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博士論文

4D Effective Action from Non-Abelian DBI Theory with Magnetic Background

(背景磁場の存在する非可換 DBI 理論の次元削減による 4 次元有効理論)

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Contents

Abstract

In this paper, we derive 4 dimensional supersymmetric effective theory from 10 dimensional non-Abelian Dirac-Born-Infeld (NDBI) action corresponding to D9-brane by compactifying on 6 dimensional facorizable tori $T^2 \times T^2 \times T^2$. In order to preserve the only 4D $N = 1$ supersymmetry, we introduce the magnetic fluxes and impose the no-tachyon constraint which is just the supersymmetric condition. We especially consider U(3) non-abelian gauge group and break U(3) to $\mathrm{U}(1)_a \times \mathrm{U}(1)_b \times \mathrm{U}(1)_c$ by magnetic fluxes. We adapt a symmetirized trace prescription NDBI action and focus on the only bosonic part since fermionic part can be derived by supersymmetry. We expand the NDBI action in terms of F up to $\mathcal{O}(F^4)$ order in case that the compact scale is enough larger than the flux scale. We rewrite the 4 dimensional effective theory as a supergravity formulation. And then we obtain a new type of matter Kähler metric, gauge kinetic function and superpotential. We can show that the gauge kineti function and superpotential are consistent with previous studies. Also we derive an F-term scalar potential by the two ways from NDBI reduction and Kähler metric and then we confirm that it is consistent with a supergravity formulation.

Notation

Here is a summary for notation. The standard conventional mathematical and physical notations are adopted.

$\mu, \nu, ... = 0, 1, 2, 3,$	four dimenaional coordinate indices
$m, n, ... = 4, ..., 9$	extra dimensional coordinate indices
$i, j, ...$	extra dimensional complex coordinate indices
$\alpha, \beta, ...$	spinor indices
$M, N, ...$	higher dimensional coordinate indices
$\underline{\mu}, \underline{\nu}, ...$	four dimensional tangent space coordinate indices
$\underline{m}, \underline{n}, ...$	extra dimensional tangent coordinate indicies
$\underline{i}, \underline{j}, ...$	extra dimensional tangent complex coordinate indices
$\underline{\alpha}, \underline{\beta}, ...$	tangent space spinor indices
$A, \overline{B}, ...$	gauge indices
x^μ	four dimensional coordinate
y^m	compact space coordinate
tr	trace of gauge indices
$(i), (r), ...$	torus indices.
γ^μ	four dimensional Clifford generator
Γ^M	higher dimensional Clifford generator
σ^μ	Pauli matrix
α'	string scale

Abbreviation

Abbreviation list in this paper.

SUSY	Supersymmetry
YM	Yang-Mills theory
SYM	Super Yang-Mills theory
SUGRA	Supergravity
DBI	Dirac-Born-Infeld
NDBI	Non-abelian Dirac-Born Infeld
CS	Chern-Simon
WZ	Wess-Zumino
KK	Kalza-Klein
op	Operator
vev	Vacuum Expectation Value
R sector	Ramond sector
NS sector	Neveu-Schwarz sector
RNS	Ramond-Neveu-Schwarz
GS	Green-Schwarz
DDF	E. Del Giudice, P. Di Vecchia and S. Fubini

1 Introduction

Elementary particles are the smallest units of matter, and particle physics is the study of describing the behavior of elementary particles. Standard Model is the most accurate theory describing microscopic phenomena, and consistent with the experimental results with very high accuracy. However, the Standard Model is not the most unified theory that can describe the universe. For example, the Standard Model cannot describe gravity, and the clasical gravity is described by Einstein's theory of relativity. It also cannot explain why the particles are classified for three generations. In addition, there are many parameters such as mass and coupling constant that are determined only by experiments. Because of these problems, it is believed that there is a more fundamental and unified theory than the Standard Model which is called Beyond the Standard Model. Beyond the Standard Model must not oonly include the Standard Model but also predict and explain phenomena that could not be explained before.

Although up to now the many theories have been proposed, string theory is the most promising. String theory is an attractive candidate for a theory with all elementary forces including gravity and it requires only one parameter to construct the theory. Considering the supersymmetric string theory (superstring theory), we can obtain fermions which can be regarded as the matters such as quarks and leptons. Thus it provide us the descriptions of all the interactions and matter such as quarks, leptons and Higgs. In this sense, it is considered to be a theory that goes beyond the Standard Model of elementary particles.

Depending on how we introduce the supersymmetry, there are 5 types of superstring theory which are typeI, type IIA, type IIB, S0(32) heterotic string and $E_8 \times E_8$ heterotic string. These theories are defined so that the gravitational anomaly is canceled. Especially, type I and two type Ⅱstring theories lead to the type I and two typeII supergravity respectively at low energy. These supergravity theories have the solitonic solutions, which implys the superstring theory leads to the solitons called D-brane. From this fact, string theory naturally incorporates non-abelian gauge groups which derived from stacks of D-branes in type I and type II string theories. Thus the dynamics of low energy excitation on D-branes describe the gauge theory.

However string theory is consisitent at 10 dimensional spacetime because of a conformal anomaly and so this fact predicts extra dimensions. Thus we compactify the extra 6-dimensions with certain compact spaces. One of the simple compact space is a torus. But the simple torus compactification is not realistic because it leads to four dimensional non-chiral theory while Standard Model is chiral theory since left handed spinors and right handed spinors interact differently under the Standard model gauge group $SU(3) \times SU(2) \times U(1)$. In order to realize the 4D chiral theory, it is necessary to preserve only $N = 0$ or $N = 1$ supersymmetry because $N \geq 2$ supersymmetry gives a relation between left handed spinors and right handed spinors and these spinors belong to the same representation of Standard model gauge group.

One of ways to obtain a chiral theory in the torus compactification is to introduce non-trivial gauge backgrounds such as gauge fluxes on the compact space which breakdown the supersymmetry. Such compactification with magnetic fluxes can realize 4D chiral theory even though the compact space is torus [1–4]. We explain why the supersymmetries are broken in Section 5. The number of generations is determined by the magnitude of the magnetic flux on torus.

The other way to break the supersymmetry is orbifold projection. By considering the orbifolding and magnetic flux same time, the several interesting mdoels have been proposed [5–7]. Indeed, the phenomenologically interesting models which have three generations have been classified [8–10]. In these models, Yukawa couplings play an important role to lead to the mass of 4D matters [11, 12]. Realization of quarks and lepton masses and their mixing angels was studied [13–16]. Recently, their flavor structure controlled by modular symmetry is actively researched [17–22]. Thus, torus compactification with magnetic fluxes is one of the interesting models to lead to realistic particle physics.

The low energy effective open string theory is a ten dimensional super Yang-Mills (SYM) theory. So far, higher dimensional SYM models with magnetic fluxes have been studied [11, 23]. On the other hand, the dynamics of open string and single D-brane is described by the Dirac-Born-Infeld (DBI) action [24–26] which corresponds to Maxwell action at low energy, and the Chern-Simons action [27–33]. However the DBI action includes more stringy and D-brane corrections. Thus, it is important to study the compactification of DBI action in order to ivestigate higher order corrections.

The dynamics of the stack of D-branes with open string is described by non-abelian Dirac-Born -Infeld (NDBI) action [34, 35]. They are proposed at the point of consistency such as T-duality [36–38]. At the lowest order, the NDBI action is equivalant to Yang-Mills theory. However the corrections of NDBI action are less-known due to the non-commutativity of gauge representation. Futhermore the effective thoery derived from NDBI action is less-known. These facts give us a motivation to study dimensional reduction of the NDBI action.

In this paper, we derive the 4D effective action from NDBI action with magnetic fluxes. In the following discussions, we set the field strength H of NS-NS 2-form B to zero for simplicity. Also we ignore CS terms because they contribute to topological terms mainly. We focus on the bosonic part of NDBI action since fermionic terms can be found by supersymmetry.

In paticular, we derive these corrections and examine them from the viewpoint of the supergravity. In other words, we study Kähler potential, gauge kinetic functions and superpotential including D-terms and F-terms. It turns out that the magnetic fluxes contribute to the Kähler potential and gauge kinetic functions in 4 dimensional effective theory derived from NDBI action. Whereas there are no additional flux contributions in holomorphic Yukawa couplings in the superpotential comparing with that from super

Yang-Mills theory. The superpotential can be read from qurtic coupling terms and the contribution of Kähler metric to the scalar potential is shown to be consistent with supergravity formulation.

In Section 2 and 3, we give a brief review on supersymmetric theory including super Yang-Mills and supergravity. In Section 4, we explain the string theory and show that the supergravity theories correspond to the superstring theory at low energy. In Section 5, we explain the magnetized compactification and show the some simple models. In Section 6, we construct the higher dimensional supergravity thoery explicitly and show that there are many solitonic solutions known as D-brane. In Section 7, we study 4D low energy effective action derived by dimensional reduction of the NDBI action with magnetic fluxes. The detailed calculation results are in appendix B. section 8 is devoted to our summary and diacussion.

2 Supersymmetry

In this section, we review supersymmetric theory that will be enough in this paper. We are interested in the algebras, their representation, the transformation of the fields and the invariant actions. This section follows refs [26, 39–46]

2.1 Wess-Zumino Model

2.1.1 Wess-Zumino Action

Supersymmetry is a symmetry in which bosons and fermions are interchanged. In this part, before considering supersymmetric algebra, we consider a Lagrangian containing free bosons and free fermions, and find a symmetry that swaps bosons and fermions in such a way as to keep the Lagrangian invariant.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 + \frac{1}{2}\overline{\Psi} i\gamma^\mu \partial_\mu \Psi \tag{2.1.1}$$

We take ψ to be the Mayorana spinor. This action is called Wess-Zumino model. Since A and B are real scalar fields, we will use them to define the complex scalar field:

$$\phi = \frac{A - iB}{\sqrt{2}} \tag{2.1.2}$$

Using this complex scalar, the Lagrangian can be rewritten as follows:

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* + \frac{1}{2}\overline{\Psi} i\gamma^\mu \partial_\mu \Psi \tag{2.1.3}$$

This Lagrangian is invariant under global O(2) symmetry because the real scalar field A, B is symmetrically contained. In other words, the Lagrangian has global U(1) symmetry. This fact plays an important role for the Lagrangian to have supersymmetry.

Now we consider a transformation that swaps bosons and fermions, with the grassmanian parameter of the transformation being θ:

$$\begin{aligned}
\delta A &= \overline{\varepsilon}\Psi \\
\delta B &= i\overline{\varepsilon}\gamma^5\Psi \\
\delta\Psi &= -i\gamma^\mu\varepsilon(\partial_\mu A) + \gamma^\mu\gamma^5\varepsilon(\partial_\mu B) \\
\delta\overline{\Psi} &= i\gamma^\mu\overline{\varepsilon}(\partial_\mu A) - \gamma^5\gamma^\mu\overline{\varepsilon}(\partial_\mu B)
\end{aligned} \tag{2.1.4}$$

The Lagrangian is invariant under this transformation except for the total derivative.

The symmetry of the Lagrangian forms a group structure, so the algebra generated by this transformation must be closed. To confirm this, we compute the two-fold supersymmetric transformation of the fields.

$$[\delta_1, \delta_2]A = -2i\bar{\varepsilon}_2\gamma^\rho\varepsilon_1\partial_\rho A \tag{2.1.5}$$

$$[\delta_1, \delta_2]B = -2i\bar{\varepsilon}_2\gamma^\rho\varepsilon_1\partial_\rho B \tag{2.1.6}$$

These results show that the two-fold supersymmetric transformation for scalar fields gives a translation. Similarly, if we consider a double transformation for the spinor field,

$$[\delta_1, \delta_2]\Psi = -2i\bar{\varepsilon}_2\gamma^\mu\varepsilon_1\partial_\mu\Psi + i\bar{\varepsilon}_2\gamma^\rho\varepsilon_1\gamma_\rho\gamma^\mu\partial_\mu\Psi. \tag{2.1.7}$$

The first term on the right hand is a translation as well as a scalar field. But the second term is not a translation. However, if the spinor field satisfies the Dirac equation $\gamma^\mu\partial_\mu\Psi = 0$, the second term becomes zero, the same result as for the scalar field.

Since invariance is a property that holds without equations of motion, it is unnatural that the algebra is closed only if it is on-shell. Therefore, we introduce a new scalar field F, G and extend the supersymmetric transformation of fermions.

$$\delta\Psi = -i\gamma^\rho\varepsilon(\partial_\rho A) + \gamma^\rho\gamma^5\varepsilon(\partial_\rho B) - \varepsilon F - i\gamma^5\varepsilon G \tag{2.1.8}$$

The transformation low of F and G are taken as follows:

$$\begin{aligned} \delta F &= i\bar{\varepsilon}\gamma^\rho\partial_\rho\psi \\ \delta G &= -\bar{\varepsilon}\gamma^5\gamma^\rho\partial_\rho\psi \end{aligned} \tag{2.1.9}$$

By dermining the transformation rule in this way, the transformation rule of $[\delta_1, \delta_2]$ for A and B is invariant. Furthermore, the extra second term in the transformation of $[\delta_1, \delta_2]$ for ψ is zero without equation of motion. In other words, the supersymmetric transformation for the fields originally included in the Lagrangian is a closed algebra. However, since we have introduced new fields F and G into the theory, we have to make sure that the algebra of transformations for these fields is closed. Indeed, calculating the algebara, we confirm the algebra is closed for F and G.

$$[\delta_1, \delta_2]F = -2\bar{\varepsilon}_2\gamma^\rho\varepsilon_1\partial_\rho F \tag{2.1.10}$$

$$[\delta_1, \delta_2]G = -2\bar{\varepsilon}_2\gamma^\rho\varepsilon_1\partial_\rho G \tag{2.1.11}$$

In the above discussions, the Lagrangian is invariant under supersymmetry except for the total derivative. In other words, at the level of the action, it is supersymmetric invariant by imposing appropriate boundary conditions, but strictly speaking, the Lagrangian is not invariant. We have introduced a new field F, G so that the algebra generated by the supersymmetric transformation is closed. So, if we consider adding a new term to the Lagrangian to cancel this total derivative term using F and G, we can easily calculate that we can add the following terms;

$$\frac{1}{2}F^2 + \frac{1}{2}G^2 \tag{2.1.12}$$

Thus, the Lagrangian that is supersymmetric invariant and closes the algebra of its transformations and its transsformation low are as follows:

$$S = \int \left[\frac{1}{2}(\partial_\mu A)^2 + \frac{1}{2}(\partial_\mu B)^2 + \frac{1}{2}\overline{\Psi} i\gamma^\mu \partial_\mu \Psi + \frac{1}{2}F^2 + \frac{1}{2}G^2\right] d^4x \tag{2.1.13}$$

$$\begin{aligned}
\delta A &= \tilde{\varepsilon}\Psi \\
\delta B &= \mathrm{i}\bar{\varepsilon}\gamma^5\Psi \\
\delta F &= \mathrm{i}\bar{\varepsilon}\gamma^\rho\partial_\rho\Psi \\
\delta G &= -\bar{\varepsilon}\gamma^5\gamma^\rho\partial_\rho\Psi \\
\delta\Psi &= -\mathrm{i}\gamma^\rho\varepsilon\left(\partial_\rho A\right) + \gamma^\rho\gamma^5\varepsilon\left(\partial_\rho B\right) - \varepsilon F - \mathrm{i}\gamma^5\varepsilon G \\
\delta\bar{\Psi} &= \mathrm{i}\bar{\varepsilon}\gamma^\rho\left(\partial_\rho A\right) - \bar{\varepsilon}\gamma^5\gamma^\rho\left(\partial_\rho B\right) - \bar{\varepsilon}F - \mathrm{i}\bar{\varepsilon}\gamma^5 G
\end{aligned} \tag{2.1.14}$$

One of characteristic features of this supersymmetric Lagrangian is that it contains the same number of fermions and bosons. Since supersymmetry is a symmetry that swaps bosons and fermions, it is intuitively consistent to have the same number of bosons and fermions.

Furthermore, the newly introduced fields F, G do not contain any kinetic terms in their action. Only the mass term is included in the Lagrangian. Solving the equation of motion for F, G, we find that $F = 0, G = 0$. Therefore, felds such as F and G are called auxiliary fields. Solving the equations of motion for all fields, F, G is zero, but the algebra generated by the transformation law of the spinor field is also closed, so it is consistent.

We consider the mass terms for scalar and spinor fields. The quadratic terms of such fields that preserve supersymmetry are written as follows:

$$\mathcal{L}_m = \left(\frac{1}{2}\overline{\Psi}\Psi + AF + BG\right) m \tag{2.1.15}$$

AF and BG are the mass terms of A and B using the auxiliary field equations of motion $F = -mA, G = -mB$. If we add such terms to the Lagrangian, we are again left with the contribution of the total derivative.

The model considered above describes two scalar fields and one spinor field, and both masses of each have to be m.

The equation of motion for the Lagrangian with the introduction of the interaction term can be summarized:

$$\Box A + mF^* = 0, \quad i\overline{\sigma}^\mu \partial_\mu \psi + m\overline{\psi} = 0, \quad F^* + mA = 0 \tag{2.1.16}$$

Since this Lagrangian is of normal order, the vacuum expectation value is vanished. In other words, even if an interaction term is introduced, there will be an equal number of bosons and fermions, and the divergence cancels out. The Wess-Zumino model with such an interaction term can be constructed relatively easily by considering a superfield.

2.1.2 Superalgebra of Wess-Zumno Model

In the previous section, we constructed a Wess-Zumino model in 4-dimension and found the supersymmetric transformation laws of the fields are given by (2.1.14). These transformation laws form a closed algebra, called the extended Lie group.

The smallest spinor in four-dimensions has four real degrees of freedom. We can describe the spinor as a Weyl spinor which has two complex degree of freedom in 4 demensions or Majorana spinor with four components imposed a reality condition.

Supersymmetric transformations are the symmetry that swaps bosons and fermions, and from the point of view of group theory, they are transformations that swap representations of the Lorentz group. In other words, we can expect that the Lorentz group and supersymmetry are non-commutative. For this reason, we first consider the Lorentz group. The spinor field ψ of spin s is the $2s+1$-dimensional representation of SU(2). Since there are three independent generators of SU(2), the required parameters are also three. Let us write the parameters as $\theta_x, \theta_y, \theta_z$ respectively, and the SU(2) transformation rule for the spinor field is given by

$$\psi \to e^{i\theta_n J_n}\psi, \tag{2.1.17}$$

where J_n is a $(2s+1)\times(2s+1)$ matrix, which generates the SU(2) Lie algebra. Considering the spin $1/2$, J is a 2×2 matrix, and its representation can be written in terms of Pauli matrices. Since J_n is a generator of SU(2), the following commutation relation is satisfied

$$[J_i, J_j] = i\varepsilon_{ijk}J_k. \tag{2.1.18}$$

The algebra generated by the supersymmetric transformation can be written as

$$[\delta_1, \delta_2] = -2i\bar{\varepsilon}_2\gamma^\rho\varepsilon_1\partial_\rho \tag{2.1.19}$$

Defining $P_\rho = -i\partial_\rho$, we obtain the following algebra:

$$[\delta_1, \delta_2] = 2i\bar{\varepsilon}_2\gamma^\rho\varepsilon_1 P_\rho \tag{2.1.20}$$

Since the P_ρ is a generator of translation, supersymmetry is a nontrivial algebra not only with the Lorentz group but also with the translational group. In other words, supersymmetry extends the Poincaré group. In the following, we will discuss the general supersymmetric algebra.

2.2 Superalgebra

The following important theorem is proved by Coleman and Mandula [47].

No-Go Theorem

The only symmetries that make the S-matrix invariant are the following:

1. Poincaré invariance
2. Discrete symmetry C, P, T
3. Internal symmetry

Here, internal symmetry is a symmetry that does not affect spacetime, such as the phase transformation U(1) or gauge symmetry. The assertion of this theorem seems to prohibit symmetries such as swapping bosons and fermions, as we saw in the previous section. However, this theorem holds only if Lie algebras has product structure defined by commutation relation. In other words, Clifford algebras with anti-commutative relations do not violate this No-Go theorem. In fact, the following theorem was proven by Haag, Lopuszanski, and Sohnius [48]

Haag-Lopuszanski-Sohnius theorem

. The only algebra with a product structure of anti-commutative relations that is consistent with Poincaré symmetry is supersymmetry.

By considering the Clifford algebra, we can construct a supersymmetry algebra in several dimensions [26, 49].

2.2.1 $N = 1,\ D = 4$ Superalgebra

The smallest $d = 4$ supersymmetry algebara would have one Weyl or Majorana spinor of supercharges. In global supersymmetry (SUSY), quantum field theory with Poincaré and internal symmetry whose charges are denoted by $M_{[\mu\nu]}, P_\mu$ and T^A respectivly is extended to include spinor supercharges Q^i_α, where α is a spacetime spinor index and $i = 1, ..., N$ is a label of distinct supercharges.

Since the 4D spinors can be taken to be Weyl spinors, we decompose ε and Q as Weyl spinors.

$$\varepsilon^A = \begin{pmatrix} \varepsilon^\alpha \\ \overline{\varepsilon}_{\dot{\beta}} \end{pmatrix}, \quad Q_A = \begin{pmatrix} Q_\alpha \\ \overline{Q}^{\dot{\beta}} \end{pmatrix} \tag{2.2.1}$$

Then we can obtain the super-Poincaré algebra from the commutation relation of the transformations:

$$\begin{aligned} &\{Q_\alpha, Q_\beta\} = 0 \qquad\qquad \{\overline{Q}^{\dot{\alpha}}, \overline{Q}^{\dot{\beta}}\} = 0 \\ &\{Q_\alpha, \overline{Q}_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu. \end{aligned} \tag{2.2.2}$$

This algebra is called an extended Lie algebra since it includes not only commutation relations but also anti-commutation relations.

Next, we derive the commutation relation between Q^α and $M_{\mu\nu}$ and P_μ. $M_{\mu\nu}$ is a generator of the Lorentz group, and if we take the spinor representation, $M_{\mu\nu}$ is represented as

$$M_{\mu\nu} = i \begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \overline{\sigma}_{\mu\nu} \end{pmatrix}. \tag{2.2.3}$$

Since Q_α is a spinor, it obeys the same transformation rule as the spinor field:

$$[M_{\mu\nu}, Q_\alpha] = -i(\sigma_{\mu\nu})_\alpha^\beta Q_\beta, \quad [M_{\mu\nu}, \overline{Q}_{\dot{\alpha}}] = -i(\overline{\sigma}_{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}} \overline{Q}_{\dot{\beta}}. \tag{2.2.4}$$

Similarly, the commutation relation with P_μ is given by

$$[P_\mu, Q_\alpha] = 0. \tag{2.2.5}$$

This algebra can also be seen from the fact that the supersymmetric transformation does not change the mass of the fields. These algebras can be derived by using the Jacobi identity.

From the above, the generators of the super-Poincaré group satisfy the following algebra:

$$[M_{\mu\nu}, M_{\rho\sigma}] = \mathrm{i}\,(\eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\sigma} M_{\mu\rho}) \tag{2.2.6}$$

$$[P_\mu, J_{\rho\sigma}] = \mathrm{i}\left(\eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho\right) \tag{2.2.7}$$

$$[P_\mu, P_v] = 0 \tag{2.2.8}$$

$$\{Q_\alpha, Q_\beta\} = 0 \tag{2.2.9}$$

$$\left\{\bar{Q}^{\dot{\alpha}}, \bar{Q}^{\dot{\beta}}\right\} = 0 \tag{2.2.10}$$

$$\left\{Q_\alpha, \bar{Q}_\beta\right\} = 2\sigma^\mu_{\alpha\dot{\beta}}P_\mu \tag{2.2.11}$$

$$\left\{Q^a, \bar{Q}^{\dot{\beta}}\right\} = 2\bar{\sigma}^{\mu\hat{\beta}\alpha}P_\mu \tag{2.2.12}$$

$$[Q_\alpha, P_\mu] = 0 \tag{2.2.13}$$

$$[J_{\mu\nu}, Q_\alpha] = 0 \tag{2.2.14}$$

$$[J_{\mu\nu}, Q_\alpha] = -\mathrm{i}\left(\sigma_{\mu\nu}\right)_\alpha^{\ \beta} Q_\beta \tag{2.2.15}$$

$$\left[J_{\mu\nu}, \bar{Q}_{\dot{\alpha}}\right] = -\mathrm{i}\left(\bar{\sigma}_{\mu\nu}\right)_{\dot{\alpha}}^{\ \dot{\beta}} \bar{Q}_{\dot{\beta}} \tag{2.2.16}$$

The fact that these commutation relations are actually algebraic can be confirmed by calculating the Jacobi identity.

Finally, since the supersymmetric generator Q is a complex representation, there naturally exists a U(1) symmetry:

$$[Q_\alpha, R] = Q_\alpha, \quad [\overline{Q}_{\dot{\beta}}, R] = -\overline{Q}_{\dot{\beta}} \tag{2.2.17}$$

This symmetry is called R-symmetry.

2.3 Representation

Next, we construct the representation of superalgebra.

2.3.1 Super-Poincaré Algebra and its Casimir Operator

There are only two Casimir operators in the Poincaré group. One is the mass operator $P_\mu P^\mu$ and the other is $W_\mu W^\mu$ where W_μ is the Pauli-Lubanski operator. Even if we extend the Poincaré group to the supe-Poincaré group, there are only two Casimir operators, one of which is the mass operator $P_\mu P^\mu$, the same as the Poincaré group. The other Casimir operator is the Pauli-Lubanski operator with corrections from supersymmetric generators:

$$C_{\mu\nu} = B_\mu P_\nu - B_\nu P_\mu, \quad B_\mu = W_\mu - \frac{1}{4}\overline{Q}_{\dot{\alpha}}\overline{\sigma}_\mu^{\dot{\alpha}\beta}Q_\beta \tag{2.3.1}$$

Using this operator, the Casimir operator is given by $C_{\mu\nu}C^{\mu\nu}$.

As is clear from the supersymmetric algebra, in order to define a representation of a supersymmetric algebra, we must define the number of supersymmetries, massive or massless states, and existence of the central charges. In the following, we focus on $N = 1$ SUSY and the massless states. Using Lorentz symmetry, we choose the light cone system $p_\mu = (E, 0, 0, E)$ for massless states.

2.3.2 $N = 1$ Massless Supersymmetry Representations

In $N = 1$ supersymmetry, there are no central charges. Since the only supersymmetric generators are $Q, \overline{Q}$, there are only two states, $|\phi\rangle$ and $\overline{Q}_{\dot{\alpha}} |\phi\rangle$. If we consider even CPT conjugates, there are four. We focus on the massless states. If the helicity is written by λ, their helicities are given by

$$(\lambda, \lambda + 1/2), \quad (-\lambda - 1/2, -\lambda). \tag{2.3.2}$$

Since λ is an arbitrary, we can consider λ such that $\lambda, \lambda + 1/2$ are interested physycally. Since spin-1/2 fermions, scalar particles, and vector gauge fields are of physical interest, we take λ to be 0 or 1/2. There is no problem in setting $\lambda = 3$, but whether it is physically important is another matter. Massless particle with helicities greater than 2 are believed to be impossible to coupling to gravity, and have not arisen in string theory.

First, we consider the $\lambda = 0$ case. The massless supermultiplet in this case is given by

$$(0, 1/2), \quad (-1/2, 0). \tag{2.3.3}$$

The chiral fermion is included in this multiplet. Hence this multiplet is called chiral multiplet or matter multiplet. The spin 1/2 particle is a Weyl spinor, which has two-components, so the spin-0 scalar particle must be a complex scalar.

Next, we consider the $\lambda = 1/2$ case. In this case, the multiplet is given by

$$(1/2, 1), \quad (0, -1/2). \tag{2.3.4}$$

Since these four states are massless and the spin one particle correspond to vector field, this multiplet contains a gauge field A_μ. This is why it is called a gauge multiplet. The spin 1/2 state, which is supersymmetric with the spin-1 state, is called the gaugeino. A_μ is a adjoint representation of the gauge group, and so the gaugino is also adjoint representation.

In order to obtain the spin-2 state, we consider the $\lambda = 3/2$ case. The multiplet is given by

$$(3/2, 2), \quad (-1, -3/2) \tag{2.3.5}$$

Since spin-2 state describe a graviton, this multiplet is called graviton multiplet.

2.4 Superspace

In this section, in order to construct a supersymmetric action easily, we define the sparspace and sperfields in next section.

The coordinates x^μ is a representation of the Lorentz group and describes spacetime. On the other hand, the spinor θ is a representation of SU(2), the covering group of the Lorentz group. Supersymmetry, as we have seen in the previous discussion, is a symmetry that swaps bosons and fermions, which can be regarded as a rotation of the coordinates x^μ and the spinor θ. Therefore, by considering θ as a coordinate in space in the same way as x^μ, supersymmetry is regard as just a rotation in this coordinate and we can easily extend a theory with no supersymmetry to a theory with supersymmetry. Thus, the space-time coordinate x^μ plus the θ direction, which is the representation of SU(2), is called the superspace.

The sperspace coordinate is written as

$$z^m = (x^\mu, \theta^\alpha, \overline{\theta}_{\dot{\alpha}}) \tag{2.4.1}$$

where since θ and $\overline{\theta}$ are two component spinor respectively, z^m has 8 components.

Extending spacetime in this way, the original Poincaré symmetry of spacetime is extended to a super-Poincaré symmetry. The super-Poincaré generators are given by a product of the Lorentz generator $L(\omega)$ and the supertranslational generator $S(x, \theta, \overline{\theta})$:

$$g_L(x, \theta, \overline{\theta}) = S(x, \theta, \overline{\theta}) L(\omega) \tag{2.4.2}$$

Futhermore, the supertranslational generator S is given by the supersymmetry generator Q and $\overline{Q}$ and translation generator P:

$$S(x, \theta, \overline{\theta}) = \exp\Big[ix^\mu P_\mu + i\theta^\alpha Q_\alpha + i\overline{\theta}_{\dot{\alpha}}\overline{Q}^{\dot{\alpha}}\Big] \tag{2.4.3}$$

where the reason for treating the translational group and supersymmetry as a pair instead of the Lorentz group and supersymmetry is that the products of two supersymmetric transformations is translation.

Calculating the product of two supertranslation, the transformation law of the parameters can be read from the coefficients of the generators and then we find

$$\begin{aligned}
x'^\mu &= x^\mu + a^\mu + i(\theta\sigma^\mu\overline{\xi} - \xi\sigma^\mu\overline{\theta}) \\
\theta'_\alpha &= \theta_\alpha + \xi_\alpha \\
\overline{\theta}'_{\dot{\alpha}} &= \overline{\theta}_{\dot{\alpha}} + \overline{\xi}_{\dot{\alpha}}
\end{aligned} \tag{2.4.4}$$

The translation group is an infinite dimensional group and there is no finite dimensional unitary representation. One of the most common representations is to express translation generator by spatial differentiation:

$$P_\mu = -i\partial_\mu \tag{2.4.5}$$

Since the spacetime is extended to the superspace, the generator Q is also seemed to be given by the coordinate derivative of the Grassmann direction $\partial/\partial\theta$. However, from the supersymmetric algebra, the product of two translations in the Grassmann direction lead to a translation in the space-time direction. Therefore, it is necessary to modify the derivative along the Grassmann direction by adding a derivative term in the spacetime as follows.

$$\begin{aligned} iQ_\alpha &= \partial_\alpha - i\sigma^\mu_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_\mu \\ i\overline{Q}^{\dot{\alpha}} &= \overline{\partial}^{\dot{\alpha}} + i\theta^\alpha\sigma^{\mu\dot{\alpha}}_\alpha\partial_\mu \end{aligned} \tag{2.4.6}$$

Indeed, by using these representations to compute the commutation relations, we can confirm that super-Poincaré algebra are satisfied.

2.5 Superfields

In the quantum field theory, fields are given by the function of space-time coordinates that obeys a transformation law under the Poincaré group. Since we are now considering a superspace, we get the function obeyed the transformation law under the super-Poincaré group. Since the functions depend on the superspace coordinates, the functions can be written as

$$f(z) = f(x, \theta, \overline{\theta}). \tag{2.5.1}$$

A supertranslation on sperspace is described by

$$f'(z) = e^{i(aP + \xi Q + \overline{\xi Q})} f(z). \tag{2.5.2}$$

The infinitesimal transformation corresponding to this transformation is given by

$$\delta f = i(aP + \xi Q + \overline{\xi Q})f. \tag{2.5.3}$$

In a field theory, fields are distinguished by a representation of the Poincaré symmetry. The fields on a superspace can likewise be defined by a representation of the super-Poincaré symmetry.

First of all, we explain some general properties on superspace. Since we consider the functions on a superspace, Φ is a function of $x, \theta, \overline{\theta}$. Since $\theta, \overline{\theta}$ are Grassmann, the second order is vanished. Therefore, we expand the field Φ in $\theta, \overline{\theta}$.

$$\begin{aligned}\Phi(z) =& C(x) + \theta^\alpha \chi_\alpha(x) + \overline{\theta}_{\dot{\alpha}} \overline{\eta}^{\dot{\alpha}}(x) + \theta^2 M(x) + \overline{\theta}^2 N(x) \\ &+ \theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \overline{\theta}^{\dot{\alpha}} A_\mu(x) + \overline{\theta}^2 \theta^\alpha \lambda_\alpha(x) + \theta^2 \overline{\theta}_{\dot{\alpha}} \overline{\xi}^{\dot{\alpha}} + \theta^2 \overline{\theta}^2 \Delta(x)\end{aligned} \tag{2.5.4}$$

Since the fields in each terms depends only on x, it is a field on spacetime. Also, since $\theta, \overline{\theta}$ is a spinor, we can find the representation of the Lorentz group.

$$\begin{aligned}&\text{Scalar} \quad C(x),\ M(x),\ N(x),\ \Delta(x) \\ &\text{Spinor} \quad \chi(x),\ \overline{\eta}(x),\ \lambda(x),\ \overline{\xi}(x) \\ &\text{Vector} \quad A_\mu(x)\end{aligned} \tag{2.5.5}$$

Since these fields are components of Φ that behave as scalars under the super-Poincaré group, they swap each other under the super-Poincaré group. In other words, they belong to the same super-Poincaré group representation. Since the component fields swap each other, the degrees of freedom of the fermions and bosons must coincide. In fact, the degrees of freedom of the boson and fermion are 8 respectively, so they coincide. However, we can see that the number of fields in the Lagrangian is larger than the number of fields in Wess-Zumino model. This implies that the field Φ is a reducible representation of SUSY and the constraints are needed.

2.5.1 Scalar Superfields

In order to construct the supersymmetric action easily, we define the scalar field. A scalar is a quantity that transforms trivialy under a symmetry. Since we consider a superfield on a superspace, a scalar superfield is a field invariant under the super-Poincaré group. Therefore, the scalar superfield is the field satisfying the following conditions:

$$\Phi'(z') = \Phi(z) \tag{2.5.6}$$

With this condition, we restrict the component fields derived from the expansion in (2.5.4). The parameters of the transformation are regarded as small quantities, and by expanding to the first order, we obtain the field variates as follows:

$$\delta\Phi(z) = -i(a^\mu P_\mu + \xi^\alpha Q_\alpha + \overline{\xi}_{\dot{\alpha}} \overline{Q}^{\dot{\alpha}})\Phi \tag{2.5.7}$$

Similarly, since $\partial_\mu \Phi$ is also a superfield, the infinitesimal transformation is given by

$$\delta(\partial_\mu \Phi(z)) = -i(a^\mu P_\mu + \xi^\alpha Q_\alpha + \overline{\xi}_{\dot{\alpha}} \overline{Q}^{\dot{\alpha}})\partial_\mu \Phi \tag{2.5.8}$$

In other words, Φ and $\partial_\mu \Phi$ follow the same transformation rule. However, the derivative along Grassmann direction ∂_α and Q is not commutative. Therefore, $\partial_\alpha \Phi$ is not a superfield. This corresponds to the fact that if a supersymmetric transformation is applied twice, it becomes a translational transformation of spacetime. In other words, the Grassmann direction is twisted. Therefore, as in relativity or gauge theory, we need a covariant derivative:

$$D_\alpha \overline{Q}^{\dot{\beta}} = -\overline{Q}^{\dot{\beta}} D_\alpha \tag{2.5.9}$$

The derivative satisfied this property is determined by

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}} \overline{\theta}^{\dot{\alpha}} \partial_\mu . \tag{2.5.10}$$

Next, we consider the supersymmetric transformation law of the field that makes the supermultiplet contained in the scalar superfield. Focusing on only the supersymmetric transformations proportional to ξ in (2.5.8), and substituting the expression by the differential operator of Q, we obtain the supersymmetry transformations:

$$\begin{aligned}
&\delta C = \xi\chi + \overline{\xi}\overline{\eta}, \quad \delta\chi_\alpha = 2\xi_\alpha M + (\sigma^\mu)_{\alpha\dot{\beta}}(i\partial_\mu C + A_\mu), \quad \delta\overline{\eta}^{\dot{\alpha}} = 2\overline{\xi}^{\dot{\alpha}} N + \chi^\beta (\sigma^\mu)_{\beta\dot{\gamma}} \epsilon^{\dot{\gamma}\dot{\alpha}} (i\partial_\mu C - A_\mu) \\
&\delta M = \overline{\xi\psi} - \frac{i}{2}\partial_\mu \chi \sigma^\mu \overline{\xi}, \quad \delta N = \xi\lambda + \frac{i}{2}\xi\sigma^\mu \partial_\mu \overline{\eta} \\
&\delta A_\mu = \xi\sigma_\mu \overline{\psi} + \lambda\sigma_\mu \overline{\xi} + \frac{i}{2}\xi\partial_\mu\chi - \frac{i}{2}\partial_\mu \overline{\eta}\overline{\xi} - \xi\sigma^{\mu\nu}\partial_\nu\chi - \partial_\nu \overline{\eta}\overline{\sigma^{\mu\nu}}\overline{\xi}, \quad \delta\Delta = \frac{i}{2}\partial_\mu(\xi\sigma\overline{\psi} - \lambda\sigma^\mu\overline{\xi}) \\
&\delta\lambda_\alpha = 2\xi_\alpha \Delta + i(\sigma^\mu \overline{\xi}_\alpha \partial_\mu N - \frac{i}{2}(\partial^\mu A_\mu \xi_\alpha - \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\xi\sigma^{\mu\nu})^\beta \epsilon_{\alpha\beta} \\
&\delta\overline{\psi}^{\dot{\alpha}} = 2\overline{\xi}^{\dot{\alpha}}\Delta + i\xi^\beta (\sigma^\mu)_{\beta\dot{\gamma}} \epsilon^{\dot{\gamma}\dot{\alpha}} \partial_\mu M + \frac{i}{2}\partial^\mu A_\mu \overline{\xi}^{\dot{\alpha}} - \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\overline{\sigma}^{\mu\nu}\overline{\xi})^{\dot{\alpha}}
\end{aligned} \tag{2.5.11}$$

Finally, the scalar superfield defined in this way is not an irreducible representation of the N = 1 supersymmetric algebra. This can be seen from the discussion in the previous section, where there are three fields included in the supermultiplets, including the auxiliary field, but there are nine fields included in the scalar superfield. In addition, the supersymmetric transformation of the component field Δ, which has the highest order among the component fields of the scalar superfield, is total differentiation. This means that the addition of Δ to the Lagrangian is a supersymmetric invariant.

2.5.2 Chiral Superfields

Next, we consider the field on superspace including spinor.

The Chiral multiplet including additional auxiliary degree of freedom, which eventually dissappear on-shell is given by

$$(C, \chi, N), \tag{2.5.12}$$

where N is a complex scalar auxiliary field. These fields are included as a θ term in the general superfield (2.5.4). Therefore, we impose the constraint defined by $\partial/\partial\overline{\theta}$:

$$\overline{D}_{\dot{\alpha}}\Phi = 0 \tag{2.5.13}$$

In order to find the chiral superfield satisfy this condition, we specifically expand Φ and write it down in term of the componet field, then we get the following fields:

$$\begin{aligned} &\overline{\eta}_{\dot{\alpha}} = 0, \quad N = 0, \quad A_\mu = i\partial_\mu C \\ &\lambda = 0, \quad \overline{\psi}_{\dot{\alpha}} = -\frac{i}{2}\partial_\mu \phi^\beta (\sigma^\mu)_{\beta\dot{\alpha}}, \quad \Delta = -\frac{1}{4}\partial^2 C \end{aligned} \tag{2.5.14}$$

Forthermore, substituting this component fields in the supersymmetry transformations, the transformation laws for the chiral multiplets can be obtained as

$$\begin{aligned} &\delta C = \xi\phi, \quad \delta\chi_\alpha = 2\xi_\alpha M + 2i(\sigma^\mu)_{\alpha\dot{\beta}}\overline{\xi}^{\dot{\beta}}\partial_\mu C, \quad \delta\overline{\psi}^{\dot{\alpha}} = 0 \\ &\delta N = 0, \quad \delta A_\mu = i\partial_\mu\chi\xi - \partial_\nu\chi\sigma^{\mu\nu}\xi - \xi\sigma^{\mu\nu}\partial_\nu\chi, \quad \delta\lambda = 0 \\ &\delta\overline{\psi}^{\dot{\alpha}} = \partial_\mu(i\xi^\beta(\sigma_\mu)_{\beta\dot{\gamma}}\epsilon^{\dot{\gamma}\dot{\alpha}}M - \partial_\mu C\overline{\xi}^{\dot{\alpha}}), \quad \delta\Delta = -\frac{1}{4}\partial^2\chi\xi \end{aligned} \tag{2.5.15}$$

From the above, by substituting these conditions into the general superfield, the chiral suprefield is given by

$$\Phi(x,\theta,\overline{\theta}) = C + \theta\chi + \theta^2 M + i(\theta\sigma^\mu\overline{\theta}\partial_\mu C - \frac{i}{2}\theta^2(\partial_\mu\chi\sigma^\mu\overline{\theta}) - \frac{1}{4}\theta^2\overline{\theta}^2\partial^2 C \tag{2.5.16}$$

For more simple discription, we define the new coordinate y

$$y^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta}, \quad \overline{y}^\mu = x^\mu - i\theta\sigma^\mu\overline{\theta}, \tag{2.5.17}$$

and this coordinate satisfies

$$\overline{D}_{\dot{\alpha}} y^\mu = 0 = D_\alpha \overline{y}^\mu. \tag{2.5.18}$$

Using this coordinate, we can write down the chiral super fields as

$$\Phi(y,\theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta^2 F \tag{2.5.19}$$

If we expand these fields in term of x, we can realize (2.5.16).

2.5.3 Vector Superfields

In order to describe gauge interactions in SUSY theories, we introduce gauge bosons. Gauge bosons are contained in vector multiplets and we define superfields satisfying reality condition:

$$V = V^\dagger \tag{2.5.20}$$

Expanding this condition with component fields, we can obtain the conditions for the component fields

$$C = C^\dagger, \quad \chi = \eta, \quad M = N^\dagger, \quad A_\mu = A_\mu^\dagger, \quad \lambda = \psi, \quad \Delta = \Delta^\dagger, \tag{2.5.21}$$

and write down the vector superfield in term of component fields:

$$V = C + \theta\chi + \overline{\theta}\overline{\chi} + \theta^2 N^\dagger + \overline{\theta}^2 N + (\theta\sigma^\mu\overline{\theta})A_\mu + \theta^2\overline{\theta}\overline{\lambda} + \overline{\theta}^2\theta\lambda + \theta^2\overline{\theta}^2\Delta \tag{2.5.22}$$

The fields contained in the vector superfield (2.5.22) are more numerous than those contained in the N=1 vector supermultiplet, so it is not seems to be an irreducible representation of the supersymmetric algebra. However, some of the fields in the vector superfield are gauge degrees of freedom that appear by gauge transformations, and the physical component fields in the vector superfield coincide with the vector multiplet. In order to see this, we define the superfield corresponding to gauge degree of freedom, satisfying reality condition trivialy, and which is given by

$$\Phi + \Phi^\dagger. \tag{2.5.23}$$

In other words, the gauge transformation can be described as

$$V \to V' = V + \Phi + \Phi^\dagger \tag{2.5.24}$$

This transformation can be written down in the component field as follows:

$$
\begin{aligned}
C &\to C + A + A^\dagger \\
\chi &\to \chi + \sqrt{2}\psi \\
N &\to N + F^\dagger \\
V_\mu &\to i\partial_\mu (A - A^\dagger) \\
\lambda &\to \lambda + \frac{i}{\sqrt{2}}\sigma^\mu \partial_\mu \overline{\psi} \\
\Delta &\to \Delta - \frac{1}{4}\partial^2 (A + A^\dagger)
\end{aligned}
\tag{2.5.25}
$$

Indeed, the transformations for A_μ are the same usual gauge transformations for gauge fields. We can also read from these transformation laws that some of the fields in the vector superfield are not physical degrees of freedom.

By taking the field of the gauge degree of freedom as

$$
C + A + A^\dagger = 0, \quad \chi + \sqrt{2}\psi = 0, \quad N + F^\dagger = 0, \tag{2.5.26}
$$

the vector field can be written more simplily as

$$
V' = (\theta\sigma^\mu\overline{\theta})A'_\mu + \theta^2\overline{\theta}\overline{\lambda}' + \overline{\theta}^2\theta\lambda' + \theta^2\overline{\theta}^2\Delta' \tag{2.5.27}
$$

These component fields are the same as vector multiplet. The gauge of (2.5.26) such that the component fields of the vector superfield are only physical fields is called the Wess-Zumino gauge, and we write this vector field as V_{WZ}.

The supersymmetry transformation for the V_{WZ} is given by

$$
\delta_\xi V_{WZ} = -\theta\sigma^\mu\overline{\theta}\delta_\xi v_\mu + i\theta^2\overline{\theta}\delta_\xi\overline{\lambda} - i\overline{\theta}^2\theta\delta_\xi\lambda
$$

Using the fact that $\delta_\xi = (Q\xi + \overline{Q\xi})$, we have

$$
(\xi Q + \overline{\xi Q})V_{WZ} \neq \delta_\xi V_{ZW}
$$

This equation implys that WZ gauge breaks supersymmetry because the left-hand side of θ has a first-order term of θ, but no such term appears in $\delta_\xi V_{WZ}$.

2.5.4 Spinor Superfields

Next, in order to construct a gauge invariant action, we need to define a superfield corresponding to field strength in the abelian case. Since the superfield containing the field strength must be supersymmetric invariant, the differential operator must be described by the covariant differential operator D_α and from the anticommutation relation $\{D_\alpha, \overline{D}_\beta\} = 2i\sigma_{\alpha\beta}\partial_\mu$, it seems that we can apply DD to the vector superfield. In addition, from the condition for a chiral superfield, D or $\overline{D}$ must act once more because $DD\overline{D}$ and $\overline{DD}D$ have at most θ or $\overline{\theta}$ terms respectively, and the chiral condition is trivialy satisfied from $DDD\overline{D} = \overline{DDD}D = 0$. If we impose the chiral condition on DDV, the condition is a nontrivial, and it imposes additional restrictions on the vector superfield. However, since the vector multiplet is a irreducible representation of the supersymmetry, even if we impose a restriction on the irreducible representation, $D\overline{D}V = 0$ because there is no other irreducible representation other than the trivial one. Therefore, the chiral condition must be satisfied trivially and the superfield corresponding to the field strength is defined by

$$W_\alpha = -\frac{1}{4}(\overline{DD})D_\alpha V \tag{2.5.28}$$

Similarly, we can define an anti-chiral superfield $\overline{W}_{\dot{\alpha}}$ by replacing $D \leftrightarrow \overline{D}$. These superfields are satisfied with the chiral condition trivially and called spinor supefieds. These spinor fields are satisfied with the following identity:

$$\overline{D}_{\dot{\alpha}}\overline{W}^{\dot{\alpha}} = D^\alpha W_\alpha \tag{2.5.29}$$

We show that the spinor superfield defined in this way does indeed contain a field strength. If the spinor superfield contains a field strength, it must be gauge invariant. In fact, if we consider the gauge transformation of the spinor superfield, the gauge degrees of freedom are described by $\Phi + \Phi^\dagger$, and since W_α contains both D and $\overline{D}$, by using the chiral condition $\overline{D}\Phi = 0$, the gauge degrees of freedom vanish. Therefore, $W' = W$ is clearly satisfied.

Therefore, the action described by W_α is gauge invariant and supersymmetric invariant. This means that even if W_α is fixed in gauge, it will not affect the action, so for simplicity we take the WZ gauge fixation. Furthermore, we use y instead of x as the coordinate system. The spinor superfield can be written down in terms of the component fields as follows:

$$W_\alpha = \lambda_\alpha + 2\theta_\alpha\Delta + (\sigma^{\mu\nu}\theta)_\alpha(\partial A_\mu - \partial_\nu A_\mu) - i\theta^2(\sigma^\mu\partial/\mu)(\sigma^\mu\partial_\mu\overline{\lambda}_\alpha) \tag{2.5.30}$$

2.5.5 Non-abelian Chiral, Vector and Spinner Superfield

In the previous discussions, the superfields were the representation of U(1) which is U(1) R-symmetry, but more generally, we can consider superfields that are representations of non-abelian gauge groups.

The gauge transformation in the vector superfield is given by (2.5.24), and the finite transformation of this transformation can be written as

$$e^V \to e^{V'} = e^{\Phi^\dagger} e^V e^\Phi. \tag{2.5.31}$$

Since from the group theoretical point this transformation rule is in general, the non-abelian gauge transformation rule is also given by like this.

Let T^a be the generator of a non-abelian gauge group and assume the algebra for this gauge group is given by

$$[T^a, T^b] = i f^{abc} T^c, \tag{2.5.32}$$

where we choose the following normalization:

$$\operatorname{tr} T^a T^b = \delta^{ab} \tag{2.5.33}$$

If we expand (2.5.31) in first order, we can obtain the infinitesimal non-abelian gauge transformation rule

$$\delta V = \frac{1}{2}[V, \Phi - \Phi^\dagger] + (\Phi + \Phi^\dagger) \tag{2.5.34}$$

Expanding this transformation in the component fields, we realize the transformation rules such as Yang-Mills theory.

$$\begin{aligned} \frac{1}{2}\delta A_\mu &= \frac{1}{2}[A_\mu, A] + i\partial_\mu A \\ \delta\lambda_\alpha &= [\lambda_\alpha, A] \\ \delta\overline{\lambda}^{\dot{\alpha}} &= [\overline{\lambda}^{\dot{\alpha}}, A] \\ \delta\Delta &= [\Delta, A] \end{aligned} \tag{2.5.35}$$

We define spinor superfield for the non-abelian gauge vector superfields.

$$W_\alpha = -\frac{1}{4}\overline{DD}e^{-V} D_\alpha e^V \tag{2.5.36}$$

Indeed, the spinor field defined in this way is transformed covariantly.

Finally, non-abelian spinor fields in the WZ gauge are given by

$$W_\alpha = \lambda_\alpha + 2\theta_\alpha\Delta + (\sigma^{\mu\nu})_\alpha^\beta\theta_\beta \left(\partial_\mu A_\nu - \partial_\nu A_\mu - \frac{i}{2}[A_\mu, A_\nu]\right) - i\theta^2(\sigma^\mu)_{\alpha\dot{\beta}} \left(\partial_\mu\overline{\lambda}^{\dot{\beta}} - \frac{i}{2}[A_\mu, \overline{\lambda}^{\dot{\beta}}]\right). \tag{2.5.37}$$

2.6 $D = 4,\ N = 1$ Supersymmetric Action

In this section, we construct supersymmetric action with superfields.

2.6.1 Superpotential and Kähler Potential

Since the d-term of a vector multiplet is the total derivative under supersymmetry transformation, the following action is supersymmetric invariant:

$$\int d^4x d^4\theta V(x,\theta,\overline{\theta}) \tag{2.6.1}$$

Since the vector superfield is a superfield that satisfies the real conditions, if we take Φ as the most direct vector field and $V = \Phi\Phi^\dagger$, we can derive the kinetic terms for the component fields of the chiral superfield. Therefore, $\Phi^\dagger\Phi$ is called the Kähler potential.

Similarly, the F-term existed as a supersymmetric invariant component of the chiral superfield. Since the F-term is a θ^2-component field of the chiral superfield Φ, the following action is supersymmetry invariant:

$$\int d^4x d^2\theta\Phi(y,\theta) + \int d^4x d^2\overline{\theta}\overline{\Phi}^\dagger(y^\dagger,\overline{\theta}) \tag{2.6.2}$$

In general, the field formed by the product and sum of chiral superfields is also a chiral superfield, so any function $W(\Phi)$ is also a chiral superfield. Therefore the following action is also supersymmetric:

$$\int d^4x \left[\int d^2\theta W(\Phi) + h.c\right] \tag{2.6.3}$$

Performing the θ^2 and $\overline{\theta}^2$ integrals of this action, we can obtain the mass terms and the interaction terms for the component fields of Φ. $W(\Phi)$ is called the superpotential.

From the above discussions, since the kinetic term is given by the Kähler potential and the mass and interaction terms by the superpotential, the general action can be written as

$$S = \int d^4x \int d^4\theta \Phi^\dagger\Phi - \int d^4x \left[\int d^2\theta W(\Phi) + h.c\right] \tag{2.6.4}$$

If there are some chiral superfields Φ which written as Φ^I, the general $N = 1$ supersymmetric action is given by

$$S = \int d^4x d^4\theta^4 K(\Phi^{I\dagger},\Phi^J) - \int d^4x \left[\int d^2\theta W(\Phi^I) + h.c\right]. \tag{2.6.5}$$

In order to write down this Lagrangian explicitly, we expand K and W in terms of Φ such as

$$\begin{aligned} K(\Phi^I, \Phi^{J\dagger}) &= \sum_{I_1\dots I_N, J_1\dots J_M} \Phi^{I_1}\cdots\Phi^{I_N}\Phi^{J_1\dagger}\cdots\Phi^{J_M\dagger} \\ W(\Phi^I) &= \sum d_{I_1\dots I_N}\Phi^{I_1}\cdots\Phi^{I_N}. \end{aligned} \tag{2.6.6}$$

We also expand K and W in terms of θ to find the Lagrangian for the component fields:

$$W(\Phi^I) = W(A) + \sqrt{w}\theta\chi^I\frac{\partial W(A)}{\partial A^I} + \theta^2\left[F^I\frac{\partial W(A)}{\partial A^I} - \frac{1}{2}\chi^I\chi^J\frac{\partial^2 W(A)}{\partial A^I\partial A^J}\right], \tag{2.6.7}$$

$$\begin{aligned} K|_{\theta^2\overline{\theta}^2} =& g_{I\overline{J}}F^I F^{J*} - \frac{1}{2}g_{I\overline{M}}\Gamma^{\overline{M}}_{\overline{JK}}F^I\overline{\chi}^J\overline{\chi}^K - \frac{1}{2}G_{M\overline{I}}\Gamma^M_{JK}F^{\overline{I}}\chi^J\chi^K + \frac{1}{4}g_{I\overline{J},K\overline{L}}\chi^I\chi^K\overline{\chi}^J\overline{\chi}^L \\ &- g_{I\overline{J}}\partial_m A^I\partial^m A^{J*} - ig_{I\overline{J}}\overline{\chi}^J\overline{\sigma}^m\partial_m\chi^I - ig_{M\overline{K}}\Gamma^M_{IJ}\overline{\chi}^K\overline{\sigma^m}\chi^I\partial_m A^I, \end{aligned} \tag{2.6.8}$$

where all the component fields are the function of y^m and we have defined

$$\begin{aligned} g_{I\overline{J}} &= \frac{\partial}{\partial A^I}\frac{\partial}{\partial A^{J*}}K|_{\theta=\overline{\theta}=0} \\ g_{I\overline{J},K} &= \frac{\partial}{\partial A^K}g^{I\overline{J}} = g_{M\overline{J}}\Gamma^M_{IK} \end{aligned} \tag{2.6.9}$$

Substituting these expansions into (2.6.5) and using the equation of motion of F^I, we obtain the Lagrangian for the component fields:

$$\begin{aligned} \mathcal{L} =& - g_{I\overline{J}}\partial_m A^I\partial^m A^{J*} - ig_{I\overline{J}}\overline{\chi}^J\overline{\sigma}^m D_m\chi^I + \frac{1}{4}R_{I\overline{J}K\overline{L}}\chi^I\chi^K\overline{\chi}^J\overline{\chi}^L \\ &- \frac{1}{2}D_I D_J W\chi^I\chi^J - \frac{1}{2}D_{\overline{I}}D_{\overline{J}}W^*\overline{\chi}^I\overline{\chi}^J - g^{I\overline{J}}D_I W D_{\overline{J}}W^*, \end{aligned} \tag{2.6.10}$$

where we have defined

$$\begin{aligned} D_m\chi^I &= \partial_m\chi^I + \Gamma^I_{JK}\partial_m A^I\chi^K \\ D_I W &= \frac{\partial W}{\partial A^I} \\ D_I D_J W &= \frac{\partial^2 W}{\partial A^I\partial A^J} - \Gamma^K_{IJ}\frac{\partial W}{\partial A^K}. \end{aligned} \tag{2.6.11}$$

2.6.2 Super Yang-Mills Theory

In this section, we explain $N=1, D=4$ Super Yang-Mills theory.

Since spinor field W_α is a chiral superfield, $W_\alpha W^\alpha$ is also a chiral superfield. Thus, a supersymmetric invariant action can be constructed by using the F-term of the chiral superfield $W_\alpha W^\alpha$:

$$\begin{aligned}[\mathrm{tr}\, W^\alpha W_\alpha]_F\theta^2 =&\epsilon^{\alpha\beta}\,\mathrm{tr}\Big[-2i\lambda_\beta\theta^2(\sigma^\mu)_{\alpha\dot\gamma}D_\mu\overline{\lambda}^{\dot\gamma}+2\theta_\beta\Delta 2\theta_\alpha\Delta \\ &+4\theta_\beta(\sigma^{\mu\nu})_\alpha^\gamma\theta_\gamma F_{\mu\nu}+(\sigma^{\mu\nu})_\beta^\rho\theta_\rho F_{\mu\nu}(\sigma^{\mu'\nu'})_\alpha^\gamma\theta_\gamma F_{\mu'\nu'}\Big]\end{aligned} \tag{2.6.12}$$

By using Firtz identities and trace properties, (2.6.12) can be rewritten as

$$[\mathrm{tr}\, W^\alpha W_\alpha]_F=\mathrm{tr}\left[-2i\lambda\sigma^\mu D_\mu\overline{\lambda}+4\Delta^2-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}-\frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right]. \tag{2.6.13}$$

We define the complex coupling constant as

$$\tau=\frac{4\pi i}{g^2}+\frac{\Theta}{2\pi}, \tag{2.6.14}$$

where g is the gauge coupling constant and Θ is the Yang-Mills theta angle

From the above, the N=1 super Yang-Mills action without matter field is given by

$$\begin{aligned}S_{YM}^{N=1}&=\frac{1}{8\pi}\int d^4x\,\mathrm{Im}\left(\tau\int d^2\theta\,\mathrm{tr}\, W^\alpha W_\alpha\right)\\ &=\frac{1}{8\pi}\int d^4x\,\mathrm{Im}[\tau\,\mathrm{tr}\, W^\alpha W_\alpha]_F],\end{aligned} \tag{2.6.15}$$

and in the component fields

$$S_{YM}^{N=1}=\frac{1}{g^2}\int d^4x\,\mathrm{tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-i\lambda\sigma^\mu D_\mu\overline{\lambda}+2\Delta^2\right]-\frac{\Theta}{32\pi^2}\int d^4x\,\mathrm{Im}[\tau\,\mathrm{tr}\, W^\alpha W_\alpha]_F \tag{2.6.16}$$

Next, we consider the Yang-Mills theory coupled to the matter fields. The matter fields are described by the chiral superfield. Since we would like to consider a system in which the chiral superfield Φ interacts with the gauge field, we assume that Φ is an R-representation of the gauge group, and the transformation laws are given by

$$\Phi\to e^{-\Lambda}\Phi,\quad \Phi^\dagger\to\Phi^\dagger e^{-\Lambda^\dagger}. \tag{2.6.17}$$

Since the interaction between the gauge field and the matter field is described by 3-point function , the following quantity contains the 3-point functions, and the coupling term, which is gauge invariant, can be written as

$$\Phi^\dagger e^V\Phi. \tag{2.6.18}$$

Indeed expanding with component fields, this term is rewritten as

$$\begin{aligned} S_M &= \int d^4xd^2\theta\Phi^\dagger e^V\Phi - \int d^4xd^2\theta\left[\frac{m_{ab}}{2}\Phi_a\Phi_b + \frac{g_{abc}}{3}\Phi_a\Phi_b\Phi_c + h.c\right] \\ &= \int d^4x[\Phi^\dagger e^V\Phi]_d - \int d^4x\left[\frac{m_{ab}}{2}\Phi_a\Phi_b + \frac{g_{abc}}{3}\Phi_a\Phi_b\Phi_c + h.c\right]_F, \end{aligned} \tag{2.6.19}$$

where m_{ab} and g_{abc} are the mass matrix and the coupling constant matrix, which are fully symmetric matrices.

From the above discussions, the $N=1$ super Yang-Mills theory coupled to the matter field is described by the following action:

$$S_{N=1}\frac{1}{8\pi}\int d^4x\,\mathrm{Im}[\tau\,\mathrm{tr}\,W^\alpha W_\alpha]_F + \int d^4x[\Phi^\dagger e^V\Phi]_d - \int d^4x[W(\Phi)+h.c]_F \tag{2.6.20}$$

The $F^\dagger F$ derived from the superpotential contributes to the scalar potential:

$$V_F(A^\dagger, A) = F^\dagger F = (mA+gA^2)^\dagger(mA+gA^2) \tag{2.6.21}$$

This potential is called F-term potential. Since the d-term of W_α is also an auxiliary field, it leads to a potential

$$V_D(A^\dagger, A) = \frac{2}{g^2}\Delta^2 = \frac{g^2}{8}[A, A^\dagger]^2, \tag{2.6.22}$$

which is called d-term potential. Thus, the potential is given by

$$V(A^\dagger, A) = V_F + V_D = F^\dagger F + \frac{2}{g^2}\Delta^2. \tag{2.6.23}$$

We construct a more general $N=1$ supersymmetry theory. The renormalizable superpotential in four dimensions includes Φ^3 as a maximal order, but without accounting for renormalizability, it is possible to include the infinite order. Furthermore, we assume that the chiral superfield Φ is a representation of gauge symmetry as well as a representation of flavor symmetry. In other words, the chiral superfield has a flavor symmetry index I, and for each I there is a different gauge representation R_I. Since the kinetic term is obtained by the d-term of the vector superfield generated from the chiral superfield, the kinetic term can be generalized to be written in terms of the Kähler potential K. Furthermore, since the gauge kinetic term $W_\alpha W^\alpha$ is given by the F-term of the chiral superfield, there is a degree of freedom to insert an arbitrary function $f(\Phi)$ of the chiral superfield. Thus, the most general N = 1 supersymmetry theory is given by

$$S^{N=1} = \frac{1}{8\pi}\int d^4x\left[\mathrm{Im}[\tau\,\mathrm{tr}\,f(\Phi^I)W^\alpha W_\alpha]_F + [K(\Phi^{I\dagger}e^V, \Phi^J]_d - [W(\Phi) + h.c]_F\right]. \tag{2.6.24}$$

Since $f(\Phi)$ is the coefficient of the gauge kinetic term, it is called the gauge kinetic function. Also, in order for the gauge kinetic term to be gauge invariant, the gauge kinetic function must be a adjoint representation of the gauge group:

$$f(\Phi^I) \to e^{-\Lambda}f(\Phi^I)e^\Lambda \tag{2.6.25}$$

Finally, the supersymmetric invariant action need only be constructed by the F-term of the chiral superfield and the d-term of the vector superfield, and the F-term of the general chiral superfield is given by the superpotential. However, there is no term in (2.6.24) which is constructed by the d-term of the general vector superfield. Therefore, the following terms can be added:

$$S_{FI} = \int d^4x\xi[V]_d \tag{2.6.26}$$

This term is called the Fayet-Illiopoulos term.

3 Supergravity

In this section, we explain the supergravity theory formally and briefly for preliminaries [45, 50–54]. We omit the explicit calculations and the reviews of Type I and Type IIA/IIB supergravity theory.

3.1 Noether Method

In order to obtain supergravity theory, we consider the local supersymmetry. In other words, the parameter θ of the transformation is dependent on spacetime coordinate x,

$$[\overline{\theta}_1(x)Q, \overline{\theta}_2(x)Q] = 2i\overline{\theta}_1(x)\gamma^\mu\theta_2(x)P_\mu \tag{3.1.1}$$

Where, Q is a generator of supersymmetry, γ^μ is a generator of Clifford algebra and P^μ is a translation generator. The commutator of two local supersymmetry transformation is a local translation, which is a general coodinate transformation. Thus local supersymmetry leads to a theory of gravity and this theory known as supergravity.

Let us consider the kinetic terms of Wess-Zumino model.

$$S_0 = \int d^4x \left[\partial_\mu\phi^*\partial_\mu\phi + \frac{i}{2}\overline{\Psi}\gamma^\mu\Psi\right] \tag{3.1.2}$$

where, ϕ is a complex scalar field defined by two real scalar fields A and B as $\phi = (A + iB)/\sqrt{2}$ and Ψ is a Majorana supinor. This action is invariant under the glocal supersymmetry.

$$\delta A = \overline{\theta}\Psi, \quad \delta B = i\overline{\theta}\gamma_5\Psi, \quad \delta\Psi_\alpha = -i(\gamma^\mu[\partial_\mu(A + i\gamma_5 B)]\theta)_\alpha \tag{3.1.3}$$

If we consider the local supersymmetry, S_0 is no longer invariant;

$$\delta S_0 = \int d^4x\partial_\mu\overline{\theta}\gamma^\rho\gamma^\mu[\partial_\rho(A + i\gamma_5 B)]\Psi \equiv \int d^4x(\partial_\mu\overline{\theta}_\alpha)J^\mu_\alpha(x) \tag{3.1.4}$$

In order to obtain an invariant action, we introduce a new field $\psi_{\mu\alpha}(x)$ with vector and spinor indices and a new interaction term

$$S_1 = -\frac{\kappa}{2}\int d^4x\overline{\psi}_\mu\gamma^\rho\gamma^\mu[\partial_\rho(A + i\gamma_5 B)]\Psi \tag{3.1.5}$$

where κ is a coupling constant between Ψ and $\psi_{\mu\alpha}$. The supersymmetry transformation of $\psi_{\mu\alpha}$ is chosen in order to cancel (3.1.4)

$$\delta\psi^\mu_\alpha = \frac{2}{\kappa}\partial^\mu\theta_\alpha(x). \tag{3.1.6}$$

However, $S_0 + S_1$ is also not invariant under local supersymmetry. The transformation of S_1 yields the following extra terms;

$$\delta S_1 = i\kappa \int d^4x \overline{\psi}_\mu(x)\gamma_\nu\theta(x)\left[\partial^\mu A\partial^\nu A - \frac{1}{2}g^{\mu\nu}\partial^\rho A\partial_\rho A + \cdots\right] \tag{3.1.7}$$

Here, the terms between the bracket correspond to energy-momentum tesor $\Theta^{\mu\nu}$ for the scalar field which is obtained by varying with metric

$$\delta S = \int d^4x \frac{1}{2}\delta g_{\mu\nu}\Theta^{\mu\nu}. \tag{3.1.8}$$

So, we regard the coefficents of bracket in (3.1.7) as a supersymmetry transformation of a local metric $g_{\mu\nu}(x)$,

$$\delta g_{\mu\nu}(x) = -i\kappa(\overline{\psi}_\mu\gamma_\nu\theta(x) + \overline{\psi}_\nu\gamma_\mu\theta(x)) \tag{3.1.9}$$

The corresponding spin 2 field which is graviton is a super-partner of gravitino ψ_μ. Thus coupling constant κ is a gravitational coupling constant.

Continuing the above procedure until the action becames invariant, we can obtain the local supersymmetry transformation laws

$$\delta g_{\mu\nu}(x) = -i\kappa(\overline{\psi}_\mu\gamma_\nu\theta(x) + \overline{\psi}_\nu\gamma_\mu\theta(x)) \tag{3.1.10}$$

$$\delta\psi_\mu(x) = \frac{2}{\kappa}D^\mu\theta(x), \quad D^\mu = \partial^\mu + \frac{1}{2}\omega_\mu^{ab}\sigma_{ab}. \tag{3.1.11}$$

If we introduce the kinetic terms for gravity $g_{\mu\nu}$ and gravitino $\psi_{\mu\alpha}$

$$S = \int d^4x\sqrt{g}\left[-\frac{1}{2\kappa^2}R - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\overline{\psi}_\mu\gamma_5\gamma_\nu\partial_\rho\psi_\sigma\right], \tag{3.1.12}$$

we obtain the supergravity action with coupling matters. Here, ω_μ^{ab} is a spin connction and R is a curvature which leads to Einstein equation.

The following sections, we construct a general SUGRA action with the superfields.

3.2 Superspace for Curved Space

In order to deal with the quantities on the curved spacetime, it is useful to consider the tangent space of the curved spacetime and differential forms on it.

3.2.1 Differential Forms in Superspace

The SUGRA theory was formulated by considering local supersymmetry, which leads to the local translation and they are regarded as the general coordinate transformations in superspace. This fact means that we can simplily construct the SUGRA action by using the differential forms since they are defined on the local coordinate system.

We introduce the global superspace coordinate z^M, where $M = (\mu, \alpha, \dot{\alpha})$. Since the superspace coordinates z^M includes the fermionic coordinate θ and $\overline{\theta}$, exterior products in superspace are defined by

$$dz^M \wedge dz^N = -(-)^{NM} dz^N \wedge dz^M \tag{3.2.1}$$

$$(-)^{MN} = \begin{cases} -1 & \text{Both } M \text{ and } N \text{ are the Grassmann direction.} \\ 1 & \text{the others.} \end{cases} \tag{3.2.2}$$

Then we can define the differential forms in superspace

$$\Omega = dz^{M_1} \wedge \cdots \wedge dz^{M_p} W_{M_p \ldots, M_1}(z). \tag{3.2.3}$$

If we assume that the coefficient functions W with an odd number spinorial indeces are fermionic and that those with an even number of spinorial indices are bosonic, we can realize the same multiplication rules and exterior derivative as those on space-time manifold.

Next we consider the two coordinate systems y and z which are related by

$$y^M = y^M(z) \tag{3.2.4}$$

Functions of y have a natural mapping into functions of z:

$$F(y) = F(y(z)) = \phi^* F(z) \tag{3.2.5}$$

If y and z represent same point in superspace, a certain quantity should take the same value. Similary, this map ϕ induces a natural mapping between two p-forms:

$$\Omega(y) = \phi^* \Omega(z) \tag{3.2.6}$$

The map ϕ^* satisfies the following properties:

1. $\phi^*(\Omega + \Sigma) = \phi^* \Omega + \phi^* \Sigma$

2. $\phi^*(\Omega\Sigma) = (\phi^*\Omega)(\phi^*\Sigma)$

3. $d(\phi^*\Omega) = \phi^*(d\Omega)$

We consider the infinitesimal transformations:

$$z^M = y^M + \xi^M \tag{3.2.7}$$

In this transfornation for 1-form W_M, we find

$$\delta W_M(z) = -\phi^* W_M(z) - W_M(z) = -\xi^N \partial_N W_M(z) - \frac{\partial \xi^N}{\partial z^M} W_N(z). \tag{3.2.8}$$

This can be easily generalized for arbitrary p-forms.

In the gauge theory, the differential forms are not only covariant under general coordinate transformations but also under gauge group which is a compact Lie group or Lorentz group. If the representations of the element of the gauge group are given by X^a_b where a and b are the gauge indices, p-forms Ω^a transform as

$$\Omega'^a = \Omega^b X^a_b(x), \tag{3.2.9}$$

where $a = 1, ..., L = \dim G$ is a gauge index.

But, exterior derivatives do not transform as tensor:

$$d\Omega' = \Omega dX + d\Omega X \tag{3.2.10}$$

Thus we must introduce a connection in order to cancel out the first term. The connection are defined as Lie algebra valued 1-form

$$\phi = dz^M \phi^r_M(z) i T^r, \tag{3.2.11}$$

where T^r is a generator of Lie algebra corresponding to the gauge group G. The connections obey the following transformation law:

$$\phi' = X^{-1}\phi X - X^{-1}dX. \tag{3.2.12}$$

Thus we define the covariant derivatives as

$$\begin{aligned} \mathcal{D}\Omega &\equiv d\Omega + \Omega\phi \\ &= dz^{M_1} \wedge \cdots \wedge dz^{M_p} \wedge dz^N \left(\frac{\partial}{\partial z^N} + \phi^r_N i T^r \right) W_{M_p \ldots, M_1}(z) \end{aligned} \tag{3.2.13}$$

The curvature and the covariant derivative of a tensor are the only tensorial quantities. Higher derivatives lead to identities because of the property $dd = 0$ which called Bianchi identities. The curvature is given by

$$F = d\phi + \phi \wedge \phi, \tag{3.2.14}$$

and is a Lie algebra valued 2-form

$$F = \frac{1}{2} dz^M \wedge dz^N F^r_{NM}(z) iT^r \tag{3.2.15}$$

Since the tensorial quantities are only the cuvature and the covariant derivative of a tensor, there are two ways to obtain Bianchi identities. First type Bianchi identities are found from the covariant derivative of a tensor such as

$$d\mathcal{D}\Omega = \Omega F - \mathcal{D}\Omega\phi. \tag{3.2.16}$$

These can be rewritten as

$$\mathcal{D}\mathcal{D}\Omega = \Omega F. \tag{3.2.17}$$

The second type Bianchi identities are found from the curvature such as

$$dF = \phi F - F\phi, \tag{3.2.18}$$

which can be written as

$$\mathcal{D}F = 0. \tag{3.2.19}$$

If we rewrite (3.2.19) in terms of the components, we find

$$\mathcal{D}_L F_{NM} + (-)^{L(N+M)} \mathcal{D}_N F_{ML} + (-)^{M(N+L)} \mathcal{D}_M F_{LN} = 0. \tag{3.2.20}$$

3.2.2 Local Coordinate System

Next we localize the supersymmetry to obtain SUGRA theory. The supersymmetric transformations depend on space-time coordinate:

$$\begin{aligned} x^\mu &\to x^\mu + i(\theta\sigma^\mu\overline{\xi}(x) - \xi(x)\sigma^\mu\overline{\theta}) \\ \theta^\alpha &\to \theta^\alpha + \xi^\alpha(x). \end{aligned} \tag{3.2.21}$$

These transformations generate a general coordinate transformations in supersupace:

$$z^M \to z'^M = z^M - \xi^M(z) \tag{3.2.22}$$

The gauge field corresponding to local Lorentz is vielbein $E^{\underline{M}}$ defined by

$$E^{\underline{M}} = dz^M E_M{}^{\underline{M}}, \tag{3.2.23}$$

where index $\underline{M}$ represent the representation of local Lorentz which is a vector index in local superspace coordinate system. Since this is a 1-form, the SUSY transformation law is given by

$$\begin{aligned} z'^M &= z^M - \xi^M(z) \\ \delta E_M{}^{\underline{M}} &= -\xi^L \partial_L E_M{}^{\underline{M}} - (\partial_M \xi^L) E_L{}^{\underline{M}}. \end{aligned} \tag{3.2.24}$$

Also $E^{\underline{M}}$ transforms under the local Lorentz

$$\delta E^{\underline{M}} = E^{\underline{N}} L_{\underline{N}}{}^{\underline{M}}(z). \tag{3.2.25}$$

The vielbein and its invese satisfy

$$\begin{aligned} E_M{}^{\underline{M}} E_{\underline{M}}{}^N &= \delta_M{}^N \\ E_{\underline{M}}{}^M E_M{}^{\underline{N}} &= \delta_{\underline{M}}{}^{\underline{N}}. \end{aligned} \tag{3.2.26}$$

With the vielbein, we can rewrite the superspace coordinate to its local coordinate and wherever possible we choose the local coordinate since it is flat.

The connection of the local Lorentz ϕ has a Lorentz indices defined by

$$\phi = dz^M \phi_M, \quad \phi_M = \phi_{M\underline{N}}{}^{\underline{L}}. \tag{3.2.27}$$

This connection transform under the local Lorentz group as

$$\delta\phi = [\phi, L] + dL. \tag{3.2.28}$$

From the previous section, there are two tensorial quantities. One of them is the covariant derivative of the vielbein

$$T^{\underline{M}} \equiv \mathcal{D}E^{\underline{M}} = dE^{\underline{M}} + E^{\underline{N}} \phi_{\underline{N}}{}^{\underline{M}}, \tag{3.2.29}$$

which is called torsion and we can rewrite explicitly with the coefficient

$$T_{NM}{}^{\underline{M}} = \partial_N E_M{}^{\underline{M}} - (-)^{NM} \partial_M E_N{}^{\underline{M}} + (-)^{N(\underline{N}+M)} E_M{}^{\underline{N}} \phi_{N\underline{N}}{}^{\underline{M}} - (-)^{M\underline{N}} E_N{}^{\underline{N}} \phi_{M\underline{N}}{}^{\underline{M}}. \tag{3.2.30}$$

We also rewrite the torsion to the local coordinate

$$T_{\underline{N}\underline{L}}{}^{\underline{M}} = (-)^{\underline{N}(M+\underline{L})} E_{\underline{L}}{}^M E_{\underline{N}}{}^N T_{NM}{}^{\underline{M}}. \tag{3.2.31}$$

Whereas the other of them is curvature tensor defined by

$$R = d\phi + \phi\phi, \tag{3.2.32}$$

and its components are given by

$$\begin{aligned} R_{NM\underline{M}}{}^{\underline{N}} &= \partial_N \phi_{M\underline{M}}{}^{\underline{N}} \\ &- (-)^{NM} \partial_M \phi_{N\underline{M}}{}^{\underline{N}} + (-)^{N(M+\underline{M}+\underline{L})} \phi_{M\underline{M}}{}^{\underline{L}} \phi_{N\underline{L}}{}^{\underline{N}} - (-)^{M(\underline{M}+\underline{L})} \phi_{N\underline{M}}{}^{\underline{L}} \phi_{M\underline{L}}{}^{\underline{N}}. \end{aligned} \tag{3.2.33}$$

The curvature tensor and the derivative of tensor should satisfy the Bianchi identities

$$\begin{aligned} \mathcal{D}T^{\underline{M}} &= \mathcal{D}\mathcal{D}E^{\underline{M}} = E^{\underline{N}} R_{\underline{N}}{}^{\underline{M}} \\ \mathcal{D}R &= 0 \end{aligned} \tag{3.2.34}$$

First, we focus on the first Bisnchi identity. Since the local coordinate is a flat space, the connection ϕ equals to zero and the torsion along spacetime coordinete becomes 0. Thus the non-vanishing comoponents of the torsion are given by

$$T_{\underline{\alpha\dot{\beta}}}{}^{\underline{\mu}} = 2i\sigma_{\underline{\alpha\dot{\beta}}}{}^{\underline{\mu}}. \tag{3.2.35}$$

We notice that these solutions are in the local coordinate system not in superspace.

