博士学位論文

Two Constructions of Hopf Algebroids Based on the FRT Construction and Their Relations
（FRT 構成法に基づくホップ亜代数の 2 つの構成法とそれらの関係性）

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Two Constructions of Hopf Algebroids Based on the FRT Construction and Their Relations

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Abstract

We introduce two constructions of Hopf algebroids as generalizations of the FRT construction. We construct the Hopf algebroid A_σ by using a rigid family σ of elements in an arbitrary algebra L. If L is not Frobenius-separable, then this Hopf algebroid A_σ is not a weak Hopf algebra. The rigid family w of elements in an arbitrary algebra R gives birth to another Hopf algebroid $\mathfrak{U}(w)$. The setting of this $\mathfrak{U}(w)$ is similar to that of Hayashi's (Hopf) face algebras $\mathfrak{A}(w)$ and of $\mathrm{Hc}(\mathfrak{A}(w))$. We can show that the Hopf algebroid $\mathfrak{U}(w)$ is a partial generalization of A_σ by constructing a strict Hopf algebroid isomorphism $\Phi\colon \mathfrak{U}(w) \to A_\sigma$.

Part I
Introduction

Background

Many generalizations of bialgebras and Hopf algebras [14, 28] have been proposed. A pioneering work is Takeuchi's $\times_L$-bialgebra [29]. These bialgebras are modules over commutative rings with comultiplications and counits, while $\times_L$-bialgebras are defined by the utilization of modules over non-commutative rings called base rings. Later Lu [16] also introduced the notion of left bialgebroids (he called bialgebroids simply), which turned out to be equivalent to the notion of $\times_L$-bialgebras in [4]. Schauenburg [22] proposed the Hopf algebraic structure on the left bialgebroid, which is called a $\times_L$-Hopf algebra. Although this $\times_L$-Hopf algebra does not have the antipode, the Hopf algebroid, introduced by Böhm and Szlachányi [3], is a left bialgebroid with a bijective antipode. This Hopf algebroid is a special case of the $\times_L$-Hopf algebra.

Hayashi introduced the notion of (Hopf) face algebras in [11, 12]. The paper [21] showed that this is a special case of the left bialgebroid with a commutative separable base ring. Later Böhm, Nill, and Szlachányi [2] generalized (Hopf) face algebras to weak bialgebras (weak Hopf algebras). Schauenburg [23] showed that weak bialgebras are left bialgebroids with (non-commutative) Frobenius-separable base rings. Moreover, if the left bialgebroid has a Frobunius-separable base ring, then this is a weak bialgebra.

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We can construct bialgebras by using solutions to the quantum Yang-Baxter equation (QYBE for short [1, 18, 30]). This way is called the FRT construction and introduced by Faddeev, Reshetikhin, and Takhtajan [8]. Bialgebras by this construction and their quotients give birth to all coquasitriangular bialgebras (for example, see [13]).

There are many generalizations of this construction.

The quantum dynamical Yang-Baxter equation (QDYBE for short) was introduced by Gervais and Neveu [10] in the study of the string field theory. After the study of elliptic quantum groups by Felder [9], Etingof and Varchenko [7] showed that solutions to the QDYBE, called dynamical R-matrices, give birth to $\mathfrak{h}$-bialgebroids which are special cases of left bialgebroids. They also studied a condition that this $\mathfrak{h}$-bialgebroid becomes a $\mathfrak{h}$-Hopf algebroid. This condition is called rigidity.

As a set-theoretical analogue of the QYBE, Drinfel'd [5] proposed the Yang-Baxter map (YBM for short). The FRT construction for the YBM, a special case of the ordinary FRT construction, was mentioned in [6]

Shibukawa [24, 25] generalized the YBM to the dynamical Yang-Baxter map (DYBM for short) whose solutions depend on the dynamical parameter. In [27], Shibukawa and Takeuchi constructed the left bialgebroid A_σ by using the DYBM. Similar to $\mathfrak{h}$-bialgebroids by R-matrices, if the DYBM σ satisfies rigidity, then this left bialgebroid A_σ becomes a Hopf algebroid with a bijective antipode [26].

Hayashi [11] introduced construction of face algebras by using solutions to the face-type QYBE which is defined by the utilization of the two-fold fiber product of the quiver. Similar to ordinary FRT bialgebras, these face algebras $\mathfrak{A}(w)$ are coquasitriangular face algebras. In [13], Hayashi showed that the Hopf closure $\mathrm{Hc}(\mathfrak{H})$ exists for a coquasitriangular face algebra $\mathfrak{H}$ satisfying closurability. The Hopf closure $\mathrm{Hc}(\mathfrak{H})$ is a Hopf face algebra satisfying a certain universal property. If a solution w to the quiver-theoretical QYBE satisfies closurability, then the face algebra $\mathfrak{A}(w)$ also has the Hopf closure $\mathrm{Hc}(\mathfrak{A}(w))$.

It is interesting to investigate relations between the two algebras A_σ and $\mathfrak{A}(w)$. Matsumoto and Shimizu [17] showed that the DYBM σ with a finite parameter set Λ gives birth to a solution w_σ to the quiver-theoretical QYBE. Because the base ring $\overline{M}$ consisting of all maps from Λ to the field k is Frobenius-separable, the left bialgebroid A_σ is a weak bialgebra [15]. In [17], a weak bialgebra homomorphism ϕ from $\mathfrak{A}(w_\sigma)$ to A_σ was also constructed. Although a weak Hopf algebra homomorphism $\phi' \colon \mathrm{Hc}(\mathfrak{A}(w_\sigma)) \to A_\sigma$ is given if the solution w_σ is closuable and σ is rigid, we do not know properties of ϕ' and relations between rigidity and closuability.

Shibukawa-Takeuchi's construction of left bialgebroids has one problem. Because the base ring is Frobenius-separable if the DYBM has a finite parameter set Λ, it is difficult to get a left bialgebroid A_σ that is not a weak bialgebra.

In this thesis, we have two purposes. The first is to gain a simpler construction of the Hopf algebroid A_σ that does not have a weak Hopf algebra structure, even if the parameter set Λ is finite. This research is based on a joint work [20] with Youichi Shibukawa. We try to generalize the base ring $\overline{M}$ to an algebra L over a commutative ring k. If R is a not Frobenius-separable k-algebra, then the Hopf algebroid A_σ whose base ring M is a k-algebra consisting of all maps from Λ to R does not have a weak Hopf algebra structure.

The second is to construct a Hopf algebroid $\mathfrak{U}(w)$ and to show relations between $\mathfrak{U}(w)$ and A_σ. In order to construct this $\mathfrak{U}(w)$, we use similar settings to the construction of the (Hopf) face algebras $\mathfrak{A}(w)$ and $\mathrm{Hc}(\mathfrak{A}(w))$ in [11, 13]. We also induce a left and right bialgebroid $\mathfrak{U}(w_\sigma) := \mathfrak{U}(w)$ from the DYBM σ and construct a left and right bialgebroid isomorphism $\Phi\colon \mathfrak{U}(w_\sigma) \to A_\sigma$ (see [17]). $\mathfrak{U}(w_\sigma)$ becomes a quiver-theoretical generalization of A_σ with the base ring M. We can show that $\mathfrak{U}(w_\sigma)$ is a Hopf algebroid if and only if so is A_σ.

In [19], we generalize Hayashi's weak bialgebra $\mathfrak{A}(w)$ in [13] and Matsumoto-Shimizu's homomorphism ϕ in [17]. The left and right bialgebroid $\mathfrak{U}(w)$ in this thesis has two kinds of generators $\{\mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}\}$ and $\{\widetilde{\mathbf{e}}\begin{bmatrix} p \\ q \end{bmatrix}\}$, while $\mathfrak{A}(w)$ in [19] consists of one kind of generators $\{\mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}\}$. As a result, the homomorphism ϕ in [19] is not necessarily bijective.

Organization of this thesis

The thesis is organized as follows. In Part II, by using a family σ of elements in an arbitrary algebra L, we discuss the construction of the Hopf algebroid A_σ and its example based on [20], which is a joint work with Youichi Shibukawa. In Section 1, we give a brief introduction of left and right bialgebroids following [3]. If the left bialgebroid has a ring anti-automorphism satisfying certain conditions, then this left bialgebroid becomes a Hopf algebroid. The Hopf algebroid has an algebraic structure with both left and right bialgebroid structures. In Section 2, the left bialgebroid A_σ is constructed as a generalization of [26]. We need a family σ of elements in the base ring L for this construction. The condition that σ satisfies is based on the invariance condition of the DYBM. In Section 3, we show that A_σ is a right bialgebroid if σ has a stronger condition than that given in Section 2. We introduce the rigidity condition with respect to the family σ in Section 4. By virtue of this condition, we can construct an algebra anti-homomorphism S, which makes the pair (A_σ, S) a Hopf algebroid. Since the rigidity condition is not effective in constructing a Hopf algebroid A_σ, we give a sufficient condition for σ to be rigid. In Section 5, we give examples of the rigid σ. In order to show the rigidity condition, we use a sufficient condition given in the previous section. We also introduce an example of the base ring L that is not Frobenius-separable.

In Part III, we clarify conditions of a family w of elements in an arbitrary algebra in order to construct the Hopf algebroid $\mathfrak{U}(w)$. Moreover, we discuss relations between the left and right bialgebroids $\mathfrak{U}(w)$ and A_σ. In Section 6, we introduce a left bialgebroid $\mathfrak{U}(w)$. Let k be a commutative ring, R a k-algebra, and Q a finite quiver over a non-empty finite set Λ. We denote by M the k-algebra consisting of all maps from Λ to R. If the family w of elements in R satisfies a certain condition based on the quiver homomorphism on the two-fold fiber product, the left bialgebroid $\mathfrak{U}(w)$ can be constructed whose base ring is M. In Section 7, we show this $\mathfrak{U}(w)$ has a right bialgebroid structure if w satisfies a similar condition in Section 6. The rigidity condition for w is given in Section 8. Similar to the rigidity for σ, this w induces an algebra anti-homomorphism S of $\mathfrak{U}(w)$ and the pair $(\mathfrak{U}(w), S)$ is consequently a Hopf algebroid. In Section 9, we show that the family w can be induced from the family σ of elements in M which satisfies a stronger condition than that of Section 3. By virtue of these w and σ, the left and right bialgebroids $\mathfrak{U}(w_\sigma) := \mathfrak{U}(w)$ and A_σ can be constrcted. Moreover, we can construct a left and right bialgebroid isomorphism $\Phi\colon \mathfrak{U}(w_\sigma) \to A_\sigma$. By virtue of this isomorphism, the rigidity of w can induce that of σ and vise versa.

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Part II
Hopf algebroid A_σ

In this part, we construct a Hopf algebroid A_σ by using a rigid family σ consisting of elements in the base ring. This is a joint work with Youichi Shibukawa [20].

1 Preliminaries

In this section, we review the notion of left and right bialgebroids and Hopf algebroids following [3].

Definition 1.1. Let A and L be associative and unital rings. A sextuplet $\mathcal{A}_L = (A, L, s_L, t_L, \Delta_L, \pi_L)$ is called a left bialgebroid if the following conditions are satisfied:

1. The maps $s_L\colon L \to A$ and $t_L\colon L^{op} \to A$ are ring homomorphisms such that
$$s_L(l)t_L(l') = t_L(l')s_L(l) \ \ (\forall l, l' \in L). \tag{1.1}$$
Here L^{op} stands for the opposite ring of L. By using these maps s_L and t_L, we can give the ring A an (L, L)-bimodule structure ${}_LA_L$. The left and right actions ${}_LA$ and A_L can be defined by the following:
$${}_LA\colon l \cdot a = s_L(l)a; \ \ A_L\colon a \cdot l = t_L(l)a \ \ (l \in L, a \in A). \tag{1.2}$$

2. The triple $({}_LA_L, \Delta_L, \pi_L)$ is a comonoid in the category of (L, L)-bimodules satisfying
$$a_{[1]}t_L(l) \otimes a_{[2]} = a_{[1]} \otimes a_{[2]}s_L(l); \tag{1.3}$$
$$\Delta_L(1_A) = 1_A \otimes 1_A; \tag{1.4}$$
$$\Delta_L(ab) = \Delta_L(a)\Delta_L(b); \tag{1.5}$$
$$\pi_L(1_A) = 1_L; \tag{1.6}$$
$$\pi_L(as_L(\pi_L(b))) = \pi_L(ab) = \pi(at_L(\pi_L(b))) \tag{1.7}$$
for all $l \in L$ and $a, b \in A$. Here we use Sweedler's notation written by $\Delta_L(a) = a_{[1]} \otimes a_{[2]}$. The right-hand-side of (1.5) makes sense because of (1.3).

The ring A and L are called the total ring and the base ring, respectively. In order to avoid ambiguity, we may write $\mathcal{A}_L = (A, L, s_L^A, t_L^A, \Delta_L^A, \pi_L^A)$.

Definition 1.2. Let $\mathcal{A}_L = (A, L, s_L, t_L, \Delta_L, \pi_L)$ and $\mathcal{A}'_{L'} = (A', L', s_{L'}, t_{L'}, \Delta_{L'}, \pi_{L'})$ be left bialgebroids. A left bialgebroid homomorphism is a pair of ring homomorphisms $(\Phi\colon A \to A', \phi\colon L \to L')$ such that

$$s_{L'} \circ \phi = \Phi \circ s_L; \tag{1.8}$$
$$t_{L'} \circ \phi = \Phi \circ t_L; \tag{1.9}$$
$$\pi_{L'} \circ \Phi = \phi \circ \pi_L; \tag{1.10}$$
$$\Delta_{L'} \circ \Phi = (\Phi \otimes \Phi) \circ \Delta_L. \tag{1.11}$$

Proposition 1.3. The map $\Phi \otimes \Phi : A \otimes_L A \ni a \otimes b \mapsto \Phi(a) \otimes \Phi(b) \in A' \otimes_{L'} A'$ makes sense by virtue of (1.8) and (1.9).

Proof. The map $\overline{\Phi \otimes \Phi}\colon A \times A \to A' \otimes_{L'} A'$ is defined by

$$\overline{\Phi \otimes \Phi}(a, b) = \Phi(a) \otimes \Phi(b) \ \ (a, b \in A).$$

For any $l \in L$, $a, b \in A$,

$$\begin{aligned}
\overline{\Phi \otimes \Phi}(a \cdot l, b) &= \Phi(t_L(l)a) \otimes \Phi(b) \\
&= \Phi(t_L(l))\Phi(a) \otimes \Phi(b) \\
&\underset{(1.9)}{=} t_{L'}(\phi(l))\Phi(a) \otimes \Phi(b) \\
&= \Phi(a) \otimes s_{L'}(\phi(l))\Phi(b) \\
&\underset{(1.8)}{=} \Phi(a) \otimes \Phi(s_L(l))\Phi(b) \\
&= \Phi(a) \otimes \Phi(s_L(l)b) \\
&= \overline{\Phi \otimes \Phi}(a, l \cdot b).
\end{aligned}$$

Hence there exists a $\mathbb{Z}$-module homomorphism $\Phi \otimes \Phi\colon A \otimes_L A \to A' \otimes_{L'} A'$ such that $(\Phi \otimes \Phi)(a \otimes b) = \Phi(a) \otimes \Phi(b)$ $(a, b \in A)$. □

We next introduce the notion of right bialgebroids.

Definition 1.4. Let A and N be associative and unital rings. A sextuplet $\mathcal{A}_N = (A, N, s_N, t_N, \Delta_N, \pi_N)$ is called a right bialgebroid if the following conditions are satisfied:

1. The maps $s_N\colon N \to A$ and $t_N\colon N^{op} \to A$ are ring homomorphisms such that

   $$s_N(n)t_N(n') = t_N(n')s_N(n) \ \ (\forall n, n' \in N). \tag{1.12}$$

   By using these maps s_N and t_N, we can give the ring A an (N, N)-bimodule structure ${}^N A^N$. The left and right actions ${}^N A$ and A^N can be defined by the following:

   $${}^N A\colon n \cdot a = at_N(n); \ \ A^N\colon a \cdot n = as_N(n) \ \ (n \in N, a \in A). \tag{1.13}$$

2. The triple $({}^N A^N, \Delta_N, \pi_N)$ is a comonoid in the category of (N, N)-bimodules satisfying

$$
\begin{aligned}
&s_N(n)a^{[1]} \otimes a^{[2]} = a^{[1]} \otimes t_N(n)a^{[2]}; && (1.14)\\
&\Delta_N(1_A) = 1_A \otimes 1_A; && (1.15)\\
&\Delta_N(ab) = \Delta_N(a)\Delta_N(b); && (1.16)\\
&\pi_N(1_A) = 1_N; && (1.17)\\
&\pi_N(s_N(\pi_N(a))b) = \pi_N(ab) = \pi_N(t_N(\pi_N(a))b) && (1.18)
\end{aligned}
$$

for all $n \in N$ and $a, b \in A$. Here we use Sweedler's notation written by $\Delta_N(a) = a^{[1]} \otimes a^{[2]}$. The right-hand-side of (1.16) makes sense because of (1.14).

In order to avoid ambiguity, we may write $\mathcal{A}_N = (A, N, s_N^A, t_N^A, \Delta_N^A, \pi_N^A)$.

Definition 1.5. Let $\mathcal{A}_N = (A, N, s_N, t_N, \Delta_N, \pi_N)$ and $\mathcal{A}'_{N'} = (A', N', s_{N'}, t_{N'}, \Delta_{N'}, \pi_{N'})$ be right bialgebroids. A right bialgebroid homomorphism is a pair of ring homomorphisms $(\Phi\colon A \to A', \phi\colon N \to N')$ such that

$$
\begin{aligned}
&s_{N'} \circ \phi = \Phi \circ s_N; && (1.19)\\
&t_{N'} \circ \phi = \Phi \circ t_N; && (1.20)\\
&\pi_{N'} \circ \Phi = \phi \circ \pi_N; && (1.21)\\
&\Delta_{N'} \circ \Phi = (\Phi \otimes \Phi) \circ \Delta_N. && (1.22)
\end{aligned}
$$

The map $\Phi \otimes \Phi : A \otimes_N A \ni a \otimes b \mapsto \Phi(a) \otimes \Phi(b) \in A' \otimes_{N'} A'$ makes sense by virtue of (1.19) and (1.20).

Let $\mathcal{A}_L := (A, L, s_L, t_L, \Delta_L, \pi_L)$ be a left bialgebroid and S an anti-automorphism of the ring A such that

$$S \circ t_L = s_L. \tag{1.23}$$

Let N be a ring isomorphic to L^{op} and we fix a ring isomorphism $\omega\colon L^{op} \to N$. By the utilization of (1.23), this A becomes an (N, N)-bimodule whose left action ${}^N A$ and right action A^N are given by

$${}^N A\colon n \cdot a = a(s_L \circ \omega^{-1})(n);\;\; A^N : a \cdot n = a(S \circ s_L \circ \omega^{-1})(n) \;\; (a \in A, n \in N). \tag{1.24}$$

Moreover, we suppose that the map $S\colon A \to A$ satisfies the following condition:

$$S(a_{[1]})a_{[2]} = (t_L \circ \pi_L \circ S)(a) \;\; (\forall a \in A). \tag{1.25}$$

The left-hand-side of (1.25) is well defined because of (1.23). By virtue of (1.23) and (1.25), we can define two $\mathbb{Z}$-module homomorphisms $S_{A\otimes_L A}$ and $S_{A\otimes_N A}$ by

$$
\begin{aligned}
&S_{A\otimes_L A}\colon A \otimes_L A \ni a \otimes b \mapsto S(b) \otimes S(a) \in A \otimes_N A; && (1.26)\\
&S_{A\otimes_N A}\colon A \otimes_N A \ni a \otimes b \mapsto S(b) \otimes S(a) \in A \otimes_L A. && (1.27)
\end{aligned}
$$

Definition 1.6. Let $\mathcal{A}_L$ be a left bialgebroid and S an anti-automorphism of the ring A satisfying (1.23) and (1.25). We assume that the $\mathbb{Z}$-module homomorphism $S_{A\otimes_N A}$ is bijective with the inverse $S^{-1}_{A\otimes_N A}$ and satisfies

$$S_{A\otimes_L A} \circ \Delta_L \circ S^{-1} = S^{-1}_{A\otimes_N A} \circ \Delta_L \circ S. \tag{1.28}$$

We define $\Delta_N = S_{A\otimes_L A} \circ \Delta_L \circ S^{-1}$. A Hopf algebroid is a pair $(\mathcal{A}_L, S)$ satisfying the following conditions:

$$(\Delta_L \otimes \mathrm{id}_A) \circ \Delta_N = (\mathrm{id}_A \otimes \Delta_N) \circ \Delta_L; \tag{1.29}$$
$$(\Delta_N \otimes \mathrm{id}_A) \circ \Delta_L = (\mathrm{id}_A \otimes \Delta_L) \circ \Delta_N. \tag{1.30}$$

The map S, called an antipode, is unique if there exists (see Proposition 1.8).

Let $(\mathcal{A}_L, S_A)$ and $(\mathcal{A}'_{L'}, S_{A'})$ be Hopf algebroids. A left bialgebroid homomorphism $(\Phi\colon A \to A', \phi\colon L \to L')$ is called a Hopf algebroid homomorphism. This pair (Φ, ϕ) is called strict if $\Phi \circ S_A = S_{A'} \circ \Phi$ is satisfied.

Hopf algebroids have some equivalent conditions. We introduce one of those.

Proposition 1.7. (See [3, Proposition 4.2(iii)].) Let $\mathcal{A}_L$ be a left bialgebroid. The following conditions are equivalent:

1. $(\mathcal{A}_L, S)$ is a Hopf algebroid.
2. There exist a right bialgeboid structure $\mathcal{A}_N$ such that N is isomorphic to L^{op} as rings and a $\mathbb{Z}$-module automorphism $S\colon A \to A$ satisfying the following conditions:

$$s_L(L) = t_N(N); \quad t_L(L) = s_N(N) \ \text{(as subrings of } A); \tag{1.31}$$
$$(\Delta_L \otimes \mathrm{id}_A) \circ \Delta_N = (\mathrm{id}_A \otimes \Delta_N) \circ \Delta_L; \tag{1.32}$$
$$(\Delta_N \otimes \mathrm{id}_A) \circ \Delta_L = (\mathrm{id}_A \otimes \Delta_L) \circ \Delta_N; \tag{1.33}$$
$$S(t_L(l) a t_L(l')) = s_L(l') S(a) s_L(l); \tag{1.34}$$
$$S(t_N(n) a t_N(n')) = s_N(n') S(a) s_N(n); \tag{1.35}$$
$$S(a_{[1]}) a_{[2]} = (s_N \circ \pi_N)(a); \quad a^{[1]} S(a^{[2]}) = (s_L \circ \pi_L)(a) \tag{1.36}$$

   for any $l, l' \in L, n, n' \in N, a \in A$.

Proposition 1.8. The map $S\colon A \to A$ satisfying (1.31)-(1.36) is unique if there exists.

Proof. We suppose that S' is another map satisfying (1.31)-(1.36). For any $a \in A$, the counitality of (Δ_L, π_L) and (Δ_N, π_N) induces that

$$\begin{aligned}
S(a) =& S((t_L \circ \pi_L)(a_{[2]}) a_{[1]})) \\
\underset{(1.34)}{=}& S(a_{[1]})(s_L \circ \pi_L)(a_{[2]}) \\
\underset{(1.36)}{=}& S(a_{[1]}) a_{[2]}^{[1]} S'(a_{[2]}^{[2]}) \\
\underset{(1.32)}{=}& S(a_{[1]}^{[1]}) a_{[2]}^{[1]} S'(a^{[2]}) \\
\underset{(1.36)}{=}& (s_N \circ \pi_N)(a^{[1]}) S'(a^{[2]}) \\
\underset{(1.35)}{=}& S'(a^{[2]}(t_N \circ \pi_N)(a^{[1]})) \\
=& S'(a).
\end{aligned}$$

This is the desired conclusion. □

2 Left bialgebroid A_σ

In this section, we introduce a left bialgebroid A_σ as a generalization of that in [26].

Let k be a commutative ring and L a k-algebra. We denote by G a group and let $T_\alpha\colon L \to L$ ($\alpha \in G$) be a k-algebra automorphism such that

$$T_\alpha \circ T_{\alpha^{-1}} = \mathrm{id}_L \tag{2.1}$$

for any $\alpha \in G$. The symbol X means a finite set. We define the set Gen as follows:

$$Gen := (L \otimes_k L^{op}) \bigsqcup \{L_{ab} \mid a, b \in X\} \bigsqcup \{(L^{-1})_{ab} \mid a, b \in X\}.$$

We denote by $k\langle Gen\rangle$ the free k-algebra generated by the set Gen. Let A_σ denote the quotient of the k-algebra $k\langle Gen\rangle$ by the two-sided ideal I_σ generated by the following elements:

(1) $\xi + \xi' - (\xi +_{L\otimes_k L^{op}} \xi')$, $c\xi - (c \cdot_{L\otimes_k L^{op}} \xi)$, $\xi\xi' - (\xi \cdot_{L\otimes_k L^{op}} \xi')$ $(\forall c \in k, \forall \xi, \xi' \in L \otimes_k L^{op})$.
Here the notation $\xi + \xi'$ means the addition in the k-algebra $k\langle Gen\rangle$. On the other hand, the notation $(\xi +_{L\otimes_k L^{op}} \xi')(\in Gen)$ is that of the k-algebra $L \otimes_k L^{op}$. The other two symbols for the scalar multiplication and the multiplication are similar.

(2) $\displaystyle\sum_{c\in X} L_{ac}(L^{-1})_{cb} - \delta_{a,b}\emptyset$, $\displaystyle\sum_{c\in X}(L^{-1})_{ac}L_{cb} - \delta_{a,b}\emptyset$ $(\forall a, b \in X)$.
Here $\emptyset$ means the empty word and the symbol $\delta_{a,b} \in k$ $(a, b \in X)$ stands for Kronecker's delta symbol.

(3) $(T_{\deg(a)}(f) \otimes 1_L)L_{ab} - L_{ab}(f \otimes 1_L)$,
$(1_L \otimes T_{\deg(b)}(f))L_{ab} - L_{ab}(1_L \otimes f)$,
$(f \otimes 1_L)(L^{-1})_{ab} - (L^{-1})_{ab}(T_{\deg(b)}(f) \otimes 1_L)$,
$(1_L \otimes f)(L^{-1})_{ab} - (L^{-1})_{ab}(1_L \otimes T_{\deg(a)}(f))$ $(\forall f \in L, \forall a, b \in X)$.
Here $\deg(a)$ $(\forall a \in X)$ means a collection of elements in the group G.

(4) $\displaystyle\sum_{x,y\in X}(\sigma^{xy}_{ac} \otimes 1_L)L_{yd}L_{xb} - \sum_{x,y\in X}(1_L \otimes \sigma^{bd}_{xy})L_{cy}L_{ax}$ $(\forall a, b, c, d \in X)$.
Here σ^{ab}_{cd} $(\forall a, b, c, d \in X)$ stands for a collection of elements in the k-algebra L.

(5) $\emptyset - 1_L \otimes 1_L$.

Theorem 2.1. Suppose that the elements $\deg(a) \in G$ and $\sigma^{ab}_{cd} \in L$ satisfy

$$\rho_l(\sigma^{bd}_{ac}) \circ T_{\deg(d)} \circ T_{\deg(b)} = \rho_r(\sigma^{bd}_{ac}) \circ T_{\deg(c)} \circ T_{\deg(a)} \tag{2.2}$$

for all $a, b, c, d \in X$. Then A_σ is a left bialgebroid with the base ring L. Here $\rho_l(f)$ and $\rho_r(f)$ $(f \in L)$ are maps defined by $\rho_l(f)\colon L \ni g \mapsto fg \in L$ and $\rho_r(f)\colon L \ni g \mapsto gf \in L$.

We will prove Theorem 2.1 by constructing the maps s_L, t_L, Δ_L, and π_L in Definition 1.1. We first define two maps $s_L\colon L \to A_\sigma$ and $t_L\colon L^{op} \to A_\sigma$ as follows:

$$s_L(l) = l \otimes 1_L + I_\sigma; \tag{2.3}$$

$$t_L(l) = 1_L \otimes l + I_\sigma \quad (l \in L). \tag{2.4}$$

These are k-algebra homomorphisms satisfying (1.1). Therefore A_σ has an (L, L)-bimodule structure by the scalar multiplications as in (1.2).

The next task is to construct the (L, L)-bimodule homomorphism Δ_L. We denote by I_2 the right ideal of $A_\sigma \otimes_k A_\sigma$ generated by the set $\{t_L(l) \otimes 1_{A_\sigma} - 1_{A_\sigma} \otimes s_L(l) \mid l \in L\}$. The k-algebra homomorphism $\overline{\Delta}\colon k\langle Gen\rangle \to A_\sigma \otimes_k A_\sigma$ is defined by

$$\overline{\Delta}(\xi) = (s_L \otimes t_L)(\xi) \quad (\xi \in L \otimes_k L^{op}); \tag{2.5}$$

$$\overline{\Delta}(L_{ab}) = \sum_{c \in X} L_{ac} + I_\sigma \otimes L_{cb} + I_\sigma \quad (a, b \in X); \tag{2.6}$$

$$\overline{\Delta}((L^{-1})_{ab}) = \sum_{c \in X} (L^{-1})_{cb} + I_\sigma \otimes (L^{-1})_{ac} + I_\sigma. \tag{2.7}$$

Proposition 2.2. $\overline{\Delta}(I_\sigma) \subset I_2$.

Proof. We first show that $\overline{\Delta}(g) \in I_2$ for every generator g in I_σ. For the first generator in (2), we can induce that

$$\begin{aligned}
&\overline{\Delta}(\sum_{c \in X} L_{ac}(L^{-1})_{cb} - \delta_{a,b}\emptyset) \\
=&\sum_{c,d,e \in X} L_{ad}(L^{-1})_{eb} + I_\sigma \otimes L_{dc}(L^{-1})_{ce} + I_\sigma - \delta_{a,b}(1_{A_\sigma} \otimes 1_{A_\sigma}) \\
=&\sum_{d,e \in X} L_{ad}(L^{-1})_{eb} + I_\sigma \otimes \delta_{d,e}\emptyset + I_\sigma - \delta_{a,b}(1_{A_\sigma} \otimes 1_{A_\sigma}) \\
=&\sum_{d \in X} L_{ad}(L^{-1})_{db} + I_\sigma \otimes 1_{A_\sigma} - \delta_{a,b}(1_{A_\sigma} \otimes 1_{A_\sigma}) \\
=&\delta_{a,b}(1_{A_\sigma} \otimes 1_{A_\sigma} - 1_{A_\sigma} \otimes 1_{A_\sigma}) \\
=&0
\end{aligned}$$

for all $a, b \in X$. The proof for the other generator in (2) is similar.

If g is a generator in (3), its image of $\overline{\Delta}$ can be calculated as follows:

$$\begin{aligned}
&\overline{\Delta}((T_{\deg(a)}(f) \otimes 1_L)L_{ab} - L_{ab}(f \otimes 1_L)) \\
=&(s_L \otimes t_L)(T_{\deg(a)}(f) \otimes 1_L)(\sum_{c \in X} L_{ac} + I_\sigma \otimes L_{cb} + I_\sigma) \\
&- (\sum_{c \in X} L_{ac} + I_\sigma \otimes L_{cb} + I_\sigma)(s_L \otimes t_L)(f \otimes 1_L) \\
=&\sum_{c \in X} ((T_{\deg(a)}(f) \otimes 1_L)L_{ac} + I_\sigma \otimes L_{cb} + I_\sigma) - (L_{ac}(f \otimes 1_L) + I_\sigma \otimes L_{cb} + I_\sigma) \\
=&\sum_{c \in X} ((T_{\deg(a)}(f) \otimes 1_L)L_{ac} + I_\sigma \otimes L_{cb} + I_\sigma) - ((T_{\deg(a)}(f) \otimes 1_L)L_{ac} + I_\sigma \otimes L_{cb} + I_\sigma) \\
=&0
\end{aligned}$$

for all $a, b \in X$ and $f \in L$. The proof for the other three generators in (3) is similar.

Let g be an arbitrary generator (4) in the two-sided ideal I_σ. For any $a, b, c, d \in X$,

$$
\begin{aligned}
&\overline{\Delta}\Big(\sum_{x,y\in X}(\sigma^{xy}_{ac}\otimes 1_L)L_{yd}L_{xb} - \sum_{x,y\in X}(1_L\otimes\sigma^{bd}_{xy})L_{cy}L_{ax}\Big)\\
&= \sum_{v,w,x,y\in X}(\sigma^{xy}_{ac}\otimes 1_L)L_{yv}L_{xw} + I_\sigma \otimes L_{vd}L_{wb} + I_\sigma\\
&\quad - \sum_{q,r,x,y\in X} L_{cq}L_{ar} + I_\sigma \otimes (1_L\otimes\sigma^{bd}_{xy})L_{qy}L_{rx} + I_\sigma\\
&= \sum_{v,w,x,y\in X} t_L(\sigma^{wv}_{xy})L_{cy}L_{ax} + I_\sigma \otimes L_{vd}L_{wb} + I_\sigma\\
&\quad - \sum_{q,r,x,y\in X} L_{cq}L_{ar} + I_\sigma \otimes s_L(\sigma^{xy}_{rq})L_{yd}L_{xb} + I_\sigma\\
&= \sum_{q,r,x,y\in X}(t_L(\sigma^{xy}_{rq})\otimes 1_{A_\sigma} - 1_{A_\sigma}\otimes s_L(\sigma^{xy}_{rq}))(L_{cq}L_{ar} + I_\sigma\otimes L_{yd}L_{xb} + I_\sigma) \in I_2.
\end{aligned}
$$

The proof for the generators (1) and (5) is straightforward.

In order to complete the proof, we need to show that $\overline{\Delta}(k\langle Gen\rangle)I_2 \subset I_2$. It suffices to check that $\overline{\Delta}(\alpha)(t_L(l)\otimes 1_{A_\sigma} - 1_{A_\sigma}\otimes s_L(l)) \in I_2$ for any $\alpha \in Gen$ and $l \in L$.

If $\alpha = (L^{-1})_{ab}$, (2.1) and the generators (3) in I_σ induce that

$$
\begin{aligned}
&\overline{\Delta}(\alpha)(t_L(l)\otimes 1_{A_\sigma} - 1_{A_\sigma}\otimes s_L(l))\\
&= \sum_{c\in X}(L^{-1})_{cb}(1_L\otimes l) + I_\sigma \otimes (L^{-1})_{ac} + I_\sigma\\
&\quad - \sum_{c\in X}(L^{-1})_{cb} + I_\sigma \otimes (L^{-1})_{ac}(l\otimes 1_L) + I_\sigma\\
&= \sum_{c\in X}(1_L\otimes T_{\deg(c)^{-1}}(l))(L^{-1})_{cb} + I_\sigma\otimes (L^{-1})_{ac} + I_\sigma\\
&\quad - \sum_{c\in X}(L^{-1})_{cb} + I_\sigma \otimes (T_{\deg(c)^{-1}}(l)\otimes 1_L)(L^{-1})_{ac} + I_\sigma\\
&= \sum_{c\in X}(t_L(T_{\deg(c)^{-1}}(l))\otimes 1_{A_\sigma} - 1_{A_\sigma}\otimes s_L(T_{\deg(c)^{-1}}(l)))\\
&\qquad \times ((L^{-1})_{cb} + I_\sigma \otimes (L^{-1})_{ac} + I_\sigma) \in I_2.
\end{aligned}
$$

The proof for the other elements α in Gen is similar. This completes the proof. □

By using this proposition, we can define the k-module homomorphism $\tilde{\Delta}\colon A_\sigma \to A_\sigma \otimes_k A_\sigma/I_2$ by $\tilde{\Delta}(\alpha + I_\sigma) = \overline{\Delta}(\alpha) + I_2$ ($\alpha \in k\langle Gen\rangle$).

Lemma 2.3. As $\mathbb{Z}$-modules,

$$A_\sigma \otimes_k A_\sigma/I_2 \cong A_\sigma \otimes_L A_\sigma.$$

Proof. Since s_L and t_L are k-algebra homomorphisms, there exists a $\mathbb{Z}$-module map $\overline{F}\colon A_\sigma \otimes_k A_\sigma \to A_\sigma \otimes_L A_\sigma$ such that $\overline{F}(\alpha \otimes \beta) = \alpha \otimes \beta$ ($\forall \alpha, \beta \in A_\sigma$). Moreover, for any $l \in L$, $\alpha, \beta \in A_\sigma$,

$$\begin{aligned}\overline{F}((t_L(l) \otimes 1_{A_\sigma} - 1_{A_\sigma} \otimes s_L(l))(\alpha \otimes \beta)) =& t_L(l)\alpha \otimes \beta - \alpha \otimes s_L(l)\beta \\ =& \alpha \cdot l \otimes \beta - \alpha \otimes l \cdot \beta \\ =& \alpha \otimes l \cdot \beta - \alpha \otimes l \cdot \beta \\ =& 0.\end{aligned}$$

Therefore we can conclude that $\overline{F}(I_2) = \{0\}$ and there exists a $\mathbb{Z}$-module homomorphism $F\colon A_\sigma \otimes_k A_\sigma / I_2 \to A_\sigma \otimes_L A_\sigma$ such that $F(\alpha \otimes \beta + I_2) = \overline{F}(\alpha \otimes \beta)$.

In order to construct the inverse of F, we define a map $\overline{F'}\colon A_\sigma \times A_\sigma \to A_\sigma \otimes_k A_\sigma / I_2$ as follows:

$$\overline{F'}(\alpha, \beta) = \alpha \otimes \beta + I_2$$

for any $\alpha, \beta \in A_\sigma$. This $\overline{F'}$ satisfies that

$$\begin{aligned}\overline{F'}(t_L(l)\alpha, \beta) =& t_L(l)\alpha \otimes \beta + I_2 \\ =& t_L(l)\alpha \otimes \beta - (t_L(l) \otimes 1_{A_\sigma} - 1_{A_\sigma} \otimes s_L(l))(\alpha \otimes \beta) + I_2 \\ =& \alpha \otimes s_L(l)\beta \\ =& \overline{F'}(\alpha, s_L(l)\beta)\end{aligned}$$

Thus there exists a $\mathbb{Z}$-module homomorphism $F'\colon A_\sigma \otimes_L A_\sigma \to A_\sigma \otimes_k A_\sigma / I_2$ satisfying $F'(\alpha \otimes \beta) = \alpha \otimes \beta + I_2$ ($\forall \alpha, \beta \in A_\sigma$).

By the definition of F and F', it is clear that F' is the inverse of F. This completes the proof. □

By the utilization of the map $\tilde{\Delta}\colon A_\sigma \to A_\sigma \otimes_k A_\sigma / I_2$, the above lemma induces the $\mathbb{Z}$-module homomorphism $\Delta_L\colon A_\sigma \to A_\sigma \otimes_L A_\sigma$.

Proposition 2.4. The map Δ_L is an (L, L)-bimodule homomorphism.

Proof. For any $\alpha \in A_\sigma$, we fix an element $\overline{\alpha} \in k\langle Gen \rangle$ satisfying $\alpha = \overline{\alpha} + I_\sigma$. We write $\overline{\Delta}(\overline{\alpha}) = \overline{\alpha}_{[1]} \otimes \overline{\alpha}_{[2]}$. By the definition of $\overline{\Delta}$,

$$\begin{aligned}\Delta_L(l \cdot \alpha \cdot l') =& \Delta_L((l \otimes l')\overline{\alpha} + I_\sigma) \\ =& F(\overline{\Delta}((l \otimes l')\overline{\alpha}) + I_2) \\ =& F(s_L(l)\overline{\alpha}_{[1]} \otimes t_L(l')\overline{\alpha}_{[2]} + I_2) \\ =& s_L(l)\overline{\alpha}_{[1]} \otimes t_L(l')\overline{\alpha}_{[2]} \\ =& l \cdot (\overline{\alpha}_{[1]} \otimes \overline{\alpha}_{[2]}) \cdot l' \\ =& l \cdot F(\overline{\alpha}_{[1]} \otimes \overline{\alpha}_{[2]} + I_2) \cdot l' \\ =& l \cdot \Delta_L(\alpha) \cdot l'\end{aligned}$$

for any $l, l' \in L$. This completes the proof. □

Let us construct the map $\pi_L\colon A_\sigma \to L$. We define a k-algebra homomorphism $\overline{\varepsilon}\colon k\langle Gen \rangle \to \mathrm{End}_k(L)$ as follows:

$$\begin{aligned}&\overline{\varepsilon}(l \otimes l') = \rho_l(l)\rho_r(l') \ (l, l' \in L); \\ &\overline{\varepsilon}(L_{ab}) = \delta_{a,b} T_{\deg(a)}; \\ &\overline{\varepsilon}((L^{-1})_{ab}) = \delta_{a,b} T_{\deg(a)^{-1}} \ \ (a, b \in X).\end{aligned}$$

Proposition 2.5. $\overline{\varepsilon}(I_\sigma) = \{0\}$.

Proof. Let g be an arbitrary generator of I_σ. It suffices to show that $\overline{\varepsilon}(g) = 0$. We give the proof only for the genarators (3) and (4). For any $a, b \in X$ and $f \in L$,

$$\begin{aligned}&\overline{\varepsilon}((f \otimes 1_L)(L^{-1})_{ab} - (L^{-1})_{ab}(T_{\deg(b)}(f) \otimes 1_L))\\ =&\delta_{a,b}(\rho_l(f)T_{\deg(a)^{-1}} - T_{\deg(a)^{-1}}\rho_l(T_{\deg(b)}(f))).\end{aligned}$$

If $a = b$, we can induce that

$$\begin{aligned}&(\rho_l(f)T_{\deg(a)^{-1}} - T_{\deg(a)^{-1}}\rho_l(T_{\deg(b)}(f)))(l)\\ =&(\rho_l(f)T_{\deg(a)^{-1}} - T_{\deg(a)^{-1}}\rho_l(T_{\deg(a)}(f)))(l)\\ =&fT_{\deg(a)^{-1}}(l) - (T_{\deg(a)^{-1}}T_{\deg(a)})(f)T_{\deg(a)^{-1}}(l)\\ =&0\end{aligned}$$

for all $l \in L$. Here we use the assumption (2.1) for the third equality. The proof for the other three generators in (3) is similar.

Let us show that $\overline{\varepsilon}(g) = 0$ if g is a generator (4). By the utilization of (2.2),

$$\begin{aligned}&\overline{\varepsilon}(\sum_{x,y\in X}(\sigma_{ac}^{xy} \otimes 1_L)L_{yd}L_{xb} - \sum_{x,y\in X}(1_L \otimes \sigma_{xy}^{bd})L_{cy}L_{ax})\\ =&\sum_{x,y\in X}\rho_l(\sigma_{ac}^{xy})\delta_{y,d}T_{\deg(y)}\delta_{x,b}T_{\deg(x)} - \sum_{x,y\in X}\rho_r(\sigma_{xy}^{bd})\delta_{c,y}T_{\deg(c)}\delta_{a,x}T_{\deg(a)}\\ =&\rho_l(\sigma_{ac}^{bd})T_{\deg(d)}T_{\deg(b)} - \rho_r(\sigma_{ac}^{bd})T_{\deg(c)}T_{\deg(a)}\\ =&0\end{aligned}$$

for all $a, b, c, d \in X$. This is the desired conclusion. □

From this proposition, the map $\varepsilon(\alpha + I_\sigma) = \overline{\varepsilon}(\alpha)$ ($\alpha \in k\langle Gen\rangle$) is a well-defined k-algebra homomorphism. The map π_L can be defined by

$$\pi_L \colon A_\sigma \ni a \mapsto \varepsilon(a)(1_L) \in L. \tag{2.8}$$

Proposition 2.6. The map π_L is an (L, L)-bimodule homomorphism.

Proof. For any $a \in A_\sigma$ and $l, l' \in L$,

$$\begin{aligned}\pi_L(l \cdot a \cdot l') =&\pi_L((l \otimes l' + I_\sigma)a)\\ =&\varepsilon(l \otimes l' + I_\sigma)(\varepsilon(a)(1_L))\\ =&\varepsilon(l \otimes l' + I_\sigma)(\pi_L(a))\\ =&l\pi_L(a)l'.\end{aligned}$$

This is the desired conclusion. □

Proposition 2.7. The triple $(A_\sigma, \Delta_L, \pi_L)$ is a comonoid in the tensor category of (L, L)-bimodules.

