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On the Degrees of Ignorance: via Epistemic Logic and μ-Calculus

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Epistemic Logic normally discourses on knowledge, belief, and related concepts. We here study ignorance instead. With the help of the μ-calculus, we analyze the degrees of ignorance in which an agent doesn't know whether or not a given proposition is true. Building on the study by Stalnaker [14], we argue that logics "closer" to S4.2 allow greater degrees of ignorance, compared to logics "closer" to S5.

We consider Epistemic Logic with the modal operators K (for knowledge) and B (for belief). We will focus on the case where there is only one agent, following Hintikka [7], Lenzen [8], and Stalnaker [14]. We suppose that K satisfies (at least) S4 and B satisfies KD45. By belief, we mean strong belief, and suppose that the agent's beliefs has no contradiction and they have introspection about their own beliefs:

$$\begin{aligned} &K\varphi \to B\varphi, \\ &B\varphi \to KB\varphi, \text{ and} \\ &\neg B\varphi \to K\neg B\varphi. \end{aligned}$$

Lenzen [8] showed that the interaction axioms above imply that $B\varphi \leftrightarrow \hat{K}K\varphi$, that is, belief can be defined by knowledge. In practice, we consider B as a defined modality. Lenzen's proof also implies that K satisfies S4.2.

The concept of ignorance was also studied by van der Hoek and Lomuscio [6]. They defined a modal operator for ignorance by

$$I\varphi :\leftrightarrow \neg K\varphi \wedge \neg K\neg\varphi.$$

They also define a logic for ignorance Ig and prove that it is sound and complete over the class of all frames. Fine [5] studied the n^{th}-order ignorance $I^n\varphi$, where $I^{n+1}\varphi :\leftrightarrow I(I^n\varphi)$ and $I^0\varphi :\leftrightarrow \varphi$. In particular, it is shown that $I^2\varphi$ is equivalent to the so-called Rumsfeld ignorance "the unknown unknown", $I\varphi \wedge \neg KI\varphi$. Fine also showed that, for any φ, $\neg KI^2\varphi$ is valid on any frame of S4.2. Therefore, the knowledge of second-order ignorance is unobtainable. Note that the ignorance modality I is a particular case of the contingency modality ∇ (see Montgomery and Routley [10]). ∇ is defined by

$$\nabla\varphi :\leftrightarrow \Diamond\varphi \wedge \Diamond\neg\varphi.$$

We build on Stalnaker's [14] analysis and consider the logics S4.2, S4.3, S4.3.2, S4.4, and S5 for knowledge. They are defined using the axioms in Table 1 with the necessitation rule. Aucher [2] characterized these logics by axioms relating knowledge and belief/conditional belief similar to Lenzen's characterization of S4.2.

Axiom Name	Axiom	Frame conditions
K	$K(\varphi \to \psi) \to (K\varphi \to K\psi)$	(no condition)
D	$K\varphi \to \hat{K}\varphi$	Serial
T	$K\varphi \to \varphi$	Reflexive
4	$K\varphi \to KK\varphi$	Transitive
5	$\hat{K}\varphi \to K\hat{K}\varphi$	Euclidean
.2	$\hat{K}K\varphi \to K\hat{K}\varphi$	Convergent
.3	$K(K\varphi \to \psi) \vee K(K\psi \to \varphi)$	Weakly Connected
.3.2	$(\hat{K}\varphi \wedge \hat{K}K\psi) \to K(\hat{K}\varphi \vee \psi)$	Semi-Euclidean
.4	$(\varphi \wedge \hat{K}K\varphi) \to K\varphi$	(no particular name)

Table 1: Modal axioms for K.

