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Persistent Inequality and Private Provision of Public Goods

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Abstract

This paper examines the relationship between the incentive to free ride and inequality by studying the case in which agents invest in human capital and then provide public goods privately. An agent's stock of human capital is affected by his parental stock; the more human capital a parent has, the more effectively his child can learn. Then, the incentives to free ride at provision of public goods in the old period are different among agents. We find that an agent born of a well-educated parent studies harder than an agent born of a less-educated parent, which induces persistent inequality.

Key Words: Human Capital, Incentives to Free Ride, Inequality.

JEL Classifications: H41, O15

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1 Introduction

In a standard neoclassical growth model, inequality among individuals will disappear in the long run. Because the poor grow faster than the rich due to concavity of production function. However, inequality does not have general tendency to decline over time in the real world. Why does inequality persist? This paper analyzes the persistence of inequality in human capital by introducing private provision of public goods into a simple overlapping generations model.

Generally, differences in income and wealth among individuals are ignored in the standard growth models. Because they assume perfect competition; existence of perfect markets, no externalities, and so on, the differences in individual incomes have no effects on macroeconomic performance, and hence representative agent models are justified. Therefore, recent researches on the inequality assume imperfection of markets. Galor and Zeira (1993), Banerjee and Newman (1993), Aghion and Bolton (1997), and Piketty (1997) focus on capital market imperfection and technological indivisibility. The presence of imperfection prevents the poor from borrowing to invest enough. So, the poor remain poor.

Another approach emphasizes human capital formation which is affected by public education or externality. Bénabou (1993) and Durlauf (1996) study the effect of local externalities in education and show the emergence of endogenous segregation.

In this paper, we focus on the effect of incentive to free ride on human capital formation. The poorer is an individual, the stronger is his incentive to free ride, and hence the less he invests in human capital. That is, the poor invest less than the rich, so inequality persists. The basic mechanism is as follows. When young, an individual learns to accumulate human capital, and when old, he provides public goods voluntarily. Assuming that a child born of well-educated parent is given a better talent or better environment to study than one born of less-educated parent, that is, a talented child can accumulate more human capital and hence provide more public goods than a less-talented child. As a result, a talented child expects that a less-talented child will not provide public goods enough when they are old. Thus, a talented child studies hard to provide for future public activities. On the other hand, a less-talented child has a prospect that a talented child will be a highly educated adult and do public activities a lot, and thus he will need not to do so. Therefore, a less-talented child enjoys much leisure and doesn't study harder than a well-talented child. This difference of investment rates induces the disparity of growth rates of human capitals and thus makes the economy unequal.

As a public good, we think of a factor which is provided voluntarily by old agents and which benefits the learning environments for young agents; e.g. knowledge, public order, and education which is received by all children. That is, the learning technologies have a common factor among all young agents. A common factor in production function has an equalizing force. In Tamura (1991), a spillover effect of human capital in the investment technology provides the below-average human capital agents with a higher rate of return on investment than the above-average human capital agents. Thus the below-average human capital agents grow faster than the above-average human capital agents. In our model, education to the young has a similar effect. The long-run states of the economy, therefore, are determined by the balance of the equalizing force and the unequalizing force; difference of investment rates that the rich invest more than the poor. This paper has the same implication as Eeckhout and Jovanovic (2002), which says that knowledge spillovers promote inequality by inducing the technological followers to free ride. They assume that the less knowledge a firm has, the more knowledge it can access from others. Thus the technological followers invest less than the leaders.

This paper regards the disparity between two classes as a definition of inequality. In our model, therefore, the persistence of inequality means that the poor dynasties remain relatively poor and the rich dynasties remain relatively rich over time. That is, the intergenerational mobility in this economy may be very low. Recent empirical researches find that the intergenerational mobility in America is much lower than it has been thought (see e.g. Solon, 1992, 1999; Zimmerman, 1992). Our model supports these empirical findings.

Following Glomm and Ravikumar (1992), we use simple functional forms for preferences and learning technologies. Preferences are logarithmic and learning technologies are Cobb-Douglas. Generally, these functional forms imply that all agents invest at the same rate (Glomm and Ravikumar, 1992; Tamura, 1991; Bénabou, 1996). However, our analysis shows the disparity in investment rates between the rich and the poor. With such a specification, we can highlight the influence of incentives to free ride on investment decisions.

Section 2 of the paper presents the model. Section 3 analyses individuals' behaviors, and shows the differences of investment decisions among agents. Section 4 describes the long-run dynamics of the economy. Section 5 concludes.

2 Model

We analyze an effect of private provision of public goods to accumulation of human capital. Consider an overlapping generations model in which agents accumulate human capital when young and produce private and public goods when old.

There are M agents in the economy. Each agent has a parent and a child, so that there is no population growth. The economy is divided into two classes; a well-educated and a less-educated class. A well-educated class has N agents and a less-educated class $M-N$ agents, where $M \geq N$, $M \geq 2$, and $N \geq 1$. Difference between two classes is the stock of human capital, which is the only factor of production in this economy. At the initial period, an agent in a well-educated class has more human capital than a less-educated class. We use subscript $i = w$ for a well-educated agent and l for a less-educated agent. The preferences of an agent i born at time t are represented by

$$U_{it} = \ln n_{it} + \ln c_{it+1} + \ln E_{t+1}, \tag{1}$$

where n_{it} is leisure at time t, c_{it+1} is consumption at time $t+1$, and E_{t+1} is an educational activity to children at time $t+1$. The term $\ln E_{t+1}$ represents intergenerational altruism. An old agent's utility depends on the level of education received by his child. [1] In this community, all children are gathered at one place and educated same contents (E_{t+1}) by old agents who take part in this activity voluntarily. If an old agent think that his child is not educated enough (E_{t+1} is too small), he participates in education. If he is satisfied with educational activities by other old agents, he does nothing for children. So, E_{t+1} is a public good for each old agent.

