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# Craig Interpolation for a Sequent Calculus for Combining Intuitionistic and Classical Propositional Logic

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## 1 Introduction

This paper establishes the Craig interpolation for a multi-succedent sequent calculus for a combination of intuitionistic and classical propositional logic, denoted by  $G(\mathbf{C} + \mathbf{J})$ . The calculus was provided in [16] and is based on the semantics offered in [4, 5]. The logic, called  $\mathbf{C} + \mathbf{J}$ , has two implications: intuitionistic and classical one<sup>1</sup>. They are interpreted in the Kripke semantics as follows (cf. [4, 5]):

$$\begin{aligned} w \models_M A \rightarrow_i B & \text{ iff } \text{ for all } v \in W, (wRv \text{ and } v \models_M A \text{ jointly imply } v \models_M B), \\ w \models_M A \rightarrow_c B & \text{ iff } w \models_M A \text{ implies } w \models_M B, \end{aligned}$$

where  $M$  is an intuitionistic Kripke model,  $w$  is a possible world in  $M$ , and  $R$  is a preorder equipped in  $M$ . However this semantic treatment breaks one feature of intuitionistic logic called *heredity*, which is defined as:  $w \models A$  and  $wRv$  jointly imply  $v \models A$  for all Kripke models  $M$  and all states  $w$  and  $v$  in  $M$ . It is a well-known fact that this feature corresponds to an intuitionistically valid formula  $A \rightarrow_i (B \rightarrow_i A)$ . Therefore, the formula is not valid in the Kripke semantics of  $\mathbf{C} + \mathbf{J}$ . In order to avoid the formula being derivable in  $G(\mathbf{C} + \mathbf{J})$ , the right rule for the intuitionistic implication should be restricted as follows:

$$\frac{A, C_1 \rightarrow_i D_1, \dots, C_m \rightarrow_i D_m, p_1, \dots, p_n \Rightarrow B}{C_1 \rightarrow_i D_1, \dots, C_m \rightarrow_i D_m, p_1, \dots, p_n \Rightarrow A \rightarrow_i B} (\Rightarrow \rightarrow_i).$$

The resulting calculus is sound and complete and a conservative extension of both an intuitionistic and a classical propositional sequent calculus (see [16]).

It is well-known that classical propositional logic and intuitionistic propositional logic enjoy the Craig interpolation theorem:

If  $A \rightarrow B$  is derivable, then there exists a formula  $C$  such that both  $\Rightarrow A \rightarrow C$  and  $\Rightarrow C \rightarrow B$  are also derivable and that  $\text{Prop}(C) \subseteq \text{Prop}(A) \cap \text{Prop}(B)$ ,

where  $\text{Prop}(D)$  denotes the set of all propositional variables in a formula  $D$ . The theorem can be shown in terms of a classical sequent calculus  $\mathbf{LK}$  by Maehara's method in [9]. In multi-succedent intuitionistic sequent calculus  $\mathbf{mLJ}$ , the theorem can also be shown, though some modification of the ways is needed, as is noted in [10]. Since  $\mathbf{C} + \mathbf{J}$  contains the two kinds of implication, the two types of Craig interpolation theorem can be considered in  $G(\mathbf{C} + \mathbf{J})$ .

<sup>1</sup>In addition to  $\mathbf{C} + \mathbf{J}$ , other attempts to combine intuitionistic and classical logic are displayed in [1, 2, 3, 6, 7, 11, 12, 13, 14].

## 2 Syntax, Kripke Semantics and Sequent Calculus

### 2.1 Syntax and Kripke Semantics

This section reviews the syntax and the Kripke semantics of  $\mathbf{C} + \mathbf{J}$ . The syntax is defined in [16], and the Kripke semantics is based on the ones in [4, 5]. The syntax  $\mathcal{L}$  consists of a countably infinite set  $\text{Prop}$  of propositional variables and the following logical connectives: falsum  $\perp$ , disjunction  $\vee$ , conjunction  $\wedge$ , intuitionistic implication  $\rightarrow_i$ , and classical implication  $\rightarrow_c$ . The set  $\text{Form}$  of all formulas in our syntax is defined inductively as follows:

$$A ::= p \mid \perp \mid A \vee A \mid A \wedge A \mid A \rightarrow_i A \mid A \rightarrow_c A,$$

where  $p \in \text{Prop}$ . We define  $\top := \perp \rightarrow_i \perp$ ,  $\neg_c A := A \rightarrow_c \perp$  and  $\neg_i A := A \rightarrow_i \perp$ .