With $R, G_{\alpha\dot{\alpha}}, W_{\alpha\beta\gamma}$ defined by

$$\overline{\mathcal{D}}_{\dot{\alpha}} R = 0 \tag{3.2.36}$$

$$\mathcal{D}^{\alpha} G_{\alpha\dot{\beta}} = \overline{\mathcal{D}}_{\dot{\beta}} R^{\dagger} \tag{3.2.37}$$

$$\overline{\mathcal{D}}_{\dot{\alpha}} W_{\beta\gamma\delta} = 0 \tag{3.2.38}$$

$$\mathcal{D}^{\alpha} W_{\alpha\beta\delta} + \frac{1}{2} i (\mathcal{D}_{\beta\dot{\beta}} G_{\delta}^{\dot{\beta}} + \mathcal{D}_{\delta\dot{\beta}} G_{\beta}^{\dot{\beta}}) = 0 \tag{3.2.39}$$

$$(G_{\alpha\dot{\alpha}})^{\dagger} = G_{\alpha\dot{\alpha}} \tag{3.2.40}$$

$$(W_{\alpha\beta\gamma})^{\dagger} = \overline{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}, \tag{3.2.41}$$

the general solutions of the first Bianchi identity in the supersupace are given by the following

$$\begin{aligned} &T_{\underline{\mu\nu}}{}^{\underline{\dot{\alpha}}} = \frac{1}{4} \overline{\sigma}_{\underline{\mu}}{}^{\dot{\delta}\delta} \overline{\sigma}_{\underline{\nu}}{}^{\dot{\gamma}\gamma} T_{\underline{\delta\dot{\delta}\gamma\dot{\gamma}}}{}^{\underline{\dot{\alpha}}}, \quad T_{\underline{\gamma\dot{\epsilon}}}{}^{\underline{\mu}} = 2i\sigma_{\underline{\gamma\dot{\epsilon}}}{}^{\underline{\mu}} \\ &T_{\underline{\dot{\beta}\mu}}{}^{\underline{\alpha}} = -\frac{1}{2} \overline{\sigma}_{\underline{\mu}}{}^{\dot{\epsilon}\epsilon} T_{\underline{\dot{\beta}\epsilon\dot{\epsilon}}}{}^{\underline{\alpha}}, \quad T_{\underline{\dot{\delta}\epsilon\dot{\epsilon}\alpha}} = -2i\epsilon_{\underline{\dot{\delta}\dot{\epsilon}}} \epsilon_{\underline{\epsilon\alpha}} R, \quad T_{\underline{\delta\mu}}{}^{\underline{\dot{\alpha}}} = -\frac{1}{2} \overline{\sigma}_{\underline{\mu}}{}^{\dot{\epsilon}\epsilon} T_{\underline{\delta\epsilon\dot{\epsilon}}}{}^{\underline{\dot{\epsilon}}} \\ &T_{\underline{\delta\epsilon\dot{\epsilon}\dot{\alpha}}} = -2i\epsilon_{\underline{\delta\epsilon}} \epsilon_{\underline{\dot{\epsilon}\dot{\alpha}}} R^{\dagger}, \quad T_{\underline{\delta\mu}}{}^{\underline{\alpha}} = -\frac{1}{2} \overline{\sigma}_{\underline{\mu}}{}^{\dot{\epsilon}\epsilon} T_{\underline{\delta\epsilon\dot{\epsilon}}}{}^{\underline{\alpha}}, \quad T_{\underline{\delta\epsilon\dot{\epsilon}\alpha}} = \frac{i}{4} (\epsilon_{\underline{\epsilon\alpha}} G_{\underline{\delta\dot{\epsilon}}}) - 3\epsilon_{\underline{\delta\alpha}} G_{\underline{\epsilon\dot{\epsilon}}} - 3\epsilon_{\underline{\delta\epsilon}} G_{\underline{\alpha\dot{\epsilon}}} \end{aligned}$$

$$
\begin{aligned}
&T_{\underline{\dot\delta\epsilon}}{}^{\underline{\dot\alpha}} = -\frac{1}{2}\overline{\sigma}_{\underline{\mu}}{}^{\underline{\dot\epsilon\epsilon}} T_{\underline{\dot\delta\epsilon\dot\epsilon}}{}^{\underline{\dot\alpha}}, \quad T_{\underline{\dot\delta\epsilon\dot\epsilon\dot\alpha}} = \frac{i}{5}(\epsilon_{\underline{\dot\epsilon\dot\alpha}} - 3\epsilon_{\underline{\dot\delta\dot\alpha}} G_{\underline{\epsilon\dot\epsilon}} - 3\epsilon_{\underline{\dot\delta\dot\epsilon}} G_{\underline{\epsilon\dot\alpha}}), \quad T_{\underline{\mu\nu}}{}^{\underline{\alpha}} = \frac{1}{4}\overline{\sigma}_{\underline{\mu}}{}^{\underline{\dot\delta\delta}}\overline{\sigma}_{\underline{\nu}}{}^{\underline{\dot\gamma\gamma}} T_{\underline{\delta\dot\delta\gamma\dot\gamma}}{}^{\underline{\alpha}} \\
&T_{\underline{\delta\dot\delta\gamma\dot\gamma\alpha}} = -2\epsilon_{\underline{\dot\delta\dot\gamma}} W_{\underline{\delta\gamma\alpha}} - \frac{1}{2}\epsilon_{\underline{\dot\delta\dot\gamma}}(\epsilon_{\underline{\delta\alpha}}\overline{\mathcal{D}}_{\underline{\dot\phi}} G_{\underline{\gamma}}{}^{\underline{\dot\phi}} + \epsilon_{\underline{\gamma\alpha}}\overline{\mathcal{D}}_{\underline{\dot\phi}} G_{\underline{\delta}}{}^{\underline{\dot\phi}}) + \frac{1}{2}\epsilon_{\underline{\delta\gamma}}(\overline{\mathcal{D}}_{\underline{\dot\delta}} G_{\underline{\alpha\dot\gamma}} + \overline{\mathcal{D}}_{\underline{\dot\gamma}} G_{\underline{\alpha\dot\delta}}) \\
&T_{\underline{\delta\dot\delta\gamma\dot\gamma\dot\alpha}} = -2\epsilon_{\underline{\delta\gamma}} \overline{W}_{\underline{\dot\delta\dot\gamma\dot\alpha}} - \frac{1}{2}\epsilon_{\underline{\delta\gamma}}(\epsilon_{\underline{\dot\delta\dot\alpha}}\mathcal{D}_{\underline{\beta}} G_{\underline{\dot\gamma}}{}^{\underline{\beta}} + \epsilon_{\underline{\dot\gamma\dot\alpha}}\mathcal{D}_{\underline{\beta}} G_{\underline{\dot\delta}}{}^{\underline{\beta}}) + \frac{1}{2}\epsilon_{\underline{\dot\delta\dot\gamma}}(\mathcal{D}_{\underline{\delta}} G_{\underline{\gamma\dot\alpha}} + \mathcal{D}_{\underline{\gamma}} G_{\underline{\delta\dot\alpha}})
\end{aligned} \tag{3.2.42}
$$

$$
\begin{aligned}
&R_{\underline{\delta\gamma\epsilon\alpha}} = 4(\epsilon_{\underline{\delta\epsilon}}\epsilon_{\underline{\gamma\alpha}} + \epsilon_{\gamma\epsilon}\epsilon_{\delta\alpha})R^\dagger, \quad R_{\underline{\delta\gamma\dot\epsilon\dot\alpha}} = R_{\underline{\dot\delta\dot\gamma\epsilon\alpha}} = 0 \\
&R_{\underline{\delta\dot\gamma\epsilon\alpha}} = -(\epsilon_{\underline{\delta\epsilon}} G_{\underline{\alpha\dot\gamma}} + \epsilon_{\underline{\delta\alpha}} G_{\underline{\epsilon\dot\gamma}}), \quad R_{\underline{\epsilon\mu\delta\alpha}} = -\frac{1}{2}\overline{\sigma}_{\underline{\mu}}{}^{\underline{\dot\gamma\gamma}} R_{\underline{\epsilon\gamma\dot\gamma\delta\alpha}} \\
&R_{\underline{\epsilon\gamma\dot\gamma\delta\alpha}} = i(\epsilon_{\underline{\epsilon\delta}}\epsilon_{\underline{\gamma\alpha}} + \epsilon_{\underline{\epsilon\alpha}}\epsilon_{\underline{\gamma\delta}})\mathcal{D}_{\underline{\beta}} G_{\underline{\dot\gamma}}{}^{\underline{\beta}} + \frac{i}{2}(\epsilon_{\underline{\epsilon\gamma}}\mathcal{D}_{\underline{\delta}} + \epsilon_{\underline{\epsilon\delta}}\mathcal{D}_{\underline{\gamma}})G_{\underline{\alpha\dot\gamma}} + \frac{i}{2}(\epsilon_{\underline{\epsilon\gamma}}\mathcal{D}_{\underline{\alpha}} + \epsilon_{\underline{\epsilon\alpha}}\mathcal{D}_{\underline{\gamma}})G_{\underline{\delta\dot\gamma}} \\
&R_{\underline{\epsilon\mu\dot\delta\dot\alpha}} = -\frac{1}{2}\overline{\sigma}_{\underline{\mu}}{}^{\underline{\dot\gamma\gamma}} R_{\underline{\epsilon\gamma\dot\gamma\dot\delta\dot\alpha}}, \quad R_{\underline{\epsilon\gamma\dot\gamma\dot\delta\dot\alpha}} = 4i\epsilon_{\underline{\epsilon\gamma}}\overline{W}_{\underline{\dot\gamma\dot\delta\dot\alpha}} + \frac{i}{2}(\epsilon_{\underline{\dot\gamma\dot\delta}}\mathcal{D}_{\underline{\epsilon}} G_{\underline{\gamma\dot\alpha}} + \epsilon_{\underline{\dot\gamma\dot\alpha}}\mathcal{D}_{\underline{\epsilon}} G_{\underline{\gamma\dot\delta}}) \\
&R_{\underline{\dot\epsilon\mu\delta\alpha}} = -\frac{1}{2}\overline{\sigma}_{\underline{\mu}}{}^{\underline{\dot\gamma\gamma}} R_{\underline{\dot\epsilon\gamma\dot\gamma\delta\alpha}}, \quad R_{\underline{\dot\epsilon\gamma\dot\gamma\delta\alpha}} = 4i\epsilon_{\underline{\dot\epsilon\dot\gamma}} W_{\underline{\delta\gamma\alpha}} + \frac{i}{2}(\epsilon_{\underline{\gamma\delta}}\overline{\mathcal{D}}_{\underline{\dot\epsilon}} G_{\underline{\alpha\dot\gamma}} + \epsilon_{\underline{\gamma\alpha}}\overline{\mathcal{D}}_{\underline{\dot\epsilon}} G_{\underline{\delta\dot\gamma}}) \\
&R_{\underline{\dot\epsilon\mu\dot\delta\dot\alpha}} = -\frac{1}{2}\overline{\sigma}_{\underline{\mu}}{}^{\underline{\dot\gamma\gamma}} R_{\underline{\dot\epsilon\gamma\dot\gamma\dot\delta\dot\alpha}} \\
&R_{\underline{\dot\epsilon\gamma\dot\gamma\dot\delta\dot\alpha}} = i(\epsilon_{\underline{\dot\epsilon\dot\delta}}\epsilon_{\underline{\dot\gamma\dot\alpha}} + \epsilon_{\underline{\dot\epsilon\dot\alpha}}\epsilon_{\underline{\dot\gamma\dot\delta}})\overline{\mathcal{D}}_{\underline{\dot\beta}} G_{\underline{\gamma}}{}^{\underline{\dot\beta}} + \frac{i}{2}(\epsilon_{\underline{\dot\epsilon\dot\gamma}}\overline{\mathcal{D}}_{\underline{\dot\delta}} + \epsilon_{\underline{\dot\epsilon\dot\delta}}\overline{\mathcal{D}}_{\underline{\dot\gamma}})G_{\underline{\gamma\dot\alpha}} + \frac{i}{2}(\epsilon_{\underline{\dot\epsilon\dot\gamma}}\overline{\mathcal{D}}_{\underline{\dot\alpha}} + \epsilon_{\underline{\dot\epsilon\dot\alpha}}\overline{\mathcal{D}}_{\underline{\dot\gamma}})G_{\underline{\gamma\dot\delta}} \\
&R_{\underline{\mu\nu\gamma\alpha}} = \frac{1}{4}\overline{\sigma}_{\underline{\mu}}{}^{\underline{\dot\epsilon\epsilon}}\overline{\sigma}_{\underline{\nu}}{}^{\underline{\dot\delta\delta}} R_{\underline{\epsilon\dot\epsilon\delta\dot\delta\gamma\alpha}}, \quad R_{\underline{\epsilon\dot\epsilon\delta\dot\delta\gamma\alpha}} = 2\epsilon_{\underline{\dot\epsilon\dot\delta}} X_{\underline{\epsilon\delta\gamma\alpha}} - 2\epsilon_{\underline{\epsilon\delta}}\overline{\Psi}_{\underline{\dot\epsilon\dot\delta\gamma\alpha}} \\
&X_{\underline{\gamma\delta\epsilon\alpha}} = -\frac{1}{4}(\mathcal{D}_{\underline{\gamma}} W_{\underline{\delta\epsilon\alpha}} + \text{cycric perm}) \\
&+ (\epsilon_{\underline{\gamma\alpha}}\epsilon_{\underline{\epsilon\delta}} + \epsilon_{\underline{\delta\alpha}}\epsilon_{\underline{\epsilon\gamma}})\left\{-2RR^\dagger + \frac{1}{8}G_{\underline{\rho\dot\rho}}G^{\underline{\rho\dot\rho}} + \frac{1}{16}(\overline{\mathcal{D}}_{\underline{\dot\rho}}\overline{\mathcal{D}}^{\underline{\dot\rho}}R^\dagger + \mathcal{D}^{\underline{\rho}}\mathcal{D}_{\underline{\rho}}R)\right\} \\
&\Psi_{\underline{\epsilon\alpha\dot\gamma\dot\delta}} = \frac{1}{4}(G_{\underline{\epsilon\dot\delta}}G_{\underline{\alpha\dot\gamma}} + G_{\underline{\alpha\dot\delta}}G_{\underline{\epsilon\dot\gamma}}) + \frac{i}{8}(\mathcal{D}_{\underline{\alpha\dot\gamma}}G_{\underline{\epsilon\dot\delta}} + \text{cycric perm}) \\
&+ \frac{1}{8}(\overline{\mathcal{D}}_{\underline{\dot\gamma}}(\mathcal{D}_{\underline{\alpha}}G_{\underline{\epsilon\dot\delta}} + \mathcal{D}_{\underline{\epsilon}}G_{\underline{\alpha\dot\delta}}) + (\underline{\dot\delta} \leftrightarrow \underline{\dot\gamma}))
\end{aligned} \tag{3.2.43}
$$

(3.2.44)

and the other components of the first Bisnchi identities vanish. Thus the torsion and the curvature are given by W, G, R and from those definitions, W is a chiral superfield with symmetric indices and R is a chiral superfield. We notice that the second Bianchi identities are automatically satisfied.

3.3 Superfield on the Curved Space

3.3.1 Supergravity Multiplet

The vielbein transforms under the general coordinate transformation and local Lorentz transformation as

$$
\delta E_M{}^{\underline{M}} = -\mathcal{D}_M\xi^{\underline{M}} - \xi^{\underline{N}}T_{\underline{NM}}{}^{\underline{M}} + E_M{}^{\underline{N}}L_{\underline{N}}{}^{\underline{M}}, \tag{3.3.1}
$$

where the $\theta = \overline{\theta} = 0$ component of ξ corresponds to gauged supersymmetry. As well as Wess-Zumono gauge, by using this gauge transformation we gauge away the degree of freedom of $E_M{}^{\underline{M}}$ and find $\theta = \overline{\theta} = 0$ components are given by

$$E_M{}^{\underline{M}}(z)|_{\theta=\overline{\theta}=0} = \begin{pmatrix} e_\mu{}^{\underline{\mu}}(x) & \frac{1}{2}\psi_\mu{}^{\underline{\alpha}}(x) & \frac{1}{2}\overline{\psi}_{\mu\underline{\dot{\alpha}}}(x) \\ 0 & \delta_\alpha{}^{\underline{\alpha}} & 0 \\ 0 & 0 & \delta_{\underline{\dot{\alpha}}}{}^{\dot{\alpha}} \end{pmatrix}. \tag{3.3.2}$$

Thus e and ψ generate a supergravity multiplet.

Similarly, from the transformation law of the connection

$$\delta\phi_{M\underline{M}}{}^{\underline{N}} = -\xi R_{\underline{L}M\underline{M}}{}^{\underline{N}} + \phi_{M\underline{M}}{}^{\underline{L}} L_{\underline{L}}{}^{\underline{N}} - (-)^{M(\underline{M}+\underline{L})} L_{\underline{M}}{}^{\underline{L}} \phi_{M\underline{L}}{}^{\underline{N}} - \partial_M L_{\underline{M}}{}^{\underline{N}}, \tag{3.3.3}$$

we find the $\theta = \overline{\theta} = 0$ component of the connection is

$$\phi_{\mu\underline{M}}{}^{\underline{N}}(z)|_{\theta=\overline{\theta}=0} = \omega_{\mu\underline{M}}{}^{\underline{N}}(x), \quad \phi_{\alpha\underline{A}}{}^{\underline{N}}|_{\theta=\overline{\theta}=0} = 0 \tag{3.3.4}$$

If we assume tortion free geometry, the connection $\omega_{\mu\underline{M}}{}^{\underline{N}}$ can be expressed by veilbein e and we can realize the general relativity.

Futhermore by considering the transformation of $\theta = \overline{\theta} = 0$ components ζ of (3.3.1), we find the local supersymmetry transformations

$$\begin{aligned}
\delta e_\mu^{\underline{\mu}} &= i(\psi_\mu \sigma^{\underline{\mu}} \overline{\zeta} - \zeta \sigma^{\underline{\mu}} \overline{\psi}_\mu) \\
\delta\psi_\mu{}^{\underline{\alpha}} &= -2\mathcal{D}_\mu \zeta^{\ \underline{\alpha}} + i e_\mu^{\underline{\mu}} \left\{ \frac{1}{3} M (\epsilon \sigma_{\underline{\mu}} \overline{\zeta})^{\underline{\alpha}} + b_{\underline{\mu}} \zeta^{\underline{\alpha}} + \frac{1}{3} b^{\underline{\nu}} (\zeta \sigma_{\underline{\nu}} \overline{\sigma}_{\underline{\mu}})^{\underline{\alpha}} \right\} \\
\delta M &= -\zeta(\sigma^{\underline{\mu}} \overline{\sigma}^{\underline{\nu}} \psi_{\underline{\mu\nu}} + i b^{\underline{\mu}} \psi_{\underline{\mu}} - i \sigma^{\underline{\mu}} \overline{\psi}_{\underline{\mu}} M) \\
\delta b_{\underline{\alpha\dot{\alpha}}} &= \zeta^{\underline{\delta}} \left\{ \frac{3}{4} \overline{\psi}_{\underline{\alpha\delta\dot{\gamma}\dot{\alpha}}}{}^{\underline{\dot{\gamma}}} + \frac{1}{4} \epsilon_{\underline{\delta\alpha}} \overline{\psi}_{\underline{\gamma\dot{\alpha}\dot{\gamma}}}{}^{\underline{\gamma\dot{\gamma}}} - \frac{i}{2} M^* \psi_{\underline{\alpha\dot{\alpha}\delta}} + \frac{i}{4} (\overline{\psi}_{\underline{\alpha\dot{\rho}}}{}^{\underline{\dot{\rho}}} b_{\underline{\delta\dot{\alpha}}} + \overline{\psi}_{\underline{\delta\dot{\rho}}}{}^{\underline{\dot{\rho}}} b_{\underline{\alpha\dot{\alpha}}} - \overline{\psi}_{\underline{\delta\dot{\alpha}}}{}^{\underline{\dot{\rho}}} b_{\underline{\alpha\dot{\rho}}}) \right\} \\
&\quad - \overline{\zeta}^{\underline{\dot{\delta}}} \left\{ \frac{3}{4} \psi_{\underline{\dot{\delta}\gamma\dot{\alpha}\alpha}}{}^{\underline{\gamma}} + \frac{1}{4} \epsilon_{\underline{\dot{\delta}\dot{\alpha}}} \psi_{\underline{\alpha\dot{\gamma}\gamma}}{}^{\underline{\dot{\gamma}\gamma}} + \frac{i}{2} M \overline{\psi}_{\underline{\alpha\dot{\alpha}\dot{\delta}}} - \frac{i}{4} (\psi_{\underline{\rho\dot{\alpha}}}{}^{\underline{\rho}} b_{\underline{\alpha\dot{\delta}}} + \psi_{\underline{\rho\dot{\delta}}}{}^{\underline{\rho}} b_{\underline{\alpha\dot{\alpha}}} - \psi_{\underline{\dot{\delta}\alpha}}{}^{\underline{\rho}} b_{\underline{\rho\dot{\alpha}}}) \right\},
\end{aligned} \tag{3.3.5}$$

where M and b are the auxiliary fields which correspond to θ^2 or $\overline{\theta}^2$ component fields of E, in other words, the $\theta = \overline{\theta} = 0$ component fields of R and G_a respectively.

3.3.2 Superfields on the Curved Space

Chiral Superfield

Chiral superfields were defined by SUSY covariant derivative D_α in (2.5.13) in flat space. In the curved superspace, we define chiral sperfield by replacing D_α to $\mathcal{D}_{\underline{\alpha}}$

$$\overline{\mathcal{D}}_{\dot{\alpha}}\Phi = 0, \tag{3.3.6}$$

where this definition is local since we define the chiral suprefield with local operator $\overline{\mathcal{D}}_{\underline{\dot{\alpha}}}$. We can find the component fields from the coefficients of θ:

$$\begin{aligned} A &= \Phi|_{\theta=\overline{\theta}=0} \\ \chi_{\underline{\alpha}} &= \frac{1}{\sqrt{2}}\mathcal{D}_{\underline{\alpha}}\Phi|_{\theta=\overline{\theta}=0} \\ F &= -\frac{1}{4}\mathcal{D}_{\underline{\alpha}}\mathcal{D}^{\underline{\alpha}}\Phi|_{\theta=\overline{\theta}=0} \end{aligned} \tag{3.3.7}$$

From

$$\delta\Phi = -\xi^{\underline{M}}\mathcal{D}_{\underline{M}}\Phi, \tag{3.3.8}$$

we can also obtain the supersymmetry tansformation laws

$$\begin{aligned} \delta A &= -\sqrt{2}\zeta^{\underline{\alpha}}\chi_{\underline{\alpha}} \\ \chi_{\underline{\alpha}} &= -\sqrt{2}\zeta_{\underline{\alpha}}F - i\sqrt{2}\sigma^{\mu}_{\underline{\alpha}\underline{\dot{\beta}}}\overline{\zeta}^{\underline{\dot{\beta}}}\hat{D}_{\underline{\mu}}A \\ \delta F &= -\frac{1}{3}\sqrt{2}M^{*}\zeta^{\underline{\alpha}}\chi_{\underline{\alpha}} + \overline{\zeta}^{\underline{\dot{\alpha}}}\left(\frac{1}{6}\sqrt{2}b_{\underline{\alpha}\underline{\dot{\alpha}}}\chi^{\underline{\alpha}} - i\sqrt{2}\hat{D}_{\underline{\alpha}\underline{\dot{\alpha}}}\chi^{\underline{\alpha}}\right), \end{aligned} \tag{3.3.9}$$

where we have defined

$$\hat{D}_{\underline{\mu}}A \equiv e_{\underline{\mu}}{}^{\mu}\left(\partial_\mu A - \frac{1}{\sqrt{2}}\psi_{\mu}{}^{\alpha}\chi_\alpha\right) \tag{3.3.10}$$

Vector Superfield

Next, we define vector superfield. Since vector superfield is defined without derivative, its definition is the same as flat space:

$$V = V^{\dagger}. \tag{3.3.11}$$

As well as the definition of chiral superfield, the component fields of vector field can be obtained such as those of flat space by replacing the covariant derivatives:

$$\begin{aligned} C &= V|_{\theta=\bar{\theta}=0} \\ \phi_{\underline{\alpha}} &= -i\mathcal{D}_{\underline{\alpha}}V|_{\theta=\bar{\theta}=0} \\ \bar{\phi}_{\underline{\dot{\alpha}}} &= i\overline{\mathcal{D}}_{\underline{\dot{\alpha}}}V|_{\theta=\bar{\theta}=0} \\ M &= \frac{i}{4}(\mathcal{D}^{\underline{\alpha}}\mathcal{D}_{\underline{\alpha}} - \overline{\mathcal{D}}_{\underline{\dot{\alpha}}}\overline{\mathcal{D}}^{\underline{\dot{\alpha}}})V|_{\theta=\bar{\theta}=0} \\ N &= \frac{1}{4}(\mathcal{D}^{\underline{\alpha}}\mathcal{D}_{\underline{\alpha}} + \overline{\mathcal{D}}_{\underline{\dot{\alpha}}}\overline{\mathcal{D}}^{\underline{\dot{\alpha}}})V|_{\theta=\bar{\theta}=0} \\ v_{\underline{\alpha\dot{\alpha}}} &= -\frac{1}{2}[\mathcal{D}_{\underline{\alpha}}, \overline{\mathcal{D}}_{\underline{\dot{\alpha}}}]V|_{\theta=\bar{\theta}=0} \end{aligned} \tag{3.3.12}$$

Spinor Superfield

Finally we would like to define spinor superfield. Since spinor superfield is a chiral superfield, it should satisfy the chiral condition

$$\overline{\mathcal{D}}_{\underline{\dot{\beta}}}W_{\underline{\alpha}} = 0. \tag{3.3.13}$$

Futhermore in a flat space, spinor superfield is defined with three covariant derivatives and the covariant derivative satisfies anti-commutation relation

$$\{\mathcal{D}_{\underline{\alpha}}, \mathcal{D}^{\underline{\beta}}\} = -R_{\underline{\alpha\gamma}}{}^{\underline{\gamma\beta}}. \tag{3.3.14}$$

Thus spinor field is defined by

$$W_{\underline{\alpha}} = -\frac{1}{4}(\overline{\mathcal{D}}_{\underline{\dot{\beta}}}\overline{\mathcal{D}}^{\underline{\dot{\beta}}} - 8R)\mathcal{D}_{\underline{\alpha}}V, \tag{3.3.15}$$

whose component fields are given by

$$\begin{aligned} \lambda_{\underline{\alpha}} &= iW_{\underline{\alpha}}|_{\theta=\bar{\theta}=0} \\ D &= -\frac{1}{2}\mathcal{D}^{\underline{\alpha}}W_{\underline{\alpha}}|_{\theta=\bar{\theta}=0}. \end{aligned} \tag{3.3.16}$$

The gauge degree of freedom is given by a chiral field Λ and the gauge transfrmation is

$$\delta V = \Lambda + \Lambda^{\dagger}, \tag{3.3.17}$$

and this transformation law keeps W invariant.

Global Grassmann Variable

In the above discussion, we define the superfields and those transformation by using Grassmann coordinates θ. In the following discussion, in order to simplify the calculations we introduce a new Grassmann coordinates Θ defined by

$$\Phi = A(x) + \sqrt{2}\Theta^\alpha \chi_\alpha(x) + \Theta^\alpha\Theta_\alpha F(x) \tag{3.3.18}$$

and define the supersymetry transformation as

$$\delta\Phi = -\eta^M(x,\Theta)\partial_M\Phi. \tag{3.3.19}$$

This is a same formulation as the flat spacetime supersymmetry in 2.5.13. The components fields A, χ and F are defined by covariant derivative $\mathcal{D}$ and so if we expand the superfield in term of θ, (**??**) is not derived. Therefore we redefine the global superspace as $x^M = (x^\mu, \Theta^\alpha, \overline{\Theta}_{\dot\alpha})$. Since Θ is also a grassmanian variable, we expand η^M in terms of Θ:

$$\eta^M(x,\Theta) = \eta^M_{(0)}(x) + \Theta^\alpha \eta^M_{(a)\alpha}(x) + \Theta^\alpha\Theta_\alpha \eta^M_{(2)}(x). \tag{3.3.20}$$

Substituting this expression into (3.3.19) and comparing with (3.3.9), we find the relations between η and ξ:

$$\begin{aligned}\eta^\mu &= 2i\Theta\sigma^\mu\overline{\zeta} + \Theta\Theta\overline{\psi}_\nu\overline{\sigma}^\mu\sigma^\nu\overline{\zeta} \\ \eta^\alpha &= \zeta^\alpha - i\Theta\sigma^\mu\overline{\zeta}\psi^\alpha_\mu \\ &\quad + \Theta^2\left\{\frac{M^*}{3}\zeta^\alpha + \frac{1}{6}b_{\underline{\mu}}(\epsilon\sigma^{\underline{\mu}}\overline{\zeta})^\alpha - i\omega^{\alpha\beta}_\mu(\sigma^\mu\overline{\zeta})_\beta - \frac{1}{2}\psi^\alpha_\nu(\overline{\psi}_\mu\overline{\sigma}^\nu\sigma^\mu\overline{\zeta})\right\}.\end{aligned} \tag{3.3.21}$$

In order to construct local supersymmetric invariant action, we define the superfield Δ called chiral density transforming to total derivative:

$$\delta\Delta = -\partial_M(\eta^M\Delta(-)^M). \tag{3.3.22}$$

This is useful because the product of chiral density and chiral superfields becames also chiral density:

$$\delta\Delta\Phi = -\partial_M(\eta^M\Delta\Phi(-)^M). \tag{3.3.23}$$

Thus we can easily construct a local supersymmetric action such as

$$S = \int d^4x d^2\Theta \Delta f(\Phi) \tag{3.3.24}$$

where f is a arbitrary function of chiral superfields Φ.

3.4 Supergravity Action

3.4.1 Minimal Chiral Supergravity

In a flat space, the general renormalizable Lagrangian for the only chiral superfields is given by

$$\mathcal{L} = \int d^2\theta d^2\overline{\theta}^2 \Phi_I^\dagger \Phi_I + \left[\int d^2\theta \left(a_I \Phi_I + \frac{1}{2} m_{IJ} \Phi_I \Phi_J + \frac{1}{3} g_{IJK} \Phi_I \Phi_J \Phi_K \right) + \text{h.c.} \right]. \tag{3.4.1}$$

where if we carry out the θ intagration, the first term is rewritten as $\overline{DD}\Phi_I^\dagger \Phi_I$. In this section, we extend this model to curved space which is called the minimal chiral supergravity model.

First we consider the invariant action involving the only supergravity multiplet which is given by the product of chiral density $\mathcal{E}$ and the curvature R

$$\begin{aligned} \mathcal{L}_{\text{SUGRA}} &= -\frac{6}{\kappa^2} \int d^2\Theta \mathcal{E} R + \text{h.c.} \\ &= e \left[-\frac{1}{2}\mathcal{R} - \frac{1}{3}|M|^2 + \frac{1}{3} b^{\underline{\mu}} b_{\underline{\mu}} + \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \left(\overline{\psi}_\mu \overline{\sigma}_\nu \tilde{D}_\rho \psi_\sigma - \psi_\mu \sigma_\nu \tilde{D}_\rho \overline{\psi}_\sigma \right) \right], \end{aligned} \tag{3.4.2}$$

where $\kappa^2 = 8\pi G_N$ is a gravitational coupling constant and in the following we set κ to 1, $\tilde{D}_\mu$ has been defined by

$$\tilde{D}_\nu \psi_\mu{}^\alpha = \partial_\nu \psi_\mu{}^\alpha + \psi_\mu{}^\beta \omega_{\nu\beta}{}^\alpha, \tag{3.4.3}$$

and $\mathcal{R}$ is a curvature tensor for the 4 dimensional spacetime.

We can easily extend (3.4.1) to the supergravity action by replacing

$$\theta \to \Theta, \quad d^2\theta \to d^2\Theta 2\mathcal{E}, \quad \overline{DD} \to \overline{\mathcal{D}\mathcal{D}} - 8R. \tag{3.4.4}$$

This gives the action in curved superspace:

$$\begin{aligned} \mathcal{L} = \int d^2\Theta 2\mathcal{E} \Big[&-3R - \frac{1}{8}(\overline{\mathcal{D}\mathcal{D}} - 8R)\Phi_I^\dagger \Phi_I - \frac{1}{8}(\overline{\mathcal{D}\mathcal{D}} - 8R)(c_I \Phi_I + \overline{c}_I \Phi_I^\dagger) \\ &+ d + a_I \Phi_I + \frac{m_{IJ}}{2} \Phi_I \Phi_J + \frac{g_{IJK}}{3} \Phi_I \Phi_J \Phi_K \Big] + \text{h.c.} \end{aligned} \tag{3.4.5}$$

This Lagrangian describes the minimal chiral supergravity model.

3.4.2 General Chiral Model

In this section, by extending (2.6.5) to local supersymmetry, we find the general chiral supergravity model. Applying (3.4.4) to (2.6.5), we obtain the general chiral supergravity model

$$\mathcal{L} = \frac{1}{\kappa^2}\int d^2\Theta 2\mathcal{E}\left[\frac{3}{8}(\overline{\mathcal{D}\mathcal{D}} - 8R)e^{-\frac{\kappa^2}{3}K(\Phi,\Phi^\dagger)} + \kappa^2 W(\Phi)\right] + \text{h.c.} \tag{3.4.6}$$

where if we expand with κ^2, we realize the minimal chiral model. If we carry out the Θ integration and substitute the equation of motion of F^I, we find

$$V_F(\Phi^I, \Phi^{J*}) = e^{\kappa_4^2 K}(K^{I\overline{J}} D_I W D_{\overline{J}} W^* - 3\kappa_4^2 |W|^2), \tag{3.4.7}$$

which is called F-term potential.

The function $g_{I\overline{J}}$ can be regarded as a geometric way: if we regard the scalar fields $A^I, \overline{A}^J$ parametrize a complex manifold, the Kähler metric is the metric on this manifold.

Kähler metric determins kinetic terms and it is defined by the holomorphic and anti-holomorphic derivatives of the Kähler potential. Thus Kähler potential has a uncertainty of holomorphic and anti-holomorphic functions

$$\delta K(\Phi^I, \overline{\Phi}^J) = F(\Phi^I) + \overline{F}(\overline{\Phi}^I) \tag{3.4.8}$$

which $F(\Phi^i)$ is an arbitrary holomorphic function of Φ^I and this deformation is called Kähler deformation. Since the Christoffel symbol and the curvature tensor of Kähler manifold are defined by Kähler metric, those are invariant under the Kähler deformation. However the U(1) connection is not invariant:

$$\begin{aligned} \mathcal{D}_\mu \chi^I &\to \mathcal{D}_\mu - \frac{i}{2}\partial_\mu(\operatorname{Im} F)\chi^I \\ \tilde{\mathcal{D}}_\mu \psi_\nu &\to \tilde{\mathcal{D}}_\mu \psi_\nu + \frac{i}{2}\partial_\mu(\operatorname{Im} F)\psi_\nu \end{aligned} \tag{3.4.9}$$

In order to remain the supergravity action invariant, the Kähler deformation must induce Weyl rotation for the spinor fields,

$$\begin{aligned} \chi^I &\to e^{\frac{i}{2}\operatorname{Im} F}\chi^I \\ \psi_\mu &\to e^{-\frac{i}{2}\operatorname{Im} F}\psi_\mu. \end{aligned} \tag{3.4.10}$$

With these rules, kinetic terms are invariant under the Kähler-Weyl deformation.

Since the supergravity Lagrangian has a contribution e^K, their invariance is not automatic under the Kähler-Weyl deformations. If we impose Kähler-Weyl deformation invariant to the scalar potential, the superpotential must transform as

$$\delta W(\Phi^I) = e^{-\kappa^2 F(\Phi^I)} W(\Phi^I). \tag{3.4.11}$$

Since Lagrangian includes cuvature tensor of curved superspace, we need to cancel out the additional terms induced by the Kähler-Weyl deformations. For this purpose, We define super-Weyl transformation

$$\begin{aligned}\delta E_M{}^{\mu} &= (\Sigma + \overline{\Sigma}) E_M{}^{\mu} \\ \delta E_M{}^{\alpha} &= (2\overline{\Sigma} - \Sigma) E_M{}^{\alpha} + \frac{i}{2} E_M{}^{\mu} (\epsilon \sigma_{\underline{\mu}})^{\alpha}_{\underline{\dot{\alpha}}} \overline{\mathcal{D}}^{\dot{\alpha}} \overline{\Sigma},\end{aligned} \tag{3.4.12}$$

where Σ are chiral superfields.

The super-Weyl transformations of the matter fields are also given by Σ:

$$\begin{aligned}\delta \Phi &= w \Sigma \Phi \\ \delta V &= w' (\Sigma + \overline{\Sigma}) V,\end{aligned} \tag{3.4.13}$$

where w and w' are the Weyl weight for the superfields. Using these super-Weyl transformations, the Lagrangian transforms to

$$\delta \mathcal{L} = \int d^2\Theta 2\mathcal{E} \left[\frac{3}{4} (\overline{\mathcal{D}\mathcal{D}} - 8R)(\Sigma + \overline{\Sigma}) e^{-K/3} + 6\Sigma W \right] + \text{h.c.} \tag{3.4.14}$$

Whereas the Lagrangian transforms under Kähler-Weyl deformations

$$\delta \mathcal{L} = \int d^2\Theta 2\mathcal{E} \left[-\frac{1}{8} (\overline{\mathcal{D}\mathcal{D}} - 8R)(F + F^*) e^{-K/3} - FW \right] + \text{h.c.} \tag{3.4.15}$$

Therefore if $F = 6\Sigma$, the supergravity Lagrangian is ivnariant. This fact means that superspace Kähler-Weyl deformations induce local Weyl rotations of the component fields.

We take $F(\Phi^I)$ to

$$\kappa^2 F(\Phi^I) = \log W(\Phi^I), \tag{3.4.16}$$

then supergravity action only depends on one function

$$\mathfrak{G} = \kappa^2 K + \log |W|^2. \tag{3.4.17}$$

This function is called Kähler function.

3.4.3 Gauge Invariant Supersymmetric Theory

The general supersymmetric Lagrangian for chiral superfields in flat superspace is given by (2.6.5). In this section, we extend this Lagrangian to gauge invariant Lagrangian. The Lagrangian (2.6.5) is Kähler-Weyl invariant and this deformation induces super-Weyl transformations which are defined by chiral superfield Σ. Thus by gauging Kähler-Weyl transformation we introduce the gauge symmetry.

Under the isometry group G, K and W transform as

$$\begin{aligned} \delta K &= [\epsilon^a X^a + \epsilon^{*(a)} X^{*(a)}]K \\ \delta W &= \epsilon^{(a)} X^{(a)} W, \end{aligned} \tag{3.4.18}$$

where $X^{(a)}$ are the Killing vectors.In order to keep the action invariant, the variation of W must vanish. Whereas the variation of K dose not need to vanish and it can be written as

$$\delta K = \epsilon^{(a)} F^{(a)} + \epsilon^{*(a)} - i\epsilon(\epsilon^{(a)} - \epsilon^{*(a)}) D^{(a)}, \tag{3.4.19}$$

where $F^{(a)} = X^{(a)} K + iD^{(a)}$. From this expression, the real part of (3.4.19) is just a Kähler transformation, while the imaginary part correspnds to Killing potential $D^{(a)}$ which is not invariant. Indeed introducing superfield $\Lambda^{(a)}$ whose lowest component $\epsilon^{(a)}$, the variation of the Lagrangian is given by

$$\delta \mathcal{L} = -i \int d^2\theta d^2\overline{\theta}^2 (\Lambda^{(a)} - \Lambda^{\dagger(a)}) D^{(a)}. \tag{3.4.20}$$

In order to cancel out this term, we introduce the following additional term:

$$\delta \Gamma = i(\Lambda^{(a)} - \Lambda^{\dagger(a)}) D^{(a)}. \tag{3.4.21}$$

Thus gauge invariant Lagrangian is given by

$$\mathcal{L} = \int d^2\theta d^2\overline{\theta} \left(K(\Phi^I, \Phi^{\dagger J}) + \Gamma(\Phi^I, \Phi^{\dagger J}, V^{(a)}) \right) + \int d^2\theta W(\Phi^I) + \text{h.c.} \tag{3.4.22}$$

Considering the constraint that Γ transforms as (3.4.21) and if there is no gauge invariance, $\Gamma(\Phi^I, \Phi^{\dagger J}, 0) = 0$, we find the Γ explicitly

$$\Gamma(\Phi^I, \Phi^{\dagger J}, V^{(a)}) = \int_0^1 d\alpha e^{(1/2)\alpha V^{(a)}(X^{(a)} - X^{*(a)})} V^{(b)} D^{(b)}. \tag{3.4.23}$$

The transformation laws under the local isometries G are given by

$$\delta \Phi^I = \Lambda^{(a)} X^{I(a)}(\Phi^J) \tag{3.4.24}$$

$$\delta e^V = -\Lambda^{\dagger(a)} T^{(a)} e^V + ie^V \Lambda^{(a)} T^{(a)}. \tag{3.4.25}$$

The Lagrangian (3.4.22) leads to the following scalar potential:

$$V_{\text{scalar}} = \frac{1}{2}g^2 D^{(a)2} + g^{I\overline{J}} D_I W D_{\overline{J}} W^*. \tag{3.4.26}$$

The first term is known as D-term potential.

$X^{(a)}$ and $D^{(a)}$ depend on the gauge group. If $G = U(n)$, they are given by $X^{I(a)} = -iT_J^{(a)I} a^J$ and $D^{(a)} = a^{*I} T_J^{(a)I} a^J$, and hence Γ can be written as

$$\Gamma(\Phi^I, \Phi^{\dagger J}, V^{(a)}) = \int d^2\theta d^2\overline{\theta} \Phi^\dagger (e^V - 1)\Phi. \tag{3.4.27}$$

3.4.4 Gauge Invariant Supergravity

In the previous section, we fonud that the gauge invariant theory is constructed by introducing the counterterm Γ to Kähler potential K

$$\begin{aligned}\mathcal{L} = \int d^2\Theta 2\mathcal{E} \Big[&\frac{3}{8}(\mathcal{D}^2 - 8R) e^{-\frac{1}{3}[K(\Phi,\Phi^\dagger) + \Gamma(\Phi,\Phi^\dagger,V)]} \\ &+ \frac{1}{16g^2} H_{(ab)}(\Phi) W^{(a)} W^{(b)} + W(\Phi) \Big] + \text{h.c.}\end{aligned} \tag{3.4.28}$$

where we have defined the field strength as

$$W_{\underline{\alpha}} \equiv W_{\underline{\alpha}}^{(a)} T^{(a)} = -\frac{1}{4}(\overline{\mathcal{D}}^2 - 8R) e^{-V} \mathcal{D}_{\underline{\alpha}} e^V. \tag{3.4.29}$$

Since the Lagrangian is described by superfields, the local supersymmetric invariance is manifest.

However the gauge symmetry is not manifest in this Lagrangian. In order to confirm the gauge invariance, we consider the gauge transformations

$$\begin{aligned}\delta K &= \Lambda^{(a)} F^{(a)} + \Lambda^{\dagger(a)} F^{\dagger(a)} - i[\Lambda^{(a)} - \Lambda^{\dagger(a)}] D^{(a)} \\ \delta\Gamma &= i[\Lambda^{(a)} - \Lambda^{\dagger(a)}] D^{(a)} \\ \delta W &= \Lambda^a X^{(a)} W.\end{aligned} \tag{3.4.30}$$

where $F^{(a)} = X^{(a)} K + iD^{(a)}$. Then we find the non-vanishing terms

$$\delta_G \mathcal{L} = \int d^2\Theta 2\mathcal{E} \left[-\frac{1}{8}(\overline{\mathcal{D}}^2 - 8R)[\Lambda^{(a)} F^{(a)} + \Lambda^{\dagger(a)} F^{\dagger(a)}] e^{-(K+\Gamma)/3} + \Lambda^{(a)} X^{(a)} W \right] + \text{h.c.} \tag{3.4.31}$$

The gauge transformation is derived from Kähler transformation and hence in order to keep the Lagrangian invariant, we should carry out the super-Weyl transformation same time

$$\delta_W \mathcal{L} = \int d^2\Theta 2\mathcal{E} \left[\frac{3}{4}(\overline{\mathcal{D}}^2 - 8R)[\Sigma + \overline{\Sigma}]e^{-(K+\Gamma)/3} + 6\Sigma P\right] + \text{h.c.} \tag{3.4.32}$$

Thus we find that the non-vanishing terms are canceled if

$$\Sigma = \frac{1}{6}\Lambda^{(a)} F^{(a)}, \tag{3.4.33}$$

$$X^{(a)} = -F^{(a)}. \tag{3.4.34}$$

The second condition is derived from the superpotential term and this is a non-triviral constraint for the gauge invariance.

The scalar potential is given by

$$V_{\text{scalar}} = \frac{1}{2} g^2 F^{(a)2} + e^K [g^{I\overline{J}}(D_I W)(D_J W)^* - 3|W|^2]. \tag{3.4.35}$$

The first term is D-term potential depending on the gauge group, and the second term is F-term potential. In this paper, this formula for the potential plays an impoptant role.

4 String theory

In this section we review a string theory briefly [26,55].

4.1 Point Particle Action

4.1.1 Classical Action

First, we consider a point particle propageting in D-dimensional Minkowski spacetime. The trajectory of the particle is a one-dimensional object, which is called wirld line. The relativistically invariant action is given by the length of the world line:

$$S = -m \int ds, \tag{4.1.1}$$

where m is the mass of the particle. Let the spacetime coordinate be x^μ,$\mu = 0, 1, ..., D-1$, then the line segment is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \tag{4.1.2}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, ..., 1)$ is the Minkowski metric. Furthermore, if we take τ to be the parameter along the world line, the action is rewirtten as

$$S = -m \int d\tau \sqrt{-\dot{x^\mu} \dot{x^\nu} \eta_{\mu\nu}}. \tag{4.1.3}$$

The motion of the particle is determined by maximizing the action. This can be seen by examining $\delta^2 L / \delta x^\nu \delta x^\nu$.

We consider the reparameterisation $\tau \to \tau' = f(\tau)$ in this action. Since this transformation only changes the way to measure the coordinate, the coordinates x^μ should be invariant:

$$x^\mu \to x^{\mu\prime}(\tau') = x^\mu(\tau). \tag{4.1.4}$$

That is, $x^\mu(\tau)$ behaves as a scalar under this transformation and the action should be invariant. In fact, if we consider the infinitesimal reparameterisation

$$\tau' = \tau - f(\tau), \tag{4.1.5}$$

the infinitesimal transformation of x^μ is given by

$$\delta x^\mu = x'^\mu(\tau) - x^\mu(\tau) = f(\tau) \dot{x}^\mu. \tag{4.1.6}$$

If we use this transformation rule and calculate δS, we find that $\delta S = 0$. Such a transformation where the parameters of the transformation (in this case $f(\tau)$) depend on the coordinates is called a local transformation, and the symmetry is called local symmetry. If there are the local symmetries, the action has a non-trivial relation (constraint) between the variables. The reparameterisation is a local symmetry, and the corresponding global symmetry is a translational symmetry along the world line.

The constraint is a relation between coordinates. The canonical form is used as a method in order to find the constraints. By performing canonical transformation, we can find whether there are constraints or not. If there are some relations between the variables, there is a loss of rank in the Jaccobi matrix, and the determinant becomes zero. In other words, when the determinant is zero, a constraint exists. Therefore, we use the canonical form of the Lagrangian.

The canonical conjugate corresponding to x^μ is determined by

$$p^\mu = \frac{\partial L}{\partial \dot{x}_\mu(\tau)} = \frac{m\dot{x}^\mu}{\sqrt{-\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu}}} \tag{4.1.7}$$

This system contains one constraint, because Jacobi matrix has the zero-determinant. Indeed, this p^μ satisfies a non-trivial relation

$$p^\mu p_\mu + m^2 = 0. \tag{4.1.8}$$

This relation is a constraint corresponding to the reparameterization. The Hamiltonian in this system is

$$H = p^\mu \dot{x}_\mu - L = 0. \tag{4.1.9}$$

Since the reparameterisation is a local translational symmetry and the corresponding color symmetry is a time translational symmetry, the Hamiltonian is included in the constraint. This means that the motion of the particle is completely determined by the constraint.

Next, we construct an action that is classically equivalent to this nonlinear action (4.1.3). The action (4.1.3) has the field x^μ, Lorentz symmetry of spacetime, and reparameterization invariance. If these symmetries are present and the fields satisfy the same equations of motion, the action is classically equivalent. Here we use Noether method to construct a gauge-invariant theory. In this method, we consider the action which has the global symmetry and make the global symmetry parameter dependent on spacetime. Finally we construct the gauge theory. Since the color symmetry of the reparameterization is a time-translational symmetry, we first write down the action with these symmetries.

$$L_0 = \frac{1}{2}\dot{x}^\mu \dot{x}_\mu \tag{4.1.10}$$

This Lagrangian is invariant under the Lorentz and time-translational symmetries as requested:

$$\tau \to \tau + a; \quad x^\mu \to x^\mu + a\dot{x}^\mu. \tag{4.1.11}$$

Next, let the parameter a^μ be $a^\mu(\tau)$. In fact, if we perform this transformation

$$\delta L_0 = \frac{1}{2}\dot{a}\dot{x}^\mu\dot{x}_\mu + \frac{d}{d\tau}\left(\frac{1}{2}a\dot{x}^\mu\dot{x}_\mu\frac{d}{d\tau}\right). \tag{4.1.12}$$

Thus it is no longer invariant. So, we introduce the new term consisted of the infinitesimal quantity κ and the new field ψ:

$$\frac{1}{2}\kappa\psi\dot{x}^\mu\dot{x}_\mu. \tag{4.1.13}$$

Futhermore we define the new field transformation

$$\delta\psi = \dot{a}/\kappa. \tag{4.1.14}$$

Then

$$\delta L_1 = \left(\delta L_0 + \frac{1}{2}\kappa\psi\dot{x}^\mu\dot{x}_\mu\right) = -\frac{1}{2}\kappa\psi\dot{a}\dot{x}^\mu\dot{x}_\mu - \frac{1}{2}\kappa\psi\frac{d}{d\tau}(a\dot{x}^\mu\dot{x}_\mu). \tag{4.1.15}$$

Since κ is an infinitesimal quantity, L_1 is now invariant at the zero order of κ. To make it invariant to the first order of κ, add $a\dot{\psi}$ to $\delta\psi$ and add L_1 to

$$\frac{1}{4}\kappa^2\psi^2\dot{x}^\mu\dot{x}_\mu \tag{4.1.16}$$

and if we write this as L_2, then

$$\delta L_2 = \frac{1}{4}\kappa^2\psi^2\dot{a}\dot{x}^\mu\dot{x}_\mu \tag{4.1.17}$$

and became invariant at first order in κ. This sequence of processes has a certain regularity. By repeating this infinitely many times and defining $\exp\{(-\kappa\psi)\} = e$, we get

$$L_f = \frac{1}{2}e^{-1}\dot{x}^\mu\dot{x}_\mu \tag{4.1.18}$$

This Lagrangian is completely invariant under reparameterization. The transformation rule for the field e follows from the transformation rule for ψ:

$$\delta e = e\dot{a} + a\dot{e}. \tag{4.1.19}$$

Since this action describe a massless particle, we can introduce a mass term

$$S=\int\left(\frac{1}{2}e^{-1}\dot{x}^{\mu}\dot{x}_{\mu}-em^{2}\right). \tag{4.1.20}$$

By substituting the equation of motion for the field e into the action (4.1.20), and eliminating e, we realize the original action (4.1.3). Also, as is clear from the form of this action, this action can describe massless particles.

This action is also invariant under the following transformation:

$$\delta x^{\mu}=w^{\mu}\quad\delta e=2e\partial_{\nu}w^{\nu}, \tag{4.1.21}$$

which is called a conformal transformation. This symmetry is obvious since it is included reparameterization symmetry.

4.1.2 BRST Quantization

In quantum theory, two variables form a pair and satisfy a commutation relation. If there are the constraints in classical theory, the variables are not independent and which means that one variable appears in multiple pairs. Therefore, if we carry out a quantization, it is necessary to handle constraints well. One way is to write down the classical theory with completely independent variables. The other way is to derive all the states in the quantum theory and pick up only those states that satisfy the constraints. Since the former is generally difficult because it requires solving the constraints, the latter is used here.

The system we are considering now has reparameterisation invariance, and so the corresponding constraints exist. The corresponding color symmetry is time-translation invariant, which means that the Klein-Goldon equation is included in the constraints. Therefore, the constraint condition is sufficient to impose on the state in quantum theory.

Ghost Action

In the following, we explain a way of quantization by replacing the local symmetry to a global symmetry by introducing ghost fields. The original action is given by (4.1.20). This action has a reparameterization invariant (4.1.6), (4.1.19). The symmetry parameter f depends on τ. So we formally consider this parameter $f(\tau)$ as a field

$$f(\tau)\rightarrow\Lambda c(\tau) \tag{4.1.22}$$

Since f is real and has no index, $c(\tau)$ is a real scalar field. Also, since f is Grassmann even, the product of Λ and $c(\tau)$ should be Grassmann even. We take each of them to be a Grassman-odd. Thus, $c(\tau)$ is a Grassmann-odd scalar field, which is called a ghost field. This replacement allows us to define a new transformation.

$$\delta x^{\mu} = \Lambda c(\tau)\dot{x}^{\mu}, \quad \delta e = \frac{d}{d\tau}(\Lambda c(\tau)e) \tag{4.1.23}$$

This transformation is a global symmetry, called BRST symmetry. In other words, by introducing a new field $c(\tau)$, we are able to create a global symmetry corresponding to the original local symmetry. While the local symmetry must be fixed, the BRST symmetry is a global symmetry and so can be treated as such in quantum theory. Since the BRST transformation is regarded the reparameterisation symmetry, the action is BRST-invariant automatically. Also, since the form of the action remains unchanged, the action still has a reparameterisation symmetry. Therefore, in order to fix the reparameterisation symmetry, we introduce a gauge fixing term.

$$S^{gf} = \int d\tau \lambda \log e \tag{4.1.24}$$

This term breaks the reparameterisation symmetry.

Since new fields c and λ are introduced in the procedure here, we consider the properties of these fields. Since the ghost field c is related to x by the BRST transformation, the ghost field c is dynamical. In order to describe the kinetic terms, we introduce a b-field that is Hermitian by product with c. Since c is grassmann odd, it obeys the anti-commutation relation by statistical properties, and b is also a grassmann odd scalar field. Using c and b, we write down the kinetic terms as in the spinor field

$$b\partial c \tag{4.1.25}$$

Since the ghost field is coupled to gravity from the BRST transformation of e, we replace it with the covariant derivative

$$S^{gh} = -\int d\tau b D_{\tau} c \tag{4.1.26}$$

$$D_{\tau} c = \dot{c} + \frac{d \log e}{d\tau} c \tag{4.1.27}$$

From the above discussion, the action for the ghost field is

$$S^{BRST} = S^{orig} + S^{gf} + S^{gh} = \int d\tau \left[\frac{1}{2}(e^{-1}\dot{x}^{\mu}\dot{x}_{\mu} - m^2 e) + \lambda \log e - b D_{\tau} c \right] \tag{4.1.28}$$

This action is no longer reparameterisation invariant. But the existence of a BRST symmetry corresponding to this local symmetry is also nontrivial due to the kinetic term for the ghost field and gauge fixing term.

We now consider the BRST transformation law of the field. First, the transformations for x and e are determined by replacing the parameter f with c. Second, since λ is a Lagrangian multiplier, it does not lead to the dynamical fields as a transformation. Therefore we set

$$\delta_B \lambda = 0. \tag{4.1.29}$$

Since the ghost field c is a parameter of local symmetry in origin, we only need to look at the change in the parameter. If we carry out the double reparameterisation to x, we find

$$f_{12} = -f_1 \dot{f}_2 + f_2 \dot{f}_1. \tag{4.1.30}$$

Therefore the transformation law for the ghost c is given by

$$\delta_B c = (\Lambda c)\dot{c}, \tag{4.1.31}$$

and this can be rewritten as

$$\delta_B c = -\frac{\Lambda}{2}\{c, \dot{c}\} \tag{4.1.32}$$

The remaining field b is uniquely determined by keeping the action BRST invariant:

$$\delta_B b = \Lambda \lambda. \tag{4.1.33}$$

From those transformation laws, we can rewrite the action $S^{gf} + S^{gh}$ to

$$S^{gf} + S^{gh} = \int d\tau \delta_B (b \log e). \tag{4.1.34}$$

Since S^{orig} is BRST invariant due to reparameterisation invariance, and $S^{gf} + S^{gh}$ is clearly invariant due to nilpotency, S^{BRST} is BRST invariant.

If we calculate the equation of motion for λ and take it on-shell, we obtain

$$S^{final} = \int d\tau \left(\frac{1}{2}\dot{x}^\mu \dot{x}_\mu - \frac{1}{2}m^2 - b\dot{c} \right). \tag{4.1.35}$$

Ghost Number Symmetry

Here, if we consider the transformation to replace the fields b and c

$$c \to b, \quad b \to c, \tag{4.1.36}$$

the action is invariant. This is because the field c and b are arbitrarily introduced. Thus we can suppose $O(2) \simeq U(1)$ symmetry and scale transformation is inclided in $O(2)$ symmetry:

$$b \to e^{-\theta} b, \quad c \to e^{\theta} c. \tag{4.1.37}$$

Indeed the action is invariant under this transformation. If we examine the Poisson brackets of c and b, we find

$$\{c, b\} = 1, \quad \{b, b\} = 0 = \{c, c\} \tag{4.1.38}$$

If we set $a_1 = c, a_2 = b$, then

$$\{a_i, a_j\} = g_{ij} \tag{4.1.39}$$

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.1.40}$$

This algebra form a Clifford algebra if the metric is δ_{ij}. By diagonalizing g_{ij} and complexfing b and c, we realize U(1) algebra. Thus the conserved charge Q_{fp} corresponding to this transformation is called ghost charge and defines the ghost number of the field.

BRST Charge

The conserved charge corresponding to the BRST symmetry is given by

$$Q_B = c(p^2 + m^2). \tag{4.1.41}$$

This means that the constraint for the reparameterisation is actually a conserved charge. Also, $Q_B^2 = 0$ is satisfied, and the BRST charge satisfies nilpotency. In other words, the fields have the property of becoming zero after double BRST transformations.

Quantization

Quantization is performed by the Derac rule. The Poisson brackets for the fields are

$$\{x^\mu, p^\nu\} = \eta^{\mu\nu}, \{c, b\} = 1. \tag{4.1.42}$$

The Hamiltonian is

$$H = p^2 + m^2. \tag{4.1.43}$$

This is proportional to Q_B since Hamiltonian is the conserved charge for translation which is a color symmetry corresponding to the reparameterisation. Replace the Poisson brackets with the commutation relation

$$[x^\mu, p^\nu] = \eta^{\mu\nu}, \{c, b\} = i \tag{4.1.44}$$

and we adopt following expressions for these relations:

$$\begin{aligned} x^\mu &\to x^\mu, \qquad p^\mu \to -i\frac{\partial}{\partial x^\mu} \\ c &\to c, \qquad\quad b \to i\frac{\partial}{\partial c}. \end{aligned} \tag{4.1.45}$$

Futhermore by using these expressions, we can rewrite the BRST charge and define a quantum state:

$$Q_B = c(-\partial^2 + m^2), \tag{4.1.46}$$

$$\Psi = \Psi(x, c). \tag{4.1.47}$$

Expanding by c and using Grassmann property of the ghost field, we find

$$\Psi(x, c) = \psi(x) + c\phi(x). \tag{4.1.48}$$

The constraint is given by the BRST charge because it is inherently a constraint:

$$Q\Psi = 0. \tag{4.1.49}$$

That is

$$(-\partial^2 + m^2)\psi = 0. \tag{4.1.50}$$

From the nilpotency, the wave function Ψ has a ambiguities:

$$\Psi \to \Psi + Q_B\chi. \tag{4.1.51}$$

Therefore, the degrees of freedom of ϕ can be eliminated using χ. In fact, the degrees of freedom of $Q_B\chi$ correspond to the gauge degrees of freedom, and the arbitrariness of the gauge symmetry is expressed by the nilpotency.

The ghost fields are the Grassmann fields and BRST transformation swaps Grassmann even fields and Grasmann odd fields, so we regard BRST symmetry as a kind of supersymmetry. Thus we can use the superspace formalism [56].

4.2 Super Particle

Here, we introduce supersymmetry to describe fermions. At this point, we find that there are two ways to introduce supersymmetry. We explain them respectively.

4.2.1 Supersymmetry for the World Line

We introduce fermions by imposing supersymmetry on the point-particle action (4.1.10) [57,58]. There are two bosonic parameters, τ and μ. Here, we impose a supersymmetry on τ and introduce the corresponding fermion χ^μ. Since τ is one-dimensional, the Clifford algebra is one-dimension and its representation is also one-dimensional. In other words, the spinor χ^μ is one dimensional spinor. The supersymmtric action is given by

$$S^0 = \frac{1}{2}\int d\tau(\dot{x}^\mu \dot{x}_\mu - i\chi^\mu \dot{\chi}_\mu). \tag{4.2.1}$$

There two global symmetries in this action. One of them is a global supersymmetry

$$\delta x^\mu = i\varepsilon\chi^\mu, \quad \delta\chi^\mu = \dot{x}^\mu\varepsilon. \tag{4.2.2}$$

The other is a time translation symmetry

$$\delta x^\mu = a\dot{x}^\mu, \quad \delta\chi^\mu = a\dot{\chi}^\mu. \tag{4.2.3}$$

These two transformations form a closed algebra

$$[\delta_{s1}, \delta_{s2}] = 2i\varepsilon_2\varepsilon_1\frac{d}{d\tau} \tag{4.2.4}$$

In other words, by applying the double supersymmetric transformation, a time translation is induced. Since the parameter τ is not physical and the supersymmetry is also not physical, this supersymmetry is not observable.

Local Supersymmetry

By using the Noethor method for the supersymmetry, we obtain the local supersymmetric action:

$$S^F = \frac{1}{2}\int d\tau(e^{-1}\dot{x}^\mu \dot{x}_\mu - i\chi^\mu \dot{\chi}_\mu - \kappa e^{-1} i\psi\chi^\mu \dot{x}_\mu), \tag{4.2.5}$$

where from (4.2.4) time-translation is automatically localized and so e and ψ are the gauge fields for the reparameterisation and local supersymmetry. Thus there are two local symmetries: One of them is a local supersymmetry

$$\begin{aligned} \delta x^{\mu} &= i\varepsilon\chi^{\mu}, \quad \delta\chi = (\dot{x}^{\mu} - i\kappa\psi\chi^{\mu}/2)\varepsilon e^{-1} \\ \delta e &= i\kappa\varepsilon\psi, \quad \delta\psi = 2\dot{\varepsilon}\kappa \end{aligned} \tag{4.2.6}$$

The other is a Reparameterisation

$$\begin{aligned} \delta x^{\mu} &= k\dot{x}^{\mu}, \quad \delta\chi = k\dot{\chi}^{\mu} \\ \delta e &= \frac{d}{d\tau}(ke), \quad \delta\psi = \frac{d}{d\tau}(k\psi). \end{aligned} \tag{4.2.7}$$

4.2.2 Brink-Schwarz Particle

In the previous discussion, we introduced supersymmetry for the bosonic parameter τ. Here we introduce supersymmetry for the μ [59, 60]. The supersymmetric transformation is

$$\delta x^{\mu} = i\bar{\varepsilon}^{a}\gamma^{\mu}\chi^{a} \quad , \delta\chi^{a}_{A} = \varepsilon^{a}_{A}, \quad \delta e = 0. \tag{4.2.8}$$

A is the spinor index and runs to $2^{2/D}$. We assume that the spinor is a Majorana. The index a is the index for internal symmetry, and for N supersymmetry, $a = 1, ..., N$. These transformations are spacetime supersymmetry and so obversable. In order to construct an invariant action under spacetime supersymmetry, we introduce a invariant quantity under supersymmetric transformation:

$$\Pi^{\mu} = \dot{x}^{\mu} - i\overline{\chi}^{a}\gamma^{\mu}\dot{\chi}^{a}. \tag{4.2.9}$$

Using this quantity, we can construct the super-Poincaré invariant action easily:

$$S = \frac{1}{2}\int d\tau \Pi^{\mu}\Pi^{\nu}\eta_{\mu\nu}. \tag{4.2.10}$$

Since this action can be constructed by just replacing the x^{μ} with Π, we can easily obtain the action with reparameterisation symmetry and local supersymmetry.

$$S = \frac{1}{2}\int d\tau e^{-1}\Pi^{\mu}\Pi^{\nu}\eta_{\mu\nu} \tag{4.2.11}$$

The equations of motion are given by

$$\Pi^2 = 0, \quad \gamma\cdot\Pi\dot{\chi} = 0, \tag{4.2.12}$$

From $(\gamma \cdot \Pi)^2 = \Pi^2 = 0$, the degree of freedom of $\gamma \cdot \Pi$ becomes half, i.e., the matrix $\gamma \cdot \Pi$ is a $(2^{D/2}/2) \times (2^{D/2}/2)$ matrix. Furthermore, from $\gamma \cdot \Pi \dot{\chi} = 0$ and (4.2.9), $\dot{\chi}$ always appears as a quantity multiplied by γ, the physical degrees of freedom of χ are also halved. This implies that there is some constraint on χ.

In fact Π^μ is not only supersymmetric invariant variable. Futhermore we can use the different supersymmetric invariant variable at each point because the action is trivially invariant as long as Π^μ is supersymmetric. In other words, Π^μ should be invariant under certain local symmetries. Indeed, this action is invariant under this symmetry

$$\delta x^\mu = i\overline{\chi}^a \gamma^\mu \delta\chi^a, \quad \delta\chi^a = i\gamma^\mu \Pi^a_\kappa, \quad \delta e = -4e\dot{\overline{\chi}}^a \kappa^a, \tag{4.2.13}$$

which is called the κ symmetry [61]. From these transformation laws, $\delta\Pi^\mu = -2\dot{\overline{\chi}}\gamma^\mu \not{\Pi}\kappa$. Thus the action constructed by new supersymmetric variable $\Pi'^\mu = \Pi^\mu + \delta\Pi^\mu$ is also invariant trivially. Since κ is a Grassmann parameter, the transformation is a kind of supersymmetry. However, since χ^a_A is a spacetime spinor, there is no spinor on the worldline. In other words, it is not an ordinary local world line supersymmetry. This κ symmetry is a common transformation between spacetime supersymmetry and world line supersymmetry. Indeed some part of the reparameterisation and κ symmetry form a closed algebra.

In the following, we calculate the algebra more explicitly. Given the second-order transformation of the κ symmetry

$$[\delta_1, \delta_2]\chi^a = (2i\gamma_\mu \kappa^a_2 \dot{\overline{\chi}}^b \gamma \cdot p \gamma^\mu \kappa^b_1 + 4i\gamma \cdot p \kappa^a_2 \dot{\overline{\chi}}^b \kappa^b_1) - (1 \leftrightarrow 2) \tag{4.2.14}$$

and we can see that the algebra is not closed in the off-shell. If we impose the equation of motion $\gamma \cdot p\dot{\chi} = 0$, the first term becomes zero, so

$$[\delta_1, \delta_2]\chi^a = i\gamma^\mu p_\mu \kappa'^a + (\text{equation of motion}) \tag{4.2.15}$$

$$\kappa'^a = 4\kappa^a_2 \dot{\overline{\chi}}^b \kappa^b_1 - (1 \leftrightarrow 2) \tag{4.2.16}$$

Since the κ symmetry is a subgroup of local supersymmetry, there is a reparameterisation corresponding to κ symmetry:

$$\delta\chi^a = \lambda\dot{\chi}^a, \quad \delta x^\mu = i\overline{\chi}\gamma^\mu \delta\chi^a, \quad \delta e = 0. \tag{4.2.17}$$

the canonical momentum for χ is

$$\pi^a_\chi = i\gamma^\mu \pi_x \chi^a \tag{4.2.18}$$

This variable satisfies $\pi^a_\chi = 0$ in the on-shell. This means that there is no canonical variable for χ, and no canonical commutation relation can be imposed on χ. This is why the Brink-Schwarz supersymmetric particle is difficult to carry out canonical quantization. However, it can be canonically quantized by light cone quantization, where the constraint is completely removed.

4.3 Classical Bosonic String

4.3.1 Bosonic String Action

We extend the action for the point particle to an arbitrary dimensional object. If the dimension of the object is n, the action is given by the $n+1$ dimensional world volume generated by the object through time evolution. Since the area is given by the outer product of the tangent vectors, the action is

$$S = T\int_M d^{n+1}\xi\sqrt{-\det(\partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu})}, \tag{4.3.1}$$

where T is a parameter of mass dimension $[M]^{n+1}$.

A simple extension of the linearized action of a point particle is

$$S = -\frac{T}{2}\int d^{n+1}\xi\sqrt{g}g^{\alpha\beta}(\xi)G_{\mu\nu}(x)\partial_\alpha x^\mu \partial_\beta x^\nu \tag{4.3.2}$$

The $g_{\alpha\beta}$ is the metric on the world volume, and the $G_{\mu\nu}$ is the metric on spacetime.

This action has a reparameterisation symmetry, and $g_{\alpha\beta}$ is the corresponding gauge field. The $g_{\alpha\beta}$ has $(n+1)(n+2)/2$ components, and the parameter of the reparameterisation invariance is the $(n+1)$ components. In other words, the independent degrees of freedom of $g_{\alpha\beta}$ are $n(n+1)/2$. Therefore, if $n>0$, the gauge field $g_{\alpha\beta}$ cannot be removed. However, if there are other gauge symmetries, $g_{\alpha\beta}$ can be fixed completely.

But there is a another local symmetry for the metric called Weyl symmetry:

$$g_{\alpha\beta} \to \Lambda(\sigma)g_{\alpha\beta}. \tag{4.3.3}$$

and the action transforms

$$\sqrt{g}g_{\alpha\beta} \to \Lambda^{\frac{1}{2}(n+1)-1}\sqrt{g}g^{\alpha\beta}. \tag{4.3.4}$$

Thus, if $n=1$, the action is Weyl invariant. This means that the gauge field $g_{\alpha\beta}$ can be fixed comletely.

String Action

Therefore we consider the action describing the motion of a one-dimensional object which is just a string.

For simplicity, we consider a flat spacetime.

$$S = -\frac{1}{2\pi\alpha'}\int d^2\xi\sqrt{-\mathrm{det}\partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}} \tag{4.3.5}$$

This action is known as Nambu-Goto action. Here ξ^α gives the coordinates of the world sheet, $\alpha = 0, 1$. We rewrite it in canonical form for quantization. First, we define the canonical variable for x^μ:

$$p^\mu = \frac{\delta L}{\delta\partial_\tau x_\mu}. \tag{4.3.6}$$

Examining the Jacobian $\delta L/\delta\partial_\tau x^\mu \delta p^\nu$ of this transformation, we find its determinant is zero. Therefore, the system has local symmetry.

In order to rewrite this action linearly, we examine the symmetry of the action. The local symmetry is a reparameterisation

$$\xi^\alpha \to \xi'^\alpha(\xi) \tag{4.3.7}$$

Under this transformation, the space-time coordinates are scalar. Now let's consider a certain reparameterisation:

$$\tau \pm \sigma \to f(\tau \pm \sigma). \tag{4.3.8}$$

In this transformation, the action is invariant trivialy, and this transformation is called conformal transformation.

$$x^{\mu\prime}(\xi') = x^\mu(\xi) \tag{4.3.9}$$

The global symmetry is a Poincaré symmetry

$$x^{\mu\prime}(\xi) = \Lambda^\mu_\nu x^\nu(\xi) + a^\mu. \tag{4.3.10}$$

a^μ and Λ^μ_ν are the parameter for the translation and Lorentz transformation respectively and Λ^μ_ν satisfies

$$\eta_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\tau = \eta_{\rho\tau}. \tag{4.3.11}$$

In order to obtain the action which is equivalent to Nambu-Goto action (4.3.5), we consider the action with global Poincaré symmetry

$$S = -\alpha\int d^2\xi\partial_\alpha x^\nu \partial_\beta x_\nu \eta^{\alpha\beta} \tag{4.3.12}$$

and by using Noether method we can obtain [1] :

$$S = -\frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{-g} g^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}. \tag{4.3.13}$$

This action is called the Polyakov action, and there is Weyl symmetry:

$$g'_{\alpha\beta}(\xi) = \Lambda(\xi) g_{\alpha\beta}(\xi). \tag{4.3.14}$$

$$x^{\mu\prime}(\xi) = x^\mu(\xi) \tag{4.3.15}$$

4.3.2 Equation of Motion and Constraints

The equations of motion and the boundary term are given by

$$\partial_\alpha \left(\sqrt{-g} g^{\alpha\beta} \partial_\beta x^\mu \right) = 0 \tag{4.3.16}$$

$$\partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} - \frac{1}{2} g_{\alpha\beta} g^{\gamma\sigma} \partial_\gamma x^\mu \partial_\sigma x^\nu \eta_{\mu\nu} = 0 \tag{4.3.17}$$

$$\left[\sqrt{-g} g^{1\beta} \partial_\beta x^\mu \right]_0^\pi = 0. \tag{4.3.18}$$

We define the constraint as $T_{\alpha\beta}$:

$$T_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} - \frac{1}{2} g_{\alpha\beta} g^{\gamma\sigma} \partial_\gamma x^\mu \partial_\sigma x^\nu \eta_{\mu\nu} = 0. \tag{4.3.19}$$

Also, since reparameterisation includes time translation, $T_{\alpha\beta}$ is also a generator of time translation. Furthermore, since $g_{\alpha\beta}$ is also the gauge field for the Weyl symmetry, the constraint for the Weyl symmetry can be described by $T_{\alpha\beta}$:

$$g^{\alpha\beta} T_{\alpha\beta} = 0. \tag{4.3.20}$$

The action has reparameterisation invariance and Weyl invariance, and there are three gauge parameters. Therefore, we can impose three gauge conditions. Since $g^{\alpha\beta}$ is symmetric, it has 3 degrees of freedom, so we take the metric of the world sheet to be flat.

$$g_{\alpha\beta} = \eta_{\alpha\beta} \tag{4.3.21}$$

[1] In general, using Noether method, in addition to the symmetric metric $g^{\alpha\beta}$, a term proportional to the antisymmetric tensor $\epsilon^{\alpha\beta}$ appears, but it disappears because the spacetime metric is flat and symmetric.

With this condition, the equations of motion and constraints become

$$\partial^2 x^\mu = 0, \tag{4.3.22}$$

$$T_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} - \frac{1}{2}\eta_{\alpha\beta}\partial_\gamma x^\mu \partial^\gamma x^\nu \eta_{\mu\nu} = 0. \tag{4.3.23}$$

Here, the conformal transformation cannot be fixed by this gauge-fixing condition because it keeps the action invariant trivially. Those transformations are the $\tau \pm \sigma \to f(\tau \pm \sigma)$ transformations, and each transformations are commutable. Since we can regard $\tau + \sigma$ directional local transformation as the $\tau - \sigma$ directional global transformation, thus these symmetries are called semi-local symmetry. In fact, in the Lorentz gauge, we can see that only one gauge can be fixed using these two semi-local transformations. Therefore, even though the gauge is fixed, the constraint remains.

$$T_{00} = T_{11} = \frac{1}{2}(\dot{x}^\mu \dot{x}^\nu + x'^\mu \dot{x}^\nu + x'^\mu x'^\nu)\eta_{\mu\nu} \tag{4.3.24}$$

$$T_{01} = T_{10} = \dot{x}^\mu x'^\nu \eta_{\mu\nu} \tag{4.3.25}$$

The Poisson brackets are given by

$$\{x^\mu(\sigma), x^\nu(\sigma')\} = 0 = \{p^\mu(\sigma), p^\nu(\sigma')\} \tag{4.3.26}$$

$$\{x^\mu(\sigma), p^\nu(\sigma')\} = \eta^{\mu\nu}\delta(\sigma - \sigma'). \tag{4.3.27}$$

The Solutions

Next, we solve the equations of motion. The boundary condition is

$$\left[\frac{\partial L}{\partial(\partial_\sigma x^\mu)}\delta x^\mu\right]_0^\pi = 0. \tag{4.3.28}$$

There are two possible boundary conditions which are the Lorentz covariant. In the following discussion, let $u = \tau + \sigma$ and $v = \tau - \sigma$.

One of them is

$$x^\mu(\sigma) = x(\sigma + 2\pi). \tag{4.3.29}$$

which describe the closed string. Here we have taken the string to be $-\pi \sim \pi$. The solution is independent in the u and v directions.

$$x^\mu = x^\mu_L(u) + x^\mu_R(v) \tag{4.3.30}$$

$$x^\mu_L(u) = \frac{q^\mu}{2} + \alpha' \frac{p^\mu}{2} u + i\frac{\alpha'}{2}\sum \frac{\alpha^\mu_n}{n}\exp\{(-inu)\} \tag{4.3.31}$$

$$x_R^\mu(v) = \frac{q^\mu}{2} + \alpha' \frac{p^\mu}{2} v + i\sqrt{\frac{\alpha'}{2}} \sum \frac{\overline{\alpha}_n^\mu}{n} \exp\{(-inv)\} \tag{4.3.32}$$

q^μ is the center of the string and p^μ is the momentum of the center. The third term is a sum of the oscillators and represents the amplitude from the center. Since x^μ is real, $\alpha_{-n}^\mu = \alpha_n^{\mu\dagger}$. Substituting these solutions into the Poisson brackets, we obtain

$$\{\alpha_n^\mu, \alpha_m^\nu\} = -in\delta_{n+m,0}\eta^{\mu\nu} \tag{4.3.33}$$

$$\{\overline{\alpha}_n^\mu, \overline{\alpha}_m^\nu\} = -in\delta_{n+m,0}\eta^{\mu\nu} \tag{4.3.34}$$

$$\{\alpha_n^\mu, \overline{\alpha}_m^\nu\} = 0. \tag{4.3.35}$$

The other posibility is

$$\partial_\alpha x^\mu = 0|_0^\pi, \tag{4.3.36}$$

which describe the open string. This boundary condition is known as the Neumann boundary condition and indicates that a right-moving oscillations bounce off the boundary and move to the left. Therefore, left-moving and right-moving are not independent.

$$x^\mu = q^\mu + \frac{1}{2}\sqrt{2\alpha'}\alpha_0^\mu(u+v) + i\sqrt{\frac{\alpha'}{2}} \sum \frac{1}{2}\alpha_n^\mu(e^{-inu} + e^{-inv}) \tag{4.3.37}$$

Similarly, substituting into Poisson brackets

$$\{\alpha_n^\mu, \alpha_m^\nu\} = -in\delta_{n+m,0}\eta^{\mu\nu} \tag{4.3.38}$$

In addition to this, the Dirichlet boundary condition is also possible, but it is not Poincaré covariant.

4.3.3 Conformal Algebra

Rewrite the constraint condition $T_{\alpha\beta}$ and find the algebra generated by the constraints.

$$T_{10} = T_{01} = \dot{x}x' = 0 \tag{4.3.39}$$

$$T_{00} = T_{11} = \frac{1}{2}(\dot{x}^2 + x'^2) = 0 \tag{4.3.40}$$

Rewriting with u and v, we get

$$T_{\pm\pm} = \frac{1}{2}(T_{00} \pm T_{01}) = \partial_\pm x \partial_\pm x \tag{4.3.41}$$

$T_{\pm\mp}$ is 0 from the constraint condition for Weyl symmetry . The constraints for reparameterisation are

$$\dot{x}_R^2 = \dot{x}_L^2 = 0 \tag{4.3.42}$$

In general, the energy conservation law in a two-dimensional field theory is

$$\partial_- T_{++} + \partial_+ T_{-+} = 0 \tag{4.3.43}$$

$$\partial_+ T_{--} + \partial_- T_{+-} = 0 \tag{4.3.44}$$

If the Weyl invariance exists, $T_{+-} = 0$. Therefore, the conservation law is $\partial_- T_{++} = 0$. Here, if we consider the function $f(u)$, it satisfies $\partial_- f = 0$. Therefore, since $\partial_- f T_{++} = 0$ is satisfied for any function $f(u)$, fT_{++} is also preserved. The corresponding charge is

$$Q_f = \int d\sigma f(u) T_{++} \tag{4.3.45}$$

and since f is arbitrary, there are the infinite number of the charges.

In string theory, since T_{++} is a constraint, Q_f is also a constraint. This means that there are infinite constraints. This is because the Weyl symmetry and the gauge fixation of the reparameterisation are not complete, and the conformal symmetry remains.

We carry out Fourier expansion

$$L_n = \frac{1}{2\pi\alpha'} \int d\sigma T_{++} = \frac{1}{2} \sum \alpha_m^\mu \alpha_{n-m}^\nu \eta_{\mu\nu} \tag{4.3.46}$$

$$\overline{L}_n = \frac{1}{2\pi\alpha'} \int d\sigma T_{--} = \frac{1}{2} \sum \overline{\alpha}_m^\mu \overline{\alpha}_{n-m}^\nu \eta_{\mu\nu} \tag{4.3.47}$$

and we obtain the infinite number of constraints. In other words, the string has an infinite number of gauge symmetries. Also, these constraints can be regarded as global currents. This can be interpreted as saying that any local conformal transformation can be expressed as a sum of infinite global transformations. The L_n and $\overline{L}_n$ are independent and have the same algebraic structure, respectively.

$$\{L_n, L_m\} = -i(n-m) L_{n+m} \tag{4.3.48}$$

4.4 Quantum Bosonic String

Next we consider the quantization of string theory. As mentioned above the semi-local symmetry, conformal symmetry, could not be fixed. There are two ways to deal with this conformal symmetry.

One is to quantize the infinete chages and impose them on the quantum states. The other is to fix the gauge by combining the two semi-local symmetries into a complete local symmetry. In particular, there are two more methods for the former, old covariant quantization and BRST quantization. These two quantizations do not fix conformal symmetry, and conformal symmetry remains in quantum theory. This is why it is called gauge-covariant quantization.

4.4.1 Old Covariant Quantization

The Weyl symmetry and reparameterisation symmetry were removed by fixing the metric to flat. In this fixation, the Polyakov action is given by

$$S = -\frac{T}{2}\int d^2\xi \partial^\alpha x \partial_\alpha x. \tag{4.4.1}$$

Furthermore, an additional condition

$$(\dot{x} \pm x')^2 = 0 \tag{4.4.2}$$

This condition is a constraint corresponding conformal symmetry.

