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博士学位論文

Congruence relations for p -adic hypergeometric
functions $\widehat{\mathcal{F}}_{a,\dots,a}^{(\sigma)}(t)$ and its transformation formula

(p 進超幾何関数の合同関係と変数変換公式)

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Congruence relations for p -adic hypergeometric functions $\widehat{\mathcal{F}}_{a,\dots,a}^{(\sigma)}(t)$ and its transformation formula

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Abstract

We introduce new kinds of p -adic hypergeometric functions. We show these functions satisfy congruence relations similar to Dwork's p -adic hypergeometric functions, so they are convergent functions. We also show that there is a transformation formula between our new p -adic hypergeometric functions and p -adic hypergeometric functions of logarithmic type defined in [1] in a particular case.

Keywords— p -adic hypergeometric functions, p -adic hypergeometric functions of logarithmic type, congruence relations

MSC Classification Codes: 33E50

1 Introduction

The classical hypergeometric function of one-variable is defined to be the power series

$${}_sF_{s-1} \left(\begin{matrix} a_1, \dots, a_s \\ b_1, \dots, b_{s-1} \end{matrix}; t \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_s)_k}{(b_1)_k \cdots (b_{s-1})_k} \frac{t^k}{k!}$$

where $(\alpha)_k$ denotes the Pochhammer symbol (cf. [11]). Let $(a_1, \dots, a_s) \in \mathbb{Z}_p^s$ be a s -tuple of p -adic integers, and consider the series

$$F_{a_1, \dots, a_s}(t) = {}_sF_{s-1} \left(\begin{matrix} a_1, \dots, a_s \\ 1, \dots, 1 \end{matrix}; t \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k}{k!} \cdots \frac{(a_s)_k}{k!} t^k.$$

This is a formal power series with $\mathbb{Z}_p$ -coefficients. In the paper [4], B. Dwork introduced his p -adic hypergeometric functions. Let a' be the Dwork prime of a , which is defined to be $(a+l)/p$ where $l \in \{0, 1, \dots, p-1\}$ is the unique integer such that $a+l \equiv 0 \pmod{p}$. Then Dwork proved his p -adic hypergeometric function $\mathcal{F}_{a_1, \dots, a_s}^{\text{Dw}}(t)$ (Definition 4.16) satisfies the congruence relations

$$\mathcal{F}_{a_1, \dots, a_s}^{\text{Dw}}(t) \equiv \frac{F_{a_1, \dots, a_s}(t)_{< p^n}}{[F_{a'_1, \dots, a'_s}(tp)]_{< p^n}} \pmod{p^n \mathbb{Z}_p[[t]]}$$

where for a power series $f(t) = \sum_{n=0}^{\infty} A_n t^n$, we denote $f(t)_{<m} := \sum_{n<m} A_n t^n$ the truncated polynomial. As a consequence, his function is p -adically analytic in the sense of Krasner (i.e. an element of the Tate algebra, [5, 3.1]), and one can define the value at $t = \alpha$ with $|\alpha|_p = 1$. Dwork applied his function to the unit root formula of elliptic curve of Legendre type (e.g. [8, §7]).

Recently, M. Asakura introduced a new function which he calls p -adic hypergeometric functions of logarithmic type. Let $W = W(\overline{\mathbb{F}}_p)$ denote the Witt ring, and $K = \text{Frac} W$ its fraction field (e.g. [10, Ch II, §6]). Let $\sigma : W[[t]] \rightarrow W[[t]]$ be a p -th Frobenius given by $\sigma(t) = ct^p$ with $c \in 1 + pW$:

$$\left(\sum_i a_i t^i \right)^\sigma = \sum_i a_i^F c^i t^{ip}$$

where $F : W \rightarrow W$ is the Frobenius on the Witt ring. Then we can define Asakura's function $\mathcal{F}_{a_1, \dots, a_s}^{(\sigma)}(t)$ (see Definition 4.5 or [1, §3.1]). If we write $\mathcal{F}_{a_1, \dots, a_s}^{(\sigma)}(t) = G_{a_1, \dots, a_s}(t) / F_{a_1, \dots, a_s}(t)$, these p -adic hypergeometric functions of logarithmic type satisfy congruence relations

$$\mathcal{F}_{a_1, \dots, a_s}^{(\sigma)}(t) \equiv \frac{G_{a_1, \dots, a_s}(t)_{<p^n}}{F_{a_1, \dots, a_s}(t)_{<p^n}} \pmod{p^n W[[t]]}.$$

which are similar to Dwork's. Asakura studied his functions from the viewpoint of p -adic regulators. In particular, some values of $\mathcal{F}_{a_1, \dots, a_s}^{(\sigma)}(t)$ are expected to be the special values of p -adic L -function of elliptic curves over $\mathbb{Q}$ ([1, §5]).

In this paper, we introduce new p -adic hypergeometric functions which we denote by $\widehat{\mathcal{F}}_{a, \dots, a}^{(\sigma)}(t)$ (Definition 2.1). Let $a_1 = \dots = a_s = a$. Then our function is defined as

$$\widehat{\mathcal{F}}_{a, \dots, a}^{(\sigma)}(t) := \frac{t^{-a}}{F_{a, \dots, a}(t)} \int_0^t (t^a F_{a, \dots, a}(t) - (-1)^{se} [t^{a'} F_{a', \dots, a'}(t)]^\sigma) \frac{dt}{t}.$$

Write $\widehat{\mathcal{F}}_{a, \dots, a}^{(\sigma)}(t) = \widehat{G}_{a, \dots, a}^{(\sigma)}(t) / F_{a, \dots, a}(t)$. Our first main result is the following congruence relations (Theorem 3.1)

$$\widehat{\mathcal{F}}_{a, \dots, a}^{(\sigma)}(t) \equiv \frac{\widehat{G}_{a, \dots, a}^{(\sigma)}(t)_{<p^n}}{F_{a, \dots, a}(t)_{<p^n}} \pmod{p^n W[[t]]}.$$

The second main result is the transformation formula (=Theorem 4.14). Let $\sigma(t) = ct^p$ and $\widehat{\sigma}(t) = c^{-1}t^p$. We will prove that

$$\mathcal{F}_{a, a}^{(\sigma)}(t) = -\widehat{\mathcal{F}}_{a, a}^{(\widehat{\sigma})}(t^{-1})$$

in case that $a \in \frac{1}{N}\mathbb{Z}$, $0 < a < 1$ and $p > N$ which comes from geometric observation (see next paragraph). We then generally conjecture a formula

$$\mathcal{F}_{a,\dots,a}^{(\sigma)}(t) = -\widehat{\mathcal{F}}_{a,\dots,a}^{(\hat{\sigma})}(t^{-1})$$

between $\mathcal{F}_{a,\dots,a}^{(\sigma)}(t)$ and $\widehat{\mathcal{F}}_{a,\dots,a}^{(\hat{\sigma})}(t)$, which we refer to as the transformation formula (Conjecture 4.12) and prove our theorem as a special case.

The strategy of the proof of transformation formula is as follows. Let N, A be integers with $N \geq 2$, $p > N$, $1 \leq A < N$ and $\gcd(A, N) = 1$. We consider the fibration $f : Y \rightarrow \mathbb{P}^1$ over $K = \text{Frac}(W)$ whose general fiber $f^{-1}(t)$ is the projective nonsingular model of

$$y^N = x^A(1-x)^A(1-(1-t)x)^{N-A},$$

and put $X_0 := f^{-1}(S)$ where $S_K := \text{Spec}K[t, (t-t^2)^{-1}]$.

Let ξ be the K_2 -symbol defined in (4.14). Asakura showed that his $\mathcal{F}_{a_1,\dots,a_s}^{(\sigma)}(t)$ appears in the p -adic regulator of ξ . The key step in our proof is to provide an alternative description of the p -adic regulator by our $\widehat{\mathcal{F}}_{a,\dots,a}^{(\sigma)}(t)$. A key ingredient is the different description of X via $(z, w, s_0) = (1-x, t_0^{A-N}y, t_0^{-1})$ with

$$X : y^N = x^A(1-x)^A(1-(1-t_0^N)x)^{N-A}$$

and

$$\widehat{X} : w^N = z^A(1-z)^A(1-(1-s_0^N)z)^{N-A}$$

with $t = t_0^N$.

We also conjecture a similar transformation formula for Dwork's hypergeometric functions (Conjecture 4.18). We attach a proof in case $s = 2$, $a \in \frac{1}{N}\mathbb{Z}$, $0 < a < 1$ and $p > N$ which is based on a similar idea to the above (Theorem 4.20).

This paper is organized as follows. In §2, we give the definition of $\widehat{\mathcal{F}}_{a,\dots,a}^{(\sigma)}(t)$. In §3, we prove the congruence relations. In §4, we give the proof of the transformation formula. We also give a brief review on hypergeometric curves and some results in [1] which are necessary in the proof.

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2 Definition of $\widehat{\mathcal{F}}_{a,\dots,a}^{(\sigma)}(t)$

Let $W = W(\overline{\mathbb{F}}_p)$ denote the Witt ring, and $K = \text{Frac} W$ denote its fraction field. Let $\sigma : W[[t]] \rightarrow W[[t]]$ be a p -th Frobenius given by $\sigma(t) = ct^p$ with $c \in 1 + pW$:

$$\left(\sum_i a_i t^i \right)^\sigma = \sum_i a_i^F c^i t^{ip} \quad (2.1)$$

where $F : W \rightarrow W$ is the Frobenius on W . Let $a \in \mathbb{Z}_p$. Let a' be the Dwork prime of a , which is defined to be $(a + l)/p$ where $l \in \{0, 1, \dots, p-1\}$ is the unique integer such that $a + l \equiv 0 \pmod{p}$. We denote the i -th Dwork prime by $a^{(i)}$ which is defined to be $(a^{(i-1)})'$ with $a^{(0)} = a$.

Let

$$F_{a,\dots,a}(t) = \sum_{k=0}^{\infty} \left(\frac{(a)_k}{k!} \right)^s t^k, \quad F_{a',\dots,a'}(t) = \sum_{k=0}^{\infty} \left(\frac{(a')_k}{k!} \right)^s t^k \quad (2.2)$$

be the hypergeometric series for $a \in \mathbb{Z}_p$, where $(a)_k$ denotes the Pochhammer symbol (i.e. $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$ when $k \neq 0$ and $(\alpha)_0 = 1$).

Put

$$q := \begin{cases} 4 & p = 2 \\ p & p \geq 3. \end{cases}$$

Let $l' \in \{0, 1, \dots, q-1\}$ be the unique integer such that $a + l' \equiv 0 \pmod{q}$.

Put

$$e := l' - \left\lfloor \frac{l'}{p} \right\rfloor.$$

Define a power series

$$\begin{aligned} \widehat{G}_{a,\dots,a}^{(\sigma)}(t) &:= t^{-a} \int_0^t (t^a F_{a,\dots,a}(t) - (-1)^{se} [t^{a'} F_{a',\dots,a'}(t)]^\sigma) \frac{dt}{t} \\ &= \sum_{k=0}^{\infty} B_k t^k \end{aligned}$$

for $a \in \mathbb{Z}_p \setminus \mathbb{Z}_{\leq 0}$. Here we think t^α to be an abstract symbol with relations $t^\alpha \cdot t^\beta = t^{\alpha+\beta}$, on which σ acts by $\sigma(t^\alpha) = c^\alpha t^{p\alpha}$, where

$$c^\alpha = (1 + pu)^\alpha := \sum_{i=0}^{\infty} \binom{\alpha}{i} p^i u^i, \quad u \in W.$$

Moreover we think $\int_0^t (-) \frac{dt}{t}$ to be an operator such that

$$\int_0^t t^\alpha \frac{dt}{t} = \frac{t^\alpha}{\alpha}, \quad \alpha \neq 0.$$

Definition 2.1. *Define*

$$\widehat{\mathcal{F}}_{a,\dots,a}^{(\sigma)}(t) := \frac{\widehat{G}_{a,\dots,a}^{(\sigma)}(t)}{F_{a,\dots,a}(t)}, \quad a \in \mathbb{Z}_p \setminus \mathbb{Z}_{\leq 0}.$$

If we write $F_{a,\dots,a}(t) = \sum A_k t^k$ and $F_{a',\dots,a'}(t) = \sum A_k^{(1)} t^k$, then we have

$$B_k = \frac{1}{k+a} \left(A_k - (-1)^{se} (A_{\frac{k-l}{p}}^{(1)}) c^{\frac{k+a}{p}} \right),$$

where $A_{\frac{m}{p}}^{(1)} = 0$ if $m \not\equiv 0 \pmod{p}$ or $m < 0$. In fact, we have $\widehat{\mathcal{F}}_{a,\dots,a}^{(\sigma)}(t)$ and $\widehat{G}_{a,\dots,a}^{(\sigma)}(t)$ are power series with W -coefficients from the following lemma.

Lemma 2.2. *We have*

$$B_k \in W, \quad \forall k \in \mathbb{Z}_{\geq 0}.$$

Proof. If $k \not\equiv l \pmod{p}$, then $k+a \not\equiv 0 \pmod{p}$. Hence we have

$$B_k = \frac{A_k}{k+a} \in W.$$

For $k \equiv l \pmod{p}$, we write $k+a = p^n m$ with $p \nmid m$. Then

$$c^{\frac{k+a}{p}} = c^{p^{n-1}m} \equiv 1 \pmod{p^n}$$

since $c \in 1 + pW$. So if we can show

$$A_k \equiv (-1)^{se} A_{\frac{k-l}{p}}^{(1)} \pmod{p^n}$$

then $B_k \in W$. Indeed, for any p -adic integer $\alpha \in \mathbb{Z}_p$ and $n \in \mathbb{Z}_{\geq 1}$ we define

$$\{\alpha\}_n := \prod_{\substack{1 \leq i \leq n \\ p \nmid (\alpha+i-1)}} (\alpha+i-1).$$

Using [1, Lemma 3.6], one have

$$\frac{(a)_k}{k!} = \left(\frac{(a')_{\lfloor k/p \rfloor}}{\lfloor k/p \rfloor!} \right)^{-1} \frac{\{a\}_k}{\{1\}_k} = \frac{(a')_{\frac{k-l}{p}}}{(\frac{k-l}{p})!} \frac{\{a\}_k}{\{1\}_k}.$$

Therefore we obtain

$$A_k = A_{\frac{k-l}{p}}^{(1)} \left(\frac{\{a\}_k}{\{1\}_k} \right)^s.$$

Since $k + a \equiv 0 \pmod{p^n}$, we have that $\{a\}_k \equiv \{-k\}_k \pmod{p^n}$. Therefore

$$\begin{aligned} \frac{\{a\}_k}{\{1\}_k} &\equiv \frac{\{-k\}_k}{\{1\}_k} \\ &\equiv \frac{\prod_{\substack{1 \leq i \leq k \\ p \nmid i}} (-k + i - 1)}{\prod_{\substack{1 \leq i \leq k \\ p \nmid i}} i} \\ &\equiv \prod_{\substack{i=1 \\ p \nmid i}}^k (-1) = (-1)^{k - \lfloor k/p \rfloor} \pmod{p^n}. \end{aligned}$$

Now we claim $(-1)^{k - \lfloor k/p \rfloor} = (-1)^e$.

