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Technical Proofs and Additional Results

for the article

The Dynamics of Takeovers through Exchange Offers in the Presence of Competition

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This Online Resource collects the standard valuation arguments together with the branch-specific and repetitive derivations omitted from the main text. Some of these results are standard—in particular, the change-of-numeraire and hitting-time identities in Section S.1—and are reproduced only to keep the analysis self-contained.

Numbering convention. Sections, equations, and lemmas of this Online Resource carry the prefix S. All other numbered items—equations, propositions, the theorem, the assumption, Lemmas 1 and 2, and section numbers without the prefix S—refer to the main article, *The Dynamics of Takeovers through Exchange Offers in the Presence of Competition*.

S.1 Change of numeraire and hitting-time identities

This section records the standard change-of-numeraire and perpetual-hitting identities used from Subsection 2.1 of the main text onward. Full derivations of these standard results are in Karatzas and Shreve (1991) for the measure-change and optional-sampling steps and in Peskir and Shiryaev (2006) for the perpetual-hitting and optimal-stopping formulas; we state only what the main text uses.

Change of numeraire. Because Y_t is the ex-payout value of the target's assets, its payout yield under $\mathbb{Q}$ is $\delta_Y := r - \mu_Y > 0$, and the normalized discounted total return

$$L_t := \frac{\mathrm{e}^{-rt}\mathrm{e}^{\delta_Y t}Y_t}{y} = \exp\bigl(\sigma_Y W_t^Y - \tfrac{1}{2}\sigma_Y^2 t\bigr)$$

is a strictly positive $\mathbb{Q}$-martingale. Define $\mathbb{Q}^{(Y)}$ on $\mathcal{F}_t$ by $\mathrm{d}\mathbb{Q}^{(Y)}/\mathrm{d}\mathbb{Q}|_{\mathcal{F}_t} = L_t$. For G homogeneous of degree one with $g(z) := G(z,1)$, so that $G(X_t, Y_t) = Y_t g(Z_t)$, the density L_t absorbs the target value and yields, for every stopping time with finite expectation,

$$\mathbb{E}^{\mathbb{Q}}_{x,y}\bigl[\mathrm{e}^{-r\tau}G(X_\tau, Y_\tau)\bigr] = y\,\mathbb{E}^{\mathbb{Q}^{(Y)}}_{z}\Bigl[\mathrm{e}^{-(r-\mu_Y)\tau}g(Z_\tau)\Bigr],$$

which is (4); the same change of measure gives the running counterpart

$$\mathbb{E}^{\mathbb{Q}}_{x,y}\left[\int_0^\tau \mathrm{e}^{-rt}X_t\,\mathrm{d}t\right] = y\,\mathbb{E}^{\mathbb{Q}^{(Y)}}_{z}\left[\int_0^\tau \mathrm{e}^{-(r-\mu_Y)t}Z_t\,\mathrm{d}t\right] \tag{S.1}$$

whenever either side is finite.

Finally, because $k_X X_t$ is the ex-payout value of the bidder's assets with payout yield $\delta_X := r-\mu_X$, its ordinary standalone payouts are fully credited: for every admissible stopping time τ,

$$k_X x = \mathbb{E}^{\mathbb{Q}}_{x,y}\left[\int_0^\tau \mathrm{e}^{-rt}\delta_X k_X X_t\,\mathrm{d}t + \mathrm{e}^{-r\tau}k_X X_\tau\right]. \tag{S.2}$$

Ratio dynamics and the occupation identity. By the multidimensional Girsanov theorem (Karatzas and Shreve 1991), the change to $\mathbb{Q}^{(Y)}$ shifts the drifts of X and Y so that, by Itô's formula, the ratio $Z_t = X_t/Y_t$ follows

$$\frac{\mathrm{d}Z_t}{Z_t} = (\mu_X-\mu_Y)\,\mathrm{d}t + \sigma_Z\,\mathrm{d}W^{(Y)}_t, \qquad \sigma_Z^2 := \sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2,$$

for a standard $\mathbb{Q}^{(Y)}$-Brownian motion $W^{(Y)}$; this is (5). Writing $\widetilde{r} := r-\mu_Y > 0$, the discounted ratio $\mathrm{e}^{-\widetilde{r}t}Z_t$ solves a geometric SDE with drift $\mu_X - r < 0$ and is therefore a nonnegative supermartingale, so optional sampling gives, for every bounded stopping time τ,

$$\mathbb{E}^{\mathbb{Q}^{(Y)}}_{z}\left[\mathrm{e}^{-\widetilde{r}\tau}Z_\tau\right] + (r-\mu_X)\mathbb{E}^{\mathbb{Q}^{(Y)}}_{z}\left[\int_0^\tau \mathrm{e}^{-\widetilde{r}t}Z_t\,\mathrm{d}t\right] = z. \tag{S.3}$$