Since the string states were described in the previous section by α_n^μ, q^μ,p^μ, we replace their Poisson brackets with commutation relations.

$$[\alpha_n^\mu, \alpha_m^\nu] = n\delta_{n+m,0}\eta^{\mu\nu} \tag{4.4.3}$$

$$[q^\mu, p^\nu] = i\eta^{\mu\nu} \tag{4.4.4}$$

We use the creation and annihilation operator as a way to express these commutation relations, and take the state to be an eigenstate of p^μ:

$$\alpha_n^\mu |0,p\rangle = 0, n > 0, \quad \alpha_0^\mu |0,p\rangle = p^\mu |0,p\rangle, \tag{4.4.5}$$

where, from the algebra generated by the creation and annihilation operator, the negative norm states are exist:

$$\langle 0| \alpha_m^0 \alpha_m^{0\dagger} |0\rangle = -n. \tag{4.4.6}$$

However, we see later in the discussion that it is removed due to the constraint of conformal symmetry.

Next, we describe the constraints with operators. Unlike in classical theory, we must define the order since α_n^μ is noncommutative.

$$: \alpha_n^\mu \alpha_m^\nu := \begin{cases} \alpha_n^\mu \alpha_m^\nu & m > n \\ \alpha_m^\nu \alpha_n^\mu & n > m \end{cases} \tag{4.4.7}$$

It can be seen from the commutation relation that this kind of ambiguity peculiar to quantum theory appears only in L_0:

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_1^\infty \alpha_{-n}\alpha_n. \tag{4.4.8}$$

Under this rule, the constraint is

$$L_n = \frac{1}{2} : \sum \alpha_m^\mu \alpha_{n-m}^\nu \eta_{\mu\nu} : \tag{4.4.9}$$

With this ambiguity, L_n generates an algebra different from the classical theory.

The Virasoro algebra generated by the Poisson brackets in classical theory is

$$\{L_m, L_n\} = \frac{i}{2}\sum_k k\alpha_{m-k}\alpha_{k+n} + \frac{i}{2}\sum_k (m-k)\alpha_{m-k+n}\alpha_k = i(m-n)L_{m+n} \tag{4.4.10}$$

The Virasoro algebra in quantum theory can be obtained directly by using the Wick decomposition or argebraic properties. The Virasoro algebra in quantum theory is

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{12}n(n^2-1)\delta_{n+m,0} \tag{4.4.11}$$

L_n is called Virasoro generator. Here D denotes the dimension of spacetime. Since all of these Virasoro generators corresponded to constraints in classical theory, we should impose all generators on the states:

$$L_n\psi = 0. \tag{4.4.12}$$

However, the solution is only $\psi = 0$. In fact, the state generated by L_n, $n < 0$ represents the gauge degrees of freedom [26]. L_0 is written down

$$L_0 = \frac{1}{2}p^2 + N \tag{4.4.13}$$

$$N = \sum_0 \alpha_n^\mu \alpha_{-n}^\nu \eta_{\mu\nu} \tag{4.4.14}$$

and expressed as $p^\mu = -i\partial^\mu$, $p^2 = -\partial^2$ and then L_0 is the Klein-Gordon equation for mass N. In other words, L_0 is an energy operator. From the commutation relation between L_0 and the other generators,

$$L_0 L_n = L_n(L_0 - n) \tag{4.4.15}$$

and since the energy of the state multiplied by $L_n(n > 0)$ goes down infinitely, we impose

$$L_n \psi = 0, \quad n > 0. \tag{4.4.16}$$

Also, $L_n(n < 0)$ generate a zero-norm state, but the energy is given by

$$L_0 L_1 = L_1 L_0 + 1. \tag{4.4.17}$$

Therefore, in order to set this zero-norm state to a vacuum, we take

$$L_0 \to L_0 - 1 \tag{4.4.18}$$

The zero-norm state corresponds to the gauge degrees of freedom, and since $L_{-1} |\phi\rangle$ corresponds to the gauge degrees of freedom of the massless boson, we need to make this state massless. Therefore, the condition to be imposed on the state is

$$(L_n - \delta_{n,0})\psi = 0, n \geq 0 \tag{4.4.19}$$

This condition is called the Virasoro condition. String theory is determined by this condition.

Spectrum

Next we find the spectrum of the open string [62–64]. The states of the string are generated by α^μ_{-n} and is a function of x^μ:

$$|\psi\rangle = \{\phi(x) + iA^1_\mu \alpha^\mu_{-1} + iA^2_\mu \alpha^\mu_{-2} + h_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1} + ...\} |0\rangle . \tag{4.4.20}$$

If we impose the Virasoro conditions, we obtain the following equations:

$$\begin{aligned} L_0 - 1 &\quad (\alpha' \partial^2 + 1)\phi = 0 = (\alpha' \partial^2 - (s-1))A^s_\mu = (\alpha' \partial^2 - 1)h_{\mu\nu} = ... \\ L_{n>0} &\quad \partial^\mu A^1_\mu = 0 = -\sqrt{2\alpha'}\partial^\mu h_{\mu\nu} + A^2_\nu = 2\sqrt{2\alpha'}\partial^\mu A^2_\mu + h^\nu \nu = ... \end{aligned} \tag{4.4.21}$$

From these equations, as we expected, the constraint $L_n\psi = 0$ $(n > 0)$ corresponds to the gauge fix condition of the gauge field.

We see the first few states.

ϕ	This field is a tachyon because the mass is negative.
A_1^μ	Spin 1 massless vector field.
$H_{\mu\nu}$	Spin 2 massive tensor field. This is a massive gravity.
A_2^μ	Massive spin 1 gauge field. This field corresponds to the trace part of $h_{\mu\nu}$ by using the gauge fixation condition, and can be taken to zero using the gauge degrees of freedom.

The masses of higher-order fields are given by $m_n = (1/\alpha')(n-1)$. The tachyon appears at the lowest order. This fact is caused from the vacuum of the bosonic string being not taken correctly.

In a closed string, the theory is determined by the two Virasoro conditions generated by $L_n, \overline{L}_n$, and the two are independent. Thus, it is given by the direct product of two open strings. Since $\alpha_n^\mu, \overline{\alpha}_n^\mu$ generate the states which are the functions of the coordinates, we expand the string state in term of the basis

$$|\psi\rangle = (\phi(x) + h_\mu \alpha_{-1}^\mu + k_\mu \overline{\alpha}_{-1}^\mu + h_{\mu\nu}\alpha_{-1}^\mu \overline{\alpha}_{-1}^\nu + h_\mu^2 \alpha_{-2}^\mu + k_\mu \overline{\alpha}_{-2}^\mu + k_{\mu\nu}^2 \alpha_{-1}^\mu \alpha_{-1}\nu + l_{\mu\nu}^2 \overline{\alpha}_{-1}^\mu \alpha_{-1}^\nu + \ldots)\,|0\rangle \tag{4.4.22}$$

By imposing the Virasoro condition as in the open string, we obtain the equations:

$$h^\mu = k^\mu = h_\mu^2 = k_\mu^2 = k_{\mu\nu}^2 = l_{\mu\nu}^2 = 0 \tag{4.4.23}$$

$$(\alpha' \partial^2 + 4)\phi = \partial^2 h_{mu\nu} = 0 \tag{4.4.24}$$

$$\partial^\mu h_{\mu\nu} = \partial^\nu h_{\mu\nu} = 0. \tag{4.4.25}$$

Thus ϕ at level 0 is a tachyon and $h_{\mu\nu}$ is a massless symmetric tensor representing gravity. Higher levels are massive states.

Physical States and Virasoro Condition

The Fock space of string theory is constructed by α_m^μ, but since the metric is Minkowski, the norm is not positive definite. However, the Virasoro condition

$$(L_n - a\delta_{n,0})\,|\psi\rangle = 0 \tag{4.4.26}$$

restrict that space and lead to a physical state. Since the number of these conditions coincides with the number of α_0^μ, we can restrict it to the positive definite Fock space.

The Virasoro algebra in quantum theory is given by (4.4.11). This result shows that L_{-1}, L_1, L_0 form an closed sub-algebra with no anomaly and is isomorphic to SU(1,1) or SL(2,R).

Next, we consider the condition for removing negative norm physical states. If the space of a positive norm and a negative norm exist in Hilbert space, there exists a zero norm space at the boundary between the two of them. Therefore, we consider the zero-norm state. In the following, we only consider them for open strings. Since closed strings are generated by the direct product of open strings, the extension is easily.

Let $|0;k^\mu\rangle$ be the ground state with momentum k^μ. If we impose the Virasoro condition $L_0 = a$, we find $\alpha' k^2 = a$. Next, we consider the first excited mode. Such a state is $\zeta\cdot\alpha_{-1}\,|0;k^\mu\rangle$. Similarly, imposing the mass-shell condition, we have $\alpha' k^2 = a-1$. Furthermore, imposing $L_1 = 0$, we get $\zeta\cdot k = 0$. With this constraints, there are $D-1$ independent components of k^μ. If the momentum k^μ is physical and timelike, $k^2 \leq 0$. Therefore, $a \leq 1$ must satisfy.

In particular, if the boundary $a = 1$, the first excited mode is massless and the ground state is tachyonic. The $L_1 = 0$ condition corresponds to the covariant gauge condition $\partial^\mu A_\mu = 0$. This constraint removes the negative norm states and gives $D-2$ transverse positive norm states and longitudinal zero norm states.

It can be shown that in field theory the zero-norm states do not contribute to the S-matrix due to current conservation laws and gauge symmetry. The first excited mode at $a = 1$ is the null state, and there exits infinitely. Let ϕ be any physical state that satisfies the Virasoro conditions, and let ψ be a spurious state that is orthogonal to all physical states and satisfies the mass-shell condition.

$$\langle\phi|\psi\rangle = 0 \tag{4.4.27}$$

Since the fock space is generated by L_n, we can expand the states in term of L_n

$$|\psi\rangle = \sum_{n>0} L_{-n}\,|\chi_n\rangle\,. \tag{4.4.28}$$

This is shown in a simple way. If ψ is a spurious state, the operator $O = |\psi\rangle\langle\psi|$ annihilates all states. Therefore, if we take any operator as X_n, then

$$O = \sum_{n>0} X_{-n} L_n \tag{4.4.29}$$

Therefore, we believe that ψ can be expanded by Virasoro generators.

Since ψ satisfies the mass-shell condition,

$$(L_0 - a + n)\,|\chi_n\rangle = 0. \tag{4.4.30}$$

Here, $L_n,\ (n < -2)$ can be expressed by the Virasoro algebra by the commutation relation of L_{-1} and L_{-2}. Thus,

$$|\psi\rangle = L_{-1}\,|\chi_1\rangle + L_{-2}\,|\chi_2\rangle \tag{4.4.31}$$

Let us assume that this spurious state satisfies the Virasoro condition:

$$\langle\phi|\psi\rangle = 0, \quad (L_m - \delta_{m,0})\,|\psi\rangle = 0. \tag{4.4.32}$$

In this case, the norm of ψ is zero.

Let's calculate the spurious state.

$$|\psi\rangle = L_{-1}\,|\overline{\chi}\rangle \tag{4.4.33}$$

Then, the Virasolo condition is

$$L_m\,|\overline{\chi}\rangle = 0, \quad (L_0 - a + 1)\,|\overline{\chi}\rangle = 0 \tag{4.4.34}$$

If we take $a = 1$, the Virasoro condition for L_1 is

$$L_1\,|\psi\rangle = 2L_0\,|\overline{\chi}\rangle = 0 \tag{4.4.35}$$

Thus, ψ is a physical and zero-norm spurious state. This result does not change no matter how many times L_{-1} is applied to $\tilde{\chi}$, so there are infinite number of zero-norm states.

Next, consider the spurious state with L_{-2} applied:

$$|\psi\rangle = (L_{-2} + \gamma L_{-1}^2)\,|\overline{\chi}\rangle \tag{4.4.36}$$

Taking $a = 1$, from the mass-shell condition,

$$(L_0 + 1)\,|\overline{\chi}\rangle = 0. \tag{4.4.37}$$

In order for ψ to be the zero norm, the Virasoro condition must be satisfied. Since L_n $(n > 2)$ can be described by L_1 and L_2, all we should confirm are $L_1\,|\psi\rangle = L_2\,|\psi\rangle = 0$. From these two equations, we obtain the equations $3 - 2\gamma = 0$ and $D/2 - (4 + 6\gamma) = 0$. Solving these equations, we get $\gamma = 3/2$ and $D = 26$. From the above, there is no negative norm state when $a = 1$ and $D = 26$. Since sprious states represent gauge degrees of freedom, above process corresponds to require the gauge transformation to be consistent.

In old covariant quantization, we can remove the negative norm state from the physical state, and the

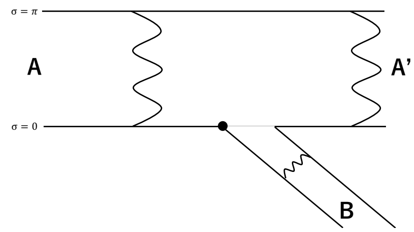


Fig. 1: The state A transits to A' with emission of the on-shell.

intercept a and dimension D are determined so that the gauge transformations is consistent. However, as can be seen from (4.4.11), the conformal anomaly has not disappeared. In fact, if we consider quantum theory, i.e., 1-loop corrections, using the physical state of the positive norm obtained by old covariant quantization, we find that the theory is not consistent. The old covariant quantization is not a consistent quantization.

Vertex Operator and Physical States

We can obtain the physical states with scattering the string. In other words, since the scattering of a string is a physical phenomenon, the states absorbed or emitted by such a phenomenon are of course physical. Futhermore vertex operator is important to construct the physical states in light cone quantization and prove the no-Ghost theorem.

The scattering of strings is described by the separation or merger of strings. For the simplest scattering, we consider the case where a propagating string separates into two and one of them propagates to infinity:

$$A \to A' + B. \tag{4.4.38}$$

This process is a scattering process where string A emits string B and transitions to string A'. String A and A' are off-shell because they are generally internal lines in the scattering process. On the other hand, string B propagates to infinity and can be observed at the infinity point, so it must be on-shell.

This scattering can be described by applying the vertex operator to the end of string A. Thus we insert the local operator at $\sigma = 0$. We map a vertex operator to each on-shell physical state $|\phi\rangle$. By considering this vertex operator, we can examine the physical state indirectly.

We consider the local operator $V(\sigma, \tau)$ on the open string Hilbert space. For the simplicity, we take $\sigma = 0$:

$$A(\tau) = A(0, \tau). \tag{4.4.39}$$

The time evolution operator is the Hamiltonian, and the Hamiltonian is included in the Virasoro constraint:

$$L_0 - a. \tag{4.4.40}$$

By using this operator, we can eliminate the time τ-dependence of the operator:

$$A(\tau) = e^{i\tau L_0} A(0) e^{-i\tau L_0}, \tag{4.4.41}$$

where $e^{-i\tau a}$ vanishes because it is clearly commutative with the operator A. The conformal transformability of a vertex operator is determined by its conformal weight, and if the conformal weight of $A(\tau)$ is J, then

$$A'(\tau) = \left(\frac{d\tau}{d\tau'}\right)^J A(\tau) \tag{4.4.42}$$

Considering the infinitesimal transformation $\tau \to \tau' = \tau + \epsilon(\tau)$,

$$\delta A(\tau) = -\epsilon \frac{dA}{d\tau} - JA\frac{d\epsilon}{d\tau}. \tag{4.4.43}$$

As an explicit conformal transformation, the parameter of the transformation of the Virasoro generator L_m is $\epsilon = ie^{im\tau}$, so

$$[L_m, A(\tau)] = e^{im\tau}\left(-i\frac{d}{d\tau} + mJ\tau\right) V(\tau). \tag{4.4.44}$$

Fourier expansion of V is

$$A(\tau) = \sum_{-\infty}^{\infty} A_m e^{-im\tau}. \tag{4.4.45}$$

Substituting this expression into the commutation relation, then we obtain

$$[L_m, A_n] = [m(J-1) - n]A_{m+n}. \tag{4.4.46}$$

This commutation relation is similar to Virasoro algebra. The vertex operator A defined in this way plays an important role in the proof of the No-gohst theorem.

In the above discussion, in order to obtain physical states, we introduced operators with certain conformal weights corresponding to those states. In the following, we construct the physical states by applying various operators to the physical state.

First, the physical state on which the vertex operator acts is written as $|\phi\rangle$, which must satisfy the Virasoro condition:

$$(L_n - a\delta_{n,0})\,|\phi\rangle = 0, \quad n \leq 0. \tag{4.4.47}$$

Considering the vertex operator $A(\tau)$ of conformal weight 1, from (4.4.46)

$$[L_m, A_0] = 0 \tag{4.4.48}$$

is satisfied. Therefore

$$|\phi'\rangle = A_0 |\phi\rangle \tag{4.4.49}$$

also satisfies the Virasoro condition and is therefore a physical state. $|\phi\rangle$ is string A in the diagram, A_0 is the vertex operator corresponding to the state of string B that is emitted from the end of $\sigma = 0$, and $|\phi'\rangle = A_0 |\phi\rangle$ corresponds to string A'. Therefore, the conformal weights of the vertex operators of the open strings are consistent with $J = 1$.

In the following, we investigate a more explicit form of the vertex operator. We consider the case where a string is emitted from $\sigma = 0$ at time τ. Let k^μ denote the momentum change at this time:

$$V(k, 0, \tau) = V(k, \tau). \tag{4.4.50}$$

This vertex operator is the operator corresponding to the state of string B to be emitted, and it changes the momentum of string A by k^μ. Applying the momentum operator P_μ to the state $|\phi\rangle$ with momentum p_μ gives

$$P_\mu |\phi\rangle = p_\mu |\phi\rangle \tag{4.4.51}$$

and the vertex operator changes the momentum of this state by k_μ, which is

$$P_\mu(V |\phi\rangle) = (p_\mu - k_\mu)V |\phi\rangle . \tag{4.4.52}$$

As V satisfying these properties, we can use the coherent state:

$$V \sim e^{ik\cdot x}. \tag{4.4.53}$$

Since V acts at time τ and position $\sigma = 0$, it emits or absorbs k^μ in spacetime coordinates $X^\mu(0, \tau)$. Since X^μ contains $\alpha_n^\mu, x^\mu, p^\mu$ in the on-shell, and not all of them are commutative, we define them with a normal ordered:

$$\begin{aligned} V(k, \tau) &=: e^{ik\cdot x(0,\tau)} : \\ &= \exp\left(k \cdot \sum_1^\infty \frac{\alpha_{-n}}{n} e^{in\tau}\right) e^{ik\cdot q(\tau)} \exp\left(-k \cdot \sum_1^\infty \frac{\alpha_{-n}}{n} e^{-in\tau}\right) \end{aligned} \tag{4.4.54}$$

If we define without normal ordering, the contribution of the constant term $e^{\alpha' k^2 \sum \frac{1}{n}}$ derived from the commutation relation is added.

Next, we derive the conformal dimension of the vertex operator $V(k,\tau)$. The conformal dimension of x^μ is zero. Therefore, the conformal weight of the function $f(x^\mu)$ is also zero in general. The exception is the case $f(x^\mu)$ has a singularity. For example, consider two functions $f(x), g(x)$. If f and g are renormalized with different energies, fg is singular. In other words, there is a short-range singularity in the operator product expansion of fg. In fact, $: x^\mu(\tau) x_\mu(\tau) :$ has no fixed conformal weight. Since the vertex operator V also contains a normal ordering, it seems that the conformal weight is not fixed, but in fact it can be uniquely determined. This means that the vertex operator has a short-range singularity.

In order to find the conformal weights, we calculate the commutation relation between the Virasoro generators L_m:

$$[L_m, V(k,\tau)] = e^{im\tau} \left(-i\frac{d}{d\tau} + \frac{1}{2} m k^2 \right) V(k,\tau) \tag{4.4.55}$$

Thus, the conformal weight of V is $k^2/2$. Here, we can obtain the anomaly dimension of $e^{ik\cdot x}$ by computing the two-point function, which is $k^2/4$ for closed strings and $k^2/2$ for open strings. It is the scattering of open strings, and it is consistent with the anomaly dimension being $k^2/2$. In the following, we consider vertex operators with certain conformal weights.

The lowest energy state is the tachyon, which has a mass of k^2. Since the vertex operator V corresponds to a physical state, we first consider the case where $k^2 = 2$. In this case, the conformal weight is $J = 1$. Of course, this vertex operator is now the operator describing the emission or absorption of a tachyon of mass $M^2 = -2$.

Next, we consider the case where $k^2 = 0$. The conformal weight of the vertex operator is $J = 0$. The condition for the state after the emission $V|\phi\rangle$ to satisfy the Virasoro condition and become physical is the case that the conformal weight of V is $J = 1$. In other words, it is not possible to obtain the physical states if $J = 0$. In order to generate a vertex operator with $J = 1$ at $k^2 = 0$, we apply an operator with conformal weight 1 to $: e^{ik\cdot x} :$. We use $\dot{x}^\mu$ as the conformal weight 1 operator. Since the vertex operator V must be Lorentz invariant, we introduce a vector ζ_μ and define it as follows:

$$V_\zeta(k,\tau) = \zeta \cdot \frac{dX}{d\tau} e^{ik\cdot x}. \tag{4.4.56}$$

Thus V_ζ does not have a short-range singularity by choosing ζ so that $\zeta \cdot k = 0$, and hence there is no need to impose a normal ordering. Since the condition $\zeta \cdot k = 0$ is satisfied, ζ^μ can be interpreted as a polarization vector. Therefore, (4.4.56) corresponds to the massless meson emission of the polarization vector ζ^μ.

The conformal weight of the operator V_0 defined by (4.4.54) is $J = -1$. Since the physical vertex operator is $J = 1$, we must apply an operator of conformal weight 2 to V_0. Since the conformal weight of $\dot{x}^\mu$ is 1, we define

$$\zeta^{\mu\nu}\dot{x}_\mu\dot{x}_\nu : \exp[ik \cdot x] : \tag{4.4.57}$$

and then we obtin the vertex operator with $J = 1$. If $k_\mu \cdot \zeta^{\mu\nu} = \mathrm{tr}\,\zeta = 0$ is not satisfied, this equation must have normal ordering. This condition is consistent with the polarization tensor condition for massive spin two states. Therefore, it is a symmetric traceless tensor for SO(D-1). Apart from this state, $\ddot{x}^\mu$ is also a $J = 2$ operator, so

$$Y_{k,\eta} = \eta_\mu \ddot{X}^\mu e^{ik\cdot X} \tag{4.4.58}$$

is also a $J = 1$ operator. The state corresponding to this vertex operator has spin 1 and is the state with the polarization vector η_μ. If $\eta_\mu \cdot k^\mu = 0$ is satisfied, the sum of this vertex operator and the vertex operator of (4.4.57) is the total derivative:

$$-i\frac{d}{d\tau}(\eta \cdot \dot{X} : e^{ik\cdot x} :) = (-i \cdot \ddot{x} + \cdot\ddot{x}k \cdot \ddot{x}) : e^{ik\cdot x} : \tag{4.4.59}$$

This vertex operator describes the emission of the zero-norm state $L_{-1}\eta \cdot \alpha_{-1} |0; k\rangle$. In other words, the vertex operator of massive spin 2 has an ambiguity of Y. It follows from the requirement that this ambiguity does not change the physics that $D = 26$.

We consider the more general case of $k^2 = -n$. The conformal weight of the vertex operator defined by (4.4.54) is $J = k^2/2$. And for the state $V|\phi\rangle$ after emission to be a physical state, it should be $J = 1$. Therefore, the product of (4.4.54) and the operator $J = 1 - k^2/2$ must be the vertex operator. Since $d^n X^\mu/d\tau^n$ is an operator of $J = n$, the vertex operator for general $k^2 = -n$ takes the following form:

$$: f(\dot{x}, \ddot{x}, ..., x^{(n)})e^{ik\cdot x} : . \tag{4.4.60}$$

Finally, we consider the vertex operator for the zero-norm state. Since the zero-norm state is a gauge transformation degree of freedom, it is a physical state. Therefore, it is described by $V(k, \tau)$ with $J = 1$. Here, in old covariant quantization, the spurious state is described as $L_{-1}|\lambda\rangle$, and the conformal weight of $|\lambda\rangle$ was zero. So we suppose that the vertex operator for spurious states can also be described by the $J = 0$ operator. Therefore, if W is an operator with conformal weight 0 and containing $e^{ik\cdot x}$, we can assume that the vertex operator for the zero-norm state is as follows:

$$V(k, \tau) = -i\frac{dW(k, \tau)}{d\tau} = [L_0, W(k, \tau)]. \tag{4.4.61}$$

Let $k^2 = 0$, $W = e^{ik\cdot X}$, then $V(k, \tau)$ is the emission operator for the massless vector meson of the polarization vector k^μ.

4.4.2 Light Cone Quantization

The Polyakov action is invariant under reparameterisation and the Weyl symmetry, and these local symmetries are fixed by taking the metric flat. However, two semi-local symmetries are not completely fixed. This is due to a special parameterisation

$$\tau \pm \sigma \to f(\tau \pm \sigma). \tag{4.4.62}$$

The Polyakov action is trivially invariant under those transformations, and those transformations are not fixed. Those transformations are called conformal symmetry. For example, the transformation $\tau + \sigma \to f(\tau + \sigma)$ is a local symmetry in the $\tau + \sigma$ direction, but it is independent of $\tau - \sigma$ and so can be regarded as a global symmetry in that direction. Also, as we will see later, the two semi-local symmetries together work the same as the local symmetry. In old covariant quantization and BRST quantization, the constraints corresponding to these semi-local symmetries (Virasoro conditions) are quantized and imposed on the states to construct the physical states. Here we completely solve the Virasoro condition at the classical level, and quantize only the phisical indipendent degrees of freedom.

Light-Cone Gauge

First, we change the coordinate system to make it easier to deal with the semi-local symmetry:

$$\tau \pm \sigma = \xi^\pm, \tag{4.4.63}$$

$$x^\pm = \frac{1}{\sqrt{2}}(x^{D-1} \pm x^0), x^i, i = 1, ...D-2. \tag{4.4.64}$$

The metric under this world sheet coordinate is

$$\eta^{ij} = 1, \quad \eta^{+-} = \eta^{-+} = -1 \tag{4.4.65}$$

and the residual symmetries $\tau \pm \sigma \to f(\tau \pm \sigma)$ are

$$\xi^\pm \to f(\xi^\pm) \tag{4.4.66}$$

which completely separates the two. From this equation, $f(\xi^+)$ can be regarded as the parameter of reparameterisation in the ξ^+ direction, while the ξ^- direction can be regarded as a constant. Thus those transformations are called semi-local symmetry. The equations of motion and constraints are

$$\frac{\partial x}{\partial \xi^{\pm}} \frac{\partial x}{\partial \xi^{\pm}} = 0 \tag{4.4.67}$$

$$\frac{\partial^2 x^{\mu}}{\partial \xi^{+} \partial \xi^{-}} \tag{4.4.68}$$

The conformal transformation for x^+ is

$$\delta x^+ = f(\xi^+) \frac{\partial x^+}{\partial \xi^+} + g(\xi^-) \frac{\partial x^+}{\partial \xi^-} \tag{4.4.69}$$

The solution to the equation of motion is

$$x^+ = x_0^+(0) + c\tau + l^+(\xi^+) + k^+(\xi^-) \tag{4.4.70}$$

From conformal symmetry, we can use f and g to fix l^+ and k^+. In other words, we impose

$$x^+(\tau, \sigma) = x_0^+(0) + c\tau. \tag{4.4.71}$$

[2] Using this condition and substituting it into the Virasoro constraint, we obtain

$$(\dot{x} \pm x')^2 = 0 \to (\dot{x}^- \pm x'^-) = (\dot{x}^i \pm x'^i)^2 / 2p^+. \tag{4.4.72}$$

From these equations, we can see that x^- is completely determined by x^i except the ambiguity of the integration constant. In other words, x^+ and x^- are removed by the light cone gauge. The solution of the equation of motion for x^- is

$$x^- = x_0^- + p^- \tau + i \sum \frac{1}{n} \alpha_n^- e^{-in\tau} \cos n\sigma. \tag{4.4.73}$$

Substituting in the constraint

$$\alpha_n^- = \frac{1}{p^+} \left(\frac{1}{2} \sum_{i=1}^{D-2} \sum_{m=-\infty}^{\infty} : \alpha_{n-m}^i \alpha_m^i : - a\delta_{n,0} \right), \quad \alpha_0^- = p_0^-. \tag{4.4.74}$$

[2] τ and x^+ satisfy the same equations of motion. With this gauge-fixing condition, α_n^+, $n \neq 0$ of x^+ is zero, and the string coordinates are considered time in this system.

Here, considering the constraint of $n = 0$, from $(L_0 - a)\psi = 0$, we find

$$\begin{array}{c} M^2 = \left(2p^+p^- - p^ip^i\right) = 2(N-a) \\ N = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n \end{array} . \tag{4.4.75}$$

This shows that α_0^- and p^- are equal. Also, $p^+\alpha_n^-$ satisfies the Virasolo condition:

$$\left[p^+\alpha_m^-, p^+\alpha_n^-\right] = (m-n)p^+\alpha_{m+n}^- + \left[\frac{D-2}{12}\left(m^3 - m\right) + 2am\right]\delta_{m+n}. \tag{4.4.76}$$

Thus, we find that the independent states of the string are given by the transverse mode α_n^i. The independent variables are

$$x^i(\tau,\sigma), P^i(\tau,\sigma), q^-(\tau), p^+(\tau), \tag{4.4.77}$$

where P^i is the canonical momentum, q^- is the center of mass, and p^+ is the total momentum in the + direction. The Hamiltonian is

$$H = -2\alpha' p^+ p^-. \tag{4.4.78}$$

Quantization

In the old covariant and BRST quantization, the equations of motion were included in the constraints, so there was no need to impose equations of motion in quantum theory, only constraints. In the light cone quantization, the constraints no longer exist, and there are no conditions to be imposed on the quantum state. However, if we consider a system with constraints, we need to impose the non-contradiction condition that the constraints do not develop in time direction:

$$\{q^+ + p^+\tau, H\} = p^+ = 0 \tag{4.4.79}$$

Rewriting p^+, we get

$$p^+ = -H/2\alpha' p^- \tag{4.4.80}$$

This non-contradiction condition is not solved and must be imposed on the state in quantum theory.:

$$H\left|\Psi\right\rangle = 0. \tag{4.4.81}$$

Poisson brackets are

$$\{x^i(\sigma), P^j(\sigma')\} = \delta^{ij}(\sigma - \sigma'), \quad \{q^-, p^+\} = 1, \tag{4.4.82}$$

and mode expansion of x^i gives

$$\{a_n^i, a_m^j\} = -in\delta^{ij}\delta_{n+m,0} \tag{4.4.83}$$

According to the Dirac rule, we replace the Poisson brackets with a commutation relation:

$$[x^i(\sigma), P^j(\sigma')] = i\delta^{ij}(\sigma - \sigma'), [q^-, p^+] = 1. \tag{4.4.84}$$

Rewriting the no-contradiction condition with the operator

$$\left(-\alpha'\partial^2 - i\partial_\tau - M^2\right)\Psi = 0. \tag{4.4.85}$$

This is just the Schrödinger equation.

Thus, by fixing the conformal symmetry in gauge, the constraint conditions are completely solved and only the independent degrees of freedom are quantized. Since the constraints include equations of motion, there should be no constraints or equations of motion imposed on states in quantum theory. However, a non-contradiction condition associated with gauge fixation must be imposed on the state in quantum theory, and this conditional expression leads to Schrödinger equation.

Lorentz symmetry

Due to the gauge fixation of x^+, x^+ and x^- have been removed, and only x^i is now an independent degree of freedom. Therefore, the Lorentz symmetry of spacetime has become nontrivial. By requiring the spacetime Lorentz symmetry, we can derive $D = 26$ and $a = 1$

There are two ways to require the Lorentz symmetry. One is to require the massless representation in Poincaré group is consistent [55]. The other is to confirm the Lorentz argebra is close [26]. Here we explain the former way.

The theory is invariant under the Lorentz group SO(D), but only the SO(D-2) symmetry is trivial for the first excited state $\alpha_n^i |0;p\rangle$ generated by independent oscillators. In general, the massless representation of the Poincaré group corresponds to the SO(D-2) irreducible representation, while the massive spin states correspond to the SO(D-1) irreducible representation. Therefore, in order to construct a Lorentz invariant string theory, the first excited state must be massless, and hence $a = 1$. From this requirement, we can obtain $D = 26$.

We see this more explicily. If we calculate the ordering of generators in L_0 specifically:

$$\frac{1}{2}\sum_{i=1}^{D-2}\sum_{n=-\infty}^{\infty}\alpha_{-n}^i\alpha_n^i = \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n=-\infty}^{\infty}\alpha_{-n}^i\alpha_n^i + \frac{D-2}{2}\sum_{n=1}^{\infty}n. \tag{4.4.86}$$

Since the second term on the right-hand side is divergent, we regularize it by the zeta function $\zeta = \sum_1^\infty n^{-s}$ (converges at Re$s > 0$). In this case, $s = -1$, and the value of the zeta function is $\zeta(-1) = -1/12$. Therefore, the second term is

$$-\frac{D-2}{24} \tag{4.4.87}$$

and this must match $-a = -1$, so $D = 26$. Since L_0 is a Hamiltonian, this means that the energy of the vacuum is divergent. However, since only the difference in energy matters, the zero point of the vacuum can be chosen freely. This regularization corresponds to setting a value with respect to an infinite value.

Physical State

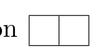
We fix the gauge symmetry in the light cone gauge and describe the physical state with physical degrees of freedom.

The ground state of open string is described by $|0;p\rangle$, which is a tachyonic state with negative mass $\alpha' M^2 = -1$.

The first excitation modes after gauge fixation is $\alpha^i_{-1}|0;p\rangle$. This state is a massless vector boson and the degrees of freedom are 24 for the transverse component. Therefore, this state belongs to the multiplet of SO(24).

The mass is $\alpha' M^2 = 1$ and the states are

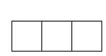
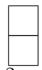
$$\alpha^i_{-2}|0;p\rangle\,, \quad \alpha^i_{-1}\alpha^j_{-1}|0;p\rangle \tag{4.4.88}$$

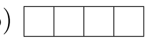
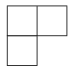
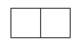
The second state has the corresponding symmetric tensor since α^i_{-1}'s are commutative each other, and so the degrees of freedom of the state are $D-2$ and $(D-2)(D-1)/2$, respectively. Since these two states belong to the same multiplet, they have a total of $(D-2)(D+1)/2$ degrees of freedom. This number is equal to the degrees of freedom of the symmetric traceless representation  of SO(D-1). This representation is called the spin-2 representation.

In the light cone gauge, the positive norm Hilbert space is constructed to be trivial. Lorentz symmetry is guaranteed at $D = 26$. In the massive case, the Lorentz invariance is nontrivial because it is an SO(D-1) multiplet. However, the SO(25) multiplet can be made from the SO(24) multiplet, and thus we find Lorentz invariance is preserved.

Next we consider the third level states. The mass of the states at this level are $\alpha' M^2 = 2$, and the states are

$$\alpha^i_{-3}|0;p\rangle\,, \quad \alpha^i_{-2}\alpha^j_{-1}|p;0\rangle\,, \quad \alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}|0;p\rangle \tag{4.4.89}$$

The total number of degrees of freedom for these states is $24 + 576 + 2600 = 3200$, which is the sum of the third-order fully symmetric traceless representation of SO(25)  (2900 representation) and the second-order antisymmetric tensor  (300 representation).

The mass of fourth level is $\alpha' M^2 = 3$ and the degrees of freedom are the sum of SO(25)  (20150 representations),  (5175 representations),  (324 representations) and 1 representation (25,620).

The maximum spin of the states belonging to mass M is given by $n = \alpha' M^2 + 1$. The n-th order symmetric traceless tensor representation of SO(25) is

$$\alpha_{-1}^{i_1} \cdots \alpha_{-1}^{i_n} |0; p\rangle \tag{4.4.90}$$

If we decompose in the SO(3) subgroup, there is only one spin-n term, which is the highest spin state at this mass level.

The states for closed string can be easily generated from the states of open string. In light-cone gauge, the independent excited modes are described by α_n^i and $\overline{\alpha}_n^i$. Since $L_0 = \overline{L}_0$ (level matching condition) is satisfied from the Virasoro condition, the energy of the right-handed and left-handed oscillations must be equal:

$$N = \overline{N}. \tag{4.4.91}$$

The mass of the closed string is

$$\alpha' M^2 = 4(N - 1) \tag{4.4.92}$$

and the closed string state is given by the direct product of the open string states.

The ground state is a tachyonic since $\alpha' M^2 = -4$.

The mass at first level is a massless $\alpha' M^2 = 0$, and the states are

$$\left|\Omega^{ij}\right\rangle = \alpha_{-1}^i \overline{\alpha}_{-1}^j |0; p\rangle \, . \tag{4.4.93}$$

Since this state can be written as a direct sum of the first excited modes of the open string, the tensor representation of this state is the tensor product of the massless vector representation of SO(24). Since the direct product representation is not necessarily irreducible, we decompose this state irreducibly. In order to obtain the irreducible representation, we check the transformability of the representation. First, it can be divided into the symmetric component $\left|\Omega^{ij}\right\rangle$ and the antisymmetric component $\left|\Omega^{[ij]}\right\rangle$ [3]. This anti-symmetric component is the antisymmetric second-order tensor of SO(24). The transformation of the symmetric component differs depending on whether the term is proportional to the metric or not. The term proportional to the metric is the trace component, which is the massless scalar (dilaton)

[3] Since the product of right-moving and left-moving state is arbitrary, there are Z_2 symmetry. This symmetry is an automorphism. Thus we can decompose the direct product representation into symmetric part and anti-symmetric part.

representation of SO(24). The last remaining symmetric traceless component is the massless spin-2 particle, which describes gravity.

The mass of second level is given by $\alpha' M^2 = 4$ and the states are given by the direct product of the second excited mode of the open string. The representation of the second excited mode of the open string is the symmetric traceless representation of SO(25) $\square\square$, and the direct product of the two is $\square\square \otimes \square\square$. By decomposing this Young diagram according to the usual rules, we can obtain the irreducible representation.

Unoriented String

The closed string are described by left-moving generator α_n^i and right-moving generator $\overline{\alpha}_n^i$. Strings described by the right- and left-moving generators are called orientable string, and such a string theory is called the Shapiro-Virasoro model. Similarly, strings that cannot distinguish between right-moving and left-moving generators are called unorientable. The quantum unorientable string state is described by a wave function $\Psi(x)$ that is invariant under the Ω dual $\sigma \to -\sigma$. The Ω dual is a transformation that makes the right-moving and left-moving coincide, actually swapping α_n^i and $\overline{\alpha}_n^i$. Also, by imposing invariance under the Ω dual, right-moving and left-moving become indistinguishable, and the antisymmetric components of the tensor vanish. The unorientable string theory can be obtained by imposing Ω duality on the orientable string theory.

Relations between Light-Cone and Old Covariant in Quantum Theory

At the classical level, the relation between the old covariant quantization and the light-cone quantization is obvious: gauge-fixing conformal symmetry or not. At the quantum level, however, the relation between these formalizations is nontrivial.

First of all, we briefly summarize the old covariant. The states generated by the harmonic oscillator α_n are imposed a constraint, the Virasoro condition. The physical state is obtained as the state that satisfies the Virasoro condition. Thus the general formula for the physical states are nontrivial.

The purpose here is to obtain a general expression for the physical states. The reasons that we have not considered the general physical states are that the states are generated by α_n, but the constraints are described by L_n, which is given by a nonlinear sum of α_n. In other words, we cannot know whether a state satisfies the Virasoro condition without actually computing it. To solve such a problem, we construct generators that are commutative with L_n. This means that we define the other Fock-space basis which are L_n and the remaining $D-1$ generators inspite of α_n^μ. In this way, the generators at a new basis have a clear commutation relation with L_n, and these generators are used to generate the physical states. These generators are called DDF (Del Giudice, Di Vecchia and Fubini) operators [55, 65]. The excited states generated by the DDF operator have a clear commutation relation with L_n, which makes it easy to determine whether they are physical or not, and hence we can obtain a general expression.

First of all, we consider a generator A_n^i such that there is a one-to-one correspondence with α_n^i, and assume that these generators produce a spectrum. Since the Virasoro conditions give one condition for each n, the algebra generated by the spectral generators (the spectral algebra) has $D-1$ generators for each n. The theory also requires a longitudinal wave mode A_n^-.

The vacuum of the bosonic string theory is a tachyon: $|0;p_0\rangle$. Also, for the gauge transformation to be consistent, it has to be $a=1, D=26$. Then the mass in the vacuum state is $p_0^2=2$. By using Lorentz symmetry, we choose the following system:

$$p_0^+ = 1, \quad p_0^- = -1, \quad p_0^i = 0. \tag{4.4.94}$$

Furthermore, we introduce the null vector k^μ

$$k_0^- = -1, \quad k_0^+ = k_0^i = 0 \tag{4.4.95}$$

Then $k_0^2 = 0,\ k_0 \cdot p_0 = 1$ is satisfied. The mass of the state is generally given by the Virasoro condition for $n=0$:

$$\alpha' M^2 = N - 1. \tag{4.4.96}$$

The momentum such that this mass-shell condition is satisfied is

$$p^\mu = p_0^\mu - N k_0^\mu \tag{4.4.97}$$

and a state with such a momentum is called a possible state.

Next we find the expression of A^i. The purpose is to construct a generator of the Fock space such that the commutation relation with L_n is tirivial. Since α_n^μ generates the states, we expect that we can replace α_n^μ by the vertex operator V corresponding to the state. The vertex operator V defined by (4.4.56) is a periodic function of 2π in τ except for the $\exp(ik\cdot p\tau)$ factor which appears from the $p^\mu\tau$ term in the expansion of the on-shell string $x^i(0,\tau)$. If $k^\mu = nk_0^\mu$, $p\cdot k = n$, and so the $\exp(ik\cdot p\tau)$ factor is also periodic. We consider the massless vector vertex operator with k^μ. The vertex operator corresponding to transverse polarization is

$$V^i = (nk_0, \tau) = \dot{x}^i(\tau) e^{inx^+(\tau)} \tag{4.4.98}$$

This operator is periodic in the subspace generated by the possible states of the Hilbert space. The V^i corresponds to the transverse state of the massless vector, and n describes the excited level of the state.

Since there is no τ-dependence in the generators that form the Fock space, we integrate (4.4.98) with τ:

$$A_n^i = \frac{1}{2\pi}\int_{-}^{2\pi} V^i(nk_0,\tau)d\tau \tag{4.4.99}$$

These operators are called DDF operator and correspond to α_n^i.

In the following, we focus on the commutation relations with L_n and A_i. First, since the vertex operator V is an operator of conformal weight $J=1$, we find

$$[L_m, V(\tau)] = -i\frac{d}{d\tau}(e^{im\tau V(\tau)}). \tag{4.4.100}$$

Integrating both sides by τ, from the periodicity of the vertex operator we find

$$[L_m, A_n^i] = 0. \tag{4.4.101}$$

And from the commutation relation with L_0, we find

$$[N, A_n^i] = -nA_n^i. \tag{4.4.102}$$

Therefore, the states defined by

$$A_{-n_1}^{i_1}\cdots A_{-n_m}^{i_m}\left|0;p_0\right\rangle \tag{4.4.103}$$

always satisfy the Virasoro condition, and the mass of this state is given by $N=\sum_j n_j$.

Next we consider the commutation relation between A_n^i. To find the commutation relation between A_n^i, we calculate the commutation relation of $\dot{x}$. The on-shell $\dot{x}$ is

$$\dot{x}^i(\tau) = \sum_{-\infty}^{\infty}\alpha_m^i e^{-im\tau}, \tag{4.4.104}$$

and then we find

$$[\dot{X}^i(\tau_1), \dot{X}^j(\tau_2)] = 2\pi\delta^{ij}\delta'(\tau_1-\tau_2). \tag{4.4.105}$$

Using this commutation relation, we obtain the following commutation relation:

$$[A_n^i, A_m^j] = n\delta_{ij}\delta_{m+n,0}. \tag{4.4.106}$$

From these commutation relations, the states generated by A_n^i from the tachyonic vacuum are physical states because these satisfiy the Virasoro condition. In light cone quantization, the physical states are generated by α_n^i, so A_n^i is the operator corresponding to α_n^i. This A_n^i simplifies the handling of the

Virasoro condition, and general physical states can be written down as states with A_n^i acting on them. This DDF operator plays an important role in the proof of the No-Ghost theorem.

As you can see, the zero-norm states correspond to the time directional component and this contribution determine the spacetime dimension. In other words, we can construct higher dimensional theory by considering multi-time configuration [66, 67].

4.4.3 BRST Quantization

Next we perform BRST quantization [68], where local symmetries are treated as global symmetry (BRST symmetry). The action of the bosonic string is

$$S^{orig} = -\frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{-g}g^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu\eta_{\mu\nu}. \tag{4.4.107}$$

This action has two-dimensional reparameterisation symmetry:

$$\delta x^\mu = f^\alpha\partial_\alpha x^\mu, \quad \delta g_{\alpha\beta} = f^\gamma\partial_\gamma g_{\alpha\beta} + \partial_\alpha f^\gamma g_{\gamma\beta} + \partial_\beta f^\gamma g_{\alpha\gamma} \tag{4.4.108}$$

This local symmetry is rewritten into a global symmetry using the BRST formulation.

Introducing BRST Symmetry

First, we replace the parameter of the local symmetry with the ghost field c and define the transformation law for x^μ and $g_{\alpha\beta}$.

$$f^\alpha \to \Lambda c^\alpha \tag{4.4.109}$$

With this replacement, the coordinate dependent parameter becomes ghost field and that transformation parameter becomes global. This global symmetry is known as BRST symmetry.

Next, since the local symmetry can be described by the global symmetry, BRST symmetry, we introduce a gauge-fixing term to fix the local symmetry:

$$S^{gf} = -\frac{1}{\pi}\int d^2\xi\lambda_{\alpha\beta}(\sqrt{-g}g^{\alpha\beta} - \sqrt{-\hat{g}}\hat{g}^{\alpha\beta}), \tag{4.4.110}$$

where $\hat{g}$ represents a fixed metric. The BRST transformation for $\lambda_{\alpha\beta}$ is 0.

To construct the kinetic term for the ghost field c, we introduce an anti-ghost field b, and we define its BRST transformation so that the action is invariant. Also from the algebra of reparameterisation, the transformation law for c is given by

$$\delta b_{\alpha\beta} = \Lambda\lambda_{\alpha\beta}, \quad \delta c = \Lambda c^\gamma\partial_\gamma c^\alpha. \tag{4.4.111}$$

With the covariant derivative as D^α, the final action is

$$S^{BRST} = S^{orig} + S^{gf} + S^{gh} \tag{4.4.112}$$

$$S^{gh} = -\frac{1}{\pi}\int d^2\xi\sqrt{-g}b_{\alpha\beta}(D^\alpha c^\beta + D^\beta c^\alpha - g^{\alpha\beta}D^\gamma c_\gamma) \tag{4.4.113}$$

If we set $\hat{g}^{\alpha\beta} = \eta^{\alpha\beta}$ and $\lambda_{\alpha\beta}$ on-shell, we can obtain

$$S^F = -\frac{1}{4\pi\alpha'}\int d^2\xi\eta^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x^\nu\eta_{\mu\nu} - \frac{1}{\pi}\int d^2\xi b_{\alpha\beta}(\partial^\alpha c^\beta + \partial^\beta c^\alpha - \eta^{\alpha\beta}\partial_\gamma c^\gamma) \tag{4.4.114}$$

Here, as before, the local symmetry is not completely fixed, and which is conformal symmetry. In other words, there are two conformal symmetries in this action: the conformal symmetry in the reparametrization invariance and the Weyl invariance, and the conformal symmetry in the BRST symmetry. Here BRST symmetry corresponding to Weyl symmetry and reparameterisation except for comformal symmetry was broken by integrating out $\lambda_{\alpha\beta}$.

Equation of Motion

The equations of motion are given by

$$\partial^2 x^\mu = 0, \quad \partial_\beta b^{\alpha\beta} = 0, \quad \partial^\alpha c^\beta + \partial^\beta c^\alpha - \eta^{\alpha\beta}\partial c = 0 \tag{4.4.115}$$

There are various possibilities that Lorentz covariant boundary condition satisfies the surface term.

One is the periodic condition corresponding to closed string:

$$x^\mu(\sigma) = x^\mu(\sigma + 2\pi), \quad b_{\alpha\beta}(\sigma) = b_{\alpha\beta}(\sigma + 2\pi), \quad c^\beta(\sigma) = c^\beta(\sigma + 2\pi) \tag{4.4.116}$$

The other is the Neumann condition corresponding to open string:

$$\partial_\tau x^\mu = 0, \quad b_{10} = 0, \quad c^1 = 0 \tag{4.4.117}$$

Calculating the Poisson brackets, we obtain

$$\begin{aligned} \{b_{01}(\sigma), c^1(\sigma')\} &= -\tfrac{i\pi}{2}\delta(\sigma' - \sigma) \\ \{c^0(\sigma), b_{00}(\sigma')\} &= -\tfrac{i\pi}{2}\delta(\sigma' - \sigma) \end{aligned} \tag{4.4.118}$$

For x^μ it is the same result in the old covariant quantization.

In order to treat open and closed strings equally, we fold the open string so that it satisfies the boundary condition at $\sigma = 0$.

$$\begin{aligned} c^0(\sigma) = c^0(-\sigma) \qquad c^1(\sigma) = -c^1(-\sigma) \\ b_{00}(\sigma) = b_{00}(-\sigma) \quad b_{01}(\sigma) = -b_{01}(-\sigma) \end{aligned} \tag{4.4.119}$$

Futhermore, if we rewrite

$$\begin{aligned} c(\sigma) = c^0 + c^1 = c^+ \\ b = -2(b_{00} + b_{01}) = -4b_{++}, \end{aligned} \tag{4.4.120}$$

the boundary condition for the open string becomes

$$c(\pi) = c(-\pi), \quad b(\pi) = b(-\pi), \tag{4.4.121}$$

and then it is just the boundary condition for closed string. To extend the range of σ, we impose

$$c(\sigma) = c(\sigma + 2\pi), \quad b(\sigma) = b(\sigma + 2\pi) \tag{4.4.122}$$

Using these expressions, the anti-commutation relations are given by

$$\begin{aligned} \{b(\sigma), c(\sigma')\} = 2i\pi\delta(\sigma' - \sigma) \\ \{b, b\} = \{c, c\} = 0 \end{aligned} \tag{4.4.123}$$

Since we have imposed periodicity on b and c, the mode expansions are

$$c(\sigma) = \sum_{-\infty}^{\infty} c_n e^{-in\sigma} \tag{4.4.124}$$

$$b(\sigma) = i\sum_{-\infty}^{\infty} b_n e^{-in\sigma}. \tag{4.4.125}$$

Substituting these mode expansions into the anti-commutation relations, we obtain

$$\{c_n, c_m\} = \{b_n, b_m\} = 0, \{c_n, b_m\} = \delta_{n+m,0}. \tag{4.4.126}$$

Quantization

Next we consider the representation that satisfies these algebras. We adopt the creation and annihilation operators and the representation in SU(2) since b_0 and c_0 form a two-dimensional Clifford algebra with indefinite metric.

$$\begin{aligned} c_0 \left|+\right\rangle = 0, b_0 \left|+\right\rangle = \left|-\right\rangle \\ c_n \left|\pm\right\rangle = b_n \left|\pm\right\rangle = 0 \end{aligned} \tag{4.4.127}$$

Since it is an indefinite metric, the norm is

$$\langle \pm | \pm \rangle = 0, \quad \langle + | - \rangle = \langle - | + \rangle = 1. \tag{4.4.128}$$

The states are generated by the creation operators α^μ_{-n}, b_{-n} and c_{-n} and the basis is given by $|\pm\rangle$. Thus

$$|\chi\rangle = \psi \,|+\rangle + \phi \,|-\rangle \,. \tag{4.4.129}$$

For closed string theory, we take the direct product of these states. Also, since b_0 and c_0 are commutative with the Hamiltonian, we can take $|\pm\rangle$ as the ground state.

Constraints and Virasoro Conditions

The above discussion so far is only to determin the properties of the state space, not the physical conditions. In quantum theory, it is the constraint conditions and the equations of motion that determine the state.

The constraint conditions can be calculated as

$$T_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} - \frac{1}{2}\eta_{\alpha\beta}\partial x^\mu \partial x^\nu \eta_{\mu\nu} + 4\alpha' \{ (b_{\alpha\gamma}\partial_\beta c^\gamma + b_{\beta\gamma}\partial_\alpha c^\gamma) + (\partial^\gamma b_{\alpha\beta})c_\gamma \} \tag{4.4.130}$$

By modal expansion, we define the Virasoro generators and calculate the commutation relation. Since the Virasoro generators for x is already given, we focus on the ghost part. The constraints for the ghost part are

$$T^{(c)}_{++} = -i \left[\tfrac{1}{2} c^+ \partial_+ b_{++} + \partial_+ c^+ b_{++} \right] \tfrac{1}{2} c^- \partial_- b_{--} + \partial_- c^- b_- \tag{4.4.131}$$

By mode-expanding these conditions, we can obtain the Virasoro generators corresponding to the ghost part:

$$L^{(c)}_m = \sum_{n=-\infty}^{\infty} [m(J-1) - n] b_{m+n} c_{-n}, \tag{4.4.132}$$

where J is the conformal dimension, which is 2 for b and -1 for c. The ordering is still required only for $m = 0$.

The commutation relation generated by the Virasoro generators for ghosts is

$$\left[L^{(c)}_m, L^{(c)}_n \right] = (m-n) L^{(c)}_{m+n} + A^c(m)\delta_{m+n} \tag{4.4.133}$$

$$A^c(m) = \frac{1}{12}\left[1 - 3k^2\right] m^3 + \frac{1}{6} m, \quad k = 2J - 1 \tag{4.4.134}$$

For $J = 2$, we have

$$A^c(m) = \frac{1}{6}(m - 13m^3) \tag{4.4.135}$$

Thus, the Virasoro algebra for the whole action is given by a combination of the conformal transformations of BRST symmetry and regidual symmetry. If we define

$$L_m = L_m^{\alpha} + L_m^{(c)} - a\delta_{m,0}, \tag{4.4.136}$$

then

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D-26}{12}n(n^2-1)\delta_{n+m,0}. \tag{4.4.137}$$

The second term on the right-hand side is a comformal anomaly, and its existence breaks conformal symmetry in quantum theory. Therefore, by taking $D = 26$, we eliminate the anomaly and make the symmetry in the classical theory and the quantum theory coincide. This means that the comformal transformation derived from the reparameterisation and the conformal transformation from the BRST symmetry can be performed spontaneously.

Physical States

In old covariant quantization, the Virasoro generators are the constraints, and the physical states satisfy

$$(L_n - a\delta_{n,1})\,|\chi\rangle = 0. \tag{4.4.138}$$

The basic idea is the same for BRST quantization, but since we introduced ghost and anti-ghost fields, we need to consider a larger Fock space.

the Virasoro algebra in general dimensions has an anomaly, so the property $Q^2 = 0$ is nontrivial. Also, the ghost number operator U has an ambiguity due to the order of the operators. In BRST quantization, the action has BRST symmetry, so the corresponding charge for open strings is

$$Q_B =: \sum c_{-n}\left(L_n - \frac{1}{2}L_{-n}^{(c)}\frac{1}{2}\delta_{n,0}\right) : . \tag{4.4.139}$$

Similarly, the ghost number operator for open is

$$U = \sum_{m=-\infty}^{\infty} : c_{-m}b_m : . \tag{4.4.140}$$

For closed strings, we can incorporate the left-moving contributions.

The BRST current is

$$J_+^B = 2c^+ \left(T_{++}^{(\alpha)} + \frac{1}{2} T_{++}^{(c)} \right). \tag{4.4.141}$$

The ghost number current is

$$J_+ = c^+ b_{++}. \tag{4.4.142}$$

Of course, these currents obey the conservation laws:

$$\partial_- J_+^B = \partial_- J_+ = 0. \tag{4.4.143}$$

The corresponding charges are given

$$Q_B = \frac{1}{2\pi} \int_0^\pi d\sigma \left(J_+^B + J_-^B \right) \tag{4.4.144}$$

$$U = \frac{1}{2\pi} \int_0^\pi d\sigma \left(J_+ + J_- \right) \tag{4.4.145}$$

In order to confirm the nilpotency, we calculate Q_B^2 explicitly:

$$Q_B^2 = \frac{1}{2}\{Q, Q\} = \frac{1}{2} \sum_{-\infty}^{\infty} \left([L_m, L_n] - (m-n) L_{m+n} \right) c_{-m} c_{-n} \tag{4.4.146}$$

Therefore, if $D = 26$ and $a = 1$, this charge satisfies the nilpotency. This means that the conformal anomaly cancels out. Indeed if we use the relation

$$L_m = \{Q, b_m\}, \tag{4.4.147}$$

and nilpotency, then

$$[L_m, Q] = [\{Q, b_m\}, Q] = 0. \tag{4.4.148}$$

So, using these equations, we find

$$\begin{aligned} [L_m, L_n] &= [L_m, \{Q, b_n\}] = \{Q, [L_m, b_n]\} \\ &= (m-n) \{Q, b_{m+n}\} = (m-n) L_{m+n} \end{aligned} \tag{4.4.149}$$

and the anomaly has disappeared.

Since a charge corresponding to a symmetry is the generator of the transformation, the transformation law for the field Y is in general

$$\delta Y = [\lambda Q, Y]. \tag{4.4.150}$$

Specifically, the transformation laws of the fields are

$$\delta X^\mu = \lambda c^+ \partial_+ X^\mu + \lambda c^- \partial_\neg X^\mu \tag{4.4.151}$$

$$\delta c^+ = \lambda c^+ \partial_+ c^+ \tag{4.4.152}$$

$$\delta b_{++} = 2i\lambda T_{++} \tag{4.4.153}$$

$$\delta T_{++} = 0. \tag{4.4.154}$$

Next, we consider the ambiguity of the ghost number operator:

$$U = \frac{1}{2}\left(c_0 b_0 - b_0 c_0\right) + \sum_{n=1}^{\infty}\left(c_{-n} b_n - b_{-n} c_n\right). \tag{4.4.155}$$

If we act on the ground state $|\pm\rangle$, we find that the eigenvalue is $\pm 1/2$. Therefore, the ghost numbers of all eigenstates are half integers.

Q is conserved current in this system and the constraint. Therefore, we impose

$$Q\left|\chi\right\rangle = 0. \tag{4.4.156}$$

Thus, the physical state is distinguished by a BRST cohomology class with a ghost number. Since the physical states do not contain ghost fields, we assume that they are proportional to the ground state $|\pm\rangle$, which has no ghost excitations. From the action, since the ghost field and the anti-ghost field do not exist symmetrically, the ground state $|+\rangle$ and $|-\rangle$ are not same states. Indeed the conformal weights of the ghost field and the anti-ghost field are different. Therefore, we need to consider which ground state should be adopted. Since $|\pm\rangle$ is a ground state, it satisfies the following condition.

$$c_n|\chi\rangle = b_n|\chi\rangle = 0, \quad n > 0 \tag{4.4.157}$$

These conditions can be interpreted as ghosts and anti-ghosts not being included in the state. Considering the state of $|-\rangle$, from $b_0\left|-\right\rangle = 0$, we have

$$0 = Q|\chi\rangle = \left(c_0\left(L_0^{(\alpha)} - 1\right) + \sum_{n>0} c_{-n} L_n^{(\alpha)}\right)|\chi\rangle \tag{4.4.158}$$

and which is equal to the Virasoro condition in old covariant quantization. On the other hand, if we adopt the ground state of $|+\rangle$, $|\chi\rangle$ vanishes due to c_0, so the first term in the above equation vanishes, and as a result, all the Virasoro conditions are no longer obtained. From the above, the physical state of the bosonic string is a cohomology class with ghost number -1/2.

From the nilpotency, there is ambiguity in the physical states:

$$|\chi\rangle + Q_B |\Lambda\rangle . \tag{4.4.159}$$

This second term corresponds to the degrees of freedom of the gauge transformation. Indeed, the second term is a physical and zero-norm state:

$$\langle\Lambda| Q^{\dagger} Q |\Lambda\rangle = 0. \tag{4.4.160}$$

4.5 Superstring Theory for RNS Formalism

We introduce fermions by imposing a supersymmetry on the theory. Supersymmetry is a symmetry that swaps spinor indices to vector indices. Polyakov action is given by (4.3.13), where there are two types of vector indices: μ for spacetime and α for the world sheet. Therefore, there are two ways to introduce supersymmetry into string theory. One is to impose supersymmetry on the spacetime index μ, called the Gree-Schwarz formalism. Since this is a spacetime symmetry, we only need to make a global supersymmetry. The other is for the world sheet index α and is called the RNS formalism. Since the world sheet is a local object, we need to impose local supersymmetry. However, since the symmetry we observe is that of spacetime, in RNS formalism we must impose a certain projection. Under this projection, the two theories coincide. In this section, we start with the RNS form, which is relatively easy to handle.

4.5.1 Classical RNS Formalism

Global World Sheet Supersymmetry

In RNS formalism we impose a supersymmetry on the index α on the world sheet. Since the world sheet is two-dimensional, the dimension of the Clifford algebra is two-dimensional, and the dimension of the spinor representation is also two-dimensional:

$$\gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{4.5.1}$$

Since $\mu = 1, ..., D$, we introduce D two-dimensional spinors. Futhermore in order to match the off-shell degrees of freedom of the bosons and fermions, we introduce D bosons N^{μ}. The action with global supersymmetry is

$$\int d^2\xi \left(-\frac{1}{2}\partial_\alpha x^{\frac{i}{2}} \partial^\alpha x^\nu - \frac{i}{2}\overline{\chi}^\mu \not{\gamma}\chi^\nu + \frac{1}{2}N^\mu N^\nu \right). \tag{4.5.2}$$

where χ_μ is the spinor representation of the world sheet Lorentz group SO(1,1) and the vector representation of the space-time Lorentz symmetry SO(1,D-1). The super partner χ_μ can be taken to Majorana spinor:

$$\overline{\chi} = \chi^\dagger \gamma^0 \tag{4.5.3}$$

The action describes a field theory in two dimensions, and in this sense SO(1,D-1) is an internal symmetry. Also, the action is invariant under global off-shell supersymmetric transformations of the world sheet.

$$\delta x^\mu = \bar{\epsilon}\chi^\mu, \quad \delta\chi^\mu = -i\gamma^\alpha \partial_\alpha x^\mu \epsilon, \tag{4.5.4}$$

The ϵ is a parameter for supersymmetry and Majorana fermion. This action is invariant under world sheet global supersymmetry and two dimensional Poincaré symmentry, and the conserved currents for these symmetries are

$$T_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x_\mu + \frac{i}{4}\overline{\psi}^\mu \gamma_{\{\alpha}\partial_{\beta\}}\psi_\mu - (\text{trace of first term and second term}) \tag{4.5.5}$$

$$J_\alpha = \frac{1}{2}\gamma^\beta \gamma_\alpha \psi^\mu \partial_\beta x_\mu. \tag{4.5.6}$$

Since supersymmetry and two dimensional Poincaré symmetry act on the world sheet, the index of current is for the world sheet. Also, $T_{\alpha\beta}$ is clearly traceless, and if we choose light cone coordinates, we obtain

$$T_{+-} = T_{-+} = 0 \tag{4.5.7}$$

Local Supersymmetry

Next we use consider the local supersymmetry. By using Noether method, we obtain the local supersimmetric action:

$$S = \frac{1}{2\pi\alpha'}\int d^2\xi e \left\{ -\frac{1}{2}\partial_\alpha x^\mu e^{\alpha\beta}\partial_\beta x^\nu - \frac{i}{2}\overline{\chi}^\mu \gamma^\rho \partial_\rho \chi^\nu + \frac{1}{2}N^\mu N^\nu + \frac{i\kappa}{2}\overline{\psi}_\alpha \gamma^\rho \partial_\rho x^\mu \gamma^\alpha \chi\nu - \frac{\kappa^2}{16}\overline{\psi}_\alpha \gamma^\beta \gamma^\alpha \psi_\beta \overline{\chi}^\mu \chi^\nu \right\} \eta_{\alpha\beta}. \tag{4.5.8}$$

Also we obtain the transformation laws:

$$\begin{gathered} \delta x^\mu = \bar{\epsilon}\psi^\mu, \quad \delta\chi^\mu = -i\gamma^\alpha \epsilon(\partial_\alpha x^\mu - \overline{\chi}^\mu \psi_\alpha \\ \delta e^\alpha_\mu = -2i\bar{\epsilon}\gamma^\alpha \psi_\alpha, \quad \delta\psi_\alpha = \nabla_\alpha \epsilon \end{gathered}, \tag{4.5.9}$$

where ψ_α is a gauge field for local supersymmetry, $e_\mu^{\ \alpha}$ is a gauge field for the reparameterisation and ∇_α is the spin connection.

Symmetries

We used the Noether method for the global world sheet supersymmetry and introduce the local supersymmetry for the world sheet:

$$\begin{array}{ccc} \delta x^\mu = i\bar{\epsilon}\chi^\mu & \delta\chi = (\gamma^\alpha D_\alpha x^\mu + N^\mu)\epsilon & \delta N^\mu = i\bar{\epsilon}\gamma^\alpha D_\alpha \chi^\mu \\ \delta e_\alpha^a = i\kappa\bar{\epsilon}\gamma^\alpha\psi_\alpha & \delta\psi_\alpha = \frac{2}{\kappa}D_\alpha\epsilon & \end{array} \tag{4.5.10}$$

$$D_\alpha x^\mu = \partial x^\mu - \frac{i\kappa}{2}\overline{\psi}_\alpha\chi^\mu \quad D_\beta\chi\mu = \left(D_\beta x^\mu - \frac{\kappa}{2}\gamma^\alpha D_\alpha x^\mu + N^\mu\right)\psi_\beta \tag{4.5.11}$$

Since the double supersymmetric transformations induce a reparameterisation symmetry, there is a two-dimensional reparameterisation invariant:

$$\begin{array}{ccc} \delta e_\alpha^a = f^\beta\partial_\beta e_\alpha^a + \partial_\alpha f^\beta e_\beta^a & \delta\psi_{\alpha A} = f^\beta\partial_\beta\psi_{\alpha A} + \partial_\alpha f^\beta\psi_{\beta A} & \\ \delta x^\mu = f^\beta\partial_\beta x^\mu & \delta\chi_A^\mu = f^\beta\partial_\beta\chi_A^\mu & \delta N^\mu = f^\beta\partial_\beta N^\mu \end{array}. \tag{4.5.12}$$

Also from the super algebra, there is a two-dimensional local Lorentz transformation

$$\begin{array}{cc} \delta e_\alpha^a = -w_b^a e_\alpha^b & \delta\psi_{\alpha A} = \frac{1}{4}(w^{ab}\gamma_{ab})_A^B\psi_{\alpha B} \\ \delta\chi_A^\mu = \frac{1}{4}(w^{ab}\gamma_{ab})_A^B\chi_B^\mu & \end{array} \tag{4.5.13}$$

As bosonic string theory, this action has also Weyl symmetry

$$\begin{array}{ccc} \delta e_\alpha^a = \Lambda e_\alpha^a & \delta\psi_\alpha = \frac{1}{2}\Lambda\psi_\alpha & \\ \delta x^\mu = 0 & \delta\chi^\mu = \frac{1}{2}\Lambda\chi^\mu & \delta N^\mu = -\Lambda N^\mu \end{array} \tag{4.5.14}$$

Finally, the following symmetry is induced:

$$\delta\psi_\alpha = \gamma_\alpha\xi \tag{4.5.15}$$

This symmetry follows from the identity of the two-dimensional Clifford algebra, and is necessary for the superconformal symmetry to form a closed algebra. This symmetry is called S-symmetry. This action is not invariant under the spacetime supersymmetry.

Using these symmetries, we can set

$$e_\alpha^a = \delta_\alpha^a, \quad \psi = 0. \tag{4.5.16}$$

Equation of Motion

The equations of motion and constraints are given by

$$\partial^2 x^\mu = 0, \quad \partial^\alpha \gamma_\alpha \chi^\mu = 0 \tag{4.5.17}$$

$$\begin{gathered} T_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu} + \tfrac{i}{2}\overline{\chi}^\mu \gamma_\beta \partial_\alpha \chi^\nu \eta_{\mu\nu} + \tfrac{\eta_{\alpha\beta}}{2}(\partial_\gamma x^\mu \partial^\gamma x^\nu + i\overline{\chi}^\mu \partial^\sigma \gamma_\sigma \chi^\nu)\eta_{\mu\nu} = 0 \\ J^\alpha = \partial^\beta \gamma_\beta x^\mu \gamma^\alpha \chi^\nu \eta_{\mu\nu} = 0 \end{gathered} . \tag{4.5.18}$$

And the boundary term is

$$\overline{\chi}^\mu \gamma^1 \delta \chi^\nu \eta_{\mu\nu}|_0^\pi = 0. \tag{4.5.19}$$

Since χ is a representation of the two-dimensional Clifford algebra, we can write it down in Weyl spinors:

$$(\chi_-^\mu \delta \chi_-^\nu - \chi_+^\mu \delta \chi_+^\nu)\eta_{\mu\nu}|_0^\pi = 0. \tag{4.5.20}$$

Although there are several possibilities of boundary conditions that satisfy this boundary term, there are only two that do not give a non-trivial solution. Those are the Ramond sector and the Nieve-Schwarz sector. For open string,

$$\begin{aligned} (R) &\quad \chi_+(\pi) = \chi_-(\pi) \\ (NS) &\quad \chi_+ = -\chi_-(\pi) \end{aligned} \tag{4.5.21}$$

and for closed string

$$\begin{aligned} \chi_\pm^\mu(\pi) &= \chi_\pm^\mu(-\pi) \quad (R) \\ \chi_\pm^\mu(\pi) &= -\chi_\pm^\mu(-\pi) \quad (NS). \end{aligned} \tag{4.5.22}$$

As with bosonic strings, closed strings are given by the direct product of open strings, so we focus on only the open string

We define the following field:

$$\Psi^\mu(\sigma) = \frac{1}{\alpha'} \begin{cases} \chi_+^\mu(\sigma) & 0 < \sigma < \pi \\ \chi_-^\mu(-\sigma) & -\pi < \sigma < 0 \end{cases} . \tag{4.5.23}$$

Using this notation, the boundary condition is

$$\begin{aligned} \Psi^\mu(\pi) &= \Psi^\mu(-\pi) \quad (R) \\ \Psi^\mu(\pi) &= -\Psi(\pi) \quad (NS) \end{aligned} \tag{4.5.24}$$

From these conditions, the solution is a mode expansion of Ψ using the light cone coordinates:

$$\begin{aligned} \Psi^\mu &= \sum_n d_n^\mu e^{-in\xi^+} \quad (R) \\ \Psi^\mu &= \sum_r b_r^\mu e^{-in\xi^-} \quad (NS). \end{aligned} \tag{4.5.25}$$

and then the Poisson brackets and constraints are

$$\begin{aligned} \{d_n^\mu, d_m^\nu\} &= -i\eta^{\mu\nu}\delta_{n+m,0} \quad (R) \\ \{b_r^\mu, b_s^\nu\} &= -i\eta^{\mu\nu}\delta_{r+s,0} \quad (NS) \end{aligned} \tag{4.5.26}$$

$$\begin{aligned} G_r &= \sum \alpha_{-n} b_{r+n} \quad (NS) \\ F_n &= \sum \alpha_{-m} d_{n+m} \quad (R) \end{aligned} \tag{4.5.27}$$

The algebra generated by the constraints are given by

$$\begin{aligned} \{L_n, L_m\} = -i(n-m)L_{n+m} \quad \{L_n, F_m\} = -i\left(-m + \tfrac{n}{2}\right) F_{n+m} \\ \{F_n + F_m\} = -2iL_{n+m} \end{aligned} \tag{4.5.28}$$

for Ramond sector and

$$\begin{aligned} \{L_n, L_m\} = -i(n-m)L_{n+m} \quad \{L_n, G_m\} = -i\left(\tfrac{n}{2} - r\right) G_{n+r} \\ \{G_r, G_s\} = -2iL_{r+s} \end{aligned} \tag{4.5.29}$$

for NS sector.

For the closed string, we use these sectors independently for left-moving and right-moving.

4.5.2 Old Covariant Quantization

Virasoro conditions

First, we replace the Poisson bracket of the bosonic field in the commutation relation and the Poisson bracket of the fermionic field in the anti-commutation relation:

$$\begin{aligned} [\alpha_n^\mu, \alpha_m^\nu] &= n\delta_{n+m,0}\eta^{\mu\nu} \\ \{d_n^\mu, d_m^\nu\} &= -i\eta^{\mu\nu}\delta_{n+m,0} \quad (R) \\ \{b_r^\mu, b_s^\nu\} &= -i\eta^{\mu\nu}\delta_{r+s,0} \quad (NS). \end{aligned} \tag{4.5.30}$$

Since these operators are non-commutative, we can redefine the constraint using the normal order and recompute the commutation relation. For R sector, we find

$$\begin{array}{c}[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{8}n(n^2-1)\delta_{n+m,0} \quad [L_n, F_m] = -\left(-m+\frac{n}{2}\right)F_{n+m} \\ \{F_n + F_m\} = 2L_{n+m} + \frac{D}{2}n^2\delta_{n+m,0}\end{array}, \tag{4.5.31}$$

where the anomaly term is different from the bosonic string anomaly $D/12$. This is because there is a contribution from the local supersymmetry.

Similarly for NS Sector, we find

$$\begin{array}{c}[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{8}n(n^2-1)\delta_{n+m,0}, \quad [L_n, G_m] = -\left(\frac{n}{2}-r\right)G_{n+r} \\ \{G_r, G_s\} = 2L_{r+s} + \frac{D}{2}\left(r^2-\frac{1}{4}\right)\delta_{r+s,0}\end{array}. \tag{4.5.32}$$

From these commutation relations, $L_\pm, L_0$ and $G_{\pm 1/2}$ make an OSp(1|2) subalgebra.

These equations form an algebra different from the classical theory because of the superconformal anomaly. The existence of this anomaly shows that the symmetry that existed in classical theory is broken in quantum theory.

Spectrum

Next we derive the spectrum of the open string for each sector. The Virasoro condition for the zero mode is

$$L_0 = L_0^{(\alpha)} + L_0^{(b)}, \quad \text{or} \quad L_0^{(\alpha)} + L_0^{(d)} \tag{4.5.33}$$

and we use

$$N = \sum_1^\infty \alpha_{-m}\alpha_m, \quad N^b = \sum_{1/2}^\infty r b_{-r} b_r, \quad N^d = \sum_1^\infty m d_{-m} d_m, \tag{4.5.34}$$

$$p^2 = -M^2, \tag{4.5.35}$$

then the Virasoro condition for these zero modes is

$$\alpha' M^2 = N + \text{const.} \tag{4.5.36}$$

This condition is called the mass shell condition because it corresponds to the equation of motion.

From $[\alpha_0, \alpha_0] \sim -1$ and $\{d_0, d_0\} \sim -1$, there are ghost modes, and these modes are the cause of the superconformal anomaly. Since the anomaly for gauge symmetry breaks the unitarity of the theory, we must remove the superconformal anomaly by setting D and a appropriately as well as the bosonic string. In particular, if a and D are critical values enough to just eliminate the ghost, a zero-norm state exists, and this zero-norm state plays the role of a gauge degrees of freedom.

In the following, for simplicity, we consider only the NS sector. The R sector can be calculated in the same way as the NS sector. Let $|0,k\rangle$ be a ground state satisfying $k^2 = 2a$ which is followed from mass shell condition. Consider the excited state $|\phi\rangle = G_{-1/2}|0,k\rangle$. if we impose the mass-shell condition, we find $k^2/2 = a - 1/2$. Here, the basis of the state space is $\alpha_{-n},\ d_{-n}$ and b_{-r}, so $|\phi\rangle$ is not the first excited state. If we calculate the norm of this state, we find

$$\langle\phi|\phi\rangle = 0. \tag{4.5.37}$$

Therefore, $|\phi\rangle$ is a zero-norm state (spurious state). We impose the remaining $L_{n\neq 0}$ and G_r Virasoro conditions on this state. The condition with the smallest value in n and r is $r = 1/2$. Thus, the Virasoro condition of $G_{1/2}$ is imposed on the first excited state:

$$G_{1/2}(G_{-1/2}|0,k\rangle) = 0. \tag{4.5.38}$$

From this condition, we find $a = 1/2$. Now, if $\left|\tilde{\phi}\right\rangle$ is a physical state that satisfies all the Virasoro conditions, we can see that the same argument holds if we replace $|0,k\rangle$ with $\left|\tilde{\phi}\right\rangle$.

Next, we consider $|\phi\rangle = (G_{-3/2} + \lambda G_{-1/2}L_{-1})\left|\tilde{\phi}\right\rangle$. If we impose Virasoro condition

$$G_{3/2}|\phi\rangle = 0, \quad G_{1/2}|\phi\rangle = 0, \tag{4.5.39}$$

we find $\lambda = 2$ and $D = 10$

Higher-order states and Virasoro conditions can be generated by the product of $G_{\pm 1/2}$ and $G_{\pm 3/2}$, so there is no need to consider them. The $G_{-1/2}\left|\tilde{\phi}\right\rangle$ and $G_{-3/2}\left|\tilde{\phi}\right\rangle$ used in this calculation correspond to the gauge degrees of freedom. Also since $\left|\tilde{\phi}\right\rangle$ is an arbitrary physical state, the gauge degrees of freedom are infinite dimensional, which is the same as the gauge transformations in field theory.

Next we derive the Virasoro conditions for the states generated by the basis $\alpha,\ d$ and b of the Fock space. [4] We discuss the states in each sectors sepalately. For NS sector, the constraints are

$$\begin{aligned} G_r|\Phi\rangle &= 0 \qquad & r > 0 \\ (L_n - \delta_{n,0})|\Phi\rangle &= 0 \qquad & n \geq 0 \end{aligned} \tag{4.5.40}$$

Since the only independent variables are $x_0^\mu,\ p^\mu,\ \alpha_n^\mu$ and b_r^μ, we reparesent p^μ as derivative and α_n^μ as creation and annihilation operators. Expanding the state of the general string with the creation operators,

[4]The previous discussion was about the Virasoro condition for zero-norm (spurious) states, and the discussion that follows is about the Virasoro condition for non-zero-norm states. The difference is whether they are generated by $\alpha,\ b$ and d or by $L,\ G$ and F.

we find

$$|\Phi\rangle = (\phi(x) + i\alpha^{\mu}_{-1}B_{\mu} + ib^{\mu}_{-1/2}A_{\mu} + \frac{1}{2}b^{\mu}_{-1/2}b^{\nu}_{-1/2}S_{\mu\nu} + ...)\,|0\rangle \tag{4.5.41}$$

The conditions for the Fock space are

$$\begin{aligned} \alpha^{\mu}_{n}\,|0,p\rangle = 0 \quad n > 0 \\ b^{\mu}_{r}\,|0,p\rangle = 0 \quad r > 0 \end{aligned} \tag{4.5.42}$$

We impose constraints on this state. Focus on the lower order, for $L_0 - 1/2$

$$\begin{aligned} (\alpha'\partial^2 + \tfrac{1}{2})\phi = 0 \qquad & \partial^2 A_{\mu} = 0 \\ (\alpha'\partial^2 - \tfrac{1}{2})S_{\mu\nu} = 0 \quad & (\alpha'\partial^2 - \tfrac{1}{2})B_{\mu} = 0 \end{aligned} \tag{4.5.43}$$

and gives the Klein-Gordon equations for each field. From these conditions, ϕ is a tachyon and A^{μ} is a massless vector. For $G_{1/2}$, we find

$$\begin{aligned} \partial^{\mu}A_{\mu} = 0 \\ \sqrt{2\alpha'}\partial S_{\mu\nu} - B_{\nu} = 0 \end{aligned} . \tag{4.5.44}$$

These conditions are regarded as the gauge condition. The calculations we performed to find D and a are, for example, for the first excited mode, that the gauge transformation is given by

$$i\alpha^{\mu}_{-1}B_{\mu}\,|0,k\rangle \to i\alpha^{\mu}_{-1}B_{\mu}\,|0,k\rangle + G_{-1/2}\left|\tilde{\phi}\right\rangle \tag{4.5.45}$$

and we have calculated the conditions for this gauge-fixed state to be physical.

Next we consider the R sector. The states are constructed in the same way as in the NS sector, but it cannot be expressed by the creation and annihilation operators because d^{μ}_{0} cannot be paired with creation and annihilation. Therefore, we need to express the algebra generated by d^{μ}_{0} in a different way.

$$\{d^{\mu}_{0}, d^{\nu}_{0}\} = \eta^{\mu\nu} \tag{4.5.46}$$

Since this algebra has the same form as the 10-dimensional Clifford algebra, we represent it with a spin on the state. That is, the degrees of freedom generated by d^{μ}_{0} correspond to the spinor degrees of freedom so the R sector is a fermion. The constraints are

$$\begin{aligned} F_n\,|\psi\rangle = 0 \quad n \geq 0 \\ L_n\,|\psi\rangle = 0 \quad n > 0 \end{aligned} \tag{4.5.47}$$

Since the general string state has the spinor index ϵ, we expand this state like

$$|\phi\rangle_\epsilon = \{\lambda_\epsilon(x) + \alpha^\mu_{-1}\psi^1_{\mu\epsilon}(x) + d^\mu_{-1}\psi_{\mu\epsilon}(x) + ...\}\,|0;p\rangle\,. \tag{4.5.48}$$

The conditions for the Fock space are

$$\begin{aligned} \alpha^\mu_n\,|0;p\rangle = 0 \quad n > 0 \\ d^\mu_n\,|0;p\rangle = 0 \quad n > 0 \end{aligned}\,. \tag{4.5.49}$$

From the condition on F_0, we find

$$\begin{aligned} \partial_\mu d^\mu_0 \lambda = 0 \\ \sqrt{\alpha'}\partial_\nu d^\nu_0 \psi^1_\mu = \psi^2_\mu \quad -\sqrt{\alpha'}\partial_\nu d^\nu_0 \psi^2_\mu = \psi^1_\mu \end{aligned}\,. \tag{4.5.50}$$

These are now the Dirac equations for each field.