[1] E_t is a public factor which affects the learning environment for children. So, we can think of E_t not only as education but also as a stock of knowledge or public order, and so on.

Each agent is endowed a unit time when young, and allocates n_{it} units of time toward leisure and $1-n_{it}$ units toward accumulating human capital. The stock of human capital of an agent i born at time t, h_{it+1}, is determined by not only learning time, but also the stock of human capital of his parent, h_{it}, and an educational activity by old generation, E_t. The dependence on h_{it} means that a child born of well-educated parents is given better talents or environment to learn. The learning technology is represented by

$$h_{it+1} = A(1-n_{it})h_{it}^{\beta}E_t^{1-\beta}, \quad 0<\beta<1, \tag{2}$$

where A is a constant parameter. In this learning technology, community education, E_t, is a common factor among agents, which has the effect that less-educated agents have a greater rate of return to human capital than well-educated agents[2)]. That is, a common factor implies the existence of the equalizing force.

When old, each agent i produces private and public goods from human capital accumulated in his youth. That is, old agent's activities are producing private consumption goods, c_{it+1}, and education to children in the community, e_{it+1}. Human capital is transformed into c_{it+1} and e_{it+1} directly, so that following relationship holds.

$$c_{it+1}+e_{it+1} = h_{it+1}, \quad c_{it+1},\ e_{it+1} \geq 0 \tag{3}$$

Total education, E_{t+1}, is equal to the sum of all individuals' educational activities.

$$E_{t+1} = \sum_{k=1}^{M} e_{kt+1} \tag{4}$$

Given h_{it}, E_t and e_{-it+1}, an agent $i=u,l$ born at time t chooses n_{it}, c_{it+1}, and e_{it+1} to maximize (1) subject to (2), (3), and (4).

3 Short-Run Equilibrium

We solve agent i's optimization problem in two steps. First, we solve for optimal consumption and educational activity; choices when old. Substituting (3) into objective function U_{it} and solving for c_{it+1}, we have a following first order condition;

$$c_{it+1} = \frac{1}{2}(h_{it+1}+e_{-it+1}), \tag{5}$$

where $e_{-it+1} = E_{t+1} - e_{it+1}$. This condition implies $c_{it+1} = E_{t+1}$. That is, given educational activities by the others e_{-it+1}, an old agent i allocate human capital to equate consumption and total education. With this relationship, we can rewrite objective function as

$$U_{it} = \ln n_{it} + 2\ln\left[\frac{1}{2}(h_{it+1}+e_{-it+1})\right]. \tag{6}$$

In the next step, substituting (2) into (6), we solve for optimal leisure n_{it}; a choice when young.

$$n_{it} = \frac{1}{3} + \frac{e_{-it+1}}{3Ah_{it}^{\beta}E_t^{1-\beta}} \tag{7}$$

2) In order to see this effect, rewrite (2) as $h_{it+1}/h_{it} = A(1-n_{it})(E_t/h_{it})^{1-\beta}$. See Tamura (1991).

We see from equation (7) that in what situation agent i reduces leisure; (i) when other agents supply less education[3], and (ii) when the stock of human capital of his parent is large. That is, an agent born of a well-educated family studies harder than an agent born of a less-educated family. These two are obviously related. In our model, education is a public good, i.e. an agent's utility depends on not only his own contribution but also contributions by the others. Then, we can rewrite utility function as $\ln(e_i + e_{-i})$. From this representation, it is clear that the larger is e_{-i}, the lower is the marginal utility of e_i[4]. Then, agent i reduces an educational activity and increases leisure and private consumption. That is, the more other agents supply public goods, the stronger the incentives to free ride are. Well-educated agents supply education more than less-educated agents, so that $e_{-u} < e_{-l}$. Less-educated agents, therefore, have stronger incentive to free ride and study less than well-educated agents. Because of this mechanism, inequality persists.

Generally, in models with log preference and Cobb-Douglas production function, investment rates are equal among agents [5]. That is, every agent studies for same hours. But, in our model, times devoted to learning are different among agents because of presence of public goods.[6]

Using (2), (5), and (7), we obtain e_{it+1} as a reaction function to education by the others.

$$e_{it+1} = -\frac{2}{3}e_{-it+1} + \frac{Ah_{it}^{\beta}E_t^{1-\beta}}{3} \tag{8}$$

Because there exist M agents, we must solve M equations to determine optimal contributions; the Cournot-Nash equilibrium of this economy.

$$e_{it+1} = \frac{AE_t^{1-\beta}}{2M+1}\left[(2M+1)h_{it}^{\beta} - 2\sum_{k=1}^{M} h_{kt}^{\beta}\right] \tag{9}$$

Considering that a well-educated class has N agents and a lower has $M-N$ agents, we can rewrite terms in the square brackets as $(2M+1)h_{it}^{\beta} - 2[Nh_{wt}^{\beta} + (M-N)h_{lt}^{\beta}]$. Substituting h_{wt} and h_{lt} into h_{it} , we obtain optimal education of well-educated and less-educated agents.

$$e_{wt+1} = \frac{AE_t^{1-\beta}}{2M+1}\left[2(M-N)\left(h_{wt}^{\beta} - h_{lt}^{\beta}\right) + h_{wt}^{\beta}\right] \tag{10}$$

$$e_{lt+1} = \frac{AE_t^{1-\beta}}{2M+1}\left[(2N+1)h_{lt}^{\beta} - 2Nh_{wt}^{\beta}\right] \tag{11}$$

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[3] If $e_{-it+1} = 0$, then $n_{it} = 1/3$, which agents enjoy when education is privately supplied.

