Doctoral Thesis

Functional Renormalization Group Study for Continuous Symmetry Breaking Phases
（汎関数繰り込み群に基づく連続対称性の破れた相の研究）

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March, 2022

Acknowledgements

I would like to express my sincere gratitude to Professor Takafumi Kita for his research ideas, critical discussions, and many valuable suggestions through collaborative research. I would like to acknowledge Professor Koji Nemoto and Professor Hiroyuki Yamase for the opportunity to gain new knowledge from their discussions during the review process of this thesis. Finally, I would like to thank my parents for supporting me in my daily life.

Contents

Chapter 1

Introduction

1.1 Spontaneous symmetry breaking and Nambu–Goldstone mode

Symmetry is a universal concept that underlies physics, and it leads to the law of conservation of systems. For example, uniformity of space (translational symmetry) conserves the momentum of a system, while isotropy (rotational symmetry) leads to the law of conservation of angular momentum.

When symmetry is broken, physical properties change qualitatively. Iron is paramagnetic above the Curie temperature but becomes ferromagnetic below it, and the direction of magnetization does not change even when the external magnetic field is set to zero. In this case, the rotational symmetry in the spin space is broken from the macroscopic point of view, while the microscopic Hamiltonian has symmetry. Therefore, the system spontaneously breaks its symmetry.

Spontaneous symmetry breaking (SSB) is closely related to phase transition phenomena. In the ordered phase, where the symmetry is lowered, a gapless excitation (massless mode) called Nambu–Goldstone (NG) mode appears based on Goldstone's theorem [1]. Goldstone's theorem is a fundamental theorem on excitations in systems with spontaneously broken continuous symmetry. In systems with broken continuous symmetry, the physics remains nontrivial in the whole low-temperature phase due to the presence of NG modes, which implies that correlations decay algebraically. The emergence of NG modes leads to singularities in various physical quantities. For example, the divergence of the transverse component of the susceptibility at zero magnetic fields corresponds to the infrared (IR) divergence of the transverse fluctuation of the ordered variable, which can be explained as a consequence of Goldstone's theorem. On the other hand, the longitudinal component is considered finite as in the disordered phase. However, Schwabl et al. [2], and Patashinskii et al. [3] have shown that the longitudinal component of the susceptibility also diverges at zero fields. It was later found that this behavior is because the magnetic field applied parallel to the spontaneous magnetization is also coupled with phase fluctuations [4]. This indicates that in the ordered phase where NG modes appear, the correlation length defined in the disordered phase always diverges to infinity.

1.2 Functional renormalization group with Goldstone's theorem I

The perturbation expansion, which is widely used to analyze many-body problems, becomes useless with the appearance of the NG modes. For example, in the vicinity of second-order phase transition, it is well known that the perturbation expansion from the mean-field approximation breaks down due to the infrared (IR) divergence [5]. It is necessary to incorporate the effects of fluctuations beyond Gaussian fluctuations for describing the long-distance behavior of the correlation functions, as is the case in renormalization group theory.

Functional renormalization group (FRG) [6, 7, 8, 9, 10, 11] is one of the methods to incorporate fluctuations for systems with avoiding IR divergence. In this method, we first introduce an infrared cutoff Λ in the momentum part of the Euclidean action. Next, we derive the renormalization group equations obeyed by the thermodynamic potential and the n-point vertex when a small amount varies Λ. By solving the differential equation down to $\Lambda = 0$, we obtain the exact thermodynamic potential and n-point vertex.

In the spontaneously broken phase, the emergence of finite order parameters produces the non-trivial relation between vertices, and this can be concluded according to the proof of Goldstone's theorem I. In recent years, Kita [12] formulated the FRG combined with Goldstone's theorem I [1] for studying the BEC phase at finite temperatures. The advantages of this formulation are that (i) it guarantees that the excitation in the system is in massless mode and (ii) it dramatically reduces the number of renormalization group parameters. Although this method was initially formulated for the finite-temperature BEC phase, it can be extended to other systems with continuous symmetry. Therefore, it is possible to analyze the ordered phase in a non-perturbative manner.

1.3 Motivation and Present Study

The introduction of cutoff in the FRG can effectively study the ordered phase where NG modes appear. We will investigate the role of spontaneous symmetry breaking and NG mode. In this thesis, we consider the classical $O(N)$ model for $N > 1$ close to the critical point of the second-order phase transition. The order parameter and the three-point vertices are finite in the ordered phase. How will their existence affect the fixed points? We will find that the fixed points emerge such that they have different coupling constants between the longitudinal and transverse fluctuations. Next, as we have already seen above, the zero-field susceptibility (static susceptibility) and correlation length diverge even in the temperature range away from the critical point in the ordered phase with NG modes. It follows that the critical exponents associated with them cannot be defined. How can we recover the scaling laws that are usually held in this case? We will see that by introducing a length scale, called the Josephson length [13], the conventional scaling laws are restored.

This thesis is organized as follows. Chapter 2 summarized the effective action and the functional renormalization group that we use in this thesis. Goldstone's theorem is the key, and we determine the renormalization group parameters according to the derived identities. In chapter 3, we investigate the critical fixed points of the classical $O(N)$ model. Numerical analysis of the renormalization group equations in the spontaneous symmetry-breaking phase shows that, in dimensions lower than the upper critical dimension, (i)fixed points appear for which the three-point vertex function is finite, and (ii)the value of the critical exponent is different from that of the fixed points obtained in the symmetric phase. Finally, we conclude and summarize those results in chapter 4.

Chapter 2

Formulation

This chapter summarizes the functional renormalization group that satisfies Goldstone's theorem I [12].

2.1 Effective Action

Here, we consider a system described by action $S[\phi]$ with a bosonic field $\phi_j(x)$. In chapter 3, $\phi_j(x)$ will be an N-component real field ($j = 1, \cdots, N$) and x is d-dimensional space coordinate. The following will be abbreviated using $\xi = (j, x)$ as in $\phi(\xi)$. The grand partition function is given by

$$Z[\eta] = \int \mathcal{D}[\phi] \exp\left[-S[\phi] - \beta \int d\xi \eta(\xi)\phi(\xi)\right]. \tag{2.1}$$

Successive differentiation of $-\beta^{-1} \ln Z[\eta]$ in terms of η yield

$$-\beta^{-1} \frac{\delta \ln Z}{\delta \eta(\xi)} = \langle \phi(\xi) \rangle_\eta \equiv \tilde{\Phi}(\xi) \tag{2.2}$$

and

$$-\beta^{-1} \frac{\delta^2 \ln Z}{\delta \eta(\xi)\eta(\xi')} = -\beta \left(\langle \phi(\xi)\phi(\xi') \rangle_\eta - \tilde{\Phi}(\xi)\tilde{\Phi}(\xi') \right) \equiv \tilde{G}(\xi, \xi') \equiv (\underline{\tilde{G}})_{\xi,\xi'}. \tag{2.3}$$

The notation $\langle \cdots \rangle_\eta$ represents the expectation value in the presence of the external field $\eta(\xi)$. Subsequently, we perform the Legendre transformation

$$\Gamma[\tilde{\Phi}] = -\beta^{-1} \ln Z[\eta] - \int d\xi \tilde{\Phi}(\xi)\eta(\xi). \tag{2.4}$$

We obtain the first derivative of $\Gamma[\tilde{\Phi}]$ with respect to $\tilde{\Phi}$ as

$$\frac{\delta \Gamma[\tilde{\Phi}]}{\delta \tilde{\Phi}(\xi)} = -\eta(\xi), \tag{2.5}$$

where we have used the chain rule

$$\frac{\delta}{\delta \tilde{\Phi}(\xi)} = \int d\xi' \frac{\delta \eta(\xi')}{\delta \tilde{\Phi}(\xi)} \frac{\delta}{\delta \eta(\xi')}. \tag{2.6}$$

Substitution of Eq. (2.5) into Eq. (2.6) yields

$$-\int d\xi'' \frac{\delta^2 \Gamma[\tilde{\Phi}]}{\delta \tilde{\Phi}(\xi')\delta \tilde{\Phi}(\xi'')} \frac{\delta}{\delta \eta(\xi'')} = \frac{\delta}{\delta \tilde{\Phi}(\xi')} \tag{2.7}$$

where we have replaced (ξ, ξ') by (ξ', ξ'') for later convenience. Operating it on Eq. (2.2), we obtain

$$\int d\xi'' \beta^{-1} \frac{\delta^2 \Gamma[\tilde{\Phi}]}{\delta \tilde{\Phi}(\xi')\delta \tilde{\Phi}(\xi'')} \frac{\delta^2 \ln Z}{\delta \eta(\xi'')\delta \eta(\xi)} = \delta(\xi, \xi'). \tag{2.8}$$

It is convenient for later purposes to introduce the matrix $\underline{\tilde{\Gamma}}$ by

$$\left(\underline{\tilde{\Gamma}}^{(2)}\right)_{\xi,\xi'} \equiv \tilde{\Gamma}^{(2)}(\xi,\xi') \equiv \frac{\delta^2\Gamma[\tilde{\Phi}]}{\delta\tilde{\Phi}(\xi)\delta\tilde{\Phi}(\xi')}. \tag{2.9}$$

Using Eqs. (2.3) and (2.9), we can express Eq. (2.8) concisely as

$$\underline{\tilde{\Gamma}}^{(2)} = -\underline{\tilde{G}}^{-1} \tag{2.10}$$

The emergence of a finite expectation at $\eta = 0$, an order parameter,

$$\Phi(\xi) \equiv \tilde{\Phi}(\xi)\Big|_{\eta=0} \tag{2.11}$$

signals a spontaneously broken symmetry. It is obtained from the stationary condition of the effective action,

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi(\xi)} = 0. \tag{2.12}$$

Writing

$$\tilde{\Phi}(\xi) = \Phi(\xi) + \tilde{\phi}(\xi) \tag{2.13}$$

we expand $\Gamma[\tilde{\Phi}]$ in terms of $\tilde{\phi}$ as

$$\Gamma[\tilde{\Phi}] = \Gamma[\Phi] + \sum_{n=1}^{\infty} \int d\xi_1 \cdots \int d\xi_n \frac{\Gamma^{(n)}(\xi_1,\cdots,\xi_n)}{n!}\tilde{\phi}(\xi_1)\cdots\tilde{\phi}(\xi_n) \tag{2.14}$$

with

$$\Gamma^{(n)}(\xi_1,\cdots,\xi_n) \equiv \frac{\delta^n\Gamma[\Phi]}{\delta\Phi(\xi_1)\cdots\delta\Phi(\xi_n)}. \tag{2.15}$$

Therefore, $\Gamma[\Phi]$ is the generating functional of the one-particle irreducible vertices. We note that $\Gamma^{(n)}$ has a symmetry under the permutation for arguments $(\xi_1,\cdots,\xi_n)$ by definition. It follows from Eq. (2.5) that

$$\Gamma^{(1)}(\xi) = 0 \tag{2.16}$$

holds. From Eq. (2.10), we also obtain

$$\underline{\Gamma}^{(2)} = -\underline{G}^{-1}, \tag{2.17}$$

where G^{-1} is the inverse matrix of $\tilde{G}$ at $\eta = 0$ given in Eq.(2.3). With Eq. (2.17), it is convenient to express $\underline{\tilde{\Gamma}}^{(2)}$ in Eq. (2.10) as

$$\underline{\tilde{G}} = -\left[\underline{\tilde{\Gamma}}^{(2)}\right]^{-1} = \left(\underline{G}^{-1} - \delta\underline{\Gamma}^{(2)}\right)^{-1} \tag{2.18}$$

Substituting Eq. (2.14) into Eq. (2.5) and noting Eqs. (2.13) and (2.16), we obtain

$$\eta(\xi_1) = -\sum_{n=1}^{\infty} \int d\xi_2 \cdots \int d\xi_{n+1} \frac{\Gamma^{(n+1)}(\xi_1,\xi_2,\cdots,\xi_{n+1})}{n!}\tilde{\phi}(\xi_2)\cdots\tilde{\phi}(\xi_{n+1}) \tag{2.19}$$

which vanishes for $\tilde{\Phi}(\xi) \to \Phi(\xi)$. Also substituting Eq. (2.14) into Eq. (2.9), we obtain an expansion of $\delta\underline{\Gamma}^{(2)}$ in Eq. (2.18),

$$\delta\Gamma^{(2)}(\xi_1,\xi_2) = \sum_{n=1}^{\infty} \int d\xi_3 \cdots \int d\xi_{n+2} \frac{\Gamma^{(n+2)}(\xi_1,\xi_2,\cdots,\xi_{n+2})}{n!}\tilde{\phi}(\xi_3)\cdots\tilde{\phi}(\xi_{n+2}). \tag{2.20}$$

2.2 Goldstone's Theorem I

Suppose the action $S[\phi]$ has a specific continuous symmetry X. The effective action $\Gamma[\tilde{\Phi}]$ is then invariant under the operation of the field $\tilde{\Phi}$ belonging to X [1]. By imposing this symmetry to $\Gamma[\tilde{\Phi}]$, we derive the identity between vertices [12]. We will use these constraints to parametrize in the renormalization group study. Here we focus on the case of O(3) symmetry for simplicity (the following results are easily extended to $O(N)$ for any $N > 1$). Rotation of $\boldsymbol{\Phi}$ can be written generally as $R_{ij}\Phi_j$, where summation over repeated arguments is implied. Particularly, infinitesimal rotation by $\delta\theta_k$ around the axis k can be described by $\delta\mathcal{R}^k_{ij} \equiv -\epsilon_{ijk}\delta\theta_k$.

Action $\Gamma[\tilde{\Phi}]$ is invariant under the transformation

$$\tilde{\Phi}_i \to \tilde{\Phi}'_i \equiv \tilde{\Phi}_i + \delta\tilde{\Phi}_i \tag{2.21}$$

with

$$\delta\tilde{\Phi}_i = -\epsilon_{ijk}\tilde{\Phi}_j\delta\theta_k = -\epsilon_{ijk}(\Phi_j + \tilde{\phi}_j)\delta\theta_k \equiv -\delta\theta_k\mathcal{R}^k_{ij}(\Phi_j + \tilde{\phi}_j). \tag{2.22}$$

This invariance can be written as $-\partial\Gamma[\tilde{\Phi}']/\partial\theta_k\big|_{\theta_k=0} = 0$, i.e.,

$$\sum_{i_1 j_1 \boldsymbol{R}_1} \frac{\delta\Gamma[\tilde{\Phi}]}{\delta\Phi_{i_1}(\boldsymbol{R}_1)}\mathcal{R}^k_{i_1 j_1}[\Phi_{j_1}(\boldsymbol{R}_1) + \tilde{\phi}_{j_1}(\boldsymbol{R}_1)] = 0. \tag{2.23}$$

Substituting Eq. (2.14) into this equation, we obtain

$$\begin{aligned}0 = &\sum_{i_1 j_1 \boldsymbol{R}_1} \Gamma^{(1)}_{j_1}(\boldsymbol{R}_1)\mathcal{R}^k_{j_1 i_1}[\Phi_{i_1}(\boldsymbol{R}_1) + \tilde{\phi}_{i_1}(\boldsymbol{R}_1)] + \sum_{n=1}^{\infty}\sum_{i_1 j_1 \boldsymbol{R}_1}\sum_{i_2 \boldsymbol{R}_2}\cdots\sum_{i_{n+1}\boldsymbol{R}_{n+1}} \frac{\Gamma^{(n+1)}_{j_1 i_2\cdots i_{n+1}}(\boldsymbol{R}_1, \boldsymbol{R}_2, \cdots, \boldsymbol{R}_{n+1})}{n!} \\ &\times \mathcal{R}^k_{j_1 i_1}[\Phi_{i_1}(\boldsymbol{R}_1) + \tilde{\phi}_{i_1}(\boldsymbol{R}_1)]\tilde{\phi}_{i_2}(\boldsymbol{R}_2)\cdots\tilde{\phi}_{i_{n+1}}(\boldsymbol{R}_{n+1}).\end{aligned} \tag{2.24}$$

Noting that $\tilde{\phi}$ can be chosen arbitrarily, we conclude that the coefficient of the terms nth order in $\tilde{\phi}$ should be equal to zero. The zeroth-order identity is given by

$$\sum_{i_1 j_1 \boldsymbol{R}_1} \Gamma^{(1)}_{j_1}(\boldsymbol{R}_1)\mathcal{R}^k_{j_1 i_1}\Phi_{\boldsymbol{R}_1, i_1} = 0,$$

which is certainly met due to Eq. (2.16). The first-order identity is

$$\sum_{j_1}\Gamma^{(1)}_{j_1}(\boldsymbol{R}_1)\mathcal{R}^k_{j_1 i_1} + \sum_{i_2 j_2 \boldsymbol{R}_2}\Gamma^{(2)}_{i_1 j_2}(\boldsymbol{R}_1, \boldsymbol{R}_2)\mathcal{R}^k_{j_2 i_2}\Phi_{\boldsymbol{R}_2, i_2} = 0. \tag{2.25}$$

Substitution of Eqs. (2.16) and (2.17) yields

$$\sum_{\boldsymbol{R}_2}\underline{G}^{-1}(\boldsymbol{R}_1, \boldsymbol{R}_2)\underline{\mathcal{R}}^k\boldsymbol{\Phi}(\boldsymbol{R}_2) = \boldsymbol{0}. \tag{2.26}$$

This is a direct consequence of Goldstone's theorem. Let us differentiate Eq. (2.25) with respect to $\Phi_{\boldsymbol{R}_2, i_2}$. The resulting equation can be written using Eq. (2.15) as

$$\sum_{j_1}\Gamma^{(2)}_{j_1 i_2}(\boldsymbol{R}_1, \boldsymbol{R}_2)\mathcal{R}^k_{j_1 i_1} + \sum_{j_2}\Gamma^{(2)}_{i_1 j_2}(\boldsymbol{R}_1, \boldsymbol{R}_2)\mathcal{R}^k_{j_2 i_2} + \sum_{i_3 j_3 \boldsymbol{R}_3}\Gamma^{(3)}_{i_1 i_2 j_3}(\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3)\mathcal{R}^k_{j_3 i_3}\Phi_{\boldsymbol{R}_3, i_3} = 0. \tag{2.27}$$

Similarly, differentiation of Eq. (2.27) for $\Phi_{\boldsymbol{R}_3, i_3}$ yields

$$\begin{aligned}&\sum_{j_1}\Gamma^{(3)}_{j_1 i_2 i_3}(\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3)\mathcal{R}^k_{j_1 i_1} + \sum_{j_2}\Gamma^{(3)}_{i_1 j_2 i_3}(\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3)\mathcal{R}^k_{j_2 i_2} + \sum_{j_3}\Gamma^{(3)}_{i_1 i_2 j_3}(\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3)\mathcal{R}^k_{j_3 i_3} \\ &+ \sum_{i_4 j_4 \boldsymbol{R}_4}\Gamma^{(4)}_{i_1 i_2 i_3 j_4}(\boldsymbol{R}_1, \boldsymbol{R}_2, \boldsymbol{R}_3, \boldsymbol{R}_4)\mathcal{R}^k_{j_4 i_4}\Phi_{\boldsymbol{R}_4, i_4} = 0.\end{aligned} \tag{2.28}$$

We obtain identities for $n(> 4)$-point vertices are obtained by successive differentiation of Eq. (2.24) with respect to Φ.

2.3 Functional Renormalization Group

Now we introduce cutoff Λ. The partition function with cutoff Λ can be expressible as

$$Z_\Lambda[\eta] = \int \mathcal{D}[\phi] \exp\left[-S_\Lambda[\phi] - \beta \int d\xi \eta(\xi)\phi(\xi)\right]. \tag{2.29}$$

The effective action is given by

$$\Gamma_\Lambda[\tilde{\Phi}] = -\beta^{-1} \ln Z_\Lambda[\eta] - \int d\xi \tilde{\Phi}_\Lambda(\xi)\eta(\xi), \tag{2.30}$$

where $\eta(\xi) \equiv \eta_j(x)$ obeys

$$\frac{\delta\Gamma_\Lambda[\tilde{\Phi}]}{\delta\tilde{\Phi}_\Lambda(\xi)} = -\eta(\xi) \tag{2.31}$$

so that it also has Λ dependence. However, the dependence can be dropped when considering the first-order derivative $\partial_\Lambda\Gamma_\Lambda[\tilde{\Phi}]$. S_Λ in Eq. 2.29 is expressed as

$$S_\Lambda[\phi] = S[\phi] + \frac{1}{2}\int d\xi \int d\xi' \phi(\xi) R_\Lambda(\xi,\xi')\phi(\xi'). \tag{2.32}$$

The second term is an additional cutoff term and R_Λ denotes a regulator function. Although the choice of regulator is arbitrary, it has the following common properties [6, 7]:

$$\lim_{\Lambda\to\infty} R_\Lambda = \infty \tag{2.33a}$$

$$\lim_{\Lambda\to 0} R_\Lambda = 0. \tag{2.33b}$$

The first property implies

$$\lim_{\Lambda\to\infty} \Gamma_\Lambda[\Phi] = S[\Phi], \tag{2.34}$$

that is, it coincides with the classical action in the UV regime. On the other hand, the second property implies

$$\lim_{\Lambda\to 0} \Gamma_\Lambda[\Phi] = \Gamma[\Phi]. \tag{2.35}$$

Therefore, solving the partial differential equation for the functional Γ_Λ down to $\Lambda = 0$, we obtain an effective action describing the low-energy state. We also consider $R_\Lambda(\xi,\xi') = R_\Lambda(\xi',\xi)$. The introduction of the regulator creates a cutoff dependence for the free-propagator:

$$G_{0\Lambda}^{-1}(\xi,\xi') = G_0^{-1}(\xi,\xi') - R_\Lambda(\xi,\xi'). \tag{2.36}$$

To derive the RG equation for $\Gamma_\Lambda[\tilde{\Phi}]$, let us differentiate Eq. (2.30) with respect to Λ,

$$\partial_\Lambda\Gamma_\Lambda[\tilde{\Phi}] = -\beta^{-1}\partial_\Lambda \ln Z_\Lambda - \int d\xi \eta(\xi)\partial_\Lambda\Phi_\Lambda(\xi) \tag{2.37}$$

The first term on the r.h.s. of the equation can be expressible in terms of the functional $\delta\underline{\Gamma}_\Lambda$ as

$$-\beta^{-1}\partial_\Lambda \ln Z_\Lambda = -\frac{1}{2\beta}\mathrm{Tr}\underline{\dot{G}}_\Lambda\left[\underline{G}_\Lambda^{-1} + \sum_{n=0}^{\infty}\left(\delta\underline{\Gamma}_\Lambda^{(2)}\underline{G}_\Lambda\right)^n \delta\underline{\Gamma}_\Lambda^{(2)}\right] - \frac{1}{2}(\boldsymbol{\Phi}_\Lambda + \tilde{\boldsymbol{\phi}})^{\mathrm{T}}\left(\partial_\Lambda \underline{G}_{0\Lambda}^{-1}\right)(\boldsymbol{\Phi}_\Lambda + \tilde{\boldsymbol{\phi}}) \tag{2.38}$$

with $\underline{\dot{G}} \equiv -\underline{G}_\Lambda\left(\partial_\Lambda\underline{G}_{0\Lambda}^{-1}\right)\underline{G}_\Lambda$. Substituting Eq. (2.38) into Eq. (2.37), we obtain

$$-\partial_\Lambda\Gamma_\Lambda = \frac{1}{2\beta}\mathrm{Tr}\underline{\dot{G}}_\Lambda\left[\underline{G}_\Lambda^{-1} + \sum_{n=0}^{\infty}\left(\delta\underline{\Gamma}_\Lambda^{(2)}\underline{G}_\Lambda\right)^n \delta\underline{\Gamma}_\Lambda^{(2)}\right] + \frac{1}{2}(\boldsymbol{\Phi}_\Lambda + \tilde{\boldsymbol{\phi}})^{\mathrm{T}}\left(\partial_\Lambda\underline{G}_{0\Lambda}^{-1}\right)(\boldsymbol{\Phi}_\Lambda + \tilde{\boldsymbol{\phi}}) + \int d\xi \eta(\xi)\partial_\Lambda\Phi_\Lambda(\xi). \tag{2.39}$$