As you have seen in bosonic string theory, by considering the multi-time configuration, we can realize less than 10 dimensional superstring thoery. In [69], the author proposed the 4+2 dimensional superstring theory as two-time physics. Also Since local supersymmetry reduces the comformal anomaly, by introducing the extended local supersymmetry such as $N = 2$ or $N = 4$, we can also realize less than 10 dimensional superstring theory [55].

4.5.3 Light-Cone Quantization

Superstring theory is an invariant theory under reparameterisation, Weyl symmetry, and world sheet local supersymmetry, and we fixed these local symmetries using gauge-fixing conditions. However, this gauge fixing could not fix the superconformal symmetry which makes the action trivially invariant. In old covariant quantization, we dealt with this superconformal symmetry by quantizing it and imposing constraints on the quantum states. Here we quantize only the physical degrees of freedom by imposing an light cone gauge condition to solve the super-Virasoro condition. In the old covariant quantization, $D = 10$ and $a = 1/2$ was found by imposing the consistency of the gauge transformation of the quantum state after the quantization. In the light cone quantization, the dimension D and the intercept a could not be determined in the same way as in the old covariant quantization, because there is no superconformal symmetry in quantum theory, but they could be determined by imposing consistency with the spacetime Lorentz symmetry in quantum theory.

Light Cone Gauge

As in the bosonic string theory, we fix the reparameterisation invariance and the Weyl invariance by imposing a conformal gauge for the world sheet metric g: $g_{\alpha\beta} = \eta_{\alpha\beta}$. Since this gauge condition leaves the theory with conformal symmetry, we further impose the light cone gauge condition:

$$X^+(\sigma, \tau) = x^+ + p^+\tau \tag{4.5.51}$$

In addition to these symmetries, the action is invariant under local world sheet supersymmetry. Therefore, one more gauge condition must be imposed:

$$\psi^+ = 0 \tag{4.5.52}$$

Since local supersymmetry forms a closed algebra with conformal symmetry which is superconformal algebra, we have to confirm the consistency with the gauge condition for conformal symmetry (4.5.51). If we consider the global supersymmetric transformation (4.5.51), we find

$$\delta X^+ \bar{\epsilon}\psi^+ = 0 \tag{4.5.53}$$

and thus there is no contradiction.

Next, in order to simplify the calculation under the light cone gauge, we use the light cone coordinates:

$$\sigma^\pm = \tau \pm \sigma, \quad \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \tag{4.5.54}$$

$$\sigma^\pm = \tau \pm \sigma, \quad \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \tag{4.5.55}$$

Also in order to treat fields with $\sigma^\pm$ as a variable equivalently, we define the following field:

$$\psi^\mu(\sigma^\pm) = \begin{cases} \psi^\mu_+(\sigma^+) & \sigma > 0 \\ \psi^\mu_-(\sigma^-) & \sigma < 0 \end{cases} \tag{4.5.56}$$

The constraints for the superconformal symmetry are

$$\begin{aligned} J_+ &= \psi \cdot \partial_+ X = 0 \\ T_{++} &= (\partial_+ X)^2 + \tfrac{i}{2}\psi \cdot \partial_+ \psi = 0. \end{aligned} \tag{4.5.57}$$

Substituting the gauge condition (4.5.51), (4.5.52) into these constraints, we find

$$\begin{aligned} \psi^- &= \tfrac{2}{p^+}\psi^i \partial_+ X^i \\ \partial_+ X^- &= \tfrac{1}{p^+}(\partial_+ X^i \partial_+ X^i + \tfrac{i}{2}\psi^i \partial_+ \psi^i) \end{aligned} \tag{4.5.58}$$

Furthermore, Fourier expansions of X and ψ are

$$\begin{aligned} b_r^- &= \frac{1}{p^+}\sum_{i=1}^{D-2}\sum_{s=-\infty}^{\infty}\alpha_{r-s}^i b_s^i \\ \alpha_n^- &= \frac{1}{2p^+}\sum_{i=1}^{D-2}\left(\sum_{m=-\infty}^{\infty} :\alpha_{n-m}^i\alpha_m^i: + \sum_{r=-\infty}^{\infty}\left(r-\frac{n}{2}\right) :b_{n-r}^i b_r^i;\right) - \frac{a\delta_{n,0}}{2p^+}. \end{aligned} \tag{4.5.59}$$

From these expressions, $p^+\alpha_n^-$ and $p^+b_r^-$ have the same form as the super-Virasoro generators, so we find the super-Virasoro algebra obtained by old covariant quantization. However, since the old covariant quantization is made with α^μ instead of α^i, we have to replace D in the anomaly term in the old covariant with $D-2$:

$$\begin{aligned} A(m) &= \frac{D-2}{8}(m^3-m) + 2am \\ B(r) &= \frac{D-2}{2}\left(r^2-\frac{1}{4}\right) + 2a \end{aligned} \tag{4.5.60}$$

As in the bosonic string theory, we require the D dimensional Lorentz symmetry, we obtain $D=10, a=1/2$.

4.5.4 Spacetime Supersymmetry and GSO Projection

We derived the spectrum of the string in RNS formalism, and all of these spectra are supersymmetric invariants on the world sheet. However, the world sheet is not physical. In other words, the supersymmetry of the world sheet is not observable if considering the effective field theory of point particles at low-energy. In order for supersymmetry to exist in the low-energy effective theory, string theory must be supersymmetric not only in the world sheet but also in spacetime. In order to impose spacetime supersymmetry on RNS string theory, we extract a spectrum from the spectrum of RNS string theory that satisfies the spacetime supersymmetry. In other words, by imposing a projection on the spectrum, we can construct a theory in which spacetime supersymmetry and world sheet supersymmetry are compatible. The NS sector represents bosons, the R sector represents fermions, and these degrees of freedom must be equal on-shell due to the spacetime supersymmetry. With the above manifest, we construct a spacetime supersymmetric string theory.

Secondly, the RNS form of string theory is not a consistent theory even for $D=10$ and $a=1/2$ at the following points.

One of them is the exist of tachyon. Since the NS sector contains tachyon and the vacuum is unstable, it is better to remove them by this projection. In principle it is possible to extract the state so that it contains a tachyon as long as the spacetime supersymmetry can exist.

The other is the statistical nature of particles. The ψ^μ is a spinor on the world sheet and is a Grassmann odd. However, ψ^μ is a boson of spacetime Lorentz symmetry. In other words, the boson state $|\phi\rangle$ is a commutative state, but $\psi^\mu |\phi\rangle$ is anti-commutative, even though it is a boson. To solve this problem, we can keep only the even-acting states of ψ^μ and remove the odd-acting states of ψ^μ by a projection. This requirement leads to the κ symmetry.

In order to satisfy these two requests, the projection must contain an operator that counts the number of ψ^μ which means the number of b^μ, such a measure is G-parity. We define the operator $(-1)^F$ to count the G-parity. Since we want to consider the low energy effective theory, we require that the projection leaves massless particles. Therefore, we define $(-1)^F = 1$ for massless particles, and remove the state where $(-1)^F = -1$ and then

$$(-1)^F(\psi^{\mu_1}\cdots\psi^{\mu_n}|\phi\rangle) = (-1)^{-n}(\psi^{\mu_1}\cdots\psi^{\mu_n}|\phi\rangle). \tag{4.5.61}$$

This projection is called GSO projection. In fact the RNS string theory imposed the GSO projection is invariant under spacetime supersymmetry. Indeed, under the GSO projection, a spin 3/2 particle remains, which is coupled to a spin 1/2 charge. Since the spin 1/2 charge is a supersymmetric charge, this spin 3/2 particle is a gauge field for the spacetime supersymmetry. In the following, we specifically investigate RNS superstring theory, which imposed the GSO projection.

The massless spectrum in the NS sector is $b^\mu_{-1}|0;k\rangle$, so it is a vector particle. On the other hand, the massless spinor is a state that satisfies the super-Virasoro condition $F_0|\psi\rangle = 0$. The ground state $|\psi\rangle$ is a spinor. Since $F_0|\psi\rangle = 0$ is a Dirac equation, we write $u^a(k)$ as the solution for the Dirac equation where a is the spinor index, and k is the momentum. If supersymmetry exists in the theory, these states must form a multiplet. If we denote the coefficient of $b^\mu_{-1}|0;k\rangle$ as A_μ, A_μ has 10 components. Of these, the physical degrees of freedom are the eight transverse components. On the other hand, $|\psi\rangle$ is a 10-dimensional spinor, and its degrees of freedom are complex $2^{10/2} = 32$. However, for $D = 10$, the spinor can be taken to be a Mayorana-Weyl spinor, so the degrees of freedom are real 16 components. Furthermore, by the Dirac equation, only 8 degrees of freedom are physical degrees of freedom. Therefore, in the on-shell, the degrees of freedom of bosons and fermions coincide in 8 components.

In order to confirm this, we first consider the Majorana condition. The massless Dirac equation is $\Gamma^\mu\partial_\mu\psi = 0$. If all components of the gamma matrix are real or imaginary, we can impose a condition that makes ψ real and this spinor is called a Majorana representation.

If $D = 10$, we can construct a representation such that all the components of the gamma matrix are imaginary. In this case, Γ^0 is antisymmetric and the remaining gamma matrices are symmetric. In general, we can create a Majorana fermion for $D = 2, 3, 4 \bmod 8$.

Of the 10 gamma matrices, the 8 components Γ^i correspond to the transverse components. These form the Clifford algebra for SO(8), which is a subalgebra of SO(1,9). If we denote the generators of the Clifford algebra as $\gamma^i, i = 1, ...,$ the algebra is $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$. Here, γ^i is a real symmetric 16×16 matrix. By using the Clifford algebra for SO(8), we construct the Clifford algebra for SO(1,9), $\{\Gamma^\mu, \Gamma^\nu\} = -2\eta^{\mu\nu}$. Using the 16×16 matrix γ^i and the 2×2 Pauli matrix σ, the 32×32 matrix Γ^μ can be constructed as follows:

$$\Gamma^0 = \sigma_2 \otimes 1_{16}, \quad \Gamma^i = \gamma_1 \otimes \gamma^i, \quad \Gamma^9 = i\sigma^3 \otimes 1_{16}. \tag{4.5.62}$$

We adopt the representation of the Pauli matrix such that σ_1, σ_3 are real and σ_2 are imaginary. Then all components of Γ^μ are imaginary. Therefore, we can impose the Majorana condition.

Next, we confirm that we can impose the Weyl condition in 10 dimension . Whenever the spacetime dimension D is even, we can define a chiral matrix:

$$\Gamma_{11} = \Gamma^0 \cdots \Gamma^9. \tag{4.5.63}$$

This chiral matrix satisfies the following properties:

$$\{\Gamma_{11}, \Gamma^\mu\} = 0, \quad (\Gamma_{11})^2 = 1 \tag{4.5.64}$$

The eigenvalues of this matrix are ± 1 from $(\Gamma_{11})^2 = 1$:

$$\Gamma_{11}\psi = \pm\psi \tag{4.5.65}$$

The eigenvalue ± 1 is called the chirality. Since the eigenvalues are given like that, we can construct the projection operators

$$\frac{1 \pm \Gamma_{11}}{2} \tag{4.5.66}$$

which leave only spinors with a certain chirality. Spinors with a defined chirality are called Weyl spinors.

Next, we consider the conditions under which the Weyl condition can be imposed on the Majorana spinors. Since the gamma matrix has all components as imaginary numbers, Γ_{11}, which is created by multiplying an imaginary number by an even number of times, is a real matrix. This shows that the Majorana and Weyl conditions are consistent. In other words, we can take ψ to be a real spinor with positive chirality. In contrast to the 10-dimensional case, in the 4-dimensional case, the Majorana and Weyl spinor can be defined, but they cannot be taken to be real Weyl spinor because γ_5 is imaginary. In this way, the only dimension where the Weyl and Majorana conditions are consistent is if $D = 2 \bmod 8$ from the general theory of Clifford algebra.

If $D = 10$, the Majorana-Weyl spinor is a real $32/2 = 16$ component. If we impose the Dirac equation

$\Gamma \cdot \partial\psi = 0$ on this Majorana-Weyl spinor, the 8 component is related to the other 8 components by the Dirac equation. Therefore, the physical degree of freedom of on-shell ψ is 8 components. Since this degree of freedom is equal to the degree of freedom of the on-shell gauge field A^μ, ψ and A_μ can form a supermultiplet.

From the above discussion, we found that the $D = 10$ massless fermion takes on the Majorana-Weyl. This means that it forms a massless supermultiplet. The Weyl condition means that the spinor, which is an R-sector ground state, is an eigenstate of Γ_{11}. The extension of this Weyl condition to the Weyl condition for fermions at an arbitrary mass level can be defined by

$$P_R = \Gamma_{11}(-1)^{\Sigma_1^\infty d_n^\dagger d_n} \equiv (-1)^F. \tag{4.5.67}$$

Since this definition is a Weyl condition that can be imposed on fermions of an arbitrary mass level, P_R satisfies

$$\{P_R, d_n^\mu\} = 0. \tag{4.5.68}$$

Also, since ψ^μ in the R sector is linear in d_n^μ, we find

$$\{P_R, \psi^\mu\} = 0 \tag{4.5.69}$$

Futhermore, P_R satisfies $(P_R)^2 = 1$, indicating that it is a projection operator.

On the other hand, the GSO conditions imposed on the state of the NS sector are

$$P_{NS} \left|\phi\right\rangle = \left|\phi\right\rangle, \quad P_{NS} = (-1)^{\sum_{r=1/2}^\infty b_r^\dagger b_r - 1} \equiv (-1)^F. \tag{4.5.70}$$

In fact, the even-order mass in the NS sector is given by $\sqrt{(2s-1)/2\alpha'}$, which is not present in the R sector. This operator satisfies $P_{NS}^2 = 1$, and P_{NS} makes the masses equal for bosons and fermions, and removes the tachyons.

From the above, the massless sector forms a supermultiplet, and in order for the theory to have supersymmetry, the Weyl condition must be imposed to the R sector, which is, if we write $\left|\psi\right\rangle$ for any R-sector state, given by

$$P_R \left|\psi\right\rangle = \left|\psi\right\rangle \tag{4.5.71}$$

Of course, we can also impose a Weyl condition of $P_R \left|\psi\right\rangle = -\left|\psi\right\rangle$ that leaves the opposite chirality. This Weyl condition is a GSO projection, as it is an operator that removes half of the chirality.

Here, the Weyl condition of (4.5.71) does not imply the removal of positive chirality at all mass levels.

Any state belonging to the R sector is written as

$$d^{i_1}_{-m_1} \cdots d^{i_n}_{-m_n} |\alpha\rangle . \tag{4.5.72}$$

From the GSO condition (4.5.71), the chirality of the ground state $|\alpha\rangle$ is positive. If P_R is applied to this state, $P_R = (-1)^F = (-1)^n$. In other words, we are left with positive chirality for $n = 0$, negative chirality for $n = 1$.

Specifically, we consider imposing the GSO condition on massive fermions. Since Γ_{11} is not commutative with $i\Gamma\cdot\partial+m$, the massive spinor representation of the Lorentz group cannot be taken to be a Weyl spinor. However, in string theory, since the commutation relation with the Dirac operator F_0 is $\{P_R, F_0\} = 0$, the GSO projection can be imposed on massive spinors. We consider the first excited fermionic state $\alpha^i_{-1} |0\rangle u^1$ and $d^i_{-1} |0\rangle u_2$ in light-cone string theory. From the GSO condition for the R sector P_R, u_1 and u_2 must be Majorana Weyl spinors with different chirality:

$$\Gamma_{11} u_1 = u_1, \quad \Gamma_{11} u_2 = -u_2 \tag{4.5.73}$$

Putting these two spinors together, we can create a Majorana spinor, which is an irreducible massive representation of the Lorentz group.

A necessary condition for the theory to maintain supersymmetry is that the degrees of freedom of the fermions and bosons be equal at each mass level. By the Weyl condition, the degree of freedom of a massless fermion is 8, which is the same degree of freedom as the physical massless vector. By counting up the degrees of freedom of fermions and bosons at higher mass levels, we can confirm that they are consistent at each level [26].

Next, to see that the projection operators are consistent with the RNS superstring theory, we can investigate the commutation relations, since superstring theory is completely determined by constraints. Indeed we find the commutativities:

$$\begin{aligned} \{P_{NS}, G_r\} = 0 \quad [P_{NS}, L_n] = 0 \\ \{P_R, F_n\} = 0 \quad\quad [P_R, L_n] = 0 \end{aligned} \tag{4.5.74}$$

Thus, there is no contradiction. Also, the projection operator extracts a spacetime supersymmetric transformation from the world sheet local supersymmetric transformation, and this symmetry is just the κ symmetry.

Closesd String Spectra and TypeIIA / II B String Theory

Finally, we use the above results to derive the spectrum of the closed string. Since the closed string is multiplied by the direct product of the open string, there are four sectors of states: (NS,NS), (NS,R), (R,NS), and (R,R). In these sectors, fermions are (NS,R) and (R,NS). The constraint conditions for each sector are

$$\begin{aligned} G_r \left|\phi\right\rangle - \overline{G}_r \left|\phi\right\rangle = 0 r \geq 1/2 \\ (L_n - \frac{1}{2}\delta_{n,0}) \left|\phi\right\rangle = (\overline{L}_N - \frac{1}{2}\delta_{n,0}) \left|\phi\right\rangle = 0 n \geq 0 \end{aligned} \tag{4.5.75}$$

for (NS, NS). Considering the condition for $n = 0$, we find

$$\left|\phi\right\rangle = \{\phi + b^{\mu}_{-1/2}\overline{b}^{\nu}_{-1/2} l_{\mu\nu} + ...\} \left|0\right\rangle \tag{4.5.76}$$

$$(\alpha' \partial^2 + 1)\phi = 0, \quad \partial^{\mu} l_{\mu\nu} = 0, \quad \partial^2 l_{\mu\nu} = 0 \tag{4.5.77}$$

From the conditions, ϕ is a tachyon and $l_{\mu\nu}$ is a massless second-order tensor. By decomposing $l_{\mu\nu}$, we obtain the traceless symmetry $h_{\mu\nu}$ corresponding to gravity, and the antisymmetric second-order tensor $B_{\mu\nu}$ and the scalar ϕ of the trace part, which belongs to the SO(D-2) representation.

Next we consider the (NS, R) sector and (R, NS) sector:

$$\left|\phi\right\rangle_{\epsilon} = (b^{\mu}_{-1/2}\psi_{\mu\epsilon} + ...) \left|0\right\rangle, \tag{4.5.78}$$

$$\partial_{\mu} d^{\mu}_0 \psi^{\nu} = \partial^{\mu}\psi_{\mu} = 0. \tag{4.5.79}$$

If we divide ψ_{μ} into scalar and gamma traceless parts, we obtain

$$\psi_{\mu} = \hat{\psi}_{\mu} + d_{0\mu}\lambda; \quad d^{\mu}_0 \hat{\psi}_{\mu} = 0. \tag{4.5.80}$$

Substituting in the condition

$$\begin{aligned} d^{\mu}_0 \hat{\psi}_{\mu} = \partial^{\mu}\hat{\psi}_{\mu} = 0 \\ \partial_{\mu} d^{\mu}_0 \lambda = 0 \end{aligned} \tag{4.5.81}$$

The $\hat{\psi}_{\mu}$ represents the gravitino.

Finally we consider the (R,R) sector:

$$\left|\phi\right\rangle_{\epsilon\delta} = \{S_{\epsilon\delta} + ...\} \left|0\right\rangle, \tag{4.5.82}$$

$$(\partial_{\mu} d^{\mu}_0)^{\beta}_{\epsilon} S_{\beta\delta} = (\partial_{\mu} d^{\mu}_0)^{\gamma}_{\delta} S_{\epsilon\gamma} = 0. \tag{4.5.83}$$

Next, we impose a GSO projection condition on each of the left- and right-moving objects to give the spectrum a spacetime supersymmetry. For the (NS, NS) sector, since this sector contains tachyons, we impose $P_{NS} = 1$. For the (NS, R) sector, we impose $P_{NS} = 1$ to leave the massless state. For the right-moving R sector, there is no request, so $P_R = \pm 1$ are possible.

Similarly for the (R,NS) sector, we impose $\overline{P}_{NS} = 1$ and $P_R = \pm 1$. Here, there are two ways to project the R sector in (NS,R) and (R,NS): the same chirality and different chirality in (NS,R) and (R,NS).

For the (R, R) sector, there is also no request at all for the way P_R is taken. Since $S_{\epsilon\delta}$ belongs to the representation of Clifford algebra, expanding it in the gamma matrix gives

$$\begin{aligned} S = {} & C^{-1}F + \Gamma_\mu C^{-1}F^\mu + \Gamma_{\mu\nu}C^{-1}F^{\mu\nu} + \Gamma_{\mu\nu\rho}C^{-1}F^{\mu\nu\rho} + \Gamma_{\mu\nu\rho\kappa}C^{-1}F^{\mu\nu\rho\kappa} + \Gamma_{\mu\nu\rho\kappa\epsilon}C^{-1}F^{\mu\nu\rho\kappa\epsilon} \\ & + \Gamma_{\mu\nu\rho\kappa}\Gamma_{11}C^{-1}G^{\mu\nu\rho\kappa} + \Gamma_{\mu\nu\rho}\Gamma_{11}C^{-1}G^{\mu\nu\rho} + \Gamma_{\mu\nu}\Gamma_{11}C^{-1}G^{\mu\nu} + \Gamma_\mu\Gamma_{11}C^{-1}G^\mu + \Gamma_{11}C^{-1}G \end{aligned} \tag{4.5.84}$$

If we impose the projection condition $P_R = \pm 1$ and $\overline{P}_R = \pm 1$, then we obtain

$$\begin{aligned} F = \pm G, \quad F^\mu = \mp G^\mu, \quad F^{\mu\nu} = \pm G^{\mu\nu}, \quad F^{\mu\nu\rho} = \mp G^{\mu\nu\rho} \\ F^{\mu\nu\rho\kappa} = \pm G^{\mu\nu\rho\kappa}, \quad F^{\mu_1\cdots\mu_5} = \pm\frac{1}{5}\epsilon^{\mu_1\cdots\mu_{10}}F_{\mu_6\cdots\mu_{10}} \end{aligned}. \tag{4.5.85}$$

$$\begin{aligned} F = \mp G, \quad F^\mu = \mp G^\mu, \quad F^{\mu\nu} = \mp G^{\mu\nu}, \quad F^{\mu\nu\rho} = \mp G^{\mu\nu\rho} \\ F^{\mu\nu\rho\kappa} = \mp G^{\mu\nu\rho\kappa}, \quad F^{\mu_1\cdots\mu_5} = \pm\tfrac{1}{5}\epsilon^{\mu_1\cdots\mu_{10}}F_{\mu_6\cdots\mu_{10}} \end{aligned} \tag{4.5.86}$$

There are two possible combinations of the values of P_R and $\overline{P}_R$: the same chirality or different chirality. The first possibility is $(P_R, \overline{P}_R) = (1, 1)$. The fields that are consistent under this condition are $F^\mu, F^{\mu\nu\rho}, F^{\mu_1\cdots\mu_5}$. The constraints are

$$\begin{aligned} \partial_\mu F^{\mu_1\cdots\mu_n} &= 0 \\ \partial_\nu F_{\mu_1\cdots\mu_n} &= 0, \quad n = 1, 3, 5 \end{aligned} \tag{4.5.87}$$

The superstring theory constructed under this projection condition is called the TypeIIB string.

Second Possibility is $(P_R, \overline{P}_R) = (1, -1)$. A consistent field is $F, F^{\mu\nu}, F^{\mu\nu\rho\kappa}$. The constraint conditions for these fields are

$$\begin{aligned} \partial_\mu F^{\mu_1\cdots\mu_n} &= 0 \\ \partial_\nu F_{\mu_1\cdots\mu_n} &= 0, \quad n = 0, 2, 4 \end{aligned} \tag{4.5.88}$$

The superstring theory constructed using this projection condition is called the TypeIIA string.

TypeIIA/IIB string is invariant under 10-dimensional $N = 2$ supersymmetry.

In the above discussions, we distinguish the left-moving and right-moving modes. But since the world sheet is not observable, there is no requirement to make a distinction. Therefore, in addition to the GSO projection, we can impose an orienting symmetry which is Ω-duality on the RNS string theory.

$$\alpha_n^\mu,\ b_r^\mu,\ d_n^\mu \simeq \overline{\alpha}_n^\mu,\ \bar{b}_r^\mu,\ \overline{d}_n^\mu \tag{4.5.89}$$

Due to this symmetry, the left-moving and right-moving sectors must have equal chirality. Therefore, this symmetry can only be imposed on TypeIIB string theory. And the theory resulting from this is called a TypeI string.

From the above discussions, we have constructed three types of supersymmetric string theories in spacetime. In order for these theories to be consistent as string theories, they must not contain superconformal anomaly and also must be modular invariant. Type IIA/IIB theories satisfy these requirements, and Type I theories satisfy the requirements by including open strings in the theory. It is also possible to create a theory that satisfies these requirements without spacetime supersymmetry as a condition for GSO projection, and it is called Type0A/0B theory [26]. Phenomenological model from Type0A/0B theory was proposed [70].

4.6 Superstring in Green-Schwartz Formalism

In the RNS formalism, we introduced a spinor for the world sheet index α. In low energy effective theories, where the string is regarded as a point particle, the world sheet can no longer be observed and the supersymmetry is no longer visible. Therefore, the supersymmetry in the low-energy effective theory is nontrivial. This corresponds to the fact that spacetime supersymmetry does not exist in the RNS formalism and thus we imposed the GSO condition on the RNS formalism and realize tachyon-free theory. This operation is equivalent to imposing a spacetime supersymmetry on the RNS formalism, which means the spacetime supersymmetry exists in the final string theory. In other wards, in RNS string theory, the spectra are the representation of the world sheet local supersymmetry, and the these states is also a representation of the spacetime supersymmetry by GSO projection. This corresponds to extracting the states satisfying spacetime supersymmetry from the states satisfying world sheet local supersymmetry. As a result, the spectrum of RNS string theory becomes a theory with space-time local supersymmetry.

In this section, we construct a theory with spacetime local supersymmetry at first step [71]. The procedure is similar to that of the Brink-Schwartz superparticle theory. First, we impose a global spacetime supersymmetry, and then we find the local supersymmetry can be exist. The superstring theory constructed in this way is called the Green-Schwartz formalism, and we see that it is equivalent to the RNS formalism from group theoretic reasons.

4.6.1 Green-Schwartz Formalism

This formalism imposes a global sapace upersymmetry on the Polyakov action. First, the fermions of spacetime are denoted by θ^i_A, where A is the spinor index of spacetime and i is an internal index where we impose the maximum supersymmetry, $i = 1, 2$. The supersymmetric transformations are

$$\delta\theta^i = \epsilon^i, \quad \delta x^\mu = i\bar{\epsilon}^j \gamma^\mu \theta^j \tag{4.6.1}$$

where ϵ^i is a parameter of symmetry. We define the invariant variable under SUSY thistransformation:

$$\Pi^\mu_\alpha = \partial_\alpha x^\mu - i\overline{\theta}^j \gamma^\mu \partial_\alpha \theta^j. \tag{4.6.2}$$

If we replace the $\partial_\alpha x^\mu$ of the Polyakov action by this variable, we can obtain trivially a supersymmetric invariant action.

$$S_1 = -\frac{1}{2\pi}\int d^2\xi \sqrt{-g} g^{\alpha\beta} \Pi^\mu_\alpha \Pi^\nu_\beta \eta_{\mu^\nu} \tag{4.6.3}$$

Here, the Π we just defined is not unique as a global supersymmetric invariant variable. Also we can choose a different variable Π' for each point as long as it is globaly supersymmetric invariant. That is, Π has a certain gauge symmetry $\Pi \to \Pi'$, and should have a symmetry that mixes x^μ and θ^i in Π. Thus we define the following transformation:

$$\begin{aligned} \delta x^\mu &= i\overline{\theta}^j \gamma^\mu \delta\theta^j \\ \delta\theta^i &= 2i\gamma^\mu \Pi_{\mu\alpha} \kappa^{i\alpha}. \end{aligned} \tag{4.6.4}$$

This symmetry is called κ symmetry. And the corresponding symmetry for translation is of course the local Poincaré symmetry. This forms a closed algebra with κ-symmetry. In order for the action to have this symmetry, it needs to be modified

$$S_2 = \frac{1}{\pi}\int d^\xi \epsilon^{\alpha\beta}\{-i\partial_\alpha x^\mu(\overline{\theta}^1\gamma^\nu\partial_\beta\theta^1 - \overline{\theta}^2\gamma^\nu\partial_\beta\theta^2) + \overline{\theta}^1\gamma^\mu\partial_\alpha\theta^1\overline{\theta}^2\gamma^\nu\partial_\beta\theta^2\}\eta_{\mu\nu}. \tag{4.6.5}$$

The action $S = S_1 + S_2$ describes the TypeIIA superstring theory. We can also see that S_2 does not contribute to the energy-momentum tensor $T_{\alpha\beta}$ of the world sheet because it does not depend on $\sqrt{h}$.

Since we introduce the new additional term S_2, we should confirm the Lorentz symmetry and the global spacetime $N = 2$ supersymmetry. Indeed, if $D = 3$ and θ is Majorana, $D = 4$ and θ is Majorana or Weiyl, $D = 6$ and θ is Weyl, or $D = 10$ and θ is Majorana-Weyl, the superstring theory can exist. This result is different from bosonic strings, which can exist in any dimension. By considering quantum theory, we can see that superstring theory can exist only in $D = 10$ as well as in RNS formalism.

κ Symmetry

We constructed the action in GS form using an invariant variable Π under a global supersymmetric transformation. This supersymmetric invariant field Π is not unique, and there are different supersymmetric invariant fields Π'. This naturally induce a gauge symmetry, $\Pi \to \Pi'$, since either Π or Π' can be used at each point. This symmetry is called the κ symmetry, and this symmetry was a fermionic symmetry. The transformation rule for this symmetry was also given by (4.6.4).

From the above discussion, the parameter of κ symmetry has the three indices: the spacetime spinor index a, the world sheet vector index α because it is an SO(2) vector on the world sheet, and the R-symmetry index $i = 1, 2$. Since the two-dimensional Lorentz group SO(2) is isomorphic to U(1), the SO(2) vector is a reducible representation and the irreducible representation is a one-dimensional representation. The irreducible representation is the self-dual and anti-self-dual part of the vector. So, we define the following projection operator:

$$P_{\pm}^{\alpha\beta} = \frac{1}{2}\left(h^{\alpha\beta} \pm \frac{\epsilon^{\alpha\beta}}{\sqrt{h}}\right). \tag{4.6.6}$$

We can confirm the following properties and $P_{\pm}^{\alpha\beta}$ are the projection:

$$P_{\pm}^{\alpha\beta} h_{\beta\gamma} P_{\pm}^{\gamma\delta} = P_{\pm}^{\alpha\delta}, \tag{4.6.7}$$

$$P_{\pm}^{\alpha\beta} h_{\beta\gamma} P_{\mp}^{\gamma\delta} = 0. \tag{4.6.8}$$

$P_{\pm}^{\alpha\beta}$ extract the irreducible representations from the reducible representation. The first property means that the representation remains the same even if the projection operator is applied to the irreducible representation. The second property means that it is not possible to extract the other irreducible representation from the irreducible representation.

Let us apply this projection operator to $\kappa^{ia\alpha}$. Here, there are two world sheet vectors, $\kappa^{1\alpha}$ and $\kappa^{2\alpha}$, because of the doublet under the R-symmetry. Since κ^1 and κ^2 are equivalent under R-symmetry, we impose the anti-self duality condition on κ^1 and the self duality condition on κ^2 without losing generality:

$$\kappa^{1\alpha} = P_{-}^{\alpha\beta} \kappa_{\beta}^{1}, \tag{4.6.9}$$

$$\kappa^{2\alpha} = P_{+}^{\alpha\beta} \kappa_{\beta}^{2}. \tag{4.6.10}$$

From these conditions, $i = 1$ corresponds to the right-moving mode and $i = 2$ to the left-moving mode.

Next we confirm that the κ symmetry is a special spacetime local supersymmetry. In general, supersymmetry forms a close algebra with local Poincaré symmetry. Therefore, there exists a bosonic symmetry in the local Poincaré symmetry, i.e., the reparameterisation that forms a close algebra with κ symmetry. Such a symmetry can be found as follows:

$$\begin{aligned} \delta\theta^1 &= \sqrt{h}P_-^{\alpha\beta}\partial_\beta\theta^1\lambda_\alpha, & \delta\theta^2 &= \sqrt{h}P_+^{\alpha\beta}\partial_\beta\theta^2\lambda_\alpha \\ \delta X^\mu &= i\theta^A\gamma^\mu\delta\theta^A, & \delta(\sqrt{h}h^{\alpha\beta}) &= 0 \end{aligned} \tag{4.6.11}$$

Equation of motion

The equations of motion derived from the action in GS formalism are obtained as

$$\Pi_\alpha \cdot \Pi_\beta = \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\Pi_\gamma \cdot \Pi_\delta \tag{4.6.12}$$

$$\gamma \cdot \Pi_\alpha P_-^{\alpha\beta}\partial_\beta\theta^1 = 0 \tag{4.6.13}$$

$$\gamma \cdot \Pi_\alpha P_+^{\alpha\beta}\partial_\beta\theta^2 = 0 \tag{4.6.14}$$

$$\partial_\alpha[\sqrt{h}(h^{\alpha\beta}\partial_\beta X^\mu - 2iP_-^{\alpha\beta}\overline{\theta}^1\gamma^\mu\partial_\beta\theta^1 - 2iP_+^{\alpha\beta}\overline{\theta}^2\gamma^\mu\partial_\beta\theta^2)] = 0 \tag{4.6.15}$$

The first equation corresponds to $T_{\alpha\beta} = 0$. Although these equations are also nonlinear,we can derive a free field theory by light cone gauge fixing.

Types of Superstring Theory

In D=10 superstring theory, the spacetime fermion coordinates θ has to be Mayorana-Weil spinors. This fact means that the specific chiralities are defined for θ^1, θ^2. There are two ways to define the chiralities. The first is if the two Majorana-Weyl spinors have the same chirality, and the other is if they have different chiralities. In closed string theory, the boundary condition is the periodic boundary condition. Therefore, in closed strings, θ^1 and θ^2 are independent of each other, and two different ways of taking the chirality are possible. On the other hand, when considering open strings, θ^1 and θ^2 must coincide at the end of the string and therefore must have the same chirality.

In the case of open string, the supersymmetry drops from $N = 2$ to $N = 1$, since the two fermionic coordinates coincide at the boundary. This is why it is called Type I superstring theory. By assigning a charge to the end of the string, we can construct the $N = 1$ super Yang-Mills theory with quantum numbers for any classical group. This method of assigning charge to the end is called Chan-Paton's method. It will be shown in later section that in classical theory any gauge group can be realized, but in quantum theory the only consistent gauge group is SO(32). Also, is the gauge group is an orthogonal or symplectic group, the string is unorientable. Therefore, the open string superstring theory is the Type I theory with SO(32) gauge symmetry. This theory describes the interaction between open and closed string that are not orientable. Type I theory are inconsistent in quantum theory without the existence of closed string. This is because if the end of open string join together, it forms closed string. Since the quantum number exists at the end of the open string, the closed string cannot have quantum numbers, and thus form a singlet of the gauge group.

Next, we consider the theory of closed string. Since θ^1 and θ^2 are independent, they can have different chirality. Since the propagation directions of θ^1 and θ^2 are opposite, the orientation can be determined by the chirality. Therefore, the closed string can be oriented. Furthermore, $N = 2$ supersymmetry is preserved since the two fermionic coordinates are independent. Since the supersymmetric transformation is given by $\delta\theta^i = \epsilon^i$, the two supercharges have different chirality from each other. The closed string theory that takes the opposite chirality is called Type IIAtheory. This Type IIA theory is a non-chiral theory because it treats the two chirality symmetrically. Also, since string do not have end, it is not possible to introduce a gauge group.

Finally, consider the case where two Majorana-Weyl spinors are taken to the same chirality. In this case, since we cannot distinguish between θ^1 and θ^2, we can construct two theories. One is a unorientable closed string theory by Ω duality. The other is to construct an orientable theory without imposing Ω duality. The unorientable theory corresponds to the closed string theory that emerges from Type I theory. In other words, unorientable closed string theory are conosistent with Type I superstring theory. On the other hand, there is no such restriction in the orientable case. In the orientable theory, there are now two supercharges with the same chirality. This is why it is called Type IIB superstring theory. This Type IIB theory is a chiral theory because it deals with only one chirality. Also, like Type IIA theories, Type IIB theories cannot introduce a gauge group.

4.6.2 Light Cone Quantization

As in the RNS formalism, we can carry out the three ways to quantize this theory. In old covariant quantization, the conformal symmetry left by gauge fixation is quantized without gauge fixation, and the physical state is obtained by imposing the Virasoro constraints on the quantum state. In such quantization, the state had gauge symmetry in quantum theory and the gauge degrees of freedom were constructed by the Virasoro generators. In the BRST quantization, the reparameterisaton was replaced to BRST symmetry. Both of these quantizations are called covariant quantizations because they can treat gauge symmetry in quantum theory. On the other hand, in light cone quantization, the conformal symmetry is fixed by the light cone gauge, and only the physical degrees of freedom are quantized. In this kind of quantization, conformal symmetry does not exist in quantum theory.

As can be seen from the example of Brink-Schwartz superparticle, the GS formalism contains nonlinear terms in the action, and the canonical commutation relations cannot be well set up. Therefore, the GS formalism cannot be quantized covariantly, and can only be quantized by taking an light cone gauge.

Using the results of the RNS formalism, we assume that $D = 10$. In this case, the two spinors θ^1, θ^2 in the action are the Mayorana-Weyl spinors. Superstring theory in GS formalism for $D = 3, 4, 6$ can be discussed in the same way as for $D = 10$. However, in these dimensions, $[J^{i-}, J^{j-}] \neq 0$, and then it breaks the Lorentz symmetry. In the case of $D = 10$, we can consider two cases: one where θ^1, θ^2 are taken to the same chirality, and the other where they are taken to different chirality.

The local symmetries of the GS formalism are reparameterisation, Weyl invariance, and κ symmetry. We use these gauge symmetries to fix the degrees of freedom of the field. First, using reparameterisation and Weyl invariance, we set

$$g_{\alpha\beta} = \eta_{\alpha\beta}. \tag{4.6.16}$$

The equations of motion cannot be linearized by adopting such a gauge fixation condition. The remaining gauge symmetries after this gauge fixation are conformal symmetry and κ symmetry.

First of all, we consider fixing the κ symmetry. Since the κ symmetry is fermionic, we can drop the degrees of freedom of the fermionic fields θ^1, θ^2. In addition, since the parameters κ^1, κ^2 of the κ symmetry satisfy the anti-self-duality and self-duality conditions respectively, we have half the independent degrees of freedom, respectively. Therefore, the degrees of freedom of the fermionic field θ^A can also be fixed to half

$$\Gamma^+\theta^1 = 0 = \Gamma^+\theta^2 \tag{4.6.17}$$

$$\Gamma^\pm = \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^9) \tag{4.6.18}$$

where $\Gamma^\pm$ is the light cone component of the gamma matrix and satisfy

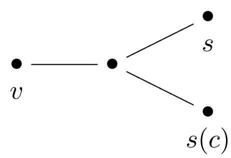
$$(\Gamma^+)^2 = 0 = (\Gamma^-)^2. \tag{4.6.19}$$

These gauge-fixed conditions make the equations of motion of x^+ and x^i free field equations.

Finally, the conformal symmetry is fixed by the light cone gauge

$$x^+(\tau, \sigma) = q^+ + p^+\tau \tag{4.6.20}$$

The 10 dimensional Dirac spinor has 32 complex components, and by the Majorana-Weyl condition it has 16 real components. In addition, the light cone gauge fixation for κ symmetry reduces θ^i components to real 8 degrees. On the other hand, the symmetry that is trivially preserved in the light cone gauge fixation is the rotations for the transverse eight components. In other words, θ^i can be regarded as an 8 dimensional spinor representation of the transverse rotation group SO(8). More precisely, it is the **8** representation of Spin(8). The rank of the SO(8) Lie algebra is 4, and the Dynkin diagram is as follows:



where v represents the SO(8) vector representation and s and c represent the spinor and anti-spinor representation respectively. This Dynkin diagram is invariant under the central point axis-rotation. This means that the physics is invariant even if we regard the vector representation as a spinor representation. Similarly, it means that we can consider spinors as vectors. In fact, both spinors and vectors are real 8-components, and there is no contradiction. Such an interchange of representations is an automorphism, and the symmetry is called a triplicity. There are $3! = 6$ ways to interchange spinors and vectors due to triplicity, interchanging the vector index i, j, k with the spinor index a, b, c and the anti-spinor index $\dot{a}, \dot{b}, \dot{c}$.

In the following, we denote S for the eight independent components of the 16-component Majorana spinor θ^A that satisfy the fixed condition. The Γ^+ appearing in the fixed condition always contributes to the action and the equations of motion together with p_+. Also since the spinor index is contracted as $\overline{\theta}\Gamma^\mu\theta$, it depends on the equations of motion in the form of $\sqrt{p^+}\theta^i$. Therefore, if we denote $\sqrt{p^+}\theta$ as S, from the fixiation condition, we can replace as

$$\begin{aligned} \sqrt{p^+}\theta^1 &\to S^{1a} \text{ or } S^{1\dot{a}} \\ \sqrt{p^+}\theta^2 &\to S^{2a} \text{ or } S^{2\dot{a}}. \end{aligned} \tag{4.6.21}$$

A theory with S^1 and S^2 at the same chirality is a Type I or Type IIB theory, and a theory with different chirality is a Type IIA theory. In the following calculations, we focus on Type I or Type IIB theory with the same chirality. Type IIA can be calculated in the same way.

The equations of motion are rewritten using the above gauge fixiation conditions. From $\Gamma^+\theta = 0$, we have

$$\overline{\theta}\Gamma^\mu\partial_\alpha\theta = 0, \quad \mu \neq - \tag{4.6.22}$$

This equation is trivial for $\mu = +$. For $\mu = i$, by insertin $1 = (\Gamma^+\Gamma^-\Gamma^-\Gamma^-\Gamma^+)/2$, we confirm this equation. Substituting these equations into the equations of motion, we find

$$(\partial_\sigma^2 - \partial_\tau^2)X^i = 0, \tag{4.6.23}$$

$$(\partial_\tau + \partial_\sigma)S^{1a} = 0, \tag{4.6.24}$$

$$(\partial_\tau - \partial_\sigma)S^{2a} = 0. \tag{4.6.25}$$

These equations are consistent with the equations for X^i, ψ^i_- and ψ^i_+ in RNS formalism. This fact is obvious from the triplicity in the eight representations of SO(8). In other words, the SO(8) vector representation $\psi^i_\pm$ is simply replaced by the SO(8) spinor representation $S^{1(2)}_a$.

The action in the light cone gauge that leads to these equations of motion is given by

$$S_{l.c} = -\frac{1}{2}\int d^2\sigma \left(T\partial_\alpha X^i \partial^\alpha X^i - \frac{i}{\pi}\overline{S}^a \gamma^\alpha \partial_\alpha S^a \right), \tag{4.6.26}$$

where, we have defined $S^a = (S^{1a}, S^{2a})$ and describe in the world sheet two-component Majorana spinor. The S^{1a} and S^{2a} are one-component world sheet Majorana spinors, corresponding to the right- and left-moving modes, respectively. Here, θ^{ia} is a spacetime spinor and a world sheet scalar. Therefore, S must be a world sheet scalar in general. However, after the light cone gauge fixation, the remaining world sheet scalar components for a world sheet spinor.

The quantum theory for the spacetime coordinate x^i can be constructed in the same way as before. The quantization for S^{ia} can be carried out by imposing the canonical anti-commutation relation:

$$\{S^{ia}(\sigma,\tau), S^{jb}(\sigma',\tau)\} = \pi\delta^{ab}\delta^{ij}\delta(\sigma' - \sigma) \tag{4.6.27}$$

Next, we expand S in a mode such that the boundary conditions are satisfied. In the open string case, the two fermions are related to each other at the end of the string. In this case, $N = 2$ supersymmetry is broken to $N = 0$ or $N = 1$ supersymmetry. Since we hope there are zero modes in the mode expansion of S, we require that $N = 1$ supersymmetry be preserved. The boundary condition for S satisfying such a requiring is

$$S^{1a}(0,\tau) = S^{2a}(\tau, 0), \tag{4.6.28}$$

$$S^{1a}(\pi,\tau) = S^{2a}(\pi,\tau). \tag{4.6.29}$$

Since the global supersymmetry transformation is given by (4.6.1), ϵ^1 and ϵ^2 are related to each other by the supersymmetry transformation of this boundary condition. In other words, the number of independent supersymmetric parameters is reduced and supersymmetry is broken. Therefore, only the $N = 1$ supersymmetry is preserved, giving the Type I theory. Furthermore, from the boundary conditions, we know that S^1 and S^2 must have the same chirality. We can also impose a boundary condition that changes the sign of the right-hand side of the boundary condition. However, imposing such a boundary condition would completely break the supersymmetry.

Modal expansions of S satisfing the open string boundary condition are

$$S^{1a}(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{-\infty}^{\infty} S_n^a e^{-in(\tau-\sigma)}, \tag{4.6.30}$$

$$S^{2a}(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{-\infty}^{\infty} S_n^a e^{-in(\tau+\sigma)}. \tag{4.6.31}$$

θ is a Majorana-Weyl spinor, so it satisfies the real condition

$$S_{-m}^a = (S_m^a)^\dagger \tag{4.6.32}$$

Substituting these modal expansions into the anti-commutation relation, we find

$$\{S_m^a, S_n^b\} = \delta^{ab}\delta_{m+n,0}. \tag{4.6.33}$$

Similarly, we can also consider closed string by considering the periodic boundary conditions:

$$S^{Aa}(\sigma,\tau) = S^{Aa}(\sigma+\pi,\tau). \tag{4.6.34}$$

Expanding to satisfy this boundary condition, we find

$$S^{1a}(\sigma,\tau) = \sum S_n^a e^{-2in(\tau-\sigma)}, \tag{4.6.35}$$

$$S^{2a} = (\sigma,\tau) = \sum \tilde{S}_n^a e^{-2in(\tau+\sigma)}. \tag{4.6.36}$$

As you can see from these expressions, S^1 and S^2 are consisted from independent S and $\tilde{S}$, respectively. Since S^1 and S^2 are independent, they can have different chirality. In this case, it is a Type IIA theory. Of course, right-moving and left-moving can be distinguished by their chirality, so they are always orientable. In the case of Type IIB, since left-moving and right-moving have the same chirality, we can eliminate the orientation by imposing Ω duality, and the resulting theory is a Type I closed string theory.

4.6.3 Equivalence between RNS Formalism and GS Formalism

The equivalence of the RNS form and the GS form can be shown as follows. First, the action under the light cone gauge fixation in the RNS formalism is given by

$$S'_{l.c} = -\frac{1}{2}\int d^2\sigma(\partial_\alpha X^i \partial^\alpha X^i - i\overline{\psi}^i\gamma^\alpha\partial_\alpha\psi^i). \tag{4.6.37}$$

In the GS formalism, the fermionic degrees of freedom are described by the spacetime spinor S^a. On the other hand, in the RNS formalism, they are described by the spacetime vector ψ^i. In other words, the difference is a representation of the group.

The relation between these two formalizations can be said to be the relation between the spacetime spinor S^a and the spacetime vector ψ^i. Therefore, we consider the bosonization of ψ^i.

The first step is the bosonization of the world sheet spinor ψ^i. The bosonization can be performed by expressing the fermionic current by the derivative of a scalar field:

$$\begin{aligned}
\frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\phi_1 &= \overline{\psi}^1\gamma^\alpha\psi^2, \\
\frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\phi_2 &= \overline{\psi}^3\gamma^\alpha\psi^4, \\
\frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\phi_3 &= \overline{\psi}^5\gamma^\alpha\psi^6, \\
\frac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\phi_4 &= \overline{\psi}^7\gamma^\alpha\psi^8.
\end{aligned} \tag{4.6.38}$$

Next, we rewrite it by a linear combination of scalar fields ϕ:

$$\begin{aligned}
\sigma_1 &= \frac{1}{2}(\phi_1+\phi_2+\phi_3+\phi_4), \\
\sigma_1 &= \frac{1}{2}(\phi_1+\phi_2-\phi_3-\phi_4), \\
\sigma_1 &= \frac{1}{2}(\phi_1-\phi_2+\phi_3-\phi_4), \\
\sigma_1 &= \frac{1}{2}(\phi_1-\phi_2-\phi_3+\phi_4).
\end{aligned} \tag{4.6.39}$$

Finally, since σ is a spacetime boson, we fermionize it to a spacetime fermion:

$$\begin{aligned}
\tfrac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\sigma_1 &= \overline{S}^1\gamma^\alpha S^2, \\
\tfrac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\sigma_2 &= \overline{S}^3\gamma^\alpha S^4, \\
\tfrac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\sigma_3 &= \overline{S}^5\gamma^\alpha S^6, \\
\tfrac{1}{\sqrt{\pi}}\epsilon^{\alpha\beta}\partial_\beta\sigma_4 &= \overline{S}^7\gamma^\alpha S^8.
\end{aligned} \tag{4.6.40}$$

The spin(8) transformability of S^a obtained in this way is non-trivial. The SO(8) currents used in this bosonization are the four on the right-hand side of (4.6.38). These four generators are commutative. Therefore, these four generators are maximal commutative subalgebra, which is Cartan subalgebras. By the 1/2 factor in (4.6.39), we find that the quantum number, which is an eigenvalue of the Cartan subalgebra of S^a defined by (4.6.40), is a half-integer. Therefore, S^a transforms as a spinor under SO(8). In fact, S^a transforms as the spinor representation $\mathbf{8}_s$, while the $\mathbf{8}_c$ representation can be made by changing some of the signs of (4.6.40). This fact can be rigorously proven by examining the bosonization formula for $\psi \sim e^{i\phi^\dagger}$ [26].

The bosonization of fermions in infinite volume can be done relatively directly. However, if the volume is finite, it becomes more complicated because of the existence of boundaries. In other words, bosonization and fermionization do not change the expression of the Lagrangian, but they do change the boundary conditions. By examining the representations of the S^a and ψ^i, we find the Lagrangian in GS formalism automatically describes the boson and fermion in the same supermultiplets. On the other hand, in RNS formalism, these bosons and fermions correspond to the states generated by the quantization of the field satisfying the boundary conditions. Thus, in the GS formalism, the string spectra can be obtained directly from the Lagrangian field, while in the RNS formalis, it is obtained as a state produced by a field satisfying the boundary conditions.

4.6.4 Super-Poincaré Group

In the RNS formalism, the action do not have spacetime supersymmetry, but it is derived by imposing a GSO projection on the states in the quantum theory. On the other hand, the GS formalism imposes spacetime supersymmetry on the action, which makes the spacetime supersymmetry easier to understand than the RNS formalism. In the following discussion, we omit the R-symmetry index i for the simplicity.

In the light cone gauge, the ten dimensional spacetime Lorentz symmetry is not obvious. Thus the spacetime supersymmetry is not also obvious after light cone gauge fixation. Indeed, if we consider the supersymmetric transformation $\delta\theta = \epsilon$ for the gauge fixing condition $\Gamma^+\theta = 0$, we obtain $\Gamma^+\epsilon \neq 0$ and so does not preserve the supersymmetry. In other words, the gauge-fixing condition is a condition for fixing the κ symmetry, but the fixing condition may also fix the supersymmetry of the spacetime, which is global. Therefore, we divide ϵ into two components, one satisfying $\Gamma^+\epsilon = 0$ and the other not, and investigate the supersymmetric transformation law

First we consider the ϵ satisfying $\Gamma^+\epsilon = 0$. A ϵ that satisfies $\Gamma^+\epsilon = 0$ also satisfies $\bar{\epsilon}\gamma^i\theta = 0$. There are eight components of ϵ that satisfy these conditions, and if we write them as η, then

$$\begin{aligned} \delta\theta = \epsilon \to \delta S^a \sqrt{2p^+}\eta^a \\ \delta x^\mu = i\bar{\epsilon}\Gamma^\mu\theta \to \delta X^i = 0 \end{aligned} \tag{4.6.41}$$

Therefore, the action after the light cone gauge (4.6.26) is invariant under this transformation. This means that the symmetry which cannot be fixed by the gauge-fixing condition $\Gamma^+\theta = 0$, is given by η. In fact, if we consider $\theta \to \theta + \epsilon$, then $\Gamma^+\theta' = \Gamma^+\epsilon = 0$, and the degrees of freedom of θ given by η are not fixed.

Next we consider the ϵ for $\Gamma^+\epsilon \neq 0$ Similarly, ϵ, which does not satisfy this condition, also has eight components, and these correspond to the spinors $\epsilon^{\dot{a}}$ in the $\mathbf{8}_c$ representation. These components are the part ot the global spacetime supersymmetry which is broken by the fixation condition on the gauge κ symmetry. Since we hope the spacetime supersymmetry to exist in the theory after the gauge fixation, we have to construct a global spacetime supersymmetry that is not broken by the fixation condition of $\Gamma^+\theta = 0$. Then we use the κ symmetry. Since the theory is invariant under κ symmetry, it is also invariant under global κ symmetry (color symmetry), where κ is trivially regarded as a constant. This color symmetry is used to modify the spacetime supersymmetry:

$$\delta\theta = \epsilon + 2i\Gamma \cdot \Pi_\alpha \kappa^\alpha. \tag{4.6.42}$$

By acting Γ^+ on both sides and setting the constant κ to be zero, we can obtain a global spacetime supersymmetry that cannot be broken by gauge fixation. In order to see the transformation rule in more detail, we rewrite (4.6.42) in terms of S^a satisfying the fixation condition as follows:

$$\delta S^a = -i\rho \cdot \partial x^i \gamma^i_{a\dot{a}} \epsilon^{\dot{a}} \sqrt{2p^+}, \tag{4.6.43}$$

$$\delta x^i = 2\gamma^i_{a\dot{a}} \bar{\epsilon}^{\dot{a}} S^a / \sqrt{2p^+}. \tag{4.6.44}$$

Here, γ^i is not the gamma matrix, but the Clebsch–Gordan coefficients that relates the three $8D$ representations of spin(8). Of course, the action after gauge fixation is invariant under the spacetime supersymmetric transformation given in this way. Also, by considering $\epsilon^{\dot{a}}$ as a parameter of the 8-dimensional supercharge, the transformation law for x^i in (4.6.44) can be regarded as a supersymmetric transformation law in two-dimensional field theory.

By the above discussions, we have shown that the GS formalism in the light cone gauge is a spacetime supersymmetric invariant, and we have obtained its transformation law. Next, we investigate the symmetry algebra by using the transformation rule of spacetime supersymmetry. From the general theory of supersymmetry algebra, we can obtain a translational transformation law by performing two supersymmetric transformations. Of course, since the supersymmetric transformation is a transformation law after fixing the light cone gauge, the resulting translational transformation law is also after the light cone gauge:

$$[\delta_1, \delta_2] = \xi^\alpha \partial_\alpha x^i + a^i, \tag{4.6.45}$$

$$[\delta_1, \delta_2] S^a = \xi^\alpha \partial_\alpha S^a. \tag{4.6.46}$$

Here, we have used the equation of motion for S^a, $\gamma \cdot \partial S^a = 0$, in deriving these transformation laws, so these variational observations are on-shell. Also, the translational parameter ξ^α is defined by the supersymmetric parameter

$$\xi^\alpha = -2i\epsilon^{(1)}\gamma^\alpha\epsilon^{(2)} \tag{4.6.47}$$

And the constant term a^i represents the translation of the transverse coordinates:

$$a^i = \sqrt{2}\eta^{(2)}\gamma^i\epsilon^{(1)} - \sqrt{2}\eta^{(1)}\gamma^i\epsilon^{(2)}. \tag{4.6.48}$$

The supercharge corresponding to the supersymmetry $\eta^a, \epsilon^{\dot{a}}$, which is also conserved in the light cone gauge fixation, can be derived using Noether's theorem. The supercharge corresponding to η^a is

$$Q^a = (2p^+)^{1/2}S_0^a, \tag{4.6.49}$$

and the supercharge for $\epsilon^{\dot{a}}$ is

$$Q^{\dot{a}} = (p^+)^{-1/2}\gamma^i_{a\dot{a}}\sum_{-\infty}^{\infty} S^a_{-n}\alpha^i_m n. \tag{4.6.50}$$

Each of these is an 8-component Majorana-Weyl spinor. Combining these to form a 16-component Majorana spinor Q, and calculating the algebra:

$$\{Q, Q\} \sim (1 \pm \Gamma_{11})\Gamma \cdot p. \tag{4.6.51}$$

Computing the algebra in Majorana-Weyl spinor, which is an irreducible representation of spin(8), we have the following

$$\{Q^a, Q^b\} = 2p^+\delta^{ab} \tag{4.6.52}$$

$$\{Q^a, Q^{\dot{a}}\} = \sqrt{2}\gamma^i_{a\dot{a}}p^i \tag{4.6.53}$$

$$\{Q^{\dot{a}}, Q^{\dot{b}}\} = 2H\delta^{\dot{a}\dot{b}} \tag{4.6.54}$$

$$H = \frac{1}{p^+}\left((p^i)^2 + 2N\right) \tag{4.6.55}$$

$$N = \sum_{m=1}^{\infty}\left(\alpha^i_{-m}\alpha^i_m + mS^a_{-m}S^a_m\right) \tag{4.6.56}$$

As can be seen from these expressions, H is the Hamiltonian under the light cone gauge. Since the vacuum expectation values of the boson and fermion cancel because of supersymmetry, the normal ordering is obvious and is omitted from the notation. Furthermore, $H = p^-$ from the mass-shell condition.

Next, we consider the spacetime Lorentz generators. The covariant Lorentz generators for the spinor

θ before gauge fixation are $b_{\mu\nu}J^{\mu\nu} = b_{\mu\nu}\gamma^{\mu\nu}$:

$$\theta \to \theta' = b_{\mu\nu}\gamma^{\mu\nu}\theta. \tag{4.6.57}$$

This transformation rule keeps the gauge fixing condition invariant for J^{ij}, J^{+i}, J^{+-}. Therefore, it can be broken by gauge fixation only for J^{-i}. Then, by using the color symmetry corresponding to the global spacetime translation symmetry ξ and the κ symmetry, we derive Lorentz generators that are preserved under the gauge-fixing condition. First, we find the covariant Lorentz generators before gauge fixation

$$J^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu} + K^{\mu\nu} \tag{4.6.58}$$

$$l^{\mu\nu} = x^{\mu}p^{\nu} - x^{\nu}p^{\mu} \tag{4.6.59}$$

$$E^{\mu\nu} = -i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^{\mu}_{-n}\alpha^{\nu}_{n} - \alpha^{\nu}_{-n}\alpha^{\mu}_{n}) \tag{4.6.60}$$

If we take the light cone gauge, the x^+ direction is fixed and therefore $\alpha^+_n = 0$. Therefore, $E^{\mu+} = 0$. Furthermore, solving the Virasoro condition for x^- in the light cone gauge gives

$$\alpha = \frac{1}{2p_+}\sum_{-\infty}^{\infty}\left(\alpha^i_{n-m}\alpha^i_m + \left(m - \frac{N}{2}\right)S^a_{n-m}S^a_m\right). \tag{4.6.61}$$

is obtained. As can be seen from this expression, $\alpha^-_0 = p^- = H$ is satisfied. Similarly, if we calculate $K^{\mu\nu}$, we obtain

$$\begin{aligned} K^{\mu+} &= 0, \\ K^{ij} &= K^{ij}_0, \\ K^{i-} &= \frac{1}{p^+}\sum_{-\infty}^{\infty}K^{ij}_{-n}\alpha^j_n, \\ K^{ij}_n &= -\frac{i}{4}\sum_{-\infty}^{\infty}S^a_{n-m}\gamma^{ij}_{ab}S^b_m. \end{aligned} \tag{4.6.62}$$

If we calculate the algebra of α^-_n and K^{ij}_n, we see that they form the same algebra. This result is the same as the result in the RNS formalism if S^a_n is replaced by d^i_n. This is because the RNS formalism and the GS formalism are related by the triplicity of SO(3). In fact, if we calculate $[J^{i-}, J^{j-}]$, it is nonzero in general, but by taking $D = 10$, it becomes zero and we recover the space-time Lorentz algebra.

Using the above results, we derive the super-Poincaré algebra. In the previous discussion, the algebra between supersymmetric generators and the Lorentz algebra have been calculated. The remaining commutation relation for super-Poincaré algebra is that between Lorentz generators and supercharges. Since the commutation relation between Lorentz generators and supercharges conserved after the light cone gauge is consistent with the result in the classical theory, we only need to calculate the commutation relation between J^{-i} and supercharges:

$$[J^{i-}, Q^a] = \frac{i}{\sqrt{2}}\gamma^i_{a\dot{a}}Q^{\dot{a}}. \tag{4.6.63}$$

Finally, we list the functional forms of the super-Poincaré generators. In particular, the Type IIB case is enumerated and given as follows:

$$p^i = \int_0^\pi d\sigma P^i_\tau(\sigma,\tau), \tag{4.6.64}$$

$$H = \frac{1}{2\pi p^+}\int_0^\pi d\sigma[\pi^2(P^i_\tau)^2 + (X^{i\prime})^2 - iS^1S^{1\prime} + iS^2S^{2\prime}] \tag{4.6.65}$$

$$Q^a_A = \frac{1}{\pi}(2p^+)^{1/2}\int_0^\pi d\sigma S^a_A, \tag{4.6.66}$$

$$Q^{\dot{a}}_1 = \frac{1}{\pi}(p^+)^{-1/2}\int_0^\pi d\sigma(\gamma^i S_1)^{\dot{a}}(\pi P^i_\tau - X^{i\prime}), \tag{4.6.67}$$

$$Q^{\dot{a}}_2 = \frac{1}{\pi}(p^+)^{-1/2}\int_0^\pi d\sigma(\gamma^i S_2)^{\dot{a}}(\pi P^i_\tau + X^{i\prime}), \tag{4.6.68}$$

$$P^i_\tau(\sigma,\tau) = \frac{1}{\pi}\dot{X}^i(\sigma,\tau) = -i\frac{\delta}{\delta X^i(\sigma,\tau)}. \tag{4.6.69}$$

4.6.5 Spectrum

Since the quantum theory of the GS formalism is quantized by light-cone quantization, all quantum states are physical.

Spectrum of Open String

Since the open string boundary condition leads to only one boson-fermion pair, only $N = 1$ supersymmetry is preserved. The introduction of the Chan-Paton factor allows us to have a charge at the end of the open string, which introduces gauge symmetry into the theory. The U(n) group is introduced for orientable open strings, while the SO(n) or USp(n) group is introduced for non-orientable strings. The massless states are the adjoint representation of these gauge groups. The specific gauge groups will be introduced in the next section. We do not consider these gauge symmetries here. In the following, we investigate each mass level.

First we consider the massless sector. In the RNS formalism, the lowest order was the tachyon, since there was no spacetime supersymmetry. However, in the GS formalism, there is a spacetime supersymmetry, so there are no tachyons. Therefore, the ground state is a massless state. In the RNS formalism of light-cone quantization, the massless boson is given by $b^i_{-1/2}\left|0\right\rangle$. On the other hand, the fermionic ground state was described by a 16-component Majorana-Weyl spinor by degeneracy of d^i_0. Furthermore, the equation of motion leaves only the 8_c representation of spin(8) as a physical degree of freedom. Thus, both bosons and fermions have 8 degrees of freedom, corresponding to the $D = 10,\ N = 1$ super Yang-Mills theory.

Similarly, we investigate the massless sector in the GS formalism. In spin(8) notation, the ground state is degenerate by $\{S^a_0, S^b_0\} = \delta^{ab}$. From the triplicity of SO(8), the representation of this algebra can be expressed in Clifford algebra

$$S^a_0 \sim \begin{pmatrix} 0 & \gamma^a_{i\dot{a}} \\ \gamma^a_{\dot{a}i} & 0 \end{pmatrix} \tag{4.6.70}$$

As can be seen from the fact that the index is $i, \dot{a}$ and can be expressed in block diagonal, this representation space is $8_v + 8_c$. In other words, it can be obtained by two supermultiplets in RNS formalism. We write $\left|\phi_0\right\rangle$ for the 16-dimensional multiplet of massless ground states in the Fock space. The massless ground state $\left|\phi_0\right\rangle$ consists of eight boson degrees of freedom 8_v and eight fermionic degrees of freedom 8_c. We write each degree of freedom as $\left|i\right\rangle$ and $\left|\dot{a}\right\rangle$. The normalization of the ground state of these bosons and fermions is defined as follows:

$$\langle i|j\rangle = \delta_{ij}, \quad \langle \dot{a}|\dot{b}\rangle = \delta_{\dot{a}\dot{b}}. \tag{4.6.71}$$

Using the ground States, we can create the identity operator:

$$I = \left|i\right\rangle \left\langle i\right| + \left|\dot{a}\right\rangle \left\langle \dot{a}\right|. \tag{4.6.72}$$

To investigate the properties of the operator S^a_0, we use the Filtz identity:

$$\begin{aligned} S^a_0 S^b_0 &= \frac{1}{2}\{S^a_0, S^b_0\} + \frac{1}{2}[S^a_0, S^b_0], \\ &= \frac{1}{2}\delta^{ab} + \frac{1}{16} S^c_0 \gamma^{ij}_{cd} S^d_0 \gamma^{ij}_{ab}. \end{aligned} \tag{4.6.73}$$

From this identity, the independent tensors that can be created by S^a_0 are the only δ^{ij} and

$$R^{ij}_0 \equiv \frac{1}{4} S^a_0 \gamma^{ij}_{ab} S^b_0. \tag{4.6.74}$$

It can be seen that the tensor R^{ij}_0 defined in this way is equal to the expression of iK^{ij}_0. Therefore, the commutation relation generated by R^{ij}_0 can be calculated as follows:

$$[R^{ij}_0, R^{kl}_0] = \delta^{il} R^{jk}_0 - \delta^{ik} R^{jl}_0 + \delta^{jk} R^{il}_0 - \delta^{jl} R^{ik}_0. \tag{4.6.75}$$

This is obvious from the fact that K_0^{ij} is a Lorentzian generator. Since R_0^{ij} is a Lorentzian generator for spin, it rotates the spin state. In particular, since the spin generated by S_0^a is the direct sum of the vector i and the spinor index $\dot{a}$, it rotates the degenerate states of the boson and fermion, respectively

$$R_0^{ij} |k\rangle = \delta^{jk} |i\rangle - \delta^{ik} |j\rangle \tag{4.6.76}$$

$$R_0^{ij} |\dot{a}\rangle = -\frac{1}{2}\gamma^{ij}_{\dot{a}\dot{b}} \left|\dot{b}\right\rangle \tag{4.6.77}$$

Since S_0^a is the spinor generation operator, we can change the boson index i to the fermionic index $\dot{a}$

$$\begin{aligned} S_0^a |\dot{a}\rangle &= \frac{1}{\sqrt{2}}\gamma^i_{a\dot{a}} |i\rangle \\ S_0^a |i\rangle &= \frac{1}{\sqrt{2}}\gamma^i_{a\dot{a}} |\dot{a}\rangle \end{aligned} \tag{4.6.78}$$

To make the calculation easier to perform, we map the wavefunction to these states. In the light cone coordinate system, the 16-component Majorana-Weyl spinor can be represented by $(u^a, u^{\dot{a}})$. The Dirac equation in the light cone coordinate system can be written as follows:

$$\begin{aligned} k^+ u^a + \gamma^i_{a\dot{a}} k^i u^{\dot{a}} &= 0 \\ k^- u^{\dot{a}} + \gamma^i_{a\dot{a}} k^i u^a &= 0 \end{aligned} \tag{4.6.79}$$

Here, since $k^+ = i\partial/\partial x^-$ from the quantization condition, it is not the time derivative in the light cone coordinate system. Therefore, the first Dirac equation should be regarded as a constraint condition, not an equation of motion. Solving the first conditional equation for u^a, we obtain

$$u^a = -\frac{1}{k^+}\gamma^i_{a\dot{a}} k^i u^{\dot{a}} \tag{4.6.80}$$

From this equation, u^a is represented by $u^{\dot{a}}$, so actually imposing the Dirac equation halves the degrees of freedom of the Majorana-Weyl spinors. Substituting the equation (4.6.80) into the remaining Dirac equation yields the Klein-Gordon equation. The state $|u\rangle$ is given by the ground state $|\dot{a}\rangle$ and the wave function $u^{\dot{a}}$ from the constraint:

$$|u\rangle = \frac{u^{\dot{a}}(k)}{\sqrt{k^+}} |\dot{a}\rangle . \tag{4.6.81}$$

On the other hand, the wavefunction for a vector state with vector index i is the Klein-Gordon equation $k^2 = 0$ and the polarization condition

$$\zeta^\mu(k) k_\mu = 0 \tag{4.6.82}$$

is described by the polarization vector $\zeta^{\mu}(k)$, which satisfies Light cone gauge

$$\zeta^{+} = 0 \tag{4.6.83}$$

Solving the polarization condition for ζ^{-}, we obtain

$$\zeta^{-} = \frac{1}{k^{+}}\zeta^{i}(k)k^{i}. \tag{4.6.84}$$

Therefore, the only independent gauge field components are the transverse components. Also, from this equation, the vector state $|\zeta\rangle$ is

$$|\zeta\rangle = |i\rangle\, \zeta^{i}(k). \tag{4.6.85}$$

From the general super Yang-Mills theory, we can find the supersymmetric transformation law for the massless field

$$\delta A^{a}_{\mu} = \frac{i}{2}\bar{\epsilon}\Gamma_{\mu}\psi^{a} \tag{4.6.86}$$

Since the field is now a free field in the light cone gauge, the linear limit of the supersymmetric transformation law corresponds to the supersymmetric transformation law for the physical wave function. So, in order to keep the light cone gauge fixed $A^{+} = 0$ under the supersymmetric transformation invariant, we consider performing the gauge transformation simultaneously with the supersymmetric transformation:

$$\delta A^{\mu} = \frac{i}{2}\bar{\epsilon}\Gamma^{\mu}\psi + \partial^{\mu}\Lambda, \tag{4.6.87}$$

$$\delta\psi = -\frac{1}{4}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon. \tag{4.6.88}$$

Choosing the gauge parameter Λ so that $\delta A^{+} = 0$, we find

$$\Lambda = -\frac{1}{2p^{+}}\bar{\epsilon}\Gamma^{+}\psi. \tag{4.6.89}$$

Thus, the supersymmetric transformation for the physical gauge field A^{i} is as follows:

$$\delta A^{i} = \frac{i}{2}\bar{\epsilon}\Gamma^{i}\psi - i\frac{p^{i}}{2p^{+}}\bar{\epsilon}\Gamma^{+}\psi. \tag{4.6.90}$$

On the other hand, the transformation rule for ψ does not change at the linear level.

Consider the above discussion from the SO(8) perspective. The 10-dimensional spinor could be taken as a real 16-component Majorana-Weyl spinor. The 16-component Maynor-Weil spinor, which is a supersymmetric parameter, can be divided into two real 8-component spinors $\eta^{a}, \epsilon^{\dot{a}}$, from the requirement that the gauge-fixing condition is consistent with spacetime supersymmetry. We found in the previous discussions that the supersymmetric transformation corresponding to $\epsilon^{\dot{a}}$ need a correction from the global κ symmetry. From such a splitting of the SO(10) spinor in terms of SO(8), the SO(8) Dirac spinor is the $2^{8/2} = 16$ component, so the SO(10) Majorana-Weyl spinor can be regarded as an SO(8) Dirac spinor. The $\eta^a$and $\epsilon^{\dot{a}}$ are the SO(8) Majorana-Weyl spinor. From the real 16-component SO(8) spinor ϵ, the real 8-component spinor η and ϵ can be obtained by the projection condition

$$\frac{1}{2}\Gamma^+\Gamma^-\epsilon \sim \eta^a, \quad \frac{1}{2}\Gamma^-\Gamma^+\epsilon \sim \epsilon^{\dot{a}}. \tag{4.6.91}$$

Similarly, the gaugino ψ can be decomposed. Then the supersymmetric transformation (4.6.88), (4.6.90), which is consistent with the light cone gauge fixation condition, can be decomposed into the eight-component spinors η^a and $\epsilon^{\dot{a}}$ of SO(8). The transformation rules for the wave functions ζ and u are obtained from identifying of $\zeta \sim A$ and $u \sim \psi$.

These transformation rules can be derived by considering how supersymmetry is realized in the massless multiplet states $|\zeta\rangle$ and $|u\rangle$. First, we found that the supersymmetric generators are

$$Q^a = (2p^+)^{1/2} S_0^a, \tag{4.6.92}$$

$$Q^{\dot{a}} = (p^+)^{-1/2}\gamma^i_{a\dot{a}} p^i S_0^a. \tag{4.6.93}$$

Thus the supersymmetric transformation corresponding to η^a can be calculated as follows

$$\eta^a Q^a |u\rangle = \eta^a \gamma^i_{a\dot{a}} |i\rangle u^{\dot{a}}(k) = \left|\tilde{\zeta}\right\rangle, \tag{4.6.94}$$

$$\eta^a Q^a |\zeta\rangle = \eta^a (2k^+)^{1/2} \gamma^i_{a\dot{a}} |\dot{a}\rangle \zeta^i(k) = |\tilde{u}\rangle . \tag{4.6.95}$$

The $\tilde{\zeta}$ and $\tilde{u}$ are wave functions obtained by the η supersymmetric transformation and are defined from the calculation results as follows:

$$\tilde{\zeta}^i = \gamma^i_{a\dot{a}} \eta^a u^{\dot{a}}(k), \tag{4.6.96}$$

$$\tilde{u}^{\dot{a}} = \eta^a k^+ \gamma^i_{a\dot{a}} \zeta^i(k). \tag{4.6.97}$$

These expressions are consistent with the results obtained from the super-Yang-Mills theory. Similarly, computing the transformation for $Q^{\dot{a}}$, we obtain the following result:

$$\epsilon^{\dot{a}} Q^{\dot{a}} |\zeta\rangle = |\tilde{\tilde{u}}\rangle , \tag{4.6.98}$$

$$\epsilon^{\dot{a}}Q^{\dot{a}}\left|u\right\rangle = \left|\tilde{\tilde{\zeta}}\right\rangle, \tag{4.6.99}$$

$$\tilde{\tilde{u}}^{\dot{a}} = \frac{1}{2}(\epsilon\gamma_{ij})^{\dot{a}}k^i\zeta^j + \frac{1}{\sqrt{2}}\epsilon^{\dot{a}}\zeta^i k^i, \tag{4.6.100}$$

$$\tilde{\tilde{\zeta}}^j = \frac{1}{\sqrt{2}}\epsilon^{\dot{a}}\gamma^j_{a\dot{a}}u^a + \frac{\sqrt{2}}{k^+}\epsilon^{\dot{a}}u^{\dot{a}}k^j. \tag{4.6.101}$$

These results are likewise consistent with those in the super-Yang-Mills theory.

Next we consider the massive sector. The excited open string state is obtained by applying the creation operators α^i_{-n} and S^a_{-n} to the ground state $|\phi_0\rangle$. The number of fermions at all levels is the same as in the RNS formalism. Half of the states at each level are fermions, so the number of states is equal to the number of states obtained by applying α^i_{-n} and d^i_{-n} to the eight-component ground state. Since the spacetime supersymmetry in the GS formalism is required in classical theory, the number of bosons is equal to the number of fermions, which is consistent with the GSO projected states.

In the following, we concretely construct the states. The first excited states are

$$\alpha^i_{-1}\left|\phi_0\right\rangle, \quad S^a_{-1}\left|\phi_0\right\rangle. \tag{4.6.102}$$

The $|\phi_0\rangle$ had a total of 16 components because it contains the spin(8) vector representation $\mathbf{8}_v$ and the spinor representation $\mathbf{8}_c$. Therefore, the bosons and fermions in the first excited modes have $8 \times 16 = 128$ degrees of freedom each. The massive state of SO(10) is a representation of the small group SO(9) of the Poincaré group. Decomposing the first excited state into an irreducible representation of SO(9), the boson is

$$128 = 44 \oplus 84 \tag{4.6.103}$$

and the fermions are spin 3/2 multiplet. These supermultiplets generated by bosons and fermions are the same as the multiplet in 11D supergravity theory. This is because the on-shell massless multiplet in 11D supergravity theory belongs to the S0(9) representation. In general, the D-dimensional massive multiplet is in the SO(D-1) representation, which is equivalent to the D+1 dimensional massless representation.

The second excited states are

$$\alpha^i_{-2}\left|\phi_0\right\rangle, \quad S^a_{-2}\left|\phi_0\right\rangle, \quad \alpha^i_{-1}\alpha^j_{-1}\left|\phi_0\right\rangle, \quad S^a_{-1}S^b_{-1}\left|\phi_0\right\rangle, \quad \alpha^i_{-1}S^a_{-1}\left|\phi_0\right\rangle. \tag{4.6.104}$$

The total number of degrees of freedom for these states is 2304. Expressing as a direct product of the spin(9) representation, we find

$$\mathbf{23049} \otimes ((\mathbf{44} + \mathbf{84}) + \mathbf{128}). \tag{4.6.105}$$

This is an $N = 1$ irreducibe massive supermultiplet. [5]. In general, an irreducible supermultiplet can be described by a tensor product of fundamental supermultiplet containing any irreducible spin representation. In this present case, $(\mathbf{44} + \mathbf{84} + \mathbf{128})$ is the fundamental massive supermultiplet.

If we calculate the degrees of freedom for the third excited state as well as the first and second excited states, we find that it has 15,360. These states are described by two supermultiplets

$$(\mathbf{44} + \mathbf{16}) \otimes (\mathbf{44} + \mathbf{84} + \mathbf{128}). \tag{4.6.106}$$

Thus, as the mass level increases, the number of states increases exponentially. On the other hand, the magnitude of the maximum supermultiplet increases linearly.

Spectrum of Closed String

The closed-string spectra are described by the left-moving and right-moving sectors.

The massless sector is given by the direct product of two ground states $|\phi_0\rangle \times \left|\tilde{\phi}_0\right\rangle$. Since left-moving and right-moving are independent in closed string theory, two theories can be constructed by taking whether two Majorana-Weyl spinors are the same chirality or at different chirality.

If the two chirality are different, we can distinguish between left-moving and right-moving, so $16 \times 16 = 256$ modes appear. These spin(8) representations are given by the direct product of two super Yang-Mills multiplet with different chirality

$$(\mathbf{8}_v + \mathbf{8}_c) \otimes (\mathbf{8}_v + \mathbf{8}_s) = (\mathbf{1} + \mathbf{28} + \mathbf{35}_v + \mathbf{8}_v + \mathbf{56}_v)_B + (\mathbf{8}_s + \mathbf{8}_c + \mathbf{56}_s + \mathbf{56}_c)_F \tag{4.6.107}$$

These are of course consistent with the particles contained in Type IIA supergravity theory. Also, Type IIA theory is obtained by the S^1 compactification of $D = 11$ supergravity theories.

If we take the left-moving and right-moving sectors to have the same chirality and do not impose a Ω duality on the left and right sectors, we obtain 256 modes. These modes are given by the direct product of two super-Yang-Mills multiplet with the same chiralities

$$(\mathbf{8}_v + \mathbf{8}_c) \otimes (\mathbf{8}_v + \mathbf{8}_c) = (\mathbf{1} + \mathbf{28} + \mathbf{35}_v + \mathbf{1} + \mathbf{28} + \mathbf{35}_c)_B + (\mathbf{8}_s + \mathbf{8}_s + \mathbf{56}_s + \mathbf{56}_s)_F \tag{4.6.108}$$

These expressions are consistent with the particles contained in the Type IIB supergravity theory. Unlike Type IIA theory, they cannot be obtained by dimensional reduction in higher dimensional theory because type IIB theory has a same chirality. The $\mathbf{35}_v$ is gravity, and the $\mathbf{35}_c$ is a self-adjoint fourth-order antisymmetric tensor.