- For $p \geq 3$, write $k = l + bp$. Then

$$k - \lfloor k/p \rfloor = l + bp - \lfloor l/p + b \rfloor = e + b(p - 1) \equiv e \pmod{2}.$$

Therefore $(-1)^{k - \lfloor k/p \rfloor} = (-1)^e$.

- For $p = 2$, write $k + a = 2^n m$, $2 \nmid m$.

(1) If $n = 1$, we have $(-1)^{k - \lfloor k/p \rfloor} = (-1)^e$.

(2) If $n \geq 2$, we have $k + a \equiv 0 \pmod{4}$. Therefore $k \equiv L \pmod{4}$ where $L \in \{0, 1, 2, 3\}$ is the unique integer such that $a + L \equiv 0 \pmod{4}$. Write $k = L + 4b$, then

$$k - \lfloor k/2 \rfloor = L + 4b - \lfloor L/2 \rfloor - 2b \equiv e \pmod{2}.$$

Again, we obtain $(-1)^{k - \lfloor k/p \rfloor} = (-1)^e$.

Therefore

$$A_k \equiv A_{\frac{k-l}{p}}^{(1)} (-1)^{es} \pmod{p^n}.$$

This completes the proof. $\square$

3 Congruence Relations

For a power series $f(t) = \sum_{n=0}^{\infty} A_n t^n$, we denote $f(t)_{<m} := \sum_{n < m} A_n t^n$ the truncated polynomial. Then we have the following theorem which we call the congruence relations of $\widehat{\mathcal{F}}_{a,\dots,a}^{(\sigma)}(t)$.

Theorem 3.1. *Let $a \in \mathbb{Z}_p \setminus \mathbb{Z}_{\leq 0}$ and suppose $c \in 1 + qW$. Then*

$$\widehat{\mathcal{F}}_{a,\dots,a}^{(\sigma)}(t) \equiv \frac{\widehat{G}_{a,\dots,a}^{(\sigma)}(t)_{<p^n}}{F_{a,\dots,a}(t)_{<p^n}} \pmod{p^n W[[t]]}$$

for all $n \in \mathbb{Z}_{\geq 0}$.

Corollary 3.2. *If $a^{(r)} = a$ for some $r > 0$, where $(-)^{(r)}$ denote the r -th Dwork prime, then*

$$\widehat{\mathcal{F}}_{a,\dots,a}^{(\sigma)}(t) \in W\langle t, t^{-1}, h(t)^{-1} \rangle, \quad h(t) := \prod_{i=0}^{r-1} F_{a^{(i)}, \dots, a^{(i)}}(t)_{<p}$$

is a convergent function, where $W\langle t, t^{-1}, h(t)^{-1} \rangle := \varprojlim_n (W/p^n[t, t^{-1}, h(t)^{-1}])$.

Proof. It suffices to show that modulo p^n

$$\frac{1}{F_{a,\dots,a}(t)_{<p^n}}$$

lies in $W/p^n[t, t^{-1}, h(t)^{-1}]$. From Dwork congruence [4, p.37, Theorem 2], we have

$$F_{a,\dots,a}(t)_{<p} \equiv \frac{F_{a,\dots,a}(t)_{<p^n}}{[F_{a',\dots,a'}(t^p)]_{<p^n}} \pmod{p\mathbb{Z}_p[[t]]}.$$

Thus one has

$$F_{a,\dots,a}(t)_{<p^n} \equiv F_{a,\dots,a}(t)_{<p} [F_{a',\dots,a'}(t^p)]_{<p^n} \pmod{p\mathbb{Z}_p[[t]]}.$$

Using Dwork congruence again, we obtain

$$\begin{aligned} & F_{a,\dots,a}(t)_{<p} [F_{a',\dots,a'}(t^p)]_{<p^n} \\ & \equiv F_{a,\dots,a}(t)_{<p} [F_{a',\dots,a'}(t^p)]_{<p^2} [F_{a^{(2)},\dots,a^{(2)}}(t^{p^2})]_{<p^n} \pmod{p\mathbb{Z}_p[[t]]}. \end{aligned}$$

So by applying Dwork congruence inductively, we have $F_{a,\dots,a}(t)_{<p^n}$ is congruent to

$$F_{a,\dots,a}(t)_{<p} \cdots [F_{a^{(i)},\dots,a^{(i)}}(t^{p^i})]_{<p^{i+1}} \cdots [F_{a^{(n-1)},\dots,a^{(n-1)}}(t^{p^{n-1}})]_{<p^n}$$

modulo $p\mathbb{Z}_p[[t]]$. Since both sides of the equation are polynomials, $p\mathbb{Z}_p[[t]]$ can be replaced by $p\mathbb{Z}_p[t]$. Now we use the fact that for any $F(x) \in \mathbb{Z}_p[[t]]$, one has

$$F(x)_{<p^{n-1}}|_{x=t^p} \equiv \left(F(x)_{<p^{n-1}}|_{x=t} \right)^p = \left(F(t)_{<p^{n-1}} \right)^p \pmod{p\mathbb{Z}_p[t]}.$$

Then by induction, we have

$$[F_{a^{(i)},\dots,a^{(i)}}(t^{p^i})]_{<p^{i+1}} \equiv [F_{a^{(i)},\dots,a^{(i)}}(t)_{<p}]^{p^i} \pmod{p\mathbb{Z}_p[t]}$$

Therefore we have $F_{a,\dots,a}(t)_{<p^n}$ is congruent to

$$F_{a,\dots,a}(t)_{<p} \cdots [F_{a^{(i)},\dots,a^{(i)}}(t)_{<p}]^{p^i} \cdots [F_{a^{(n-1)},\dots,a^{(n-1)}}(t)_{<p}]^{p^{n-1}} \pmod{p\mathbb{Z}_p[t]}.$$

So we can write

$$F_{a,\dots,a}(t)_{<p^n} = F_{a,\dots,a}(t)_{<p} \cdots \left[F_{a^{(i)},\dots,a^{(i)}}(t)_{<p} \right]^{p^i} \cdots \left[F_{a^{(n-1)},\dots,a^{(n-1)}}(t)_{<p} \right]^{p^{n-1}} + pg(t)$$

for some $g(t) \in \mathbb{Z}_p[t]$. Then considering $(F_{a,\dots,a}(t)_{<p^n})^{-1}$ modulo p^n , we see that it lies in $W/p^n[t, t^{-1}, h(t)^{-1}]$. $\square$

3.1 Proof of Congruence Relations: Lipschitz Functions and Congruence Relations for B_k/A_k

Here we introduce a notion called Lipschitz functions.

Definition 3.3. Let W be the Witt ring of $\overline{\mathbb{F}}_p$. A function $f : \mathbb{Z}_{\geq 0} \rightarrow \text{Frac}(W)$ is called Lipschitz if it takes values in W and $p^s | (n - m)$ implies $p^s | (f(n) - f(m))$ for any $n, m, s \in \mathbb{Z}_{\geq 0}$.

Remark 3.4. Generally, one can define Lipschitz functions as functions from X to K where K denote a complete extension of $\mathbb{Q}_p$ and $X \subseteq K$ is a subset with no isolated point and satisfy

$$|f(x) - f(y)|_p \leq M|x - y|_p$$

for some constant M and $x, y \in X$ (See [9, §5.1]). In our case, $M = 1$.

In this section, we fix $a \in \mathbb{Z}_p \setminus \mathbb{Z}_{\leq 0}$. The main result of this section is the following lemma.

Lemma 3.5. The function defined by $k \mapsto B_k/A_k$ is Lipschitz.

Before proving Lemma 3.5, we need some preliminary lemmas.

Lemma 3.6. Polynomial functions are Lipschitz. The sums and products of Lipschitz functions are Lipschitz.

Lemma 3.7. Lemma 3.5 can be reduced to the case $\sigma(t) = t^p$, i.e., $c = 1$.

Proof. Suppose the function defined by $n \mapsto B_n^\circ/A_n$ is Lipschitz. Let B_n° be the coefficients of $\widehat{G}_{a,\dots,a}^{(\sigma)}(t)$ with respect to $c = 1$. Then

$$n \mapsto \frac{A_n^{(1)\frac{n-1}{p}}}{A_n} = (-1)^{se+1} \left((n+a) \frac{B_n^\circ}{A_n} - 1 \right)$$

is also Lipschitz by Lemma 3.6. Observe that

$$\frac{B_n}{A_n} = \frac{B_n^\circ}{A_n} + (-1)^{se+1} \left(\frac{c^{\frac{n+a}{p}} - 1}{n+a} \right) \frac{A_{\frac{n-l}{p}}^{(1)}}{A_n}.$$

So to prove $n \mapsto B_n/A_n$ is Lipschitz, it suffices to prove $n \mapsto \frac{c^{n/p}-1}{n}$ is Lipschitz (here the value is 0 when $p \nmid n$). Write $c = 1 + kq$ for some $k \in W$. Then for those n with $p|n$, we have

$$\frac{c^{n/p} - 1}{n} = \frac{(1 + kq)^{n/p} - 1}{n} = \frac{1}{n} \sum_{i=1}^{\infty} \binom{n/p}{i} k^i q^i.$$

So to prove $n \mapsto \frac{c^{n/p}-1}{n}$ is Lipschitz, it suffices to show that

$$\frac{1}{n_1} \binom{n_1/p}{i} k^i q^i \equiv \frac{1}{n_2} \binom{n_2/p}{i} k^i q^i \pmod{p^m}$$

for $n_1 \equiv n_2 \pmod{p^m}$. For $i = 1$, they are equal. Suppose $i \geq 2$, we have

$$\frac{1}{n_1} \binom{n_1/p}{i} k^i q^i = \left(\frac{n_1}{p} - 1 \right) \cdots \left(\frac{n_1}{p} - i + 1 \right) \frac{k^i q^i}{i!p}.$$

By [13, §5.1 p.49], we know

$$v_p(i!) < \frac{i}{p-1}.$$

Thus we have

$$v_p\left(\frac{k^i q^i}{i!p}\right) \geq 1.$$

Therefore

$$\begin{aligned} \left(\frac{n_1}{p} - 1 \right) \cdots \left(\frac{n_1}{p} - i + 1 \right) \frac{k^i q^i}{i!p} &\equiv \left(\frac{n_2}{p} - 1 \right) \cdots \left(\frac{n_2}{p} - i + 1 \right) \frac{k^i q^i}{i!p} \pmod{p^m} \\ &= \frac{1}{n_2} \binom{n_2/p}{i} k^i q^i. \end{aligned}$$

Therefore, the result follows. $\square$

Lemma 3.8. *Assuming $\sigma(t) = t^p$, lemma 3.5 can be reduced to $s = 1$.*

Proof. Assume that $s = 1$, $\sigma(t) = t^p$ and suppose we have proved that $n \mapsto B_n/A_n$ is Lipschitz. Then $n \mapsto A_{\frac{n-l}{p}}^{(1)}/A_n$ is also Lipschitz (see the proof of Lemma 3.7). Now for arbitrary $s \in \mathbb{Z}_{>0}$, we have the function

$$\begin{aligned} n \mapsto & \frac{1}{n+a} \left(\frac{A_n^s - ((-1)^e A_{\frac{n-l}{p}}^{(1)})^s}{A_n^s} \right) \\ &= \frac{1}{n+a} \left(\frac{A_n - (-1)^e A_{\frac{n-l}{p}}^{(1)}}{A_n} \right) \sum_{j=0}^{s-1} \left(\frac{(-1)^e A_{\frac{n-l}{p}}^{(1)}}{A_n} \right)^j \\ &= \frac{B_n}{A_n} \sum_{j=0}^{s-1} \left(\frac{(-1)^e A_{\frac{n-l}{p}}^{(1)}}{A_n} \right)^j \end{aligned}$$

is also Lipschitz by Lemma 3.6. $\square$

By Lemma 3.7 and Lemma 3.8, from now on, we assume that $\sigma(t) = t^p$ and $s = 1$.

Lemma 3.9. *If $a + l = cp^n$ with $p \nmid c$, then*

$$\frac{B_l}{A_l} \equiv \psi_p(a+l) - \psi_p(1+l) \equiv -(\psi_p(1+l) + \gamma_p) \pmod{p^n},$$

where $\psi_p(z)$ is the p -adic digamma function and γ_p is the p -adic Euler constant ([1, §2.2]).

Proof. Case I : $p \neq 2$

First, if $l = 0$, then $e = l = 0$. Therefore we have

$$\frac{B_l}{A_l} = \frac{B_0}{A_0} = \frac{1}{a}(1 - (-1)^e) = 0$$

and

$$\psi_p(a+l) - \psi_p(1+l) \equiv \psi_p(0) - \psi_p(1) = 0 \pmod{p^n}.$$

If $l \neq 0$, then

$$\begin{aligned} \frac{B_l}{A_l} &= \frac{1}{a+l} \left(1 - (-1)^e \frac{1}{A_l} \right) \\ &= \frac{1}{cp^n} \left(1 - (-1)^e \frac{l!}{(a)_l} \right). \end{aligned}$$

Arranging the formula above and using $a + l = cp^n$, we obtain

$$\begin{aligned}
(-1)^e \left(1 - cp^n \frac{B_l}{A_l}\right) &= (-1)^l \frac{1 \cdot 2 \cdots l}{(l - cp^n) \cdots (1 - cp^n)} \\
&= (-1)^l \prod_{1 \leq k \leq l} \left(\frac{1}{1 - cp^n/k} \right) \\
&\equiv (-1)^l \left(1 + \sum_{1 \leq k \leq l} \frac{cp^n}{k} \right) \pmod{p^{2n}} \\
&\equiv (-1)^l \left(1 + cp^n \sum_{1 \leq k \leq l} \frac{1}{k} \right) \pmod{p^{2n}} \\
&= (-1)^l \left(1 + cp^n (\psi_p(1 + l) + \gamma_p) \right).
\end{aligned}$$

Therefore, we have

$$1 - cp^n \frac{B_l}{A_l} \equiv (-1)^{(e-l)} \left(1 + cp^n (\psi_p(1 + l) + \gamma_p) \right) \pmod{p^{2n}}.$$

Since we assume p is odd, e is equal to l . Hence

$$\frac{B_l}{A_l} \equiv -(\psi_p(1 + l) + \gamma_p) \equiv \psi_p(a + l) - \psi_p(1 + l) \pmod{p^n}$$

from the formula in [1, (2.13) and Theorem 2.4].