2.4 Equation for Vertices

Let us substitute Eqs. (2.14), (2.19), and (2.20) into Eq. (2.39) and equate the coefficients of $\mathrm{O}((\delta\Phi)^n)$ on both sides. In accordance with $\Gamma^{(n)}$ on the left-hand side, we should symmetrize the right-hand side in terms of the arguments in the process. We thereby obtain

$$-\partial_\Lambda \Gamma_\Lambda^{(n)}(1,\cdots,n) = W_\Lambda^{(n)}(1,\cdots,n) \tag{2.40}$$

with $1 \equiv \xi_1$. Functions $W^{(n)}$ for $1 \le n \le 4$ are given by

$$W_\Lambda^{(1)}(1) = -\Gamma_\Lambda^{(2)}(1,1')\partial_\Lambda\Phi_\Lambda(1') + \frac{1}{2\beta}\mathrm{Tr}\underline{\Gamma}_\Lambda^{(3)}(1)\underline{\dot{G}}_\Lambda + \left(\partial_\Lambda \underline{G}_{0\Lambda}^{-1}\right)\mathbf{\Phi}_\Lambda \tag{2.41a}$$

$$\begin{aligned} W_\Lambda^{(2)}(1,2) =& -\Gamma_\Lambda^{(3)}(1,2,1')\partial_\Lambda\Phi_\Lambda(1') + \frac{1}{2\beta}\mathrm{Tr}\underline{\Gamma}_\Lambda^{(4)}(1,2)\underline{\dot{G}}_\Lambda \\ &+ \frac{1}{2\beta}\mathrm{Tr}\sum_P \underline{\Gamma}_\Lambda^{(3)}(p_1)\underline{G}_\Lambda\underline{\Gamma}_\Lambda^{(3)}(p_2)\underline{\dot{G}}_\Lambda + \partial_\Lambda G_{0\Lambda}^{-1}(1,2) \end{aligned} \tag{2.41b}$$

$$\begin{aligned} W_\Lambda^{(3)}(1,2,3) =& -\Gamma_\Lambda^{(4)}(1,2,3,1')\partial_\Lambda\Phi_\Lambda(1') + \frac{1}{2\beta}\mathrm{Tr}\underline{\Gamma}_\Lambda^{(5)}(1,2,3)\underline{\dot{G}}_\Lambda \\ &+ \frac{1}{2\beta}\mathrm{Tr}\frac{1}{2!}\sum_P \left[\underline{\Gamma}_\Lambda^{(4)}(p_1,p_2)\underline{G}_\Lambda\underline{\Gamma}_\Lambda^{(3)}(p_3) + \underline{\Gamma}_\Lambda^{(3)}(p_1)\underline{G}_\Lambda\underline{\Gamma}_\Lambda^{(4)}(p_2,p_3)\right]\underline{\dot{G}}_\Lambda \\ &+ \frac{1}{2\beta}\mathrm{Tr}\sum_P \underline{\Gamma}_\Lambda^{(3)}(p_1)\underline{G}_\Lambda\underline{\Gamma}_\Lambda^{(3)}(p_2)\underline{G}_\Lambda\underline{\Gamma}_\Lambda^{(3)}(p_3)\underline{\dot{G}}_\Lambda \end{aligned} \tag{2.41c}$$

$$\begin{aligned} W_\Lambda^{(4)}(1,2,3,4) =& -\Gamma_\Lambda^{(5)}(1,2,3,4,1')\partial_\Lambda\Phi_\Lambda(1') + \frac{1}{2\beta}\mathrm{Tr}\underline{\Gamma}_\Lambda^{(6)}(1,2,3,4)\underline{\dot{G}}_\Lambda \\ &+ \frac{1}{2\beta}\mathrm{Tr}\frac{1}{3!}\sum_P \left[\underline{\Gamma}_\Lambda^{(5)}(p_1,p_2,p_3)\underline{G}_\Lambda\underline{\Gamma}_\Lambda^{(3)}(p_4) + \underline{\Gamma}_\Lambda^{(3)}(p_1)\underline{G}_\Lambda\underline{\Gamma}_\Lambda^{(5)}(p_2,p_3,p_4)\right]\underline{\dot{G}}_\Lambda \\ &+ \frac{1}{2\beta}\mathrm{Tr}\frac{1}{(2!)^2}\sum_P \underline{\Gamma}_\Lambda^{(4)}(p_1,p_2)\underline{G}_\Lambda\underline{\Gamma}_\Lambda^{(4)}(p_3,p_4)\underline{\dot{G}}_\Lambda \\ &+ \frac{1}{2\beta}\mathrm{Tr}\frac{1}{2!}\sum_P \Big[\underline{\Gamma}_\Lambda^{(3)}(p_1)\underline{G}_\Lambda\underline{\Gamma}_\Lambda^{(3)}(p_2)\underline{G}_\Lambda\underline{\Gamma}_\Lambda^{(4)}(p_3,p_4) + \underline{\Gamma}_\Lambda^{(3)}(p_1)\underline{G}_\Lambda\underline{\Gamma}_\Lambda^{(4)}(p_2,p_3)\underline{G}_\Lambda\underline{\Gamma}_\Lambda^{(3)}(p_4) \\ &+\underline{\Gamma}_\Lambda^{(4)}(p_1,p_2)\underline{G}_\Lambda\underline{\Gamma}_\Lambda^{(3)}(p_3)\underline{G}_\Lambda\underline{\Gamma}_\Lambda^{(3)}(p_4)\Big]\underline{\dot{G}}_\Lambda \\ &+ \frac{1}{2\beta}\mathrm{Tr}\sum_P \underline{\Gamma}_\Lambda^{(3)}(p_1)\underline{G}_\Lambda\underline{\Gamma}_\Lambda^{(3)}(p_2)\underline{G}_\Lambda\underline{\Gamma}_\Lambda^{(3)}(p_3)\underline{G}_\Lambda\underline{\Gamma}_\Lambda^{(3)}(p_4)\underline{\dot{G}}_\Lambda. \end{aligned} \tag{2.41d}$$

Here we denote $[\underline{\Gamma}^{(n+2)}(\xi_1,\cdots,\xi_n)]_{\xi,\xi'} \equiv \Gamma^{(n+2)}(\xi_1,\cdots,\xi_n,\xi,\xi')$. They are expressible diagrammatically as Fig. 2.1.

In the following chapters, we will analyze the spontaneously broken phase by combining the renormalization group equations for vertices given by Eq. (2.40) with Goldstone's theorem I from the previous section.

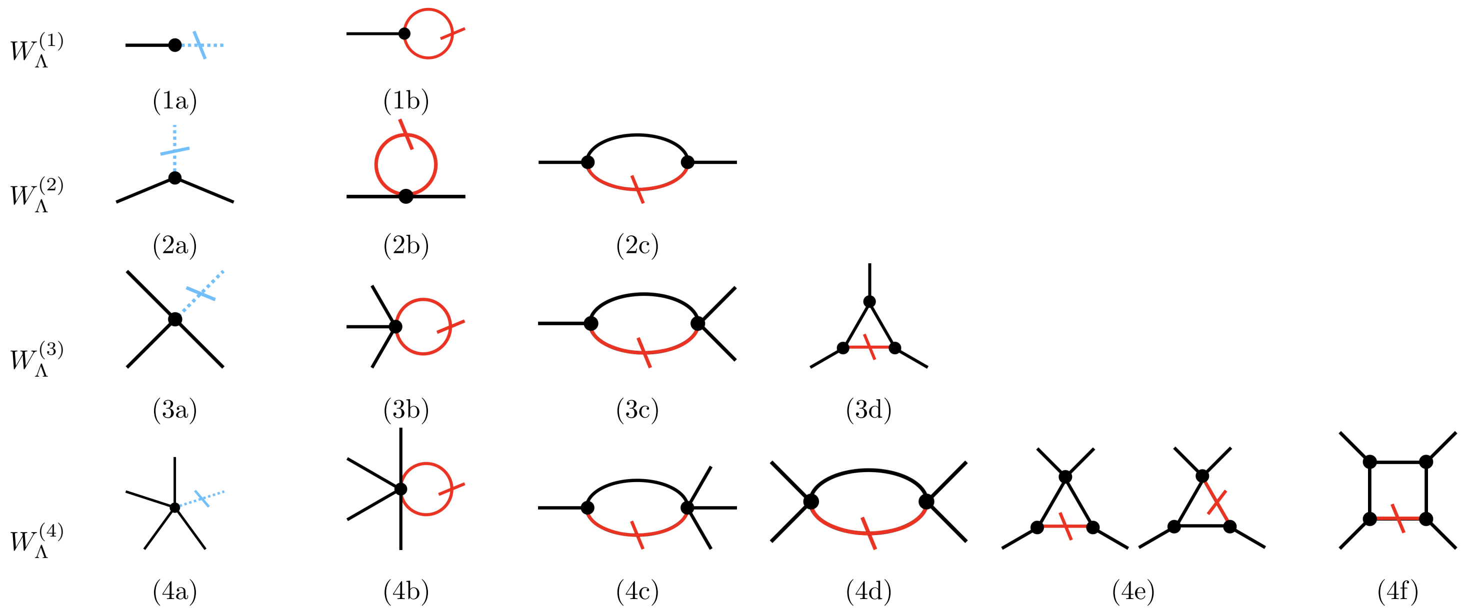

Figure 2.1: Diagrammatic expressions of $W^{(n)}_{\Lambda,j_1\cdot j_n}$ for $1 \leq n \leq 4$. A line with a dash (dotted line with a dash) denotes $\underline{\dot{G}}_\Lambda$ $(\partial_\Lambda \Phi_\Lambda)$.

Chapter 3

Classical $O(N)$ model at Critical Point

Using dimensional analysis, n-point vertices $\Gamma^{(n)}$ have a dimension of $\dim[\Gamma^{(n)}] = d - n \times \frac{d-2}{2}$ (we assign the scaling dimension 1 to momenta), where d is a dimension of the system. If $\dim[\Gamma^{(n)}] > 0$, the vertex is considered relevant in the context of the renormalization group. In particular, in the case of $d = 3$, since $\Gamma^{(6)}$ is marginal term $\dim[\Gamma^{(6)}] = 0$, the lower order vertices will be relevant. This implies that, whereas only the four-point vertices are relevant in the symmetric phase, three-point vertices also become relevant in the spontaneously broken phase. New fixed points are expected to emerge with the increase in the number of relevant parameters. We apply the FRG formulated in the previous chapter to the $O(N)$ symmetric model to see the prediction.

We consider a d-dimensional system with $O(N)$ symmetry defined by the action

$$S = \beta \int d^d r \left(\frac{1}{2} |\nabla \boldsymbol{\phi}|^2 + \frac{r_0}{2} |\boldsymbol{\phi}|^2 + \frac{g_0}{4} |\boldsymbol{\phi}|^4 \right) \tag{3.1}$$

where $\boldsymbol{\phi}$ is an N-component real field, r_0 is a mass parameter, and $g_0 > 0$ is a coupling constant. We assume $N > 2$ and $d > 2$. Expanding ϕ_j such as

$$\phi_j(\boldsymbol{R}) = \frac{1}{\sqrt{N_\mathrm{a}}} \sum_{\boldsymbol{k}} \phi_j(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{R}}, \tag{3.2}$$

where N_a is the number of site, and substituting them into Eq. (3.1), we obtain

$$S = \beta \left[-\frac{1}{2} \sum_{j,\boldsymbol{k}} \phi_j(-\boldsymbol{k}) H^{(0)}(\boldsymbol{k}) \phi_j(\boldsymbol{k}) + \frac{g_0}{4N_\mathrm{a}} \sum_{jj'} \sum_{\boldsymbol{k}_1 \cdots \boldsymbol{k}_4} \phi_j(\boldsymbol{k}_1) \phi_{j'}(\boldsymbol{k}_2) \phi_{j'}(\boldsymbol{k}_3) \phi_j(\boldsymbol{k}_4) \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4,\boldsymbol{0}} \right]. \tag{3.3}$$

with $H^{(0)}(\boldsymbol{k}) = -(k^2 + r_0)$.

3.1 Mean-Field Description

Let us express the field $\phi_j(\boldsymbol{k})$ as

$$\phi_j(\boldsymbol{k}) = \Phi_j(\boldsymbol{k}) + \tilde{\phi}_j(\boldsymbol{k}). \tag{3.4}$$

Here $\Phi_j(\boldsymbol{k})$ is a Fourier component for $\Phi(\xi)$ defined by Eq. (2.11) with $\xi \equiv (j, \boldsymbol{r})$. In homogeneous system, we can choose the order parameter as

$$\Phi_j(\boldsymbol{k}) \equiv N_\mathrm{a}^{1/2} \delta_{jN} \delta_{\boldsymbol{k},\boldsymbol{0}} \Phi, \tag{3.5}$$

where factor $\sqrt{N_\mathrm{a}}$ has been incorporated for $\boldsymbol{k} = \boldsymbol{0}$ so as to cancel the prefactor in Eq. (3.2). Φ is defined by

$$\Phi \equiv -\frac{\delta \ln Z[\eta]}{\delta \eta_N(\boldsymbol{0})} \bigg|_{\eta=0} \tag{3.6}$$

and corresponds to the spontaneous magnetization in spin systems. The corresponding action can be written in powers of Φ as

$$S = \sum_{n=0}^{4} S_n \tag{3.7}$$

with $S_1 = 0$ due to the momentum conservation law. We can write down the other terms on the right-hand side as

$$\frac{S_0}{\beta} = \frac{N_\mathrm{a}}{2}\left[-H^{(0)}(\mathbf{0})\Phi^2 + \frac{g_0}{2}\Phi^4\right] \tag{3.8a}$$

$$\frac{S_2}{\beta} = -\frac{1}{2}\sum_{jj'}\sum_{\boldsymbol{k}} \tilde{\phi}_j(-\boldsymbol{k})\tilde{H}^{(0)}_{jj'}(\boldsymbol{k})\tilde{\phi}_{j'}(\boldsymbol{k}) \tag{3.8b}$$

$$\frac{S_3}{\beta} = \frac{g_0\Phi}{N_\mathrm{a}^{1/2}}\sum_{j}\sum_{\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3} \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3,\mathbf{0}}\tilde{\phi}_j(\boldsymbol{k}_1)\tilde{\phi}_j(\boldsymbol{k}_2)\tilde{\phi}_N(\boldsymbol{k}_3) \tag{3.8c}$$

$$\frac{S_4}{\beta} = \frac{g_0}{4N_\mathrm{a}}\sum_{jj'}\sum_{\boldsymbol{k}_1\cdots\boldsymbol{k}_4} \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4,\mathbf{0}}\tilde{\phi}_j(\boldsymbol{k}_1)\tilde{\phi}_j(\boldsymbol{k}_2)\tilde{\phi}_{j'}(\boldsymbol{k}_3)\tilde{\phi}_{j'}(\boldsymbol{k}_4) \tag{3.8d}$$

with

$$\tilde{H}^{(0)}_{jj'}(\boldsymbol{k}) = H^{(0)}(\boldsymbol{k})\delta_{jj'} - g_0\Phi^2\left(\delta_{jj'} + 2\delta_{jN}\delta_{j'N}\right) \tag{3.9}$$

The mean-field description of the phase transition is given solely in terms of S_0 which can be written as

$$S_0 = \frac{\beta g_0 N_\mathrm{a}}{4}\left[\left(\Phi^2 + \frac{r_0}{g_0}\right)^2 - \left(\frac{r_0}{g_0}\right)^2\right]. \tag{3.10}$$

Near the second-order phase transition, r_0 is expressible as

$$r_0 = \frac{2}{a^2}\frac{T - T_\mathrm{c0}}{T_\mathrm{c0}}, \tag{3.11}$$

where a is the lattice spacing and T_c0 is the transition temperature given by the mean-field approximation. Minimizing Eq. (3.10), we obtain the mean-field order parameter Φ_0,

$$\Phi_0 = \begin{cases} 0 & : r_0 > 0 \\ \sqrt{\dfrac{|r_0|}{g_0}} & : r_0 < 0 \end{cases}. \tag{3.12}$$

We also introduce the non-interacting Green's function $\hat{G}_0 = (G_{0,jj'})$,

$$\begin{aligned} G^{-1}_{0,jj'}(\boldsymbol{k}) &\equiv H^{(0)}(\boldsymbol{k})\delta_{jj'} - g_0\Phi_0^2\left(\delta_{jj'} + 2\delta_{jN}\delta_{j'N}\right) \\ &= \begin{cases} -(k^2 + r_0)\delta_{jj'} & : r_0 > 0 \\ -(k^2 + 2|r_0|\delta_{jN})\delta_{jj'} & : r_0 < 0 \end{cases}. \end{aligned} \tag{3.13}$$

We also define the longitudinal and transverse propagators for the spontaneously broken phase ($r_0 < 0$) as follows:

$$\begin{aligned} G_{0,\mathrm{l}} &\equiv -\frac{1}{k^2 + 2|r_0|}, \\ G_{0,\mathrm{t}} &\equiv -\frac{1}{k^2}. \end{aligned} \tag{3.14}$$

In terms of perturbation expansion, the zeroth order of self-energy can be regarded as $\Sigma_\mathrm{l}^{(0)}(\boldsymbol{k}) \sim |r_0|$. Now, we consider the one-loop correction to the self-energy, which comes from S_3 and S_4. It is shown diagrammatically in Fig. 3.1. The left-hand side diagram has a finite contribution, while the right one gives a contribution to the self-energy for the longitudinal component $\Sigma_\mathrm{l}(\boldsymbol{k})$ in the infrared limit $\boldsymbol{k} \to \mathbf{0}$ as follows [14, 5]:

$$\begin{aligned} \Sigma_\mathrm{l}^{(1)}(\boldsymbol{k}) &\sim (N-1)\frac{g_0^2\Phi_0^2}{N_\mathrm{a}}\sum_{\boldsymbol{q}} G_{0,\mathrm{t}}(\boldsymbol{q})G_{0,\mathrm{t}}(\boldsymbol{k}+\boldsymbol{q}) \\ &\sim (N-1)g_0^2\Phi_0^2|\boldsymbol{k}|^{d-4} \end{aligned} \tag{3.15}$$

and diverges due to transverse fluctuations for $d \leq 4$. This fact gives a scale at which the perturbation expansion would break down. The wavelength scale k_G can then be determined from $\Sigma_\mathrm{l}^{(0)}(\boldsymbol{k}) \sim \Sigma_\mathrm{l}^{(1)}(\boldsymbol{k})$ for $|\boldsymbol{k}| \sim k_G$, and the correlation length $\xi_G \equiv k_G^{-1}$ is called the Ginzburg correlation length [14].

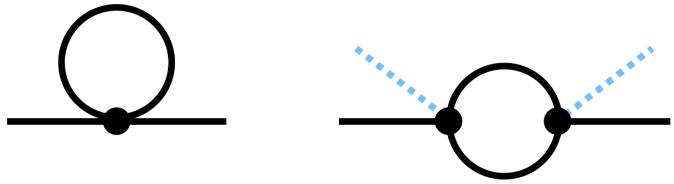

Figure 3.1: Diagrammatic representation for one-loop correction to the self-energy. The solid (dashed) lines are $G_{0,jj}(\boldsymbol{k})$ (Φ_0) and vertices are g_0. The left (right) hand side diagram comes from $S_4(S_3)$.

3.2 Parametrization

Let us expand the vertices as

$$\Gamma^{(n)}_{j_1\cdots j_n}(\boldsymbol{R}_1,\cdots,\boldsymbol{R}_n)=\frac{1}{N_{\mathrm{a}}^{n-1}}\sum_{\boldsymbol{k}_1\cdots\boldsymbol{k}_n}\Gamma^{(n)}_{j_1\cdots j_n}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\,\mathrm{e}^{-i(\boldsymbol{k}_1\cdot\boldsymbol{R}_1+\cdots+\boldsymbol{k}_n\cdot\boldsymbol{R}_n)} \tag{3.16}$$

The order parameter $\Phi_{\boldsymbol{R},j}$ can also be written in terms of Φ in Eq. (3.4) as $\Phi_{\boldsymbol{R},j}=\Phi\delta_{jn}$. The Fourier coefficients satisfy $\Gamma^{(n)}_{j_1\cdots j_n}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)\propto\delta_{\boldsymbol{k}_1+\cdots+\boldsymbol{k}_n,\boldsymbol{0}}$ with $\Gamma^{(1)}_{j_1}(\boldsymbol{0})=0$ from Eq. (2.16). It follows from Eqs. (2.14) and (3.16) that the zeroth-order expressions of the four-point vertices are given in terms of Eq. (3.3) by

$$\begin{aligned}\Gamma^{(40)}_{j_1j_2j_3j_4}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)&\equiv N_{\mathrm{a}}\frac{\delta^4 S}{\delta\phi_{j_1}(\boldsymbol{k}_1)\delta\phi_{j_2}(\boldsymbol{k}_2)\delta\phi_{j_3}(\boldsymbol{k}_3)\delta\phi_{j_4}(\boldsymbol{k}_4)}\\&=2g_0\mathcal{S}_{j_1j_2j_3j_4}\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4,\boldsymbol{0}}.\end{aligned} \tag{3.17}$$

with

$$\mathcal{S}_{j_1j_2j_3j_4}=\delta_{j_1j_2}\delta_{j_3j_4}+\delta_{j_1j_3}\delta_{j_2j_4}+\delta_{j_1j_4}\delta_{j_2j_3}. \tag{3.18}$$

Noting this fact and incorporating possible anisotropy between the longitudinal and transverse components, we model $\Gamma^{(4)}_{j_1j_2j_3j_4}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)$ as

$$\begin{aligned}\Gamma^{(4)}_{j_1j_2j_3j_4}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)=&2[g_{\mathrm{tt}}(1-\delta_{j_1N})(1-\delta_{j_2N})(1-\delta_{j_3N})(1-\delta_{j_4N})\\&+g_{\mathrm{ll}}\delta_{j_1N}\delta_{j_2N}\delta_{j_3N}\delta_{j_4N}\\&+g_{\mathrm{lt}}\sum_P(1-\delta_{j_{p_1}N})(1-\delta_{j_{p_2}N})\delta_{j_{p_3}N}\delta_{j_{p_4}N}]\mathcal{S}_{j_1j_2j_3j_4}\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4,\boldsymbol{0}}.\end{aligned} \tag{3.19}$$

In the following, we denote the longitudinal component $j=N$ as z, while the transverse components $j\neq N$ as $x,y,\cdots$. We also focus on the case of $N=3$ as well Sect. 2.2.1. Finite elements of $\Gamma^{(4)}$ are

$$\Gamma^{(4)}_{zzzz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)=6g_{\mathrm{ll}}\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4,\boldsymbol{0}}, \tag{3.20a}$$

$$\Gamma^{(4)}_{xxxx}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)=\Gamma^{(4)}_{yyyy}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)=6g_{\mathrm{tt}}\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4,\boldsymbol{0}}, \tag{3.20b}$$

$$\Gamma^{(4)}_{xxyy}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)=2g_{\mathrm{tt}}\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4,\boldsymbol{0}}, \tag{3.20c}$$

$$\Gamma^{(4)}_{xxzz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)=\Gamma^{(4)}_{yyzz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)=2g_{\mathrm{lt}}\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4,\boldsymbol{0}}. \tag{3.20d}$$

Equations (2.25), (2.27), and (2.28) are transformed into

$$\sum_{j_2}\Gamma^{(2)}_{j_1j_2}(\boldsymbol{0},\boldsymbol{0})\mathcal{R}^k_{j_2n}\Phi=0, \tag{3.21a}$$

$$\sum_{j_1'}\Gamma^{(2)}_{j_1'j_2}(-\boldsymbol{k},\boldsymbol{k})\mathcal{R}^k_{j_1'j_1}+\sum_{j_2'}\Gamma^{(2)}_{j_1j_2'}(-\boldsymbol{k},\boldsymbol{k})\mathcal{R}^k_{j_2'j_2}=-\sum_{j_3'}\Gamma^{(3)}_{j_1j_2j_3'}(-\boldsymbol{k},\boldsymbol{k},\boldsymbol{0})\mathcal{R}^k_{j_3'n}\Phi, \tag{3.21b}$$

$$\begin{aligned}&\sum_{j_1'}\Gamma^{(3)}_{j_1'j_2j_3}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\mathcal{R}^k_{j_1'j_1}+\sum_{j_2'}\Gamma^{(3)}_{j_1j_2'j_3}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\mathcal{R}^k_{j_2'j_2}+\sum_{j_3'}\Gamma^{(3)}_{j_1j_2j_3'}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\mathcal{R}^k_{j_3'j_3}\\&=-\sum_{j_4'}\Gamma^{(4)}_{j_1j_2j_3j_4'}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{0})\mathcal{R}^k_{j_4'n}\Phi.\end{aligned} \tag{3.21c}$$

We also consider the infinitesimal rotation $\mathcal{R}^{j_3}_{j_1j_2} = \epsilon_{j_1j_2j_3}$ with $j_3 = y$ in Eq. (3.21). Equation (3.21c) gives the finite three-point vertices as follows

$$\Gamma^{(3)}_{xxz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = \Gamma^{(3)}_{yyz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = 2g_{\rm tt}\Phi\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3,\boldsymbol{0}}, \tag{3.22a}$$

$$\Gamma^{(3)}_{zzz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = 2(2g_{\rm tt}+g_{\rm lt})\Phi\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3,\boldsymbol{0}}, \tag{3.22b}$$

whereas the other elements vanish, i.e.,

$$\begin{aligned}0 =&\Gamma^{(3)}_{xxx}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = \Gamma^{(3)}_{yyy}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = \Gamma^{(3)}_{zzx}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = \Gamma^{(3)}_{zzy}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = \Gamma^{(3)}_{xxy}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\\ =&\Gamma^{(3)}_{yyx}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = \Gamma^{(3)}_{xyz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\end{aligned} \tag{3.22c}$$

Equation (3.21b) with Eqs. (3.20) gives the two-point vertices as follows

$$\Gamma^{(2)}_{xx}(\boldsymbol{k}_1,\boldsymbol{k}_2) = \Gamma^{(2)}_{yy}(\boldsymbol{k}_1,\boldsymbol{k}_2), \qquad \Gamma^{(2)}_{zz}(\boldsymbol{k}_1,\boldsymbol{k}_2) = 2g_{\rm tt}\Phi^2\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{0}} + \Gamma^{(2)}_{xx}(\boldsymbol{k}_1,\boldsymbol{k}_2) \tag{3.23a}$$

$$\Gamma^{(2)}_{xy}(\boldsymbol{k}_1,\boldsymbol{k}_2) = \Gamma^{(2)}_{yz}(\boldsymbol{k}_1,\boldsymbol{k}_2) = \Gamma^{(2)}_{zx}(\boldsymbol{k}_1,\boldsymbol{k}_2) = 0. \tag{3.23b}$$

From Eq. (3.21a), we obtain

$$\Gamma^{(2)}_{xx}(\boldsymbol{0},\boldsymbol{0}) = \Gamma^{(2)}_{yx}(\boldsymbol{0},\boldsymbol{0}) = \Gamma^{(2)}_{yy}(\boldsymbol{0},\boldsymbol{0}) = \Gamma^{(2)}_{xy}(\boldsymbol{0},\boldsymbol{0}) = 0, \tag{3.23c}$$

which means that the transverse propagator is gapless, in agreement with Goldstone's theorem.