[5](4.6.105) is of course not an irreducible representation of spin(9), but it is an irreducible representation of the N=1 supersymmetric algebra

n the case of Type ⅡB theory, the distinction between left and right can be eliminated because left-moving and right-moving have the same chirality. In other words, we can impose the interchange symmetry (Ω-duality) on $|\phi_0\rangle$ and $\left|\tilde{\phi}_0\right\rangle$ on TypeⅡB. If we take the direct product of two super-Yang-Mills multiplet with the same chiral and impose the left-right symmetry as in Type ⅡB

$$[(\mathbf{8}_v+\mathbf{8}_c)\otimes(\mathbf{8}_v+\mathbf{8}_c)]_{sym}=(\mathbf{8}_v\times\mathbf{8}_v)_{sym}+(\mathbf{8}_v\times\mathbf{8}_c)+(\mathbf{8}_c\times\mathbf{8}_c)_{antisym}=(\mathbf{1}+\mathbf{28}+\mathbf{35}_v)_B+(\mathbf{8}_s+\mathbf{56}_s)_F \tag{4.6.109}$$

These expressions are consistent with the particles included in Type I supergravity theory.

Next, we calculate the massive closed string spectra. From the Virasoro condition $L_0=\tilde{L}_0$, the left-moving and right-moving excitations must be the same. This means that the position of the origin of the closed string coordinate σ is arbitrary. Therefore, the closed string at level n can be obtained by the tensor product of the two open string states of level n. Here, the massive open string multiplet has no chiral, so the massive multiplet of Type IIA and Type IIB are completely same. The only difference between these two theories is the massless sector.

Specifically, if we calculate the first excited state,

$$(\mathbf{44}+\mathbf{84}+\mathbf{128})\otimes(\mathbf{44}+\mathbf{84}+\mathbf{128}). \tag{4.6.110}$$

Therefore, the degree of freedom of the closed string first excited mode is $(256)^2$. Since each factor is an $N=1$ massive multiplet, the tensor product is an $N=2$ massive multiplet. This means that the $N=2$ supersymmetry of the closed string acts independently on the two factors, left-moving and right-moving.

Since there is no chiral distinction between left and right in the massive sector, we can impose the Ω duality and calculate the Type I multiplet. In this case, the total number of degrees of freedom of the field obtained is $(256)^2/2$. Of course, since left-moving and right-moving are considered identical, only the $N=1$ supersymmetry is preserved.

4.6.6 Covariant Quantization

If we consider covariant quantization in GS formalism, a problem arises. In the GS formalism, the structure of the phase space is complicated by the presence of κ symmetry and the constraint for reparameterisation that form a closed algebra with it. However, it can be quantized by using Dirac's quantization method.

In order to quantize by using Dirac's method, it is necessary to classify the constraints into two types [56]. Since the constraint conditions are current, they form an algebra. If the algebra created by the constraints is closed by itself, it is called a first-class constraint condition. Virasolo generators and super-Virasoro generators are first-class constraints because they form algebras that are closed by themselves. RNS formalism and bosonic string can be covariantly quantized by old covariant quantization and BRST quantization. In old covariant quantization, the constraints are quantized and imposed on the string state space as weak conditions [6]. In BRST quantization, which is a modern covariant quantization, gauge symmetry is treated covariantly by introducing FP ghosts. These covariant quantizations are generally effective only for systems with first-class constraints. On the other hand, constraint conditions that do not for a closed algebra are called second-class constraint conditions. The product structure of the algebra of constraint conditions in classical theory is a Poisson bracket. In Dirac's method, the product structure is achieved by replacing Poisson brackets with Dirac brackets.

In the GS formalism, there are second-class of constraints in addition to the first-class of constraints. The first-class of constraints is the super-Virasoro generators. The second-class of constraints are derived from the κ symmetry, and the constraints give an expression for the relation between the Grassmann coordinate θ^i and its conjugate momentum P^i_θ. This can be done by considering the map from $\dot{\theta}^A$ to P^A_θ and making sure that the map is not a bijection. Therefore, all we whould is to perform a Legendre transformation from the Lagrangian to the Hamiltonian. The conjugate momentum P^i_θ is

$$P^i_\theta = \frac{\delta S}{\delta \partial_\tau \theta^i} \tag{4.6.111}$$

From this definition, P^i_θ is given by the other variables X^μ, P^μ, θ^i and their derivatives. If we calculate the fermionic constraints, we find that half of those fermionic constraints are first-class constraints. Those first-class constraints are non-trivial commutation relations with the Virasoro generators $T_{\alpha\beta}$, resulting in an expantended Virasoro algebra. In other words, it is a super-Virasoro algebra. The other half of the fermionic constraints is the second-class constraint. If we do not take these facts into account, and replace the commutation relations generated by all fermionic constraints by Dirac brackets, that bacames singular. In fact, if we calculate the commutation relations of the fermionic constraints in Dirac brackets, the equations of motion appear in the denominator. Therefore, the constraint conditions must be treated separately in the first and second-class.

In handling the constraint conditions, we found that we have to perform quantization by dividing them into the first and second-class, but the problem arises that we cannot perform such division of the constraint conditions covariantly. In other words, even if we try to perform the quantization covariantly, the covariance is lost due to the classification of the constraints. Specifically, the parameter of the fermionic transformation is a real 16-component Majorana-Weyl spinor, and there is no 8-dimensional irreducible representation at $D = 10$. However, half of the fermionic constraints are of the second-class,

[6] The constraint is on all n Virasoro generators, but only $n \geq 0$ Virasoro generators are imposed as constraints on the quantum state

and the transformation parameters corresponding to these constraints are real 8-component spinors. Therefore, in $D = 10$, the constraint conditions cannot be divided into first and second-class constraint conditions, and the parameters corresponding to these constraint conditions are not covariant unless they are dropped into the spin(8) of the transverse components.

4.7 Non-Abelian Gauge Symmetry

The unified theory that best describes the real world is the Standard Model, which is an SU(3)×SU(2)×U(1) gauge theory. Therefore, for superstring theory to describe the real world, the theory must have non-commutative gauge symmetry. However, the local symmetries of string theory are the only reparameterisation invariance and local supersymmetry. In other words, there is no noncommutative gauge symmetry in superstring theory that we have constructed so far.

There are two possibilities for superstring theory to describe the real world. The first is that the 4 dimensional low-energy effective action derived from the string theory has non-commutative gauge symmetry. The second is that there is a theory that allows the string theory to have non-commutative gauge symmetry in 10-dimension. In the latter case, the gauge symmetry existing in ten dimensions is broken into its subgroups by compactification to a four-dimensional theory. In this section, we consider theories where string theory has non-commutative gauge symmetry. The breaking of the gauge symmetry by compactification will be discussed later.

There are two more types of theories where the superstring theory has non-commutative gauge symmetry. The first is the method of inducing noncommutative gauge symmetry by adding charge to the end of open string. This method does not allow for a charge on the closed string. The other method is to add charge to the string itself, which is called a heterotic string.

4.7.1 Chan-Paton Method

Whether or not it is possible to have a charge on the string depends on the type of string theory. In the superstring theories, open strings exist only in Type I theories, so Type I theories are the only theories that can introduce noncommutative gauge symmetry. On the other hand, Type IItheories have $N = 2$ supersymmetry and cannot be coupled to open string because of having $N = 1$ supersymmetry. This method to give U(2) or U(3) gauge symmetry to open strings in bosonic string theory was proposed by Chan and Paton.

Let G be any semi-simple Lie group and R be any representation of it. Let N be the dimension of the representation R. Assume that one end of the open string is transformed by the representation R and the other end is transformed by the complex conjugate representation $\overline{R}$. Since the dimension of the representation R is n, each end of the open string is degenerate to n-weight. Therefore, there are

n^2 possibilities for open strings. If R is a complex representation, it is an orientable open string because each end can be distinguished. On the other hand, if R is a real representation, we can take the open string to be unoriented.

We calculate the spectra produced by such an open string in detail. From the results of the bosonic string, the mode expansion of x^i is

$$x^i(\sigma,\tau) = q^i + p^i\tau + i\sum_{n\neq 0}\frac{1}{n}(-1)^n\alpha_n^i e^{in\tau}\cos n\sigma, \tag{4.7.1}$$

and it is normalized by $\sigma \in [0,\pi]$. By reversing the parametrization by σ, the end of $x^i(\sigma,\tau)$ are swapped:

$$x^i(\pi-\sigma,\tau) = q^i + p^i\tau + i\sum_{n\neq 0}\frac{1}{n}(-1)^n\alpha_n^i e^{-in\tau}\cos n\sigma \tag{4.7.2}$$

Therefore, the reversal of the direction of the open string is

$$\alpha_n^i \to (-1)^n\alpha_n^i \tag{4.7.3}$$

If there is no Chan-Paton factor, the string state $|\Lambda\rangle$ is simply a state in Hilbert space. However, if each end is a representation R or $\overline{R}$, the transformation index $a, \overline{b} = 1, ..., n$ of the gauge group G is required, and the string state is written as $\left|\Lambda; a, \overline{b}\right\rangle$. We take the $\sigma = 0$ point of the open string to be R and the $\sigma = \pi$ point to be $\overline{R}$.

Thus, the string spectrum is given by $\left|\Lambda; a, \overline{b}\right\rangle$ and is degenerate to n^2 weight. The massless vector particle has to be transformed into a adjoint representation under the gauge group. Thus, if $\left|\Lambda; a, \overline{b}\right\rangle$ describes a massless state, the quantum number a and $\overline{b}$ do not take all values in n^2 ways, but are restricted to the adjoint representation. If $G = U(N)$, $R = N$, a and $\overline{b}$ are index of the N-representation and $\overline{N}$, respectively, and the $N \times \overline{N}$-representation is automatically the adjoint representation. However, for the general gauge group G, the $R \times \overline{R}$ representation must be imposed requirements since it contains other representations than the adjoint representation. If R is a real representation, there is no longer any need to distinguish between the end of open string, and we can impose invariance under $\sigma \to \pi - \sigma$;

$$\left|\Lambda(\alpha_n^i); b, a\right\rangle = \epsilon\left|\Lambda((-1)^n\alpha_n^i); a, b\right\rangle \tag{4.7.4}$$

Here, $\epsilon = \pm$. A string satisfying this condition is called a non-orientable string. Since the state $\left|\Lambda(\alpha_n^i); b, a\right\rangle$ is generated by α_n^i, if we write the number operator as N and the eigenvalue of N for the massless state as N_0, then

$$N\left|\Lambda(\alpha_n^i);b,a\right\rangle = (n-N_0)\left|\Lambda(\alpha_n^i);b,a\right\rangle \tag{4.7.5}$$

Thus, rewriting $(-1)^n$ with the operator

$$|\Lambda;a,b\rangle = \pm(-1)^{N-N_0}|\Lambda;b,a\rangle \tag{4.7.6}$$

In the following, we consider a concrete semi-simple Lie group.

First we considering the case $G = SO(N)$ and $R = N$, and the fundamental representation N is a real representation, so $N = \overline{N}$. The coefficients of the condition (4.7.6) must be negative because the adjoint representation is an antisymmetric matrix. In the bosonic string, the ground state is a tachyon, so the massless state was given by $N = 1$. Therefore, $\epsilon = 1$. On the other hand in superstring theory, the ground state is a massless state, so $N = 0$. Therefore, $\epsilon = -1$. Consider the states at general level N, or mass $\alpha' M$. At even level N =even, $\epsilon(-1)^{N-1} = -1$ from $N_0 = 1$ in bosonic string, while $\epsilon(-1)^{N-0} = -1$ in superstring theory. Similarly, considering N =odd, both bosonic string and superstring theory transform as symmetric tensors under SO(N).

Next we consider the case where $G = USp(N)$. Taking R as the fundamental representation, $R \neq \overline{R}$ since R is a complex representation. The adjoint representation of $USp(N)$ is the symmetric part of the two direct product representations of the R-representation; $[R \times R]_{sym}$. Therefore, contrary to the case of SO(N), we can take $\epsilon = -1$ for bosonic string and $\epsilon = 1$ for superstring. Also, contrary to the SO(N) case, it is symmetric at N =even and antisymmetric at N =odd.

Thus, for a gauge theory to be consistent, the massless vector particle must be a adjoint representation of the gauge group. So we write $\lambda^i_{a\bar{b}}$ for the $N \times N$ anti-Hermitian matrix representing the algebra of the gauge group G. Here, $i = 1, ..., n$. In general, the matrix λ is a subset of a set of all anti-Hermitian matrices because of some restriction. All $N \times N$ anti-Hermitian matrices form a U(N) group. If the anti-Hermitian matrix is a real matrix, then it is SO(N). Therefore, the gauge group G represented by an anti-Hermitian matrix is a subgroup of U(N). In particular, we assume that the gauge group G is a normal subgroup of the U(N) group.

The condition on λ representing the gauge group G is obtained from the consistency of the theory. We use the fact that massless particles appear in any M-particle tree amplitude. The massless vector particle obtained by scattering must of course be a adjoint representation. In the Yang-Mills theory, if the algebra $[\lambda^i, \lambda^j]$ becomes λ^k again, then the massless vector particle obtained by scattering is also a adjoint representation. This means that λ represents the algebra of the gauge group G. On the other hand, in the case of Type I superstring theory, restrictions are imposed on G and the representation R.

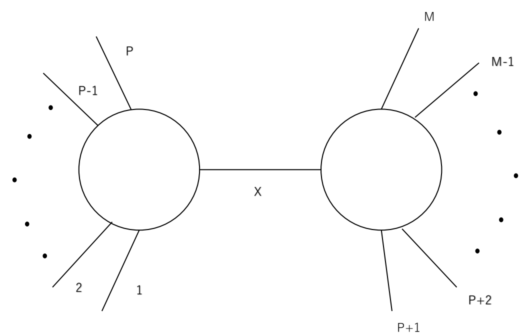


Fig. 2: M-point function

To see this concretely, let $A(1, ..., M)$ be the M-particle tree amplitude for open strings. We assume that the open strings have no quantum numbers for the gauge group. The amplitude A has a cyclic symmetry that replaces the M external lines. Let the gauge quantum numbers of the particles be the matrices $\lambda_1, ..., \lambda_M$ The tree amplitudes for open strings with quantum numbers can be written as follows

$$T(1, ..., M) = \sum \mathrm{tr}(\lambda_1 ... \lambda_M) A(1, ..., M) \tag{4.7.7}$$

Here, the summation is performed about the different factors under the $(M-1)!$ replacement. The fundamental amplitude $A(1, ..., M)$ has poles in the channels consisting of cyclic continuous external lines and no poles in the other channels. The residues of these poles have a simple factorization property

$$A(1, ..., M) \sim \frac{1}{m^2 - s} \sum_X A(1, ..., P, X) A(X, P+1, ..., M) \tag{4.7.8}$$

$$s = -(k_1 + \cdots + k_P)^2 \tag{4.7.9}$$

where X can take all possible masses m. The Feynman diagram for this channel can also be written as (4.7.1) The factorization of (4.7.8) must be unitary in order for the theory to be consistent. This requirement impose a restriction on the matrix λ. In the amplitude T, there is a term $A(P, ..., 1, P+1)$ in addition to the term (4.7.8) that gives the pole of the mass s. Specifically, the terms that give the mass s are the following four cases:

$$\begin{aligned}
&\mathrm{tr}(\lambda_1...\lambda_P\lambda_{P+1}...\lambda_M)A(1,...,P,X)A(X,P+1,...,M),\\
&\mathrm{tr}(\lambda_P...\lambda_1\lambda_{P+1}...\lambda_M)A(P,...,1,X)A(X,P+1,...,M),\\
&\mathrm{tr}(\lambda_1...\lambda_P\lambda_M...\lambda_{P+1})A(1,...,P,X)A(X,M,...,P+1),\\
&\mathrm{tr}(\lambda_P...\lambda_1\lambda_M...\lambda_{P+1})A(P,...,1,X)A(X,M,...,P+1).
\end{aligned} \tag{4.7.10}$$

Because of the cyclic symmetry of the amplitude A, the Feynman diagrams are all on the same world sheet. Therefore, the amplitudes of these terms all have the same value. However, since the Type I theory satisfies the condition (4.7.4), the only difference between each amplitude due to the difference in ϵ is the sign. Since each external line is a string spectrum and each is generated by α_n^i, from (4.7.4)

$$A(1,...,N)=\prod_{i=1}^{N}(\epsilon(-1)^{N_i})A(N,...,1) \tag{4.7.11}$$

must be satisfied.

Assuming that all M external lines are massless vector particles, $N_i=0$, and since this is superstring theory, $\epsilon=-1$, then we find

$$A(P,...,1,X)=(-1)^{P+1}A(1,...,P,X). \tag{4.7.12}$$

Therefore, the sum of the four terms in (4.7.10) is as follows:

$$\mathrm{tr}[(\lambda_1...\lambda_P-(-1)^P\lambda_P...\lambda_1)(\lambda_{P+1}...\lambda_M-(-1)^{M-P}\lambda_M...\lambda_P)]A(1,...,P,X)A(X,P+1,...,M). \tag{4.7.13}$$

Using the trace property, we decompose this trace into two parts for particles $1,...,P$ and for particles $P+1,...,M$:

$$\begin{aligned}
&\mathrm{tr}[(\lambda_1...\lambda_P-(-1)^P\lambda_P...\lambda_1)(\lambda_{P+1}...\lambda_M-(-1)^{M-P}\lambda_M...\lambda_P)]\\
&=\sum_\alpha \mathrm{tr}[(\lambda_1...\lambda_P-(-1)^P\lambda_P...\lambda_1)\lambda_\alpha]\,\mathrm{tr}[\lambda_\alpha^T(\lambda_{P+1}...\lambda_M-(-1)^{M-P}\lambda_M...\lambda_{P+1})].
\end{aligned} \tag{4.7.14}$$

Such a decomposition is only possible if λ_α is a complex $N\times N$ matrix normalized by $\mathrm{tr}(\lambda_\alpha\lambda\beta^T)=\delta_{\alpha\beta}$. If $\lambda=(\lambda_1...\lambda_P-(-1)^P\lambda_P...$ forms a subspace of the all $N\times N$ matrices, the validity of (4.7.14) becomes more complicated. For example, if $\lambda_1,...,\lambda_P$ are the arbitrary real antisymmetric matrices, λ is also an antisymmetric matrix. Then, since $\lambda_\alpha\neq 0$, λ_α is also restricted to be an antisymmetric matrix.

The unitarity condition restricts the number of particles of massless vector particles appearing as poles of T to the adjoint representation of the gauge group to which the external line belongs. More generally, the quantum number of any pole of T should correspond to a particle of the spectrum. Moreover, two factors in the residues should describe the amplitude T. From the above considerations, the non-contradiction condition is that all matrices defined as

$$\lambda = (\lambda_1 ... \lambda_P - (-1)^P \lambda_P ... \lambda_1). \tag{4.7.15}$$

are satisfied by being the $N \times N$ anti-Hermitian matrices λ^i that represent the gauge group.

Consider the case where $P = 2$. Then $\lambda = \lambda_1\lambda_2\lambda_2\lambda_1 = [\lambda_1, \lambda_2]$, and as long as λ is a representation of the gauge group, $\lambda = f_{12i}\lambda_i$, so the no-consistency condition is satisfied.

In the general case $P > 2$, the restriction is stronger, and the gauge group G and the representation R are restricted. Such restrictions on groups and representations due to unitarity do not occur in field theories, due to the existence of cyclic symmetry in string theory.

Restrictions on Gauge Groups and Representations

In the above discussions, we found a restriction on the representation matrix λ of the gauge group. Here, we derive a general solution to this condition. We denote by L_a the linear space created by the anti-Hermitian matrix λ_i, which is a closed algebra. The algebra formed by λ is formally written as $[\lambda, \lambda] = \lambda$ without the index. The anti-commutation relation of anti-Hermitian matrix $\{\lambda, \lambda\} = \mu$ is Hermitian matrix. These matrices μ describe the possible quantum numbers at mass level N=odd [7] . More generally, the possible Hermitian matrix μ can be written as follows

$$\mu = \lambda_1 ... \lambda_P + (-1)^P \lambda_P ... \lambda_1 \tag{4.7.16}$$

We denote L_h for the linear space formed by this Hermitian matrix. Also, the anti-commutation relation of the Hermitian matrix is a Hermitian matrix, so $\{\mu, \mu\} = \mu$,forming a closed algebra.

Next, consider the linear space L formed by all matrices ρ, which can be decomposed into $L_a + L_h$. In other words

$$\rho = a\mu + b\lambda. \tag{4.7.17}$$

The condition for an open string to have a quantum number of a noncommutative gauge group was that λ form a closed algebra. Since λ forms a closed algebra by the commutation relation, but the anti-commutation relation forms a Hermitian matrix μ, it is non-trivial whether λ forms a closed algebra

[7] The $(-1)^P$ factor in the definition of λ is originally given by $\prod_{i=1}^{N}(\epsilon(-1)^{N_i})$, and in the massless case $\epsilon(-1)^N = -1$, so $\prod_{i=1}^{P+1}(-1)^i = (-1)^{P+1}$. In superstring theory, the massless state is $N = 0$, so for N=odd, the $(-1)^P$ factor in the definition of λ is replaced by $(-1)^{P+1}$.

by the anti-commutation relation. So we can say that λ is a closed algebra if the L is closed. Thus we comfirm

$$\rho_1\rho_2 \in L, \quad {}^\forall\rho_1, \rho_2 \in L. \tag{4.7.18}$$

In other words, not only L_a, but also L must be a Lie algebra.

From the above, we can find the possible Lie algebras as L. The general solution to this problem is given by Wedderburn's theorem [72].

Theorem. Wedderburn's Theorem

The irreducible complex closed algebra is the matrix algebra GL(n,C).

Thus, if $a, b \in \mathbb{C}$, the anti-Hermitian matrix λ generates a Lie algebra U(N). If $a, b \in \mathbb{R}$, it is a real form of GL(n,C). By this theorem, it is known that there are only two possible Lie algebras in real form. One is SO(N), the anti-Hermitian part L_a of the real form GL(n,C). The other is $U(2N)$ [8]. The anti-Hermitian part of $U(2N)$ is $USp(2N)$ [9]. From the above, the only possible matrices are $N \times N$ matrices such that the elements are real, complex, and quaternions.

From the above analysis, it is clear that only classical Lie groups are consistent with Chan-Paton's method. Exceptional groups such as E_8 are not included. Furthermore, since U(N) for complex matrices, SO(2N) for real matrices, and USp(2N) for pseudo-real matrices, these are all fundamental representations. Thus, strings are not orientable in SO(2N) and USp(2N), but are orientable in U(N). [10] .

In fact the U(N) case is not consistent with the quantum theory at one-loop order in superstring theory. If we consider a nonplanar 1-loop diagram, this diagram shows the transition from open strings to closed strings. In other words, the ends of the open strings join to form the closed strings. In the case of U(N) open strings, the string has an orientation, so the closed string obtained by the transition also has an orientation. From the discussion of superstring theory, orientable closed strings have $N = 2$ supersymmetry, while unorientable closed strings have $N = 1$ supersymmetry. Since an open string can only have $N = 1$ supersymmetry, it cannot be coupled to a closed string with $N = 2$ supersymmetry. In other words, closed strings cannot be created consistently from orientable open strings. Therefore, there is an anomaly in U(N) open string theory and we have to choose SO(2N) or USp(2N) as the noncommutative gauge group.

[8] U(2N) is complex in general, but is noncompact if taken real

[9] USp(2N) is the algebra of $N \times N$-matrices whose field is quaternions

[10] In the real representation, $R = \overline{R}$, which can be taken as unorientable, but in the complex representation, $R \neq \overline{R}$, which always determines the orientation. In other words, the orientation of the string is determined by the U(1) charge

4.8 Heterotic String Theory

String Theory is completely deteremined by the Virasoro Constrains. These constrants are independent as left- and right-moving. Thus we construct the left-moving string theory and right-moving string separetely and consistently. In other word, we can construct the theory whose left-moving sector is a bosonic string and right-moving sector is a superstring heterotically. In order to obtain the gauge theory, we need a additional bosonic degrees. So we can use the extra 16 dimensional coordinate $x^i,\ i = 1, ..., 16$ from bosonic strgin to realize the gauge symmetry.

We begin by considering the simplest compactification, the S^1 compactification of bosonic string theory. Then, we consider a more general d-dimensional compactification, and construct a $(26 - d)$ dimensional theory. We will also see that the theory obtained by the compactification automatically reproduces the noncommutative gauge symmetry. The extra 16 degrees of freedom of the left-moving boson coordinates can be torus-compactified to reproduce SO(32) symmetry and $E_8 \times E_8$ symmetry.

The construction method using this compactification makes the $E_8 \times E_8$ symmetry easier to understand than the construction method using current algebra [55]. Furthermore, it can be shown that SO(32) theory and $E_8 \times E_8$ correspond to two even-self dual lattices on a compactified 16-dimensional space.

4.8.1 S^1-compactification

First of all, we consider the simplest S^1-compactification.

The bosonic string theory is a consistent theory at $D = 26$, and we consider compactifying the spatial dimension of this theory by S^1. Since the spatial dimension one is S^1, the coordinate satisfying equivalence relation

$$x = x + 2\pi Rn, \tag{4.8.1}$$

where R is a radius of S^1 and n is an arbitrary number. In this relation, the solotion of the equation of motion for the periodic boundary condition is given by

$$X(\sigma,\tau) = x + p\tau + 2L\sigma + \frac{i}{2}\sum_{n=0}\frac{1}{n}\left[\alpha_n e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n e^{-2in(\tau+\sigma)}\right], \tag{4.8.2}$$

where

$$p = \frac{m}{R}, \quad L = nR, \quad n \in \mathbb{Z}. \tag{4.8.3}$$

The constant m can only take integer values due to the uniqueness of the wave function [11].

Also, the n introduced by the equivalence relation represents the winding number. If $n \neq 0$, then

[11] Considering the wavefunction e^{ipx}, by the equivalence relation of $x \sim x + 2\pi R$, we have $e^{ipx} \sim e^{ipx}e^{2\pi ipR}$. Since x and $x + 2\pi R$ represent the same space-time point on S^1, the wave function at that point must be uniquely determined. Thus, $pR = m \in \mathbb{Z}$

the string X^μ is a soliton. In other words, in an uncompactified theory, there is no integer n and no soliton like state. In fact, if we consider the limit of $R \to \infty$, the compactified space S^1 can no longer be regarded as a compact flat space and is therefore equivalent to the original non-compactified space, but the soliton state diverges in energy due to this limit. This can be physically interpreted as the string tension increases as the string tightening space increases, and the energy diverges. Such a solitonic state exists even if the compact space is not S^1, and it exists if the manifold exists after a non-contractible loop. In other words, it is enough if the first fundamental group π_1 is nontrivial.

The mode expansion (4.8.2) can be decomposed into left-moving and right-moving modes:

$$X(\sigma,\tau) = X_R(\tau-\sigma) + X_L(\tau+\sigma), \tag{4.8.4}$$

$$X_R(\tau-\sigma) = x_r + \left(\frac{p}{2} - L\right)(\tau-\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n e^{-2in(\tau-\sigma)}, \tag{4.8.5}$$

$$X_L(\tau+\sigma) = x_L + \left(\frac{p}{2} + L\right)(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n e^{-2in(\tau+\sigma)}. \tag{4.8.6}$$

Calculating the Virasoro condition $T_{++} = 0 = T_{--}$, and in particular the $n = 0$ Virasoro generator corresponding to the equation of motion is calculated as

$$L_0 = \frac{1}{2}\left(\frac{1}{2}p - L\right)^2 + N + \frac{p^\mu p_\mu}{8}, \tag{4.8.7}$$

$$\tilde{L}_0 = \frac{1}{2}\left(\frac{1}{2}p + L\right) + \tilde{N} + \frac{p^\mu p_\mu}{8}, \tag{4.8.8}$$

where

$$N = \sum_{n=1}^{\infty}(\alpha^\mu_{-n}\alpha_{n\mu} + \alpha_{-n} + \alpha_n), \tag{4.8.9}$$

$$\tilde{N} = \sum_{n=1}^{\infty}(\tilde{\alpha}^\mu_{-n}\tilde{\alpha}_{n\mu} + \tilde{\alpha}_{-n}\tilde{\alpha}_n). \tag{4.8.10}$$

and since the Virasolo condition is for a bosonic string theory

$$L_0 = 1 = \tilde{L}_0, \tag{4.8.11}$$

where the index μ is the index of the uncompactified spacetime Lorentz symmetry, $\mu = 0, ..., 24$. On the other hand, α_m and $\tilde{\alpha}_m$ are oscillators in the compact space direction. We can rewrite the Virasoro condition using the relation (4.8.3) and recombine the linear combination of the left and right Virasoro conditions as follows:

$$\frac{1}{4}M^2 = N + \tilde{N} - 2 + \frac{m^2}{4R^2} + n^2R^2, \tag{4.8.12}$$

$$N - \tilde{N} = pL = mn. \tag{4.8.13}$$

The fourth term is the momentum in the compact space direction. The momentum of the string in the compact space direction can no longer be observed, it is observed as the mass of the particle. The fifth term depends on the number of wraps and represents the energy due to the wraps, which means that the increased tension due to the wrapping of the string in the compact space direction is converted into the mass of the particle. In a closed string, there is no reference point for the string length parameter, σ. This means that any point on the closed string can be parameterized as $\sigma = 0$. In fact, consider an operator that translates the parameter σ by σ_0:

$$U(\sigma_0) = \exp\left[2i(N - \tilde{N} - pL)\sigma_0\right] \tag{4.8.14}$$

Then $U(\sigma_0) = 1$ for physical string states satisfying the Virasoro condition, and the physical state does not depend on the way the origin of σ is taken.

Consider the 25-dimensional theory obtained by compactification. The vacuum in the original 26-dimensional theory is labeled by the Casimir operator of the Poincaré group. In the 25-dimensional theory, the vacuum is also labeled by the Casimir operator of the Poincaré group in 25-dimensional spacetime. Since p^{25} is given by the integer m, the vacuum in the 25-dimensional theory is labeled by n, m, p^μ. As before, the momentum p^μ label is omitted and written as $|m, n\rangle$.

First we consider the massless states;

$$N + \tilde{N} - 2 + \frac{m^2}{4R^2} + n^2R^2 = 0 \tag{4.8.15}$$

As can be seen from this equation, there are various possibilities of $N, \tilde{N}, m, n$ such that the state is massless. One of them is the case $N + \tilde{N} = 2$. There are the three patterns $N = 2, \tilde{N} = 2$ and $(N, \tilde{N}) = (1, 1)$, and the momentum and number of wraps are $m = n = 0$ from the Virasoro condition. Then, from the other Virasoro condition (4.8.13), only the case $(N, \tilde{N}) = (1, 1)$ is possible. Furthermore, from the number operators (4.8.9) and (4.8.10), we can see that there are four possible combinations such that $N = \tilde{N} = 1$;

Considering the 25-dimensional oscillator $\alpha^\mu_{-1}, \tilde{\alpha}^\mu_{-1}$ for both left and right as a combination of generators with $N = \tilde{N} = 1$;.

$$\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|0, 0\rangle \tag{4.8.16}$$

This state is the spectrum of a typical closed string, as it has appeared in previous discussions. That is, it can be decomposed into symmetric, antisymmetric, and trace components, with the symmetric part being 25 dimensional gravity.

Next we can use $\alpha^\mu_{-n}, \tilde{\alpha}_{-n}$ or $\alpha_{-n}, \tilde{\alpha}^\mu_{-1}$. The states generated in this case is

$$\alpha^\mu_{-1}\tilde{\alpha}_{-1}\left|0,0\right\rangle, \quad \alpha_{-1}\tilde{\alpha}^\mu_{-1}\left|0,0\right\rangle. \tag{4.8.17}$$

To make it easier to understand the left-right symmetry, we can rewrite the linear combination

$$(\alpha^\mu_{-1}\tilde{\alpha}_{-1} \pm \alpha_{-1}\tilde{\alpha}^\mu_{-1})\left|0,0\right\rangle. \tag{4.8.18}$$

These states are included in $\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}\left|0\right\rangle$ in the 26-dimensional theory before compactification. In other words, it is $g^{25\mu}, B^{25,\mu}$ if written as a component of a field in 26 dimensions.

Finally, consider the particles produced by the oscillators in the compact space direction together:

$$\alpha_{-1}\tilde{\alpha}_{-1}\left|0,0\right\rangle \tag{4.8.19}$$

This state is the g^{2525} component in the 26-dimensional theory.

Second possibility is $(N, \tilde{N}) = (1, 0)$ From the second Virasoro condition (4.8.13)

$$1 - 0 = mn \tag{4.8.20}$$

The integer solution satisfying this equation is

$$(m, n) = (\pm 1, \pm 1) \tag{4.8.21}$$

Therefore, the massless states in the (1,0) case are

$$\alpha^\mu_{-1}\left|\pm 1, \pm 1\right\rangle, \quad \alpha_{-1}\left|\pm 1, \pm 1\right\rangle. \tag{4.8.22}$$

Furthermore, substituting the first one into Virasoro condition (4.8.12), we find

$$0 = 1 - 2 + \frac{1}{4R^2} + R^2 \tag{4.8.23}$$

Solving this equation, we have $R^2 = 1/2$. Here, we used a system of units where $\alpha' = 1/2$, so in general, $R^2 = \alpha'$. Thus, the size of the compact space can be taken to be arbitrary, but only for a special size

$R = \sqrt{\alpha'}$ does such a massless mode appear. If R deviates from $\sqrt{\alpha'}$, these states become massive modes or tachyonic modes [12].

The third possibility is $(N, \tilde{N}) = (0, 1)$. Calculating this case in the same way as (1,0), we find $mn = \pm 1$, so the massless states are

$$\tilde{\alpha}^{\mu}_{=1} \left|\pm 1, \mp 1\right\rangle, \quad \tilde{\alpha}_{-1} \left|\pm 1. \mp 1\right\rangle. \tag{4.8.24}$$

Also, $m^2 = n^1 = 1$, so $R^2 = \alpha'$.

Final possibility is $(N, \tilde{N}) = (0, 0)$ Substituting (0,0) into the two Virasoro conditions, we find

$$\frac{m^2 2}{4R^2} + n^2 R^2 = 2, \tag{4.8.25}$$

$$mn = 0. \tag{4.8.26}$$

Therefore, $m = 0$ or $n = 0$. Substituting this solution into the first Virasoro condition, we find

$$2 = \frac{m^2}{4R^2}, \quad \text{or} \quad 2 = n^2 R^2. \tag{4.8.27}$$

Such m, n is determined depending on R. In particular, for $R^2 = 1/2$, $(m = \pm 2, n = 0)$ or $(m = 0, n = \pm 2$, so the states are

$$\left|\pm 2, 0\right\rangle, \quad \left|0, \pm 2\right\rangle. \tag{4.8.28}$$

Here, from a physical point of view, massless vectors must be coupled to conserved currents, and for any R, there are two kinds of massless vectors appearing in (1,1), which are commutative massless vector fields, and thus $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ symmetry appears. Since these massless vector states are states included in gravity in 26-dimensional theory, the $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ symmetry is a subgroup of the 26-dimensional general coordinate transformation. Furthermore, in the case of $R^2 = \alpha'$, there are 8 more degrees of freedom for massless scalars and 4 more degrees of freedom for massless vectors, so the corresponding symmetries ppear. Specifically, by computing the current algebra, we can find that the $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ symmetry is extended to $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry [13].

In fact in the case $R^2 = \alpha'$ the massless scalar field is also in the representation of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. There are a total of 9 degrees of freedom in the massless scalar field, and these form a (3,3)-representation of SU(2)×SU(2).

[12] Substituting $R^2 = 1/2 + \delta$ into (4.8.12) gives $M \simeq 1 - 2\delta$, so if the compact scale is smaller than the string scale, tachyons remain. In other words, the negative vacuum energy in the bosonic string is canceled out by the energy due to the tension of the string wrapped around the compact space

[13] Compact spaces are fiber bundles on manifolds, and in the general R case, the fibers are S^1, so the U(1) symmetry with S^1 group manifold is realized

If we consider non-orientable (Z_2-projective or Ω-duality) closed strings, only SU(2) appears because the distinction between left-moving and right-moving sectors disappears. Also, only 6 degrees of freedom remain for massless scalars due to this Ω duality, and these are the (1+5) representations of SU(2).

4.8.2 General Dimension Compactification

We extend the S^1 compactification to the arbitrary dimensional compactification:

$$\begin{aligned} x_L^I &= q_L^I + \frac{\sqrt{2\alpha'}}{2}\alpha_0^I(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum \frac{1}{n}\alpha_n^I e^{-in(\tau+\sigma} \\ x_R^{I'} &= q_R^{I'} + \frac{\sqrt{2\alpha'}}{2}\overline{\alpha}_0^I(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum \frac{1}{n}\overline{\alpha}_n^I e^{-in(\tau-\sigma} \end{aligned} \tag{4.8.29}$$

Let the respective compact spaces be $T_L^D = \mathrm{R}^D/2\pi\Lambda_L$ and $T_R^{D'} = \mathrm{R}^{D'}/2\pi\Lambda_R$. Also, if we set the number of wrappings as L_L^I and $L_R^{I'}$, respectively, the boundary conditions are

$$\begin{aligned} x_L^I(\tau+\sigma+2\pi) &= x_L^I(\tau+\sigma) + 2\pi L_L^I \\ x_R^{I'}(\tau-\sigma-2\pi) &= x_R^{I'}(\tau-\sigma) + 2\pi L_R^{I'}. \end{aligned} \tag{4.8.30}$$

By substituting the solutions into these conditions, we obtain

$$\frac{\alpha'}{2}p_{L,R} \in \Lambda_{L,R}. \tag{4.8.31}$$

Also, from the univalence of the wave function,

$$\frac{\alpha'}{2}p_{L,R} \in \Lambda_{L,R}^*. \tag{4.8.32}$$

Thus, each lattice is self-dual. The Viasoro condition for L_0 is

$$\begin{aligned} M^2 &= \frac{1}{2}\sum^{D}(p_L)^2 + \frac{1}{2}\sum^{D'}(p_R)^2 + \frac{2}{\alpha'}(N+\overline{N}-2) \\ \frac{\alpha'}{4}[\sum^{D}(p_L)^2 - \sum^{D'}(p_R)^2] &= -(N-\overline{N}) \end{aligned} \tag{4.8.33}$$

The massless states in 4D theory are given as the following. The scalar massless states are given by

$$\alpha_{-1}^I\overline{\alpha}_{-1}^{J'}\left|0,0\right\rangle \tag{4.8.34}$$

On the other hand, the vector massless states are

$$\alpha^I_{-1}\overline{\alpha}^{\nu}_{-1}\left|0,0\right\rangle \quad \alpha^{\mu}_{-1}\overline{\alpha}^{J'}_{-1}\left|0,0\right\rangle \tag{4.8.35}$$

These states are gauge fields of $[U(1)_L]^D \times [U(1)_R]^{D'}$.

$$\overline{\alpha}^{\mu}_{-1}\left|w_L^2 = 2, w_R^2 = 0\right\rangle \quad \alpha^{\mu}_{-1}\left|w_L^2 = 0, w_R^2 = 2\right\rangle \tag{4.8.36}$$

These states are gauge fields for the noncommutative group.

4.8.3 Heterotic String

We take the left-moving sector as a bosonic string and the right-moving sector as a superstring theory, and construct a 10-dimensional string theory by compactifying the left-moving sector in 16 dimensions. In the right-moving sector, there are scalars x_R^{μ} and fermions ψ_R^{μ}, and in the left-moving sector, there are scalars x_L^{μ} and x_L^I. The compact space is a vector bundle, which leads to a gauge group with the torus lattice as the root lattice. The dimension of the vector bundle is 16, and the lattice is a self-dual Euclidean even lattice. The only such root lattice is SO(32) or $E_8 \times E_8$, and on such a lattice the gravitational anomaly is canceled by Green-Schwartz mechanism [55, 73–76]. In order for x_L^{μ} and x_R^{μ} to constitute a single consitente theory, we take their zero modes to be equal.

$$\begin{aligned} x_L^{\mu} &= q_L^{\mu} + \tfrac{\sqrt{2\alpha'}}{2}\alpha_0^{\mu}(\tau+\sigma) + i\sqrt{\tfrac{\alpha'}{2}}\sum \tfrac{1}{n}\alpha_n^{\mu}e^{-in(\tau+\sigma} \\ x_R^{\mu} &= q_R^{\mu} + \tfrac{\sqrt{2\alpha'}}{2}\overline{\alpha}_0^{\mu}(\tau-\sigma) + i\sqrt{\tfrac{\alpha'}{2}}\sum \tfrac{1}{n}\overline{\alpha}_n^{\mu}e^{-in(\tau-\sigma} \end{aligned} \tag{4.8.37}$$

$$q_L^{\mu} = q_R^{\mu} = x_0^{\mu}(0), \alpha_0^{\mu} = \overline{\alpha}_0^{\mu}, x^{\mu} = x_L^{\mu} + x_R^{\mu} \tag{4.8.38}$$

Since ψ_R^{μ} is a 10-dimensional spinor, it can be taken to be a Majonara-Weyl. Therefore, the Weyl condition

$$\gamma_5\psi_R^{\mu} = -\psi_R^{\mu} \tag{4.8.39}$$

is satisfied. The compact space coordinate x_L^I exists on a SO(32) or $E_8 \times E_8$ root lattice on a 16-dimensional self-dual Euclidean even lattice and

$$x_L^I(\tau+\sigma+2\pi) = x_L^I(\tau+\sigma) + 2\pi L_L^I \tag{4.8.40}$$

is satisfied. The solution is

$$x_L^I = q_L^I + \frac{\sqrt{2\alpha'}}{2}\alpha_0^I(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\Sigma\frac{1}{n}\alpha_n^I e^{-in(\tau+\sigma)} \tag{4.8.41}$$

Since the constraints are independent for the left- and right-moving sectors, we can use the constraints for each of them. The result shows that there is no tachyon in the lowest level state.

5 Magnetized D-Brane

Here we summarize the torus compactification with magnetic fluxes. Our model can correspond to the D5-brane system as wel as D9-brane. In this section, we note 10 dimensional coordinate as $x^M, M = 0, ..., 9$, 4 dimensional coordinate label as $\mu = 0, ..., 3$, and extra dimensional coordinate as $m = 4, ..., 9$ since torus has a flat metric.

5.1 Kalza-Klein Reduction

The computation of 4 dimensional spectrum in compactifications is based on the Kalza-Klein approach [77, 78]. Since massless fields are counted by topological invariants of compactification (Index Theorem, see also [79]), their number is unchanged under the continuous deformations. In the following we consider the KK compactification of various 10d fields to 4d on a compact 6d manifold $\mathcal{M}_6$. We keep the discussion general and restrit to the torus, magnetized torus case in the next section.

5.1.1 Scalar Fields

Consider a free 10d scalar $\phi(x^M)$ on $\mathcal{M}_4 \times \mathcal{M}_6$. The generalized Fourier decomposition is an expansion

$$\phi(x^\mu, y^m) = \sum_k \phi^k_{6d}(y^m)\phi^k_{4d}(x^\mu), \tag{5.1.1}$$

where the 6d scalars $\phi^k_{6d}(y^m)$ are the eigenfunctions of the 6d laplacian Δ_{X_6} with eigencalues $-\lambda^{(k)}$. Since 10 dimensional d'Alembertian can be decomposed to $\Box_{10d} = \Delta_{6d} + \Box_{4d}$, the equation of motion in 10 dimension for free scalar fields becomes

$$(\Box_{4d} - \lambda^{(k)})\phi^k_{4d}(x^\mu) = 0 \tag{5.1.2}$$

Hence, the eigenvalue $\lambda^{(k)}$ plays the role of the squared 4d scalar mass. If $\lambda_{k'} = 0$, $\phi^{k'}_{4d}$ is a massless field.

5.1.2 p-Form Fields

Next we consider the p-form fields $C_p(x^M)$ like gauge fields with gauge invarince $C_p \to C_p + d\Lambda_{p-1}$. The gauge invariant field strength associated with gauge field C_p is given by

$$\hat{F} = dC, \tag{5.1.3}$$

and Maxwell action is

$$S = \frac{p+1}{2p!}\int_M g^{i_1 j_1}\cdots g^{i_{p+1}j_{p+1}}\hat{F}_{i_1\ldots i_{p+1}}\hat{F}_{j_1\ldots j_{p+1}} \equiv \frac{\langle dC, dC\rangle}{2(p!)^2} \tag{5.1.4}$$

where we define the inner product $\langle , \rangle$ for (p+1)-forms. The field equation derived from (5.1.4) is

$$d^\dagger dC = 0 \tag{5.1.5}$$

where $d^\dagger$ is the adjoint of d.

We assume the following decomposition:

$$C_p(x^M) = c_q(y^m)c_n(x^\mu), \quad p = n+q \tag{5.1.6}$$

where we implicitly denote the wedge product, KK labels k and subindices 10d, 6d and 4d since we can find them from each coordinate. The kinetic term operator for 10d p-form can be decomposed to

$$\Delta_{10d} = (dd^\dagger + d^\dagger d)_{X_6} + \Delta_{4d} \tag{5.1.7}$$

In order to obtain massless fields, we solve the zero mode equation of 6d q-form:

$$(dd^\dagger + d^\dagger d)_{X_6} c_q(y^m) = 0 \tag{5.1.8}$$

Since this kinetic operator is positive define, zero mode must satisfy $dc_q = 0, d^\dagger c_q = 0$. This fact implies that the q-form fields c_q with zero eigenvalue are harmonic forms. There are one-to-one correspondence between harmonic q-form and q-cohomology class. Thus the number of zero modes of q-forms in X_6 is given by Betti number $b_q(X_6)$ which is defined topologically.

The bottom line is that since the number of massless fields is defined topologically, the counting cannot change by continuous deformations like the adding interactions. Thus it is sufficient that we consider the only free field.

Substituting (5.1.6) into (5.1.4) with (5.1.7) and (5.1.8), we obtain the 4d action:

$$S^{(4)} = \frac{n+1}{2n!}\int g^{i_1 j_1}\cdots g^{i_{n+1}j_{n+1}} F_{i_1\ldots i_{n+1}} F_{j_1\ldots j_{n+1}} \equiv \frac{\langle dC_n, dC_n\rangle}{2(n!)^2} \tag{5.1.9}$$

where $F = dc_n$ propagating on $\mathcal{M}_4$.

5.1.3 Spinors

We consider the 10d spinors ψ_{10d}. In compactification ansatz there are components of the form

$$\psi_{10d}(x^M) = \chi_{6d}(y^m)\psi_{4d}(x^\mu). \tag{5.1.10}$$

Their representations of 4d and 6d Lorentz group follow from the below decomposition:

$$SO(10) \to SO(6) \times SO(3,1), \quad \mathbf{16} = (\mathbf{4}, \mathbf{2}) + (\overline{\mathbf{4}}, \mathbf{2'}) \tag{5.1.11}$$

where $\mathbf{2}, \mathbf{2'}$ are the left- and right-handed spinor respectively in the 4d Lorentz group.

The 10d kinetic operator is the Dirac operator $/\!\!D_{10d}$ including the 10d spacetime derivative, spin connection and gauge connection. Focussing on $(\mathbf{4}, \mathbf{2})$ component, we can decompose the 10 Dirac operator to

$$\not{D}_{10d} = \not{D}_{6d} + \not{D}_{4d}. \tag{5.1.12}$$

Thus the number of 4d massless left-handed Weyl fermions is determined by the 6d zero mode equation

$$\not{D}_{6d}\chi_{6d}(y^m) = 0. \tag{5.1.13}$$

Similary, in $(\overline{\mathbf{4}}, \mathbf{2'})$ the Dirac op can be fuctorized as

$$\not{D}_{10d} = \not{D}^\dagger_{6d} + \not{D}^\dagger_{4d}, \tag{5.1.14}$$

and the number of 4d massless right-handed Weyl fermion is given by the 6d zero mode equation for $\not{D}^\dagger_{6d}$.

5.1.4 Metric

In KK compactifications the 10d metric g_{MN} can be decomposed like

$$g_{MN} = \left(\begin{array}{c|c} g_{\mu\nu} & g_{\mu n} \\ \hline g_{m\nu} & g_{mn} \end{array}\right), \tag{5.1.15}$$

and leads to various 4d fields.

The componet $g_{\mu\nu}$ corresponds to a scalar field in X_6. Thus the zero mode equation can be obtained as well as a scalar fields case. Therefore the 10d graviton leads to a unique 4d graviton. The components $g_{\mu n}$ lead to 4d vector bosons which are also 6d vector boson. The component g_{mn} is correspossnts to 4d scalar fields.

5.2 $N = 1$ SYM and its Compactification

Let us consider $D = 10, N = 1$ supersymmetric Yang-Mills theory (SYM), which corresponds to low energy effective action of the open string of superstring theory. Its Lagrangian density is given by

$$\mathcal{L}_{\text{SYM}} = -\frac{1}{4g^2}\,\text{tr}\{F^{MN}F_{MN}\} + \frac{i}{2g^2}\,\text{tr}\{\bar{\lambda}\Gamma^M D_M \lambda\} \tag{5.2.1}$$

where the tr is trace for the adjoint representation of gauge group G. The field strength F_{MN} and covariant derivative D_M are defined as

$$F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N] \tag{5.2.2}$$

$$D_M\lambda = \partial_M\lambda - i[A_M, \lambda]. \tag{5.2.3}$$

A_M is 10 dimensional geuge field and λ is a Majorana-Weyl spinor whose degree of freedom is $2^{10/2}/2 = 16$ and transforms as the adjoint representation of G. This action is also invriant under the gauge transformations

$$A_M \to A_M + \partial_M\Lambda + i[\Lambda, A_M] \tag{5.2.4}$$

$$\lambda \to \lambda + i[\Lambda, \lambda] \tag{5.2.5}$$

where Λ is a gauge degree of freedom on adjoint of G.

5.2.1 Dimensional Reduction

In order to obtain a $D = 4$ theory at low energies, we should compactify the D-4 dimension with compact manifold $\mathcal{M}_{D-4}$, so that we obtain Standard Model at lower energy than compactification scale M_c.

In order to construct the symmetry breaking model $G \to H$, we denote the elements of Cartan subalgebra in G as U and the other elements as e, and the basis of G can be chosen to $(U_a)^i_j = \delta_{ai}\delta_{aj}$ and $(e_{ab})_{ij} = \delta_{ai}\delta_{bj}$. By using these notations, the gauge field A_M and the spinor field λ decomposed to

$$A_M = B^a_M U_a + W^{ab}_M e_{ab} \tag{5.2.6}$$

$$\lambda = \eta^a U_a + \Psi^{ab} e_{ab}. \tag{5.2.7}$$

Due to hermiticity of the U(N) generators, B_M and η to be real and $W^{ab}_M = (W^{ba}_M)^*, \Psi^{ab} = (\Psi^{ba})^*$. By substituting these decompositions into the Lagrangian (5.2.1), we find

$$\mathcal{L}_{\rm B} = \mathcal{L}_{\rm B}^{(1)} + \mathcal{L}_{\rm B,4\text{-point}} + \mathcal{L}_{\rm B}^{(2)} \tag{5.2.8}$$

$$\mathcal{L}_{\rm B}^{(1)} = -\frac{1}{2g^2}\,{\rm tr}\{D_M W_N D^M W^N - D_M W_N D^N W^M - iG_{MN}[W^M, W^N] \tag{5.2.9}$$

$$\mathcal{L}_{\rm B,4\text{-point}} = \frac{1}{4g^2}\,{\rm tr}\{[W_M, W_N][W^M, W^N]\} \tag{5.2.10}$$

$$\mathcal{L}_{\rm B}^{(2)} = \frac{i}{2g^2}\,{\rm tr}\{(D_M W_N - D_N W_M)[W^M, W^N]\} - \frac{1}{4g^2}\,{\rm tr}\{G_{MN}G^{MN}\} \tag{5.2.11}$$

and

$$\mathcal{L}_{\rm F} = \mathcal{L}_{\rm F}^{(1)} + \mathcal{L}_{\rm F,Y} + \mathcal{L}_{\rm F}^{(2)} \tag{5.2.12}$$

$$\mathcal{L}_{\rm F}^{(1)} = \frac{i}{2g^2}\,{\rm tr}\{\overline{\Psi}\gamma^M\partial_M\Psi - \overline{\Psi}\Gamma^M[B_M, \Psi] \tag{5.2.13}$$

$$\mathcal{L}_{\rm F,Y} = \frac{1}{2g^2}\,{\rm tr}\{\overline{\Psi}\gamma^M[W_M, \Psi]\} \tag{5.2.14}$$

$$\mathcal{L}_{\rm F}^{(2)} = \frac{i}{2g^2}\,{\rm tr}\{\overline{\chi}\Gamma^M\partial_M\chi - i\overline{\chi}\Gamma^M[W_M, \Psi] - i\overline{\Psi}\Gamma^M[W_M, \chi]\}, \tag{5.2.15}$$

where we have defined

$$G_{MN} = \partial_M B_N - \partial_N B_M \tag{5.2.16}$$

$$D_M W_N = \partial_M W_N - i[B_M, W_N]. \tag{5.2.17}$$

Let us compactify this theory with a $D-4$ dimensional manifold $\mathcal{M}_{D-4}$ with dimensional reductions in the following two steps. First we decompose the D-dimensional fields into 4 dimensional Poincaré representations like $B_M^a \to B_\mu^a, B_i^a$. Although the Poincaré invariance restricts a non-vanishing vevs along the μ directions, we can give non-vanishing vevs along the i directions:

$$B_i^a(x^M) = \overline{B}_i^a(y) + C_i^a(x^M) \tag{5.2.18}$$

$$W_i^{ab}(x^M) = \overline{W_i^{ab}}(y) + \Phi_i^{ab}(x^M) \tag{5.2.19}$$

where we define the fluctuations around the vevs $\overline{B}_i^a(y), \overline{W}_i^{ab}(y)$ as $C_i^a(x^M), \Phi_i^{ab}(x^M)$ respectively. These non-vanishing vevs generally break gauge invariance. In the following discussions, we consider the case $\overline{W}_i^{ab}(y) = 0$.

Since we are interested in the Yukawa couplings in the lower dimensional theory, we focus on the following terms:

$$\mathcal{L}_{\rm B}^{(1)} = \frac{i}{2g^2}(G^a_{MN} - G^b_{MN})\left((W^{Mab})^* W^{Nab} - (W^{Nab})^* W^{Mab}\right) \tag{5.2.20}$$

$$- \frac{1}{2g^2}\left[(D_M W_N)^{ab*}(D^M W^N)^{ab} - (D_M W_N)^{ab*}(D^N W^M)^{ab}\right] \tag{5.2.21}$$

$$\mathcal{L}_{\rm B,4\text{-}point} = \frac{1}{2g^2}\left[W^{ab}_M W^{bc}_N W^{Mcd} W^{Nda} - W^{ab}_M W^{bc}_N W^{Ncd} W^{Mda}\right] \tag{5.2.22}$$

$$\mathcal{L}_{\rm F}^{(1)} = \frac{i}{2g^2}\overline{\Psi}_{ba}\Gamma^M (D_M \Psi)^{ab} \tag{5.2.23}$$

$$\mathcal{L}_{\rm F,Y} = \frac{1}{2g^2}\left(\overline{\Psi}^{ab}\Gamma^M W^{bd}_M \Psi^{da} - \overline{\Psi}^{ab}\Gamma^M W^{ca}_M \Psi^{bc}\right) \tag{5.2.24}$$

Substituting (5.2.18) and (5.2.19) into these terms, they can be rewritten as

$$\begin{aligned}
\mathcal{L}_{\rm B}^{(1)} &= \mathcal{L}_{\rm B,kin} + \tilde{\mathcal{L}}_{\rm B}^{(1)} \\
&= \frac{i}{2g^2}(G^a_{ij} - G^b_{ij})\left((\Phi^{iab})^*\Phi^{jab} - (\Phi^{jab})^*\Phi^{iab}\right) \\
&- \frac{1}{2g^2}\Big[(D_\mu \Phi^{ab})^*(D^\mu \Phi^{ab}) + (\tilde{D}_i \Phi^{ab}_j)^*(\tilde{D}^j \Phi^{iab}) - (D_\mu \Phi^{ab}_i)^*(\tilde{D}^i W^{\mu ab}) \\
&-(\tilde{D}_i \Phi^{ab}_j)^*(\tilde{D}^j \Phi^{iab})\Big] + \tilde{\mathcal{L}}_{\rm B}^{(1)}
\end{aligned} \tag{5.2.25}$$

$$\mathcal{L}_{\rm B,4\text{-}point} = \mathcal{L}_{\rm 4\text{-}point} + \tilde{\mathcal{L}}_{\rm B,4\text{-}point} = \frac{1}{2g^2}\left[\Phi^{ab}_i \Phi^{bc}_j \Phi^{icd}\Phi^{jda} - \Phi^{ab}_i \Phi^{bc}_j \Phi^{jcd}\Phi^{ida}\right] + \tilde{\mathcal{L}}_{\rm B,4\text{-}point} \tag{5.2.26}$$

$$\mathcal{L}_{\rm F}^{(1)} = \frac{i}{2g^2}\overline{\Psi}^{ba}\gamma^\mu D_\mu \Psi^{ab} + \frac{i}{2g^2}\overline{\Psi}^{ba}\Gamma^i \tilde{D}_i \Psi^{ab} \tag{5.2.27}$$

$$\mathcal{L}_{\rm F,Y} = \mathcal{L}_{\rm Y} + \tilde{\mathcal{L}}_{\rm Y} = \frac{1}{2g^2}\left(\overline{\Psi}^{ab}\Gamma^i \Phi^{bd}_i \Psi^{da} - \overline{\Psi}^{ab}\Gamma^i \Phi^{ca}_i \Psi^{bc}\right) + \tilde{\mathcal{L}}_{\rm Y}, \tag{5.2.28}$$

where the $\tilde{\mathcal{L}}$ terms are irrelevant terms for the following discussions and we have defined $\tilde{D}_i = \partial_i - ig\overline{B}_i$. Notice that since Φ^{ab} and Ψ^{ab} are off-diagonal components in the adjoint representation, these fields are bifundamental representations of gauge group $U(1)_a \times U(1)_b$. This means that the covariant derivative act as

$$\tilde{D}_i \Phi^{ab}_j = \partial_j \Phi^{ab}_j - i\overline{B}^a_i W^{ab}_j + i\overline{B}^b_i W^{ab}_j. \tag{5.2.29}$$

$\mathcal{L}_{\rm B,kin}$ and $\mathcal{L}_{\rm F}^{(1)}$ contain all the possible terms in Φ_i and Ψ that lead to effective mass terms.

The second step is KK expansion of D dimensional fields and splits the 4 dimensional coordinates and the compact space coordinates dependence. Spinor fields can be decomposed into the product of 4 dimensional spinor and the extra dimensional spinor:

$$\Psi^{ab}(x^M) = \sum_n \chi_n^{ab}(x) \otimes \psi_n^{ab}(y) \tag{5.2.30}$$

As well as spinor fields, the scalar fields can be decomposed into

$$\Phi_i^{ab}(x^M) = \sum_n \varphi_{n,i}^{ab}(x) \otimes \phi_{n,i}^{ab}(y). \tag{5.2.31}$$

We choose the y-dipendent fields to the eigenfunctions for the internal wave operators:

$$i \not{D}_{D-4}\psi_n^{ab} = m_n \psi_n^{ab} \tag{5.2.32}$$

$$\Delta_{D-4}\phi_{n,i}^{ab} = M_{n,i}^2 \phi_{n,i}^{ab} \tag{5.2.33}$$

From the discussions in 5.1, the eigenvalues m_n and $M_{n,i}$ are directly related to the 4 dimensional masses for the fields χ_n^{ab} and $\varphi_{n,i}^{ab}$. Indeed by applying this equations, we find

$$i\gamma^5 \not{D}_4 \chi_n^{ab} = -m_n \chi_n^{ab} \tag{5.2.34}$$

$$\Delta_4 \varphi_{n,i}^{ab} = M_{n,i}^2 \varphi_{n,i}^{ab} - 2i \int_{\mathcal{M}} \langle G_i^{aj} - G_i^{bj} \rangle \varphi_{n,j}^{ab}, \tag{5.2.35}$$

where in order to get canonical kinetic terms, the $D-4$ dimensional fields must satisfy

$$g^{-2} \int_{\mathcal{M}} d^{D-4}y \phi_{n,i}^{ab}(y)^* \phi_{n,i}^{cd}(y) = \delta_{ac}\delta_{bd} \tag{5.2.36}$$

$$g^{-2} \int_{\mathcal{M}} d^{D-4}y \psi_n^{ab}(y)^\dagger \psi_n^{ab}(y) = \delta_{ac}\delta_{bd}. \tag{5.2.37}$$

Since the scalar mass has the corrections from non-vanishing vevs, the correspondence between the scalar and the spinor with same masses are broaken. Thus the fluxes break the supersymmetry.

By functional integration of massive fields, we can obtain the 4 dimensional effective field theory including the lightest modes. The massless fields in 4 dimensional theory are given by $U(n_\alpha)$ gauge bosons $A_\mu^\alpha(x)$ which derived from KK expansion of B_μ^α, the gaugino λ_α derived from η^α, the scalar c_i^α derived from C_i^α and bifundamental spinor χ_0^{ab}. From the mass term in (5.2.35), the remained fields $\varphi_{n,i}^{ab}$ may be massive, massless or tachyonic depending on the fluxes.

Substituting the above reductions into $\mathcal{L}_{\mathrm{F,Y}}$, 4 dimensional Yukawa coupling terms can be written by

$$S_Y \equiv \frac{1}{2g^2} \sum_{I,J,K} \left[\left(\int_{\mathcal{M}} d^{D-4} y \psi_I^{ab\dagger} \phi_{J,i}^{bd} \Gamma^i \psi_K^{da} \right) \int d^4 x \overline{\chi}_I^{ab} \varphi_{J,i}^{bd} \chi_K^{da} - \left(\int_{\mathcal{M}} d^{D-4} y \psi_I^{ab\dagger} \phi_{J,i}^{ca} \Gamma^i \psi_K^{bc} \right) \int d^4 x \overline{\chi}_I^{ab} \varphi_{J,i}^{ca} \chi_K^{bc} \right] \tag{5.2.38}$$

5.2.2 Magnetic Fluxes and Supersymmetry

The above discussion is general in the sence that it dose not depend whether the supersymmetric or not. If we would like to consider SYM theory coupled to gravity, it may be useful to consider the compactifications preserving $N = 1$ supersymmetry in the effective theory. In this part, we derive the conditions for preserving the supersymmetries.

We consider $D = 10$ SYM theory compactified in $\mathcal{M}_{2n}$ coupled to $N = 1$ SUGRA and put the assumptions $H = 0 = d\phi$ for simplicity, where H is a field strength of 2-form field in NS-NS sector and ϕ is a dilaton. Since the SUSY parameters are spinor, in order to get the supersymmetric theory in the lower dimensional effective theory, it is enough that spinor exists without contradiction whose statement leads to SUSY conditions.

In order to get the spinor consistently in 4 dimensions, this spinor should be defined uniquely. But since 10 dimensional spinor ξ is non-trivialy transformed under SO(10) and η is also transformed under SO(6) non-trivialy, there are many posibilities for the supercharges at each point in 4 dimensions. Thus we need the covariantly constant spinor in $\mathcal{M}_{2n}$:

$$\nabla_{\mathcal{M}_{2n}} \xi(y^m) = 0 \tag{5.2.39}$$

This condition implies that the compact manifold $\mathcal{M}_{2n}$ is a Ricci-flat Kähler manifold.

Futhermore, if the supersymmetry is preserved, the supersymmetry keeps the vacuum invariant:

$$Q \left| \Omega \right\rangle = 0 \tag{5.2.40}$$

This condition leads to

$$0 = \left\langle \Omega \right| \{ Q, \psi \} \left| \Omega \right\rangle = \left\langle \Omega \right| \delta\psi \left| \Omega \right\rangle , \tag{5.2.41}$$

where ψ is an arbitrary fermionic operator. Thus in classical limit, we find

$$\delta\psi = 0. \tag{5.2.42}$$

If we regard ψ as a gaugino, this condition becomes

$$\delta\psi \sim \Gamma^{mn} F_{mn} = 0, \tag{5.2.43}$$

and this implies

$$F_{ij} = F_{\overline{ij}} = 0, \quad g^{i\bar{j}} F_{i\bar{j}} = 0. \tag{5.2.44}$$

5.2.3 Yukawa Couplings

These unwanted feature in torus compactification can be avoided by two approaches. First, we introduce a non-trivial expectation values (vev) for the gauge fields. Second, we impose an obifold projection. In this section, we choose the first approach.

Since we interested in preserving Poincaré invariance in the four-dimensional spacetime, we introduce non-vanishing vevs $\langle A_m(x)\rangle$. These vevs reduce gauge group G to the subgroup $H \subset G$ commuting with the subgroup J which contains vevs. Also, these vevs modify the wave operator. Thus supersymmetries are broaken and we obtain chiral theory. Therefore we find that magnetized $\mathcal{M}_6$ compactifications with $\langle F_{mn}\rangle \neq 0$ lead to $D = 4$ chiral theories with reduced gauge group.

In addition, we obtain the degenerated chiral fermions ψ_j which can be regarded as generations of the fermion. In order to obtain canonical kinetic terms, we impose normalization condition for the internal wavefunctions

$$\int_{\mathcal{M}_/} d^6 x \psi_j(y^m)^\dagger \psi_k(y^m) = \delta_{jk} \tag{5.2.45}$$

as well as bosonic wavefunction.

Finally, the kinetic term of spinor λ in $D = 10$ SYM Lagrangian leads to four-dimensional Yukawa couplings

$$Y_{ijk} = \int_{\mathcal{M}_6} \psi_i^{a\dagger} \gamma^m \psi_j^b \phi_{k,m}^c f_{abc} \tag{5.2.46}$$

where f_{abc} are structure constant of the initial gauge group G.

In order to compute the Yukawa couplings (5.2.46), we need explicit expression for the internal wavefunctionψ_j. This wavefunction corresponds to massless modes.

In the following sections, we consider the 8 cases which are clasified with whether abelian or non-abelian gauge group, including Abelian Wilson line or non-Abelian ones and T^2 or general tori T^{2n}. Also we discuss the masses of the zero-modes in each cases.

5.3 Magnetized Torus with Abelian Wilson Lines

In the following, we compute the wavefunction involving Wilson lines on magnetized T^2 in U(1) and U(N) gauge theory.

5.3.1 Abelian Gauge Theory

First, we explain T^2 geometry. T^2 is obtained by the identification

$$z \sim z + 1, \quad z \sim z + \tau \tag{5.3.1}$$

where, $z \equiv y_4 + \tau y_5$ and $\tau \in \mathbb{C}$ is the complex structure moduli of T^2. We introduce an abelian magnetic flux such that $\int_{T^2} F = b$,

$$F = \frac{b}{\tau_I}\frac{i}{2}dz \wedge d\bar{z}, \quad \tau_I = \mathrm{Im}\,\tau \tag{5.3.2}$$

This flux can be derived from

$$A(z) = \frac{b}{2\tau_I}\,\mathrm{Im}((\bar{z} + \bar{\xi})dz). \tag{5.3.3}$$

These expressions are general solution and ζ is a constant called Wilson line. This vector field should be well-defined on the torus, but its torus translations are

$$\begin{aligned} A(z+1) &= A(z) + \frac{b}{2\tau_I}\,\mathrm{Im}\,dz & &\equiv A(z) + d\chi_1 \\ A(z+\tau) &= A(z) + \frac{b}{2\tau_I}\,\mathrm{Im}\,\bar{\tau}dz & &\equiv A(z) + d\chi_2. \end{aligned} \tag{5.3.4}$$

So, we identify these translations with the gauge transformations $A \to A + d\chi_i$.

Next we consider any complex field $\phi(z)$ with U(1) charge q. Its torus tranlations are given by

$$\begin{aligned} \phi(z+1) &= e^{iq\chi_1(z)}\phi(z) \\ \phi(z+\tau) &= e^{iq\chi_2(z)}\phi(z) \end{aligned} \tag{5.3.5}$$

and for the consistency, these translations are also regarded as the gauge transformations.

Considering a contractible loop in T^2, we obtain Dirac's charge quantization condition

$$\frac{b}{2\pi} = M \in \mathbb{Z}. \tag{5.3.6}$$

In order to obtain the explicit wavefunction, we solve the Dirac equation. Dirac operator and two-dimensional spinor with U(1) charge q are given by

$$i \not{D}_2 = \frac{i}{\pi R} \begin{pmatrix} 0 & \partial - q\frac{\pi M}{2\tau_I}(\bar{z}+\bar{\zeta}) \\ \bar{\partial} + q\frac{\pi M}{2\tau_I}(z+\zeta) & \end{pmatrix} \equiv i \begin{pmatrix} 0 & -D^\dagger \\ D & 0 \end{pmatrix} \tag{5.3.7}$$

$$\Psi(z,\bar{z}) = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \tag{5.3.8}$$

So, we obtain the two Dirac equations

$$D\psi_+ = 0, \quad D^\dagger \psi_- = 0. \tag{5.3.9}$$

If $M > 0$, ψ_+ is only well-defined and if $M < 0$, ψ_- is so. Thus by introducing a non trivial flux M, one chirality spinor is autmatically selected. In the following, we assume $M > 0$.

The wave functions satisfying with Dirac equation and the boundary condition (5.3.5) are given by

$$\psi_+^j(z) = \mathcal{N}_j e^{i\pi qM(z+\zeta)\frac{\mathrm{Im}(z+\zeta)}{\tau}I}\,\theta \begin{bmatrix} \frac{j}{qM} \\ 0 \end{bmatrix} (qM(z+\zeta), qM\tau) \quad j = 0, ..., M-1 \tag{5.3.10}$$

where, $\mathcal{N}_j$ is a normalization factor

$$\mathcal{N}_j = \left(\frac{2\tau_I M}{\mathcal{A}^2}\right)^{1/4} \tag{5.3.11}$$

and θ is a Jacobbi theta-function defined by

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (\nu,\tau) = \sum_{l\in\mathbb{Z}} e^{\pi i(a+l)^2\tau} e^{2\pi i(a+l)(\nu+b)} \tag{5.3.12}$$

Since ψ_+^j is degenerate, we realize a generation structure.

If $M < 0$, we get the same result for ψ_-

5.3.2 Non-Abelian Gauge Theory

Here, we consider a more general setup. Explicitly, we describe a U(N) gauge theory broken to $\prod_i U(N_i)$ by introducing the following magnetic fluxes

$$F = \frac{\pi}{\mathrm{Im}\,\tau}(dz \wedge d\bar{z}) \begin{pmatrix} M_1 I_{N_1\times N_1} & & \\ & \ddots & \\ & & M_n I_{N_n\times N_n} \end{pmatrix} \tag{5.3.13}$$

where, $I_{N_a\times N_a}, a = 1, ..., n$ denotes the $N_a \times N_a$ identity matrix with $\sum_{a=1}^{n} N_a = N$. Due to the boundary condition, M_a must be integer. This flux is induced by the gauge field,

$$A(z) = \frac{\pi}{\operatorname{Im}\tau} \operatorname{Im}(\bar{z}dz) \begin{pmatrix} M_1 I_{N_1\times N_1} & & \\ & \ddots & \\ & & M_n I_{N_n\times N_n} \end{pmatrix} \tag{5.3.14}$$

In the following, we focus on $N = 2$ case:

$$F_{z\bar{z}} = \frac{\pi i}{\tau_I} \begin{pmatrix} m_a & 0 \\ 0 & m_b \end{pmatrix}. \tag{5.3.15}$$

In this case, the gauge group U(2) is broken to U(1)$_a\times$U(1)$_b$. The corresponding vector fields are given by

$$A_{\bar{z}} = \frac{\pi}{2\tau_I} \begin{pmatrix} m_a(z + \zeta_a) & 0 \\ 0 & m_b(z + \zeta_b) \end{pmatrix} \tag{5.3.16}$$

as well as A_z. The Dirac operator is given by

$$i\not{D} = i \begin{pmatrix} 0 & -D^\dagger \\ D & 0 \end{pmatrix} = \frac{i}{\pi R} \begin{pmatrix} 0 & \partial + A_z \\ \bar{\partial} + A_{\bar{z}} & 0 \end{pmatrix} \tag{5.3.17}$$

Since a two-deimensional spinor is a adjoint representation of U(2), each of chiral spinors is given by the matrices,

$$\Psi(z, \bar{z}) = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad \psi_\pm = \begin{pmatrix} A_\pm & B_\pm \\ C_\pm & D_\pm \end{pmatrix} \tag{5.3.18}$$

Hence we obtain the Dirac equations for zero-modes

$$D\psi_+ = (\pi R)^{-1} \begin{pmatrix} \bar{\partial} A_+ & \left(\bar{\partial} + \frac{\pi I_{ab}}{2\tau_I}(z + \zeta_{ab})\right) B_+ \\ \left(\bar{\partial} - \frac{\pi I_{ab}}{2\tau_I}(z + \zeta_{ab})\right) C_+ & \bar{\partial} D_+ \end{pmatrix} = 0 \tag{5.3.19}$$

where we defined

$$\begin{aligned} I_{ab} &\equiv m_a - m_b \neq 0 \\ \zeta_{ab} &\equiv (m_a\zeta_a - m_b\zeta_b)/I_{ab} \end{aligned} \tag{5.3.20}$$

ψ_- is given by $\psi_- = \psi_+^\dagger$. From (5.3.19), A_+ and D_+ should be holomorphic functions. Furthermore due to the periodic boundary condition, $A_\pm, D_\pm$ must be constant. Since Dirac equations for B_+ and C_+ are almost the same as U(1) case, they have to be of the form

$$B_+ = \psi^{j,I_{ab}}(\tau, z+\zeta_{ab}), \quad C_+ = \psi^{j,I_{ab}}(\tau, z+\zeta ab). \tag{5.3.21}$$

If $I_{ab} > 0$, B_+ can only be present as well as C_-

5.3.3 Eigenfunctions of the Laplace Equation

In this part, we show that the eigenfunction (5.3.21) is not only for Dirac equation but also for the Laplace equation. Thus these eigenvalues correspond to scalar mass and depend on the Kähler moduli of compactification. In order to prove this, we compute the squared Dirac op:

$$(i\not{D})^2 = \Delta + \begin{pmatrix} \frac{2iF_{z\bar{z}}}{(2\pi R)^2} & 0 \\ 0 & \frac{2iF_{\bar{z}z}}{(2\pi R)^2} \end{pmatrix} \tag{5.3.22}$$

Applying the laplacian to the ab sector wavefunctions, we obtain these eigenvalues

$$\Delta\psi^{j,\pm I_{ab}} = \frac{2\pi|I_{ab}|}{\mathcal{A}}\psi^{j,\pm I_{ab}}. \tag{5.3.23}$$

By defining the harmonic operators

$$a = \sqrt{\frac{\mathcal{A}}{4\pi|I_{ab}|}}D, \tag{5.3.24}$$

the eigenfunctions and the eigenvalues for the laplacian are given by

$$\psi^{j,\pm I_{ab}} = (D^\dagger)^r\psi^{j,\pm I_{ab}} \tag{5.3.25}$$

$$\lambda_r = 2\pi\frac{|I_{ab}|}{\mathcal{A}}(2r+1) > 0 \tag{5.3.26}$$

However, this is not just the mass eigenvalues. From (5.2.35), the mass matrix is given by

$$M^2 = \begin{pmatrix} \Delta & -4i\pi\frac{I_{ab}}{\mathcal{A}} \\ 4i\pi\frac{I_{ab}}{\mathcal{A}} & \Delta \end{pmatrix}, \tag{5.3.27}$$

and its eigenvalues are given by

$$\tilde{\lambda}_r = \lambda_r \pm 4\pi\frac{I_{ab}}{\mathcal{A}}. \tag{5.3.28}$$

From this expression, there is one tachyonic scalar mode which corresponds to no fermionic spectrum. This fact means that the supersymmetry is broken completely.

5.4 Magnetized T^{2n} with Abelian Wilson Lines

In this section, we attempt to extend magnetized torus compactifications to higher dimensional torus T^{2n} which is factorized as

$$T^{2n} \simeq T_1^2 \times \cdots \times T_n^2. \tag{5.4.1}$$

Let us consider the constant magnetic flux F breaking $U(N) \to U(1)^N$. This implies the magnetic flux on T_r^2 is given by

$$F_{z^r \bar{z}^r} \equiv F_{z\bar{z}}^{(r)} = \frac{\pi}{\operatorname{Im}\tau^{(r)}} \begin{pmatrix} m_a^{(r)} & & \\ & m_b^{(r)} & \\ & & \ddots \end{pmatrix} \tag{5.4.2}$$

In the following, we focus on $N = 2$ case. As well as the T^2 case, we decompose the Dirac operator into the holomorphic part and anti-holomorphic part:

$$i \not{D} = i \sum_{\bar{r}} \Gamma^{\bar{r}} D_r - i \sum_r \Gamma^r D_r^\dagger \tag{5.4.3}$$

$$D_r = \bar{\partial}_r + \frac{\pi}{2\operatorname{Im}\tau^{(r)}} \begin{pmatrix} m_a^{(r)}(z^r + \zeta_a^r) & \\ & m_b^{(r)}(z^r + \zeta_b^r) \end{pmatrix} \tag{5.4.4}$$

The spinor Ψ in the compact spase T^{2n} is a $2^{2n/2} = 2^n$ components spinor. We decompose this spinor to the spinors on each torus T_r^2 which are denoted $\psi_{\epsilon^1,\ldots,\epsilon^n}$ where $\epsilon^r = \pm 1$. In this notation, the zero-mode equation on T_r^2 is given by

$$\begin{aligned} D_r \psi_{\epsilon^1,\ldots,\epsilon^n} &= 0, \quad \epsilon^r = 1 \\ D_r^\dagger \psi_{\epsilon^1,\ldots,\epsilon^n} &= 0, \quad \epsilon^r = -1 \end{aligned} \tag{5.4.5}$$

The wavefunctions of $2n$-dimensional fermions can be obtained by solving the equation (5.4.5) and the torus boundary conditions, which are given by

$$\psi_{\epsilon^1,\ldots,\epsilon^n} = \begin{pmatrix} \text{const} & \prod_r \delta_{\epsilon^r,\text{sign}(I_{ab}^{(r)})} \psi_{\epsilon^r}^{j^{(r)},|I_{ab}^{(r)}|} \\ \prod_r \delta_{-\epsilon^r,\text{sign}(I_{ab}^{(r)})} \psi_{\epsilon^r}^{j^{(r)},|I_{ab}^{(r)}|} & \text{const} \end{pmatrix} \tag{5.4.6}$$

where $\text{sign}(I_{ab}^{(r)})$ is a sign of $I_{ab}^{(r)}$ and $\psi_{\epsilon^r}^{j^{(r)},|I_{ab}^{(r)}|}$ are defined by

$$\psi_+^{j^{(r)},I_{ab}^{(r)}} = \psi^{j^{(r)},I_{ab}^{(r)}}(\tau^{(r)}, z^r + \zeta_{ab}^r), \quad \psi_-^{j^{(r)},I_{ab}^{(r)}} = \left(\psi^{j^{(r)},I_{ab}^{(r)}}(\tau^{(r)}, z^r + \zeta_{ab}^r)\right)^*. \tag{5.4.7}$$

Therefore, we find 2^{n-1} numbers of $U(1)_a \times U(1)_b$ gauginos associated to the constant wavefunctions, and $I_{ab} \equiv \prod_r |I_{ab}^{(r)}|$ numbers of chiral fermions associated to chiral fermions in the bifundamental representation of $U(1)_a \times U(1)_b$.

Just as in the T^2 case, the wavefunctions (5.4.6) are not only the eigenfunctions of Dirac operator, but also of Laplacian. In contrast T^2 case, there are no tachyonic mode discussed the following section.

5.5 Magnetized Torus with Non-Abelian Wilson Lines

In this section, we consider the general symmetry breaking $U(N) \to \prod_i (n_i), \sum_i n_i < N$.

5.5.1 Gauge Symmetry Breaking

First let us consider the T^2 case and U(N) gauge group. The magnetic flux F and the gauge field A transform in the adjoint represention of U(N) and F is given by

$$F_{z\bar{z}} = \frac{\pi i}{\operatorname{Im}\tau} \frac{M}{N} \cdot I_N. \tag{5.5.1}$$

Consider a field Φ in the fundamental representation of U(N). In contrast to the abelian Wilson line cases, Wilson lines can be arbitrary elements of U(N). Thus the most general transformation is given by

$$\Phi(z) \to \Omega_i(z)\Phi(z) = e^{i\chi_i(z)}\omega_i\Phi(z) \tag{5.5.2}$$

where

$$\chi_1(z) = \frac{\pi M}{N \operatorname{Im}\tau} \operatorname{Im}(z+\zeta) I_N \tag{5.5.3}$$

$$\chi_2(z) = \frac{\pi M}{N \operatorname{Im}\tau} \operatorname{Im}\bar{\tau}(z+\zeta) I_N \tag{5.5.4}$$

and ω_i are constant element of SU(N).