Proof. We first prove the coassociativity of Δ_L. We assume the following two lemmas for a moment:

Lemma 2.8. As $\mathbb{Z}$-modules,

$$A_\sigma \otimes_k A_\sigma \otimes_k A_\sigma / I_3 \cong A_\sigma \otimes_L A_\sigma \otimes_L A_\sigma.$$

Here I_3 means the right ideal of $A_\sigma \otimes_k A_\sigma \otimes_k A_\sigma$ generated by $t_L(l) \otimes 1_{A_\sigma} \otimes 1_{A_\sigma} - 1_{A_\sigma} \otimes s_L(l) \otimes 1_{A_\sigma}$ and $1_{A_\sigma} \otimes t_L(l) \otimes 1_{A_\sigma} - 1_{A_\sigma} \otimes 1_{A_\sigma} \otimes s_L(l)$ $(\forall l \in L)$.

Lemma 2.9. For any $\alpha \in A_\sigma$, we fix $\overline{\alpha} \in k\langle Gen\rangle$ satisfying $\alpha = \overline{\alpha} + I_\sigma$. We write $\overline{\Delta}(\overline{\alpha}) = \overline{\alpha}_{[1]} \otimes \overline{\alpha}_{[2]}$. Then it follows that

$$\begin{aligned}
&((\Delta_L \otimes \mathrm{id}_{A_\sigma}) \circ \Delta_L)(\alpha) = ((\mathrm{id}_{A_\sigma} \otimes \Delta_L) \circ \Delta_L)(\alpha) \\
\Leftrightarrow & \overline{\overline{\alpha}_{[1]}}_{[1]} \otimes \overline{\overline{\alpha}_{[1]}}_{[2]} \otimes \overline{\alpha}_{[2]} - \overline{\alpha}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[2]} \in I_3; \qquad (2.9)\\
&((\Delta_L \otimes \mathrm{id}_{A_\sigma}) \circ \Delta_L)(\alpha\beta) = ((\mathrm{id}_{A_\sigma} \otimes \Delta_L) \circ \Delta_L)(\alpha\beta) \\
\Leftrightarrow & (\overline{\overline{\alpha}_{[1]}}_{[1]} \otimes \overline{\overline{\alpha}_{[1]}}_{[2]} \otimes \overline{\alpha}_{[2]})(\overline{\overline{\beta}_{[1]}}_{[1]} \otimes \overline{\overline{\beta}_{[1]}}_{[2]} \otimes \overline{\beta}_{[2]}) \\
& - (\overline{\alpha}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[2]})(\overline{\beta}_{[1]} \otimes \overline{\overline{\beta}_{[2]}}_{[1]} \otimes \overline{\overline{\beta}_{[2]}}_{[2]}) \in I_3 \qquad (2.10)
\end{aligned}$$

for $\alpha, \beta \in A_\sigma$. Moreover, if α and $\beta \in A_\sigma$ satisfy (2.9), then these elements α and β also satisfy (2.10).

By virtue of Lemma 2.8 and 2.9, it is sufficient to show (2.9) for the following $\alpha \in A_\sigma$:

$$\alpha = \begin{cases} l \otimes l' + I_\sigma \quad (\forall l, l' \in L); \\ L_{ab} + I_\sigma; \\ (L^{-1})_{ab} + I_\sigma \quad (\forall a, b \in X). \end{cases} \qquad (2.11)$$

We give the proof only for $\alpha = (L^{-1})_{ab} + I_\sigma$ $(\forall a, b \in X)$. By the definition of $\overline{\Delta}$,

$$\begin{aligned}
\overline{\alpha}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[2]} &= \sum_{c \in X} (L^{-1})_{cb} + I_\sigma \otimes \overline{\Delta}((L^{-1})_{ac}) \\
&= \sum_{c,d \in X} (L^{-1})_{cb} + I_\sigma \otimes (L^{-1})_{dc} + I_\sigma \otimes (L^{-1})_{ad} + I_\sigma \\
&= \sum_{d \in X} \overline{\Delta}((L^{-1})_{db}) \otimes (L^{-1})_{ad} + I_\sigma \\
&= \overline{\overline{\alpha}_{[1]}}_{[1]} \otimes \overline{\overline{\alpha}_{[1]}}_{[2]} \otimes \overline{\alpha}_{[2]}.
\end{aligned}$$

Thus Δ_L is coassociative.

In order to complete the proof, let us check that the maps Δ_L and π_L satisfy the counitality. Let $\alpha = (f \otimes g) w_1 \cdots w_n + I_\sigma \in A_\sigma$ for any $f, g \in L$ and $n \in \mathbb{N}$. Here w_i $(i = 1, 2, \cdots, n)$ means that

$$w_i = \begin{cases} L_{a_i b_i}; \\ (L^{-1})_{a_i b_i} \quad (a_i, b_i \in X). \end{cases} \qquad (2.12)$$

The definition of $\overline{\Delta}$ induces that

$$\overline{\Delta}(w_i) = \begin{cases} \sum_{c_i \in X} L_{a_i c_i} + I_\sigma \otimes L_{c_i b_i} + I_\sigma, & (w_i = L_{a_i b_i}); \\ \sum_{c_i \in X} (L^{-1})_{c_i b_i} + I_\sigma \otimes (L^{-1})_{a_i c_i} + I_\sigma, & (w_i = (L^{-1})_{a_i b_i}). \end{cases} \qquad (2.13)$$

We write (2.13) as

$$\overline{\Delta}(w_i) = \sum_{c_i \in X} w_{[1]c_i} + I_\sigma \otimes w_{[2]c_i} + I_\sigma. \tag{2.14}$$

Thus,

$$\begin{aligned}\tilde{\Delta}(\alpha) + I_2 =& \overline{\Delta}(f \otimes g)\overline{\Delta}(w_1) \cdots \overline{\Delta}(w_n) + I_2 \\ =& \sum_{c_1, \cdots, c_n \in X} ((f \otimes 1_L) w_{[1]c_1} \cdots w_{[1]c_n} + I_\sigma \\ & \otimes (1_L \otimes g) w_{[2]c_1} \cdots w_{[2]c_n} + I_\sigma) + I_2.\end{aligned}$$

Since $\pi_L(L_{ab} + I_\sigma) = \pi_L((L^{-1})_{ab} + I_\sigma) = \delta_{a,b} 1_L$ for any $a, b \in X$, it follows that

$$\begin{aligned}\pi_L(\alpha_{[1]}) \cdot \alpha_{[2]} =& \sum_{c_1, \cdots, c_n \in X} \delta_{d_1, c_1} \cdots \delta_{d_n, c_n} s_L(f)((1_L \otimes g) w_{[2]c_1} \cdots w_{[2]c_n} + I_\sigma) \\ =& (f \otimes g) w_{[2]d_1} \cdots w_{[2]d_n} + I_\sigma \\ =& \alpha\end{aligned}$$

Here we write

$$d_i = \begin{cases} a_i, & (w_i = L_{a_i b_i}); \\ b_i, & (w_i = (L^{-1})_{a_i b_i}). \end{cases}$$

The proof for $\alpha_{[1]} \cdot \pi_L(\alpha_{[2]}) = \alpha$ is similar. This is the desired conclusion. □

Proof of Lemma 2.8. Since s_L and t_L are k-algebra homomorphisms, there exists a $\mathbb{Z}$-module homomorphism $\overline{G}\colon A_\sigma \otimes_k A_\sigma \otimes_k A_\sigma \to A_\sigma \otimes_L (A_\sigma \otimes_L A_\sigma)$ such that $\overline{G}(\alpha \otimes \beta \otimes \gamma) = \alpha \otimes (\beta \otimes \gamma)$ $(\forall \alpha, \beta, \gamma \in A_\sigma)$. For any $l \in L$, $\alpha, \beta, \gamma \in A_\sigma$,

$$\begin{aligned}& \overline{G}((t_L(l) \otimes 1_{A_\sigma} \otimes 1_{A_\sigma} - 1_{A_\sigma} \otimes s_L(l) \otimes 1_{A_\sigma})(\alpha \otimes \beta \otimes \gamma)) \\ =& \alpha \cdot l \otimes (\beta \otimes \gamma) - \alpha \otimes (s_L(l)\beta \otimes \gamma) \\ =& \alpha \otimes l \cdot (\beta \otimes \gamma) - \alpha \otimes (s_L(l)\beta \otimes \gamma) \\ =& \alpha \otimes (s_L(l)\beta \otimes \gamma) - \alpha \otimes (s_L(l)\beta \otimes \gamma) \\ =& 0.\end{aligned}$$

Similarly, we can also prove that $\overline{G}((1_{A_\sigma} \otimes t_L(l) \otimes 1_{A_\sigma} - 1_{A_\sigma} \otimes 1_{A_\sigma} \otimes s_L(l))(\alpha \otimes \beta \otimes \gamma)) = 0$. Thus $\overline{G}(I_3) = \{0\}$ and there exists a $\mathbb{Z}$-module map $G\colon A_\sigma \otimes_k A_\sigma \otimes_k A_\sigma / I_3 \to A_\sigma \otimes_L A_\sigma \otimes_L A_\sigma$ such that $G(\alpha \otimes \beta \otimes \gamma + I_3) = \overline{G}(\alpha \otimes \beta \otimes \gamma)$ $(\forall \alpha, \beta, \gamma \in A_\sigma)$.

In order to construct the inverse of G, we define a map $\overline{\tilde{G}'_\alpha}\colon A_\sigma \times A_\sigma \to A_\sigma \otimes_k A_\sigma \otimes_k A_\sigma / I_3$ $(\forall \alpha \in A_\sigma)$ as follows:

$$\overline{\tilde{G}'_\alpha}(\beta, \gamma) = \alpha \otimes \beta \otimes \gamma + I_3 \ \ (\beta, \gamma \in A_\sigma).$$

This $\overline{\tilde{G}'_\alpha}$ satisfies that

$$\begin{aligned}\overline{\tilde{G}'_\alpha}(t_L(l)\beta, \gamma) =& \alpha \otimes t_L(l)\beta \otimes \gamma + I_3 \\ =& \alpha \otimes t_L(l)\beta \otimes \gamma \\ & - (1_{A_\sigma} \otimes t_L(l) \otimes 1_{A_\sigma} - 1_{A_\sigma} \otimes 1_{A_\sigma} \otimes s_L(l))(\alpha \otimes \beta \otimes \gamma) + I_3 \\ =& \alpha \otimes \beta \otimes s_L(l)\gamma + I_3 \\ =& \overline{\tilde{G}'_\alpha}(\beta, s_L(l)\gamma)\end{aligned}$$

for all $\beta, \gamma \in A_\sigma$. Thus there exists a $\mathbb{Z}$-module homomorphism $\overline{G'_\alpha}\colon A_\sigma \otimes_L A_\sigma \to A_\sigma \otimes_k A_\sigma \otimes_k A_\sigma / I_3$ ($\forall \alpha \in A_\sigma$) such that $\overline{G'_\alpha}(\beta \otimes \gamma) = \alpha \otimes \beta \otimes \gamma + I_3$ ($\forall \beta, \gamma \in A_\sigma$). By this fact, the map $\overline{G}\colon A_\sigma \times (A_\sigma \otimes_L A_\sigma) \ni (\alpha, \beta \otimes \gamma) \mapsto \alpha \otimes \beta \otimes \gamma + I_3 \in A_\sigma \otimes_k A_\sigma \otimes_k A_\sigma / I_3$ makes sense. We have

$$
\begin{aligned}
\overline{G}(t_L(l)\alpha, \beta \otimes \gamma) =& t_L(l)\alpha \otimes \beta \otimes \gamma + I_3 \\
=& t_L(l)\alpha \otimes \beta \otimes \gamma \\
& - (t_L(l) \otimes 1_{A_\sigma} \otimes 1_{A_\sigma} - 1_{A_\sigma} \otimes s_L(l) \otimes 1_{A_\sigma})(\alpha \otimes \beta \otimes \gamma) + I_3 \\
=& \alpha \otimes s_L(l)\beta \otimes \gamma + I_3 \\
=& \overline{G}(\alpha, s_L(l)\beta \otimes \gamma)
\end{aligned}
$$

for all $\alpha, \beta, \gamma \in A_\sigma$. By the above calculation, there exists a $\mathbb{Z}$-module homomorphism $G'\colon A_\sigma \otimes_L (A_\sigma \otimes_L A_\sigma) \to A_\sigma \otimes_k A_\sigma \otimes_k A_\sigma / I_3$ such that $G'(\alpha \otimes (\beta \otimes \gamma)) = \alpha \otimes \beta \otimes \gamma + I_3$ ($\forall \alpha, \beta, \gamma \in A_\sigma$). Since it is clear that $G' = G^{-1}$, this lemma is proved. $\square$

Proof of Lemma 2.9. Since $\Delta_L(\alpha) = \overline{\alpha}_{[1]} \otimes \overline{\alpha}_{[2]}$ for all $\alpha \in A_\sigma$, it is easy to prove (2.9). Here we fix an element $\overline{\alpha} \in k\langle Gen \rangle$ satisfying $\alpha = \overline{\alpha} + I_\sigma$.

Let us check (2.10). For any $\alpha, \beta \in A_\sigma$, it follows that $\Delta_L(\alpha\beta) = \overline{\alpha}_{[1]}\overline{\beta}_{[1]} \otimes \overline{\alpha}_{[2]}\overline{\beta}_{[2]}$ because $\alpha\beta = \overline{\alpha\beta} + I_\sigma = \overline{\alpha}\overline{\beta} + I_\sigma$. By using this fact repeatedly, we can induce that $\Delta_L(\overline{\alpha}_{[1]}\overline{\beta}_{[1]}) = \overline{\overline{\alpha}_{[1]}}_{[1]}\overline{\overline{\beta}_{[1]}}_{[1]} \otimes \overline{\overline{\alpha}_{[1]}}_{[2]}\overline{\overline{\beta}_{[1]}}_{[2]}$ and $\Delta_L(\overline{\alpha}_{[2]}\overline{\beta}_{[2]}) = \overline{\overline{\alpha}_{[2]}}_{[1]}\overline{\overline{\beta}_{[2]}}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[2]}\overline{\overline{\beta}_{[2]}}_{[2]}$. Thus we conclude that

$$
\begin{aligned}
& ((\Delta_L \otimes \mathrm{id}_{A_\sigma}) \circ \Delta_L)(\alpha\beta) = ((\mathrm{id}_{A_\sigma} \otimes \Delta_L) \circ \Delta_L)(\alpha\beta) \\
\Leftrightarrow & \overline{\overline{\alpha}_{[1]}}_{[1]}\overline{\overline{\beta}_{[1]}}_{[1]} \otimes \overline{\overline{\alpha}_{[1]}}_{[2]}\overline{\overline{\beta}_{[1]}}_{[2]} \otimes \overline{\alpha}_{[2]}\overline{\beta}_{[2]} - \overline{\alpha}_{[1]}\overline{\beta}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[1]}\overline{\overline{\beta}_{[2]}}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[2]}\overline{\overline{\beta}_{[2]}}_{[2]} \in I_3 \\
\Leftrightarrow & (\overline{\overline{\alpha}_{[1]}}_{[1]} \otimes \overline{\overline{\alpha}_{[1]}}_{[2]} \otimes \overline{\alpha}_{[2]})(\overline{\overline{\beta}_{[1]}}_{[1]} \otimes \overline{\overline{\beta}_{[1]}}_{[2]} \otimes \overline{\beta}_{[2]}) \\
& - (\overline{\alpha}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[2]})(\overline{\beta}_{[1]} \otimes \overline{\overline{\beta}_{[2]}}_{[1]} \otimes \overline{\overline{\beta}_{[2]}}_{[2]}) \in I_3.
\end{aligned}
$$

Suppose that two elements α and $\beta \in A_\sigma$ satisfy (2.9). For any $\alpha \in A_\sigma$, we write $i_\alpha = \overline{\overline{\alpha}_{[1]}}_{[1]} \otimes \overline{\overline{\alpha}_{[1]}}_{[2]} \otimes \overline{\alpha}_{[2]} - \overline{\alpha}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[2]} \in I_3$. It follows that

$$
\begin{aligned}
& (\overline{\overline{\alpha}_{[1]}}_{[1]} \otimes \overline{\overline{\alpha}_{[1]}}_{[2]} \otimes \overline{\alpha}_{[2]})(\overline{\overline{\beta}_{[1]}}_{[1]} \otimes \overline{\overline{\beta}_{[1]}}_{[2]} \otimes \overline{\beta}_{[2]}) \\
& - (\overline{\alpha}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[2]})(\overline{\beta}_{[1]} \otimes \overline{\overline{\beta}_{[2]}}_{[1]} \otimes \overline{\overline{\beta}_{[2]}}_{[2]}) \\
=& (\overline{\alpha}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[2]} + i_\alpha)(\overline{\beta}_{[1]} \otimes \overline{\overline{\beta}_{[2]}}_{[1]} \otimes \overline{\overline{\beta}_{[2]}}_{[2]} + i_\beta) \\
& - (\overline{\alpha}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[2]})(\overline{\beta}_{[1]} \otimes \overline{\overline{\beta}_{[2]}}_{[1]} \otimes \overline{\overline{\beta}_{[2]}}_{[2]}) \\
=& (\overline{\alpha}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[2]})i_\beta + i_\alpha(\overline{\beta}_{[1]} \otimes \overline{\overline{\beta}_{[2]}}_{[1]} \otimes \overline{\overline{\beta}_{[2]}}_{[2]}) + i_\alpha i_\beta.
\end{aligned}
$$

According to the above calculation, we need to prove that $(\overline{\alpha}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[2]})i_\beta \in I_3$. It suffices to prove this fact when

$$
\alpha = \begin{cases} l \otimes l' + I_\sigma \quad (\forall l, l' \in L); \\ L_{ab} + I_\sigma; \\ (L^{-1})_{ab} + I_\sigma \quad (\forall a, b \in X). \end{cases}
$$

Let $i_\beta = t_L(l)\otimes 1_{A_\sigma}\otimes 1_{A_\sigma} - 1_{A_\sigma}\otimes s_L(l)\otimes 1_{A_\sigma}$ $(\forall l \in L)$. If $\alpha = (L^{-1})_{ab}+I_\sigma$, the identity $(T_{\deg(b)^{-1}}(f)\otimes T_{\deg(a)^{-1}}(g))(L^{-1})_{ab}+I_\sigma = (L^{-1})_{ab}(f\otimes g)+I_\sigma$ $(\forall f, g \in L)$ induces that

$$
\begin{aligned}
&(\overline{\alpha}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[1]} \otimes \overline{\overline{\alpha}_{[2]}}_{[2]})i_\beta \\
&= \sum_{c,d\in X} ((L^{-1})_{cb} + I_\sigma \otimes (L^{-1})_{dc} + I_\sigma \otimes (L^{-1})_{ad} + I_\sigma) \\
&\quad \times (t_L(l) \otimes 1_{A_\sigma} \otimes 1_{A_\sigma} - 1_{A_\sigma} \otimes s_L(l) \otimes 1_{A_\sigma}) \\
&= \sum_{c,d\in X} (L^{-1})_{cb}(1_L \otimes l) + I_\sigma \otimes (L^{-1})_{dc} + I_\sigma \otimes (L^{-1})_{ad} + I_\sigma \\
&\quad - (L^{-1})_{cb} + I_\sigma \otimes (L^{-1})_{dc}(l \otimes 1_L) + I_\sigma \otimes (L^{-1})_{ad} + I_\sigma \\
&= \sum_{c,d\in X} (1_L \otimes T_{\deg(c)^{-1}}(l))(L^{-1})_{cb} + I_\sigma \otimes (L^{-1})_{dc} + I_\sigma \otimes (L^{-1})_{ad} + I_\sigma \\
&\quad - (L^{-1})_{cb} + I_\sigma \otimes (T_{\deg(c)^{-1}}(l) \otimes 1_L)(L^{-1})_{dc} + I_\sigma \otimes (L^{-1})_{ad} + I_\sigma \\
&= \sum_{c,d\in X} (t_L(T_{\deg(c)^{-1}}(l)) \otimes 1_{A_\sigma} \otimes 1_{A_\sigma} - 1_{A_\sigma} \otimes s_L(T_{\deg(c)^{-1}}(l)) \otimes 1_{A_\sigma}) \\
&\quad \times ((L^{-1})_{cb} + I_\sigma \otimes (L^{-1})_{dc} + I_\sigma \otimes (L^{-1})_{ad} + I_\sigma) \in I_3.
\end{aligned}
$$

The proof for the other generators is similar. This is the desired conclusion. □

Proposition 2.10. The maps Δ_L and π_L satisfy the conditions (1.3)-(1.7).

Proof. We first show (1.3). Let $\alpha = (f\otimes g)w_1\cdots w_n + I_\sigma \in A_\sigma$ $(\forall f, g \in L)$ defined by (2.12). By the utilization of the notation (2.14) and the fact that $A_\sigma \otimes_k A_\sigma / I_2 \cong A_\sigma \otimes_L A_\sigma$ as $\mathbb{Z}$-modules, the image of α under Δ_L can be denoted by

$$\Delta_L(\alpha) = \sum_{c_1,\cdots,c_n\in X} (f\otimes 1_L)w_{[1]c_1}\cdots w_{[1]c_n} + I_\sigma \otimes (1_L\otimes g)w_{[2]c_1}\cdots w_{[2]c_n} + I_\sigma.$$

Since the generators (3) induce that $(1_L\otimes T_{\deg(a)^{-1}}(f))(L^{-1})_{ab} - (L^{-1})_{ab}(1_L\otimes f) \in I_\sigma$ for all $a, b \in X$, we have

$$
\begin{aligned}
&\alpha_{[1]}t_L(l)\otimes \alpha_{[2]} \\
&= \sum_{c_1,\cdots,c_n\in X} (f\otimes 1_L)w_{[1]c_1}\cdots w_{[1]c_n}(1_L\otimes l) + I_\sigma \otimes (1_L\otimes g)w_{[2]c_1}\cdots w_{[2]c_n} + I_\sigma \\
&= \sum_{c_1,\cdots,c_n\in X} (f\otimes 1_L)w_{[1]c_1}\cdots w_{[1]c_{n-1}}(1_L\otimes T_{\deg(c_n)^{q_n}}(l))w_{[1]c_n} + I_\sigma \\
&\qquad \otimes (1_L\otimes g)w_{[2]c_1}\cdots w_{[2]c_n} + I_\sigma \\
&= \sum_{c_1,\cdots,c_n\in X} t_L((T_{\deg(c_1)^{q_1}}\circ\cdots\circ T_{\deg(c_n)^{q_n}})(l))(f\otimes 1_L)w_{[1]c_1}\cdots w_{[1]c_n}(1_L\otimes l) + I_\sigma \\
&\qquad \otimes (1_L\otimes g)w_{[2]c_1}\cdots w_{[2]c_n} + I_\sigma \\
&= \sum_{c_1,\cdots,c_n\in X} (f\otimes 1_L)w_{[1]c_1}\cdots w_{[1]c_n}(1_L\otimes l) + I_\sigma \\
&\qquad \otimes s_L((T_{\deg(c_1)^{q_1}}\circ\cdots\circ T_{\deg(c_n)^{q_n}})(l))(1_L\otimes g)w_{[2]c_1}\cdots w_{[2]c_n} + I_\sigma
\end{aligned}
$$

$$
\begin{aligned}
=& \sum_{c_1,\cdots,c_n \in X} (f \otimes 1_L) w_{[1]c_1} \cdots w_{[1]c_n} (1_L \otimes l) + I_\sigma \\
& \qquad \otimes (1_L \otimes g) w_{[2]c_1} ((T_{\deg(c_2)^{q_2}} \circ \cdots \circ T_{\deg(c_n)^{q_n}})(l) \otimes 1_L) w_{[2]c_2} \cdots w_{[2]c_n} + I_\sigma \\
=& \sum_{c_1,\cdots,c_n \in X} (f \otimes 1_L) w_{[1]c_1} \cdots w_{[1]c_n} + I_\sigma \otimes (1_L \otimes g) w_{[2]c_1} \cdots w_{[2]c_n} (l \otimes 1_L) + I_\sigma \\
=& \alpha_{[1]} \otimes \alpha_{[2]} s_L(l)
\end{aligned}
$$

for any $l \in L$. Here q_i $(i = 1, \cdots, n)$ means that

$$
q_i = \begin{cases} 1, & (w_i = L_{a_i b_i}); \\ -1, & (w_i = (L^{-1})_{a_i b_i}). \end{cases}
$$

Thus (1.3) is proved.

Since $\overline{\Delta}$ is a k-algebra homomorphism, it is easy to check (1.4).

Let us show (1.5). For any $\alpha \in A_\sigma$, we fix $\overline{\alpha} \in k\langle Gen \rangle$ such that $\alpha = \overline{\alpha} + I_\sigma$ and write $\overline{\Delta}(\overline{\alpha}) = \overline{\alpha}_{[1]} \otimes \overline{\alpha}_{[2]}$. It follows that $\Delta_L(\alpha) = \overline{\alpha}_{[1]} \otimes \overline{\alpha}_{[2]}$. Since $\alpha\beta = \overline{\alpha\beta} + I_\sigma = \overline{\alpha}\overline{\beta} + I_\sigma$ for any $\alpha, \beta \in A_\sigma$, the right-hand-side of (1.5) can be calculated as follows:

$$
\begin{aligned}
\Delta_L(\alpha)\Delta_L(\beta) =& \overline{\alpha}_{[1]}\overline{\beta}_{[1]} \otimes \overline{\alpha}_{[2]}\overline{\beta}_{[2]} \\
=& F(\overline{\alpha}_{[1]}\overline{\beta}_{[1]} \otimes \overline{\alpha}_{[2]}\overline{\beta}_{[2]} + I_2) \\
=& F(\overline{\Delta}(\overline{\alpha}\overline{\beta}) + I_2) \\
=& F(\tilde{\Delta}(\alpha\beta)) \\
=& \Delta_L(\alpha\beta).
\end{aligned}
$$

Thus Δ_L is multiplicative.

Because ε is a k-algebra homomorphism, the proof for (1.6) is straightforward.

Finally we show (1.7). The first equality of (1.7) can be calculated as follows:

$$
\begin{aligned}
\pi_L(\alpha s_L(\pi_L(\beta))) =& (\varepsilon(\alpha)\varepsilon(\pi_L(\beta) \otimes 1_L + I_\sigma))(1_L) \\
=& \varepsilon(\alpha)(\pi_L(\beta)) \\
=& \varepsilon(\alpha)(\varepsilon(\beta)(1_L)) \\
=& \pi_L(\alpha\beta)
\end{aligned}
$$

for all $\alpha, \beta \in A_\sigma$. The proof for the second equality of (1.7) is similar. This is the desired conclusion. □

By virtue of Propositions 2.7 and 2.10, the sextuplet $A_\sigma = (A_\sigma, L, s_L, t_L, \Delta_L, \pi_L)$ is a left bialgebroid.

3 Right bialgebroid A_σ

In this section, we show that A_σ is a right bialgebroid under the condition (3.1) implying (2.2).

Theorem 3.1. Suppose that the elements $\deg(a) \in G$ and $\sigma_{cd}^{ab} \in L$ satisfy

$$T_{\deg(a)^{-1}} \circ T_{\deg(c)^{-1}} \circ \rho_l(\sigma_{ac}^{bd}) = T_{\deg(b)^{-1}} \circ T_{\deg(d)^{-1}} \circ \rho_r(\sigma_{ac}^{bd}) \tag{3.1}$$

for all $a, b, c, d \in X$. Then A_σ is a right bialgebroid with the base ring L^{op}.

The k-algebra homomorphisms are defined by $s_{L^{op}} = t_L$ and $t_{L^{op}} = s_L$ (for s_L and t_L, see (2.3) and (2.4)). Since these maps satisfy (1.12), the k-algebra A_σ becomes an (L^{op}, L^{op})-bimodule via (1.13).

Let us construct the map $\Delta_{L^{op}}$. We denote by I_2' the left ideal of $A_\sigma \otimes_k A_\sigma$ whose generators are $s_{L^{op}}(l) \otimes 1_{A_\sigma} - 1_{A_\sigma} \otimes t_{L^{op}}(l)$ ($\forall l \in L^{op}$). For the k-algebra homomorphism $\overline{\Delta}$ defined by (2.5)-(2.7), we can also induce the proposition below.

Proposition 3.2. $\overline{\Delta}(I_\sigma) \subset I_2'$.

Proof. Let us prove that $\overline{\Delta}(g) = 0$ for every generator g in I_σ. We give the proof only for the generators (4). The calculation of Proposition 2.2 induces that

$$\begin{aligned}
&\overline{\Delta}\Big(\sum_{x,y\in X} (\sigma_{ac}^{xy} \otimes 1_L) L_{yd} L_{xb} - \sum_{x,y \in X} (1_L \otimes \sigma_{xy}^{bd}) L_{cy} L_{ax}\Big) \\
&= \sum_{q,r,x,y\in X} (1_L \otimes \sigma_{rq}^{xy}) L_{cq} L_{ar} + I_\sigma \otimes L_{yd} L_{xb} + I_\sigma \\
&\quad - \sum_{q,r,x,y\in X} L_{cq} L_{ar} + I_\sigma \otimes (\sigma_{rq}^{xy} \otimes 1_L) L_{yd} L_{xb} + I_\sigma \\
&= \sum_{q,r,x,y\in X} L_{cq} L_{ar} (1_L \otimes (T_{\deg(r)^{-1}} \circ T_{\deg(q)^{-1}})(\sigma_{rq}^{xy})) + I_\sigma \otimes L_{yd} L_{xb} + I_\sigma \\
&\quad - \sum_{q,r,x,y\in X} L_{cq} L_{ar} + I_\sigma \otimes L_{yd} L_{xb} ((T_{\deg(x)^{-1}} \circ T_{\deg(y)^{-1}})(\sigma_{rq}^{xy}) \otimes 1_L) + I_\sigma \\
&= \sum_{q,r,x,y\in X} L_{cq} L_{ar} (1_L \otimes (T_{\deg(x)^{-1}} \circ T_{\deg(y)^{-1}})(\sigma_{rq}^{xy})) + I_\sigma \otimes L_{yd} L_{xb} + I_\sigma \\
&\quad - \sum_{q,r,x,y\in X} L_{cq} L_{ar} + I_\sigma \otimes L_{yd} L_{xb} ((T_{\deg(x)^{-1}} \circ T_{\deg(y)^{-1}})(\sigma_{rq}^{xy}) \otimes 1_L) + I_\sigma \\
&= \sum_{q,r,x,y\in X} (L_{cq} L_{ar} + I_\sigma \otimes L_{yd} L_{xb} + I_\sigma) \\
&\quad \times (s_{L^{op}}((T_{\deg(x)^{-1}} \circ T_{\deg(y)^{-1}})(\sigma_{rq}^{xy})) \otimes 1_{A_\sigma} \\
&\quad - 1_{A_\sigma} \otimes t_{L^{op}}((T_{\deg(x)^{-1}} \circ T_{\deg(y)^{-1}})(\sigma_{rq}^{xy}))) \in I_2'
\end{aligned}$$

for any $a, b, c, d \in X$. Here we use the generators (3) for the second equality and (3.1) for the third equality.

The proof for $I_2' \overline{\Delta}(k\langle Gen \rangle) \subset I_2'$ is similar to that for $\overline{\Delta}(k\langle Gen \rangle) I_2 \subset I_2$. This completes the proof. □

Therefore the k-module homomorphism $\tilde{\Delta}' : A_\sigma \ni \alpha + I_\sigma \mapsto \overline{\Delta}(\alpha) + I_2' \in A_\sigma \otimes_k A_\sigma / I_2'$ is well defined. Similar to the proof for Lemma 2.3, we can induce the following lemma.

Lemma 3.3. As $\mathbb{Z}$-modules,

$$A_\sigma \otimes_k A_\sigma / I_2' \cong A_\sigma \otimes_{L^{op}} A_\sigma.$$

By using this lemma, we can construct the $\mathbb{Z}$-module homomorphism $\Delta_{L^{op}} \colon A_\sigma \to A_\sigma \otimes_{L^{op}} A_\sigma$. This $\Delta_{L^{op}}$ is an (L^{op}, L^{op})-bimodule homomorphism.

We next construct the map $\pi_{L^{op}} \colon A_\sigma \to L^{op}$. The k-algebra anti-homomorphism $\overline{\varepsilon'} \colon k\langle Gen\rangle \to \mathrm{End}_k(L^{op})$ is defined by

$$\begin{aligned}
&\overline{\varepsilon'}(l \otimes l') = \rho_l(l')\rho_r(l)\ (l, l' \in L);\\
&\overline{\varepsilon'}(L_{ab}) = \delta_{a,b} T_{\deg(a)^{-1}};\\
&\overline{\varepsilon'}((L^{-1})_{ab}) = \delta_{a,b} T_{\deg(a)} \quad (a, b \in X).
\end{aligned}$$

Proposition 3.4. $\overline{\varepsilon'}(I_\sigma) = \{0\}$.

Proof. We show that $\overline{\varepsilon'}(g) = 0$ if g is a generator (4) of I_σ. By the utilization of the condition (3.1),

$$\begin{aligned}
&\overline{\varepsilon}\Big(\sum_{x,y\in X} (\sigma^{xy}_{ac} \otimes 1_L) L_{yd} L_{xb} - \sum_{x,y\in X} (1_L \otimes \sigma^{bd}_{xy}) L_{cy} L_{ax}\Big)\\
=&\sum_{x,y\in X} \delta_{x,b} T_{\deg(x)^{-1}} \delta_{y,d} T_{\deg(y)^{-1}} \rho_r(\sigma^{xy}_{ac}) - \sum_{x,y\in X} \delta_{a,x} T_{\deg(a)^{-1}} \delta_{c,y} T_{\deg(c)^{-1}} \rho_l(\sigma^{bd}_{xy})\\
=&T_{\deg(b)^{-1}} T_{\deg(d)^{-1}} \rho_r(\sigma^{bd}_{ac}) - T_{\deg(a)^{-1}} T_{\deg(c)^{-1}} \rho_l(\sigma^{bd}_{ac})\\
=&0
\end{aligned}$$

for $a, b, c, d \in X$. This is the desired conclusion. □

The above proposition induce the k-algebra anti-homomorphism by $\varepsilon'(\alpha + I_\sigma) = \overline{\varepsilon'}(\alpha)$ $(\alpha \in k\langle Gen\rangle)$. The map $\pi_{L^{op}} \colon A_\sigma \to L^{op}$ is defined by

$$\pi_{L^{op}} \colon A_\sigma \ni a \mapsto \varepsilon'(a)(1_L) \in L^{op}. \tag{3.2}$$

This is an (L^{op}, L^{op})-bimodule homomorphism.

The triple $(A_\sigma, \Delta_{L^{op}}, \pi_{L^{op}})$ is a comonoid in the tensor category of (L^{op}, L^{op})-bimodules satisfying (1.14)-(1.18). The proof is similar to that for Propositions 2.7 and 2.10. Therefore the sextuplet $A_\sigma = (A_\sigma, L^{op}, s_{L^{op}}, t_{L^{op}}, \Delta_{L^{op}}, \pi_{L^{op}})$ is a right bialgebroid.

Proposition 3.5. The right bialgebroid A_σ has the left bialgebroid structure in Section 2.

Proof. We have only to prove that (3.1) induces (2.2). Let a, b, c, and d be arbitrary elements in X. By the utilization of (3.1),

$$(T_{\deg(a)^{-1}} \circ T_{\deg(c)^{-1}} \circ \rho_l(\sigma^{bd}_{ac}))(1_L) = (T_{\deg(b)^{-1}} \circ T_{\deg(d)^{-1}} \circ \rho_r(\sigma^{bd}_{ac}))(1_L).$$

This formula is equivalent to

$$(T_{\deg(a)^{-1}} \circ T_{\deg(c)^{-1}})(\sigma^{bd}_{ac}) = (T_{\deg(b)^{-1}} \circ T_{\deg(d)^{-1}})(\sigma^{bd}_{ac}). \tag{3.3}$$

By virtue of (3.1) and (3.3), we can caluculate that

$$\begin{aligned}
&T_{\deg(c)} \circ T_{\deg(a)} \circ T_{\deg(a)^{-1}} \circ T_{\deg(c)^{-1}} \circ \rho_l(\sigma^{bd}_{ac}) \circ T_{\deg(d)} \circ T_{\deg(b)}\\
=&T_{\deg(c)} \circ T_{\deg(a)} \circ T_{\deg(b)^{-1}} \circ T_{\deg(d)^{-1}} \circ \rho_r(\sigma^{bd}_{ac}) \circ T_{\deg(d)} \circ T_{\deg(b)}\\
=&T_{\deg(c)} \circ T_{\deg(a)} \circ \rho_r(T_{\deg(b)^{-1}} \circ T_{\deg(d)^{-1}}(\sigma^{bd}_{ac}))\\
=&\rho_r(T_{\deg(c)} \circ T_{\deg(a)} \circ T_{\deg(b)^{-1}} \circ T_{\deg(d)^{-1}}(\sigma^{bd}_{ac})) \circ T_{\deg(c)} \circ T_{\deg(a)}\\
=&\rho_r(\sigma^{bd}_{ac}) \circ T_{\deg(c)} \circ T_{\deg(a)}.
\end{aligned}$$

This completes the proof. □

4 Hopf algebroid A_σ

This section deals with a sufficient condition under which the left and right bialgebroid A_σ becomes a Hopf algebroid with a bijective antipode.

Suppose that the k-algebra A_σ satisfies the condition (3.1). Then this A_σ has the left and right bialgebroid structures introduced in Sections 2 and 3. Let $\sigma^{ab}_{cd} \in L$ $(a, b, c, d \in X)$ be a collection of elements which composes the two-sided ideal I_σ of A_σ.

Definition 4.1. The family $\sigma = \{\sigma^{ab}_{cd}\}_{a,b,c,d\in X}$ is called rigid if the following conditions are satisfied:

For any $a, b \in X$, there exist $x_{ab}, y_{ab} \in A_\sigma$ such that

$$\begin{aligned}\sum_{c\in X}((L^{-1})_{cb} + I_\sigma)x_{ac} &= \sum_{c\in X} x_{cb}((L^{-1})_{ac} + I_\sigma)\\ &= \sum_{c\in X}(L_{cb} + I_\sigma)y_{ac}\\ &= \sum_{c\in X} y_{cb}(L_{ac} + I_\sigma)\\ &= \delta_{a,b}1_{A_\sigma}.\end{aligned}$$

Proposition 4.2. The elements x_{ab} and y_{ab} are unique if there exist.

Proof. For any $a, b \in X$, let x_{ab} and x'_{ab} be elements in A_σ satisfying

$$\sum_{c\in X}((L^{-1})_{cb} + I_\sigma)x_{ac} = \sum_{c\in X} x'_{cb}((L^{-1})_{ac} + I_\sigma) = \delta_{a,b}1_{A_\sigma}.$$

It follows that

$$\begin{aligned}x'_{ab} &= \sum_{d\in X}\delta_{a,d}x'_{db}\\ &= \sum_{c,d\in X} x'_{db}((L^{-1})_{cd} + I_\sigma)x_{ac}\\ &= \sum_{c\in X}\delta_{b,c}x_{ac}\\ &= x_{ab}.\end{aligned}$$

The proof for the uniqueness of y_{ab} is similar. □

We can construct a k-algebra anti-automorphism $S\colon A_\sigma \to A_\sigma$ by using the rigid σ.

Proposition 4.3. The following conditions are equivalent:

1. σ is rigid;
2. There exists a unique $\mathbb{K}$-algebra anti-automorphism $S\colon A_\sigma \to A_\sigma$ such that

$$\begin{cases} S(f\otimes g + I_\sigma) = g\otimes f + I_\sigma \;\; (f, g \in L);\\ S(L_{ab} + I_\sigma) = (L^{-1})_{ab} + I_\sigma \;\; (a, b \in X);\end{cases} \tag{4.1}$$

Proof. We first show that the condition 2 implies 1. For any $a, b \in X$, the elements x_{ab} and y_{ab} are defined by $x_{ab} = S((L^{-1})_{ab} + I_\sigma)$ and $y_{ab} = S^{-1}(L_{ab} + I_\sigma)$. By the utilization of the generators (2) in I_σ,

$$\begin{aligned}\delta_{a,b}1_{A_\sigma} &= \sum_{c\in X} S((L_{ac} + I_\sigma)((L^{-1})_{cb} + I_\sigma))\\ &= \sum_{c\in X} S((L^{-1})_{cb} + I_\sigma)S(L_{ac} + I_\sigma)\\ &= \sum_{c\in X} x_{cb}((L^{-1})_{ac} + I_\sigma).\end{aligned}$$

The proof for the other identities in Definition 4.1 is similar.

Let us suppose that the condition 1. The k-algebra homomorphism $\overline{S}\colon k\langle Gen\rangle \to A_\sigma^{op}$ is defined by

$$\begin{aligned}&\overline{S}(\xi) = \psi(\xi) \ \ (\xi \in L \otimes_k L^{op});\\ &\overline{S}(L_{ab}) = (L^{-1})_{ab} + I_\sigma;\\ &\overline{S}((L^{-1})_{ab}) = x_{ab} \ \ (a, b \in X).\end{aligned}$$

Here ψ means the k-algebra homomorphism defined by $\psi\colon L \otimes_k L^{op} \ni f \otimes g \mapsto g \otimes f + I_\sigma \in A_\sigma^{op}$. We prove that $\overline{S}(I_\sigma) = \{0\}$. Since this map S is a k-algebra homomorphism, it is sufficient to show that $\overline{S}(\alpha) = 0$ for every generator α in I_σ.

Let α be an arbitrary generator (3) in I_σ. Since $(T_{\deg(b)^{-1}}(f) \otimes 1_L)(L^{-1})_{ab} - (L^{-1})_{ab}(f \otimes 1_L) \in I_\sigma$ for any $a, b \in X$ and $f \in L$,

$$\begin{aligned}0 = &\sum_{a,b\in X} x_{bd}((T_{\deg(b)^{-1}}(f) \otimes 1_L)(L^{-1})_{ab} + I_\sigma - (L^{-1})_{ab}(f \otimes 1_L) + I_\sigma)x_{ca}\\ =&\sum_{b\in X} \delta_{c,b}x_{bd}(T_{\deg(b)^{-1}}(f) \otimes 1_L + I_\sigma) - \sum_{a\in X} \delta_{a,d}(f \otimes 1_L + I_\sigma)x_{ca}\\ =&x_{cd}(T_{\deg(c)^{-1}}(f) \otimes 1_L + I_\sigma) - (f \otimes 1_L + I_\sigma)x_{cd}\end{aligned}$$

for all $c, d \in X$. Thus, for any $a, b \in X$ and $f \in L$,

$$\begin{aligned}&\overline{S}((1_L \otimes f)(L^{-1})_{ab} - (L^{-1})_{ab}(1_L \otimes T_{\deg(a)}(f)))\\ =&x_{ab}(f \otimes 1_L + I_\sigma) - (T_{\deg(a)}(f) \otimes 1_L + I_\sigma)x_{ab}\\ =&0.\end{aligned}$$

The proof for the other three generators in (3) is similar.

