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Sharp-Interface Limit of the Allen-Cahn Action Functional in One Space Dimension

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Abstract

We analyze the sharp-interface limit of the action minimization problem for the stochastically perturbed Allen-Cahn equation in one space dimension. The action is a deterministic functional which is linked to the behavior of the stochastic process in the small noise limit. Previously, heuristic arguments and numerical results have suggested that the limiting action should "count" two competing costs: the cost to nucleate interfaces and the cost to propagate them. In addition, constructions have been used to derive an upper bound for the minimal action which was proved optimal on the level of *scaling*. In this paper, we prove that for $d = 1$, the upper bound achieved by the constructions is in fact sharp. Furthermore, we derive a lower bound for the functional itself, which is in agreement with the heuristic picture. To do so, we characterize the sharp-interface limit of the space-time energy measures. The proof relies on an extension of earlier results for the related elliptic problem.

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Contents

1 Introduction

It is a well-known result of Modica and Mortola (see, for instance, [17, 18]) that the Ginzburg Landau energy

$$E_\varepsilon[u] := \int_\Omega \left(\frac{\varepsilon}{2}(u_x)^2 + \frac{V(u)}{\varepsilon} \right) \, dx \tag{1.1}$$

Γ-converges to the perimeter functional as $\varepsilon \to 0$. Our goal is to understand a similar phenomenon, namely, the sharp-interface limit of the Allen-Cahn action minimization problem:

$$\inf_{\substack{u(\cdot,0)=-1 \\ u(\cdot,T)=+1}} S_\varepsilon[u] =: S_\varepsilon, \tag{1.2}$$

where $S_\varepsilon[u]$ is the action functional,

$$S_\varepsilon[u] := \frac{1}{4} \int_0^T \int_0^1 \left(\varepsilon^{1/2}\dot{u} + \varepsilon^{-1/2}(-\varepsilon u_{xx} + \varepsilon^{-1}V'(u)) \right)^2 \, dx \, dt. \tag{1.3}$$

Although action minimization is different from energy minimization, the problems are related; see (1.10) and the accompanying discussion, below. For simplicity, we restrict attention throughout the paper to the standard potential, $V(u) = (1 - u^2)^2/4$; generalization to other nondegenerate, equal-well potentials is possible.

The scientific motivation for studying this problem has to do with probabilistic estimates for the stochastically perturbed Allen-Cahn equation, as we will soon explain. Our method, on the other hand, has close links to the Modica-Mortola problem and a long standing conjecture of DeGiorgi. Although we analyze the action problem in one space dimension, the essential ingredient in any dimension is the connection between bounded action and the behavior of the limiting energy measures. In Subsection 1.3, we comment on the higher dimensional problem and links to recent work by Bellettini and Mugnai [1] and Moser [19].

The stochastically perturbed Allen-Cahn equation on $[0,1] \times [0,T]$ is

$$\dot{u} = \varepsilon u_{xx} - \varepsilon^{-1} V'(u) + \sqrt{2\gamma}\, \eta, \tag{1.4}$$

where η is a space-time white noise. Under the influence of stochastic perturbations, "rare events" which are never seen in the deterministic setting become possible. For instance, the deterministic Allen-Cahn equation (set $\gamma = 0$ in (1.4)) is the L^2-gradient flow for the energy (1.1). The energy has two global minimizers, $u_- \equiv -1$ and $u_+ \equiv +1$ which are, of course, stable under the deterministic gradient flow. Under the influence of a small stochastic forcing, however, even if $u(\cdot, 0) = u_-$, it is almost certain that the process eventually surmounts the energy barrier which surrounds it and switches to a neighborhood of the other minimizer, u_+. Thus, the switching problem for Allen-Cahn involves a barrier-crossing event in function space.

Questions about rare events - such as the mean time for occurrence, the probability of occurrence within time T, and the typical mechanism by which they occur - can be studied using a large deviation action functional [12]. Faris and Jona-Lasinio [10] proved that $S_\varepsilon[u]$ is the action functional for (1.4). This implies, for instance, that the exponential factor in the probability of switching is controlled by (1.2) when the noise is small. Furthermore, the minimizer of the action functional is the "most likely switching path," meaning that when the stochastic process undergoes switching, its pathway through configuration space stays within an arbitrarily small neighborhood of the action minimizing path, with probability one in the zero-noise limit. See [16] for additional discussion.

The sharp-interface limit. It is well-known [12, 10] that for fixed ε and $T \to \infty$, the most likely pathway consists of following (in reverse time) the
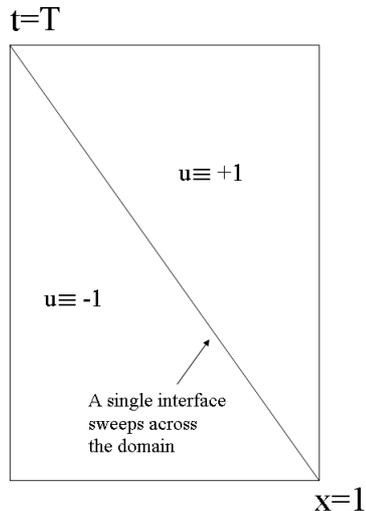
3

Figure 1: When the domain is sufficiently small, the sharp-interface action minimizer nucleates a single interface at time $t = 0$ and propagates it across the domain at a constant velocity.

deterministic orbit connecting u_- and the minimum energy saddle point, followed by the downhill flow from the saddle to u_+. The sharp-interface action minimizing pathway, on the other hand, is forced to switch at a faster rate. A host of different short-time action minimizers are discovered in [9] (in one and two space dimensions), through numerical investigations and partially heuristic arguments. The conclusion is that constraining the system to switch at a faster rate generates a competition between nucleation costs and propagation costs in the action. Nucleation costs dominate when the spatial scale is small compared to the time scale; in this case, the minimizing pathway (for Neumann boundary conditions) consists of the generation of a single wall at one edge of the domain, which then sweeps across the domain (see Figure 1). For larger domains, propagation costs become important, and minimizers generate additional spatial structure in order to move the ± 1 interfaces across the domain quickly enough. Figure 2 illustrates a sharp-interface switching path in which two pairs of interior walls are generated. Similar conclusions are drawn in [11], where a phase space approach is used to analyze the one-dimensional problem. See also [5], in which a competition between nucleation and propagation costs are studied in the context of a (different but related) one-dimensional model for the motion of an interface between two solid phases.

4
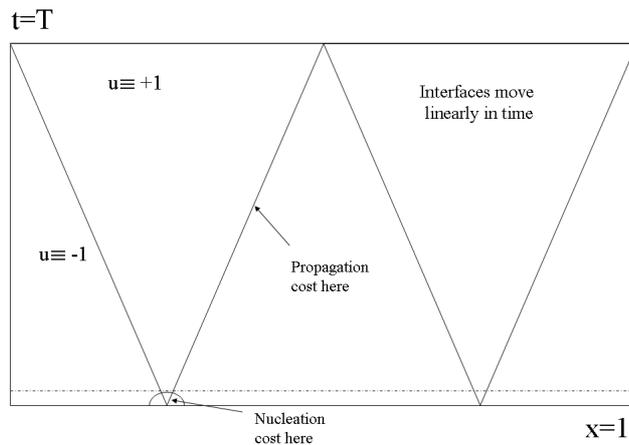
Figure 2: Given a shorter time to switch or a bigger domain, the action minimizer generates more than one interface in order to reduce the "propagation cost." Heuristics suggest that the optimal pathway generates all interfaces immediately, and propagates each with the same, constant speed. Theorem 1.3 proves that such behavior does indeed lead to the optimal action in the limit $\varepsilon \to 0$. Theorem 1.4 proves that under the assumption of single-multiplicity interfaces, any optimal path *must* behave in this way.

Implicit in Figure 2 are two assumptions about the optimal switching path: that all of the interfaces are generated immediately at $t = 0$, and that each wall propagates at the same, constant speed. These assumptions can be justified heuristically [9, 16]. Roughly, if the optimal path has N nucleations, they might as well happen right away. Furthermore, minimizing the first term in the action,

$$\frac{\varepsilon}{4} \int_0^T \int_0^1 \dot{u}^2 dx\, dt, \tag{1.5}$$

for functions with sharp walls suggests a constant speed of propagation, and there is no reason why one wall should have to move more quickly than the others.

In [16], these heuristics are used in a constructive proof of an upper bound for the action minimization problem (1.2). The construction uses functions which transition from u_- to u_+ in time T by nucleating N interfaces at time $t = 0$ and propagating the interfaces at a constant speed. The interfaces have the energy-optimal hyperbolic tangent profile. For Neumann boundary conditions, the resultant upper bound on the action (1.2) is

$$\limsup_{\varepsilon \to 0} S_\varepsilon \leq c_0 \min_{N \in \mathbb{Z}^+} \left(N + \frac{1}{4NT} \right), \tag{1.6}$$

where the constant $c_0 := \int_{-1}^{+1} \sqrt{2V(u)}\, du = 2\sqrt{2}/3$ depends only on the potential, V. An elementary argument produces a lower bound which matches the upper bound *in terms of scaling* [16].

In Theorem 1.3 below, we prove an ansatz-free lower bound which is sharp (i.e. which matches the r.h.s. of (1.6)). In so doing, we prove that no switching pathway can do a better job, asymptotically, than the construction in [16]. It is not immediately clear that this is the case. For instance, why should energy-optimal profiles be "action-optimal?" Of course, $\varepsilon u_{xx} - \varepsilon^{-1} V'(u)$ must be driven to zero in the limit (by the boundedness of the action), but rather than setting it identically to zero, one might expect a trade-off between the cost associated with the shape of the wall and the transportation of the wall. The lower bound rules out the possibility of beating the upper bound in (1.6).

The next step, after understanding the limit of the action, is to understand the limit of the functional itself. Formal manipulations lead to a limiting action on the space of interface functions (not just action minimizers):

$$c_0 N + \sum_{j=1}^N \frac{c_0}{4} \int_0^T \dot{g}_j(t)^2\, dt, \tag{1.7}$$

Figure 3: In the sharp-interface limit, the action is expected to count two costs: the cost of *interface nucleation* and the cost of *interface propagation*. See Theorem 1.4.

where the interface is a finite collection of time-dependent points $\{g_j(t)\}_{j=1}^N$, with $g_j(t)$ representing the location of the jth interface at time t. This expression again suggests a cost for nucleation (the first term) and a cost for propagation (the second term). See Figure 3 for an illustration. Theorem 1.4 places this previously formal calculation on a rigorous basis (under the assumption of single-multiplicity interfaces), by showing that the limiting action can be no smaller than (1.7).

Recall the probabilistic interpretation, that the action of the minimizing pathway reveals the probability of switching. More broadly, the action functional induces a measure on the space of continuous functions: not only does it determine the probability of the optimal path, but it says "how likely" other paths are, as well. In this spirit, the limit of the action functional suggests a measure on the space of interface functions, although to say more requires studying an exchange of limits. Here, we restrict attention to the distinguished limit in which we take the noise strength to zero first to arrive at (1.3), and then take $\varepsilon \to 0$.