In order to confirm the consistency between gauge transformations and the homology of T^2, we consider the contractible loop:

$$\Omega_2^{-1}(z+\tau)\Omega_1^{-1}(z+1+\tau)\Omega_2(z+1)\Omega_1(z)\Phi(z) = \Phi(z). \tag{5.5.5}$$

The left hand of this condition is rewritten as

$$\omega_2^{-1}\omega_1^{-1}\omega_2\omega_2 = e^{2\pi i k/N} I_N, \quad k \in \mathbb{Z}. \tag{5.5.6}$$

Thus $M = k \bmod N$ must be integer which correspond to Dirac's charge quantization condition.

The ω_i is the representation of SU(N) and can be chosen to

$$\omega_1 = Q \equiv \mathrm{diag}(1, e^{2\pi i/N, \dots, e^{2\pi i(N-1)/N}}), \quad \omega_2 = P^M \equiv \begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & \ddots \\ 1 & & & \end{pmatrix}^M \tag{5.5.7}$$

or

$$\omega_1 = Q^M, \quad \omega_2 = P. \tag{5.5.8}$$

The gauge field A_μ transforms in the adjoint

$$A_\mu = \omega_1 A_\mu \omega_1^{-1} = \omega_2 A_\mu \omega_2^{-1}. \tag{5.5.9}$$

Thus A_μ must be a diagonal $N \times N$ matrix with only $P = \text{g.c.m.}(N, M)$ independent elements and we find that magnetic flux breaks U(N) gauge group to U(P).

If $M = SN, S \in \mathbb{Z}$, $k = 0$ and we can take $\omega_1 = \omega_2 = 1$. Also if $P = N$, the gauge group is going to be U(N) that is no rank reduction.

5.5.2 Wavefunctions on T^2

Let us consider F to be sum of two irreducible representations

$$F_{z\bar{z}} = \frac{\pi i}{\operatorname{Im}\tau} \mathrm{diag}\left(\frac{m_a}{n_a} I_{n_a}, \frac{m_b}{n_b} I_{n_b}\right), n_a + n_b = N. \tag{5.5.10}$$

This flux can be regarded as two fluxes F_a and F_b on the same T^2 which correponds to the gauge group $U(n_a)$ and $U(n_b)$ respectively, and magnetic quanta m_a and m_b respectively.

Thus due to the magnetic fluxes, the gauge group U(N) is broken to $U(p_a) \times U(p_b)$ where $p_\alpha = \text{g.c.d.}(n_\alpha, m_\alpha)$. If $m_\alpha \propto n_\alpha$, then there are no non-abelian Wilson lines and the gauge group is give by $U(n_a) \times U(n_b)$. From the discussion of the previous section, the wavefunction of fermion in this case transforms in $(n_a, \overline{n}_b)$ if $I_{ab} > 0$. In the following discussions, we focus on $p_\alpha < n_\alpha$ and $p_a = p_b = 1$.

The gauge fields corresponding to (5.5.10) are given by

$$\begin{aligned} A_{\bar{z}} = A^\alpha_{\bar{z}} T^\alpha &= \frac{\pi}{2\operatorname{Im}\tau} \begin{pmatrix} \frac{m_a}{n_a}(z+\zeta_a) I_{n_a} & \\ & \frac{m_b}{n_b}(z+\zeta_b) I_{n_b} \end{pmatrix} \\ A_z = A^\alpha_z T^\alpha &= -\frac{\pi}{2\operatorname{Im}\tau} \begin{pmatrix} \frac{m_a}{n_a}(\bar{z}+\bar{\zeta}) I_{n_a} & \\ & \frac{m_b}{n_b}(\bar{z}+\bar{\zeta}_b) I_{n_b} \end{pmatrix}. \end{aligned} \tag{5.5.11}$$

The Dirac equation for the spinorial component ψ_+ is

$$D\psi_+ = (\pi R)^{-1} \begin{pmatrix} \overline{\partial} A_+ & \left(\overline{\partial} + \frac{\pi \tilde{I}_{ab}}{2\,\mathrm{Im}\,\tau}(z+\zeta_{ab})\right) B_+ \\ \left(\overline{\partial} - \frac{\pi \tilde{I}_{ab}}{2\,\mathrm{Im}\,\tau}(z+\zeta_{ab})\right) C_+ & \overline{\partial} D_+ \end{pmatrix} = 0 \tag{5.5.12}$$

where we have defined

$$I_{ab} \equiv -n_a m_b + n_b m_a \neq 0 \tag{5.5.13}$$

$$\tilde{I}_{ab} \equiv \frac{I_{ab}}{n_a n_b} \tag{5.5.14}$$

$$\zeta_{ab} \equiv \frac{(n_b m_a \zeta_a - n_a m_b \zeta_b)}{I_{ab}}. \tag{5.5.15}$$

The Dirac equation for ψ_- is quite same in the previous section without the exchanging $I_{ab} \to \tilde{I}_{ab}$. The boundary conditions and Dirac equation restrict A_+ and D_+ to have to be constant matrices associated to $U(1)_a \times U(1)_b$ gauginos. On the other hand, B_+ and C_+ have to be of the form

$$N e^{\pm i \frac{\pi \tilde{I}_{ab}}{\mathrm{Im}\,\tau}(z+\zeta_{ab})\,\mathrm{Im}(z+\zeta_{ab})} \xi(z) \tag{5.5.16}$$

respectively, where $\xi(z)$ is an arbitrary holomorphic matrix-valued function and N is a normalization factor. Futhermore by imposing the periodic boundary condition and computing the normalization factor, we find the fermionic zero-mode wavefunctions for chiral fields transforming in the bifundamental representation:

$$B_+ = \Phi^{j,I_{ab}}, \quad C_+ = \Phi^{j,-I_{ab}}, \quad j = 1, ..., |I_{ab} \tag{5.5.17}$$

$$\left(\Phi^{j,I_{ab}}\right)_{k_a,k_b} = \left(\frac{2\,\mathrm{Im}\,\tau |\tilde{I}_{ab}|}{\mathcal{A}}\right) 1/4 e^{i \frac{\pi \tilde{I}_{ab}}{\mathrm{Im}\,\tau}(z+\zeta_{ab})\,\mathrm{Im}(z+\zeta_{ab})} \xi(z)^{j,I_{ab}}_{k_a,k_b} \tag{5.5.18}$$

$$\left(\xi^{j,I_{ab}}\right)_{l.l} = \theta \begin{bmatrix} \frac{j}{I_{ab}} + \frac{l}{n_a n_b} \\ 0 \end{bmatrix} ((z+\zeta_{ab}) I_{ab}, \tau I_{ab} n_a n_b) \tag{5.5.19}$$

These solutions are exclusive and chosen by the sign of I_{ab}:

$$\begin{aligned} B_+ < \infty \Leftrightarrow I_{ab} > 0 \text{Left-handed fermions in (1,-1) of} U(1)_a \times U(1)_b \\ C_+ < \infty \Leftrightarrow I_{ab} < 0 \text{Left-handed fermions in (-1,1) of} U(1)_a \times U(1)_b \end{aligned} \tag{5.5.20}$$

The solutions of anti-particles ψ_- are given by the hermitian conjugate respectively.

5.5.3 Eigenfunctions for Laplacian

As well as the T^2 case, the wavefunctions (5.5.18) are also the eigenfunctions for the Laplace operator. By simillar computation in the previous section, we find

$$\Delta\Phi^{j,\pm I_{ab}} = \pm\frac{2\pi\tilde{I}_{ab}}{\mathcal{A}}\Phi^{j,\pm I_{ab}} = \frac{2\pi|\tilde{I}_{ab}|}{\mathcal{A}}, \tag{5.5.21}$$

and same eigenvalue without the replacement $I_{ab} \to \tilde{I}_{ab}$. Computing the mass matrix, we find the lightest scalar mass is tachyonic:

$$m_0^2 = -2\pi\frac{|\tilde{I}_{ab}|}{\mathcal{A}} \tag{5.5.22}$$

Thus there is also no supersymmetry as well as abelian Wilson line cases.

5.6 Magnetized T^{2n} with Non-Abelian Wilson Lines

In this section, we extend T^2 with non-abelian Wilson lines to the higher dimensional tori.

5.6.1 Gauge Symmetry Breaking

The most general constant magnetix flux associated to U(N) gauge group is given by

$$F_{ij} = 2\pi\frac{n_{ij}}{Na_i a_j}I_N \tag{5.6.1}$$

where $n_{ij} = -n_{ji}$. T^{2n} is equivalent to the quotient space $\mathbb{R}^{2n}/\Lambda$, where $\Lambda = \{x \in \mathbb{R}^{2n} | x = n_i a^i ; n \in \mathbb{Z}^{2n}\}$ and we have defined $\{x \in \mathbb{R}^{2n} | 0 < x_i \leq a_i\}$ and $a_i = ||a^i||$. From (5.6.1), the boundary conditions on a field Φ transforming in the fundamental representation of U(N) are of the form

$$\Phi(x) \to \Omega_i(x)\Phi(x) = e^{i\chi_i(x)}\omega_i\Phi(x), \tag{5.6.2}$$

where we have defined

$$\chi_i(x) = e^{\pi\sum_j \frac{n_{ij}x^j}{Na_j}} I_N, \tag{5.6.3}$$

and ω_i are the constant elements of SU(N). The consistency condition is given by contractible loop:

$$\Omega_j^{-1}(x+a_i)\Omega_i^{-1}(x+a_i+a_j)\Omega_j(x+a_i)\Omega_i(x)\Phi(x) = \Phi(x) \tag{5.6.4}$$

This condition leads to

$$\omega_j^{-1}\omega_i^{-1}\omega_j\omega_i = e^{2\pi i c_{ij}/N}I_N, \quad c_{ij} \in \mathbb{Z}, \quad n_{ij} = c_{ij} \mod N. \tag{5.6.5}$$

By using SU(N) constant matrices P and Q such that $PQ = QPe^{2\pi i/N}$, ω_i can be chosen to

$$\omega_i = P^{s_i} Q^{t_i}, \quad s_i, t_i \in \mathbb{Z} \tag{5.6.6}$$

and the condition (5.6.5) can be rewritten as

$$t_i s_j - t_j s_i = n_{ij} \mod N. \tag{5.6.7}$$

If we consider the fluxes which don't satisfy this consistency condition, the initial gauge group U(N) is broken to $\prod_i U(P_i)$. Then flux (5.6.1) can be splited to a direct sum of more fundamental fluxes corresponding to gauge group U(P_i) and each flux satisfies (5.6.7).

5.6.2 Factorized Torus

Let us consider the factorized tori and a constant magnetic flux F on them. Also we deal with F to be a (1,1)-from on each T^2. This allows us to specify F in terms of $2n$ integer numbers $(N^{(r)}, M^{(r)}), r = 1, ..., n$ such that

$$\begin{aligned} N &= \prod_r N^{(r)} \\ n_{2r-1,2r} &= N^{(1)} \cdots N^{(r-1)} M^{(r)} N^{(r+1)} \cdots N^{(n)}. \end{aligned} \tag{5.6.8}$$

Then the components of the magnetic flux are written as

$$F_{z^r \bar{z}^{\bar{r}}} = \frac{\pi}{\operatorname{Im}\tau^{(r)}} \frac{M^{(r)}}{N^{(r)}} I_N \tag{5.6.9}$$

and the boundary conditions for a scalar field Φ in fundamental representation are given by $\Omega_r = e^{i\chi_r}\omega_r$ defined by

$$\begin{aligned} \chi_{2r-1}(z) &= \frac{\pi M^{(r)}}{N^{(r)} \operatorname{Im}\tau^{(r)}} \operatorname{Im}(z^r + \zeta^r) I_N \\ \chi_{2r}(z) &= \frac{\pi M^{(r)}}{N^{(r)} \operatorname{Im}\tau^{(r)}} \operatorname{Im}\bar{\tau}^{(r)}(z^r + \zeta^r) I_N. \end{aligned} \tag{5.6.10}$$

Since the rank of gauge group U(N) is decomposed to (5.6.8), Φ_k is decompoed to the bifundamental representations $\Phi_{k^{(1)},...,k^{(n)}}$ where $k = 1, ..., N$ and $k^{(r)} = 1, ..., N^{(r)}$ are the label of the field components. Also the constant elements ω_i of SU(N) are splited to the constant elements of each gauge group U(P_r):

$$\begin{aligned} \omega_{2r-1} &= I_{N^{(1)}} \otimes \cdots I_{N^{(r-1)}} \otimes Q^{M^{(r)}} \otimes I_{N^{(r+1)}} \otimes \cdots \otimes I_{N^{(N)}} \\ \omega_{2r} &= I_{N^{(1)}} \otimes \cdots \otimes I_{N^{(r-1)}} \otimes P \otimes I_{N^{(r+1)}} \otimes \cdots \otimes I_{N^{(n)}} \end{aligned} \tag{5.6.11}$$

Here Q and P are the generalization of (5.5.7) to $N^{(r)} \times N^{(r)}$ matrices. By introducing such flux, the gauge group U(N) is broken to $\prod_r U(P^{(r)})$, where $P^{(r)} = \text{g.c.d.}(N^{(r)}, M^{(r)})$.

In the following discussions, we focus on the case that F is a direct sum of two fundamental fluxes:

$$F_{z^r\bar{z}^r} = \frac{\pi i}{\operatorname{Im}\tau^{(r)}} \operatorname{diag}\left(\frac{m_a^{(r)}}{n_a^{(r)}} I_{N_a}, \frac{m_b^{(r)}}{n_b^{(r)}} I_{N_b}\right), \quad N_\alpha = \prod_r n_\alpha^{(r)} \tag{5.6.12}$$

Then the Dirac operator is given by

$$i \not{D} = i \sum_{\bar{r}} \Gamma^{\bar{r}} D_r - i \sum_r \Gamma^r D_r^\dagger \tag{5.6.13}$$

$$D_r = \bar{\partial}_r + \frac{\pi}{2\operatorname{Im}\tau^{(r)}} \operatorname{diag}\left(\frac{m_a^{(r)}}{n_a^{(r)}}(z^r + \zeta^r) I_{N_a}, \frac{m_b^{(r)}}{n_b^{(r)}}(z^r + \zeta_b^r) I_{N_b}\right). \tag{5.6.14}$$

Also we decompose the $2n$ dimensional spinor to the tensor product $\psi_{\epsilon^1,\ldots,\epsilon^n}$. We can find the solution of the Dirac equation as well as the previous section:

$$\psi_{\epsilon^1,\ldots,\epsilon^n} = \begin{pmatrix} \text{const} & \prod_i \delta_{\epsilon^r, s(I_{ab}^{(r)})} \bigotimes_r \Phi_{\epsilon^r}^{j^{(r)},|I_{ab}^{(r)}|} \\ \prod_i \delta_{-\epsilon^r, s(I_{ab}^{(r)})} \bigotimes_r \Phi_{\epsilon^r}^{j^{(r)},|I_{ab}^{(r)}|} & \text{const} \end{pmatrix} \tag{5.6.15}$$

where we have defined

$$\Phi_+^{j^{(r)},I_{ab}^{(r)}} = \Phi^{j^{(r)},I_{ab}^{(r)}}, \quad \Phi_-^{j^{(r)},I_{ab}^{(r)}} = \left(\Phi^{j^{(r)},I_{ab}^{(r)}}\right)^\dagger, \quad j^{(i)} = 1, \ldots, I_{ab}^{(i)}, \tag{5.6.16}$$

and $\Phi^{j^{(r)},I_{ab}^{(r)}}$ as in (5.5.10). The tensor product means

$$\left(\bigotimes_r \Phi_{\epsilon^r}^{j^{(r)},|I_{ab}^{(r)}|}\right)_{(k_a^{(r)};k_b^{(r)})} = \prod_r \left(\Phi_{\epsilon^r}^{j^{(r)},|I_{ab}^{(r)}|}\right)_{k_a^{(r)};k_b^{(r)}}. \tag{5.6.17}$$

5.6.3 Laplace Eigenvalues and Masses

As well as the previous sections, $\Phi_{\epsilon^r}^{j^{(r)},|I_{ab}^{(r)}|}$ is not only the eigenfunction of Dirac operator, but also of Laplace operator:

$$\Delta\left(\Phi_{\epsilon^r}^{j^{(r)},|I_{ab}^{(r)}|}\right) = \left(\sum_r \frac{2\pi|\tilde{I}_{ab}^{(r)}|}{\mathcal{A}^{(r)}}\right) \bigotimes_r \Phi_{\epsilon^r}^{j^{(r)},|I_{ab}^{(r)}|} \tag{5.6.18}$$

By considering the harmonic operator, the higher order eigenvalues are given by

$$\lambda_{\{s^{(r)}\}_r} = 2\pi \sum_r \frac{|\tilde{I}_{ab}^{(r)}|}{\mathcal{A}^{(r)}}(2s^{(r)} + 1), \quad s^{(r)} \in \mathbb{N}. \tag{5.6.19}$$

On the other hand the eigenvalues of mass matrix are given by

$$\tilde{\lambda}_{\{s^{(r)}\}_r} = 2\pi \left(\sum_r \frac{|\tilde{I}_{ab}^{(r)}|}{\mathcal{A}^{(r)}} (2s^{(r)} + 1) \pm 2 \frac{\tilde{I}_{ab}^{(i)}}{\mathcal{A}^{(i)}} \right). \tag{5.6.20}$$

The lightest scalar modes are obtained for $s^{(r)} = 0$. In the T^2 case, these modes correspond to tachyons. This fact implies that there are no $N = 1$ SUSY configurations. But from the expression (5.6.20), since the mass is given by the sum of the contributions from the fluxes on each torus, the scalar can be massless, massive or tachyonic, depending on the values of the fluxes. In the T^4 case, the lightest scalars are massless or tachyonic. Finally in the $T^2 \times T^2 \times T^2$ case, the lightest scalar mode can be massive, massless or tachyonic. Thus by choosing the fluxes such that (5.6.20) equals to 0, we can realize the $N = 1$ SUSY, which is just a supersymmetric conditions.

5.7 Yukawa Couplings

In the previous sections, we find the wavefunctions in the various cases. In this section, we compute the 3-point functions by using (5.2.46) in the various models.

5.7.1 T^2 Models with Abelian Wilson Lines

In order to obtain non-trivial Yukawa couplings, we need three gauge factors allowing for three different types of bifundamental matter fields. For the simplicities, we consider the T^2 model first and following magnetic fluxe:

$$F_{z\bar{z}} = \frac{\pi i}{\operatorname{Im}\tau} \operatorname{diag}\left(\frac{m_a}{n_a} I_{n_a}, \frac{m_b}{n_b} I_{n_b}, \frac{m_c}{n_c} I_{n_c} \right) \tag{5.7.1}$$

where $n_\alpha \in \mathbb{N}^+, m_\alpha \in \mathbb{Z}, \alpha = a, b, c$. If $p_\alpha = \text{g.c.d.}(n_\alpha, m_\alpha)$, the initial gauge group U(N) is broken to $U(p_\alpha) \times U(p_b) \times U(p_c)$. If we define

$$\begin{aligned} I_{\alpha\beta} &\equiv -n_\alpha m_\beta + n_\beta m_\alpha \\ \tilde{I}_{\alpha\beta} &\equiv \frac{I_{\alpha\beta}}{n_\alpha n_\beta}, \end{aligned} \tag{5.7.2}$$

the $\tilde{I}_{\alpha\beta}$'s trivialy follow the relation

$$\tilde{I}_{ab} + \tilde{I}_{bc} + \tilde{I}_{ca} = 0 \tag{5.7.3}$$

This relation implies that one of $|\tilde{I}_{\alpha\beta}|$ is bigger than the other two. In the following, we take this to $\tilde{I}_{bc}$.

There are two posibilities of the wavefunction depending on whether the sign of $\tilde{I}_{bc}$ is positive or negative. From the results of the previous sections, the wavefunction Ψ which is gaugino is given by

$$\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{5.7.4}$$

$$\begin{cases} \psi_+ = \begin{pmatrix} \text{const} & \Phi^{i,I_{ab}} & 0 \\ 0 & \text{const} & 0 \\ \Phi^{j,I_{ca}} & \Phi^{k,I_{cb}} & \text{const} \end{pmatrix} & \tilde{I}_{bc} < 0 \\ \psi_+ = \begin{pmatrix} \text{const} & 0 & \Phi^{j,I_{ac}} \\ \Phi^{i,I_{ba}} & \text{const} & \Phi^{k,I_{bc}} \\ 0 & 0 & \text{const} \end{pmatrix} & \tilde{I}_{bc} > 0 \end{cases} \tag{5.7.5}$$

where $\psi_- = (\psi_+)^\dagger$ and const $= \mathcal{A}^{-1/2}$. The chiral wavefunctions $\Phi^{j,I_{\alpha\beta}}, j = 0, ..., |I_{\alpha\beta}| - 1$ have been found.

By applying the general form (5.2.46) to T^2 model, we obtain the Yukawa couplings involving chiral massless fermions in this model:

$$\int_{T^2} dz d\bar{z}\, \mathrm{tr}\{\psi_+[\phi_-, \psi_+]\}, \quad \int_{T^2} dz d\bar{z}\, \mathrm{tr}\{\psi_-[\phi_+, \psi_-]\} \tag{5.7.6}$$

where these couplings are CPT conjugates each other and $\phi_\pm$ are the wavefunctions of the bosonic fluctuations contained higher dimensional gauge field A_M with helicity ± 1 in the internal coordinates. Since the eigenfunctions of Dirac and Laplace operator are the same, we find the Yukawa couplings involving left-handed fermions is given by

$$Y_{ijk} = \begin{cases} \sigma_{abc} g \int_{T^2} dz d\bar{z}\, \mathrm{tr}\{\Phi^{i,I_{ab}} \Phi^{j,I_{ca}} \left(\Phi^{k,I_{cb}}\right)^\dagger\} & \tilde{I}_{bc} < 0 \\ \sigma_{abc} g \int_{T^2} dz d\bar{z}\, \mathrm{tr}\{\Phi^{i,I_{ba}} \Phi^{j,I_{ac}} \left(\Phi^{k,I_{bc}}\right)^\dagger\} & \tilde{I}_{bc} > 0 \end{cases} \tag{5.7.7}$$

where g is a gauge coupling constant and we have defined $\sigma_{abc} = \mathrm{sign}(\tilde{I}_{ab}\tilde{I}_{bc}\tilde{I}_{ca})$. In $D = 4$ physics, this term leads to the couplings between two chiral fermions with opposit chiralities transforming in the $(p_a, \overline{p}_b), (p_a, \overline{p}_c)$ bifundamental representations and complex tachyon in the $(p_b, \overline{p}_c)$.

Above discussions are general in the T^2 models. Let us first consider the abelian Wilson lines which mean $(n_\alpha, m_\alpha) = n_\alpha(1, s_\alpha), s_\alpha \in \mathbb{Z}$. Then the initial gauge group U(N) is broken to $U(n_a) \times U(n_b) \times U(n_c), n_a + n_b + n_c = N$ and the degeneracy of chiral fermions in $(n_\alpha, \overline{n}_\beta)$ is given by $\tilde{I}_{\alpha\beta} = s_\alpha - s_\beta$. Also the matrix-valued wavefunction $\Phi^{i,I_{\alpha\beta}}$ is reduced to a product of $n_\alpha \times n_\beta$ matix and the wavefunctions

$\psi^{i,\tilde{I}_{\alpha\beta}}$ defined as (??). Thus the Yukawa couplings are rewritten as

$$Y_{ijk} = \sigma_{abc} g \int_{T^2} dz d\bar{z} \psi^{i,\tilde{I}_{ab}}(z+\zeta_{ab}) \psi^{j,\tilde{I}_{ca}}(z+\zeta_{ca}) \left(\psi^{k,\tilde{I}_{cb}}(z+\zeta_{cb})\right)^* \tag{5.7.8}$$

where we have chosen $\tilde{I}_{bc}$ negative and the abelian Wilson lines $\zeta_{\alpha\beta}$ are defined by (??) with the substitutions $m_\alpha \to s_\alpha, I_{\alpha\beta} \to \tilde{I}_{\alpha\beta}$.

In order to compute (5.7.8), we use the following formula:

$$\begin{aligned} &\theta \begin{bmatrix} \frac{r}{N_1} \\ 0 \end{bmatrix} (z_1, \tau N_1) \cdot \theta \begin{bmatrix} \frac{s}{N_2} \\ 0 \end{bmatrix} (z_2, \tau N_2) \\ &= \sum_{m \in \mathbb{Z}_{N_1+N_2}} \theta \begin{bmatrix} \frac{r+s+N_1 m}{N_1+N_2} \\ 0 \end{bmatrix} (z_1+z_2, \tau(N_1+N_2)) \times \theta \begin{bmatrix} \frac{N_2 r - N_1 s + N_1 N_2 m}{N_1 N_2 (N_1+N_2)} \\ 0 \end{bmatrix} (z_1 N_2 - z_2 N_1, \tau N_1 N_2 (N_1+N_2)) \end{aligned} \tag{5.7.9}$$

In this case, by substituting

$$r = i, \quad s = j, \quad N_1 = \tilde{I}_{ab}, \quad N_2 = \tilde{I}_{ca}, z_1 = (z+\zeta_{zb})\tilde{I}_{ab}, \quad z_2 = (z+\zeta_{ca})\tilde{I}_{ca} \tag{5.7.10}$$

and using the relation (5.7.3) we find

$$\begin{aligned} \psi^{i,\tilde{I}_{ab}}(z)\psi^{j,\tilde{I}_{ca}}(z) =& \frac{(2\,\mathrm{Im}\,\tau)^{1/4}}{\mathcal{A}^{1/2}} \left| \frac{\tilde{I}_{ab}\tilde{I}_{ca}}{\tilde{I}_{bc}} \right| \sum_{m \in \mathbb{Z}_{\tilde{I}_{bc}}} \psi^{i+j+\tilde{I}_{ab}m, \tilde{I}_{cb}}(z) \\ &\times \theta \begin{bmatrix} \frac{\tilde{I}_{ca} i - \tilde{I}_{ab} j + \tilde{I}_{ab}\tilde{I}_{ca} m}{-\tilde{I}_{ab}\tilde{I}_{bc}\tilde{I}_{ca}} \\ 0 \end{bmatrix} (0, \tau|\tilde{I}_{ab}\tilde{I}_{bc}\tilde{I}_{ca}). \end{aligned} \tag{5.7.11}$$

where we set $\zeta = 0$ for simplicity. In this expression, the theta factor in the right hand side is no longer dependent on z, hence it may be fatored out from the integration. Futhermore the theta functions form orthonomality system. Thus the integrand in Yukawa couplings (5.7.8) is given by two theta functions and it can be carried out easily:

$$Y_{ijk}^{\zeta=0} = \left(\frac{2\,\mathrm{Im}\,\tau}{\mathcal{A}^2} \right)^{1/4} \left| \frac{\tilde{I}_{ab}\tilde{I}_{ca}}{\tilde{I}_{bc}} \right|^{1/4} \theta \begin{bmatrix} -\left(\frac{j}{\tilde{I}_{ca}} + \frac{k}{\tilde{I}_{bc}} \right) / \tilde{I}_{ab} \\ 0 \end{bmatrix} (0, \tau|\tilde{I}_{ab}\tilde{I}_{bc}\tilde{I}_{ca}|) \tag{5.7.12}$$

Similary, we can obtain the Yukawa coplings with abelian Wilson lines. Here we define the following quantities:

$$\tilde{\zeta}_\alpha = s_\alpha \zeta_\alpha, \quad \tilde{\zeta}_{\alpha\beta} = \tilde{\zeta}_\alpha - \tilde{\zeta}_\beta, \quad \tilde{\zeta} = \tilde{I}_{ab}\tilde{\zeta}_c + \tilde{I}_{bc}\tilde{\zeta}_a + \tilde{I}_{ca}\tilde{\zeta}_b$$
$$\delta_{ijk} = \frac{i}{\tilde{I}_{ab}} + \frac{j}{\tilde{I}_{ca}} + \frac{k}{\tilde{I}_{bc}}, \quad H(\tilde{\zeta}, \tau) = 2\pi i |\tilde{I}_{ab}\tilde{I}_{bc}\tilde{I}_{ca}|^{-1} \frac{\tilde{\zeta}\,\mathrm{Im}\,\tilde{\zeta}}{\mathrm{Im}\,\tau} \tag{5.7.13}$$

Then the Yukawa couplings with non-abelian Wilson lines are given by

$$Y_{ijk} = \sigma_{abc} g \left(\frac{2\,\mathrm{Im}\,\tau}{\mathcal{A}^2}\right)^{1/4} \left|\frac{\tilde{I}_{ab}\tilde{I}_{ca}}{\tilde{I}_{bc}}\right|^{1/4} e^{H(\tilde{\zeta},\tau)/2} \theta \begin{bmatrix} \delta_{ijk} \\ 0 \end{bmatrix} (\tilde{\zeta}, \tau | \tilde{I}_{ab}\tilde{I}_{bc}\tilde{I}_{ca}|) \tag{5.7.14}$$

We can obtain similar results for $\tilde{I}_{bc} > 0$.

5.7.2 T^2 Models with Non-Abelian Wilson Lines

Next, we consider the non-abelian Wilson lines such that $p_\alpha = \text{g.c.d.}(n_\alpha, m_\alpha) = 1 < n_\alpha$. The chiral fields are given by (5.5.18) and the integral (**??**) are rewritten as

$$Y_{ijk} = \sigma_{abc} g \int_{T^2} dz d\bar{z} \sum_{l=1}^{n_a n_b n_c} \phi^{i,I_{ab}} \phi^{j,I_{ca}} \left(\phi^{k,I_{cb}}\right)^*, \tag{5.7.15}$$

where $\phi^{i,I_{\alpha\beta}}_{k_a,k_b}$ is the componets of $\Phi^{i,I_{\alpha\beta}}$ and we have used the boundary conditions and assumed g.c.d.$(n_a, n_b, n_c) = 1$. However since $\phi^{i,I_{\alpha\beta}}_{l,l}$ the extra z-dependent factors except theta function, the computation of this integral is harder than abelian case. But if we fix $l = 0$ by using the boundary condtions and integrate over a torus of complex structure $n_a n_b n_c \tau$ instead of performing the summation over l, we can find

$$Y_{ijk} = \sigma_{abc} g \left(\frac{2\,\mathrm{Im}\,\tau}{\mathcal{A}^2}\right)^{1/4} \left|\frac{\tilde{I}_{ab}\tilde{I}_{ca}}{\tilde{I}_{bc}}\right|^{1/4} e^{H/2} \theta \begin{bmatrix} \delta_{ijk} \\ 0 \end{bmatrix} (\tilde{\zeta}, \tau | I_{ab} I_{bc} I_{ca}|) \tag{5.7.16}$$

where we have defined $\tilde{\zeta}_\alpha = m_\alpha \zeta_\alpha$ This expression (5.7.16) is quite simillar to the abelian Wilson lines case.

5.7.3 General Tori

Finally, let us compute the Yukawa couplings for $2n$ dimensional tori. In the particular case that T^{2n} can be factorazed as T^2, the chiral matter wavefunctions are given by (5.4.6) in abelian Wilson lines or (**??**) in non-abelian Wilson lines. Futhermore, the integral (5.2.46) is also decomposed to the integrals over the each torus T^2. Thus the Yukawa couplings for factorizable T^{2n} are given by

$$Y_{ijk} = \sigma_{abc} \prod_{r=1}^{n} \left(\frac{2\,\mathrm{Im}\,\tau^{(r)}}{(\mathcal{A}^{(r)})^2} \right)^{1/4} \left| \frac{\tilde{I}_1^{(r)} \tilde{I}_2^{(r)}}{\tilde{I}_1^{(r)} + \tilde{I}_2^{(r)}} \right|^{1/4} e^{H^{(r)}/2} \theta \begin{bmatrix} \delta_{ijk}^{(r)} \\ 0 \end{bmatrix} \left(\tilde{\zeta}^{(r)} (I_{ab}^{(r)} I_{bc}^{(r)} I_{ca}^{(r)}) \right) \tag{5.7.17}$$

where if $\sigma_{abc}^{(s)} = -1$, we must perform the substitutions $J^{(s)} \to \overline{J}^{(s)}$ and $\tilde{\zeta}^{(s)} \to \overline{\tilde{\zeta}}^{(s)}$. The quantities with the label (r) defined the quantities without (r) on T^2.

5.7.4 Yukawa Couplings in Supersymmetric Description

In this section, we attempt to understand Yukawa couplings as a 3-point function in a $N = 1, D = 4$ supersymmetric theory. The Yukawa couplings can be given by the factors and the trilinear coupling W_{ijk} of the superpotential:

$$Y_{ijk} = (K_{ab} K_{bc} K_{ca})^{-1/2} e^{K/2} W_{ijk} \tag{5.7.18}$$

where K is the Kähler potential and we have defined $K_{\alpha\beta} \equiv \partial_{\alpha\beta} \overline{\partial}_{\alpha\beta} K$ which are come from the normalization of the kinetic terms of chiral fields in the $\alpha\beta$ sector.

We apply the general formula(5.7.18) to (5.7.16) and we find

$$W_{ijk} = \prod_{r=1}^{n} \theta \begin{bmatrix} \delta_{ijk}^{(r)} \\ 0 \end{bmatrix} (\tilde{\zeta}^{(r)}, \tau^{(r)} |I_{ab}|^{(r)} I_{bc}^{(r)} I_{ca}^{(r)}) \tag{5.7.19}$$

$$(K_{ab} K_{bc} K_{ca})^{-1} e^{K} = g^2 \prod_{r=1}^{n} \frac{(2\,\mathrm{Im}\,\tau^{(r)})^{1/2}}{\mathcal{A}^{(r)}} \left| \frac{\tilde{I}_1^{(r)} \tilde{I}_2^{(r)}}{\tilde{I}_1^{(r)} + \tilde{I}_2^{(r)}} \right|^{1/2} e^{\tilde{H}^{(r)}} \tag{5.7.20}$$

where in (5.7.20) we have ignored phase factors and defined

$$\tilde{H}^{(r)} = -2\pi |I_{ab}^{(r)} I_{bc}^{(r)} I_{ca}^{(r)}|^{-1} \frac{(\mathrm{Im}\,\tilde{\zeta}^{(r)})^2}{\mathrm{Im}\,\tau^{(r)}}. \tag{5.7.21}$$

Futhermore if we define the supergravity fields as

$$\mathrm{Re}\,S = (2\pi)^{-1} g^{-2} \prod_{r=1}^{n} \frac{\mathcal{A}^{(r)}}{4\pi} \tag{5.7.22}$$

$$\mathrm{Re}\,T^{(r)} = (2\pi)^{-1} g^{-2} \frac{\mathcal{A}^{(r)}}{4\pi} \tag{5.7.23}$$

$$\mathrm{Re}\,U^{(r)} = \mathrm{Im}\,\tau^{(r)}, \tag{5.7.24}$$

then (5.7.20) can be rewritten to

$$(K_{ab}K_{bc}K_{ca})^{-1}e^{K} = \frac{2}{2\pi)^3}(S+\overline{S})^{-1}\prod_{r=1}^{n}\left(U^{(r)}+\overline{U}^{(r)}\right)^{1/2}\left|\frac{\tilde{I}_1^{(r)}\tilde{I}_2^{(r)}}{\tilde{I}_1^{(r)}+\tilde{I}_2^{(r)}}\right|^{1/2} e^{\tilde{H}^{(r)}} \tag{5.7.25}$$

We can see that is are no $\operatorname{Re} T^{(r)}$ dependence in (5.7.25) and the only explicit dependence on the Wilson lines $\zeta_\alpha^{(r)}$ through the $\tilde{H}^{(r)}$ factor. These Wilson lines can be regard as the vev's of the scalar fields in the adjoint representation of each gauge group and hence belong to $D = 4, N = 1$ chiral multiplets.

6 D-Brane

Supergravity theory is a supersymmetric theory that includes gravity as the highest spin. String theory is closely related to supergravity theory because it contains gravity and is invariant under the spacetime supersymmetry. The supersymmetric algebra is determined by only the dimension of spacetime and the type of spinors. Thus supergravity theory must correspond to the superstring theory because of supersymmetry algebra. Since string theory is ten-dimensional, only three types of supersymmetric algebra in ten dimensions are possible, and there are three types of supergravity theories. There are three kinds of supergravity theories, and each of them corresponds to Type IIA, Type IIB, and Type I superstring theories. Due to the strong restriction of supersymmetry, the solutions of supergravity theories coincide with those of string theories. As a result, we can obtain solitonic solutions of the string as a solution of the supergravity theory. This solitonic solutions are known as brane. In this section, we construct the higher dimensional supergravity theory, and show that there exist strings and branes as solutions of supergravity theories.

6.1 High Dimensinal Supergravity

6.1.1 Eleven Dimensinal Supergravity

The theory of supergravity consists of representations of supersymmetric algebras that contain gravity. Consider a four-dimensional supersymmetric algebra, and a representation such that the highest spin is 2. Since the supersymmetric algebra corresponds to a Clifford algebra, we can give a representation with up or down helicity 1/2. Since the highest spin is 2, the maximum helicity is 2 and the minimum is -2. Therefore, there must be at most eight supercharges. From the irreducible representation of the four-dimensional Clifford algebra, the spinors are the four Majorana components. Therefore, the four-dimensional maximum supergravity theory has $N = 8$ and a total spinor components is 32. Since the lower dimensional supergravity theory is given by the compactification of the higher dimensional supergravity theory, the spinors of the higher dimensional theory must have at most 32 components. Since the highest dimension with a real 32-component spinor is 11-dimensional, the highest dimensional supergravity theory is 11-dimensional.

The 11-dimensional supersymmetric algebra is

$$\{Q_\alpha, Q_\beta\} = (\Gamma^M C^{-1})_{\alpha\beta} P_M + (\Gamma^{MN} C^{-1})_{\alpha\beta} Z_{MN} + (\Gamma^{MNPQR} C^{-1})_{\alpha\beta} Z_{MNPQR}. \tag{6.1.1}$$

The local symmetries of 11D supergravity theories are general coordinate transformation and supersymmetry, and the corresponding gauge fields are gravity $e^{\overline{\mu}}{}_{\mu}$ and gravitino ψ_μ. We consider the respective degrees of freedom. The irreducible representation of the Poincaré group is determined by the equation of motion, which determines the mass of the particle, and coincides with the irreducible representation of the small group, which keeps it invariant. Since these fields are massless, the irreducible representation of the Poincaré group is given by the irreducible representation of the small group SO(D-2). The gravity $g_{\mu\nu}$ is a symmetric traceless tensor, and the on-shell degrees of freedom are $(D-2)(D-1)/2-1$ in D dimensions. On the other hand, the gravitino has a vector and a real spinor component and is given by $(D-3)2^{D/2-1}r$, where r takes 2 for the Dirac spinor, 1 for the Majorana, and 1/2 for the Majorana-Weyl. Thus, If $D=11$, gravity has 44 degrees of freedom and the gravitino has 128 degrees of freedom. Because of supersymmetry, the degrees of freedom of the boson and fermion must match on-shell, so we need to add the boson by 128-44=84 degrees of freedom. The only representation of SO(11-2) with 84 boson degrees of freedom is the third-order antisymmetric tensor $A_{\mu_1\mu_2\mu_3}$. Therefore, the action of 11-dimensional supergravity theory is

$$\begin{aligned}L =&\frac{e}{4\kappa_{11}^2}R(\Omega(e,\psi)) - \frac{e}{48}F_{\mu_1\ldots\mu_4}F^{\mu_1\ldots\mu_4} - \frac{e}{2}\overline{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu(\frac{1}{2}(\Omega+\hat{\Omega}))\psi_\rho \\ &- \frac{1}{192}e\kappa_{11}(\overline{\psi}_{\mu_1}\gamma^{\mu_1\ldots\mu_6}\psi_{\mu_2} + 12\overline{\psi}^{\mu_3}\gamma^{\mu_4\mu_5}\psi^{\mu_6})(F_{\mu_3\ldots\mu_6}+\hat{F}_{\mu_3\ldots\mu_6}) \\ &+ \frac{2\kappa_{11}}{(12)^4}\epsilon^{\mu_1\ldots\mu_{11}}F_{\mu_1\ldots\mu_4}F_{\mu_5\ldots\mu_8}A_{\mu_9\mu_{10}\mu_{11}}\end{aligned} \tag{6.1.2}$$

$$\begin{aligned}F_{\mu_1\ldots\mu_4} &= 4\partial_{[\mu_1}A_{\mu_2\mu_3\mu_4]} \\ \hat{F}_{\mu_1\ldots\mu_4} &= F_{\mu_1\ldots\mu_4} + 3\kappa_{11}\overline{\psi}_{[\mu_1}\gamma_{\mu_2\mu_3}\psi_{\mu_4]} \\ \Omega_{\mu mn} &= \hat{\Omega}_{\mu mn} - \frac{1}{4}\kappa_{11}^2\overline{\psi}_\nu\gamma_{\mu mn}^{\nu\lambda}\psi_\lambda \\ \hat{\Omega} &= \Omega_{\mu mn}^{(0)}(e) + \frac{1}{2}\kappa_{11}^2(\overline{\psi}_\nu\gamma_n\psi_m - \overline{\psi}_\nu\gamma_m\psi_n + \overline{\psi}_n\gamma_\nu\psi_m),\end{aligned} \tag{6.1.3}$$

where $\Omega^{(0)}$ is the usual spin connection. Also we have defined

$$D_\mu(\Omega)\psi_\nu = (\partial_\mu - \frac{1}{4}\Omega_\mu^{mn}\gamma_{mn})\psi_\nu. \tag{6.1.4}$$

The local supersymmetric transformations are

$$\begin{aligned}\delta e_\mu^m &= \kappa_{11}\bar{\epsilon}\gamma^m\psi_\mu \\ \delta A_{\mu_1\mu_2\mu_3} &= -\frac{3}{2}\bar{\epsilon}\gamma_{[\mu_1\mu_2}\psi_{\mu_3]} \\ \delta\psi_\mu &= \frac{1}{\kappa_{11}}D_\mu(\hat{\Omega})\epsilon + \frac{1}{12^2}(\gamma_\mu^{\nu_1\ldots\nu_4} - 8\delta_\mu^{\nu_1}\gamma^{\nu_2\nu_3\nu_4})\hat{F}_{\nu_1\ldots\nu_4}\epsilon.\end{aligned} \tag{6.1.5}$$

The gauge transformation is

$$\delta A_{\mu_1\mu_2\mu_3} = 3\partial_{[\mu_1}\Lambda_{\mu_2\mu_3]}. \tag{6.1.6}$$

This action can be constructed by the several methods [26,80]. The only coupling constant in this action has is κ_{11}, and its dimension is $[M]^{-9/2}$. We can also remove κ_{11} from the supersymmetric transformation rule by the following scale transformation.

$$\psi_\mu \to \kappa_{11}^{-1}\psi_\mu, \quad A_{\mu_1\mu_2\mu_3} \to \kappa_{11}^{-1}A_{\mu_1\mu_2\mu_3} \tag{6.1.7}$$

Since the action is invariant under this transformation except for the overall factor, κ_{11} does not contribute to the equation of motion and has no physical meaning at the classical level.

6.1.2 TypeIIA Supergravity Theory

Next we construct a 10-dimensional supergravity theory. There are two types of maximal $N = 2$ supersymmetric algebras in 10 dimensions, TypeIIA or TypeIIB. In addition, TypeI existed as an $N = 1$ supersymmetric algebra. The representation dimension of the 10-dimensional Clifford algebra is 32 components. Therefore, the spinors can be taken to be Majorana-Weyl, and the spinors are real 16-component. Since we have constructed the 11 dimensional supergravity, we use the method of dimensional reduction to obtain the action of 10-dimensional supergravity theory. It can be seen that all 32 components of the supercharge are conserved, which leads to a 10D $N = 2$ supergravity theory. Also, since chirality is induced by compactification, the theory derived is Type IIA.

We compactify the x^{10} direction of the 11-dimensional supergravity theory to S^1.

$$x'^{10} \sim x^{10} \leftrightarrow x'^{10} = x^{10} + 2\pi nR \tag{6.1.8}$$

The notation for space-time indexing is $\hat{\mu} = 0, ..., 10$ and $\mu = 0, ..., 9$. Fourier decompose the x^{10} direction according to the Kalza-Klein method, leaving only the zero modes:

$$\begin{aligned}
D = 11: &\quad e_{\hat{\mu}}^{\hat{m}}, &&A_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3},\ \psi_{\hat{\mu}\alpha} \\
D = 10: &\quad e_\mu^m, B_\mu,\ \sigma, &&A_{\mu_1\mu_2\mu_3},\ A_{\mu_1\mu_2},\ \psi_{\mu\alpha},\ \lambda_\alpha.
\end{aligned} \tag{6.1.9}$$

Decomposing the action of the 11D supergravity theory under this expansion, we obtain

$$L = L^B + L^F \tag{6.1.10}$$

$$
\begin{aligned}
L^B = \frac{1}{2\kappa_{10}^2}\,[\,& eR(\omega(e)) - \frac{1}{12}ee^{\sigma/2}\hat{F}_{\mu_1\ldots\mu_4}\hat{F}^{\mu_1\ldots\mu_4} - \frac{1}{3}ee^{-\sigma}F_{\mu_1\ldots\mu_3}F^{\mu_1\ldots\mu_3} \\
& - \frac{1}{4}ee^{3\sigma/4}F_{\mu_1\mu_2}F^{\mu_1\mu_2} - \frac{1}{2}e\partial_\mu\sigma\partial^\mu\sigma \\
& + \frac{1}{2(12)^2}\epsilon^{\mu_1\cdots\mu_{10}}F_{\mu_1\ldots\mu_4}F_{\mu_5\ldots\mu_8}A_{\mu_9\mu_{10}}\ ]
\end{aligned} \tag{6.1.11}
$$

$$
\begin{aligned}
F_{\mu_1\mu_2} &= 2\partial_{[\mu_1}A_{\mu_2]} \\
F_{\mu_1\mu_2\mu_3} &= 3\partial_{[\mu_1}A_{\mu_2\mu_2]} \\
\hat{F}_{\mu_1\ldots\mu_4} &= 4(\partial_{[\mu_1}A_{\ldots\mu_4]} + A_{[\mu_1}F_{\mu_2\mu_3\mu_4]})
\end{aligned} \tag{6.1.12}
$$

$$
\begin{aligned}
L^F = \frac{4e}{2\kappa_{10^2}}\Bigg[& -\frac{1}{2}\overline{\psi}_{\mu_1}\gamma^{\mu_1\mu_2\mu_3}D_{\mu_2}\psi_{\mu_3} - \frac{1}{2}\overline{\lambda}\gamma^\mu D_\mu\lambda + \frac{\sqrt{2}}{4}\overline{\lambda}\gamma^{11}\gamma^{\mu_1\mu_2}\psi_{\mu_1}\partial_{\mu_2}\sigma \\
& + \frac{1}{96}e^{\sigma/4}\left\{-\overline{\psi}_{\mu_1}\gamma^{\mu_1\cdots\mu_6}\psi_{\mu_2} - 12\overline{\psi}^{\mu_3}\gamma^{\mu_4\mu_5}\psi^{\mu_6} + \frac{1}{\sqrt{2}}\overline{\lambda}\gamma^{11}\gamma^{\mu_1\mu_3\cdots\mu_6}\psi_{\mu_1} + \frac{3}{4}\overline{\lambda}\gamma^{11}\gamma^{\mu_3\cdots\mu_6}\lambda\right\}F_{\mu_3\ldots\mu_6} \\
& - \frac{1}{24}e^{-\sigma/2}\left\{\overline{\psi}_{\mu_1}\gamma^{11}\gamma^{\mu_1\cdots\mu_4}\psi_{\mu_2} - 6\overline{\psi}^{\mu_3}\gamma^{11}\gamma^{\mu_4}\psi^{\mu_5} - \sqrt{2}\overline{\lambda}\gamma^{\mu_1\mu_3\mu_4\mu_5}\psi_{\mu_1}\right\}F_{\mu_3\mu_4\mu_5} \\
& - \frac{1}{16}e^{3\sigma/4}\left\{\overline{\psi}_{\mu_1}\gamma^{11}\gamma^{\mu_1\cdots\mu_4}\psi_{\mu_2} + 2\overline{\psi}^{\mu_3}\gamma^{11}\psi^{\mu_4} + \frac{3}{\sqrt{2}}\overline{\lambda}\gamma^{\mu_1\mu_3\mu_4}\psi_{\mu_1} - \frac{5}{4}\overline{\lambda}\gamma^{11}\gamma^{\mu_3\mu_4}\lambda\right\}F_{\mu_3\mu_4} \\
& +\text{4-th order}\ ]
\end{aligned} \tag{6.1.13}
$$

The supersymmetric transformation rules are also given by decomposing the 11-dimensional transformation rules

$$\delta e^n_\mu = \bar{\epsilon}\gamma^n\psi_\mu \tag{6.1.14}$$

$$\delta\sigma = -\sqrt{2}\overline{\lambda}\gamma^{11}\epsilon \tag{6.1.15}$$

$$\delta A_\mu = e^{13\sigma/4}\{\overline{\psi}_\mu\gamma^{11}\epsilon + \frac{3\sqrt{2}}{4}\overline{\lambda}\gamma_\mu\epsilon\}A_{\mu_1\mu_2} = e^{\sigma/2}\{-\overline{\psi}_{[\mu_1}\gamma_{\mu_2]}\gamma^{11}\epsilon + \frac{1}{2\sqrt{2}\overline{\lambda}\gamma_{\mu_1\mu_2}\epsilon}\} \tag{6.1.16}$$

$$\delta A_{\mu_1\mu_2\mu_3} = \frac{e^{-\sigma/4}}{2}\{3\overline{\psi}_{\mu_1}\gamma_{\mu_2\mu_3}\epsilon + \frac{1}{2\sqrt{2}}\overline{\lambda}\gamma^{11}\gamma_{\mu_1\mu_2\mu_3}\epsilon\} + 3e^{\sigma/2}\{-A_{[\mu_1}\overline{\psi}_{\mu_2}\gamma_{\mu_3]}\gamma^{11}\epsilon + \frac{1}{2\sqrt{2}}A_{[\mu_1}\overline{\lambda}\gamma_{n\mu_2\mu_3]}\epsilon\} \tag{6.1.17}$$

$$
\begin{aligned}
\delta\lambda =& \frac{\sqrt{2}}{4}\hat{D}_\mu\sigma\gamma^\mu\gamma^{11}\epsilon - \frac{3}{16\sqrt{2}}e^{3\sigma/4}\gamma^{\nu_1\nu_2}\epsilon F_{\nu_1\nu_2} + \frac{1}{12\sqrt{2}}e^{\sigma/2}\gamma^{\nu_1\nu_2\nu_3}\epsilon F_{\nu_1\nu_2\nu_3} \\
&+ \frac{1}{96\sqrt{2}}e^{\sigma/4}\gamma^{\nu_1\nu_2\nu_3\nu_4}\gamma^{11}\epsilon F_{\nu_1\nu_2\nu_3\nu_4} + (\,4\,\text{次})
\end{aligned} \tag{6.1.18}
$$

$$
\begin{aligned}
\delta\psi_\mu =& D_\mu\epsilon - \frac{1}{64}e^{3\sigma/4}(\gamma_\mu^{\nu_1\nu_2} - 14\delta_\mu^{\nu_1}\gamma^{\nu_2})\gamma^{11}\epsilon F_{\nu_1\nu_2} - \frac{1}{48}e^{-\sigma/2}(\gamma_\mu^{\nu_1\nu_2\nu_3} - 9\delta_\mu^{\nu_1}\gamma^{\nu_2\nu_3}\gamma^{11}\epsilon F_{\nu_1\nu_2\nu_3} \\
&+ \frac{1}{128}e^{\sigma/4}(\gamma_\mu^{\nu_1\ldots\nu_4} - \frac{2\sigma}{3}\delta_\mu^{\nu_1}\gamma^{\nu_2\nu_3\nu_4})\gamma^{11}\epsilon F_{\nu_1\ldots\nu_4} + (\,4\,\text{次})
\end{aligned} \tag{6.1.19}
$$

We now consider the fields that exists in a TypeⅡA theory. The field strengths that exist are of order 2, 3, and 4. If we consider the duality of the field strengths, they are 6th, 7th, and 8th order. In other words, the TypeⅡA theory contains $p = 1, 2, 3, 5, 6$ and 7-order gauge fields. In the TypeⅡA string theory, $p = 2, 6$ appear in the NS×NS sector, while $p = 1, 3, 5, 7$, appear in the R×R sector. If we consider the scalar field σ as a $p = 0$-order gauge field, the strength of the field is first-order, and the dual gauge field is eighth-order. Therefore, it contains gauge fields other than the 4th order.

6.1.3 TypeⅡB Supergravity Theory

There existed a TypeⅡB supersymmetric algebra created by Majorana-Weyl spinor with the same chirality. Therefore, we can construct a supergravity theory corresponding to this algebra. The supersymmetric algebra is

$$\begin{aligned}\{Q^i_\alpha, Q^j_\beta\} &= (P_-\gamma_m C^{-1})_{\alpha\beta}\delta^{ij}P^m + (P_-\gamma_m C^{-1})_{\alpha\beta}Z^{ij}_M \\ &\quad + (P_-\gamma_{m_1m_2m_3}C^{-1})_{\alpha\beta}\epsilon^{ij}Z^{m_1m_2m_3} + (P_-\gamma_{m_1\ldots m_5}C^{-1})_{\alpha\beta}Z^{m_1\ldots m_5 ij},\end{aligned} \tag{6.1.20}$$

where, $Z^{ij}_m, Z^{ij}_{m_1..m_5}$ is symmetric about i, j. Also, $i, j = 1, 2$ are the index of SO(2) symmetry because it mixes two Majorana supercharges. In other words, the supersymmetric algebra with SU(1,1) symmetry is broken to SO(2) symmetry by the Majorana condition. In the following, we treat Q^1 and Q^2 as $SO(2) \simeq U(1)$ by combining them into a complex supercharge. The off-shell component fields are an irreducible representation of the supersymmetric algebra:

$$e^m_\mu \quad A_{\mu\nu} \quad \varphi \quad B_{\mu\nu\rho\kappa} \quad \psi_{\alpha\mu} \quad \lambda_\alpha. \tag{6.1.21}$$

φ is a complex scalar, $A_{\mu\nu}$ is a complex vector, and $B_{\mu\nu\rho\kappa}$ is a self-dual real field and so its action cannot be written down. Therefore, there is no action in TypeⅡB supergravity theory, and it is described by equations of motion. The equations of motion are invariant under general coordinate and supersymmetric transformations, and all terms in each of the equation of motion must have the same U(1) charge. The U(1) charge of the component fields are

Component Fields	e^m_μ	$A_{\mu\nu}$	φ	$B_{\mu\nu\rho\kappa}$	$\psi_{\alpha\mu}$	λ_α
U(1) Charge	0	2	4	0	1	3

(6.1.22)

Firs we consider the scalar field φ. Since the U(1) symmetry comes from SU(1,1), the scalar field is given by the nonlinear representation:

$$-\frac{1}{2}\int d^Dx P_\mu \overline{P}^\mu = -\frac{1}{2}\int d^Dx \frac{\partial_\mu\varphi\partial^\mu\varphi^*}{(1-\varphi\varphi^*)^2}. \tag{6.1.23}$$

Therefore, the equation of motion is

$$D_\mu P^\mu = 0. \tag{6.1.24}$$

Consider the contributions from other fields. The U(1) charge of φ is 4, and the geometric dimension is 2. The only pair that has the same U(1) charge and geometric dimensions is $F_{\mu_1\mu_2\mu_3}F^{\mu_1\mu_2\mu_3}$. By determining the coefficients by invariance, we obtain

$$D^\mu P_\mu = \frac{1}{6}F_{\mu_1\mu_2\mu_3}F^{\mu_1\mu_2\mu_3}. \tag{6.1.25}$$

Next we consider the vector field $A_{\mu\nu}$. The equation of motion for $A_{\mu\nu}$ involves $D^{\mu_3}F_{\mu_1\mu_2\mu_3}$. The U(1) charge of this quantity is 2 and its geometric dimension is 2. Quantities with the same dimension and charge are $\overline{F}_{\mu_1\mu_2\mu_3}P^{\mu_3}$ and $G_{\mu_1\ldots\mu_5}F^{\mu_3\mu_4\mu_5}$. Therefore, the equation of motion is

$$D^{\mu_3}F_{\mu_1\mu_2\mu_3} = -\overline{F}_{\mu_1\mu_2\mu_3}P^{\mu_3} - \frac{i}{6}G_{\mu_1\ldots\mu_5}F^{\mu_3\mu_4\mu_5}. \tag{6.1.26}$$

Finally we consider the vector field $B_{\mu_1\ldots,\mu_4}$. This field is self-dual and no quantity has the same U(1) charge and dimension. Therefore, $B_{\mu_1\ldots\mu_4}$ is only subject to the self-duality condition.

$$G_{\mu_1\ldots\mu_5} = *G_{\mu_1\ldots\mu_5}, \quad G = dB \tag{6.1.27}$$

The remaining fields such a gravitational belong to the NS×NS sector in the string theory and are the same as in Type ⅡA. By using Noether method, we can obtain the transformation law for TypeⅡB supergravity theory.

By considering 11-dimensional E_{11} gauge theory, we can easily realize typeⅡA and typeⅡB supergravity theory [26, 81–84]. Also its reduction theory was given in [85].

6.1.4 TypeI supergravity theory

TypeⅡA/ⅡB theories are based on $N = 2$ supersymmetry in 10 dimensions. There is an N=1 algebra as a supersymmetric algebra in 10 dimensions.

$$\{Q_\alpha, Q_\beta\} = (P_\pm\gamma_m C^{-1})_{\alpha\beta}P^m + (P_\pm\gamma_{m_1\ldots m_5}C^{-1})_{\alpha\beta}Z^{m_1\ldots m_5} \tag{6.1.28}$$

Q_α is the Majorana-Weyl spinor, and the component fields are

$$e^a_\mu, \phi, A_{\mu\nu}, \psi_{\alpha\mu}, \lambda_\alpha. \tag{6.1.29}$$

The action of this theory can be seen by considering the relationship between TypeIIB string theory and TypeI string theory. In other wards, we can apply the Ω projection to the spectrum of the TypeIIB theory without distinguishing the orientations.

6.2 Brane Solutions

As mentioned at the beginning of this section, it can be seen that due to the strong restriction of supersymmetry, supergravity theory gives solutions of superstring theory. This is because the solution in string theory is perturbative, whereas the supergravity theory solution is non-perturbative. Therefore, the string's non-perturbative effect can be given by supergravity theory. One such solution is the brane solution.

Supergravity theory is a theory that describes the interaction between gravity and scalars, gauge fields, fermions, etc. The action is generally given by

$$S = \frac{1}{2\kappa_D^2}\int d^Dx\sqrt{-g}(R - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \Sigma_i\frac{1}{2\times n_i!}e^{a_i\phi}F_{\mu_1\dots\mu_i}F^{\mu_1\dots\mu_i}). \tag{6.2.1}$$

The equations of motion are

$$R^\mu_\nu = \frac{1}{2}\partial^\mu\phi\partial_\nu\phi + \Sigma_i\frac{1}{2n_i!}e^{a_i\phi}(n_iF^{\mu\lambda_2\dots\lambda_{n_i}}F_{\nu\lambda_2\dots\lambda_{n_i}} - \frac{n_i-1}{D-2}\delta^\mu_\nu F_{\lambda_1\dots\lambda_{n_i}}F^{\lambda_1\dots\lambda_{n_i}} \tag{6.2.2}$$

$$\partial_\mu(\sqrt{-g}e^{a_i\phi}F^{\mu\lambda_2\dots\lambda_{n_i}} = 0 \tag{6.2.3}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) - \Sigma_i\frac{a_i}{2n_i!}e^{a_i\phi}F_{\lambda_1\dots\lambda_{n_i}}F^{\lambda_1\dots\lambda_{n_i}} = 0. \tag{6.2.4}$$

This action does not include the Chern-Simons term, but we can see that it does not contribute to the brane solution. Also, there is a dual field in the strength of each field:

$$\sqrt{-g}e^{a_i\phi}F^{\mu_1\dots\mu_{n_i}} = \frac{1}{(p-n_i)!}\epsilon^{\mu_1\dots\mu_{n_i}\nu_1\dots\nu_{D-n_i}}\tilde{F}_{\nu_1\dots\nu_{D-n_i}}. \tag{6.2.5}$$

The equation of motion for the dual field can be obtained by substituting (6.2.5) into the corresponding equation of motion.

In the following, we investigate the p-brane solution with $p+1$-dimensional volume. Let $t,\ x_j,\ j = 0,...,p$ be the coordinates on the p-brane. Also let the coordinates perpendicular to the p-brane be $y_b, b = 1,....,d,\ d = D-p-1$. In order to find the p-brane solutions, we impose the following assumptions: (1) The brane is stationary in space-time. (2) The brane has SO(p) symmetry. (3) The brane is time-reversal invariant.

From the assumption (1), we find that the metric is independent of t and x_j. Since the spacetime has

Lorentz symmetry, the direction orthogonal to the brane has SO(D-p-1) symmetry from assumption (2). Therefore, the spacetime metric has the following form:

$$ds^2 = -B^2 dt^2 + C^2 \Sigma_{j=1}^{p} dx_j^2 + E^2 \Sigma_a^d dy_a^2 + F^2 (\Sigma_1 y_a dy_a)^2. \tag{6.2.6}$$

$B,\ C,\ E$ and F are functions of $r = \sqrt{y^2}$ only. Since the orthogonal direction is isotropic, it is usefull to use polar coordinates:

$$\begin{aligned} y_a &= r \sin\theta_1 ... \sin\theta_{a-1} \cos\theta_a, \quad a = 1, ..., d-1 \\ y_d &= r \sin\theta_1 ... \sin\theta_{d-1}. \end{aligned} \tag{6.2.7}$$

Substituting into the line segment (6.2.6), we find

$$ds^2 = -B^2 dt^2 + C^2 \sum_{j=1}^{j=p} dx_j^2 + E^2 \left(dr^2 + r^2 d\Omega_{d-1}^2\right) + F^2 r^2 dr^2. \tag{6.2.8}$$

By performimg the general coordinate transformation for r and using the contribution of the third term, we can eliminate the fourth term:

$$ds^2 = -B^2 dt^2 + C^2 \sum_{j=1}^{j=p} dx_j^2 + E^2 \sum_{a=1}^{d} (dy_a)^2 . \tag{6.2.9}$$

Since the p-brane couples to the (p+1)-form gauge field, it is subject to the same restrictions as the metric:

$$F_{ti_1 \cdots i_p a} = \epsilon_{i_1 \cdots i_p} \partial_a A(r). \tag{6.2.10}$$

Next we consider the p-brane solution with these properties. The Ricci tensor for this metric is

$$R_t' = E^{-2} \left\{ -\sum_a \partial_{y_a} \partial_{y_a} \ln B - \sum_a \partial_{y_a} \ln B \partial_{y_a} \Psi \right\} \tag{6.2.11}$$

$$R_{x_i}^{x_i} = E^{-2} \delta_i^j \left\{ -\sum_a \partial_{y_a} \partial_{y_a} \ln C - \sum_a \partial_{y_a} \ln C \partial_{y_a} \Psi \right\} \tag{6.2.12}$$

$$\begin{aligned} R_{y_i}^{y_h} =& E^{-2} \left\{ -\sum_a \partial_{y_a} \partial_{y_a} \Psi + \partial_{y_a} \ln E \partial_{y_b} \Psi + \partial_{y_h} \ln E \partial_{y_a} \Psi - \partial_{y_a} \ln B \partial_{y_b} \ln B \right. \\ & - p \partial_{y_a} \ln C \partial_{y_b} \ln C - (d-2) \partial_{y_a} \ln E \partial_{y_b} \ln E - \delta_b^a \sum_c \partial_{y_c} \partial_{y_c} \ln E \\ & \left. - \delta_b^a \sum_c \partial_{y_c} \ln E \partial_{y_a} \Psi \right\}, \end{aligned} \tag{6.2.13}$$

where we have defined $\Psi = \ln B + p \ln C + (d-2) \ln E$. We will see later that this Ψ is a quantity related to the conservation condition of supersymmetry.

For simplicity, we first consider the case of $\Psi = 0$. The Ricci tensor in this case is

$$R_t^t = \frac{1}{E^2}\left\{-(\ln B)'' - \frac{d-1}{r}(\ln B)' - (\ln B)'\Psi'\right\} \tag{6.2.14}$$

$$R_{x_i}^{x_j} = \frac{1}{E^2}\left\{-(\ln C_i)'' - \frac{d-1}{r}(\ln C)' - (\ln C)'\Psi'\right\}\delta \tag{6.2.15}$$

$$\begin{aligned}R_{y_a}^{y_b} =&\delta_a^b\frac{1}{E^2}\left\{-(\ln E)'' - \frac{d-1}{r}(\ln E)' - (\ln E)'\Psi' - \frac{1}{r}\Psi'\right\}\\ &+\frac{y_a y_b}{r^2}\frac{1}{E^2}\left\{-\Psi'' + \frac{1}{r}\Psi' + 2(\ln E)'\Psi' - (\ln B)'^2\right.\\ &\left.- p(\ln C)'^2 - (d-2)(\ln E)'^2\right\}\end{aligned} \tag{6.2.16}$$

In the polar coordinate,

$$R_r^r = \frac{\partial y_a}{\partial r}\frac{\partial r}{y_b}R_a^b = R_1 + R_2, R_{\theta_\alpha}^{\theta_\beta} = R_1\delta_\alpha^\beta, \tag{6.2.17}$$

where $\alpha, \beta = 1, ..., d-1$. Maxwell's equations are then

$$\frac{d}{dr}\left(e^{a\phi}\left(BC^pE\right)^{-1}(Er)^{d-1}A(r)'\right) = 0 \tag{6.2.18}$$

This equation is easily solved:

$$F_{ti_1\cdots i_p r} = \epsilon_{i_1\cdots i_p}BC^pEe^{-a\phi}\frac{Q}{(Er)^{d-1}}. \tag{6.2.19}$$

Substituting this solution into the dilaton and Einstein equations, we find

$$\frac{d}{dr}\left\{r^{(d-1)}(\ln B)'\right\} = \frac{d-2}{2(D-2)}e^{-a\phi}E^{-2(d-2)}\frac{Q^2}{(r)^{d-1}}, \tag{6.2.20}$$

$$\frac{d}{dr}\left\{r^{(d-1)}(\ln C)'\right\} = \frac{d-2}{2(D-2)}e^{-a\phi}E^{-2(d-2)}\frac{Q^2}{(r)^{d-1}}, \tag{6.2.21}$$

$$\begin{aligned}l - \frac{1}{r^{d-1}}\frac{d}{dr}\left(r^{d-1}(\ln E)'\right) - (d-2)\left((\ln E)'\right)^2 - \left((\ln B)'\right)^2 - p\left((\ln C)'\right)^2\\ = \frac{1}{2}\phi'^2 - \frac{d-2}{2(D-2)}e^{-a\phi}E^{-2(d-2)}\frac{Q^2}{(r)^{2(d-1)}}\end{aligned} \tag{6.2.22}$$

$$\frac{d}{dr}\left\{\left(r^{(d-1)}(\ln E)'\right\} = -\frac{p+1}{2(D-2)}E^{-2(d-2)}e^{-a\phi}\frac{Q^2}{(r)^{d-1}} \tag{6.2.23}$$

$$\frac{d}{dr}\left\{r^{(d-1)}\phi'\right\} = -\frac{a}{2}E^{-2(d-2)}e^{-a\phi}\frac{Q^2}{(r)^{d-1}} \tag{6.2.24}$$

In order to solve these equations, we assume that the p-brane has not only the Lorentzian symmetry SO(p) in space, but also the ISO(1,p) symmetry. This assumption leads to $B = C$. Also, from $\Psi = 0$, the equations for B and E are equal, and the resulting equations are

$$\frac{d}{dr}\left\{\left(r^{(d-1)}(\ln E)'\right)\right\} = -\frac{p+1}{2(D-2)}E^{-2(d-2)}e^{-a\phi}\frac{Q^2}{(r)^{d-1}}, \tag{6.2.25}$$

$$-\frac{(D-2)(d-2)}{(p+1)}\left((\ln E)'\right)^2 = \frac{1}{2}\left(\phi'\right)^2 - \frac{1}{2}e^{-a\phi}E^{-2(d-2)}\frac{Q^2}{(r)^{2(d-1)}}, \tag{6.2.26}$$

$$\frac{d}{dr}\left\{r^{(d-1)}\phi'\right\} = -\frac{a}{2}E^{-2(d-2)}e^{-a\phi}\frac{Q^2}{(r)^{d-1}}, \tag{6.2.27}$$

and then we obtain the solutions:

$$\begin{aligned} B &= N^{-(l-2)/\nabla} E = N^{(p+1)/\nabla} \\ e^{\phi} &= N^{a(D-2)/\nabla} F_{i_1 \dots i_p} = \pm\sqrt{\frac{2(D-2)}{\nabla}}\frac{d}{dr}N^{-1}, \end{aligned} \tag{6.2.28}$$

where we have defined

$$\begin{aligned} \nabla &\equiv (p+1)(d-2) + \frac{1}{2}a^2(D-2) \\ N &\equiv 1 + \frac{1}{d-2}\sqrt{\frac{\Delta}{2(D-2)}}\frac{|Q|}{r^{d-2}}. \end{aligned} \tag{6.2.29}$$

The symmetry in this theory is $ISO(1,p) \times SO(d)$.

In the case of the dual field $\tilde{F}$, the coupling is a $(D-p-4)$-brane, and the solution can be obtained by the replacements $a \to -a,\ p \to D-p-4,\ d \to p+3$.

Next we consaider the more general case such $\Psi \neq 0$. In this case, we also assume $ISO(1,p) \times SO(d)$:

$$C = B, \quad e^{\Psi} = B^{p+2}E^{d-2} \tag{6.2.30}$$

Using Einstein's and dilaton's equations, we can construct the equations for Ψ only

$$\begin{aligned} \frac{d}{du}\left(\ln\frac{de^{\Psi}}{du}\right) &= \frac{1}{u} \\ u &= -\frac{1}{(d-2)r^{d-2}} \end{aligned} \tag{6.2.31}$$

This equations are easily solved, and if we impose asymptotic flatness as a boundary condition, we obtain

$$\Psi = \ln\left(1 - \left(\frac{u}{u_0}\right)^2\right) = \ln\left(1 - \left(\frac{r_0}{r}\right)^{2(d-2)}\right). \tag{6.2.32}$$

Substituting this Ψ into the dilaton and Einstein equations, we find B and E:

$$\begin{aligned}\ln B &= -\frac{(d-2)}{a(D-2)}\phi + \ln\frac{\left(1-(r_0/r)^{(d-2)}\right)}{\left(1+(r_0/r)^{(d-2)}\right)}\\ \ln E &= -\frac{p}{(d-2)}\ln\left(1-\left(\frac{r_0}{r}\right)^{(d-2)}\right)+\frac{(p+2)}{(d-2)}\ln\left(1+\left(\frac{r_0}{r}\right)^{(d-2)}\right)+\frac{(p+1)}{a(D-2)}\phi\end{aligned} \tag{6.2.33}$$

Also by substituting these equations into the dilaton equation and solving

$$e^{\phi} = e^{\frac{(d-2)}{2\nabla}T(v,c_\phi)}e^{-\frac{2(p+1)a(D-2)}{\nabla}\frac{v}{u_0}}, \tag{6.2.34}$$

where we have defined

$$\begin{aligned}v &\equiv \frac{u_0}{2}\ln\left(\frac{1+(u/u_0)}{1-(u/u_0)}\right)\\ c_\phi &\equiv \frac{\nabla Q^2}{D-2}.\end{aligned} \tag{6.2.35}$$

and $T(v,c)$ is a differential equation

$$\left(\frac{dT}{dv}\right)^2 = 2ce^T + \text{const.} \tag{6.2.36}$$

These are the general p-brane solution.

6.2.1 Brane Solutions for 11D Supergravity Theory

Eleven-dimensional supergravity theory has 3-form gauge fields and do not contain dilaton. The Chern-Simons term was not taken into account, but since the contribution of this term is ineffective because of the ISO(p)×SO(d) symmetry assumption, the results of the previous discussions can be used. Therefore, since there are 2-brane solutions and no dilaton, $a = 0$. Substituting $D = 11, d = 8, p = 2, a = 0$ into the previous result, we obtain $\nabla = 18$, so

$$ds^2 = N_2^{-2/3}(-dt^2 + dx_1^2 + dx_2^2) + N_2^{1/3}(dy_1^2 + ... + dy_8^2), \tag{6.2.37}$$

where

$$F_{t12a} = \pm\frac{\partial}{\partial y_a}N_2^{-1}, i = 1, ..., 8 \tag{6.2.38}$$

$$N_2 = 1 + \frac{|Q|}{6r^6}. \tag{6.2.39}$$

This solution is called the M2 brane solution.

The solution for the dual field strength $\tilde{F}$ is 5-brane, and substituting $D = 11, d = 5, p = 5, a = 0$, we obtain

$$ds^2 = N_5^{-1/3}(-dt^2 + dx_1^2 + ... + dx_5^2) + N_5^{2/3}(dy_1^2 + ... + dy_5^2), \tag{6.2.40}$$

where

$$\tilde{F}_{1...6a} = \frac{\partial}{\partial y_a} N_5^{-1} \tag{6.2.41}$$

$$N_5 = 1 + \frac{|Q|}{3r^3}. \tag{6.2.42}$$

Other solutions to the 11D supergravity theory are the pp-wave solutions, K-K monopole solutions, and $AdS_4 \times S^7$ solutions.

6.2.2 Brane Solutions for 10D Supergravity Theory

Since there are two types of 10D supergravity theories, we investigate them separately.

TypeIIA supergravity theory

There are $p = 2, 4, 6$, and 8-rank field strengths in the R×R sector, including the strength of the dual fields, and $p = 3$ and 7-rank field strengths in the NS×NS sector. The bosonic action of TypeIIA is

$$\begin{aligned} L^B = \frac{1}{2\kappa_{10}^2}[eR(\omega(e)) &- \frac{1}{12} e e^{\sigma/2} \hat{F}_{\mu_1 \dots \mu_4} \hat{F}^{\mu_1 \dots \mu_4} - \frac{1}{3} e e^{-\sigma} F_{\mu_1 \dots \mu_3} F^{\mu_1 \dots \mu_3} \\ &- \frac{1}{4} e e^{3\sigma/4} F_{\mu_1 \mu_2} F^{\mu_1 \mu_2} - \frac{1}{2} e \partial_\mu \sigma \partial^\mu \sigma \\ &+ \frac{1}{2(12)^2} \epsilon^{\mu_1 \dots \mu_{10}} F_{\mu_1 \dots \mu_4} F_{\mu_5 \dots \mu_8} A_{\mu_9 \mu_{10}}] \end{aligned} \tag{6.2.43}$$

From this action, the dilaton coefficients a are for gauge fields with $p = 1, 2$ and 3,

$$a_{p+2} = \eta \frac{(3-p)}{2}, \eta = \begin{cases} 1, & \text{R - R .} \\ -1 & \text{N S - N S} \end{cases} \tag{6.2.44}$$

Thus, $D = 10,\ p = 0, ..., 6$, and using a_{p+2}, we find the brane solutions using the results of the previous section. For all p in this case, $\nabla = 16$, and the brane solution is given by

$$\begin{aligned} ds^2 =& N_p^{-(7-p)/8} \left(-(dt)^2 + (dx_1)^2 + \cdots + (dx_p)^2\right) \\ &+ N_p^{(p+1)/8} \, (dy_1)^2 + \cdots + (dy_d)^2) \end{aligned} \tag{6.2.45}$$

where

$$N_p = 1 + \frac{|Q|}{(7-p)r^{7-p}} \tag{6.2.46}$$

$$A_{1\ldots P+1} = \pm(N_p^{-1} - 1). \tag{6.2.47}$$

Since the gauge field in the NS×NS sector is $p = 2, 6$, there are 1-brane and 5-brane solutions. The 1-brane is coupled to the 2-form gauge field and is a string solution. In other words, string theory is a theory that describes the motion of a 1-brane solution coupled to a 2-form gauge field (gravity). This 1-brane is called the FI brane. The 5-brane is also called the NS5-brane. Since there are $p = 1, 3, 5, 7$-form gauge fields in the R×R sector, there are $p = 0, 2, 4, 6$-branes. The brane that couples to the R×R sector gauge field is called the Dp-brane. Here, the 8-brane has not been derived, but is included in the framework of TypeIIA theory. Since the 8-brane couples to the 9-form gauge field, the action is

$$\int d^{10}x \det e\left(F_{\mu_1\cdots\mu_{10}}F^{\mu_1\cdots\mu_{10}}\right). \tag{6.2.48}$$

If we take the dual of the strength of this field, then

$$F_{\mu_1\cdots\mu_{10}} = \epsilon_{\mu_1\cdots\mu_{10}}c. \tag{6.2.49}$$

and so this action is a constant. In other words, by including the cosmological constant term, we can derive the 8-brane. This theory is called Ramond theory.

TypeIIB supergravity theory

Since the only difference between IIA and IIB is how the chirality of the R sector is chosen by the GSO projection NS×NS sector is the same as in 2A theory. The R×R sector contains $p = 0, 2, 4, 6, 8$-form gauge fields including its dual. Therefore, there are $p = -1, 1, 3, 5, 7$-brane solutions [14]. As well as the TypeIIA theory, we find

$$ds^2 = (N_p)^{-(7-p)8}\left(-(dt)^2 + (dx_1)^2 + \cdots + (dx_p)^2\right) + (N_p)^{(p+1)/8}\left((dy_1)^2 + \cdots + (dy_d)^2\right), \tag{6.2.50}$$

where

$$N_p = 1 + \frac{|Q|}{(7-p)r^{7-p}} \tag{6.2.51}$$

[14] The brane for $p = -1$ is just a 0 dimensional object because p represents only a spacial dimension and do not include a time direction.

$$A_{1...p+1} = \pm N_p^{-1} - 1 \tag{6.2.52}$$

A brane that exists in the R×R sector is called a Dp-brane, as is TypeⅡA. The $p = -1$ brane is an instanton because it exists only at one point in spacetime. There are two types of 1-branes, the FI brane and the D1-brane. These two branes are coupled to each other with different 2-form gauge fields, and these two gauge fields are complex conjugate. Therefore, the two branes form a SU(1,1) doublet. The D9 brane has not been derived, but the it is just a spacetime.

6.3 Brane Dynamics

The brane solutions were obtained by solving the supergravity equations. For the 11-dimensional supergravity theory, there are M5-branes and M2 branes. For the 10D supergravity theory, there were F1 brane, NS5 brane, and D branes. From the point of view of supergravity theory, string theory is a theory that describes the motion of the 1-brane solution, and its quantum corrections describe the other brane solutions. In the following we consider the motion of brane solutions. First, we consider the dynamics of brane solutions by the NS×NS sector, and then we consider brane solutions for the R×R sector.

6.3.1 Bosonic Brane

As in string theory, we consider the case of bosons only. The construction of the action can be done by generalizing the construction method of string theory. Let the background spacetime manifold be $\mathcal{M}$ and its coordinates be $X^{\underline{n}}, \underline{n} = 0, ..., D-1$. The p-brane moves through $\mathcal{M}$, creating a world volume of dimension $p+1$. Let this manifold be M and its coordinates be $\xi^m,\ m = 0, ..., p$, and the index of the tangent space coordinates of M be $a, b, c, ...$ and the tangent space coordinates of $\mathcal{M}$ are $\underline{a}$,$\underline{b}$,$\underline{c}$, ... Since the world volume created by the p-brane is given by $X^{\underline{n}}(\xi^n)$ and its action is given by the volume

$$S = -T \int d^{p+1}\xi \sqrt{-\det\{g_{mn}\}} \tag{6.3.1}$$

$$g_{mn} = \partial_n X^{\underline{n}} \partial_m X^{\underline{m}} g_{\underline{mn}}, \tag{6.3.2}$$

weher T is the tension of the brane. This action has the reparameterization invariant on each of the manifolds M and $\mathcal{M}$.

$$X^{\underline{m}} \to X^{\underline{m}'}(X^{\underline{n}})\xi^n \to \xi^n \tag{6.3.3}$$

$$\xi^n \to \xi'^n(\xi) X'^{\underline{m}} = X^{\underline{m}}(\xi) \tag{6.3.4}$$

Also, since the p-brane couples to the p+1-form gauge field

$$g \int A \tag{6.3.5}$$

can be added. This term has gauge symmetry in addition to the previous symmetry. The number of parameters in the world sheet reparameterisation is p+1, and we take the static gauge using this local symmetry. As $\underline{n} = (n, n')$.

$$X^n(\xi) = \xi^n, \quad n = 0, ..., p \tag{6.3.6}$$

In other words, we divide the background spacetime into coordinates parallel to the brane and coordinates orthogonal to the brane. The dimension of the direction perpendicular to the brane is $D - p - 1$, and the field depends on $X^{n'}$, which followed from the assumption used to find the brane solution in the previous section and determines the position of the brane. Also, due to the gauge fixation by the p-brane, the translational symmetry is broken and $X^{n'}$ is regarded as the corresponding Gold boson.

