{C}_0)$ is a regular neighborhood of $\tilde{C}_0$ in $\tilde{S}_0$. We will first construct a section of the S^1-bundle $\partial\tilde{\nu}_0$ over $\tilde{S}_0 \setminus \mathrm{Int}N(\tilde{C}_0)(\subset \tilde{X}_0)$ as follows.

Take a point $\tilde{x} \in \tilde{S}_0 \setminus \mathrm{Int}N(\tilde{C}_0)$ and set $x = f(\tilde{x}) \in S_0$. Then $h_0(x, 1) \in P$ is the image of a unique indefinite fold point. Therefore, $f^{-1}(h_0(x, 1/2))$ can be considered as a nearby fiber of the fiber over $h_0(x, 1)$, which is of type I^1, and hence we can take a pair of two antipodal points on $f^{-1}(h_0(x, 1/2))$ canonically as in Figure 3.9. We can thus construct a continuous section of $\partial\tilde{\nu}_0$ over each component of $\tilde{S}_0 \setminus \mathrm{Int}N(\tilde{C}_0)$.

Let us consider the twelve bands embedded in $N(\tilde{C}_0)$ as in Figure 3.10, where each band is homeomorphic to $[-1, 1] \times [-1, 1]$. Each band is also considered to be a 1-handle attached to $\tilde{S}_0 \setminus \mathrm{Int}N(\tilde{C}_0)$ along $\{-1, 1\} \times [-1, 1]$. We orient the core of each band as in the figure, where a core is an arc properly embedded in a band corresponding to $[-1, 1] \times \{0\}$. The subset $\tilde{X}_0$ is the union of $\tilde{S}_0 \setminus \mathrm{Int}N(\tilde{C})$ and the twelve bands. Let us extend the section of $\partial\tilde{\nu}_0$ over $\tilde{S}_0 \setminus \mathrm{Int}N(\tilde{C}_0)$ constructed above through the twelve bands as follows.

Take a band and let $\tilde{\alpha}$ be its core. Set $\alpha = f(\tilde{\alpha})$. Since $\alpha' = h_0(\alpha \times \{1/2\})$ is close to the transverse double point of $f(S(f))$ as in Figure 1.2 (2), the regular fibers over the two points $\partial\alpha'$ are close to a II^2-fiber. Furthermore, the pairs of antipodal points on the circle fibers over the two points $\partial\tilde{\alpha}$ associated with the

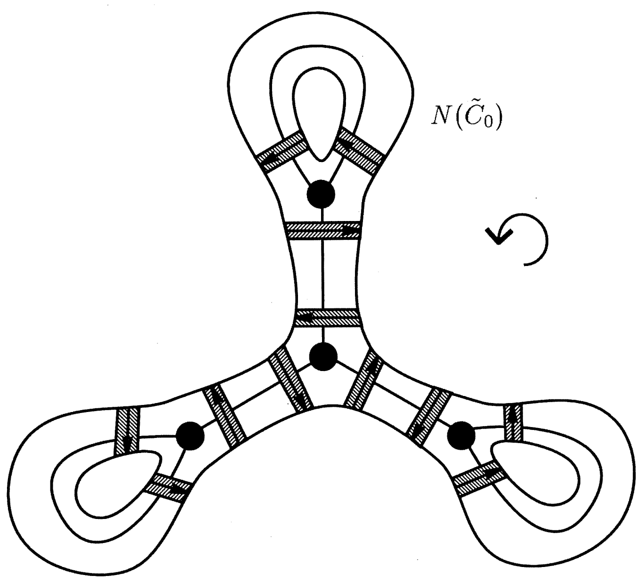

FIGURE 3.10. Twelve bands in $N(\tilde{C}_0)$

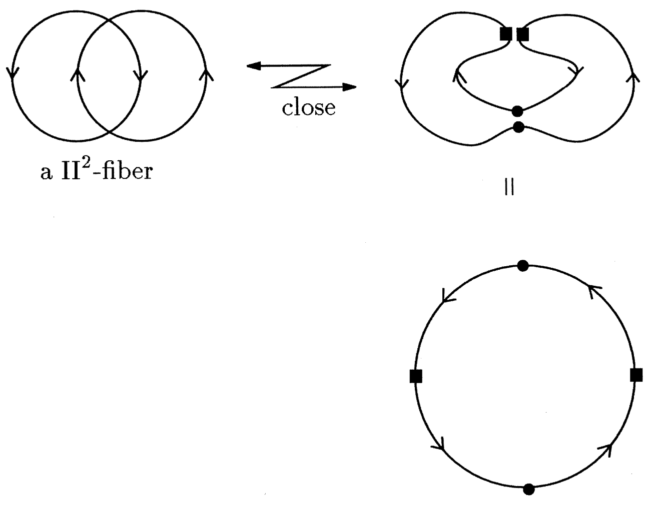

FIGURE 3.11. Two pairs of antipodal points

above-constructed section of $\partial\tilde{\nu}_0$ over $\tilde{S}_0 \setminus \mathrm{Int}N(\tilde{C}_0)$ are situated as in Figure 3.11. Let us extend the section of $\partial\tilde{\nu}_0$ over $\tilde{S}_0 \setminus \mathrm{Int}N(\tilde{C}_0)$ through $\tilde{\alpha}$ so that when one goes along $\tilde{\alpha}$ in the direction indicated as in Figure 3.10, the pair of antipodal points on the circle fibers of $\partial\nu_0$ gets the rotation through the angle $\pi/2$ in the positive direction of regular fibers. Then we can naturally extend the section to the band. We apply this construction to all the twelve bands to get a section of $\partial\tilde{\nu}_0$ over $\tilde{X}_0$.