Case II : $p = 2$

Again, we have

$$\frac{B_l}{A_l} = \frac{1}{a + l} \left(1 - (-1)^e \frac{1}{A_l} \right).$$

(1) For $a = 2^n c$ with $2 \nmid c$, we have $l = 0$. Hence

$$\frac{B_l}{A_l} = \frac{B_0}{A_0} = \frac{1}{a} (1 - (-1)^e) = \frac{1}{2^n c} (1 - (-1)^e).$$

If $n \geq 2$, then $l' = 0$ and $e = 0$. Therefore, we obtain

$$\frac{B_0}{A_0} = 0, \quad \psi_p(a) - \psi_p(1) \equiv 0 \pmod{p^n}$$

by [1, (2.13) Case II].

If $n = 1$, then $l' = 2$ and $e = 1$. This implies

$$\frac{B_0}{A_0} = \frac{1}{2c} (1 - (-1)) \equiv 1 \pmod{2}$$

and

$$\psi_p(a) - \psi_p(1) = \psi_p(2c) - \psi_p(0) \equiv c \equiv 1 \pmod{2}.$$

Hence, in both cases, we have

$$\frac{B_l}{A_l} \equiv \psi_p(a+l) - \psi_p(1+l) \pmod{p^n}.$$

(2) For $a+1 = 2^n c$ with $2 \nmid c$, we have $l = 1$. This implies

$$\frac{B_l}{A_l} = \frac{B_1}{A_1} = \frac{1}{a+1} (1 - (-1)^e \frac{1}{A_1}) = \frac{1}{2^n c} (1 - (-1)^e \frac{1}{a}).$$

If $n \geq 2$, then $l' = 1$ and $e = 1$. We obtain

$$\begin{aligned} 2^n c \frac{B_1}{A_1} &= (1 + \frac{1}{a}) \\ &= (1 - \frac{1}{1 - 2^n c}) \\ &\equiv 1 - (1 + 2^n c) = -2^n c \pmod{2^{2n}}, \end{aligned}$$

which implies

$$\frac{B_1}{A_1} \equiv -1 \pmod{2^n}.$$

Also, we have

$$\psi_p(a+1) - \psi_p(2) \equiv \psi_p(0) - \psi_p(2) \equiv -1 \pmod{2^n}.$$

If $n = 1$, then $l' = 3$ and $e = 2$. We have

$$\begin{aligned} 2c \frac{B_1}{A_1} &= 1 + \left(\frac{1}{1 - 2c} \right) \\ &\equiv 1 + (1 + 2c) \pmod{4} \\ &\equiv 0 \pmod{4}. \end{aligned}$$

Hence, we get

$$\frac{B_1}{A_1} \equiv 0 \pmod{2}.$$

On the other hand, we have

$$\begin{aligned} \psi_p(a+1) - \psi_p(2) &\equiv \psi_p(2c) - \psi_p(2) \\ &\equiv c - 1 \\ &\equiv 0 \pmod{2}. \end{aligned}$$

□

Lemma 3.10. *If $k = l + bp^m$ with $p \nmid b$, then*

$$\frac{B_k}{A_k} \equiv \frac{B_l}{A_l} \pmod{p^m}.$$

Proof. Again, we write $a + l = cp^n$ with $p \nmid c$.

Case I : $m \neq n$

We have

$$\begin{aligned} & 1 - (a + l + bp^m) \frac{B_{l+bp^m}}{A_{l+bp^m}} \\ &= (-1)^e \frac{A_{bp^{m-1}}^{(1)}}{A_{l+bp^m}} \\ &\stackrel{(*)}{=} (-1)^e \frac{\{1\}_{l+bp^m}}{\{a\}_{l+bp^m}} \\ &= (-1)^e \frac{\{1\}_l}{\{a\}_l} \cdot \frac{\{1+l\}_{bp^m}}{\{1\}_{bp^m}} \frac{\{1\}_{bp^m}}{\{a+l\}_{bp^m}} \\ &\stackrel{(**)}{\equiv} (-1)^e \frac{1}{A_l} \left(1 - bp^m(\psi_p(a+l) - \psi_p(1+l)) \right) \pmod{p^{2m}} \\ &= \left(1 - (a+l) \frac{B_l}{A_l} \right) \left(1 - bp^m(\psi_p(a+l) - \psi_p(1+l)) \right) \pmod{p^{2m}} \end{aligned}$$

where $(*)$ and $(**)$ follow from [1, Lemma 3.6] and [1, Lemma 3.8], respectively. By arranging the equation above, we obtain

$$1 - (a + l + bp^m) \frac{B_{l+bp^m}}{A_{l+bp^m}} \equiv 1 - (a + l) \frac{B_l}{A_l} - bp^m(\psi_p(a+l) - \psi_p(1+l))$$

modulo p^{2m} when $n > m$ (modulo p^{n+m} when $n < m$). Hence we obtain

$$(a + l + bp^m) \left(\frac{B_l}{A_l} - \frac{B_{l+bp^m}}{A_{l+bp^m}} \right) \equiv bp^m \left(\frac{B_l}{A_l} - (\psi_p(a+l) - \psi_p(1+l)) \right).$$

Therefore, by Lemma 3.9, we have

$$\begin{aligned} (a + l + bp^m) \left(\frac{B_l}{A_l} - \frac{B_{l+bp^m}}{A_{l+bp^m}} \right) &\equiv 0 \pmod{p^{2m}}, \text{ when } n > m \\ &\pmod{p^{n+m}}, \text{ when } n < m. \end{aligned}$$

$$\implies \frac{B_{l+bp^m}}{A_{l+bp^m}} \equiv \frac{B_l}{A_l} \equiv 0 \pmod{p^m}$$

in both cases since $\text{ord}_p(a + l + bp^m)$ is m if $m < n$ (n if $n < m$).

Case II : $m = n$

We write $a + l = cp^m$, $k = l + bp^m$ and $a + k = (b + c)p^m = dp^{m'+m}$ with $p \nmid d$. Here we can assume $m' \geq 1$; otherwise, use the method in Case I. Then

$$\frac{B_{l+bp^m}}{A_{l+bp^m}} - \frac{B_l}{A_l} = \frac{c[1 - (-1)^e \frac{A_{bp^{m-1}}^{(1)}}{A_{bp^m+l}}] - dp^{m'}[1 - (-1)^e \frac{1}{A_l}]}{cdp^{m+m'}}.$$

We claim that the numerator

$$c[1 - (-1)^e \frac{A_{bp^{m-1}}^{(1)}}{A_{bp^m+l}}] - dp^{m'}[1 - (-1)^e \frac{1}{A_l}] \equiv 0 \pmod{p^{2m+m'}}.$$

(1) p is odd or $m \geq 2$.

First, we calculate $A_{bp^{m-1}}^{(1)}/A_{bp^m+l}$. We have that

$$\frac{A_{bp^{m-1}}^{(1)}}{A_{bp^m+l}} = \frac{\{1\}_{l+bp^m}}{\{a\}_{l+bp^m}} = \frac{\{1\}_l \{1+l\}_{bp^m}}{\{a\}_l \{a+l\}_{bp^m}} = \frac{1}{A_l} \frac{\{1+l\}_{bp^m}}{\{a+l\}_{bp^m}}$$

and

$$\begin{aligned} & \frac{\{1+l\}_{bp^m}}{\{a+l\}_{bp^m}} \\ &= \frac{\prod_{\substack{1+l \leq i \leq bp^m+l \\ p \nmid i}} i}{\prod_{\substack{1 \leq i \leq bp^m \\ p \nmid i}} (cp^m + i)} \\ &= \frac{\prod_{\substack{1 \leq i \leq bp^m \\ p \nmid i}} i \cdot \prod_{\substack{bp^m \leq i \leq bp^m+l \\ p \nmid i}} i}{\prod_{\substack{1 \leq i \leq l \\ p \nmid i}} i} \cdot \frac{1}{\prod_{\substack{1 \leq i \leq bp^m \\ p \nmid i}} (cp^m + i)} \\ &= \prod_{\substack{1 \leq i \leq bp^m \\ p \nmid i}} \left(\frac{1}{1 + cp^m/i} \right) \frac{\prod_{1 \leq i \leq l} (dp^{m+m'} - a - l + i)}{\prod_{1 \leq i \leq l} i}. \end{aligned}$$

Since p is odd, or $m \geq 2$ and $p \nmid i$, we have that

$$\prod_{\substack{1 \leq i \leq bp^m \\ p \nmid i}} \left(\frac{1}{1 + cp^m/i} \right) = \prod_{\substack{1 \leq i < \frac{bp^m}{2} \\ p \nmid i}} \left(\frac{1}{1 + cp^m/i} \right) \left(\frac{1}{1 + cp^m/(bp^m - i)} \right)$$

where $bp^m - i \neq i$. Since

$$\prod_{\substack{1 \leq i < \frac{bp^m}{2} \\ p \nmid i}} \left(\frac{1}{1 + cp^m/i} \right) \left(\frac{1}{1 + cp^m/(bp^m - i)} \right) = \prod_i \frac{1}{1 + \frac{cp^m \cdot p^m(b+c)}{i(bp^m - i)}} \equiv 1$$

modulo $p^{2m+m'}$, we obtain

$$\begin{aligned} \frac{\{1+l\}_{bp^m}}{\{a+l\}_{bp^m}} &\equiv \frac{\prod_{1 \leq i \leq l} (dp^{m+m'} - a - l + i)}{\prod_{1 \leq i \leq l} i} \pmod{p^{2m+m'}} \\ &= \frac{(-1)^l \prod_{0 \leq i \leq l-1} (a + i - dp^{m+m'})}{l!} \\ &\equiv \frac{(-1)^l}{l!} \left[\prod_{i=0}^{l-1} (a + i) + (-dp^{m+m'}) \sum_{i=0}^{l-1} (a + i) \right] \\ &\equiv (-1)^l \left[(-dp^{m+m'}) \frac{la + l(l-1)/2}{l!} + \frac{(a)_l}{l!} \right] \pmod{p^{2m+m'}}. \end{aligned}$$

Multiplying $\frac{1}{A_l}$ on both sides, we get

$$\begin{aligned} \frac{A_{bp^{m-1}}^{(1)}}{A_{bp^m+l}} &\equiv \frac{(-1)^l}{A_l} \left[A_l + (-dp^{m+m'}) \frac{la + l(l-1)/2}{l!} \right] \pmod{p^{2m+m'}} \\ &= (-1)^l \left[1 + \frac{l!}{(a)_l} (-dp^{m+m'}) \frac{la + l(l-1)/2}{l!} \right] \\ &\equiv (-1)^l \left[1 + (-1)^l (-dp^{m+m'}) \frac{la + l(l-1)/2}{l!} \right] \pmod{p^{2m+m'}} \end{aligned}$$

since $\frac{l!}{(a)_l} \equiv (-1)^l \pmod{p^m}$ (use $a \equiv -l \pmod{p^m}$).

For $1/A_l$, since

$$\frac{(a)_l}{l!} \equiv \frac{-cp^m(l+1)l/2 + (-1)^l l!}{l!} = \frac{-cp^m(l+1)}{2(l-1)!} + (-1)^l \pmod{p^{2m}},$$

we have

$$\frac{1}{A_l} = \frac{l!}{(a)_l} \equiv \frac{(-1)^l}{1 + (-1)^{l+1} \frac{cp^m(l+1)}{2(l-1)!}} \equiv (-1)^l + \frac{cp^m(l+1)}{2(l-1)!} \pmod{p^{2m}}.$$

Hence the numerator

$$\begin{aligned}
& c[1 - (-1)^e \frac{A_{bp^{m-1}}^{(1)}}{A_{bp^m+l}}] - dp^{m'}[1 - (-1)^e \frac{1}{A_l}] \\
&= -b + (-1)^e \left(\frac{dp^{m'}}{A_l} - c \frac{A_{bp^{m-1}}^{(1)}}{A_{bp^m+l}} \right) \\
&\equiv -b + (-1)^e \left((-1)^l b + dp^{m'} \frac{cp^m(l+1)}{2[(l-1)!]} + c(dp^{m+m'}) \frac{2la + l(l-1)}{2(l!)} \right) \\
&\equiv -b + (-1)^e \left((-1)^l b + \frac{dp^{m+m'}c}{2(l!)} (2l(a+l)) \right) \\
&\equiv b((-1) + (-1)^{(e+l)}) \pmod{p^{2m+m'}}.
\end{aligned}$$

Since p is odd or $p = 2(m \geq 2)$, we have $e = l$. Hence $(-1) + (-1)^{(e+l)} = 0$. So we have the numerator is congruent to 0 modulo $p^{2m+m'}$. And this implies

$$\frac{B_k}{A_k} \equiv \frac{B_l}{A_l} \pmod{p^m}.$$

(2) $p = 2$ and $m = 1$.

Write $k = l + 2b, a + l = 2c, a + k = 2(b + c) = 2^{n+1}d$ with $2 \nmid bc$. Then

$$\begin{aligned}
\frac{A_{b2^{m-1}}^{(1)}}{A_{b2^m}} &= \frac{1}{A_l} \frac{\{1+l\}_{b2^m}}{\{a+l\}_{b2^m}} \\
&= \frac{1}{A_l} \left[\prod_{\substack{1 \leq i \leq b2^m \\ 2 \nmid i}} \frac{1}{1 + c2^m/i} \right] \left[\frac{(-1)^l (a - d2^{m+n})_l}{l!} \right] \\
&\equiv \frac{1}{A_l} \left(\frac{1}{1 + 2c/b} \right) (-1)^l \left(\frac{(a - d2^{m+n})_l}{l!} \right) \pmod{2^{2+n}}.
\end{aligned}$$

(Note: Only the term $\left(\frac{1}{1+2c/b} \right)$ is different from case (1)).