We give the proof for the generators (4) in I_σ. By virtue of the generators (2), (3), and the condition (3.3),

$$\begin{aligned}0 = &\sum_{a,b,c,d\in X} ((L^{-1})_{x''a} + I_\sigma)((L^{-1})_{y''c} + I_\sigma)\\ &\times ((\sum_{x,y\in X} (\sigma_{ac}^{xy} \otimes 1_L)L_{yd}L_{xb} - \sum_{x,y\in X} (1 \otimes \sigma_{xy}^{bd})L_{cy}L_{ax}) + I_\sigma)\\ &\times ((L^{-1})_{bx'} + I_\sigma)((L^{-1})_{dy'} + I_\sigma)\\ =&\sum_{a,c,x,y\in X} \delta_{x,x'}\delta_{y,y'}(L^{-1})_{x''a}(L^{-1})_{y''c}(\sigma_{ac}^{xy} \otimes 1_L) + I_\sigma\end{aligned}$$

$$- \sum_{b,d,x,y \in X} \delta_{x,x''}\delta_{y,y''}(1_L \otimes (T_{\deg(x)^{-1}} \circ T_{\deg(y)^{-1}})(\sigma_{xy}^{bd}))(L^{-1})_{bx'}(L^{-1})_{dy'} + I_\sigma$$

$$= \sum_{a,c \in X} (L^{-1})_{x''a}(L^{-1})_{y''c}(\sigma_{ac}^{x'y'} \otimes 1_L) + I_\sigma$$

$$- \sum_{b,d \in X} ((L^{-1})_{bx'}(L^{-1})_{dy'} + I_\sigma)$$

$$\times (1_L \otimes (T_{\deg(d)} \circ T_{\deg(b)} \circ T_{\deg(x'')^{-1}} \circ T_{\deg(y'')^{-1}})(\sigma_{x''y''}^{bd}) + I_\sigma)$$

$$= \sum_{a,c \in X} (L^{-1})_{x''a}(L^{-1})_{y''c}(\sigma_{ac}^{x'y'} \otimes 1_L) + I_\sigma$$

$$- \sum_{b,d \in X} ((L^{-1})_{bx'}(L^{-1})_{dy'}(1_L \otimes \sigma_{x''y''}^{bd}) + I_\sigma$$

for all $x', y', x'', y'' \in X$. Thus,

$$\begin{aligned} 0 = &\sum_{a,c \in X} (L^{-1})_{x''a}(L^{-1})_{y''c}(\sigma_{ac}^{x'y'} \otimes 1_L) + I_\sigma \\ &- \sum_{b,d \in X} (L^{-1})_{bx'}(L^{-1})_{dy'}(1_L \otimes \sigma_{x''y''}^{bd}) + I_\sigma \\ =&\overline{S}(\sum_{a,c \in X} (1_L \otimes \sigma_{ac}^{x'y'})L_{y''c}L_{x''a} - \sum_{b,d \in X} (\sigma_{x''y''}^{bd} \otimes 1_L)L_{dy'}L_{bx'}). \end{aligned}$$

The proof for the generators (1), (2), and (5) is straightforward. Therefore the k-algebra homomorphism $S(\alpha + I_\sigma) = \overline{S}(\alpha)$ is well defined.

In order to construct the inverse of S, we define a k-algebra homomorphism $\overline{S'} \colon k\langle Gen \rangle \to A_\sigma^{op}$ as follows:

$$\begin{aligned} &\overline{S'}(\xi) = \psi(\xi) \ \ (\xi \in L \otimes_k L^{op}); \\ &\overline{S'}(L_{ab}) = y_{ab}; \\ &\overline{S'}((L^{-1})_{ab}) = L_{ab} + I_\sigma \ \ (a, b \in X). \end{aligned}$$

Similar to the proof for $\overline{S}$, we can induce that $\overline{S'}(I_\sigma) = \{0\}$. Thus the k-algebra homomorphism $S'(\alpha + I_\sigma) = \overline{S'}(\alpha)$ makes sense.

Let us check that $S' \circ S = S \circ S' = \mathrm{id}_{A_\sigma}$. We give the proof only for $(S' \circ S)((L^{-1})_{ab} + I_\sigma) = (L^{-1})_{ab} + I_\sigma$ $(\forall a, b \in X)$. Since $\delta_{a,b}1_{A_\sigma} = \sum_{c \in X} S'(x_{ac})(L_{cb} + I_\sigma)$, it follows that

$$\begin{aligned} S'(x_{ab}) &= \sum_{c \in X} S'(x_{ac})\delta_{b,c} \\ &= \sum_{c,d \in X} S'(x_{ac})(L_{cd}(L^{-1})_{db} + I_\sigma) \\ &= \sum_{c,d \in X} S'(((L^{-1})_{cd} + I_\sigma)x_{ac})((L^{-1})_{db} + I_\sigma) \\ &= \sum_{d \in X} \delta_{a,d}((L^{-1})_{db} + I_\sigma) \\ &= (L^{-1})_{ab} + I_\sigma \end{aligned}$$

for any $a, b \in X$. Thus the map S is a k-algebra anti-automorphism.

The uniqueness of S can be induced by the uniqueness of x_{ab}. This is the desired conclusion. □

The left and right bialgebroid A_σ becomes a Hopf algebroid with the antipode S constructed by Proposition 4.3.

Theorem 4.4. Let S be a k-algebra anti-automorphism of A_σ defined by the rigid σ. Then the pair (A_σ, S) is a Hopf algebroid for $N = L^{op}$ and $\omega = \mathrm{id}_L$.

Proof. The proof for (1.23) is straightforward by the definition of S.

Let us show (1.25). For any $f, g \in L$, let $a = w_1 w_2 \cdots w_n(f \otimes g) + I_\sigma \in A_\sigma$ (for the definition of w_i, see (2.12)). We write

$$\Delta_L(w_i + I_\sigma) = \sum_{c_i \in X} w_{[1]c_i} + I_\sigma \otimes w_{[2]c_i} + I_\sigma. \tag{4.2}$$

Here $w_{[1]c_i}$ and $w_{[2]c_i}$ stand for

$$w_{[1]c_i} = \begin{cases} L_{a_i c_i}, \\ (L^{-1})_{c_i b_i}, \end{cases} \quad w_{[1]c_i} = \begin{cases} L_{c_i b_i}, & (w_i = L_{a_i b_i}), \\ (L^{-1})_{a_i c_i}, & (w_i = (L^{-1})_{a_i b_i}). \end{cases}$$

By using the notation of (4.2), we can calculate the left-hand-side of (1.25) as follows:

$$\begin{aligned} S(a_{[1]})a_{[2]} = &\sum_{c_1,\dots,c_n \in X} S(f \otimes 1_L + I_\sigma) S(w_{[n]c_n} + I_\sigma) \dots S(w_{[1]c_1} + I_\sigma) \\ &\times (w_{[2]c_1} \dots w_{[2]c_n}(1_L \otimes g) + I_\sigma). \end{aligned}$$

Since σ is rigid, it follows that $\sum_{c_i \in X} S(w_{[1]c_i}) w_{[2]c_i} = \delta_{a_i, b_i} 1_{A_\sigma}$. Thus we conclude that $S(a_{[1]})a_{[2]} = \delta_{a_1,b_1} \cdots \delta_{a_n,b_n}(1_L \otimes gf) + I_\sigma$. For any $a, b \in X$,

$$\begin{aligned} \delta_{a,b} T_{\deg(a)} =& \delta_{a,b} \mathrm{id}_L T_{\deg(a)} \\ =& \varepsilon(\sum_{c \in X} x_{cb}((L^{-1})_{ac} + I_\sigma)) T_{\deg(a)} \\ =& \varepsilon(x_{ab}) T_{\deg(a)^{-1}} T_{\deg(a)} \\ =& \varepsilon(x_{ab}). \end{aligned}$$

Thus the right-hand-side of (1.25) can be calculated as follows:

$$\begin{aligned} (t_L \circ \pi_L \circ S)(a) =& t_L(\varepsilon(g \otimes f + I_\sigma)\varepsilon(S(w_n + I_\sigma)) \cdots \varepsilon(S(w_1 + I_\sigma))(1_L)) \\ =& \delta_{a_1,b_1} \cdots \delta_{a_n,b_n} t_L(gf) \\ =& \delta_{a_1,b_1} \cdots \delta_{a_n,b_n}(1_L \otimes gf) + I_\sigma. \end{aligned}$$

Therefore (1.25) is proved.

We next check (1.28). It is clear that $S^{-1}_{A_\sigma \otimes_N A_\sigma} \colon A_\sigma \otimes_L A_\sigma \ni a \otimes b \mapsto S^{-1}(b) \otimes S^{-1}(a) \in A_\sigma \otimes_N A_\sigma$ is an inverse of $S_{A_\sigma \otimes_N A_\sigma}$ by virtue of (1.23). We assume the following lemma for a moment:

Lemma 4.5. For any $a, b \in X$,

$$\Delta_L(x_{ab}) = \sum_{c \in X} x_{ac} \otimes x_{cb}; \tag{4.3}$$

$$\Delta_L(y_{ab}) = \sum_{c \in X} y_{cb} \otimes y_{ac}. \tag{4.4}$$

For any $f, g \in L$, let $a = w_1 w_2 \cdots w_n (f \otimes g) + I_\sigma \in A_\sigma$. Here w_i is defined by (2.12). By the utilization of (4.4) in Lemma 4.5,

$$\Delta_L(S^{-1}(w_i + I_\sigma)) = \begin{cases} \sum_{c_i \in X} y_{c_i b_i} \otimes y_{a_i c_i}, & (w_i = L_{a_i b_i}); \\ \sum_{c_i \in X} L_{a_i c_i} + I_\sigma \otimes L_{c_i b_i} + I_\sigma, & (w_i = (L^{-1})_{a_i b_i}). \end{cases}$$

We write $\Delta_L(S^{-1}(w_i + I_\sigma)) = \sum_{c_i \in X} z_{[1]c_i} \otimes z_{[2]c_i}$. For the left-hand-side of (1.28),

$$\begin{aligned} & (S_{A_\sigma \otimes_L A_\sigma} \circ \Delta_L \circ S^{-1})(a) \\ = & \sum_{c_1, \cdots c_n \in X} S_{A_\sigma \otimes_L A_\sigma}((g \otimes 1_L + I_\sigma) z_{[1]c_n} \cdots z_{[1]c_1} \otimes (1_L \otimes f + I_\sigma) z_{[2]c_n} \cdots z_{[2]c_1}) \\ = & \sum_{c_1, \ldots c_n \in X} Z_{[2]c_1} \cdots Z_{[2]c_n} (f \otimes 1_L + I_\sigma) \otimes Z_{[1]c_1} \cdots Z_{[1]c_n} (1_L \otimes g + I_\sigma). \end{aligned}$$

Here $Z_{[j]c_i} = S(z_{[j]c_i})$ $(j = 1, 2)$. This $Z_{[j]c_i}$ satisfies that

$$Z_{[1]c_i} = \begin{cases} L_{c_i b_i} + I_\sigma, \\ (L^{-1})_{a_i c_i} + I_\sigma, \end{cases} \quad Z_{[2]c_i} = \begin{cases} L_{a_i c_i} + I_\sigma, & (w_i = L_{a_i b_i}), \\ (L^{-1})_{c_i b_i} + I_\sigma, & (w_i = (L^{-1})_{a_i b_i}). \end{cases}$$

On the other hand, because we assume (4.3) in Lemma 4.5,

$$\Delta_L(S(w_i + I_\sigma)) = \begin{cases} \sum_{c_i \in X} (L^{-1})_{c_i b_i} + I_\sigma \otimes (L^{-1})_{a_i c_i} + I_\sigma, & (w_i = L_{a_i b_i}); \\ \sum_{c_i \in X} x_{a_i c_i} \otimes x_{c_i b_i}, & (w_i = (L^{-1})_{a_i b_i}). \end{cases}$$

Let $\Delta_L(S(w_i + I_\sigma)) = \sum_{c_i \in X} z'_{[1]c_i} \otimes z'_{[2]c_i}$. The right-hand-side of (1.28) can be calculated as follows:

$$\begin{aligned} & (S^{-1}_{A_\sigma \otimes_N A_\sigma} \circ \Delta_L \circ S)(a) \\ = & \sum_{c_1, \cdots c_n \in X} S^{-1}_{A_\sigma \otimes_R A_\sigma}((g \otimes 1_L + I_\sigma) z'_{[1]c_n} \cdots z'_{[1]c_1} \otimes (1_L \otimes f + I_\sigma) z'_{[2]c_n} \cdots z'_{[2]c_1}) \\ = & \sum_{c_1, \ldots c_n \in X} Z'_{[2]c_1} \cdots Z'_{[2]c_n} (f \otimes 1_L + I_\sigma) \otimes Z'_{[1]c_1} \cdots Z'_{[1]c_n} (1_L \otimes g + I_\sigma). \end{aligned}$$

Here $Z'_{[j]c_i} = S^{-1}(z'_{[j]c_i})$ $(j = 1, 2)$. For the element $Z'_{[j]c_i}$,

$$Z'_{[1]c_i} = \begin{cases} L_{c_i b_i} + I_\sigma, \\ (L^{-1})_{a_i c_i} + I_\sigma, \end{cases} \quad Z'_{[2]c_i} = \begin{cases} L_{a_i c_i} + I_\sigma, & (w_i = L_{a_i b_i}); \\ (L^{-1})_{c_i b_i} + I_\sigma, & (w_i = (L^{-1})_{a_i b_i}). \end{cases}$$

Thus we have proved (1.28).

The proof for the identities (1.29) and (1.30) is similar to that for the comultiplicativity of Δ_L. This completes the proof. □

Proof of Lemma 4.5. We prove (4.3). Let a and b be arbitrary elements in

X. The rigid σ induces that

$$\begin{aligned}
&\sum_{c,d,e\in X}((L^{-1})_{db}+I_\sigma)x_{ae}\otimes((L^{-1})_{cd}+I_\sigma)x_{ec}\\
=&\sum_{d,e\in X}((L^{-1})_{db}+I_\sigma)x_{af}\otimes\delta_{d,e}1_{A_\sigma}\\
=&\sum_{d\in X}((L^{-1})_{db}+I_\sigma)x_{ad}\otimes 1_{A_\sigma}\\
=&\delta_{a,b}(1_{A_\sigma}\otimes 1_{A_\sigma}).
\end{aligned}$$

We fix $\overline{x_{ab}}\in k\langle Gen\rangle$ such that $x_{ab}=\overline{x_{ab}}+I_\sigma$. Since $\sum_{c\in X}\overline{x_{cb}}(L^{-1})_{ac}-\delta_{a,b}\emptyset\in I_\sigma$, Proposition 2.2 induces that

$$\begin{aligned}
I_2\ni&\overline{\Delta}(\sum_{c\in X}\overline{x_{cb}}(L^{-1})_{ac}-\delta_{a,b}\emptyset)\\
=&\sum_{c,d\in X}\overline{\Delta}(\overline{x_{cb}})((L^{-1})_{dc}+I_\sigma\otimes(L^{-1})_{ad}+I_\sigma)-\delta_{a,b}(1_{A_\sigma}\otimes 1_{A_\sigma}).
\end{aligned}$$

Thus we can conclude that

$$\begin{aligned}
\overline{\Delta}(\overline{x_{ab}})+I_2=&\sum_{c\in X}\delta_{a,c}\overline{\Delta}(\overline{x_{cb}})+I_2\\
=&\sum_{c,c',d',e\in X}\overline{\Delta}(\overline{x_{cb}})(((L^{-1})_{d'c}+I_\sigma)x_{ae}\otimes((L^{-1})_{c'd'}+I_\sigma)x_{ec'})+I_2\\
=&\sum_{c,c',e\in X}\overline{\Delta}(\overline{x_{cb}}(L^{-1})_{c'c})(x_{ae}\otimes x_{ec'})+I_2\\
=&\sum_{c',e\in X}(\delta_{b,c'}(1_{A_\sigma}\otimes 1_{A_\sigma})+\alpha_{c'})(x_{ae}\otimes x_{ec'})+I_2\\
=&\sum_{e\in X}x_{ae}\otimes x_{eb}+I_2
\end{aligned}$$

Here $\alpha_{c'}=\overline{\Delta}(\sum_{c\in X}\overline{x_{cb}}(L^{-1})_{c'c}-\delta_{b,c'}\emptyset)\in I_2$. Since $A_\sigma\otimes_k A_\sigma/I_2\cong A_\sigma\otimes_L A_\sigma$ as $\mathbb{Z}$-modules, (4.3) is proved.

The proof for (4.4) is similar. □

Definition 4.1 is not a practical condition to construct examples of Hopf algebroids A_σ. We introduce a sufficient condition for rigidity. For all $a,b,c,d\in X$, we write

$$\tilde{\sigma}^{ab}_{cd}=T_{\deg(d)^{-1}}(\sigma^{ab}_{cd})\in L. \tag{4.5}$$

Proposition 4.6. If the following conditions are satisfied, then σ is rigid.

1. For any $a,b,c,d\in X$, there exists $i_*(\tilde{\sigma})^{ab}_{cd}\in L$ such that

$$\sum_{a,b\in X}i_*(\tilde{\sigma})^{wa}_{zb}\,\tilde{\sigma}^{by}_{ax}=\sum_{a,b\in X}\tilde{\sigma}^{wa}_{zb}\,i_*(\tilde{\sigma})^{by}_{ax}=\delta_{w,x}\delta_{y,z}1_L. \tag{4.6}$$

2. For any $a,b\in X$, we define Q_{ab}, Q'_{ab}, Q''_{ab}, and $Q'''_{ab}\in L$ by

$$\begin{aligned}
Q_{ab}&=\sum_{u\in X}i_*(\tilde{\sigma})^{ub}_{ua};\quad Q'_{ab}=\sum_{u\in X}i_*(\tilde{\sigma})^{bu}_{au};\\
Q''_{ab}&=\sum_{u\in X}T_{\deg(b)}(i_*(\tilde{\sigma})^{bu}_{au});\quad Q'''_{ab}=\sum_{u\in X}T_{\deg(a)^{-1}}(i_*(\tilde{\sigma})^{ua}_{ub}).
\end{aligned}$$

Then there exist Q_{ab}^{-1}, $Q_{ab}'^{-1}$, $Q_{ab}''^{-1}$, and $Q_{ab}'''^{-1} \in L$ such that

$$\begin{aligned} &\sum_{b\in X} Q_{ab}Q_{bc}^{-1} = \sum_{b\in X} Q_{bc}'^{-1}Q_{ab}' \\ =&\sum_{b\in X} Q_{ab}''^{-1}Q_{bc}'' = \sum_{b\in X} Q_{ab}'''Q_{bc}'''^{-1} = \delta_{a,c}1_L. \end{aligned} \tag{4.7}$$

Proof. For any $a, b \in X$, we define x_{ab} and y_{ab} as follows:

$$x_{ab} = \sum_{c,d\in X} (Q_{ac} \otimes Q_{db}^{-1})L_{cd} + I_\sigma \tag{4.8}$$

$$= \sum_{c,d\in X} (Q_{ac}''^{-1} \otimes Q_{db}'')L_{cd} + I_\sigma; \tag{4.9}$$

$$y_{ab} = \sum_{c,d\in X} (Q_{db}'^{-1} \otimes Q_{ac}')(L^{-1})_{cd} + I_\sigma \tag{4.10}$$

$$= \sum_{c,d\in X} (Q_{bd}''' \otimes Q_{ca}'''^{-1})(L^{-1})_{cd} + I_\sigma. \tag{4.11}$$

It suffices to prove that

$$\sum_{c\in X} T_{cb}((L^{-1})_{ac} + I_\sigma) = \delta_{a,b}1_{A_\sigma}; \tag{4.12}$$

$$\sum_{c\in X} ((L^{-1})_{cb} + I_\sigma)T_{ac}' = \delta_{a,b}1_{A_\sigma}; \tag{4.13}$$

$$\sum_{c\in X} S_{cb}(L_{ac} + I_\sigma) = \delta_{a,b}1_{A_\sigma}; \tag{4.14}$$

$$\sum_{c\in X} (L_{cb} + I_\sigma)S_{ac}' = \delta_{a,b}1_{A_\sigma} \tag{4.15}$$

for all $a, b \in X$. Here T_{ab}, T_{ab}', S_{ab}, and S_{ab}' $(a, b \in X)$ stand for the right-hand-side of (4.8)-(4.11), respectively. We give the proof only for (4.12). The following lemma plays an important role in this proof.

Lemma 4.7. For any $a, b, c, d \in X$,

$$\sum_{x,y\in X} (\tilde{\sigma}_{ax}^{by} \otimes 1_L)(L^{-1})_{cx}L_{yd} + I_\sigma = \sum_{x,y\in X} (1_L \otimes \tilde{\sigma}_{xc}^{yd})L_{ax}(L^{-1})_{yb} + I_\sigma$$

By using (4.6) and Lemma 4.7, we can induce that

$$\begin{aligned} &\sum_{a,b,x,y\in X} (i_*(\tilde{\sigma})_{ub}^{va} \otimes \tilde{\sigma}_{xc}^{yd})L_{ax}(L^{-1})_{yb} + I_\sigma \\ =&\sum_{a,b,x,y\in X} (i_*(\tilde{\sigma})_{ub}^{va}\tilde{\sigma}_{ax}^{by} \otimes 1_L)(L^{-1})_{cx}L_{yd} + I_\sigma \\ =&\sum_{x,y\in X} \delta_{v,x}\delta_{u,y}(L^{-1})_{cx}L_{yd} + I_\sigma \\ =&(L^{-1})_{cv}L_{ud} + I_\sigma \end{aligned}$$

for any $c, d, u, v \in X$. The generators (2) in I_σ induces that

$$\begin{aligned}
\delta_{a,b} 1_{A_\sigma} &= \sum_{u \in X} (L^{-1})_{au} L_{ub} + I_\sigma \\
&= \sum_{c,d,u,x,y \in X} (i_*(\tilde{\sigma})^{uc}_{ud} \otimes 1_L)(1_L \otimes \tilde{\sigma}^{yb}_{xa}) L_{cx} (L^{-1})_{yd} + I_\sigma \\
&= \sum_{c,d,x,y \in X} (Q_{dc} \otimes 1_L)(1_L \otimes \tilde{\sigma}^{yb}_{xa}) L_{cx} (L^{-1})_{yd} + I_\sigma
\end{aligned}$$

for all $a, b \in X$. Thus, by the utilization of (4.6),

$$\begin{aligned}
1_L \otimes Q_{vu} + I_\sigma &= \sum_{a,b \in X} \delta_{a,b} (1_L \otimes i_*(\tilde{\sigma})^{au}_{bv} + I_\sigma) \\
&= \sum_{a,b,c,d,x,y \in X} (1_L \otimes \tilde{\sigma}^{yb}_{xa} i_*(\tilde{\sigma})^{au}_{bv})(Q_{dc} \otimes 1_L) L_{cx} (L^{-1})_{yd} + I_\sigma \\
&= \sum_{c,d,x,y \in X} \delta_{u,x} \delta_{v,y} (Q_{dc} \otimes 1_L) L_{cx} (L^{-1})_{yd} + I_\sigma \\
&= \sum_{c,d \in X} (Q_{dc} \otimes 1_L) L_{cu} (L^{-1})_{vd} + I_\sigma
\end{aligned}$$

for all $u, v \in X$. Therefore, for any $a, b \in X$, (4.7) induces that

$$\begin{aligned}
\delta_{a,b} 1_{A_\sigma} &= \sum_{u \in X} (1_L \otimes Q^{-1}_{ub})(1_L \otimes Q_{au}) + I_\sigma \\
&= \sum_{c,d,u \in X} (1_L \otimes Q^{-1}_{ub})(Q_{dc} \otimes 1_L) L_{cu} (L^{-1})_{ad} + I_\sigma \\
&= \sum_{d \in X} \Big(\sum_{c,u \in X} (Q_{dc} \otimes Q^{-1}_{ub}) L_{cu} + I_\sigma \Big)((L^{-1})_{ad} + I_\sigma) \\
&= \sum_{d \in X} T_{db} ((L^{-1})_{ad} + I_\sigma).
\end{aligned}$$

This is the desired conclusion. □

Proof of Lemma 4.7. Let a, b, c, and d be arbitrary elements in X. By the utilization of the generators (2) and (4) in I_σ,

$$\begin{aligned}
&\sum_{x,y \in X} (L^{-1})_{cx} (\sigma^{by}_{ax} \otimes 1_L) L_{yd} + I_\sigma \\
=& \sum_{x,x',y,b' \in X} (L^{-1})_{cx} (\sigma^{x'y}_{ax} \otimes 1_L) L_{yd} L_{x'b'} (L^{-1})_{b'b} + I_\sigma \\
=& \sum_{x,x',y,b' \in X} (L^{-1})_{cx} (1_L \otimes \sigma^{b'd}_{x'y}) L_{xy} L_{ax'} (L^{-1})_{b'b} + I_\sigma
\end{aligned}$$

Thus the generators (2) and (3) induce that

$$\begin{aligned}
&\sum_{x,y\in X}(\tilde{\sigma}^{by}_{ax}\otimes 1_L)(L^{-1})_{cx}L_{yd}+I_\sigma\\
&=\sum_{x,y\in X}(L^{-1})_{cx}(\sigma^{by}_{ax}\otimes 1_L)L_{yd}+I_\sigma\\
&=\sum_{x,x',y,b'\in X}(L^{-1})_{cx}(1_L\otimes \sigma^{b'd}_{x'y})L_{xy}L_{ax'}(L^{-1})_{b'b}+I_\sigma\\
&=\sum_{x,x',y,b'\in X}(1_L\otimes T_{\deg(c)^{-1}}(\sigma^{b'd}_{x'y}))(L^{-1})_{cx}L_{xy}L_{ax'}(L^{-1})_{b'b}+I_\sigma\\
&=\sum_{x',b'\in X}(1_L\otimes \tilde{\sigma}^{b'd}_{x'c})L_{ax'}(L^{-1})_{b'b}+I_\sigma.
\end{aligned}$$

This completes the proof. □

5 Examples

In this section, we construct a rigid σ by using the sufficient condition in Proposition 4.6. Moreover, we present a ring R that is not Frobenius-separable. By virtue of these σ and R, A_σ becomes a Hopf algebroid that does not have a weak Hopf algebra structure.

Definition 5.1. A non-empty set QG endowed with a binary operation is called a quasigroup if the following conditions are satisfied:

1. For any $b, c \in QG$, there uniquely exists the element $a \in QG$ such that $ab = c$;
2. for any $a, c \in QG$, there uniquely exists the element $b \in QG$ such that $ab = c$.

Here the juxtaposition stands for the binary operation on QG. We write c/b for the unique element $a \in QG$ in the condition 1, and $a\backslash c$ for the unique $b \in QG$ in the condition 2.

Let QG be a finite quasigroup satisfying $|QG| \geq 2$ and M a set with a ternary operation $\mu\colon M \times M \times M \to M$. We assume the following conditions:

(QG1) $|QG| = |M|$;

(QG2) $\mu(a, \mu(a, b, c), \mu(\mu(a, b, c), c, d)) = \mu(a, b, \mu(b, c, d))$ $(\forall a, b, c, d \in M)$;

(QG3) $\mu(\mu(a, b, c), c, d) = \mu(\mu(a, b, \mu(b, c, d)), \mu(b, c, d), d)$ $(\forall a, b, c, d \in M)$;

(QG4) for any $b, c, d \in M$, there uniquely exists $a \in M$ such that $\mu(a, b, c) = d$;

(QG5) for any $a, c, d \in M$, there uniquely exists $b \in M$ such that $\mu(a, b, c) = d$;

(QG6) for any $a, b, d \in M$, there uniquely exists $c \in M$ such that $\mu(a, b, c) = d$.

Example 5.2. Let M be an abelian group satisfying $|M| \geq 2$. The ternary operation $\mu\colon M \times M \times M \to M$ is defined by

$$\mu(a, b, c) = a - b + c \ \ (a, b, c \in M).$$

This μ satisfies (QG2)-(QG6).

Let k be a commutative ring satisfying $0 \neq 1_k$ and R a k-algebra. We denote by L the k-algebra consisting of all maps from $H := QG$ to R. The multiplication on L is defined by the pointwise multiplication.

We fix a bijection $\pi\colon QG \to M$. For all $a, b, c, d \in X$, the map $\sigma^{ab}_{cd} \in L$ is defined by

$$\sigma^{ab}_{cd}(\lambda) = \begin{cases} 1_R, & (c = \pi^{-1}(\mu(\pi(\lambda), \pi(\lambda b), \pi((\lambda b)a)))\backslash((\lambda b)a)) \\ & \text{and } d = \lambda\backslash\pi^{-1}(\mu(\pi(\lambda), \pi(\lambda b), \pi((\lambda b)a))); \\ 0_R, & (\text{otherwise}). \end{cases} \tag{5.1}$$

We denote by G the opposite group consisting of all bijections on the set H. Let $X := QG$ and the element $\deg(a) \in G$ $(a \in X)$ is defined by $\deg(a)(\lambda) = \lambda a$ $(\lambda \in H)$. For any $\alpha \in G$, we define the k-algebra automorphism by $T_\alpha(f) = f \circ \alpha$.

For any $a, b, c, d \in X$, the map $\sigma R^{ab}_{cd} \in L$ is defined by $\sigma R^{ab}_{cd}(\lambda) = \sigma^{ab}_{cd}(\lambda) I^{ab}_{cd}(\lambda d)$. Here $I^{ab}_{cd}(\lambda)$ stands for an invertible element in the center of R.

Proposition 5.3. The elements $\tilde{\sigma}^{ab}_{cd} \in L$ defined by (4.5) satisfy the following conditions.

1. For any $\lambda \in H$ and $(c, d) \in X \times X$, there uniquely exists $(a_0, b_0) \in X \times X$ such that
$$\tilde{\sigma}^{bc}_{ad}(\lambda) = \begin{cases} 1_R, & ((a, b) = (a_0, b_0)); \\ 0_R, & (\text{otherwise}). \end{cases}$$
Here $(a_0, b_0) = (\lambda\backslash\pi^{-1}(x), ((\lambda/d)c)\backslash\pi^{-1}(x))$, and $x \in M$ means the unique solution to the equation $\mu(\pi(\lambda/d), \pi((\lambda/d)c), x) = \pi(\lambda)$ (see (QG6)).

2. For any $\lambda \in H$ and $(a, b) \in X \times X$, there uniquely exists $(c_0, d_0) \in X \times X$ such that
$$\tilde{\sigma}^{bc}_{ad}(\lambda) = \begin{cases} 1_R, & ((c, d) = (c_0, d_0)); \\ 0_R, & (\text{otherwise}). \end{cases}$$
Here $(c_0, d_0) = (\pi^{-1}(y)\backslash((\lambda a)/b), \pi^{-1}(y)\backslash\lambda)$, and $y \in M$ is the unique solution to the equation $\mu(y, \pi((\lambda a)/b), \pi(\lambda a)) = \pi(\lambda)$ (see (QG4)).

Proof. We give the proof only for 1.

Let us check that $\tilde{\sigma}^{b_0 c}_{a_0 d}(\lambda) = 1_R$ if $(a_0, b_0) = (\lambda\backslash\pi^{-1}(x), ((\lambda/d)c)\backslash\pi^{-1}(x))$. The fact that $b_0 = ((\lambda/d)c)\backslash\pi^{-1}(x)$ deduces $\pi^{-1}(x) = ((\lambda/d)c)b_0$. Thus it follows that $\mu(\pi(\lambda/d), \pi((\lambda/d)c), \pi(((\lambda/d)c)b_0)) = \mu(\pi(\lambda/d), \pi((\lambda/d)c), x) = \pi(\lambda)$. For $a_0 \in X$, we can calculate that

$$\begin{aligned} a_0 =& \lambda\backslash\pi^{-1}(x) \\ =& \lambda\backslash(((\lambda/d)c)b_0) \\ =& \pi^{-1}(\mu(\pi(\lambda/d), \pi((\lambda/d)c), \pi(((\lambda/d)c)b_0)))\backslash(((\lambda/d)c)b_0). \end{aligned}$$

The equality $(\lambda/d)d = \lambda$ induces that $d = (\lambda/d)\backslash\lambda$. By using this fact, we compute

$$\begin{aligned} d =&(\lambda/d)\backslash\lambda \\ =&(\lambda/d)\backslash\pi^{-1}(\mu(\pi(\lambda/d), \pi((\lambda/d)c), \pi(((\lambda/d)c)b_0))). \end{aligned}$$

Thus $\tilde{\sigma}^{b_0 c}_{a_0 d}(\lambda) = \sigma^{b_0 c}_{a_0 d}(\lambda/d) = 1_R$ is satisfied.

In order to complete the proof, we need to prove that $(a, b) = (a_0, b_0)$ if $\tilde{\sigma}^{bc}_{ad}(\lambda) \neq 0_R$. The definition of $\tilde{\sigma}^{bc}_{ad}$ deduces that $\tilde{\sigma}^{bc}_{ad}(\lambda) = 1_R$ if $\tilde{\sigma}^{bc}_{ad}(\lambda) \neq 0_R$. From (5.1), it is easy to induce that $\lambda = (\lambda/d)d = \pi^{-1}(\mu(\pi(\lambda/d), \pi((\lambda/d)c),$ $\pi(((\lambda/d)c)b)))$. Moreover,

$$\begin{aligned} a =&\pi^{-1}(\mu(\pi((\lambda/d)), \pi((\lambda/d)c), \pi(((\lambda/d)c)b)))\backslash(((\lambda/d)c)b)) \\ =&\lambda\backslash(((\lambda/d)c)b). \end{aligned}$$

By the utilization of the assumption (QG6), the unique elements $x \in M$ satisfying $\mu(\pi(\lambda/d), \pi((\lambda/d)c), x) = \pi(\lambda)$ is $x = \pi(((\lambda/d)c)b)$. Therefore we can conclude that $b = ((\lambda/d)c)\backslash\pi^{-1}(x) = b_0$ and $a = \lambda\backslash(((\lambda/d)c)b) = \lambda\backslash\pi^{-1}(x) = a_0$. This is the desired conclusion. □

Theorem 5.4. The family $\sigma_R = \{\sigma_R{}^{ab}_{cd}\}_{a,b,c,d\in X}$ satisfies (3.1). Moreover, this σ_R is rigid.

Proof. We first prove (3.1). In order to show this fact, the following lemma plays an important role.

Lemma 5.5. For $a, b, c, d, f, \lambda \in QG$, we suppose that $(a, c) = (f\backslash\lambda, ((\lambda/b)/d)\backslash f)$. Then $(\lambda/a)/c = (\lambda/b)/d$ is satisfied.

Proof. By the utilizaition of the assumption that $(a, c) = (f\backslash\lambda, ((\lambda/b)/d)\backslash f)$,

$$\begin{aligned} (((\lambda/b)/d)c)a =&(((\lambda/b)/d)(((\lambda/b)/d)\backslash f))(f\backslash\lambda) \\ =&f(f\backslash\lambda) \\ =&\lambda. \end{aligned}$$

Here we use $a'(a'\backslash b') = b'$ $(\forall a', b' \in QG)$ for the second and third equality. Thus $(\lambda/a)/c = (\lambda/b)/d$. □

Since $I^{bd}_{ac}((\lambda/a)/c)$ is a centeral element of the k-algebra R, we have

$$\begin{aligned} (T_{\deg(a)^{-1}} \circ T_{\deg(c)^{-1}} \circ \rho_l(\sigma_R{}^{bd}_{ac}))(f)(\lambda) = \sigma^{bd}_{ac}((\lambda/a)/c)I^{bd}_{ac}((\lambda/a)/c)f((\lambda/a)/c), \\ (T_{\deg(b)^{-1}} \circ T_{\deg(d)^{-1}} \circ \rho_r(\sigma_R{}^{bd}_{ac}))(f)(\lambda) = \sigma^{bd}_{ac}((\lambda/b)/d)I^{bd}_{ac}((\lambda/b)/d)f((\lambda/b)/d) \end{aligned}$$

for all $f \in L$ and $\lambda \in H$. Thus it is sufficient to show the following facts:

$$\begin{cases} \sigma^{bd}_{ac}((\lambda/a)/c) = 1_R \Rightarrow (\lambda/a)/c = (\lambda/b)/d; \\ \sigma^{bd}_{ac}((\lambda/a)/c) = 0_R \Rightarrow \sigma^{bd}_{ac}((\lambda/b)/d) = 0_R. \end{cases} \tag{5.2}$$

Let us show the first assertion in (5.2). By the utilization of $\sigma^{bd}_{ac}((\lambda/a)/c) = 1_R$ and (5.1),

$$\begin{aligned} a = \pi^{-1}(\mu(\pi((\lambda/a)/c), \pi(((\lambda/a)/c)d), \pi((((\lambda/a)/c)d)b)))\backslash((((\lambda/a)/c)d)b), \\ c = ((\lambda/a)/c))\backslash\pi^{-1}(\mu(\pi((\lambda/a)/c), \pi(((\lambda/a)/c)d), \pi((((\lambda/a)/c)d)b))). \end{aligned}$$

Lemma 5.5 induces that $(((((\lambda/a)/c)d)b)/a)/c = (\lambda/a)/c$. Therefore $(\lambda/a)/c = (\lambda/b)/d$.

For the proof of the second claim in (5.2), we have only to check the following: $\sigma_{ac}^{bd}((\lambda/a)/c) = 1_R$, if $\sigma_{ac}^{bd}((\lambda/b)/d) = 1_R$. The proof is similar to that for the first one.

Finally we show that σ_R is rigid. It suffices to check that σ_R satisfies (4.6) and (4.7). By the definition (4.5), $\tilde{\sigma}_R{}_{cd}^{ab}(\lambda) = \tilde{\sigma}_{cd}^{ab}(\lambda) I_{cd}^{ab}(\lambda)$. The elements $i_*(\tilde{\sigma}_R)_{ad}^{bc} \in L$ $(a, b, c, d \in X)$ are defined by $i_*(\tilde{\sigma}_R)_{ad}^{bc}(\lambda) = \tilde{\sigma}_{cb}^{da}(\lambda) I_{cb}^{da}(\lambda)^{-1}$.

In order to complete the proof of (4.6), the following lemma plays an important role.

Lemma 5.6. 1. Let $c, d, c', d', \lambda \in QG$. We denote by a_0, b_0, a_0', and b_0' the elements in QG satisfying $\tilde{\sigma}_{a_0 d}^{b_0 c}(\lambda) = \tilde{\sigma}_{a_0' d'}^{b_0' c'}(\lambda) = 1_R$. If $(c, d) \neq (c', d')$, then $(a_0, b_0) \neq (a_0', b_0')$.

2. Let $a, b, a', b', \lambda \in QG$. We denote by c_0, d_0, c_0', and d_0' the elements in QG satisfying $\tilde{\sigma}_{ad_0}^{bc_0}(\lambda) = \tilde{\sigma}_{a'd_0'}^{b'c_0'}(\lambda) = 1_R$. If $(a, b) \neq (a', b')$, then $(c_0, d_0) \neq (c_0', d_0')$.

Proof. We give the proof only for 1 and show its contraposition. We suppose that $(a_0, b_0) = (a_0', b_0')$. The elements a_0, b_0, a_0', and b_0' can be denoted by

$$\begin{aligned}(a_0, b_0) =&(\lambda\backslash\pi^{-1}(x), ((\lambda/d)c)\backslash\pi^{-1}(x));\\ (a_0', b_0') =&(\lambda\backslash\pi^{-1}(x'), ((\lambda/d')c')\backslash\pi^{-1}(x')).\end{aligned}$$

Here x and x' are unique solutions to the following equations:

$$\mu(\pi(\lambda/d), \pi((\lambda/d)c), x) = \pi(\lambda) = \mu(\pi(\lambda/d'), \pi((\lambda/d')c'), x').$$

The fact that $a_0 = a_0'$ induces that $x = x'$. In addition, we compute that $(\lambda/d)c = (\lambda/d')c'$ because $b_0 = b_0'$ and $((\lambda/d)c)b_0 = \pi^{-1}(x) = \pi^{-1}(x') = ((\lambda/d')c')b_0'$. Thus,

$$\begin{aligned}\pi(\lambda) &= \mu(\pi(\lambda/d), \pi((\lambda/d)c), x)\\ &= \mu(\pi(\lambda/d'), \pi((\lambda/d')c'), x')\\ &= \mu(\pi(\lambda/d'), \pi((\lambda/d)c), x).\end{aligned}$$

By using the condition (QG4), we can conclude that $\lambda/d = \lambda/d'$. Thus we calculate that $\lambda = (\lambda/d)d = (\lambda/d')d' = (\lambda/d)d'$. By the definition of quasigroups, it follows that $d = d'$. For $c = c'$, we compute

$$\begin{aligned}c =&(\lambda/d)\backslash((\lambda/d)c)\\ =&(\lambda/d')\backslash((\lambda/d')c')\\ =&c'.\end{aligned}$$

This is the desired conclusion. □

By the utilization of Proposition 5.3 and Lemma 5.6, the condition (4.6) is satisfied: $\sum_{a,b\in X} i_*(\tilde{\sigma}_R)_{zb}^{wa}\ \tilde{\sigma}_R{}_{ax}^{by} = \sum_{a,b\in X} \tilde{\sigma}_R{}_{zb}^{wa}\ i_*(\tilde{\sigma}_R)_{ax}^{by} = \delta_{w,x}\delta_{y,z}1_L$. For the proof of (4.7), we use the following lemma.

Lemma 5.7. 1. Let $c, d, \lambda \in QG$. We denote by a_0 and b_0 the elements in QG satisfying $\tilde{\sigma}^{b_0 c}_{a_0 d}(\lambda) = 1_R$. $a_0 = b_0$ is equivalent to $c = d$.

2. Let $a, b, \lambda \in QG$. We denote by c_0 and d_0 the elements in QG satisfying $\tilde{\sigma}^{bc_0}_{ad_0}(\lambda) = 1_R$. $c_0 = d_0$ is equivalent to $a = b$.

Proof. We will show 1. Suppose that $c = d$ and we denote by $x(\in X)$ the unique solution to the equation $\mu(\pi(\lambda/c), \pi(\lambda), x) = \pi(\lambda)$. Then the element b_0 can be calculated as follows:

$$\begin{aligned} b_0 =& ((\lambda/c)c)\backslash\pi^{-1}(x) \\ =& \lambda\backslash\pi^{-1}(x). \end{aligned}$$

Thus we conclude $a_0 = b_0$. Conversely, we assume that $a_0 = b_0$. This fact induces $\lambda = (\lambda/d)c$. Since $(\lambda/d)\backslash\lambda = (\lambda/d)\backslash((\lambda/d)c)$, $d = c$ is proved.

For the proof of 2, we can calculate that $c_0 = d_0 = \pi^{-1}(y)\backslash\lambda$ when $a = b$. Here $y \in M$ means the unique solution to the equation $\mu(y, \pi(\lambda), \pi(\lambda a)) = \pi(\lambda)$. If $c_0 = d_0$, then $\pi^{-1}(y)\backslash((\lambda a)/b) = \pi^{-1}(y)\backslash\lambda$. Thus we can induce that $(\lambda a)/b = \pi^{-1}(y)(\pi^{-1}(y)\backslash((\lambda a)/b)) = \pi^{-1}(y)(\pi^{-1}(y)\backslash\lambda) = \lambda$. By using the identity $\lambda a = \lambda b$, we can induce that $a = \lambda\backslash(\lambda b) = b$. This completes the proof. □

By virtue of Lemma 5.7 and its proof, we can calculate that

$$\begin{aligned} Q_{Ra,b}(\lambda) &= \sum_u i_*(\tilde{\sigma}_R)^{ub}_{ua}(\lambda) = \delta_{a,b} I^{a\ \pi^{-1}(y)\backslash\lambda}_{a\ \pi^{-1}(y)\backslash\lambda}(\lambda)^{-1}, \\ Q'_{Ra,b}(\lambda) &= \sum_u i_*(\tilde{\sigma}_R)^{bu}_{au}(\lambda) = \delta_{a,b} I^{\lambda\backslash\pi^{-1}(x)\ a}_{\lambda\backslash\pi^{-1}(x)\ a}(\lambda)^{-1}, \\ Q''_{Ra,b}(\lambda) &= \sum_u T_{\deg(b)}(i_*(\tilde{\sigma}_R)^{bu}_{au})(\lambda) = \delta_{a,b} I^{(\lambda a)\backslash\pi^{-1}(x')\ a}_{(\lambda a)\backslash\pi^{-1}(x')\ a}(\lambda a)^{-1}, \\ Q'''_{Ra,b}(\lambda) &= \sum_u T_{\deg(a)^{-1}}(i_*(\tilde{\sigma}_R)^{ua}_{ub})(\lambda) = \delta_{a,b} I^{a\ \pi^{-1}(y')\backslash(\lambda/a)}_{a\ \pi^{-1}(y')\backslash(\lambda/a)}(\lambda/a)^{-1}. \end{aligned}$$

Here $x, x', y, y' \in M$ stand for the unique solutions to the following equations, respectively:

$$\begin{aligned} &\mu(\pi(\lambda/a), \pi(\lambda), x) = \pi(\lambda); \quad \mu(\pi(\lambda), \pi(\lambda a), x') = \pi(\lambda a); \\ &\mu(y, \pi(\lambda), \pi(\lambda a)) = \pi(\lambda); \quad \mu(y', \pi(\lambda/a), \pi(\lambda)) = \pi(\lambda/a). \end{aligned}$$

Thus the four elements $Q_{R}{}^{-1}_{a,b}(\lambda) = \delta_{a,b} I^{a\ \pi^{-1}(y)\backslash\lambda}_{a\ \pi^{-1}(y)\backslash\lambda}(\lambda)$, $Q'_{R}{}^{-1}_{a,b}(\lambda) = \delta_{a,b} I^{\lambda\backslash\pi^{-1}(x)\ a}_{\lambda\backslash\pi^{-1}(x)\ a}(\lambda)$, $Q''_{R}{}^{-1}_{a,b}(\lambda) = \delta_{a,b} I^{(\lambda a)\backslash\pi^{-1}(x')\ a}_{(\lambda a)\backslash\pi^{-1}(x')\ a}(\lambda a)$, and $Q'''_{R}{}^{-1}_{a,b}(\lambda) = \delta_{a,b} I^{a\ \pi^{-1}(y')\backslash(\lambda/a)}_{a\ \pi^{-1}(y')\backslash(\lambda/a)}(\lambda/a)$ satisfy the conditions (4.7) in Proposition 4.6. □

Corollary 5.8. A_{σ_R} is a Hopf algebroid.