The (modal) μ-calculus is obtained by adding to modal logic the fixed-point operators μ and ν, for least and greatest fixed-points. The μ-formulas are generated by the grammar

$$\varphi := P \mid \neg P \mid X \mid \varphi \wedge \varphi \mid K\varphi \mid \mu X.\varphi \mid \nu X.\varphi.$$

We denote the dual operators of K and B by $\hat{K}$ and $\hat{B}$. For reasons that will become clear later, we consider only alternation-free formulas, that is, formulas with no nested alternation of μ and ν operators. More rigorously, a μ-formula is alternation-free if it has no subformula of the form $\mu X.\varphi$ (or $\nu X.\varphi$) such that φ has a subformula $\nu Y.\psi$ (or $\mu Y.\psi$) with a free occurrence of X in ψ.

The relational semantics for the μ-calculus is defined as follows. Given a model M and a μ-formula φ, we will define $\|\varphi\|^M$ to be the set of worlds w where φ holds. Propositional operators and modal operators are treated as usual. For fixed-point operators, letting $\Gamma_\varphi(X) = \|\varphi(X)\|^M$, we have

$$\|\mu X.\varphi(X)\|^M \text{ is the least fixed point of } \Gamma_\varphi, \text{ and}$$
$$\|\nu X.\varphi(X)\|^M \text{ is the greatest fixed point of } \Gamma_\varphi.$$

For an example of the use of fixed-point operators, suppose we have modality E for "everyone knows". Then a formula φ is common knowledge iff the following formula holds:

$$\nu X.(\varphi \wedge EX).$$

That is, φ is common knowledge iff it is true, everybody knows that φ is true, everybody knows that "everybody knows that φ is true", and so on.

The operators μ and ν induce a (syntactical) hierarchy of the μ-formulas, measuring the entanglement of least and greatest fixed-point operators. Bradfield [4] showed that, in general, the hierarchy is strict: for all natural number

n there is a formula with alternation depth $n+1$ which is not equivalent to any formula of alteration depth n. But this strictness may fail in a restricted class of models. In fact, Alberucci and Facchini [1] showed that on a frame satisfying S4, the hierarchy collapses to its alternation-free fragment: every μ-formula is equivalent to an alternation-free μ-formula. This justifies our restriction to alternation-free formulas in the definition of our μ-calculus. They also showed that the hierarchy collapses to modal logic on frames of S5: every μ-formula is equivalent to a modal formula. Also note that we can define a *weak* alternation hierarchy on the alternation-free fragment. The authors have shown the strictness of the weak alternation hierarchy on recursive frames [11].

In [12], the authors show that the alternation hierarchy collapses to its alternation-free fragment over frames of

S4.2 and S4.3;

and collapses to modal logic over frames of

S4.3.2, S4.4 and KD45.

Therefore there must be an (alternation-free) formula φ which is not equivalent to any modal formula over S4.2 and S4.3; but *is* equivalent to a modal formula over S4.3.2, S4.4 and KD45. While this abstract uses only relational semantics, the collapses of the alternation hierarchy can be transferred to topological semantics by a result of Baltag *et al.* [3].

We analyze a formula which is not equivalent to any modal formula over S4.2 and S4.3. Let φ be any μ-formula, and define

$$\alpha_\varphi(X) := \hat{K}(\varphi \wedge X) \wedge \hat{K}(\neg\varphi \wedge X).$$

We study $\alpha^\infty_\varphi := \nu X.\alpha_\varphi$ and its approximants $\alpha^1_\varphi := \alpha_\varphi(T)$, $\alpha^{n+1}_\varphi := \alpha_\varphi(\alpha^n_\varphi)$; they will be used to measure the agent's degree of ignorance with respect to φ. Each α^i_φ will represent a degree of ignorance. Over S4.2, any degree implies the weaker degrees but the converse may not hold. That is, if $i, j \in \mathbb{N} \cup \{\infty\}$ and $i < j$, then α^j_φ implies α^i_φ; and the converse doesn't hold as $\alpha^i_\varphi \wedge \neg\alpha^j_\varphi$ is satisfiable. Therefore we have, in general, infinitely many degrees of ignorance. Our first degree of ignorance α^1_φ is equivalent to $I\varphi$, and all the α^i_φ can be thought of as generalizations of $I\varphi$.

