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# On the Degrees of Ignorance: via Epistemic Logic and $\mu$ -Calculus

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Epistemic Logic normally discourses on knowledge, belief, and related concepts. We here study ignorance instead. With the help of the  $\mu$ -calculus, we analyze the degrees of ignorance in which an agent doesn't know whether or not a given proposition is true. Building on the study by Stalnaker [14], we argue that logics “closer” to **S4.2** allow greater degrees of ignorance, compared to logics “closer” to **S5**.

We consider Epistemic Logic with the modal operators  $K$  (for knowledge) and  $B$  (for belief). We will focus on the case where there is only one agent, following Hintikka [7], Lenzen [8], and Stalnaker [14]. We suppose that  $K$  satisfies (at least) **S4** and  $B$  satisfies **KD45**. By belief, we mean strong belief, and suppose that the agent's beliefs has no contradiction and they have introspection about their own beliefs:

$$\begin{aligned} K\varphi &\rightarrow B\varphi, \\ B\varphi &\rightarrow KB\varphi, \text{ and} \\ \neg B\varphi &\rightarrow K\neg B\varphi. \end{aligned}$$

Lenzen [8] showed that the interaction axioms above imply that  $B\varphi \leftrightarrow \hat{K}K\varphi$ , that is, belief can be defined by knowledge. In practice, we consider  $B$  as a defined modality. Lenzen's proof also implies that  $K$  satisfies **S4.2**.

The concept of ignorance was also studied by van der Hoek and Lomuscio [6]. They defined a modal operator for ignorance by

$$I\varphi := \leftrightarrow \neg K\varphi \wedge \neg K\neg\varphi.$$

They also define a logic for ignorance **lg** and prove that it is sound and complete over the class of all frames. Fine [5] studied the  $n^{\text{th}}$ -order ignorance  $I^n\varphi$ , where  $I^{n+1}\varphi := \leftrightarrow I(I^n\varphi)$  and  $I^0\varphi := \leftrightarrow \varphi$ . In particular, it is shown that  $I^2\varphi$  is equivalent to the so-called Rumsfeld ignorance “the unknown unknown”,  $I\varphi \wedge \neg KI\varphi$ . Fine also showed that, for any  $\varphi$ ,  $\neg KI^2\varphi$  is valid on any frame of **S4.2**. Therefore, the knowledge of second-order ignorance is unobtainable. Note that the ignorance modality  $I$  is a particular case of the contingency modality  $\nabla$  (see Montgomery and Routley [10]).  $\nabla$  is defined by

$$\nabla\varphi := \leftrightarrow \diamond\varphi \wedge \diamond\neg\varphi.$$

We build on Stalnaker’s [14] analysis and consider the logics S4.2, S4.3, S4.3.2, S4.4, and S5 for knowledge. They are defined using the axioms in Table 1 with the necessitation rule. Aucher [2] characterized these logics by axioms relating knowledge and belief/conditional belief similar to Lenzen’s characterization of S4.2.

Axiom Name	Axiom	Frame conditions
$K$	$K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$	(no condition)
$D$	$K\varphi \rightarrow \hat{K}\varphi$	Serial
$T$	$K\varphi \rightarrow \varphi$	Reflexive
4	$K\varphi \rightarrow KK\varphi$	Transitive
5	$\hat{K}\varphi \rightarrow K\hat{K}\varphi$	Euclidean
.2	$\hat{K}K\varphi \rightarrow K\hat{K}\varphi$	Convergent
.3	$K(K\varphi \rightarrow \psi) \vee K(K\psi \rightarrow \varphi)$	Weakly Connected
.3.2	$(\hat{K}\varphi \wedge \hat{K}K\psi) \rightarrow K(\hat{K}\varphi \vee \psi)$	Semi-Euclidean
.4	$(\varphi \wedge \hat{K}K\varphi) \rightarrow K\varphi$	(no particular name)

Table 1: Modal axioms for  $K$ .

The (modal)  $\mu$ -calculus is obtained by adding to modal logic the fixed-point operators  $\mu$  and  $\nu$ , for least and greatest fixed-points. The  $\mu$ -formulas are generated by the grammar

$$\varphi := P \mid \neg P \mid X \mid \varphi \wedge \psi \mid K\varphi \mid \mu X.\varphi \mid \nu X.\varphi.$$

We denote the dual operators of  $K$  and  $B$  by  $\hat{K}$  and  $\hat{B}$ . For reasons that will become clear later, we consider only alternation-free formulas, that is, formulas with no nested alternation of  $\mu$  and  $\nu$  operators. More rigorously, a  $\mu$ -formula is alternation-free if it has no subformula of the form  $\mu X.\varphi$  (or  $\nu X.\varphi$ ) such that  $\varphi$  has a subformula  $\nu Y.\psi$  (or  $\mu Y.\psi$ ) with a free occurrence of  $X$  in  $\psi$ .