The complement of $\mathrm{Int}\tilde{X}_0$ in $\tilde{S}_0$ consists of ten 2-disks: six rectangular disks corresponding to the edges of $\tilde{C}_0$ and four hexagonal disks corresponding to the vertices of $\tilde{C}_0$. Note that the bundle $\tilde{\nu}_0$ over each 2-disk Δ is trivial, and using a trivialization, we can define the degree of the above-constructed section over $\partial\Delta$. The self-intersection number of the zero section of $\tilde{\nu}_0$ is then equal to the sum of

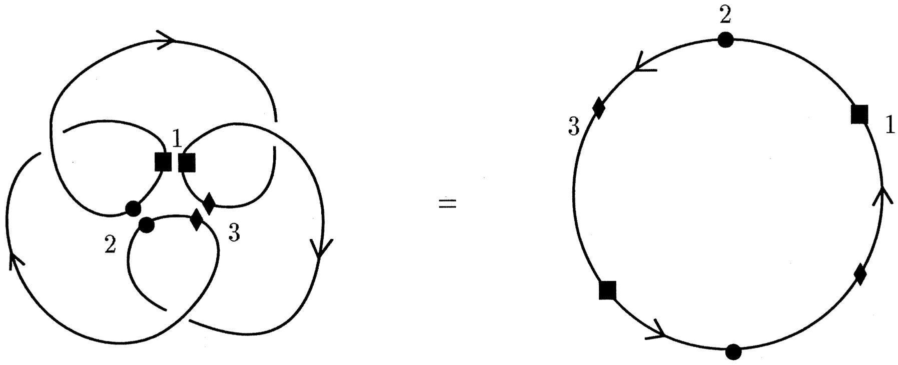

FIGURE 3.12. Three pairs of antipodal points

these "degrees" over the boundaries of the ten 2-disks. Note that here $\tilde{S}_0$ should be oriented so that its orientation together with the orientation of the 2-disk fiber gives the orientation of the total space. Since we have oriented the source 4-manifold by the "fiber first" convention, we see that the induced orientation $\mathcal{O}_{S_0}$ on S_0 should satisfy that the "outward normal" plus $\mathcal{O}_{S_0}$ is consistent with the right-handed orientation of $\mathbb{R}^3$. Hence $\mathcal{O}_{S_0}$ is the "left-handed" orientation when viewed from outside (see Figure 3.10). Here we adopt the convention that the figure of $N(\tilde{C}_0)$ (Figure 3.10) is consistent with that of $N(C_0)$ viewed from outside.

It is easy to see that for each of the six rectangular 2-disks of $\tilde{S}_0 \setminus \mathrm{Int}\tilde{X}_0$, the degree of the section of $\partial\tilde{\nu}_0$ over its boundary is equal to -1, since when we go around its boundary in its positive direction, the pair of antipodal points on the circle fibers of $\partial\nu_0$ rotates through the angle $-\pi$.

Let us now consider the contribution of each of the other four regions of $\tilde{S}_0 \setminus \mathrm{Int}\tilde{X}_0$ that are hexagonal. The image H by $h_0(*, 1/2)$ of its f-image is close to the triple point y of $f(S(f))$ and it lies in a 1-octant (see Figure 3.5). When we go along the boundary of the hexagonal disk H from a point near the sheet $f(D_1)$ to another point near the sheet $f(D_2)$, the pair of antipodal points on a circle fiber of $\partial\nu$ corresponding to the sheet $f(D_1)$ makes a rotation through the angle $\pi/3$ as in Figure 3.12, since the sign of the III^8 type fiber is positive. Therefore, the degree of the section of $\partial\tilde{\nu}_0$ over the boundary of the hexagonal disk is equal to $+1$, since when we go around its boundary in its positive direction, the pair of antipodal points on the circles fibers of $\partial\nu_0$ rotates through the angle π (see Figure 3.12).

Thus the sum of the degrees is equal to

$$(-1) \cdot 6 + (+1) \cdot 4 = -2.$$

Therefore, the self-intersection number of the zero section of $\tilde{\nu}_0$ is equal to -2 and that of ν_0 is equal to $-2/2 = -1$.

The self-intersection numbers $\tilde{S}_1 \cdot \tilde{S}_1$ and $\tilde{S}_2 \cdot \tilde{S}_2$ can be computed by a similar method as follows. Let ν_i be the normal disk bundle to $\tilde{S}_i$ in M, $i = 1, 2$. We can construct the continuous map h_i, $i = 1, 2$, for $f(\tilde{S}_i) = S_i$ satisfying the properties similar to those for h_0. Then we take the 1-dimensional complexes C_i and $\tilde{C}_i$, their regular neighborhoods $N(C_i)$ and $N(\tilde{C}_i)$ respectively, and the twelve bands in $\tilde{S}_i$, $i = 1, 2$, as above, and define $\tilde{X}_i$ to be the union of $\tilde{S}_i \setminus \mathrm{Int}N(\tilde{C}_i)$ and the twelve bands. Then we can construct a section of $\partial\nu_i$, $i = 1, 2$, over $\tilde{X}_i$ by using an argument similar to that for $\partial\tilde{\nu}_0$. (Note that in the present case, we use ν_i itself, rather than its associated $\mathbb{R}P^1$-bundle.) More precisely, let us consider the regular fiber over the point $h_i(x, 1/2)$ for a point $\tilde{x} \in \tilde{S}_i \setminus \mathrm{Int}N(\tilde{C}_i)$ with $x = f(\tilde{x})$.

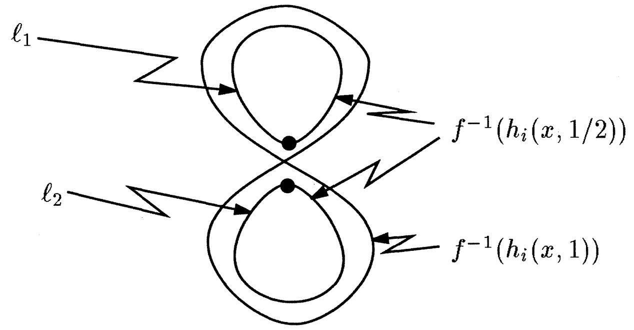

FIGURE 3.13. Two points on a regular fiber which are on distinct connected components

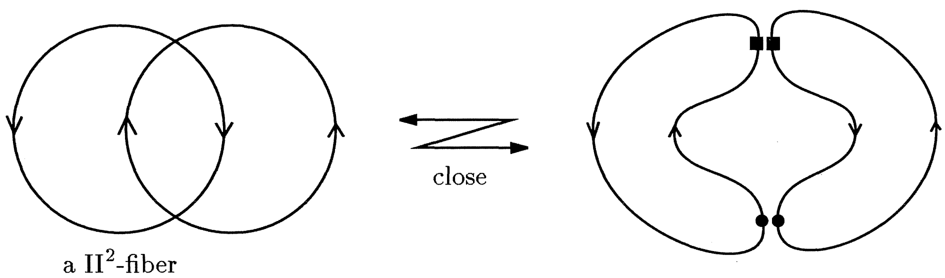

FIGURE 3.14. Two pairs of points

It consists of two circles ℓ_1 and ℓ_2 and we can take a pair of points which lie on distinct components as in Figure 3.13.