Again, we have

$$\begin{aligned}
& (-b) + (-1)^e \left(\frac{d2^n}{A_l} - c \frac{A_b^{(1)}}{A_{2b}} \right) \\
& \equiv (-b) + (-1)^e \left[(-1)^l d2^n + d2^n \frac{c2(l+1)}{2(l-1)!} - c \left(\frac{b}{b+2c} \right) (-1)^l - \right. \\
& \quad \left. c \left(\frac{b}{b+2c} \right) (-d2^{1+n}) \frac{2la+l(l-1)}{2(l!)} \right] \pmod{4 \cdot 2^n} \\
& \equiv (-b) + (-1)^e \left[(-1)^l d2^n - c \left(\frac{b}{b+2c} \right) (-1)^l + d2^n \frac{c2(l+1)}{2(l-1)!} \right. \\
& \quad \left. + c(d2^{1+n}) \frac{2la+l(l-1)}{2(l!)} \right] \pmod{4 \cdot 2^n} \text{ (since } \frac{b}{b+2c} \equiv 1 \pmod{2} \text{)} \\
& \equiv (-b) + (-1)^e \left[(-1)^l d2^n - c \left(\frac{b}{b+2c} \right) (-1)^l \right] \\
& = (-b) + (-1)^{(e+l)} \left[d2^n - c \left(\frac{b}{b+2c} \right) \right]. \quad (*)
\end{aligned}$$

Observe that $e+l$ is odd. This is because if $l=0$, then $l'=2$ and $e=1$. If $l=1$, then $l'=3$ and $e=2$. Hence, we have

$$\begin{aligned}
(*) & = (-b) - \left(b + c - \frac{bc}{b+2c} \right) \\
& = \frac{(-2)(b+c)^2}{(b+2c)} \\
& \equiv 0 \pmod{4 \cdot 2^n}.
\end{aligned}$$

Therefore, we have

$$\frac{B_k}{A_k} \equiv \frac{B_l}{A_l} \pmod{2}.$$

□

Now we can start the proof of Lemma 3.5 with the assumption $\sigma(t) = t^p$ and $s=1$.

(proof of Lemma 3.5). If $k \not\equiv l \pmod{p}$, then

$$\frac{B_k}{A_k} = \frac{1}{k+a}.$$

So the result follows. Suppose $k \equiv l \pmod{p}$. We write $a+l = cp^n$, $k = l + bp^m$ with $p \nmid bc$. It is enough to prove the lemma for the case $k' = k + p^{m'}$ and for any $m' \in \mathbb{N}$.

Case I: $m' \leq m$

This follows from Lemma 3.10 since we have $k \equiv k' \equiv l \pmod{p^{m'}}$.

Case II: $m' > m$ and $n \neq m$ or $n = m$ with $\text{ord}_p(k+a) = m$

We have

$$\begin{aligned}
& 1 - (k' + a) \frac{B_{k'}}{A_{k'}} \\
&= (-1)^e \frac{A_{k'}^{(1)} \left[\frac{k'}{p} \right]}{A_{k'}} \\
&= (-1)^e \frac{\{1\}_{l+bp^m+pm'}}{\{a\}_{l+bp^m+pm'}} \\
&= (-1)^e \frac{\{1\}_{l+bp^m} \{1+l+bp^m\}_{p^{m'}}}{\{a\}_{l+bp^m} \{a+l+bp^m\}_{p^{m'}}} \\
&= (-1)^e \left(\frac{A_{bp^m-1}^{(1)}}{A_{l+bp^m}} \right) \cdot \left(\frac{\{1+l+bp^m\}_{p^{m'}}}{\{1\}_{p^{m'}}} \frac{\{1\}_{p^{m'}}}{\{a+l+bp^m\}_{p^{m'}}} \right) \\
&\equiv \left(1 - (k+a) \frac{B_k}{A_k} \right) \left(1 + p^{m'} (\psi_p(1+l+bp^m) + \gamma_p) \right) \\
&\quad \left(1 - p^{m'} (\psi_p(a+l+bp^m) + \gamma_p) \right) \pmod{p^{2m'}} \\
&\equiv \left(1 - (k+a) \frac{B_k}{A_k} \right) \left(1 + p^{m'} (\psi_p(1+l+bp^m) - \psi_p(a+l+bp^m)) \right) \\
&\stackrel{(*)}{\equiv} \left(1 - (k+a) \frac{B_k}{A_k} \right) \left(1 - p^{m'} \frac{B_l}{A_l} \right) \pmod{p^{m+m'}}
\end{aligned}$$

where $(*)$ follows from [1, (2.13)] and Lemma 3.9 when $p \neq 2$ or $m \geq 2$. For $p = 2, m = 1$, $(*)$ follows from

$$\psi_p(1+l+2b) - \psi_p(a+l+2b) \equiv \psi_p(1+l) - \psi_p(a+l) \pmod{2}.$$

Therefore one has

$$\begin{aligned}
& 1 - (k+a+p^{m'}) \frac{B_{k'}}{A_{k'}} \\
&\equiv 1 - (k+a) \frac{B_k}{A_k} - p^{m'} \frac{B_l}{A_l} + p^{m'} (k+a) \frac{B_k}{A_k} \frac{B_l}{A_l} \pmod{p^{m+m'}} \\
&\equiv 1 - (k+a) \frac{B_k}{A_k} - p^{m'} \frac{B_l}{A_l} \pmod{p^{n^*+m'} (n^* := \min\{n, m\})}.
\end{aligned}$$

We get

$$(k+a) \left(\frac{B_k}{A_k} - \frac{B_{k'}}{A_{k'}} \right) \equiv p^{m'} \left(\frac{B_{k'}}{A_{k'}} - \frac{B_l}{A_l} \right) \equiv 0 \pmod{p^{n^*+m'}}.$$

By assumption in Case II, we have $\text{ord}_p(k+a) = n^*$. Hence we obtain our result

$$\frac{B_k}{A_k} \equiv \frac{B_{k'}}{A_{k'}} \pmod{p^{m'}}.$$

Case III: $m' > m$ and $n = m$ with $\text{ord}_p(k+a) > m$

In this case, we write $a+l = cp^m, k = l + bp^m, k+a = (b+c)p^m = dp^{n+m}$ and $k' = k + p^{m'}$ with $m \geq 1, n \geq 1, p \nmid bcd$. Then

$$\frac{B_{k'}}{A_{k'}} - \frac{B_k}{A_k} = \frac{-p^{m'} + (-1)^e \left[(dp^{m+n} + p^{m'}) \frac{A_{bp^{m-1}}^{(1)}}{A_{l+bp^m}} - (dp^{m+n}) \frac{A_{bp^{m-1}+p^{m'-1}}^{(1)}}{A_{l+bp^m+p^{m'}}} \right]}{(dp^{m+n} + p^{m'})dp^{m+n}}$$

We claim that the numerator is congruent to 0 modulo $(dp^{m+n} + p^{m'})p^{m+n}$. For simplicity, we denote the numerator by $(*)$. First, we consider

(1) For $m' \leq m+n$, we have that

$$\frac{A_{bp^{m-1}+p^{m'-1}}^{(1)}}{A_{l+bp^m+p^{m'}}} = \frac{\{1\}_{l+bp^m+p^{m'}}}{\{a\}_{l+bp^m+p^{m'}}}$$

and

$$\begin{aligned} & \frac{\{1\}_{l+bp^m+p^{m'}}}{\{a\}_{l+bp^m+p^{m'}}} \\ &= \frac{\prod_{\substack{1 \leq i \leq l+bp^m+p^{m'} \\ p \nmid i}} i}{\prod_{\substack{1 \leq i \leq l+bp^m+p^{m'} \\ p \nmid i}} a + l + bp^m + p^{m'} - i} \\ &= \left(\prod_{\substack{1 \leq i \leq l+bp^m+p^{m'} \\ p \nmid i}} \frac{1}{1 - (dp^{m+n} + p^{m'})/i} \right) (-1)^{l+bp^m+p^{m'} - \lfloor \frac{l+bp^m+p^{m'}}{p} \rfloor} \end{aligned}$$

This is congruent to

$$\left(1 + (dp^{m+n} + p^{m'}) \sum_{\substack{1 \leq i \leq l+bp^m+p^{m'} \\ p \nmid i}} \frac{1}{i} \right) (-1)^{l+bp^{m-1}(p-1)}$$

modulo $(dp^{m+n} + p^{m'})p^{m'}$ since $m' \leq m+n$ and $p^{m'-1}(p-1) \equiv 0 \pmod{2}$.

It equals

$$\left[1 + (dp^{n+m} + p^{m'}) (\psi_p(1+l+bp^m+p^{m'}) + \gamma_p) \right] (-1)^{l+bp^{m-1}(p-1)}.$$

Therefore we have

$$\frac{A_{bp^{m-1}+p^{m'-1}}^{(1)}}{A_{l+bp^m+p^{m'}}} \equiv \left[1 + (dp^{n+m} + p^{m'}) (\psi_p(1+l+bp^m+p^{m'}) + \gamma_p) \right] (-1)^{l+bp^{m-1}(p-1)}$$

$$\text{mod } (dp^{m+n} + p^{m'})p^{m'}.$$

Similarly, we can derive

$$\frac{A_{bp^{m-1}}^{(1)}}{A_{l+bp^m}} \equiv \left[1 + dp^{n+m}(\psi_p(1+l+bp^m) + \gamma_p) \right] (-1)^{l+bp^{m-1}(p-1)} \text{ mod } (dp^{m+n})p^{m'}.$$

Hence, (*) is congruent to

$$\begin{aligned} & -p^{m'} + (-1)^{(e+l+bp^{m-1}(p-1))} \cdot \left(p^{m'} + (dp^{m+n} + p^{m'}) dp^{m+n} (\psi_p(1+l+bp^m) \right. \\ & \quad \left. - \psi_p(1+l+bp^m + p^{m'})) \right) \end{aligned}$$

modulo $p^{m+n+m'}(dp^{m+n} + p^{m'})$.

Furthermore, since $m' \geq 2$ we have

$$\psi_p(1+l+bp^m) \equiv \psi_p(1+l+bp^m + p^{m'}) \text{ mod } p^{m'}.$$

Therefore, (*) is congruent to

$$p^{m'} \left(1 - (-1)^{(e+l+bp^{m-1}(p-1))} \right).$$

Now we discuss case by case.

- If p is odd, then $(-1)^{(e+l+bp^{m-1}(p-1))} = 1$.
- If $p = 2, m \geq 2$, then $(-1)^{(e+l+bp^{m-1}(p-1))} = (-1)^{(e+l)} = 1$ ($e = l$ since $m \geq 2$).
- If $p = 2, m = 1$, then $(-1)^{(e+l+b)} = 1$ since it can again be divided into the following two cases:

$$\begin{cases} l = 0, l' = 2, e = 1, b \equiv 1 \pmod{2} \\ l = 1, l' = 3, e = 2, b \equiv 1 \pmod{2}. \end{cases}$$

Hence, (*) is congruent to 0 modulo $p^{m+n+m'}(dp^{m+n} + p^{m'})$, and this implies

$$\frac{B_k}{A_k} \equiv \frac{B_{k'}}{A_{k'}} \text{ mod } p^{m'}.$$

We obtain the first case.

(2) For $m' > m + n$, we have

$$\begin{aligned}
& \frac{A_{bp^{m-1}+p^{m'-1}}^{(1)}}{A_{l+bp^m+p^{m'}}} \\
&= \frac{\{1\}_{l+bp^m+p^{m'}}}{\{a\}_{l+bp^m+p^{m'}}} \\
&= \frac{A_{bp^{m-1}}^{(1)}}{A_{l+bp^m}} \left(\frac{\{1+l+bp^m\}_{p^{m'}}}{\{a+l+bp^m\}_{p^{m'}}} \right) \\
&= \frac{A_{bp^{m-1}}^{(1)}}{A_{l+bp^m}} \left(\frac{\{1+l+bp^m\}_{p^{m'}}}{\{1\}_{p^{m'}}} \frac{\{1\}_{p^{m'}}}{\{a+l+bp^m\}_{p^{m'}}} \right) \\
&\equiv \frac{A_{bp^{m-1}}^{(1)}}{A_{l+bp^m}} \left(1 + p^{m'} \left(\psi_p(1+l+bp^m) + \gamma_p \right) \right) \left(1 - p^{m'} \left(\psi_p(a+l+bp^m) + \gamma_p \right) \right)
\end{aligned}$$

modulo $p^{2m'}$. Since $m' > m + n$ and $\psi_p(a+l+bp^m) + \gamma_p \equiv 0 \pmod{dp^{m+n}}$, the formula above is congruent to

$$\frac{A_{bp^{m-1}}^{(1)}}{A_{l+bp^m}} \left(1 + p^{m'} \left(\psi_p(1+l+bp^m) + \gamma_p \right) \right)$$

modulo $(dp^{m+n} + p^{m'})p^{m'}$.

Substituting this result into (*), we have

$$(*) \equiv -p^{m'} + (-1)^e \frac{A_{bp^{m-1}}^{(1)}}{A_{l+bp^m}} p^{m'} \left(1 - dp^{m+n} \left(\psi_p(1+l+bp^m) + \gamma_p \right) \right)$$

modulo $(dp^{m+n} + p^{m'})p^{m'}$.

Now we calculate the term $A_{bp^{m-1}}^{(1)}/A_{l+bp^m}$.

Since we have

$$\begin{aligned}
& \frac{A_{bp^{m-1}}^{(1)}}{A_{l+bp^m}} \\
&= \frac{\{1\}_{l+bp^m}}{\{a\}_{l+bp^m}} \\
&= \prod_{\substack{1 \leq i \leq l+bp^m \\ p \nmid i}} \left(\frac{i}{a+l+bp^m-i} \right) \\
&= \left(\prod_{\substack{1 \leq i \leq l+bp^m \\ p \nmid i}} \frac{1}{1-dp^{m+n}/i} \right) (-1)^{l+bp^{m-1}(p-1)} \\
&= \left(\prod_{\substack{1 \leq i \leq l+bp^m \\ p \nmid i}} \frac{1}{1-dp^{m+n}/i} \right) (-1)^e \quad \left((-1)^{l+bp^{m-1}(p-1)} = (-1)^e \forall p \right) \\
&\equiv (-1)^e \left(1 + dp^{m+n} \left(\psi_p(1+l+bp^m) + \gamma_p \right) \right) \pmod{(p^{m+n})^2},
\end{aligned}$$

we obtain

$$\begin{aligned}
(*) &\equiv -p^{m'} + p^{m'} \left(1 + dp^{m+n} \left(\psi_p(1+l+bp^m) + \gamma_p \right) \right) \left(1 - \right. \\
&\quad \left. dp^{m+n} \left(\psi_p(1+l+bp^m) + \gamma_p \right) \right) \\
&\equiv 0 \pmod{(dp^{m+n} + p^{m'})p^{m+n}p^{m'}}.
\end{aligned}$$

Hence, again, we obtain

$$\frac{B_k}{A_k} \equiv \frac{B_{k'}}{A_{k'}} \pmod{p^{m'}}.$$

Now we prove that $B_k/A_k \in W$. For $k \not\equiv l \pmod{p}$, $B_k/A_k = 1/(k+a) \in W$. For $k \equiv l \pmod{p}$, it follows from $A_l \in \mathbb{Z}_p^\times$ and

$$\frac{B_k}{A_k} \equiv \frac{B_l}{A_l} \pmod{p}.$$

□

3.2 Proof of Congruence Relations: End of the Proof

Here we follow the method used in the paper [1, §3.5] where M. Asakura prove the congruence relations of p -adic hypergeometric functions of logarithmic type.