Method. Our method involves extending some results for the related elliptic problem to the time-dependent case. Before proceeding, we explain the relationship between the problems. Fix Neumann boundary conditions and

let f_ε denote minus the functional derivative of the energy,

$$f_\varepsilon := \varepsilon u_{xx} - \varepsilon^{-1} V'(u). \tag{1.8}$$

Although action minimizers are not, a priori, energy minimizers, the estimate:

$$\frac{1}{4} \int_0^T \int_0^1 (\varepsilon^{1/2} \dot{u} - \varepsilon^{-1/2} f_\varepsilon)^2 \, dx \, dt \tag{1.9}$$

$$\geq \frac{1}{4} \int_0^s \int_0^1 (\varepsilon^{1/2} \dot{u} - \varepsilon^{-1/2} f_\varepsilon)^2 \, dx \, dt$$

$$= \frac{1}{4} \int_0^s \int_0^1 \left(\left(\varepsilon^{1/2} \dot{u} + \varepsilon^{-1/2} f_\varepsilon \right)^2 - 4 \dot{u} f_\varepsilon \right) dx \, dt$$

$$= (\text{positive term}) + \int_0^s \frac{d}{dt} E_\varepsilon[u] \, dt$$

$$\geq E_\varepsilon[u(\cdot, s)] - E_\varepsilon[u(\cdot, 0)], \tag{1.10}$$

says that the action controls the energy for all $s \leq T$, so bounded initial energy (zero, in our case) and a bounded action imply an L^∞-bound on the energy. Similarly, one may observe that the action decouples:

$$S_\varepsilon[u] = \frac{1}{4} \int_0^T \int_0^1 \left(\varepsilon \dot{u}^2 + \varepsilon^{-1} f_\varepsilon^2 \right) dx \, dt.$$

Thus, an action bound gives a space-time L^2-bound on $\varepsilon^{-1/2} f_\varepsilon$.

These facts connect the action minimization problem with the work of Hutchinson and Tonegawa [14] and Tonegawa [20] on the following elliptic problem: They consider (1.8) with $u = u(x)$ independent of time, and study the properties of the limiting energy and discrepancy measures as $\varepsilon \to 0$, when f_ε is a bounded sequence of constants, or a sequence of functions which is uniformly bounded in $W^{1,d}$. Our problem is simpler in being restricted to one space dimension, but more complicated because of the time-dependence and the weaker bound on f_ε.

1.1 Main Results: Limiting Measures

In order to analyze the time-dependent action problem, we study the limits of the space-time energy measures μ_ε defined by

$$d\mu_\varepsilon := \left(\frac{\varepsilon}{2} (u_x)^2 + \frac{V(u)}{\varepsilon} \right) dx \, dt,$$

in combination with the action measures ν_ε defined by

$$d\nu_\varepsilon := \frac{1}{4}\left(\varepsilon \dot{u}^2 + \varepsilon^{-1} f_\varepsilon^2\right) dx\, dt.$$

We start with a relatively easy result (Theorem 1.1) concerning the time-continuity of the limiting energy measure. We then prove a more involved result (Theorem 1.2) concerning the structure of the limiting energy measure. The latter says that apart from a finite set of "singular times," the energy measures at *fixed time* defined by

$$d\mu_\varepsilon^t := \left(\frac{\varepsilon}{2}(u_x)^2 + \frac{V(u)}{\varepsilon}\right) dx$$

must tend to a finite sum of delta masses with an integer constraint on the coefficients. The theorems are:

Theorem 1.1. *Consider any sequence of smooth functions on $[0,1] \times [0,T]$ which have uniformly bounded action, bounded initial and final energy, and Neumann boundary conditions. Choose any subsequence such that the corresponding measures μ_ε and ν_ε converge as measures to μ and ν in the limit $\varepsilon \to 0$. Let S be the set of times at which $\eta := \int_{[0,1]} d\nu$ has a point mass. Then*

(1) *For all t in $[0,T] \setminus S$, μ_ε^t converges in the sense of measures to a limit, μ^t.*

(2) *For all t in $[0,T] \setminus S$, μ^t is continuous as a function of t with values in $(W^{1,\infty})^*$.*

(3) *$\mu(\Psi) = \int_0^T \mu^t(\Psi)\, dt$ for all $\Psi \in C([0,T] \times [0,1])$.*

Remark 1. *This theorem may actually be stated for any space dimension. The proof goes through in the same way.*

Theorem 1.2. *For any subsequence as in Theorem 1.1, there exists a* finite *set of "singular times" $F_{sing} := \{T_1, T_2, \ldots, T_M\}$ such that for all $t \in [0,T] \setminus F_{sing}$,*

$$\mu_\varepsilon^t \underset{\varepsilon \to 0}{\to} \mu^t = \sum_{\ell=1}^m c_0\, \theta(x_\ell)\, \delta_{\{x_\ell\}}, \quad where\ \theta(x_\ell) \in \mathbb{Z}^+, \tag{1.11}$$

and $\delta_{\{x_\ell\}}$ is the delta-function at x_ℓ. Furthermore, $\theta(x_\ell)$ and x_ℓ are continuous in time.

Remark 2. *The singular times are defined to be times at which η has a point mass with weight at least $c_0/16$. It follows that there can be only finitely many such times, since η is a bounded measure. Moreover, we show that any time at which x_ℓ or $\theta(x_\ell)$ is discontinuous must be a singular time.*

Combining the two theorems, we see that the limiting energy measure, μ, consists of time-dependent delta masses in space, which are generated at the M singular times and which move continuously during the rest of the time interval. We also prove a time-dependent version of equipartition of energy:

Corollary 1. *Consider any subsequence as in Theorem 1.2 and any interval $[s, t] \subset [0, T]$ which does not contain any singular time of μ. Then*

$$\lim_{\varepsilon \to 0} \int_s^t \int_0^1 \frac{V(u(\cdot, t'))}{\varepsilon} \, dx \, dt' = \lim_{\varepsilon \to 0} \frac{1}{2} \int_s^t \int_0^1 \left(\frac{\varepsilon}{2}(u_x)^2 + \frac{V(u(\cdot, t'))}{\varepsilon} \right) dx \, dt'$$

$$= \frac{\mu^t([0, 1]) \times (t - s)}{2}.$$

This result is relevant for the action problem. We remark that Theorem 1.2 says that $\mu^{\tilde{t}}([0, 1])$ is constant for all $\tilde{t} \in [s, t]$, so the second equality follows immediately from the first.

1.2 Main Results: The Action Functional

The control on the limiting measures allows us to solve the action minimization problem. The following theorem gives a sharp, ansatz-free lower bound. Thus, the theorem proves that the upper bound construction from [16] is a good one, in the sense that no other construction can achieve a lower action.

Theorem 1.3. *Let $u_\varepsilon : [0, 1] \times [0, T] \to \mathbb{R}$ be a sequence of smooth functions which have uniformly bounded action and satisfy Neumann boundary conditions as well as the initial and final conditions $u_\varepsilon(\cdot, 0) = -1$, $u_\varepsilon(\cdot, T) = +1$. Then*

$$\liminf_{\varepsilon \to 0} S_\varepsilon[u_\varepsilon] \geq c_0 \min_{N \in \mathbb{Z}^+} \left(N + \frac{1}{4NT} \right).$$

Furthermore, under the assumption of multiplicity-one interfaces, we can go beyond the study of action minimizers and give a general lower bound for any action-bounded sequence:

Theorem 1.4. *Let u_ε be as in Theorem 1.3. Suppose without loss that μ_ε and ν_ε converge, and let $\{T_j\}_{j=1}^M$ be as in Theorem 1.2. Assume that for almost*

every $t \in [0, T]$, $x_\ell \in sppt(\mu^t)$ implies that $x \in \partial\{u_0 = 1\}$ and that the corresponding weight, $\theta(x_\ell) = 1$. Then:

$$\liminf_{\varepsilon \to 0} S_\varepsilon[u_\varepsilon] \geq \sum_{j=1}^{M-1} \sup_{0 \leq \phi \leq 1} \left(\mu^{T_j^+}(\phi) - \mu^{T_j^-}(\phi) \right)^+ + \frac{c_0}{4} \sum_{j=1}^{M} \int_{T_j}^{T_{j+1}} \sum_{k=1}^{N(t)} \dot{g}_k(t)^2 \, dt$$
$$=: S_0[u_0].$$

The first term measures the action cost of nucleating new interfaces in terms of the local energy jump. The second term measures the cost of propagating the interfaces. Here, $g_k(t)$ is the location of the kth interface and $N(t)$ (the number of interfaces) is defined via $\mu^t([0,1]) = c_0 N(t)$ and is constant a.e. on each (T_j, T_{j+1}). By definition, $\mu^{0^-}([0,1]) = 0$.

Theorem 1.4 resembles the lower bound associated with Γ-convergence of the action functional, however it falls short of a real Γ-convergence theorem, because the hypothesis ("simple interfaces") is not formulated solely in terms of the limiting function, u_0. We discuss this point in Subsection 3.2, after the proof of the theorem.

1.3   Higher Space Dimensions

The action minimization problem in higher space dimensions has also been explored in [9, 16], revealing geometrically interesting switching pathways and a connection with motion-by-curvature. The action minimization problem in $d \geq 2$ is

$$\inf_{\substack{u(\cdot,0)=-1 \\ u(\cdot,T)=+1}} \frac{1}{4} \int_0^T \int_\Omega \left(\varepsilon \dot{u}^2 + \varepsilon^{-1}(\tau_\varepsilon(u))^2 \right) \, dx \, dt, \tag{1.12}$$

where

$$\tau_\varepsilon(u) := \varepsilon \Delta u + \varepsilon^{-1}(u - u^3).$$

As explained in [16], the natural candidate for the Γ-limit of (1.12) is:

$$\frac{c_0}{4} \int_0^1 \int_{\partial S(t)} (v_n + \kappa)^2 d\sigma \, dt + 2c_0 \sum_j \mathcal{H}^{d-1}(\partial S_j).$$

(Here, v_n and κ are the normal velocity and curvature of $\partial S(t)$, and ∂S_j is the jth connected component of the interface at the time of "nucleation" of that component.) A rigorous analysis of the sharp-interface limit in higher dimensions remains open; however, as in one dimension, the essential ingredients are

the limiting equipartition of energy and the structure of the limiting energy measures [16]. The analytical challenge is to derive the equipartition based only on the action bound.

Recently, progress has been made on a conjecture of DeGiorgi which is closely related to the action problem. The conjecture is, roughly speaking, the Γ-convergence:

$$\varepsilon^{-1} \int_\Omega (\tau_\varepsilon(u))^2 \, dx \xrightarrow[\varepsilon \to 0]{\Gamma} c_0 \int_{\partial S} \kappa^2 \, d\mathcal{H}^{d-1}. \tag{1.13}$$

Here, Ω is the domain of $u(x)$, and the set S is defined by

$$\lim_{\varepsilon \to 0} u = \begin{cases} +1 & x \in S \\ -1 & x \in S^c. \end{cases}$$

Constrained minimizers of the functional on the l.h.s. of (1.13) are studied in [6]. On the analytical side, [1] proves the Γ-convergence for $\Omega \subset \mathbb{R}^2$ with an assumption of radial symmetry; [19] proves Γ-convergence for $\Omega \subset \mathbb{R}^3$ (with a technical assumption that $u_{x_3} \geq 0$). These results may be useful in proving the Γ-convergence of the action functional in $d \geq 2$.

On another front, research motivated by image processing also touches on related ideas, although the goal is in some sense the opposite of ours: one begins with a sharp-interface model and seeks a diffuse interface approximation. Several recent papers motivated by image processing explore Γ-limits with curvature-dependent functionals; see, for instance, [2, 4, 7].

1.4 Organization

The article is organized as follows. In Section 2, we study the limiting measures. Theorem 1.1, the general theorem about the continuity of the limiting energy measure, is proved in Subsection 2.1. In Subsection 2.2, we study the measures μ^t at "good times." We use these results in Subsection 2.3 to prove Theorem 1.2, the structure theorem about the limiting measures. In Section 3, we consider the action functional, proving Theorem 1.3 (the lower bound for minimizers) and Theorem 1.4 (the stronger theorem about the structure of the limiting functional).