We now see whether the n-point vertices with $n > 4$ are relevant or not. The equation for $n = 5$ is given by

$$\begin{aligned}&\sum_{j_1}\Gamma^{(4)}_{j_1i_2i_3i_4}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)R^k_{j_1i_1} + \sum_{j_2}\Gamma^{(4)}_{i_1j_2i_3i_4}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\mathcal{R}^k_{j_2i_2} + \sum_{j_3}\Gamma^{(4)}_{i_1i_2j_3i_4}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\mathcal{R}^k_{j_3i_3}\\ &+\sum_{j_4}\Gamma^{(4)}_{i_1i_2i_3j_4}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\mathcal{R}^k_{j_4i_4} + \sum_{i_5j_5\boldsymbol{k}_5}\Gamma^{(5)}_{i_1i_2i_3i_4j_5}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4,\boldsymbol{0})\mathcal{R}^k_{j_5i_5}\Phi_{i_5} = 0.\end{aligned} \tag{3.24}$$

Setting $k = y$ and $(i_1, i_2, i_3, i_4) = (x, x, x, z), (y, y, x, z), (x, z, z, z)$ with $\Phi_i = \delta_{iz}\Phi$, we obtain finite elements of $\Gamma^{(5)}$ as follows:

$$\begin{aligned}\Gamma^{(5)}_{xxxxz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4,\boldsymbol{0}) =& 3\Gamma^{(5)}_{xxyyz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4,\boldsymbol{0}) = \frac{6}{\Phi}(g_{\rm lt}-g_{\rm tt})\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4,\boldsymbol{0}},\\ \Gamma^{(5)}_{xxzzz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4,\boldsymbol{0}) =& \frac{6}{\Phi}(g_{\rm ll}-g_{\rm lt})\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4,\boldsymbol{0}}.\end{aligned} \tag{3.25}$$

Next, we consider the equation for $n = 6$ and set $k = y$ and $(i_1, i_2, i_3, i_4, i_5) = (x, x, x, x, x)$, (y, y, x, x, x), (y, y, y, y, x), (x, x, y, y, j), (x, x, x, z, z), (y, y, x, z, z), (x, z, z, z, z) with $\Phi_i = \delta_{iz}\Phi$ and $j \neq x, y, z$. Finite elements of $\Gamma^{(6)}$ can be written as

$$\begin{aligned}\Gamma^{(6)}_{xxxxxx}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4,\boldsymbol{0},\boldsymbol{0}) =& \frac{30}{\Phi^2}(g_{\rm lt}-g_{\rm tt})\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4,\boldsymbol{0}},\\ \Gamma^{(6)}_{xxxxyy}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4,\boldsymbol{0},\boldsymbol{0}) =& \frac{6}{\Phi^2}(g_{\rm lt}-g_{\rm tt})\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4,\boldsymbol{0}},\\ \Gamma^{(6)}_{xxyyjj}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4,\boldsymbol{0},\boldsymbol{0}) =& \frac{2}{\Phi^2}(g_{\rm lt}-g_{\rm tt})\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4,\boldsymbol{0}},\\ \Gamma^{(6)}_{xxxxzz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4,\boldsymbol{0},\boldsymbol{0}) =& 3\Gamma^{(6)}_{xxyyzz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4,\boldsymbol{0},\boldsymbol{0})\\ =& \frac{6}{\Phi^2}(3g_{\rm ll}-5g_{\rm lt}+2g_{\rm tt})\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4,\boldsymbol{0}},\\ \Gamma^{(6)}_{xxzzzz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4,\boldsymbol{0},\boldsymbol{0}) =& -\frac{24}{\Phi^2}(g_{\rm ll}-g_{\rm lt})\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4,\boldsymbol{0}}.\end{aligned} \tag{3.26}$$

3.3 Functional Renormalization Group

We now introduce a cut off Λ so that $G_{0\Lambda,jj'}(\boldsymbol{k})$ is given in terms of Eq. (3.13) by

$$G_{0\Lambda,jj'}(\boldsymbol{k}) = G_{0,jj'}(\boldsymbol{k})\Theta(k-\Lambda). \tag{3.27}$$

We still have degrees of freedom in the parameters for $\Gamma^{(2)}_{\Lambda,jj}(-\boldsymbol{k},\boldsymbol{k})$ for $j\neq N$. We parametrize two-point verices as

$$\Gamma^{(2)}_{\Lambda,jj'}(-\boldsymbol{k},\boldsymbol{k})=\begin{cases}-(z_\Lambda^{-1}k^2+\sigma_\Lambda)\delta_{jj'} & \text{: symmetric phase}\\ -(z_\Lambda^{-1}k^2+2\Delta_\Lambda\delta_{jN})\delta_{jj'} & \text{: symmetry broken phase}\end{cases} \tag{3.28}$$

with $\Delta_\Lambda\equiv g_{\mathrm{tt}\Lambda}\Phi_\Lambda^2$. z_Λ is a renormalized factor and defined by

$$z_\Lambda^{-1}\equiv-\frac{1}{2}\frac{\partial^2\Gamma^{(2)}_{\Lambda,xx}(-\boldsymbol{k},\boldsymbol{k})}{\partial k^2}\bigg|_{\boldsymbol{k}=\boldsymbol{0}}. \tag{3.29}$$

This parameterization in the spontaneously broken phase satisfies Goldstone's theorem I. Since the system is isotropic in the symmetric phase of $\Phi=0$, we parametrize vertices according to Eq. (3.17) as

$$\Gamma^{(4)}_{\Lambda,j_1j_2j_3j_4}=2g_\Lambda\mathcal{S}_{j_1j_2j_3j_4}\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4,\boldsymbol{0}}. \tag{3.30}$$

in the symmetric phase.

3.3.1 Equations for vertices in momentum space

The functional renormalization group equations are given by

$$-\partial_\Lambda\Gamma^{(n)}_{\Lambda,j_1\cdots j_n}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)=W^{(n)}_{\Lambda,j_1\cdots j_n}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n). \tag{3.31}$$

Noting that $W^{(n)}$'s are proportional to $\delta_{\boldsymbol{k}_1+\cdots+\boldsymbol{k}_n,\boldsymbol{0}}$ and $W^{(2)}_{j_1j_2}$ additionally to $\delta_{j_1j_2}$, we can write down their non-zero elements for $n=1,\cdots,4$ as follows

$$W^{(1)}_{\Lambda,j}(\boldsymbol{0})=-\Gamma^{(2)}_{\Lambda,jz}(\boldsymbol{0},\boldsymbol{0})\partial_\Lambda\Phi_\Lambda+\frac{1}{2\beta N_\mathrm{a}}\mathrm{Tr}\hat{\Gamma}^{(3)}_{\Lambda,j}(\boldsymbol{0})\hat{\dot{G}}_\Lambda \tag{3.32a}$$

$$\begin{aligned}W^{(2)}_{\Lambda,jj}(-\boldsymbol{k},\boldsymbol{k})=&-\Gamma^{(3)}_{\Lambda,jjz}(-\boldsymbol{k},\boldsymbol{k},\boldsymbol{0})\partial_\Lambda\Phi_\Lambda+\frac{1}{2\beta N_\mathrm{a}}\mathrm{Tr}\hat{\Gamma}^{(4)}_{\Lambda,jj}(-\boldsymbol{k},\boldsymbol{k})\hat{\dot{G}}_\Lambda+\frac{1}{\beta N_\mathrm{a}}\mathrm{Tr}\hat{\Gamma}^{(3)}_{\Lambda,j}(-\boldsymbol{k})\hat{G}_\Lambda\hat{\Gamma}^{(3)}_{\Lambda,j}(\boldsymbol{k})\hat{\dot{G}}_\Lambda\\&+\partial_\Lambda G^{-1}_{0\Lambda}(\boldsymbol{k})\end{aligned} \tag{3.32b}$$

$$\begin{aligned}W^{(3)}_{\Lambda,j_1j_2j_3}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)=&-\Gamma^{(4)}_{\Lambda,j_1j_2j_3z}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{0})\partial_\Lambda\Phi_\Lambda+\frac{1}{2\beta N_\mathrm{a}}\mathrm{Tr}\hat{\Gamma}^{(5)}_{\Lambda,j_1j_2j_3}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\hat{\dot{G}}_\Lambda\\&+2\frac{1}{2\beta N_\mathrm{a}}\mathrm{Tr}\frac{1}{2!}\sum_P\hat{\Gamma}^{(4)}_{\Lambda,j_{p_1}j_{p_2}}(\boldsymbol{k}_{p_1},\boldsymbol{k}_{p_2})\hat{G}_\Lambda\hat{\Gamma}^{(3)}_{\Lambda,j_{p_3}}(\boldsymbol{k}_{p_3})\hat{\dot{G}}_\Lambda\\&+\frac{1}{2\beta N_\mathrm{a}}\mathrm{Tr}\sum_P\hat{\Gamma}^{(3)}_{\Lambda,j_{p_1}}(\boldsymbol{k}_{p_1})\hat{G}_\Lambda\hat{\Gamma}^{(3)}_{\Lambda,j_{p_2}}(\boldsymbol{k}_{p_2})\hat{G}_\Lambda\hat{\Gamma}^{(3)}_{\Lambda,j_{p_3}}(\boldsymbol{k}_{p_3})\hat{\dot{G}}_\Lambda\end{aligned} \tag{3.32c}$$

$$\begin{aligned}&W^{(4)}_{\Lambda,j_1j_2j_3j_4}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\\=&-\Gamma^{(5)}_{\Lambda,j_1j_2j_3j_4z}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4,\boldsymbol{0})\partial_\Lambda\Phi_\Lambda+\frac{1}{2\beta N_\mathrm{a}}\mathrm{Tr}\hat{\Gamma}^{(6)}_{\Lambda,j_1j_2j_3j_4}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\hat{\dot{G}}_\Lambda\\&+\frac{1}{2\beta N_\mathrm{a}}\mathrm{Tr}\frac{1}{3!}\sum_P[\hat{\Gamma}^{(5)}_{\Lambda,j_{p_1}j_{p_2}j_{p_3}}(\boldsymbol{k}_{p_1},\boldsymbol{k}_{p_2},\boldsymbol{k}_{p_3})\hat{G}_\Lambda\hat{\Gamma}^{(3)}_{\Lambda,j_{p_4}}(\boldsymbol{k}_{p_4})+\hat{\Gamma}^{(3)}_{\Lambda,j_{p_1}}(\boldsymbol{k}_{p_1})\hat{G}_\Lambda\hat{\Gamma}^{(5)}_{\Lambda,j_{p_2}j_{p_3}j_{p_4}}(\boldsymbol{k}_{p_2},\boldsymbol{k}_{p_3},\boldsymbol{k}_{p_4})]\hat{\dot{G}}_\Lambda\\&+\frac{1}{2\beta N_\mathrm{a}}\mathrm{Tr}\frac{1}{(2!)^2}\sum_P\hat{\Gamma}^{(4)}_{\Lambda,j_{p_1}j_{p_2}}(\boldsymbol{k}_{p_1},\boldsymbol{k}_{p_2})\hat{G}_\Lambda\hat{\Gamma}^{(4)}_{\Lambda,j_{p_3}j_{p_4}}(\boldsymbol{k}_{p_3},\boldsymbol{k}_{p_4})\hat{\dot{G}}_\Lambda\\&+\frac{1}{2\beta N_\mathrm{a}}\mathrm{Tr}\frac{1}{2!}\sum_P\Big[\hat{\Gamma}^{(4)}_{\Lambda,j_{p_1}j_{p_2}}(\boldsymbol{k}_{p_1},\boldsymbol{k}_{p_2})\hat{G}_\Lambda\hat{\Gamma}^{(3)}_{\Lambda,j_{p_3}}(\boldsymbol{k}_{p_3})\hat{G}_\Lambda\hat{\Gamma}^{(3)}_{\Lambda,j_{p_4}}(\boldsymbol{k}_{p_4})+\hat{\Gamma}^{(3)}_{\Lambda,j_{p_1}}(\boldsymbol{k}_{p_1})\hat{G}_\Lambda\hat{\Gamma}^{(4)}_{\Lambda,j_{p_2}j_{p_3}}(\boldsymbol{k}_{p_2},\boldsymbol{k}_{p_3})\hat{G}_\Lambda\hat{\Gamma}^{(3)}_{\Lambda,j_{p_4}}(\boldsymbol{k}_{p_4})\\&+\hat{\Gamma}^{(3)}_{\Lambda,j_{p_1}}(\boldsymbol{k}_{p_1})\hat{G}_\Lambda\hat{\Gamma}^{(3)}_{\Lambda,j_{p_2}}(\boldsymbol{k}_{p_2})\hat{G}_\Lambda\hat{\Gamma}^{(4)}_{\Lambda,j_{p_3}j_{p_4}}(\boldsymbol{k}_{p_3},\boldsymbol{k}_{p_4})\Big]\hat{\dot{G}}_\Lambda\\&+\frac{1}{2\beta N_\mathrm{a}}\mathrm{Tr}\sum_P\hat{\Gamma}^{(3)}_{\Lambda,j_{p_1}}(\boldsymbol{k}_{p_1})\hat{G}_\Lambda\hat{\Gamma}^{(3)}_{\Lambda,j_{p_2}}(\boldsymbol{k}_{p_2})\hat{G}_\Lambda\hat{\Gamma}^{(3)}_{\Lambda,j_{p_3}}(\boldsymbol{k}_{p_3})\hat{G}_\Lambda\hat{\Gamma}^{(3)}_{\Lambda,j_{p_4}}(\boldsymbol{k}_{p_4})\hat{\dot{G}}_\Lambda\end{aligned} \tag{3.32d}$$

3.3.2 RG flows for parameters

Now we derive the RG flow equations for parameters. The derivation of the renormalization group equations for the parameters is given in appendix A.

Symmetric Phase

First, we consider the symmetric phase of $\Phi_\Lambda = 0$, where functions $W_\Lambda^{(n)}$ for $n = 1, 3, \cdots$ vanish identically. The flow equations for σ_Λ and g_Λ are derived from $W_{jj}^{(2)}(\mathbf{0},\mathbf{0})$ and $W_{j_1j_2j_3j_4}^{(4)}(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$, respectively. It is convenient for discussing the fixed point to introduce the scaling parameters defined by

$$\tilde{\sigma}_l \equiv \Lambda^{-2} z_\Lambda \sigma_\Lambda,\ \ \tilde{g}_l \equiv \Lambda^{d-4}\frac{K_d z_\Lambda^2}{\beta} g_\Lambda \tag{3.33}$$

with $K_d = S_d/(2\pi)^d$ (S_d: surface area of d-dimensional unit sphere). RG flow equations for $\tilde{\sigma}_l$ and $\tilde{g}_l$ are then given by

$$\partial_l \tilde{\sigma}_l = (2-\eta_l)\tilde{\sigma}_l - \tilde{g}_l(N+2)\tilde{G}_l, \tag{3.34a}$$

$$\partial_l \tilde{g}_l = (4-d-2\eta_l)\tilde{g}_l - \tilde{g}_l^2(N+8)\tilde{G}_l^2, \tag{3.34b}$$

where $\partial_l \equiv -\Lambda\partial_\Lambda,\ \tilde{G}_l \equiv (1+\tilde{\sigma}_\Lambda)^{-1}$, and η_Λ is a flowing anomalous dimension defined by

$$\eta_l = -\partial_l \ln z_\Lambda, \tag{3.35}$$

which gives an anomalous dimension $\eta = \lim_{l\to\infty} \eta_\Lambda$. The expression for η_l is given as follows:

$$\eta_l = \int_0^l dt K(l,t)\tilde{g}_{l-t}^2 \exp\left(-2\int_{l-t}^l dt' \eta_{t'}\right) \tag{3.36}$$

with

$$K(l,t) = 6(N+2)\frac{1}{d(1+\tilde{\sigma}_l)}\left[(d-1)e^{-(d-3)t}\tilde{\chi}'_{l-t}(e^{-t}) + e^{-(d-2)t}\tilde{\chi}''_{l-t}(e^{-t})\right]. \tag{3.37}$$

The derivation of η_l and η is discussed in the Appendix B.

Spontaneously Broken Phase

Next, we focus on the spontaneously broken phase. To convenient we introduce the dimensionless parameters defined by

$$\tilde{\Phi}_l \equiv \Lambda^{-d+2}\frac{\beta}{K_d z_\Lambda}\Phi_\Lambda^2,\ \ \tilde{g}_{\mathrm{ij},l} \equiv \Lambda^{d-4}\frac{K_d z_\Lambda^2}{\beta} g_{\mathrm{ij}\Lambda}. \tag{3.38}$$

RG flows for parameters are given as follows:

$$\partial_l \tilde{\Phi}_l^2 = (d-2+\eta_l)\tilde{\Phi}_l^2 - (N-1)\tilde{G}_{\mathrm{t},l} - (2+\alpha_l)\,\tilde{G}_{\mathrm{l},l}, \tag{3.39a}$$

$$\partial_l \tilde{g}_{\mathrm{tt},l} = (4-d-2\eta_l)\tilde{g}_{\mathrm{tt},l} - \frac{1}{2\tilde{\Delta}_l}\tilde{g}_{\mathrm{tt},l}^2\left[\alpha_l(2+\alpha_l) - 3\beta_l\right]\tilde{G}_{\mathrm{l},l} - \tilde{g}_{\mathrm{tt},l}^2[(N-1)\tilde{G}_{\mathrm{t},l}^2 + (2+\alpha_l)^2\tilde{G}_{\mathrm{l},l}^2], \tag{3.39b}$$

$$\begin{aligned}\partial_l \tilde{g}_{\mathrm{lt},l} = {}&(4-d-2\eta_l)\tilde{g}_{\mathrm{lt},l} + \frac{1}{2\tilde{\Delta}_l}\tilde{g}_{\mathrm{tt},l}^2[2(N-1)(1-\alpha_l)\tilde{G}_{\mathrm{t},l} + [(2-3\beta_l+3\alpha_l)(2+\alpha_l) - 6\beta_l]\tilde{G}_{\mathrm{l},l}]\\ &- \tilde{g}_{\mathrm{tt},l}^2[(N-1)(3\alpha_l-2)\tilde{G}_{\mathrm{t},l}^2 + (9\beta_l - 2\alpha_l - 4)(2+\alpha_l)\tilde{G}_{\mathrm{l},l}^2]\\ &+ 4\tilde{g}_{\mathrm{tt},l}^2\tilde{\Delta}_l[(N-1)\tilde{G}_{\mathrm{t},l}^3 + (2+\alpha_l)^3\tilde{G}_{\mathrm{l},l}^3],\end{aligned} \tag{3.39c}$$

$$\begin{aligned}\partial_l \tilde{g}_{\mathrm{ll},l} = {}&(4-d-2\eta_l)\tilde{g}_{\mathrm{ll},l} - 2\frac{\tilde{g}_{\mathrm{tt},l}^2}{\tilde{\Delta}_l}(N-1)(\beta_l-\alpha_l)\tilde{G}_{\mathrm{t},l} - \tilde{g}_{\mathrm{tt},l}^2[(N-1)\left[4(\beta_l-\alpha_l)+\alpha_l^2\right]\tilde{G}_{\mathrm{t},l}^2 + 9\beta_l^2\tilde{G}_{\mathrm{l},l}^2]\\ &+ 8\tilde{g}_{\mathrm{tt},l}^2\tilde{\Delta}_l[(N-1)\alpha_l\tilde{G}_{\mathrm{t},l}^3 + 3(2+\alpha_l)^2\beta_l\tilde{G}_{\mathrm{l},l}^3] - 8\tilde{g}_{\mathrm{tt},l}^2\tilde{\Delta}_l^2[(N-1)\tilde{G}_{\mathrm{t},l}^4 + (2+\alpha_l)^4\tilde{G}_{\mathrm{l},l}^4],\end{aligned} \tag{3.39d}$$

where $\alpha_l \equiv \tilde{g}_{\mathrm{lt},l}/\tilde{g}_{\mathrm{tt},l},\ \beta_l \equiv \tilde{g}_{\mathrm{ll},l}/\tilde{g}_{\mathrm{tt},l}\ \tilde{\Delta}_l \equiv \tilde{g}_{\mathrm{tt},l}\tilde{\Phi}_l^2$, and $\tilde{G}_{\mathrm{t},l} \equiv 1,\ \tilde{G}_{\mathrm{l},l} \equiv (1+2\tilde{\Delta}_l)^{-1}$. Consider the RG flow in the vicinity of the symmetric point $g_{\mathrm{tt}} = g_{\mathrm{lt}} = g_{\mathrm{ll}}$. Substituting $\alpha_l = \beta_l = 1$ into the r.h.s. of Eqs. (3.40), (3.40), and (3.40), we obtain

$$\partial_l \tilde{g}_{\mathrm{tt},l} = (4-d-2\eta_l)\tilde{g}_{\mathrm{tt},l} - \tilde{g}_{\mathrm{tt},l}^2[(N-1)\tilde{G}_{\mathrm{t},l}^2 + 9\tilde{G}_{\mathrm{l},l}^2],$$

$$
\begin{aligned}
\partial_l \tilde{g}_{\mathrm{lt},l} =& (4-d-2\eta_l)\tilde{g}_{\mathrm{lt},l} - \tilde{g}_{\mathrm{tt},l}^2[(N-1)\tilde{G}_{\mathrm{t},l}^2 + 9\tilde{G}_{\mathrm{l},l}^2] \\
&+ 4\tilde{g}_{\mathrm{tt},l}^2\tilde{\Delta}_l[(N-1)\tilde{G}_{\mathrm{t},l}^3 + 27\tilde{G}_{\mathrm{l},l}^3], \\
\partial_l \tilde{g}_{\mathrm{ll},l} =& (4-d-2\eta_l)\tilde{g}_{\mathrm{ll},l} - \tilde{g}_{\mathrm{tt},l}^2[(N-1)\tilde{G}_{\mathrm{t},l}^2 + 9\tilde{G}_{\mathrm{l},l}^2] \\
&+ 8\tilde{g}_{\mathrm{tt},l}^2\tilde{\Delta}_l[(N-1)\tilde{G}_{\mathrm{t},l}^3 + 27\tilde{G}_{\mathrm{l},l}^3] - 8\tilde{g}_{\mathrm{tt},l}^2\tilde{\Delta}_l^2[(N-1)\tilde{G}_{\mathrm{t},l}^4 + 81\tilde{G}_{\mathrm{l},l}^4],
\end{aligned}
$$

Comparing with Fig. 2.1, we can see that the RG flow of $\tilde{g}_{\mathrm{tt},l}$ is dominated by the contribution from (4d), while the flow of $\tilde{g}_{\mathrm{lt},l}$ ($\tilde{g}_{\mathrm{ll},l}$) has also contribution(s) from (4e) (and (4f)). From the diagrammatic representation of (4e) and (4f), the difference between the coupling constants may be caused by the three-point vertices.

3.4 Results and Discussion

3.4.1 Fixed point

First, we find the non-trivial (not Gaussian) fixed points. The fixed point is obtained by setting the l.h.s. of RG equations zero. We have to also obtain the expressions for $\eta = \lim_{l\to\infty} \eta_l$. In the symmetric phase, only (4d) in Fig. 2.1 contributes to the anomalous dimension, while there are contributions from (3c), (3d), (4e), and (4f) in addition to (4d) in the spontaneously broken phase. For the latter, we will partially incorporate the contributions from (3c), (3d), and (4d) (see Appendix B for expressions of each term). The critical fixed point can be determined by whether $\tilde{\Delta}_*$ is finite or not. Here the subscript * denotes the value at the fixed point.

First, we consider the case of $d \lesssim 4$. Using ϵ-expansion for $\epsilon \equiv 4-d$, we obtain

$$
\begin{aligned}
\tilde{\Delta}_* =& \frac{N+2}{2(N+8)}\epsilon\left[1 + \frac{(N+2)^2(8N^3+370N^2+1117N+5498)}{4(N+8)^3(8N^2+2N+17)}\epsilon\right] \\
\tilde{g}_{\mathrm{tt}*} =& \frac{1}{N+8}\epsilon\left[1 + \frac{N+2}{(N+8)^2}\frac{152N^2+542N+2681}{8N^2+2N+17}\epsilon\right], \\
\begin{bmatrix}\tilde{g}_{\mathrm{lt}*}\\ \tilde{g}_{\mathrm{ll}*}\end{bmatrix} =& \tilde{g}_{\mathrm{tt}*} + \frac{2(N+26)(N+2)^2}{(8N^2+2N+17)(N+8)^3}\begin{bmatrix}4N-37\\ 8N-34\end{bmatrix}\epsilon^3, \\
\eta =& \frac{N+2}{2(N+8)^2}\epsilon^2.
\end{aligned}
\tag{3.41}
$$

This result shows that the difference between g_{ij} emerges at the order of $O(\epsilon^3)$. Since the contribution of (4e) in Fig. 2.1 is of the order of $O(\epsilon^3)$ below $d=4$, this difference is considered to be caused by the three-point vertices. In this model, we also found that there is no isotropic fixed point satisfying $\tilde{g}_{\mathrm{tt}*} = \tilde{g}_{\mathrm{lt}*} = \tilde{g}_{\mathrm{ll}*} \neq 0$ (so-called $O(N)$ symmetric fixed point) with $\tilde{\Phi}_* \neq 0$. On the other hand, the fixed point in the symmetric phase can be described as follows:

$$
\begin{aligned}
\tilde{\sigma}_* =& -\frac{N+2}{2(N+8)}\epsilon\left[1 - \frac{(N+2)(N+10)}{2(N+8)^2}\epsilon\right], \\
\tilde{g}_* =& \frac{1}{N+8}\epsilon\left[1 - \frac{(N+2)(N+9)}{(N+8)^2}\epsilon\right], \\
\eta =& \frac{N+2}{2(N+8)^2}\epsilon^2.
\end{aligned}
\tag{3.42}
$$

Thus, the anomalous dimension coincides with that of the symmetric phase on the order of ϵ^2.