We can extend this section through the twelve bands as in the case of $\tilde{S}_0$. Let us compute the sum of the degrees of the section over the components of $\partial\tilde{X}_i$. First note that the orientation on S_i is the "left-handed" orientation when viewed *from inside.* Then the contribution of each rectangular region is equal to $+1$, since when we go along the core of a band, the chosen point on a connected component of a regular fiber gets the rotation through the angle π (see Figure 3.14).

As to the other four hexagonal regions, they correspond to the triple point of $f(S(f))$. When we go around the boundary, the chosen point gets the rotation through the angle -2π (see Figure 3.15). Hence its contribution to the self-intersection number is equal to -1.

Therefore, the self-intersection number $\tilde{S}_i \cdot \tilde{S}_i$ is equal to

$$(+1) \cdot 6 + (-1) \cdot 4 = 2$$

for $i = 1, 2$.

Thus the self-intersection number of $S_0(f)$ in M is equal to

$$S_0(f) \cdot S_0(f) = \tilde{S}_0 \cdot \tilde{S}_0 + \tilde{S}_1 \cdot \tilde{S}_1 + \tilde{S}_2 \cdot \tilde{S}_2 = -1 + 2 + 2 = 3.$$

Therefore, the signature of the source 4-manifold is equal to $+1$ according to the formula (7.1). In other words, the source 4-manifold M is oriented diffeomorphic to $2\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$. This completes the proof of Lemma 7.1. $\square$

Let us now proceed to the proof of our main theorem of this paper.

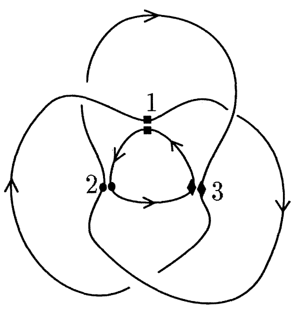

FIGURE 3.15. Three pairs of points

PROOF OF THEOREM 5.5. Let us fix a 3-manifold N. For a closed oriented 4-manifold M and a C^∞ stable map $f : M \to N$ into N, let us consider the integer

$$\psi(M, f) = ||\mathrm{III}^8(f)||.$$

By virtue of Proposition 6.7, ψ depends only on the oriented cobordism class of M. Therefore, ψ induces a map

$$\overline{\psi} : \Omega_4 \to \mathbb{Z}$$

of the 4-dimensional oriented cobordism group to the additive group of integers. This is clearly a homomorphism.

Recall that Ω_4 is an infinite cyclic group generated by the class of oriented 4-manifolds with signature $+1$. In other words, the signature induces an isomorphism

$$\sigma : \Omega_4 \to \mathbb{Z}.$$

Let us consider the explicit example $f : 2\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2} \to \mathbb{R}^3$ given in Lemma 7.1. By composing it with an embedding $\mathbb{R}^3 \hookrightarrow N$, we get a C^∞ stable map of an oriented 4-manifold of signature $+1$ into N. This stable map has exactly one III^8 type fiber, whose sign is equal to $+1$. Therefore, the homomorphism

$$\overline{\psi} \circ \sigma^{-1} : \mathbb{Z} \to \mathbb{Z}$$

sends $+1$ to $+1$ and hence is the identity. Thus we have $\sigma = \overline{\psi}$. This completes the proof. □

We have a direct consequence of our main theorem as follows.

COROLLARY 7.2. *Let M be a closed oriented 4-manifold and N a 3-manifold. Then every C^∞ stable map of M into N has at least $|\sigma(M)|$ singular fibers of type III^8.*

For example, if M is the underlying real 4-dimensional manifold of the complex K3 surface, then every C^∞ stable map of M into a 3-manifold has at least 16 fibers of type III^8, although no explicit example of such a map is known. Construction of an explicit example can be an interesting problem.

Our study of the explicit example gives a new proof to the following formula, which has been proved in [**35, 37**].

COROLLARY 7.3. *Let $f : M \to N$ be a C^∞ stable map of a closed oriented 4-manifold M into an orientable 3-manifold N. Then we have*

$$S(f) \cdot S(f) = 3\sigma(M),$$

where $S(f) \cdot S(f)$ is the self-intersection number of the surface of singular point set of f in M.

Note that the formula (7.1) follows from Corollary 7.3. We give a proof to the above corollary without using the formula (7.1).

PROOF OF COROLLARY 7.3. It is not difficult to show that the self-intersection number $S(f) \cdot S(f)$ is an oriented cobordism invariant of the source 4-manifold M (for this, use the argument as in [**37**, Proof of Lemma 3.2] together with a result of [**7**] as in the proof of Proposition 6.7). Therefore, there exists a constant c such that

$$S(f) \cdot S(f) = 3c\sigma(M)$$

holds for any C^∞ stable map f of a closed oriented 4-manifold M into an orientable 3-manifold N.

For the explicit example f studied above, we have[1] $S(f) \cdot S(f) = \pm 3$ and $3\sigma(M) = \pm 3$. Therefore, the constant c must be equal to ± 1.

On the other hand, by a result of Sakuma [**47**] we have

$$S(f) \cdot S(f) \equiv 3\sigma(M) \pmod 4.$$

Therefore, the constant c must be equal to $+1$. This completes the proof. □

Recall that Sadykov [**37**] proved Corollary 7.3 by a characteristic class argument together with Sakuma's result [**47**].

Now some remarks concerning our result are in order.

Remark 7.4. For a closed oriented 4-manifold M, let n_+ and n_- be arbitrary integers such that $\sigma(M) = n_+ - n_-$. Then does there exist a C^∞ stable map $f : M \to N$ that has exactly n_+ positive singular fibers of III^8 type and n_- negative ones? The authors do not know the answer to the question.

Remark 7.5. In [**39**] Saeki proved that if $f : M \to N$ is a C^∞ stable map of a closed oriented 4-manifold into a 3-manifold with only definite fold singularities, then $\sigma(M) = 0$. This follows from our main theorem as well.

Remark 7.6. The technique used in this section to determine the self-intersection number of the surface of definite fold points may be generalized in a more general setting. This might give a new direct proof of our main theorem.

Remark 7.7. It is known that there exist oriented surface bundles over oriented surfaces with non-zero signatures (for example, see [**29**]). This means that we cannot expect a similar signature formula for C^∞ stable maps of closed oriented 4-manifolds into surfaces.