Here we prove it in a more general setting.

Lemma 3.11. *Let $a \in \mathbb{Z}_p \setminus \mathbb{Z}_{\leq 0}$ and let $B_n \in \mathbb{Z}_p$ for $n \in \mathbb{Z}_{\geq 0}$. Suppose the map $n \mapsto B_n/A_n$ is a Lipschitz function. Put $B(t) := \sum_{n=0}^{\infty} B_n t^n$. Then there are Dwork's congruences for $B(t)/F_{a,\dots,a}(t)$; that is*

$$\frac{B(t)}{F_{a,\dots,a}(t)} \equiv \frac{B(t)_{<p^n}}{F_{a,\dots,a}(t)_{<p^n}} \pmod{p^n W[[t]]}$$

for all $n \in \mathbb{Z}_{\geq 0}$.

If we can prove this lemma, then by Lemma 3.5 we obtain the congruence relations for our hypergeometric functions.

Put $S_m := \sum_{i+j=m} A_{i+p^n} B_j - A_i B_{j+p^n}$ for $m \in \mathbb{Z}_{\geq 0}$. We claim that $S_m \equiv 0 \pmod{p^n}$ for $n \in \mathbb{Z}_{\geq 0}$ since Lemma 3.11 is equivalent to for any $n, m \in \mathbb{Z}_{\geq 0}$, $S_m \equiv 0 \pmod{p^n}$. First, we need the following lemmas.

Lemma 3.12. *We have*

$$S_m \equiv \sum_{i+j=m} (A_{i+p^n} A_j - A_i A_{j+p^n}) \frac{B_j}{A_j} \pmod{p^n}.$$

Proof. We compute

$$\begin{aligned} S_m &= \sum_{i+j=m} A_{i+p^n} B_j - A_i A_{j+p^n} \frac{B_{j+p^n}}{A_{j+p^n}} \\ &\equiv \sum_{i+j=m} A_{i+p^n} B_j - A_i A_{j+p^n} \frac{B_j}{A_j} \pmod{p^n} \quad (\text{Lipschitz}) \\ &= \sum_{i+j=m} (A_{i+p^n} A_j - A_i A_{j+p^n}) \frac{B_j}{A_j}. \end{aligned}$$

□

Lemma 3.13. *We have*

$$S_m \equiv \sum_{i+j=m} (A_{[j/p]}^{(1)} A_{[i/p]+p^{n-1}}^{(1)} - A_{[i/p]}^{(1)} A_{[j/p]+p^{n-1}}^{(1)}) \frac{A_i}{A_{[i/p]}^{(1)}} \frac{A_j}{A_{[j/p]}^{(1)}} \frac{B_j}{A_j} \pmod{p^n}.$$

Proof. By Lemma 3.12, we know that

$$\begin{aligned} S_m &\equiv \sum_{i+j=m} (A_{i+p^n} A_j - A_i A_{j+p^n}) \frac{B_j}{A_j} \pmod{p^n} \\ &= \sum_{i+j=m} \left(\frac{A_{i+p^n} A_{[j/p]}^{(1)} A_{[j/p]}^{(1)}}{A_i} - \frac{A_{j+p^n} A_{[i/p]}^{(1)} A_{[j/p]}^{(1)}}{A_j} \right) \frac{A_i}{A_{[i/p]}^{(1)}} \frac{A_j}{A_{[j/p]}^{(1)}} \frac{B_j}{A_j}. \end{aligned}$$

By [1, Lemma 3.8], we have

$$\frac{A_{i+p^n} A_{[i/p]}^{(1)}}{A_i} \equiv A_{[i/p]+p^{n-1}}^{(1)}, \quad \frac{A_{j+p^n} A_{[j/p]}^{(1)}}{A_j} \equiv A_{[j/p]+p^{n-1}}^{(1)} \pmod{p^n}.$$

Since $A_i/A_{[i/p]}^{(1)}$, $A_j/A_{[j/p]}^{(1)}$ and B_j/A_j are all in W ([1, Lemma 3.8] and assumption), we have S_m is congruent to

$$\sum_{i+j=m} (A_{[j/p]}^{(1)} A_{[i/p]+p^{n-1}}^{(1)} - A_{[i/p]}^{(1)} A_{[j/p]+p^{n-1}}^{(1)}) \frac{A_i}{A_{[i/p]}^{(1)}} \frac{A_j}{A_{[j/p]}^{(1)}} \frac{B_j}{A_j} \pmod{p^n}.$$

□

Now we start the proof of Lemma 3.11.

(proof of Lemma 3.11). Put

$$q_i := \frac{A_i}{A_{[i/p]}^{(1)}}, \quad A(i, j) := A_i^{(1)} A_j^{(1)}, \quad A^*(i, j) := A(j, i+p^{n-1}) - A(i, j+p^{n-1})$$

$$B(i, j) := A^*([i/p], [j/p]).$$

Put $m = pt + l$ with $l \in \{0, 1, \dots, p-1\}$. Then we have

$$\begin{aligned} S_m &\equiv \sum_{i+j=m} B(i, j) q_i q_j \frac{B_j}{A_j} \pmod{p^n} \\ &= \sum_{i=0}^{p-1} \sum_{k=0}^{\lfloor (m-i)/p \rfloor} B(i+kp, m-(i+kp)) q_{i+kp} q_{m-(i+kp)} \frac{B_{m-(i+kp)}}{A_{m-(i+kp)}} \\ &= \sum_{k=0}^t \sum_{i=0}^l B(i+kp, m-(i+kp)) q_{i+kp} q_{m-(i+kp)} \frac{B_{m-(i+kp)}}{A_{m-(i+kp)}} \\ &\quad + \sum_{k=0}^{t-1} \sum_{i=l+1}^{p-1} B(i+kp, m-(i+kp)) q_{i+kp} q_{m-(i+kp)} \frac{B_{m-(i+kp)}}{A_{m-(i+kp)}} \\ &= \sum_{k=0}^t A^*(k, t-k) \overbrace{\left(\sum_{i=0}^l q_{i+kp} q_{m-(i+kp)} \frac{B_{m-(i+kp)}}{A_{m-(i+kp)}} \right)}^{P_k} \\ &\quad + \sum_{k=0}^{t-1} A^*(k, t-k-1) \underbrace{\left(\sum_{i=l+1}^{p-1} q_{i+kp} q_{m-(i+kp)} \frac{B_{m-(i+kp)}}{A_{m-(i+kp)}} \right)}_{Q_k}. \end{aligned}$$

We show that the first term vanishes modulo p^n .

It follows from the assumption and [1, Lemma 3.7] that $B_k/A_k, q_k \in W$ for all $k \in \mathbb{Z}_{\geq 0}$ and

$$k \equiv k' \pmod{p^i} \implies \frac{B_k}{A_k} \equiv \frac{B_{k'}}{A_{k'}}, q_k \equiv q_{k'} \pmod{p^i}.$$

Therefore, we have

$$k \equiv k' \pmod{p^i} \implies P_k \equiv P_{k'} \pmod{p^{i+1}}. \quad (3.1)$$

Then one can write

$$\sum_{k=0}^s A^*(k, s-k) P_k \equiv \sum_{i=0}^{p^{n-1}-1} P_i \overbrace{\left(\sum_{k \equiv i \pmod{p^{n-1}}} A^*(k, s-k) \right)}^{(*)} \pmod{p^n}.$$

Let us recall Lemma 3.12 in [1].

Lemma 3.14. *For all $m, k, s \in \mathbb{Z}_{\geq 0}$ and $0 \leq l \leq n$, then*

$$\sum_{\substack{i+j=m \\ i \equiv k \pmod{p^{n-l}}}} A_i A_{j+p^{n-1}} - A_j A_{i+p^{n-1}} \equiv 0 \pmod{p^l}.$$

Using this lemma, we obtain $(*)$ is 0 modulo p . Hence, by (3.1) again, we can write

$$\sum_{k=0}^s A^*(k, s-k) P_k \equiv \sum_{i=0}^{p^{n-2}-1} P_i \overbrace{\left(\sum_{k \equiv i \pmod{p^{n-2}}} A^*(k, s-k) \right)}^{(**)} \pmod{p^n}.$$

It follows from Lemma 3.9 that $(**)$ is 0 modulo p^2 , so

$$\sum_{k=0}^s A^*(k, s-k) P_k \equiv \sum_{i=0}^{p^{n-3}-1} P_i \left(\sum_{k \equiv i \pmod{p^{n-3}}} A^*(k, s-k) \right) \pmod{p^n}.$$

Continuing the same discussion, we have

$$\sum_{k=0}^s A^*(k, s-k) P_k \equiv \sum_{k=0}^s A^*(k, s-k) P_0 = 0 \pmod{p^n}.$$

Similarly, we can show the vanishing of the second term

$$\sum_{k=0}^{s-1} A^*(k, s-k) Q_k \equiv 0 \pmod{p^n}.$$

Hence $S_m \equiv 0 \pmod{p^n}$. This is the end of the proof. $\square$

4 Transformation Formulas

In this section, we will introduce two conjectures and prove these conjectures in a particular case. The first one is “Transformation Formulas between p -adic hypergeometric functions $\widehat{\mathcal{F}}_{a,\dots,a}^{(\sigma)}(t)$ and p -adic hypergeometric functions of logarithmic type” which will be discussed in Section 4.3. The second one is “Transformation formula on Dwork’s p -adic Hypergeometric Functions” which is discussed in Section 4.4.

4.1 Hypergeometric Curves and Hypergeometric Curves of Gauss Type

The main reference of this section is [1, §4.1, §4.2, §4.6]

4.1.1 Hypergeometric Curves

Let $W = W(\overline{\mathbb{F}}_p)$ be the Witt ring of $\overline{\mathbb{F}}_p$ and $R = W[t, (t - t^2)^{-1}]$. Let $N \geq 2$ be an integer and a prime $p > N$. We denote by $\mathbb{P}_R^1(Z_0, Z_1)$ the projective line over R with homogeneous coordinate (Z_0, Z_1) . Then we define U to be

$$\mathrm{Spec} R[u, v] / ((1 - u^N)(1 - v^N) - t) \subset Y := \overline{U}$$

where Y is the closure in $\mathbb{P}_R^1 \times \mathbb{P}_R^1$. It is called a *hypergeometric curve* over R . The morphism $Y \rightarrow \mathrm{Spec} R$ is smooth and projective with connected fibers of relative dimension one and the genus of a geometric fiber is $(N - 1)^2$.

Lemma 4.1. *There is a morphism*

$$\overline{f} : \overline{Y} \rightarrow \mathbb{P}_W^1 = \mathbb{P}_W^1(T_0, T_1)$$

of smooth projective W -schemes satisfying

(1) *Let $S := \mathrm{Spec} R = \mathrm{Spec} W[t, (t - t^2)^{-1}] \subset \mathbb{P}_W^1$ with $t := T_1/T_0$. Then $Y = \overline{f}^{-1}(S) \rightarrow S$ is the hypergeometric curve.*

(2) *The morphism $\overline{f}$ has a semistable reduction at $t = 0$. The fiber $D := \overline{f}^{-1}(t = 0)$ is a relative simple NCD, and the multiplicities of the components are one.*

Proof. This is [1, Lemma 4.1]. □

Let $K = \mathrm{Frac} W(\overline{\mathbb{F}}_p)$. For a W -scheme Z and a W -algebra R , we write $Z_R = Z \times_W R$. The group $\mu_N(K) \times \mu_N(K)$ acts on Y in the following way

$$[\zeta, \nu] \cdot (x, y, t) = (\zeta x, \nu y, t), \quad (\zeta, \nu) \in \mu_N \times \mu_N.$$

One has the eigen decomposition

$$H_{\text{dR}}^1(Y_K/S_K) = \bigoplus_{i=1}^{N-1} \bigoplus_{j=1}^{N-1} H_{\text{dR}}^1(Y_K/S_K)(i, j)$$

where $H_{\text{dR}}^1(Y_K/S_K)(i, j)$ denotes the subspace on which (ζ, ν) acts by multiplication by $\zeta^i \nu^j$ for all (ζ, ν) . Then each eigenspace $H_{\text{dR}}^1(Y_K/S_K)(i, j)$ is free of rank 2 over $\mathcal{O}(S_K)$. Put

$$a_i := 1 - \frac{i}{N}, \quad b_j := 1 - \frac{j}{N}, \quad (4.1)$$

$$\omega_{i,j} := N \frac{x^{i-1} j^{j-N}}{1-x^N} dx = -N \frac{x^{i-N} y^{j-1}}{1-y^N} dy, \quad (4.2)$$

and

$$\eta_{i,j} := \frac{1}{x^N - 1 + t} \omega_{i,j} = N t^{-1} x^{i-N} y^{j-N-1} dy \quad (4.3)$$

for integers i, j such that $1 \leq i, j \leq N-1$. Then they forms a $\mathcal{O}(S_K)$ -free basis of $H_{\text{dR}}^1(Y_K/S_K)(i, j)$ ([2, Lemma 2.2]).

Let

$$F_{a,b}(t) = \sum_{i=0}^{\infty} \frac{(a)_i}{i!} \frac{(b)_i}{i!} t^i \in K[[t]]$$

be the hypergeometric series. Put

$$\tilde{\omega}_{i,j} := \frac{1}{F_{a_i, b_j}(t)} \omega_{i,j}, \quad \tilde{\eta}_{i,j} := -t(1-t)^{a_i+b_j} (F'_{a_i, b_j}(t) \omega_{i,j} + b_j F_{a_i, b_j}(t) \eta_{i,j}). \quad (4.4)$$

Lemma 4.2. *Let $\nabla_{i,j}$ be the connection on the eigen component $H_{i,j} := K((t)) \otimes_{\mathcal{O}_S} H_{\text{dR}}^1(Y_K/S_K)(i, j)$. Then $\text{Ker} \nabla_{i,j} = K \tilde{\eta}_{i,j}$. Moreover let $\bar{\nabla}_{i,j}$ be the connection on $H_{i,j}/K((t)) \tilde{\eta}_{i,j}$ induced from $\nabla_{i,j}$. Then $\text{Ker} \bar{\nabla}_{i,j} = K \tilde{\omega}_{i,j}$.*

Proof. See [1, Proposition 4.3, Corollary 4.4]. $\square$

Let $(i, j) = (q/r, q'/r') \in \mathbb{Q}^2$ such that $\gcd(r, N) = \gcd(r', N) = 1$ and q, q' are not divided by N . Then we define

$$H_{\text{dR}}^1(Y_K/S_K)(i, j) = H_{\text{dR}}^1(Y_K/S_K)(i_0, j_0), \quad \omega_{i,j} = \omega_{i_0, j_0}, \dots, \tilde{\eta}_{i,j} = \tilde{\eta}_{i_0, j_0} \quad (4.5)$$

where i_0, j_0 are the unique integers such that $i_0 \equiv i \pmod{N}$, $j_0 \equiv j \pmod{N}$ and $1 \leq i_0, j_0 < N$.