Upper hitting times. Let $0 < z < a$ and $\tau_a := \inf\{t \ge 0 : Z_t \ge a\}$. The generator of Z under $\mathbb{Q}^{(Y)}$ is

$$\mathcal{L}f(z) = (\mu_X-\mu_Y)zf'(z) + \tfrac{1}{2}\sigma_Z^2 z^2 f''(z), \tag{S.4}$$

and $\theta > 1$ in (6) is the positive root of $(\mathcal{L}-\widetilde{r})z^m = 0$. Since $\mathrm{e}^{-\widetilde{r}t}Z_t^\theta$ is then a local martingale, the standard perpetual-hitting argument (Peskir and Shiryaev 2006)—localizing at $\tau_a \wedge n$ and letting $n\to\infty$, with $Z_n < a$ on $\{\tau_a > n\}$—gives

$$\mathbb{E}^{\mathbb{Q}^{(Y)}}_{z}\left[\mathrm{e}^{-\widetilde{r}\tau_a};\,\tau_a<\infty\right] = \left(\frac{z}{a}\right)^\theta, \tag{S.5}$$

which is (7). Applying (S.3) to $\tau_a \wedge n$ under the same localization yields the occupation identity

$$\mathbb{E}^{\mathbb{Q}^{(Y)}}_{z}\left[\int_0^{\tau_a} \mathrm{e}^{-\widetilde{r}t}Z_t\,\mathrm{d}t\right] = \frac{1}{r-\mu_X}\left\{z - a\left(\frac{z}{a}\right)^\theta\right\}, \tag{S.6}$$

and combining (S.1) with (S.6) gives the verification-cost identity used in Subsection 2.2 of the main text:

$$\mathbb{E}^{\mathbb{Q}}_{x,y}\left[\int_0^{\tau_a} \mathrm{e}^{-rt}\lambda k_X X_t\,\mathrm{d}t\right] = y\,\frac{\lambda k_X}{r-\mu_X}\left\{z - a\left(\frac{z}{a}\right)^\theta\right\}.$$

Affine threshold payoffs. Continuity of Z and (S.5) imply that a reduced payoff $h(a)$ received at τ_a has value $h(a)(z/a)^\theta$ for $0 < z < a$. For an affine call-type payoff $h(u) = \beta u - \gamma$ with $\beta, \gamma > 0$, maximizing $h(a)/a^\theta$ gives the unique threshold

$$z_h = \frac{\theta}{\theta - 1}\frac{\gamma}{\beta}, \tag{S.7}$$

and the standard value-matching and smooth-pasting verification (Peskir and Shiryaev 2006) gives

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}_z^{\mathbb{Q}^{(Y)}}\left[\mathrm{e}^{-\widetilde{r}\tau} h(Z_\tau)\right] = \begin{cases} h(z_h)\left(\dfrac{z}{z_h}\right)^\theta, & 0 < z < z_h, \\ h(z), & z \geq z_h, \end{cases} \tag{S.8}$$

the option of never stopping being included and yielding zero. Equations (S.7)–(S.8) are specialized to the takeover payoff components in Section S.2.

S.2 Primitive thresholds and ordering

This section derives the primitive thresholds stated in Subsection 2.2 of the main text and records their complete ordering. The generic affine-payoff formula (S.7) applied to g_E, g_M, g_M^Y, and g_M^X gives equations (9), (10b), and (15). The restriction $\xi > \xi_{\min}$ is equivalent to positivity of the slope of $g_M^X(\cdot, \xi)$.

Lemma S.1 (Primitive threshold ordering)**.** *For $\xi \in \Xi$,*

$$\frac{\mathrm{d}z_M^X(\xi)}{\mathrm{d}\xi} < 0, \qquad \frac{\mathrm{d}z_M^Y(\xi)}{\mathrm{d}\xi} > 0, \qquad \frac{\mathrm{d}z_E(\xi)}{\mathrm{d}\xi} > 0. \tag{S.9}$$

Moreover,

$$z_E(\xi) \leq z_M^Y(\xi) \leq z_M \leq z_M^X(\xi), \qquad \xi_{\min} < \xi \leq \widetilde{\xi}, \tag{S.10}$$

whereas

$$z_M^X(\xi) < z_M < z_M^Y(\xi) < z_E(\xi), \qquad \widetilde{\xi} < \xi < 1. \tag{S.11}$$

At $\xi = \widetilde{\xi}$, all four thresholds coincide as in (17).

Proof. Direct differentiation of (9) and (15) gives the three signs in (S.9), and substitution of (16) gives the alignment

$$z_M^Y(\widetilde{\xi}) = z_M^X(\widetilde{\xi}) = z_M = z_E(\widetilde{\xi}), \tag{S.12}$$

so z_M^Y and z_M^X lie on the stated sides of z_M. Finally,

$$z_M^Y(\xi) - z_E(\xi) = \frac{\theta}{\theta - 1}\frac{k_Y\{\alpha k_Y + k_X(\alpha + \varphi)\}}{k_X(\alpha k_Y + k_X)}\frac{\widetilde{\xi} - \xi}{1 - \xi}.$$

This difference is nonnegative for $\xi \leq \widetilde{\xi}$ and negative for $\xi > \widetilde{\xi}$. Combining these comparisons proves

(S.10)–(S.11). □

S.3 Full fixed-share stopping problems

This section provides the complete target-acceptance and bidder-proposal values for a fixed retained share $\xi \in \Xi$. These calculations preserve the branches omitted from the main text and verify the standard optimal-stopping majorants.

Target acceptance. Let

$$H_A(z,\xi) := (1-\xi)v_M(z) + g_E(z,\xi)$$

be the reduced reward in (18).