Note that if the fiber genus $g \leq 2$, then there is a signature formula for Lefschetz fibrations in terms of singular fibers [**27, 28**]. We also have a similar formula for hyperelliptic Lefschetz fibrations of any genus [**10**]. It may be possible to prove these formulas by using our main theorem as follows. Let M be a Lefschetz fibration over a surface B, and consider a line bundle N over B. Note that N is a 3-manifold. Then we can consider a generic map $f : M \to N$ which makes the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ B & \xrightarrow{=} & B \end{array}$$

commutative, where the vertical arrows are relevant fibration maps. In other words, we consider a generic family of fiberwise functions. By applying our Theorem 5.5 to f, we might be able to get a signature formula for certain Lefschetz fibrations.

[1]We know that $S(f) \cdot S(f) = 3\sigma(M) = 3$: however, in order to show this, we used the formula (7.1). Here we are proving the corollary without using (7.1).

8. Universal complex of chiral singular fibers

In [**43**], the universal complex of singular fibers was introduced as a refinement of Vassiliev's universal complex of multi-singularities (see [**50**] or [**15, 34**]), and it was shown that its cohomology classes give invariants of cobordisms of singular maps in the sense of Rimányi and Szűcs [**36**]. In this section, we study the universal complex (with integer coefficients) of singular fibers corresponding to chiral singular fibers and give an interpretation of our main theorem in terms of the theory of universal complex of singular fibers.

We can define the universal complex of chiral singular fibers for proper C^∞ stable maps of oriented 5-manifolds into 4-manifolds by exactly the same procedure as in [**43**] as follows.

For κ with $3 \leq \kappa \leq 4$, let C^κ be the free $\mathbb{Z}$-module generated by the C^0 equivalence classes modulo regular fibers of chiral singular fibers of codimension κ. Note that $\mathrm{rank} C^3 = 3$ and $\mathrm{rank} C^4 = 14$ according to Proposition 6.1. Since there exist no chiral singular fibers of codimension $\kappa \neq 3, 4$, we put $C^\kappa = 0$ for $\kappa \neq 3, 4$. Note that for $\kappa = 4$, we take the C^0 equivalence classes modulo regular fibers of chiral singular fibers with positive signs as generators, and we consider those with negative signs to be -1 times the corresponding class with positive sign.

The coboundary homomorphism $\delta_3 : C^3 \to C^4$ is defined by

$$\delta_3(\mathfrak{G}) = \sum_{\kappa(\mathfrak{F})=4} [\mathfrak{G} : \mathfrak{F}]\mathfrak{F}$$

for every generator $\mathfrak{G}$ of C^3, where $[\mathfrak{G} : \mathfrak{F}] \in \mathbb{Z}$ is the incidence coefficient which can be defined by exactly the same method as for $[\mathrm{III}^8 : \mathfrak{F}]$ (see §6). Note that all the other coboundary homomorphisms δ_κ, $\kappa \neq 3$, are necessarily trivial.

We call the resulting cochain complex $(C^\kappa, \delta_\kappa)_\kappa$ the *the universal complex of chiral singular fibers* for proper C^∞ stable maps of oriented 5-manifolds into 4-manifolds. Note that its unique cohomology group that makes sense is its third cohomology group, and is nothing but the kernel of the coboundary homomorphism δ_3.

Then we get the following.

PROPOSITION 8.1. *The* 3*-dimensional cohomology group of the universal complex of chiral singular fibers for proper C^∞ stable maps of oriented* 5*-manifolds into* 4*-manifolds is an infinite cyclic group generated by the C^0 equivalence class modulo regular fibers of* III^8 *type fibers.*

PROOF. Recall that we have exactly three C^0 equivalence classes modulo regular fibers of chiral singular fibers of codimension 3, namely, III^5, III^7 and III^8, by Proposition 6.1. By Lemma 6.3, the incidence coefficients involving III^8 are all zero and hence III^8 is a cocycle. On the other hand, for the other two equivalence classes of chiral singular fibers, we have, for example,

$$[\mathrm{III}^5, \mathrm{IV}^{11}] \neq 0, \quad [\mathrm{III}^5, \mathrm{IV}^{10}] = 0,$$
$$[\mathrm{III}^7, \mathrm{IV}^{11}] = 0, \quad [\mathrm{III}^7, \mathrm{IV}^{10}] \neq 0.$$

Therefore, a linear combination of III^5, III^7 and III^8 is a cocycle if and only if the coefficients of III^5 and III^7 both vanish. Therefore, the kernel of the coboundary homomorphism δ_3 is infinite cyclic and is generated by III^8. This completes the proof. □

According to Proposition 8.1, we can interpret our main theorem (Theorem 5.5) as follows. The 3-dimensional cohomology class represented by the cocycle III^8 of the universal complex of chiral singular fibers for proper C^∞ stable maps of oriented 5-manifolds into 4-manifolds gives a complete invariant of the oriented cobordism class of the source 4-manifold. In particular, for $N = \mathbb{R}^3$, it gives a complete invariant of the oriented bordism class of a C^∞ stable map of a closed oriented 4-manifold into $\mathbb{R}^3$.

For related discussions, see [**43**].

We also see that the fiber which satisfies the property as in Theorem 5.5 should necessarily be the fiber of type III^8. This explains the reason why the III^8 type fiber appeared in the modulo two Euler characteristic formula in [**43**] (see Corollary 5.6 of the present paper).

Remark 8.2. If we can realize the proof of our main theorem as mentioned in Remark 7.6, then that would imply that III^8 is a cocycle of the universal complex (see [**43**, §12.2]). That is, it might be possible to prove that III^8 is a cocycle without even classifying the singular fibers.

Chapter 4

Singular fibers and characteristic classes

CHAPTER 4

Singular fibers and characteristic classes

(joint work with O.Saeki)

1. Introduction

Let $f : M \to N$ be a smooth map between smooth manifolds. For a singular value $y \in N$ of f, the *singular fiber* over y means the map germ

$$f : (M, f^{-1}(y)) \to (N, y) \tag{1.1}$$

along the set $f^{-1}(y)$. Note that $f^{-1}(y)$ has positive dimension in general if the codimension $k = \dim N - \dim M$ of f is negative.