Let us move to the semantics for the syntax  $\mathcal{L}$ .

**Definition 1.** A *model* is a tuple  $M = (W, R, V)$  where

- $W$  is a non-empty set of possible worlds,
- $R$  is a preorder on  $W$ , i.e.,  $R$  satisfies reflexivity and transitivity,
- $V : \text{Prop} \rightarrow \mathcal{P}(W)$  is a valuation function satisfying the following *heredity* condition:  $w \in V(p)$  and  $wRv$  jointly imply  $v \in V(p)$  for all worlds  $w, v \in W$ .

**Definition 2.** Given a model  $M = (W, R, V)$ , a world  $w \in W$  and a formula  $A$ , the *satisfaction relation*  $w \models_M A$  is inductively defined as follows:

$$\begin{aligned} w \models_M p & \quad \text{iff} \quad w \in V(p), \\ w \not\models_M \perp, \\ w \models_M A \wedge B & \quad \text{iff} \quad w \models_M A \text{ and } w \models_M B, \\ w \models_M A \vee B & \quad \text{iff} \quad w \models_M A \text{ or } w \models_M B, \\ w \models_M A \rightarrow_i B & \quad \text{iff} \quad \text{for all } v \in W, (wRv \text{ and } v \models_M A \text{ jointly imply } v \models_M B). \\ w \models_M A \rightarrow_c B & \quad \text{iff} \quad w \models_M A \text{ implies } w \models_M B. \end{aligned}$$

Let us say that a formula  $A$  is a *semantic consequence* of a set of formulas  $\Gamma$ , represented as  $\Gamma \models A$ , if  $w \models_M C$  for any formula  $C \in \Gamma$ , then  $w \models_M A$  for all models  $M = (W, R, V)$  and all worlds  $w \in W$ . We use  $\Gamma \models \Delta$  if  $\Gamma \models A$  for some formula  $A \in \Delta$ . We say that  $A$  is *valid* if  $\emptyset \models A$  holds. We say a formula  $A$  satisfies *heredity* if the following holds:  $w \models A$  and  $wRv$  jointly imply  $v \models A$  for all Kripke models  $M$  and all states  $w$  and  $v$  in  $M$ .

**Proposition 1.** A formula  $\neg_c p$  does not satisfy heredity.

**Proposition 2.** Neither  $\neg_c p \rightarrow_i (\top \rightarrow_i \neg_c p)$  nor  $\neg_c p \rightarrow_c (\top \rightarrow_i \neg_c p)$  is valid.

Proposition 2 implies that an intuitionistic tautology  $A \rightarrow_i (B \rightarrow_i A)$ , which is known for the correspondence to heredity in intuitionistic logic, is no longer valid.

### 2.2 Multi-succedent sequent calculus $\mathbf{G}(\mathbf{C} + \mathbf{J})$

This section reviews the sequent calculus  $\mathbf{G}(\mathbf{C} + \mathbf{J})$  provided in [16]. In what follows, we use the ordinary notion of multi-succedent sequent. A *sequent* is a pair of finite multisets denoted by  $\Gamma \Rightarrow \Delta$ , which is read as ‘‘if all formulas in  $\Gamma$  hold then some formulas in  $\Delta$  hold.’’ Table 1 provides our multi-succedent sequent calculus  $\mathbf{G}(\mathbf{C} + \mathbf{J})$ , where the notion of derivability is defined as an existence of a

Table 1: Sequent Calculus  $G(\mathbf{C} + \mathbf{J})$ **Axioms**

$$\frac{}{A \Rightarrow A} (Id) \quad \frac{}{\perp \Rightarrow} (\perp)$$

**Structural Rules**

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} (\Rightarrow w) \quad \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} (w \Rightarrow) \quad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A} (\Rightarrow c) \quad \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} (c \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} (Cut)$$