4.1.2 Hypergeometric Curves of Gauss Type

Let A, B be integers such that $1 \leq A, B < N$ and $\gcd(A, N) = \gcd(B, N) = 1$.

Let $f : Y \rightarrow \mathbb{P}^1$ be a fibration over K ($K = \text{Frac}W(\overline{\mathbb{F}}_p)$) whose general fiber $Y_t = f^{-1}(t)$ is the projective nonsingular curve associated to the affine curve

$$y^N = x^A(1-x)^B(1-(1-t)x)^{N-B}.$$

We call X a hypergeometric curve of Gauss type. This is a fibration of curves of genus $N-1$, smooth outside $t = 0, 1, \infty$. Put $S_K := \text{Spec}K[t, (t-t^2)^{-1}]$ and $X_0 := f^{-1}(S_K)$.

Let $[\zeta] : X_0 \rightarrow X_0$ be the automorphism given by

$$[\zeta](x, y, t) = (x, \zeta^{-1}y, t)$$

for any N -th root of unity $\zeta \in \mu_N = \mu_N(K)$. We denote

$$H_{\text{dR}}^1(X_0/S_K)(n) := \{x \in H_{\text{dR}}^1(X_0/S_K) \mid [\zeta]x = \zeta^n x, \forall \zeta \in \mu_N\}.$$

Then one has the eigen decomposition

$$H_{\text{dR}}^1(X_0/S_K) = \bigoplus_{n=1}^{N-1} H_{\text{dR}}^1(X_0/S_K)(n)$$

of $\mathcal{O}(S_K)$ -module and each eigen space is free of rank 2.

A basis of $H_{\text{dR}}^1(X_0/S_K)(n)$ ([2, Lemma 2.5]) is given by

$$\omega_n := x^{A_n}(1-x)^{B_n}(1-(1-t)x)^{n-1-B_n} \frac{dx}{y^n}, \quad \eta_n := \frac{x}{1-(1-t)x} \omega_n$$

where

$$A_n := \lfloor \frac{nA}{N} \rfloor, \quad B_n := \lfloor \frac{nB}{N} \rfloor.$$

Let

$$a_n := \left\{ \frac{-nA}{N} \right\}, \quad b_n := \left\{ \frac{-nB}{N} \right\}$$

where $\{x\} := x - \lfloor x \rfloor$. Put

$$\tilde{\omega}_n := \frac{1}{F_{a_n, b_n}(t)} \omega_n, \quad \tilde{\eta}_n := -t(1-t)^{a_n+b_n} (F'_{a_n, b_n}(t) \omega_n + b_n F_{a_n, b_n}(t) \eta_n). \quad (4.6)$$

which form a $K((t))$ -basis of $K((t)) \otimes H_{\text{dR}}^1(X_0/S_K)$.

Let Y/S be the hypergeometric curve in §4.1.1. Then we have the following lemma.

Lemma 4.3. ([1, §4.6]) Let $g_{A,B}$ be the automorphism of Y_K/S_K given by $g_{A,B} : (u, v) \mapsto (\zeta_N^B u, \zeta_N^{-A} v)$ for a fixed primitive N -root of unity in $\mu_N(K)$. Let $G = \langle g_{A,B} \rangle \subset \text{Aut}(Y_K/S_K)$ be the cyclic group of order N . Then the quotient $Y_G := Y_K/G$ by G is isomorphic to the curve $X : y^N = x^A(1-x)^B(1-(1-t)x)^{N-B}$. The quotient map $\rho : Y_K \rightarrow X$ is given by

$$(u, v) \mapsto (x, y) = (u^{-N}, u^{-A}(1 - u^{-N})v^{N-B})$$

which is a S -morphism (i.e. $t \mapsto t$).

Put

$$\widehat{\omega}_n = \sum_{g \in G} g^* \omega_{nA, nB}, \quad \widehat{\eta}_n = \sum_{g \in G} g^* \eta_{nA, nB}.$$

Lemma 4.4. Using the notation above, we have

$$\omega_{nA, nB} = \rho^*(\omega_n), \quad \eta_{nA, nB} = \rho^*(\eta_n) \quad (4.7)$$

$$\widetilde{\omega}_{nA, nB} = \rho^*(\widetilde{\omega}_n), \quad \widetilde{\eta}_{nA, nB} = \rho^*(\widetilde{\eta}_n) \quad (4.8)$$

$$\widehat{\omega}_n = N\rho^*(\omega_n), \quad \widehat{\eta}_n = N\rho^*(\eta_n). \quad (4.9)$$

Then we see that the pull-back ρ^* satisfies

$$\rho^*(H_{\text{dR}}^1(X_0/S_K)(n)) = H_{\text{dR}}^1(Y_K/S_K)(nA, nB), \quad 0 < n < N \quad (4.10)$$

and the push-forward ρ_* satisfies

$$\rho_*(H_{\text{dR}}^1(Y_K/S_K)(i, j)) = \begin{cases} H_{\text{dR}}^1(X_0/S_K)(n) & (i, j) \equiv (nA, nB) \pmod{N} \\ 0 & \text{otherwise} \end{cases} \quad (4.11)$$

for $0 < i, j < N$. One has

$$\rho_*(\omega_{nA, nB}), \quad \rho_*(\eta_{nA, nB}) \quad (4.12)$$

form a basis of $H_{\text{dR}}^1(X_0/S_K)(n)$ and

$$\rho^* \rho_*(\omega_{nA, nB}) = N\omega_{nA, nB}.$$

from (4.10), (4.11) and $\rho_* \rho^* = N$.

4.2 Review of paper [1]

In this part, we will give a brief review of what we need in the paper [1]. First, let us recall the definition of p -adic hypergeometric functions of logarithmic type.

Definition 4.5 (*p*-adic hypergeometric functions of logarithmic type).

Let $s \geq 1$ be a positive integer. Let $(a_1, \dots, a_s) \in \mathbb{Z}_p$ and $(a'_1, \dots, a'_s)$ where a'_i denotes the Dwork prime. Let $\sigma : W[[t]] \rightarrow W[[t]]$ be the p -th Frobenius endomorphism given by $\sigma(t) = ct^p$ with $c \in 1 + pW$. Then we define the p -adic hypergeometric functions of logarithmic type $\mathcal{F}_{a_1, \dots, a_s}^{(\sigma)}(t)$ to be

$$\frac{1}{F_{a_1, \dots, a_s}(t)} \left[\psi_p(a_1) + \dots + \psi_p(a_s) + s\gamma_p - p^{-1} \log(c) + \int_0^t (F_{a_1, \dots, a_s}(t) - F_{a'_1, \dots, a'_s}(t^\sigma)) \frac{dt}{t} \right]$$

where $\psi_p(z)$ is the p -adic digamma function ([1, §2.2]), γ_p is the p -adic Euler constant ([1, §2.2]) and $\log(z)$ is the Iwasawa logarithmic function.

Definition 4.6 (cf. [6, 3.1.1]). Let $W = W(\bar{k})$ where $\bar{k}$ is an algebraic closed field with $\text{char } \bar{k} > 0$. Define

$$\begin{aligned} W\langle t_1, \dots, t_n \rangle &:= \varprojlim_n W/p^n[t_1, \dots, t_n] \\ &= \left\{ \sum_I a_I t^I \in W[[t_1, \dots, t_n]] \mid |a_I|_p \rightarrow 0 \text{ as } |I| \rightarrow \infty \right\} \\ W[t_1, \dots, t_n]^\dagger &:= \left\{ \sum_I a_I t^I \in W[[t_1, \dots, t_n]] \mid \exists r > 1 \text{ such that} \right. \\ &\quad \left. |a_I|_p r^{|I|} \rightarrow 0 \text{ as } |I| \rightarrow \infty \right\}. \end{aligned}$$

Furthermore, if

$$A = W[t_1, \dots, t_n]/(f_1, \dots, f_r),$$

we define the weak completion of A to be

$$A^\dagger := W[t_1, \dots, t_n]^\dagger/(f_1, \dots, f_r).$$

Definition 4.7. Let $F : W \rightarrow W$ be the p -th Frobenius. Then $\sigma : A^\dagger \rightarrow A^\dagger$ is called a p -th Frobenius if

- σ is F -linear ($\sigma(\alpha x) = F(\alpha)\sigma(x)$, $\alpha \in W, x \in A^\dagger$)
- $\sigma \bmod p$ on $A^\dagger/pA^\dagger \simeq A/pA$ is given by $x \rightarrow x^p$.

Let σ be a p -th Frobenius on $W[t, (t - t^2)^{-1}]^\dagger$ which extends on $K[t, (t - t^2)^{-1}]^\dagger := K \otimes W[t, (t - t^2)^{-1}]^\dagger$. Write $X_{\bar{\mathbb{F}}_p} := X_W \times_W \bar{\mathbb{F}}_p$ and $S_{\bar{\mathbb{F}}_p} := S_W \times_W \bar{\mathbb{F}}_p$. Then the rigid cohomology groups

$$H_{\text{rig}}^\bullet(X_{\bar{\mathbb{F}}_p}/S_{\bar{\mathbb{F}}_p})$$

are defined. We refer the book [7] for the theory of rigid cohomology.

The required properties are the following.

- The cohomology group $H_{\text{rig}}^\bullet(X_{\overline{\mathbb{F}}_p}/S_{\overline{\mathbb{F}}_p})$ is a finitely generated $\mathcal{O}(S_K)^\dagger$ -module.
- (Frobenius) The p -th Frobenius Φ on $H_{\text{rig}}^\bullet(X_{\overline{\mathbb{F}}_p}/S_{\overline{\mathbb{F}}_p})$ (depending on σ) is defined. This is a σ -linear endomorphism:

$$\Phi(f(t)x) = \sigma(f(t))\Phi(x), \text{ for } x \in H_{\text{rig}}^\bullet(X_{\overline{\mathbb{F}}_p}/S_{\overline{\mathbb{F}}_p}), f(t) \in \mathcal{O}(S_K)^\dagger.$$

- (Comparison) There is the comparison isomorphism with algebraic de Rham cohomology,

$$c : H_{\text{rig}}^\bullet(X_{\overline{\mathbb{F}}_p}/S_{\overline{\mathbb{F}}_p}) \cong H_{\text{dR}}^\bullet(X_0/S_K) \otimes_{\mathcal{O}(S_K)} \mathcal{O}(S_K)^\dagger.$$

Theorem 4.8. *Let σ be a p -Frobenius on $\mathcal{O}(S_K)^\dagger$ such that $\sigma(t) = ct^p$ with $c \in 1 + pW$. Then there exists an exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathcal{O}(S_K)^\dagger \otimes_{\mathcal{O}(S_K)} H_{\text{dR}}^1(X_0/S_K) &\longrightarrow \mathcal{O}(S_K)^\dagger \otimes_{\mathcal{O}(S_K)} M_\xi(X_0/S_K) \\ &\longrightarrow \mathcal{O}(S_K)^\dagger \longrightarrow 0 \end{aligned}$$

endowed with

- Frobenius Φ -action which is σ -linear
- $\text{Fil}^i \subseteq M_\xi(X_0/S_K)$ (Hodge filtration) with

$$\mathcal{O}(S_K)^\dagger \otimes_{\mathcal{O}(S_K)} \text{Fil}^0 M_\xi(X_0/S_K) \xrightarrow{\sim} \mathcal{O}(S_K)^\dagger$$

In particular, from this isomorphism, there exists a unique lifting e_ξ in $\mathcal{O}(S_K)^\dagger \otimes_{\mathcal{O}(S_K)} \text{Fil}^0 M_\xi(X_0/S_K)$ of $1 \in \mathcal{O}(S_K)^\dagger$.

Proof. Recall $U = \text{Spec} R[u, v] / ((1 - u^N)(1 - v^N) - t) \subset Y := \overline{U}$ where Y is the closure in $\mathbb{P}_R^1 \times \mathbb{P}_R^1$. For $(\nu_1, \nu_2) \in \mu_N(W) \times \mu_N(W)$, put

$$\xi(\nu_1, \nu_2) = \left\{ \frac{u-1}{u-\nu_2}, \frac{v-1}{v-\nu_2} \right\} \in K_2^M(\mathcal{O}(U)).$$

Then according to [3, §2.6], we have the 1-extension

$$0 \longrightarrow H^1(Y/S)(2) \longrightarrow M_{\xi(\nu_1, \nu_2)}(Y/S) \longrightarrow \mathcal{O}_S \longrightarrow 0 \quad (4.13)$$

in the category of Fil- F -MIC(S, σ).

Let $\xi := \sum_{i=0}^{N-1} (g_{A,B}^i)^* \xi(\nu_1, \nu_2) \in K_2^M(\mathcal{O}(U))$ which is fixed under the action of G , so that G acts on $M_\xi(Y/S)$. Taking the fixed part of (4.13) by $\langle g_{A,B} \rangle$, we have a 1-extension

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(Y/S)^G(2) & \longrightarrow & M_\xi(Y/S)^G & \longrightarrow & \mathcal{O}_S \longrightarrow 0 \\ & & \parallel & & \parallel & & \\ & & H & & M_\xi & & \end{array}$$

By Lemma 4.3, it follows

- $H_{\text{dR}} \simeq H_{\text{dR}}^1(X_0/S_K)$
- $H_{\text{rig}} \simeq \mathcal{O}(S_K)^\dagger \otimes H_{\text{dR}}^1(X_0/S_K)$
- $\text{Fil}^i H_{\text{dR}} = \text{Fil}^{i+2} H_{\text{dR}}^1(X_0/S_K)$ where $\text{Fil}^\bullet$ in the right denotes the Hodge filtration. In particular, $\text{Fil}^0 H_{\text{dR}} = 0$.

Therefore there is a unique element $e_\xi \in \text{Fil}^0 M_\xi$ which is a lifting of $1 \in \mathcal{O}(S_K)$. $\square$

Let $1 \leq n \leq N-1$ be an integer and A, B be integers such that $1 \leq A, B < N$ and $\gcd(A, N) = \gcd(B, N) = 1$.