Instead of considering the map germ (1.1) along the whole inverse image of a singular value, we can also consider the multi-germ

$$f : (M, S_y) \to (N, y) \tag{1.2}$$

along $S_y = f^{-1}(y) \cap S(f)$, where $S(f)$ is the set of singular points of f. Note that if f is proper and generic, then S_y is a finite set of points. For a given (contact) equivalence class $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ of multi-germs, let $\underline{\alpha}(f)$ be the closure of the set of points x_1 in the source manifold M such that for some points $x_2, x_3, \ldots, x_r$ in $f^{-1}(f(x_1)) \subset M$ with $x_i \neq x_j$ for $1 \leq i < j \leq r$, the map germ $f : (M, \{x_1, x_2, \ldots, x_r\}) \to (N, f(x_1))$ is in the equivalence class $\underline{\alpha}$. Furthermore, set $\overline{\alpha}(f) = f(\underline{\alpha}(f))$. Then, according to Kazarian [**19**], $\underline{\alpha}(f)$ (or $\overline{\alpha}(f)$) represents a $\mathbb{Z}_2$-homology class of closed support and its Poincaré dual is expressed as a polynomial of the characteristic classes of the forms $w_i(f^*TN - TM)$ and $f^*f_!w_I(f^*TN - TM)$ (resp. $f_!w_I(f^*TN - TM)$), where w_i is the i-th Stiefel-Whitney class, $w_I = w_{i_1}w_{i_2}\cdots w_{i_k}$ for a multi-index $I = (i_1, i_2, \ldots, i_k)$, $f^*TN - TM$ is the formal difference bundle, TM (resp. TN) is the tangent bundle of M (resp. N), and $f_!$ is the Gysin homomorphism in the cohomology. Furthermore, the polynomial expressions are unique in the sense that they do not depend on a particular proper generic map f. Note that the proof of this fact depends on the existence of a universal space, whose cohomology ring plays the key role.

In this Chapter, we consider the corresponding cohomology classes determined by the topological type of map germs of the form (1.1), instead of multi-germs as in (1.2). Recall that in [**43**] Saeki has developed a theory of singular fibers of generic differentiable maps. Since the existence of a universal space has not been clarified until now in the context of singular fibers, we do not know if the corresponding universal expression in terms of the characteristic classes exists for singular fibers. However, for some explicit singular fibers, we will show that such universal expressions do exist.

For the terminology of this Chapter, refer to Chapter 1, 2 and 3.

This Chapter is organized as follows.

In §2, we first recall the signature formula we generalize this formula for proper Thom-Boardman generic maps $f : M \to N$ of codimension -1. In order to obtain a generalization, we should work with integer coefficients rather than the $\mathbb{Z}_2$-coefficients. For this reason, we will work with oriented maps. Recall that a smooth map between smooth manifolds of negative codimension is an oriented map if the regular parts of the fibers are consistently oriented (see also [**2**]). In this situation, we will show that the Poincaré dual to the homology class represented by the closure of the III^8-locus coincides with $f_! p_1(M)$ modulo torsion, where p_1 is the first Pontrjagin class (Theorem 2.5). See also Remark 2.7.

In §3, we give some speculations concerning the general behavior of singular fibers and the associated characteristic classes, based on Kazarian's results [**19**].

The authors would like to thank Toru Ohmoto for various important comments and stimulating discussions. They would also like to thank András Szűcs for indicating Atiyah's paper [**2**] to them.

2. A generalization of the signature formula

Recall the following definition.

DEFINITION 2.1. A smooth map $f : M \to N$ between smooth manifolds is said to be *Thom-Boardman generic* if its jet extension $j^r f : M \to J^r(M, N)$ is transverse to all the Thom-Boardman strata for all r (for details, see [**4**]), and f restricted to its Thom-Boardman singular sets is in general position (for details, see [**14**, Chap. VI, §5]).

Let us first consider singular fibers of proper C^∞ stable maps of oriented 4-manifolds into 3-manifolds. In Chapter 3, we have seen that the singular fiber of type III^8 (see Figure 3.1) is chiral. In fact, we can define a sign $(= \pm 1)$ for each singular fiber of type III^8 by using the orientation of the source 4-manifold. For a C^∞ stable map f of a closed oriented 4-manifold into a 3-manifold, we call the sum of the signs over all singular fibers of f of type III^8 the *algebraic number of* III^8*-type fibers of* f and denote it by $||\mathrm{III}^8(f)||$. Note also that each point of $\mathrm{III}^8(f) \subset N$ has its own sign, and then $\mathrm{III}^8(f)$ can naturally be regarded as a 0-dimensional $\mathbb{Z}$-cycle of N.

In Chapter 3, we have proved the following.

THEOREM 2.2. *Let M be a closed oriented 4-manifold and N a 3-manifold. Then, for any C^∞ stable map $f : M \to N$, the algebraic number of III^8-type fibers of f coincides with the signature of M. In particular, if N is also oriented, then three times the Poincaré dual to the 0-dimensional $\mathbb{Z}$-homology class represented by $\mathrm{III}^8(f)$ coincides with $f_! p_1(M)$ in $H^3_c(N; \mathbb{Z})$.*

Remark 2.3. By using the universal complex of chiral singular fibers (see §8 in Chapter 3) we see that the fiber which satisfies the property as in Theorem 2.2 should necessarily be the fiber of type III^8.

Let us recall the following definition.

DEFINITION 2.4. Let $f : M \to N$ be a smooth map between smooth manifolds with $\dim N - \dim M < 0$. Such a map f is *oriented* if the fibers of f restricted to the complement of the singular point set is consistently oriented. For another equivalent definition, which works also for nonnegative codimension maps, see [**2**].

In what follows, by III^8 we also denote the equivalence class with respect to the C^0 equivalence modulo regular fibers represented by a singular fiber obtained by an iterated suspension of a singular fiber of type III^8.

Let $f : M \to N$ be a proper oriented map of an n-dimensional manifold into an $(n-1)$-dimensional manifold which is Thom-Boardman generic. Then we see that the set $\mathrm{III}^8(f)$ consisting of those points in N over which lies a singular fiber of III^8-type is a codimension three regular submanifold of N. Furthermore, its normal bundle in N is naturally oriented in the following manner. Let B^3 be a normal slice of $\mathrm{III}^8(f)$; i.e. it is a closed 3-disk fiber of a closed tubular neighborhood of $\mathrm{III}^8(f)$ in N. Let y be the intersection point of B^3 and $\mathrm{III}^8(f)$. We may assume that B^3 is transverse to f. Then $\tilde{B} = f^{-1}(\mathrm{Int}B^3)$ is an orientable 4-manifold, since f is an oriented map, and $f|_{\tilde{B}} : \tilde{B} \to \mathrm{Int}B^3$ is a proper C^∞ stable map whose fiber over y is of type III^8, where $\mathrm{Int}B^3 = B^3 \setminus \partial B^3$. We orient $\tilde{B}$ so that the III^8-type fiber over y gets the sign $+1$. Since the regular parts of the fibers of $f|_{\tilde{B}}$ are consistently oriented, this induces an orientation of B^3. This means that $\mathrm{III}^8(f)$ is co-oriented.