6.3.2 Superbrane

In order to impose supersymmetry on bosonic p-branes, we use a superspace formalism. Let M be the world volume created by the superp-brane, the coordinates of M be $\xi^n, n = 0, ...$, the coordinates of $\underline{M}$ be $Z^{\underline{N}} = (X^{\underline{n}}, \theta^{\underline{\alpha}}$, and the tangent space coordinates be $\underline{N}, \underline{M}$, $\underline{A}, \underline{B}$ for the tangent space coordinates. The motion of the brane is given by $Z^{\underline{N}}(\xi^n)$. The brane that is given only by $Z^{\underline{N}}(\xi^n)$ is called a TypeI brane. In general, there are vector fields and so on in the world volume, and SO(1,p) symmetry is nontrivial. An arbitrary superbrane action is

$$\begin{aligned} S &= S_1 + S_2 \\ S_1 &= -T \int d^{p+1}\xi \sqrt{-\,\mathrm{detg}_{mn}} + \cdots \end{aligned} \tag{6.3.7}$$

$$g_{mn} = \partial_n Z^N \partial_m Z^{\underline{M}} g_{\underline{N}M} g_{NM} = E^{\underline{a}}_{\underline{N}} E^{\underline{b}}_{\underline{M}} \eta_{\underline{a}\underline{b}} \tag{6.3.8}$$

where, $g_{\underline{NM}}$ is not the metric for the background spacetime, since the super field E has a Grassmann odd component. The coupling term with the gauge is

$$S_2 = q \int d^{p+1}\xi \epsilon^{n_1 \cdots n_{p+1}} \partial_{n_1} X^{\underline{m}_1} \ldots \partial_{n_{p+1}} X^{\underline{m}_{p+1}} A_{\underline{m}_1 \cdots \underline{m}_{p+1}} + \cdots \tag{6.3.9}$$

Since supersymmetry is imposed on the background spacetime, this action is invariant under reparameterisation and local supersymmetry on $\mathcal{M}$ and reparameterisation on M. In addition, it is supersymmetric in the world volume as well as in the string theory of GS formalism, so it has κ symmetry. Due to the κ symmetry, half of the spinor component is zero by gauge fixing. Such a gauge fixation is called a super-static gauge. Since the theory has supersymmetry on the world volume, the boson and fermion degrees of freedom coincide.

Eleven-dimensional supergravity theory

The branes in this theory are M2 and M5 branes. For the M2 brane, the number of scalar fields is 11-2-1=8 and The number of fermionic fields is 32/2/2=8 due to the super-static gauge and the on-shell condition. Therefore, the degrees of freedom coincide. Also, there are no other fields in the world volume, so it is a TypeI brane.

On the other hand, for M5 branes, the boson field is 11-5-1=5 and the fermion field is 32/2/2=8. Therefore, the boson has 3 degrees of freedom less. This is consistent with the degrees of freedom if there is a second-order self-dual antisymmetric tensor in the 6-dimensional brane volume. Therefore, the M5 brane is not a TypeI brane.

TypeIIA Supergravity Theory

The branes that exist in this theory are $p = 0, 1, 2, 4, 5, 6$-branes. The $p = 1, 5$-brane couples to the gauge field in the NS×NS sector, while the $p = 0, 2, 4, 6$ brane couples to the gauge field in the R×R sector. By counting the degrees of freedom, F1 brane is the TypeI brane, and is called the fundamental string in particular.

NS5 branes are obtained by compactifying the M5-branes in 11D supergravity theory. The remaining $p = 0, 2, 4,$ and 6-branes are D-branes, which require a gauge field in the world volume.

TypeIIB supergravity theory

NS×NS sector is the same as TypeIIA. The branes that couple to the gauge field in the R×R sector are $p = -1, 1, 3, 5, 7$-branes, all D-branes.

6.3.3 TypeI Brane

First, we consider the TypeI brane. The corresponding branes are a M2 brane in 11 dimension and a F1 brane of dimension 10. The motion of these branes is given only by $Z^{\underline{N}}(X^{\underline{n}}, \Theta^{\underline{\alpha}})$. The action is

$$S = S_1 + S_2 \tag{6.3.10}$$

$$\begin{aligned} S_1 &= -T \int d^{p+1}\xi \sqrt{-\det g_{mn}} \\ S_2 &= -\frac{T}{(p+1)!} \int d^{p+1}\xi \epsilon^{n_1 \cdots n_{p+1}} \partial_{n_1} Z^{\underline{M}_1} E_{\underline{M}_1}^{A_1} \ldots \partial_{n_{p+1}} Z^{M_{p+1}} E_{\underline{M}_{p+1}}^{A_{p+1}} B_{\underline{A}_1 \cdots A_{p+1}} \end{aligned} \tag{6.3.11}$$

$$g_{mn} = \partial_n Z^N \partial_m Z^{\underline{M}} g_{NM} \tag{6.3.12}$$

$$g_{\underline{N}M} = E^{\underline{a}}_{\underline{N}} E^{\underline{b}}_{\underline{M}} \eta_{\underline{a}b} \tag{6.3.13}$$

$$B = \frac{1}{(p+1)!} E^{A_1} \wedge \cdots \wedge E^{A_{p+1}} B_{\underline{A}_1 \cdots \underline{A}_{p+1}} = dZ^{N_{p+1}} \wedge \cdots \wedge dZ^{N_1} B_{\underline{N}_1 \cdots N_{p+1}} \tag{6.3.14}$$

This action has a reparameterisation invariance in superspace. That is, it is invariant under spacetime reparameterisation and local supersymmetric transformations of spacetime. This action describes only the motion of the brane and does not determine the geometry of the background spacetime. In addition to the field E and the spin connection, the field strength for the gauge field B

$$H = dB \tag{6.3.15}$$

contributes to the geometry of the background. The curvature and torsion of the background superspace satisfy the Bianchi identities, and likewise the field strength satisfies the Bianchi identities. Consider the case where the background superspace is flat. The components of torsion and field strength are

$$\begin{aligned} T^{\underline{a}}_{\alpha\underline{\beta}} &= i\left(\gamma^{\underline{a}} C^{-1}\right)_{\alpha B} \\ H_{\underline{\alpha}\underline{\beta}\underline{a}_1 \cdots \underline{a}_p} &= -i(-1)^{\frac{1}{4}p(p-1)} \left(\gamma_{\underline{a}_1 \cdots a_p} C^{-1}\right)_{\underline{\alpha}\underline{\beta}}, \end{aligned} \tag{6.3.16}$$

the other components are vanished. Since the field strength is a complete form,

$$dH = 0 \tag{6.3.17}$$

is satisfied. From this equation,

$$d\overline{\Theta}\gamma_{\underline{a}} d\Theta d\overline{\Theta}\gamma^{\underline{a}b_1 \cdots b_{ip-1}} d\Theta = 0 \tag{6.3.18}$$

must hold regardless of the type of spin. In other words, a TypeI brane exists if the spinor satisfies this condition. For example, if $p = 2,\ D = 11$ for the Majorana spinor, and $p = 1,\ D = 10$ for the Majorana-Weyl spinors.

Next we consider the κ symmetry of the action.

$$\begin{aligned} \delta Z^N E^{\underline{a}}_{\underline{N}} &= 0 \\ \delta Z^N E^{\underline{\alpha}}_{\underline{N}} &= (1+\Gamma)^{\underline{\alpha}}_{\underline{\beta}} \kappa^{\underline{\beta}}, \end{aligned} \tag{6.3.19}$$

where

$$\Gamma = \frac{(-1)^{\frac{1}{4}(p-2)(p-1)}}{(p+1)!\sqrt{-\det g_{im}}} \epsilon^{n_1 \cdots n_{p+1}} \partial_{n_1} Z^{N_1} E^{\underline{a}_1}_{\underline{N}_1} \ldots \partial_{n_{p+1}} Z^{N_{p+1}} E^{\underline{a}_{p+1}}_{\underline{N}_{p+1}} \gamma_{a_1 \cdots a_{p+1}} \tag{6.3.20}$$

Since this matrix Γ satisfies $\Gamma^2 = 1$, $(1 \pm \Gamma)/2$ are the projection operator. There is a $(1+\Gamma)\kappa$ in the transformation law, which indicates that only half of the components of the parameter κ contribute to the transformation law. In other words, always half of the spinor component can be removed by gauge fixing of the κ symmetry.

6.3.4 D-Brane

We consider D-brane that exist in 10 dimensions. In the case of TypeI branes, the degrees of freedom coincide with $X^{\underline{n}}$ and $\theta^{\underline{\alpha}}$ in the on-shell. However, the D-brane coincides by introducing a gauge field A_n on the world volume. Therefore, we extend the TypeI brane action to include a gauge field on the world volume. The action was given by Dirac [86] and, Born and Infeld [26]:

$$S_{DBI} = -T \int d^{p+1}\xi \sqrt{-\det(g_{nm} + cF_{nm})}. \tag{6.3.21}$$

This action is invariant under general coordinate transformations. Expanding the action, we obtain

$$\sqrt{-\det(g_{nm} + cF_{nm})} = \sqrt{-\det(g_{nm})}\left(1 + \frac{c^2}{4}F_{nm}F^{nm} - \frac{c^4}{8}F^n_pF^p_qF^q_rF^r_n + \frac{c^4}{32}(F_{nm}F^{nm})^2 + ...\right). \tag{6.3.22}$$

In addition to the usual Maxwell term, there is a higher order term.

Actions such as κ symmetry are

$$S = S_1 + S_2 \tag{6.3.23}$$

$$S_1 = -T_p \int d^{p+1}\xi e^{-\phi} \sqrt{-\det(g_{nm} + \mathcal{F}_{mn})}, \tag{6.3.24}$$

where ϕ is the dilaton and

$$\begin{aligned} \mathcal{F}_{nm} &= 2\pi\alpha' F_{nm} + B_{nm} \\ B_{nm} &= (\partial X^n/\partial\xi^n)(\partial X^{\underline{m}}/\partial\xi^m). \end{aligned} \tag{6.3.25}$$

B_{nm} is a pullback of the massless 2-form belonging to the NS×NS sector in the background spacetime. Also, S_1 contains only all the fields in the NS×NS sector, which is the same in TypeIIA and TypeIIB. Since A_n is a U(1) gauge field, the transformation rule is $\delta A_n = \partial_n\Lambda$. On the other hand, $\delta B_{nm} = \partial_{[n}\Lambda_{m]}$, and we can take the gauge transformation of $\mathcal{F}_{mm}$ to zero.

The coupling term between the brane and the gauge field is C as a pullback of the gauge field in the R×R sector

$$S_2 = q \int \left\{C \wedge e^{\mathcal{F}}\right\}_{p+1} \wedge \mathcal{G}, \tag{6.3.26}$$

where q is the charge that the brane has, and $\mathcal{G}$ is the quantity created by the Riemann curvature.

6.3.5 Non-Abelian DBI Theory

The DBI action (6.3.21) describes the dynamics of open string whose ends exist on a single D-brane. The DBI action for abelian gauge leads to abelian gauge theory. We extend the single brane action (6.3.21) to the stack of D-branes. We consider the stacks of N D-branes. This configuration corresponds to the Chan-Paton factor. Thus the open string is a representation of a non-Abelian gauge group.

But it is ambigious for non-abelian gauge. In [34], the author proposed the DBI action for the non-abelian gauge. We call this action the non-abelian DBI (NDBI) action.

Non-Local string theory which means there a massive states tower implies that the low energy effective theory for massless modes is an infinite power series of all order in Regge slope α'. This applies to the tree-level Lagrangian $\mathcal{L}_{\text{eff}}$ for the gauge fields in the open bosonic or type I string theory. In the abelian case, all terms in the action depending on the field strength F_{MN} without its derivatives sum up into the Dirac-Born-Infeld (BI) action:

$$\mathcal{L}_{DBI} = -T_p e^{-\phi}\sqrt{\det(g_{MN} + T^{-1}F_{MN})}, \quad T^{-1} = 2\pi\alpha'. \tag{6.3.27}$$

where $M, N, ..$ stand for the $(p+1)$-dimensional world volume indices, g_{MN} is the pull-back of the bulk metric to the D-brane and ϕ is the 10 dimensional dilaton field. T_p is the brane tension which satisfies

$$T_p = \frac{2\pi}{l_s^{p+1}} = \frac{2\pi}{(2\pi\alpha'^{1/2})^{p+1}}$$

where l_s is the string length. The ablian case corresponds to the open string on a single D-brane derivative corrections have been discussed.

In non-abelian case which corresponts to a stuck of D-brane, the tree-level effective theory from the open string theory can be expressed as an expansion of the field strength and its covariant derivatives:

$$\mathcal{L}_{\text{eff}} = \text{tr}\left(a_0 F^2 + a_1 F D^2 F + a_2 F^4 + a_3 F^2 D^2 F + \cdots\right) \equiv \mathcal{L}_F + O(DF) \tag{6.3.28}$$

where $F_{MN} = \partial_M A_N - \partial_N A_M + i[A_M, A_N]$ and we have defined L_F is the part of containing no covariant derivatives. In the following discussions, we address the L_F corresponding to the DBI action.

In the abelian case, the derivative terms and non-derivative terms can be separated with no unambiguity. Whereas in the non-abelian case, this is not true since due to the relation $[D_M, D_N]F_{KL} = [F_{MN}, F_{KL}]$, some derivative terms may become non-derivative terms and vice versa. In order to resolve this unambiguity, we deal with all $[F, F]$ terms as DF terms. In other words, $[F, F]$ terms are not in L_F. Futermore, by assuming DF terms much smaller than the magnitude of F, we focus on $\mathcal{L}_F$ part. We call $\mathcal{L}_F$ Non-abelian Dirac-Born-Infeld (NDBI) action.

We streightforwardly extend the DBI action to NDBI action like

$$\mathcal{L}_F \equiv \mathcal{L}_{\mathrm{NDBI}} = -T_p e^{-\phi}\,\mathrm{Str}\sqrt{\det(g_{MN} + T^{-1}F_{MN})}, \tag{6.3.29}$$

where Str is the symmetric trace in the fundamental representation of the gauge group and defined by

$$\mathrm{str}(T_1 \cdots T_n) \equiv \frac{1}{n!}\,\mathrm{tr}[T_1 \cdots T_n + (\text{permutations})]. \tag{6.3.30}$$

T_p is the brane tension and defined by $T_p = 2\pi/l_s^{p+1}$ for the type II string theories. Although we can also consider the antisymmetric trace part, it gives the sum of traces of odd powers of F which always contain a factor of $[F, F]$. Thus anti-symmetric part is not included in NDBI Lagrangian.

In the α'^4order, the action becomes

$$\begin{aligned} S_{\mathrm{NDBI}} = &- T_p \int d^{p+1}\xi e^{-\phi}\frac{(2\pi\alpha')^2}{4}\,\mathrm{tr}\Big[F_{MN}^2 \\ &- \frac{(2\pi\alpha')^2}{3}\bigg(F_{MN}F_{NL}F_{KM}F_{LK} + \frac{1}{2}F_{MN}F_{NL}F_{LK}F_{KM} \\ &- \frac{1}{4}F_{MN}F_{NM}F_{LK}F_{KL} - \frac{1}{8}F_{MN}F_{LK}F_{MN}F_{LK}\bigg) + \mathcal{O}(\alpha'^4)\Big] \end{aligned} \tag{6.3.31}$$

where we have defined $F_{MN}F_{MN} \equiv g^{MK}g^{NL}F_{MN}F_{KL}$ for the simplicity. We adopt $\mathrm{tr}\big(T^aT^b\big) = \delta^{ab}$ as the normalization of gauge group generator.

6.3.6 M-theory

In 11-dimensional supergravity theories, there were M2 brane and M5 brane. The M2 brane is a TypeI brane because the degrees of freedom coincided only with scalar and fermionic fields and did not require a vector field on the world volume. However, the M5 brane, coupled with its dual field strength, requires a vector field in the world volume. Because of the translational symmetry and supersymmetry breaking, the fields of the M5 brane have a self-dual second-rank antisymmetric gauge field in the world volume in addition to the 5 scalar fields and the 16-component spinors as Goldstone particles. We decompose the bosonic index of the M5 brane field into the brane direction and the direction orthogonal to the brane:

$$\underline{n} = (n, n'). \tag{6.3.32}$$

With this gauge fixation, SO(1,10) is decomposed into SO(1,5)×SO(5). The spin group Spin(1,10) becomes Spin(1,5)×Spin(5). From Spin(5)∼USp(4), the M5 brane has Spin(1,5)×USp(4) symmetry.

The motion of the M5 brane is described by the embedded coordinates $X^{\underline{n}}$ and $\Theta^{\underline{\alpha}}$ plus the 2-form gauge field B_{mn}. Taking the stationary gauge, $X^n = \xi^n,\ n = 0, ..., 5$, and there are five scalar fields $X^{n'}$. These scalar fields are the singlet of SO(5) and Spin(1,5), and is represented by the 2-rank antisymmetric tensor representation ϕ_{ij} of USp(4). Also, B_{mn} is a singlet of SO(5) because it exists on the world volume. Consider the decomposition of the spinor $\Theta^{\underline{\alpha}}$. Since it is 11-dimensional, it is a 32-component spinor, and if we divide the index into Spin(1,5) and Spin(5), then

$$\underline{\alpha} = (\alpha, \alpha'). \tag{6.3.33}$$

The α and α' have 16 components each. Since α' is an internal symmetry index, we use the following notation

$$\alpha \to \alpha, \quad \alpha' \to i. \tag{6.3.34}$$

$\alpha = 1, 2, 3, 4$ is a index of Spin(1,5) and $i = 1, 2, 3, 4$ is a index of USp(4). Using κ symmetry, we take $\Theta^\alpha = 0$. That is, the remaining spinor components are Θ^i_α. This fixation eliminates the spinors on the M5 branes and leaves only the orthogonal spinors. Since the orthogonal direction is six-dimensional, the spinors are symplectic-Majorana with the index USp(4). It can also be taken to be a Weyl spinor since it is even dimensional. Therefore, the spinor has eight components. The equations of motion for the M5 branes with taking a super stationary gauge fixation are

$$\begin{aligned} G^{mn}\nabla_m\nabla_n X^{a'} &= 0 \\ G^{mn}\nabla_m H_{npq} &= 0 \end{aligned} \tag{6.3.35}$$

Here, $a, b, c = 0, ..., 5$ are the indexes of the tangent space to the world volume, $a', b' = 6, ..., 10$ are the indexes of the tangent space in the orthogonal direction. If the background spacetime is flat and the stationary gauge is taken, the world volume metric is

$$g_{mn} = \eta_{mn} + \partial_m X^{a'} \partial_n X^{b'} \delta_{a'b'}. \tag{6.3.36}$$

The covariant derivative is defined by the Levitivita connection for the vector field T_n

$$\begin{aligned} \nabla_m T_n &= \partial_m T_n - \Gamma^p_{mn} T_p \\ \Gamma^p_{mn} &= \partial_m \partial_n X^{a'} \partial_r X^{b'} g^{rs} \delta_{a'b'} \end{aligned} \tag{6.3.37}$$

The G^{mn} is not the inverse of the metric, but contains the contribution of the gauge field:

$$\begin{aligned} G^{m} &= \left(e^{-1}\right)_{c}^{m} \eta^{ca} m_{a}^{d} m_{d}^{b} \left(e^{-1}\right)_{b}^{n} \\ m_{a}^{b} &= \delta_{a}^{b} - 2h_{acd}h^{bcd} \\ H_{mnp} &= e_{m}^{a} e_{n}^{b} e_{p}^{c} \left(m^{-1}\right)_{c}^{d} h_{abd} \end{aligned} \tag{6.3.38}$$

H_{mnp} is the field strength of the background gauge field B_{ab} and is not self-dual, but H_{abc} induced into the world volume is self-dual in the world volume. These equations of motion and fields can be derived by considering the superembedding form.

6.4 String on D-Brane

The string we have considered in section 4 only with the condition satisfying the Poincaré invariance. However, it turns out that the boundary term can also be satisfied by using the Direchlet boundary condition, where the end of the string is fixed. This boundary condition breaks the translational symmetry.

6.4.1 Bosonic D-Brane

Classical String

First, for simplicity, we consider the bosonic string.

$$S = -\frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{-g} g^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x^v \eta_{\mu\nu} \tag{6.4.1}$$

By taking a variant of this action, we obtain the constraints, equations of motion, and boundary terms:

$$\begin{aligned} T_{\alpha\beta} &= 0 \\ \delta A / \delta \left(\partial_\alpha x^\mu\right) &= 0 \\ \partial_\alpha \left(\delta x^\mu \frac{\delta A}{\delta \left(\partial_\alpha \cdot x^\mu\right)}\right) &= 0. \end{aligned} \tag{6.4.2}$$

As a condition for satisfying this boundary term, we can take the Direchlet boundary condition $x^m u|pi_0 =$ const.

Now consider the p-brane and let $\mu = 0, ..., p$. In this case, the orthogonal direction is $\mu = p+1, ...D-1$. If both ends of the open string are fixed to p-brane, the orthogonal motion of the string is fixed. The boundary conditions correspond to the Neumann boundary condition (N) in the brane direction and the

Direchlet boundary condition (D) in the orthogonal direction:

$$\begin{aligned} &\frac{\partial x^{\mu}}{\partial \sigma}(\tau,0)=0=\frac{\partial x^{\mu}}{\partial \sigma}(\tau,\pi); \qquad \mu=0,1\ldots,p \\ &\frac{\partial x''}{\partial \tau}(\tau,0)=0=\frac{\partial x''}{\partial \tau}(\tau,\pi); \qquad \mu=p+1,\ldots,D-1 \end{aligned} \tag{6.4.3}$$

Since the orthogonal direction coordinates represent the position of the brane, we denote its coordinates as a^{μ}. If the ends of a string are on the same brane, the Direchlet condition is

$$x''(\tau,0)=a^{\mu}=x^{\mu}(\tau,\pi), \quad \mu=p+1\ldots\ldots D-1. \tag{6.4.4}$$

The existence of this p-brane breaks the Lorentz symmetry SO(1,D-1) into SO(1,p)×SO(D-p-1). Also, if there are two parallel branes and the ends of the string are on each brane:

$$x''(\tau,0)=a_1'', \quad x^{\mu}(\tau,\pi)=a_2^{\mu}, \mu=p+1\ldots.D-1 \tag{6.4.5}$$

We can also consider other cases where the two branes are not parallel. For example, if four branes D_1 exist in the 01236 direction and four branes D_2 exist in the 01456 direction, the boundary condition is as follows:

$$\begin{aligned} &\frac{\partial x^{\mu}}{\partial \sigma}(\tau,0)=0, \qquad \mu=0,1,2,3,6 \\ &\frac{\partial x^{\mu}}{\partial \tau}(\tau,0)=0; \qquad \mu=4.5.7\ldots,D-1 \end{aligned} \tag{6.4.6}$$

$$\begin{aligned} &\frac{\partial x^{\mu}}{\partial \sigma}(\tau,\pi)=0, \qquad \mu=0,1,4,5,6 \\ &\frac{\partial x^{\mu}}{\partial \tau}(\tau,\pi)=0: \qquad \mu=2,3,7\ldots,D-1 \end{aligned} \tag{6.4.7}$$

In the following, we investigate the solutions to the general equations of motion. First, we introduce the distance $f^{\mu}(\tau)$ between the two branes:

$$x^{\mu}(\tau,\pi)-x^{\mu}(\tau,0)\equiv -f^{\mu}(\tau)\pi. \tag{6.4.8}$$

Also we transform the spacetime coordinates:

$$\hat{x}^{\mu}(\tau,\sigma)=x^{\mu}(\tau,\sigma)+f^{\mu}(\tau)\sigma. \tag{6.4.9}$$

The boundary condition for this field is

$$\hat{x}^{\mu}(\tau,0)=\hat{x}^{\mu}(\tau,\pi) \tag{6.4.10}$$

and can be treated as a closed string. From this condition, we can carry out Fourier expansion and rewrite it in the original field:

$$x^\mu(\tau,\sigma) = x_0^\mu(\tau) + \sum_{\substack{n\in Z\\ n\neq 0}} x_n^\mu(\tau)e^{in\sigma} + \sum_{r\in Z+\frac{1}{2}} y_r^\mu(\tau)e^{ir\sigma} + f^\mu(\tau)\sigma. \tag{6.4.11}$$

Substitute this into the equation of motion we obtain the solution:

$$\begin{aligned} x^\mu(\tau,\sigma) =& a^\mu(0) + k^\mu(0)z + l^\mu(0)\overline{z} + i\sqrt{\frac{\alpha}{2}}\sum_{\substack{n\in 2\\ n\neq 0}} \left(\alpha_n^\mu(0)e^{-inz} + \hat{\alpha}_n^\mu(0)e^{-in\overline{z}}\right)\frac{1}{n} \\ &+ i\sqrt{\frac{\alpha'}{2}}\sum_{r<z+\}} \left(c_r^\mu(0)e^{-ir\Sigma} + \hat{c}_r^\mu(0)e^{-ir\overline{z}}\right)\frac{1}{r} \end{aligned} \tag{6.4.12}$$

Since we have already obtained the solution wiith both ends of the string are (N) in a certain spatial direction, in the following we investigate the solution if Direchlet boundary conditions are included in the boundary conditions. In the following, we denote the direction having Direchlet boundary condition at $\sigma = 0,\ \pi$ as D_0 and D_π

First we consider the case $\mu \in D_0, \mu \in D_\pi$. This simply shows that both ends of the string are Direchlet boundary condition. From $a_2^\mu - a_1^\mu = -f^\mu\pi$, we have $\alpha_n^\mu = -\hat{\alpha}_n^\mu, c_r^\mu = \hat{c}_r^\mu = 0, k^\mu = -l^\mu$, so

$$x^\mu(\tau,\sigma) = a_1^\mu + \frac{1}{2\pi}(a_2^\mu - a_1^\mu)(z-\overline{z}) + i\sqrt{\frac{\alpha'}{2}}\Sigma\frac{1}{n}\alpha_n^\mu(e^{-inz} - e^{-in\overline{z}}) \tag{6.4.13}$$

From this solution, there is no central momentum because there is no term proportional to τ, and the center of mass is fixed.

Next we consider $\mu \in N_0, \mu \in D_\pi$ At this time, $k^\mu = l^\mu = \alpha_n^\mu = \hat{\alpha}_n^\mu = 0$, so

$$x^\mu(\tau,\sigma) = a_2^\mu + i\sqrt{\frac{\alpha'}{2}}\sum_{r\in Z+\frac{1}{2}}\frac{1}{r}c_r(0)\left(e^{-irz} + e^{-ir\overline{z}}\right) \tag{6.4.14}$$

Finally we consider $\mu \in N_0, \mu \in D_\pi$, then $k^\mu = l^\mu = \alpha_n^\mu = \hat{\alpha}_n^\mu = 0$, and so

$$x^\mu(\tau,\sigma) = a_1^\mu + i\sqrt{\frac{\alpha'}{2}}\sum_{r\in Z+\frac{1}{2}}\frac{1}{r}c_r(0)\left(e^{-irz} - e^{-ir\overline{z}}\right). \tag{6.4.15}$$

These solutions only consider the case where the two branes are parallel or orthogonal. In general, the boundary conditions for cases where the two branes intersect are more complicated. There are several realistic models, which called intersect D-brane model, are proposed and its theory is dual with the magnetized compactification model [78].

Quantum String

From the above discussions, we obtain the classical solutions. In order to consider the quantum theory, we calculate the Poisson brackets and replace them with commutation relations:

$$[\alpha_n^\mu, \alpha_m^v] = n\eta^{\mu v}\delta_{n+m,0}, \quad [c_r^\mu, c_s^v] = r\eta^{\mu v}\delta_{r+s,0}. \tag{6.4.16}$$

The constraints are

$$L_n\psi = 0 \quad n \geq 1, \quad (L_0 - a)\,\psi = 0, \tag{6.4.17}$$

where the contribution of the half-integer oscilations exists due to the boundary condition:

$$\begin{aligned} L_n =& \frac{1}{2}\sum_{\mu.v\in N-N or D-D}\sum_{m\in Z} : \alpha_m^\mu \alpha_{n-m}^v \eta_{\mu\nu} \\ &+ \frac{1}{2}\sum_{\mu,v\in N-D or D-N}\sum_{s\in Z+\frac{1}{2}} : c_s^\mu c_{n-s}^v \eta_{\mu\nu} : . \end{aligned} \tag{6.4.18}$$

Considering an N-N or D-D string described by integer modes only:

$$\psi = \left\{\phi\left(x^\mu\right) + \sum_{\nu\in N-N}\alpha_{-1}^\nu A_\nu + \sum_{\nu\in D-D}\alpha_{-1}^\nu \varphi_\nu(x^\mu) + ...\right\}\psi_0. \tag{6.4.19}$$

The x^μ in the D-D direction is a constant and since the brane is in a stationary gauge, so $x^\mu = \xi^\mu, \mu \in$ (N-N direction). Therefore, ϕ, A_μ, and φ^μ are the fields on the D-brane. Futhermore, since μ is split into D-D and N-N direction, the Lorentz group SO(1,D-1) is split and SO(D-p-1)×SO(1,p). Therefore, these fields on the brane have the internal symmetry of SO(D-p-1). If we impose the Virasoro condition on these states, we find that ϕ is a tachyon, A_μ is a massless vector on the world sheet, and φ_μ is $D-p-1$ massless scalas. We also obtain a Lorentz covariant gauge-fixing condition $\partial^\mu A_\mu = 0$ from $L\psi = 0$.

If there are N parallel D-branes labeled $i, j = 1, ..., N$, there are N^2 possible open strings. We denote $[i, j]$ when the end at $\sigma = 0$ is D_i and the end at $\sigma = \pi$ is D_j, If we consider an unoriented string such as a TypeI string, $[i, j] = [j, i]$. Also, If there are two ends on a same brane, $a_1^\mu = a_2^\mu$. This theory is the Yang-Mills theory of U(N) gauge group coupled to $26 - p - 1$ scalars.

6.4.2 Super D-brane

We extend the discussion to superstring theory. From section 4, κ-symmetry is important for finding the supra-D brane. In the RNS formalism, supersymmetry is introduced to the world sheet, and by imposing the GSO projection, the theory obtain κ symmetric. On the other hand in the GS formalism, κ symmetry is naturally introduced to the classical action. For this reason, we use the GS formalism in the following discussion. Also, for convenience, we treat the RNS formalism as the classification of the string spectrum.

Since the dimension of superstring theory is 10-dimensional, we take the spinors to be Majorana-Weyl. Ifn we take the gauge fixed condition $g_{\alpha\beta} = \eta_{\alpha\beta} e^{\phi}$, the boundary conditions are

$$\begin{aligned} l\delta x^{\mu} \left(\Pi_{1\mu} + i \left(\theta^1 \gamma_\mu \partial_0 \theta^1 - \theta^2 \gamma_\mu \partial_0 \theta^2 \right) \right) \\ + \left\{ -\Pi_{1\mu} \left(-i\overline{\theta}^j \gamma^\mu \delta\theta^j \right) - i\partial_0 x^\mu \left(\overline{\theta}^1 \gamma_\mu \delta\theta^1 - \overline{\theta}^2 \gamma_\mu \delta\theta^2 \right) \right. \\ + \overline{\theta}^1 \gamma^\mu \partial_0 \theta^1 \overline{\theta}^2 \gamma^v \delta\theta^2 - \overline{\theta}^1 \gamma^\mu \delta\theta^1 \overline{\theta}^2 \gamma_\mu \partial_0 \theta^2 \} = 0, \end{aligned} \tag{6.4.20}$$

where

$$\Pi_\alpha^\mu \equiv \partial_\alpha x^\mu - i\overline{\theta}^j \gamma^\mu \partial_\alpha \theta^j \tag{6.4.21}$$

We investigate the boundary conditions for each term. The first term is

$$\delta x^\mu \left(\Pi_{1\mu} + i \left(\theta^1 \gamma_\mu \partial_0 \theta^1 - \theta^2 \gamma_\mu \partial_0 \theta^2 \right) \right). \tag{6.4.22}$$

The first term in the bracket, $\delta x^\mu \partial_\alpha x_\mu$, equals the boundary condition of the bosonic string and vanishes in Direchlet boundary condition because the whole is proportional to δx^μ. Imposing Nuemann baoundary condition at $\sigma = 0$, we obtain

$$\left(\theta^1 \gamma^\mu \partial_0 \theta^1 - \theta^2 \gamma^\mu \partial_0 \theta^2 \right) - \left(\theta^1 \gamma^\mu \partial_1 \theta^1 + \theta^2 \gamma^\mu \partial_1 \theta^2 \right) = 0, \quad \mu \in N_{(1)} \tag{6.4.23}$$

The second term is

$$-\Pi_{1\mu} \left(-i\overline{\theta}^j \gamma^\mu \delta\theta^j \right) - i\partial_0 x^\mu \left(\overline{\theta}^1 \gamma_\mu \delta\theta^1 - \overline{\theta}^2 \gamma_\mu \delta\theta^2 \right) + \overline{\theta}^1 \gamma^\mu \partial_0 \theta^1 \overline{\theta}^2 \gamma^v \delta\theta^2 - \overline{\theta}^1 \gamma^\mu \delta\theta^1 \overline{\theta}^2 \gamma_\mu \partial_0 \theta^2 \tag{6.4.24}$$

The conditions at $\sigma = 0$ are

$$\begin{aligned} l\overline{\theta}^1 \gamma^\mu \delta\theta^1 + \overline{\theta}^2 \gamma^\mu \delta\theta^2 = 0, \qquad \mu \in D_0 \\ \overline{\theta}' \gamma^\mu \delta\theta' - \overline{\theta}^1 \gamma^\mu \delta\theta^2 = 0, \qquad \mu \in N_0 \end{aligned} \tag{6.4.25}$$

Since θ^i has an internal index i, let $\theta^2 = P_0 \theta^1$:

$$\begin{aligned} l\gamma^\mu + \hat{P}_0 \gamma^\mu P_0 = 0, \qquad \mu \in D_0 \\ \gamma^\mu - \hat{P}_0 \gamma^\mu P_0 = 0, \qquad \mu \in N_0 \end{aligned} \tag{6.4.26}$$

Here, we use the Majorana condition.

If we set the direction $\mu = 0, ..., p$ to Nuemann conditions and $P_0 = \gamma^{0 \cdots p}$, it is automatically satisfied.

Therefore, we only need to consider the first term. At $\sigma = 0$, the first term of Nuemann condition is vanished by imposing

$$\partial_\sigma \theta^2 = -P_0 \partial_\sigma \theta^1. \tag{6.4.27}$$

The same is true for $\sigma = \pi$. From the above, we find that the end of the open string exists on the D-brane and couples to the R×R sector.

Supersymmetry

From the point of view of the supergravity, we have seen that supersymmetry was broken by branes. To confirm this, we will investigate supersymmetric transformations at the boundary of an open string. The supersymmetric transformation is

$$\delta\theta^i = \epsilon^i. \tag{6.4.28}$$

If we carry out this to the boundary conditions,

$$\epsilon' = P_0^{-1} P_\pi \epsilon^1, \tag{6.4.29}$$

where ϵ^1 is a 10-dimensional Majorana Weyl spinor, so it has 16 components. From this condition, if the branes are parallel, $P_0 = P_\pi$, so the condition is obviously satisfied. However, if the branes are different, the conditional is nontrivial, so the parameter degrees of freedom are halved to 8 components. Applying the supersymmetric transformation to the boundary of x^μ gives

$$\begin{aligned} \delta\, x^\mu\big|_{\sigma=0} &= i\epsilon^{-1}\left[\gamma^\mu + \hat{P}_0\gamma^\mu P_0\right]\theta^1\big|_{\sigma=0} &&= 0, \quad \mu \in D_0 \\ \delta\partial_\sigma\, x^\mu\big|_{\sigma=0} &= i\epsilon^{-1}\left[\gamma^\mu - \hat{P}_0\gamma^\mu P_0\right]\partial_\sigma\,\theta^1\big|_{\sigma=0} &&= 0, \quad \mu \in N_0 \end{aligned} \tag{6.4.30}$$

These conditions are satisfied by setting $P_0 = \gamma^{0\cdots p}$.

Next, we consider the κ symmetry. If we carry out the transformation to the boundary condition,

$$P_0\gamma^\mu \Pi_{\mu\alpha}\kappa^{1\alpha} = \gamma^\mu \Pi_{\mu\alpha}\kappa^{2\alpha}. \tag{6.4.31}$$

This condition is satisfied if

$$\kappa^{10} = eP_0\kappa^{20}, \kappa^{11} = -eP_0\kappa^{21}. \tag{6.4.32}$$

Thus, the transformation rule for κ symmetry becomes simple

$$\delta\theta^1 = 2l\gamma^\mu \Pi_{\mu\alpha}\kappa^{1\alpha} \tag{6.4.33}$$

$$\delta x^\mu = \begin{cases} 0 & \mu \in D \\ 2i\bar{\theta}^1 \gamma^\mu \delta\theta^1 & \mu \in N \end{cases} \tag{6.4.34}$$

To find the degrees of freedom of the open string states, we take the light cone gauge. In other words, using conformal symmetry,

$$x^+(\tau,\sigma) = x^+ + 2\alpha' p^+ \tau. \tag{6.4.35}$$

Also, as a gauge fixation for the κ symmetry,

$$\gamma^{01}\theta^i = \theta^i. \tag{6.4.36}$$

The spinor is then decomposed into $S^i = \frac{1}{2}\left(1+\gamma^{01}\right)\theta^i$. Here, there is no sign arbitrariness for this gauge fixation because of the condition on the symmetry parameter κ.

The equation of motion is

$$\begin{aligned} \left(\partial_0^2 - \partial_1^2\right) x^\mu = 0, &\qquad \mu \neq 0,1 \\ \left(\partial_0 + \partial_1\right)\theta^1 = 0 &\qquad . \left(\partial_0 - \partial_1\right)\theta^2 = 0 \end{aligned} \tag{6.4.37}$$

The solution for x^μ is already given. The solutions for the spinors are

$$\begin{aligned} S' &= \sum_n S_n^1 e^{-in(\tau-\sigma)} + \sum_r S_r^! e^{-in(\tau-\sigma)} \\ S^2 &= \sum_n S_n^2 e^{-in(\tau+\sigma)} + \sum_r S_r^2 e^{-in(t+\sigma)} \end{aligned} \tag{6.4.38}$$

If we impose a boundary condition on these solutions, we obtain

$$\left.\begin{aligned} S_n^2 &= P_0 S_n^1 \\ S_r^2 &= P_0 S_r^1 \end{aligned}\right\} \text{ at } \sigma = 0, \qquad \left.\begin{aligned} S_n^2 &= P_\pi S_n^1 \\ S_r^2 &= -P_\pi S_r^1 \end{aligned}\right\} \text{ at } \sigma = \pi. \tag{6.4.39}$$

Putting these together, we find

$$S_n^1 = P_0^{-1} P_\pi S_n^1, \quad S_r^1 = -P_0^{-1} P_\pi S_r^1. \tag{6.4.40}$$

From $(P_0 P_\pi^{-1})^2 = 1$, $P_0 P_\pi^{-1}$ is given by the gamma matrix or $\pm I$. Also if it is given by a gamma matrix, $\mathrm{Tr}(P_0 P_\pi^{-1}) = 0$, which can be divided into three ways as follows: (1) $P_0 P_\pi^{-1} = I$. (2) $P_0 P_\pi^{-1} = -I$. (3) Half of the eigenvalues of $P_0 P_\pi^{-1}$ are +1 the others are -1. In (1) and (2), the D-branes are parallel. In (3), they are not parallel and there is a D-N or N-D boundary condition. The quantum states are generated by $\alpha_n^\mu, S_{-r}^1, S_{-n}^1$, and from their commutation relations, these states are represented by the generated annihilation operator and the Clifford algebra representation. Here the half of the spinor is fixed by the gauge fixation of the κ symmetry, and these generators cannot generate the states with all helicities. This means that they are not CPT self-conjugate. Therefore, we need to introduce other supergravity multiplet.

First we consider the $P_0 = P_\pi$. This means that the two ends of the string exist on D-branes that are parallel to each other. The conservation condition for supersymmetry is trivially satisfied, and all 16 components of the parameters are remained. At this point, the possible boundary conditions are N-N or D-D. The independent oscillation modes of x^μ have 8 components, and the spinors have $16/2 = 8$ degrees of freedom from the gauge-fixing condition of κ symmetry in Majorana-Weyl. Therefore, the degrees of freedom are matched in the boson and fermion, and so the vacuum energy is zero.

In $P_0 = -P_\pi$, the supersymmetry conservation condition only has a trivial solution: $\epsilon^1 = 0$. Therefore, the theory has no spacetime supersymmetry. The scalar field has 8 components as before, and the spinors are generated by S^1_r and have 8 components.

Final case is that $P_0 P_\pi^{-1}$ is given by gamma matrix. In this case, in addition to N-N and D-D, there are also N-D and D-N as boundary conditions. The N-N and D-D directions are gormed by α^μ_n, and the N-D and D-N directions are formed by c^μ_r. The fermionic oscillator has 8 components each of S^1_n, S^1_r but they are reduced to 4 components each due to the boundary conditions. From the supersymmetry conservation condition, the supersymmetry parameter is 16/2=8 components because there is one conditional expression. However, since S^1_0 has 4 components, the dimension of the Clifford algebra generated by its anti-commutation relation is 4, and the spinor component generated by S^1_0 has only 4 components. This is due to the fact that half of the spinors are fixed by the κ symmetry, as mentioned before. To form the supermultiplet, all we need is to add a conjugate multiplet.

7 NDBI Action with Magnetic Flux

In this section, we perform the flux compactification of NDBI action, and show that the 4 dimensional effective theory is consistent with the supergravity theory.

7.1 Magnetized Torus Compactification

First we consider the NDBI action for N stack D9-brane on $M_4 \times T^6$. In addtion, we assume the factorizable tori $T^6 = \prod_{i=1}^{3} T_i^2$. Then ten dimensional metric is given by

$$ds_{10}^2 = e^{2\Phi} ds_4^2 + l_s^2 \sum_{i=1}^{3} g_{mn}^{(i)} dy^m dy^n \tag{7.1.1}$$

$$g_{mn}^{(i)} = e^{2\sigma_i} \begin{pmatrix} 1 & \tau_R^{(i)} \\ \tau_R^{(i)} & |\tau^{(i)}|^2 \end{pmatrix} \tag{7.1.2}$$

where $\tau^{(i)}$ is the modulus on i-th torus T_i^2 and $\tau_R^{(i)}$ and $\tau_I^{(i)}$ are the real part and imaginary part of $\tau^{(i)}$ respectively. This torus is normalized by the string length

$$0 \leq y_{2i+2}, y_{2i+3} \leq 1 \tag{7.1.3}$$

We choose the complex coordinate on T_i^2;

$$z_i = y_{2i+2} + \tau^{(i)} y_{2i+3} \tag{7.1.4}$$

Then, ten dimensional metric is rewritten as

$$ds_{10}^2 = e^{2\Phi} ds_4^2 + l_s^2 \sum_{i=1} e^{2\sigma_i} dz_i dz_{\bar{i}}, \tag{7.1.5}$$

and we can read the metric on the T_i^2 from (7.1.5)

$$g_{i\bar{j}} = l_s^2 \frac{e^{2\sigma_i}}{2} \delta_{ij}. \tag{7.1.6}$$

The area of i-th torus is given by

$$\mathcal{A}^{(i)} \equiv e^{2\sigma_i} \tau_I^{(i)} = \sqrt{g^{(i)}}. \tag{7.1.7}$$

Thus the factor $e^{2\sigma_i}$ is the volume of T_i^2.

In order to describe the Einstein frame in 4 dimensions, we introduce the 4 dimensional dilation Φ:

$$\Phi = \phi - \frac{1}{2}\log\prod_i e^{2\sigma_i}\tau_I^{(i)}. \tag{7.1.8}$$

where ϕ is the 10D dilaton.

In the following part in this paper, we focus on the D9-brane and factorized torus $\prod_{i=1}^{3} T^2_{(i)}$ with magnetic fluxes. D9-brane NDBI action is given by

$$S_{\mathrm{NDBI}} = -T_9 \int_{\mathcal{M}_4\times\prod_i^3 T^2_{(i)}} d^{10}X e^{-\phi}\,\mathrm{Str}\sqrt{-\det(g_{MN}+2\pi\alpha' F_{MN})}, \tag{7.1.9}$$

and the flux of each torus takes integer M,

$$\int_{T^2} \hat{F}_{2i+2,2i+3} = 2\pi M^{(i)} \tag{7.1.10}$$

Thus, the backgraund field strength is given by

$$\hat{F}_{i\bar{j}} = \frac{i(2\pi M^{(i)})}{2\tau_I^{(i)} l_s^2}\delta_{ij}. \tag{7.1.11}$$

This backgraound field flux can be rewritten as the differential form

$$\hat{F}_2 = \frac{i(2\pi M^{(i)})}{2\tau_I^{(i)} l_s^2}(dz_i \wedge d\bar{z}_i) \tag{7.1.12}$$

and corresponding gauge field is given by

$$\hat{A}_1(z) = \frac{2\pi M}{2\tau_I^i l_s^2}\,\mathrm{Im}(\bar{z}_i dz_i) \tag{7.1.13}$$

In this paper, we chose 3-stack D9-brane and so U(3) geuge. In order to break $U(3) \to U(1)_a \times U(1)_b \times U(1)_c$, we introduce the following fluxes:

$$M^{(i)} = \begin{pmatrix} M_a^{(i)} & & \\ & M_b^{(i)} & \\ & & M_c^{(i)} \end{pmatrix}. \tag{7.1.14}$$

So, the non-abelian DBI action reduced to

$$S_{NDBI} = -\frac{2\pi}{l_s^4}\int d^4x\sqrt{-g_4}e^{4\Phi}\left(\prod_{i=1}^{3}\int_{T^2} dy_{2i+2}dy_{2i+3}e^{2\sigma_i}\tau_I^i\right) \times e^{-\phi}\,\mathrm{Str}\sqrt{-\det(g_{MN}+2\pi\alpha' F_{MN})}, \tag{7.1.15}$$

where, the integrand has been reduced to

$$\sqrt{-\det(G_{MN}+2\pi\alpha' F_{MN})} = \sqrt{-g_4}\prod\sqrt{g_2^{(i)}}\,\mathrm{Str}\sqrt{-\det\left(\delta_N^M+2\pi\alpha' G^{MK}F_{KN}\right)}, \tag{7.1.16}$$

and we have used

$$\sqrt{g_2^{(i)}} = e^{2\sigma_i}\tau_I^{(i)}. \tag{7.1.17}$$

This action can be expanded by (6.3.31).

7.2 Supersymmetry Condition

In order to obtain supersymmetric gauge theory in 4 dimension, we must impose the supersymmetry condition for the background fields on the complex manifold:

$$g^{i\bar{j}}F_{i\bar{j}} = 0, \quad F_{ij} = F_{\bar{i}\bar{j}} = 0, \tag{7.2.1}$$

where these conditions are defined as up to $O(F^2)$. In other words these expressions are not sufficient at $O(F^n), n > 2$. But in this paper we focus on $O(F^4)$ and (7.2.1) are correct at this order.

By substituting (7.1.11) into first condition in (7.2.1), we find

$$\sum_{i=1}^{3}\frac{M_\alpha^{(i)}}{\mathcal{A}^{(i)}} \tag{7.2.2}$$

The second condition is automatically satisfied if we consider the diagonal flux (7.1.14).

From these conditions, we can obtain the supersymmetry condition:

$$\sum_{i=1}^{3}\pm\frac{M_a^{(i)}-M_b^{(i)}}{\mathcal{A}^{(i)}} = 0. \tag{7.2.3}$$

This condition is same as the no-tachyon condition. Then the massless scalar modes exist and 4D $N = 1$ supersymmetry remains unbroken at least in the $a - b$ sector. For the simplicity, we introduce the flux per unit area;

$$m^i_\alpha \equiv \frac{M^{(i)}_\alpha}{\mathcal{A}^{(i)}} \tag{7.2.4}$$

7.3 The Zero Mode Wavefunctions

In order to obtain the zero mode wave functions for the gauge boson, we consider the 10 dimensional spinor which is a super partner of the gauge boson A_M:

$$\Gamma\lambda^{(10)} = \lambda^{(10)}. \tag{7.3.1}$$

By the compactification with $T^2 \times T^2 \times T^2$, the SO(10) spinor is decomoposed into the irreducible representations of SO(2):

$$\lambda_0 \equiv \lambda_{+++}, \quad \lambda_1 \equiv \lambda_{+--}, \quad \lambda_2 \equiv \lambda_{-+-}, \quad \lambda_3 \equiv \lambda_{--+}, \tag{7.3.2}$$

where $\pm$ denotes the eigenvalues of each SO(2). Also the 10 deimensional gauge field A_M can be decomposed into

$$A_\mu, \quad A_{z_1}, \quad A_{z_2}, \quad A_{z_3}. \tag{7.3.3}$$

If $N = 1$ SUSY is preserved in 4 dimensional theory, A_μ and λ_0 form a vector multiplet V. Futhermore the A_{z_i} and λ_i form a chiral multiplets Φ_i. The zero mode wavefunctions for the bosons have the same functional forms as the super partner fermions. Thus the zero mode functions for the chiral multiplets Φ_i are given by

$$\Phi_i(x,y) = \sum_{\mathbb{A}} \Phi_i^{\mathbb{A},I_{\alpha\beta}}(x) \otimes \phi_i^{\mathbb{A},I_{\alpha\beta}}(y) + \text{massive modes}, \tag{7.3.4}$$

$$\phi_i^{\mathbb{A},I_{\alpha\beta}}(y) = \prod_{r=1}^{3} \phi_{i,T^2_r}^{A^{(r)},I^{(r)}_{\alpha\beta}}(y_r), \tag{7.3.5}$$

and $\phi_{i,T^2_r}^{A^{(r)},I^{(r)}_{\alpha\beta}}(y_r)$ is given

$$\phi_{i,T^2_r}^{A^{(r)},I^{(r)}_{\alpha\beta}} = \begin{cases} \Theta^{A^{(r)},I^{(r)}_{\alpha\beta}}(z_r) & (r = i, I^{(r)}_{\alpha\beta} > 0) \\ \overline{\Theta^{A^{(r)},|I_{\alpha\beta}|^{(r)}}}(z_r) & (r \neq i, I^{(r)}_{\alpha\beta} < 0) \\ 0 & (\text{others}) \end{cases} \tag{7.3.6}$$

where Θ and $\overline{\Theta}$ are B_+ and C_+ with $\zeta_{\alpha\beta=0}$ respectively in the Section 5. $A^{(r)} = 0, 1, ..., |I^{(r)}_{\alpha\beta} - 1|$ is the index of flavor on each torus and $\mathbb{A} = 0, ..., |I_{\alpha\beta}| - 1$ is the total flavor index.

In the following discussion, we fix the sign of the fluxes without loss generality:

$$\begin{aligned} I_{ab}^{(1)} &> 0, \quad I_{ab}^{(2),(3)} < 0 \\ I_{bc}^{(2)} &> 0, \quad I_{bc}^{(1),(3)} < 0 \\ I_{ca}^{(3)} &> 0, \quad I_{ca}^{(1),(2)} < 0. \end{aligned} \tag{7.3.7}$$

7.4 U(3) Gauge Theory

In this paper, we focus on the bosonic parts due to supersymmetry. Due to the flux, the U(3) gauge symmetry breaks down to $U(1)_a \times U(1)_b \times U(1)_c$, and the following fields which are the fluctuations of 10 dimensional gauge fields remain massless fields:

$$a_\mu = \begin{pmatrix} a_\mu^a & & \\ & a_\mu^b & \\ & & a_\mu^c \end{pmatrix}, \quad a_m = \begin{pmatrix} & a_m^{ab} & \\ & & a_m^{bc} \\ a_m^{ca} & & \end{pmatrix}. \tag{7.4.1}$$

$a_\mu^{\alpha\beta}$ depends on not only 4 dimensional coordinate x but also the compact space coordinates y. Thus we carry out Kalza-Kelin reductions such as

$$a_m = \begin{pmatrix} & A_i(x)\phi_i^{ab}(y)\delta_{i1} & \\ & & B_i(x)\phi_i^{bc}(y)\delta_{i2} \\ C_i(x)\phi_i^{ca}(y)\delta_{i3} & & \end{pmatrix}, \tag{7.4.2}$$

where $\phi_i^{\alpha\beta} \equiv \phi_i^{I_{\alpha\beta}}$. Since $a_m^{a,b,c}$'s are in diagonal component, these fields associated with $\mathrm{U}(1)_{a,b,c}$respectively. Since A_m, B_m and C_m are in off-diagonal component, these are bi-fundamental scalar fields under $U(1)_a \times U(1)_b \times U(1)_c$. We asign the charge to these scalar fields like $(Q_a, Q_b, Q_c) = (1, -1, 0), (0, 1, -1), (-1, 0, 1)$.

From the fact in privious section, we focus on the zero modes and the remaining zero modes in 4 dimension are only

$$A_1^{\mathbb{A}}, \quad B_2^{\mathbb{B}}, \quad C_3^{\mathbb{C}}. \tag{7.4.3}$$

Futhermore according to (7.3.7), the zero mode of $\phi_i^{\alpha\beta}$ on the compact space are given by

$$\begin{aligned} \phi_1^{\mathbb{A},ab} &= \Theta^{A^{(1)},I_{ab}^{(1)}}(z_1) \otimes \overline{\Theta^{A^{(2)},I_{ab}^{(2)}}}(z_2) \otimes \overline{\Theta^{A^{(3)},I_{ab}^{(3)}}}(z_3) \\ \phi_2^{\mathbb{B},bc} &= \overline{\Theta^{B^{(1)},I_{bc}^{(1)}}}(z_1) \otimes \Theta^{B^{(2)},I_{bc}^{(2)}}(z_2) \otimes \overline{\Theta^{B^{(3)},I_{bc}^{(3)}}}(z_3) \\ \phi_3^{\mathbb{C},ca} &= \overline{\Theta^{C^{(1)},I_{ca}^{(1)}}}(z_1) \otimes \overline{\Theta^{C^{(2)},I_{ca}^{(2)}}}(z_2) \otimes \Theta^{C^{(3)},I_{ca}^{(3)}}(z_3). \end{aligned} \tag{7.4.4}$$

7.5 Effective Action

In this section, we derive 4 dimensional effective action from 10 dimensional NDBI action with magnetic flux compactification.

According to Ref [87], we introduce the closed string moduli fields as

$$
\begin{aligned}
s &= e^{-\phi} \prod_i e^{2\sigma_i} \tau_I^{(i)} \\
t_i &= e^{-\phi} e^{2\sigma_i} \tau_I^{(i)} \\
u_i &= \tau_I^{(i)}
\end{aligned}
\tag{7.5.1}
$$

where ϕ denotes the 10D dilaton field. From (7.1.8), s is the 4 dimensional dilaton, t_i are the area which means those are the Kähler moduli and u_i are the imaginary part of the complex structure moduli of T^2. In combination with axions decended from R-R tensors, the above moduli are complexified and we denote each moduli as dilaton S, Kähler moduli T_i and complex structure moduli U_i. In this notation, the Kähler potential for these closed string moduli $K^{(0)}$ is given by [78]

$$
K^{(0)} = -\log\big(S + \overline{S}\big) - \sum_{i=1}^{3} \log\big(T_i + \overline{T_i}\big) - \sum_{i=1}^{3} \log\big(U_i + \overline{U}_i\big). \tag{7.5.2}
$$

Thus, (7.1.15) is rewritten as

$$
S_{NDBI} = -\frac{2\pi}{l_s^4} \int d^4x \sqrt{-g_4} e^{4\Phi} \left(\prod_{i=1}^{3} \int_{T^2} dy_{2i+2} dy_{2i+3} \right) s \, \mathrm{Str} \sqrt{-\det\big(\delta_N^M + 2\pi\alpha' g^{MK} F_{KN}\big)} \tag{7.5.3}
$$

7.5.1 Gauge Couplings

$U(1)_{a,b,c}$ gauge couplings is obtained from the coefficient of the gauge kinetic term such as

$$
S_{4D} \supset -\frac{1}{2\pi} \int d^4x \sqrt{g_4} \frac{1}{4g_a^2} (f_{\mu\nu}^a)^2 \tag{7.5.4}
$$

where $f_{\mu\nu}$ is the field strength for the 4 dimensional gauge fields a_μ. According to appendix B, the gauge coupling is given by

$$
\begin{aligned}
\frac{1}{g_a^2} &= s \left[1 + \frac{1}{2} \sum_{i=1}^{3} (m_a^i)^2 \right] \\
&= s - T_R^1 M_a^2 M_a^3 - T_R^1 M_a^1 M_a^3 - T_r^3 M_a^1 M_a^2,
\end{aligned}
\tag{7.5.5}
$$

where we use the SUSY condition in the second line. This result is equal to the gauge coupling [88–91], because the contribution of the flux for the gauge kinetic term is factorized as $\mathrm{tr}\, F_i^2 f_{\mu\nu}^2$. This result is regarded as the real part of a holomorphic gauge coupling f_a:

$$f_a = S - T_1 M_a^{(2)} M_a^{(3)} - T_2^{(1)} M_a^{(3)} - T_3 M_a^{(1)} M_a^{(2)}. \tag{7.5.6}$$

So the situation is same to the brane models, which the gauge coupling is given as the volume of the magnetized extra dimension.

We notice that this expansion in fluxes is assumed that the compact scale is enough larger than the flux scale which means $s > t_i |M_a^{(j)} M_a^{(k)}| (i \neq j \neq k)$. Thus the gauge coupling becomes weak for the large vevs of moduli.

7.5.2 Kähler Metric

The Kähler metric is obtained from the coefficient of the charged matters kinetic terms;

$$2\pi S_{NDBI} \supset - \int d^4x \sqrt{-g_4} Z_{ab}^i |D_\mu A|^2 \tag{7.5.7}$$

where A is a any charged matter and 4D covariant derivatives have been defined as $D_\mu A_i^{\mathbb{A}} = (\partial_\mu + i a_\mu^a - i a_\mu^b) A_i^{\mathbb{A}}$. According to Appendix B, the explicit form is given by

$$\begin{aligned} Z_{ab}^i &= \frac{1}{2t_i} \left(\prod_{k=1}^{3} \frac{1}{\sqrt{2u_k}} \right) \sqrt{\frac{|I_{ab}^{(i)}}{\prod_{i\neq j} |I_{ab}^{(j)}}} \left[1 - \frac{1}{6} (2m_a^{(j)} m_a^{(k)} + 2m_b^{(j)} m_b^{(k)} + m_a^{(j)} m_b^{(k)} + m_b^{(j)} m_a^{(k)}) \right] \\ &\equiv G_{ab}^i \times \left[1 - \frac{1}{6} (2m_a^{(j)} m_a^{(k)} + 2m_b^{(j)} m_b^{(k)} + m_a^{(j)} m_b^{(k)} + m_b^{(j)} m_a^{(k)}) \right], \quad (i \neq j \neq k) \end{aligned} \tag{7.5.8}$$

G_{ab^i} is a normalization factor and defined by

$$G_{ab}^i \equiv \frac{1}{2t_i} \left(\prod_{k=1}^{3} \frac{1}{\sqrt{2u_k}} \right) \sqrt{\frac{|I_{ab}^{(i)}}{\prod_{i\neq j} |I_{ab}^{(j)}}} \equiv \frac{2u_i}{t_i s} e^{\phi} \left(\alpha_{ab}^{(i)}) \right)^2. \tag{7.5.9}$$

where $\alpha_{ab}^{(i)}$ is defined by

$$\alpha_{ab}^{(i)} \equiv \frac{1}{2^2 u_i} \frac{s e^{-\phi}}{(2^3 u_1 u_2 u_3)^{1/4}} \left(\frac{|I_{ab}^{(i)}|}{\prod_{r\neq i} |I_{ab}^{(r)}|} \right)^{1/4}. \tag{7.5.10}$$

$\alpha_{ab}^{(i)}$ is a scale factor for the matter field A_i such as $A_i \to \alpha_{ab}^{(i)} A_i$. G_{ab}^i is the metric for the effective theory derived from 10D SYM with magnetized torus. As expected, the contributions from the each T_i^2 are symmetric with the torus labels.

This result can be also written with comlexfied moduli and intersection numbers as

$$Z^i_{ab} = G^i_{ab} \times \left[1 + \frac{(T_i + \overline{T}_i)}{6(S+\overline{S})}(I^{(j)}_{ab} I^{(k)}_{ab} - 3M^{(j)}_a M^{(k)}_a - 3M^{(j)}_b M^{(k)}_b)\right] \quad (i \neq j \neq k) \tag{7.5.11}$$

$$G^i_{ab} = \frac{1}{T_i + \overline{T}_i}\left(\prod_{k=1}^{3} \frac{1}{\sqrt{U_k + \overline{U}_k}}\right)\sqrt{\frac{|I^{(i)}_{ab}}{\prod_{j\neq i}|I^{(j)}_{ab}}}. \tag{7.5.12}$$

Since $s > t_i|M^{(j)}M^{(k)}|$, the Kähler metric Z^i_{ab} is positive definite in SUSY theory.

7.6 Quartic Couplings in Supergravity Description

We confirm that the quartic couplings obeys the SUGRA description. From the privious section, the F-term potential is given by

$$V_{\rm F} = e^K[g^{I\overline{J}}(D_I W)(D_J W)^* - 3|W|^2] \tag{7.6.1}$$

If we assume that the Kähler metric (7.5.8) is correct, $A_1 B_2 \overline{A}_1 \overline{B_2}$ terms are included in F-term potential $V_{\rm F}$.

D-term potential in U(N) gauge theory is given by

$$V_{\rm D} = \frac{1}{2}g^2\left(a^{*I} T^{(a)I}_J a^J\right)^2. \tag{7.6.2}$$

In the 4D effective thery, the U(N) gauge group was broken to $U(1)_a \times U(1)_b \times U(1)_c$ and so the generator T is a diagonal matrix. Thus $|A_1|^4$ terms are included in $V_{\rm D}$.

Since the F-term potential is given by the superpotential which is restricted as a holomorphic function, it is also restricted. On the other hands D-term is given by (7.6.2) and so it is less restricted. Thus we focus on the only $A_1 B_2 \overline{A_1 B_2}$ scalar potential term which is related to the Yulawa coupling.

As seen (7.5.7), the matter fields kinetic terms are not canonical. In order to get physical coupling constants, we redefine the scalar fields as $A_i \to A_i/\sqrt{Z^i_A}$.

Appendics B, we can obtain the F-term pottential:

$$\begin{aligned} V_F \supset & 2\frac{e^\phi}{s^2} g^{1\overline{1}} g^{2\overline{2}} (\alpha^{(1)}_{ab})^2 (\alpha^{(2)}_{bc})^2 \int d^6y \sqrt{g_6}\, \phi_1^{\mathbb{A},ab} \phi_1^{\mathbb{B},bc} \overline{\phi_1^{\mathbb{A}',ab} \phi_1^{\mathbb{B}',bc}} \\ & \times \left[1 + \frac{1}{6}\left\{2m^{(1)}_a m^{(2)}_a + 2m^{(1)}_c m^{(1)}_c + m^{(1)}_a m^{(2)}_c + m^{(1)}_c m^{(2)}_a\right\}\right] A_1^{\mathbb{A}} \overline{A}_1^{\mathbb{A}'} \overline{B}_2^{\mathbb{B}} B_2^{\mathbb{B}'}, \end{aligned} \tag{7.6.3}$$

where we have carried out the rescale $A_1^{\mathbb{A}} \to \alpha^{(1)}_{ab} A_1^{\mathbb{A}}$ and $B_2^{\mathbb{B}} \to \alpha^{(2)}_{bc} B_2^{\mathbb{B}}$. The contributions of fluxes can be described by Kähler metric in (7.5.11) and (7.5.12):

$$V_F \supset \left[\frac{2G_{ca}^3}{Z_{ca}^3} \frac{e^{\phi}}{s^2} g^{1\bar{1}} g^{2\bar{2}} (\alpha_{ab}^{(1)})^2 (\alpha_{bc}^{(2)})^2 \int d^6 y \sqrt{g_6} \phi_1^{\mathbb{A},ab} \phi_1^{\mathbb{B},bc} \overline{\phi_1^{\mathbb{A}',ab} \phi_1^{\mathbb{B}',bc}} \right] A_1^{\mathbb{A}} \overline{A}_1^{\mathbb{A}'} \overline{B}_2^{\mathbb{B}} B_2^{\mathbb{B}'} \tag{7.6.4}$$

Futhermore in order to rewrite this expression as SUGRA formula (7.6.1), we define the lowest Kähler potential as

$$e^{K^{(0)}} \equiv \frac{1}{2^7 s t_1 t_2 t_3 u_1 u_2 u_3}, \tag{7.6.5}$$

and then (7.6.4) can be written as

$$V_F \supset \left[\frac{e^{K^{(0)}}}{Z_{ca}^3} \left(\sqrt{2} e^{-K^{(0)}/2} e^{3\Phi - \phi} \frac{\alpha_{ab}^{(1)} \alpha_{bc}^{(2)} \alpha_{ca}^{(3)}}{\sqrt{g_{1\bar{1}} g_{2\bar{2}} g_{3\bar{3}}}} \right)^2 \int d^6 y \sqrt{g_6} \phi_1^{\mathbb{A},ab} \phi_1^{\mathbb{B},bc} \overline{\phi_1^{\mathbb{A}',ab} \phi_1^{\mathbb{B}',bc}} \right] A_1^{\mathbb{A}} \overline{A}_1^{\mathbb{A}'} \overline{B}_2^{\mathbb{B}} B_2^{\mathbb{B}'} \tag{7.6.6}$$

In order to construct superpotential W, we evaluate the integral in (7.6.6). First, we split the integral to two integrals such as

$$\begin{aligned} &\int d^6 y \sqrt{g_6} \phi_1^{\mathbb{A},ab} \phi_1^{\mathbb{B},bc} \overline{\phi_1^{\mathbb{A}',ab} \phi_1^{\mathbb{B}',bc}} \\ &= \int d^6 y \sqrt{g_6} \phi_1^{\mathbb{A},ab} \phi_1^{\mathbb{B},bc} \int d^6 y' \overline{\phi_1^{\mathbb{A}',ab} \phi_1^{\mathbb{B}',bc}} \times \frac{1}{\sqrt{g_6}} \delta y - y'. \end{aligned} \tag{7.6.7}$$

Since $\phi_1^{\mathbb{A},ab}$ and $\phi_1^{\mathbb{B},bc}$ are given by theta function, we can use the formula for the theta finction (**??**), and thus (7.6.6) can be rewritten as the squared form:

$$V_F \supset \frac{e^{K^{(0)}}}{Z_{ca}^3} \left| \sqrt{2} e^{-K^{(0)}/2} e^{3\Phi - \phi} \frac{\alpha_{ab}^{(1)} \alpha_{bc}^{(2)} \alpha_{ca}^{(3)}}{\sqrt{g_{1\bar{1}} g_{2\bar{2}} g_{3\bar{3}}}} \int d^6 y \sqrt{g_6} \phi_1^{\mathbb{A},ab} \phi_1^{\mathbb{B},bc} \phi_3^{\mathbb{C}',ca} \right|^2 A_1^{\mathbb{A}} \overline{A}_1^{\mathbb{A}'} \overline{B}_2^{\mathbb{B}} B_2^{\mathbb{B}'}. \tag{7.6.8}$$

By defining the Yukawa coupling such as

$$\begin{aligned} Y_{\mathbb{ABC}} &\equiv \sqrt{2} e^{-K^{(0)}/2} e^{3\Phi - \phi} \frac{\alpha_{ab}^{(1)} \alpha_{bc}^{(2)} \alpha_{ca}^{(3)}}{\sqrt{g_{1\bar{1}} g_{2\bar{2}} g_{3\bar{3}}}} \int d^6 y \sqrt{g_6} \phi_1^{\mathbb{A},ab} \phi_1^{\mathbb{B},bc} \phi_3^{\mathbb{C}',ca} \\ &\equiv 2 \prod_{r=1}^{3} W_{A^{(r)} B^{(r)} C^{(r)}}, \end{aligned} \tag{7.6.9}$$

we find the superpotential

$$W = \sum_{\mathbb{A},\mathbb{B},\mathbb{C}} Y_{\mathbb{A},\mathbb{B},\mathbb{C}} A_1^{\mathbb{A}} B_2^{\mathbb{B}} C_3^{\mathbb{C}}. \tag{7.6.10}$$

Then the F-term potential (7.6.8) is

$$V_F \supset \frac{e^{K^{(0)}}}{Z_{ca}^3} \sum_{\mathbb{C}} (\partial_{C_3^{\mathbb{C}}} W)(\overline{\partial_{C_3^{\mathbb{C}}} W}). \tag{7.6.11}$$

This expression is just the same as (7.6.1).

Since ϕ's are given by the theta function, from the discussion in Section 5 we can find the Yukawa coupling explicitly:

$$W_{A^{(1)}B^{(1)}C^{(1)}} = \overline{\theta} \begin{bmatrix} \frac{B^{(1)}|I_{ca}^{(1)}| - C^{(1)}|I_{bc}^{(1)}| + m^{(1)} I_{bc}^{(1)} I_{ca}^{(1)}}{|I_{ab}^{(1)} I_{bc}^{(1)} I_{ca}^{(1)}} \\ 0 \end{bmatrix} (0, i\overline{U}_1 |I_{ab}^{(1)} I_{bc}^{(1)} I_{ca}^{(1)}|) \tag{7.6.12}$$

$$W_{A^{(2)}B^{(2)}C^{(2)}} = \overline{\theta} \begin{bmatrix} \frac{C^{(2)}|I_{ab}^{(2)}| - A^{(2)}|I_{ca}^{(2)}| + m^{(2)} I_{ab}^{(2)} I_{ca}^{(2)}}{|I_{ab}^{(2)} I_{bc}^{(2)} I_{ca}^{(2)}} \\ 0 \end{bmatrix} (0, i\overline{U}_2 |I_{ab}^{(2)} I_{bc}^{(2)} I_{ca}^{(2)}|) \tag{7.6.13}$$

$$W_{A^{(3)}B^{(3)}C^{(3)}} = \overline{\theta} \begin{bmatrix} \frac{A^{(3)}|I_{bc}^{(3)}| - B^{(3)}|I_{ab}^{(3)}| + m^{(3)} I_{ab}^{(3)} I_{bc}^{(3)}}{|I_{ab}^{(3)} I_{bc}^{(3)} I_{ca}^{(3)}} \\ 0 \end{bmatrix} (0, i\overline{U}_3 |I_{ab}^{(3)} I_{bc}^{(3)} I_{ca}^{(3)}|), \tag{7.6.14}$$

and

$$\begin{aligned} A^{(1)} &= B^{(1)} + C^{(1)} + m^{(1)}|I_{bc}^{(1)}|, \quad m^{(1)} = 0, ..., I_{ab}^{(1)} - 1 \\ B^{(2)} &= A^{(2)} + C^{(2)} + m^{(2)}|I_{ca}^{(2)}|, \quad m^{(2)} = 0, ..., I_{bc}^{(2)} - 1 \\ C^{(3)} &= A^{(3)} + B^{(3)} + m^{(3)}|I_{ab}^{(3)}|, \quad m^{(3)} = 0, ..., I_{ca}^{(3)} - 1. \end{aligned} \tag{7.6.15}$$

Thus the Kähler metric is consistent with supergravity theory.

8 Summary and discussions

We have considered the model which the initial gauge group U(3) breaks to $\mathrm{U}(1)_a \times \mathrm{U}(1)_b \times \mathrm{U}(1)_c$ by magnetized brane model described non-abelian Dirac-Born-Infeld (NDBI) action. We carried out the dimensional reduction and obtained the series of fluxes up to $\mathcal{O}(F^4)$ order. Then we found that flux corrections gave the symmetric contributions and its correction to the gauge couplings, the matter Kähler metrices and scalar quartic couplings. The gauge couplings realized the privous works. The additional flux corrections appear in the Kähler metrices in a flavor independent way. And also we confirmed that the contributions of the matter Kähler metric to the F-term potential derived from NDBI action is consistent with the supergravity theory. Futhermore we derived the holomorphic superpotential from the F-term potential, and we confirmed it is consistent with the privious work.

Phenomenologically the Standard model is realized with the multiple stacks of D-branes on which quarks or leptons are living defferently, and the matter Kähler metric on a D-brane is defferent from a matter Kähler metric on the other D-branes. This differences could explain mass difference between quarks and leptons.

Calculating the higher order flux corrections for the D-term, we found that there are SUSY breaking terms in NDBI action. We propose several posibilities to explain why supersymmetry is broken.

1. Kähler potential (7.5.8) is not suitable.

   In this calculation of D-term, we assumed the Kähler potential is correct.

2. Non-perturbative effect is needed.

3. There are further corrections.

4. NDBI action proposed by [34] is not correct.

   This NDBI action is supposed that DF terms not affect the D-brane dynamics.

In this paper, although we considered the magnetized compactification model, there are several possibilities of manifold such as Calabi-Yau, Orbifold and so on. Beside these famous manifolds, we can also consider the non-compact space comopactification. In [92], 5-dimensional Yang-Mills theory is compactified by one dimensional interval. It may be interesting model derived from 6 dimensional Yang-Mills thoery by compactifing by Disck.

We constructed the compactification model by using the NDBI action. In the section 4, we have proposed several higher dimensional unified thories such as Type0A/0B, multi-time theory and E-theory. By using these higher dimensional theory, we can construct new phenomenological models.

A Formulas for supersymmetry

A.1 Integration and derivabtive on supersupace

The differential operator for fermionic coordinates is defined as follows, as is the operator for spacetime coordinates:

$$\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha}, \quad \overline{\partial}^{\dot{\alpha}} \equiv \frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} \tag{A.1.1}$$

The raising and lowering of the indexes of these differential operators are performed by Levi-Chivita symbol $\epsilon_{\alpha\beta}$.

Next, we define the integral with respect to the fermionic coordinates. The integral is defined by Berezin integral. With θ as the one-component Grassmann, the Berezin integral is defined as follows:

$$\int d\theta\theta = 1, \quad \int d\theta = 0 \tag{A.1.2}$$

The any functions depend on θ can be expanded to

$$f(\theta) = f_0 + \theta f_1 \tag{A.1.3}$$

Integrating this function, we obtain

$$\int d\theta f(\theta) = f_1 \tag{A.1.4}$$

This result is equal to the θ derivative of $f(\theta)$. In other words, integration is equivalent to differentiation. Also, since arbitrary functions on superspace do not contain terms of $\theta\theta$, $df/d\theta$ is always of zero order in θ. Thus the following equation is satisfied:

$$\int d\theta \frac{df(\theta)}{d\theta} = 0 \tag{A.1.5}$$

Futhermore, the Berezin integral is invariant under any translation of the Grassmann coordinate θ:

$$\int d\theta f(\theta) = \int d(\theta+\xi) f(\theta+\xi) \tag{A.1.6}$$

The delta function $\delta(\theta)$ is defined by

$$\delta(\theta) = \begin{cases} 1 & \theta = 0 \\ 0 & \theta \neq 0 \end{cases} \tag{A.1.7}$$

Since any function $f(\theta)$ is a linear function of θ, and $\theta = 0$ leaves only a constant term that is not proportional to θ, delta function is expressed to

$$\delta(\theta) = \theta. \tag{A.1.8}$$

Using the Berezin integral defined above, we define the Grassmann integral in $N = 1$ superspace. In superspace, there are a total of four Grassmann coordinates, $\theta_\alpha, \overline{\theta}_{\dot{\alpha}}$. Therefore, each of the integral measures can be considered up to the second order.

A.2 Isometries and Kähler manifold

In this appendix, we explain the isometries of Kähler manifolds.

Let us consider a curve λ on the manifold $\mathcal{M}$ described by real coordinates $x^i = x^i(t)$ where t is a parameter along the curve and a differentiable function $f : \mathcal{M} \to \mathbb{R}$. Then the directional derivative of f along the curve is defined by

$$\frac{df}{dt} = \frac{dx^i}{dt}\frac{\partial f}{\partial x^i} \equiv Xf \tag{A.2.1}$$

The operator X has a direction as $\partial/\partial x^i$ where dx^i/dt are its components and so X is called vector. This vector can be defined for arbitrary curves on $\mathcal{M}$. Thus the components dx^i/dt are the functions on $\mathcal{M}$. The other words, if a set of continuous functions X^i on $\mathcal{M}$, we always possible to find a set of curves $x^i(t)$ as solutions to

$$\frac{dx^i}{dt} = X^i \tag{A.2.2}$$

Since the solutions for these equations are unique, these curves can never cross.

Therefore we have seen that sets of space-filling curves are one-to-tone correspondence with vector fields X.

If we consider the automorphism $\phi_t : \mathcal{M} \to \mathcal{M}$, it induces a map between vectors in tangent space. The induced map allow us to compare the vectors ar different points along the curves. In order to see this statment, we consider the two points x^i and x'^i which denote the coordinates at p and p' respectively and vector field Y which is described with the components $Y^i(x)$ at p' and $\tilde{Y}^i(x')$. Then the relation between $Y^i(x)$ and $\tilde{Y}(x')$ is given by

$$\tilde{Y}^i(x') = \frac{\partial x'^i}{\partial x^j} Y^j(x(x')), \tag{A.2.3}$$

which is called Lie transport. The Lie-transported vector field

$$\tilde{Y} \equiv \tilde{Y}^i(x')\frac{\partial}{\partial x'^i} \tag{A.2.4}$$

is defined for all points p' along the curves. If we consider the infinitesimal transport $x'^i = x^i + X^i \delta t$, (A.2.3) can be written as

$$\tilde{Y}^i(x') = Y^i(x') + \frac{\partial X^i}{\partial x^j} Y^j \delta t - \frac{\partial Y^i}{\partial x^j} X^j \delta t. \tag{A.2.5}$$

Since Y and $\tilde{Y}$ are defined ar the same points, we can define the Lie derivattive:

$$(\mathcal{L}_X Y)^i \equiv \lim_{\delta t \to 0} \frac{Y^i(x') - \tilde{Y}^i(x')}{\delta t} = X^j \frac{\partial}{\partial x^j} Y^i - Y^j \frac{\partial}{\partial x^j} X^i \equiv [X, Y]^i \tag{A.2.6}$$

This definition can be extended to the general tensorial quantities. The Lie derivative of the metric is given by

$$(\mathfrak{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i, \tag{A.2.7}$$

where we have defined $\nabla_i X_j = \partial_i - \Gamma^k_{ij} X_k$.

Since Lie derivative describes the variation of the Lie transport, the vanishing Lie derivative means that the vector fields are invariant. Thus if the metric is invariant, then

$$(\mathfrak{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i = 0. \tag{A.2.8}$$

In this case, X generates an isometry of the manifold $\mathcal{M}$ and X is called Killing vector field and (**??**) is known as Killing equation.

Killing vectors generate the continuous symmetries on tha manifold which close into the isometry. Indeed, we can see that the Lie bracket of two Killing vectors becomes another killing vector

$$[X^{(a)}, X^{(b)}] = -f^{abc} X^{(c)}, \tag{A.2.9}$$

where f^{abc} are the sturucture constants of the isometry group G.

In the following, we focus on Kähler manifolds and denote the metric as $g_{i\bar{j}}$ and its coordinetes as $a^i a^{*j}$. Then the vector fields are given by

$$\begin{aligned} X^(b) &= X^{i(b)}(a) \frac{\partial}{\partial a^i} \\ X^{*(b)} &= X^{*i(b)}(a^*) \frac{\partial}{\partial a^{*i}}. \end{aligned} \tag{A.2.10}$$

Since the $X^{(a)}$ is a holomorphic function, the non-trivial Killing equations are given by

$$\begin{aligned} \nabla_i X^{(a)}_j + \nabla_j X^{(a)}_i &= 0 \\ \nabla_{\bar{i}} X^{(a)}_j + \nabla_j X^{*(a)}_i &= 0. \end{aligned} \tag{A.2.11}$$

On the Kähler manifolds, $\nabla_i X_j = \partial_i X_j - g^{k\bar{l}} \frac{\partial g_{j\bar{l}}}{\partial a^i} X_k = 0$ and so the first equation is automatically satisfied. The second equation is written as

$$g_{i\bar{l}} \partial_j (g^{k\bar{l}} X_k^{*(a)}) + g_{j\bar{l}} \partial_{\bar{i}} (g^{k\bar{l}} X_k) = 0, \tag{A.2.12}$$

and the solution can be given by

$$\begin{aligned} g_{i\bar{j}} X^{*j(a)} &= i \frac{\partial D^{(a)}}{\partial a^i} \\ g_{i\bar{j}} X^{i(a)} &= -i \frac{\partial D^{(a)}}{\partial a^{*i}}. \end{aligned} \tag{A.2.13}$$

$D^{(a)}$ is a function of a and a^* and called Killing potential. Since the Killing vector X is determined by the derivative of $D^{(a)}$, there is a constant arbitrarity $D^{(a)} \to D^{(a)} + c^{(a)}$. This transformation is related to the Fayet-Iliopoulos D-term.