Lemma S.2 (Complete target-acceptance value)**.** *The optimal acceptance threshold is* (19)*. If* $\xi_{\min} < \xi \le \widetilde{\xi}$*, then*

$$v_A(z,\xi) = \begin{cases} (1-\xi)g_M(z_M)\left(\dfrac{z}{z_M}\right)^\theta + g_E(z_E(\xi),\xi)\left(\dfrac{z}{z_E(\xi)}\right)^\theta, & 0 < z < z_E(\xi), \\ (1-\xi)g_M(z_M)\left(\dfrac{z}{z_M}\right)^\theta + g_E(z,\xi), & z_E(\xi) \le z < z_M, \\ g_M^Y(z,\xi), & z \ge z_M. \end{cases} \tag{S.13}$$

If $\widetilde{\xi} < \xi < 1$*, then*

$$v_A(z,\xi) = \begin{cases} g_M^Y(z_M^Y(\xi),\xi)\left(\dfrac{z}{z_M^Y(\xi)}\right)^\theta, & 0 < z < z_M^Y(\xi), \\ g_M^Y(z,\xi), & z \ge z_M^Y(\xi). \end{cases} \tag{S.14}$$

The two formulas coincide at $\xi = \widetilde{\xi}$*.*

Proof. First suppose $\xi_{\min} < \xi \le \widetilde{\xi}$, so $z_E(\xi) \le z_M$ by Lemma S.1. On $(0, z_M)$ the discounted merger-option value is a martingale up to the first hitting of z_M, so the timing choice before exercise concerns only the exchange component, with optimal threshold $z_E(\xi)$; adding the retained merger-option value to the call value of g_E gives (S.13), and at z_M the reward becomes g_M^Y with value matching and smooth pasting.

Now suppose $\widetilde{\xi} < \xi < 1$, so $z_M < z_M^Y(\xi) < z_E(\xi)$. Once z_M is reached the merger option is exercised and the reward is the affine payoff $g_M^Y(\cdot,\xi)$, whose call threshold $z_M^Y(\xi)$ makes acceptance and completion simultaneous; (S.8) then gives (S.14), the smallest $(r-\mu_Y)$-excessive majorant of H_A. At $\xi = \widetilde{\xi}$ all thresholds coincide by (S.12), so the branches agree. □

Immediate proposal payoff. Let $a := z_A(\xi)$. Combining delayed and immediate acceptance in (23) gives the complete formula

$$\Pi(z,\xi) = \begin{cases} \{\xi v_M(a) - g_E(a,\xi)\} \left(\dfrac{z}{a}\right)^{\theta} - v_C(z,a), & 0 < z < a, \\ \xi v_M(z) - g_E(z,\xi), & a \le z < z_M, \\ g_M^X(z,\xi), & z \ge a \vee z_M. \end{cases} \tag{S.15}$$

The middle branch is present only when $a < z_M$. Thus (S.15) reduces to (24) when $\xi \le \widetilde{\xi}$; when $\xi > \widetilde{\xi}$, the first branch extends through the region below $a = z_M^Y(\xi) > z_M$ and acceptance and completion occur together at a.

Lemma S.3 (Complete fixed-share proposal value)**.** *For $\xi \in \Xi$, define*

$$z_P(\xi) := z_A(\xi) \vee z_M^X(\xi) = \begin{cases} z_M^X(\xi), & \xi_{\min} < \xi \le \widetilde{\xi}, \\ z_M^Y(\xi), & \widetilde{\xi} < \xi < 1. \end{cases} \tag{S.16}$$

Then the proposal value in (25) is

$$v_P(z,\xi) = \begin{cases} g_M^X(z_P(\xi),\xi) \left(\dfrac{z}{z_P(\xi)}\right)^{\theta}, & 0 < z < z_P(\xi), \\ g_M^X(z,\xi), & z \ge z_P(\xi). \end{cases} \tag{S.17}$$

In particular, $z_P(\xi) \ge z_M$ for every $\xi \in \Xi$.

Proof. Write $\widetilde{r} := r - \mu_Y$ and let $\mathcal{L}$ be the generator in (S.4). For notational convenience, set

$$g_M^X(z,\xi) = \beta_1 z - \beta_2, \qquad \beta_1 := \xi\alpha k_Y - (1-\xi)k_X, \qquad \beta_2 := \xi k_Y(\alpha + \varphi - 1).$$

Because $\xi \in \Xi$, $\beta_1, \beta_2 > 0$, and

$$z_M^X(\xi) = \frac{\theta}{\theta - 1}\frac{\beta_2}{\beta_1}.$$

Let $a := z_A(\xi)$, $b := a \vee z_M^X(\xi)$, and consider

$$\widehat{v}(z,\xi) := \begin{cases} g_M^X(b,\xi) \left(\dfrac{z}{b}\right)^{\theta}, & 0 < z < b, \\ g_M^X(z,\xi), & z \ge b. \end{cases}$$

We first show that $\widehat{v}$ dominates the immediate-proposal payoff.

For every $0 < p < a$, equation (S.15) gives

$$\Pi(p,\xi) = \Pi(a,\xi)\left(\frac{p}{a}\right)^{\theta} - v_C(p,a). \tag{S.18}$$

Thus a proposal strictly below the acceptance boundary is dominated by waiting until that boundary when $\lambda > 0$, as shown more explicitly in Lemma 1 of the main text.

Suppose first that $\widetilde{\xi} < \xi < 1$. Lemma S.1 gives

$$a = b = z_M^Y(\xi) > z_M > z_M^X(\xi).$$

At a, acceptance and completion are simultaneous, so $\widehat{v}(a,\xi) = \Pi(a,\xi) = g_M^X(a,\xi)$. For $z < a$, equations (S.18) and (22) imply

$$\Pi(z,\xi) < \Pi(a,\xi)\left(\frac{z}{a}\right)^{\theta} = \widehat{v}(z,\xi)$$

when $\lambda > 0$, while weak inequality holds at $\lambda = 0$. For $z \geq a$, $\Pi(z,\xi) = g_M^X(z,\xi) = \widehat{v}(z,\xi)$.