Therefore, if the closure of $\mathrm{III}^8(f)$ is a co-oriented cycle of codimension 3 of N, then it represents a homology class of closed support in $H^c_{n-4}(N; \mathcal{O}_N)$, where $\mathcal{O}_N$ denotes the orientation local system of N. Note that then its Poincaré dual lies in the usual cohomology group $H^3(N; \mathbb{Z})$.

Then we have the following.

THEOREM 2.5. *Let $f : M \to N$ be a proper C^∞ map of an n-dimensional manifold into an $(n-1)$-dimensional manifold which is Thom-Boardman generic. Furthermore, we assume that f is an oriented map. Then the closure of $\mathrm{III}^8(f)$ forms a co-oriented cycle of closed support in N, and three times the Poincaré dual to the homology class represented by it coincides with $f_! p_1(M)$ modulo torsion in $H^3(N; \mathbb{Z})$ (when M is closed, in $H^3_c(N; \mathbb{Z})$), where $p_1(M) \in H^4(M; \mathbb{Z})$ denotes the first Pontrjagin class of M.*

PROOF. Let T denote the closure of $\mathrm{III}^8(f)$. The fact that T represents a co-oriented cycle of N of closed support follows from [**43**, Proposition 12.11] and Theorem 2.2 together with the fact that the signature vanishes for oriented null-cobordant 4-manifolds.

Since the cycle T is co-oriented, we see that it represents a homology class $[T] \in H^c_{n-4}(N; \mathcal{O}_N)$. We denote by $[T]^* \in H^3(N; \mathbb{Z})$ its Poincaré dual. In order to prove that

$$3[T]^* = f_! p_1(M) \quad \text{(modulo torsion)}$$

in $H^3(N; \mathbb{Z}_2)$, it suffices to show

$$\langle 3[T]^*, \gamma \rangle = \langle f_! p_1(M), \gamma \rangle$$

for every $\gamma \in H_3(N; \mathbb{Z})$ by virtue of the universal coefficient theorem.

By [**49**], there exists a closed oriented 3-manifold A and a smooth map $a : A \to N$ such that $\gamma = a_*[A]$, where $[A] \in H_3(A; \mathbb{Z})$ is the $\mathbb{Z}$-fundamental class of A. We may assume that the image of A does not intersect the strata of T of codimension greater than or equal to 4, and that A is transverse to the strata of T of codimension 3. Furthermore, we may assume that a is transverse to f.

Set

$$\tilde{M} = \{(x, t) \in M \times A : f(x) = a(t)\},$$

which is a smooth closed 4-manifold, since f and a are transverse to each other. Let us consider the following commutative diagram:

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{a}} & M \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{a} & N, \end{array}$$

where $\tilde{a}$ (or $\tilde{f}$) is the projection to the first (resp. second) factor restricted to $\tilde{M} \subset M \times A$. Note that A is oriented and the regular fibers of $\tilde{f}$ are also oriented, since f is an oriented map. Therefore, $\tilde{M}$ is also oriented.

Furthermore, if a is generic enough with respect to T, then $\tilde{f}$ is a C^∞ stable map. Therefore, by Theorem 2.2 we have

$$\tilde{f}_! p_1(\tilde{M}) = 3[\mathrm{III}^8(\tilde{f})]^* \tag{2.1}$$

in $H^3(A;\mathbb{Z})$, where $[\mathrm{III}^8(\tilde{f})]^* \in H^3(A;\mathbb{Z})$ is the cohomology class Poincaré dual to the 0-dimensional $\mathbb{Z}$-homology class represented by $\mathrm{III}^8(\tilde{f})$.

On the other hand, since

$$\langle 3[T]^*, \gamma\rangle = \langle 3[T]^*, a_*[A]\rangle \tag{2.2}$$

is equal to the algebraic intersection number of $3[T]$ and a, we see that (2.2) is equal to $3||\mathrm{III}^8(\tilde{f})||$. By (2.1), this is equal to $\langle \tilde{f}_! p_1(\tilde{M}), [A]\rangle$.

We need the following lemma.

LEMMA 2.6. *We have*

$$\tilde{a}^* p_1(M) = p_1(\tilde{M})$$

in $H^4(\tilde{M};\mathbb{Z})$.

PROOF. Note that the product map

$$f \times a : M \times A \to N \times N$$

is transverse to the diagonal

$$\Delta_N = \{(y,y) \in N \times N \,:\, y \in N\}$$

and that $(f \times a)^{-1}(\Delta_N) = \tilde{M}$. Let us denote by ν_{Δ_N} the normal bundle to Δ_N in $N \times N$. Then we have

$$T(M \times A)|_{\tilde{M}} \cong T\tilde{M} \oplus ((f \times a)|_{\tilde{M}})^* \nu_{\Delta_N}$$

and hence

$$p_1(T(M \times A))|_{\tilde{M}} = p_1(\tilde{M}) + ((f \times a)|_{\tilde{M}})^* p_1(\nu_{\Delta_N}) \in H^4(\widetilde{M};\mathbb{Z})$$

On the other hand, we see that

$$p_1(T(M \times A))|_{\tilde{M}} = \tilde{a}^* p_1(M) + \tilde{f}^* p_1(A) \in H^4(\widetilde{M},\mathbb{Z})$$

holds. Note that the cohomology class $p_1(A)$ vanishes, since A is a 3-dimensional manifold and $H^4(A;\mathbb{Z}) = 0$.

Let us consider the following commutative diagram:

$$\begin{array}{ccc}
\tilde{M} & \xrightarrow{(f\times a)|_{\tilde{M}}} & \Delta_N \\
{\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle \pi_2} \\
A & \xrightarrow{\quad a \quad} & N,
\end{array}$$

where $\pi_2 : \Delta_N \to N$ is defined by $\pi_2(y,y) = y$ for $(y,y) \in \Delta_N$. Since π_2 is a diffeomorphism, there exists a unique class $x \in H^4(N;\mathbb{Z})$ such that $p_1(\nu_{\Delta_N}) = \pi_2^* x$. Then we have

$$\begin{aligned}
((f \times a)|_{\tilde{M}})^* p_1(\nu_{\Delta_N}) &= ((f \times a)|_{\tilde{M}})^* \pi_2^* x \\
&= \tilde{f}^* a^* x = 0,
\end{aligned}$$

since $a^* x \in H^4(A;\mathbb{Z})$ vanishes.