Notation 1. *We use $\mathcal{A}$ to denote the bound on the action (and also, therefore, the energy). We use the small constant c to denote a generic constant (which may change from line to line).*

Notation 2. *For the rest of the paper, we use subscripts on ε and ε-dependent quantities (e.g. ε_i, u_i, f_i) to keep track of the sequences.*

2 The Limiting Measures

We begin by proving Theorem 1.1, using the action measures ν_i to deduce the continuity in time. In Subsection 2.2, we restrict to "good times," at which the f_i are controlled. We use the methods of [15, 14] to conclude among other things that at such times, μ_i^t converges to a finite sum of delta masses with an integer constraint on the coefficients. Finally, in Subsection 2.3, we string together the good times (using the time-continuity) to characterize the space-time measure, μ, and deduce a time-integrated version of equipartition of energy. Because of the integer constraint on the delta masses, we see that in fact, there are at most finitely many "singular times" at which the energy measures can jump.

2.1 Proof of Theorem 1.1

Proof. Choose any subsequence such that μ_i and ν_i converge as measures to μ and ν, respectively. Then $\eta := \int_{[0,1]} d\nu$ is a bounded measure, implying that S is at most countable.

Choose a dense set of times from S^c, denoted $\{t_j\}_{j=1}^\infty$. By a diagonal argument, we can choose a subsequence (still labelled by i for simplicity) such that $\mu_i^{t_j}$ converges as a measure to some limit, denoted μ^{t_j}, for all t_j. We will show that for all $t \in S^c$,

$$\mu^t := \lim_{t_j \to t} \mu^{t_j} \tag{2.1}$$

is well defined, and continuous in $(W^{1,\infty})^*$. Next, we will verify that in fact, $\mu^t = \lim_{i \to \infty} \mu_i^t$, and that *(3)* holds. Finally, we will show that the limiting measures, μ^t, are uniquely defined (by convergence of μ_i), implying that the whole sequence converges.

We now show that the limit on the r.h.s. of (2.1) exists. Consider any two subsequences μ^{t_j} and $\mu^{\tilde{t}_j}$ (where $t_j \to t$ and $\tilde{t}_j \to t$), such that they converge as measures to μ^t and $\tilde{\mu}^t$, respectively. We show that necessarily, $\mu^t = \tilde{\mu}^t$ in $(W^{1,\infty})^*$ and, therefore, as measures. Fix any $\Psi \in W^{1,\infty}([0,1])$

with $||\Psi||_{W^{1,\infty}} \leq 1$. Then

$$\int_0^1 \Psi \left(\frac{\varepsilon_i}{2}(u_{i,x})^2 + \frac{V}{\varepsilon_i} \right) \Big|_{t_j} dx - \int_0^1 \Psi \left(\frac{\varepsilon_i}{2}(u_{i,x})^2 + \frac{V}{\varepsilon_i} \right) \Big|_{\tilde{t}_j} dx$$

$$= \int_0^1 \int_{t_j}^{\tilde{t}_j} \Psi \frac{d}{dt} \left(\frac{\varepsilon_i}{2}(u_{i,x})^2 + \frac{V}{\varepsilon_i} \right) dt\, dx$$

$$= \int_0^1 \int_{t_j}^{\tilde{t}_j} \Psi \left(\varepsilon_i u_{i,x} \dot{u}_{i,x} + \frac{V'}{\varepsilon_i} \dot{u}_i \right) dt\, dx$$

$$= \int_0^1 \int_{t_j}^{\tilde{t}_j} \left(-\Psi' \varepsilon_i u_{i,x} \dot{u}_i - \Psi \varepsilon_i u_{i,xx} \dot{u}_i + \Psi \frac{V'}{\varepsilon_i} \dot{u}_i \right) dt\, dx$$

$$\leq \frac{1}{2} \int_{t_j}^{\tilde{t}_j} \int_0^1 \left(|\Psi'| \varepsilon_i (u_{i,x})^2 + |\Psi'| \varepsilon_i \dot{u}_i^2 + |\Psi| \left(\varepsilon_i \dot{u}_i^2 + \varepsilon_i^{-1} f_i^2 \right) \right) dx\, dt$$

$$\leq \frac{|\tilde{t}_j - t_j|}{2} \sup_t \left[\int_0^1 \varepsilon_i (u_{i,x})^2 dx \right] + \int_{t_j}^{\tilde{t}_j} \int_0^1 \left(\varepsilon_i \dot{u}_i^2 + \varepsilon_i^{-1} f_i^2 \right) dx\, dt.$$

Sending first $i \to \infty$ and then $j \to \infty$ confirms that $\mu^t = \tilde{\mu}^t$ and thus, (2.1) is a good definition. The calculation also shows continuity in the sense that for all $t \in S^c$ and $\gamma > 0$, there exists a $\delta > 0$ such that $|t - s| \leq \delta$ implies $|\mu^t(\Psi) - \mu^s(\Psi)| \leq \gamma ||\Psi||_{W^{1,\infty}}$.

Next, we show that in fact, for all $t \in S^c$, $\mu^t = \lim_{i \to \infty} \mu_i^t$. Choose any subsequence such that μ_i^t converges to a limit, $\tilde{\mu}^t$. Choose any set of times from $\{t_j\}$ such that $t_j \to t$, and any function Ψ, as above. Estimating as above, we see that

$$\left| \mu^{t_j}(\Psi) - \tilde{\mu}^t(\Psi) \right| = \lim_{i \to \infty} \left| \mu_i^{t_j}(\Psi) - \mu_i^t(\Psi) \right|$$

$$\leq c|t_j - t| + 4 \int_t^{t_j} d\eta,$$

which goes to zero as $j \to \infty$. Therefore, the whole sequence μ_i^t converges and the limit is μ^t.

Item *(3)* follows from the Dominated Convergence Theorem:

$$\int \Psi d\mu = \lim_{i \to \infty} \int \Psi d\mu_i = \lim_{i \to \infty} \int_0^T \left(\int_0^1 \Psi \left(\frac{\varepsilon_i}{2}(u_{i,x})^2 + \frac{V}{\varepsilon_i} \right) dx \right) dt$$

$$= \int_0^T \int_0^1 \Psi d\mu^t\, dt.$$

Finally, we claim that $\mu = \lim_{i \to \infty} \mu_i$ uniquely determines the limit μ^t for each $t \in S^c$. This is true because for any continuous function Ψ, $\mu^t(\Psi)$ is

14

uniquely defined for a.e. $t \in [0, T]$, and by (2.1), the definition extends to all $t \in S^c$. $\qquad \square$

2.2 Limits of the Energy Measures at "Good Times"

In this subsection, we study the limit μ^t for a fixed t which is a "good time." By good time, we will mean one at which

$$\limsup_{i \to \infty} \|\varepsilon_i^{-1/2} f_i\|_{L_x^2([0,1])} < \infty. \tag{2.2}$$

We will see that the limit (up to a subsequence) of the measures μ_i^t is a finite sum of delta functions, with weights which are integer multiples of c_0. Furthermore, the discrepancy measures $|\xi_i^t|$ defined via

$$d|\xi_i^t| := \left| \frac{\varepsilon_i}{2}(u_{i,x})^2 - \frac{V(u_i)}{\varepsilon_i} \right| dx \tag{2.3}$$

converge to the zero measure. These facts will be useful for the proof of Theorem 1.2 (Subsection 2.3), where we use Theorem 1.1 and apply Theorem 2.1 at a countable number of good times to get a precise characterization of the measure μ.

Notation 3. *In the present subsection, time is fixed. Therefore, for simplicity of notation, we write $u_i = u_i(x)$ (although we retain the t-superscript on μ^t and μ_i^t to distinguish them from the space-time measures of Subsection 2.1). All integrals are over $[0, 1]$ unless otherwise specified.*

Theorem 2.1 (Limits at "good times"). *Let $u_i : [0, 1] \to \mathbb{R}$ be a sequence of smooth functions which have uniformly bounded energy, and satisfy Neumann boundary conditions as well as condition (2.2). Then for any subsequence such that μ_i^t converges to the measure μ^t,*

(i) The support of μ^t is at most a finite number of points.

(ii) The discrepancy measures converge to the zero measure, and

$$\frac{\varepsilon_i}{2}|u_{i,x}|^2 - \sqrt{V(u_i)/2}|u_{i,x}| \quad \text{and} \quad \varepsilon_i^{-1}V(u_i) - \sqrt{V(u_i)/2}|u_{i,x}|$$

converge to zero in L^1, as well.

(iii) Furthermore, μ^t is of the form

$$\mu^t = c_0 \sum_{\ell=1}^{m} \theta(x_\ell)\, \delta_{\{x_\ell\}}, \quad \text{with } \theta(x_\ell) \in \mathbb{Z}^+.$$

15

Proof. *(i)* Consider any convergent subsequence. The first step is to show that for all $x_0 \in \operatorname{sppt} \mu^t$, there exists a sequence $x_i \to x_0$ with $|u_i(x_i)| \leq 3/4$. We show it by contradiction, demonstrating that if $|u_i|$ is "large" on an interval around x_0 then the energy vanishes in the limit. Indeed, suppose there exists $r > 0$ such that

$$\inf_{B_r(x_0)} |u_i| > \frac{3}{4} \quad \text{for } i \text{ sufficiently large.}$$

Let $\Psi \in C_0^2(B_r(x_0))^+$ be such that $\Psi = 1$ on $B_{r/2}(x_0)$ and $|\Psi| \leq 1$. Multiply (1.8) by $\Psi^2 u_{i,xx}$ and integrate by parts to get

$$\int \varepsilon_i (u_{i,xx})^2 \Psi^2 + \frac{V''(u_i)}{\varepsilon_i}(u_{i,x})^2 \Psi^2 + \frac{V'(u_i)}{\varepsilon_i} u_{i,x}(\Psi^2)_x = \int f_i u_{i,xx} \Psi^2.$$

Integrate by parts once more and use Young's inequality to deduce

$$\int \left(\varepsilon_i (u_{i,xx})^2 + \frac{V''(u_i)}{\varepsilon_i}(u_{i,x})^2 \right) \Psi^2 = \int f_i u_{i,xx} \Psi^2 + \frac{V(u_i)}{\varepsilon_i}(\Psi^2)_{xx}$$

$$\leq \int \frac{f_i^2}{2\varepsilon_i} \Psi^2 + \frac{\varepsilon_i}{2}(u_{i,xx})^2 \Psi^2 + \frac{V(u_i)}{\varepsilon_i}(\Psi^2)_{xx}.$$

Combining terms,

$$\int \left(\frac{\varepsilon_i}{2}(u_{i,xx})^2 + \frac{V''(u_i)}{\varepsilon_i}(u_{i,x})^2 \right) \Psi^2 \leq \int \frac{f_i^2}{2\varepsilon_i} \Psi^2 + \frac{V(u_i)}{\varepsilon_i}(\Psi^2)_{xx}.$$

Since the r.h.s. is bounded by a constant (depending on r) and $V''(t) \geq \kappa$ for $|t| > 3/4$, this shows

$$\int_{B_{r/2}(x_0)} (u_{i,x})^2 \leq c\varepsilon_i.$$

Therefore, the gradient energy is small on $B_{r/2}(x_0)$. This also shows that

$$|u_i(x) - u_i(y)| \leq c\varepsilon_i^{1/2}|x-y|^{1/2} \text{ for } x, y \in B_{r/2}(x_0), \tag{2.4}$$

so u_i is close to 1 or -1 entirely on $B_{r/2}(x_0)$. Assume without loss that it is close to 1. We will use this fact as we now show that the potential term of the energy is also small on $B_{r/4}(x_0)$. Choose $\Phi \in C_0^1(B_{r/2}(x_0))^+$ with $\Phi = 1$ on $B_{r/4}(x_0)$ and $|\Phi| \leq 1$, and multiply (1.8) by $\Phi^2(u_i - 1)$. Integrating,

$$\int \varepsilon_i u_{i,xx}(u_i - 1)\Phi^2 - \frac{V'(u_i)}{\varepsilon_i}(u_i - 1)\Phi^2 = \int f_i(u_i - 1)\Phi^2.$$

Using the fact that $V'(t)(t-1) \geq cV(t)$ for $t \geq 3/4$,

$$c \int \frac{V(u_i)}{\varepsilon_i} \Phi^2 \leq \int \varepsilon_i u_{i,xx}(u_i - 1)\Phi^2 + |f_i||u_i - 1|\Phi^2$$

$$= \int -\varepsilon_i(u_{i,x})^2\Phi^2 - \varepsilon_i u_{i,x}(u_i - 1)(\Phi^2)_x + |f_i||u_i - 1|\Phi^2$$

$$\leq \int -\varepsilon_i u_{i,x}(u_i - 1)(\Phi^2)_x + |f_i||u_i - 1|\Phi^2,$$

where we have dropped the negative term. Using Hölder's inequality and recalling the bounds on f_i and $|u_i|$ (from (2.4)),

$$\int_{B_{r/4}(x_0)} \frac{V(u_i)}{\varepsilon_i} \leq c \left(\int \varepsilon_i(u_{i,x})^2 |(\Phi^2)_x| \right)^{1/2} \left(\int \varepsilon_i(u_i - 1)^2 |(\Phi^2)_x| \right)^{1/2} + o(1),$$

which tends to zero as $i \to \infty$. Together with the smallness of the gradient energy on the ball, this contradicts $x_0 \in \text{sppt}\,\mu^t$, and we have shown the existence of a sequence $\{x_i\}$ with $x_i \to x_0$ and $|u_i(x_i)| \leq 3/4$.