Now suppose $\xi_{\min} < \xi \leq \widetilde{\xi}$. Then

$$a = z_E(\xi) \leq z_M \leq b = z_M^X(\xi).$$

For $a \leq z < z_M$, the immediate-proposal payoff is $\Pi(z,\xi) = \xi v_M(z) - g_E(z,\xi)$. Since $z_E(\xi)$ maximizes $g_E(u,\xi)/u^{\theta}$,

$$g_E(z,\xi) \geq g_E(z_M,\xi)\left(\frac{z}{z_M}\right)^{\theta}.$$

Consequently,

$$\begin{aligned}\Pi(z,\xi) &\leq \{\xi g_M(z_M) - g_E(z_M,\xi)\}\left(\frac{z}{z_M}\right)^{\theta} \\ &= g_M^X(z_M,\xi)\left(\frac{z}{z_M}\right)^{\theta} \leq \widehat{v}(z,\xi),\end{aligned}$$

where the final inequality uses the standard majorant property of the call payoff g_M^X. In particular, $\Pi(a,\xi) \leq \widehat{v}(a,\xi)$. Equation (S.18) then gives $\Pi(z,\xi) \leq \widehat{v}(z,\xi)$ for $z < a$, with strict inequality when $\lambda > 0$. For $z_M \leq z < b$, equation (S.15) gives $\Pi(z,\xi) = g_M^X(z,\xi) \leq \widehat{v}(z,\xi)$, and equality holds for $z \geq b$. Hence

$$\widehat{v}(z,\xi) \geq \Pi(z,\xi) \qquad \text{for all } z > 0. \tag{S.19}$$

It remains to verify the optimal-stopping majorant. On $(0,b)$, $\widehat{v}$ is proportional to z^{θ}, so

$$(\mathcal{L} - \widetilde{r})\widehat{v}(z,\xi) = 0, \qquad z < b.$$

On $[b,\infty)$, $\widehat{v} = g_M^X$, and $b \geq z_M^X(\xi)$ implies

$$(\mathcal{L} - \widetilde{r})\widehat{v}(z,\xi) = (\mathcal{L} - \widetilde{r})g_M^X(z,\xi) \leq 0.$$

At b, the derivative jump is

$$\partial_z^+ \widehat{v}(b,\xi) - \partial_z^- \widehat{v}(b,\xi) = \frac{\beta_1(\theta-1)}{b}\{z_M^X(\xi) - b\} \leq 0.$$

The generalized Itô–Tanaka formula, localization, and Fatou's lemma therefore yield

$$\mathbb{E}_z^{\mathbb{Q}^{(Y)}}\left[\mathrm{e}^{-\widetilde{r}\tau}\widehat{v}(Z_\tau,\xi)\right] \leq \widehat{v}(z,\xi)$$

for every stopping time τ. Together with (S.19), this makes $\widehat{v}$ an upper bound for the proposal problem. The first hitting time of b attains the bound because $\Pi(b,\xi) = g_M^X(b,\xi)$ and no verification cost is incurred at immediate acceptance. Therefore $v_P = \widehat{v}$ and $z_P = b$. Applying Lemma S.1 yields the piecewise expression in (S.16) and $z_P(\xi) \geq z_M$. □

S.4 Auxiliary properties and monopoly share choice

The following properties are repeatedly used in the competitive analysis.

Lemma S.4 (Auxiliary fixed-share properties)**.** *The acceptance threshold $z_A(\xi)$ is strictly increasing on Ξ. For every fixed $z > 0$, the map $\xi \mapsto v_A(z,\xi)$ is continuous and strictly decreasing. On the exchange-threshold branch,*

$$g_E(z_E(\xi),\xi) = \frac{\xi k_Y}{\theta - 1}, \tag{S.20a}$$

$$\Pi(z_E(\xi),\xi) = \xi\left\{v_M(z_E(\xi)) - \frac{k_Y}{\theta-1}\right\}. \tag{S.20b}$$

Whenever an offer is accepted immediately before the merger threshold,

$$\frac{\partial}{\partial\xi}\{\xi v_M(z) - g_E(z,\xi)\} = v_M(z) + k_X z + k_Y > 0. \tag{S.21}$$

Proof. Equation (19), Lemma S.1, and continuity at $\widetilde{\xi}$ establish strict increase of z_A. For every fixed exercise state u,

$$\frac{\partial}{\partial\xi}\{(1-\xi)v_M(u) + g_E(u,\xi)\} = -v_M(u) - k_X u - k_Y < 0.$$

The explicit values (S.13)–(S.14) then establish continuity and strict decrease of $v_A(z,\cdot)$ on both branches and at their common endpoint. Substitution of (9) into (8b) gives (S.20a); equation (S.20b) follows because acceptance is immediate and $v_C(z_E,z_E) = 0$. Finally, differentiating the immediate-acceptance payoff gives (S.21). □

Lemma S.5 (Location of a monopoly-optimal retained share)**.** *Fix an initial state $0 < z < z_M$. The problem*

$$\sup_{\xi \in \Xi} v_P(z, \xi)$$

admits a nonempty compact set of maximizers, and every maximizer lies in $[\widetilde{\xi}, 1)$. Hence, under a monopoly-optimal share, proposal, target acceptance, and completion of the back-end merger occur at the same state and no verification cost is incurred on the equilibrium path.