The relational semantics for the  $\mu$ -calculus is defined as follows. Given a model  $M$  and a  $\mu$ -formula  $\varphi$ , we will define  $\|\varphi\|^M$  to be the set of worlds  $w$  where  $\varphi$  holds. Propositional operators and modal operators are treated as usual. For fixed-point operators, letting  $\Gamma_\varphi(X) = \|\varphi(X)\|^M$ , we have

$$\begin{aligned} \|\mu X.\varphi(X)\|^M &\text{ is the least fixed point of } \Gamma_\varphi, \text{ and} \\ \|\nu X.\varphi(X)\|^M &\text{ is the greatest fixed point of } \Gamma_\varphi. \end{aligned}$$

For an example of the use of fixed-point operators, suppose we have modality  $E$  for “everyone knows”. Then a formula  $\varphi$  is common knowledge iff the following formula holds:

$$\nu X.(\varphi \wedge EX).$$

That is,  $\varphi$  is common knowledge iff it is true, everybody knows that  $\varphi$  is true, everybody knows that “everybody knows that  $\varphi$  is true”, and so on.

The operators  $\mu$  and  $\nu$  induce a (syntactical) hierarchy of the  $\mu$ -formulas, measuring the entanglement of least and greatest fixed-point operators. Bradfield [4] showed that, in general, the hierarchy is strict: for all natural number

$n$  there is a formula with alternation depth  $n + 1$  which is not equivalent to any formula of alternation depth  $n$ . But this strictness may fail in a restricted class of models. In fact, Alberucci and Facchini [1] showed that on a frame satisfying S4, the hierarchy collapses to its alternation-free fragment: every  $\mu$ -formula is equivalent to an alternation-free  $\mu$ -formula. This justifies our restriction to alternation-free formulas in the definition of our  $\mu$ -calculus. They also showed that the hierarchy collapses to modal logic on frames of S5: every  $\mu$ -formula is equivalent to a modal formula. Also note that we can define a *weak* alternation hierarchy on the alternation-free fragment. The authors have shown the strictness of the weak alternation hierarchy on recursive frames [11].

In [12], the authors show that the alternation hierarchy collapses to its alternation-free fragment over frames of

S4.2 and S4.3;

and collapses to modal logic over frames of

S4.3.2, S4.4 and KD45.

Therefore there must be an (alternation-free) formula  $\varphi$  which is not equivalent to any modal formula over S4.2 and S4.3; but *is* equivalent to a modal formula over S4.3.2, S4.4 and KD45. While this abstract uses only relational semantics, the collapses of the alternation hierarchy can be transferred to topological semantics by a result of Baltag *et al.* [3].

We analyze a formula which is not equivalent to any modal formula over S4.2 and S4.3. Let  $\varphi$  be any  $\mu$ -formula, and define

$$\alpha_\varphi(X) := \hat{K}(\varphi \wedge X) \wedge \hat{K}(\neg\varphi \wedge X).$$

We study  $\alpha_\varphi^\infty := \nu X.\alpha_\varphi$  and its approximants  $\alpha_\varphi^1 := \alpha_\varphi(T)$ ,  $\alpha_\varphi^{n+1} := \alpha_\varphi(\alpha_\varphi^n)$ ; they will be used to measure the agent's degree of ignorance with respect to  $\varphi$ . Each  $\alpha_\varphi^i$  will represent a degree of ignorance. Over S4.2, any degree implies the weaker degrees but the converse may not hold. That is, if  $i, j \in \mathbb{N} \cup \{\infty\}$  and  $i < j$ , then  $\alpha_\varphi^j$  implies  $\alpha_\varphi^i$ ; and the converse doesn't hold as  $\alpha_\varphi^i \wedge \neg\alpha_\varphi^j$  is satisfiable. Therefore we have, in general, infinitely many degrees of ignorance. Our first degree of ignorance  $\alpha_\varphi^1$  is equivalent to  $I\varphi$ , and all the  $\alpha_\varphi^i$  can be thought of as generalizations of  $I\varphi$ .