Next, we claim that there exists a constant c_1 such that any $x_0 \in \text{sppt}\,\mu^t$ contributes a delta mass of at least weight c_1. To see this, fix any $r > 0$. There exists an i_1 such that $|x_i - x_0| < r$ for all $i \geq i_1$. Also, since $u_i \to \pm 1$ a.e. (by the energy bound), there exists $y \in [x_0 - r, x_0 + r]$ such that $u_i(y) \to \pm 1$. Suppose without loss that the limit is $+1$. There is an i_2 such that $u_i(y) > 0.9$ for all $i \geq i_2$. Choose $i_3 := \max(i_1, i_2)$. Then for all $i \geq i_3$,

$$\int_{B_r(x_0)} \left(\frac{\varepsilon_i}{2}(u_{i,x})^2 + \frac{V(u_i)}{\varepsilon_i} \right) dx \geq \sqrt{2} \int_{3/4}^{0.9} \sqrt{V(u)}\, du =: c_1.$$

Since the energy is bounded, this completes the proof of the fact that the support of μ^t is at most a finite number of points.

(ii) It is easy to deduce equipartition of energy in the one dimensional case. From *(i)*, the energy measure is supported on points. By absolute continuity of $|\xi^t|$ with respect to μ^t, so is $|\xi^t|$. The following lemma shows that

$$\left| \frac{\varepsilon_i}{2}(u_{i,x})^2 - \frac{V(u_i)}{\varepsilon_i} \right|$$

is uniformly bounded, however, so concentration is ruled out and $|\xi^t|$ must be the zero measure.

Lemma 2.1 (Uniform bound on discrepancy). *Let $\{u_i\}$ be a sequence which satisfies the hypotheses of Theorem 2.1. Then there exists a constant c_2 such that*

$$\sup_{x \in [0,1]} \left| \frac{\varepsilon_i}{2}(u_{i,x})^2 - \frac{V(u_i)}{\varepsilon_i} \right| \leq c_2. \tag{2.5}$$

Proof. Recalling that the action bound $S_{\varepsilon_i}[u_i] \leq \mathcal{A}$ and zero initial energy induce a uniform in time bound on the energy (cf. (1.10)), we deduce

$$\inf_{x \in [0,1]} \left| \frac{\varepsilon_i}{2}(u_{i,x})^2 - \frac{V(u_i)}{\varepsilon_i} \right| \leq \inf_{x \in [0,1]} \left(\frac{\varepsilon_i}{2}(u_{i,x})^2 + \frac{V(u_i)}{\varepsilon_i} \right)$$

$$\leq \int_0^1 \left(\frac{\varepsilon_i}{2}(u_{i,x})^2 + \frac{V(u_i)}{\varepsilon_i} \right) dx$$

$$\leq \mathcal{A}. \tag{2.6}$$

On the other hand, the variation is also bounded, as:

$$\left| \int \partial_x \left(\frac{\varepsilon_i}{2}(u_{i,x})^2 - \frac{V(u_i)}{\varepsilon_i} \right) dx \right| = \left| \int f_i u_{i,x} \, dx \right|$$

$$\leq ||\varepsilon_i^{-1/2} f_i||_{L^2} ||\varepsilon_i^{1/2} u_{i,x}||_{L^2}$$

$$\leq c. \tag{2.7}$$

Together, (2.6) and (2.7) imply (2.5). $\qquad\qquad\square$

The last part of *(ii)* follows from completing the square:

$$\left| \frac{\varepsilon_i}{2}|u_{i,x}|^2 + \varepsilon_i^{-1}V(u_i) - 2\sqrt{V(u_i)/2}|u_{i,x}| \right| = \left(\sqrt{\frac{\varepsilon_i}{2}}|u_{i,x}| - \sqrt{\frac{V(u_i)}{\varepsilon_i}} \right)^2$$

$$\leq \left| \frac{\varepsilon_i}{2}|u_{i,x}|^2 - \frac{V(u_i)}{\varepsilon_i} \right|.$$

(iii) Finally, we show that for $x_0 \in \operatorname{sppt} \mu^t$, $\mu^t(\{x_0\}) = c_0 N$ for some $N \in \mathbb{Z}^+$. The technique is developed in Hutchinson and Tonegawa [14]. The main idea is that on $\{u_i \approx \pm 1\}$ there is no energy contribution, and away from ± 1, u_i is monotone, so the energy can be calculated explicitly, yielding a multiple of c_0 per transition. We make these statements precise in two lemmas.

Lemma 2.2 (No energy away from transition layers). *Given any $\delta > 0$, there exists an $s > 0$ and an $\varepsilon_0 > 0$ such that*

$$\int_{\{|u_i| \geq 1-s\}} \left(|u_{i,x}|\sqrt{|V(u_i)|} \right) dx \leq \delta,$$

for all $\varepsilon_i < \varepsilon_0$.

Proof. The meaning of the lemma is that energy does not accumulate away from the "transition layers." The proof of the lemma is somewhat technical. We proceed by steps, breaking $\{|u_i| \geq 1 - s\}$ into subsets and studying the gradient term and the potential term in the energy. In each case, we prove a claim which shows that one term is small (and the other is always bounded). This is enough by Hölder's inequality, since

$$\int_A |u_{i,x}| \sqrt{|V(u_i)|} \leq \left(\int_A \varepsilon_i (u_{i,x})^2 \right)^{1/2} \left(\int_A \frac{V(u_i)}{\varepsilon_i} \right)^{1/2}. \qquad (2.8)$$

We now state Claims 1 through 3, which we will use for the proof. (For ease of notation, we drop the subscript i.)

Claim 1 (Small gradient energy on $|u| > 1$). *There exists a constant c such that*

$$\int_{\{|u| \geq 1\}} \varepsilon u_x^2 \leq c\varepsilon^2.$$

Proof. Multiply (1.8) by $(u - 1)_+$, integrate over $\{u \geq 1\}$, and integrate by parts to get

$$\int_{\{u \geq 1\}} \varepsilon u_x^2 + \frac{V'(u)}{\varepsilon}(u - 1) = \int_{\{u \geq 1\}} f(1 - u).$$

Use $V'(t) \geq c(t - 1)$ for $t \geq 1$ and Young's inequality to deduce

$$\int_{\{u \geq 1\}} \varepsilon u_x^2 + \frac{c(u - 1)^2}{\varepsilon} \leq \frac{\varepsilon}{2c} \int_{\{u \geq 1\}} f^2 + \frac{c}{2\varepsilon} \int_{\{u \geq 1\}} (u - 1)^2.$$

Thus, by the action bound on $||\varepsilon_i^{-1/2} f_i||_{L_x^2}$,

$$\int_{\{u \geq 1\}} \varepsilon u_x^2 \leq c\varepsilon^2.$$

The same applies to $\{u < -1\}$. $\qquad \square$

Claim 2. *There exist $c_4, \varepsilon_0 > 0$ with the following property. Assume that $0 < \beta \leq 3/4$, $u(0) \leq 1 - \varepsilon^\beta$, and $\varepsilon^\beta \leq 1$. Then for all $\varepsilon \leq \varepsilon_0$,*

$$\min_{|x| \leq r_0} u(x) \leq \frac{3}{4},$$

where $r_0 := \beta c_4 \varepsilon \ln(1/\varepsilon)$.

Proof. We use the fact that for $t \geq 3/4$, $V''(t) \geq \kappa$. Set $c_4 = 2/\sqrt{\kappa}$. Assume for a contradiction that for r_0 defined above, the minimum on the ball is larger than $3/4$. Set

$$v(x) := 1 - u(x) - \frac{\varepsilon^\beta}{2} \cosh\left(\frac{x\sqrt{\kappa}}{\varepsilon}\right).$$

First,

$$v(0) = 1 - u(0) - \frac{\varepsilon^\beta}{2} \geq \frac{\varepsilon^\beta}{2}, \tag{2.9}$$

by the assumption on $u(0)$. Next, for $|x| = \beta c_4 \varepsilon \ln(1/\varepsilon)$,

$$\frac{1}{2} \cosh\left(\frac{x\sqrt{\kappa}}{\varepsilon}\right) \geq \frac{1}{4} \exp\left(\beta c_4 \sqrt{\kappa} \ln(1/\varepsilon)\right) \geq \frac{1}{4} \varepsilon^{-\beta c_4 \sqrt{\kappa}}.$$

With the choice $c_4 = 2/\sqrt{\kappa}$,

$$v(x) \leq 1 - u(x) - \frac{1}{4}\varepsilon^{-\beta}.$$

Recalling $\varepsilon^{-\beta} \geq 1$ and $u \geq 3/4$, we conclude that

$$v(\pm r_0) \leq 0. \tag{2.10}$$

Next, the mean value theorem implies $V'(u) \leq \kappa(u-1)$ for $u \in [3/4, 1]$, which we use to conclude

$$-\varepsilon v'' + \frac{\kappa}{\varepsilon} v = \varepsilon u'' - \frac{\kappa}{\varepsilon}(u-1) \leq \varepsilon u'' - \frac{V'(u)}{\varepsilon} = f,$$

whenever $3/4 \leq u \leq 1$. Multiply by v_+ (which restricts to $u \leq 1$) and integrate by parts on $|x| \leq r_0$, with the boundary terms vanishing because of (2.10). We have

$$\int_{B_{r_0}(0)} \varepsilon(v'_+)^2 + \frac{\kappa}{\varepsilon} v_+^2 \leq \int_{B_{r_0}(0)} |f||v_+| \leq \frac{\varepsilon}{2\kappa} \int_{B_{r_0}(0)} f^2 + \frac{\kappa}{2\varepsilon} \int_{B_{r_0}(0)} v_+^2,$$

and conclude in the usual way (cf. Claim 1) that

$$\int_{B_{r_0}(0)} \varepsilon(v'_+)^2 \leq c\varepsilon^2.$$

Together with (2.9) and (2.10), this implies:

$$\frac{\varepsilon^{\beta}}{2} \le v_+(0) - v_+(-r_0) \le r_0^{1/2} \left(\int_{B_{r_0}(0)} v_+'^2 \right)^{1/2}$$
$$\le (\beta c_4 \varepsilon \ln(1/\varepsilon))^{1/2} (\varepsilon c)^{1/2}$$
$$\le c\varepsilon \ln(1/\varepsilon),$$

which is a contradiction for ε sufficiently small, since $\beta \le 3/4$. □

Claim 3 (Measure of the set close to $\{|u| \le 3/4\}$). *Let*

$$T_r := \{x | d(x, \{|u| \le 3/4\}) < r\}.$$

Then there exist constants c_5, c_6 such that

$$r \ge c_5 \varepsilon \Rightarrow m(T_r) \le c_6 \, r.$$

Proof. By Vitali's covering lemma, we may choose $x_j \in \{|u| \le 3/4\}$ such that $\{[x_j - r, x_j + r]\}_{j=1}^N$ are mutually disjoint and $\bigcup_{j=1}^N [x_j - 5r, x_j + 5r] \supset T_r$. We claim that there exists a $c > 0$ depending only on V such that

$$\int_{[x_j - r, x_j + r]} \left(\frac{\varepsilon}{2} u_x^2 + \frac{V(u)}{\varepsilon} \right) \ge c, \tag{2.11}$$

which implies that since the intervals are mutually disjoint, the number N has an upper bound, depending on c and the total energy. Consequently,

$$m(T_r) \le m \left(\bigcup_{j=1}^N [x_j - 5r, x_j + 5r] \right) \le 10rN \le c_6 r.$$