Next, we focus on the case of $d=3$. Figure 3.2 shows the N dependence of the fixed point at $d=3$. This result clearly shows an anisotropy between coupling constant for finite N. In particular, the anisotropy is enhanced with decreasing N because the contribution of longitudinal fluctuations increases relatively. For large N, the fixed point is given by

$$
\begin{aligned}
&\tilde{\Delta}_* = \frac{4-d-2\eta}{d-2+\eta},\ \tilde{g}_{\mathrm{tt}*} = \frac{4-d-2\eta}{N}, \\
&\tilde{g}_{\mathrm{lt}*} = \tilde{g}_{\mathrm{tt}*}\left(1 + \frac{4\tilde{\Delta}_*^2}{1+2\tilde{\Delta}_*}\right),\ \tilde{g}_{\mathrm{ll}*} = \tilde{g}_{\mathrm{tt}*}\left(1 + \frac{16(1+\tilde{\Delta}_*)^3\tilde{\Delta}_*^2}{(2+3\tilde{\Delta}_*)(1+2\tilde{\Delta}_*)^2}\right),
\end{aligned}
\tag{3.43}
$$

that is, all coupling constants vanish in $N\to\infty$, while $\tilde{\Delta}_*$ is finite. Figure 3.3 shows the N dependence of η. We can see that the value of η increases with N decreasing. This enhancement is also seen in the fixed point

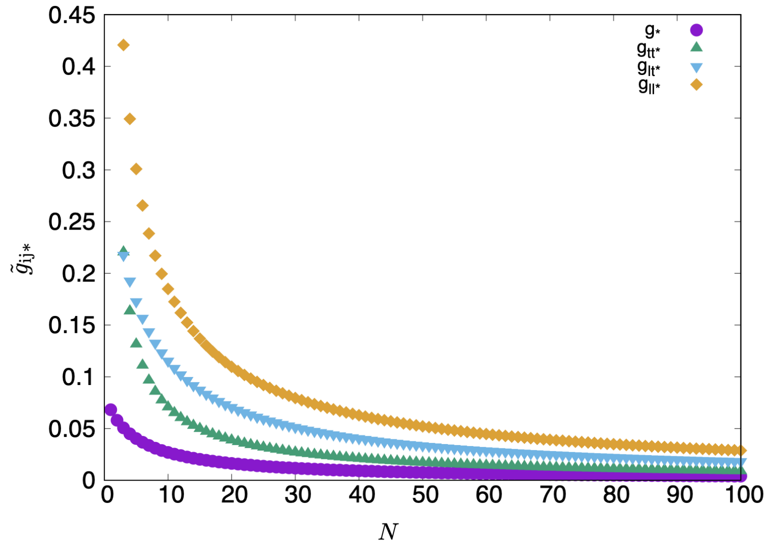

Figure 3.2: N dependence of the non-trivial fixed point at $d = 3$. We also plot the fixed point g_* for symetric phase.

describing the ordered phase ($\tilde{\Phi}_* \to \infty$) [15]. On the other hand, it is known that in the BEC phase starting from $U(1)$ symmetric model, the sum of the contributions of (4e) and (4f) in Fig. 2.1 is negative for $d \lesssim 4$. Therefore, incorporating these effects may change the behavior of the anomalous dimension and may be consistent with the results obtained in the symmetric phase. We have also checked that η vanishes with $O(1/\sqrt{N})$ at $d = 3$ both analytically and numerically due to the term proportional to η^{-1} (see Appendix B). This behavior is different from the result of the $1/N$ expansion method used in calculating critical exponents [16, 17]. However, it turns out that this behavior is regulator-derived (See Appendix C). We also plot the results for $N = 1, 2, 3$ obtained by other field theoretical approaches that reproduce the values of universality class with high accuracy. The anomalous dimensions we obtained are very different from the results of the universality class, even for the symmetric phase. However, the latter result is close to the value of the universality class by considering $\Gamma^{(6)}$ and the linear term of $\Gamma^{(4)}$, which are marginal terms in the case of $d = 3$ [18].

3.4.2 Flow of correlation functions

Here, we solve the renormalization group equations and investigate the momentum dependence of the correlation function from its behavior. One can retrieve the momentum dependence of $G_{\Lambda=0}(\boldsymbol{k})$ at finite $\boldsymbol{k}$ by stopping the RG flow at $\Lambda = |\boldsymbol{k}|$; that is, $G_{\Lambda=0}(\boldsymbol{k}) \simeq G_{\Lambda\sim|\boldsymbol{k}|}(\boldsymbol{k})$ [14].

Figure 3.4 shows $\Lambda = \Lambda_0 e^{-l}$ dependence of $G_\Lambda(\Lambda) = -(z_\Lambda^{-1}\Lambda^2 + \sigma_\Lambda)$ and η_l. When approaching a critical fixed point, η_l becomes constant, giving an anomalous dimensional η. In this regime, $G_\Lambda(\Lambda)$ behaves as

$$G_\Lambda^{-1}(\Lambda) \sim \Lambda^{2-\eta}. \tag{3.44}$$

As the RG flow leaves the critical fixed point and approaches the disordered fixed point, η_l begins to decay. The scale Λ at which the RG flow leaves the critical fixed point gives the correlation length of the system: $\Lambda = \xi^{-1}$. Since $\xi < \infty$ in the disordered phase, G_Λ approaches a constant value. ξ diverges at $T = T_c$ and we can define the critical exponent as $\xi \propto |T - T_c|^{-\nu}$.

On the other hand, it was complicated to evaluate the flowing anomalous dimension using the sharp momentum cutoff in the ordered phase, so we re-derived the renormalization group equation using the additive type cutoff and solved the RG equation (See appendix C). Figure 3.5 shows $\Lambda = \Lambda_0 e^{-l}$ dependence of $G_{i\Lambda}(\Lambda) = -(z_\Lambda^{-1}\Lambda^2 + 2\delta_{i,l}\Delta_\Lambda)$ and η_l. In the critical regime, G_t and G_l shows the same momentum dependence such as

$$G_{t\Lambda}^{-1}(\Lambda) \sim G_{l\Lambda}^{-1}(\Lambda) \sim \Lambda^{2-\eta}. \tag{3.45}$$

However, the RG flow leaves the critical fixed point and approaches the ordered fixed point, the behavior of both

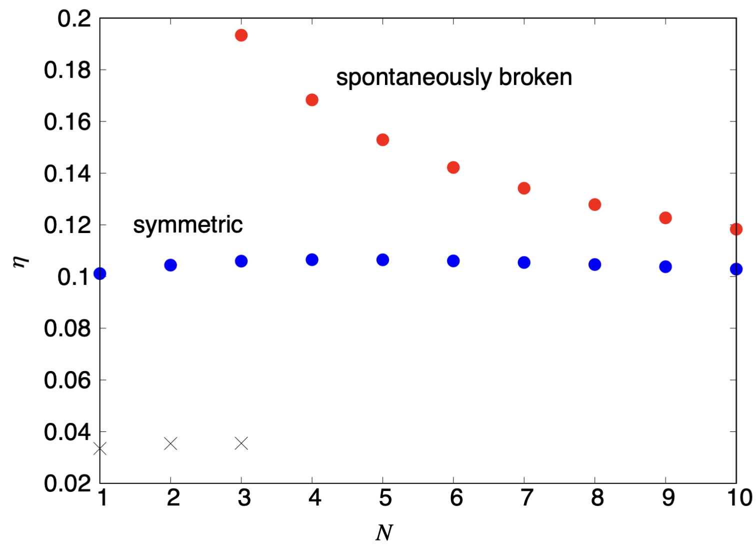

Figure 3.3: The anomalous dimension η for $O(N)$ model at $d = 3$. The red (blue) points are results with (without) NG mode. The cross symbols are the result of [19] using the Borel summation.

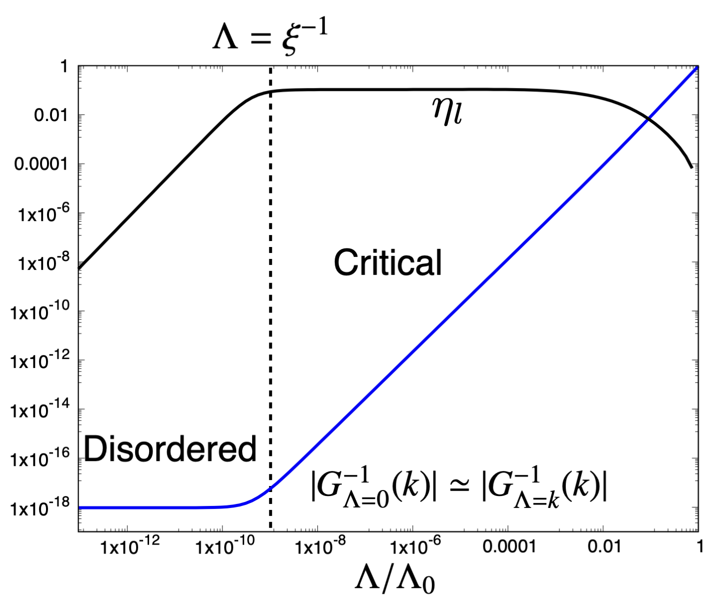

Figure 3.4: Flow of the correlation function $G_l(\Lambda)$ for $O(3)$ at $d = 3$ in the disordered phase. The black solid line is the flowing anomalous dimension. The parallel dotted line is $\Lambda = \xi^{-1}$, where ξ is the correlation length.

changes as

$$G_{\mathrm{t}\Lambda}^{-1}(\Lambda) \sim \Lambda^2,\ G_{\mathrm{l}\Lambda}^{-1}(\Lambda) \sim \Lambda^{4-d}. \tag{3.46}$$

and G_{l} also diverges in the IR region. This region is called the Goldstone regime, where the effective action between NG modes vanish as seen Fig. 3.6. This behavior has been obtained in previous studies of FRGs in the $O(N)$ model [20, 21]. As in the case of the disordered phase, the length scale in the ordered phase ξ' can be defined by the momentum scale where the behavior of the correlation function changes. This length scale $\xi' \equiv \xi_J$ is called the Josephson length, and it separates the critical regime from a regime dominated by NG modes [13, 14].

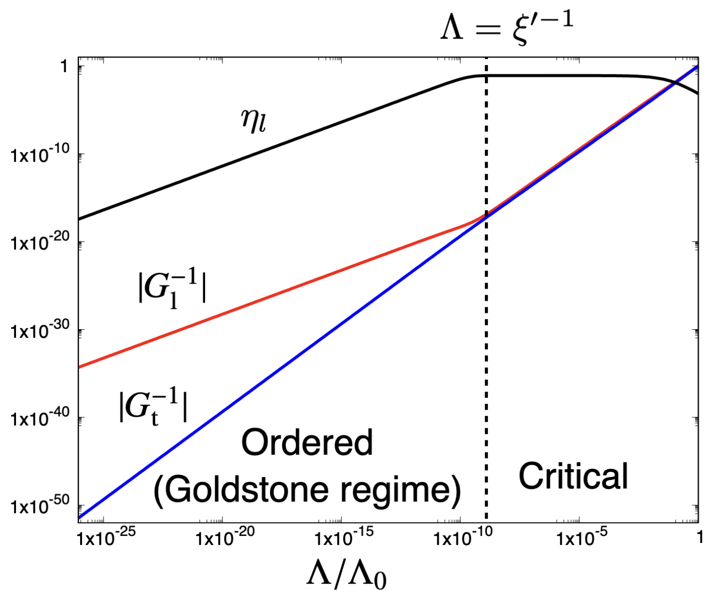

Figure 3.5: Flow of the correlation function $G_{\mathrm{l}\Lambda}(\Lambda)$ and $G_{\mathrm{t}\Lambda}(\Lambda)$ for $O(3)$ at $d = 3$ in the ordered phase. The parallel dotted line is $\Lambda = \xi'^{-1}$, where $\xi' \equiv \xi_J$ is the Josephson length [13].

3.4.3 Scaling laws in ordered phase

As we have seen, although the longitudinal correlation function diverges in the IR region in the spontaneously broken phase [3], we can define the characteristic length scale ξ_J in the critical regime of the spontaneously broken phase [14]. Since ξ_J diverges at $T = T_{\mathrm{c}}$, we can define the critical exponent for ξ_J as $\xi_J \sim |T - T_{\mathrm{c}}|^{-\nu'}$ [13].

We now investigate how the conventional scaling law is given when the critical exponent of the Josephson length can be defined. The critical exponent for the order parameter is defined by $\Phi(T) \sim |T - T_{\mathrm{c}}|^{\beta}$. The scaling law can be derived by the context of the renormalization group [22, 23]. The scale dependence of Φ has been given by

$$\Phi_\Lambda^2 \sim \frac{z_\Lambda}{\Lambda^{-d+2}} \tilde{\Phi}_l(\xi'\Lambda).$$

Here, we consider $\Lambda \to 0$ with $\xi'\Lambda = 1$. Since z_Λ behaves as $z_\Lambda \sim \Lambda^\eta$ in the critical regime, we obtain

$$\Phi_\Lambda^2 \propto \Lambda^{d-2+\eta}. \tag{3.47}$$

Assuming that $\xi' = \xi_J \propto |T - T_{\mathrm{c}}|^{-\nu'}$,

$$\Phi_\Lambda^2 \sim \xi_J^{-(d-2+\eta)} \propto |T - T_{\mathrm{c}}|^{(d-2+\eta)\nu'}. \tag{3.48}$$

Finally, we obtain

$$2\beta = (d - 2 + \eta)\nu' \tag{3.49}$$

and this is a well-known hyper-scaling law.

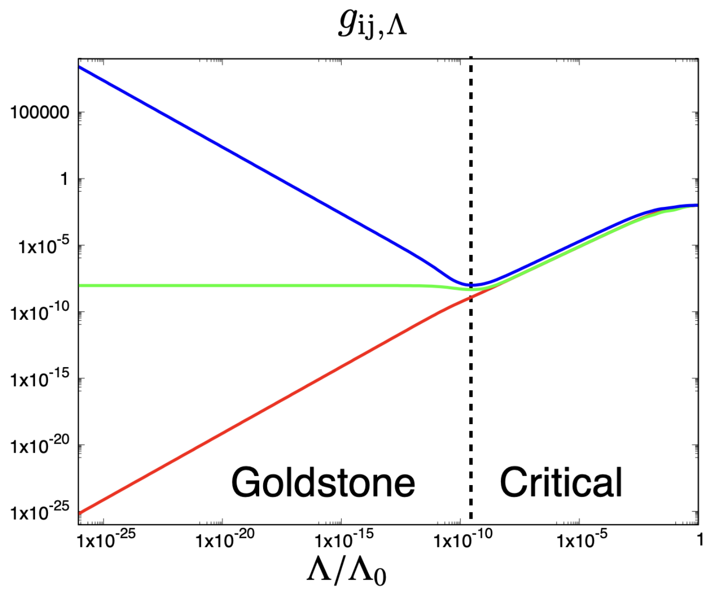

Figure 3.6: Flow of the coupling constants $g_{\mathrm{tt}\Lambda}$ (red), $g_{\mathrm{lt}\Lambda}$ (green), and $g_{\mathrm{ll}\Lambda}$ (blue) for $O(3)$ at $d = 3$ in the ordered phase. The parallel dotted line is $\Lambda = \xi'^{-1}$, with $\xi' \equiv \xi_J$.

3.4.4 Correlation length exponent

It can be seen that the conventional scaling law can be recovered by defining the critical exponent of the Josephson length. Now we analyze the RG flow for $\tilde{\Delta}_l$ in the ordered phase near the fixed point obtained before subsection and focus on the Josephson length exponent. This exponent can be obtained from the behavior of $\tilde{\Delta}_l$ by (i) linearizing the RG equation around a fixed point and (ii) finding the eigenvalues and eigenvectors of the linearized equation [5].

For $d \lesssim 4$, we obtain

$$\nu' \approx \nu \approx \frac{1}{2} + \frac{1}{4}\frac{N+2}{N+8}\epsilon \tag{3.50}$$

in the leading order of ϵ. Since the anisotropy between coupling constants appears on the order of ϵ^3, the values of the critical exponents at lower orders are consistent with that of the symmetric phase.

Figure 3.7 shows the result of ν, ν' at $d = 3$. We find that the results for anisotropic fixed points are different from those obtained for isotropic fixed points appearing in the symmetric phase. For large N, ν' is close to $(d-2)^{-1}$ and reproduces the result of the non-linear sigma model (or spherical model) [24] and large-N limit [14]. Our results for the spontaneously broken phases are also close to those obtained by FRG with derivative expansion [21, 25]. On the other hand, although the results for the symmetric phase are far from the universality class values [5], they can be improved by considering higher-order couplings [26].

The effect of "anisotropy" in spin systems on critical phenomena has been studied in the past. In the Heisenberg magnet on a cubic lattice, it exists due to the effect of the crystal field. In general, the critical exponents are identical in both phases because the RG equations are identical in the symmetric and spontaneously broken phases [5]. Nelson [27] was the first to propose a counterexample using the $O(2)$ model at $d = 3$. He argued that anisotropic such as cubic [27, 28] or hexagonal [27, 29] perturbations induce the modification of the exponent of the susceptibility in the low-temperature phase even though they are irrelevant. F. Léonard and B. Delamotte [30] argued further and found that the asymmetry of the critical exponent may appear when the continuous symmetry of the system is spontaneously broken and reduced to a discrete symmetry, where there are no NG modes so that the correlation length is finite even in the spontaneously broken phases. However, in FRG, the RG equations obtained according to the vertex expansion are different for each phase. Therefore, the inconsistent of critical exponents on both sides may be universal, regardless of whether the symmetry drops to discrete or not.

The exponents of the correlation length and magnetic susceptibility in the Heisenberg model are obtained with high accuracy by Monte Carlo (MC) simulation [31]. Due to the divergence of the correlation length in the low-temperature phase [3], it is difficult to evaluate the exponents in the low-temperature phase using finite-size Monte

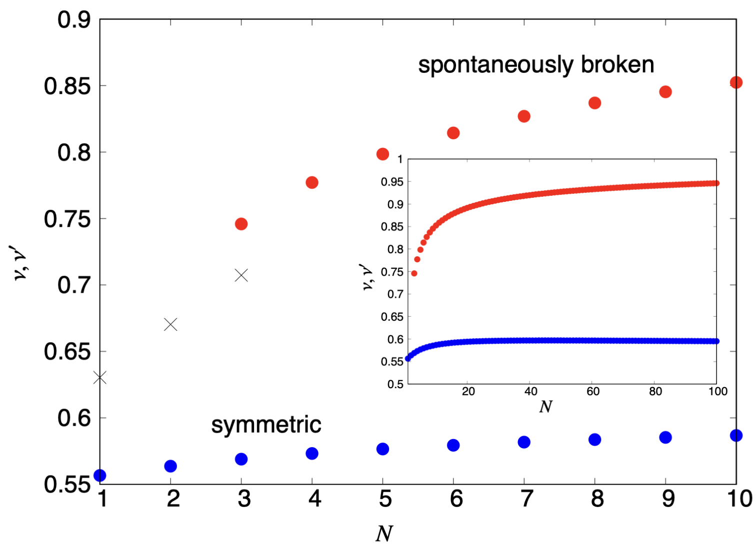

Figure 3.7: N dependence of critical exponent ν and ν' at $d = 3$. The cross symbols are the result of [19] using the Borel summation. Inset shows the results for large N.

Carlo simulations. Indeed, MC simulations in Ref. [31] are only available for the disordered phase.

Chapter 4

Summary and Conclusion

In this thesis, we have analyzed the phases with spontaneous symmetry breaking. The appearance of NG modes leads to singularities in various physical quantities, such as the divergence of susceptibility.

In chapter 2, we have summarized a formulation of FRG that satisfies Goldstone's Theorem I. Goldstone's Theorem I gives an identity between the vertices of a spontaneously broken phase, leading to a reduction in the number of RG parameters.

In chapter 3, we have studied the $O(N)$ symmetric model. The transverse fluctuations correspond to the NG modes in the ordered phase, which is extremely difficult to analyze using the usual perturbation expansion method due to IR divergence. We derived the RG equations to include the fluctuations in the ordered phase. Solving the RG equation in the ordered phase, we found that an anisotropic fixed point appears. This anisotropy is enhanced with decreasing $O(N)$. The longitudinal component of the correlation function also diverges due to the coupling to the phase fluctuation. Although the correlation length cannot be defined in the ordered phase in the same way as in the disordered phase, the characteristic length scale ξ_J emerged. This length scale is called the Josephson length, and it diverges at the critical point. It was found that the conventional scaling law can be recovered by introducing a critical exponent of Josephson length. The analysis around this fixed point revealed that the critical exponent has a different value from the result obtained from the RG equation in the symmetric phase. This result implies that the spontaneous symmetry breaking itself is the mechanism that gives rise to the asymmetry of the critical exponents in both sides of the second-order phase transition.

Appendix A

Derivation of RG Equations in $O(N)$ Model

Here we derive the RG equations for parameters defined in chapter 3.

A.1 Symmetric phase

In the symmetric phase of $\Phi = 0$, the system is isotropic, and functions $W^{(n)}$ for $n = 1, 3, \cdots$ vanish identically. The vertex $\Gamma^{(4)}_{\Lambda}$ can be parametrized as

$$\Gamma^{(4)}_{\Lambda,j_1j_2j_3j_4}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4) = 2g_\Lambda \mathcal{S}_{j_1j_2j_3j_4} \delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4,\boldsymbol{0}} \tag{A.1}$$

with $\mathcal{S}_{j_1j_2j_3j_4} \equiv \delta_{j_1j_2}\delta_{j_3j_4} + \delta_{j_1j_3}\delta_{j_2j_4} + \delta_{j_1j_4}\delta_{j_2j_3}$. Note that it satisfies

$$\sum_{j_3} \mathcal{S}_{j_1j_2j_3j_3} = (N+2)\delta_{j_1j_2} \tag{A.2a}$$

$$\sum_{j_5j_6} \mathcal{S}_{j_1j_2j_5j_6}\mathcal{S}_{j_5j_6j_3j_4} = (N+4)\delta_{j_1j_2}\delta_{j_3j_4} + 2\delta_{j_1j_3}\delta_{j_2j_4} + 2\delta_{j_1j_4}\delta_{j_2j_3} \tag{A.2b}$$

$$\sum_{j_5j_6} (\mathcal{S}_{j_1j_2j_5j_6}\mathcal{S}_{j_5j_6j_3j_4} + \mathcal{S}_{j_1j_3j_5j_6}\mathcal{S}_{j_5j_6j_2j_4} + \mathcal{S}_{j_1j_4j_5j_6}\mathcal{S}_{j_5j_6j_2j_3}) = (N+8)\mathcal{S}_{j_1j_2j_3j_4} \tag{A.2c}$$

Substituting Eq. (A.1) into Eq. (3.32b), we can transform $W^{(2)}$ as

$$W^{(2)}_{\Lambda,jj}(\boldsymbol{0}, \boldsymbol{0}) = (N+2)g_\Lambda \frac{K_d \Lambda^{d-1}}{\beta} \frac{z_\Lambda}{\Lambda^2 + z_\Lambda \sigma_\Lambda} \tag{A.3}$$

Substituting Eq. (A.1) into Eq. (3.32d) and using Eq. (A.2), we can also transform $W^{(4)}_{\Lambda,jjjj}(\boldsymbol{0},\boldsymbol{0},\boldsymbol{0},\boldsymbol{0})$ as

$$W^{(4)}_{\Lambda,jjjj}(\boldsymbol{0},\boldsymbol{0},\boldsymbol{0},\boldsymbol{0}) = 12(N+8)g_\Lambda^2 \frac{1}{\beta N_\mathrm{a}} \sum_{\boldsymbol{k}_1} \dot{G}_\Lambda(\boldsymbol{k}_1) G_\Lambda(\boldsymbol{k}_1). \tag{A.4}$$

Substituting Eq. (A.3) into $-\partial_\Lambda \Gamma^{(2)}_{\Lambda,j_1j_1}(\boldsymbol{k}, -\boldsymbol{k}) = W^{(2)}_{\Lambda,j_1j_1}(-\boldsymbol{k}, \boldsymbol{k})$ with $\Gamma^{(2)}_{\Lambda,j_1j_1}(-\boldsymbol{k}, \boldsymbol{k}) = -G^{-1}_{0\Lambda,j_1j_1}(\boldsymbol{k}) + \Sigma_{\Lambda,j_1j_1}(\boldsymbol{k})$, we obtain

$$-\partial_\Lambda \sigma_\Lambda = (N+2)g_\Lambda \frac{K_d z_\Lambda}{\beta} \frac{\Lambda^{d-1}}{\Lambda^2 + z_\Lambda \sigma_\Lambda}. \tag{A.5}$$

Also substituting it together with Eq. (A.1) into $-\partial_\Lambda \Gamma^{(4)}_{\Lambda,jjjj}(\boldsymbol{0},\boldsymbol{0},\boldsymbol{0},\boldsymbol{0}) = W^{(4)}_{\Lambda,jjjj}(\boldsymbol{0},\boldsymbol{0},\boldsymbol{0},\boldsymbol{0})$, we obtain

$$-\partial_\Lambda g_\Lambda = \frac{1}{6} W^{(4)}_{\Lambda,jjjj}(\boldsymbol{0},\boldsymbol{0},\boldsymbol{0},\boldsymbol{0}) = -(N+8)g_\Lambda^2 \frac{K_d \Lambda^{d-1}}{\beta} \frac{z_\Lambda^2}{(\Lambda^2 + z_\Lambda \sigma_\Lambda)^2}. \tag{A.6}$$