[4] Geometrically, this means that the graph of $\ln(e_i + e_{-i})$ shifts toward left as e_{-i} becomes large.

[5] For example of this specification, see Tamura (1991), Glomm and Ravikumar (1992), and Bénabou (1996).

[6] Even if there are no public goods, investment rates vary among agents with general preferences of constant relative risk aversion. For example, when utility function is $U_{it} = n_{it}^{1-\sigma}/(1-\sigma) + c_{it+1}^{1-\sigma}/(1-\sigma)$ and production function is $h_{it+1} = (1-n_{it})h_{it}^{\beta}$, then optimal leisure becomes

$$n_{it} = h_{it}^{\beta\frac{\sigma-1}{\sigma}} \Big/ \left(1 + h_{it}^{\beta\frac{\sigma-1}{\sigma}}\right).$$

We can see that $\partial n_{it}/\partial h_{it} > 0$ when $\sigma > 1$, and that $\partial n_{it}/\partial h_{it} < 0$ when $\sigma < 1$.

If the sign of terms in the square brackets is not positive, then non-negativity constraint is binding, so that $e_{it+1} = 0$; that is, an agent i free rides perfectly. Clearly, because $h_{wt} \geq h_{lt}$, e_{wt+1} is positive, i.e. well-educated agents never become perfect free riders. On the other hand, terms in the square brackets of (11) are negative when following relationship between h_{wt} and h_{lt} is hold.

$$\frac{h_{wt}}{h_{lt}} > \left(\frac{2N+1}{2N}\right)^{\frac{1}{\beta}} \tag{12}$$

Define L.H.S. of this inequality as I_t, and R.H.S. as $\bar{I}$. I_t means a degree of inequality. Since $h_{wt} \geq h_{lt}$ from definition, the minimum value of I_t is unity and the more unequal is the community, the larger becomes I_t. $\bar{I}$ represents a critical value of inequality. If the community is so unequal that $I_t \geq \bar{I}$, then agents in a less-educated class become perfect free-riders. We call such a situation the *free-ride phase*, and the another situation the *normal phase*, in which less-educated agents also supply public goods. It follows from (12) that $\partial \bar{I}/\partial N < 0$ and $\partial \bar{I}/\partial \beta < 0$. That is, in an economy in which a number of well-educated agents or β is large, less-educated agents are more likely to be perfect free riders. That a number of well-educated agents is large means that contributions by others are large for a less-educated agent. So he has a large incentive to free ride.

Now, we derive human capital accumulation in normal phase. Using (9) to obtain e_{-ut+1} and e_{-lt+1}, the substitution of them and (7) into (2) yields

$$h_{wt+1} = \frac{1}{2M+1} A E_t^{1-\beta} \left[(2M - N + 1) h_{wt}^{\beta} - (M - N) h_{lt}^{\beta}\right] \tag{13}$$

$$h_{lt+1} = \frac{1}{2M+1} A E_t^{1-\beta} \left[(M + N + 1) h_{lt}^{\beta} - N h_{wt}^{\beta}\right]. \tag{14}$$

These equations describe the transition of human capitals and we use them in order to analyze aggregate behavior of the economy.

Optimal Decisions in Free-Ride Phase

In the free-ride phase, since less-educated agents do not contribute at all, well-educated agents' choices are not affected by less-educated agents' behavior. Therefore each agent in well-educated class considers interaction among N well-educated agents. That is, in this case, we can know well-educated agents' behaviors by solving a voluntary contribution model with N homogeneous agents described by (1), (2), (3), and (4). Well-educated agents' contribution and accumulation of human capital are as follows.

$$e_{wt+1} = \frac{1}{2N+1} A h_{wt}^{\beta} E_t^{1-\beta} \tag{15}$$

$$h_{wt+1} = \frac{N+1}{2N+1} A h_{wt}^{\beta} E_t^{1-\beta} \tag{16}$$

From these equations, we can see a standard result of voluntary provision of public goods. The larger is N, the less is the supply of education by each agent. That is, as a number of contributers becomes large, each agent free rides more strongly. [7]。

[7] For example, see Chamberlin (1974), McGuire (1974), and Andoreoni (1988).

Next, we describe optimal decisions of less-educated agents. In the free-ride phase, since $e_{lt+1} = 0$, less-educated agents' utility functions are as follows.

$$U_{lt} = \ln n_{lt} + \ln c_{lt+1} + \ln N e_{wt+1} \tag{17}$$

Less-educated agents can not control the third term in R.H.S. so that they maximize the first two terms with respect to leisure and consumption. They don't supply education, so they use human capital only to produce consumption goods; $c_{lt+1} = h_{lt+1}$. Then agents choose only n_{lt}. From f.o.c., we can see that optimal leisure is $n_{lt} = 1/2$. Substituting this value into (2), the stock of human capital is determined.

$$h_{lt+1} = \frac{1}{2} A h_{lt}^{\beta} E_t^{1-\beta} \tag{18}$$

Comparing (16) and (18), it is clear that less-educated agents study less than well-educated agents; $(N+1)/(2N+1) > 1/2$. This is the same result as in the normal phase.

4 Dynamics

In this section, we will analyze dynamic behavior of the economy, that is, transition of the degree of inequality and long-run growth rates.

4.1 Transition of the degree of inequality

First, we will derive transition equations of I_t in normal and free-ride phases. Then, global dynamics of I_t will be analyzed.