**Propositional Logical Rules**

$$\frac{A, C_1 \rightarrow_i D_1, \dots, C_m \rightarrow_i D_m, p_1, \dots, p_n \Rightarrow B}{C_1 \rightarrow_i D_1, \dots, C_m \rightarrow_i D_m, p_1, \dots, p_n \Rightarrow A \rightarrow_i B} (\Rightarrow \rightarrow_i) \quad \frac{\Gamma_1 \Rightarrow \Delta_1, A \quad B, \Gamma_2 \Rightarrow \Delta_2}{A \rightarrow_i B, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} (\rightarrow_i \Rightarrow)$$

$$\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow_c B} (\Rightarrow \rightarrow_c) \quad \frac{\Gamma_1 \Rightarrow \Delta_1, A \quad B, \Gamma_2 \Rightarrow \Delta_2}{A \rightarrow_c B, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} (\rightarrow_c \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} (\Rightarrow \wedge) \quad \frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} (\wedge \Rightarrow_1) \quad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} (\wedge \Rightarrow_2)$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B} (\Rightarrow \vee_1) \quad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B} (\Rightarrow \vee_2) \quad \frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} (\vee \Rightarrow)$$

finite tree, which is called a *derivation*, generated by inference rules of Table 1 from initial sequents  $(Id)$  and  $(\perp)$  of Table 1.

Our basic strategy of constructing  $G(\mathbf{C} + \mathbf{J})$  is to add classical implication to the propositional fragment of multi-succedent sequent calculus  $\mathbf{mLJ}$  of intuitionistic propositional logic, proposed by Maehara [8]. However, if the ordinary left and right rules of classical implication were added, the soundness of the resulting calculus would fail, because a formula  $\neg_c p \rightarrow_c (\top \rightarrow_i \neg_c p)$ , which is not valid by Proposition 2, would be derivable. This is the reason why the original right rule

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow_i B}$$

of intuitionistic implication of  $\mathbf{mLJ}$  is restricted to the right rule given in Table 1. Based on the abbreviation defined in Section 2.1, the following rules for negations are obtained respectively:

$$\frac{A, C_1 \rightarrow_i D_1, \dots, C_m \rightarrow_i D_m, p_1, \dots, p_n \Rightarrow}{C_1 \rightarrow_i D_1, \dots, C_m \rightarrow_i D_m, p_1, \dots, p_n \Rightarrow \neg_i A} (\Rightarrow \neg_i) \quad \frac{\Gamma \Rightarrow \Delta, A}{\neg_i A, \Gamma \Rightarrow \Delta} (\neg_i \Rightarrow)$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg_c A, \Delta} (\Rightarrow \neg_c) \quad \frac{\Gamma \Rightarrow \Delta, A}{\neg_c A, \Gamma \Rightarrow \Delta} (\neg_c \Rightarrow).$$

**Proposition 3.** For any  $\Gamma \cup \Delta \subseteq \text{Form}$ ,  $\Gamma \Rightarrow \Delta$  is derivable in  $G(\mathbf{C} + \mathbf{J})$  iff  $\Gamma \models \Delta$  holds.

**Proposition 4.** If  $\Gamma \Rightarrow \Delta$  is derivable in  $G(\mathbf{C} + \mathbf{J})$ , then  $\Gamma \Rightarrow \Delta$  is derivable in  $G^-(\mathbf{C} + \mathbf{J})$ , where  $G^-(\mathbf{C} + \mathbf{J})$  is the calculus obtained by removing the rule  $(Cut)$  from  $G(\mathbf{C} + \mathbf{J})$ .

By Proposition 4, the subformula property is obtained, which ensures the calculus is a conservative extension of both intuitionistic and classical propositional logic.

### 3 Craig Interpolation

In this section, we establish two types of Craig interpolation theorem for  $G(\mathbf{C} + \mathbf{J})$ , based on Maehara's partition argument in [9]. This argument is originally for classical sequent calculus  $\mathbf{LK}$ , and is dependent on the fact that the cut elimination holds in the calculus. Since cut elimination holds also in  $G(\mathbf{C} + \mathbf{J})$ , as is guaranteed by Proposition 4, this method can be employed. In the following part of this section,  $\text{Prop}(D)$  denotes the set of all propositional variables in a formula  $D$ . And if  $\Gamma$  is a finite multiset of formulas, we define  $\text{Prop}(\Gamma) = \bigcup\{\text{Prop}(D) \mid D \in \Gamma\}$ . Especially, we have  $\text{Prop}(\perp) = \emptyset$ . We call  $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$  a *partition* of a sequent  $\Gamma \Rightarrow \Delta$ , if  $\Gamma$  is  $\Gamma_1, \Gamma_2$  and  $\Delta$  is  $\Delta_1, \Delta_2$ . Let us say that  $C$  is an *interpolant* of  $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$  if  $\Gamma_1 \Rightarrow \Delta_1, C$  and  $C, \Gamma_2 \Rightarrow \Delta_2$  are derivable and  $\text{Prop}(C) \subseteq \text{Prop}(\Gamma_1, \Delta_1) \cap \text{Prop}(\Gamma_2, \Delta_2)$ .