Finally, we give a k-algebra R that is not Frobenius-separable. This R makes A_σ a Hopf algebroid that is not a weak Hopf algebra.

Let k be a field and $M_2(k)$ the matrix algebra consisting of all 2×2 matrices. We denote by $R \subset M_2(k)$ the k-subspace spanned by the basis $\{1 =$

$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, e_1 = \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}, \alpha = \begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\}$. For all $a, b, c, d, f, g \in k$, we can induce that

$$\begin{aligned}&(a1 + be_1 + c\alpha)(d1 + fe_1 + g\alpha)\\ =&ad1 + (af + bd + bf)e_1 + (ag + bg + cd)\alpha.\end{aligned} \tag{5.3}$$

Thus this R becomes a k-subalgebra of $M_2(k)$. We note that the k-algebra R is not commutative because $\alpha e_1 = 0 \neq \alpha = e_1\alpha$.

Proposition 5.9. This R is not a Frobenius-separable k-algebra.

Proof. Let us check that R is not a Frobenius k-algebra. It suffices to prove that there exists a non-zero left ideal included in $\mathrm{Ker}(\phi)$ for any k-linear map $\phi : R \to k$.

Since (5.3) is satisfied, $J_{a,b} := k(ae_1 \oplus b\alpha)$ is a left ideal of R for any $a, b \in k$. We will prove that $\mathrm{Ker}(\phi) \cap J_{1,1} \neq \{0\}$ for any k-linear map $\phi : R \to k$. Let us suppose that this assertion is false. Because $\mathrm{Ker}(\phi)$ is a k-submodule of R,

$$\begin{aligned}3 =& \dim_k R\\ \geq& \dim_k \mathrm{Ker}(\phi) \oplus J_{1,1}\\ \geq& 2 + 2 = 4 > \dim_k R.\end{aligned}$$

This is a contradiction.

For $\beta \in \mathrm{Ker}(\phi) \cap J_{1,1} \setminus \{0\}$, $k\beta(\subset \mathrm{Ker}(\phi))$ is thus a non-zero left ideal of R. This completes the proof. □

Part III

Hopf algebroid $\mathfrak{U}(w)$ and relations with A_σ

In this part, we construct a Hopf algebroid $\mathfrak{U}(w)$ by using a rigid family w consisting of elements in the base ring. Moreover, the family σ in Section 4 satisfying the condition (9.1) induces the family w. We can construct a left and right bialgebroid $\mathfrak{U}(w_\sigma) := \mathfrak{U}(w)$ and a left and right bialgebroid isomorphism $\Phi \colon \mathfrak{U}(w_\sigma) \to A_\sigma$ by using this w. The map Φ induces that the rigidity of σ is equivalent to that of w.

6 Left bialgebroid $\mathfrak{U}(w)$

In this section, we introduce a left bialgebroid $\mathfrak{U}(w)$ (cf. [13, 19]).

Definition 6.1. Let Λ be a non-empty set. A quiver over Λ is a set Q endowed with two maps $\mathfrak{s}, \mathfrak{t} \colon Q \to \Lambda$. These maps $\mathfrak{s}$ and $\mathfrak{t}$ are called the source map and the target map, respectively.

For any $m \in \mathbb{Z}_{\geq 0}$, the fiber product $Q^{(m)}$ is defined by $Q^{(0)} := \Lambda$, $Q^{(1)} := Q$, and $Q^{(m)} := \{q = (q_1, \ldots, q_m) \in Q^m \mid \mathfrak{t}(q_i) = \mathfrak{s}(q_{i+1}), 1 \leq \forall i \leq m-1\}$ $(m > 1)$. Each $Q^{(m)}$ $(m > 0)$ is also a quiver over Λ with $\mathfrak{s}(q) = \mathfrak{s}(q_1)$, $\mathfrak{t}(q) = \mathfrak{t}(q_m)$. For $Q^{(0)}$, we define $\mathfrak{s}$ and $\mathfrak{t}$ by $\mathfrak{s} = \mathfrak{t} = \mathrm{id}_\Lambda$.

Let R be a k-algebra and Q a finite quiver over a non-empty finite set Λ. We define the set ΛQ as follows:

$$\Lambda Q := (R \otimes_k R^{op}) \bigsqcup \{\mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} \mid m \in \mathbb{Z}_{\geq 0}, p, q \in Q^{(m)}\} \bigsqcup \{\widetilde{\mathbf{e}}\begin{bmatrix} p \\ q \end{bmatrix} \mid m \in \mathbb{Z}_{\geq 0}, p, q \in Q^{(m)}\}.$$

We denote by $k\langle \Lambda Q \rangle$ the free k-algebra generated by ΛQ. Let $\mathbf{w}\left[\begin{smallmatrix} & a & \\ c & & b \\ & d & \end{smallmatrix}\right]$ $(\forall (a,b),(c,d) \in Q^{(2)})$ be a collection of elements in R. The symbol $\mathfrak{J}_w$ stands for the two-sided ideal of $k\langle \Lambda Q \rangle$ whose generators are:

(A) $\xi + \xi' - (\xi +_{R\otimes_k R^{op}} \xi')$, $c\xi - (c \cdot_{R\otimes_k R^{op}} \xi)$, $\xi\xi' - (\xi \cdot_{R\otimes_k R^{op}} \xi')$ $(\forall c \in k, \forall \xi, \xi' \in R \otimes_k R^{op})$.
Here the notation $\xi + \xi'$ means the addition in the k-algebra $k\langle \Lambda Q \rangle$. On the other hand, the notation $(\xi +_{R\otimes_k R^{op}} \xi')(\in Gen)$ is that of the k-algebra $R \otimes_k R^{op}$. The other two symbols for the scalar multiplication and the multiplication are similar.

(B) $\mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}\mathbf{e}\begin{bmatrix} p' \\ q' \end{bmatrix} - \delta_{\mathfrak{t}(p),\mathfrak{s}(p')}\delta_{\mathfrak{t}(q),\mathfrak{s}(q')}\mathbf{e}\begin{bmatrix} pp' \\ qq' \end{bmatrix}$, $\widetilde{\mathbf{e}}\begin{bmatrix} p \\ q \end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix} p' \\ q' \end{bmatrix} - \delta_{\mathfrak{t}(p'),\mathfrak{s}(p)}\delta_{\mathfrak{t}(q'),\mathfrak{s}(q)}\widetilde{\mathbf{e}}\begin{bmatrix} p'p \\ q'q \end{bmatrix}$ $(\forall m, n \in \mathbb{Z}_{\geq 0}, \forall p, q \in Q^{(m)}, \forall p', q' \in Q^{(n)})$.

(C) $\sum_{u\in Q^{(m)}} \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix} u \\ q \end{bmatrix} - \delta_{p,q}\sum_{\lambda\in\Lambda} \mathbf{e}\begin{bmatrix} \mathfrak{s}(p) \\ \lambda \end{bmatrix}$, $\sum_{u\in Q^{(m)}} \widetilde{\mathbf{e}}\begin{bmatrix} p \\ u \end{bmatrix}\mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} - \delta_{p,q}\sum_{\lambda\in\Lambda} \mathbf{e}\begin{bmatrix} \lambda \\ \mathfrak{t}(p) \end{bmatrix}$ $(\forall m \in \{0,1\}, \forall p, q \in Q^{(m)})$.

(D) $(r \otimes r')\mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} - \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}(r \otimes r')$, $(r \otimes r')\widetilde{\mathbf{e}}\begin{bmatrix} p \\ q \end{bmatrix} - \widetilde{\mathbf{e}}\begin{bmatrix} p \\ q \end{bmatrix}(r \otimes r')$ $(\forall r, r' \in R, \forall m \in \mathbb{Z}_{\geq 0}, \forall p, q \in Q^{(m)})$.

(E) $\sum_{(x,y)\in Q^{(2)}} (\mathbf{w}\left[\begin{smallmatrix} & x & \\ a & & y \\ & b & \end{smallmatrix}\right] \otimes 1_R)\mathbf{e}\begin{bmatrix} x \\ c \end{bmatrix}\mathbf{e}\begin{bmatrix} y \\ d \end{bmatrix} - \sum_{(x,y)\in Q^{(2)}} (1_R \otimes \mathbf{w}\left[\begin{smallmatrix} & c & \\ x & & d \\ & y & \end{smallmatrix}\right])\mathbf{e}\begin{bmatrix} a \\ x \end{bmatrix}\mathbf{e}\begin{bmatrix} b \\ y \end{bmatrix}$ $(\forall (a,b),(c,d) \in Q^{(2)})$.

(F) $\emptyset - \sum_{\lambda,\mu\in\Lambda} \mathbf{e}\begin{bmatrix} \lambda \\ \mu \end{bmatrix}$, $\emptyset - 1_R \otimes 1_R$.

We denote by M the k-algebra consisting of all maps from Λ to R. The following theorem is satisfied:

Theorem 6.2. If the following conditions are satisfied, then the quotient $\mathfrak{U}(w) := k\langle \Lambda Q \rangle / \mathfrak{J}_w$ is a left bialgebroid with the base ring M.

$$\begin{cases} \mathbf{w}\left[\begin{smallmatrix} & a & \\ c & & b \\ & d & \end{smallmatrix}\right] \in Z(R); \\ \mathfrak{s}(a) \neq \mathfrak{s}(c) \text{ or } \mathfrak{t}(b) \neq \mathfrak{t}(d) \Rightarrow \mathbf{w}\left[\begin{smallmatrix} & a & \\ c & & b \\ & d & \end{smallmatrix}\right] = 0. \end{cases} \tag{6.1}$$

Here $Z(R)$ means the center of R.

We will prove Theorem 6.2 by constructing the maps s_M, t_M, Δ_M, and π_M in Definition 1.1. We first define the maps s_M and t_M as follows:

$$s_M(f) = \sum_{\lambda,\mu\in\Lambda} (f(\lambda)\otimes 1_R)\mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} + \mathfrak{I}_w; \tag{6.2}$$

$$t_M(f) = \sum_{\lambda,\mu\in\Lambda} (1_R\otimes f(\lambda))\mathbf{e}\begin{bmatrix}\mu\\ \lambda\end{bmatrix} + \mathfrak{I}_w \quad (f\in M). \tag{6.3}$$

It is easy to check that these s_M and t_M are k-algebra homomorphisms and satisfy (1.1). Thus $\mathfrak{U}(w)$ becomes an (M,M)-bimodule by virtue of the action (1.2).

Let $\mathfrak{I}_2$ denote the right ideal of $\mathfrak{U}(w)\otimes_k \mathfrak{U}(w)$ generated by $t_M(f)\otimes 1_{\mathfrak{U}(w)} - 1_{\mathfrak{U}(w)}\otimes s_M(f)$ $(\forall f\in M)$. In order to construct Δ_M, we define the k-algebra homomorphism $\overline{\nabla}\colon k\langle\Lambda Q\rangle \to \mathfrak{U}(w)\otimes_k \mathfrak{U}(w)$ as follows:

$$\overline{\nabla}(\xi) = \chi(\xi) \quad (\xi\in R\otimes_k R^{op}); \tag{6.4}$$

$$\overline{\nabla}(\mathbf{e}\begin{bmatrix}p\\ q\end{bmatrix}) = \sum_{u\in Q^{(m)}} \mathbf{e}\begin{bmatrix}p\\ u\end{bmatrix} + \mathfrak{I}_w \otimes \mathbf{e}\begin{bmatrix}u\\ q\end{bmatrix} + \mathfrak{I}_w; \tag{6.5}$$

$$\overline{\nabla}(\widetilde{\mathbf{e}}\begin{bmatrix}p\\ q\end{bmatrix}) = \sum_{u\in Q^{(m)}} \widetilde{\mathbf{e}}\begin{bmatrix}u\\ q\end{bmatrix} + \mathfrak{I}_w \otimes \widetilde{\mathbf{e}}\begin{bmatrix}p\\ u\end{bmatrix} + \mathfrak{I}_w \quad (m\in\mathbb{Z}_{\geq 0}, p,q\in Q^{(m)}). \tag{6.6}$$

Here the symbol χ means the k-algebra homomorphism defined by $\chi\colon R\otimes_k R^{op} \ni r\otimes r' \mapsto (r\otimes 1_R + \mathfrak{I}_w)\otimes(1_R\otimes r' + \mathfrak{I}_w) \in \mathfrak{U}(w)\otimes_k \mathfrak{U}(w)$.

Proposition 6.3. $\overline{\nabla}(\mathfrak{I}_w)\subset \mathfrak{I}_2$

Proof. We first prove that $\overline{\nabla}(\alpha)\in\mathfrak{I}_2$ for any generator α of $\mathfrak{I}_w$.

Since $\overline{\nabla}$ is a k-algebra homomorphism, it is easy to check this fact if α is an arbitrary generator of (A).

For the generators (B), we can calculate that

$$\begin{aligned}
&\overline{\nabla}(\widetilde{\mathbf{e}}\begin{bmatrix}p\\ q\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p'\\ q'\end{bmatrix})\\
=&\sum_{\substack{u\in Q^{(m)}\\ u'\in Q^{(n)}}} \delta_{\mathfrak{t}(u'),\mathfrak{s}(u)}\delta_{\mathfrak{t}(q'),\mathfrak{s}(q)}\widetilde{\mathbf{e}}\begin{bmatrix}u'u\\ q'q\end{bmatrix} + \mathfrak{I}_w \otimes \delta_{\mathfrak{t}(p'),\mathfrak{s}(p)}\delta_{\mathfrak{t}(u'),\mathfrak{s}(u)}\widetilde{\mathbf{e}}\begin{bmatrix}p'p\\ u'u\end{bmatrix} + \mathfrak{I}_w\\
=&\delta_{\mathfrak{t}(p'),\mathfrak{s}(p)}\delta_{\mathfrak{t}(q'),\mathfrak{s}(q)}\sum_{v\in Q^{(m+n)}} \widetilde{\mathbf{e}}\begin{bmatrix}v\\ q'q\end{bmatrix} + \mathfrak{I}_w\otimes \widetilde{\mathbf{e}}\begin{bmatrix}p'p\\ v\end{bmatrix} + \mathfrak{I}_w\\
=&\overline{\nabla}(\delta_{\mathfrak{t}(p'),\mathfrak{s}(p)}\delta_{\mathfrak{t}(q'),\mathfrak{s}(q)}\widetilde{\mathbf{e}}\begin{bmatrix}p'p\\ q'q\end{bmatrix})
\end{aligned}$$

for any $m,n\in\mathbb{Z}_{\geq 0}, p,q\in Q^{(m)}, p',q'\in Q^{(n)}$. The proof for the other generator is similar.

Let us check that $\overline{\nabla}(\alpha)=0$ if α is a generator (C). We assume the following lemma for a moment:

Lemma 6.4. For any $\lambda,\mu\in\Lambda$,

$$\mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} + \mathfrak{I}_w = \widetilde{\mathbf{e}}\begin{bmatrix}\mu\\ \lambda\end{bmatrix} + \mathfrak{I}_w.$$

For any $m \in \{0,1\}, p, q \in Q^{(m)}$,

$$
\begin{aligned}
&\overline{\nabla}\Big(\sum_{u \in Q^{(m)}} \widetilde{\mathbf{e}}\begin{bmatrix} p \\ u \end{bmatrix} \mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} - \delta_{p,q} \sum_{\lambda \in \Lambda} \mathbf{e}\begin{bmatrix} \lambda \\ \mathfrak{t}(p) \end{bmatrix}\Big) \\
=& \sum_{u,v,v' \in Q^{(m)}} \widetilde{\mathbf{e}}\begin{bmatrix} v \\ u \end{bmatrix} \mathbf{e}\begin{bmatrix} u \\ v' \end{bmatrix} + \mathfrak{J}_w \otimes \widetilde{\mathbf{e}}\begin{bmatrix} p \\ v \end{bmatrix} \mathbf{e}\begin{bmatrix} v' \\ q \end{bmatrix} + \mathfrak{J}_w \\
& - \delta_{p,q} \sum_{\lambda,\mu \in \Lambda} \mathbf{e}\begin{bmatrix} \lambda \\ \mu \end{bmatrix} + \mathfrak{J}_w \otimes \mathbf{e}\begin{bmatrix} \mu \\ \mathfrak{t}(p) \end{bmatrix} + \mathfrak{J}_w \\
=& \sum_{\substack{v \in Q^{(m)} \\ \tau \in \Lambda}} \mathbf{e}\begin{bmatrix} \tau \\ \mathfrak{t}(v) \end{bmatrix} + \mathfrak{J}_w \otimes \widetilde{\mathbf{e}}\begin{bmatrix} p \\ v \end{bmatrix} \mathbf{e}\begin{bmatrix} v \\ q \end{bmatrix} + \mathfrak{J}_w \\
& - \delta_{p,q} \sum_{\lambda,\mu \in \Lambda} \mathbf{e}\begin{bmatrix} \lambda \\ \mu \end{bmatrix} + \mathfrak{J}_w \otimes \mathbf{e}\begin{bmatrix} \mu \\ \mathfrak{t}(p) \end{bmatrix} + \mathfrak{J}_w
\end{aligned}
$$

By the definition of the right ideal $\mathfrak{J}_2$,

$$
\begin{aligned}
& \sum_{\lambda,\mu \in \Lambda} (\mathbf{e}\begin{bmatrix} \lambda \\ \mu \end{bmatrix} + \mathfrak{J}_w) \otimes (\mathbf{e}\begin{bmatrix} \mu \\ \mathfrak{t}(p) \end{bmatrix} + \mathfrak{J}_w) + \mathfrak{J}_2 \\
=& \sum_{\mu \in \Lambda} t_M(\delta_\mu) \otimes (\mathbf{e}\begin{bmatrix} \mu \\ \mathfrak{t}(p) \end{bmatrix} + \mathfrak{J}_w) + \mathfrak{J}_2 \\
=& \sum_{\mu \in \Lambda} 1_{\mathfrak{U}(w)} \otimes s_M(\delta_\mu)(\mathbf{e}\begin{bmatrix} \mu \\ \mathfrak{t}(p) \end{bmatrix} + \mathfrak{J}_w) + \mathfrak{J}_2 \\
=& \sum_{\mu \in \Lambda} 1_{\mathfrak{U}(w)} \otimes (\mathbf{e}\begin{bmatrix} \mu \\ \mathfrak{t}(p) \end{bmatrix} + \mathfrak{J}_w) + \mathfrak{J}_2.
\end{aligned}
$$

Here $\delta_\lambda \in M$ $(\forall \lambda \in \Lambda)$ is defined by $\delta_\lambda(\mu) = \delta_{\lambda,\mu}$ $(\mu \in \Lambda)$. On the other hand, Lemma 6.4 induces that

$$
\begin{aligned}
& \sum_{\substack{v \in Q^{(m)} \\ \tau \in \Lambda}} (\mathbf{e}\begin{bmatrix} \tau \\ \mathfrak{t}(v) \end{bmatrix} + \mathfrak{J}_w) \otimes (\widetilde{\mathbf{e}}\begin{bmatrix} p \\ v \end{bmatrix} \mathbf{e}\begin{bmatrix} v \\ q \end{bmatrix} + \mathfrak{J}_w) + \mathfrak{J}_2 \\
=& \sum_{v \in Q^{(m)}} t_M(\delta_{\mathfrak{t}(v)}) \otimes (\widetilde{\mathbf{e}}\begin{bmatrix} p \\ v \end{bmatrix} \mathbf{e}\begin{bmatrix} v \\ q \end{bmatrix} + \mathfrak{J}_w) + \mathfrak{J}_2 \\
=& \sum_{v \in Q^{(m)}} 1_{\mathfrak{U}(w)} \otimes s_M(\delta_{\mathfrak{t}(v)})(\widetilde{\mathbf{e}}\begin{bmatrix} p \\ v \end{bmatrix} \mathbf{e}\begin{bmatrix} v \\ q \end{bmatrix} + \mathfrak{J}_w) + \mathfrak{J}_2 \\
=& \sum_{\substack{v \in Q^{(m)} \\ \lambda \in \Lambda}} 1_{\mathfrak{U}(w)} \otimes (\widetilde{\mathbf{e}}\begin{bmatrix} \lambda \\ \mathfrak{t}(v) \end{bmatrix} \widetilde{\mathbf{e}}\begin{bmatrix} p \\ v \end{bmatrix} \mathbf{e}\begin{bmatrix} v \\ q \end{bmatrix} + \mathfrak{J}_w) + \mathfrak{J}_2 \\
=& \sum_{v \in Q^{(m)}} 1_{\mathfrak{U}(w)} \otimes (\widetilde{\mathbf{e}}\begin{bmatrix} p \\ v \end{bmatrix} \mathbf{e}\begin{bmatrix} v \\ q \end{bmatrix} + \mathfrak{J}_w) + \mathfrak{J}_2 \\
=& \delta_{p,q} \sum_{\lambda \in \Lambda} 1_{\mathfrak{U}(w)} \otimes (\mathbf{e}\begin{bmatrix} \lambda \\ \mathfrak{t}(p) \end{bmatrix} + \mathfrak{J}_w) + \mathfrak{J}_2.
\end{aligned}
$$

Thus we can conclude that $\overline{\nabla}(\sum_{u\in Q^{(m)}} \widetilde{\mathbf{e}}\begin{bmatrix} p \\ u \end{bmatrix}\mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} - \delta_{p,q}\sum_{\lambda\in\Lambda}\mathbf{e}\begin{bmatrix} \lambda \\ \mathfrak{t}(p) \end{bmatrix}) \in \mathfrak{I}_2$. The proof for the other generator is similar.

We will show that $\overline{\nabla}(\alpha) = 0$ for any α in the generators (D). For all $r, r' \in R$, $m \in \mathbb{Z}_{\geq 0}$, $p, q \in Q^{(m)}$,

$$\begin{aligned}
&\overline{\nabla}((r\otimes r')\mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} - \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}(r\otimes r')) \\
=&\sum_{u\in Q^{(m)}} (r\otimes 1_R)\mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{J}_w \otimes (1_R\otimes r')\mathbf{e}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{J}_w \\
&-\sum_{v\in Q^{(m)}} \mathbf{e}\begin{bmatrix} p \\ v \end{bmatrix}(r\otimes 1_R) + \mathfrak{J}_w \otimes \mathbf{e}\begin{bmatrix} v \\ q \end{bmatrix}(1_R\otimes r') + \mathfrak{J}_w \\
=&0.
\end{aligned}$$

The same proof works for the other generator.

For the generators (E), we can calculate that

$$\begin{aligned}
&\overline{\nabla}(\sum_{(x,y)\in Q^{(2)}} (\mathbf{w}\begin{bmatrix} a \ {}^{x}_{b}\ y \end{bmatrix}\otimes 1_R)\mathbf{e}\begin{bmatrix} x \\ c \end{bmatrix}\mathbf{e}\begin{bmatrix} y \\ d \end{bmatrix} - \sum_{(x,y)\in Q^{(2)}} (1_R\otimes \mathbf{w}\begin{bmatrix} x \ {}^{c}_{y}\ d \end{bmatrix})\mathbf{e}\begin{bmatrix} a \\ x \end{bmatrix}\mathbf{e}\begin{bmatrix} b \\ y \end{bmatrix}) \\
=&\sum_{(x,y),(u,v)\in Q^{(2)}} (\mathbf{w}\begin{bmatrix} a \ {}^{x}_{b}\ y \end{bmatrix}\otimes 1_R)\mathbf{e}\begin{bmatrix} x \\ u \end{bmatrix}\mathbf{e}\begin{bmatrix} y \\ v \end{bmatrix} + \mathfrak{J}_w \otimes \mathbf{e}\begin{bmatrix} u \\ c \end{bmatrix}\mathbf{e}\begin{bmatrix} v \\ d \end{bmatrix} + \mathfrak{J}_w \\
&-\sum_{(x,y),(u,v)\in Q^{(2)}} \mathbf{e}\begin{bmatrix} a \\ u \end{bmatrix}\mathbf{e}\begin{bmatrix} b \\ v \end{bmatrix} + \mathfrak{J}_w \otimes (1_R\otimes \mathbf{w}\begin{bmatrix} x \ {}^{c}_{y}\ d \end{bmatrix})\mathbf{e}\begin{bmatrix} u \\ x \end{bmatrix}\mathbf{e}\begin{bmatrix} v \\ y \end{bmatrix} + \mathfrak{J}_w \\
=&\sum_{(x,y),(u,v)\in Q^{(2)}} (1_R\otimes \mathbf{w}\begin{bmatrix} x \ {}^{u}_{y}\ v \end{bmatrix})\mathbf{e}\begin{bmatrix} a \\ x \end{bmatrix}\mathbf{e}\begin{bmatrix} b \\ y \end{bmatrix} + \mathfrak{J}_w \otimes \mathbf{e}\begin{bmatrix} u \\ c \end{bmatrix}\mathbf{e}\begin{bmatrix} v \\ d \end{bmatrix} + \mathfrak{J}_w \\
&-\sum_{(x,y),(u,v)\in Q^{(2)}} \mathbf{e}\begin{bmatrix} a \\ u \end{bmatrix}\mathbf{e}\begin{bmatrix} b \\ v \end{bmatrix} + \mathfrak{J}_w \otimes (\mathbf{w}\begin{bmatrix} u \ {}^{x}_{v}\ y \end{bmatrix}\otimes 1_R)\mathbf{e}\begin{bmatrix} x \\ c \end{bmatrix}\mathbf{e}\begin{bmatrix} y \\ d \end{bmatrix} + \mathfrak{J}_w \\
=&\sum_{(x,y),(u,v)\in Q^{(2)}} t_M(\mathbf{w}\begin{bmatrix} x \ {}^{u}_{y}\ v \end{bmatrix}_\sharp)\mathbf{e}\begin{bmatrix} a \\ x \end{bmatrix}\mathbf{e}\begin{bmatrix} b \\ y \end{bmatrix} + \mathfrak{J}_w \otimes \mathbf{e}\begin{bmatrix} u \\ c \end{bmatrix}\mathbf{e}\begin{bmatrix} v \\ d \end{bmatrix} + \mathfrak{J}_w \\
&-\sum_{(x,y),(u,v)\in Q^{(2)}} \mathbf{e}\begin{bmatrix} a \\ x \end{bmatrix}\mathbf{e}\begin{bmatrix} b \\ y \end{bmatrix} + \mathfrak{J}_w \otimes s_M(\mathbf{w}\begin{bmatrix} x \ {}^{u}_{y}\ v \end{bmatrix}_\sharp)\mathbf{e}\begin{bmatrix} u \\ c \end{bmatrix}\mathbf{e}\begin{bmatrix} v \\ d \end{bmatrix} + \mathfrak{J}_w \\
=&\sum_{(x,y),(u,v)\in Q^{(2)}} (t_M(\mathbf{w}\begin{bmatrix} x \ {}^{u}_{y}\ v \end{bmatrix}_\sharp)\otimes 1_{\mathfrak{U}(w)} - 1_{\mathfrak{U}(w)}\otimes s_M(\mathbf{w}\begin{bmatrix} x \ {}^{u}_{y}\ v \end{bmatrix}_\sharp)) \\
&\times(\mathbf{e}\begin{bmatrix} a \\ x \end{bmatrix}\mathbf{e}\begin{bmatrix} b \\ y \end{bmatrix} + \mathfrak{J}_w \otimes \mathbf{e}\begin{bmatrix} u \\ c \end{bmatrix}\mathbf{e}\begin{bmatrix} v \\ d \end{bmatrix} + \mathfrak{J}_w) \in \mathfrak{I}_2
\end{aligned}$$

for any $(a,b),(c,d) \in Q^{(2)}$. Here $r_\sharp \in M$ $(\forall r \in R)$ is defined by $r_\sharp(\lambda) = r$ $(\lambda \in \Lambda)$. Thus $\overline{\nabla}(\alpha) \in \mathfrak{I}_2$ is satisfied for any α in the generators (E).

The first generator of (F) induces that

$$\overline{\nabla}(\emptyset - \sum_{\lambda,\mu\in\Lambda}\mathbf{e}\begin{bmatrix} \lambda \\ \mu \end{bmatrix}) = 1_{\mathfrak{U}(w)}\otimes 1_{\mathfrak{U}(w)} - \sum_{\lambda,\mu,\tau\in\Lambda}\mathbf{e}\begin{bmatrix} \lambda \\ \tau \end{bmatrix} + \mathfrak{J}_w \otimes \mathbf{e}\begin{bmatrix} \tau \\ \mu \end{bmatrix} + \mathfrak{J}_w.$$

By using the definition of $\mathfrak{I}_2$, we can calculate that

$$
\begin{aligned}
&\sum_{\lambda,\mu,\tau\in\Lambda}(\mathbf{e}\begin{bmatrix}\lambda\\ \tau\end{bmatrix}+\mathfrak{I}_w)\otimes(\mathbf{e}\begin{bmatrix}\tau\\ \mu\end{bmatrix}+\mathfrak{I}_w)+\mathfrak{I}_2\\
=&\sum_{\mu,\tau\in\Lambda}t_M(\delta_\tau)\otimes(\mathbf{e}\begin{bmatrix}\tau\\ \mu\end{bmatrix}+\mathfrak{I}_w)+\mathfrak{I}_2\\
=&\sum_{\mu,\tau\in\Lambda}1_{\mathfrak{U}(w)}\otimes s_M(\delta_\tau)(\mathbf{e}\begin{bmatrix}\tau\\ \mu\end{bmatrix}+\mathfrak{I}_w)+\mathfrak{I}_2\\
=&\sum_{\lambda,\tau,\mu\in\Lambda}1_{\mathfrak{U}(w)}\otimes(\mathbf{e}\begin{bmatrix}\tau\\ \lambda\end{bmatrix}\mathbf{e}\begin{bmatrix}\tau\\ \mu\end{bmatrix}+\mathfrak{I}_w)+\mathfrak{I}_2\\
=&\sum_{\mu,\tau\in\Lambda}1_{\mathfrak{U}(w)}\otimes(\mathbf{e}\begin{bmatrix}\tau\\ \mu\end{bmatrix}+\mathfrak{I}_w)+\mathfrak{I}_2\\
=&1_{\mathfrak{U}(w)}\otimes 1_{\mathfrak{U}(w)}+\mathfrak{I}_2.
\end{aligned}
$$

Here we use the generators (B) for the fourth equality and the generators (F) for the fifth equality. The proof for the second generator of (F) is straightforward. Therefore $\overline{\nabla}(\alpha)=0$ for any generator α of $\mathfrak{I}_w$.

In order to complete the proof, we need to check that $\overline{\nabla}(k\langle\Lambda Q\rangle)\mathfrak{I}_2\subset\mathfrak{I}_2$. It suffices to check that $\overline{\nabla}(\alpha)(t_M(f)\otimes 1_{\mathfrak{U}(w)}-1_{\mathfrak{U}(w)}\otimes s_M(f))\in\mathfrak{I}_2$ for any $\alpha\in\Lambda Q$ and $f\in M$.

We assume that $\alpha=r\otimes r'$ ($\forall r,r'\in R$). The generators (F) and the identity (1.1) induce that

$$
\begin{aligned}
&\overline{\nabla}(\alpha)(t_M(f)\otimes 1_{\mathfrak{U}(w)}-1_{\mathfrak{U}(w)}\otimes s_M(f))\\
=&((r\otimes 1_R+\mathfrak{I}_w)\otimes(1_R\otimes r'+\mathfrak{I}_w))(t_M(f)\otimes 1_{\mathfrak{U}(w)}-1_{\mathfrak{U}(w)}\otimes s_M(f))\\
=&(s_M(r_\sharp)\otimes t_M(r'_\sharp))(t_M(f)\otimes 1_{\mathfrak{U}(w)}-1_{\mathfrak{U}(w)}\otimes s_M(f))\\
=&s_M(r_\sharp)t_M(f)\otimes t_M(r'_\sharp)-s_M(r_\sharp)\otimes t_M(r'_\sharp)s_M(f)\\
=&t_M(f)s_M(r_\sharp)\otimes t_M(r'_\sharp)-s_M(r_\sharp)\otimes s_M(f)t_M(r'_\sharp)\\
=&(t_M(f)\otimes 1_{\mathfrak{U}(w)}-1_{\mathfrak{U}(w)}\otimes s_M(f))(s_M(r_\sharp)\otimes t_M(r'_\sharp))\in\mathfrak{I}_2
\end{aligned}
$$

for any $f\in M$.

If $\alpha=\widetilde{\mathbf{e}}\begin{bmatrix}p\\ q\end{bmatrix}$ ($\forall m\in\mathbb{Z},\forall p,q\in Q^{(m)}$), then it follows from Lemma 6.4 and the generators (D) and (F) that

$$
\begin{aligned}
&\overline{\nabla}(\alpha)(t_M(f)\otimes 1_{\mathfrak{U}(w)}-1_{\mathfrak{U}(w)}\otimes s_M(f))\\
=&(\sum_{u\in Q^{(m)}}\widetilde{\mathbf{e}}\begin{bmatrix}u\\ q\end{bmatrix}+\mathfrak{I}_w\otimes\widetilde{\mathbf{e}}\begin{bmatrix}p\\ u\end{bmatrix}+\mathfrak{I}_w)\\
&\times(\sum_{\lambda,\mu\in\Lambda}(1_R\otimes f(\lambda))\widetilde{\mathbf{e}}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}+\mathfrak{I}_w\otimes 1_{\mathfrak{U}(w)}-\sum_{\tau,\gamma\in\Lambda}1_{\mathfrak{U}(w)}\otimes(f(\tau)\otimes 1_R)\widetilde{\mathbf{e}}\begin{bmatrix}\gamma\\ \tau\end{bmatrix}+\mathfrak{I}_w)\\
=&\sum_{u\in Q^{(m)}}(1_R\otimes f(\mathfrak{s}(u)))\widetilde{\mathbf{e}}\begin{bmatrix}u\\ q\end{bmatrix}+\mathfrak{I}_w\otimes\widetilde{\mathbf{e}}\begin{bmatrix}p\\ u\end{bmatrix}+\mathfrak{I}_w\\
&-\sum_{u\in Q^{(m)}}(\widetilde{\mathbf{e}}\begin{bmatrix}u\\ q\end{bmatrix}+\mathfrak{I}_w\otimes(f(\mathfrak{s}(u))\otimes 1_R)\widetilde{\mathbf{e}}\begin{bmatrix}p\\ u\end{bmatrix}+\mathfrak{I}_w
\end{aligned}
$$

$$= \sum_{u\in Q^{(m)}} (t_M(f(\mathfrak{s}(u))_\sharp)\otimes 1_{\mathfrak{U}(w)} - 1_{\mathfrak{U}(w)}\otimes s_M(f(\mathfrak{s}(u))_\sharp))(\widetilde{\mathbf{e}}\begin{bmatrix}u\\q\end{bmatrix} + \mathfrak{J}_w \otimes \widetilde{\mathbf{e}}\begin{bmatrix}p\\u\end{bmatrix} + \mathfrak{J}_w)$$

$\in \mathfrak{J}_2$.

The proof for $\alpha = \mathbf{e}\begin{bmatrix}p\\q\end{bmatrix}$ is similar. This is the desired conclusion. □

Proof of Lemma 6.4. By virtue of the generators (C) and (F), it follows that $1_{\mathfrak{U}(w)} = \sum_{\tau,\gamma\in\Lambda} \mathbf{e}\begin{bmatrix}\tau\\\gamma\end{bmatrix} + \mathfrak{J}_w = \sum_{\tau,\gamma\in\Lambda} \mathbf{e}\begin{bmatrix}\tau\\\gamma\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}\gamma\\\tau\end{bmatrix} + \mathfrak{J}_w$. For any $\lambda,\mu\in\Lambda$,

$$\begin{aligned}
\mathbf{e}\begin{bmatrix}\lambda\\\mu\end{bmatrix} + \mathfrak{J}_w &= \sum_{\tau,\gamma\in\Lambda} \mathbf{e}\begin{bmatrix}\lambda\\\mu\end{bmatrix}\mathbf{e}\begin{bmatrix}\tau\\\gamma\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}\gamma\\\tau\end{bmatrix} + \mathfrak{J}_w \\
&= \sum_{\tau,\gamma\in\Lambda} \delta_{\lambda,\tau}\delta_{\mu,\gamma}\mathbf{e}\begin{bmatrix}\lambda\\\mu\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}\gamma\\\tau\end{bmatrix} + \mathfrak{J}_w \\
&= \mathbf{e}\begin{bmatrix}\lambda\\\mu\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}\mu\\\lambda\end{bmatrix} + \mathfrak{J}_w \\
&= \sum_{\tau,\gamma\in\Lambda} \delta_{\lambda,\tau}\delta_{\mu,\gamma}\mathbf{e}\begin{bmatrix}\tau\\\gamma\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}\mu\\\lambda\end{bmatrix} + \mathfrak{J}_w \\
&= \sum_{\tau,\gamma\in\Lambda} \mathbf{e}\begin{bmatrix}\tau\\\gamma\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}\gamma\\\tau\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}\mu\\\lambda\end{bmatrix} + \mathfrak{J}_w \\
&= \widetilde{\mathbf{e}}\begin{bmatrix}\mu\\\lambda\end{bmatrix} + \mathfrak{J}_w.
\end{aligned}$$

This completes the proof. □

By using this proposition, we can induce that the k-module homomorphism $\tilde{\nabla}(\alpha+\mathfrak{J}_w) = \overline{\nabla}(\alpha) + \mathfrak{J}_2$ ($\alpha \in k\langle\Lambda Q\rangle$) is well defined.

Lemma 6.5. As $\mathbb{Z}$-modules,

$$\mathfrak{U}(w)\otimes_k \mathfrak{U}(w)/\mathfrak{J}_2 \cong \mathfrak{U}(w)\otimes_M \mathfrak{U}(w).$$

Proof. We can prove this isomorphism by using the two maps $J\colon \mathfrak{U}(w)\otimes_k \mathfrak{U}(w)/\mathfrak{J}_2 \ni \alpha\otimes\beta + \mathfrak{J}_2 \mapsto \alpha\otimes\beta \in \mathfrak{U}(w)\otimes_M \mathfrak{U}(w)$ and $K\colon \mathfrak{U}(w)\otimes_M \mathfrak{U}(w) \ni \alpha\otimes\beta \mapsto \alpha\otimes\beta + \mathfrak{J}_2 \in \mathfrak{U}(w)\otimes_k \mathfrak{U}(w)/\mathfrak{J}_2$. The proof is similar to that of Lemma 2.3. □

By virtue of the map $\tilde{\nabla}$ and the above lemma, we can induce the $\mathbb{Z}$-module homomorphism $\Delta_M\colon \mathfrak{U}(w) \to \mathfrak{U}(w)\otimes_M \mathfrak{U}(w)$.

Proposition 6.6. The map Δ_M is an (M,M)-bimodule homomorphism.

Proof. For any $\alpha\in\mathfrak{U}(w)$, we fix an element $\overline{\alpha}\in k\langle\Lambda Q\rangle$ such that $\alpha = \overline{\alpha} + \mathfrak{J}_w$.

We write $\overline{\nabla}(\overline{\alpha}) = \overline{\alpha}_{[1]} \otimes \overline{\alpha}_{[2]}$. For any $\alpha \in \mathfrak{U}(w)$ and $f, g \in M$,

$$\begin{aligned}
&\Delta_M(f \cdot \alpha \cdot g)\\
=&\sum_{\lambda,\mu,\tau,\nu\in\Lambda} \Delta_M(((f(\lambda)\otimes 1_R)\mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} + \mathfrak{I}_w)((1_R\otimes g(\tau))\mathbf{e}\begin{bmatrix}\nu\\ \tau\end{bmatrix} + \mathfrak{I}_w)\alpha)\\
=&\sum_{\lambda,\mu\in\Lambda} \Delta_M(((f(\lambda)\otimes g(\mu))\mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} + \mathfrak{I}_w)\alpha)\\
=&\sum_{\lambda,\mu,\tau\in\Lambda} J(((f(\lambda)\otimes 1_R)\mathbf{e}\begin{bmatrix}\lambda\\ \tau\end{bmatrix} + \mathfrak{I}_w)\overline{\alpha}_{[1]} \otimes ((1_R\otimes g(\mu))\mathbf{e}\begin{bmatrix}\tau\\ \mu\end{bmatrix} + \mathfrak{I}_w)\overline{\alpha}_{[2]} + \mathfrak{I}_2)\\
=&\sum_{\lambda,\mu,\tau\in\Lambda} ((f(\lambda)\otimes 1_R)\mathbf{e}\begin{bmatrix}\lambda\\ \tau\end{bmatrix} + \mathfrak{I}_w)\overline{\alpha}_{[1]} \otimes ((1_R\otimes g(\mu))\mathbf{e}\begin{bmatrix}\tau\\ \mu\end{bmatrix} + \mathfrak{I}_w)\overline{\alpha}_{[2]}\\
=&\sum_{\lambda,\mu,\tau,\nu\in\Lambda} (t_M(\delta_\tau)(f(\lambda)\otimes 1_R)\mathbf{e}\begin{bmatrix}\lambda\\ \nu\end{bmatrix} + \mathfrak{I}_w)\overline{\alpha}_{[1]} \otimes ((1_R\otimes g(\mu))\mathbf{e}\begin{bmatrix}\tau\\ \mu\end{bmatrix} + \mathfrak{I}_w)\overline{\alpha}_{[2]}\\
=&\sum_{\lambda,\mu,\tau,\nu\in\Lambda} ((f(\lambda)\otimes 1_R)\mathbf{e}\begin{bmatrix}\lambda\\ \nu\end{bmatrix} + \mathfrak{I}_w)\overline{\alpha}_{[1]} \otimes s_M(\delta_\tau)((1_R\otimes g(\mu))\mathbf{e}\begin{bmatrix}\tau\\ \mu\end{bmatrix} + \mathfrak{I}_w)\overline{\alpha}_{[2]}\\
=&s_M(f)\overline{\alpha}_{[1]} \otimes t_M(g)\overline{\alpha}_{[2]}\\
=&f\cdot\Delta_M(\alpha)\cdot g.
\end{aligned}$$

This completes the proof. □

The next task is to construct $\pi_M \colon \mathfrak{U}(w) \to M$. We define the k-algebra homomorphism $\tilde{\zeta} \colon k\langle \Lambda Q\rangle \to \mathrm{End}_k(M)$ as follows:

$$\tilde{\zeta}(\xi) = \psi(\xi) \quad (\xi \in R\otimes_k R^{op}); \tag{6.7}$$

$$\tilde{\zeta}(\mathbf{e}\begin{bmatrix}p\\ q\end{bmatrix})(f) = \delta_{p,q} f(\mathfrak{t}(q))_\sharp \delta_{\mathfrak{s}(q)}; \tag{6.8}$$

$$\tilde{\zeta}(\widetilde{\mathbf{e}}\begin{bmatrix}p\\ q\end{bmatrix})(f) = \delta_{p,q} f(\mathfrak{s}(q))_\sharp \delta_{\mathfrak{t}(q)} \quad (f\in M, m\in\mathbb{Z}_{\geq 0}, p,q\in Q^{(m)}). \tag{6.9}$$

Here the map $\psi \colon R\otimes_k R^{op} \to \mathrm{End}_k(M)$ is a k-algebra homomorphism defined by $\psi(r\otimes r')(f) = r_\sharp f r'_\sharp$ $(r, r' \in R)$.