Van der Hoek and Lomuscio [6] state that the ignorance modality I is not intended to capture degrees of ignorance, while our α^i_φ's are intended to do so. Furthermore, $K\alpha^i_\varphi$ is satisfiable for any $i \in \mathbb{N} \cup \{\infty\}$. Therefore the α^i_φ are different from second-order ignorance $I^2\varphi$, and not obtainable by iterations of I.

If we change our settings, we may have finitely many degrees of ignorance. Consider S4.4, the logic of knowledge as true belief. We can show here that $\alpha^1_\varphi \wedge \neg\alpha^2_\varphi$ is equivalent to the agent having a false belief and that α^2_φ is equivalent to α^∞_φ. That is, we have only two non-equivalent degrees of ignorance: α^1_φ, where the agent's belief is false; and α^2_φ, where the agent believes neither φ nor $\neg\varphi$.

The same analysis holds for S4.3.2, which has the same two degrees. This logic is used, for example, in [13].

In S5, the standard logic for multi-agent epistemic logic, we have only one degree of ignorance. In this setting, α^1_φ is equivalent to α^∞_φ. We also have that belief is equivalent to knowledge, $B\varphi \leftrightarrow K\varphi$, so the agent has no wrong beliefs, and being ignorant of φ also means that they believe neither φ nor $\neg\varphi$.

Now consider the interpretation of $\alpha^i_\varphi \wedge \neg\alpha^{i+1}_\varphi$ over S4.2 and S4.3. S4.2 is the logic of knowledge according to Lenzen [8] and Stalnaker [14]. S4.3 is Lehrer and Paxon's undefeated justified true belief [9]. Here, $\alpha^1_\varphi \wedge \neg\alpha^2_\varphi$ is equivalent to the agent's belief being false and the agent knowing whether φ holds in every world other than the real world. Likewise, $\alpha^2_\varphi \wedge \neg\alpha^3_\varphi$ holds exactly when the agent has a true belief but considers it possible that their belief is false. Symbolically,

$$\alpha^2_\varphi \wedge \neg\alpha^3_\varphi \equiv [\varphi \wedge B\varphi \wedge \hat{K}(\neg\varphi \wedge B\varphi)] \vee [\neg\varphi \wedge B\neg\varphi \wedge \hat{K}(\varphi \wedge B\neg\varphi)].$$

This analysis can be extended to other $\alpha^i_\varphi \wedge \neg\alpha^{i+1}_\varphi$ to show that each degree of ignorance expresses a higher level of the agent's self-doubt.

Still in S4.2 and S4.3, note that, for $n \in \mathbb{N}$, α^n_φ implies the agent has a belief (which may be true or false) and the agent not having a belief implies α^∞_φ. In other words, having no belief implies a high degree of ignorance, but a high degree of ignorance does not deny the agent having a belief.

From the point of view of our degrees of ignorance, S4.3 and S4.3.2 are very different: S4.3 has infinitely many degrees of ignorance; while S4.3.2 has only two degrees. This contrasts with Stalnaker's critics of S4.3 and S4.3.2, which argues that in both logics false belief can deny knowledge: in S4.3 a false belief can deny some knowledge the agent may be justified in having; and in S4.3.2 a false belief denies all non-trivial knowledge.

At last, we can do a similar analysis to belief, using

$$\delta_\varphi(X) := \hat{B}(\varphi \wedge X) \wedge \hat{B}(\neg\varphi \wedge X),$$

a belief variant of $\alpha_\varphi(X)$. Define $\delta^1_\varphi := \delta_\varphi(\top)$, $\delta^{n+1}_\varphi := \delta_\varphi(\delta^n_\varphi)$ and $\delta^\infty = \nu X.\delta(X)$. Then, for $i \in \mathbb{N} \cup \{\infty\}$, δ^1_φ is equivalent to δ^i_φ over KD45, similar to the case where knowledge satisfies S5. Therefore we can only define one degree of disbelief by our approach.

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