Proof. By Lemma S.3, $z_P(\xi) \geq z_M$ for every $\xi \in \Xi$, so the maintained initial state is in the continuation region and

$$v_P(z, \xi) = g_M^X(z_P(\xi), \xi) \left(\frac{z}{z_P(\xi)} \right)^{\theta}. \tag{S.22}$$

On $(\xi_{\min}, \widetilde{\xi}]$, $z_P(\xi) = z_M^X(\xi)$. Substitution into (S.22) shows, up to a positive factor independent of ξ, that

$$v_P(z, \xi) \propto \{\xi(\alpha k_Y + k_X) - k_X\}^{\theta} \xi^{1-\theta}.$$

Therefore

$$\frac{\partial}{\partial \xi} \log v_P(z, \xi) = \frac{\xi(\alpha k_Y + k_X) + (\theta - 1) k_X}{\xi\{\xi(\alpha k_Y + k_X) - k_X\}} > 0.$$

No maximizer can lie below $\widetilde{\xi}$.

On $[\widetilde{\xi}, 1)$, the two branch formulas agree at $\widetilde{\xi}$, where the value is strictly positive. For $\xi > \widetilde{\xi}$, $z_P(\xi) = z_M^Y(\xi) \to \infty$ as $\xi \uparrow 1$. Because $g_M^X(\cdot, \xi)$ is affine and $\theta > 1$, equation (S.22) implies $v_P(z, \xi) \to 0$. Continuity therefore gives a compact argmax on a subinterval of $[\widetilde{\xi}, 1)$. Since z^{θ} is a common positive factor in (S.22) for every $0 < z < z_M$, the argmax set, its target-favorable minimum, and the proposal boundary are independent of the initial state in this region. Finally, for every maximizing share Lemma S.3 gives $z_P = z_A \geq z_M$, so proposal, acceptance, and completion coincide with an empty verification interval. □

S.5 Formal bidding histories and stationary strategies

This section formalizes the protocol of Subsection 2.3 of the main text, making the strategy space, off-path bidding rounds, and the theorem's domain explicit. Theorem 1 is stated for initial no-offer states $0 < z < z_L$. Its relevant continuation domain also includes no-offer states $p \geq z_L$ reached if bidder ℓ delays an effective challenge; at those states the delayed-challenge response objects defined in Appendix A.4 close the continuation game. Post-selection states are unrestricted. The theorem does not characterize histories in which players disregard the earliest-effective-proposal refinement indefinitely.

Let $\mathcal{I} := \{w, \ell\}$ and let $j \neq i$ denote bidder i's rival. Besides the ratio z, the observed state includes a phase variable: N (competitive no-offer), $\mathsf{R}_i(\xi_i)$ (response node after an initial offer (i, ξ_i)), $\mathsf{S}_i(\xi_i)$ (selected offer awaiting acceptance), and $\mathsf{A}_i(\xi_i)$ (accepted offer awaiting bidder i's completion). The merger-completed state is absorbing.

Initiation and response. In phase N, a stationary initiation strategy for bidder i is an initiation region $D_i \subset (0,\infty)$ and an offer rule $\chi_i^I : D_i \to \Xi$ with

$$\chi_i^I(z) \in \mathcal{C}_i(z) \qquad \text{for every } z \in D_i.$$

The next bidding round begins at the first entrance of Z into $D_w \cup D_\ell$; if both regions are entered at the same state, bidder ℓ's offer is processed first and bidder w retains the immediate counter-opportunity. A threshold initiation strategy is the case where this entrance time is an upper hitting time. At a response node $\mathsf{R}_i(\xi_i)$, bidder j chooses

$$\chi_j^R(z; i, \xi_i) \in \{\varnothing\} \cup \mathcal{C}_j(z),$$

where $\varnothing$ denotes passing; the initial bidder cannot revise its offer. Since initiation, response, and selection occur within an instantaneous bidding round, Z is unchanged throughout the round.

An initiation time is *effective* if the induced response and selection decisions move the game from N to some selected-offer phase $\mathsf{S}_k(\xi_k)$. Among effective initiation times yielding the same maximal primary continuation payoff, the timing tie-break selects the earliest one. A round that returns immediately to N with no selected offer and no elapsed calendar time is identified with inactivity.

Delayed-challenge states. For $p \geq z_L$, bidder ℓ's target-favorable credible challenge is the immediately acceptable break-even offer $\widehat{\xi}_\ell(p)$ in (39). Appendix A.4 defines bidder w's compact delayed-response set $\mathcal{D}_w(p)$, its target-favorable payoff-maximizing response $\xi_w^D(p)$, and the associated target value $A_C^D(p)$ in (A14)–(A16). The stationary off-path rules are

$$\chi_\ell^I(p) = \widehat{\xi}_\ell(p), \qquad \chi_w^R(p; \ell, \xi_\ell) = \xi_w^D(p), \qquad p \geq z_L,$$

for every credible initial offer ξ_ℓ whenever the response affects the target's choice. Equations (40)–(43) imply that $\xi_w^D(p)$ is credible, gives bidder w a strictly positive payoff, and gives the target at least as much as any credible offer by bidder ℓ. At $p = z_L$, these rules reduce to the on-path challenge ξ_ℓ^0 and response ξ_w^R. Thus a delay by bidder ℓ is evaluated through another competitive challenge–response round.