Proposition 6.7. $\tilde{\zeta}(\mathfrak{I}_w) = \{0\}$.

Proof. Let f be an arbitrary element in M. Since the map $\tilde{\zeta}$ is a k-algebra homomorphism, it suffices to show that $\tilde{\zeta}(\alpha) = 0$ for any generator α of $\mathfrak{I}_w$.

It is easy to prove $\tilde{\zeta}(\alpha) = 0$ for an arbitrary generator α in (A).

We suppose that α is a generator (B). For any $m \in \mathbb{Z}_{\geq 0}$, $p, q \in Q^{(m)}$,

$$\begin{aligned}
\tilde{\zeta}(\widetilde{\mathbf{e}}\begin{bmatrix}p\\ q\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p'\\ q'\end{bmatrix})(f) =&\delta_{p',q'}\tilde{\zeta}(\widetilde{\mathbf{e}}\begin{bmatrix}p\\ q\end{bmatrix})(f(\mathfrak{s}(q'))_\sharp\delta_{\mathfrak{t}(q')})\\
=&\delta_{\mathfrak{t}(q'),\mathfrak{s}(q)}\delta_{p,q}\delta_{p',q'} f(\mathfrak{s}(q'))_\sharp \delta_{\mathfrak{t}(q)};\\
\tilde{\zeta}(\delta_{\mathfrak{t}(p'),\mathfrak{s}(p)}\delta_{\mathfrak{t}(q'),\mathfrak{s}(q)}\widetilde{\mathbf{e}}\begin{bmatrix}p'p\\ q'q\end{bmatrix})(f) =&\delta_{\mathfrak{t}(p'),\mathfrak{s}(p)}\delta_{\mathfrak{t}(q'),\mathfrak{s}(q)}\delta_{p,q}\delta_{p',q'} f(\mathfrak{s}(q'))_\sharp\delta_{\mathfrak{t}(q)}.
\end{aligned}$$

If $p = q$ and $p' = q'$, then we can deduce that $\mathfrak{s}(p) = \mathfrak{s}(q)$ and $\mathfrak{t}(p') = \mathfrak{t}(q')$. It is easy to show that $\delta_{p,q}\delta_{p',q'} = 0$ if $p \neq q$ or $p' = q'$. Thus $\tilde{\zeta}(\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p'\\q'\end{bmatrix} - \delta_{\mathfrak{t}(p'),\mathfrak{s}(p)}\delta_{\mathfrak{t}(q'),\mathfrak{s}(q)}\widetilde{\mathbf{e}}\begin{bmatrix}p'p\\q'q\end{bmatrix}) = 0$. The proof for the other generator is similar.

For the generators in (C), we can calculate that

$$\begin{aligned}
&\tilde{\zeta}(\sum_{u\in Q^{(m)}} \mathbf{e}\begin{bmatrix}p\\u\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}u\\q\end{bmatrix} - \delta_{p,q}\sum_{\lambda\in\Lambda}\mathbf{e}\begin{bmatrix}\mathfrak{s}(p)\\\lambda\end{bmatrix})(f)\\
=&\sum_{u\in Q^{(m)}} \delta_{u,q}\tilde{\zeta}(\mathbf{e}\begin{bmatrix}p\\u\end{bmatrix})(f(\mathfrak{s}(q))_\sharp\delta_{\mathfrak{t}(q)}) - \delta_{p,q}\sum_{\lambda\in\Lambda}\delta_{\mathfrak{s}(p),\lambda}f(\lambda)_\sharp\delta_\lambda\\
=&\sum_{u\in Q^{(m)}} \delta_{\mathfrak{t}(u),\mathfrak{t}(q)}\delta_{p,u}\delta_{u,q}f(\mathfrak{s}(q))_\sharp\delta_{\mathfrak{s}(u)} - \delta_{p,q}f(\mathfrak{s}(p))_\sharp\delta_{\mathfrak{s}(p)}\\
=&\delta_{p,q}(f(\mathfrak{s}(q))_\sharp\delta_{\mathfrak{s}(q)} - f(\mathfrak{s}(p))_\sharp\delta_{\mathfrak{s}(p)})
\end{aligned}$$

for any $m \in \{0, 1\}$, $p, q \in Q^{(m)}$. If $p = q$, then $\mathfrak{s}(p) = \mathfrak{s}(q)$. Thus we can deduce that $f(\mathfrak{s}(p)) = f(\mathfrak{s}(q))$ and $\delta_{\mathfrak{s}(p)} = \delta_{\mathfrak{s}(q)}$. Similarly, we can show that $\tilde{\zeta}(\sum_{u\in Q^{(m)}} \widetilde{\mathbf{e}}\begin{bmatrix}p\\u\end{bmatrix}\mathbf{e}\begin{bmatrix}u\\q\end{bmatrix} - \delta_{p,q}\sum_{\lambda\in\Lambda}\mathbf{e}\begin{bmatrix}\lambda\\\mathfrak{t}(p)\end{bmatrix}) = 0$.

Let α be the first generator of (D). For all $r, r' \in R$, $m \in \mathbb{Z}_{\geq 0}$, $p, q \in Q^{(m)}$,

$$\begin{aligned}
\tilde{\zeta}(\alpha)(f) =&\tilde{\zeta}((r\otimes r')\mathbf{e}\begin{bmatrix}p\\q\end{bmatrix} - \mathbf{e}\begin{bmatrix}p\\q\end{bmatrix}(r\otimes r'))(f)\\
=&\delta_{p,q}r_\sharp f(\mathfrak{t}(q))_\sharp\delta_{\mathfrak{s}(q)}r'_\sharp - \delta_{p,q}(r_\sharp f r'_\sharp)(\mathfrak{t}(q))_\sharp\delta_{\mathfrak{s}(q)}\\
=&\delta_{p,q}((rf(\mathfrak{t}(q))r')_\sharp\delta_{\mathfrak{s}(q)} - (rf(\mathfrak{t}(q))r')_\sharp\delta_{\mathfrak{s}(q)})\\
=&0.
\end{aligned}$$

The proof for the second generator of (D) is similar.

We assume that α is an arbitrary generator in (E). Since $\mathbf{w}\left[{}_{c}\,{}^{a}_{d}\,{}_{b}\right]$ is a central element in R for any $(a, b), (c, d) \in Q^{(2)}$,

$$\begin{aligned}
\tilde{\zeta}(\alpha)(f) =&\sum_{(x,y)\in Q^{(2)}}\tilde{\zeta}((\mathbf{w}\left[{}_{a}\,{}^{x}_{b}\,{}_{y}\right]\otimes 1_R)\mathbf{e}\begin{bmatrix}xy\\cd\end{bmatrix})(f)\\
&-\sum_{(x,y)\in Q^{(2)}}\tilde{\zeta}((1_R\otimes\mathbf{w}\left[{}_{x}\,{}^{c}_{y}\,{}_{d}\right])\mathbf{e}\begin{bmatrix}ab\\xy\end{bmatrix})(f)\\
=&\sum_{(x,y)\in Q^{(2)}}\delta_{x,c}\delta_{y,d}(\mathbf{w}\left[{}_{a}\,{}^{x}_{b}\,{}_{y}\right]f(\mathfrak{t}(d)))_\sharp\delta_{\mathfrak{s}(c)}\\
&-\sum_{(x,y)\in Q^{(2)}}\delta_{a,x}\delta_{b,y}(f(\mathfrak{t}(y))\mathbf{w}\left[{}_{x}\,{}^{c}_{y}\,{}_{d}\right])_\sharp\delta_{\mathfrak{s}(a)}\\
=&(\mathbf{w}\left[{}_{a}\,{}^{c}_{b}\,{}_{d}\right]f(\mathfrak{t}(d)))_\sharp\delta_{\mathfrak{s}(c)} - (\mathbf{w}\left[{}_{a}\,{}^{c}_{b}\,{}_{d}\right]f(\mathfrak{t}(b)))_\sharp\delta_{\mathfrak{s}(a)}.
\end{aligned}$$

Here we use the calculation of the generators (B) for the first equality. If $\mathbf{w}\left[{}_{a}\,{}^{c}_{b}\,{}_{d}\right] \neq 0$, then $\mathfrak{s}(a) = \mathfrak{s}(c)$ and $\mathfrak{t}(b) = \mathfrak{s}(d)$ are satisfied. Thus the proof for the generators (E) is completed.

For the generators (F), it follows that

$$\begin{aligned}\tilde{\zeta}(\emptyset - \sum_{\lambda,\mu\in\Lambda} \mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix})(f) =& f - \sum_{\lambda,\mu\in\Lambda} \delta_{\lambda,\mu} f(\mu)_\sharp \delta_\mu \\ =& f - \sum_{\lambda\in\Lambda} f(\lambda)_\sharp \delta_\lambda \\ =& f - f \\ =& 0.\end{aligned}$$

The proof for $\emptyset - 1_R \otimes 1_R$ is straightforward. This is the desired conclusion. □

Therefore the k-algebra homomorphism $\zeta\colon \mathfrak{U}(w) \ni \alpha + \mathfrak{J}_w \mapsto \tilde{\zeta}(\alpha) \in \mathrm{End}_k(M)$ makes sense. The map π_M is defined by $\pi_M(a) = \zeta(a)(1_M)$ $(a \in \mathfrak{U}(w))$.

Proposition 6.8. The map π_M is an (M, M)-bimodule homomorphism.

Proof. For any $\alpha \in \mathfrak{U}(w)$ and $f, g \in M$,

$$\begin{aligned}\pi_M(f\cdot\alpha\cdot g) =& \sum_{\lambda,\mu\in\Lambda} \pi_M(((f(\lambda)\otimes g(\mu))\mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} + \mathfrak{J}_w)\alpha) \\ =& \sum_{\lambda,\mu\in\Lambda} \zeta((f(\lambda)\otimes g(\mu))\mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} + \mathfrak{J}_w)(\pi_M(\alpha)) \\ =& \sum_{\lambda,\mu\in\Lambda} \delta_{\lambda,\mu} f(\lambda)_\sharp \pi_M(\alpha)(\mu)_\sharp \delta_\mu g(\mu)_\sharp \\ =& \sum_{\lambda\in\Lambda} f(\lambda)_\sharp \pi_M(\alpha)(\lambda)_\sharp \delta_\lambda g(\lambda)_\sharp \\ =& \sum_{\lambda\in\Lambda} ((f\pi_M(\alpha)g)(\lambda))_\sharp \delta_\lambda \\ =& f\pi_M(\alpha)g.\end{aligned}$$

This is the desired conclusion. □

Proposition 6.9. The triple $(\mathfrak{U}(w), \Delta_M, \pi_M)$ is a comonoid in the tensor category of (M, M)-bimodules.

Proof. The proof for the coassociativity of Δ_M is similar to that for A_σ.

Let us show that Δ_M and π_M satisfy the counitality. It is sufficient to prove that $((\pi_M \otimes \mathrm{id}_{\mathfrak{U}(w)}) \circ \Delta_M)(\alpha) = \alpha = ((\mathrm{id}_{\mathfrak{U}(w)} \otimes \pi_M) \circ \Delta_M)(\alpha)$. Here α stands for the element in $\mathfrak{U}(w)$ written by

$$\alpha = \begin{cases} (r\otimes r')\mathbf{e}\begin{bmatrix}p_1\\ q_1\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_2\\ q_2\end{bmatrix}\cdots\mathbf{e}\begin{bmatrix}p_{2n-1}\\ q_{2n-1}\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_{2n}\\ q_{2n}\end{bmatrix} + \mathfrak{J}_w; & (6.10) \\ (r\otimes r')\mathbf{e}\begin{bmatrix}p_1\\ q_1\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_2\\ q_2\end{bmatrix}\cdots\mathbf{e}\begin{bmatrix}p_{2n-1}\\ q_{2n-1}\end{bmatrix} + \mathfrak{J}_w; & (6.11) \\ (r\otimes r')\widetilde{\mathbf{e}}\begin{bmatrix}p_1\\ q_1\end{bmatrix}\mathbf{e}\begin{bmatrix}p_2\\ q_2\end{bmatrix}\cdots\widetilde{\mathbf{e}}\begin{bmatrix}p_{2n-1}\\ q_{2n-1}\end{bmatrix}\mathbf{e}\begin{bmatrix}p_{2n}\\ q_{2n}\end{bmatrix} + \mathfrak{J}_w; & (6.12) \\ (r\otimes r')\widetilde{\mathbf{e}}\begin{bmatrix}p_1\\ q_1\end{bmatrix}\mathbf{e}\begin{bmatrix}p_2\\ q_2\end{bmatrix}\cdots\widetilde{\mathbf{e}}\begin{bmatrix}p_{2n-1}\\ q_{2n-1}\end{bmatrix} + \mathfrak{J}_w & (6.13) \end{cases}$$

for any $r, r' \in R$, $i \in \{1, 2, \cdots, 2n\}$, $m_i \in \mathbb{Z}_{\geq 0}$, $p_i, q_i \in Q^{(m_i)}$. We give the proof only for (6.10). Since $\overline{\nabla}$ is a k-algebra homomorphism, it follows that

$$\begin{aligned}
&\overline{\nabla}((r \otimes r')\mathbf{e}\begin{bmatrix}p_1\\q_1\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_2\\q_2\end{bmatrix}\cdots\mathbf{e}\begin{bmatrix}p_{2n-1}\\q_{2n-1}\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_{2n}\\q_{2n}\end{bmatrix})\\
=&\overline{\nabla}((r \otimes r'))\overline{\nabla}(\mathbf{e}\begin{bmatrix}p_1\\q_1\end{bmatrix})\overline{\nabla}(\widetilde{\mathbf{e}}\begin{bmatrix}p_2\\q_2\end{bmatrix})\cdots\overline{\nabla}(\mathbf{e}\begin{bmatrix}p_{2n-1}\\q_{2n-1}\end{bmatrix})\overline{\nabla}(\widetilde{\mathbf{e}}\begin{bmatrix}p_{2n}\\q_{2n}\end{bmatrix})\\
=&\sum_{u_1,\cdots,u_{2n}}(r \otimes 1_R)\mathbf{e}\begin{bmatrix}p_1\\u_1\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}u_2\\q_2\end{bmatrix}\cdots\mathbf{e}\begin{bmatrix}p_{2n-1}\\u_{2n-1}\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}u_{2n}\\q_{2n}\end{bmatrix}+\mathfrak{I}_w\\
&\otimes(1_R \otimes r')\mathbf{e}\begin{bmatrix}u_1\\q_1\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_2\\u_2\end{bmatrix}\cdots\mathbf{e}\begin{bmatrix}u_{2n-1}\\q_{2n-1}\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_{2n}\\u_{2n}\end{bmatrix}+\mathfrak{I}_w.
\end{aligned}$$

By the definition of $\tilde{\zeta}$, we can induce that

$$\tilde{\zeta}(\mathbf{e}\begin{bmatrix}p\\q\end{bmatrix})(\delta_\lambda) = \delta_{p,q}\delta_{\lambda,\mathfrak{t}(q)}\delta_{\mathfrak{s}(q)}; \tag{6.14}$$

$$\tilde{\zeta}(\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix})(\delta_\lambda) = \delta_{p,q}\delta_{\lambda,\mathfrak{s}(q)}\delta_{\mathfrak{t}(q)} \tag{6.15}$$

for any $m \in \mathbb{Z}_{\geq 0}$, $p, q \in Q^{(m)}$, $\lambda \in \Lambda$. By using (6.14) and (6.15) repeatedly, we can calculate that

$$\begin{aligned}
&\pi_M(\alpha_{[1]})\cdot\alpha_{[2]}\\
=&\sum_{u_1,\cdots,u_{2n}}(\tilde{\zeta}(r\otimes 1_R)\tilde{\zeta}(\mathbf{e}\begin{bmatrix}p_1\\u_1\end{bmatrix})\tilde{\zeta}(\widetilde{\mathbf{e}}\begin{bmatrix}u_2\\q_2\end{bmatrix})\cdots\tilde{\zeta}(\mathbf{e}\begin{bmatrix}p_{2n-1}\\u_{2n-1}\end{bmatrix})\tilde{\zeta}(\widetilde{\mathbf{e}}\begin{bmatrix}u_{2n}\\q_{2n}\end{bmatrix}))(1_M)\\
&\cdot(1_R\otimes r')\mathbf{e}\begin{bmatrix}u_1\\q_1\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_2\\u_2\end{bmatrix}\cdots\mathbf{e}\begin{bmatrix}u_{2n-1}\\q_{2n-1}\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_{2n}\\u_{2n}\end{bmatrix}+\mathfrak{I}_w\\
=&\sum_{u_1,\cdots,u_{2n-1}}(\tilde{\zeta}(r\otimes 1_R)\tilde{\zeta}(\mathbf{e}\begin{bmatrix}p_1\\u_1\end{bmatrix})\tilde{\zeta}(\widetilde{\mathbf{e}}\begin{bmatrix}u_2\\q_2\end{bmatrix})\cdots\tilde{\zeta}(\mathbf{e}\begin{bmatrix}p_{2n-1}\\u_{2n-1}\end{bmatrix}))(\delta_{\mathfrak{t}(q_{2n})})\\
&\cdot(1_R\otimes r')\mathbf{e}\begin{bmatrix}u_1\\q_1\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_2\\u_2\end{bmatrix}\cdots\mathbf{e}\begin{bmatrix}u_{2n-1}\\q_{2n-1}\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_{2n}\\q_{2n}\end{bmatrix}+\mathfrak{I}_w\\
=&\sum_{u_1,\cdots,u_{2n-2}}\delta_{\mathfrak{t}(q_{2n}),\mathfrak{t}(p_{2n-1})}(\tilde{\zeta}(r\otimes 1_R)\tilde{\zeta}(\mathbf{e}\begin{bmatrix}p_1\\u_1\end{bmatrix})\tilde{\zeta}(\widetilde{\mathbf{e}}\begin{bmatrix}u_2\\q_2\end{bmatrix})\cdots\tilde{\zeta}(\widetilde{\mathbf{e}}\begin{bmatrix}u_{2n-2}\\q_{2n-2}\end{bmatrix}))(\delta_{\mathfrak{s}(q_{2n-1})})\\
&\cdot(1_R\otimes r')\mathbf{e}\begin{bmatrix}u_1\\q_1\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_2\\u_2\end{bmatrix}\cdots\mathbf{e}\begin{bmatrix}p_{2n-1}\\q_{2n-1}\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_{2n}\\q_{2n}\end{bmatrix}+\mathfrak{I}_w\\
=&\sum_{u_1,\cdots,u_{2n-3}}\delta_{\mathfrak{t}(q_{2n}),\mathfrak{t}(p_{2n-1})}\delta_{\mathfrak{s}(p_{2n-1}),\mathfrak{s}(q_{2n-2})}\\
&\times(\tilde{\zeta}(r\otimes 1_R)\tilde{\zeta}(\mathbf{e}\begin{bmatrix}p_1\\u_1\end{bmatrix})\tilde{\zeta}(\widetilde{\mathbf{e}}\begin{bmatrix}u_2\\q_2\end{bmatrix})\cdots\tilde{\zeta}(\mathbf{e}\begin{bmatrix}p_{2n-3}\\u_{2n-3}\end{bmatrix}))(\delta_{\mathfrak{t}(q_{2n-2})})\\
&\cdot(1_R\otimes r')\mathbf{e}\begin{bmatrix}u_1\\q_1\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_2\\u_2\end{bmatrix}\cdots\widetilde{\mathbf{e}}\begin{bmatrix}p_{2n-2}\\q_{2n-2}\end{bmatrix}\mathbf{e}\begin{bmatrix}p_{2n-1}\\q_{2n-1}\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_{2n}\\q_{2n}\end{bmatrix}+\mathfrak{I}_w\\
=&\delta_{\mathfrak{t}(q_{2n}),\mathfrak{t}(p_{2n-1})}\delta_{\mathfrak{s}(p_{2n-1}),\mathfrak{s}(q_{2n-2})}\cdots\delta_{\mathfrak{s}(p_3),\mathfrak{s}(q_2)}\delta_{\mathfrak{t}(q_2),\mathfrak{t}(p_1)}s_M(r_\sharp\delta_{\mathfrak{s}(p_1)})\\
&\times(1_R\otimes r')\mathbf{e}\begin{bmatrix}p_1\\q_1\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_2\\q_2\end{bmatrix}\cdots\mathbf{e}\begin{bmatrix}p_{2n-1}\\q_{2n-1}\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_{2n}\\q_{2n}\end{bmatrix}+\mathfrak{I}_w
\end{aligned}$$

$$=\delta_{\mathfrak{t}(q_{2n}),\mathfrak{t}(p_{2n-1})}\delta_{\mathfrak{s}(p_{2n-1}),\mathfrak{s}(q_{2n-2})}\cdots\delta_{\mathfrak{s}(p_3),\mathfrak{s}(q_2)}\delta_{\mathfrak{t}(q_2),\mathfrak{t}(p_1)}$$
$$\times (r\otimes r')\mathbf{e}\begin{bmatrix}p_1\\q_1\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_2\\q_2\end{bmatrix}\cdots\mathbf{e}\begin{bmatrix}p_{2n-1}\\q_{2n-1}\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_{2n}\\q_{2n}\end{bmatrix}+\mathfrak{J}_w.$$

For any $m,n\in\mathbb{Z}_{\geq 0}$, $p,q\in Q^m$, $p',q'\in Q^{(n)}$, Lemma 6.4 induces that

$$\begin{aligned}
\mathbf{e}\begin{bmatrix}p\\q\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p'\\q'\end{bmatrix}+\mathfrak{J}_w =&\mathbf{e}\begin{bmatrix}p\\q\end{bmatrix}(\sum_{\lambda\in\Lambda}\mathbf{e}\begin{bmatrix}\mathfrak{t}(p)\\\lambda\end{bmatrix})(\sum_{\mu\in\Lambda}\widetilde{\mathbf{e}}\begin{bmatrix}\mu\\\mathfrak{t}(q')\end{bmatrix})\widetilde{\mathbf{e}}\begin{bmatrix}p'\\q'\end{bmatrix}+\mathfrak{J}_w\\
=&\mathbf{e}\begin{bmatrix}p\\q\end{bmatrix}(\sum_{\lambda\in\Lambda}\mathbf{e}\begin{bmatrix}\mathfrak{t}(p)\\\lambda\end{bmatrix})(\sum_{\mu\in\Lambda}\mathbf{e}\begin{bmatrix}\mathfrak{t}(q')\\\mu\end{bmatrix})\widetilde{\mathbf{e}}\begin{bmatrix}p'\\q'\end{bmatrix}+\mathfrak{J}_w\\
=&\delta_{\mathfrak{t}(p),\mathfrak{t}(q')}\mathbf{e}\begin{bmatrix}p\\q\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p'\\q'\end{bmatrix}+\mathfrak{J}_w;\\
\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix}\mathbf{e}\begin{bmatrix}p'\\q'\end{bmatrix}+\mathfrak{J}_w =&\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix}(\sum_{\lambda\in\Lambda}\widetilde{\mathbf{e}}\begin{bmatrix}\lambda\\\mathfrak{s}(q)\end{bmatrix})(\sum_{\mu\in\Lambda}\mathbf{e}\begin{bmatrix}\mathfrak{s}(p')\\\mu\end{bmatrix})\mathbf{e}\begin{bmatrix}p'\\q'\end{bmatrix}+\mathfrak{J}_w\\
=&\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix}(\sum_{\lambda\in\Lambda}\mathbf{e}\begin{bmatrix}\mathfrak{s}(q)\\\lambda\end{bmatrix})(\sum_{\mu\in\Lambda}\mathbf{e}\begin{bmatrix}\mathfrak{s}(p')\\\mu\end{bmatrix})\mathbf{e}\begin{bmatrix}p'\\q'\end{bmatrix}+\mathfrak{J}_w\\
=&\delta_{\mathfrak{s}(q),\mathfrak{s}(p')}\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix}\mathbf{e}\begin{bmatrix}p'\\q'\end{bmatrix}+\mathfrak{J}_w.
\end{aligned}$$

Thus we can conclude that $\alpha=\pi_M(\alpha_{[1]})\cdot\alpha_{[2]}$. The same proof works for $\alpha=\alpha_{[1]}\cdot\pi_M(\alpha_{[2]})$. This completes the proof. □

Proposition 6.10. The maps Δ_M and π_M satisfy the conditions (1.3)-(1.7).

Proof. We first show (1.3). It is equivalent to show that

$$(\overline{\alpha}_{[1]}\otimes\overline{\alpha}_{[2]})(t_M(f)\otimes 1_{\mathfrak{U}(w)}-1_{\mathfrak{U}(w)}\otimes s_M(f))\in\mathfrak{J}_2 \tag{6.16}$$

for any $\alpha\in\mathfrak{U}(w)$. Here we fix $\overline{\alpha}\in k\langle\Lambda Q\rangle$ satisfying $\alpha=\overline{\alpha}+\mathfrak{J}_w$ and write $\overline{\nabla}(\overline{\alpha})=\overline{\alpha}_{[1]}\otimes\overline{\alpha}_{[2]}$. Because $\overline{\nabla}$ is a k-algebra homomorphism, it is sufficient to show (6.16) for any $\alpha\in\Lambda Q$. Thus the proof is similar to that of $\overline{\nabla}(k\langle\Lambda Q\rangle)\mathfrak{J}_2\subset\mathfrak{J}_2$.

Since $\overline{\nabla}$ and ζ are k-algebra homomorphisms, it is easy to prove (1.4) and (1.6).

The proof for (1.5) is similar to that for A_σ.

Let us check (1.7). For any $\alpha,\beta\in\mathfrak{U}(w)$, we fix elements $\overline{\alpha}$ and $\overline{\beta}$ in $k\langle\Lambda Q\rangle$ such that $\alpha=\overline{\alpha}+\mathfrak{J}_w$ and $\beta=\overline{\beta}+\mathfrak{J}_w$.

$$\begin{aligned}
\pi_M(\alpha s_M(\pi_M(\beta)))=&\sum_{\lambda,\mu\in\Lambda}\pi_M(\overline{\alpha}(\pi_M(\beta)(\lambda)\otimes 1_R)\mathbf{e}\begin{bmatrix}\lambda\\\mu\end{bmatrix}+\mathfrak{J}_w)\\
=&\sum_{\lambda,\mu\in\Lambda}(\tilde{\zeta}(\overline{\alpha})\tilde{\zeta}(\pi_M(\beta)(\lambda)\otimes 1_R)\tilde{\zeta}(\mathbf{e}\begin{bmatrix}\lambda\\\mu\end{bmatrix}))(1_M)\\
=&\sum_{\lambda,\mu\in\Lambda}\delta_{\lambda,\mu}\tilde{\zeta}(\overline{\alpha})(\pi_M(\beta)(\lambda)_\sharp\delta_\mu)\\
=&\sum_{\lambda\in\Lambda}\tilde{\zeta}(\overline{\alpha})(\pi_M(\beta)(\lambda)_\sharp\delta_\lambda)
\end{aligned}$$

$$
\begin{aligned}
&=\tilde{\zeta}(\overline{\alpha})(\pi_M(\beta)) \\
&=\pi_M(\alpha\beta).
\end{aligned}
$$

The proof for $\pi(\alpha t_M(\pi_M(\beta))) = \pi(\alpha\beta)$ is similar. This is the desired conclusion. $\square$

It follows from Propositions 6.9 and 6.10 that the sextuplet $\mathfrak{U}(w) = (\mathfrak{U}(w), M, s_M, t_M, \Delta_M, \pi_M)$ is a left bialgebroid.

7 Right bialgebroid $\mathfrak{U}(w)$

In this section, we show that the left bialgebroid $\mathfrak{U}(w)$ has a right bialgebroid structure under the condition (6.1) in Theorem 6.2.

Theorem 7.1. If the condition (6.1) is satisfied, then $\mathfrak{U}(w)$ is a right bialgebroid.

Let M denote the k-algebra consisting of all maps from Λ to R. We define two maps $s_{M^{op}}$ and $t_{M^{op}}$ by $s_{M^{op}} = t_M$ and $t_{M^{op}} = s_M$ (for s_M and t_M, see (6.2) and (6.3)). Because these $s_{M^{op}}$ and $t_{M^{op}}$ satisfy (1.12), the k-algebra $\mathfrak{U}(w)$ has an (M^{op}, M^{op})-bimodule structure via (1.13).

We next construct the map $\Delta_{M^{op}}$. The symbol $\mathfrak{I}'_2$ means the left ideal of $\mathfrak{U}(w)\otimes_k\mathfrak{U}(w)$ whose generators are $s_{M^{op}}(f)\otimes 1_{\mathfrak{U}(w)} - 1_{\mathfrak{U}(w)}\otimes t_{M^{op}}(f)$ $(\forall f \in M^{op})$. The k-algebra homomorphism $\overline{\nabla}$ defined by (6.4)-(6.6) satisfies the following proposition.

Proposition 7.2. $\overline{\nabla}(\mathfrak{J}_w) \subset \mathfrak{I}'_2$

Proof. We first prove that $\overline{\nabla}(\alpha) \in \mathfrak{I}'_2$ for any generator α of $\mathfrak{J}_w$.

The proof for the generators (A), (B), (C), (D), and (F) is similar to that for Proposition 6.3.

For the generators (E), we can use the calculation of Proposition 6.3 and it follows that

$$
\begin{aligned}
&\overline{\nabla}\Big(\sum_{(x,y)\in Q^{(2)}} (\mathbf{w}\left[\begin{smallmatrix} & x & \\ a & & y \\ & b & \end{smallmatrix}\right] \otimes 1_R)\mathbf{e}\begin{bmatrix} x \\ c \end{bmatrix}\mathbf{e}\begin{bmatrix} y \\ d \end{bmatrix} - \sum_{(x,y)\in Q^{(2)}} (1_R \otimes \mathbf{w}\left[\begin{smallmatrix} & c & \\ x & & d \\ & y & \end{smallmatrix}\right])\mathbf{e}\begin{bmatrix} a \\ x \end{bmatrix}\mathbf{e}\begin{bmatrix} b \\ y \end{bmatrix}\Big) \\
=& \sum_{(x,y),(u,v)\in Q^{(2)}} (1_R \otimes \mathbf{w}\left[\begin{smallmatrix} & u & \\ x & & v \\ & y & \end{smallmatrix}\right])\mathbf{e}\begin{bmatrix} a \\ x \end{bmatrix}\mathbf{e}\begin{bmatrix} b \\ y \end{bmatrix} + \mathfrak{J}_w \otimes \mathbf{e}\begin{bmatrix} u \\ c \end{bmatrix}\mathbf{e}\begin{bmatrix} v \\ d \end{bmatrix} + \mathfrak{J}_w \\
&- \sum_{(x,y),(u,v)\in Q^{(2)}} \mathbf{e}\begin{bmatrix} a \\ u \end{bmatrix}\mathbf{e}\begin{bmatrix} b \\ v \end{bmatrix} + \mathfrak{J}_w \otimes (\mathbf{w}\left[\begin{smallmatrix} & x & \\ u & & y \\ & v & \end{smallmatrix}\right] \otimes 1_R)\mathbf{e}\begin{bmatrix} x \\ c \end{bmatrix}\mathbf{e}\begin{bmatrix} y \\ d \end{bmatrix} + \mathfrak{J}_w \\
=& \sum_{(x,y),(u,v)\in Q^{(2)}} \mathbf{e}\begin{bmatrix} a \\ x \end{bmatrix}\mathbf{e}\begin{bmatrix} b \\ y \end{bmatrix}(1_R \otimes \mathbf{w}\left[\begin{smallmatrix} & u & \\ x & & v \\ & y & \end{smallmatrix}\right]) + \mathfrak{J}_w \otimes \mathbf{e}\begin{bmatrix} u \\ c \end{bmatrix}\mathbf{e}\begin{bmatrix} v \\ d \end{bmatrix} + \mathfrak{J}_w \\
&- \sum_{(x,y),(u,v)\in Q^{(2)}} \mathbf{e}\begin{bmatrix} a \\ u \end{bmatrix}\mathbf{e}\begin{bmatrix} b \\ v \end{bmatrix} + \mathfrak{J}_w \otimes \mathbf{e}\begin{bmatrix} x \\ c \end{bmatrix}\mathbf{e}\begin{bmatrix} y \\ d \end{bmatrix}(\mathbf{w}\left[\begin{smallmatrix} & x & \\ u & & y \\ & v & \end{smallmatrix}\right] \otimes 1_R) + \mathfrak{J}_w \\
=& \sum_{(x,y),(u,v)\in Q^{(2)}} (\mathbf{e}\begin{bmatrix} a \\ x \end{bmatrix}\mathbf{e}\begin{bmatrix} b \\ y \end{bmatrix} + \mathfrak{J}_w)s_{M^{op}}(\mathbf{w}\left[\begin{smallmatrix} & u & \\ x & & v \\ & y & \end{smallmatrix}\right]_\sharp) \otimes \mathbf{e}\begin{bmatrix} u \\ c \end{bmatrix}\mathbf{e}\begin{bmatrix} v \\ d \end{bmatrix} + \mathfrak{J}_w
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{(x,y),(u,v)\in Q^{(2)}} \mathbf{e}\begin{bmatrix} a \\ x \end{bmatrix}\mathbf{e}\begin{bmatrix} b \\ y \end{bmatrix} + \mathfrak{I}_w \otimes (\mathbf{e}\begin{bmatrix} u \\ c \end{bmatrix}\mathbf{e}\begin{bmatrix} v \\ d \end{bmatrix} + \mathfrak{I}_w) t_{M^{op}}(\mathbf{w}\begin{bmatrix} & u & \\ x & & v \\ & y & \end{bmatrix}_\sharp) \\
=& \sum_{(x,y),(u,v)\in Q^{(2)}} (\mathbf{e}\begin{bmatrix} a \\ x \end{bmatrix}\mathbf{e}\begin{bmatrix} b \\ y \end{bmatrix} + \mathfrak{I}_w \otimes \mathbf{e}\begin{bmatrix} u \\ c \end{bmatrix}\mathbf{e}\begin{bmatrix} v \\ d \end{bmatrix} + \mathfrak{I}_w) \\
\times& (s_{M^{op}}(\mathbf{w}\begin{bmatrix} & u & \\ x & & v \\ & y & \end{bmatrix}_\sharp) \otimes 1_{\mathfrak{U}(w)} - 1_{\mathfrak{U}(w)} \otimes t_{M^{op}}(\mathbf{w}\begin{bmatrix} & u & \\ x & & v \\ & y & \end{bmatrix}_\sharp)) \in \mathfrak{I}'_2
\end{aligned}
$$

for any $(a,b),(c,d) \in Q^{(2)}$. Therefore we conclude that $\overline{\nabla}(\alpha) \in \mathfrak{I}'_2$ for any generator of $\mathfrak{I}_w$.

In order to complete the proof, we need to prove that $\mathfrak{I}'_2\overline{\nabla}(k\langle \Lambda Q\rangle) \subset \mathfrak{I}'_2$. It suffices to check that $(s_{M^{op}}(f) \otimes 1_{\mathfrak{U}(w)} - 1_{\mathfrak{U}(w)} \otimes t_{M^{op}}(f))\overline{\nabla}(\alpha) \in \mathfrak{I}'_2$ for any $\alpha \in \Lambda Q$ and $f \in M$.

The proof for $\alpha = r \otimes r'$ $(\forall r, r' \in R)$ is similar to that of Proposition 6.3.

If $\alpha = \widetilde{\mathbf{e}}\begin{bmatrix} p \\ q \end{bmatrix}$ $(\forall m \in \mathbb{Z}_{\geq 0}, \forall p, q \in Q^{(m)})$, the generators (D) and Lemma 6.4 induce that

$$
\begin{aligned}
&(s_{M^{op}}(f) \otimes 1_{\mathfrak{U}(w)} - 1_{\mathfrak{U}(w)} \otimes t_{M^{op}}(f))\overline{\nabla}(\alpha) \\
=&(\sum_{\lambda,\mu\in\Lambda} (1_R \otimes f(\lambda))\widetilde{\mathbf{e}}\begin{bmatrix} \lambda \\ \mu \end{bmatrix} + \mathfrak{I}_w \otimes 1_{\mathfrak{U}(w)} - \sum_{\tau,\gamma\in\Lambda} 1_{\mathfrak{U}(w)} \otimes (f(\tau) \otimes 1_R)\widetilde{\mathbf{e}}\begin{bmatrix} \gamma \\ \tau \end{bmatrix} + \mathfrak{I}_w) \\
&\times (\sum_{u\in Q^{(m)}} \widetilde{\mathbf{e}}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_w \otimes \widetilde{\mathbf{e}}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_w) \\
=& \sum_{u\in Q^{(m)}} (1_R \otimes f(\mathsf{t}(u)))\widetilde{\mathbf{e}}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_w \otimes \widetilde{\mathbf{e}}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_w \\
&- \sum_{u\in Q^{(m)}} (\widetilde{\mathbf{e}}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_w \otimes (f(\mathsf{t}(u)) \otimes 1_R)\widetilde{\mathbf{e}}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_w \\
=& \sum_{u\in Q^{(m)}} (\widetilde{\mathbf{e}}\begin{bmatrix} u \\ q \end{bmatrix} + \mathfrak{I}_w \otimes \widetilde{\mathbf{e}}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{I}_w) \\
&\times (s_{M^{op}}(f(\mathsf{t}(u))_\sharp) \otimes 1_{\mathfrak{U}(w)} - 1_{\mathfrak{U}(w)} \otimes t_{M^{op}}(f(\mathsf{t}(u))_\sharp)) \in \mathfrak{I}'_2.
\end{aligned}
$$

The same proof works for $\alpha = \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}$ $(\forall m \in \mathbb{Z}_{\geq 0}, \forall p, q \in Q^{(m)})$. This completes the proof. □

By using the above proposition, we can define the k-module homomorphism $\tilde{\nabla}' \colon \mathfrak{U}(w) \ni \alpha + \mathfrak{I}_w \mapsto \overline{\nabla}(\alpha) + \mathfrak{I}'_2 \in \mathfrak{U}(w) \otimes_k \mathfrak{U}(w)/\mathfrak{I}'_2$. Similar to the proof of Lemma 3.3, we can induce the following lemma.

Lemma 7.3. As $\mathbb{Z}$-modules,

$$
\mathfrak{U}(w) \otimes_k \mathfrak{U}(w)/\mathfrak{I}'_2 \cong \mathfrak{U}(w) \otimes_{M^{op}} \mathfrak{U}(w).
$$

This lemma induces the $\mathbb{Z}$-module homomorphism $\Delta_{M^{op}} \colon \mathfrak{U}(w) \to \mathfrak{U}(w) \otimes_{M^{op}} \mathfrak{U}(w)$. This $\Delta_{M^{op}}$ is an (M^{op}, M^{op})-bimodule homomorphism.

In order to construct the map $\pi_{M^{op}}\colon A_\sigma \to M^{op}$, we need the k-algebra anti-homomorphism $\overline{\zeta'}\colon k\langle \Lambda Q\rangle \to \mathrm{End}_k(M^{op})$ defined by

$$\tilde{\zeta}'(\xi) = \psi'(\xi) \ \ (\xi \in R \otimes_k R^{op}) \tag{7.1}$$

$$\tilde{\zeta}'(\mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix})(f) = \delta_{p,q} f(\mathfrak{s}(q))_\sharp \delta_{\mathfrak{t}(q)} \tag{7.2}$$

$$\tilde{\zeta}'(\widetilde{\mathbf{e}}\begin{bmatrix} p \\ q \end{bmatrix})(f) = \delta_{p,q} f(\mathfrak{t}(q))_\sharp \delta_{\mathfrak{s}(q)} \ \ (f \in M, m \in \mathbb{Z}_{\geq 0}, p, q \in Q^{(m)}) \tag{7.3}$$

Here the map $\psi'\colon R \otimes_k R^{op} \to \mathrm{End}_k(M^{op})$ is a k-algebra anti-homomorphism defined by $\psi'(r \otimes r')(f) = r'_\sharp f r_\sharp$ $(r, r' \in R)$.

Proposition 7.4. $\tilde{\zeta}'(\mathfrak{J}_w) = \{0\}$.

Proof. Let f be an arbitrary element in M. It suffices to show that $\tilde{\zeta}'(\alpha) = 0$ for any generator of $\mathfrak{J}_w$.

The proof for the generators (A) is straightforward.

We give the proof for the generators (B). We have

$$\begin{aligned}
\tilde{\zeta}'(\widetilde{\mathbf{e}}\begin{bmatrix} p \\ q \end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix} p' \\ q' \end{bmatrix})(f) =& \delta_{p,q}\tilde{\zeta}'(\widetilde{\mathbf{e}}\begin{bmatrix} p' \\ q' \end{bmatrix})(f(\mathfrak{t}(q))_\sharp \delta_{\mathfrak{s}(q)}) \\
=& \delta_{\mathfrak{t}(q'),\mathfrak{s}(q)}\delta_{p,q}\delta_{p',q'} f(\mathfrak{t}(q))_\sharp \delta_{\mathfrak{s}(q')}; \\
\tilde{\zeta}'(\delta_{\mathfrak{t}(p'),\mathfrak{s}(p)}\delta_{\mathfrak{t}(q'),\mathfrak{s}(q)}\widetilde{\mathbf{e}}\begin{bmatrix} p'p \\ q'q \end{bmatrix})(f) =& \delta_{\mathfrak{t}(p'),\mathfrak{s}(p)}\delta_{\mathfrak{t}(q'),\mathfrak{s}(q)}\delta_{p,q}\delta_{p',q'} f(\mathfrak{t}(q))_\sharp \delta_{\mathfrak{s}(q')}
\end{aligned}$$

for any $m \in \mathbb{Z}_{\geq 0}$, $p, q \in Q^{(m)}$. If $p = q$ and $p' = q'$, it follows that $\mathfrak{s}(p) = \mathfrak{s}(q)$ and $\mathfrak{t}(p') = \mathfrak{t}(q')$. We suppose that $p \neq q$ or $p' \neq q'$. Then it follows that $\delta_{p,q}\delta_{p',q'} = 0$. Therefore we conclude that $\tilde{\zeta}'(\widetilde{\mathbf{e}}\begin{bmatrix} p \\ q \end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix} p' \\ q' \end{bmatrix} - \delta_{\mathfrak{t}(p'),\mathfrak{s}(p)}\delta_{\mathfrak{t}(q'),\mathfrak{s}(q)}\widetilde{\mathbf{e}}\begin{bmatrix} p'p \\ q'q \end{bmatrix}) = 0$. Similar considerations can apply to the other generator of (B).

Let α be an arbitrary generator in (C). For any $m \in \{0, 1\}$, $p, q \in Q^{(m)}$,

$$\begin{aligned}
&\tilde{\zeta}'(\sum_{u \in Q^{(m)}} \mathbf{e}\begin{bmatrix} p \\ u \end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix} u \\ q \end{bmatrix} - \delta_{p,q}\sum_{\lambda \in \Lambda} \mathbf{e}\begin{bmatrix} \mathfrak{s}(p) \\ \lambda \end{bmatrix})(f) \\
=&\tilde{\zeta}'(\widetilde{\mathbf{e}}\begin{bmatrix} p \\ q \end{bmatrix})(f(\mathfrak{s}(p))_\sharp \delta_{\mathfrak{t}(p)}) - \delta_{p,q} f(\mathfrak{s}(p))_\sharp \delta_{\mathfrak{s}(p)} \\
=&\delta_{\mathfrak{t}(p),\mathfrak{t}(q)}\delta_{p,q} f(\mathfrak{s}(p))_\sharp \delta_{\mathfrak{s}(q)} - \delta_{p,q} f(\mathfrak{s}(p))_\sharp \delta_{\mathfrak{s}(p)}.
\end{aligned}$$

If $p = q$, then $\mathfrak{s}(p) = \mathfrak{s}(q)$ and $\mathfrak{t}(p) = \mathfrak{t}(q)$ are satisfied. We can deduce that $\delta_{p,q} = 0$ if $p \neq q$. Thus $\tilde{\zeta}'(\alpha) = 0$ is proved. The proof for the other generatos is similar.