To show (2.11), we use the estimate

$$|u_x|_{C^0} \le \frac{c}{\varepsilon},$$

from (2.2), cf. the proof of Lemma 2.3, below. From here we conclude that if

$$|y - x_j| \le \frac{\varepsilon}{8c},$$

then

$$|u(y)| \le |u(x_j)| + \frac{1}{8} \le \frac{7}{8}.$$

Therefore, letting $\gamma := \varepsilon/(8c)$,

$$\int_{B_\gamma(x_j)} \frac{V}{\varepsilon} \ge \frac{1}{8c} \min_{|t| \le 7/8} V(t),$$

which verifies (2.11) and completes the proof of the claim. □

Putting it all together, we decompose $\{u > 1 - s\}$ into four sets,

$$\{u \geq 1\}, \qquad\qquad \{1 - \varepsilon^{3/4} \leq u < 1\},$$
$$\{1 - s \leq u < 1 - \varepsilon^{1/2}\}, \qquad\qquad \{1 - \varepsilon^{1/2} \leq u < 1 - \varepsilon^{3/4}\}.$$

We show that by choice of s and ε_0, we can guarantee that on the first set, the gradient energy is small, and on the other three sets, the potential energy term is small. (Recall that by (2.8), this is sufficient.) Claim 1 gives us the control we need on the first set. On $\{1 - \varepsilon^{3/4} \leq u < 1\}$, it is easy to control the potential term in the energy, since $V(u) \leq c\varepsilon^{3/2}$, so

$$\int_{\{1 - \varepsilon^{3/4} \leq u < 1\}} \frac{V(u)}{\varepsilon} \leq c\varepsilon^{1/2}.$$

The third and fourth sets are more involved.

Consider $\{1 - s \leq u < 1 - \varepsilon^{1/2}\}$. Let

$$A_j := \{1 - \varepsilon^{1/2^{(j+1)}} \leq u \leq 1 - \varepsilon^{1/2^j}\}, \text{ for } j = 1 \ldots N,$$

where N is chosen so that

$$1 - \varepsilon^{1/2^{(N+1)}} \leq 1 - s < 1 - \varepsilon^{1/2^N}.$$

We will use Claims 2 and 3 to estimate the potential term in the energy, summed over all the A_j. First, by Claim 2,

$$\operatorname{dist}(\{|u| \leq 3/4\}, A_j) \leq 2^{-j} c\varepsilon \ln(1/\varepsilon),$$

so $A_j \subset T_{2^{-j}c\varepsilon \ln(1/\varepsilon)}$. By Claim 3,

$$m(A_j) \leq m(T_{2^{-j}c\varepsilon \ln(1/\varepsilon)}) \leq c2^{-j}\varepsilon \ln(1/\varepsilon),$$

as long as s is small enough that $2^{-N}c\ln(1/\varepsilon) > c_5$. Therefore,

$$\int_{A_j} \frac{V(u)}{\varepsilon} \leq c\frac{(\varepsilon^{2^{-(j+1)}})^2}{\varepsilon}m(A_j)$$
$$\leq c\varepsilon^{-1+1/2^j}2^{-j}\varepsilon \ln(1/\varepsilon)$$
$$= c\varepsilon^{1/2^j}2^{-j}\ln(1/\varepsilon).$$

Since $\bigcup_{j=1}^N A_j = \{1 - \varepsilon^{1/2^{(N+1)}} \leq u < 1 - \varepsilon^{1/2}\}$ and $1 - \varepsilon^{1/2^{(N+1)}} \leq 1 - s$,

$$\int_{\{1 - s \leq u < 1 - \varepsilon^{1/2}\}} \frac{V(u)}{\varepsilon} \leq c\ln(1/\varepsilon)\sum_{j=1}^N 2^{-j}\varepsilon^{1/2^j}.$$

Because $g(x) := x^{-1}\varepsilon^{x^{-1}}$ is increasing on $[1, \ln(1/\varepsilon)]$, we can estimate the sum using the integral test, as long as $s \leq \exp(-2)$, so that $2^{(N+1)} \leq \ln(1/\varepsilon)$. Therefore,

$$\int_{\{1-s \leq u < 1-\varepsilon^{1/2}\}} \frac{V(u)}{\varepsilon} \leq c \ln(1/\varepsilon) \int_1^{N+1} 2^{-t} \varepsilon^{2^{-t}} dt$$
$$= c((\varepsilon^{2^{-N}})^{1/2} - \sqrt{\varepsilon})$$
$$\leq c\sqrt{s}.$$

Thus, by choosing s small, we control the integral of the potential term on $\{1 - s \leq u < 1 - \varepsilon^{1/2}\}$.

Finally, consider $A_0 := \{1 - \varepsilon^{1/2} \leq u < 1 - \varepsilon^{3/4}\}$. Here, $V(u) \leq c\varepsilon$. Also, by Claim 3, $m(A_0) \leq c\varepsilon \ln(1/\varepsilon)$. Thus

$$\int_{A_0} \frac{V(u)}{\varepsilon} \leq c\, m(A_0) \leq c\varepsilon \ln(1/\varepsilon).$$

$\square$

Lemma 2.3 (Monotone transition layers). *For every $s \in (0,1)$, there exists an i_0 such that u_i is monotone on connected components of $\{|u_i| \leq 1-s\}$ for $i \geq i_0$.*

Proof of lemma. The proof is in two steps. First, we bound $||u_{i,x}||_{C^{0,1/2}}$. Then we use the bound to argue by contradiction.

It is convenient to rescale (1.8), letting $y = x/\varepsilon_i$, $v(y) := u_i(\varepsilon_i y)$, and $g(y) := \varepsilon_i f_i(\varepsilon_i y)$. Then

$$v_{yy} - V'(v) = g_i, \tag{2.12}$$

and $||g_i||_{L^2} \leq c\varepsilon_i$. In one space dimension, the energy bound gives a uniform bound on $|u_i|$ and, therefore, on v. This L^∞ control on v together with equation (2.12) gives a uniform bound on $||v_{yy}||_{L^2}$, leading in turn to the uniform bound:

$$||v_y||_{C^{0,1/2}} \leq c.$$

This can be expressed in the original variables:

$$|x_1 - x_2| \leq \eta\varepsilon \implies |u_{i,x}(x_1) - u_{i,x}(x_2)| \leq \varepsilon^{-1} c\, \eta^{1/2}. \tag{2.13}$$

Now suppose for a contradiction that for every i_0, there exists an $i > i_0$ and a corresponding point x_i such that

$$\begin{cases} |u_i(x_i)| \leq 1 - s, \\ u_{i,x}(x_i) = 0. \end{cases} \tag{2.14}$$

23

Given any $s \in (0, 1)$, there exists a constant c_1 such that

$$|u| \leq 1 - s/2 \Rightarrow V(u) \geq c_1. \tag{2.15}$$

Fix

$$\eta := \min\left\{\left(\frac{s}{2c}\right)^{2/3}, \left(\frac{c_1}{c^2}\right)\right\}.$$

Equation (2.13) and the second condition in (2.14) imply:

$$|x - x_i| \leq \varepsilon\eta \quad \Rightarrow \quad |u_{i,x}(x)| \leq \varepsilon^{-1}c\eta^{1/2}. \tag{2.16}$$

From (2.16) and the first condition in (2.14),

$$|x - x_i| \leq \varepsilon\eta \quad \Rightarrow \quad |u_i(x)| \leq |u_i(x_i)| + (\varepsilon^{-1}c\eta^{1/2})(\varepsilon\eta)$$
$$\leq 1 - s + c\eta^{3/2}$$
$$\leq 1 - s/2, \tag{2.17}$$

by choice of η. We conclude by observing that (2.17), (2.15), (2.16), and the choice of η contradict the vanishing of the discrepancy measure:

$$\int_{x_1 - \eta\varepsilon_i}^{x_1 + \eta\varepsilon_i} \left| \frac{V(u_i)}{\varepsilon_i} - \frac{\varepsilon_i}{2}(u_{i,x})^2 \right| dx \geq \left(\frac{c_1}{\varepsilon_i} - \frac{\varepsilon_i}{2}\left(\frac{c\eta^{1/2}}{\varepsilon_i}\right)^2 \right)(2\eta\varepsilon_i)$$
$$= \left(\frac{c_1}{\varepsilon_i} - \frac{c^2\eta}{2\varepsilon_i} \right)(2\eta\varepsilon_i)$$
$$\geq c_1\eta.$$

$\square$

Returning to the proof of *(iii)*, fix $\delta > 0$ and find the corresponding s, per Lemma 2.2. Let N_i denote the number of transitions of u_i from $(-1 + s)$ to $(1 - s)$ and vice versa. Using the equipartition from *(ii)*,

$$\int \frac{\varepsilon_i}{2}(u_{i,x})^2 + \frac{V(u_i)}{\varepsilon_i} \, dx$$
$$= \sqrt{2} \int \sqrt{V(u_i)} \, |u_{i,x}| \, dx + o(1)_{(i\to\infty)}$$
$$= \sqrt{2} \sum_{j=1}^{N_i} \int_{-1+s}^{1-s} \sqrt{V(u_i)} \, du + \sqrt{2} \int_{|u_i|>1-s} \sqrt{V(u_i)} \, |u_{i,x}| \, dx + o(1)_{(i\to\infty)}$$
$$= c_0 \, N_i + o(1)_{(s\to0)} + o(1)_{(i\to\infty)},$$

where for the second equality, we have applied the monotonicity from Lemma 2.3 to integrate on connected components of $\{|u_i| \leq 1 - s\}$, and in the last equality we have applied Lemma 2.2. Now we send $i \to \infty$, then $s \to 0$. $\square$

2.3   The Structure of μ and Equipartition

We now combine the continuity from Theorem 1.1 and the delta-mass structure at good times from Theorem 2.1 to look more closely at the space-time measure, μ. The picture which emerges in Theorem 1.2 is that $d\mu = d\mu^t dt$ consists of delta masses which move continuously (except, perhaps, at the singular times). Finally, we prove Corollary 1, a time-integrated version of equipartition of energy.

2.3.1   Proof of Theorem 1.2

Proof. By Theorem 1.1, $\mu_i^t \to \mu^t$ for all $t \in [0, T] \setminus S$, where S is the (at most countable) set at which η has a point mass. We will find a subsequence such that μ_i^t converges to a "bounded sum" of delta masses of the form (1.11) for all t in a dense set $\{t_j\}_{j=1}^\infty$. (By a "bounded sum," we mean that $\mu^t([0,1]) \le c$, so both the number and weights of the masses are bounded.)