Therefore, we have

$$\tilde{a}^* p_1(M) = p_1(\tilde{M}) \quad \text{(modulo torsion)}.$$

This completes the proof of Lemma 2.6. □

By the above lemma we have

$$\begin{aligned}
\langle \tilde{f}_! p_1(\tilde{M}), [A]\rangle &= \langle \tilde{f}_! \tilde{a}^* p_1(M), [A]\rangle \\
&= \langle a^* f_! p_1(M), [A]\rangle \\
&= \langle f_! p_1(M), a_*[A]\rangle.
\end{aligned}$$

Therefore, we have

$$\langle 3[T]^*, \gamma\rangle = \langle f_! p_1(M), \gamma\rangle.$$

This completes the proof of Theorem 2.5. □

Remark 2.7. By a straightforward calculation, we see that

$$f_! p_1(M) = -f_! p_1(f^*TN - TM),$$

where $f^*TN - TM$ denotes the formal difference bundle.

3. Universal complex of singular fibers and some speculations

In [**43**], Saeki introduced the notion of a universal complex of singular fibers. In this section, we give several conjectures related to a "theory" suggested by the results of the previous sections.

For the theory of the universal complex of the chiral singular fiber with coefficients in $\mathbb{Z}$, refer to Chapter 3 §8 or [**44**, §8].

In the following, for a multi-index $I = (i_1, i_2, \dots, i_n)$, we put

$$w_I = w_1^{i_1} w_2^{i_2} \cdots w_n^{i_n} \quad \text{and} \quad p_I = p_1^{i_1} p_2^{i_2} \cdots p_n^{i_n}.$$

Theorems 2.5 suggest the following.

Conjecture 3.1. For any cohomology class α of the universal complex of chiral singular fibers for proper oriented τ-maps of $(n+1)$-dimensional manifolds into $(p+1)$-dimensional manifolds, there exists a universal polynomial $P_\alpha(\tilde{p}_I, p_j)$ such that for any proper oriented τ-map $f : M \to N$ of an n-dimensional manifold into a p-dimensional manifold, $\varphi_f(s^*\alpha)$ coincides with $P_\alpha(f_! p_I(M), p_j(N))$ in $H^*(N;\mathbb{Z})$ (when M is closed, in $H^*_c(N;\mathbb{Z})$) modulo torsion, s^* is the homomorphism induced by suspension.

Note that the cohomology classes of the forms

$$P_\alpha(f_! p_I(M), w_j(N))$$

are easily seen to be bordism invariants for maps f into a fixed target manifold N.

In fact, we have the following.

LEMMA 3.2. *Let N be a smooth manifold. Two continuous maps $f_i : M_i \to N$, $i = 0, 1$, of closed n-dimensional manifolds are unoriented bordant if and only if $(f_0)_! w_I(M_0) = (f_1)_! w_I(M_1)$ in $H^*_c(N;\mathbb{Z}_2)$ for every multi-index I.*

PROOF. It is known that f_0 and f_1 are unoriented bordant if and only if

$$\langle f_0^*\zeta \smile w_I(M_0), [M_0]_2\rangle = \langle f_1^*\zeta \smile w_I(M_1), [M_1]_2\rangle \tag{3.1}$$

holds for all multi-indices I and for all $\zeta \in H^*(N;\mathbb{Z}_2)$, where $[M_i]_2 \in H_n(M_i;\mathbb{Z}_2)$ denotes the $\mathbb{Z}_2$-fundamental class of M_i, $i = 0, 1$ (see, for example, [**7**, (17.2) Theorem]). Note that we have

$$\begin{aligned}\langle f_i^*\zeta \smile w_I(M_i), [M_i]_2\rangle &= \langle f_i^*\zeta, w_I(M_i) \frown [M_i]_2\rangle \\ &= \langle \zeta, (f_i)_*(w_I(M_i) \frown [M_i]_2)\rangle \\ &= \langle \zeta, ((f_i)_! w_I(M_i)) \frown [N]_2\rangle,\end{aligned}$$

where $[N]_2 \in H^c_p(N;\mathbb{Z}_2)$ is the $\mathbb{Z}_2$-fundamental class of N of closed support, and $(f_i)_! w_I(M_i) \in H^*_c(N;\mathbb{Z}_2)$. Therefore, (3.1) holds for all $\zeta \in H^*(N;\mathbb{Z}_2)$ if and only if

$$(f_0)_! w_I(M_0) = (f_1)_! w_I(M_1)$$

in $H^*_c(N;\mathbb{Z}_2)$. This completes the proof. □

We also have a corresponding lemma for oriented bordisms (see [**7**, (17.5) Theorem]).

Compare Conjecture 3.1 with the results obtained by Kazarian in [**19**]. In view of Kazarian's results together with Remarks 2.7, it may be conjectured[1] that P_α is a polynomial of $f_! p_I(f^*TN - TM)$. Note that these types of characteristic classes are called *Landweber-Novikov classes* and play an important role in the theory of bordisms (for details, see [**19**] and the references therein).

Let $f : M \to N$ be a Thom-Boardman generic map. Let us denote by $\tilde{3}^2(f)$ ($\subset M$) the set of singular points which are contained in a fiber of type III^8. In other words, $\tilde{3}^8(f) = f^{-1}(\mathrm{III}^8(f)) \cap S(f)$, where $S(f)$ denotes the set of singular points of f.

Conjecture 3.3. Let $f : M \to N$ be a proper C^∞ map of an n-dimensional manifold into an $(n-1)$-dimensional manifold which is Thom-Boardman generic. Furthermore, we assume that f is an oriented map. Then the closure of $3^8(f)$ forms a co-oriented cycle of closed support in M, and the Poincaré dual to the homology class represented by it coincides with

$$p_1(M) = -p_1(f^*TN - TM)$$

modulo torsion in $H^4(M; \mathbb{Z})$.

It is probable that we can construct a universal *source* complex of singular fibers corresponding to the above types of strata in the source manifolds, taking into account the singular fibers in which the relevant points are contained.

[1]For this conjecture, probably we need to assume that the equivalence relation among the singular fibers in question should be based on the contact equivalence.

Chapter 5

Further developments

CHAPTER 5

Further developments

In this Chapter, we state the further research topics of the singular fibers of differentiable maps.