In order to see transition of inequality, we must know how agents in each class accumulate human capital. Dividing (13) by (14), we have dynamic equation of I_t in normal phase;

$$I_{t+1} = \frac{(2M - N + 1) I_t^{\beta} - (M - N)}{M + N + 1 - N I_t^{\beta}} \equiv f(I_t). \tag{19}$$

It follows that $f'(I) > 0$ and $f(1) = 1$, that is, $f(I_t)$ is monotonically increasing and has a stationary point; $I_{t+1} = I_t = 1$. If this stationary point is stable, we call it the *equal equilibrium*. From a second derivative of $f(I_t)$, it is found that the graph of $f(I_t)$ may have an inflection point. If an inflection point exists, $f(I_t)$ is concave when I_t is small, and then convex. Therefore, $f(I_t)$ may have a stable stationary point which is larger than unity as in figure 1-(b), and we call it the *unequal equilibrium*. From the form of $f(I_t)$, we can see that $f(I_t)$ has at most one stable equilibrium (see figure 1).

Next, the law of motion of I_t in the free-ride phase can be represented as a ratio of (16) and (18).

$$I_{t+1} = \frac{2N + 2}{2N + 1} I_t^{\beta} \equiv g(I_t) \tag{20}$$

Clearly from figure 1-(f), this dynamic equation has a stable stationary point as

$$I_{\infty}^{g} \equiv \left(\frac{2N + 2}{2N + 1} \right)^{\frac{1}{1-\beta}}. \tag{21}$$

We call this stationary point the *free-ride equilibrium.*

Global motion of I_t is determined by $f(I_t)$ where $I_t \leq \bar{I}$, and by $g(I_t)$ where $I_t > \bar{I}$. Define the function $Q(I_t)$ as a global dynamic equation of I_t.

$$Q(I_t) = \begin{cases} f(I_t) & \text{if} \quad I_t < \bar{I}, \\ g(I_t) & \text{if} \quad I_t \geq \bar{I}. \end{cases} \tag{22}$$

Forms of $Q(I_t)$ are described in figure 2, 3, 4. We have following lemma about a fundamental property of $Q(I_t)$.

Lemma 1 *$Q(I_t)$ has at least one globally stable equilibrium.*

Proof See Appendix.

Lemma 1 states that the long-run degree of inequality doesn't diverge, but converges to some value and remains constant over time.

As concrete conditions which characterize the dynamics of the economy, we focus on $f'(1)$ and relationship between I^g_∞ and $\bar{I}$. First, whether $f'(1)$ is larger than unity is related to the stability of a stationary point $I_t = 1$; that is, the existence of equal equilibrium. Second, relationship between I^g_∞ and $\bar{I}$ determines the existence of free-ride equilibrium. Because $f(I_t)$ describes the motion of I_t in the region $I_t < \bar{I}$, if $I^g_\infty < \bar{I}$ as in figure 2, I^g_∞ is not a solution of $Q(I_t) = 1$. From these conditions, we obtain following a proposition.

Proposition 1 *Define* $\beta_s(M) \equiv (M+1)/(2M+1)$ *and* $\beta_g(N) \equiv [\ln(2N+1) - \ln 2N]/[\ln(N+1) - \ln N]$. (i) *If* $\beta < \beta_s(M)$, *$Q(I_t)$ has the equal equilibrium.* (ii) *If* $\beta > \beta_g(N)$, *$Q(I_t)$ has the free-ride equilibrium.*

Proof (i) Rewriting a condition $f'(1) < 1$, which means that a stationary point $I_t = 1$ is stable, we get $\beta < (M+1)/(2M+1)$. Therefore, $\beta < \beta_s(M)$ is the condition which ensure the existence of equal equilibrium. (ii) Rewriting the condition $I^g_\infty > \bar{I}$, we get $\beta > [\ln(2N+1) - \ln 2N]/[\ln(N+1) - \ln N]$[8]. Q.E.D.

Proposition 1-(ii) is a main result of the paper; the persistence of inequality in a long-run equilibrium. Proposition 1 asserts that, as β gets larger, the equal equilibrium disappears and the free-ride equilibrium appears, that is, roughly speaking, the economy gets more unequal. Also we can see similar properties about the degree of inequality at the unequal and free-ride equilibria. Equation (21) implies $\partial I^g_\infty/\partial\beta > 0$, that is, the larger is β, the more unequal is the free-ride equilibrium. Moreover, first derivative of $f(I_t)$ with respect to β is strictly positive except at $I_t = 1$, which means that $f(I_t)$ shifts upward with an increase in β. Therefore, if there exists the unequal equilibrium, an increase in β makes it more unequal. These implications are as follows. $(1-\beta)$ is a weight of community education in production functions. Therefore, an increase in β implies the decrease in the weight of community education and thus weakens the equalizing force, which makes the economy more unequal.

Remark 1 An increase in β makes the economy more unequal.

[8] We also obtain this inequality from rewriting $\bar{I} < f(\bar{I})$, which means that a intersection of $f(I_t)$ with $g(I_t)$ is above 45 degrees line.

Now we classify the long-run states of the economy with various configurations of parameters. Obviously from definition, $\beta_s(M)$ and $\beta_g(N)$ depend on M and N, respectively. Therefore, as M and N vary, the relationship between $\beta_s(M)$ and $\beta_g(N)$ also changes. We see from figure 5, when the difference between M and N is small, $\beta_s(M)$ is larger than $\beta_g(N)$ and vice versa. So, in order to characterize the dynamics of the economy, we focus on the two cases in which $\beta_g(N) < \beta_s(M)$ and which $\beta_s(M) < \beta_g(N)$[9)].