If α is a generator in (D), we can calculate that

$$\begin{aligned}
&\tilde{\zeta}'((r \otimes r')\mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix} - \mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}(r \otimes r'))(f) \\
=&\tilde{\zeta}'(\mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix})(r'_\sharp f r_\sharp) - \delta_{p,q}\tilde{\zeta}'(r \otimes r')(f(\mathfrak{s}(q))_\sharp \delta_{\mathfrak{t}(q)}) \\
=&\delta_{p,q}((r' f(\mathfrak{s}(q)) r)_\sharp \delta_{\mathfrak{t}(q)} - (r' f(\mathfrak{s}(q)) r)_\sharp \delta_{\mathfrak{t}(q)}) \\
=&0.
\end{aligned}$$

for any $m \in \{0,1\}$, $p, q \in Q^{(m)}$. The same proof works for the other generator of (D).

Let α be an arbitrary generator in (E). For any $(a,b),(c,d) \in Q^{(2)}$, the central element $\mathbf{w}\left[\begin{smallmatrix} & a & \\ c & & b \\ & d & \end{smallmatrix}\right] \in R$ induces that

$$\begin{aligned}
\tilde{\zeta}'(\alpha)(f) = & \sum_{(x,y)\in Q^{(2)}} \tilde{\zeta}'((\mathbf{w}\left[\begin{smallmatrix} & x & \\ a & & y \\ & b & \end{smallmatrix}\right] \otimes 1_R)\mathbf{e}\begin{bmatrix} xy \\ cd \end{bmatrix})(f) \\
& - \sum_{(x,y)\in Q^{(2)}} \tilde{\zeta}'((1_R \otimes \mathbf{w}\left[\begin{smallmatrix} & c & \\ x & & d \\ & y & \end{smallmatrix}\right])\mathbf{e}\begin{bmatrix} ab \\ xy \end{bmatrix})(f) \\
= & \sum_{(x,y)\in Q^{(2)}} \delta_{x,c}\delta_{y,d}(f(\mathfrak{s}(c))\mathbf{w}\left[\begin{smallmatrix} & x & \\ a & & y \\ & b & \end{smallmatrix}\right])_\sharp \delta_{\mathfrak{t}(d)} \\
& - \sum_{(x,y)\in Q^{(2)}} \delta_{a,x}\delta_{b,y}(\mathbf{w}\left[\begin{smallmatrix} & c & \\ x & & d \\ & y & \end{smallmatrix}\right] f(\mathfrak{s}(x)))_\sharp \delta_{\mathfrak{t}(y)} \\
= & (\mathbf{w}\left[\begin{smallmatrix} & c & \\ a & & d \\ & b & \end{smallmatrix}\right] f(\mathfrak{s}(c)))_\sharp \delta_{\mathfrak{t}(d)} - (\mathbf{w}\left[\begin{smallmatrix} & c & \\ a & & d \\ & b & \end{smallmatrix}\right] f(\mathfrak{s}(a)))_\sharp \delta_{\mathfrak{t}(b)}.
\end{aligned}$$

Here we use the calculation of the generators (B) for the first equality. If $\mathbf{w}\left[\begin{smallmatrix} & c & \\ a & & d \\ & b & \end{smallmatrix}\right] \neq 0$, then $\mathfrak{s}(a) = \mathfrak{s}(c)$ and $\mathfrak{t}(b) = \mathfrak{t}(d)$ are satisfied. Thus the proof for the generators (E) is completed.

The proof for the generators (F) is similar to that for Proposition 6.7. This completes the proof. $\square$

By using the above proposition, we can define the k-algebra anti-homomorphism $\zeta' \colon \mathfrak{U}(w) \ni \alpha + \mathfrak{I}_w \mapsto \tilde{\zeta}'(\alpha) \in \mathrm{End}_k(M^{op})$ $(\alpha \in k\langle \Lambda Q \rangle)$. The (M^{op}, M^{op})-bimodule homomorphism $\pi_{M^{op}} \colon \mathfrak{U}(w) \to M^{op}$ is defined by

$$\pi_{M^{op}}(a) = \zeta'(a)(1_M) \quad (a \in \mathfrak{U}(w)).$$

The triple $(\mathfrak{U}(w), \Delta_{M^{op}}, \pi_{M^{op}})$ is a comonoid in the tensor category of (M^{op}, M^{op})-bimodules satisfying (1.14)-(1.18). The proof is similar to that for Propositions 6.9 and 6.10. Therefore the sextuplet $\mathfrak{U}(w) = (\mathfrak{U}(w), M^{op}, s_{M^{op}}, t_{M^{op}}, \Delta_{M^{op}}, \pi_{M^{op}})$ is a right bialgebroid.

8 Hopf algebroid $\mathfrak{U}(w)$

In this section, we introduce a condition with respect to $\mathbf{w}\left[\begin{smallmatrix} & a & \\ c & & b \\ & d & \end{smallmatrix}\right] \in R$ $(\forall (a,b), (c,d) \in Q^{(2)})$. This condition makes $\mathfrak{U}(w)$ a Hopf algebroid with a bijective antipode.

Definition 8.1. The family $w = \{\mathbf{w}\left[\begin{smallmatrix} & a & \\ c & & b \\ & d & \end{smallmatrix}\right]\}_{(a,b),(c,d)\in Q^{(2)}}$ is called rigid if the following conditions are satisfied:

For any $m \in \mathbb{Z}_{\geq 0}$ and $p, q \in Q^{(m)}$, there exist $X_{p,q}, Y_{p,q} \in A_\sigma$ such that

$$\sum_{u\in Q^{(m)}}(\widetilde{\mathbf{e}}\begin{bmatrix}u\\q\end{bmatrix}+\mathfrak{J}_w)X_{p,u}=\sum_{u\in Q^{(m)}}Y_{u,q}(\mathbf{e}\begin{bmatrix}p\\u\end{bmatrix}+\mathfrak{J}_w)$$

$$=\delta_{p,q}\sum_{\lambda\in\Lambda}\mathbf{e}\begin{bmatrix}\mathfrak{t}(p)\\\lambda\end{bmatrix}+\mathfrak{J}_w; \quad (8.1)$$

$$\sum_{u\in Q^{(m)}}X_{u,q}(\widetilde{\mathbf{e}}\begin{bmatrix}p\\u\end{bmatrix}+\mathfrak{J}_w)=\sum_{u\in Q^{(m)}}(\mathbf{e}\begin{bmatrix}u\\q\end{bmatrix}+\mathfrak{J}_w)Y_{p,u}$$

$$=\delta_{p,q}\sum_{\lambda\in\Lambda}\mathbf{e}\begin{bmatrix}\lambda\\\mathfrak{s}(p)\end{bmatrix}+\mathfrak{J}_w; \quad (8.2)$$

$$\sum_{u,v\in Q^{(m)}}X_{u,q}(\widetilde{\mathbf{e}}\begin{bmatrix}v\\u\end{bmatrix}+\mathfrak{J}_w)X_{p,v}=X_{p,q}; \quad (8.3)$$

$$\sum_{u,v\in Q^{(m)}}Y_{u,q}(\mathbf{e}\begin{bmatrix}v\\u\end{bmatrix}+\mathfrak{J}_w)Y_{p,v}=Y_{p,q}. \quad (8.4)$$

Proposition 8.2. The elements $X_{p,q}, Y_{p,q} \in \mathfrak{U}(w)$ satisfying (8.1)-(8.4) are unique if there exist.

Proof. For any non-negative integer m, let p and q be arbitrary elements in $Q^{(m)}$. We give the proof only for the uniqueness of $X_{p,q}$. Suppose that the elements $X_{p,q}, X'_{p,q} \in \mathfrak{U}(w)$ satisfy the conditions (8.1)-(8.3). By the utilization of Lemma 6.4, (8.1), and (8.3),

$$\begin{aligned}
&\sum_{u,v,y,z\in Q^{(m)}}X_{u,q}(\widetilde{\mathbf{e}}\begin{bmatrix}v\\u\end{bmatrix}+\mathfrak{J}_w)X'_{y,v}(\widetilde{\mathbf{e}}\begin{bmatrix}z\\y\end{bmatrix}+\mathfrak{J}_w)X_{p,z}\\
=&\sum_{u,y,z\in Q^{(m)}}X_{u,q}(\sum_{\lambda\in\Lambda}\delta_{y,u}\mathbf{e}\begin{bmatrix}\mathfrak{t}(y)\\\lambda\end{bmatrix}+\mathfrak{J}_w)(\widetilde{\mathbf{e}}\begin{bmatrix}z\\y\end{bmatrix}+\mathfrak{J}_w)X_{p,z}\\
=&\sum_{u,z\in Q^{(m)}}X_{u,q}(\sum_{\lambda\in\Lambda}\widetilde{\mathbf{e}}\begin{bmatrix}\lambda\\\mathfrak{t}(u)\end{bmatrix}+\mathfrak{J}_w)(\widetilde{\mathbf{e}}\begin{bmatrix}z\\u\end{bmatrix}+\mathfrak{J}_w)X_{p,z}\\
=&\sum_{u,z\in Q^{(m)}}X_{u,q}(\widetilde{\mathbf{e}}\begin{bmatrix}z\\u\end{bmatrix}+\mathfrak{J}_w)X_{p,z}\\
=&X_{p,q}.
\end{aligned}$$

On the other hand, the identities (8.1) and (8.2) induce that

$$\begin{aligned}
&\sum_{u,v,y,z\in Q^{(m)}}X_{u,q}(\widetilde{\mathbf{e}}\begin{bmatrix}v\\u\end{bmatrix}+\mathfrak{J}_w)X'_{y,v}(\widetilde{\mathbf{e}}\begin{bmatrix}z\\y\end{bmatrix}+\mathfrak{J}_w)X_{p,z}\\
=&\sum_{\substack{\lambda,\mu\in\Lambda\\v,y\in Q^{(m)}}}\delta_{q,v}\delta_{p,y}(\mathbf{e}\begin{bmatrix}\lambda\\\mathfrak{s}(v)\end{bmatrix}+\mathfrak{J}_w)X'_{y,v}(\mathbf{e}\begin{bmatrix}\mathfrak{t}(p)\\\mu\end{bmatrix}+\mathfrak{J}_w)\\
=&\sum_{\lambda,\mu\in\Lambda}\mathbf{e}\begin{bmatrix}\lambda\\\mathfrak{s}(q)\end{bmatrix}+\mathfrak{J}_w)X'_{p,q}(\mathbf{e}\begin{bmatrix}\mathfrak{t}(p)\\\mu\end{bmatrix}+\mathfrak{J}_w)
\end{aligned}$$

Thus we can conclude that $X_{p,q}=\sum_{\lambda,\mu\in\Lambda}\mathbf{e}\begin{bmatrix}\lambda\\\mathfrak{s}(q)\end{bmatrix}+\mathfrak{J}_w)X'_{p,q}(\mathbf{e}\begin{bmatrix}\mathfrak{t}(p)\\\mu\end{bmatrix}+\mathfrak{J}_w)$. The proof for $X'_{p,q} = \sum_{\lambda,\mu\in\Lambda} \mathbf{e}\begin{bmatrix}\lambda\\ \mathfrak{s}(q)\end{bmatrix} + \mathfrak{J}_w)X'_{p,q}(\mathbf{e}\begin{bmatrix}\mathfrak{t}(p)\\ \mu\end{bmatrix} + \mathfrak{J}_w)$ is similar. Therefore $X_{p,q} = X'_{p,q}$. □

Proposition 8.3. The following conditions are equivalent:

1. w is rigid;
2. There exists a unique k-algebra anti-automorphism $S\colon \mathfrak{U}(w) \to \mathfrak{U}(w)$ such that

$$\begin{cases} S(r\otimes r' + \mathfrak{J}_w) = r'\otimes r + \mathfrak{J}_w \quad (\forall r, r' \in R);\\ S(\mathbf{e}\begin{bmatrix}p\\ q\end{bmatrix} + I_\sigma) = \widetilde{\mathbf{e}}\begin{bmatrix}p\\ q\end{bmatrix} + \mathfrak{J}_w \quad (\forall m \in \mathbb{Z}_{\geq 0},\ \forall p, q \in Q^{(m)}). \end{cases} \tag{8.5}$$

Proof. We first suppose that the condition 2 holds. For any $m \in \mathbb{Z}_{\geq 0}$, $p, q \in Q^{(m)}$, the elements $X_{p,q}$ and $Y_{p,q} \in \mathfrak{U}(w)$ are defined by $X_{p,q} = S(\widetilde{\mathbf{e}}\begin{bmatrix}p\\ q\end{bmatrix} + \mathfrak{J}_w)$ and $Y_{p,q} = S^{-1}(\mathbf{e}\begin{bmatrix}p\\ q\end{bmatrix} + \mathfrak{J}_w)$. We write $p = (p_1, \cdots, p_m)$ for any $m \in \mathbb{Z}_{\geq 0}$ and $p \in Q^{(m)}$. By using Lemma 6.4 and the generators (B) and (C) repeatedly, we can calculate that

$$\begin{aligned}
&\sum_{u\in Q^{(m)}} (\widetilde{\mathbf{e}}\begin{bmatrix}u\\ q\end{bmatrix} + \mathfrak{J}_w)X_{p,u}\\
=&\sum_{u\in Q^{(m)}} S(\widetilde{\mathbf{e}}\begin{bmatrix}p\\ u\end{bmatrix}\mathbf{e}\begin{bmatrix}u\\ q\end{bmatrix} + \mathfrak{J}_w)\\
=&\sum_{u_1,\cdots,u_m\in Q} S(\widetilde{\mathbf{e}}\begin{bmatrix}p_m\\ u_m\end{bmatrix}\cdots\widetilde{\mathbf{e}}\begin{bmatrix}p_1\\ u_1\end{bmatrix}\mathbf{e}\begin{bmatrix}u_1\\ q_1\end{bmatrix}\cdots\mathbf{e}\begin{bmatrix}u_m\\ q_m\end{bmatrix} + \mathfrak{J}_w)\\
=&\delta_{p_1,q_1}\sum_{\substack{\lambda\in\Lambda\\ u_1,\cdots,u_m\in Q}} S(\widetilde{\mathbf{e}}\begin{bmatrix}p_m\\ u_m\end{bmatrix}\cdots\widetilde{\mathbf{e}}\begin{bmatrix}p_2\\ u_2\end{bmatrix}\mathbf{e}\begin{bmatrix}\lambda\\ \mathfrak{t}(p_1)\end{bmatrix}\mathbf{e}\begin{bmatrix}u_2\\ q_2\end{bmatrix}\cdots\mathbf{e}\begin{bmatrix}u_m\\ q_m\end{bmatrix} + \mathfrak{J}_w)\\
=&\delta_{p_1,q_1}\sum_{u_2,\cdots,u_m\in Q} S(\widetilde{\mathbf{e}}\begin{bmatrix}p_m\\ u_m\end{bmatrix}\cdots\widetilde{\mathbf{e}}\begin{bmatrix}p_2\\ u_2\end{bmatrix}\mathbf{e}\begin{bmatrix}u_2\\ q_2\end{bmatrix}\cdots\mathbf{e}\begin{bmatrix}u_m\\ q_m\end{bmatrix} + \mathfrak{J}_w)\\
=&\delta_{p_1,q_1}\cdots\delta_{p_m,q_m}\sum_{\lambda\in\Lambda} S(\mathbf{e}\begin{bmatrix}\lambda\\ \mathfrak{t}(p_m)\end{bmatrix} + \mathfrak{J}_w)\\
=&\delta_{p,q}\sum_{\lambda\in\Lambda}\mathbf{e}\begin{bmatrix}\mathfrak{t}(p)\\ \lambda\end{bmatrix} + \mathfrak{J}_w
\end{aligned}$$

Similar considerations can apply to the proof of (8.2). In order to prove (8.3), we can calculate that

$$
\begin{aligned}
&\sum_{u,v\in Q^{(m)}} X_{u,q}(\widetilde{\mathbf{e}}\begin{bmatrix}v\\u\end{bmatrix}+\mathfrak{J}_w)X_{p,v}\\
=&\sum_{u,v\in Q^{(m)}} S(\widetilde{\mathbf{e}}\begin{bmatrix}p\\v\end{bmatrix}\mathbf{e}\begin{bmatrix}v\\u\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}u\\q\end{bmatrix}+\mathfrak{J}_w)\\
=&\sum_{\substack{\lambda\in\Lambda\\u\in Q^{(m)}}} \delta_{p,u}S(\mathbf{e}\begin{bmatrix}\lambda\\\mathfrak{t}(p)\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}u\\q\end{bmatrix}+\mathfrak{J}_w)\\
=&\sum_{\lambda\in\Lambda} S(\widetilde{\mathbf{e}}\begin{bmatrix}\mathfrak{t}(p)\\\lambda\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix}+\mathfrak{J}_w)\\
=&\sum_{\lambda\in\Lambda} S(\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix}+\mathfrak{J}_w)\\
=&X_{p,q}.
\end{aligned}
$$

The proof for $Y_{p,q}$ is similar. Thus w is rigid by these $X_{p,q}$ and $Y_{p,q}$.

Conversely, we suppose that the condition 1 holds. The k-algebra homomorphism $\overline{S}\colon k\langle\Lambda Q\rangle\to\mathfrak{U}(w)^{op}$ is defined by

$$
\begin{aligned}
&\overline{S}(\xi)=H(\xi)\ \ (\xi\in R\otimes R^{op});\\
&\overline{S}(\mathbf{e}\begin{bmatrix}p\\q\end{bmatrix})=\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix}+\mathfrak{J}_w;\\
&\overline{S}(\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix})=X_{p,q}\ \ (m\in\mathbb{Z}_{\geq 0},p,q\in Q^{(m)}).
\end{aligned}
$$

Here H means the k-algebra homomorphism defined by $H\colon R\otimes R^{op}\ni r\otimes r'\mapsto r'\otimes r+\mathfrak{J}_w\in\mathfrak{U}(w)^{op}$. We will show that $\overline{S}(\mathfrak{J}_w)=\{0\}$. It suffices to prove that $\overline{S}(\alpha)=0$ for any generator α of $\mathfrak{J}_w$.

The proof for the generators (A) is straightforward.

For the generators (B), we can calculate that

$$
\begin{aligned}
&\overline{S}(\mathbf{e}\begin{bmatrix}p\\q\end{bmatrix}\mathbf{e}\begin{bmatrix}p'\\q'\end{bmatrix}-\delta_{\mathfrak{t}(p),\mathfrak{s}(p')}\delta_{\mathfrak{t}(q),\mathfrak{s}(q')}\mathbf{e}\begin{bmatrix}pp'\\qq'\end{bmatrix})\\
=&\widetilde{\mathbf{e}}\begin{bmatrix}p'\\q'\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix}-\delta_{\mathfrak{t}(p),\mathfrak{s}(p')}\delta_{\mathfrak{t}(q),\mathfrak{s}(q')}\widetilde{\mathbf{e}}\begin{bmatrix}pp'\\qq'\end{bmatrix}+\mathfrak{J}_w\\
=&0.
\end{aligned}
$$

for any $m\in\mathbb{Z}_{\geq 0}$, $p,q\in Q^{(m)}$. On the other hand,

$$
\begin{aligned}
&\overline{S}(\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p'\\q'\end{bmatrix}-\delta_{\mathfrak{t}(p'),\mathfrak{s}(p)}\delta_{\mathfrak{t}(q'),\mathfrak{s}(q)}\widetilde{\mathbf{e}}\begin{bmatrix}p'p\\q'q\end{bmatrix})\\
=&X_{p',q'}X_{p,q}-\delta_{\mathfrak{t}(p'),\mathfrak{s}(p)}\delta_{\mathfrak{t}(q'),\mathfrak{s}(q)}X_{p'p,q'q}.
\end{aligned}
$$

It is easy to see that $X_{p,q}=\sum_{\lambda\in\Lambda}X_{p,q}(\mathbf{e}\begin{bmatrix}\mathfrak{t}(p)\\\lambda\end{bmatrix}+\mathfrak{J}_w)=\sum_{\lambda\in\Lambda}(\mathbf{e}\begin{bmatrix}\lambda\\\mathfrak{s}(q)\end{bmatrix}+\mathfrak{J}_w)X_{p,q}$ for any $m\in\mathbb{Z}_{\geq 0}$, $p,q\in Q^{(m)}$. For non-negative integer m and n, let $p,q\in Q^{(m)}$

and $p', q' \in Q^{(n)}$. If $\mathfrak{t}(p') = \mathfrak{s}(p)$ and $\mathfrak{t}(q') = \mathfrak{s}(q)$, we have

$$\begin{aligned}
&\sum_{\substack{u,v\in Q^{(m)}\\ u',v'\in Q^{(n)}}} \delta_{\mathfrak{t}(u'),\mathfrak{s}(u)}\delta_{\mathfrak{t}(v'),\mathfrak{s}(v)} X_{u'u,q'q}(\widetilde{\mathbf{e}}\begin{bmatrix} v'v \\ u'u \end{bmatrix} + \mathfrak{J}_w)X_{p',v'}X_{p,v} \\
=& \sum_{\substack{u,v\in Q^{(m)}\\ u',v'\in Q^{(n)}}} \delta_{\mathfrak{t}(u'),\mathfrak{s}(u)} X_{u'u,q'q}(\widetilde{\mathbf{e}}\begin{bmatrix} v \\ u \end{bmatrix} + \mathfrak{J}_w)(\widetilde{\mathbf{e}}\begin{bmatrix} v' \\ u' \end{bmatrix} + \mathfrak{J}_w)X_{p',v'}X_{p,v} \\
\underset{(8.1)}{=}& \sum_{\substack{\lambda\in\Lambda\\ u,v\in Q^{(m)}}} \delta_{\mathfrak{t}(p'),\mathfrak{s}(u)} X_{p'u,q'q}(\widetilde{\mathbf{e}}\begin{bmatrix} v \\ u \end{bmatrix} + \mathfrak{J}_w)(\mathbf{e}\begin{bmatrix} \mathfrak{t}(p') \\ \lambda \end{bmatrix} + \mathfrak{J}_w)X_{p,v} \\
\underset{\text{Lemma 6.4}}{=}& \sum_{\substack{\lambda\in\Lambda\\ u,v\in Q^{(m)}}} \delta_{\mathfrak{t}(p'),\mathfrak{s}(u)} X_{p'u,q'q}(\widetilde{\mathbf{e}}\begin{bmatrix} v \\ u \end{bmatrix} + \mathfrak{J}_w)X_{p,v} \\
\underset{(8.1)}{=}& \sum_{\lambda\in\Lambda} \delta_{\mathfrak{t}(p'),\mathfrak{s}(p)} X_{p'p,q'q}(\mathbf{e}\begin{bmatrix} \mathfrak{t}(p) \\ \lambda \end{bmatrix} + \mathfrak{J}_w) \\
=& \delta_{\mathfrak{t}(p'),\mathfrak{s}(p)} X_{p'p,q'q} \\
=& X_{p'p,q'q}.
\end{aligned}$$

In addition, it follows from (8.2) that

$$\begin{aligned}
&\sum_{\substack{u,v\in Q^{(m)}\\ u',v'\in Q^{(n)}}} \delta_{\mathfrak{t}(u'),\mathfrak{s}(u)}\delta_{\mathfrak{t}(v'),\mathfrak{s}(v)} X_{u'u,q'q}(\widetilde{\mathbf{e}}\begin{bmatrix} v'v \\ u'u \end{bmatrix} + \mathfrak{J}_w)X_{p',v'}X_{p,v} \\
=& \sum_{\substack{u,v\in Q^{(m)}\\ u',v'\in Q^{(n)}\\ \mathfrak{t}(u')=\mathfrak{s}(u),\mathfrak{t}(v')=\mathfrak{s}(v)}} X_{u'u,q'q}(\widetilde{\mathbf{e}}\begin{bmatrix} v'v \\ u'u \end{bmatrix} + \mathfrak{J}_w)X_{p',v'}X_{p,v} \\
=& \sum_{\lambda\in\Lambda} (\mathbf{e}\begin{bmatrix} \lambda \\ \mathfrak{s}(q') \end{bmatrix} + \mathfrak{J}_w)X_{p',q'}X_{p,q} \\
=& X_{p',q'}X_{p,q}.
\end{aligned}$$

Suppose that $\mathfrak{t}(p') \neq \mathfrak{s}(p)$ or $\mathfrak{t}(q') \neq \mathfrak{s}(q)$. By using (8.2) and Lemma 6.4, we can induce that

$$\begin{aligned}
&\sum_{\substack{u,v\in Q^{(m)}\\ u',v'\in Q^{(n)}}} X_{u',q'}X_{u,q}(\widetilde{\mathbf{e}}\begin{bmatrix} v \\ u \end{bmatrix} + \mathfrak{J}_w)(\widetilde{\mathbf{e}}\begin{bmatrix} v' \\ u' \end{bmatrix} + \mathfrak{J}_w)X_{p',v'}X_{p,v} \\
=& \sum_{\substack{\lambda\in\Lambda\\ u',v'\in Q^{(n)}}} X_{u',q'}(\mathbf{e}\begin{bmatrix} \lambda \\ \mathfrak{s}(q) \end{bmatrix} + \mathfrak{J}_w)(\widetilde{\mathbf{e}}\begin{bmatrix} v' \\ u' \end{bmatrix} + \mathfrak{J}_w)X_{p',v'}X_{p,q} \\
=& \sum_{u',v'\in Q^{(n)}} \delta_{\mathfrak{t}(v'),\mathfrak{s}(q)} X_{u',q'}(\widetilde{\mathbf{e}}\begin{bmatrix} v' \\ u' \end{bmatrix} + \mathfrak{J}_w)X_{p',v'}X_{p,q} \\
=& \delta_{\mathfrak{t}(q'),\mathfrak{s}(q)} \sum_{\lambda\in\Lambda} (\mathbf{e}\begin{bmatrix} \lambda \\ \mathfrak{s}(q') \end{bmatrix} + \mathfrak{J}_w)X_{p',q'}X_{p,q}
\end{aligned}$$

$$=\delta_{\mathfrak{t}(q'),\mathfrak{s}(q)}X_{p',q'}X_{p,q}.$$

Similarly, the following identity is satisfied:

$$\sum_{\substack{u,v\in Q^{(m)}\\ u',v'\in Q^{(n)}}} X_{u',q'}X_{u,q}(\widetilde{\mathbf{e}}\begin{bmatrix}v\\u\end{bmatrix}+\mathfrak{J}_w)(\widetilde{\mathbf{e}}\begin{bmatrix}v'\\u'\end{bmatrix}+\mathfrak{J}_w)X_{p',v'}X_{p,v}=\delta_{\mathfrak{t}(p'),\mathfrak{s}(p)}X_{p',q'}X_{p,q}.$$

We can also calculate that

$$\begin{aligned}
&\sum_{\substack{\lambda\in\Lambda\\ u',v'\in Q^{(n)}}} X_{u',q'}(\mathbf{e}\begin{bmatrix}\lambda\\ \mathfrak{s}(q)\end{bmatrix}+\mathfrak{J}_w)(\widetilde{\mathbf{e}}\begin{bmatrix}v'\\u'\end{bmatrix}+\mathfrak{J}_w)X_{p',v'}X_{p,q}\\
\underset{(8.1)}{=}&\sum_{\lambda,\mu\in\Lambda} X_{p',q'}(\mathbf{e}\begin{bmatrix}\lambda\\ \mathfrak{s}(q)\end{bmatrix}+\mathfrak{J}_w)(\mathbf{e}\begin{bmatrix}\mathfrak{t}(p')\\ \mu\end{bmatrix}+\mathfrak{J}_w)X_{p,q}\\
=&X_{p',q'}X_{p,q}.
\end{aligned}$$

Therefore $X_{p',q'}X_{p,q}-\delta_{\mathfrak{t}(p'),\mathfrak{s}(p)}\delta_{\mathfrak{t}(q'),\mathfrak{s}(q)}X_{p'p,q'q}=0$ is satisfied.

By the definition of rigidity of w, $\overline{S}(\alpha)=0$ is clear for any generator (C).

Let α be an arbitrary generator in (D). By the definition of $\mathfrak{J}_w$,

$$\begin{aligned}
\overline{S}((r\otimes r')\mathbf{e}\begin{bmatrix}p\\q\end{bmatrix}-\mathbf{e}\begin{bmatrix}p\\q\end{bmatrix}(r\otimes r'))=&\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix}(r'\otimes r)-(r'\otimes r)\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix}+\mathfrak{J}_w\\
=&0
\end{aligned}$$

for any $r,r'\in R$, $m\in\mathbb{Z}_{\geq 0}$. For the second generator of (D), we have

$$\begin{aligned}
&\overline{S}((r\otimes r')\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix}-\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix}(r\otimes r'))\\
=&X_{p,q}(r'\otimes r+\mathfrak{J}_w)-(r'\otimes r+\mathfrak{J}_w)X_{p,q}\\
=&\sum_{\lambda\in\Lambda}X_{p,q}(\mathbf{e}\begin{bmatrix}\mathfrak{t}(p)\\ \lambda\end{bmatrix}+\mathfrak{J}_w)(r'\otimes r+\mathfrak{J}_w)-\sum_{\lambda\in\Lambda}(r'\otimes r+\mathfrak{J}_w)(\mathbf{e}\begin{bmatrix}\lambda\\ \mathfrak{s}(q)\end{bmatrix}+\mathfrak{J}_w)X_{p,q}\\
=&\sum_{\substack{\lambda\in\Lambda\\ u\in Q^{(m)}}} X_{u,q}(r'\otimes r+\mathfrak{J}_w)(\delta_{p,u}\mathbf{e}\begin{bmatrix}\mathfrak{t}(p)\\ \lambda\end{bmatrix}+\mathfrak{J}_w)\\
&-\sum_{\substack{\lambda\in\Lambda\\ v\in Q^{(m)}}}(\delta_{v,q}\mathbf{e}\begin{bmatrix}\lambda\\ \mathfrak{s}(v)\end{bmatrix}+\mathfrak{J}_w)(r'\otimes r+\mathfrak{J}_w)X_{p,v}\\
\underset{(8.1),(8.2)}{=}&\sum_{u,v\in Q^{(m)}} X_{u,q}(r'\otimes r+\mathfrak{J}_w)(\widetilde{\mathbf{e}}\begin{bmatrix}v\\u\end{bmatrix}+\mathfrak{J}_w)X_{p,v}\\
&-\sum_{u,v\in Q^{(m)}} X_{u,q}(\widetilde{\mathbf{e}}\begin{bmatrix}v\\u\end{bmatrix}+\mathfrak{J}_w)(r'\otimes r+\mathfrak{J}_w)X_{p,v}\\
=&\sum_{u,v\in Q^{(m)}} X_{u,q}((r'\otimes r)\widetilde{\mathbf{e}}\begin{bmatrix}v\\u\end{bmatrix}-\widetilde{\mathbf{e}}\begin{bmatrix}v\\u\end{bmatrix}(r'\otimes r)+\mathfrak{J}_w)X_{p,v}\\
=&0.
\end{aligned}$$

Let us prove that $\overline{S}(\alpha) = 0$ is satisfied if α is a generator in (E). We define a collection of elements $\underline{\mathbf{w}}\left[\begin{smallmatrix} & a & \\ c & & b \\ & d & \end{smallmatrix}\right]$ $(a, b, c, d \in Q)$ in R as follows:

$$\underline{\mathbf{w}}\left[\begin{smallmatrix} & a & \\ c & & b \\ & d & \end{smallmatrix}\right] = \begin{cases} \mathbf{w}\left[\begin{smallmatrix} & a & \\ c & & b \\ & d & \end{smallmatrix}\right], & ((a,b),(c,d) \in Q^{(2)}); \\ 0, & (\text{otherwise}). \end{cases}$$

By the definition of rigidity of w,

$$\begin{aligned}
0 = & \sum_{a,b,c,d \in Q} (\widetilde{\mathbf{e}}\begin{bmatrix} y'' \\ b \end{bmatrix} \widetilde{\mathbf{e}}\begin{bmatrix} x'' \\ a \end{bmatrix} + \mathfrak{I}_w) \\
& \times \Big(\sum_{x,y \in Q} (\underline{\mathbf{w}}\left[\begin{smallmatrix} & x & \\ a & & y \\ & b & \end{smallmatrix}\right] \otimes 1_R) \mathbf{e}\begin{bmatrix} x \\ c \end{bmatrix} \mathbf{e}\begin{bmatrix} y \\ d \end{bmatrix} - \sum_{x,y \in Q} (1_R \otimes \underline{\mathbf{w}}\left[\begin{smallmatrix} & c & \\ x & & d \\ & y & \end{smallmatrix}\right]) \mathbf{e}\begin{bmatrix} a \\ x \end{bmatrix} \mathbf{e}\begin{bmatrix} b \\ y \end{bmatrix} + \mathfrak{I}_w \Big) \\
& \times (\widetilde{\mathbf{e}}\begin{bmatrix} d \\ y' \end{bmatrix} \widetilde{\mathbf{e}}\begin{bmatrix} c \\ x' \end{bmatrix} + \mathfrak{I}_w) \\
= & \sum_{\substack{\lambda \in \Lambda \\ a,b,c,x \in Q}} (\underline{\mathbf{w}}\left[\begin{smallmatrix} & x & \\ a & & y' \\ & b & \end{smallmatrix}\right] \otimes 1_R) \widetilde{\mathbf{e}}\begin{bmatrix} y'' \\ b \end{bmatrix} \widetilde{\mathbf{e}}\begin{bmatrix} x'' \\ a \end{bmatrix} \mathbf{e}\begin{bmatrix} x \\ c \end{bmatrix} \mathbf{e}\begin{bmatrix} \mathfrak{s}(y') \\ \lambda \end{bmatrix} \widetilde{\mathbf{e}}\begin{bmatrix} c \\ x' \end{bmatrix} + \mathfrak{I}_w \\
& - \sum_{\substack{\lambda \in \Lambda \\ b,c,d,y \in Q}} (1_R \otimes \underline{\mathbf{w}}\left[\begin{smallmatrix} & c & \\ x'' & & d \\ & y & \end{smallmatrix}\right]) \widetilde{\mathbf{e}}\begin{bmatrix} y'' \\ b \end{bmatrix} \mathbf{e}\begin{bmatrix} \lambda \\ \mathfrak{t}(x'') \end{bmatrix} \mathbf{e}\begin{bmatrix} b \\ y \end{bmatrix} \widetilde{\mathbf{e}}\begin{bmatrix} d \\ y' \end{bmatrix} \widetilde{\mathbf{e}}\begin{bmatrix} c \\ x' \end{bmatrix} + \mathfrak{I}_w \\
= & \sum_{\substack{\lambda \in \Lambda \\ a,b \in Q}} \delta_{\mathfrak{t}(x'),\mathfrak{s}(y')} (\underline{\mathbf{w}}\left[\begin{smallmatrix} & x' & \\ a & & y' \\ & b & \end{smallmatrix}\right] \otimes 1_R) \widetilde{\mathbf{e}}\begin{bmatrix} y'' \\ b \end{bmatrix} \widetilde{\mathbf{e}}\begin{bmatrix} x'' \\ a \end{bmatrix} \mathbf{e}\begin{bmatrix} \mathfrak{s}(x') \\ \lambda \end{bmatrix} + \mathfrak{I}_w \\
& - \sum_{\substack{\lambda \in \Lambda \\ c,d \in Q}} \delta_{\mathfrak{t}(x''),\mathfrak{s}(y'')} (1_R \otimes \underline{\mathbf{w}}\left[\begin{smallmatrix} & c & \\ x'' & & d \\ & y'' & \end{smallmatrix}\right]) \mathbf{e}\begin{bmatrix} \lambda \\ \mathfrak{t}(y'') \end{bmatrix} \widetilde{\mathbf{e}}\begin{bmatrix} d \\ y' \end{bmatrix} \widetilde{\mathbf{e}}\begin{bmatrix} c \\ x' \end{bmatrix} + \mathfrak{I}_w \\
= & \sum_{a,b \in Q} \delta_{\mathfrak{t}(x'),\mathfrak{s}(y')} \delta_{\mathfrak{s}(a),\mathfrak{s}(x')} (\underline{\mathbf{w}}\left[\begin{smallmatrix} & x' & \\ a & & y' \\ & b & \end{smallmatrix}\right] \otimes 1_R) \widetilde{\mathbf{e}}\begin{bmatrix} y'' \\ b \end{bmatrix} \widetilde{\mathbf{e}}\begin{bmatrix} x'' \\ a \end{bmatrix} + \mathfrak{I}_w \\
& - \sum_{c,d \in Q} \delta_{\mathfrak{t}(x''),\mathfrak{s}(y'')} \delta_{\mathfrak{t}(y''),\mathfrak{t}(d)} (1_R \otimes \underline{\mathbf{w}}\left[\begin{smallmatrix} & c & \\ x'' & & d \\ & y'' & \end{smallmatrix}\right]) \widetilde{\mathbf{e}}\begin{bmatrix} d \\ y' \end{bmatrix} \widetilde{\mathbf{e}}\begin{bmatrix} c \\ x' \end{bmatrix} + \mathfrak{I}_w.
\end{aligned}$$

for any x', y', x'', and $y'' \in Q$. By virtue of the generators (D), we conclude that

$$\begin{aligned}
& \overline{S}\Big(\sum_{(x,y) \in Q^{(2)}} (\mathbf{w}\left[\begin{smallmatrix} & x & \\ a & & y \\ & b & \end{smallmatrix}\right] \otimes 1_R) \mathbf{e}\begin{bmatrix} x \\ c \end{bmatrix} \mathbf{e}\begin{bmatrix} y \\ d \end{bmatrix} - \sum_{(x,y) \in Q^{(2)}} (1_R \otimes \mathbf{w}\left[\begin{smallmatrix} & c & \\ x & & d \\ & y & \end{smallmatrix}\right]) \mathbf{e}\begin{bmatrix} a \\ x \end{bmatrix} \mathbf{e}\begin{bmatrix} b \\ y \end{bmatrix} \Big) \\
= & \sum_{(x,y) \in Q^{(2)}} (1_R \otimes \mathbf{w}\left[\begin{smallmatrix} & x & \\ a & & y \\ & b & \end{smallmatrix}\right]) \widetilde{\mathbf{e}}\begin{bmatrix} y \\ d \end{bmatrix} \widetilde{\mathbf{e}}\begin{bmatrix} x \\ c \end{bmatrix} + \mathfrak{I}_w \\
& - \sum_{(x,y) \in Q^{(2)}} (\mathbf{w}\left[\begin{smallmatrix} & c & \\ x & & d \\ & y & \end{smallmatrix}\right] \otimes 1_R) \widetilde{\mathbf{e}}\begin{bmatrix} b \\ y \end{bmatrix} \widetilde{\mathbf{e}}\begin{bmatrix} a \\ x \end{bmatrix} + \mathfrak{I}_w \\
= & \sum_{x,y \in Q} \delta_{\mathfrak{t}(a),\mathfrak{s}(b)} \delta_{\mathfrak{t}(b),\mathfrak{t}(y)} (1_R \otimes \underline{\mathbf{w}}\left[\begin{smallmatrix} & x & \\ a & & y \\ & b & \end{smallmatrix}\right]) \widetilde{\mathbf{e}}\begin{bmatrix} y \\ d \end{bmatrix} \widetilde{\mathbf{e}}\begin{bmatrix} x \\ c \end{bmatrix} + \mathfrak{I}_w \\
& - \sum_{x,y \in Q} \delta_{\mathfrak{t}(c),\mathfrak{s}(d)} \delta_{\mathfrak{s}(x),\mathfrak{s}(c)} (\underline{\mathbf{w}}\left[\begin{smallmatrix} & c & \\ x & & d \\ & y & \end{smallmatrix}\right] \otimes 1_R) \widetilde{\mathbf{e}}\begin{bmatrix} b \\ y \end{bmatrix} \widetilde{\mathbf{e}}\begin{bmatrix} a \\ x \end{bmatrix} + \mathfrak{I}_w
\end{aligned}$$

$= 0.$

for any $(a, b), (c, d) \in Q^{(2)}$.

The proof for the generators (F) is easy because $\widetilde{\mathbf{e}}\begin{bmatrix}\lambda \\ \mu\end{bmatrix} + \mathfrak{J}_w = \mathbf{e}\begin{bmatrix}\mu \\ \lambda\end{bmatrix} + \mathfrak{J}_w$ is satisfied for all $\lambda, \mu \in \Lambda$. Therefore the map $S(\alpha + \mathfrak{J}_w) = \overline{S}(\alpha)$ $(\alpha \in k\langle \Lambda Q\rangle)$ is well defined.

In order to construct the inverse of S, the k-algebra homomorphism $\overline{S'}\colon k\langle \Lambda Q\rangle \to \mathfrak{U}(w)$ is defined by

$$\begin{aligned}
&\overline{S'}(\xi) = H(\xi) \ \ (\xi \in R \otimes R^{op});\\
&\overline{S'}(\mathbf{e}\begin{bmatrix}p \\ q\end{bmatrix}) = Y_{p,q};\\
&\overline{S'}(\widetilde{\mathbf{e}}\begin{bmatrix}p \\ q\end{bmatrix}) = \mathbf{e}\begin{bmatrix}p \\ q\end{bmatrix} + \mathfrak{J}_w \ \ (m \in \mathbb{Z}_{\geq 0}, p, q \in Q^{(m)}).
\end{aligned}$$

Similar to the proof for $\overline{S}$, we can show that $\overline{S'}(\mathfrak{J}_w) = \{0\}$. Hence the map $S'(\alpha + \mathfrak{J}_w) = \overline{S'}(\alpha)$ $(\alpha \in k\langle \Lambda Q\rangle)$ also makes sense.

Finally, let us prove that this S' is the inverse of S. We give the proof only for $(S' \circ S)(\widetilde{\mathbf{e}}\begin{bmatrix}p \\ q\end{bmatrix} + \mathfrak{J}_w) = \widetilde{\mathbf{e}}\begin{bmatrix}p \\ q\end{bmatrix} + \mathfrak{J}_w$ $(\forall m \in \mathbb{Z}_{\geq 0}, \forall p, q \in Q^{(m)})$.

$$\begin{aligned}
(S' \circ S)(\widetilde{\mathbf{e}}\begin{bmatrix}p \\ q\end{bmatrix} + \mathfrak{J}_w) =& S'(X_{p,q})\\
=& \sum_{\substack{\lambda \in \Lambda \\ v \in Q^{(m)}}} \delta_{v,q} S'((\mathbf{e}\begin{bmatrix}\lambda \\ \mathfrak{s}(v)\end{bmatrix} + \mathfrak{J}_w) X_{p,v})\\
=& \sum_{v \in Q^{(m)}} S'(X_{p,v})(\delta_{v,q} \sum_{\lambda \in \Lambda} \mathbf{e}\begin{bmatrix}\mathfrak{s}(v) \\ \lambda\end{bmatrix} + \mathfrak{J}_w)\\
=& \sum_{u,v \in Q^{(m)}} S'(X_{p,v})(\mathbf{e}\begin{bmatrix}v \\ u\end{bmatrix} \widetilde{\mathbf{e}}\begin{bmatrix}u \\ q\end{bmatrix} + \mathfrak{J}_w)\\
=& \sum_{u,v \in Q^{(m)}} S'((\widetilde{\mathbf{e}}\begin{bmatrix}v \\ u\end{bmatrix} + \mathfrak{J}_w) X_{p,v})(\widetilde{\mathbf{e}}\begin{bmatrix}u \\ q\end{bmatrix} + \mathfrak{J}_w)\\
=& \sum_{\lambda \in \Lambda} S'(\mathbf{e}\begin{bmatrix}\mathfrak{t}(p) \\ \lambda\end{bmatrix} + \mathfrak{J}_w)(\widetilde{\mathbf{e}}\begin{bmatrix}p \\ q\end{bmatrix} + \mathfrak{J}_w)\\
=& \sum_{\lambda \in \Lambda} \mathbf{e}\begin{bmatrix}\lambda \\ \mathfrak{t}(p)\end{bmatrix} \widetilde{\mathbf{e}}\begin{bmatrix}p \\ q\end{bmatrix} + \mathfrak{J}_w\\
=& \widetilde{\mathbf{e}}\begin{bmatrix}p \\ q\end{bmatrix} + \mathfrak{J}_w.
\end{aligned}$$

The same proof works for $(S \circ S')(\mathbf{e}\begin{bmatrix}p \\ q\end{bmatrix} + \mathfrak{J}_w) = \mathbf{e}\begin{bmatrix}p \\ q\end{bmatrix} + \mathfrak{J}_w$. This is the desired conclusion. □

By the above proposition, we can construct a k-algebra anti-automorphism S if w is rigid. The left and right bialgebroid $\mathfrak{U}(w)$ has a Hopf algebroid structure by virtue of this S.