A.2 Spontaneously broken phase

Equation (3.32a) for the spontaneously broken phase can be transformed as

$$W^{(1)}_{\Lambda,j}(\mathbf{0}) = \delta_{jz} g_{\mathrm{tt}\Lambda}\Phi_\Lambda \left[-\partial_\Lambda \Phi_\Lambda^2 + \frac{N-1}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1} \dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) + \left(2 + \frac{g_{\mathrm{lt}\Lambda}}{g_{\mathrm{tt}\Lambda}}\right)\frac{1}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)\right]$$

It follows from the stationarity condition $W^{(1)}_{\Lambda,z}(\mathbf{0}) = 0$ that the order parameter obeys

$$-\partial_\Lambda \Phi_\Lambda^2 = -\frac{N-1}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1} \dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) - \left(2 + \frac{g_{\mathrm{lt}\Lambda}}{g_{\mathrm{tt}\Lambda}}\right)\frac{1}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) \tag{A.7a}$$

Next, Eq. (3.32b) can be transformed as

$$\begin{aligned}
&W^{(2)}_{\Lambda,jj}(-\boldsymbol{k},\boldsymbol{k}) \\
=&-\delta_{jz}(2g_{\mathrm{tt}\Lambda} + g_{\mathrm{lt}\Lambda})\partial_\Lambda\Phi_\Lambda^2 \\
&+\delta_{jz}\frac{1}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}[(N-1)g_{\mathrm{lt}\Lambda}\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) + 3g_{\mathrm{ll}\Lambda}\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)] \\
&+\delta_{jz}\frac{4\Phi_\Lambda^2}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\Big[(N-1)g_{\mathrm{tt}\Lambda}^2\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{t}\Lambda}(\boldsymbol{k}+\boldsymbol{k}_1) + (2g_{\mathrm{tt}\Lambda}+g_{\mathrm{lt}\Lambda})^2\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{l}\Lambda}(\boldsymbol{k}+\boldsymbol{k}_1)\Big] \\
&-(1-\delta_{jz})g_{\mathrm{tt}\Lambda}\partial_\Lambda\Phi_\Lambda^2 \\
&+(1-\delta_{jz})\frac{1}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\Big[(N+1)g_{\mathrm{tt}\Lambda}\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) + g_{\mathrm{lt}\Lambda}\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)\Big] \\
&+(1-\delta_{jz})\frac{4g_{\mathrm{tt}\Lambda}^2\Phi_\Lambda^2}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\Big[\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{l}\Lambda}(\boldsymbol{k}+\boldsymbol{k}_1) + \dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{t}\Lambda}(\boldsymbol{k}+\boldsymbol{k}_1)\Big] \\
&+\partial_\Lambda G^{-1}_{0\Lambda}(\boldsymbol{k})
\end{aligned} \tag{A.7b}$$

$$\begin{aligned}
&W^{(2)}_{\mathrm{l}\Lambda}(-\boldsymbol{k},\boldsymbol{k}) \\
=&-(2g_{\mathrm{tt}\Lambda} + g_{\mathrm{lt}\Lambda})\partial_\Lambda\Phi_\Lambda^2 \\
&+\frac{1}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}[(N-1)g_{\mathrm{lt}\Lambda}\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) + 3g_{\mathrm{ll}\Lambda}\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)] \\
&+\frac{4\Phi_\Lambda^2}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\Big[(N-1)g_{\mathrm{tt}\Lambda}^2\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{t}\Lambda}(\boldsymbol{k}+\boldsymbol{k}_1) + (2g_{\mathrm{tt}\Lambda}+g_{\mathrm{lt}\Lambda})^2\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{l}\Lambda}(\boldsymbol{k}+\boldsymbol{k}_1)\Big] \\
&+\partial_\Lambda G^{-1}_{0\Lambda}(\boldsymbol{k}) \\
\xrightarrow[\boldsymbol{k}\to\mathbf{0}]{}&-(2g_{\mathrm{tt}\Lambda} + g_{\mathrm{lt}\Lambda})\partial_\Lambda\Phi_\Lambda^2 \\
&+\frac{1}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}[(N-1)g_{\mathrm{lt}\Lambda}\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) + 3g_{\mathrm{ll}\Lambda}\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)] \\
&+\frac{4\Phi_\Lambda^2}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\Big[(N-1)g_{\mathrm{tt}\Lambda}^2\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) + (2g_{\mathrm{tt}\Lambda}+g_{\mathrm{lt}\Lambda})^2\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)\Big]
\end{aligned} \tag{A.7c}$$

$$\begin{aligned}
&W^{(2)}_{\mathrm{t}\Lambda}(-\boldsymbol{k},\boldsymbol{k}) \\
=&-g_{\mathrm{tt}\Lambda}\partial_\Lambda\Phi_\Lambda^2 \\
&+\frac{1}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\Big[(N+1)g_{\mathrm{tt}\Lambda}\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) + g_{\mathrm{lt}\Lambda}\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)\Big] \\
&+\frac{4g_{\mathrm{tt}\Lambda}^2\Phi_\Lambda^2}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\Big[\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{l}\Lambda}(\boldsymbol{k}+\boldsymbol{k}_1) + \dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{t}\Lambda}(\boldsymbol{k}+\boldsymbol{k}_1)\Big] \\
&+\partial_\Lambda G^{-1}_{0\Lambda}(\boldsymbol{k})
\end{aligned} \tag{A.7d}$$

We can transform $\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}) - \dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k})$ in Eq. (A.7b) as

$$\begin{aligned}
\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}) - \dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}) =& \frac{z_\Lambda}{k^2 + 2g_{\mathrm{tt}\Lambda}\Phi_\Lambda^2 z_\Lambda}\delta(k-\Lambda) - \frac{z_\Lambda}{k^2}\delta(k-\Lambda) \\
=& -\frac{2g_{\mathrm{tt}\Lambda}\Phi_\Lambda^2 z_\Lambda^2}{k^2(k^2 + 2g_{\mathrm{tt}\Lambda}\Phi_\Lambda^2 z_\Lambda)}\delta(k-\Lambda) \\
=& -2g_{\mathrm{tt}\Lambda}\Phi_\Lambda^2 \mathcal{G}_{\mathrm{t}\Lambda}(\boldsymbol{k})\mathcal{G}_{\mathrm{l}\Lambda}(\boldsymbol{k})\delta(k-\Lambda) \\
=& 2g_{\mathrm{tt}\Lambda}\Phi_\Lambda^2 \left[G_{\mathrm{t}\Lambda}(\boldsymbol{k})\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}) + \dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k})G_{\mathrm{t}\Lambda}(\boldsymbol{k})\right]
\end{aligned}$$

Hence, we obtain

$$G_{\mathrm{t}\Lambda}(\boldsymbol{k})\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}) = G_{\mathrm{l}\Lambda}(\boldsymbol{k})\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}) = \frac{\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}) - \dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k})}{4g_{\mathrm{tt}\Lambda}\Phi_\Lambda^2} \tag{A.8}$$

Using it, one can show that $W_{\mathrm{t}\Lambda}^{(2)}(\mathbf{0},\mathbf{0}) = 0$ is certainly satisfied by Eq. (A.7b) as follows

$$\begin{aligned}
& W_{\mathrm{t}\Lambda}^{(2)}(\mathbf{0},\mathbf{0}) \\
=& -\frac{(N-1)g_{\mathrm{tt}\Lambda}}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) - \frac{(2g_{\mathrm{tt}\Lambda} + g_{\mathrm{lt}\Lambda})}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) \\
&+ \frac{1}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\left[(N+1)g_{\mathrm{tt}\Lambda}\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) + g_{\mathrm{lt}\Lambda}\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)\right] \\
&+ \frac{8g_{\mathrm{tt}\Lambda}^2\Phi_\Lambda^2}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) \\
=& 0.
\end{aligned} \tag{A.9}$$

The equation for the parallel component is given by $-\partial_\Lambda \Gamma_{\mathrm{l}\Lambda}^{(2)}(-\boldsymbol{k},\boldsymbol{k}) = W_{\mathrm{l}\Lambda}^{(2)}(-\boldsymbol{k},\boldsymbol{k})$. Substituting $\Gamma_{\mathrm{l}\Lambda}^{(2)}(\mathbf{0},\mathbf{0}) = 2g_{\mathrm{tt}\Lambda}\Phi_\Lambda^2$ and Eq. (A.7b), we obtain

$$\begin{aligned}
& -\partial_\Lambda g_{\mathrm{tt}\Lambda} \\
=& -\frac{g_{\mathrm{lt}\Lambda}}{2\Phi_\Lambda^2}\partial_\Lambda \Phi_\Lambda^2 + \frac{(N-1)g_{\mathrm{lt}\Lambda}}{2\Phi_\Lambda^2\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) + \frac{3g_{\mathrm{ll}\Lambda}}{2\Phi_\Lambda^2\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) \\
&+ \frac{2(N-1)g_{\mathrm{tt}\Lambda}^2}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) + \frac{2(2g_{\mathrm{tt}\Lambda} + g_{\mathrm{lt}\Lambda})^2}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) \\
=& \left[3g_{\mathrm{ll}\Lambda} - \frac{g_{\mathrm{lt}\Lambda}}{g_{\mathrm{tt}\Lambda}}(2g_{\mathrm{tt}\Lambda} + g_{\mathrm{lt}\Lambda})\right]\frac{1}{2\Phi_\Lambda^2\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) \\
&+ \frac{2(N-1)g_{\mathrm{tt}\Lambda}^2}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) + \frac{2(2g_{\mathrm{tt}\Lambda} + g_{\mathrm{lt}\Lambda})^2}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{l}\Lambda}(\boldsymbol{k}_1).
\end{aligned} \tag{A.10}$$

We note that the same conclusion as obtained below can be reached by Eq. (3.32c) with $(j_1, j_2, j_3) = (x, x, z)$, (3.32d) with $(j_1, j_2, j_3, j_4) = (x, x, x, x),\ (x, x, y, y)$.

Next, we consider Eq. (3.32c) for $(j_1, j_2, j_3) = (x, x, z)$ and $(j_1, j_2, j_3) = (z, z, z)$ with $(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$.

$$\begin{aligned}
&W^{(3)}_{\Lambda,xxz}(\mathbf{0},\mathbf{0},\mathbf{0}) \\
&= -\frac{(N-1)g_{\mathrm{tt}\Lambda}}{\Phi_\Lambda \beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} \dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) \\
&\quad + \left[3g_{\mathrm{ll}\Lambda} - \left(5 + \frac{g_{\mathrm{lt}\Lambda}}{g_{\mathrm{tt}\Lambda}}\right) g_{\mathrm{lt}\Lambda}\right] \frac{1}{\Phi_\Lambda \beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} \dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) + \frac{2(g_{\mathrm{lt}\Lambda} - g_{\mathrm{tt}\Lambda})}{\Phi_\Lambda \beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} \dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) \\
&\quad + 4\frac{\Phi_\Lambda}{\beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} \left[(N+1)g^2_{\mathrm{tt}\Lambda} \dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) G_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) + g_{\mathrm{lt}\Lambda}(2g_{\mathrm{tt}\Lambda} + g_{\mathrm{lt}\Lambda}) \dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) G_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)\right] \\
&\quad + 16\frac{g_{\mathrm{tt}\Lambda} g_{\mathrm{lt}\Lambda} \Phi_\Lambda}{\beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} \dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) G_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) \\
&\quad + 24\frac{g^3_{\mathrm{tt}\Lambda} \Phi^3_\Lambda}{\beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} \dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) G^2_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) + 24\frac{g^2_{\mathrm{tt}\Lambda}(2g_{\mathrm{tt}\Lambda} + g_{\mathrm{lt}\Lambda})\Phi^3_\Lambda}{\beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} \dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) G^2_{\mathrm{l}\Lambda}(\boldsymbol{k}_1),
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
&W^{(3)}_{\Lambda,zzz}(\mathbf{0},\mathbf{0},\mathbf{0}) \\
&= -\frac{3}{\Phi_\Lambda \beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} \left[(N-1) g_{\mathrm{lt}\Lambda} \dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) + \frac{g_{\mathrm{ll}\Lambda}}{g_{\mathrm{tt}\Lambda}}(2g_{\mathrm{tt}\Lambda} + g_{\mathrm{lt}\Lambda}) \dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)\right] \\
&\quad + \frac{12\Phi_\Lambda}{\beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} \left[(N-1) g_{\mathrm{tt}\Lambda} g_{\mathrm{lt}\Lambda} \dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) G_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) + 3g_{\mathrm{ll}\Lambda}(2g_{\mathrm{tt}\Lambda} + g_{\mathrm{lt}\Lambda}(\boldsymbol{k}_1)) \dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) G_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)\right] \\
&\quad + \frac{24\Phi^3_\Lambda}{\beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} \left[(N-1) g^3_{\mathrm{tt}\Lambda} \dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) G^2_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) + (2g_{\mathrm{tt}\Lambda} + g_{\mathrm{lt}\Lambda})^3 \dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) G^2_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)\right].
\end{aligned} \tag{A.12}$$

Three products of Green's functions in Eq. (A.11) can be transformed as

$$G_{\mathrm{l}\Lambda}(\boldsymbol{k}) G_{\mathrm{t}\Lambda}(\boldsymbol{k}) \dot{G}_{\mathrm{j}\Lambda}(\boldsymbol{k}) = \pm \left[\frac{G_{\mathrm{j}\Lambda}(\boldsymbol{k}) \dot{G}_{\mathrm{j}\Lambda}(\boldsymbol{k})}{3g_{\mathrm{tt}\Lambda}\Phi^2_\Lambda} - \frac{\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}) - \dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k})}{12 g^2_{\mathrm{tt}\Lambda} \Phi^4_\Lambda}\right], \tag{A.13}$$

where $+(-)$ is result for $\mathrm{j} = \mathrm{l}(\mathrm{t})$. On the other hand, we have

$$-\partial_\Lambda \left[\Gamma^{(3)}_{zzz}(\mathbf{0},\mathbf{0},\mathbf{0}) - 2\Gamma^{(3)}_{xxz}(\mathbf{0},\mathbf{0},\mathbf{0})\right] = -2\Phi_\Lambda \partial_\Lambda g_{\mathrm{lt}\Lambda} - \frac{g_{\mathrm{lt}\Lambda}}{\Phi_\Lambda} \partial_\Lambda \Phi^2_\Lambda. \tag{A.14}$$

Substituting Eqs. (A.11), (A.12) and (A.14) into Eq. (2.40) for $n = 3$, we obtain

$$\begin{aligned}
&-\partial_\Lambda g_{\mathrm{lt}\Lambda} \\
&= \frac{(N-1)(g_{\mathrm{tt}\Lambda} - g_{\mathrm{lt}\Lambda})}{\Phi^2_\Lambda \beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} \dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) \\
&\quad + \frac{1}{2}\left[4g_{\mathrm{tt}\Lambda} + 5g_{\mathrm{lt}\Lambda} - 9g_{\mathrm{ll}\Lambda} + 3(g_{\mathrm{lt}\Lambda} - g_{\mathrm{ll}\Lambda})\frac{g_{\mathrm{lt}\Lambda}}{g_{\mathrm{tt}\Lambda}}\right] \frac{1}{\Phi^2_\Lambda \beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} \dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) \\
&\quad + 2(N-1)(3g_{\mathrm{lt}\Lambda} - 2g_{\mathrm{tt}\Lambda}) g_{\mathrm{tt}\Lambda} \frac{1}{\beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} \dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) G_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) \\
&\quad + 2(9g_{\mathrm{ll}\Lambda} - 2g_{\mathrm{lt}\Lambda} - 4g_{\mathrm{tt}\Lambda})(2g_{\mathrm{tt}\Lambda} + g_{\mathrm{lt}\Lambda}) \frac{1}{\beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} \dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) G_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) \\
&\quad + \frac{12(N-1) g^3_{\mathrm{tt}\Lambda} \Phi^2_\Lambda}{\beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} \dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) G^2_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) + \frac{12(2g_{\mathrm{tt}\Lambda} + g_{\mathrm{lt}\Lambda})^3 \Phi^2_\Lambda}{\beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} \dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) G^2_{\mathrm{l}\Lambda}(\boldsymbol{k}_1).
\end{aligned} \tag{A.15}$$

The same equation is obtained from Eq. (3.32d) with $(j_1, j_2, j_3, j_4) = (x, x, z, z)$. Finally, we consider Eq. (3.32d) for $(j_1, j_2, j_3, j_4) = (z, z, z, z)$ and $(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4) = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$. Noting that the left-hand side of Eq. (3.31) for this case is $-\partial_\Lambda \Gamma^{(4)}_{\Lambda,zzzz}(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) = -6\partial_\Lambda g_{\mathrm{ll}\Lambda}$, we consider $\frac{1}{6}W^{(4)}_{\Lambda,zzzz}(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$ and transform it as follows,

$$
\begin{aligned}
&W^{(4)}_{\Lambda,zzzz}(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) \\
&= -\frac{12(N-1)(g_{\mathrm{ll}\Lambda}-g_{\mathrm{lt}\Lambda})}{\Phi_\Lambda^2 \beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} \dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) \\
&\quad + \frac{12(N-1)\left[4g_{\mathrm{tt}\Lambda}(g_{\mathrm{ll}\Lambda}-g_{\mathrm{lt}\Lambda}) + g_{\mathrm{lt}\Lambda}^2\right]}{\beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} G_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) \\
&\quad + \frac{108 g_{\mathrm{ll}\Lambda}^2}{\beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} G_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) \\
&\quad + \frac{144(N-1)\Phi_\Lambda^2}{\beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} g_{\mathrm{tt}\Lambda}^2(\boldsymbol{k}_1) g_{\mathrm{lt}\Lambda}(\boldsymbol{k}_1) G_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) G_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) \\
&\quad + \frac{432\Phi_\Lambda^2}{\beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} [g_{\mathrm{lt}\Lambda}(\boldsymbol{k}_1) + 2g_{\mathrm{tt}\Lambda}(\boldsymbol{k}_1)]^2 g_{\mathrm{ll}\Lambda}(\boldsymbol{k}_1) G_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) G_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) \\
&\quad + \frac{192(N-1)\Phi_\Lambda^4}{\beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} g_{\mathrm{tt}\Lambda}^4(\boldsymbol{k}_1) G_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) G_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) G_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) \\
&\quad + \frac{192\Phi_\Lambda^4}{\beta N_{\mathrm{a}}} \sum_{\boldsymbol{k}_1} [g_{\mathrm{lt}\Lambda}(\boldsymbol{k}_1) + 2g_{\mathrm{tt}\Lambda}(\boldsymbol{k}_1)]^4 G_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) G_{\mathrm{l}\Lambda}(\boldsymbol{k}_1) G_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)
\end{aligned}
\tag{A.16}
$$

Substituting Eq. (A.16) into $-\partial_\Lambda g_{\mathrm{ll}\Lambda} = \frac{1}{6}W^{(4)}_{\Lambda,xxzz}$, we obtain

$$
\begin{aligned}
-\partial_\Lambda g_{\mathrm{ll}\Lambda} =& -\frac{2(N-1)K_d z_\Lambda}{\Phi_\Lambda^2 \beta}[g_{\mathrm{ll}\Lambda} - g_{\mathrm{lt}\Lambda}]\Lambda^{d-3} \\
& - \frac{(N-1)K_d z_\Lambda^2}{\beta}\left[4g_{\mathrm{tt}\Lambda}[g_{\mathrm{ll}\Lambda} - g_{\mathrm{lt}\Lambda}] + g_{\mathrm{lt}\Lambda}^2\right]\Lambda^{d-5} \\
& - \frac{9K_d z_\Lambda^2 \Lambda^{d-5}}{\beta} g_{\mathrm{ll}\Lambda}^2 \left[\frac{\Lambda^2}{\Lambda^2 + 2g_{\mathrm{tt}\Lambda}\Phi_\Lambda^2 z_\Lambda}\right]^2 \\
& + \frac{8(N-1)\Phi_\Lambda^2 K_d z_\Lambda^3}{\beta} g_{\mathrm{tt}\Lambda}^2 g_{\mathrm{lt}\Lambda}\Lambda^{d-7} \\
& + \frac{24\Phi_\Lambda^2 K_d z_\Lambda^3 \Lambda^{d-7}}{\beta}[g_{\mathrm{lt}\Lambda} + 2g_{\mathrm{tt}\Lambda}]^2 g_{\mathrm{ll}\Lambda}\left[\frac{\Lambda^2}{\Lambda^2 + 2g_{\mathrm{tt}\Lambda}\Phi_\Lambda^2 z_\Lambda}\right]^3 \\
& - \frac{8(N-1)\Phi_\Lambda^4 K_d z_\Lambda^4}{\beta} g_{\mathrm{tt}\Lambda}^4 \Lambda^{d-9} \\
& - \frac{8\Phi_\Lambda^4 K_d z_\Lambda^4 \Lambda^{d-9}}{\beta}[g_{\mathrm{lt}\Lambda} + 2g_{\mathrm{tt}\Lambda}]^4 \left[\frac{\Lambda^2}{\Lambda^2 + 2g_{\mathrm{tt}\Lambda}\Phi_\Lambda^2 z_\Lambda}\right]^4
\end{aligned}
\tag{A.17}
$$

Appendix B

Evaluation of Anomalous Dimension for $O(N)$ Model

RG equations include the flow of the anomalous dimension. To evaluate it, we have to include the momentum dependences of $\Gamma^{(n)}$.

The flowing anomalous dimension is defined by

$$\eta_\Lambda \equiv -\Lambda\partial_\Lambda \ln z_\Lambda^{-1} = \frac{\Lambda z_\Lambda}{2}\frac{\partial^2 W^{(2)}_{\Lambda,xx}(-\boldsymbol{k},\boldsymbol{k})}{\partial k^2}\bigg|_{\boldsymbol{k}=\boldsymbol{0}}. \tag{B.1}$$

The key quantity $W^{(2)}_{\Lambda,xx}(-\boldsymbol{k},\boldsymbol{k})$ is given by

$$W^{(2)}_{\Lambda,xx}(-\boldsymbol{k},\boldsymbol{k}) = \sum_{\alpha=a,b,c} W^{(2\alpha)}_{\Lambda,xx}(-\boldsymbol{k},\boldsymbol{k}) + \partial_\Lambda G^{-1}_{0\Lambda}(\boldsymbol{k}). \tag{B.2}$$

with

$$\begin{aligned} W^{(2a)}_{\Lambda,xx}(-\boldsymbol{k},\boldsymbol{k}) &\equiv -\Gamma^{(3)}_{\Lambda,xxz}(-\boldsymbol{k},\boldsymbol{k},\boldsymbol{0})\partial_\Lambda\Phi_\Lambda \\ &= -\frac{\Gamma^{(3)}_{\Lambda,xxz}(-\boldsymbol{k},\boldsymbol{k},\boldsymbol{0})}{2\beta N_{\mathrm{a}}\Phi_\Lambda}\sum_{\boldsymbol{k}_1}\left[(N-1)\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) + \left(2+\frac{g_{\mathrm{lt}\Lambda}}{g_{\mathrm{tt}\Lambda}}\right)\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)\right], \end{aligned} \tag{B.3a}$$

$$\begin{aligned} W^{(2b)}_{\Lambda,xx}(-\boldsymbol{k},\boldsymbol{k}) &\equiv \frac{1}{2\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\sum_{j_1}\Gamma^{(4)}_{\Lambda,xxj_1j_1}(-\boldsymbol{k},\boldsymbol{k},-\boldsymbol{k}_1,\boldsymbol{k}_1)\dot{G}_{\Lambda,j_1j_1}(\boldsymbol{k}_1) \\ &= \frac{1}{2\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\left[\sum_{j_1(\neq z)}\Gamma^{(4)}_{\Lambda,xxjj}(-\boldsymbol{k},\boldsymbol{k},-\boldsymbol{k}_1,\boldsymbol{k}_1)\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1) + \Gamma^{(4)}_{\Lambda,xxzz}(-\boldsymbol{k},\boldsymbol{k},-\boldsymbol{k}_1,\boldsymbol{k}_1)\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)\right], \end{aligned} \tag{B.3b}$$

$$\begin{aligned} W^{(2c)}_{\Lambda,xx}(-\boldsymbol{k},\boldsymbol{k}) &\equiv \frac{1}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\sum_{j_1j_2}\Gamma^{(3)}_{\Lambda,xj_1j_2}(-\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}-\boldsymbol{k}_1)\Gamma^{(3)}_{\Lambda,xj_1j_2}(\boldsymbol{k},-\boldsymbol{k}_1,-\boldsymbol{k}+\boldsymbol{k}_1)\dot{G}_{\Lambda,j_1j_1}(\boldsymbol{k}_1)G_{\Lambda,j_2j_2}(\boldsymbol{k}-\boldsymbol{k}_1) \\ &= \frac{1}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\Gamma^{(3)}_{\Lambda,xxz}(-\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}-\boldsymbol{k}_1)\Gamma^{(3)}_{\Lambda,xxz}(\boldsymbol{k},-\boldsymbol{k}_1,-\boldsymbol{k}+\boldsymbol{k}_1)\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{l}\Lambda}(\boldsymbol{k}-\boldsymbol{k}_1) \\ &\quad + \frac{1}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\Gamma^{(3)}_{\Lambda,xxz}(-\boldsymbol{k},\boldsymbol{k}-\boldsymbol{k}_1,\boldsymbol{k}_1)\Gamma^{(3)}_{\Lambda,xxz}(\boldsymbol{k},-\boldsymbol{k}+\boldsymbol{k}_1,-\boldsymbol{k}_1)\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{t}\Lambda}(\boldsymbol{k}-\boldsymbol{k}_1). \end{aligned} \tag{B.3c}$$