To deduce the desired structure at a dense set of times, we will apply Theorem 2.1. For any subsequence, Fatou's lemma says

$$\int_0^T \frac{1}{\varepsilon_i} \int_0^1 f_i^2 \, dx \, dt \le \mathcal{A} \;\Rightarrow\; \int_0^T \liminf_i \frac{1}{\varepsilon_i} \int_0^1 f_i^2 dx \, dt \le \mathcal{A},$$

so for a.e. t, $\exists$ subsequence s.t. $\limsup_i \left(\frac{1}{\varepsilon_i} \int_0^1 f_i^2 dx \right)\Big|_t < \infty$. Choose a dense set $\{t_j\}_{j=1}^\infty$ and (after a diagonal argument) a subsequence such that

$$\limsup_i \left(\frac{1}{\varepsilon_i} \int_0^1 f_i^2 dx \right)\Big|_{t_j} \le c(t_j).$$

By Theorem 2.1, we may again choose a subsequence such that, for each t_j,

$$\left(\frac{\varepsilon_i}{2}(u_{i,x})^2 + \frac{V}{\varepsilon_i} \right)\Big|_{t_j} dx \underset{i\to\infty}{\to} \mu^{t_j}$$

in the sense of measures, with $\mu^{t_j} = \sum_{l=1}^N c_0 \theta(x_\ell)\delta_{\{x_\ell\}}$, $\theta \in \mathbb{Z}^+$. Moreover, the discrepancy measures vanish. Also, the μ^t are all "bounded measures" in the sense above, because of the uniform energy bound. Extending the limit to all $t \in S^c$ as in the proof of Theorem 1.1, we have identified the limit (of the whole sequence) for all but countably many times.

The form (1.11) is preserved for all $t \in S^c$ after extending the definition by continuity (as in the proof of Theorem 1.1), since the limit of a bounded

sum of delta masses with positive integer coefficients is again a bounded sum of delta masses with positive integer coefficients. The continuity of $\theta(x_\ell)$ and x_ℓ for all $t \in S^c$ follows from the continuity of μ^t in $(W^{1,\infty})^*$ (cf. Theorem 1.1).

Finally, because of the integer-coefficients of the delta masses, we are able to extend the continuity to all but *finitely* many times. Fix $b := c_0/4$. We define a "singular time" to be the times $T_1 < T_2 < \cdots < T_M$ be such that

$$\eta(\{T_k\}) \geq b/4.$$

Since $\eta([0, T]) \leq \mathcal{A}$, there are at most $4\mathcal{A}/b$ such T_k. Let $t_0 \notin \{T_k\}_{k=1}^M$, and suppose for a contradiction that there is a sequence $t_j \uparrow t_0$ and a sequence $\tilde{t}_j \downarrow t_0$ such that x_ℓ is in the support of

$$\mu_L^{t_0} := \lim_{t_j \uparrow t_0} \mu^{t_j} \quad \text{and of} \quad \mu_R^{t_0} := \lim_{\tilde{t}_j \downarrow t_0} \mu^{\tilde{t}_j},$$

but that

$$\lim_{t_j \uparrow t_0} \theta(x_\ell) \quad \text{and} \quad \lim_{\tilde{t}_j \downarrow t_0} \theta(x_\ell)$$

exist and are not equal. (A similar argument rules out being able to find limits $\mu_L^{t_0}$ and $\mu_R^{t_0}$ that have different supports.) Choose $\Phi \in W^{1,\infty}$ such that $\Phi = 1$ in a neighborhood of $x_\ell(t_0)$, $|\Phi(x)| \leq 1$, and $x_\ell(t_0)$ is the only point in $\mathrm{sppt}\Phi \cap (\mathrm{sppt}\mu_L^{t_0} \cup \mathrm{sppt}\mu_R^{t_0})$. Because $\theta \in \mathbb{Z}^+$, $|\mu_L^{t_0}(\Phi) - \mu_R^{t_0}(\Phi)| \geq c_0$. However, the estimate from the proof of Theorem 1.1 implies:

$$
\begin{aligned}
c_0 &\leq |\mu_L^{t_0}(\Phi) - \mu_R^{t_0}(\Phi)| \\
&= \lim_{j \to \infty} |\mu^{t_j}(\Phi) - \mu^{\tilde{t}_j}(\Phi)| \\
&= \lim_{j \to \infty} \lim_{i \to \infty} \left| \mu_i^{t_j}(\Phi) - \mu_i^{\tilde{t}_j}(\Phi) \right| \\
&\leq \lim_{j \to \infty} \lim_{i \to \infty} \int_0^1 \int_{t_j}^{\tilde{t}_j} \left(-\Phi' \varepsilon_i u_{i,x} \dot{u}_i - \Phi \varepsilon_i u_{i,xx} \dot{u}_i + \Phi \frac{V'}{\varepsilon_i} \dot{u}_i \right) dt \, dx \\
&\leq \lim_{j \to \infty} \lim_{i \to \infty} \frac{1}{2} \int_0^1 \int_{t_j}^{\tilde{t}_j} \left((\Phi')^2 \varepsilon_i (u_{i,x})^2 + \varepsilon_i \dot{u}_i^2 + \Phi \varepsilon_i \dot{u}_i^2 + \Phi \varepsilon_i^{-1} f_i^2 \right) dx \, dt \\
&\leq \lim_{j \to \infty} \frac{|\tilde{t}_j - t_j| \, ||\Phi||_{W^{1,\infty}}^2}{2} \sup_t \int_0^1 \varepsilon_i (u_{i,x})^2 \, dx + b \\
&= \frac{c_0}{4}.
\end{aligned}
$$

This contradiction proves that μ^t is well-defined and continuous at all times which are not singular times.

$\square$

2.3.2 Proof of Corollary 1

Proof. The proof is similar to the time-independent case. By absolute continuity of $|\xi^t|$ with respect to μ^t and Theorem 1.2, we conclude that $|\xi^t|$ is supported on points, but we will use the following bound to rule out concentration.

The heart of the matter is the fact that for any subsequence as in Theorem 1.1,

$$\sup_{x \in [0,1]} \left| \frac{\varepsilon_i}{2}(u_{i,x})^2 - \frac{V(u_i)}{\varepsilon_i} \right| \leq \mathcal{A} + ||\varepsilon_i^{-1/2} f_i||_{L_x^2} ||\varepsilon_i^{1/2} u_{i,x}||_{L_x^2}. \tag{2.18}$$

(This is proved by exactly the same argument we used to prove Lemma 2.1.) If $|\xi^t| dt$ is not the zero measure, then there exist points $y(t)$ such that

$$\lim_{r \to 0} \lim_{\varepsilon_i \to 0} \int_0^T \int_{B_r(y(t))} |\xi_{\varepsilon_i}^t| \, dx \, dt \; > \; 0.$$

However, by (2.18),

$$
\begin{aligned}
\int_0^T \int_{B_r(y(t))} |\xi_{\varepsilon_i}^t| \, dx \, dt \; &\leq \; 2r \int_0^T \sup_{B_r(y(t))} \left| \frac{\varepsilon_i}{2}(u_{i,x})^2 - \frac{V(u_i)}{\varepsilon_i} \right| \, dt \\
&\leq \; 2r \int_0^T \left(\mathcal{A} + ||\varepsilon_i^{-1/2} f_i||_{L_x^2} ||\varepsilon_i^{1/2} u_{i,x}||_{L_x^2} \right) dt \\
&\leq \; 2r \left(\mathcal{A}T + ||\varepsilon_i^{-1/2} f_i||_{L_t^2(L_x^2)} ||\varepsilon_i^{1/2} u_{i,x}||_{L_t^2(L_x^2)} \right) \\
&\leq \; 2r \left(\mathcal{A}T + cT^{1/2} \right),
\end{aligned}
$$

which vanishes as $r \to 0$. $\qquad\square$

3   Bounding the Action Cost and Functional

Using the results of Section 2, we return to question of the sharp-interface limit of the action functional. Recall that the heuristic argument predicts that action minimizers should jump immediately to their maximal energy (nucleate all the interfaces at time $t = 0$), and then propagate sharp, optimal-profile interfaces at a constant speed. Theorem 1.3, the sharp lower bound for action minimizers, confirms that this behavior leads to the optimal action; we present its proof in Subsection 3.1. Then in Subsection 3.2, we generalize to a lower bound for the functional itself, under an assumption of single-multiplicity interfaces. The result is Theorem 1.4, which bounds below the action of a sequence u_i by the appropriately defined reduced action of its sharp-interface limit, u_0. See the end of Subsection 3.2 for additional discussion.

3.1 The Lower Bound for the Minimal Action

In this subsection, we prove the sharp lower bound for the limiting action. The singular times of the measure μ contribute to the "nucleation cost" of the action. On the remainder of the time interval, the "propagation cost" is calculated, using the continuity of the measure (Theorem 1.1), the structure of the measure (Theorem 1.2), and the limiting equipartition of energy (Corollary 1) to get the sharp constant. We first state two lemmas which we will use in the proof of the theorem.

Lemma 3.1 (Control in time). *Consider any sequence of smooth, action-bounded functions $u_i : [0,1] \times [0,T] \to \mathbb{R}$, with Neumann boundary conditions and uniformly bounded initial energy. Then there is a constant c such that for any times t, s, for which $u_i(\cdot, t) \xrightarrow{L^3} u_0(\cdot, t)$ and $u_i(\cdot, s) \xrightarrow{L^3} u_0(\cdot, s)$, we have*

$$|I(t; r, \hat{x}) - I(s; r, \hat{x})| \leq c|t - s|^{1/2},$$

for

$$I(t; r, \hat{x}) := \int_{B_r(\hat{x})} \left(u_0 - \frac{u_0^3}{3} \right)(\cdot, t) dx. \tag{3.1}$$

Remark 3. *Notice that (3.1) is equal to*

$$\frac{2}{3} \left(m(\{x \in B_r(\hat{x}); u_0(x, t) = 1\}) - m(\{x \in B_r(\hat{x}); u_0(x, t) = -1\}) \right),$$

since $u_0 = \pm 1$ almost everywhere, by the action bound of the energy.

Proof. By the L^3 convergence of u_i,

$$I(t; r, \hat{x}) = \lim_{i \to \infty} \int_{B_r(\hat{x})} \left(u_i(x, t) - \frac{u_i^3}{3}(x, t) \right) dx.$$

For any two times t, s, we write

$$|I(t; r, \hat{x}) - I(s; r, \hat{x})|$$
$$= \left| \int_{B_r(\hat{x})} \left(u_i(x, t) - \frac{u_i^3}{3}(x, t) - u_i(x, s) + \frac{u_i^3}{3}(x, s) \right) dx \right|$$
$$= \left| \int_s^t \int_{B_r(\hat{x})} \dot{u}_i(1 - u_i^2) \, dx \, dt' \right|$$
$$\leq \left(\int_s^t \int_{B_r(\hat{x})} \varepsilon_i \dot{u}_i^2 \, dx \, dt' \right)^{1/2} \left(\int_s^t \int_{B_r(\hat{x})} \frac{(1 - u_i^2)^2}{\varepsilon_i} \, dx \, dt' \right)^{1/2} \tag{3.2}$$
$$\leq c|t - s|^{1/2},$$

by the action and energy bounds, where in (3.2), we have used

$$\int_s^t \int_{B_r(\hat{x})} \varepsilon_i \dot{u}_i^2 \, dx \, dt' \leq 4S[u_i] + 2 \left| \int_0^1 \left(\frac{\varepsilon_i}{2} (u_{i,x})^2 + \frac{(1-u_i^2)^2}{4\varepsilon_i} \right) dx \right|_s^t \right| \leq c.$$

$\square$

Lemma 3.2 (Convergence of a subsequence). *Consider any sequence of smooth, action-bounded functions $u_i : [0,1] \times [0,T] \to \mathbb{R}$, which satisfy Neumann boundary conditions and the initial condition $u_i(x,0) \equiv -1$. Then there is a subsequence u_i which converges almost everywhere and in every L^p to a limit u_0. Furthermore, $u_0 = \pm 1$ almost everywhere in space for every time.*

Proof. Since the energy at any time is bounded above by the action, for any time $t' \in [0,T]$, we can use the usual method [17] to choose a subsequence such that $u_i(\cdot, t') \to u_0(\cdot, t')$ almost everywhere and in every L^p, and $u_0(\cdot, t') = \pm 1$ almost everywhere. Choose a countable, dense set $D \subset [0,T]$ and a diagonal subsequence such that u_i converges for each $t \in D$. We claim that in fact, u_i converges for every $t \in [0,T]$.