Futhermore by inverting the relations (A.2.13), we can obtain the Killing vectors in terms of the Killing potentials

$$\begin{aligned} X^{i(a)} &= -i g^{i\bar{j}} \frac{\partial}{\partial a^{*j}} D^{(a)} \\ X^{*j(a)} &= i g^{i\bar{j}} \frac{\partial}{\partial a^i} D^{(a)}. \end{aligned} \tag{A.2.14}$$

The fact that $X^{i(a)}$ must be a holommorphic imposes a constrant on the $D^{(a)}$.

The Killing vectors independently generate representations of the isometry group G which means that each generators independently obey the commutation relations.

B The calculation result of the dimensional reductions

In this appendix, we show the exact calculation results of the dimensional reduction of non-abelian DBI action:

$$\mathcal{L}_{NDBI} = -T_p e^{\phi} \frac{(2\pi\alpha')^2}{4} \operatorname{tr} \Big[F_{NM}F^{MN} - \frac{(2\pi\alpha')^2}{3} \Big(F_{MN}F^{RN}F^{ML}F_{RL} + \frac{1}{2} F_{MN}F^{RN}F_{RL}F^{ML} \\ - \frac{1}{4}(F_{MN}F^{MN})^2 - \frac{1}{8} F_{MN}F^{RL}F^{MN}F_{RL} \Big) \Big] \tag{B.0.1}$$

In this calculation, we use the following trace properties

$$\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B) \tag{B.0.2}$$

$$\operatorname{tr}(ABC) = \operatorname{tr}(BCA) \tag{B.0.3}$$

The field strength configuration and the torus metric are defined by

$$F_{\mu n} \simeq \partial_\mu a_n + ig[a_\mu, a_n] = D_\mu a_n, \quad F_{mn} \simeq f_{mn} + ig[a_m, a_n] \tag{B.0.4}$$

$$f_{i\bar{j}} = \frac{i(2\pi M^{(i)})}{2\tau_I l_s^2}\delta_{ij} \equiv \frac{\pi i e^{2\sigma_i} m^{(i)}}{l_s^2}\delta_{ij},, \quad m^{(i)} = \begin{pmatrix} m_a & & \\ & m_b & \\ & & m_c \end{pmatrix} \equiv m_\alpha^{(i)}\delta_{\alpha\beta} \tag{B.0.5}$$

$$a_m = \begin{pmatrix} & a_m^{ab} & \\ & & a_m^{bc} \\ a_m^{ca} & & \end{pmatrix} \equiv \begin{pmatrix} & A_m & \\ & & B_m \\ C_m & & \end{pmatrix} \equiv a_m^{\alpha\beta}, \quad a_\mu = \operatorname{diag}(a_\mu^a, a_\mu^b, a_\mu^c). \tag{B.0.6}$$

$$g^{i\bar{j}} = 2e^{-2\sigma_i}\delta_{ij} \tag{B.0.7}$$

We defin the $U(1)_a$ field strength as $F^a_{\mu\nu}$ as well as $U(1)_b$, $U(1)_c$.

$F_{MN}F^{MN}$

$$
\begin{aligned}
&F_{MN}F^{MN} \\
&= F_{\mu\nu}F^{\mu\nu} + 2F_{\mu n}F^{\mu n} + F_{mn}F^{mn} \\
&= |F_4|^2 + 4\sum_{i=1}^{3} e^{-2\sigma_i} D_\mu a_i D^\mu a_{\bar{i}} + 8\frac{\pi^2}{l_s^4}\sum_{i=1}^{3}(m^{(i)})^2 \\
&+ 16g\frac{\pi m^{(i)}}{l_s^2}\sum_{i=1}^{3} e^{-2\sigma_i}[a_i, a_{\bar{i}}] + 8g^2\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}[a_i, a_{\bar{j}}][a_j, a_{\bar{i}}] \\
&= |F_4|^2 + 4\sum_{i=1}^{3} e^{-2\sigma_i}\left(|D_\mu A_i|^2 + |D_\mu B_i|^2 + |D_\mu C_i|^2\right) + 8\frac{\pi^2}{l_s^4}\sum_{i=1}^{3}\sum_{\alpha=a,b,c}(m_\alpha^{(i)})^2 \\
&+ 16g\frac{\pi}{l_s^2}\sum_{i=1}^{3} e^{-2\sigma_i}\Big((m_a^{(i)} - m_b^{(i)})|A_i|^2 + (m_b^{(i)} - m_c^{(i)})|B_i|^2 + (m_c^{(i)} - m_a^{(i)})|C_i|^2\Big) \\
&+ 16g^2\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\left(|A_i|^2|A_j|^2 + |B_i|^2|B_j|^2 + |C_i|^2|C_j|^2\right. \\
&\left. - A_i\overline{A}_j\overline{B}_i B_j - B_i\overline{B}_j\overline{C}_i C_j - C_i\overline{C}_j\overline{A}_i A_j\right)
\end{aligned}
\tag{B.0.8}
$$

$F_{MN}F^{KN}F^{ML}F_{KL}$

$$F_{MN}F^{KN}F^{ML}F_{KL} \tag{B.0.9}$$

$$= F_{\mu\nu}F^{\rho\nu}F^{\mu\sigma}F_{\rho\sigma} \tag{B.0.10}$$

$$+ 2F_{\mu m}F^{\nu m}F^{\mu\rho}F_{\nu\rho} + F_{\mu m}F^{\nu\mu}F^{\rho m}F_{\nu\rho} + F_{\mu m}F_{\nu\rho}F^{\rho m}F^{\nu\mu} \tag{B.0.11}$$

$$+ 2F_{mn}\left(F^{\mu m}F^{\nu n}F_{\mu\nu} + F_{\mu\nu}F^{\mu m}F^{\nu n}\right) + F_{\mu m}\left(F^{\mu n}F^{\nu m} + F^{\nu m}F^{\mu n}\right)F_{\nu n} \tag{B.0.12}$$

$$+ 2F_{mn}F^{ln}F^{\mu m}F_{\mu l} + F_{mn}\left(F^{\mu n}F^{ml}F_{\mu l} + F_{\mu l}F^{ml}F^{\mu n}\right) \tag{B.0.13}$$

$$+ F_{mn}F^{rn}F^{ml}F_{rl} \tag{B.0.14}$$

$$= F_{\mu\nu}F^{\rho\nu}F^{\mu\sigma}F_{\rho\sigma} \tag{B.0.15}$$

$$+ 4\sum_i e^{-2\sigma_i}F_{\nu\rho}\left\{F^{\mu\rho}\left(D_\mu a_i D_\nu a_{\bar{i}} + D_\mu a_{\bar{i}} D_\nu a_i\right) + D_\mu a_i F^{\nu\mu} D^\rho a_{\bar{i}} + D^\rho a_i F^{\nu\mu} D_\mu a_{\bar{i}}\right\} \tag{B.0.16}$$

$$+ 16i\frac{\pi}{l_s^2}F^{\mu\nu}\sum_i e^{-2\sigma_i}m^{(i)}\left(D_\mu a_{\bar{i}} D_\nu a_i - D_\mu a_i D_\nu a_{\bar{i}}\right) \tag{B.0.17}$$

$$+ 8igF^{\mu\nu}\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\left\{[a_i, a_{\bar{j}}]\left(D_\mu a_{\bar{i}} D_\nu a_j - D_\mu a_j D_\nu a_{\bar{i}}\right) + \left(D_\mu a_{\bar{i}} D_\nu a_j - D_\mu a_j D_\nu a_{\bar{i}}\right)[a_i, a_{\bar{j}}]\right\} \tag{B.0.18}$$

$$+ 8ig\sum_{i,j}\left([a_i, a_j]D^\mu a_i D^\nu a_j F_{\mu\nu} + [\overline{a}_i, \overline{a}_j]D^\mu a_i D^\nu a_j F_{\mu\nu}\right) \tag{B.0.19}$$

$$+ 8ig\sum_{i,j}\left([a_i, a_j]F_{\mu\nu}D^\mu a_i D^\nu a_j + [\overline{a}_i, \overline{a}_j]F_{\mu\nu}D^\mu a_i D^\nu a_j\right) \tag{B.0.20}$$

$$+ 4\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\big(2D_\mu a_i D^\mu a_j D_\nu a_{\bar{i}} D^\nu a_{\bar{j}} + 2D_\mu a_i D_\nu a_j D^\nu a_{\bar{i}} D^\mu a_{\bar{j}} + 2D_\mu a_i D^\nu a_{\bar{i}} D^\mu a_{\bar{j}} D_\nu a_j \tag{B.0.21}$$

$$+ D_\mu a_i D_\nu a_{\bar{i}} D^\mu a_j D^\nu a_{\bar{j}} + D_\mu a_{\bar{i}} D_\nu a_i D^\mu a_{\bar{j}} D^\nu a_j\big) \tag{B.0.22}$$

$$+ 32\sum_i e^{-2\sigma_i}\frac{\pi^2}{l_s^4}(m^{(i)})^2\left(D_\mu a_i D^\mu \overline{a}_i + D_\mu \overline{a}_i D^\mu a_i\right)^2 \tag{B.0.23}$$

$$+ 16g\frac{\pi}{l_s^2}\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\Big[(m^{(j)}[a_i, a_{\bar{j}}] + [a_i, a_{\bar{j}}]m^{(i)})D^\mu a_j D_\mu a_{\bar{i}} + (m^{(j)}[a_j, a_{\bar{i}}] + [a_i, a_{\bar{j}}]m^{(i)})D^\mu a_{\bar{j}} D_\mu a_i\Big] \tag{B.0.24}$$

$$+ 16g^2\sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)}\left([a_j, a_{\bar{k}}][a_i, a_{\bar{j}}]D^\mu a_k D_\mu a_{\bar{i}} + [a_k, a_{\bar{j}}][a_j, a_{\bar{i}}D^\mu a_{\bar{k}} D_\mu a_i\right) \tag{B.0.25}$$

$$- 16g^2\sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)}\left([\overline{a}_k, \overline{a}_j][a_i, a_j]D^\mu a_k D_\mu \overline{a}_i + [a_k, a_j][\overline{a}_i, \overline{a}_k]D^\mu \overline{a}_k D_\mu a_i\right) \tag{B.0.26}$$

$$- 16g\left(\frac{\pi}{l_s^2}\right)\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\left([a_j, a_i]m^{(i)}D^\mu \overline{a}_j D_\mu \overline{a}_i - [\overline{a}_j, \overline{a}_i]m^{(i)}D^\mu a_j D_\mu a_i\right) \tag{B.0.27}$$

$$- 16g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left([\overline{a}_k, \overline{a}_j][a_j, \overline{a}_i] D^\mu a_k D_\mu a_i + [a_k, a_j][a_i, \overline{a}_j] D^\mu \overline{a}_k D_\mu \overline{a}_i\right) \tag{B.0.28}$$

$$- 8g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left([\overline{a}_k, \overline{a}_j] D^\mu a_k [a_i, a_j] D_\mu \overline{a}_i + [a_k, a_j] D^\mu \overline{a}_k [\overline{a}_i, \overline{a}_k] D_\mu a_i\right) \tag{B.0.29}$$

$$- 8g \left(\frac{\pi}{l_s^2}\right) \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \Big([a_j, a_i] D^\mu \overline{a}_j m^{(i)} D_\mu \overline{a}_i - [\overline{a}_j, \overline{a}_i] D^\mu a_j m^{(i)} D_\mu a_i\Big) \tag{B.0.30}$$

$$- 16g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left([\overline{a}_k, \overline{a}_j] D^\mu a_k [a_j, \overline{a}_i] D_\mu a_i + [a_k, a_j] D^\mu \overline{a}_k [a_i, \overline{a}_j] D_\mu \overline{a}_i\right) \tag{B.0.31}$$

$$- 8g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left([\overline{a}_k, \overline{a}_j] D^\mu \overline{a}_i [a_i, a_j] D_\mu \overline{a}_k + [a_k, a_j] D_\mu a_i [\overline{a}_i, \overline{a}_k] D^\mu \overline{a}_k\right) \tag{B.0.32}$$

$$- 8g \left(\frac{\pi}{l_s^2}\right) \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \Big([a_j, a_i] D_\mu \overline{a}_i m^{(i)} D^\mu \overline{a}_j - [\overline{a}_j, \overline{a}_i] D_\mu a_i m^{(i)} D^\mu a_j\Big) \tag{B.0.33}$$

$$- 8g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left([\overline{a}_k, \overline{a}_j] D_\mu a_i [a_j, \overline{a}_i] D^\mu a_k + [a_k, a_j] D_\mu \overline{a}_i [a_i, \overline{a}_j] D^\mu \overline{a}_k\right) \tag{B.0.34}$$

$$+ 16 \frac{\pi^2}{l_s^4} \sum_i e^{-2\sigma_i} m^{(i)} D^\mu a_i m^{(i)} D_\mu a_{\bar{i}} \tag{B.0.35}$$

$$+ 8g \frac{\pi}{l_s^2} \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \Big(m^{(k)} D^\mu a_j [a_i, a_{\bar{j}}] D_\mu a_{\bar{i}} + [a_i, a_{\bar{j}}] D^\mu a_j m^{(i)} D_\mu a_{\bar{i}}\Big) \tag{B.0.36}$$

$$+ 8g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} [a_k, a_{\bar{j}}] D^\mu a_{\bar{k}} [a_j, a_{\bar{i}}] D_\mu a_i \tag{B.0.37}$$

$$+ 16 \frac{\pi^2}{l_s^4} \sum_i e^{-2\sigma_i} m^{(i)} D_\mu a_i m^{(i)} D^\mu a_{\bar{i}} \tag{B.0.38}$$

$$+ 16g \frac{\pi}{l_s^2} \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} m^{(j)} \left(D_\mu a_i [a_j, a_{\bar{i}}] D^\mu a_{\bar{j}} + D_\mu a_{\bar{i}} [a_i, a_{\bar{j}}] D^\mu a_j\right) \tag{B.0.39}$$

$$+ 16g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} [a_k, a_{\bar{j}}] D_\mu a_i [a_j, a_{\bar{i}}] D^\mu a_{\bar{k}} \tag{B.0.40}$$

$$- 16 \left(\frac{\pi}{l_s^2}\right)^4 \sum_i (m^{(i)})^4 - 64g \left(\frac{\pi}{l_s^2}\right)^3 \sum_i e^{-2\sigma_i} (m^{(i)})^3 [a_i, a_{\bar{i}}] \tag{B.0.41}$$

$$- 16g^2 \left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{2(\sigma_i+\sigma_j)} \Big((m^{(i)})^2 + m^{(j)} m^{(i)})[a_i, a_{\bar{i}}] + (m^{(i)} m^{(j)} + (m^{(i)})^2)[a_i, a_{\bar{j}}]\Big) \tag{B.0.42}$$

$$- 32g^2 \left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} m^{(j)} [a_j, a_{\bar{i}}] m^{(j)} [a_i, a_{\bar{j}}] \tag{B.0.43}$$

$$- 16g^3 \left(\frac{\pi}{l_s}\right) \sum_{i,j,k} \Big[(m^{(k)} + m^{(j)})[a_k, a_{\bar{i}}][a_j, a_{\bar{k}}][a_i, a_{\bar{j}}] + (m^{(k)} + m^{(i)})[a_j, a_{\bar{k}}][a_k, a_{\bar{i}}][a_i, a_{\bar{j}}]\Big] \tag{B.0.44}$$

$$- 16g^4 [a_l, a_{\bar{k}}][a_k, a_{\bar{j}}][a_i, a_{\bar{l}}][a_j, a_{\bar{i}}] \tag{B.0.45}$$

$$- 16 \left(\frac{\pi}{l_s^4}\right)^4 \sum_i (m^{(i)})^4 - 64g \left(\frac{\pi}{l_s^2}\right)^3 g \sum_i e^{-2\sigma_i} (m^{(i)})^3 [a_i, a_{\bar{i}}] \tag{B.0.46}$$

$$- 16g^2 \left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \left((m^{(i)})^2 + m^{(i)}m^{(j)} + m^{(j)}m^{(i)} + (m^{(j)})^2\right) [a_j, a_{\bar{i}}][a_i, a_{\bar{j}}] \tag{B.0.47}$$

$$- 16g^2 \left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \left(m^{(j)}[a_j, a_{\bar{i}}]m^{(j)} + m^{(i)}[a_j, a_{\bar{i}}]m^{(i)}\right) [a_i, a_{\bar{j}}] \tag{B.0.48}$$

$$- 16g^3 \left(\frac{\pi}{l_s^2}\right) \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left((m^{(i)} + m^{(k)}\right) \left([a_j, a_{\bar{i}}][a_k, a_{\bar{j}}][a_i, a_{\bar{k}}] + [a_k, a_{\bar{i}}][a_i, a_{\bar{j}}][a_j, a_{\bar{k}}]\right) \tag{B.0.49}$$

$$- 16g^4 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)} [a_k, a_{\bar{l}}][a_j, a_{\bar{i}}][a_l, a_{\bar{j}}][a_i, a_{\bar{k}}] \tag{B.0.50}$$

$$+ 16g^4 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)} [\overline{a}_l, \overline{a}_k][a_j, a_k][a_l, a_i][\overline{a}_j, \overline{a}_i] \tag{B.0.51}$$

$$- 32g^2 \left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j+\sigma_k)} [\overline{a}_j, \overline{a}_i][a_i, a_j]m^{(j)}m^{(i)} \tag{B.0.52}$$

$$- 32g^3 \left(\frac{\pi}{l_s^2}\right) \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} [\overline{a}_j, \overline{a}_i][a_k, a_j] \left(m^{(j)}[a_i, \overline{a}_k] + [a_i, \overline{a}_k]m^{(j)}\right) \tag{B.0.53}$$

$$- 32g^4 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)} [\overline{a}_j, \overline{a}_i][a_l, a_k][a_j, \overline{a}_k][a_i, \overline{a}_l] \tag{B.0.54}$$

$$- 32g^2 \left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j+\sigma_k)} [a_j, a_i][\overline{a}_j, \overline{a}_i](m^{(j)})^2 \tag{B.0.55}$$

$$- 32g^3 \left(\frac{\pi}{l_s^2}\right) \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} [a_k, a_i][\overline{a}_j, \overline{a}_i] \left(m^{(k)}[a_j, \overline{a}_k] + [a_j, \overline{a}_k]m^{(j)}\right) \tag{B.0.56}$$

$$- 32g^4 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)} [a_l, a_i][\overline{a}_j, \overline{a}_i][a_k, \overline{a}_l][a_j, \overline{a}_k] \tag{B.0.57}$$

$$- 32g^2 \left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} m^{(i)}[a_j, a_i]m^{(i)}[\overline{a}_j, \overline{a}_i] \tag{B.0.58}$$

$$- 16g^3 \left(\frac{\pi}{l_s^2}\right) \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left((m^{(i)} + m^{(k)})[a_j, a_k][a_i, \overline{a}_k][\overline{a}_j, \overline{a}_i] + [a_i, \overline{a}_k][a_j, a_k](m^{(i)} + m^{(k)})[\overline{a}_j, \overline{a}_i]\right) \tag{B.0.59}$$

$$- 16g^4 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)} [a_l, \overline{a}_k][a_j, a_k][a_i, \overline{a}_l][\overline{a}_j, \overline{a}_i] \tag{B.0.60}$$

$$- 16g^4 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)} [a_l, \overline{a}_k][a_j, a_i][a_k, \overline{a}_i][\overline{a}_j, \overline{a}_l] \tag{B.0.61}$$

$$= F_{\mu\nu}F^{\rho\nu}F^{\mu\sigma}F_{\rho\sigma} \tag{B.0.62}$$

$$+ 4 \sum_i e^{-2\sigma_i} \left[\left(F^a_{\nu\rho}F^{\mu\rho a} + F^{\rho\mu a}F^b_{\rho\nu} + F^{\rho\mu b}F^a_{\rho\nu} + F^b_{\nu\rho}F^{\mu\rho b}\right) D_\mu A_i D^\nu \overline{A}_i + (\text{perm})\right] \tag{B.0.63}$$

$$- 16i \left(\frac{\pi}{l_s^2}\right) \sum_i e^{-2\sigma_i} \left[\left(m_a^{(i)} F^{\mu\nu a} + m_b^{(i)} F^{\mu\nu b}\right) D_\mu A_i D_\nu \overline{A}_i + (\text{perm})\right] \tag{B.0.64}$$

$$- 16ig \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \left[\left\{ F^{\mu\nu a} \left(A_i \overline{A}_j - \overline{C}_j C_i \right) + F^{\mu\nu b} \left(B_i \overline{B}_j - \overline{A}_j A_i \right) \right\} D_\mu A_j D_\nu \overline{A}_i + (\text{perm}) \right] \quad (B.0.65)$$

$$+ 8ig \sum_{i,j} (F^a_{\mu\nu} + F^c_{\mu\nu})(A_i B_j - A_j B_i) D^\mu \overline{B}_i D^\nu \overline{A}_j + (\text{perm}) \quad (B.0.66)$$

$$+ 8ig \sum_{i,j} (F^a_{\mu\nu} + F^c_{\mu\nu})(\overline{B}_i \overline{A}_j - \overline{B}_j \overline{A}_i) D^\mu A_i D^\nu B_+ (\text{perm}) \quad (B.0.67)$$

$$+ 8 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \left[\left(D^\mu B_j D_\nu \overline{B}_i D_\mu A_i D^\nu \overline{A}_j + D_\nu B_j D^\nu \overline{B}_i D_\mu A_i D^\mu \overline{A}_j + D^\mu B_i D^\nu \overline{B}_i D_\nu A_j D^\mu A_{\overline{j}} \right) + (\text{perm}) \right] \quad (B.0.68)$$

$$+ 8 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \left[|D_\mu A_i D^\mu A_j|^2 + (\text{perm}) \right] \quad (B.0.69)$$

$$+ 16 \sum_{i} e^{-2\sigma_i} \left(\frac{\pi}{l_s^2} \right)^2 \left[((m_a^{(i)})^2 + (m_b^{(i)})^2) |D_\mu A_i|^2 + (\text{perm}) \right] \quad (B.0.70)$$

$$+ 16g \left(\frac{\pi}{l_s^2} \right) \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \left[\left\{ (m_b^{(i)} + m_b^{(j)})(B_i \overline{B}_j - \overline{A}_j A_i) + (m_a^{(i)} + m_a^{(j)})(A_i \overline{A}_j - \overline{C}_j C_i) \right\} D_\mu A_j D^\mu \overline{A}_i + (\text{perm}) \right] \quad (B.0.71)$$

$$+ 16g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left[\left\{ 2|A_j|^2 A_i \overline{A}_k - (B_i \overline{B}_j + C_i \overline{C}_j) A_j \overline{A}_k \right. \right. \quad (B.0.72)$$

$$\left. \left. -(B_j \overline{B}_k + C_j \overline{C}_k) A_i \overline{A}_j + |B_j|^2 B_i \overline{B}_k + |C_j|^2 C_i \overline{C}_k \right\} D^\mu A_k D_\mu \overline{A}_i + (\text{perm}) \right] \quad (B.0.73)$$

$$- 32g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left(\overline{B}_j B_k \overline{C}_i C_j - |C_j|^2 B_i \overline{B}_k - |B_j|^2 C_i \overline{C}_k + B_j \overline{B}_k C_i \overline{C}_j \right) D^\mu A_i D_\mu \overline{A}_k + (\text{perm}) \quad (B.0.74)$$

$$- 16g \left(\frac{\pi}{l_s^2} \right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \left[(A_j B_i - A_i B_j) m_c^{(i)} D^\mu \overline{B}_j D_\mu \overline{A}_i + (\overline{B}_j \overline{A}_i - \overline{A}_j \overline{B}_i) m_a^{(i)} D^\mu A_j D_\mu B_i \right] + (\text{perm}) \quad (B.0.75)$$

$$- 16g^2 \sum_{i,j,k} e^{-2(\sigma_i+\overset{\sigma}{j}+\sigma_k)} \left(|A_j|^2 \overline{A}_i \overline{B}_k - \overline{A}_j \overline{B}_k \overline{C}_i C_j - \overline{A}_i A_j \overline{A}_k B_j + \overline{A}_k \overline{B}_j \overline{C}_i C_j \right) D^\mu A_k D_\mu B_i + (\text{perm}) \quad (B.0.76)$$

$$- 16g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left(A_k B_j C_i \overline{C}_j - |B_j|^2 A_k B_i - A_j B_k C_i \overline{C}_j + A_j B_i \overline{B}_j B_k \right) D_\mu \overline{B}_k D^\mu \overline{A}_i + (\text{perm}) \quad (B.0.77)$$

$$- 16g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left(A_j \overline{B}_j C_i \overline{C}_k - A_i \overline{B}_j C_j \overline{C}_k - A_j \overline{B}_k C_i \overline{C}_j + |C_j|^2 A_i \overline{B}_k \right) D_\mu B_k D^\mu \overline{A}_i + (\text{perm}) \quad (B.0.78)$$

$$- 8g \left(\frac{\pi}{l_s^2} \right) \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \left((A_j B_i - A_i B_j) D^\mu \overline{B}_j m_b^{(i)} D_\mu \overline{A}_i + (\overline{B}_j \overline{A}_i - \overline{B}_i \overline{A}_i) D^\mu A_j m_b^{(i)} D_\mu B_i \right) + (\text{perm}) \quad (B.0.79)$$

$$- 16g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left(\overline{A}_j \overline{B}_i B_j \overline{B}_k - |A_j|^2 \overline{A}_i \overline{B}_k - |B_j|^2 \overline{A}_k \overline{B}_i + \overline{A}_i A_j \overline{A}_k \overline{B}_j\right) D^\mu A_k D_\mu B_i + (\text{perm}) \tag{B.0.80}$$

$$- 16g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left(|B_j|^2 A_k B_i - A_i \overline{A}_j A_k B_j - A_j B_i \overline{B}_j B_k + |A_j|^2 A_i B_k\right) D_\mu \overline{A}_i D^\mu \overline{B}_k + (\text{perm}) \tag{B.0.81}$$

$$- 16g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left(|C_j|^2 \overline{A}_k A_i - \overline{A}_k B_j C_i \overline{C}_j - \overline{A}_j B_i C_j \overline{C}_k + \overline{A}_j B_j C_i \overline{C}_k\right) D_\mu A_k D^\mu \overline{B}_i + (\text{perm}) \tag{B.0.82}$$

$$- 8g \left(\frac{\pi}{l_s^2}\right) \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \Big((A_j B_i - A_i B_j) D_\mu \overline{B}_i m_b^{(i)} \overline{A}_j + (\overline{B}_j \overline{A}_i - \overline{B}_i \overline{A}_i) D_\mu A^i m_b^{(i)} D^\mu B_j\Big) + (\text{perm}) \tag{B.0.83}$$

$$- 8g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left(\overline{A}_j \overline{B}_i B_j \overline{B}_k - |A_j|^2 \overline{A}_i \overline{B}_k - |B_j|^2 \overline{A}_k \overline{B}_i + \overline{A}_i A_j \overline{A}_k \overline{B}_j\right) D_\mu A_i D^\mu B_k + (\text{perm}) \tag{B.0.84}$$

$$- 8g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left(A_k B_i |B_j|^2 - A_i \overline{A}_j A_k B_j - A_j B_i \overline{B}_j B_k + A_i |A_j|^2 B_k\right) D_\mu \overline{A}_k D^\mu \overline{B}_i + (\text{perm}) \tag{B.0.85}$$

$$+ 32 \left(\frac{\pi}{l_s^2}\right)^2 \sum_i e^{-2\sigma_i} m_a^{(i)} m_b^{(i)} |D_\mu A_i|^2 + (\text{perm}) \tag{B.0.86}$$

$$+ 8g \left(\frac{\pi}{l_s^2}\right) \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \Big[\Big\{(m_b^{(i)} - m_a^{(j)}) A_i \overline{A}_j + m_a^{(j)} B_i \overline{B}_j - m_b^{(i)} C_i \overline{C}_j\Big\} D_\mu A_j D^\mu \overline{A}_i + (\text{perm})\Big] \tag{B.0.87}$$

$$+ 8g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left[\left(B_i \overline{B}_j A_j \overline{A}_k - B_i \overline{B}_j C_j \overline{C}_k - |A_j|^2 A_i \overline{A}_k + A_i \overline{A}_j C_j \overline{C}_k\right) D^\mu A_k D_\mu \overline{A}_i + (\text{perm})\right] \tag{B.0.88}$$

$$+ 16g \left(\frac{\pi}{l_s^2}\right) \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \left[(B_j \overline{B}_i - \overline{A}_i A_j) D_\mu A_i D^\mu \overline{A}_j + (\text{perm})\right] \tag{B.0.89}$$

$$+ 16g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left[\left(A_i \overline{A}_j B_j \overline{B}_j - |A_j|^2 A_i \overline{A}_k - C_i \overline{C}_j C_j \overline{B}_k + C_i \overline{C}_j A_j \overline{A}_k\right) D_\mu A_k D^\mu \overline{A}_i + (\text{perm})\right] \tag{B.0.90}$$

$$- 16 \left(\frac{\pi}{l_s^2}\right)^4 \sum_i \Big[(m_a^{(i)})^4 + \text{perm}\Big] \tag{B.0.91}$$

$$- 64g \left(\frac{\pi}{l_s^2}\right)^3 \sum_i e^{-2\sigma_i} \Big[\Big((m_a^{(i)})^3 - (m_b^{(i)})^3\Big) |Ai|^2 + (\text{perm})\Big] \tag{B.0.92}$$

$$- 16g^2 \left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \left[\left\{(m_a^{(i)})^2 + (m_a^{(j)})^2 - (m_b^{(i)})^2 - (m_b^{(j)})^2 + 2m_a^{(i)}m_a^{(j)} - 2m_b^{(i)}m_b^{(j)}\right\} A_j\overline{A}_i + (\text{perm})\right] \tag{B.0.93}$$

$$- 32g^2 \left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \left[\left((m_a^{(j)})^2 + (m_b^{(j)})^2\right) |A_i|^2|A_j|^2 - 2\left((m_b^{(i)})^2 + (m_b^{(j)})^2\right) A_i\overline{A}_j\overline{B}_iB_j\right] \tag{B.0.94}$$

$$- 16g^3 \left(\frac{\pi}{l_s^2}\right) \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left[\left(m_a^{(i)} + m_a^{(j)} + 2m_a^{(k)} - m_b^{(i)} - m_b^{(j)} - 2m_b^{(k)}\right) |A_i|^2|A_j|^2|A_k|^2 \right. \tag{B.0.95}$$

$$\left. +4\left(m_b^{(i)} + m_b^{(j)} + m_b^{(k)}\right) |A_i|^2A_j\overline{A}_k\overline{B}_jB_k - 4\left(m_a^{(i)} + m_a^{(j)} + m_a^{(k)}\right) |A_i|^2A_j\overline{A}_k\overline{C}_jC_k + (\text{perm})\right] \tag{B.0.96}$$

$$- 64g^4 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)} \left[|A_i|^2|A_j|^2|A_k|^2|A_l|^2 - 2|A_i|^2|A_j|^2A_k\overline{A}_l\left(\overline{B}_kB_l + \overline{C}_kC_l\right)\right. \tag{B.0.97}$$

$$\left. +2|A_i|^2A_j\overline{A}_k|B_l|^2B_k\overline{B}_j + A_i\overline{A}_jA_k\overline{A}_l\overline{B}_iB_j\overline{B}_kB_l + (\text{perm})\right] \tag{B.0.98}$$

$$- 16\left(\frac{\pi}{l_s^2}\right)^4 \sum_i \left((m_a^{(i)})^4 + (\text{perm})\right) - 64g\left(\frac{\pi}{l_s^2}\right)^3 \sum_i e^{-2\sigma_i}\left[\left((m_a^{(i)})^3 - (m_b^{(i)})^3\right)|A_i|^2 + (\text{perm})\right] \tag{B.0.99}$$

$$- 16g^2 \left(\frac{\pi}{l_s^2}\right)^4 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \left[\left\{(m_a^{(i)} + m_a^{(j)})^2 + (m_b^{(i)} + m_b^{(j)})^2\right\} |A_i|^2|A_j|^2\right. \tag{B.0.100}$$

$$\left. -2(m_b^{(i)} + m_b^{(j)})^2A_i\overline{A}_j\overline{B}_iB_j + (\text{perm})\right] \tag{B.0.101}$$

$$- 16g^2 \left(\frac{\pi}{l_s^2}\right)^4 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \left[\left((m_a^{(i)})^2 + (m_b^{(i)})^2 + (m_a^{(j)})^2 + (m_b^{(j)})^2\right)|A_i|^2|A_j|^2\right. \tag{B.0.102}$$

$$\left. -2\left((m_b^{(i)})^2 + (m_b^{(j)})^2\right)A_i\overline{A}_j\overline{B}_iB_j + (\text{perm})\right] \tag{B.0.103}$$

$$- 32g^3 \left(\frac{\pi}{l_s^2}\right) \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left[\left(m_a^{(i)} + m_a^{(j)} + m_a^{(k)}\right)|A_i|^2|A_j|^2|A_k|^2\right. \tag{B.0.104}$$

$$\left. +2\left((m_b^{(i)} + m_b^{(j)} + m_b^{(k)})\overline{B}_jB_k - (m_a^{(i)} + m_a^{(j)} + m_a^{(k)})\overline{C}_jC_k\right)|A_i|^2A_j\overline{A}_k + (\text{perm})\right] \tag{B.0.105}$$

$$+ 16g^2 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)} \left(|A_i|^2|A_j|^2|A_k|^2|C_l|^2 + |A_i|^2|A_k|^2B_j\overline{B}_l\overline{C}_jC_l\right) + (\text{perm}) \tag{B.0.106}$$

$$- 16g^2 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)} \left(|A_i|^2A_j\overline{A}_k(B_k\overline{B}_l\overline{C}_jC_l + B_l\overline{B}_jC_k\overline{C}_l)\right. \tag{B.0.107}$$

$$\left. +|A_i|^2A_j\overline{A}_k|B_l|^2C_k\overline{C}_j + |A_i|^2A_j\overline{A}_k\overline{B}_jB_k|C_l|^2 - 2A_i\overline{A}_jA_k\overline{A}_lB_j\overline{B}_k\overline{C}_iC_l\right) + (\text{perm}) \tag{B.0.108}$$

$$- 64g^2 \left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} m_c^{(i)}m_c^{(j)} \left(|A_i|^2|B_j|^2 - A_i\overline{A}_j\overline{B}_iB_j\right) \tag{B.0.109}$$

$$- 64g^3 \left(\frac{\pi}{l_s^2}\right) \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} m_c^{(j)} \left(\overline{A}_iA_k|B_j|^2C_i\overline{C}_k - \overline{A}_iA_j\overline{B}_jB_kC_i\overline{C}_k\right. \tag{B.0.110}$$

$$-\overline{A}_i A_k \overline{B}_j B_i C_j \overline{C}_k + |A_j|^2 \overline{B}_i B_k C_i \overline{C}_k - |A_j|^2 |B_i|^2 |B_k|^2 + \overline{A}_j A_k |B_i|^2 B_j \overline{B}_k \Big) \tag{B.0.111}$$

$$- 64g^4 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)} \Big(-A_i \overline{A}_j A_k \overline{A}_l B_j \overline{B}_k \overline{C}_i C_k \tag{B.0.112}$$

$$+|A_l|^2 A_i \overline{A}_j B_j \overline{B}_k \overline{C}_i C_k + |A_l|^2 A_i \overline{A}_j B_k \overline{B}_i \overline{C}_k C_j - |A_i|^2 |A_l|^2 B_j \overline{B}_k \overline{C}_j C_k \Big) \tag{B.0.113}$$

$$+ 32g^2 \left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \Big((m_a^{(i)})^2 + (m_a^{(j)})^2 \Big) \Big(|A_i|^2 |B_j|^2 - A_i \overline{A}_j \overline{B}_i B_j \Big) + (\text{perm}) \tag{B.0.114}$$

$$- 32g^3 \left(\frac{\pi}{l_s^2}\right) \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \Big(-(m_a^{(i)} + m_a^{(j)}) |A_i|^2 |A_j|^2 |B_k|^2 + 2m_a^{(i)} |A_i|^2 A_j \overline{A}_k \overline{B}_j B_k \tag{B.0.115}$$

$$-(m_a^{(j)} + m_a^{(k)}) \left(\overline{A}_i A_j B_i \overline{B}_k \overline{C}_j C_k + A_i \overline{A}_j \overline{B}_i B_k \overline{C}_k C_j \right) \Big) + (\text{perm}) \tag{B.0.116}$$

$$- 32g^4 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)} \Big(-|A_i|^2 |A_j|^2 |A_k|^2 |B_l|^2 + A_i \overline{A}_j |A_k|^2 |A_l|^2 \overline{B}_i B_j \tag{B.0.117}$$

$$+A_j |A_k|^2 \overline{A}_l (\overline{B}_i B_l C_i \overline{C}_j + B_i \overline{B}_j \overline{C}_i C_l) - 2A_i \overline{A}_j A_k \overline{A}_l \overline{B}_i B_j \overline{C}_k C_l \Big) \tag{B.0.118}$$

$$+ 32g^2 \left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \Big(m_a^{(i)} m_c^{(i)} + m_a^{(j)} m_c^{(j)} \Big) \Big(A_i \overline{A}_j \overline{B}_i B_j - |A_i|^2 |B_j|^2 \Big) + (\text{perm}) \tag{B.0.119}$$

$$- 16g^3 \left(\frac{\pi}{l_s^2}\right) \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \Big(-(m_a^{(i)}) + m_a^{(k)} + m_c^{(i)} m_c^{(k)}) |A_j|^2 \overline{B}_i B_k C_i \overline{C}_k - (m_c^{(i)} + m_c^{(k)}) \overline{A}_i A_j |A_k|^2 C_i \overline{C}_j \tag{B.0.120}$$

$$+(m_c^{(i)} + m_c^{(k)}) |A_i|^2 |A_k|^2 |C_j|^2 + (m_a^{(i)} + m_a^{(k)}) \left(\overline{A}_i A_j \overline{B}_j B_k \overline{C}_k C_i + A_i \overline{A}_j B_j \overline{B}_k C_k \overline{C}_i \right) \Big) + (\text{perm}) \tag{B.0.121}$$

$$- 16g^4 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)} \Big(-A_i \overline{A}_j A_k \overline{A}_l \overline{B}_i B_j \overline{B}_k B_l + 2A_i \overline{A}_j A_k \overline{A}_l \overline{B}_i B_j \overline{C}_k C_l \tag{B.0.122}$$

$$-|A_l|^2 \overline{A}_k A_i (\overline{B}_i B_j \overline{C}_j C_k + \overline{C}_i C_j \overline{B}_j B_k) + |A_k|^2 A_i \overline{A}_l |B_j|^2 \overline{C}_i C_k \Big) \tag{B.0.123}$$

$$- 16g^4 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)} \Big(-A_i \overline{A}_j A_k \overline{A}_l \overline{B}_i B_j \overline{B}_k B_l + 2A_i \overline{A}_j A_k \overline{A}_l \overline{B}_i B_j \overline{C}_k C_l \tag{B.0.124}$$

$$-|A_l|^2 \overline{A}_i A_k (C_i \overline{C}_j B_j \overline{B}_k + B_i \overline{B}_j C_j \overline{C}_k) + |A_l|^2 A_i \overline{A}_k |B_j|^2 \overline{B}_i B_k \Big) \tag{B.0.125}$$

$F_{MN}F^{RN}F_{RL}F^{ML}$

$$F_{MN}F^{RN}F_{RL}F^{ML} \tag{B.0.126}$$

$$= F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} + 4F_{\mu\nu}F^{\nu\rho}F_{\rho m}F^{m\mu} \tag{B.0.127}$$

$$+ 4F_{mn}F^{n\mu}F_{\mu\nu}F^{\nu m} + 2F_{\mu m}F^{m\nu}F_{\nu n}F^{n\mu} \tag{B.0.128}$$

$$+ 4F_{\mu m}F^{mn}F_{nk}F^{k^m u} + F_{mn}F^{nk}F_{kl}F^{lm} \tag{B.0.129}$$

$$= F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} \tag{B.0.130}$$

$$- 8\sum_i e^{-2\sigma_i}F_{\mu\nu}F^{\nu\rho}\left(D_\rho a_i D^\mu \overline{a}_i + D_\rho \overline{a}_i D^\mu a_i\right) \tag{B.0.131}$$

$$+ 16i\left(\frac{\pi}{l_s^2}\right)\sum_i e^{-2\sigma_i}m^{(i)}\left(D^\mu \overline{a} i F_{\mu\nu}D^\nu a_i - D^\mu a_i F_{\mu\nu}D^\nu \overline{a}_i\right) \tag{B.0.132}$$

$$+ 16ig\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}[a_i,\overline{a}_j]\left(D^\mu \overline{a}_i F_{\mu\nu}D^\nu aj - D^\mu a_j F_{\mu\nu}D^\nu \overline{a}_i\right) \tag{B.0.133}$$

$$+ 16ig\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\left([a_i,a_j]D^\mu \overline{a}_i F_{\mu\nu}D^\nu \overline{a}_j + [\overline{a}_i,\overline{a}_j]D^\mu a_i F_{\mu\nu}D^\nu a_j\right) \tag{B.0.134}$$

$$+ 8\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\left(D_\mu a_i D_\nu \overline{a}_i D^\nu a_j D^\mu \overline{a}_j + D_\mu a_i D_\nu \overline{a}_i D_\nu \overline{a}_j D^\mu a_j + D_\mu \overline{a}_i D_\nu a_i D^\nu \overline{a}_j D^\mu a_j + D_\mu \overline{a}_i D_\nu a_i D^\nu a_j D^\mu \overline{a}_j\right) \tag{B.0.135}$$

$$+ 32\left(\frac{\pi}{l_s^2}\right)^2\sum_i e^{-2\sigma_i}(m^{(i)})^2(D_\mu a_i D^\mu \overline{a}_i + D_\mu \overline{a}_i D^\mu a_i) \tag{B.0.136}$$

$$+ 32g\left(\frac{\pi}{l_s^2}\right)\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}m^{(j)}\left\{[a_i,\overline{a}_j]D_\mu \overline{a} i D^\mu a_j + D_\mu a_i D^\mu \overline{a}^j\right\} \tag{B.0.137}$$

$$+ 32g\left(\frac{\pi}{l_s^2}\right)\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}[a_i,\overline{a}_j]\left(m^{(i)}D_\mu \overline{a}_i D^\mu a_j + m^{(j)}D_\mu a_j D^\mu \overline{a}_i\right) \tag{B.0.138}$$

$$+ 32g^2\sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)}[a_j,\overline{a}k]\left([a_i,\overline{a}_j]D_\mu \overline{a}_i D^\mu a_k + [a_k,\overline{a}_i]D^\mu a_i D_\mu \overline{a}_j\right) \tag{B.0.139}$$

$$- 32g^2\sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)}\left([\overline{a}_k,\overline{a}_j][a_i,a_j]D_\mu \overline{a}_i D^\mu a_k + [a_k,a_j][\overline{a}_i,\overline{a}_j]D_\mu a_i D^\mu \overline{a}_k\right. \tag{B.0.140}$$

$$\left.+[a_k,a_j][a_i,\overline{a}_j]D_\mu \overline{a}_i D^\mu \overline{a}_k + [a_k,\overline{a}_j][a_i,a_j]D_\mu \overline{a}_i D^\mu \overline{a}_k + [a_j,\overline{a}_k][\overline{a}_j,\overline{a}_i]D_\mu a_i D^\mu \overline{a}_k + [\overline{a}_j,\overline{a}_k][a_j,\overline{a}_i]D_\mu a_i D^\mu a_k\right) \tag{B.0.141}$$

$$- 32g\left(\frac{\pi}{l_s^2}\right)\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\left([a_j,a_i]m^{(i)}D_\mu \overline{a}_i D^\mu \overline{a}_j + m^{(j)}[a_i,a_j]D_\mu \overline{a}_i D^\mu \overline{a}_j\right. \tag{B.0.142}$$

$$\left.+m^{(j)}[\overline{a}_j,\overline{a}_i]D_\mu a_i D^\mu a_j + [\overline{a}_i,\overline{a}_j]m^{(i)}D_\mu a_i D^\mu a_j\right) \tag{B.0.143}$$

$$+16\left(\frac{\pi}{l_s^4}\right)^4\sum_i(m^{(i)})^4+64\left(\frac{\pi}{l_s^2}\right)^2\left(m^{(i)}\right)^3[a_i,\overline{a}_i] \tag{B.0.144}$$

$$+64\left(\frac{\pi}{l_s^2}\right)^2 g^2\sum_{i,j}e^{-2(\sigma_i+\sigma_j)}\left(m^{(j)}\right)^2[a_i,\overline{a}_j][a_j,\overline{a}_i] \tag{B.0.145}$$

$$+32\left(\frac{\pi}{l_s^2}\right)^2 g^2\sum_{i,j}e^{-2(\sigma_i+\sigma_j)}m^{(j)}[a_i,\overline{a}_j]m^{(i)}[a_j,\overline{a}_i] \tag{B.0.146}$$

$$+64\left(\frac{\pi}{l_s^2}\right)^2 g^2\sum_{i,j}e^{-2(\sigma_i+\sigma_j)}m^{(j)}[a_i,\overline{a}_j][a_k,\overline{a}_i][a_j,\overline{a}_k] \tag{B.0.147}$$

$$+16g^4\sum_{i,j,k,l}e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)}[a_j,\overline{a}_l][a_k,\overline{a}_j][a_i,\overline{a}_k][a_l,\overline{a}_i1] \tag{B.0.148}$$

$$+16\left(\frac{\pi}{l_s^2}\right)^4\sum_i(m^{(i)})^4 \tag{B.0.149}$$

$$+64\left(\frac{\pi}{l_s^2}\right)^3 g\sum_i(m^{(i)})[a_i,\overline{a}_i] \tag{B.0.150}$$

$$+64\left(\frac{\pi}{l_s^2}\right)^2 g^2\sum_{i,j}e^{-2(\sigma_i+\sigma_j)}(m^{(i)})^2[a_i,\overline{a}_j][a_j,\overline{a}_i] \tag{B.0.151}$$

$$+32\left(\frac{\pi}{l_s^2}\right)^2 g^2\sum_{i,j}e^{-2(\sigma_i+\sigma_j)}m^{(i)}[a_i,\overline{a}_j]m^{(j)}[a_j,\overline{a}_i] \tag{B.0.152}$$

$$+64\left(\frac{\pi}{l_s^2}\right) g^3\sum_{i,j,k}e^{-2(\sigma_i+\sigma_j+\sigma_k)}m^{(i)}[a_i,\overline{a}_j][a_j,\overline{a}_k][a_k,\overline{a}_i] \tag{B.0.153}$$

$$+16g^4\sum_{i,j,k,l}e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)}[a_i,\overline{a}_j][a_j,\overline{a}_k][a_k,\overline{a}_l][a_l,\overline{a}_i] \tag{B.0.154}$$

$$+64g^2\left(\frac{\pi}{l_s^2}\right)^2\sum_{i,j}e^{-2(\sigma_i+\sigma_j)}(m^{(i)})^2[\overline{a}_i,\overline{a}_j][a_i,a_j] \tag{B.0.155}$$

$$+64g^3\left(\frac{\pi}{l_s^2}\right)\sum_{i,j,k}e^{-2(\sigma_i+\sigma_j+\sigma_k)}\left([\overline{a}_k,\overline{a}_i][a_i,a_j]m^{(j)}[a_k,\overline{a}_j]+[\overline{a}_k,\overline{a}_i][a_i,a_j][a_k,\overline{a}_j]m^{(k)}\right) \tag{B.0.156}$$

$$+64g^4\sum_{i,j,k,l}e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)}[\overline{a}_l,\overline{a}_i][a_i,a_j][a_k,\overline{a}_j][a_l,\overline{a}_k] \tag{B.0.157}$$

$$+64g^2\left(\frac{\pi}{l_s^2}\right)^2\sum_{i,j}e^{-2(\sigma_i+\sigma_j)}[a_i,a_j]m^{(j)}[\overline{a}_i,\overline{a}_j]m^{(i)} \tag{B.0.158}$$

$$-64g^3\left(\frac{\pi}{l_s^2}\right)\sum_{i,j,k}e^{-2(\sigma_i+\sigma_j+\sigma_k)}\left([a_]i,a_j][a_k,\overline{a}_j][\overline{a}_k,\overline{a}_i]m^{(i)}+[a_i,a_j]m^{(j)}[\overline{a}_j,\overline{a}_k][a_k,\overline{a}_i]\right) \tag{B.0.159}$$

$$-64g^4\sum_{i,j,k,l}e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)}[a_i,a_j][a_k,\overline{a}_j][\overline{a}_k,\overline{a}_l][a_l,\overline{a}_i] \tag{B.0.160}$$

$$+32g^4\sum_{i,j,k,l}e^{-2(\sigma_i*\sigma_j+\sigma_k+\sigma_l)}[a_i,a_j][\overline{a}_j,\overline{a}_k][a_k,a_l][\overline{a}_l,\overline{a}_i] \tag{B.0.161}$$

$$+ 64g^2 \left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)}[a_i, a_j][\overline{a}_j, \overline{a}_i](m^{(i)})^2 \tag{B.0.162}$$

$$+ 64g^3 \left(\frac{\pi}{l_s^2}\right) \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left([a_i, a_j][\overline{a}_j, \overline{a}_k]m^{(k)}[a_k, \overline{a}_i] + [a_i, a_j][\overline{a}_j, \overline{a}_k][a_k, \overline{a}_i]m^{(i)}\right) \tag{B.0.163}$$

$$+ 64g^4 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)}[a_i, a_j][\overline{a}_j, \overline{a}_k][a_k, \overline{a}_l][a_l, \overline{a}_i] \tag{B.0.164}$$

$$= F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} \tag{B.0.165}$$

$$- 8 \sum_i e^{-2\sigma_i} \left[\left(F^a_{\mu\nu}F^{\nu\rho a} + F^b_{\mu\nu}F^{\nu\rho b}\right) D_\rho A_i D^\mu \overline{A}_i + (\text{perm})\right] \tag{B.0.166}$$

$$- 16i \left(\frac{\pi}{l_s^2}\right) \sum_i e^{-2\sigma_i} \left[\left(m_a^{(i)} F^b_{\mu\nu} + m_b^{(i)} F^a_{\mu\nu}\right) D^\mu A_i D^\nu \overline{A}_i + (\text{perm})\right] \tag{B.0.167}$$

$$+ 16ig \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \left[\left((C_i\overline{C}_j - A_i\overline{A}_j)F^b_{\mu\nu} + (A_i\overline{A}_j - B_i\overline{B}_j)F^a_{\mu\nu}\right) D^\mu A_j D^\nu \overline{A}_i\right] \tag{B.0.168}$$

$$+ 16ig \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \left((A_iB_j - A_jB_i)F^b_{\mu\nu}D^\mu\overline{B}_i D^\nu\overline{A}_j + (\overline{B}_i\overline{A}_j - \overline{B}_j\overline{A}_i)F^b_{\mu\nu}D^\mu A_i D^\nu B_j\right) + (\text{perm}) \tag{B.0.169}$$

$$+ 16 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \left[|D_\mu A_i D_\nu A_j|^2 + D_\mu A_i D_\nu \overline{A}_i D^\mu B_j D^\nu \overline{B}_j + (\text{perm})\right] \tag{B.0.170}$$

$$+ 32 \left(\frac{\pi}{l_s^2}\right)^2 \sum_i e^{-2\sigma_i} \left[((m_a^{(i)})^2 + (m_b^{(i)})^2)|D_\mu A_i|^2 + (\text{perm})\right] \tag{B.0.171}$$

$$+ 32g \left(\frac{\pi}{l_s^2}\right) \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \left[\left((m_a^{(i)} + m_a^{(j)})(\overline{A}_iA_j - \overline{C}_iC_j) + (m_b^{(i)} + m_b^{(j)})(\overline{B}_iB_j - \overline{A}_iA_j)\right) D_\mu A_i D^\mu \overline{A}_j + (\text{perm})\right] \tag{B.0.172}$$

$$- 32g \left(\frac{\pi}{l_s^2}\right) \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \left(m_c^{(j)} - m_a^{(i)}\right) \left((A_iB_j - A_jB_i)D_\mu\overline{A}_i D^\mu\overline{B}_j + (\overline{A}_i\overline{B}_j - \overline{A}_j\overline{B}_i)D_\mu A_i D^\mu B_j\right) \tag{B.0.173}$$

$$- 32g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left(\overline{A}_iA_j(\overline{B}_jB_k + \overline{C}_jC_k) + \overline{A}_jA_k(\overline{B}_iB_j + \overline{C}_iC_j)\right. \tag{B.0.174}$$

$$\left.-|A_j|^2(\overline{B}_iB_k - \overline{C}_iC_k) - \overline{A}_iA_k(|B_j|^2 - |C_j|^2)\right) D_\mu A_i D^\mu \overline{A}_k \tag{B.0.175}$$

$$- 32g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left(A_i\overline{A}_jA_kB_j - |A_j|^2A_kB_i + A_jB_i\overline{B}_jB_k - A_k|B_j|^2B_i\right. \tag{B.0.176}$$

$$\left.+(A_kB_j - A_jB_k)C_i\overline{C}_j + (A_jB_i - A_iB_j)\overline{C}_jC_k\right) D_\mu\overline{B}_i D^\mu\overline{A}_k \tag{B.0.177}$$

$$- 32g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left(\overline{A}_iA_j\overline{A}_k\overline{B}_j - \overline{A}_i|A_j|^2\overline{B}_k + \overline{A}_j\overline{B}_iB_j\overline{B}_k - \overline{A}_j|B_j|^2\overline{B}_k\right. \tag{B.0.178}$$

$$\left.+(\overline{A}_i\overline{B}_j - \overline{A}_j\overline{B}_i)C_j\overline{C}_k + (\overline{A}_j\overline{B}_k - \overline{A}_k\overline{B}_j)\overline{C}_iC_j\right) D_\mu A_i D^\mu B_k \tag{B.0.179}$$

$$+ 32g^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left[(2|A_k|^2\overline{A}_iA_j + |B_k|^2\overline{B}_iB_j + |C_k|^2\overline{C}_iC_j\right. \tag{B.0.180}$$

$$-A_j\overline{A}_k(\overline{B}_iB_k+\overline{C}_iC_k)-\overline{A}_iA_k(B_j\overline{B}_k+C_j\overline{C}_k)\big)\,D_\mu A_iD^\mu\overline{A}_j+(\mathrm{perm})\Big] \tag{B.0.181}$$

$$+\,32\left(\frac{\pi}{l_s^2}\right)^4\sum_i\Big((m_a^{(i)})^4+(\mathrm{perm})\Big) \tag{B.0.182}$$

$$+\,128\left(\frac{\pi}{l_s^2}\right)^3 g\sum_i e^{-2\sigma_i}\Big[\Big((m_a^{(i)})^3-(m_b^{(i)})^3\Big)|A_i|^2+(\mathrm{perm})\Big] \tag{B.0.183}$$

$$+\,64\left(\frac{\pi}{l_s^2}\right)^2 g^2\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\Big[\Big((m_a^{(i)})^2+m_a^{(i)}m_a^{(j)}+(m_a^{(j)})^2+(m_b^{(i)})^2+m_b^{(i)}m_b^{(j)}+(m_b^{(j)})^2\Big)|A_i|^2|A_j|^2+(\mathrm{perm})\Big] \tag{B.0.184}$$

$$-\,64\left(\frac{\pi}{l_s^2}\right)^2 g^2\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\Big[\Big((m_b^{(i)})^2+m_b^{(i)}m_b^{(j)}+(m_b^{(j)})^2\Big)A_i\overline{A}_J\overline{B}_iB_j+(\mathrm{perm})\Big] \tag{B.0.185}$$

$$+\,64\left(\frac{\pi}{l_s^2}\right) g^3\sum_{i,j} e^{-2(\sigma_i+\sigma_j+\sigma_k)}\Big[(m_a^{(i)}+m_a^{(j)}+m_b^{(i)}+m_b^{(j)})|A_i|^2|A_j|^2|A_k|^2+(\mathrm{perm})\Big] \tag{B.0.186}$$

$$+\,128\left(\frac{\pi}{l_s^2}\right) g^3\sum_{i,j} e^{-2(\sigma_i+\sigma_j+\sigma_k)}\Big[\Big((m_b^{(i)}+m_b^{(j)}+m_b^{(k)})\overline{B}_iB_j-(m_a^{(i)}+m_a^{(j)}+m_a^{(k)})\overline{C}_iC_j\Big)|A_k|^2A_i\overline{A}_j+(\mathrm{perm})\Big] \tag{B.0.187}$$

$$+\,64g^4\sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)}\left[|A_i|^2|A_j|^2|A_k|^2|A_l|^2+(\mathrm{perm})\right] \tag{B.0.188}$$

$$-\,128g^4\sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)}\left[|A_k|^2|A_l|^2A_i\overline{A}_j(\overline{B}_iB_j+\overline{C}_iC_j)+(\mathrm{perm})\right] \tag{B.0.189}$$

$$+\,128g^4\sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)}\left[|A_k|^2A_i\overline{A}_j|B_l|^2\overline{B}_iB_j+(\mathrm{perm})\right] \tag{B.0.190}$$

$$+\,64g^4\sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)}\left[A_i\overline{A}_jA_k\overline{A}_l\overline{B}_iB_j\overline{B}_kB_l+(\mathrm{perm})\right] \tag{B.0.191}$$

$$+\,64g^2\left(\frac{\pi}{l_s^2}\right)^2\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\Big((m_a^{(i)})^2+(m_a^{(j)})^2+m_a^{(i)}m_c^{(j)}m_a^{(j)}m_c^{(i)}+(m_c^{(i)})^2+(m_c^{(j)})^2\Big)\Big(|A_i|^2|B_j|^2-A_i\overline{A}_j\overline{B}_iB_j\Big) \tag{B.0.192}$$

$$+\,64g^3\left(\frac{\pi}{l_s^2}\right)\sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)}\Big[\Big((m_c^{(i)}+m_a^{(j)}-m_a^{(k)})|B_k|^2-(m_a^{(j)}+m_a^{(i)}-m_c^{(k)})|C_k|^2\Big)|A_i|^2|A_j|^2 \tag{B.0.193}$$

$$+\Big((m_a^{(i)}+m_a^{(j)}-2m_a^{(k)}-m_c^{(k)})\overline{B}_iB_j-(m_c^{(i)}+m_c^{(j)}-m_c^{(k)}-2m_a^{(k)})\overline{C}_iC_j\Big)A_i\overline{A}_j|A_k|^2 \tag{B.0.194}$$

$$+\,(m_a^{(k)}+m_b^{(k)}-2m_c^{(k)})|A_i|^2\overline{B}_jB_kC_j\overline{C}_k \tag{B.0.195}$$

$$+\Big((-m_a^{(i)}-m_a^{(j)}+2m_c^{(i)}+m_c^{(k)})B_j\overline{B}_k\overline{C}_iC_k-(m_a^{(i)}+m_a^{(k)}-m_c^{(j)})\overline{B}_iB_kC_j\overline{C}_k\Big)A_i\overline{A}_j\Big] \tag{B.0.196}$$

$$+\,64g^4\sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)}\left[|A_i|^2|A_j|^2|A_k|^2|C_l|^2+2|A_i|^2|A_j|^2|B_k|^2|B_l|^2-|A_i|^2|A_l|^2\left(|B_j|^2\overline{B}_kC_k+\overline{B}_j|C_k|^2C_j\right)\right. \tag{B.0.197}$$

$$- |A_i|^2|A_j|^2 A_k\overline{A}_l\overline{C}_k C_l + |A_i|^2\overline{A}_l|B_k|^2 B_j\overline{C}_j C_l \tag{B.0.198}$$

$$+ \left(|B_j|^2 C_k\overline{C}_l - B_k\overline{B}_l|C_j|^2 - 4|B_j|^2 B_k\overline{B}_l - \overline{B}_l|C_j|^2 C_k + |B_j|^2\overline{B}_l C_k\right)|A_i|^2\overline{A}_k A_l \tag{B.0.199}$$

$$- \left(-\overline{B}_l B_k\overline{B}_j C_j + \overline{B}_j B_k C_j\overline{C}_l + B_j\overline{B}_l\overline{C}_l C_k + \overline{B}_j C_j C_k\overline{C}_l\right)|A_i|^2\overline{A}_k A_l \tag{B.0.200}$$

$$- \left(\overline{A}_i B_j\overline{B}_k B_i\overline{C}_j C_k + A_i\overline{B}_j B_k C_j\overline{C}_i\overline{C}_k\right)|A_l|^2 \tag{B.0.201}$$

$$+ 2A_i\overline{A}_j A_k\overline{A}_l\left(\overline{B}_i B_j\overline{C}_k C_l + \overline{B}_i B_j\overline{B}_k B_l + \overline{B}_i C_j\overline{C}_k C_l - \overline{B}_i B_j\overline{B}_k C_l\right) \tag{B.0.202}$$

$$+\overline{A}_i A_k B_i\overline{B}_j\overline{B}_l C_j\overline{C}_k C_l] \tag{B.0.203}$$

Here, the octanal coupling terms like $\overline{A}_i A_k B_i\overline{B}_j\overline{B}_l C_j\overline{C}_k C_l$ are not invariant under the each U(1) symmetry which derived from $F_{ij}F_{\bar{j}k}F_{\overline{kl}}F_{l\bar{i}}$.

$(F_{MN}F^{MN})^2$

$$(F_{MN}F^{MN})^2 \tag{B.0.204}$$
$$= |F_4|^4 + 4|F_4|^2 \sum_i e^{-2\sigma_i}(D_\mu a_i D^\mu \overline{a}_i + D_\mu \overline{a}_i D^\mu a_i) \tag{B.0.205}$$
$$+ 16\left(\frac{\pi}{l_s^2}\right)^2 \sum_i (m^{(i)})^2 |F_4|^2 \tag{B.0.206}$$
$$+ 32g\left(\frac{\pi}{ls^2}\right)|F_4|^2 \sum_i e^{-2\sigma_i} m^{(i)}[a_i,\overline{a}_i] + 16g^2|F_4|^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)}[a_i,\overline{a}_j][a_j,\overline{a}_i] \tag{B.0.207}$$
$$+ 4\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}(D_\mu a_i D^\mu \overline{a}_i + D_\mu \overline{a}_i D^\mu a_i)(D_\nu a_j D^\nu \overline{a}_j + D_\nu \overline{a}_j D^\nu a_j) \tag{B.0.208}$$
$$+ 32\left(\frac{\pi}{l_s^2}\right)^2 \left(\sum_j (m^{(j)})^2\right) \sum_i e^{-2\sigma_i}(D_\mu a_i D^\mu \overline{a}_i + D_\mu \overline{a}_i D^\mu a_i) \tag{B.0.209}$$
$$+ 128g\left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} m^{(i)}[a_i,\overline{a}_i](D_\mu a_j D^\mu \overline{a}_j + D_\mu \overline{a}_j D^\mu a_j) \tag{B.0.210}$$
$$+ 32g^3 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)}[a_j,\overline{a}_k][a_k,\overline{a}_j](D_\mu a_i D^\mu \overline{a}_i + D_\mu \overline{a}_i D^\mu a_i) \tag{B.0.211}$$
$$+ 64\left(\frac{\pi}{l_s^2}\right)^4 \sum_{i,j} (m^{(i)})^2 (m^{(j)})^2 + 256g\left(\frac{\pi}{l_s^2}\right)^3 \left(\sum_j (m^{(j)})^2\right) \sum_i e^{-2\sigma_i} m^{(i)}[a_i,\overline{a}_i] \tag{B.0.212}$$
$$+ 128g^2\left(\frac{\pi}{l_s^2}\right)^2 \left(\sum_k (m^{(k)})^2\right) \sum_{i,j} e^{-2(\sigma_i+\sigma_j)}[a_i,\overline{a}_j][a_j,\overline{a}_i] \tag{B.0.213}$$
$$+ 256g^2\left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} m^{(i)}[a_i.\overline{a}_i] m^{(j)}[a_j,\overline{a}_j] \tag{B.0.214}$$
$$+ 256g^3\left(\frac{\pi}{l_s^2}\right) \sum_{i,j,k} e^{(-2(\sigma_i+\sigma_j+\sigma_k))} m^{(k)}[a_k,\overline{a}_k][a_i,\overline{a}_j][a_j,\overline{a}_i] \tag{B.0.215}$$
$$+ 64g^4 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)}[a_i,\overline{a}_j][a_j,\overline{a}_i][a_k,\overline{a}_l][a_l,\overline{a}_k] \tag{B.0.216}$$
$$= |F_4^a|^4 + 4(|F_4^a|^2 + |F_4^b|^2) \sum_i e^{-2\sigma_i}|D_\mu A|^2 + 16\left(\frac{\pi}{l_s^2}\right)^2 \sum_i (m_a^{(i)})^2 |F_4^a|^2 \tag{B.0.217}$$
$$+ 32g\left(\frac{\pi}{l_s^2}\right) \sum_i e^{-2\sigma_i}\left(|F_4^a|^2 m_a^{(i)} - |F_4^b|^2 m_b^{(i)}\right)|A|^2 \tag{B.0.218}$$
$$+ 16g^2 \sum_{i,j} e^{(-2(\sigma_i+\sigma_j)}\left(|F_4^a|^2 + |F_4^b|^2\right)|A_i|^2|A_j|^2 - 32g^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)}|F_4^b|^2 A_i \overline{A}_j \overline{B}_i B_j \tag{B.0.219}$$
$$+ 8\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}|D_\mu A_i|^2 |D_\nu A_j|^2 + 8\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}|D_\mu A_i|^2 |D_\nu B_j|^2 \tag{B.0.220}$$

$$+ 32\left(\frac{\pi}{l_s^2}\right)^2 \sum_j \left((m_a^{(j)})^2 + (m_b^{(j)})^2\right) \sum_i e^{-2\sigma_i} |D_\mu A_i|^2 \tag{B.0.221}$$

$$+ 64\left(\frac{\pi}{l_s^2}\right) \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} (m_a^{(i)} + m_b^{(i)}) |A_i|^2 |D_\mu A_j|^2 + 64\left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} (|B_i|^2 - |C_i|^2) |D_\mu A_j|^2 \tag{B.0.222}$$

$$+ 32g^3 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left(2|A_j|^2|A_k|^2 - 2A_j\overline{A}_k\overline{C}_jC_k + |C_j|^2|C_k|^2 + |B_j|^2|B_k|^2\right) |D_\mu A_i|^2 \tag{B.0.223}$$

$$+ 62\left(\frac{\pi}{l_s^2}\right)^4 \sum_{i,j} (m_a^{(i)})^2 (m_a^{(j)})^2 + 256g\left(\frac{\pi}{l_s^2}\right)^3 \sum_{i,j} e^{-2\sigma_i} \left((m_a^{(j)})^2 m_a^{(i)} - (m_b^{(j)})^2 m_b^{(i)}\right) |A_i|^2 \tag{B.0.224}$$

$$+ 128g^2\left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j)} \left((m_a^{(k)})^2 + (m_b^{(k)})^2\right) |A_i|^2|A_j|^2 - 256g^2\left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j)} (m_b^{(k)}) A_i\overline{A}_j\overline{B}_iB_j \tag{B.0.225}$$

$$+ 256g^2\left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} \left(m_a^{(i)}m_a^{(j)} + m_b^{(i)}m_b^{(j)}\right) |A_i|^2|A_j|^2 - 512g^2\left(\frac{\pi}{l_s^2}\right) \sum_{i,j} e^{-2(\sigma_i+\sigma_j)} m_b^{(i)}m_b^{(j)} |A_i|^2|B_j|^2 \tag{B.0.226}$$

$$+ 256g^3\left(\frac{\pi}{l_s^2}\right) \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} (m_a^{(k)} + m_b^{(k)}) |A_i|^2|A_j|^2|A_k|^2 + 512g^3\left(\frac{\pi}{l_s^2}\right) \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} (m_b^{(k\overline{B}_iB_j - m_a^{(k)})}\overline{C}_iC_j)|A_k|^2A_i\overline{A}_j \tag{B.0.227}$$

$$+ 128g^4 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)} \left(|A_i|^2|A_j|^2|A_k|^2|A_l|^2 + |A_i|^2|A_j^2|B_k^2|B_l|^2 - 2|A_i|^2|A_j|^2A_k\overline{A}_l(\overline{B}_kB_l + \overline{C}_kC_l) + 2A_i\overline{A}_jA_k\overline{A}_l\overline{B}_iB_j \tag{B.0.228}$$

$F_{MN}F^{RL}F^{MN}F_{RL}$

$$F_{MN}F^{RL}F^{MN}F_{RL} \tag{B.0.229}$$

$$= F_{\mu\nu}F^{\rho\sigma}F^{\mu\nu}F_{\rho\sigma} + 4F_{\mu\nu}F^{\rho m}F^{\mu\nu}F_{\rho m} \tag{B.0.230}$$

$$+ 2F^{mn}F_{\mu\nu}F_{mn}F^{\mu\nu} + 4F_{\mu m}F^{\nu n}F^{\mu m}F_{\nu n} \tag{B.0.231}$$

$$+ 4F^{mn}F^{\mu l}F_{mn}F_{\mu l} + F_{mn}F^{kl}F^{mn}F_{kl} \tag{B.0.232}$$

$$= F_{\mu\nu}F^{\rho\sigma}F^{\mu\nu}F_{\rho\sigma} \tag{B.0.233}$$

$$+ 16\sum_i e^{-2\sigma_i}F_{\mu\nu}D^\rho a_i F^{\mu\nu}D_\rho \overline{a}_i \tag{B.0.234}$$

$$+ 16\left(\frac{\pi}{l_s^2}\right)^2 \sum_i F^{\mu\nu}m^{(i)}F_{\mu\nu}m^{(i)} \tag{B.0.235}$$

$$+ 32g\left(\frac{\pi}{l_s^2}\right) \sum_i e^{-2\sigma_i}F^{\mu\nu}m^{(i)}F_{\mu\nu}[a_i,\overline{a}_i] \tag{B.0.236}$$

$$+ 16g^2\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}F^{\mu\nu}[a_i,\overline{a}_j]F_{\mu\nu}[a_j,\overline{a}_i] \tag{B.0.237}$$

$$+ 64\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}D_\mu a_i D_\nu a_j D^\mu \overline{a}_i D^\nu \overline{a}_j \tag{B.0.238}$$

$$+ 128\left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} m^{(i)}D_\mu a_j m^{(i)}D_\mu \overline{a}_j \tag{B.0.239}$$

$$+ 128g\left(\frac{\pi}{l_s^2}\right) \sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\left(m^{(i)}D_\mu a_j[a_i,\overline{a}_i]D_\mu\overline{a}_j + [a_i,\overline{a}_i]D_\mu a_j m^{(i)}D_\mu \overline{a}_j\right) \tag{B.0.240}$$

$$+ 128g^2\sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)}[a_k,\overline{a}_i]D_\mu a_j[a_i,\overline{a}_k]D_\mu\overline{a}_j \tag{B.0.241}$$

$$+ 64\left(\frac{\pi}{l_s^2}\right)^4 \left(\sum_i (m^{(i)})^2\right)\left(\sum_j (m^{(j)})^2\right) \tag{B.0.242}$$

$$+ 256g\left(\frac{\pi}{l_s^2}\right)^3 \left(\sum_j (m^{(j)})^2\right) m^{(i)}[a_i,\overline{a}_i] \tag{B.0.243}$$

$$+ 256g^2\left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)}m^{(i)}m^{(j)}[a_i,\overline{a}_i][a_j,\overline{a}_j] \tag{B.0.244}$$

$$+ 128g^2\left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j)}m^{(k)}[a_i,\overline{a}_j]m^{(k)}[a_j,\overline{a}_i] \tag{B.0.245}$$

$$+ 256g^3\left(\frac{\pi}{l_s^2}\right) \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)}m^{(i)}[a_j,\overline{a}_k][a_i,\overline{a}_i][a_k,\overline{a}_j] \tag{B.0.246}$$

$$+ 64g^4[a_i, \overline{a}_j][a_k, \overline{a}_l][a_j, \overline{a}_i][a_l, \overline{a}_l] \tag{B.0.247}$$

$$= F_{\mu\nu}F^{\rho\sigma}F^{\mu\nu}F_{\rho\sigma} \tag{B.0.248}$$

$$+ 16\sum_i e^{-2\sigma_i}\left[F^a_{\mu\nu}F^{\mu\nu b}|D_\rho A_i|^2 + (\text{perm})\right] \tag{B.0.249}$$

$$+ 16\left(\frac{\pi}{l_s^2}\right)^2 \sum_i \left[(m_a^{(i)})^2|F_4^a|^2 + (\text{perm})\right] \tag{B.0.250}$$

$$+ 32g\left(\frac{\pi}{l_s^2}\right)\sum_i e^{-2\sigma_i}\left[\left(m_a^{(i)}|F_4^a|^2 - m_b^{(i}|F_4^b|^2\right)|A_i|^2 + (\text{perm})\right] \tag{B.0.251}$$

$$+ 16g^2\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\left[\left(|F_4^a|^2 + |F_4^b|^2\right)|A_i|^2|A_j|^2 + (\text{perm})\right] \tag{B.0.252}$$

$$- 32g^2\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\left[|F_4^b|^2 A_i\overline{A}_j\overline{B}_iB_j + (\text{perm})\right] \tag{B.0.253}$$

$$+ 64\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\left[D_\mu A_i D_\nu B_j D^\mu \overline{B}_i D^\nu \overline{A}_j + (\text{perm})\right] \tag{B.0.254}$$

$$+ 128\left(\frac{\pi}{l_s^2}\right)\sum_i \left[m_a^{(i)}m_b^{(i)}|D_\mu A_j|^2 + (\text{perm})\right] \tag{B.0.255}$$

$$+ 128g\left(\frac{\pi}{l_s^2}\right)\sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\left[\left((m_b^{(i)} - m_a^{(i)})|A_i|^2 + m_a^{(i)}|B_i|^2 - m_b^{(i)}|C_i|^2\right)|D_\mu A_j|^2 + (\text{perm})\right] \tag{B.0.256}$$

$$- 128g^2\sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)}\left[\left(|A_j|^2|A_k|^2 - (\overline{B}_jB_k + \overline{C}_jC_k)A_j\overline{A}_j + B_j\overline{B}_k\overline{C}_jC_k\right)|D_\mu A_i|^2 + (\text{perm})\right] \tag{B.0.257}$$

$$+ 64\left(\frac{\pi}{l_s^2}\right)^4 \sum_{i,j}\sum_{\alpha=a,b,c}(m_\alpha^{(i)})^2(m_\alpha^{(j)})^2 \tag{B.0.258}$$

$$+ 256g\left(\frac{\pi}{l_s^2}\right)^3 \sum_i e^{-2\sigma_i}\left[\sum_j \left((m_a^{(j)})^2 m_a^{(i)} - (m_b^{(j)})^2 m_b^i\right)|A_i|^2 + (\text{perm})\right] \tag{B.0.259}$$

$$+ 256g^2\left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\left[\left(m_a^{(i)}m_a^{(j)} + m_b^{(i)}m_b^{(j)}\right)|A_i|^2|A_j|^2 + (\text{perm})\right] \tag{B.0.260}$$

$$- 512g^2\left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\left[m_b^{(i)}m_b^{(j)}|A_i|^2|B_j|^2 + (\text{perm})\right] \tag{B.0.261}$$

$$+ 256g^2\left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\left[\sum_k \left((m_a^{(k)})^2 + (m_b^{(k)})^2\right)|A_i|^2|A_j|^2 + (\text{perm})\right] \tag{B.0.262}$$

$$- 512g^2\left(\frac{\pi}{l_s^2}\right)^2 \sum_{i,j} e^{-2(\sigma_i+\sigma_j)}\left[\left(\sum_k (m_b^{(k)})^2\right) A_i\overline{A}_j\overline{B}_iB_j + (\text{perm})\right] \tag{B.0.263}$$

$$+ 256g^3\left(\frac{\pi}{l_s^2}\right)\sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)}\left[\left(m_a^{(i)} + m_b^{(i)}\right)|A_i|^2|A_j|^2|A_k|^2 + (\text{perm})\right] \tag{B.0.264}$$

$$- 516g^3 \left(\frac{\pi}{l_s^2}\right) \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left[\left(m_b^{(i)}\overline{B}_j B_k + m_a^{(i)}\overline{C}_j\overline{C}_k\right) |A_i|^2 A_j \overline{A}_k + (\text{perm})\right] \tag{B.0.265}$$

$$+ 256g^3 \left(\frac{\pi}{l_s^2}\right) \sum_{i,j,k} e^{-2(\sigma_i+\sigma_j+\sigma_k)} \left[\left(m_b^{(i)}|B_i|^2 - m_a^{(i)}|C_i|^2\right) |A_j|^2|A_k|^2 + (\text{perm})\right] \tag{B.0.266}$$

$$+ 128 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)} \left[|A_i|^2|A_j|^2|A_k|^2|A_l|^2 + (\text{perm})\right] \tag{B.0.267}$$

$$+ 256 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)} \left[A_i\overline{A}_j A_k \overline{A}_l \overline{B}_i B_j \overline{B}_k B_l + (\text{perm})\right] \tag{B.0.268}$$

$$- 256 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)} \left[\left(\overline{B}_k B_l + \overline{C}_k C_l\right) |A_i|^2|A_j|^2 A_k \overline{A}_l + (\text{perm})\right] \tag{B.0.269}$$

$$+ 128 \sum_{i,j,k,l} e^{-2(\sigma_i+\sigma_j+\sigma_k+\sigma_l)} \left[|A_i|^2|A_j|^2|B_k|^2|B_l|^2 + (\text{perm})\right] \tag{B.0.270}$$

B.1 Expressions for the Quartic Scalar Potentials

In the following, we derive the quartic scalar potentials. From the calculous results, there are three type quartic potentials such that $A_1|^2$, $|A_1|^2|B_2|^2$ and $A_1\overline{A}_2\overline{B}_1 B_2$.

B.1.1 $|A_1|^4$ Potential

The $|A_i|^4$ terms appear only in the D-tem potential like $g^4|a^{i*}t^a_{ij}a^j|^2$. Substituting Kähler potential (7.5.8) into (7.6.2), we can obtain the D-term potential;

$$V_D = \frac{1}{s}\left[1 - \frac{1}{4}\left\{(m_a^{(1)})^2 + (m_a^{(2)})^2 + (m_a^{(3)})^2 + (m_b^{(1)})^2 + (m_b^{(2)})^2 + (m_b^{(3)})^2\right\}\right] |A_1|^2 \tag{B.1.1}$$

However this D-term potential is not equivalent to $|A_1|^4$ term of NDBI reduction (**??**). In order to be same (**??**) and (B.1.1), the overall factor of (**??**) should have following non-trivial flux contributions:

$$\frac{\int d^6\hat{y}|f_1^{ab}|^4}{(\int d^6\hat{y}|f_1^{ab}|)^2} \propto 1 + \frac{1}{6}\left[(m_a^{(1)} - m_b^{(1)})^2 - (m_a^{(2)} - m_b^{(2)})^2 - (m_a^{(3)} - m_b^{(3)})^2\right] \tag{B.1.2}$$

B.1.2 $|A_1|^2|B_2|^2$ potential

We construct $|A_i|^2|B_j|^2$ potentials and compare to the result (**??**). This term appear in both F-term and D-term potential. According to general expressions () and () and Kähler potential (7.5.8), F-term and D-term are given by

$$V_F \equiv \frac{1}{2^7 s t_1 t_2 t_3 u_1 u_2 u_3} \frac{|\lambda|^2}{G_{ab}^{(1)} G_{bc}^{(2)} G_{ca}^{(3)}} \frac{1}{2} \left[2 + \frac{1}{3} \left\{ m_a^{(1)} m_b^{(1)} - m_b^{(1)} m_c^{(1)} - m_c^{(1)} m_a^{(1)} - 2(m_c^{(1)})^2 \right.\right.$$
$$- m_a^{(2)} m_b^{(2)} + m_b^{(2)} m_c^{(2)} - m_c^{(2)} m_a^{(2)} - 2(m_a^{(2)})^2$$
$$\left.\left. - m_a^{(3)} m_b^{(3)} - m_b^{(3)} m_c^{(3)} + m_c^{(3)} m_a^{(3)} - 2(m_b^{(3)})^2 \right\} \right] |A_1|^2 |B_2|^2 \tag{B.1.3}$$

and

$$V_D \equiv \frac{1}{s} \left[-1 + \frac{1}{2} \left\{ (m_b^{(1)})^2 + (m_b^{(2)})^2 + (m_b^{(3)})^2 \right\} \right] |A_1|^2 |B_2|^2 \tag{B.1.4}$$

From these potentials, the F-term potential is included in (**??**) up to the ovarall factor. But the D-term potential does not included (**??**) even if we use the SUSY conditions. Without the overall factor, the defference between D-term potential and (**??**) is

$$-2m_a^{(2)} m_a^{(3)} - 2m_c^{(1)} m_a^{(3)} - m_b^{(1)}(m_a^{(1)} - m_c^{(1)}) + m_b^{(2)}(m_a^{(2)} - m_c^{(2)}) + m_b^{(3)}(m_a^{(3)} + m_c^{(3)}). \tag{B.1.5}$$

B.1.3 $A_1 \overline{A}_2 \overline{B}_1 B_2$ potential

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