Target selection and rejection. Let $U_A^{\mathsf{N}}(z)$ be the target's phase-N continuation value. Under the equilibrium continuation specified in the main text,

$$U_A^{\mathsf{N}}(z) = \begin{cases} v_A^C(z), & 0 < z < z_L, \\ A_C^D(z), & z \geq z_L, \end{cases}$$

where $A_C^D(z_L) = A_C = v_A^C(z_L)$. Given offers (i, ξ_i) and, when present, (j, ξ_j), the target chooses from

$$\left\{ U_A^{\mathsf{N}}(z),\ v_{A,i}(z, \xi_i),\ v_{A,j}(z, \xi_j) \right\}, \tag{S.23}$$

omitting the value of a missing counteroffer. Choosing $U_A^{\mathsf{N}}(z)$ selects no offer and returns to phase N at the same state; selecting bidder k's offer moves to $\mathsf{S}_k(\xi_k)$, where the offer is irrevocable and awaits acceptance, with bidder k receiving $\Pi_k(z, \xi_k)$ and the other bidder receiving zero. The equilibrium withdrawal rule removes every unselected offer immediately. Because no calendar time elapses within the round, such an offer incurs no verification cost during that round. A withdrawn offer may be proposed again at a later no-offer state; only an immediate same-state repetition with no selection is identified with inactivity. The tie-breaking rules of Subsection 2.3 resolve ties in (S.23).

Continuation after selection. If offer (i, ξ) is selected at time s, the post-selection stopping rules are

$$\tau_A^{i,\xi}(s) := \inf\{t \geq s : Z_t \geq z_{A,i}(\xi)\}, \tag{S.24}$$

$$\tau_M^{i,\xi}(s) := \inf\{t \geq \tau_A^{i,\xi}(s) : Z_t \geq z_{M,i}\}, \tag{S.25}$$

with $\inf \varnothing = \infty$. The game is in phase $\mathsf{S}_i(\xi)$ before acceptance and $\mathsf{A}_i(\xi)$ after, with verification expenditure incurred only before acceptance; if $z_{A,i}(\xi) \geq z_{M,i}$ the two times coincide. These rules generate $v_{A,i}$ and Π_i of Subsection 2.2 of the main text.

Stationary Markov equilibrium on the relevant domain. A stationary Markov strategy profile consists of the two initiation rules on the relevant no-offer domain, the response rule at each reached response node, a withdrawal rule for each unselected offer, the target's selection rule, and the stopping rules (S.24)–(S.25). It is a Markov-perfect equilibrium of the relevant continuation game if every prescribed action maximizes the relevant continuation payoff given the other strategies and the tie-breaking rules. Credibility is built into the action sets through $\mathcal{C}_i(z)$. Primary payoffs are compared first; the target-favorable offer tie-break and target-selection tie-break act only after that comparison, while the earliest-effective-proposal tie-break selects among payoff-equivalent effective initiation times.

On the equilibrium profile recorded in Appendix A.4, neither bidder initiates for $0 < z < z_L$; bidder ℓ initiates at every reached no-offer state $p \geq z_L$ with $\widehat{\xi}_\ell(p)$, and bidder w acts through the response rule $\xi_w^D(p)$. Starting from $0 < z < z_L$, continuity therefore makes the first on-path round occur at z_L. Appendix A.4 verifies the prescribed off-path actions and sequential rationality by the one-deviation principle. Without the earliest-effective-proposal tie-break, primary payoffs do not uniquely select the timing of bidder ℓ's zero-payoff challenge.

S.6 Further details on the counteroffer problem

This section records algebraic and computational details omitted from Subsection 3.2 of the main text.

Primitive form of the strong-rival condition. Assumption 1 can hold only when $z_L < z_{M,w}$. Indeed, at $\xi = \widetilde{\xi}_w$ all bidder-w thresholds coincide, and

$$g^Y_{M,w}(z_{M,w}, \widetilde{\xi}_w) = \frac{k_Y}{\theta - 1} \frac{(\alpha + \varphi_w)(\alpha k_Y + k_X)}{\alpha k_Y + k_X(\alpha + \varphi_w)} > \frac{k_Y}{\theta - 1} = A_L.$$

Thus $z_L \geq z_{M,w}$ would imply $v_{A,w}(z_L, \widetilde{\xi}_w) > A_L$, contradicting the assumption. On the relevant region $z_L < z_{M,w}$, substitution of (35b) and the critical-share formula gives

$$v_{A,w}(z_L, \widetilde{\xi}_w) = \frac{k_Y}{\theta - 1} \frac{\alpha k_Y + k_X}{\alpha k_Y + k_X(\alpha + \varphi_w)} \left(\frac{\alpha + \varphi_\ell}{\alpha + \varphi_w} \right)^{\theta - 1}.$$

The primitive inequality below itself implies $z_L < z_{M,w}$ because its right-hand ratio is strictly smaller than $\alpha + \varphi_w$. Consequently, Assumption 1 is equivalent to

$$(\alpha k_Y + k_X) \left(\frac{\alpha + \varphi_\ell}{\alpha + \varphi_w} \right)^{\theta - 1} < \alpha k_Y + k_X(\alpha + \varphi_w),$$

or

$$\frac{\alpha + \varphi_\ell}{\alpha + \varphi_w} < \left\{ \frac{\alpha k_Y + k_X(\alpha + \varphi_w)}{\alpha k_Y + k_X} \right\}^{1/(\theta - 1)}.$$

The right-hand side exceeds one because $\alpha + \varphi_w > 1$. Thus the condition limits the challenger's relative integration-cost disadvantage and, in particular, implies $z_L < z_{M,w}$ and $\xi^0_\ell < \widetilde{\xi}_w$.