First, for any $t \in [0,T]$, choose a sequence t_j with $t_j \in D$ and $t_j \to t$. Let

$$v_j(x) := u_0(x, t_j).$$

We claim that

$$v(x) := \lim_{j \to \infty} v_j(x)$$

is well-defined. To see this, consider that by the (uniform in time) energy bound, every subsequence $u_0(x, t_{j_k})$ has a subsequence converging in $L^p([0,1])$ to a limit which is ± 1 almost everywhere. If there were two subsequences $t_{j_k} \to t$ and $\tilde{t}_{j_k} \to t$ such that the corresponding functions had distinct limits, this plus Lemma 3.1 would lead to a contradiction. Thus, v is the L^p-limit of the whole sequence.

Next, by a similar argument, we claim that

$$\lim_{i \to \infty} u_i(x, t) = v(x).$$

This is true because the energy bound implies that any subsequence has a subsequence converging in L^p to a limit which is ± 1 a.e., and as above, if the limit were not equal to v, Lemma 3.1 would give a contradiction. $\square$

We turn to the proof of the main theorem.

Proof of Theorem 1.3. Without loss, assume $S_{\varepsilon_i}[u_i]$ converges to some number, S_0. Using Theorem 1.2 and Lemma 3.2, choose a subsequence such that μ_i converges as described and such that u_i converges to u_0 a.e. and in $L^3([0,1])$ for every $t \in [0,T]$. Let

$$0 = T_1 < T_2 < \ldots T_{M-1} < T_M = T$$

denote all the singular times from Theorem 1.2 (adding, if necessary, $t = 0$ and $t = T$), and define the set $F_{sing} := \{T_1, T_2, \ldots T_M\}$. We will count the action separately near the singular times and away from the singular times, calling the former cost the nucleation cost, and the latter the propagation cost. It is convenient to introduce an artificial distance δ, which we will later send to zero. Let $\delta < \min_{k \neq j} |T_k - T_j|$, and let

$$A_\delta := \bigcup_{j=1}^{M} \left([T_j - \delta, T_j + \delta] \cap [0,T] \right).$$

We define the functionals:

$$S_{\varepsilon_i}^{nuc,\delta}[u] := \frac{1}{4} \int_{A_\delta} \int_0^1 \left(\varepsilon_i^{1/2} \dot{u}_i - \varepsilon_i^{-1/2} f_{\varepsilon_i} \right)^2 dx \, dt,$$

and

$$S_{\varepsilon_i}^{prop,\delta}[u] := \frac{1}{4} \int_{[0,T] \setminus A_\delta} \int_0^1 \left(\varepsilon_i^{1/2} \dot{u}_i - \varepsilon_i^{-1/2} f_{\varepsilon_i} \right)^2 dx \, dt.$$

We have the relation

$$
\begin{aligned}
S_0 &= \lim_{i \to \infty} S_{\varepsilon_i}[u_i] \\
&= \lim_{i \to \infty} \left(S_{\varepsilon_i}^{nuc,\delta}[u_i] + S_{\varepsilon_i}^{prop,\delta}[u_i] \right) \\
&\geq \liminf_{i \to \infty} S_{\varepsilon_i}^{nuc,\delta}[u_i] + \liminf_{i \to \infty} S_{\varepsilon_i}^{prop,\delta}[u_i] \\
&=: S^{nuc,\delta} + S^{prop,\delta}.
\end{aligned}
$$

Therefore, sending δ to zero,

$$S_0 \geq \lim_{\delta \to 0} \left(S^{nuc,\delta} + S^{prop,\delta} \right) =: S^{nuc} + S^{prop}.$$

(Nucleation Cost) To count the nucleation cost, we use the relation (1.10) to bound below the action on $[T_j - \delta, T_j + \delta]$ by

$$\lim_{i \to \infty} \left(E_{\varepsilon_i}[u_i(\cdot, T_j + \delta)] - E_{\varepsilon_i}[u_i(\cdot, T_j - \delta)] \right) = \mu^{T_j+\delta}([0,1]) - \mu^{T_j-\delta}([0,1]).$$

30

Notice that by the integer constraint and continuity of μ^t away from singular times, $\mu^t([0,1])$ is constant on (T_j, T_{j+1}) for all $j = 1 \ldots M$. We claim that

$$S^{nuc,\delta} \geq \max_{t \in F^c_{sing}} \mu^t([0,1]). \tag{3.3}$$

Indeed, suppose that the interval on which the maximum is achieved is (T_J, T_{J+1}). Then since the action on any time interval is positive, we get a lower bound by integrating over only

$$\bigcup_{j=1}^{J} \left([T_j - \delta, T_j + \delta] \cap [0,T]\right) \subseteq A_\delta,$$

so that

$$S^{nuc,\delta} \geq \sum_{j=2}^{J} \left(\mu^{T_j+\delta}([0,1]) - \mu^{T_j-\delta}([0,1])\right) + \mu^{T_1+\delta}([0,1]) - \mu^0([0,1]),$$

and since $\mu^0([0,1]) = 0$, the sum of differences telescopes to give (3.3).

We remark that according to Theorem 1.2,

$$\max_{t \in F^c_{sing}} \mu^t([0,1]) = c_0 \, N^*$$

for some $N^* \in \mathbb{Z}^+$. Furthermore, since the bound holds for all δ, we have

$$S^{nuc} \geq c_0 \, N^*.$$

(Propagation Cost) To count the propagation cost, we will use the functional

$$\hat{I}_{[s,t]}[u] := \int_s^t \int_0^1 \dot{u}(1-u^2)dx \, dt.$$

On the one hand,

$$\lim_{i \to \infty} \hat{I}_{[T_j+\delta, T_{j+1}-\delta]}[u_i] = \int_0^1 (u_0 - u_0^3/3)dx \Big|_{t=T_j+\delta}^{t=T_{j+1}-\delta}$$

$$= \frac{2}{3}\Delta m_j^\delta, \tag{3.4}$$

where Δm_j^δ compares the relative mass of $u = \pm 1$ at times $T_j + \delta$ and $T_{j+1} - \delta$:

$$\Delta m_j^\delta := \left(m(\{x; u_0(x,t) = 1\}) - m(\{x; u_0(x,t) = -1\}) \right) \Big|_{t=T_j+\delta}^{T_{j+1}-\delta}.$$

Below, we will use the fact that

$$\lim_{\delta \to 0} \sum_{j=1}^{M-1} \Delta m_j^\delta = 2, \qquad (3.5)$$

from the end conditions and the telescoping of the sum (which follows from Lemma 3.1).

From Hölder's inequality and the definition of V,

$$\hat{I}_{[s,t]}[u_i] \le 2 \left(\int_s^t \int_0^1 \varepsilon_i \dot{u}_i^2 dx\, dt' \right)^{1/2} \left(\int_s^t \int_0^1 \frac{V(u_i)}{\varepsilon_i} dx\, dt' \right)^{1/2}. \qquad (3.6)$$

The goal is to use the first term to represent the action and the second to represent the energy. For the first goal, we use the fact that $\mu^t([0,1])$ is constant on $[T_j + \delta, T_{j+1} - \delta]$.

$$\frac{1}{4} \int_{T_j+\delta}^{T_{j+1}-\delta} \int_0^1 \left(\varepsilon_i^{1/2} \dot{u}_i - \varepsilon_i^{-1/2} f_{\varepsilon_i} \right)^2 dx\, dt$$

$$\ge \frac{1}{4} \int_{T_j+\delta}^{T_{j+1}-\delta} \int_0^1 \varepsilon_i \dot{u}_i^2 dx\, dt + \frac{1}{2} \left(E[u_i(\cdot, T_{j+1} - \delta)] - E[u_i(\cdot, T_j + \delta)] \right), \quad (3.7)$$

with the energy difference vanishing in the limit $i \to \infty$ for every $\delta > 0$.

For the second goal, we apply the equipartition result from Corollary 1,

$$\lim_{i\to\infty} \int_{T_j+\delta}^{T_{j+1}-\delta} \int_0^1 \frac{V(u_i)}{\varepsilon_i}\, dx\, dt = \frac{1}{2} \mu^{T_j+\delta}([0,1]) \times (T_{j+1} - T_j - 2\delta) \qquad (3.8)$$

$$\le \frac{c_0\, N^*}{2} (T_{j+1} - T_j), \qquad (3.9)$$

by definition of N^*. Let $\Delta T_j := T_{j+1} - T_j$, and let

$$S_j^\delta := \liminf_{i\to\infty} \frac{1}{4} \int_{T_j+\delta}^{T_{j+1}-\delta} \int_0^1 \left(\varepsilon_i^{1/2} \dot{u}_i - \varepsilon_i^{-1/2} f_{\varepsilon_i} \right)^2 dx\, dt.$$

We send $i \to \infty$ and combine (3.4), (3.6), (3.7), and (3.9) to arrive at

$$\frac{2}{3} \Delta m_j^\delta \le 2 \left(4S_j^\delta \right)^{1/2} \left(\frac{c_0\, N^* \Delta T_j}{2} \right)^{1/2}.$$

Sending $\delta \to 0$, rearranging terms, and using the definition of $c_0 = 2\sqrt{2}/3$, we have

$$S_j^0 \ge \frac{c_0}{16} \frac{(\Delta m_j^0)^2}{N^* \Delta T_j}.$$

(We have let $S_j^0 := \lim_{\delta \to 0} S_j^\delta$ and $\Delta m_j^0 := \lim_{\delta \to 0} \Delta m_j^\delta$.) Using Jensen's inequality for x^2/y and (3.5), the sum is bounded below:

$$\sum_{j=1}^{M-1} \frac{(\Delta m_j^0)^2}{\Delta T_j} \geq \frac{(\sum_{j=1}^{M-1} \Delta m_i^0)^2}{\sum_{j=1}^{M-1} \Delta T_j} = \frac{4}{T}.$$

Summing the S_j^0 gives precisely S^{prop}. Thus,

$$S^{prop} \geq \frac{c_0}{4} \frac{1}{N^* T}.$$

$\square$

Remark 4. *In particular, the calculation above shows that if $\mu^t([0,1])$ is less than the maximal energy on any of the subintervals, the propagation cost is too high.*

3.2 A Lower Bound for the Action Functional

Finally, it is not hard to extend from Theorem 1.3 to Theorem 1.4 by localizing and refining the estimate of the propagation cost. Recall that we assume single-multiplicity interfaces almost everywhere. We comment on this assumption after the proof.