Van der Hoek and Lomuscio [6] state that the ignorance modality  $I$  is not intended to capture degrees of ignorance, while our  $\alpha_\varphi^i$ 's are intended to do so. Furthermore,  $K\alpha_\varphi^i$  is satisfiable for any  $i \in \mathbb{N} \cup \{\infty\}$ . Therefore the  $\alpha_\varphi^i$  are different from second-order ignorance  $I^2\varphi$ , and not obtainable by iterations of  $I$ .

If we change our settings, we may have finitely many degrees of ignorance. Consider S4.4, the logic of knowledge as true belief. We can show here that  $\alpha_\varphi^1 \wedge \neg\alpha_\varphi^2$  is equivalent to the agent having a false belief and that  $\alpha_\varphi^2$  is equivalent to  $\alpha_\varphi^\infty$ . That is, we have only two non-equivalent degrees of ignorance:  $\alpha_\varphi^1$ , where the agent's belief is false; and  $\alpha_\varphi^2$ , where the agent believes neither  $\varphi$  nor  $\neg\varphi$ .

The same analysis holds for S4.3.2, which has the same two degrees. This logic is used, for example, in [13].

In S5, the standard logic for multi-agent epistemic logic, we have only one degree of ignorance. In this setting,  $\alpha_\varphi^1$  is equivalent to  $\alpha_\varphi^\infty$ . We also have that belief is equivalent to knowledge,  $B\varphi \leftrightarrow K\varphi$ , so the agent has no wrong beliefs, and being ignorant of  $\varphi$  also means that they believe neither  $\varphi$  nor  $\neg\varphi$ .

Now consider the interpretation of  $\alpha_\varphi^i \wedge \neg\alpha_\varphi^{i+1}$  over S4.2 and S4.3. S4.2 is the logic of knowledge according to Lenzen [8] and Stalnaker [14]. S4.3 is Lehrer and Paxson's undefeated justified true belief [9]. Here,  $\alpha_\varphi^1 \wedge \neg\alpha_\varphi^2$  is equivalent to the agent's belief being false and the agent knowing whether  $\varphi$  holds in every world other than the real world. Likewise,  $\alpha_\varphi^2 \wedge \neg\alpha_\varphi^3$  holds exactly when the agent has a true belief but considers it possible that their belief is false. Symbolically,

$$\alpha_\varphi^2 \wedge \neg\alpha_\varphi^3 \equiv [\varphi \wedge B\varphi \wedge \hat{K}(\neg\varphi \wedge B\varphi)] \vee [\neg\varphi \wedge B\neg\varphi \wedge \hat{K}(\varphi \wedge B\neg\varphi)].$$

This analysis can be extended to other  $\alpha_\varphi^i \wedge \neg\alpha_\varphi^{i+1}$  to show that each degree of ignorance expresses a higher level of the agent's self-doubt.

Still in S4.2 and S4.3, note that, for  $n \in \mathbb{N}$ ,  $\alpha_\varphi^n$  implies the agent has a belief (which may be true or false) and the agent not having a belief implies  $\alpha_\varphi^\infty$ . In other words, having no belief implies a high degree of ignorance, but a high degree of ignorance does not deny the agent having a belief.

From the point of view of our degrees of ignorance, S4.3 and S4.3.2 are very different: S4.3 has infinitely many degrees of ignorance; while S4.3.2 has only two degrees. This contrasts with Stalnaker's critics of S4.3 and S4.3.2, which argues that in both logics false belief can deny knowledge: in S4.3 a false belief can deny some knowledge the agent may be justified in having; and in S4.3.2 a false belief denies all non-trivial knowledge.

At last, we can do a similar analysis to belief, using

$$\delta_\varphi(X) := \hat{B}(\varphi \wedge X) \wedge \hat{B}(\neg\varphi \wedge X),$$

a belief variant of  $\alpha_\varphi(X)$ . Define  $\delta_\varphi^1 := \delta_\varphi(\top)$ ,  $\delta_\varphi^{n+1} := \delta_\varphi(\delta_\varphi^n)$  and  $\delta^\infty = \nu X.\delta(X)$ . Then, for  $i \in \mathbb{N} \cup \{\infty\}$ ,  $\delta_\varphi^1$  is equivalent to  $\delta_\varphi^i$  over KD45, similar to the case where knowledge satisfies S5. Therefore we can only define one degree of disbelief by our approach.

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