1. Cobordism invariants in terms of the singular fibers

The author is interested in the characterization of the cobordism invariants of manifold not mod 2 Euler number because all of the mod 2 results in this thesis is mod 2 Euler number. The unoriented cobordism group of a 5-manifold does not vanish and the generator of it is not a mod 2 Euler number (for details, see [**30**]). So, this cobordism invariant may be characterized in terms of the singular fibers of stable maps. But, there are some difficulties. First difficulty is that the completely characterization of the stable maps of 6-manifolds into 5-manifolds of type Proposition 2.2 in Chapter 1 do not obtained. The second difficulty is that in order to characterize the cobordism invariant of manifolds in terms of the singular fibers of the stable maps, we must construct stable maps such that the corresponding homology class of it is not vanish. We note that the stable maps are dense in the mapping space, but the construction of stable maps is very difficult (for details, see [**21**] and [**43**, §6]). The first difficulty may overcome if we restrict the singularities of such maps. The second difficulty may overcome if we construct universal map as Rimańy and Szűcs [**36**]. If we can overcome these difficulties, the theory of the singular fibers of differentiable maps will develop further in higher dimension. We pose the following conjecture.

Conjecture 1.1. Let $f : M \to N$ be an oriented Thom-Boardman generic map of oriented n-manifold into $(n-1)$-manifold. Then the equivalence classes of $(4i-1)$-cycle fibers (as shown in Figure 5.1) are represent the cohomology classes of the cochain complex of the singular fibers of such maps. Furthermore, for the equivalence class $\mathcal{F}$ of $(4i-1)$-cycle fibers, the closure of $\mathcal{F}(f)$ forms a co-oriented cycle of closed support in N, and the Poincaré dual of such classes is coincide with $f_! P_i(M)$ modulo torsion in $H^{i-1}(N, \mathbb{Z})$: namely

$$f_! P_i(M) \equiv [\overline{\mathcal{F}(f)}]^* \in H^{i-1}(N, \mathbb{Z}) \text{ modulo torsion},$$

where $P_i(M)$ denote the i-th Pontrjagin classes of M and $f_!$ denote the Gysin homomorphism induced by f.

2. Applications to the surface bundles

In recent work with Saeki [**46**], we apply the theory of the singular fibers of differentiable maps for the characterization of the topology of oriented surface bundles. More precisely, we consider following situation. Let Σ_g be a connected closed orientable surface, $\pi : E \to B$ a fiber bundle whose fiber is Σ_g and structural group is $\mathrm{Diff}^+\Sigma_g$, where $\mathrm{Diff}^+\Sigma_g$ denote the group of orientation preserving self-diffeomorphisms on Σ_g, and let $f : E \to \mathbb{R}$ be a generic function on E. For $y \in B$, we denote by f_y the restriction of f to the fiber over y. We denote $\Sigma(f) \subset B$ the set of points $y \in B$ such that f_y is not a stable Morse function. Here, we call the

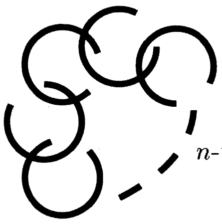

FIGURE 5.1. n-cycle fiber

function $g : S \to \mathbb{R}$ on surface S the *stable Morse function* if all critical points of f are non-degenerate, and they have distinct values. In this situation, we characterize the degeneracy of a function in terms of the singular fibers of differentiable maps. Then we have the following Theorem: Let B be a closed n-manifold, $\pi : E \to B$ be a oriented Σ_g-bundle ($g \geq 2$) over B, and $f : E \to \mathbb{R}$ be a generic function. Then the closure of $\mathrm{III}^8(\pi, f)$ forms a co-oriented cycle of closed support in B, and three times the Poincaré dual to the $(n-2)$-dimensional homology class represented by it coincides with $e_1(\pi)$ modulo torsion in $H^2(B;\mathbb{Z})$: namely

$$3e_1(\pi) \equiv [\overline{\mathrm{III}^8(\pi, f)}]^* \in H^2(B, \mathbb{Z}) \text{ modulo torsion},$$

where $\mathrm{III}^8(\pi, f)$ is the set of point $y \in B$ such that the function $f_y = f|_{\pi^{-1}(y)}$ has just one III^8 type singular fiber and $e_1(\pi)$ is the first MMM class for the Σ_g-bundle π. Here, the i-th Morita-Mumford-Miller class (or MMM class for short) $e_i(\pi)$ for oriented Σ_g-bundle $\pi : E \to B$ is defined as follows. Let $e(T_\pi) \in H^2(E;\mathbb{Z})$ be Euler class of vector bundle $T_\pi \to E$, where $T_\pi \to E$ is the oriented subbundle of $TE \to E$ which is the tangent bundle along the fiber (this is oriented vector bundle of rank 2). Then the i-th MMM class of $\pi : E \to B$ is defined by $e_i(\pi) := \pi_!(e(T_\pi)^{i+1}) \in H^{2i}(B;\mathbb{Z})$, where $\pi_! : H^k(E;\mathbb{Z}) \to H^{k-2}(B;\mathbb{Z})$ is Gysin homomorphism induced by π (for details, see **[31]** and **[32]**). This result demonstrates the wide range of applications of the theory of the singular fibers of differentiable maps. Note that for the study of the topology of the fiber bundles, the method to take the generic functions on the total space was introduced by Kazarian **[16]** **[17]** and **[18]**. For the study of Σ_g-bundles, we pose the following conjecture.

Conjecture 2.1. Let $\pi : E \to B$ be a oriented Σ_g-bundle over n-manifold, $f : E \to \mathbb{R}$ be a generic function. Then for $(2i+1)$-cycle (as shown in Figure 5.1) fiber $\mathcal{F}$ the closure of $\mathcal{F}(\pi, f)$ forms a co-oriented cycle of closed support in B. Furthermore, we have the i-th MMM class of π is coincide with the $(2i+1)$ times the class represented by Poincaré dual of $\mathcal{F}(\pi, f)$ in $H^{2i}(B^n;\mathbb{Z})$ modulo torsion: namely

$$e_i(\pi) \equiv (2i+1)[\overline{\mathcal{F}(\pi, f)}]^* \in H^{2i}(B, \mathbb{Z}) \text{ modulo torsion},$$

where $\mathcal{F}(\pi, f)$ is defined as $\mathrm{III}^8(\pi, f)$.

The author is also interested in the characterization of topology of S-bundles by the Kazarian's method, where S possibly non-orientable closed surface. Note that in that case we may consider the global condition in such bundles like the "two colorable".

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