Derivatives for computation. From (45)–(46),

$$H'_L(\xi) = -\frac{\theta}{\xi(1 - \xi)} H_L(\xi), \tag{S.26}$$

$$c'_L(\xi) = \frac{\theta k_Y}{(r - \mu_X)(1 - \xi)^2} H_L(\xi) > 0. \tag{S.27}$$

Every interior solution of (51) therefore satisfies

$$0 = \frac{\partial \psi_w(\xi; \lambda)}{\partial \xi} = v_{M,w}(z_L) + A_L H_L(\xi) \frac{\theta - 1 + \xi}{1 - \xi} - \frac{\lambda}{r - \mu_X} \frac{\theta k_Y}{(1 - \xi)^2} H_L(\xi). \tag{S.28}$$

At the lower endpoint,

$$\frac{\partial \psi_w(\xi^0_\ell; \lambda)}{\partial \xi^+} = v_{M,w}(z_L) + k_X z_L + k_Y - \frac{\lambda}{r - \mu_X} \frac{\theta k_Y}{(1 - \xi^0_\ell)^2}.$$

Thus the global solution can be the lower endpoint, an interior point satisfying (S.28), or the upper participation boundary.

Response payoff and comparative statics. For $\xi \in [\xi_\ell^0, \bar{\xi}_w]$ one has $z_L \le z_E(\xi) < z_{M,w}$, so the exchange-threshold formulas apply. Equation (20) with (S.20a) gives

$$v_{A,w}(z_L, \xi) = (1-\xi) v_{M,w}(z_L) + g_E(z_E(\xi), \xi) H_L(\xi) = (1-\xi) v_{M,w}(z_L) + \xi A_L H_L(\xi),$$

which is (49); equation (24), the continuation-region identity $v_{M,w}(z_E(\xi)) H_L(\xi) = v_{M,w}(z_L)$, and (46) give

$$\psi_w(\xi; \lambda) = \xi v_{M,w}(z_L) - \xi A_L H_L(\xi) - \lambda c_L(\xi),$$

which is (50), with $c_L' > 0$ from (S.27). For the comparative statics, write $B_w(\xi) := \xi v_{M,w}(z_L) - \xi A_L H_L(\xi)$. For $0 \le \lambda_1 < \lambda_2$ and arbitrary $\xi_j \in \mathcal{R}_w(\lambda_j)$ the two optimality inequalities give $(\lambda_2 - \lambda_1)\{c_L(\xi_2) - c_L(\xi_1)\} \le 0$, so strict monotonicity of c_L gives $\xi_2 \le \xi_1$ and hence (54). At $\lambda = 0$, $\psi_w(\cdot; 0) = v_{M,w}(z_L) - v_{A,w}(z_L, \cdot)$ is strictly increasing, so $\xi_w^R(0) = \bar{\xi}_w$. Finally, for any $\lambda_n \to \infty$ and $\xi_n \in \mathcal{R}_w(\lambda_n)$, optimality relative to ξ_ℓ^0 gives $\lambda_n c_L(\xi_n) \le B_w(\xi_n) - B_w(\xi_\ell^0)$; the right-hand side is bounded on the compact feasible interval, so $c_L(\xi_n) \to 0$ and strict monotonicity gives $\xi_n \to \xi_\ell^0$, establishing (55).

S.7 Post-selection dynamics

This section proves Proposition 5 of the main text.

Proof of Proposition 5. Proposition 3 gives

$$\xi_\ell^0 \le \xi_w^R < \widetilde{\xi}_w. \tag{S.29}$$

The bidder-specific acceptance rule in Proposition 1 therefore yields

$$z_{A,w}(\xi_w^R) = z_E(\xi_w^R) = b_A.$$

Consequently, the target's stopping rule in Theorem 1 is the first post-selection hitting time of b_A. The strict upper inequality in (S.29), together with the threshold ordering in Lemma S.1, gives $b_A < z_{M,w} = b_M$. Proposition 2 gives $z_L = z_E(\xi_\ell^0)$, and z_E is strictly increasing. Hence

$$z_L = z_E(\xi_\ell^0) \le z_E(\xi_w^R) = b_A < b_M = z_{M,w},$$

which proves (77). After acceptance, the bidder-specific back-end merger rule is the first hitting time of b_M. Continuity of Z then implies $\tau_L \le \tau_A^w < \tau_M^w$ on $\{\tau_M^w < \infty\}$.

Strict increase of z_E also gives

$$b_A = z_L \quad \Longleftrightarrow \quad \xi_w^R = \xi_\ell^0.$$

On $\{\tau_L < \infty\}$, continuity gives $Z_{\tau_L} = z_L$. Thus the preceding equality is equivalent to $\tau_A^w = \tau_L$;

when $b_A > z_L$, the acceptance time is strictly later, including the possibility $\tau_A^w = \infty$. This proves (78).