Put

$$a_n := \left\{ \frac{-nA}{N} \right\}, \quad b_n := \left\{ \frac{-nB}{N} \right\}$$

where $\{x\} := x - \lfloor x \rfloor$ denotes the fractional part. Let

$$F_n(t) := \sum_{i=0}^{\infty} \frac{(a_n)_i}{i!} \frac{(b_n)_i}{i!} t^i \in \mathbb{Z}_p[[t]]$$

be the hypergeometric power series. Put

$$e_{i,j}^{\text{unit}} := (1-t)^{-a_i-b_j} F_{a_i, b_j}(t)^{-1} \tilde{\eta}_{i,j}.$$

Using the group G in Lemma 4.3, we define

$$\hat{\omega}_n = \sum_{g \in G} g^* \omega_{nA, nB}, \quad e_n^{\text{unit}} := \sum_{g \in G} g^* e_{nA, nB}^{\text{unit}}.$$

Let e_ξ be the unique lifting of 1 in the above theorem.

Lemma 4.9.

$$\nabla(e_\xi) = - \sum_{n=1}^{N-1} \frac{(1 - \nu_1^{-nA})(1 - \nu_2^{-nB})}{N^2} \frac{dt}{t} \hat{\omega}_n.$$

Proof. Let $\xi := \sum_{i=0}^{N-1} g_{A,B}^i \xi(\nu_1, \nu_2)$ with

$$\xi(\nu_1, \nu_2) = \left\{ \frac{x-1}{x-\nu_1}, \frac{y-1}{y-\nu_2} \right\}. \quad (4.14)$$

Following from the compatibility of the connection with respect to the pull-back, we have

$$\nabla(e_\xi) = \sum_{i=0}^{N-1} (g_{A,B}^*)^i \nabla(e_{\xi(\nu_1, \nu_2)}).$$

Then by [1, §4.4 (4.25)], we have

$$\nabla(e_{\xi(\nu_1, \nu_2)}) = -d \log(\xi(\nu_1, \nu_2)) = - \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \frac{(1 - \nu_1^{-i})(1 - \nu_2^{-j})}{N^2} \frac{dt}{t} \omega_{i,j}.$$

So we have

$$\begin{aligned} \nabla(e_{\xi}) &= \sum_{i=0}^{N-1} (g_{A,B}^*)^i \nabla(e_{\xi(\nu_1, \nu_2)}) \\ &= - \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \frac{(1 - \nu_1^{-i})(1 - \nu_2^{-j})}{N^2} \frac{dt}{t} \sum_{g \in G} g^* \omega_{i,j} \\ &= - \sum_{n=1}^{N-1} \frac{(1 - \nu_1^{-nA})(1 - \nu_2^{-nB})}{N^2} \frac{dt}{t} \sum_{g \in G} g^* \omega_{nA, nB} \\ &= - \sum_{n=1}^{N-1} \frac{(1 - \nu_1^{-nA})(1 - \nu_2^{-nB})}{N^2} \frac{dt}{t} \widehat{\omega}_n. \end{aligned}$$

□

Lemma 4.10. Assume σ is given by $\sigma(t) = ct^p$ with $c \in 1 + pW$. Let $h(t) = \prod_{m=0}^s F_{a_n^{(m)}, b_n^{(m)}}(t)_{<p}$ where s is the minimal integer such that $(a_n^{(s+1)}, b_n^{(s+1)}) = (a_n, b_n)$ for all $n \in \{1, 2, \dots, N-1\}$. Then

$$e_{\xi} - \Phi(e_{\xi}) \equiv - \sum_{n=1}^{N-1} \frac{(1 - \nu_1^{-nA})(1 - \nu_2^{-nB})}{N^2} \mathcal{F}_{a_n, b_n}^{(\sigma)}(t) \widehat{\omega}_n$$

modulo $\sum_{n=1}^{N-1} K \langle t, (t - t^2)^{-1}, h(t)^{-1} \rangle e_n^{\text{unit}}$.

Proof. See [1, Theorem 4.19].

□

4.3 Transformation Formula

Let $a_i \in \mathbb{Z}_p$ ($0 \leq i \leq r-1$) and put $h(t) := \prod_{i=0}^{r-1} F_{a_i, \dots, a_i}(t)_{<p}$, where $F_{a_i, \dots, a_i}(t)$ is hypergeometric power series. Then there is an involution

$$\omega : W \langle t, t^{-1}, h(t)^{-1} \rangle \longrightarrow W \langle t, t^{-1}, h(t)^{-1} \rangle, \quad \omega(f(t)) = f(t^{-1}).$$

This follows from the following proposition.

Proposition 4.11. (1) Let $a \in \mathbb{Z}_p$. Let $F(t) := F_{a, \dots, a}(t)_{<p} \bmod p$. Then $F(t) = (-1)^{ls} t^l F(t^{-1})$ in $\mathbb{F}_p[t]$ where l is the degree of $F(t)$ which equals the unique integer in $\{0, 1, \dots, p-1\}$ such that $a + l \equiv 0 \bmod p$.

(2) Let $a_i \in \mathbb{Z}_p$ ($0 \leq i \leq r-1$) and put $h(t) := \prod_{i=0}^{r-1} F_{a_i, \dots, a_i}(t)_{<p}$. Then there is a ring homomorphism

$$\omega_n : W/p^n[t, t^{-1}, h(t)^{-1}] \rightarrow W/p^n[t, t^{-1}, h(t)^{-1}], \quad f(t) \mapsto f(t^{-1}).$$

(3) There is an involution

$$\omega : W\langle t, t^{-1}, h(t)^{-1} \rangle \longrightarrow W\langle t, t^{-1}, h(t)^{-1} \rangle, \quad \omega(f(t)) = f(t^{-1}).$$

Proof. (1) Write

$$F(t) = \sum_{i=0}^l \left(\frac{(a)_i}{i!} \right)^s t^i.$$

Since $F(t) = F_{a, \dots, a}(t)_{<p} \pmod{p}$, we have $(a)_l \not\equiv 0 \pmod{p}$ and $(a)_{l+1} \equiv 0 \pmod{p}$. That is, we have $a + l \equiv 0 \pmod{p}$.

If $l = i + j$, then

$$\frac{(a)_i}{i!} \equiv \frac{(-l)_i}{i!} = (-1)^i \binom{l}{i} = (-1)^i \binom{l}{j} = (-1)^l \frac{(-l)_j}{j!} \equiv (-1)^l \frac{(a)_j}{j!} \pmod{p}.$$

Therefore, we have

$$\left(\frac{(a)_i}{i!} \right)^s \equiv (-1)^{ls} \left(\frac{(a)_j}{j!} \right)^s \pmod{p}.$$

This implies

$$t^l F(t^{-1}) = (-1)^{ls} F(t).$$

(2) Observe that it is enough to show that

$$h(t^{-1}) \in (W/p^n[t, t^{-1}, h(t)^{-1}])^\times.$$

An element is a unit in $W\langle t, t^{-1}, h(t)^{-1} \rangle$ if and only if it is a unit modulo $pW[t, t^{-1}, h(t)^{-1}]$. So if we can prove $h(t^{-1})$ is a unit modulo $pW[t, t^{-1}, h(t)^{-1}]$, then we are done. Put

$$F_i(t) := F_{a_i, \dots, a_i}(t)_{<p} \pmod{p}.$$

From (1), we have

$$F_i(t) = \pm t^{l_i} F_i(t^{-1})$$

where l_i is the degree of $F_i(t)$. Hence we have

$$F_i(t^{-1})^{-1} = \frac{\pm t^{l_i}}{F_i(t)}$$

in $W/p[t, t^{-1}, h(t)^{-1}]$, i.e., $F_i(t)$ is a unit in $W/p[t, t^{-1}, h(t)^{-1}]$. Since $h(t)$ is the product of $F_i(t)$, it is also a unit.

(3) ω is defined by using ω_n in (2) in the following way:

$$\omega : W\langle t, t^{-1}, h(t)^{-1} \rangle \longrightarrow W\langle t, t^{-1}, h(t)^{-1} \rangle, \quad (f_n(t)) \mapsto (\omega_n f_n(t)).$$

By (2), ω is well-defined and also it is an involution. $\square$

Conjecture 4.12 (Transformation Formula between $\mathcal{F}_{a,\dots,a}^{(\sigma)}(t)$ and $\widehat{\mathcal{F}}_{a,\dots,a}^{(\hat{\sigma})}(t)$). Let $\sigma(t) = ct^p$ and $\hat{\sigma}(t) = c^{-1}t^p$. Let $a \in \mathbb{Z}_p \setminus \mathbb{Z}_{\leq 0}$ and the r th Dwork prime $a^{(r)} = a$ for some $r > 0$. Put $h(t) := \prod_{i=0}^{r-1} F_{a^{(i)},\dots,a^{(i)}}(t)_{<p}$, then

$$\mathcal{F}_{a,\dots,a}^{(\sigma)}(t) = -\widehat{\mathcal{F}}_{a,\dots,a}^{(\hat{\sigma})}(t^{-1})$$

in the ring $W\langle t, t^{-1}, h(t)^{-1} \rangle$ where $\widehat{\mathcal{F}}_{a,\dots,a}^{(\hat{\sigma})}(t^{-1})$ is defined as $\omega(\widehat{\mathcal{F}}_{a,\dots,a}^{(\hat{\sigma})}(t))$ and $\mathcal{F}_{a,\dots,a}^{(\sigma)}(t)$ is the p -adic hypergeometric functions of logarithmic type.

Before moving to the next theorem, we prove, as an example, that this conjecture is true modulo p .

Example 4.13. By Theorem 3.1 and [1, Theorem 3.3], we know that

$$\mathcal{F}_{a,\dots,a}^{(\sigma)}(t) \equiv \frac{G_{a,\dots,a}^{(\sigma)}(t)_{<p}}{F_{a,\dots,a}(t)_{<p}}, \quad \widehat{\mathcal{F}}_{a,\dots,a}^{(\hat{\sigma})}(t^{-1}) \equiv \frac{\widehat{G}_{a,\dots,a}^{(\hat{\sigma})}(t)_{<p}}{F_{a,\dots,a}(t)_{<p}} \Big|_{t^{-1}} \pmod{pW[[t]]}.$$

Let $F(t) = F_{a,\dots,a}(t)_{<p}$, $G(t) \equiv G_{a,\dots,a}^{(\sigma)}(t)_{<p}$ and $\widehat{G}(t) \equiv \widehat{G}_{a,\dots,a}^{(\hat{\sigma})}(t)_{<p} \pmod{p}$.

Then by Proposition 4.11, one has

$$\frac{G(t)}{F(t)} + \frac{\widehat{G}(t^{-1})}{F(t^{-1})} = \frac{G(t)}{(-1)^{ls}t^l F(t^{-1})} + \frac{\widehat{G}(t^{-1})}{F(t^{-1})} = \frac{G(t) + (-1)^{ls}t^l \widehat{G}(t^{-1})}{(-1)^{ls}t^l F(t^{-1})}.$$

We claim that $G(t) + (-1)^{ls}t^l \widehat{G}(t^{-1}) \equiv 0 \pmod{p}$. Indeed, since

$$B_k = A_k \frac{B_k}{A_k}, \quad \widehat{B}_k = A_k \frac{\widehat{B}_k}{A_k} \equiv 0 \pmod{p}$$

when $l+1 \leq k \leq p-1$, we have $G(t) = \sum_{i=0}^l B_k t^k$ and $\widehat{G}(t) = \sum_{i=0}^l \widehat{B}_k t^k$. Thus

$$G(t) + (-1)^{ls}t^l \widehat{G}(t^{-1}) = \sum_{i=0}^l B_k t^k + (-1)^{ls} \sum_{i=0}^l \widehat{B}_{l-k} t^k = \sum_{i=0}^l \left(B_k + (-1)^{ls} \widehat{B}_{l-k} \right) t^k.$$

Then for $1 \leq k \leq l$, we have

$$B_k + (-1)^{ls} \widehat{B}_{l-k} = \frac{A_k}{k} + (-1)^{ls} \frac{A_{l-k}}{a+l-k} \equiv \frac{1}{k} \left(A_k - (-1)^{ls} A_{l-k} \right) \equiv 0 \pmod{p}.$$

When $k = 0$, we have

$$\begin{aligned} & B_0 + (-1)^{ls} \widehat{B}_l \\ &= s(\psi_p(a) + \gamma_p) - p^{-1} \log(c) - (-1)^{ls} \frac{1}{a+l} \left(A_l - c^{\frac{l+a}{p}} (-1)^{se} \right) \\ &= s(\psi_p(a) + \gamma_p) - (-1)^{ls} \frac{A_l}{a+l} - p^{-1} \log(c) + (-1)^{s(e+l)} \frac{c^{\frac{l+a}{p}}}{a+l}. \end{aligned}$$

Write $c = 1 + pz$ and $a + l = dp^n$ with $p \nmid d$. Then

$$-p^{-1} \log(c) + (-1)^{s(e+l)} \frac{c^{\frac{l+a}{p}}}{a+l} \equiv (-1)^{s(e+l)} \frac{1}{a+l} \pmod{p}.$$

So we obtain

$$\begin{aligned} & B_0 + (-1)^{ls} \widehat{B}_l \\ & \equiv s(\psi_p(a) + \gamma_p) + (-1)^{ls} \frac{A_l - (-1)^{se}}{a+l} \\ & \stackrel{(*)}{\equiv} s(\psi_p(-l) + \gamma_p) - s(\psi_p(1+l) + \gamma_p) \pmod{p} \end{aligned}$$

where $(*)$ follows from [1, (2.13)] and imitate the proof of Lemma 3.9. Since $\psi_p(-l) = \psi_p(1+l)$ (cf. [1, Theorem 2.4 (2)]), we obtain

$$B_0 + (-1)^{ls} \widehat{B}_l \equiv 0 \pmod{p}.$$

Recall $a_n = \{-nA/N\}$ in Section 4.2. Let $r \in \mathbb{Z}_{\geq 1}$ be a number such that $a_n^{(r)} = a_n$. We will prove a special case of Conjecture 4.9 in this section.

Theorem 4.14. *Let $\sigma(t) = ct^p$ and $\widehat{\sigma}(t) = c^{-1}t^p$. Put $h(t) = \prod_{i=0}^{r-1} F_{a_n^{(i)}, a_n^{(i)}}(t)_{<p}$. Then*

$$\mathcal{F}_{a_n, a_n}^{(\sigma)}(t) = -\widehat{\mathcal{F}}_{a_n, a_n}^{(\widehat{\sigma})}(t^{-1})$$

in the ring $W\langle t, t^{-1}, h(t)^{-1} \rangle$ where $\mathcal{F}_{a_n, a_n}^{(\sigma)}(t)$ is the p -adic hypergeometric functions of logarithmic type.