When $\beta_g(N) < \beta_s(M)$, there are three possibilities as follows.

- $\beta < \beta_g(N)$; There exists the equal equilibrium only. That is, in the long run, inequality will vanish from each initial position of I_t (See figure 2).
- $\beta_g(N) \leq \beta \leq \beta_s(M)$[10)]; There exist the equal and free-ride equilibria. That is, difference in initial inequalities leads the economy to different steady states. (See figure 4.)
- $\beta_s(M) < \beta$[11)]; There exists the free-ride equilibrium, and may exist the unequal equilibrium. (See figure 3.)

When $\beta_s(M) < \beta_g(N)$, there exist following cases.

- $\beta \leq \beta_s(M)$; There exists the equal equilibrium only.
- $\beta_s(M) < \beta \leq \beta_g(N)$; There exist neither equal nor free-ride equilibria. Therefore, I_∞^f, which is greater than unity, is the only stable equilibrium.
- $\beta_g(N) < \beta$; There exists the free-ride equilibrium, and may exist the unequal equilibrium. (See figure 3.)

To sum up, the long-run dynamics are classified into five cases; the economy has (i) the equal equilibrium, (ii) the unequal equilibrium, (iii) the free-ride equilibrium, (iv) the equal and free-ride equilibria, or (v) the unequal and free-ride equilibria.

4.2 Long-Run Growth

This subsection studies the long-run growth rates. First, we consider growth rates in the normal phase. The individuals' stocks of human capital are determined by (13) and (14). The well-educated class has N members and the less-educated class $M-N$, so total stock of human capital in this economy is $H_{t+1} = Nh_{wt+1} + (M-N)h_{lt+1}$. Substituting (13), (14), and $E_t = H_t/(M+1)$, which is made up of (5), into H_{t+1}, we get

$$H_{t+1} = \frac{(M+1)^\beta}{2M+1}\frac{NI_t^\beta + M - N}{(NI_t + M - N)^\beta}AH_t. \tag{23}$$

Let $X_f(I_t)A$ denote the coefficient of H_t, which is the growth rate of H_t[12)].

[9)] Both $\beta_g(N)$ and $\beta_s(M)$ have the same limits; $\lim_{N\to\infty}\beta_g(N) = \lim_{M\to\infty}\beta_s(M) = 0.5$. Therefore, when M and N are large enough, knowing whether β is larger than 0.5 makes it possible to characterize the dynamics of the economy.

[10)] When $f''(1) > 0$, this condition is replaced with $\beta_g(N) \leq \beta < \beta_s(M)$.

[11)] When $f''(1) > 0$, this condition is replaced with $\beta_s(M) \leq \beta$.

[12)] We assume that A is large enough for the growth rate to be larger than unity.

Next, we derive the growth rate in the free-ride phase. In the long run, the degree of inequality is constant and equal to I^g_∞, which means that h_{wt} and h_{lt} grows at the same rate in the free-ride equilibrium. Therefore, the growth rate of h_{wt} is equal to that of H_t. Then, we derive the growth rate of h_{wt}. In the free-ride phase, $E_t = Nh_{ut}/(N+1)$, with which we can rewrite (16) as follows.

$$h_{wt+1} = \frac{(N+1)^\beta}{2N+1} N^{1-\beta} A h_{wt}. \tag{24}$$

Define $X_g A$ as the growth rate of h_{wt} in the free-ride equilibrium.

Now we consider growth rates in the equal and free-ride equilibria. Define $Y(x,\beta) \equiv [(x+1)^\beta/(2x+1)]x^{1-\beta}$ where x is a finite natural number, so that $Y(M,\cdot) = X_f(1)$ and $Y(N,\cdot) = X_g$. $Y(x,\cdot)$ represents the growth rate at an equilibrium in which all x contributers are identical such as the equal or free-ride equilibrium [13]. Rewriting a condition $\partial Y(x,\cdot)/\partial x > 0$, we have a following inequality.

$$\beta < (x+1)/(2x+1) \tag{25}$$

The maximum of R.H.S. is 2/3 and the minimum 1/2. Therefore, (i) if $\beta < 1/2$, an inequality (25) holds for any value of x and thus $Y(x,\cdot)$ is a monotonically increasing function of x. In this case, β is small enough so the economy has the equal equilibrium. Therefore, an economy with many contributers grows faster than that with few contributers in the equal equilibrium. (ii) if $\beta > 2/3$, an inequality (25) never hold for any value of x and thus $Y(x,\cdot)$ is a monotonically decreasing function of x. In this case, β is large enough so the economy has the free-ride equilibrium. Therefore, an economy with many contributers grows slower than that with few contributers in the free-ride equilibrium. (iii) If $1/2 < \beta < 2/3$, $Y(x,\cdot)$ has a maximum value as in figure 6.

Remark 2 (i) When β is small enough, an economy with many contributers grows faster than that with few contributers. (ii) When β is large enough, an economy with many contributers grows slower than that with few contributers.

The increase in the number of the contributers, M or N, has two meanings. First, the incentive to free ride rises with the number of contributers. So, agents reduce time to learn, which reduces the growth rate. Second, an increase in the number of contributers rises total supply of community education. In the normal phase, $E_t = M\hat{h}_t/(M+1)$, where $\hat{h}_t$ is average level of human capital of contributers. Similarly $E_t = N\hat{h}_t/(N+1)$ in the free-ride phase. From these equations, if the average level of human capital is constant, E_t increases with M or N. This effect raises the growth rate. β determines which of these two effects is larger. That β is large means that the weight of E_t at human capital formation is small, and that the weight of learning $(1-n_{it})$ is relatively large. So, when β is large, the increase in the number of contributers reduces the growth rate.