Let us express

$$\Gamma^{(3)}_{\Lambda,xxz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = 2g_{\mathrm{tt}\Lambda}\Phi_\Lambda\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3,\boldsymbol{0}} + \delta\Gamma^{(3)}_{\Lambda,xxz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3), \tag{B.4a}$$

$$\Gamma^{(4)}_{\Lambda,xxjj}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4) = [2g_{\mathrm{tt}\Lambda}(1-\delta_{jz})(1+2\delta_{jx}) + 2g_{\mathrm{tl}\Lambda}\delta_{jz}]\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3,\boldsymbol{0}} + \delta\Gamma^{(4)}_{\Lambda,xxjj}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4) \tag{B.4b}$$

with $\delta\Gamma^{(3)}_{\Lambda,xxz}(\boldsymbol{0},\boldsymbol{0},\boldsymbol{0}) = 0$ and $\delta\Gamma^{(4)}_{\Lambda,xxjj}(\boldsymbol{0},\boldsymbol{0},\boldsymbol{0},\boldsymbol{0}) = 0$ by definition. We also introduce $\delta W^{(n)}_{\Lambda,j_1\cdots j_n}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)$ for $n=3,4$ as

$$-\partial_\Lambda\delta\Gamma^{(n)}_{\Lambda,j_1\cdots j_n}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n) = \delta W^{(n)}_{\Lambda,j_1\cdots j_n}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n). \tag{B.5}$$

From this definition, $\delta W^{(n)}_{\Lambda,j_1\cdots j_n}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)$ can be expressed as

$$\delta W^{(3)}_{\Lambda,xxz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\equiv W^{(3)}_{\Lambda,xxz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)-W^{(3)}_{\Lambda,xxz}(\boldsymbol{0},\boldsymbol{0},\boldsymbol{0}), \tag{B.6a}$$

$$\delta W^{(4)}_{\Lambda,xxjj}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\equiv W^{(4)}_{\Lambda,xxjj}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)-W^{(4)}_{\Lambda,xxjj}(\boldsymbol{0},\boldsymbol{0},\boldsymbol{0},\boldsymbol{0}). \tag{B.6b}$$

Also noting $W^{(2)}_{\Lambda,xx}(\boldsymbol{0},\boldsymbol{0})=0$, we obtain expressions of $W^{(2\alpha)}_{\Lambda,xx}(-\boldsymbol{k},\boldsymbol{k})$ in terms of $\delta\Gamma^{(n)}$ as

$$W^{(2a)}_{\Lambda,xx}(-\boldsymbol{k},\boldsymbol{k})=-\frac{\delta\Gamma^{(3)}_{\Lambda,xxz}(-\boldsymbol{k},\boldsymbol{k},\boldsymbol{0})}{2\beta N_{\mathrm{a}}\Phi_\Lambda}\sum_{\boldsymbol{k}_1}\left[(N-1)\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)+\left(2+\frac{g_{\mathrm{lt}\Lambda}}{g_{\mathrm{tt}\Lambda}}\right)\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)\right] \tag{B.7a}$$

$$\begin{aligned}W^{(2b)}_{\Lambda,xx}(-\boldsymbol{k},\boldsymbol{k})=\frac{1}{2\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\Bigg\{&\sum_{j(\neq z)}\left[\delta\Gamma^{(4)}_{\Lambda,xxjj}(-\boldsymbol{k},\boldsymbol{k},-\boldsymbol{k}_1,\boldsymbol{k}_1)-\delta\Gamma^{(4)}_{\Lambda,xxjj}(\boldsymbol{0},\boldsymbol{0},-\boldsymbol{k}_1,\boldsymbol{k}_1)\right]\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)\\&+\left[\delta\Gamma^{(4)}_{\Lambda,xxzz}(-\boldsymbol{k},\boldsymbol{k},-\boldsymbol{k}_1,\boldsymbol{k}_1)-\delta\Gamma^{(4)}_{\Lambda,xxzz}(\boldsymbol{0},\boldsymbol{0},-\boldsymbol{k}_1,\boldsymbol{k}_1)\right]\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)\Bigg\}\end{aligned} \tag{B.7b}$$

$$\begin{aligned}W^{(2c)}_{\Lambda,xx}(-\boldsymbol{k},\boldsymbol{k})=&\frac{4g^2_{\mathrm{tt}\Lambda}\Phi^2_\Lambda}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\left[\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{l}\Lambda}(\boldsymbol{k}-\boldsymbol{k}_1)+\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{t}\Lambda}(\boldsymbol{k}-\boldsymbol{k}_1)-2\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)\right]\\&+\frac{4g_{\mathrm{tt}\Lambda}\Phi_\Lambda}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\Big[\delta\Gamma^{(3)}_{\Lambda,xxz}(-\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}-\boldsymbol{k}_1)\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{l}\Lambda}(\boldsymbol{k}-\boldsymbol{k}_1)\\&+\delta\Gamma^{(3)}_{\Lambda,xxz}(-\boldsymbol{k},\boldsymbol{k}-\boldsymbol{k}_1,\boldsymbol{k}_1)\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{t}\Lambda}(\boldsymbol{k}-\boldsymbol{k}_1)\\&-2\delta\Gamma^{(3)}_{\Lambda,xxz}(\boldsymbol{0},\boldsymbol{k}_1,-\boldsymbol{k}_1)\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)\Big]\\&+\frac{1}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{k}_1}\Big\{[\delta\Gamma^{(3)}_{\Lambda,xxz}(-\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}-\boldsymbol{k}_1)]^2\dot{G}_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{l}\Lambda}(\boldsymbol{k}-\boldsymbol{k}_1)\\&+[\delta\Gamma^{(3)}_{\Lambda,xxz}(-\boldsymbol{k},\boldsymbol{k}-\boldsymbol{k}_1,\boldsymbol{k}_1)]^2\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{t}\Lambda}(\boldsymbol{k}-\boldsymbol{k}_1)\\&-2[\delta\Gamma^{(3)}_{\Lambda,xxz}(\boldsymbol{0},-\boldsymbol{k}_1,\boldsymbol{k}_1)]^2\dot{G}_{\mathrm{l}\Lambda}(\boldsymbol{k}_1)G_{\mathrm{t}\Lambda}(\boldsymbol{k}_1)\Big\}\end{aligned} \tag{B.7c}$$

Now we neglect terms of second order in $\delta\Gamma^{(3)}_\Lambda$.

The Green's functions are given by

$$G_{\mathrm{t}\Lambda}(\boldsymbol{k})=-\frac{z_\Lambda}{k^2}\Theta(k-\Lambda),\ \ G_{\mathrm{l}\Lambda}(\boldsymbol{k})=-\frac{z_\Lambda}{k^2+2g_{\mathrm{tt}\Lambda}\Phi^2_\Lambda z_\Lambda}\Theta(k-\Lambda). \tag{B.8}$$

We also make a change variables

$$\begin{aligned}\boldsymbol{k}=\tilde{\boldsymbol{k}}\Lambda,\ \ l\equiv-\ln\frac{\Lambda}{\Lambda_0}\\W^{(2\alpha)}_{\Lambda,xx}(-\boldsymbol{k},\boldsymbol{k})=\frac{\Lambda}{z_\Lambda}\tilde{W}^{(2\alpha)}_{l,xx}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}),\\\delta\Gamma^{(n)}_{\Lambda,j_1\cdots j_n}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)=\frac{\beta\Phi^{4-n}_\Lambda\Lambda^{4-d}}{K_d z^2_\Lambda}\delta\tilde{\Gamma}^{(n)}_{l,j_1\cdots j_n}(\tilde{\boldsymbol{k}}_1,\cdots,\tilde{\boldsymbol{k}}_n);(n=3,4),\\\delta W^{(n)}_{\Lambda,j_1\cdots j_n}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)=\frac{\beta\Phi^{4-n}_\Lambda\Lambda^{3-d}}{K_d z^2_\Lambda}\delta\tilde{W}^{(n)}_{l,j_1\cdots j_n}(\tilde{\boldsymbol{k}}_1,\cdots,\tilde{\boldsymbol{k}}_n);(n=3,4),\\\Phi^2_\Lambda=\frac{K_d z_\Lambda\Lambda^{d-2}}{\beta}\tilde{\Phi}^2_l,\ \ g_{\mathrm{ij}\Lambda}=\frac{\beta\Lambda^{4-d}}{K_d z^2_\Lambda}\tilde{g}_{\mathrm{ij},l}.\end{aligned} \tag{B.9}$$

Using them, we can express the flowing anomalous dimension as

$$\eta_l=\frac{1}{2}\sum_{\alpha=a,b,c}\left.\frac{\partial^2\tilde{W}^{(2\alpha)}_{l,xx}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}})}{\partial\tilde{k}^2}\right|_{\tilde{k}=0}. \tag{B.10}$$

Here, we changed $\eta_\Lambda \to \eta_l$.

$$\eta \equiv \lim_{\Lambda\to 0} \eta_\Lambda = \frac{1}{2}\sum_{\alpha=a,b,c} \frac{\partial^2 \tilde{W}^{(2\alpha)}_{\infty,xx}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}})}{\partial \tilde{k}^2}\bigg|_{\tilde{k}=0}. \tag{B.11}$$

The expression of $\tilde{W}^{(2\alpha)}_{l,xx}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}})$ are obtained as

$$\tilde{W}^{(2a)}_{l,xx}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}) = -\frac{1}{2}\delta\tilde{\Gamma}^{(3)}_{l,xxz}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}},\boldsymbol{0})\left[(N-1)+(2+\alpha_l)\frac{1}{1+2\tilde{\Delta}_l}\right], \tag{B.12a}$$

$$\begin{aligned}\tilde{W}^{(2b)}_{l,xx}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}) = \frac{1}{2}\int \frac{d\tilde{k}_1}{(2\pi)^d K_d}\delta(\tilde{k}_1-1)\Bigg\{&\sum_{j\neq z}\left[\delta\tilde{\Gamma}^{(4)}_{l,xxjj}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}},-\tilde{\boldsymbol{k}}_1,\tilde{\boldsymbol{k}}_1)-\delta\tilde{\Gamma}^{(4)}_{l,xxjj}(\boldsymbol{0},\boldsymbol{0},-\tilde{\boldsymbol{k}}_1,\tilde{\boldsymbol{k}}_1)\right]\\ &+\frac{1}{1+2\tilde{\Delta}_l}\left[\delta\tilde{\Gamma}^{(4)}_{l,xxzz}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}},-\tilde{\boldsymbol{k}}_1,\tilde{\boldsymbol{k}}_1)-\delta\tilde{\Gamma}^{(4)}_{l,xxzz}(\boldsymbol{0},\boldsymbol{0},-\tilde{\boldsymbol{k}}_1,\tilde{\boldsymbol{k}}_1)\right]\Bigg\},\end{aligned} \tag{B.12b}$$

$$\begin{aligned}\tilde{W}^{(2c)}_{l,xx}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}) =& -4\tilde{g}_{\mathrm{tt},l}\tilde{\Delta}_l\int\frac{d^d\tilde{k}_1}{(2\pi)^d K_d}\delta(\tilde{k}_1-1)\left[\frac{\Theta(|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|-1)}{|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|^2+2\tilde{\Delta}_l}+\frac{1}{1+2\tilde{\Delta}_l}\frac{\Theta(|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|-1)}{|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|^2}-\frac{1}{1+2\tilde{\Delta}_l}\right]\\ &-4\tilde{\Delta}_l\int\frac{d^d\tilde{k}_1}{(2\pi)^d K_d}\delta(\tilde{k}_1-1)\Bigg[\delta\tilde{\Gamma}^{(3)}_{l,xxz}(\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}_1,-\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{k}}_1)\frac{\Theta(|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|-1)}{|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|^2+2\tilde{\Delta}_l}\\ &+\delta\tilde{\Gamma}^{(3)}_{l,xxz}(\tilde{\boldsymbol{k}},-\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{k}}_1,\tilde{\boldsymbol{k}}_1)\frac{1}{1+2\tilde{\Delta}_l}\frac{\Theta(|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|-1)}{|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|^2}\\ &-\delta\tilde{\Gamma}^{(3)}_{\Lambda,xxz}(\boldsymbol{0},-\tilde{\boldsymbol{k}}_1,\tilde{\boldsymbol{k}}_1)\frac{1}{1+2\tilde{\Delta}_l}\Bigg]\end{aligned} \tag{B.12c}$$

with $\tilde{\Delta}_l \equiv \tilde{g}_{\mathrm{tt},l}\tilde{\Phi}_l^2$ and $\alpha_l\tilde{g}_{\mathrm{lt},l}/\tilde{g}_{\mathrm{tt},l}$. The vertices $\delta\tilde{\Gamma}^{(n)}_l$ can be integrated formally

$$\delta\tilde{\Gamma}^{(3)}_l(\tilde{\boldsymbol{k}}_1,\tilde{\boldsymbol{k}}_2,\tilde{\boldsymbol{k}}_3) = \int_0^l dt\exp\left[\left(1+\frac{\epsilon}{2}\right)t-\frac{3}{2}\int_{l-t}^l dt'\eta_{t'}\right]\frac{\tilde{\Phi}_{l-t}}{\tilde{\Phi}_l}\delta\tilde{W}^{(3)}_{l-t}(e^{-t}\tilde{\boldsymbol{k}}_1,e^{-t}\tilde{\boldsymbol{k}}_2,e^{-t}\tilde{\boldsymbol{k}}_3), \tag{B.13}$$

$$\delta\tilde{\Gamma}^{(4)}_l(\tilde{\boldsymbol{k}}_1,\tilde{\boldsymbol{k}}_2,\tilde{\boldsymbol{k}}_3,\tilde{\boldsymbol{k}}_4) = \int_0^l dt\exp\left(\epsilon t-2\int_{l-t}^l dt'\eta_{t'}\right)\delta\tilde{W}^{(4)}_{l-t}(e^{-t}\tilde{\boldsymbol{k}}_1,e^{-t}\tilde{\boldsymbol{k}}_2,e^{-t}\tilde{\boldsymbol{k}}_3,e^{-t}\tilde{\boldsymbol{k}}_4). \tag{B.14}$$

$\tilde{W}^{(2\alpha)}$ can be transformed into

$$\tilde{W}^{(2a)}_{\infty,xx}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}) = -\frac{1}{2}\left[(N-1)+(2+\alpha_*)\tilde{G}_{\mathrm{l}*}\right]\int_0^1\frac{d\lambda}{\lambda^{1+\epsilon-2\eta}}\delta\tilde{W}^{(3)}_{\infty,xxz}(-\lambda\tilde{\boldsymbol{k}},\lambda\tilde{\boldsymbol{k}},\boldsymbol{0}) \tag{B.15a}$$

$$\begin{aligned}\tilde{W}^{(2b)}_{\infty,xx}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}) = \frac{1}{2}\int_0^1\frac{d\lambda}{\lambda^{1+\epsilon-2\eta}}\int\frac{d^d\tilde{k}_1}{(2\pi)^d K_d}\delta(\tilde{k}_1-1)\Bigg[&\sum_{j(\neq z)}\delta\tilde{W}^{(4)}_{\infty,xxjj}(-\lambda\tilde{\boldsymbol{k}},\lambda\tilde{\boldsymbol{k}},-\lambda\tilde{\boldsymbol{k}}_1,\lambda\tilde{\boldsymbol{k}}_1)\\ &+\tilde{G}_{\mathrm{l}*}\delta\tilde{W}^{(4)}_{\infty,xxzz}(-\lambda\tilde{\boldsymbol{k}},\lambda\tilde{\boldsymbol{k}},-\lambda\tilde{\boldsymbol{k}}_1,\lambda\tilde{\boldsymbol{k}}_1)-(\tilde{\boldsymbol{k}}\to\boldsymbol{0})\Bigg]\end{aligned} \tag{B.15b}$$

$$\begin{aligned}\tilde{W}^{(2c)}_{\infty,xx}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}) =& -4\tilde{g}_{\mathrm{tt}*}\tilde{\Delta}_*\int\frac{d^d\tilde{k}_1}{(2\pi)^d K_d}\delta(\tilde{k}_1-1)\left[\frac{\Theta(|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|-1)}{|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|^2+2\tilde{\Delta}_*}+\tilde{G}_{\mathrm{l}*}\frac{\Theta(|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|-1)}{|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|^2}-\tilde{G}_{\mathrm{l}*}\right]\\ &-4\tilde{\Delta}_*\int_0^1\frac{d\lambda}{\lambda^{1+\epsilon-2\eta}}\int\frac{d^d\tilde{k}_1}{(2\pi)^d K_d}\delta(\tilde{k}_1-1)\\ &\times\Bigg[\delta\tilde{W}^{(3)}_{\infty,xxz}(\lambda\tilde{\boldsymbol{k}},\lambda\tilde{\boldsymbol{k}}_1,-\lambda\tilde{\boldsymbol{k}}-\lambda\tilde{\boldsymbol{k}}_1)\frac{\Theta(|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|-1)}{|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|^2+2\tilde{\Delta}_*}\\ &+\delta\tilde{W}^{(3)}_{\infty,xxz}(\lambda\tilde{\boldsymbol{k}},-\lambda\tilde{\boldsymbol{k}}-\lambda\tilde{\boldsymbol{k}}_1,\lambda\tilde{\boldsymbol{k}}_1)\tilde{G}_{\mathrm{l}*}\frac{\Theta(|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|-1)}{|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|^2}\\ &-\delta\tilde{W}^{(3)}_{\infty,xxz}(\boldsymbol{0},-\lambda\tilde{\boldsymbol{k}}_1,\lambda\tilde{\boldsymbol{k}}_1)\tilde{G}_{\mathrm{l}*}\Bigg].\end{aligned} \tag{B.15c}$$

B.1 Symmetric phase

In symmetric phase, finite contributions of $\delta\tilde{W}^{(2\alpha)}$ is only $\delta\tilde{W}^{(2b4d)}$. $\tilde{W}^{(2)}_{\infty,xx}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}})$ is given by

$$\tilde{W}^{(2b)}_{l,xx}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}}) = \frac{1}{2}\frac{1}{1+\tilde{\sigma}_l}\int \frac{d\tilde{k}_1}{(2\pi)^d K_d}\delta(\tilde{k}_1-1)\sum_j \left[\delta\tilde{\Gamma}^{(4)}_{l,xxjj}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}},-\tilde{\boldsymbol{k}}_1,\tilde{\boldsymbol{k}}_1) - \delta\tilde{\Gamma}^{(4)}_{l,xxjj}(\boldsymbol{0},\boldsymbol{0},-\tilde{\boldsymbol{k}}_1,\tilde{\boldsymbol{k}}_1)\right] \tag{B.16}$$

with

$$\delta\tilde{\Gamma}^{(4)}_l(\tilde{\boldsymbol{k}}_1,\cdots,\tilde{\boldsymbol{k}}_4) = \int_0^l dt\exp\left(\epsilon t - 2\int_{l-t}^l dt'\eta_{t'}\right)\delta\tilde{W}^{(4)}_{l-t}(e^{-t}\tilde{\boldsymbol{k}}_1,\cdots,e^{-t}\tilde{\boldsymbol{k}}_4). \tag{B.17}$$

Function $W^{(4)}_{\Lambda,xxxx}(-\boldsymbol{k},\boldsymbol{k},-\boldsymbol{k}_1,\boldsymbol{k}_1)$ can be transformed as

$$W^{(4)}_{\Lambda,xxxx}(-\boldsymbol{k},\boldsymbol{k},-\boldsymbol{k}_1,\boldsymbol{k}_1) = (2g_\Lambda)^2(N+8)\left[\chi_\Lambda(\boldsymbol{k}+\boldsymbol{k}_1)+\chi_\Lambda(\boldsymbol{k}-\boldsymbol{k}_1)+\chi_\Lambda(\boldsymbol{0})\right] \tag{B.18}$$

with

$$\chi_\Lambda(\boldsymbol{k}) \equiv \frac{1}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{q}}\dot{G}_\Lambda(\boldsymbol{q})G_\Lambda(\boldsymbol{k}+\boldsymbol{q}). \tag{B.19}$$

Function $W^{(4)}_{\Lambda,xxyy}(-\boldsymbol{k},\boldsymbol{k},-\boldsymbol{k}_1,\boldsymbol{k}_1)$ can be transformed as

$$W^{(4)}_{\Lambda,xxyy}(-\boldsymbol{k},\boldsymbol{k},-\boldsymbol{k}_1,\boldsymbol{k}_1) = (2g_\Lambda)^2\left[2\chi_\Lambda(\boldsymbol{k}+\boldsymbol{k}_1)+2\chi_\Lambda(\boldsymbol{k}-\boldsymbol{k}_1)+(N+4)\chi_\Lambda(\boldsymbol{0})\right]. \tag{B.20}$$

It follows that

$$\sum_j W^{(4)}_{\Lambda,xxjj} = 3(N+2)(2g_\Lambda)^2\left[\chi_\Lambda(\boldsymbol{k}+\boldsymbol{k}_1)+\chi_\Lambda(\boldsymbol{k}-\boldsymbol{k}_1)+\frac{N+2}{3}\chi_\Lambda(\boldsymbol{0})\right]. \tag{B.21}$$

$\chi_\Lambda(\boldsymbol{k})$ can be transformed by

$$\chi_\Lambda(\boldsymbol{k}) = -\frac{K_d z_\Lambda^2\Lambda^{d-5}}{\beta}\frac{1}{1+\tilde{\sigma}_l}\frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)}\int_0^{\xi_{\tilde{k}}} d\theta\frac{\sin^{d-2}\theta}{1+\tilde{\sigma}_l+\tilde{k}^2+2\tilde{k}\cos\theta} \equiv \frac{K_d z_\Lambda^2\Lambda^{d-5}}{\beta}\tilde{\chi}_l(\tilde{k}). \tag{B.22}$$

with $\cos\tilde{\xi}_{\tilde{k}} \equiv -\tilde{k}/2$. Performing the scaling, we obtain

$$\sum_j \delta\tilde{W}^{(4)}_{l,xxjj}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}},-\tilde{\boldsymbol{k}}_1,\tilde{\boldsymbol{k}}_1) - (\boldsymbol{k}=\boldsymbol{0}) = 12(N+2)\tilde{g}_l^2\left[\tilde{\chi}_l(|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|)+\tilde{\chi}_l(|\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{k}}_1|)-2\tilde{\chi}_l(\tilde{k}_1)\right] \tag{B.23}$$

Sine $\tilde{\chi}(|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|)$ and $\tilde{\chi}(|\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{k}}_1|)$ yield the same contribution, we obtain

$$\begin{aligned}
\eta_l &= \frac{1}{4}\frac{1}{1+\tilde{\sigma}_l}\int \frac{d\tilde{k}_1}{(2\pi)^d K_d}\delta(\tilde{k}_1-1)\int_0^l dt\exp\left(\epsilon t - 2\int_{l-t}^l dt'\eta_{t'}\right) \\
&\quad\times\frac{\partial^2}{\partial\tilde{k}^2}\sum_j\left[\delta\tilde{W}^{(4)}_{l-t,xxjj}(-e^{-t}\tilde{\boldsymbol{k}},e^{-t}\tilde{\boldsymbol{k}},-e^{-t}\tilde{\boldsymbol{k}}_1,e^{-t}\tilde{\boldsymbol{k}}_1) - \delta\tilde{W}^{(4)}_{l-t,xxjj}(\boldsymbol{0},\boldsymbol{0},-e^{-t}\tilde{\boldsymbol{k}}_1,e^{-t}\tilde{\boldsymbol{k}}_1)\right]_{|\tilde{k}|=0} \\
&= 6(N+2)\frac{1}{1+\tilde{\sigma}_l}\int_0^l dt\exp\left(\epsilon t - 2\int_{l-t}^l dt'\eta_{t'}\right)\tilde{g}_{l-t}^2\frac{\partial^2}{\partial\tilde{k}^2}\left\langle\tilde{\chi}_{l-t}(e^{-t}|\tilde{\boldsymbol{k}}+\hat{\tilde{\boldsymbol{k}}}_1|)\right\rangle_{\hat{\tilde{\boldsymbol{k}}}_1}\Bigg|_{|\tilde{\boldsymbol{k}}|=0} \\
&\equiv \int_0^l dt K(l,t)\tilde{g}_{l-t}^2\exp\left(-2\int_{l-t}^l dt'\eta_{t'}\right),
\end{aligned} \tag{B.24}$$

where

$$K(l,t) \equiv 6(N+2)\frac{e^{\epsilon t}}{1+\tilde{\sigma}_l}\frac{\partial^2}{\partial\tilde{k}^2}\left\langle\tilde{\chi}_{l-t}(e^{-t}|\tilde{\boldsymbol{k}}+\hat{\tilde{\boldsymbol{k}}}_1|)\right\rangle_{\hat{\tilde{\boldsymbol{k}}}_1}\Bigg|_{|\tilde{\boldsymbol{k}}|=0} \tag{B.25}$$

with

$$\langle\cdots\rangle_{\hat{\tilde{\boldsymbol{k}}}_1} \equiv \int \frac{d\tilde{k}_1}{(2\pi)^d K_d}\delta(\tilde{k}_1-1)\cdots. \tag{B.26}$$

The integrand of the $\tilde{k}_1$ integral can be transformed as

$$\begin{aligned}
&\frac{\partial^2}{\partial \tilde{k}^2} \left\langle \tilde{\chi}_{l-t}(e^{-t}|\tilde{\boldsymbol{k}} + \hat{\tilde{\boldsymbol{k}}}_1|)\right\rangle_{\hat{\tilde{\boldsymbol{k}}}_1} \Bigg|_{|\tilde{\boldsymbol{k}}|=0} \\
&= \left\langle e^{-t}\tilde{\chi}'_{l-t}(e^{-t})(1-\cos^2\theta_1) + e^{-2t}\cos^2\theta_1 \tilde{\chi}''_{l-t}(e^{-t})\right\rangle_{\hat{\tilde{\boldsymbol{k}}}_1} \\
&= \frac{1}{d}\left[(d-1)e^{-t}\tilde{\chi}'_{l-t}(e^{-t}) + e^{-2t}\tilde{\chi}''_{l-t}(e^{-t})\right],
\end{aligned} \tag{B.27}$$

by using

$$\left\langle \cos^2\theta\right\rangle_{\hat{\tilde{\boldsymbol{k}}}_1} = \frac{1}{d}.$$

Finally, we obtain

$$K(l,t) = 6(N+2)\frac{1}{d(1+\tilde{\sigma}_l)}\left[(d-1)e^{-(d-3)t}\tilde{\chi}'_{l-t}(e^{-t}) + e^{-(d-2)t}\tilde{\chi}''_{l-t}(e^{-t})\right]. \tag{B.28}$$

In the case of $d=3$,

$$2\tilde{\chi}'_{l-t}(e^{-t}) + e^{-t}\tilde{\chi}''_{l-t}(e^{-t}) = \frac{1}{1+\tilde{\sigma}_{l-t}}\left[\frac{1}{2}\frac{1}{(1+e^{-t})^2+\tilde{\sigma}_{l-t}} - \frac{\tilde{\sigma}_{l-t}}{[(1+e^{-t})^2+\tilde{\sigma}_{l-t}]^2}\right], \tag{B.29}$$

that is,

$$K(l,t) = 2(N+2)\frac{1}{1+\tilde{\sigma}_l}\frac{1}{1+\tilde{\sigma}_{l-t}}\left[\frac{1}{2}\frac{1}{(1+e^{-t})^2+\tilde{\sigma}_{l-t}} - \frac{\tilde{\sigma}_{l-t}}{[(1+e^{-t})^2+\tilde{\sigma}_{l-t}]^2}\right]. \tag{B.30}$$

Substituting them into Eq. (B.24), taking limit $l\to\infty$ and changing the variable $t = -\ln\lambda$, we obtain

$$\begin{aligned}
\eta &= \tilde{g}_*^2\int_0^\infty dt K(\infty,t)e^{-2\eta t} \\
&= (N+2)\tilde{g}_*^2\frac{1-\tilde{\sigma}_*}{(1+\tilde{\sigma}_*)^4}\left[\frac{1}{2\eta} - \int_0^1 d\lambda\lambda^{2\eta}\frac{\lambda+2}{(1+\lambda)^2+\tilde{\sigma}_*}\right] \\
&\quad + 2(N+2)\tilde{g}_*^2\frac{\tilde{\sigma}_*}{(1+\tilde{\sigma}_*)^3}\int_0^1 d\lambda\lambda^{2\eta}\frac{\lambda+2}{[(1+\lambda)^2+\tilde{\sigma}_*]^2}.
\end{aligned} \tag{B.31}$$

Solving this nonlinear equation with RG equation, we obtain $\eta\approx 0.106$ for $n=3$, $\eta\approx 0.104$ for $n=2$, and $\eta\approx 0.101$ for $n=1$. The result for $n=1,2$ coincides with that of Refs. [32, 18]. On the other hand, just below the critical dimension $d\lesssim 4$, we reproduce the well-known result for the $O(N)$ symmetric theory,

$$\eta \approx \frac{3(N+2)\tilde{g}_*^2}{2}\int_0^1 d\lambda\left[3\tilde{\chi}'(\lambda) + \lambda\tilde{\chi}''(\lambda)\right] = \frac{N+2}{2(N+8)^2}\epsilon^2 \tag{B.32}$$

by substituting $\tilde{\sigma}_* = -\dfrac{N+2}{2(N+8)}\epsilon$ and $\tilde{g}_* = \dfrac{\epsilon}{N+8}$.