Although the main idea of giving  $G(\mathbf{C} + \mathbf{J})$  is adding classical implication to intuitionistic logic, our proof is similar to that in classical logic. For establishing the Craig interpolation theorem for  $\mathbf{mLJ}$ , we cannot employ the notion of partition of the form  $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$ . This is because we cannot find an interpolant for  $\langle(\emptyset : A); (A : \emptyset)\rangle$  as noted in [10]. Therefore, in order to show the theorem for  $\mathbf{mLJ}$ , the form of a partition should be restricted to  $\langle(\Gamma_1 : \emptyset); (\Gamma_2 : \Delta)\rangle$ . However, this restriction makes it possible to show neither of the two types of theorem in  $G(\mathbf{C} + \mathbf{J})$ . Considering this situation, it seems difficult to establish the theorem for  $G(\mathbf{C} + \mathbf{J})$ . However, the classical negation (or implication) enables us to use partitions of the form  $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$  without any restriction to calculate an interpolant by Maehara method. This fact about the way of showing Craig interpolation theorem implies that  $\mathbf{C} + \mathbf{J}$  can be regarded as the logic obtained by adding the special (intuitionistic) implication to classical logic<sup>2</sup>.

**Lemma 1.** Suppose that  $\Gamma \Rightarrow \Delta$  is derivable in  $G(\mathbf{C} + \mathbf{J})$ . Then for any partition  $\langle(\Gamma_1 : \Delta_1); (\Gamma_2 : \Delta_2)\rangle$  of the sequent, there exists an interpolant  $C$  in  $G(\mathbf{C} + \mathbf{J})$ , i.e., such that both  $\Gamma_1 \Rightarrow \Delta_1, C$  and  $C, \Gamma_2 \Rightarrow \Delta_2$  are also derivable in  $G(\mathbf{C} + \mathbf{J})$ , and  $\text{Prop}(C) \subseteq \text{Prop}(\Gamma_1, \Delta_1) \cap \text{Prop}(\Gamma_2, \Delta_2)$ .

With Lemma 1, which is the core of the proof, we can easily show the following two types of Craig interpolation theorem.

**Theorem 1.** (Intuitionistic Craig Interpolation Theorem of  $G(\mathbf{C} + \mathbf{J})$ ). If  $\Rightarrow A \rightarrow_i B$  is derivable in  $G(\mathbf{C} + \mathbf{J})$ , then there exists a formula  $C$  such that  $\Rightarrow A \rightarrow_i C$  and  $\Rightarrow C \rightarrow_i B$  are also derivable in  $G(\mathbf{C} + \mathbf{J})$  and that  $\text{Prop}(C) \subseteq \text{Prop}(A) \cap \text{Prop}(B)$ .

**Theorem 2.** (Classical Craig Interpolation Theorem of  $G(\mathbf{C} + \mathbf{J})$ ). If  $\Rightarrow A \rightarrow_c B$  is derivable in  $G(\mathbf{C} + \mathbf{J})$ , then there exists a formula  $C$  such that  $\Rightarrow A \rightarrow_c C$  and  $\Rightarrow C \rightarrow_c B$  are also derivable in  $G(\mathbf{C} + \mathbf{J})$  and that  $\text{Prop}(C) \subseteq \text{Prop}(A) \cap \text{Prop}(B)$ .

### 4 Further Direction

In [15], the first-order expansion  $G(\mathbf{FOC} + \mathbf{J})$  of  $G(\mathbf{C} + \mathbf{J})$  can be given by adding classical universal quantifier to first-order multi-succedent intuitionistic sequent calculus  $\mathbf{mLJ}$ , although the similar restriction on the right rule for the intuitionistic universal quantifier is needed. Whether Craig interpolation holds in this expansion is an open question, which deserves being inquired.

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<sup>2</sup>This interpretation of  $\mathbf{C} + \mathbf{J}$  was already noted in [4].

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