Before giving the proof, we need some settings. First, we assume $A = B$ and let $t_0^N = t$. Then we have two descriptions of the quotient curve Y_G/S_K (Lemma 4.3):

(i) $Y_G \simeq X$ with coordinates (x, y, t_0) ,

$$X : y^N = x^A(1-x)^A(1-(1-t_0^N)x)^{N-A}$$

(ii) $Y_G \simeq \widehat{X}$ with coordinates (z, w, s_0) ,

$$\widehat{X} : w^N = z^A(1-z)^A(1-(1-s_0^N)z)^{N-A}$$

where $z = 1 - x, w = t_0^{A-N}y, s_0 = t_0^{-1}$.

Lemma 4.15. *Assume $\widehat{\sigma}$ is given by $\widehat{\sigma}(t) = c^{-1}t^p$ with $c \in 1 + pW$. Let $h(t) = \prod_{m=0}^s F_{a_n^{(m)}, b_n^{(m)}}(t)_{<p}$ where s is the minimal integer such that $(a_n^{(s+1)}, b_n^{(s+1)}) = (a_n, b_n)$ for all $n \in \{1, 2, \dots, N-1\}$. Then*

$$e_\xi - \Phi(e_\xi) \equiv \sum_{n=1}^{N-1} \frac{(1 - \nu_1^{-nA})(1 - \nu_2^{-nB})}{N^2} \widehat{\mathcal{F}}_{a_n, b_n}^{(\widehat{\sigma})}(s_0^N) \widehat{\omega}_n$$

modulo $\sum_{n=1}^{N-1} K\langle t, (t - t^2)^{-1}, h(t)^{-1} \rangle e_n^{\text{unit}}$.

Proof. We consider the quotient map (Lemma 4.3) $Y_G \simeq \widehat{X}$ in coordinate (z, w, s_0) . From the coordinate (z, w, s_0) , let $e_{\widehat{\xi}}^{\widehat{X}}$ be the unique lifting in Theorem 4.8. We write $\widehat{e}_{\xi}$ for $e_{\widehat{\xi}}^{\widehat{X}}$ via the change of coordinate $(z, w, s_0) = (1 - x, t_0^{A-N}y, t_0^{-1})$, and $\widehat{\Phi}$ is the Frobenius action which is $\widehat{\sigma}$ -linear. Write

$$\widehat{e}_{\xi} - \widehat{\Phi}(\widehat{e}_{\xi}) = \sum_{n=1}^{N-1} \frac{(1 - \nu_1^{-nA})(1 - \nu_2^{-nB})}{N} \widehat{E}_1^{(n)}(t_0) \widetilde{\omega}_n + \widehat{E}_2^{(n)}(t_0) \widetilde{\eta}_n \quad (4.15)$$

in $K((t)) \otimes H_{\text{dR}}^1(X/S_K)$. We apply the Gauss-Manin connection ∇ on (4.15). Since $\nabla \widehat{\Phi} = \widehat{\Phi} \nabla$ and

$$\nabla(\widehat{e}_{\xi}) = - \sum_{n=1}^{N-1} \frac{(1 - \nu_1^{-nA})(1 - \nu_2^{-nA})}{N^2} \frac{dt}{t} t^{a_n} (N \omega_n),$$

from Lemma 4.9 and change of coordinate $(z, w, s_0) = (1 - x, t_0^{A-N}y, t_0^{-1})$, we have

$$\begin{aligned} & - \sum_{n=1}^{N-1} \frac{(1 - \nu_1^{-nA})(1 - \nu_2^{-nA})}{N} (1 - \widehat{\Phi}) \left(t^{a_n} F_n(t) \frac{dt}{t} \wedge \widetilde{\omega}_n \right) \\ &= \sum_{n=1}^{N-1} \frac{(1 - \nu_1^{-nA})(1 - \nu_2^{-nA})}{N} \nabla(\widehat{E}_1^{(n)}(t_0) \widetilde{\omega}_n + \widehat{E}_2^{(n)}(t_0) \widetilde{\eta}_n). \end{aligned} \quad (4.16)$$

By [1, Proposition 4.7], we have $\widehat{\Phi}(\widetilde{\omega}_m) \equiv p^{-1} \widetilde{\omega}_n \pmod{K((t_0)) \widetilde{\eta}_n}$ where m is the unique integer in $\{1, 2, \dots, N-1\}$ such that $pm \equiv n \pmod{N}$.

Therefore

$$\text{LHS of (4.16)} \equiv - \sum_{n=1}^{N-1} \frac{(1 - \nu_1^{-nA})(1 - \nu_2^{-nA})}{N} \left[t^{a_n} F_n(t) - (t^{a_m} F_m(t))^{\widehat{\sigma}} \right] \frac{dt}{t} \wedge \widetilde{\omega}_n.$$

On the other hand, it follows from [1, Proposition 4.3], we have

$$\text{RHS of (4.16)} \equiv \sum_{n=1}^{N-1} \frac{(1 - \nu_1^{-nA})(1 - \nu_2^{-nA})}{N} \frac{d\widehat{E}_1^{(n)}}{dt} dt \wedge \widetilde{\omega}_n \pmod{K((t_0)) \widetilde{\eta}_n}.$$

Thus

$$\frac{d\widehat{E}_1^{(n)}}{dt} = - \frac{t^{a_n} F_n(t) - (t^{a_m} F_m(t))^{\widehat{\sigma}}}{t}.$$

Namely, we have

$$-\widehat{E}_1^{(n)}(t_0) = C + \int_0^t \frac{t^{a_n} F_n(t) - (t^{a_m} F_m(t))^{\widehat{\sigma}}}{t} dt$$

for some constant $C \in K$ and with $t = t_0^N$. We claim that this constant C is 0. Indeed, since $\widehat{E}_1^{(n)}(t_0)/F_n(t_0^N)$ is an overconvergent function, $\widehat{E}_1^{(n)}(t_0)t_0^{-Na_n}/F_n(t_0^N)$ is also overconvergent.

If $C = 0$, then $\widehat{E}_1^{(n)}(t_0)t_0^{-Na_n}/F_n(t_0^N) = \widehat{\mathcal{F}}_{a_n, a_n}^{(\widehat{\sigma})}(t)$ is a convergent function by Corollary 3.2. If there is another C' such that $\widehat{E}_1^{(n)}(t_0)t_0^{-Na_n}/F_n(t_0^N)$ is a convergent function, then after subtraction, we have

$$\frac{C't_0^{-Na_n}}{F_n(t_0^N)} \in K\langle t_0, (t_0 - t_0^2)^{-1}, h(t_0^N)^{-1} \rangle.$$

This is a contradiction (as it is shown in the proof of [1, proof in Theorem 4.9]). So C must be 0. Therefore we have

$$\widehat{e}_\xi - \widehat{\Phi}(\widehat{e}_\xi) = \sum_{n=1}^{N-1} \frac{(1 - \nu_1^{-nA})(1 - \nu_2^{-nB})}{N} [-\widehat{\mathcal{F}}_{a_n, a_n}^{(\widehat{\sigma})}(t) \cdot t^{a_n}] \omega_n \mod \widetilde{\eta}_n.$$

Now we write everything back to coordinate (z, w, s_0) . Since $\text{Ker} \nabla$ is generated by $\{\widetilde{\eta}_n\}$ over K (Lemma 4.2) and $\text{Ker} \nabla$ contains in $\text{Ker} \nabla$ via change of coordinate, we have

$$e_\xi^{\widehat{X}} - \Phi(e_\xi^{\widehat{X}}) = \sum_{n=1}^{N-1} \frac{(1 - \nu_1^{-nA})(1 - \nu_2^{-nB})}{N^2} [\widehat{\mathcal{F}}_{a_n, a_n}^{(\widehat{\sigma})}(s_0^{-N})] N \omega_n \mod \widetilde{\eta}_n.$$

in coordinate (z, w, s_0) . Therefore from Lemma 4.4, we have

$$e_\xi - \Phi(e_\xi) \equiv \sum_{n=1}^{N-1} \frac{(1 - \nu_1^{-nA})(1 - \nu_2^{-nB})}{N^2} \widehat{\mathcal{F}}_{a_n, b_n}^{(\widehat{\sigma})}(t^{-1}) \widehat{\omega}_n$$

modulo $\sum_{n=1}^{N-1} K\langle t, (t - t^2)^{-1}, h(t)^{-1} \rangle e_n^{\text{unit}}$. This completes the proof. $\square$

Using these lemmas, we can prove Theorem 4.14.

(*proof of Theorem 4.14*). By Lemma 4.10, we know

$$e_\xi - \Phi(e_\xi) \equiv \sum_{n=1}^{N-1} \frac{(1 - \nu_1^{-nA})(1 - \nu_2^{-nA})}{N^2} [-\mathcal{F}_{a_n, a_n}^{(\sigma)}(t)] \widehat{\omega}_n \quad (4.17)$$

modulo $\sum_{n=1}^{N-1} K\langle t, (t - t^2)^{-1}, h(t)^{-1} \rangle e_n^{\text{unit}}$.

Also, by Lemma 4.15, we have

$$e_\xi - \Phi(e_\xi) \equiv \sum_{n=1}^{N-1} \frac{(1 - \nu_1^{-nA})(1 - \nu_2^{-nA})}{N^2} [\widehat{\mathcal{F}}_{a_n, a_n}^{(\widehat{\sigma})}(t^{-1})] \widehat{\omega}_n \quad (4.18)$$

modulo $\sum_{n=1}^{N-1} K\langle t, (t - t^2)^{-1}, h(t)^{-1} \rangle e_n^{\text{unit}}$.

Comparing equations (4.17) and (4.18), we have

$$-\widehat{\mathcal{F}}_{a_n, a_n}^{(\widehat{\sigma})}(t) = \mathcal{F}_{a_n, a_n}^{(\sigma)}(t^{-1}).$$

This completes the proof. $\square$

4.4 Transformation formula on Dwork's p -adic Hypergeometric Functions

In this section, the settings are the same as in Section 4.3. First, let us recall the definition of Dwork's p -adic hypergeometric functions.

Definition 4.16 (Dwork's p -adic hypergeometric functions). *Let $s \geq 1$ be an integer. For $(a_1, \dots, a_s) \in \mathbb{Z}_p^s$, Dwork's p -adic hypergeometric function is defined as*

$$\mathcal{F}_{a_1, \dots, a_s}^{\text{Dw}}(t) := F_{a_1, \dots, a_s}(t) / F_{a'_1, \dots, a'_s}(t^p)$$

where $F_{a_1, \dots, a_s}(t)$ and $F_{a'_1, \dots, a'_s}(t)$ are hypergeometric power series.

Remark 4.17. *We have Dwork's p -adic hypergeometric function belongs to $W\langle t, f(t)^{-1} \rangle$ where*

$$f(t) := \prod_{i=0}^N F_{a_1^{(i)}, \dots, a_s^{(i)}}(t)_{<p}$$

and N is an integer such that

$$\{\overline{F_{a_1^{(i)}, \dots, a_s^{(i)}}(t)_{<p}} \mid i \in \mathbb{Z}_{\geq 0}\} = \{\overline{F_{a_1^{(i)}, \dots, a_s^{(i)}}(t)_{<p}} \mid i = 0, 1, \dots, N\}$$

with $\overline{f(t)} = f(t) \pmod{p}$ (consequence of Dwork's congruence).

Conjecture 4.18 (Transformation formula on Dwork's p -adic Hypergeometric Functions). *Suppose $a_1 = \dots = a_s = a$. Let l is the unique integer in $\{0, 1, \dots, p-1\}$ such that $a + l \equiv 0 \pmod{p}$. Then*

$$\mathcal{F}_{a, \dots, a}^{\text{Dw}}(t) = ((-1)^s t)^l \mathcal{F}_{a, \dots, a}^{\text{Dw}}(t^{-1}).$$

Remark 4.19. *From Proposition 4.11, we have Conjecture 4.18 holds modulo p .*

We have the special case of Conjecture 4.18.

Theorem 4.20. *Let $a \in \frac{1}{N}\mathbb{Z}, 0 < a < 1$ and $p > N$. We have*

$$\mathcal{F}_{a,a}^{\text{Dw}}(t) = t^l \mathcal{F}_{a,a}^{\text{Dw}}(t^{-1}),$$

where l is the unique integer in $\{0, 1, \dots, p-1\}$ such that $a + l \equiv 0 \pmod{p}$.

Proof. Let $\sigma : W[[t]] \rightarrow W[[t]]$ be the p -th Frobenius given by $\sigma(t) = t^p$, and let Φ be the p -th Frobenius induced by σ on $K((t_0)) \otimes H_{\text{dR}}^1(Y/S)$ where Y/S is the hypergeometric curve in §4.1.1. By [1, Proposition 4.7], the p -th Frobenius Φ induces a map with

$$\Phi(\tilde{\omega}_{p^{-1}nA, p^{-1}nA}) \equiv p^{-1} \tilde{\omega}_{nA, nA} \pmod{K((t)) \tilde{\eta}_{nA, nA}}.$$

Let $m \equiv p^{-1}nA \pmod{N}$ with $1 \leq m \leq N-1$ and $k \equiv nA \pmod{N}$ with $1 \leq n \leq N-1$. Then we have

$$\Phi(\omega_{p^{-1}nA, p^{-1}nA}) \equiv p^{-1} \frac{F_m(t^\sigma)}{F_k(t)} \omega_{nA, nA} = p^{-1} \mathcal{F}_{a_n, a_n}^{\text{Dw}}(t)^{-1} \omega_{nA, nA} \quad (4.19)$$

where $a_n = \{-nA/N\}$.

Therefore we obtain

$$\Phi(\hat{\omega}_m) \equiv p^{-1} \mathcal{F}_{a_n, a_n}^{\text{Dw}}(t)^{-1} \hat{\omega}_n. \quad (4.20)$$

Recall that we have two description of Y/G . From the coordinate (z, w, s_0) , one has

$$\Phi(\tilde{\omega}_m) \equiv p^{-1} \mathcal{F}_{a_n, a_n}^{\text{Dw}}(s_0^N)^{-1} \tilde{\omega}_n. \quad (4.21)$$

By using $(z, w, s_0) = (1-x, t_0^{A-N}y, t_0^{-1})$, we obtain

$$\Phi(-t^{a_m} \tilde{\omega}_m) \equiv p^{-1} \mathcal{F}_{a_n, a_n}^{\text{Dw}}(t^{-1})^{-1} (-t^{a_n}) \tilde{\omega}_n. \quad (4.22)$$

On the other hand, we have

$$\Phi(-t^{a_m} \tilde{\omega}_m) \equiv (-t^{pa_m}) p^{-1} \mathcal{F}_{a_n, a_n}^{\text{Dw}}(t)^{-1} \tilde{\omega}_n \quad (4.23)$$

since $\Phi(t) = t^p$ and (4.20). Then comparing (4.22) and (4.23), we obtain the result. $\square$

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