Substituting $(z, a) = (z_L, b_A)$ into (22) gives (79). If $0 < z_L < b_A$, write $q = z_L/b_A \in (0, 1)$. Since $\theta > 1$,

$$z_L - b_A \left(\frac{z_L}{b_A}\right)^\theta = b_A(q - q^\theta) > 0.$$

For $\lambda > 0$, the cost therefore vanishes if and only if $b_A = z_L$, proving (80).

Finally, (54)–(55) show that $\xi_w^R(\lambda)$ is weakly decreasing and converges from $\bar{\xi}_w$ to ξ_ℓ^0. Since z_E is continuous and strictly increasing,

$$b_A(\lambda) = z_E\big(\xi_w^R(\lambda)\big)$$

is weakly decreasing,

$$b_A(0) = z_E(\bar{\xi}_w), \qquad \lim_{\lambda\to\infty} b_A(\lambda) = z_E(\xi_\ell^0) = z_L.$$

The boundary $b_M = z_{M,w}$ contains no λ. This proves (81)–(82). □

S.8 Additional value decompositions

The main text reports only the sign of the efficient bidder's loss because the joint-value result does not require its full decomposition. This section records the three economically distinct components of the efficient bidder's loss.

Fix $0 < z < z_L$, write $\xi := \xi_w^R$, and let $a := b_A = z_E(\xi)$. Define

$$\Delta_w^S(z) := v_{P,w}(z, \xi_w^*) - v_{P,w}(z, \xi), \tag{S.30}$$

$$\Delta_w^T(z) := v_{P,w}(z, \xi) - \left(\frac{z}{a}\right)^\theta \{\xi v_{M,w}(a) - g_E(a, \xi)\}, \tag{S.31}$$

$$\Delta_w^V(z) := \left(\frac{z}{z_L}\right)^\theta v_C(z_L, a) = \Delta_V(z; \lambda). \tag{S.32}$$

The first term is the loss from conceding a more target-favorable share than under monopoly. The second is the loss from responding at the challenge rather than waiting for the privately optimal proposal state associated with the same share. The third is the real verification expenditure generated by the pending selected offer.

Lemma S.6 (Efficient-bidder loss decomposition)**.** *Maintain Assumption 1, let $\lambda > 0$, and fix $0 < z < z_L$. Then*

$$\Delta_w(z) = \Delta_w^S(z) + \Delta_w^T(z) + \Delta_w^V(z), \tag{S.33}$$

with

$$\Delta_w^S(z) > 0, \qquad \Delta_w^T(z) > 0, \qquad \Delta_w^V(z) \geq 0. \tag{S.34}$$

Moreover,

$$\Delta_w^V(z) > 0 \quad \Longleftrightarrow \quad b_A > z_L \quad \Longleftrightarrow \quad \xi_w^R > \xi_\ell^0. \tag{S.35}$$

Proof. The exchange-branch proposal payoff at the challenge state is

$$\Pi_w(z_L, \xi) = \{\xi v_{M,w}(a) - g_E(a, \xi)\} \left(\frac{z_L}{a}\right)^\theta - v_C(z_L, a).$$

Multiplying by $(z/z_L)^\theta$, substituting into $v_w^C(z) = (z/z_L)^\theta \Pi_w(z_L, \xi)$, and adding and subtracting $v_{P,w}(z, \xi)$ gives (S.33).

By (53) and Proposition 1, $\xi < \widetilde{\xi}_w \leq \xi_w^*$. Lemma S.5 and its proof show that $v_{P,w}(z, \cdot)$ is strictly increasing on $(\xi_{\min}, \widetilde{\xi}_w]$. Consequently,

$$v_{P,w}(z, \xi) < v_{P,w}(z, \widetilde{\xi}_w) \leq v_{P,w}(z, \xi_w^*),$$

which proves $\Delta_w^S(z) > 0$.

For the timing component, let $p^* := z_{P,w}(\xi) = z_{M,w}^X(\xi)$. The ordering in Section S.2 gives

$$z_L \leq a < z_{M,w} < p^*.$$

Lemma 2 of the main text makes the normalized immediate-proposal payoff strictly increasing on $[a, z_{M,w})$. On $[z_{M,w}, p^*)$, write $g_{M,w}^X(p, \xi) = \beta p - \gamma$. The derivative calculation in (A18) of the main text shows that $g_{M,w}^X(p, \xi)/p^\theta$ is strictly increasing there as well. Continuity at $z_{M,w}$ therefore implies

$$v_{P,w}(z, \xi) = \left(\frac{z}{p^*}\right)^\theta g_{M,w}^X(p^*, \xi) > \left(\frac{z}{a}\right)^\theta \{\xi v_{M,w}(a) - g_E(a, \xi)\},$$

so $\Delta_w^T(z) > 0$.

Finally, nonnegativity of Δ_w^V follows from the definition of v_C. For $\lambda > 0$, this cost is strictly positive exactly when $z_L < a$. Since $a = z_E(\xi_w^R)$, $z_L = z_E(\xi_\ell^0)$, and z_E is strictly increasing, this is equivalent to $\xi_w^R > \xi_\ell^0$. This proves (S.34)–(S.35). □

The decomposition clarifies why bidder w's value falls even when equilibrium verification expenditure is zero. At the lower-corner response $\xi_w^R = \xi_\ell^0$, the verification component vanishes, but the share-concession and premature-timing components remain strictly positive. Conversely, the share concession is a transfer and therefore disappears from the joint-value comparison in Proposition 6; only the timing gain and the real verification loss remain.

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