B.2 Spontaneously broken phase

Now we evaluate the anomalous dimension in the spontaneously broken phase. We note that, in our parametrization, contributions from $W^{(3a)}$ and $W^{(3b)}$ vanish because they are independent of external momentum. We will only partially include the contributions from 2a, 2b4d, 2c0 and 2c3c. We also consider only the result for taking $l\to\infty$.

B.2.1 2a contribution

First, we consider 2a contribution given by Eq. (B.15a). We write down $\delta W^{(3)}_{\Lambda,j_1\cdots j_n}(\boldsymbol{k}_1,\cdots,\boldsymbol{k}_n)$ as follows:

$$\begin{aligned}
&\delta W^{(3c)}_{\Lambda,xxz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) \\
&= 4g_{\mathrm{tt}\Lambda}g_{\mathrm{lt}\Lambda}\Phi_\Lambda\left\{\left[\chi_{\mathrm{tl}}(\boldsymbol{k}_1)+\chi_{\mathrm{tl}}(\boldsymbol{k}_2)+\chi_{\mathrm{lt}}(\boldsymbol{k}_1)+\chi_{\mathrm{lt}}(\boldsymbol{k}_2)\right]\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3,\boldsymbol{0}} - 4\chi_{\mathrm{tl}}(\boldsymbol{0})\right\} \\
&\quad + 4(N+1)g_{\mathrm{tt}\Lambda}^2\Phi_\Lambda\left[\chi_{\mathrm{tt}}(\boldsymbol{k}_3)\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3,\boldsymbol{0}} - \chi_{\mathrm{tt}}(\boldsymbol{0})\right] \\
&\quad + 4g_{\mathrm{lt}\Lambda}(2g_{\mathrm{tt}\Lambda}+g_{\mathrm{lt}\Lambda})\Phi_\Lambda\left[\chi_{\mathrm{ll}}(\boldsymbol{k}_3)\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3,\boldsymbol{0}} - \chi_{\mathrm{ll}}(\boldsymbol{0})\right]
\end{aligned} \tag{B.33a}$$

$$
\begin{aligned}
&\delta W^{(3d)}_{\Lambda,xxz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\\
&=8g^3_{\mathrm{tt}\Lambda}\Phi^3_\Lambda\{[\chi_{\mathrm{ltt}}(\boldsymbol{k}_1,-\boldsymbol{k}_2)+\chi_{\mathrm{tlt}}(\boldsymbol{k}_2,-\boldsymbol{k}_3)+\chi_{\mathrm{tlt}}(\boldsymbol{k}_1,-\boldsymbol{k}_3)]\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3,\boldsymbol{0}}-\chi_{\mathrm{ltt}}(\boldsymbol{0},\boldsymbol{0})-2\chi_{\mathrm{tlt}}(\boldsymbol{0},\boldsymbol{0})\}\\
&\quad+8g^2_{\mathrm{tt}\Lambda}(2g_{\mathrm{tt}\Lambda}+g_{\mathrm{lt}\Lambda})\Phi^3_\Lambda\{[\chi_{\mathrm{tll}}(\boldsymbol{k}_1,-\boldsymbol{k}_2)+\chi_{\mathrm{ltl}}(\boldsymbol{k}_2,-\boldsymbol{k}_3)+\chi_{\mathrm{ltl}}(\boldsymbol{k}_1,-\boldsymbol{k}_3)]\delta_{\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3,\boldsymbol{0}}-\chi_{\mathrm{tll}}(\boldsymbol{0},\boldsymbol{0})-2\chi_{\mathrm{ltl}}(\boldsymbol{0},\boldsymbol{0})\}.
\end{aligned}
\tag{B.33b}
$$

Here we have introduced

$$
\chi_{\mathrm{ij}}(\boldsymbol{k})\equiv\frac{1}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{q}}\dot{G}_{\mathrm{i}\Lambda}(\boldsymbol{q})G_{\mathrm{j}\Lambda}(\boldsymbol{k}+\boldsymbol{q})=\chi_{\mathrm{ij}}(-\boldsymbol{k}) \tag{B.34a}
$$

$$
\chi_{\mathrm{kij}}(\boldsymbol{k}_1,\boldsymbol{k}_2)\equiv\frac{1}{\beta N_{\mathrm{a}}}\sum_{\boldsymbol{q}}\dot{G}_{\mathrm{k}\Lambda}(\boldsymbol{q})G_{\mathrm{i}\Lambda}(\boldsymbol{q}+\boldsymbol{k}_1)G_{\mathrm{j}\Lambda}(\boldsymbol{q}+\boldsymbol{k}_2)=\chi_{\mathrm{kij}}(-\boldsymbol{k}_1,-\boldsymbol{k}_2)=\chi_{\mathrm{kji}}(\boldsymbol{k}_2,\boldsymbol{k}_1), \tag{B.34b}
$$

and also used $\chi_{\mathrm{ij}}(\boldsymbol{0})=\chi_{\mathrm{ji}}(\boldsymbol{0})$.

2a-3c contribution

We have already derived the analytic expression of $\delta W^{(3c)}_{\Lambda,xxz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$. Performing the scaling and setting $(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)=(\boldsymbol{k},-\boldsymbol{k},\boldsymbol{0})$, we obtain

$$
\delta\tilde{W}^{(3c)}_{\infty,xxz}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}},\boldsymbol{0})=8\tilde{g}^2_{\mathrm{tt}*}\alpha_*\left[\tilde{\chi}_{\mathrm{tl}}(\tilde{k})+\tilde{\chi}_{\mathrm{lt}}(\tilde{k})-2\tilde{\chi}_{\mathrm{tl}}(0)\right], \tag{B.35}
$$

where

$$
\tilde{\chi}_{\mathrm{ij}}(\tilde{k})=-\tilde{G}_{\mathrm{i}*}\int_0^{\xi_{\tilde{k}}}d\theta_1 f_0(\theta_1)\frac{1}{1+\tilde{k}^2+2\tilde{k}\cos\theta_1+2\delta_{\mathrm{jl}}\tilde{\Delta}_*} \tag{B.36}
$$

with

$$
f_0(\theta)=f_0\left(\frac{\pi}{2}\right)\sin^{d-2}\theta,\ \ f_0\left(\frac{\pi}{2}\right)=\frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)}.
$$

It follows that

$$
\tilde{W}^{(2a3c)}_{\infty,xx}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}})=-4\tilde{g}^2_{\mathrm{tt}*}\alpha_*\left[(N-1)+(2+\alpha_*)\,\tilde{G}_{\mathrm{l}*}\right]\int_0^1\frac{d\lambda}{\lambda^{1+\epsilon-2\eta}}\left[\tilde{\chi}_{\mathrm{tl}}(\lambda\tilde{k})+\tilde{\chi}_{\mathrm{lt}}(\lambda\tilde{k})-2\tilde{\chi}_{\mathrm{tl}}(0)\right] \tag{B.37}
$$

with $\alpha_*\equiv\tilde{g}_{\mathrm{lt}*}/\tilde{g}_{\mathrm{tt}*}$. Differentiating twice it with respect to $\tilde{k}$ and setting $\tilde{k}=0$ subsequently, we obtain

$$
\eta^{(2a3c)}=\frac{1}{d}\frac{2}{2-\epsilon+2\eta}\tilde{g}^2_{\mathrm{tt}*}\tilde{G}_{\mathrm{l}*}\alpha_*\left[(N-1)+(2+\alpha_*)\,\tilde{G}_{\mathrm{l}*}\right]\left[4-d-d\tilde{G}_{\mathrm{l}*}+4\tilde{G}^2_{\mathrm{l}*}\right], \tag{B.38}
$$

where we have used as follows:

$$
\begin{aligned}
\tilde{\chi}_{\mathrm{ij}}(0)&=-\frac{1}{2}\tilde{G}_{\mathrm{i}*}\tilde{G}_{\mathrm{j}*},\\
\tilde{\chi}'_{\mathrm{ij}}(0)&=-\frac{1}{2}\tilde{G}_{\mathrm{i}*}\tilde{G}_{\mathrm{j}*}f_0\left(\frac{\pi}{2}\right)\left(1-\frac{4}{d-1}\tilde{G}_{\mathrm{j}*}\right),\\
\tilde{\chi}''_{\mathrm{ij}}(0)&=\tilde{G}_{\mathrm{i}*}\tilde{G}^2_{\mathrm{j}*}\left(1-\frac{4}{d}\tilde{G}_{\mathrm{j}*}\right).
\end{aligned}
$$

2a-3d contribution

Function $W^{(3d)}_{\Lambda,xxz}(-\boldsymbol{k},\boldsymbol{k},\boldsymbol{0})$ can be transformed as

$$
\begin{aligned}
&\delta W^{(3d)}_{\Lambda,xxz}(-\boldsymbol{k},\boldsymbol{k},\boldsymbol{0})\\
&=8\tilde{g}^3_{\mathrm{tt}*}\tilde{\Phi}^2_*\frac{\beta\Phi_\Lambda\Lambda^{3-d}}{K_d z^2_\Lambda}\frac{\beta}{K_d z^3_\Lambda\Lambda^{d-7}}[\chi_{\mathrm{ltt}}(\boldsymbol{k},\boldsymbol{k})+2\chi_{\mathrm{tlt}}(\boldsymbol{k},\boldsymbol{0})-\chi_{\mathrm{ltt}}(\boldsymbol{0},\boldsymbol{0})-2\chi_{\mathrm{tlt}}(\boldsymbol{0},\boldsymbol{0})]\\
&\quad+8\tilde{g}^3_{\mathrm{tt}*}\tilde{\Phi}^2_*\frac{\beta\Phi_\Lambda\Lambda^{3-d}}{K_d z^2_\Lambda}(2+\alpha_*)\frac{\beta}{K_d z^3_\Lambda\Lambda^{d-7}}[\chi_{\mathrm{tll}}(\boldsymbol{k},\boldsymbol{k})+2\chi_{\mathrm{ltl}}(\boldsymbol{k},\boldsymbol{0})-\chi_{\mathrm{tll}}(\boldsymbol{0},\boldsymbol{0})-2\chi_{\mathrm{ltl}}(\boldsymbol{0},\boldsymbol{0})].
\end{aligned}
\tag{B.39}
$$

$\chi_{\mathrm{kij}}(\boldsymbol{k}_1,\boldsymbol{k}_2)$ can be expressed as

$$
\chi_{\mathrm{kij}}(\boldsymbol{k}_1,\boldsymbol{k}_2)=\frac{K_d z^3_\Lambda\Lambda^{d-7}}{\beta}\tilde{G}_{\mathrm{k}*}\phi_{\mathrm{ij}}(\tilde{\boldsymbol{k}}_1,\tilde{\boldsymbol{k}}_2) \tag{B.40}
$$

with

$$\phi_{\mathrm{ij}}(\tilde{\boldsymbol{k}}_1,\tilde{\boldsymbol{k}}_2)\equiv\int\frac{d^d\tilde{q}_1}{(2\pi)^dK_d}\delta(\tilde{q}_1-1)\frac{\Theta(|\tilde{\boldsymbol{k}}_1+\tilde{\boldsymbol{q}}_1|-1)}{|\tilde{\boldsymbol{k}}_1+\tilde{\boldsymbol{q}}_1|^2+2\delta_{\mathrm{il}}\tilde{\Delta}_*}\frac{\Theta(|\tilde{\boldsymbol{k}}_2+\tilde{\boldsymbol{q}}_1|-1)}{|\tilde{\boldsymbol{k}}_2+\tilde{\boldsymbol{q}}_1|^2+2\delta_{\mathrm{jl}}\tilde{\Delta}_*}=\phi_{\mathrm{ji}}(\tilde{\boldsymbol{k}}_2,\tilde{\boldsymbol{k}}_1). \tag{B.41}$$

Performing the scaling,

$$\begin{aligned}\delta\tilde{W}^{(3d)}_{\infty,xxz}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}},\boldsymbol{0})=&8\tilde{g}^2_{\mathrm{tt}*}\tilde{\Delta}_*\Big\{\tilde{G}_{\mathrm{l}*}\phi_{\mathrm{tt}}(\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}})+2\phi_{\mathrm{lt}}(\tilde{\boldsymbol{k}},\boldsymbol{0})+(2+\alpha_*)\Big[\phi_{\mathrm{ll}}(\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}})+2\phi_{\mathrm{lt}}(\boldsymbol{0},\tilde{\boldsymbol{k}})\Big]\\&-(\tilde{\boldsymbol{k}}\to\boldsymbol{0})\Big\}.\end{aligned} \tag{B.42}$$

It follows that

$$\begin{aligned}\tilde{W}^{(2a3d)}_{\infty,xx}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}})=&-4\tilde{g}^2_{\mathrm{tt}*}\tilde{\Delta}_*\Big[(N-1)+(2+\alpha_*)\,\tilde{G}_{\mathrm{l}*}\Big]\\&\times\int_0^1\frac{d\lambda}{\lambda^{1+\epsilon-2\eta}}\Big\{\tilde{G}_{\mathrm{l}*}\phi_{\mathrm{tt}}(\lambda\tilde{\boldsymbol{k}},\lambda\tilde{\boldsymbol{k}})+2\phi_{\mathrm{lt}}(\lambda\tilde{\boldsymbol{k}},\boldsymbol{0})\\&+(2+\alpha_*)\Big[\phi_{\mathrm{ll}}(\lambda\tilde{\boldsymbol{k}},\lambda\tilde{\boldsymbol{k}})+2\tilde{G}_{\mathrm{l}*}\phi_{\mathrm{tl}}(\lambda\tilde{\boldsymbol{k}},\boldsymbol{0})\Big]-(\lambda\to0)\Big\}.\end{aligned} \tag{B.43}$$

Performing the differentiation with respect to $\tilde{k}$ and setting $\tilde{k}=0$, we thereby obtain

$$\begin{aligned}\eta^{(2a3d)}=&-\frac{1}{d}\frac{2}{2-\epsilon+2\eta}\tilde{g}^2_{\mathrm{tt}*}\tilde{\Delta}_*1+2\tilde{\Delta}_*\tilde{G}_{\mathrm{l}*}\Big[(N-1)+(2+\alpha_*)\,\tilde{G}_{\mathrm{l}*}\Big]\\&\times\Big[2\,(6-d)-d\tilde{G}_{\mathrm{l}*}+4\tilde{G}^2_{\mathrm{l}*}+\tilde{G}_{\mathrm{l}*}\,(2+\alpha_*)\Big[4-d-2d\tilde{G}_{\mathrm{l}*}+12\tilde{G}^2_{\mathrm{l}*}\Big]\Big].\end{aligned} \tag{B.44}$$

Result of 2a contribution

Collecting all 2a contributions Eqs. (B.38) and (B.44), we obtain

$$\begin{aligned}\eta^{(2a)}=&\frac{1}{d}\frac{2}{2-\epsilon+2\eta}\tilde{g}^2_{\mathrm{tt}*}\tilde{G}_{\mathrm{l}*}\Big[(N-1)+(2+\alpha_*)\,\tilde{G}_{\mathrm{l}*}\Big]\\&\times\Big\{\alpha_*\Big[4-d-d\tilde{G}_{\mathrm{l}*}+4\tilde{G}^2_{\mathrm{l}*}\Big]-\tilde{\Delta}_*\Big[2\,(6-d)-d\tilde{G}_{\mathrm{l}*}+4\tilde{G}^2_{\mathrm{l}*}+\tilde{G}_{\mathrm{l}*}\,(2+\alpha_*)\Big(4-d-2d\tilde{G}_{\mathrm{l}*}+12\tilde{G}^2_{\mathrm{l}*}\Big)\Big]\Big\}.\end{aligned} \tag{B.45}$$

B.2.2 2b contribution

Next, we consider 2b contribution given by Eq. (B.15b).

2b-4d contribution

Function $W^{(4d)}_{\Lambda,xxjj}(-\boldsymbol{k},\boldsymbol{k},-\boldsymbol{k}_1,\boldsymbol{k}_1)$ can be transformed as

$$\begin{aligned}W^{(4d)}_{\Lambda,xxxx}(-\boldsymbol{k},\boldsymbol{k},-\boldsymbol{k}_1,\boldsymbol{k}_1)=&(2g_{\mathrm{tt}\Lambda})^2\,\{(N+7)[\chi_{\mathrm{tt}}(\boldsymbol{k}+\boldsymbol{k}_1)+\chi_{\mathrm{tt}}(\boldsymbol{k}-\boldsymbol{k}_1)+\chi_{\mathrm{tt}}(\boldsymbol{0})]\\&+\left(\frac{g_{\mathrm{lt}\Lambda}}{g_{\mathrm{tt}\Lambda}}\right)^2[\chi_{\mathrm{ll}}(\boldsymbol{k}+\boldsymbol{k}_1)+\chi_{\mathrm{ll}}(\boldsymbol{k}-\boldsymbol{k}_1)+\chi_{\mathrm{ll}}(\boldsymbol{0})]\Bigg\}.\end{aligned} \tag{B.46a}$$

$$W^{(4d)}_{\Lambda,xxyy}(-\boldsymbol{k},\boldsymbol{k},-\boldsymbol{k}_1,\boldsymbol{k}_1)=(2g_{\mathrm{tt}\Lambda})^2\left\{2[\chi_{\mathrm{tt}}(\boldsymbol{k}+\boldsymbol{k}_1)+\chi_{\mathrm{tt}}(\boldsymbol{k}-\boldsymbol{k}_1)]+(N+3)\chi_{\mathrm{tt}}(\boldsymbol{0})+\left(\frac{g_{\mathrm{lt}\Lambda}}{g_{\mathrm{tt}\Lambda}}\right)^2\chi_{\mathrm{ll}}(\boldsymbol{0})\right\}. \tag{B.46b}$$

$$\begin{aligned}&W^{(4d)}_{\Lambda,xxzz}(\boldsymbol{k},-\boldsymbol{k},\boldsymbol{k}_1,-\boldsymbol{k}_1)\\&=(N+1)(2g_{\mathrm{tt}\Lambda})^2\frac{g_{\mathrm{lt}\Lambda}}{g_{\mathrm{tt}\Lambda}}\chi_{\mathrm{tt}}(\boldsymbol{0})+3(2g_{\mathrm{tt}\Lambda})^2\frac{g_{\mathrm{lt}\Lambda}}{g_{\mathrm{tt}\Lambda}}\frac{g_{\mathrm{ll}\Lambda}}{g_{\mathrm{tt}\Lambda}}\chi_{\mathrm{ll}}(\boldsymbol{0})\\&\quad+(2g_{\mathrm{tt}\Lambda})^2\left(\frac{g_{\mathrm{lt}\Lambda}}{g_{\mathrm{tt}\Lambda}}\right)^2[\chi_{\mathrm{tl}}(\boldsymbol{k}+\boldsymbol{k}_1)+\chi_{\mathrm{tl}}(\boldsymbol{k}-\boldsymbol{k}_1)+\chi_{\mathrm{lt}}(\boldsymbol{k}+\boldsymbol{k}_1)+\chi_{\mathrm{lt}}(\boldsymbol{k}-\boldsymbol{k}_1)]\,.\end{aligned} \tag{B.46c}$$

It follows that

$$
\begin{aligned}
&\sum_{j\neq z} W^{(4d)}_{\Lambda,xxjj}(-\boldsymbol{k},\boldsymbol{k},-\boldsymbol{k}_1,\boldsymbol{k}_1)+W^{(4d)}_{\Lambda,xxzz}(-\boldsymbol{k},\boldsymbol{k},-\boldsymbol{k}_1,\boldsymbol{k}_1)\frac{z_\Lambda\Lambda^2}{\Lambda^2+2g_{\mathrm{tt}\Lambda}\Phi_\Lambda^2 z_\Lambda}\\
=&\,3(N+1)(2g_{\mathrm{tt}\Lambda})^2\left[\chi_{\mathrm{tt}}(\boldsymbol{k}+\boldsymbol{k}_1)+\chi_{\mathrm{tt}}(\boldsymbol{k}-\boldsymbol{k}_1)+\frac{N+1}{3}\chi_{\mathrm{tt}}(\boldsymbol{0})\right]\\
&+(2g_{\mathrm{tt}\Lambda})^2\left(\frac{g_{\mathrm{lt}\Lambda}}{g_{\mathrm{tt}\Lambda}}\right)^2\left[\chi_{\mathrm{ll}}(\boldsymbol{k}+\boldsymbol{k}_1)+\chi_{\mathrm{ll}}(\boldsymbol{k}-\boldsymbol{k}_1)+(N-1)\chi_{\mathrm{ll}}(\boldsymbol{0})\right]\\
&+(N+1)(2g_{\mathrm{tt}\Lambda})^2\frac{g_{\mathrm{lt}\Lambda}}{g_{\mathrm{tt}\Lambda}}\chi_{\mathrm{tt}}(\boldsymbol{0})\frac{z_\Lambda\Lambda^2}{\Lambda^2+2g_{\mathrm{tt}\Lambda}\Phi_\Lambda^2 z_\Lambda}+3(2g_{\mathrm{tt}\Lambda})^2\frac{g_{\mathrm{lt}\Lambda}}{g_{\mathrm{tt}\Lambda}}\frac{g_{\mathrm{ll}\Lambda}}{g_{\mathrm{tt}\Lambda}}\chi_{\mathrm{ll}}(\boldsymbol{0})\frac{z_\Lambda\Lambda^2}{\Lambda^2+2g_{\mathrm{tt}\Lambda}\Phi_\Lambda^2 z_\Lambda}\\
&+(2g_{\mathrm{tt}\Lambda})^2\left(\frac{g_{\mathrm{lt}\Lambda}}{g_{\mathrm{tt}\Lambda}}\right)^2\left[\chi_{\mathrm{tl}}(\boldsymbol{k}+\boldsymbol{k}_1)+\chi_{\mathrm{tl}}(\boldsymbol{k}-\boldsymbol{k}_1)+\chi_{\mathrm{lt}}(\boldsymbol{k}+\boldsymbol{k}_1)+\chi_{\mathrm{lt}}(\boldsymbol{k}-\boldsymbol{k}_1)\right]\frac{z_\Lambda\Lambda^2}{\Lambda^2+2g_{\mathrm{tt}\Lambda}\Phi_\Lambda^2 z_\Lambda}
\end{aligned}
\tag{B.47}
$$