Proof of Theorem 1.4. First, we localize the estimate of the nucleation cost. (To see that this is necessary, consider a situation in which two delta masses collide and annihilate at time $t = T/2$ and position $x = 1/4$, while two other delta masses appear at time $t = T/2$ and position $x = 3/4$. The net energy remains constant, but there is a local jump (and corresponding contribution to the action functional) in a neighborhood of $x = 3/4$.) Choose $\phi \in C^1([0,1])$

with $0 \le \phi(x) \le 1$. Then, by the usual estimate,

$$\frac{1}{4} \int_{T_j-\delta}^{T_j+\delta} \int_0^1 (\varepsilon_i^{1/2} \dot{u}_i - \varepsilon_i^{-1/2} f_{\varepsilon_i})^2 \, dx \, dt$$

$$\ge \frac{1}{4} \int_{T_j-\delta}^{T_j+\delta} \int_0^1 (\varepsilon_i^{1/2} \dot{u}_i - \varepsilon_i^{-1/2} f_{\varepsilon_i})^2 \phi(x) \, dx \, dt$$

$$\ge \left(\int_{T_j-\delta}^{T_j+\delta} \int_0^1 -\dot{u}_i \, f_{\varepsilon_i} \, \phi(x) \, dx \, dt \right)^+$$

$$= \left(\int_{T_j-\delta}^{T_j+\delta} \int_0^1 \frac{d}{dt} \left(\frac{\varepsilon_i}{2} (u_{i,x})^2 + \frac{V(u_i)}{\varepsilon_i} \right) \phi(x) + \varepsilon_i \, u_{i,x} \dot{u}_i \, \phi'(x) \, dx \, dt \right)^+$$

$$\ge \left(\int_0^1 \phi(x) \int_{T_j-\delta}^{T_j+\delta} \frac{d}{dt} \left(\frac{\varepsilon_i}{2} (u_{i,x})^2 + \frac{V(u_i)}{\varepsilon_i} \right) dt \, dx \right)^+ - O(\delta^{1/2}) \quad (3.10)$$

$$\underset{i \to \infty}{\to} \left(\mu^{T_j+\delta}(\phi) - \mu^{T_j-\delta}(\phi) \right)^+ - O(\delta^{1/2}), \quad (3.11)$$

where the error term in (3.10) has been estimated:

$$\int_{T_j-\delta}^{T_j+\delta} \int_0^1 (\varepsilon_i \, u_{i,x} \dot{u}_i \phi'(x)) \, dx \, dt$$

$$\le ||\phi||_{C^1([0,1])} \left(\int_{T_j-\delta}^{T_j+\delta} \int_0^1 \varepsilon_i u_{i,x}^2 dx \, dt \right)^{1/2} \left(\int_{T_j-\delta}^{T_j+\delta} \int_0^1 \varepsilon_i \dot{u}_i^2 dx \, dt \right)^{1/2}$$

$$\le c(2\delta)^{1/2} ||\phi||_{C^1([0,1])},$$

by the action and energy bounds. Sending $\delta \to 0$ in (3.11) and taking a supremum over ϕ gives the desired bound for the nucleation cost:

$$S^{nuc} \ge \sum_{j=1}^{M-1} \sup_{0 \le \phi \le 1} \left(\mu^{T_j^+}(\phi) - \mu^{T_j^-}(\phi) \right)^+.$$

Now consider the propagation cost. We need a local estimate, so without loss of generality, consider the time interval between two consecutive singular times, T_1 and T_2, and two interfaces, g_1 and g_2. Since (T_1, T_2) contains no singular times, g_1 and g_2 are continuous on (T_1, T_2). Therefore, the set

$$(T_1, T_2) \setminus \{t \mid g_1(t) = g_2(t)\}$$

is open and may be decomposed into a countable, disjoint union of open intervals, $\{I_\nu\}_{\nu=1}^K$. By the single-multiplicity assumption,

$$m(\{g_1(t) = g_2(t)\}) = 0,$$

and so

$$\int_{T_1}^{T_2} \dot{g}_j^2(t)dt = \sum_{\nu=1}^{K} \int_{I_\nu} \dot{g}_j^2(t)dt, \quad j = 1, 2.$$

Thus, it is enough to show that each interval, I_ν,

$$\frac{c_0}{4} \sum_{j=1}^{2} \int_{I_\nu} \dot{g}_j^2(t) \, dt \leq \frac{1}{4} \liminf_{i \to \infty} \int_{I_\nu} \int_0^1 \varepsilon_i \dot{u}_i^2 \, dx \, dt. \tag{3.12}$$

Therefore, consider any I_ν. For each $t \in I_\nu$, $\gamma(t) := \mathrm{dist}(g_1(t), g_2(t)) > 0$. Choose $\phi_j \in C_0^1([0,1] \times I_\nu)$ such that for every $t \in I_\nu$,

$$\mathrm{sppt}\, \phi_j(\cdot, t) \subset B_{\gamma/2}(g_j(t)) =: C_j^\gamma(t).$$

We fix j and, for ease of notation, drop the subscripts on g and ϕ. It will be convenient to use the dual representation,

$$\left(\int_{I_\nu} \dot{g}^2(t) \, dt \right)^{1/2} = \sup_{\phi} \left\{ \int_{I_\nu} \phi(g(t), t) \, \dot{g}(t) \, dt \, \bigg| \int_{I_\nu} \phi^2(g(t), t) \, dt \leq 1 \right\}. \tag{3.13}$$

By the compact support of ϕ in time,

$$0 = \int_{I_\nu} \frac{d}{dt} \int_0^{g(t)} \phi \, dx \, dt = \int_{I_\nu} \int_0^{g(t)} \dot{\phi} \, dx \, dt + \int_{I_\nu} \phi(g(t), t)\dot{g}(t) \, dt,$$

and

$$0 = \int_{I_\nu} \frac{d}{dt} \int_{g(t)}^1 \phi \, dx \, dt = \int_{I_\nu} \int_{g(t)}^1 \dot{\phi} \, dx \, dt - \int_{I_\nu} \phi(g(t), t)\dot{g}(t) \, dt.$$

Combining these two identities, we deduce

$$\int_{I_\nu} \phi(g(t), t) \, \dot{g}(t) \, dt = \frac{1}{2} \left(-\int_{I_\nu} \int_0^{g(t)} \dot{\phi} \, dx \, dt + \int_{I_\nu} \int_{g(t)}^1 \dot{\phi} \, dx \, dt \right) \tag{3.14}$$

which is convenient for passing to the limit, since by assumption, $g(t) \in \partial\{u_0(t) = 1\}$ and it is the only such boundary point in the support of ϕ. Thus, we may assume without loss of generality that

$$\begin{cases} u_0(\cdot, t) = +1 & \text{on } (0, g(t)) \cap \mathrm{sppt}\phi(\cdot, t), \\ u_0(\cdot, t) = -1 & \text{on } (g(t), 1) \cap \mathrm{sppt}\phi(\cdot, t). \end{cases}$$

Therefore, we can reexpress (3.14) as a limit and calculate:

$$
\int_{I_\nu} \phi(g(t), t)\, \dot g(t)\, dt
$$
$$
= -\frac{3}{4} \lim_{i\to\infty} \int_{I_\nu} \int_0^1 \dot\phi\, (u_{\varepsilon_i} - u_{\varepsilon_i}^3/3)\, dx\, dt
$$
$$
= \frac{3}{4} \lim_{i\to\infty} \int_{I_\nu} \int_0^1 \phi\, (1 - u_{\varepsilon_i}^2) \dot u_{\varepsilon_i}\, dx\, dt
$$
$$
\leq \frac{3}{4} \lim_{i\to\infty} \left(\int_{I_\nu} \int_0^1 \phi^2 \frac{(1 - u_{\varepsilon_i}^2)^2}{\varepsilon_i}\, dx\, dt \right)^{1/2} \left(\int_{I_\nu} \int_{\mathrm{sppt}\phi(\cdot,t)} \varepsilon_i \dot u_{\varepsilon_i}^2\, dx\, dt \right)^{1/2}
$$
$$
\leq \frac{3\,(2\,c_0)^{1/2}}{4} \lim_{i\to\infty} \left(\int_{I_\nu} \int_{\mathrm{sppt}\phi(\cdot,t)} \varepsilon_i \dot u_{\varepsilon_i}^2\, dx\, dt \right)^{1/2}, \tag{3.15}
$$

where in the last step, we have used equipartition of energy (Corollary 1) and the assumption of single-multiplicity. Recalling (3.13), squaring, and using the definition of c_0, we conclude that:

$$
\frac{1}{4} \lim_{i\to\infty} \int_{I_\nu} \int_{\mathrm{sppt}\phi(\cdot,t)} \varepsilon_i \dot u_{\varepsilon_i}^2\, dx\, dt \geq \frac{c_0}{4} \int \dot g^2(t)\, dt.
$$

Because the ϕ_j have disjoint support, we may take the sum and derive the desired bound, (3.12). $\qquad\square$

Theorem 1.4 resembles the lower-bound half of a Γ-convergence argument. The matching upper bound requires proving that given any "interface-function," u_0, there is a sequence, u_ε, such that $u_\varepsilon \to u_0$ and

$$
\lim_{\varepsilon\to 0} S_\varepsilon[u_\varepsilon] = S_0[u_0].
$$

We believe that there is no problem generalizing the construction in [16] to produce such a sequence, gluing together hyperbolic tangent profiles whose zeros follow the discontinuities of u_0.

Our result falls short of a real Γ-convergence theorem because we have made the assumption of multiplicity-one interfaces. The proof of Theorem 1.4 may be generalized to higher-multiplicity interfaces, but the lower bound does not appear to be sharp. To be precise, the local propagation estimate (3.15) for an interface of multiplicity n is *inversely* proportional to n, whereas we expect that there is a sharp propagation estimate which is directly proportional to n.

Acknowledgements

References

[1] G. Bellettini and L. Mugnai, *On the approximation of the elastica functional in radial symmetry*, accepted Calc. Var. PDE (2004).

[2] A. Braides and R. March, *Approximation by Γ-convergence of a curvature-depending functional in visual reconstruction*, preprint (2004).

[3] K. Brakke, *The motion of a surface by its mean curvature*, Princeton University Press, Princeton, NJ, 1978.

[4] T. F. Chan, S. H. Kang, and J. Shen, *Euler's elastica and curvature-based inpainting*, SIAM J. Appl. Math. **63** (2002), pp. 564–592.

[5] A. De Masi, N. Dirr, and E. Presutti, *Instability of interface under forced displacements*, Max-Planck-Institut für Mathematik (2005), preprint no. 5.

[6] Q. Du, C. Liu, X. Wang, *A phase field approach in the numerical study of the elastic bending energy for vesicle membranes*, Journal of Computational Physics, **198** (2004), 450–468.

[7] S. Esedoglu and J. Shen, *Digital inpainting based on the Mumford-Shah-Euler image model*, European J. Appl. Math. **13** (2002), pp. 353–370.

[8] L. C. Evans and J. Spruck, *Motion of level sets by mean curvature*, J. Differential Geom. **33** (1991), 635–681.

[9] W. E, W. Ren, and E. Vanden-Eijnden, *Minimum action method for the study of rare events*, Comm. Pure App. Math. **57** (2004), 637–656.

[10] W. G. Faris and G. Jona-Lasinio, *Large fluctuations for a nonlinear heat equation with noise*, J. Phys. A: Math. Gen. **15** (1982), 3025–3055.

[11] H. C. Fogedby, J. Hertz, and A. Svane, *Domain wall propagation and nucleation in a metastable two-level system*, Phys. Rev. E **70** (2004), 031105.

[12] M. I. Freidlin and A. D. Wentzell, *Random Perturbations of Dynamical Systems*, Springer-Verlag, 2nd edition, New York, 1998.

[13] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom. **31** (1990), 285–299.

[14] J. Hutchinson and Y. Tonegawa, *Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory*, Calc. Var. PDE **10** (2000), 49–84.

[15] T. Ilmanen, *Convergence of the Allen Cahn equation to Brakke's motion by mean curvature*, J. Differential Geom. **38** (1993), 417–461.

[16] R. V. Kohn, F. Otto, M. G. Reznikoff, and E. Vanden-Eijnden, *Action minimization and sharp-interface limits for the stochastic Allen-Cahn equation*, submitted to Comm. Pure Appl. Math.

[17] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Rational Mech. Anal. **98** (1987), 123–142.

[18] L. Modica and S. Mortola, *Il limite nella Γ-convergenza di una famiglia di funzionali ellittici*, Boll. Un. Mat. Ital. A **14** (1977), 526–529.

[19] R. Moser, *A higher order asymptotic problem related to phase transitions*, preprint (2004).

[20] Y. Tonegawa, *Phase field model with a variable chemical potential*, Proc. Roy. Soc. Edinburgh **132A** (2002), 993–1019.