Theorem 8.4. Let S be a k-algebra anti-automorphism of $\mathfrak{U}(w)$ defined by the rigid w. Then the pair $(\mathfrak{U}(w), S)$ is a Hopf algebroid for $N = M^{op}$ and $\omega = \mathrm{id}_M$.

Proof. We prove this theorem by using Proposition 1.7.

(1.31) is obvious because $s_{M^{op}} = t_M$ and $t_{M^{op}} = s_M$.

By the definition Δ_M and $\Delta_{M^{op}}$, the proof for (1.32) and (1.33) is similar to the coassociativity of Δ_L.

Let us show (1.34) and (1.35). Because the map S is a k-algebra homomorphism, we have only to check that $S \circ t_M = s_M$ and $S \circ s_M = t_M$. By using Lemma 6.4 and the generators (D) of $\mathfrak{J}_w$, we have

$$\begin{aligned}(S \circ t_M)(f) &= \sum_{\lambda,\mu\in\Lambda} S((1_R \otimes f(\lambda))\mathbf{e}\begin{bmatrix}\mu\\ \lambda\end{bmatrix} + \mathfrak{J}_w) \\ &= \sum_{\lambda,\mu\in\Lambda} \mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}(f(\lambda) \otimes 1_R) + \mathfrak{J}_w \\ &= \sum_{\lambda,\mu\in\Lambda} (f(\lambda) \otimes 1_R)\mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} + \mathfrak{J}_w \\ &= s_M(f)\end{aligned}$$

for all $f \in M$. The proof for $S \circ s_M = t_M$ is similar.

For (1.36), we give the proof only for $a = (r \otimes r')\mathbf{e}\begin{bmatrix}p_1\\ q_1\end{bmatrix}\tilde{\mathbf{e}}\begin{bmatrix}p_2\\ q_2\end{bmatrix}\cdots\mathbf{e}\begin{bmatrix}p_{2n-1}\\ q_{2n-1}\end{bmatrix}\tilde{\mathbf{e}}\begin{bmatrix}p_{2n}\\ q_{2n}\end{bmatrix} + \mathfrak{J}_w$ $(r, r' \in R, i \in \{1, 2, \cdots, 2n\}, m_i \in \mathbb{Z}_{\geq 0}, p_i, q_i \in Q^{(m_i)})$. By using the generators (C) and (D) of $\mathfrak{J}_w$, Lemma 6.4, and the condition (8.2) repeatedly, we have

$$\begin{aligned}&S(a_{[1]})a_{[2]} \\ =&\sum_{u_1,\cdots,u_{2n}} X_{u_{2n},q_{2n}}(\tilde{\mathbf{e}}\begin{bmatrix}p_{2n-1}\\ u_{2n-1}\end{bmatrix} + \mathfrak{J}_w)\cdots X_{u_2,q_2}(\tilde{\mathbf{e}}\begin{bmatrix}p_1\\ u_1\end{bmatrix}(1_R \otimes r) + \mathfrak{J}_w) \\ &\times(\mathbf{e}\begin{bmatrix}u_1\\ q_1\end{bmatrix}\tilde{\mathbf{e}}\begin{bmatrix}p_2\\ u_2\end{bmatrix}\cdots\mathbf{e}\begin{bmatrix}u_{2n-1}\\ q_{2n-1}\end{bmatrix}\tilde{\mathbf{e}}\begin{bmatrix}p_{2n}\\ u_{2n}\end{bmatrix}(1_R \otimes r') + \mathfrak{J}_w) \\ =&\delta_{p_1,q_1}\sum_{\substack{\lambda\in\Lambda\\ u_2,\cdots,u_{2n}}} X_{u_{2n},q_{2n}}(\tilde{\mathbf{e}}\begin{bmatrix}p_{2n-1}\\ u_{2n-1}\end{bmatrix} + \mathfrak{J}_w)\cdots X_{u_2,q_2} \\ &\times(\mathbf{e}\begin{bmatrix}\lambda\\ \mathfrak{t}(p_1)\end{bmatrix}\tilde{\mathbf{e}}\begin{bmatrix}p_2\\ u_2\end{bmatrix}\cdots\mathbf{e}\begin{bmatrix}u_{2n-1}\\ q_{2n-1}\end{bmatrix}\tilde{\mathbf{e}}\begin{bmatrix}p_{2n}\\ u_{2n}\end{bmatrix}(1_R \otimes r'r) + \mathfrak{J}_w) \\ =&\delta_{p_1,q_1}\delta_{\mathfrak{t}(p_1),\mathfrak{t}(p_2)}\sum_{u_2,\cdots,u_{2n}} X_{u_{2n},q_{2n}}(\tilde{\mathbf{e}}\begin{bmatrix}p_{2n-1}\\ u_{2n-1}\end{bmatrix} + \mathfrak{J}_w)\cdots X_{u_2,q_2} \\ &\times(\tilde{\mathbf{e}}\begin{bmatrix}p_2\\ u_2\end{bmatrix}\cdots\mathbf{e}\begin{bmatrix}u_{2n-1}\\ q_{2n-1}\end{bmatrix}\tilde{\mathbf{e}}\begin{bmatrix}p_{2n}\\ u_{2n}\end{bmatrix}(1_R \otimes r'r) + \mathfrak{J}_w) \\ =&\delta_{p_1,q_1}\delta_{p_2,q_2}\delta_{\mathfrak{t}(p_1),\mathfrak{t}(p_2)} \\ &\times\sum_{\substack{\lambda\in\Lambda\\ u_3,\cdots,u_{2n}}} X_{u_{2n},q_{2n}}(\tilde{\mathbf{e}}\begin{bmatrix}p_{2n-1}\\ u_{2n-1}\end{bmatrix} + \mathfrak{J}_w)\cdots(\tilde{\mathbf{e}}\begin{bmatrix}p_3\\ u_3\end{bmatrix}\mathbf{e}\begin{bmatrix}\lambda\\ \mathfrak{s}(p_2)\end{bmatrix} + \mathfrak{J}_w) \\ &\times(\mathbf{e}\begin{bmatrix}p_3\\ u_3\end{bmatrix}\cdots\mathbf{e}\begin{bmatrix}u_{2n-1}\\ q_{2n-1}\end{bmatrix}\tilde{\mathbf{e}}\begin{bmatrix}p_{2n}\\ u_{2n}\end{bmatrix}(1_R \otimes r'r) + \mathfrak{J}_w)\end{aligned}$$

$$
\begin{aligned}
=&\delta_{p_1,q_1}\delta_{p_2,q_2}\delta_{\mathfrak{t}(p_1),\mathfrak{t}(p_2)}\delta_{\mathfrak{s}(p_2),\mathfrak{s}(p_3)}\\
&\times \sum_{u_3,\cdots,u_{2n}} X_{u_{2n},q_{2n}}(\widetilde{\mathbf{e}}\begin{bmatrix}p_{2n-1}\\u_{2n-1}\end{bmatrix}+\mathfrak{J}_w)\cdots(\widetilde{\mathbf{e}}\begin{bmatrix}p_3\\u_3\end{bmatrix}+\mathfrak{J}_w)\\
&\times(\mathbf{e}\begin{bmatrix}p_3\\u_3\end{bmatrix}\cdots\mathbf{e}\begin{bmatrix}u_{2n-1}\\q_{2n-1}\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}p_{2n}\\u_{2n}\end{bmatrix}(1_R\otimes r'r)+\mathfrak{J}_w)\\
=&\delta_{p_1,q_1}\cdots\delta_{p_{2n},q_{2n}}\delta_{\mathfrak{t}(p_1),\mathfrak{t}(p_2)}\delta_{\mathfrak{s}(p_2),\mathfrak{s}(p_3)}\cdots\delta_{\mathfrak{s}(p_{2n-2}),\mathfrak{s}(p_{2n-1})}\delta_{\mathfrak{t}(p_{2n-1}),\mathfrak{t}(p_{2n})}\\
&\times(\sum_{\lambda\in\Lambda}(1_R\otimes r'r)\mathbf{e}\begin{bmatrix}\lambda\\\mathfrak{s}(p_{2n})\end{bmatrix}+\mathfrak{J}_w).
\end{aligned}
$$

By the definition of $\tilde{\zeta'}$, we can calculate that

$$
\tilde{\zeta'}(\mathbf{e}\begin{bmatrix}p\\q\end{bmatrix})(\delta_\lambda)=\delta_{p,q}\delta_{\lambda,\mathfrak{s}(q)}\delta_{\mathfrak{t}(q)}; \tag{8.6}
$$

$$
\tilde{\zeta'}(\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix})(\delta_\lambda)=\delta_{p,q}\delta_{\lambda,\mathfrak{t}(q)}\delta_{\mathfrak{s}(q)} \tag{8.7}
$$

for any $m\in Q^{(m)}$, $p,q\in Q^{(m)}$, $\lambda\in\Lambda$. Thus (8.6) and (8.7) induce that

$$
\begin{aligned}
&(s_{M^{op}}\circ\pi_{M^{op}})(a)\\
=&s_{M^{op}}(\tilde{\zeta'}(r\otimes r')\tilde{\zeta'}(\widetilde{\mathbf{e}}\begin{bmatrix}p_{2n}\\q_{2n}\end{bmatrix})\tilde{\zeta'}(\mathbf{e}\begin{bmatrix}p_{2n-1}\\q_{2n-1}\end{bmatrix})\cdots\tilde{\zeta'}(\widetilde{\mathbf{e}}\begin{bmatrix}p_2\\q_2\end{bmatrix})\tilde{\zeta'}(\mathbf{e}\begin{bmatrix}p_1\\q_1\end{bmatrix})(1_M))\\
=&\delta_{p_1,q_1}s_{M^{op}}(\tilde{\zeta'}(r\otimes r')\tilde{\zeta'}(\widetilde{\mathbf{e}}\begin{bmatrix}p_{2n}\\q_{2n}\end{bmatrix})\tilde{\zeta'}(\mathbf{e}\begin{bmatrix}p_{2n-1}\\q_{2n-1}\end{bmatrix})\cdots\tilde{\zeta'}(\widetilde{\mathbf{e}}\begin{bmatrix}p_2\\q_2\end{bmatrix})(\delta_{\mathfrak{t}(q_1)}))\\
=&\delta_{p_1,q_1}\delta_{p_2,q_2}\delta_{\mathfrak{t}(q_1),\mathfrak{t}(q_2)}s_{M^{op}}(\tilde{\zeta'}(r\otimes r')\tilde{\zeta'}(\widetilde{\mathbf{e}}\begin{bmatrix}p_{2n}\\q_{2n}\end{bmatrix})\tilde{\zeta'}(\mathbf{e}\begin{bmatrix}p_{2n-1}\\q_{2n-1}\end{bmatrix})\cdots\tilde{\zeta'}(\mathbf{e}\begin{bmatrix}p_3\\q_3\end{bmatrix})(\delta_{\mathfrak{s}(q_2)}))\\
=&\delta_{p_1,q_1}\cdots\delta_{p_{2n},q_{2n}}\delta_{\mathfrak{t}(q_1),\mathfrak{t}(q_2)}\delta_{\mathfrak{s}(q_2),\mathfrak{s}(q_3)}\cdots\delta_{\mathfrak{s}(q_{2n-2}),\mathfrak{s}(q_{2n-1})}\delta_{\mathfrak{t}(q_{2n-1}),\mathfrak{t}(q_{2n})}\\
&\times s_{M^{op}}(\delta_{\mathfrak{s}(q_{2n})}(r'r)_\sharp).
\end{aligned}
$$

Since $s_{M^{op}}(\delta_{\mathfrak{s}(q_{2n})}(r'r)_\sharp)=t_M(\delta_{\mathfrak{s}(q_{2n})}(r'r)_\sharp)=\sum_{\lambda\in\Lambda}(1_R\otimes r'r)\mathbf{e}\begin{bmatrix}\lambda\\\mathfrak{s}(q_{2n})\end{bmatrix}+\mathfrak{J}_w=\sum_{\lambda\in\Lambda}(1_R\otimes r'r)\mathbf{e}\begin{bmatrix}\lambda\\\mathfrak{s}(p_{2n})\end{bmatrix}+\mathfrak{J}_w$ if $p_{2n}=q_{2n}$, we can conclude that $S(a_{[1]})a_{[2]}=(s_{M^{op}}\circ\pi_{M^{op}})(a)$.

The proof for $a^{[1]}S(a^{[2]})=(s_M\circ\pi_M)(a)$ is similar. Therefore $(\mathfrak{U}(w),S)$ is a Hopf algebroid. □

9 Relations between $\mathfrak{U}(w)$ and A_σ

In this section, we construct a left and right bialgebroid $\mathfrak{U}(w_\sigma):=\mathfrak{U}(w)$ by means of σ. This $\mathfrak{U}(w_\sigma)$ is isomorphic to A_σ as Hopf algebroids (cf. [17, 19]).

Let R be a k-algebra and G a subgroup of the opposite group consisting of all bijections on a finite set Λ. We can define a right group action of this group G on the set Λ: $\lambda\alpha=\alpha(\lambda)$ $(\lambda\in\Lambda,\alpha\in G)$. Let $L:=M$. For any $\alpha\in G$, we define the map $T_\alpha:M\to M$ as follows:

$$
T_\alpha(f)(\lambda)=f(\lambda\alpha)\ \ (f\in M,\lambda\in\Lambda).
$$

The map T_α ($\alpha \in G$) is a k-algebra automorphism such that $T_\alpha \circ T_{\alpha^{-1}} = \mathrm{id}_M$. We use the symbol $\sigma = \{\sigma_{cd}^{ab}\}_{a,b,c,d \in X}$ as a collection of elements in M satisfying the following conditions.

$$\begin{cases} \sigma_{cd}^{ab}(\lambda) \in Z(R) \quad (\forall \lambda \in \Lambda, \forall a,b,c,d \in X); \\ \lambda \deg(d)\deg(b) \neq \lambda \deg(c)\deg(a) \Rightarrow \sigma_{ac}^{bd}(\lambda) = 0. \end{cases} \tag{9.1}$$

Here $Z(R)$ is the center of R.

Proposition 9.1. The condition (9.1) implies (3.1).

Proof. The left hand side of (3.1) induces that

$$\begin{aligned} &(T_{\deg(a)^{-1}} \circ T_{\deg(c)^{-1}} \circ \rho_l(\sigma_{ac}^{bd}))(f)(\lambda) \\ =&(T_{\deg(a)^{-1}} \circ T_{\deg(c)^{-1}})(\sigma_{ac}^{bd} f)(\lambda) \\ =&T_{\deg(c)^{-1}}(\sigma_{ac}^{bd} f)(\lambda \deg(a)^{-1}) \\ =&(\sigma_{ac}^{bd} f)(\lambda \deg(a)^{-1}\deg(c)^{-1}) \\ =&\sigma_{ac}^{bd}(\lambda \deg(a)^{-1}\deg(c)^{-1}) f(\lambda \deg(a)^{-1}\deg(c)^{-1}) \end{aligned}$$

for any $f \in M$ and $\lambda \in \Lambda$. On the other hand, the right hand side of (3.1) can be calculated as follows:

$$\begin{aligned} &(T_{\deg(b)^{-1}} \circ T_{\deg(d)^{-1}} \circ \rho_r(\sigma_{ac}^{bd}))(f)(\lambda) \\ =&(f\sigma_{ac}^{bd})(\lambda \deg(b)^{-1}\deg(d)^{-1}) \\ =&\sigma_{ac}^{bd}(\lambda \deg(b)^{-1}\deg(d)^{-1}) f(\lambda \deg(b)^{-1}\deg(d)^{-1}) \end{aligned}$$

Here we use the first condition in (9.1) for the second equality.

If $\lambda \deg(a)^{-1}\deg(c)^{-1} = \lambda \deg(b)^{-1}\deg(d)^{-1}$, then we can deduce (3.1).

Let us suppose that $\lambda \deg(a)^{-1}\deg(c)^{-1} \neq \lambda \deg(b)^{-1}\deg(d)^{-1}$. Since the map $\deg(a)$ is bijective for any $a \in X$,

$$\begin{aligned} &\lambda \deg(a)^{-1}\deg(c)^{-1} \neq \lambda \deg(b)^{-1}\deg(d)^{-1} \\ \Leftrightarrow&\lambda \deg(a)^{-1}\deg(c)^{-1}\deg(d)\deg(b) \neq \lambda \deg(b)^{-1}\deg(d)^{-1}\deg(d)\deg(b) \\ \Leftrightarrow&\lambda \deg(a)^{-1}\deg(c)^{-1}\deg(d)\deg(b) \neq \lambda \\ \Leftrightarrow&\lambda \deg(a)^{-1}\deg(c)^{-1}\deg(d)\deg(b) \neq \lambda \deg(a)^{-1}\deg(c)^{-1}\deg(c)\deg(a). \end{aligned}$$

By using the second condition in (9.1), we can conclude that $\sigma_{ac}^{bd}(\lambda \deg(a)^{-1}\deg(c)^{-1}) = 0$. From $\lambda \neq \lambda \deg(b)^{-1}\deg(d)^{-1}\deg(c)\deg(a)$, the identity $\sigma_{ac}^{bd}(\lambda \deg(b)^{-1}\deg(d)^{-1}) = 0$ is also induced. □

From Proposition 9.1, we can construct the left and right bialgebroid A_σ by using σ in (9.1).

Now, we construct the left and right bialgebroid $\mathfrak{U}(w)$ by σ. The quiver Q over Λ is defined by

$$Q := \Lambda \times X;\ \mathfrak{s}(\lambda, x) = \lambda;\ \mathfrak{t}(\lambda, x) = \lambda \deg(x) \ \ (\lambda \in \Lambda, x \in X). \tag{9.2}$$

and we set

$$\mathbf{w}\begin{bmatrix} & (\lambda,a) & \\ (\mu,c) & & (\lambda',b) \\ & (\mu',d) & \end{bmatrix} = \delta_{\lambda,\mu}\sigma_{dc}^{ba}(\lambda) \tag{9.3}$$

for all $((\lambda,a),(\lambda',b)),((\mu,c),(\mu',d)) \in Q^{(2)}$.

Proposition 9.2. The definition (9.3) satisfies the condition (6.1).

Proof. For any $((\lambda, a), (\lambda', b)), ((\mu, c), (\mu', d)) \in Q^{(2)}$, we can deduce that $\mathbf{w}\left[\begin{smallmatrix} & (\lambda,a) & \\ (\mu,c) & & (\lambda',b) \\ & (\mu',d) & \end{smallmatrix}\right] \in Z(R)$ by virtue of $\sigma^{ba}_{dc}(\lambda) \in Z(R)$.

Let us show that $\mathbf{w}\left[\begin{smallmatrix} & (\lambda,a) & \\ (\mu,c) & & (\lambda',b) \\ & (\mu',d) & \end{smallmatrix}\right] = 0$ under the condition of $\mathfrak{s}(\lambda, a) \neq \mathfrak{s}(\mu, c)$ or $\mathfrak{t}(\lambda', b) \neq \mathfrak{t}(\mu', d)$. From Kronecker's delta symbol in (9.3), it is easy to show that $\mathbf{w}\left[\begin{smallmatrix} & (\lambda,a) & \\ (\mu,c) & & (\lambda',b) \\ & (\mu',d) & \end{smallmatrix}\right] = 0$ if $\mathfrak{s}(\lambda, a) \neq \mathfrak{s}(\mu, c)$.

We assume that $\mathfrak{t}(\lambda', b) \neq \mathfrak{t}(\mu', d)$. The definition of the fiber product of Q (9.2) induces that

$$\mathfrak{t}(\lambda', b) = \lambda \deg(a) \deg(b) \quad \text{and} \quad \mathfrak{t}(\mu', d) = \mu \deg(c) \deg(d).$$

If $\lambda = \mu$, then we can conclude that $\sigma^{ba}_{dc}(\lambda) = 0$ and $\mathbf{w}\left[\begin{smallmatrix} & (\lambda,a) & \\ (\mu,c) & & (\lambda',b) \\ & (\mu',d) & \end{smallmatrix}\right] = 0$. This completes the proof. □

By virtue of this proposition, we can construct the left and right bialgebroid $\mathfrak{U}(w_\sigma) := \mathfrak{U}(w)$ by using the setting of A_σ.

From now on, we discuss relations between A_σ and $\mathfrak{U}(w_\sigma)$. The k-algebra homomorphism $\overline{\Phi}\colon k\langle \Lambda Q\rangle \to A_\sigma$ is defined by

$$\overline{\Phi}(\xi) = \varphi(\xi) \ \ (\xi \in R \otimes_k R^{op}); \tag{9.4}$$

$$\overline{\Phi}(\mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}) = (\delta_{\mathfrak{s}(p)} \otimes \delta_{\mathfrak{s}(q)}) L_{x_1 y_1} \cdots L_{x_m y_m} + I_\sigma \ \ (m \in \mathbb{Z}_{\geq 0}, p, q \in Q^{(m)}); \tag{9.5}$$

$$\overline{\Phi}(\widetilde{\mathbf{e}}\begin{bmatrix} p \\ q \end{bmatrix}) = (L^{-1})_{x_m y_m} \cdots (L^{-1})_{x_1 y_1} (\delta_{\mathfrak{s}(q)} \otimes \delta_{\mathfrak{s}(p)}) + I_\sigma. \tag{9.6}$$

Here φ means the k-algebra homomorphism defined by $\varphi\colon R \otimes R^{op} \ni r \otimes r' \to r_\sharp \otimes r'_\sharp + I_\sigma \in A_\sigma$ and we write $p = ((\lambda_1, x_1), \cdots, (\lambda_m, x_m))$ and $q = ((\mu_1, y_1), \cdots, (\mu_m, y_m))$.

Proposition 9.3. $\overline{\Phi}(\mathfrak{I}_w) = \{0\}$.

Proof. We have only to prove that $\overline{\Phi}(\alpha) = 0$ for every generator of $\mathfrak{I}_w$.

Since $\overline{\Phi}$ is a k-algebra homomorphism, the proof for the generators (A) is straightforward.

We will give the proof for the generators (B). By using the generators (3) of I_σ repeatedly, we can induce that

$$\begin{aligned}
&\overline{\Phi}(\mathbf{e}\begin{bmatrix} p \\ q \end{bmatrix}\mathbf{e}\begin{bmatrix} p' \\ q' \end{bmatrix}) \\
=&(\delta_{\mathfrak{s}(p)} \otimes \delta_{\mathfrak{s}(q)}) L_{x_1 y_1} \cdots L_{x_m y_m} (\delta_{\mathfrak{s}(p')} \otimes \delta_{\mathfrak{s}(q')}) L_{x'_1 y'_1} \cdots L_{x'_n y'_n} + I_\sigma \\
=&(\delta_{\mathfrak{s}(p)} \otimes \delta_{\mathfrak{s}(q)})(\delta_{\mathfrak{s}(p')\deg(x_m)^{-1}\cdots\deg(x_1)^{-1}} \otimes \delta_{\mathfrak{s}(q')\deg(y_m)^{-1}\cdots\deg(y_1)^{-1}}) \\
&\times L_{x_1 y_1} \cdots L_{x_m y_m} L_{x'_1 y'_1} \cdots L_{x'_n y'_n} + I_\sigma \\
=&\delta_{\mathfrak{s}(p)\deg(x_1)\cdots\deg(x_m), \mathfrak{s}(p')} \delta_{\mathfrak{s}(q)\deg(y_1)\cdots\deg(y_m), \mathfrak{s}(q')} \\
&\times (\delta_{\mathfrak{s}(p)} \otimes \delta_{\mathfrak{s}(q)}) L_{x_1 y_1} \cdots L_{x_m y_m} L_{x'_1 y'_1} \cdots L_{x'_n y'_n} + I_\sigma
\end{aligned}$$

$$=\delta_{\mathfrak{t}(p),\mathfrak{s}(p')}\delta_{\mathfrak{t}(q),\mathfrak{s}(q')}(\delta_{\mathfrak{s}(p)}\otimes\delta_{\mathfrak{s}(q)})L_{x_1y_1}\cdots L_{x_my_m}L_{x'_1y'_1}\cdots L_{x'_ny'_n}+I_\sigma$$
$$=\overline{\Phi}(\delta_{\mathfrak{t}(p),\mathfrak{s}(p')}\delta_{\mathfrak{t}(q),\mathfrak{s}(q')}\mathbf{e}\begin{bmatrix}pp'\\qq'\end{bmatrix}).$$

for all $m,n\in\mathbb{Z}_{\geq 0}$, $p=((\lambda_1,x_1),\cdots,(\lambda_m,x_m)), q=((\mu_1,y_1),\cdots,(\mu_m,y_m))\in Q^{(m)}$, $p'=((\lambda'_1,x'_1),\cdots,(\lambda'_n,x'_n)), q'=((\mu'_1,y'_1),\cdots,(\mu'_n,y'_m))\in Q^{(n)}$. The same proof works for the other generator of (B).

Let us show that $\overline{\Phi}(\alpha)=0$ if α is any generator (C). We give the proof only for $m=1$. Let $p=(\lambda,a)$ and $q=(\mu,b)\in Q$. By virtue of the generators (2) and (3) of I_σ,

$$\overline{\Phi}(\sum_{u\in Q^{(m)}}\mathbf{e}\begin{bmatrix}p\\u\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}u\\q\end{bmatrix}-\delta_{p,q}\sum_{\lambda'\in\Lambda}\mathbf{e}\begin{bmatrix}\mathfrak{s}(p)\\\lambda'\end{bmatrix})$$
$$=\sum_{\substack{\tau\in\Lambda\\c\in X}}(\delta_\lambda\otimes\delta_\tau)L_{ac}(L^{-1})_{cb}(\delta_\mu\otimes\delta_\tau)+I_\sigma-\delta_{p,q}(\delta_\lambda\otimes 1_M+I_\sigma)$$
$$=\sum_{\substack{\tau\in\Lambda\\c\in X}}(\delta_\lambda\delta_{\mu\deg(b)\deg(a)^{-1}}\otimes\delta_\tau)L_{ac}(L^{-1})_{cb}+I_\sigma-\delta_{p,q}(\delta_\lambda\otimes 1_M+I_\sigma)$$
$$=\delta_{\lambda,\mu\deg(b)\deg(a)^{-1}}\delta_{a,b}(\delta_\lambda\otimes 1_M+I_\sigma)-\delta_{\lambda,\mu}\delta_{a,b}(\delta_\lambda\otimes 1_M+I_\sigma).$$

If $a=b$, it follows that $\mu\deg(b)\deg(a)^{-1}=\mu\deg(a)\deg(a)^{-1}=\mu$. The proof for the other generator is similar.

Since $(r_\sharp\otimes r'_\sharp)L_{ab}+I_\sigma=L_{ab}(r_\sharp\otimes r'_\sharp)+I_\sigma$ and $(r_\sharp\otimes r'_\sharp)(L^{-1})_{ab}+I_\sigma=(L^{-1})_{ab}(r_\sharp\otimes r'_\sharp)+I_\sigma$ for any $r,r'\in R$, $a,b\in X$, it is easy to give the proof for the generators (D).

Let α be an arbitrary generator in (E). We denote by $((\mu,a),(\mu',b))$ and $((\tau,c),(\tau',d))$ arbitrary elements in $Q^{(2)}$. By using the definition of w in (9.3), the generators (3), and (4) of I_σ, we have

$$\overline{\Phi}(\sum_{((\lambda,x),(\lambda',y))\in Q^{(2)}}(\mathbf{w}\begin{bmatrix} & {\scriptstyle(\lambda,x)} & \\ {\scriptstyle(\mu,a)} & & {\scriptstyle(\lambda',y)}\\ & {\scriptstyle(\mu',b)} & \end{bmatrix}\otimes 1_R)\mathbf{e}\begin{bmatrix}(\lambda,x)\\(\tau,c)\end{bmatrix}\mathbf{e}\begin{bmatrix}(\lambda',y)\\(\tau',d)\end{bmatrix})$$
$$=\sum_{\substack{\lambda\in\Lambda\\x,y\in X}}\delta_{\lambda,\mu}(\sigma^{yx}_{ba}(\lambda)_\sharp\delta_\lambda\otimes\delta_\tau)L_{xc}(\delta_{\lambda\deg(x)}\otimes\delta_{\tau\deg(c)})L_{yd}+I_\sigma$$
$$=\sum_{x,y\in X}(\sigma^{yx}_{ba}(\mu)_\sharp\delta_\mu\otimes\delta_\tau)L_{xc}L_{yd}+I_\sigma$$
$$=(\delta_\mu\otimes\delta_\tau)\sum_{x,y\in X}(\sigma^{yx}_{ba}\otimes 1_M)L_{xc}L_{yd}+I_\sigma$$
$$=(\delta_\mu\otimes\delta_\tau)\sum_{x,y\in X}(1_M\otimes\sigma^{dc}_{yx})L_{ax}L_{by}+I_\sigma.$$

On the other hand,

$$\overline{\Phi}(\sum_{((\lambda,x),(\lambda',y))\in Q^{(2)}}(1_R\otimes\mathbf{w}\begin{bmatrix} & {\scriptstyle(\tau,c)} & \\ {\scriptstyle(\lambda,x)} & & {\scriptstyle(\tau',d)}\\ & {\scriptstyle(\lambda',y)} & \end{bmatrix})\mathbf{e}\begin{bmatrix}(\mu,a)\\(\lambda,x)\end{bmatrix}\mathbf{e}\begin{bmatrix}(\mu',b)\\(\lambda',y)\end{bmatrix})$$
$$=\sum_{\substack{\lambda\in\Lambda\\x,y\in X}}\delta_{\lambda,\tau}(\delta_\mu\otimes\delta_\lambda\sigma^{dc}_{yx}(\tau)_\sharp)L_{ax}(\delta_{\mu\deg(a)}\otimes\delta_{\lambda\deg(x)})L_{by}+I_\sigma$$

$$= \sum_{x,y\in X} (\delta_\mu \otimes \delta_\tau \sigma^{dc}_{yx}(\tau)_\sharp) L_{ax}L_{by} + I_\sigma$$
$$=(\delta_\mu \otimes \delta_\tau) \sum_{x,y\in X} (1_M \otimes \sigma^{dc}_{yx}) L_{ax}L_{by} + I_\sigma.$$

Thus we can conclude that $\overline{\Phi}(\alpha) = 0$ for an arbitrary generator in (E).

Since $\sum_{\lambda\in\Lambda} \delta_\lambda = 1_M$, it is easy to give the proof for the generators (F). This is the desired conclusion. □

The above proposition induces the k-algebra homomorphism $\Phi\colon \mathfrak{U}(w_\sigma) \ni \alpha + \mathfrak{J}_w \mapsto \overline{\Phi}(\alpha) \in A_\sigma$ $(\alpha \in k\langle \Lambda Q\rangle)$.

We next construct a map from A_σ to $\mathfrak{U}(w_\sigma)$. The k-algebra homomorphism $\overline{\Theta}\colon k\langle Gen\rangle \to \mathfrak{U}(w_\sigma)$ is defined by

$$\overline{\Theta}(\xi) = \chi(\xi) \ \ (\xi \in M \otimes_k M^{op}) \tag{9.7}$$
$$\overline{\Theta}(L_{ab}) = \sum_{\lambda,\mu\in\Lambda} \mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,b)\end{bmatrix} + \mathfrak{J}_w; \tag{9.8}$$
$$\overline{\Theta}((L^{-1})_{ab}) = \sum_{\lambda,\mu\in\Lambda} \widetilde{\mathbf{e}}\begin{bmatrix}(\lambda,a)\\(\mu,b)\end{bmatrix} + \mathfrak{J}_w \ \ (a,b \in X). \tag{9.9}$$

Here χ stands for the k-algebra homomorphism defined by $\chi\colon M \otimes M^{op} \ni f\otimes g \mapsto \sum_{\lambda,\mu\in\Lambda} (f(\lambda)\otimes g(\mu))\mathbf{e}\begin{bmatrix}\lambda\\\mu\end{bmatrix} + \mathfrak{J}_w \in \mathfrak{U}(w_\sigma)$.

Proposition 9.4. $\overline{\Theta}(I_\sigma) = \{0\}$.

Proof. We prove only for the generators (1)-(5) of I_σ.

Because $\overline{\Theta}$ is a k-algebra homomorphisms, the proof for the generators (1) is straightforward.

Let us give the proof for the generators (2). By the utilization of the generators (B), (C), and Lemma 6.4,

$$\overline{\Theta}(\sum_{c\in X} L_{ac}(L^{-1})_{cb}) = \sum_{\substack{c\in X\\ \lambda,\mu,\lambda',\mu'\in\Lambda}} \mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,c)\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda',c)\\(\mu',b)\end{bmatrix} + \mathfrak{J}_w$$
$$= \sum_{\substack{c\in X\\ \lambda,\mu,\lambda',\mu'\in\Lambda}} \mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,c)\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}\lambda'\deg(c)\\\mu'\deg(b)\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda',c)\\(\mu',b)\end{bmatrix} + \mathfrak{J}_w$$
$$= \sum_{\substack{c\in X\\ \lambda,\mu,\lambda',\mu'\in\Lambda}} \mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,c)\end{bmatrix}\mathbf{e}\begin{bmatrix}\mu'\deg(b)\\\lambda'\deg(c)\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda',c)\\(\mu',b)\end{bmatrix} + \mathfrak{J}_w$$
$$= \sum_{\substack{c\in X\\ \lambda,\mu,\lambda',\mu'\in\Lambda}} \delta_{\lambda\deg(a),\mu'\deg(b)}\delta_{\mu,\lambda'}\mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,c)\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda',c)\\(\mu',b)\end{bmatrix} + \mathfrak{J}_w$$
$$= \sum_{\substack{c\in X\\ \lambda,\mu\in\Lambda}} \mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,c)\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}(\mu,c)\\(\lambda\deg(a)\deg(b)^{-1},b)\end{bmatrix} + \mathfrak{J}_w$$

$$=\delta_{a,b}\sum_{\lambda,\mu\in\Lambda}\delta_{\lambda,\lambda\deg(a)\deg(b)^{-1}}\mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}+\mathfrak{J}_w$$

for any $a,b\in X$. If $a=b$, then $\lambda\deg(a)\deg(b)^{-1}=\lambda\deg(a)\deg(a)^{-1}=\lambda$ is satisfied. Thus $\overline{\Theta}(\sum_{c\in X}L_{ac}(L^{-1})_{cb}-\delta_{a,b}\emptyset)=0$ is proved. The same proof works for the other generator of (2).

For the generators (3), we calculate that

$$\begin{aligned}
&\overline{\Theta}((f\otimes 1_M)(L^{-1})_{ab}-(L^{-1})_{ab}(T_{\deg(b)}(f)\otimes 1_M))\\
=&\sum_{\lambda,\mu,\lambda',\mu'\in\Lambda}(f(\lambda)\otimes 1_R)\mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda',a)\\ (\mu',b)\end{bmatrix}+\mathfrak{J}_w\\
&-\sum_{\tau,\nu,\tau',\nu'\in\Lambda}\widetilde{\mathbf{e}}\begin{bmatrix}(\tau,a)\\ (\nu,b)\end{bmatrix}(f(\tau'\deg(b))\otimes 1_R)\mathbf{e}\begin{bmatrix}\tau'\\ \nu'\end{bmatrix}+\mathfrak{J}_w\\
=&\sum_{\lambda,\mu,\lambda',\mu'\in\Lambda}(f(\lambda)\otimes 1_R)\widetilde{\mathbf{e}}\begin{bmatrix}\mu\\ \lambda\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda',a)\\ (\mu',b)\end{bmatrix}+\mathfrak{J}_w\\
&-\sum_{\tau,\nu,\tau',\nu'\in\Lambda}(f(\tau'\deg(b))\otimes 1_R)\widetilde{\mathbf{e}}\begin{bmatrix}(\tau,a)\\ (\nu,b)\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}\nu'\\ \tau'\end{bmatrix}+\mathfrak{J}_w\\
=&\sum_{\lambda,\mu,\lambda',\mu'\in\Lambda}\delta_{\lambda'\deg(a),\mu}\delta_{\mu'\deg(b),\lambda}(f(\lambda)\otimes 1_R)\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda',a)\\ (\mu',b)\end{bmatrix}+\mathfrak{J}_w\\
&-\sum_{\tau,\nu,\tau',\nu'\in\Lambda}\delta_{\nu',\tau}\delta_{\tau',\nu}(f(\tau'\deg(b))\otimes 1_R)\widetilde{\mathbf{e}}\begin{bmatrix}(\tau,a)\\ (\nu,b)\end{bmatrix}+\mathfrak{J}_w\\
=&\sum_{\lambda',\mu'\in\Lambda}(f(\mu'\deg(b))\otimes 1_R)\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda',a)\\ (\mu',b)\end{bmatrix}+\mathfrak{J}_w\\
&-\sum_{\tau,\nu\in\Lambda}(f(\nu\deg(b))\otimes 1_R)\widetilde{\mathbf{e}}\begin{bmatrix}(\tau,a)\\ (\nu,b)\end{bmatrix}+\mathfrak{J}_w\\
=&0.
\end{aligned}$$

Here we use Lemma 6.4 and the generators (D) for the second equality, and the generators (B) for the third equality. The proof for the other generators is similar.

Let α be an arbitrary generator in (4). By using the generators (B), (E), and the definition (9.3), we can calculate that

$$\begin{aligned}
&\overline{\Theta}(\sum_{x,y\in X}(\sigma_{ac}^{xy}\otimes 1_M)L_{yd}L_{xb})\\
=&\sum_{\substack{x,y\in X\\ \lambda,\mu,\nu,\tau,\gamma,\eta\in\Lambda}}(\sigma_{ac}^{xy}(\lambda)\otimes 1_R)\mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}\mathbf{e}\begin{bmatrix}(\nu,y)\\ (\tau,d)\end{bmatrix}\mathbf{e}\begin{bmatrix}(\gamma,x)\\ (\eta,b)\end{bmatrix}+\mathfrak{J}_w\\
=&\sum_{\substack{x,y\in X\\ \lambda,\mu\in\Lambda}}(\sigma_{ac}^{xy}(\lambda)\otimes 1_R)\mathbf{e}\begin{bmatrix}(\lambda,y)\\ (\mu,d)\end{bmatrix}\mathbf{e}\begin{bmatrix}(\lambda\deg(y),x)\\ (\mu\deg(d),b)\end{bmatrix}+\mathfrak{J}_w
\end{aligned}$$

$$= \sum_{\substack{x,y\in X \\ \beta,\lambda,\mu\in\Lambda}} \delta_{\lambda,\beta}(\sigma_{ac}^{xy}(\lambda)\otimes 1_R)\mathbf{e}\begin{bmatrix}(\lambda,y)\\(\mu,d)\end{bmatrix}\mathbf{e}\begin{bmatrix}(\lambda\deg(y),x)\\(\mu\deg(d),b)\end{bmatrix} + \mathfrak{J}_w$$

$$= \sum_{\substack{x,y\in X \\ \beta,\lambda,\mu\in\Lambda}} (\mathbf{w}\left[\begin{smallmatrix} & (\lambda,y) & \\ (\beta,c) & & (\lambda\deg(y),x) \\ & (\beta\deg(c),a) & \end{smallmatrix}\right]\otimes 1_R)\mathbf{e}\begin{bmatrix}(\lambda,y)\\(\mu,d)\end{bmatrix}\mathbf{e}\begin{bmatrix}(\lambda\deg(y),x)\\(\mu\deg(d),b)\end{bmatrix} + \mathfrak{J}_w$$

$$= \sum_{\substack{x,y\in X \\ \beta,\lambda,\mu\in\Lambda}} (1_R\otimes \mathbf{w}\left[\begin{smallmatrix} & (\mu,d) & \\ (\lambda,y) & & (\mu\deg(d),b) \\ & (\lambda\deg(y),x) & \end{smallmatrix}\right])\mathbf{e}\begin{bmatrix}(\beta,c)\\(\lambda,y)\end{bmatrix}\mathbf{e}\begin{bmatrix}(\beta\deg(c),a)\\(\lambda\deg(y),x)\end{bmatrix} + \mathfrak{J}_w.$$

for any $a,b,c,d\in X$. On the other hand,

$$\overline{\Theta}(\sum_{x,y\in X}(1_M\otimes\sigma_{xy}^{bd})L_{cy}L_{ax})$$

$$= \sum_{\substack{x,y\in X \\ \lambda,\mu,\nu,\tau,\gamma,\eta\in\Lambda}} (1_R\otimes\sigma_{xy}^{bd}(\mu))\mathbf{e}\begin{bmatrix}\lambda\\\mu\end{bmatrix}\mathbf{e}\begin{bmatrix}(\nu,c)\\(\tau,y)\end{bmatrix}\mathbf{e}\begin{bmatrix}(\gamma,a)\\(\eta,x)\end{bmatrix} + \mathfrak{J}_w$$

$$= \sum_{\substack{x,y\in X \\ \lambda,\mu\in\Lambda}} (1_R\otimes\sigma_{xy}^{bd}(\mu))\mathbf{e}\begin{bmatrix}(\lambda,c)\\(\mu,y)\end{bmatrix}\mathbf{e}\begin{bmatrix}(\lambda\deg(c),a)\\(\mu\deg(y),x)\end{bmatrix} + \mathfrak{J}_w$$

$$= \sum_{\substack{x,y\in X \\ \beta,\lambda,\mu\in\Lambda}} \delta_{\beta,\mu}(1_R\otimes\sigma_{xy}^{bd}(\beta))\mathbf{e}\begin{bmatrix}(\lambda,c)\\(\mu,y)\end{bmatrix}\mathbf{e}\begin{bmatrix}(\lambda\deg(c),a)\\(\mu\deg(y),x)\end{bmatrix} + \mathfrak{J}_w$$

$$= \sum_{\substack{x,y\in X \\ \beta,\lambda,\mu\in\Lambda}} (1_R\otimes \mathbf{w}\left[\begin{smallmatrix} & (\beta,d) & \\ (\mu,y) & & (\beta\deg(d),b) \\ & (\mu\deg(y),x) & \end{smallmatrix}\right])\mathbf{e}\begin{bmatrix}(\lambda,c)\\(\mu,y)\end{bmatrix}\mathbf{e}\begin{bmatrix}(\lambda\deg(c),a)\\(\mu\deg(y),x)\end{bmatrix} + \mathfrak{J}_w.$$

Therefore we can conclude that $\overline{\Theta}(\alpha)=0$.

The proof for the generator (5) is straightforward. This completes the proof. □

By virtue of the above proposition, we can define the k-algebra homomorphism $\Theta\colon A_\sigma \ni \alpha + I_\sigma \to \overline{\Theta}(\alpha)\in\mathfrak{U}(w_\sigma)$.

Theorem 9.5. (Φ,id_M) and (Θ,id_M) are left and right bialgebroid isomorphisms.