Next, we consider the effect of β on the growth rate. From (23) and (24), it follows that $dX_f(1)/d\beta > 0$ and $dX_g/d\beta > 0$. Therefore, the larger is β, the higher is the growth rate. We can see this fact from figure 6.

Remark 3 A long-run growth rate rises with β.

From Remark 1 and 3, it is found that the economy with large β is more unequal and grows faster than that with small β in the long run.

[13] In the free-ride equilibrium, less-educated agents are not the contributers of community educations. In the unequal equilibrium, there exist two types of contributers; well- and less-educated agents.

4.3 Multiple Equilibria

When $\beta_g(N) < \beta < \beta_s(M)$, the economy has both equal and free-ride equilibria. There exists an unstable equilibrium $\hat{I} \in (1, \bar{I})$, as in figure 4. Therefore, when an initial value of I_t is less than $\hat{I}$, the long-run inequality converges to the equal equilibrium, and when an initial value is larger than $\hat{I}$, the long-run inequality converges to the free-ride equilibrium. Generally $X_f(1)$ is different from X_g, so the long-run growth rate depends on an initial value of inequality. For example, when $M = 2$ and $N = 1$, we see that $\beta_g(N) \risingdotseq 0.585$ and $\beta_s(M) = 0.6$. If $\beta = 0.59$, then the relationship that $\beta_g(N) \leq \beta \leq \beta_s(M)$ holds so that there exist multiple equilibria and growth rates are $X_f(1) \risingdotseq 0.508$ and $X_g \risingdotseq 0.502$. In this numerical example, therefore, an initially equal economy grows faster than an initially unequal economy in the long run.

5 Conclusion

We have presented a model that links human capital formation and private provision of public goods. A child allocates his time to learn for future production of private and public goods. Learning technology is affected by parental stock of human capital; the more human capital a parent has, the more effectively his child can learn. Therefore an individual born of a well-educated parent accumulates more human capital and thus provides more public goods. Incentive to free ride for an individual is strong when contributions by the others are large. The less an individual accumulates human capital, the more contributions by the others are. So, he has stronger incentive to free ride and hence provides less public goods and allocates less time to learn than an individual born of a well-educated parent. This difference of investment rates makes the economy unequal.

In this paper, we assume a discrete distribution in the sense that initially the economy is divided into two classes. A natural extension of the model is adopting a continuous distribution. As a result, the number of well-educated agents will be determined endogenously, and thus a long-run growth rate will be considered as a function of initial inequality. That is, the economy has multiple equilibria with broader configurations of parameters than that of this paper.

Appendix

Proof of Lemma 1 Because $f(I_t)$ and $g(I_t)$ are monotonically increasing and continuous, and $f(\bar{I}) = g(\bar{I})$, $Q(I_t)$ is also monotonically increasing and continuous. $Q(I_t)$ is monotonically increasing, so I_t converge or diverge without cyclical movement in neighborhood of equilibria. Therefore, let I_∞^{max} denote the maximum I_t which satisfies $Q(I_t) = I_t$, if following relationship hold, I_t converge to a equilibrium globally.

$$Q(I_t) < I_t, \; \forall \; I_t > I_\infty^{max} \tag{26}$$

First, we consider a case in which $I_\infty^g \geq \bar{I}$. In this case, $I_\infty^{max} = I_\infty^g$. I_∞^g is a stable equilibrium, so that the condition (26) is satisfied. Next, when $I_\infty^g < \bar{I}$, I_∞^{max} exists in $[1, \bar{I})$. Rewriting the inequality, $I_\infty^g < \bar{I}$, we get $Q(\bar{I}) < \bar{I}$, which means $Q(I_t) < I_t, \; \forall \; I_t \geq \bar{I}$ from the shape of $g(I_t)$. Therefore, if we find that $Q(I_t) < I_t$ in $[I_\infty^{max}, \bar{I})$, the proof will be finished. We derive contradiction by assuming that there exists some $I' \in (I_\infty^{max}, \bar{I})$ such that $Q(I') > I'$. That $Q(I') > I'$ and $Q(\bar{I}) < \bar{I}$ means $Q(I_t)$ must intersect 45 degrees line in $(I', \bar{I})$ from the intermediate value theorem. Then, there exists a equilibrium which is larger than I_∞^{max}. This is the contradiction. Q.E.D.