Performing the scaling,

$$
\begin{aligned}
&\sum_{j\neq z}\delta\tilde W^{(4d)}_{\infty,xxjj}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}},-\tilde{\boldsymbol{k}}_1,\tilde{\boldsymbol{k}}_1)+\tilde G_{\mathrm{l}*}\delta\tilde W^{(4d)}_{\infty,xxzz}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}},-\tilde{\boldsymbol{k}}_1,\tilde{\boldsymbol{k}}_1)\\
=&\,12(N+1)\tilde g_{\mathrm{tt}*}^2\left[\tilde\chi_{\mathrm{tt}}(|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|)+\tilde\chi_{\mathrm{tt}}(|\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{k}}_1|)-2\tilde\chi_{\mathrm{tt}}(0)\right]\\
&+4g_{\mathrm{tt}*}^2\left(\frac{g_{\mathrm{lt}*}}{g_{\mathrm{tt}*}}\right)^2\left[\tilde\chi_{\mathrm{ll}}(|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|)+\tilde\chi_{\mathrm{ll}}(|\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{k}}_1|)-2\tilde\chi_{\mathrm{ll}}(0)\right]\\
&+4\tilde g_{\mathrm{tt}*}^2\alpha_*^2\tilde G_{\mathrm{l}*}\left[\tilde\chi_{\mathrm{tl}}(|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|)+\tilde\chi_{\mathrm{tl}}(|\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{k}}_1|)+\tilde\chi_{\mathrm{lt}}(|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|)+\tilde\chi_{\mathrm{lt}}(|\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{k}}_1|)-4\tilde\chi_{\mathrm{lt}}(0)\right].
\end{aligned}
\tag{B.48}
$$

From this result, we obtain

$$
\begin{aligned}
\tilde W^{(2b4d)}_{\infty,xx}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}})=&\,4g_{\mathrm{tt}*}^2\int_0^1\frac{d\lambda}{\lambda^{1+\epsilon-2\eta}}\int\frac{d^d\tilde k_1}{(2\pi)^d K_d}\delta(\tilde k_1-1)\\
&\times\left\{\left[3(N+1)+\alpha_*^2\frac{1}{(1+2\tilde\Delta_*)^2}\right]\tilde\chi_{\mathrm{tt}}(\lambda|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|)+2\alpha_*^2\tilde G_{\mathrm{l}*}\tilde\chi_{\mathrm{tl}}(\lambda|\tilde{\boldsymbol{k}}+\tilde{\boldsymbol{k}}_1|)-(\lambda\to 0)\right\}.
\end{aligned}
\tag{B.49}
$$

Here we used

$$
\tilde\chi_{\mathrm{lj}}(\lambda)=\tilde G_{\mathrm{l}*}\tilde\chi_{\mathrm{tj}}(\lambda).
\tag{B.50}
$$

This contribution to the anomalous dimension is given by

$$
\eta^{(2b4d)}=\frac{2}{d}\tilde g_{\mathrm{tt}*}^2\left[\left[3(N+1)+\alpha_*^2\frac{1}{(1+2\tilde\Delta_*)^2}\right]F^{(2)}_{\mathrm{tt}}+2\alpha_*^2\tilde G_{\mathrm{l}*}F^{(2)}_{\mathrm{tl}}\right]
\tag{B.51}
$$

with

$$
F^{(2)}_{\mathrm{ij}}\equiv\int_0^1\frac{d\lambda}{\lambda^{\epsilon-2\eta}}\left[(d-1)\tilde\chi'_{\mathrm{ij}}(\lambda)+\lambda\tilde\chi''_{\mathrm{ij}}(\lambda)\right].
\tag{B.52}
$$

B.2.3 2c contribution

Finally, we focus on the 2c contribution given by Eq. (B.15c). The first term on the r.h.s. of Eq. (B.15c) yield the following contributions:

$$
\begin{aligned}
\eta^{(2c0)}=&\,2\tilde g_{\mathrm{tt}*}\tilde\Delta_*\left[\tilde\chi''_{\mathrm{tl}}(0)+\tilde\chi''_{\mathrm{lt}}(0)\right]\\
=&-\frac{2}{d}\tilde g_{\mathrm{tt}*}\frac{\tilde\Delta_*}{1+2\tilde\Delta_*}\left[4-d-\tilde G_{\mathrm{l}*}\left(d-\frac{4}{1+2\tilde\Delta_*}\right)\right].
\end{aligned}
\tag{B.53}
$$

We focus on the second term on the r.h.s. of Eq. (B.15c).

2c-3c contribution

We have already derived the analytic expression of $\delta W^{(3c)}_{\Lambda,xxz}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$. Performing the scaling, we obtain

$$\begin{aligned}
&\delta\tilde{W}^{(3c)}_{\infty,xxz}(\tilde{\boldsymbol{k}}_1,\tilde{\boldsymbol{k}}_2,\tilde{\boldsymbol{k}}_3)\\
=&4\tilde{g}^2_{\mathrm{tt}*}\alpha_*\left\{\left[\tilde{\chi}_{\mathrm{tl}}(\tilde{k}_1)+\tilde{\chi}_{\mathrm{tl}}(\tilde{k}_2)+\tilde{\chi}_{\mathrm{lt}}(\tilde{k}_1)+\tilde{\chi}_{\mathrm{lt}}(\tilde{k}_2)\right]\delta_{\tilde{\boldsymbol{k}}_1+\tilde{\boldsymbol{k}}_2+\tilde{\boldsymbol{k}}_3,\mathbf{0}}-4\tilde{\chi}_{\mathrm{tl}}(0)\right\}\\
&+4(N+1)\tilde{g}^2_{\mathrm{tt}*}\left[\tilde{\chi}_{\mathrm{tt}}(\tilde{k}_3)\delta_{\tilde{\boldsymbol{k}}_1+\tilde{\boldsymbol{k}}_2+\tilde{\boldsymbol{k}}_3,\mathbf{0}}-\tilde{\chi}_{\mathrm{tt}}(0)\right]+4\tilde{g}^2_{\mathrm{tt}*}\alpha_*\left(2+\alpha_*\right)\left[\tilde{\chi}_{\mathrm{ll}}(\tilde{k}_3)\delta_{\tilde{\boldsymbol{k}}_1+\tilde{\boldsymbol{k}}_2+\tilde{\boldsymbol{k}}_3,\mathbf{0}}-\tilde{\chi}_{\mathrm{ll}}(0)\right].
\end{aligned}\tag{B.54}$$

The 2c-3c contribution is given by

$$\begin{aligned}
&\tilde{W}^{(2c3c)}_{\infty,xx}(-\tilde{\boldsymbol{k}},\tilde{\boldsymbol{k}})\\
=&-16\tilde{g}^2_{\mathrm{tt}*}\tilde{\Delta}_*\int_0^1\frac{d\lambda}{\lambda^{1+\epsilon-2\eta}}\left[(N+1)\int_0^{\xi_{\tilde{k}}}d\theta_1 f_{\tilde{k},\mathrm{l}}(\theta_1)\tilde{\chi}_{\mathrm{tt}}\left(\lambda\sqrt{1+\tilde{k}^2+2\tilde{k}\cos\theta_1}\right)\right.\\
&+\frac{1}{(1+2\tilde{\Delta}_*)^2}\alpha_*\int_0^{\xi_{\tilde{k}}}d\theta_1 f_{\tilde{k},\mathrm{t}}(\theta_1)\tilde{\chi}_{\mathrm{tt}}\left(\lambda\sqrt{1+\tilde{k}^2+2\tilde{k}\cos\theta_1}\right)\\
&+\tilde{G}_{\mathrm{l}*}\alpha_*\int_0^{\xi_{\tilde{k}}}d\theta_1 f_{\tilde{k},\mathrm{t}}(\theta_1)\tilde{\chi}_{\mathrm{tl}}\left(\lambda\sqrt{1+\tilde{k}^2+2\tilde{k}\cos\theta_1}\right)\\
&+\tilde{G}_{\mathrm{l}*}\alpha_*\left(2+\alpha_*\right)\int_0^{\xi_{\tilde{k}}}d\theta_1 f_{\tilde{k},\mathrm{l}}(\theta_1)\tilde{\chi}_{\mathrm{tl}}\left(\lambda\sqrt{1+\tilde{k}^2+2\tilde{k}\cos\theta_1}\right)\\
&+\left[N+1+\frac{1}{(1+2\tilde{\Delta}_*)^2}\alpha_*\left(2+\alpha_*\right)\right]\tilde{\chi}_{\mathrm{tt}}(0)\tilde{\chi}_{\mathrm{tl}}(\tilde{k})+\frac{2}{(1+2\tilde{\Delta}_*)^2}\alpha_*\tilde{\chi}_{\mathrm{tt}}(0)\tilde{\chi}_{\mathrm{tt}}(\tilde{k})\\
&-\alpha_*\left[\tilde{\chi}_{\mathrm{tl}}(\lambda)+\tilde{G}_{\mathrm{l}*}\tilde{\chi}_{\mathrm{tt}}(\lambda)-2\tilde{G}_{\mathrm{l}*}\tilde{\chi}_{\mathrm{tt}}(0)\right]\tilde{\chi}_{\mathrm{tl}}(\tilde{k})\\
&-\tilde{G}_{\mathrm{l}*}(N+1)\left[\tilde{\chi}_{\mathrm{tt}}(\lambda)-\tilde{\chi}_{\mathrm{tt}}(0)\right]\tilde{\chi}_{\mathrm{tt}}(\tilde{k})\\
&-\frac{1}{(1+2\tilde{\Delta}_*)^2}\alpha_*\left(2+\alpha_*\right)\left[\tilde{\chi}_{\mathrm{tl}}(\lambda)-\tilde{\chi}_{\mathrm{tl}}(0)\right]\tilde{\chi}_{\mathrm{tt}}(\tilde{k})\\
&-\alpha_*\left[\tilde{\chi}_{\mathrm{tl}}(\lambda\tilde{k})+\tilde{G}_{\mathrm{l}*}\tilde{\chi}_{\mathrm{tt}}(\lambda\tilde{k})-\frac{2}{1+2\tilde{\Delta}_*}\tilde{\chi}_{\mathrm{tt}}(0)\right]\tilde{\chi}_{\mathrm{tl}}(\tilde{k})\\
&-\tilde{G}_{\mathrm{l}*}\alpha_*\left[\tilde{\chi}_{\mathrm{tl}}(\lambda\tilde{k})+\tilde{G}_{\mathrm{l}*}\tilde{\chi}_{\mathrm{tt}}(\lambda\tilde{k})-2\tilde{G}_{\mathrm{l}*}\tilde{\chi}_{\mathrm{tt}}(0)\right]\tilde{\chi}_{\mathrm{tt}}(\tilde{k})\\
&\left.-\tilde{G}_{\mathrm{l}*}\left[\alpha_*\left[\tilde{\chi}_{\mathrm{tl}}(\lambda)+\tilde{\chi}_{\mathrm{lt}}(\lambda)-2\tilde{\chi}_{\mathrm{tl}}(0)\right]+(N+1)\left[\tilde{\chi}_{\mathrm{tt}}(\lambda)-\tilde{\chi}_{\mathrm{tt}}(0)\right]+\alpha_*\left(2+\alpha_*\right)\left[\tilde{\chi}_{\mathrm{ll}}(\lambda)-\tilde{\chi}_{\mathrm{ll}}(0)\right]\right]\right].
\end{aligned}\tag{B.55}$$

Let us differentiate twice it with respect to $\tilde{k}$ and setting $\tilde{k}=0$. We thereby obtain

$$\begin{aligned}
\eta^{(2c3c)}=&-\frac{8}{d}\tilde{g}^2_{\mathrm{tt}*}\frac{\tilde{\Delta}_*}{1+2\tilde{\Delta}_*}\\
&\times\left[\left(N+1+\alpha_*\tilde{G}_{\mathrm{l}*}\right)\left[4-d-\tilde{G}_{\mathrm{l}*}\left(d-\frac{4}{1+2\tilde{\Delta}_*}\right)\right]F^{(0)}_{\mathrm{tt}}\right.\\
&-2\left(N+1+\alpha_*\right)\tilde{G}_{\mathrm{l}*}F^{(1)}_{\mathrm{tt}}+\frac{1}{2}\left(N+1+\alpha_*\tilde{G}_{\mathrm{l}*}\right)F^{(2)}_{\mathrm{tt}}\\
&+\alpha_*\left[1+\left(2+\alpha_*\right)\tilde{G}_{\mathrm{l}*}\right]\left[4-d-\tilde{G}_{\mathrm{l}*}\left(d-\frac{4}{1+2\tilde{\Delta}_*}\right)\right]F^{(0)}_{\mathrm{tl}}\\
&\left.-2\alpha_*\left[1+\left(2+\alpha_*\right)\frac{1}{(1+2\tilde{\Delta}_*)^2}\right]F^{(1)}_{\mathrm{tl}}+\frac{1}{2}\alpha_*\left[1+\left(2+\alpha_*\right)\tilde{G}_{\mathrm{l}*}\right]F^{(2)}_{\mathrm{tl}}\right]\\
&+\frac{8}{d}\tilde{g}^2_{\mathrm{tt}*}\frac{\tilde{\Delta}_*}{(1+2\tilde{\Delta}_*)^2}\alpha_*\frac{1}{2-\epsilon+2\eta}\left[4-d-\tilde{G}_{\mathrm{l}*}\left(d-\frac{4}{1+2\tilde{\Delta}_*}\right)\right]\\
&+8\tilde{g}^2_{\mathrm{tt}*}\frac{\tilde{\Delta}_*}{(1+2\tilde{\Delta}_*)^2}\alpha_*\frac{1}{1-\epsilon+2\eta}\left[f_0\left(\frac{\pi}{2}\right)\right]^2\frac{1}{(d-1)^2}\left(d-3-\frac{2}{1+2\tilde{\Delta}_*}\right)^2,
\end{aligned}\tag{B.56}$$

with

$$F^{(0)}_{\rm ij} \equiv \int_0^1 \frac{d\lambda}{\lambda^{1+\epsilon-2\eta}}[\tilde{\chi}_{\rm ij}(\lambda)-\tilde{\chi}_{\rm ij}(0)], \tag{B.57a}$$

$$F^{(1)}_{\rm ij} \equiv \int_0^1 \frac{d\lambda}{\lambda^{\epsilon-2\eta}}\tilde{\chi}'_{\rm ij}(\lambda). \tag{B.57b}$$

B.2.4 Result

We partially include the contributions from $\tilde{W}^{(2\alpha)}_{\infty,xx}$ and approximate η as follows:

$$\eta \approx \eta^{(2a)}+\eta^{(2b4d)}+\eta^{(2c0)}+\eta^{(2c3c)}. \tag{B.58}$$

At $d=3$, each term of the r.h.s. are given by

$$\begin{aligned}\eta^{(2a)} =& \frac{1}{3}\frac{2}{1+2\eta}\tilde{g}^2_{\rm tt*}\tilde{G}_{\rm l*}\left[(N-1)+(2+\alpha_*)\,\tilde{G}_{\rm l*}\right]\\ &\times\left[\alpha_*\left(1-3\tilde{G}_{\rm l*}+4\tilde{G}^2_{\rm l*}\right)-\tilde{\Delta}_*\left[12-3\tilde{G}_{\rm l*}+4\tilde{G}^2_{\rm l*}+\tilde{G}_{\rm l*}\,(2+\alpha_*)\left(1-6\tilde{G}_{\rm l*}+12\tilde{G}^2_{\rm l*}\right)\right]\right],\end{aligned} \tag{B.59a}$$

$$\eta^{(2b4d)} = \frac{2}{3}\tilde{g}^2_{\rm tt*}\left[\left[3(N+1)+\alpha^2_*\tilde{G}^2_{\rm l*}\right]F^{(2)}_{\rm tt}+2\alpha^2_*\tilde{G}_{\rm l*}F^{(2)}_{\rm tl}\right] \tag{B.59b}$$

$$\eta^{(2c0)} = -\frac{2}{3}\tilde{g}_{\rm tt*}\tilde{\Delta}_*\tilde{G}_{\rm l*}\left(1-3\tilde{G}_{\rm l*}+4\tilde{G}^2_{\rm l*}\right), \tag{B.59c}$$

$$\begin{aligned}\eta^{(2c3c)} =& -\frac{8}{3}\tilde{g}^2_{\rm tt*}\tilde{\Delta}_*\tilde{G}_{\rm l*}\\ &\times\Bigg[\left(N+1+\alpha_*\tilde{G}_{\rm l*}\right)\left(1-3\tilde{G}_{\rm l*}+4\tilde{G}^2_{\rm l*}\right)F^{(0)}_{\rm tt}\\ &-2\,(N+1+\alpha_*)\,\tilde{G}_{\rm l*}F^{(1)}_{\rm tt}+\frac{1}{2}\left(N+1+\alpha_*\tilde{G}_{\rm l*}\right)F^{(2)}_{\rm tt}\\ &+\alpha_*\left[1+(2+\alpha_*)\,\tilde{G}_{\rm l*}\right]\left(1-3\tilde{G}_{\rm l*}+4\tilde{G}^2_{\rm l*}\right)F^{(0)}_{\rm tl}\\ &-2\alpha_*\left[1+(2+\alpha_*)\frac{1}{(1+2\tilde{\Delta}_*)^2}\right]F^{(1)}_{\rm tl}+\frac{1}{2}\alpha_*\left[1+(2+\alpha_*)\,\tilde{G}_{\rm l*}\right]F^{(2)}_{\rm tl}\Bigg]\\ &+\frac{8}{3}\frac{1}{1+2\eta}\tilde{g}^2_{\rm tt*}\tilde{\Delta}_*\alpha_*\tilde{G}^2_{\rm l*}\left(1-3\tilde{G}_{\rm l*}+4\tilde{G}^2_{\rm l*}\right)+\frac{1}{\eta}\tilde{g}^2_{\rm tt*}\tilde{\Delta}_*\alpha_*\tilde{G}^4_{\rm l*}.\end{aligned} \tag{B.59d}$$

$F^{(n)}_{\rm tj}$ $(n=0,1,2;{\rm j}={\rm t},{\rm l})$ defined by Eqs. (B.52), (B.57a), and (B.57b) can be expressed as

$$\begin{aligned}F^{(2)}_{\rm tt} =& \frac{1}{4}\left[\frac{1}{\eta}-(1-2\eta)\left(\psi(\eta+1)-\psi(\eta+\frac{1}{2})\right)-1\right],\\ F^{(2)}_{\rm tl} =& -\frac{1}{2}\tilde{G}_{\rm l*}\left(1-2\tilde{G}_{\rm l*}\right)\left[\frac{1}{2\eta}-\int_0^1 d\lambda\frac{\lambda^{2\eta}(\lambda+2)}{(1+\lambda)^2+2\tilde{\Delta}_*}\right]\\ &+\left(1-\tilde{G}_{\rm l*}\right)\int_0^1 d\lambda\frac{\lambda^{2\eta}(\lambda+2)}{[(1+\lambda)^2+2\tilde{\Delta}_*]^2},\\ F^{(0)}_{\rm tj} =& \frac{1}{1-2\eta}\left[\frac{1}{4}\ln\left(1+\frac{3}{1+2\delta_{\rm jl}\tilde{\Delta}_*}\right)-\frac{1}{2}\frac{1}{1+2\delta_{\rm jl}\tilde{\Delta}_*}+F^{(1)}_{\rm tj}\right],\\ F^{(1)}_{\rm tl} =& \frac{1}{2}\frac{1}{1-\eta}\left[-\frac{1}{4}\ln\left(1+\frac{3}{1+2\delta_{\rm jl}\tilde{\Delta}_*}\right)+\frac{1}{2}\frac{1}{2+\delta_{\rm jl}\tilde{\Delta}_*}+F^{(2)}_{\rm tj}\right],\end{aligned}$$

where $\psi(x)$ is a digamma function.

Appendix C

RG Equations in the Case of Litim Regulator

This appendix summarizes the renormalization group equations used to analyze the flow of correlation functions in the ordered phase. Here, we choose $R_\Lambda(k^2) = z_\Lambda^{-1}\Lambda^2 R(k^2/\Lambda^2)$ as the regulator. In particular, we adopt the Litim regulator [33]:

$$R(x) = (1-x)\theta(1-x). \tag{C.1}$$

The advantage is that the momentum integration can be carried out analytically. The cutoff dependent propagator and corresponding single-scale propagator are

$$G_{\mathrm{j}\Lambda}(\boldsymbol{k}) = -[\Gamma^{(2)}_{\mathrm{j}\Lambda}(\boldsymbol{k},-\boldsymbol{k})]^{-1} = -\frac{1}{z_\Lambda^{-1}k^2 + 2\delta_{\mathrm{j}z}\Delta_\Lambda + R_\Lambda(\boldsymbol{k})} \tag{C.2a}$$

$$\dot{G}_{\mathrm{j}\Lambda}(\boldsymbol{k}) = \partial_\Lambda R_\Lambda(\boldsymbol{k}) G^2_{\mathrm{j}\Lambda}(\boldsymbol{k}) \tag{C.2b}$$

with $\Delta_\Lambda = g_{\mathrm{tt}\Lambda}\Phi_\Lambda^2$. Using the Litim's cutoff,

$$R_\Lambda(\boldsymbol{k}) = (1-\delta_{\boldsymbol{k},\boldsymbol{0}})z_\Lambda^{-1}(\Lambda^2 - k^2)\Theta(\Lambda^2-k^2) \tag{C.3}$$

and

$$-\Lambda\partial_\Lambda R_\Lambda(\boldsymbol{k}) = (1-\delta_{\boldsymbol{k},\boldsymbol{0}})[-2+\eta_\Lambda(1-k^2/\Lambda^2)]z_\Lambda^{-1}\Lambda^2\Theta(\Lambda^2-k^2), \tag{C.4}$$

where $\eta_\Lambda = -\Lambda\partial_\Lambda \ln z_\Lambda^{-1}$ is a flowing anomalous dimension.

Since the derivation is almost the same as Appendix A, we only write down the final expressions for the dimensionless RG equations:

$$\partial_l\tilde{\Phi}_l^2 = (d-2+\eta_l)\tilde{\Phi}_l^2 - \frac{2(d+2-\eta_l)}{d(d+2)}\left[(N-1) + (2+\alpha_l)\,\tilde{G}^2_{\mathrm{l},l}\right] \tag{C.5a}$$

$$\begin{aligned}\partial_l\tilde{g}_{\mathrm{tt},l} =& (4-d-2\eta_l)\tilde{g}_{\mathrm{tt},l}\\ &- \tilde{g}^2_{\mathrm{tt},l}\frac{2(d+2-\eta_l)}{d(d+2)}\left[\frac{\alpha_l(2+\alpha_l)-3\beta_l}{2\tilde{\Delta}_l}\tilde{G}^2_{\mathrm{l},l} + 2(N-1) + 2(2+\alpha_l)^2\tilde{G}^3_{\mathrm{l},l}\right]\end{aligned} \tag{C.5b}$$

$$\begin{aligned}\partial_l\tilde{g}_{\mathrm{lt},l} =& (4-d-2\eta_l)\tilde{g}_{\mathrm{lt},l}\\ &+ \tilde{g}^2_{\mathrm{tt},l}\frac{2(d+2-\eta_l)}{d(d+2)}\Bigg[(N-1)\frac{1-\alpha_l}{\tilde{\Delta}_l} + \frac{(2+3\alpha_l-3\beta_l)(2+\alpha_l)-6\beta_l}{2\tilde{\Delta}_l}\tilde{G}^2_{\mathrm{l},l}\\ &+ 2\left[(N-1)(2-3\alpha_l) + (4+2\alpha_l-9\beta_l)(2+\alpha_l)\tilde{G}^3_{\mathrm{l},l}\right]\\ &+12\tilde{\Delta}_l\left[(N-1)+(2+\alpha_l)^3\tilde{G}^4_{\mathrm{l},l}\right]\Bigg]\end{aligned} \tag{C.5c}$$

$$\begin{aligned}\partial_l\tilde{g}_{\mathrm{ll},l} =& (4-d-2\eta_l)\tilde{g}_{\mathrm{ll},l}\\ &- 2\tilde{g}^2_{\mathrm{tt},l}\frac{2(d+2-\eta_l)}{d(d+2)}\Bigg[\frac{\beta_l-\alpha_l}{\tilde{\Delta}_l}(N-1) + \left[[4(\beta_l-\alpha_l)+\alpha_l^2](N-1) + 9\beta_l^2\tilde{G}^3_{\mathrm{l},l}\right]\\ &- 12\tilde{\Delta}_l\left[\alpha_l(N-1)+3\beta_l(2+\alpha_l)^2\tilde{G}^4_{\mathrm{l},l}\right] + 16\tilde{\Delta}_l^2\left[(N-1)+(2+\alpha_l)^4\tilde{G}^5_{\mathrm{l},l}\right]\Bigg]\end{aligned} \tag{C.5d}$$

$$\eta_l = \tilde{g}_{\mathrm{tt},l} \frac{8}{d} \tilde{\Delta}_l \tilde{G}^2_{1,l}. \tag{C.5e}$$

Now, if we find the fixed point of this RG equation when N is sufficiently large, we obtain the following result:

$$\begin{aligned} \tilde{\Phi}_*^2 &= \frac{2N}{d(d-2)},\ \tilde{g}_{\mathrm{tt}*} = \frac{d(4-d)}{4N},\ \tilde{\Delta}_* = \frac{4-d}{2(d-2)}, \\ \alpha_* &= 1 + \frac{3(4-d)^2}{(6-d)(d-2)},\ \beta_* = 1 + \frac{d(4-d)^2(d^2-16d+72)}{(8-d)(6-d)^2(d-2)^2} \end{aligned} \tag{C.5f}$$

and

$$\eta = \frac{(4-d)^2}{2N}. \tag{C.5g}$$

This result indicates that the $1/\sqrt{N}$ dependence seen in the result for the sharp momentum cutoff is a non-physical consequence of the regulator dependence.

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