Proof. We first check that $\Theta\circ\Phi = \mathrm{id}_{\mathfrak{U}(w_\sigma)}$. Since the maps Φ and Θ are k-algebra homomorphisms, it suffices to prove that $(\Theta\circ\Phi)(\alpha)=\alpha$. Here $\alpha\in\mathfrak{U}(w_\sigma)$ means that

$$\alpha = \begin{cases} r\otimes r' + \mathfrak{J}_w \quad (r,r'\in R); \\ \mathbf{e}\begin{bmatrix}p\\q\end{bmatrix} + \mathfrak{J}_w; \\ \widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix} + \mathfrak{J}_w \quad (m\in\mathbb{Z}_{\geq 0}, p,q\in Q^{(m)}). \end{cases}$$

Suppose that $\alpha = r\otimes r' + \mathfrak{J}_w$ $(\forall r,r'\in R)$. By the definition of the generators

(F) of $\mathfrak{J}_w$, we can conclude that

$$\begin{aligned}(\Theta\circ\Phi)(r\otimes r'+\mathfrak{J}_w)=&\Theta(r_\sharp\otimes r'_\sharp+I_\sigma)\\=&\sum_{\lambda,\mu\in\Lambda}(r\otimes r')\mathbf{e}\begin{bmatrix}\lambda\\\mu\end{bmatrix}+\mathfrak{J}_w\\=&r\otimes r'+\mathfrak{J}_w.\end{aligned}$$

For any $m\in\mathbb{Z}_{\geq 0}, p,q\in Q^{(m)}$, let $\alpha=\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix}$ and we write $p=((\lambda_1,x_1),\cdots,(\lambda_m,x_m))$ and $q=((\mu_1,y_1),\cdots,(\mu_m,y_m))$. The generators (B) and Lemma 6.4 induce that

$$\begin{aligned}&(\Theta\circ\Phi)(\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix}+\mathfrak{J}_w)\\
=&\Theta((L^{-1})_{x_my_m}\cdots(L^{-1})_{x_1y_1}(\delta_{\mathfrak{s}(q)}\otimes\delta_{\mathfrak{s}(p)})+I_\sigma)\\
=&(\sum_{\tau_m,\nu_m\in\Lambda}\widetilde{\mathbf{e}}\begin{bmatrix}(\tau_m,x_m)\\(\nu_m,y_m)\end{bmatrix})\cdots(\sum_{\tau_1,\nu_1\in\Lambda}\widetilde{\mathbf{e}}\begin{bmatrix}(\tau_1,x_1)\\(\nu_1,y_1)\end{bmatrix})\widetilde{\mathbf{e}}\begin{bmatrix}\lambda_1\\\mu_1\end{bmatrix}+\mathfrak{J}_w\\
=&(\sum_{\tau_m,\nu_m\in\Lambda}\widetilde{\mathbf{e}}\begin{bmatrix}(\tau_m,x_m)\\(\nu_m,y_m)\end{bmatrix})\cdots(\sum_{\tau_1,\nu_1\in\Lambda}\delta_{\lambda_1,\tau_1}\delta_{\mu_1,\nu_1}\widetilde{\mathbf{e}}\begin{bmatrix}(\tau_1,x_1)\\(\nu_1,y_1)\end{bmatrix})+\mathfrak{J}_w\\
=&(\sum_{\tau_m,\nu_m\in\Lambda}\widetilde{\mathbf{e}}\begin{bmatrix}(\tau_m,x_m)\\(\nu_m,y_m)\end{bmatrix})\cdots(\sum_{\tau_2,\nu_2\in\Lambda}\widetilde{\mathbf{e}}\begin{bmatrix}(\tau_2,x_2)\\(\nu_2,y_2)\end{bmatrix})\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda_1,x_1)\\(\mu_1,y_1)\end{bmatrix}+\mathfrak{J}_w\\
=&(\sum_{\tau_m,\nu_m\in\Lambda}\widetilde{\mathbf{e}}\begin{bmatrix}(\tau_m,x_m)\\(\nu_m,y_m)\end{bmatrix})\cdots(\sum_{\tau_2,\nu_2\in\Lambda}\delta_{\lambda_1\deg(x_1),\tau_2}\delta_{\mu_1\deg(y_1),\nu_2}\widetilde{\mathbf{e}}\begin{bmatrix}((\lambda_1,x_1),(\tau_2,x_2))\\((\mu_1,y_1),(\nu_2,y_2))\end{bmatrix})+\mathfrak{J}_w\\
=&(\sum_{\tau_m,\nu_m\in\Lambda}\widetilde{\mathbf{e}}\begin{bmatrix}(\tau_m,x_m)\\(\nu_m,y_m)\end{bmatrix})\cdots(\sum_{\tau_3,\nu_3\in\Lambda}\widetilde{\mathbf{e}}\begin{bmatrix}(\tau_3,x_3)\\(\nu_3,y_3)\end{bmatrix})\widetilde{\mathbf{e}}\begin{bmatrix}((\lambda_1,x_1),(\lambda_1\deg(x_1),x_2))\\((\mu_1,y_1),(\mu_1\deg(y_1),y_2))\end{bmatrix}+\mathfrak{J}_w\\
=&\widetilde{\mathbf{e}}\begin{bmatrix}p\\q\end{bmatrix}+\mathfrak{J}_w\end{aligned}$$

The proof for $\alpha=\mathbf{e}\begin{bmatrix}p\\q\end{bmatrix}+\mathfrak{J}_w$ is similar. Thus $\Theta\circ\Phi=\mathrm{id}_{\mathfrak{U}(w_\sigma)}$ is proved.

We next show that $\Phi\circ\Theta=\mathrm{id}_{A_\sigma}$. For any $f,g\in M$,

$$\begin{aligned}(\Phi\circ\Theta)(f\otimes g+I_\sigma)=&\sum_{\lambda,\mu\in\Lambda}\Phi((f(\lambda)\otimes g(\mu))\mathbf{e}\begin{bmatrix}\lambda\\\mu\end{bmatrix}+\mathfrak{J}_w)\\=&\sum_{\lambda,\mu\in\Lambda}f(\lambda)_\sharp\delta_\lambda\otimes g(\mu)_\sharp\delta_\mu+I_\sigma\\=&f\otimes g+I_\sigma.\end{aligned}$$

On the other hand, we can calculate that

$$\begin{aligned}(\Phi\circ\Theta)(L_{ab}+I_\sigma)=&\sum_{\lambda,\mu\in\Lambda}\Phi(\mathbf{e}\begin{bmatrix}(\lambda,a)\\(\mu,b)\end{bmatrix}+\mathfrak{J}_w)\\=&\sum_{\lambda,\mu\in\Lambda}(\delta_\lambda\otimes\delta_\mu)L_{ab}+I_\sigma\\=&L_{ab}+I_\sigma\end{aligned}$$

for all $a, b \in X$. The proof for $(\Phi \circ \Theta)((L^{-1})_{ab} + I_\sigma) = (L^{-1})_{ab} + I_\sigma$ is similar. This concludes that $\Phi \circ \Theta = \mathrm{id}_{A_\sigma}$.

Let us prove that the pair (Θ, id_M) is a left bialgebroid homomorphism.

For (1.8), we have

$$\begin{aligned}(\Theta \circ s_M^{A_\sigma})(f) =&\Theta(f \otimes 1_M + I_\sigma)\\ =& \sum_{\lambda,\mu\in\Lambda} (f(\lambda) \otimes 1_R)\mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} + \mathfrak{J}_w\\ =&s_M^{\mathfrak{U}(w_\sigma)}(f)\end{aligned}$$

for any $f \in M$. The same proof works for (1.9).

We show (1.10). Let $\alpha = (f \otimes g)w_1 \cdots w_n + I_\sigma \in A_\sigma$ ($\forall f, g \in L$) defined by (2.12). For any $a, b \in X$,

$$\begin{aligned}\sum_{\lambda,\mu\in\Lambda} \pi_M^{\mathfrak{U}(w_\sigma)}(\mathbf{e}\begin{bmatrix}(\lambda,a)\\ (\mu,b)\end{bmatrix} + \mathfrak{J}_w) =& \sum_{\lambda,\mu\in\Lambda} \delta_{\lambda,\mu}\delta_{a,b}\delta_\mu\\ =&\delta_{a,b}\sum_{\lambda\in\Lambda}\delta_\lambda\\ =&\delta_{a,b}1_M;\\ \sum_{\lambda,\mu\in\Lambda} \pi_M^{\mathfrak{U}(w_\sigma)}(\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda,a)\\ (\mu,b)\end{bmatrix} + \mathfrak{J}_w) =& \sum_{\lambda,\mu\in\Lambda} \delta_{\lambda,\mu}\delta_{a,b}\delta_{\mu\deg(b)}\\ =&\delta_{a,b}\sum_{\lambda\in\Lambda}\delta_{\lambda\deg(b)}\\ =&\delta_{a,b}1_M.\end{aligned}$$

Thus it follows that

$$\begin{aligned}(\pi_M^{\mathfrak{U}(w_\sigma)} \circ \Theta)(\alpha) =& \sum_{\lambda,\mu\in\Lambda} \delta_{a_1,b_1}\cdots\delta_{a_m,b_m}\tilde{\zeta}((f(\lambda) \otimes g(\mu))\mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix})(1_M)\\ =&\delta_{a_1,b_1}\cdots\delta_{a_m,b_m}\sum_{\lambda\in\Lambda}(fg)(\lambda)_\sharp\delta_\lambda\\ =&\delta_{a_1,b_1}\cdots\delta_{a_m,b_m}fg.\end{aligned}$$

On the other hand, it follows that $\overline{\varepsilon}(L_{ab})(1_M) = \overline{\varepsilon}((L^{-1})_{ab})(1_M) = \delta_{a,b}1_M$ and we hence conclude that $\pi_M^{A_\sigma}(\alpha) = \delta_{a_1,b_1}\cdots\delta_{a_m,b_m}fg$. Thus the map Θ preserves the counit.

We will give the proof that Θ satisfies (1.11). The following lemma plays an important role in order to complete the proof.

Lemma 9.6. Let α and β be elements in A_σ satisfying

$$((\Theta\otimes\Theta)\circ\Delta_M^{A_\sigma})(\alpha) = (\Delta_M^{\mathfrak{U}(w_\sigma)}\circ\Theta)(\alpha); \tag{9.10}$$

$$((\Theta\otimes\Theta)\circ\Delta_M^{A_\sigma})(\beta) = (\Delta_M^{\mathfrak{U}(w_\sigma)}\circ\Theta)(\beta). \tag{9.11}$$

Then these α and β also satisfy

$$((\Theta\otimes\Theta)\circ\Delta_M^{A_\sigma})(\alpha\beta) = (\Delta_M^{\mathfrak{U}(w_\sigma)}\circ\Theta)(\alpha\beta) \tag{9.12}$$

By virtue of Lemma 9.6, it is sufficient to show that $((\Theta\otimes\Theta)\circ\Delta_M^{A_\sigma})(\alpha) = (\Delta_M^{\mathfrak{U}(w_\sigma)}\circ\Theta)(\alpha)$. Here this α is defined by (2.11). If $\alpha = f\otimes g + I_\sigma$ $(\forall f, g\in M)$,

$$\begin{aligned}
&(\Delta_M^{\mathfrak{U}(w_\sigma)}\circ\Theta)(f\otimes g+I_\sigma)\\
=&\sum_{\lambda,\mu\in\Lambda}\Delta_M^{\mathfrak{U}(w_\sigma)}((f(\lambda)\otimes g(\mu))\mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix}+\mathfrak{J}_w)\\
=&\sum_{\lambda,\mu,\tau\in\Lambda}(f(\lambda)\otimes 1_R)\mathbf{e}\begin{bmatrix}\lambda\\ \tau\end{bmatrix}+\mathfrak{J}_w\otimes(1_R\otimes g(\mu))\mathbf{e}\begin{bmatrix}\tau\\ \mu\end{bmatrix}+\mathfrak{J}_w\\
=&\sum_{\lambda,\mu,\tau,\nu\in\Lambda}t_M(\delta_\tau)((f(\lambda)\otimes 1_R)\mathbf{e}\begin{bmatrix}\lambda\\ \nu\end{bmatrix}+\mathfrak{J}_w)\otimes(1_R\otimes g(\mu))\mathbf{e}\begin{bmatrix}\tau\\ \mu\end{bmatrix}+\mathfrak{J}_w\\
=&\sum_{\lambda,\mu,\tau,\nu\in\Lambda}(f(\lambda)\otimes 1_R)\mathbf{e}\begin{bmatrix}\lambda\\ \nu\end{bmatrix}+\mathfrak{J}_w\otimes s_M(\delta_\tau)((1_R\otimes g(\mu))\mathbf{e}\begin{bmatrix}\tau\\ \mu\end{bmatrix}+\mathfrak{J}_w)\\
=&\sum_{\lambda,\mu,\tau,\nu\in\Lambda}(f(\lambda)\otimes 1_R)\mathbf{e}\begin{bmatrix}\lambda\\ \nu\end{bmatrix}+\mathfrak{J}_w\otimes(1_R\otimes g(\mu))\mathbf{e}\begin{bmatrix}\tau\\ \mu\end{bmatrix}+\mathfrak{J}_w\\
=&\Theta(f\otimes 1_M+I_\sigma)\otimes\Theta(1_M\otimes g+I_\sigma)\\
=&((\Theta\otimes\Theta)\circ\Delta_M^{A_\sigma})(f\otimes g+I_\sigma).
\end{aligned}$$

Suppose that $\alpha = (L^{-1})_{ab} + I_\sigma$ $(\forall a, b\in X)$. It follows from Lemma 6.4 that

$$\begin{aligned}
(\Delta_M^{\mathfrak{U}(w_\sigma)}\circ\Theta)((L^{-1})_{ab}+I_\sigma)=&\sum_{\substack{c\in X\\ \lambda,\mu\in\Lambda}}\Delta_M^{\mathfrak{U}(w_\sigma)}(\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda,a)\\ (\mu,b)\end{bmatrix}+\mathfrak{J}_w)\\
=&\sum_{\substack{c\in X\\ \lambda,\mu,\tau\in\Lambda}}\widetilde{\mathbf{e}}\begin{bmatrix}(\tau,c)\\ (\mu,b)\end{bmatrix}+\mathfrak{J}_w\otimes\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda,a)\\ (\tau,c)\end{bmatrix}+\mathfrak{J}_w\\
=&\sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}}t_M(\delta_{\tau\deg(c)})(\widetilde{\mathbf{e}}\begin{bmatrix}(\nu,c)\\ (\mu,b)\end{bmatrix}+\mathfrak{J}_w)\otimes\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda,a)\\ (\tau,c)\end{bmatrix}+\mathfrak{J}_w\\
=&\sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}}\widetilde{\mathbf{e}}\begin{bmatrix}(\nu,c)\\ (\mu,b)\end{bmatrix}+\mathfrak{J}_w\otimes s_M(\delta_{\tau\deg(c)})(\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda,a)\\ (\tau,c)\end{bmatrix}+\mathfrak{J}_w)\\
=&\sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}}\widetilde{\mathbf{e}}\begin{bmatrix}(\nu,c)\\ (\mu,b)\end{bmatrix}+\mathfrak{J}_w\otimes\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda,a)\\ (\tau,c)\end{bmatrix}+\mathfrak{J}_w\\
=&\sum_{c\in X}\Theta((L^{-1})_{cb}+I_\sigma)\otimes\Theta((L^{-1})_{ac}+I_\sigma)\\
=&((\Theta\otimes\Theta)\circ\Delta_M^{A_\sigma})((L^{-1})_{ab}+I_\sigma).
\end{aligned}$$

The proof for $\alpha = L_{ab} + I_\sigma$ is similar. Therefore the pair (Θ, id_M) is a left bialgebroid homomorphism.

We can also show that (Θ, id_M) is a right bialgebroid homomorphism in a similar way.

Finally, let us prove that the pair (Φ, id_M) is a left and right bialgebroid homomorphism.

We give the proof for (1.8). Since the map Θ is bijective,

$$\begin{aligned}\Phi \circ s_M^{\mathfrak{U}(w_\sigma)} =&\Phi \circ \Theta \circ s_M^{A_\sigma}\\ =&s_M^{A_\sigma}.\end{aligned}$$

The proof for (1.9)-(1.11) and (1.19)-(1.22) is similar. This is the desired conclusion. □

Proof of Lemma 9.6. For all $\alpha \in A_\sigma$, we fix an element $\overline{\alpha} \in k\langle Gen\rangle$ satisfying $\alpha = \overline{\alpha} + I_\sigma$. Let $\overline{\Delta}(\overline{\alpha}) = \overline{\alpha}_{[1]} \otimes \overline{\alpha}_{[2]}$. Since $\Delta_M^{A_\sigma}(\alpha) = \overline{\alpha}_{[1]} \otimes \overline{\alpha}_{[2]}$, the left hand side of (9.10) can be calculated as follows:

$$\begin{aligned}((\Theta \otimes \Theta) \circ \Delta_M^{A_\sigma})(\alpha) =&(\Theta \otimes \Theta)(\overline{\alpha}_{[1]} \otimes \overline{\alpha}_{[2]})\\ =&\Theta(\overline{\alpha}_{[1]}) \otimes \Theta(\overline{\alpha}_{[2]})\end{aligned}$$

For any $\eta \in \mathfrak{U}(w_\sigma)$, we similarly fix elements $\overline{\eta} \in k\langle \Lambda Q\rangle$ satisfying $\eta = \overline{\eta} + \mathfrak{J}_w$. By the notation of $\overline{\nabla}(\overline{\eta}) = \overline{\eta}_{[1]} \otimes \overline{\eta}_{[2]}$, it follows that $\Delta_M^{\mathfrak{U}(w_\sigma)}(\eta) = \overline{\eta}_{[1]} \otimes \overline{\eta}_{[2]}$. For the right hand side of (9.10), we hence conclude that

$$(\Delta_M^{\mathfrak{U}(w_\sigma)} \circ \Theta)(\alpha) = \overline{\Theta(\alpha)}_{[1]} \otimes \overline{\Theta(\alpha)}_{[2]}.$$

Therefore the conditions (9.10) and (9.11) are equivalent to

$$\Theta(\overline{\alpha}_{[1]}) \otimes \Theta(\overline{\alpha}_{[2]}) - \overline{\Theta(\alpha)}_{[1]} \otimes \overline{\Theta(\alpha)}_{[2]} \in \mathfrak{J}_2; \tag{9.13}$$

$$\Theta(\overline{\beta}_{[1]}) \otimes \Theta(\overline{\beta}_{[2]}) - \overline{\Theta(\beta)}_{[1]} \otimes \overline{\Theta(\beta)}_{[2]} \in \mathfrak{J}_2. \tag{9.14}$$

In a similar way to the proof of Lemma 2.7, we can write $\Delta_M^{A_\sigma}(\alpha\beta) = \overline{\alpha}_{[1]}\overline{\beta}_{[1]} \otimes \overline{\alpha}_{[2]}\overline{\beta}_{[2]}$. Thus $((\Theta \otimes \Theta) \circ \Delta_M^{A_\sigma})(\alpha\beta) = \Theta(\overline{\alpha}_{[1]})\Theta(\overline{\beta}_{[1]}) \otimes \Theta(\overline{\alpha}_{[2]})\Theta(\overline{\beta}_{[2]})$. Since $(\Delta_M^{\mathfrak{U}(w_\sigma)} \circ \Theta)(\alpha\beta) = \overline{\Theta(\alpha)}_{[1]}\overline{\Theta(\beta)}_{[1]} \otimes \overline{\Theta(\alpha)}_{[2]}\overline{\Theta(\beta)}_{[2]}$, (9.12) is equivalent to

$$\begin{aligned}&(\Theta(\overline{\alpha}_{[1]}) \otimes \Theta(\overline{\alpha}_{[2]}))(\Theta(\overline{\beta}_{[1]}) \otimes \Theta(\overline{\beta}_{[2]}))\\ &- (\overline{\Theta(\alpha)}_{[1]} \otimes \overline{\Theta(\alpha)}_{[2]})(\overline{\Theta(\beta)}_{[1]} \otimes \overline{\Theta(\beta)}_{[2]}) \in \mathfrak{J}_2\end{aligned} \tag{9.15}$$

Let us suppose that α and $\beta \in A_\sigma$ satisfy (9.13) and (9.14). For all $\alpha \in A_\sigma$, we write $j_\alpha = \Theta(\overline{\alpha}_{[1]}) \otimes \Theta(\overline{\alpha}_{[2]}) - \overline{\Theta(\alpha)}_{[1]} \otimes \overline{\Theta(\alpha)}_{[2]} \in \mathfrak{J}_2$. It follows that

$$\begin{aligned}&(\Theta(\overline{\alpha}_{[1]}) \otimes \Theta(\overline{\alpha}_{[2]}))(\Theta(\overline{\beta}_{[1]}) \otimes \Theta(\overline{\beta}_{[2]}))\\ &- (\overline{\Theta(\alpha)}_{[1]} \otimes \overline{\Theta(\alpha)}_{[2]})(\overline{\Theta(\beta)}_{[1]} \otimes \overline{\Theta(\beta)}_{[2]})\\ =&(\Theta(\overline{\alpha}_{[1]}) \otimes \Theta(\overline{\alpha}_{[2]}))(\Theta(\overline{\beta}_{[1]}) \otimes \Theta(\overline{\beta}_{[2]}))\\ &- (\Theta(\overline{\alpha}_{[1]}) \otimes \Theta(\overline{\alpha}_{[2]}) - j_\alpha)(\Theta(\overline{\beta}_{[1]}) \otimes \Theta(\overline{\beta}_{[2]}) - j_\beta)\\ =&j_\alpha(\Theta(\overline{\beta}_{[1]}) \otimes \Theta(\overline{\beta}_{[2]})) + (\Theta(\overline{\alpha}_{[1]}) \otimes \Theta(\overline{\alpha}_{[2]}))j_\beta - j_\alpha j_\beta.\end{aligned}$$

Thus it suffices to prove that $(\Theta(\overline{\alpha}_{[1]}) \otimes \Theta(\overline{\alpha}_{[2]}))j_\beta \in \mathfrak{J}_2$. Let $j_\beta = t_M^{\mathfrak{U}(w_\sigma)}(h) \otimes 1_{\mathfrak{U}(w_\sigma)} - 1_{\mathfrak{U}(w_\sigma)} \otimes s_M^{\mathfrak{U}(w_\sigma)}(h)$ $(\forall h \in M)$. Since Θ is a k-algebra homomorphism and $\Delta_M^{A_\sigma}$ satisfies (1.5), we have only to show the above fact for $\alpha \in A_\sigma$ defined by (2.11).

Suppose that $\alpha = f \otimes g + I_\sigma$ ($\forall f, g \in M$). The identities (1.1), (1.8), and (1.9) induce that

$$
\begin{aligned}
&(\Theta(\overline{\alpha}_{[1]}) \otimes \Theta(\overline{\alpha}_{[2]}))j_\beta \\
=&(\Theta(s_M^{A_\sigma}(f)) \otimes \Theta(t_M^{A_\sigma}(g)))(t_M^{\mathfrak{U}(w_\sigma)}(h) \otimes 1_{\mathfrak{U}(w_\sigma)} - 1_{\mathfrak{U}(w_\sigma)} \otimes s_M^{\mathfrak{U}(w_\sigma)}(h)) \\
=&(s_M^{\mathfrak{U}(w_\sigma)}(f) \otimes t_M^{\mathfrak{U}(w_\sigma)}(g))(t_M^{\mathfrak{U}(w_\sigma)}(h) \otimes 1_{\mathfrak{U}(w_\sigma)} - 1_{\mathfrak{U}(w_\sigma)} \otimes s_M^{\mathfrak{U}(w_\sigma)}(h)) \\
=&s_M^{\mathfrak{U}(w_\sigma)}(f)t_M^{\mathfrak{U}(w_\sigma)}(h) \otimes t_M^{\mathfrak{U}(w_\sigma)}(g) - s_M^{\mathfrak{U}(w_\sigma)}(f) \otimes t_M^{\mathfrak{U}(w_\sigma)}(g)s_M^{\mathfrak{U}(w_\sigma)}(h) \\
=&t_M^{\mathfrak{U}(w_\sigma)}(h)s_M^{\mathfrak{U}(w_\sigma)}(f) \otimes t_M^{\mathfrak{U}(w_\sigma)}(g) - s_M^{\mathfrak{U}(w_\sigma)}(f) \otimes s_M^{\mathfrak{U}(w_\sigma)}(h)t_M^{\mathfrak{U}(w_\sigma)}(g) \\
=&(t_M^{\mathfrak{U}(w_\sigma)}(h) \otimes 1_{\mathfrak{U}(w_\sigma)} - 1_{\mathfrak{U}(w_\sigma)} \otimes s_M^{\mathfrak{U}(w_\sigma)}(h))(s_M^{\mathfrak{U}(w_\sigma)}(f) \otimes t_M^{\mathfrak{U}(w_\sigma)}(g)) \in \mathfrak{I}_2.
\end{aligned}
$$

Let $\alpha = (L^{-1})_{ab} + I_\sigma$ ($\forall a, b \in X$). Lemma 6.4 induces that

$$
\begin{aligned}
&(\Theta(\overline{\alpha}_{[1]}) \otimes \Theta(\overline{\alpha}_{[2]}))j_\beta \\
=&\sum_{c \in X}(\Theta((L^{-1})_{cb} + I_\sigma) \otimes \Theta((L^{-1})_{ac} + I_\sigma)) \\
&\times (t_M^{\mathfrak{U}(w_\sigma)}(h) \otimes 1_{\mathfrak{U}(w_\sigma)} - 1_{\mathfrak{U}(w_\sigma)} \otimes s_M^{\mathfrak{U}(w_\sigma)}(h)) \\
=&\sum_{c \in X}(\sum_{\lambda,\mu,\tau,\nu \in \Lambda} \widetilde{\mathbf{e}}\begin{bmatrix}(\lambda, c)\\(\mu, b)\end{bmatrix} + \mathfrak{I}_w \otimes \widetilde{\mathbf{e}}\begin{bmatrix}(\tau, a)\\(\nu, c)\end{bmatrix} + \mathfrak{I}_w) \\
&\times (\sum_{\gamma,\eta \in \Lambda}(1_R \otimes h(\eta))\mathbf{e}\begin{bmatrix}\gamma\\\eta\end{bmatrix} + \mathfrak{I}_w \otimes 1_{\mathfrak{U}}(w) - 1_{\mathfrak{U}(w_\sigma)} \otimes (h(\gamma) \otimes 1_R)\mathbf{e}\begin{bmatrix}\gamma\\\eta\end{bmatrix} + \mathfrak{I}_w) \\
=&\sum_{\substack{c \in X\\ \lambda,\mu,\tau,\nu \in \Lambda}} (1_R \otimes h(\lambda))\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda, c)\\(\mu, b)\end{bmatrix} + \mathfrak{I}_w \otimes \widetilde{\mathbf{e}}\begin{bmatrix}(\tau, a)\\(\nu, c)\end{bmatrix} + \mathfrak{I}_w \\
&- \widetilde{\mathbf{e}}\begin{bmatrix}(\lambda, c)\\(\mu, b)\end{bmatrix} + \mathfrak{I}_w \otimes (h(\nu) \otimes 1_R)\widetilde{\mathbf{e}}\begin{bmatrix}(\tau, a)\\(\nu, c)\end{bmatrix} + \mathfrak{I}_w.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&(t_M^{\mathfrak{U}(w_\sigma)}(T_{\deg(c)^{-1}}(h)) \otimes 1_{\mathfrak{U}(w_\sigma)} - 1_{\mathfrak{U}(w_\sigma)} \otimes s_M^{\mathfrak{U}(w_\sigma)}(T_{\deg(c)^{-1}}(h))) \\
&\times (\sum_{c \in X} \Theta((L^{-1})_{cb} + I_\sigma) \otimes \Theta((L^{-1})_{ac} + I_\sigma)) \\
=&(\sum_{\gamma,\eta \in \Lambda}(1_R \otimes h(\eta \deg(c)^{-1}))\mathbf{e}\begin{bmatrix}\gamma\\\eta\end{bmatrix} + \mathfrak{I}_w \otimes 1_{\mathfrak{U}}(w) \\
&- 1_{\mathfrak{U}(w_\sigma)} \otimes (h(\gamma \deg(c)^{-1}) \otimes 1_R)\mathbf{e}\begin{bmatrix}\gamma\\\eta\end{bmatrix} + \mathfrak{I}_w) \\
&\times (\sum_{\substack{c \in X\\ \lambda,\mu,\tau,\nu \in \Lambda}} \widetilde{\mathbf{e}}\begin{bmatrix}(\lambda, c)\\(\mu, b)\end{bmatrix} + \mathfrak{I}_w \otimes \widetilde{\mathbf{e}}\begin{bmatrix}(\tau, a)\\(\nu, c)\end{bmatrix} + \mathfrak{I}_w) \\
=&\sum_{\substack{c \in X\\ \lambda,\mu,\tau,\nu \in \Lambda}} (1_R \otimes h(\lambda))\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda, c)\\(\mu, b)\end{bmatrix} + \mathfrak{I}_w \otimes \widetilde{\mathbf{e}}\begin{bmatrix}(\tau, a)\\(\nu, c)\end{bmatrix} + \mathfrak{I}_w \\
&- \widetilde{\mathbf{e}}\begin{bmatrix}(\lambda, c)\\(\mu, b)\end{bmatrix} + \mathfrak{I}_w \otimes (h(\nu) \otimes 1_R)\widetilde{\mathbf{e}}\begin{bmatrix}(\tau, a)\\(\nu, c)\end{bmatrix} + \mathfrak{I}_w.
\end{aligned}
$$

Therefore we can conclude that $(\Theta(\overline{\alpha}_{[1]}) \otimes \Theta(\overline{\alpha}_{[2]}))j_\beta \in \mathfrak{J}_2$. The proof for $\alpha = L_{ab} + I_\sigma$ is similar. □

Theorem 9.7. σ is rigid if and only if w is rigid.

Proof. We first suppose that w is rigid. For any $a, b \in X$, the elements x_{ab} and y_{ab} are defined by

$$x_{ab} = \sum_{\lambda,\mu\in\Lambda} \Phi(X_{(\lambda,a),(\mu,b)}); \quad y_{ab} = \sum_{\lambda,\mu\in\Lambda} \Phi(Y_{(\lambda,a),(\mu,b)}).$$

For all $\lambda, \mu \in \Lambda$, Lemma 6.4 induces that

$$\begin{aligned}\sum_{\tau\in\Lambda} \widetilde{\mathbf{e}}\begin{bmatrix}\tau\\ \mu\end{bmatrix}\mathbf{e}\begin{bmatrix}\lambda\\ \tau\end{bmatrix} + \mathfrak{J}_w =& \sum_{\tau\in\Lambda} \mathbf{e}\begin{bmatrix}\mu\\ \tau\end{bmatrix}\mathbf{e}\begin{bmatrix}\lambda\\ \tau\end{bmatrix} + \mathfrak{J}_w \\ =&\delta_{\lambda,\mu} \sum_{\tau\in\Lambda} \mathbf{e}\begin{bmatrix}\lambda\\ \tau\end{bmatrix} + \mathfrak{J}_w.\end{aligned}$$

Similarly, we can prove that $\sum_{\tau\in\Lambda} \mathbf{e}\begin{bmatrix}\tau\\ \mu\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}\lambda\\ \tau\end{bmatrix} + \mathfrak{J}_w = \delta_{\lambda,\mu} \sum_{\tau\in\Lambda} \mathbf{e}\begin{bmatrix}\tau\\ \lambda\end{bmatrix} + \mathfrak{J}_w$. On the other hand,

$$\begin{aligned}\sum_{\tau,\nu\in\Lambda} \mathbf{e}\begin{bmatrix}\tau\\ \mu\end{bmatrix}\widetilde{\mathbf{e}}\begin{bmatrix}\nu\\ \tau\end{bmatrix}\mathbf{e}\begin{bmatrix}\lambda\\ \nu\end{bmatrix} + \mathfrak{J}_w =& \sum_{\tau\in\Lambda} \mathbf{e}\begin{bmatrix}\tau\\ \mu\end{bmatrix}\mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} + \mathfrak{J}_w \\ =&\mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} + \mathfrak{J}_w\end{aligned}$$

for any $\lambda, \mu \in \Lambda$. Thus we can conclude that $X_{\lambda,\mu} = \mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} + \mathfrak{J}_w$ $(\forall \lambda, \mu \in \Lambda)$. The proof for $Y_{\lambda,\mu} = \widetilde{\mathbf{e}}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} + \mathfrak{J}_w$ is similar.

Since $X_{p,q}X_{p',q'} = \delta_{\mathfrak{t}(p),\mathfrak{s}(p')}\delta_{\mathfrak{t}(q),\mathfrak{s}(q')}X_{pp',qq'}$ and $Y_{p,q}Y_{p',q'} = \delta_{\mathfrak{t}(p'),\mathfrak{s}(p)}\delta_{\mathfrak{t}(q'),\mathfrak{s}(q)}Y_{p'p,q'q}$ $(\forall m, n \in \mathbb{Z}_{\geq 0}, \forall p, q \in Q^{(m)}, \forall p', q' \in Q^{(n)})$ in the proof of Proposition 8.3, we can calculate that

$$\begin{aligned}\sum_{c\in X}((L^{-1})_{cb} + I_\sigma)x_{ac} =& \sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} \Phi((\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda,c)\\ (\mu,b)\end{bmatrix} + \mathfrak{J}_w)X_{(\tau,a),(\nu,c)}) \\ =& \sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} \Phi((\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda,c)\\ (\mu,b)\end{bmatrix}\mathbf{e}\begin{bmatrix}\tau\\ \nu\end{bmatrix} + \mathfrak{J}_w)X_{(\tau,a),(\nu,c)}) \\ =& \sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} \delta_{\nu,\lambda}\delta_{\tau,\mu}\Phi((\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda,c)\\ (\mu,b)\end{bmatrix} + \mathfrak{J}_w)X_{(\tau,a),(\nu,c)}) \\ =& \sum_{\substack{c\in X\\ \lambda,\mu\in\Lambda}} \Phi((\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda,c)\\ (\mu,b)\end{bmatrix} + \mathfrak{J}_w)X_{(\mu,a),(\lambda,c)}) \\ =&\delta_{a,b} \sum_{\lambda,\mu\in\Lambda} \Phi(\mathbf{e}\begin{bmatrix}\mu\deg(a)\\ \lambda\end{bmatrix} + \mathfrak{J}_w) \\ =&\delta_{a,b}1_{A_\sigma}\end{aligned}$$

for any $a,b\in X$. Here we use Lemma 6.4 for the third equality.

On the other hand,

$$\begin{aligned}\sum_{c\in X} y_{cb}(L_{ac}+I_\sigma) =& \sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} \Phi(Y_{(\lambda,c),(\mu,b)}(\mathbf{e}\begin{bmatrix}(\tau,a)\\(\nu,c)\end{bmatrix}+\mathfrak{J}_w))\\ =& \sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} \Phi(Y_{(\lambda,c),(\mu,b)}(\mathbf{e}\begin{bmatrix}\tau\\\nu\end{bmatrix}\mathbf{e}\begin{bmatrix}(\tau,a)\\(\nu,c)\end{bmatrix}+\mathfrak{J}_w))\\ =& \sum_{\substack{c\in X\\ \lambda,\mu,\tau,\nu\in\Lambda}} \delta_{\nu,\lambda}\delta_{\tau,\mu}\Phi(Y_{(\lambda,c),(\mu,b)}(\mathbf{e}\begin{bmatrix}(\tau,a)\\(\nu,c)\end{bmatrix}+\mathfrak{J}_w))\\ =& \sum_{\substack{c\in X\\ \lambda,\mu\in\Lambda}} \Phi(Y_{(\lambda,c),(\mu,b)}(\mathbf{e}\begin{bmatrix}(\mu,a)\\(\lambda,c)\end{bmatrix}+\mathfrak{J}_w))\\ =&\delta_{a,b}\sum_{\lambda,\mu\in\Lambda}\Phi(\mathbf{e}\begin{bmatrix}\mu\deg(a)\\\lambda\end{bmatrix}+\mathfrak{J}_w)\\ =&\delta_{a,b}1_{A_\sigma}.\end{aligned}$$

The proof for the other identities is similar.

Let us suppose that σ is rigid. We give the proof only for (8.1) and (8.3). For any $m\in\mathbb{Z}_{\geq 0}$, $p=((\lambda_1,a_1),\cdots,(\lambda_m,a_m))$, and $q=((\mu_1,b_1),\cdots,(\mu_m,b_m))\in Q^{(m)}$, we set

$$\begin{aligned}X_{p,q}=&\Theta((\delta_{\mathfrak{s}(p)}\otimes\delta_{\mathfrak{s}(q)}+I_\sigma)x_{a_1b_1}\cdots x_{a_mb_m});\\ Y_{p,q}=&\Theta(y_{a_mb_m}\cdots y_{a_1b_1}(\delta_{\mathfrak{s}(q)}\otimes\delta_{\mathfrak{s}(p)}+I_\sigma)).\end{aligned}$$

We will check (8.1). $p=q$ is equivalent to $\lambda_1=\mu_1$ and $a_i=b_i$ for all $i\in\{1,\cdots,m\}$. Thus the generators (3) induce that

$$\begin{aligned}&\sum_{u\in Q^{(m)}}(\widetilde{\mathbf{e}}\begin{bmatrix}u\\q\end{bmatrix}+\mathfrak{J}_w)X_{p,u}\\ =&\sum_{\substack{\tau\in\Lambda\\ c_1,\cdots,c_m\in X}}\Theta(((L^{-1})_{c_mb_m}\cdots(L^{-1})_{c_1b_1}(\delta_{\mathfrak{s}(q)}\otimes\delta_\tau)+I_\sigma)\\ &\times(\delta_{\mathfrak{s}(p)}\otimes\delta_\tau+I_\sigma)x_{a_1c_1}\cdots x_{a_mc_m})\\ =&\delta_{\mathfrak{s}(p),\mathfrak{s}(q)}\sum_{c_1,\cdots,c_m\in X}\Theta(((L^{-1})_{c_mb_m}\cdots(L^{-1})_{c_1b_1}(\delta_{\mathfrak{s}(q)}\otimes 1_M)+I_\sigma)x_{a_1c_1}\cdots x_{a_mc_m})\\ =&\delta_{\mathfrak{s}(p),\mathfrak{s}(q)}\sum_{c_1,\cdots,c_m\in X}\Theta(((\delta_{\mathfrak{t}(q)}\otimes 1_M)(L^{-1})_{c_mb_m}\cdots(L^{-1})_{c_1b_1}+I_\sigma)x_{a_1c_1}\cdots x_{a_mc_m})\\ =&\delta_{\lambda_1,\mu_1}\delta_{a_1,b_1}\cdots\delta_{a_m,b_m}\Theta(\delta_{\mathfrak{t}(q)}\otimes 1_M+I_\sigma)\\ =&\delta_{p,q}\sum_{\lambda\in\Lambda}\mathbf{e}\begin{bmatrix}\mathfrak{t}(q)\\\lambda\end{bmatrix}+\mathfrak{J}_w.\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\sum_{u\in Q^{(m)}} Y_{u,q}(\widetilde{\mathbf{e}}\begin{bmatrix} p \\ u \end{bmatrix} + \mathfrak{J}_w) \\
=&\sum_{\substack{\tau\in\Lambda \\ c_1,\cdots,c_m\in X}} \Theta(y_{c_m b_m}\cdots y_{c_1 b_1}(\delta_{\mathfrak{s}(q)}\otimes\delta_\tau + I_\sigma) \\
&\times(\delta_{\mathfrak{s}(p)}\otimes\delta_\tau)L_{a_1 c_1}\cdots L_{a_m c_m} + I_\sigma) \\
=&\delta_{\mathfrak{s}(p),\mathfrak{s}(q)}\sum_{c_1,\cdots,c_m\in X}\Theta(y_{c_m b_m}\cdots y_{c_1 b_1}((\delta_{\mathfrak{s}(p)}\otimes 1_M)L_{a_1 c_1}\cdots L_{a_m c_m} + I_\sigma)) \\
=&\delta_{\mathfrak{s}(p),\mathfrak{s}(q)}\sum_{c_1,\cdots,c_m\in X}\Theta(y_{c_m b_m}\cdots y_{c_1 b_1}(L_{a_1 c_1}\cdots L_{a_m c_m}(\delta_{\mathfrak{t}(p)}\otimes 1_M) + I_\sigma)) \\
=&\delta_{\lambda_1,\mu_1}\delta_{a_1,b_1}\cdots\delta_{a_m,b_m}\Theta(\delta_{\mathfrak{t}(p)}\otimes 1_M + I_\sigma) \\
=&\delta_{p,q}\sum_{\lambda\in\Lambda}\mathbf{e}\begin{bmatrix}\mathfrak{t}(p) \\ \lambda\end{bmatrix} + \mathfrak{J}_w
\end{aligned}
$$

Thus (8.1) is proved. The same proof works for (8.2).

Finally we show (8.3). By using the calculations of the proof for Proposition 4.3 and (8.1), we can calculate that

$$
\begin{aligned}
&\sum_{u,v\in Q^{(m)}} X_{u,q}(\widetilde{\mathbf{e}}\begin{bmatrix} v \\ u \end{bmatrix} + \mathfrak{J}_w)X_{p,v} \\
=&\sum_{\substack{\lambda\in\Lambda \\ u\in Q^{(m)}}} \delta_{p,u}X_{u,q}(\mathbf{e}\begin{bmatrix}\mathfrak{t}(p) \\ \lambda\end{bmatrix} + \mathfrak{J}_w) \\
=&\sum_{\lambda\in\Lambda} X_{p,q}(\mathbf{e}\begin{bmatrix}\mathfrak{t}(p) \\ \lambda\end{bmatrix} + \mathfrak{J}_w) \\
=&\sum_{\lambda\in\Lambda} X_{p,q}X_{\mathfrak{t}(p),\lambda} \\
=&\Theta((\delta_{\mathfrak{s}(p)}\otimes\delta_{\mathfrak{s}(q)} + I_\sigma)x_{a_1 b_1}\ldots x_{a_m b_m}(\delta_{\mathfrak{t}(p)}\otimes 1_M + I_\sigma)) \\
=&\Theta((\delta_{\mathfrak{s}(p)}\otimes\delta_{\mathfrak{s}(q)} + I_\sigma)x_{a_1 b_1}\ldots x_{a_m b_m}) \\
=&X_{p,q}.
\end{aligned}
$$

The proof for (8.4) is similar. This completes the proof. □

Theorem 9.8. If σ is rigid, then these (Φ, id_M) and (Θ, id_M) are strict Hopf algebroid isomorphisms.

Proof. It is sufficient to show that $\Theta\circ S_{A_\sigma} = S_{\mathfrak{U}(w_\sigma)}\circ\Theta$. For any $f, g\in M$, Lemma 6.4 induces that

$$
\begin{aligned}
(S_{\mathfrak{U}(w_\sigma)} \circ \Theta)(f \otimes g + I_\sigma) &= \sum_{\lambda,\mu\in\Lambda} S_{\mathfrak{U}(w_\sigma)}((f(\lambda) \otimes g(\mu))\mathbf{e}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} + \mathfrak{J}_w) \\
&= \sum_{\lambda,\mu\in\Lambda} (g(\mu) \otimes f(\lambda))\mathbf{e}\begin{bmatrix}\mu\\ \lambda\end{bmatrix} + \mathfrak{J}_w \\
&= \Theta(g \otimes f + I_\sigma) \\
&= (\Theta \circ S_{A_\sigma})(f \otimes g + I_\sigma).
\end{aligned}
$$

Let a and b be arbitrary elements in X. It follows that

$$
\begin{aligned}
(S_{\mathfrak{U}(w_\sigma)} \circ \Theta)((L^{-1})_{ab} + I_\sigma) &= \sum_{\lambda,\mu\in\Lambda} S_{\mathfrak{U}(w_\sigma)}(\widetilde{\mathbf{e}}\begin{bmatrix}(\lambda,a)\\ (\mu,b)\end{bmatrix} + \mathfrak{J}_w) \\
&= \sum_{\lambda,\mu\in\Lambda} X_{(\lambda,a),(\mu,b)} \\
&= \sum_{\lambda,\mu\in\Lambda} \Theta((\delta_\lambda \otimes \delta_\mu + I_\sigma)x_{ab}) \\
&= \Theta(x_{ab}) \\
&= (\Theta \circ S_{A_\sigma})((L^{-1})_{ab} + I_\sigma).
\end{aligned}
$$

The proof for $(\Theta \circ S_{A_\sigma})(L_{ab} + I_\sigma) = (S_{\mathfrak{U}(w_\sigma)} \circ \Theta)(L_{ab} + I_\sigma)$ is similar. This completes the proof. □

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