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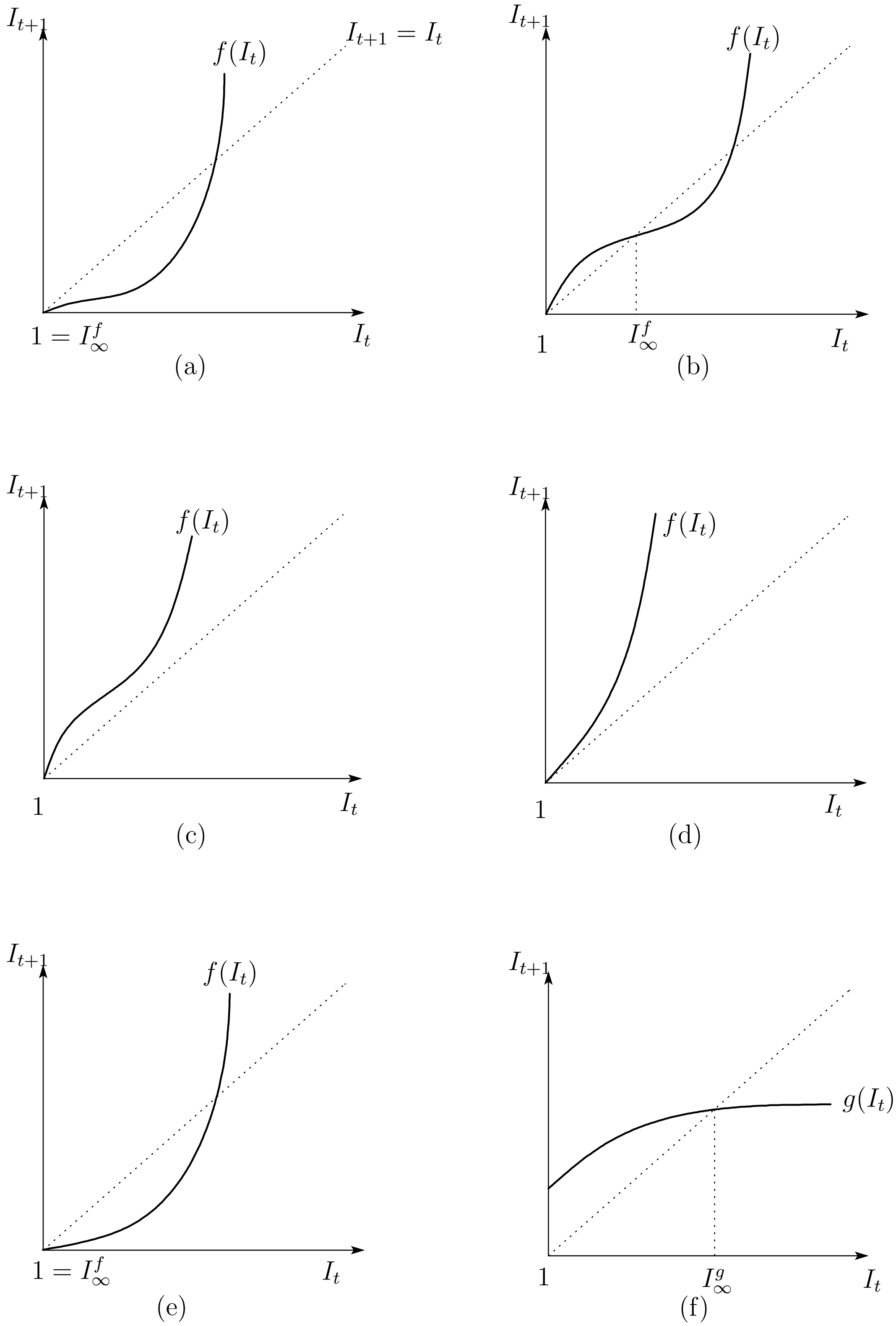

Figure 1: Forms of $f(I_t)$ and $g(I_t)$.

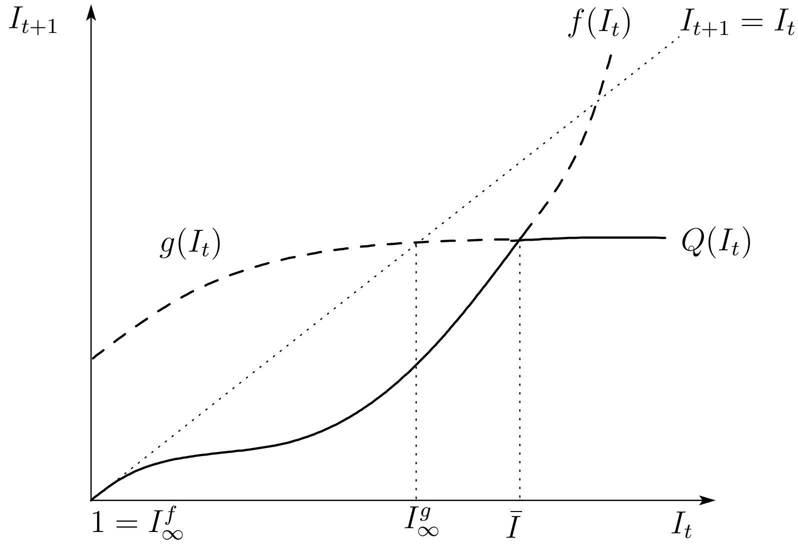

Figure 2: (a) Only equal equilibrium exists.

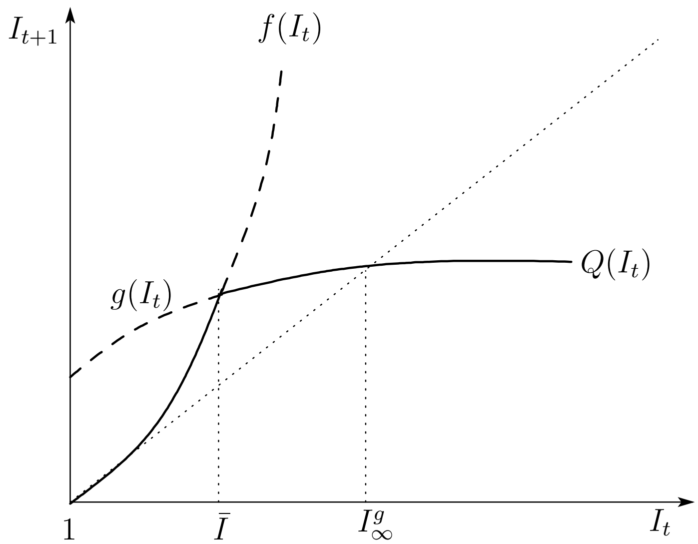

Figure 3: (b) Only free-ride equilibrium exists.

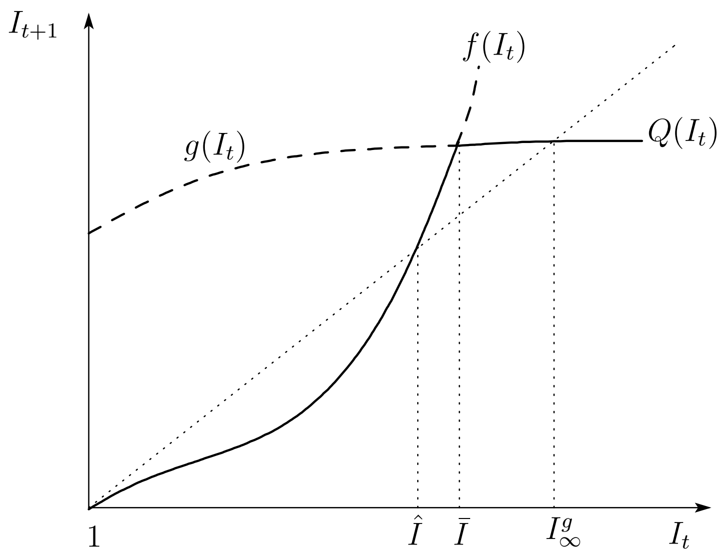

Figure 4: (c) Multiple equilibria.

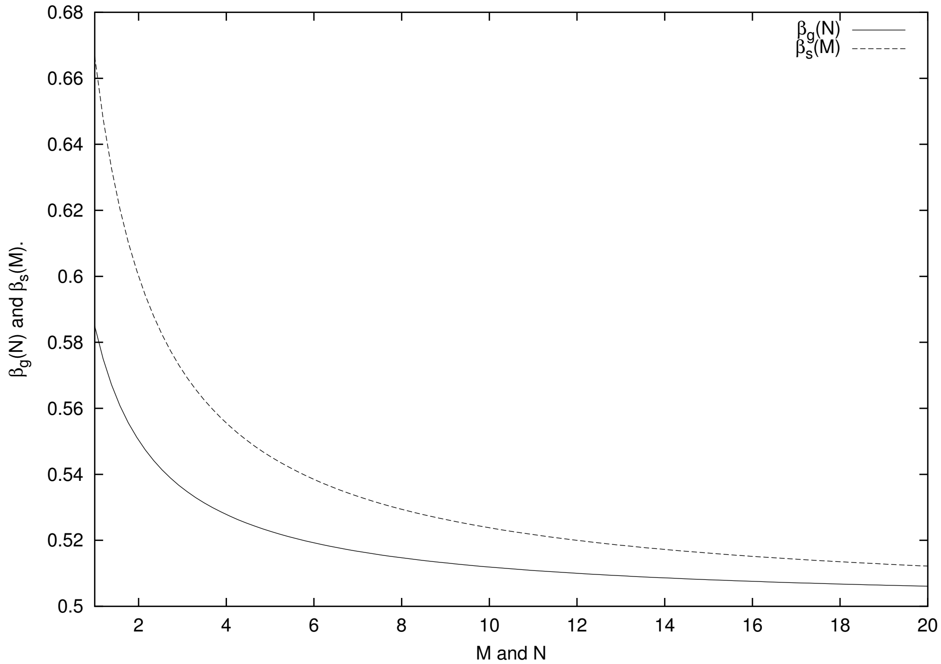

Figure 5: Critical values of β; β_g and β_s.

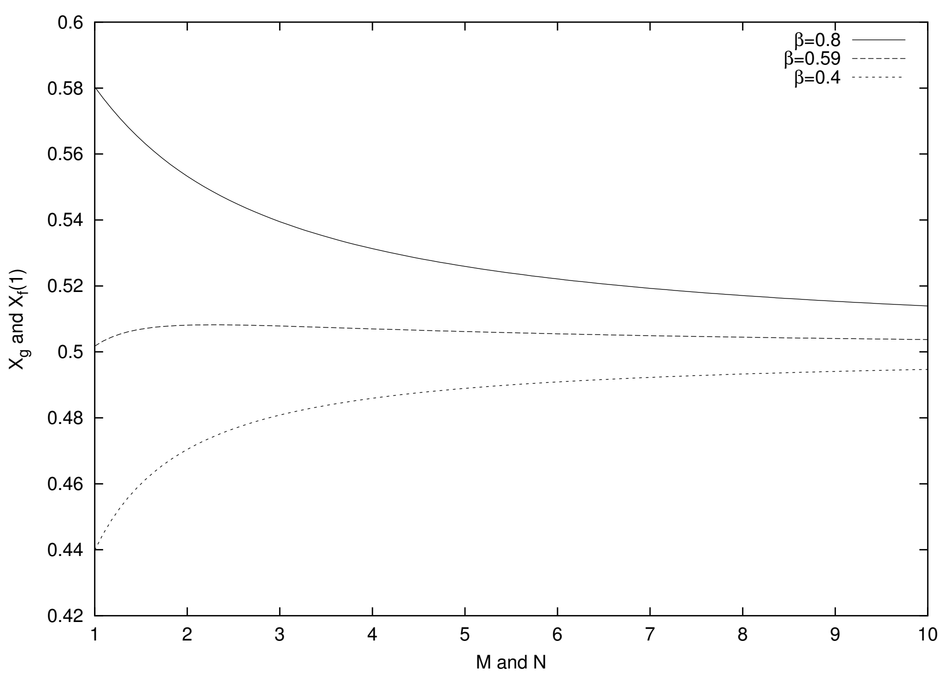

Figure 6: Growth rates; $X_f(1)$ and X_g.