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LOVE-waves in Stratified Three Layers

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Abstract

The complete expression for the amplitude of LOVE-waves is derived, by which it is possible easily to calculate numerically not only the dispersion curve but also the amplitude function. Some examples of numerical calculation for the dispersion curve, amplitude function and the amplitude distribution in the layers are exhibited.

1. Introduction

Surface waves, that is LOVE-waves and RAYLEIGH and SEZAWA-waves, play a great role in seismic records. LOVE-waves in stratified two layers have been fully studied theoretically by many students. RAYLEIGH and SEZAWA-waves in two layers have also been investigated by several authorities. In seismic prospecting RAYLEIGH and SEZAWA-waves are very important for the study of the back ground noise as well as of the refraction shooting.

However, theoretical treatments of these waves, especially the latter, in more layers are very troublesome. The present authors have undertaken theoretical considerations of LOVE-waves in three layers which will be able to represent some interesting characters common to general surface waves in multiple stratified layers, extending the treatments from two to three layers.

The notations in the present paper will follow those used in the previous paper[1]. The suffixes, 1, 2 and 3 mean respectively the quantities belonging to the first, the second and the third layers.

2. Displacements of SH-waves in three layers

When displacements of SH-waves are written

$$\psi_j(t, x, z), \qquad \text{where } j = 1, 2 \text{ and } 3, \tag{1}$$

these values must satisfy the next equation

$$\frac{\partial^2 \psi_j}{\partial t^2} = \frac{\mu_j}{\rho_j} \nabla^2 \psi_j . \tag{2}$$

Taking the space-coordinates as those illustrated in Fig. 1,

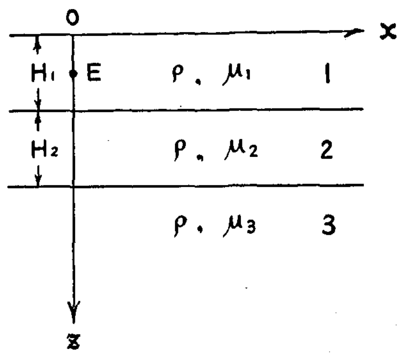

Fig. 1. The subsurface under consideration.

equation (2) can have the following particular solution :

$$\psi_j = e^{i(\omega t - \xi x)} \left(A_j e^{i\eta_j z} + B_j e^{-i\eta_j z}\right) \tag{3}$$

where

$$\xi^2 + \eta_j^2 = k_j^2, \quad k_j = \omega / v_j \text{ and } v_j = (\mu_j / \rho_j)^{1/2}, \tag{4}$$

and ω as well as ξ may have respectively any complex values.

Considering, however, the physical condition under which ψ_j might not be infinity even if t should become infinitely large, one must have the expression as to ω;

$$\omega = \bar{\omega} - i\hat{\omega} \text{ where } \bar{\omega} \text{ and } \hat{\omega} \text{ are positive real.} \tag{5}$$

There exist, moreover, two physical conditions : one is that the displacement is chosen to be the progressive wave in the direction of $r = (x^2 + z^2)^{1/2}$ and the other is that the displacement might not be infinity even if x shoud become infinitely large. Taking up these two conditions, "SOMMERFERD's radiation conditions"[2], the complex values of ξ and η_j must be expressed respectively as follows :

$$\xi = \bar{\xi} - i\hat{\xi} \text{ and } \eta_j = \bar{\eta}_j - i\hat{\eta}_j \tag{6}$$

where $\bar{\xi}$, $\hat{\xi}$ $\bar{\eta}_j$ and $\hat{\eta}_j$, are respectively positive real.

The remaining condition is that the third layer is extending infinitely in the direction of z, that is, there is no reflecting plane below the third layer. Therefore no wave proceeding in the negative direction of z can exist in the third layer, if z is larger than E, the depth of the source of SH-waves. For this reason, A_3 in (3) must be taken as zero in the present problem.

Now the displacement in each layer will be written from (3) as follows :

$$\left.\begin{aligned}\psi_1 &= e^{i(\omega t-\xi x)}\left(A_1 e^{i\eta_1 z}+B_1 e^{-i\eta_1 z}\right),\\ \psi_2 &= e^{i(\omega t-\xi x)}\left(A_2 e^{i\eta_2 z}+B_2 e^{-i\eta_2 z}\right),\\ \psi_3 &= e^{i(\omega t-\xi x)}B_3 e^{-i\eta_3 z}.\end{aligned}\right\} \quad (7)$$

3. Displacements due to the original wave from a line source

When the line source lies in $0 \leqq E \leqq H_1$, the displacement of the original SH-wave may conveniently be written by

$$\psi_0 = \int_{-\infty}^{\infty} e^{i\{\omega t-\xi x\mp\eta_1(E-z)\}}\frac{d\xi}{\eta_1}, \qquad z \lessgtr E. \qquad (8)$$

Because this is a cylindrical wave having its origin at E, (7) will also become cylindrical waves by means of the same operation as that used in (8).

If one considers the original wave at first, (7) can be taken as the secondary waves generated respectively from the boundary planes at $z=0$, H_1 and H_1+H_2.

Now the boundary conditions are stated as follows : stress must be zero on $z=0$, and displacements as well as stresses must be continuous on $z=H_1$ and H_1+H_2, namely

$$\left.\begin{aligned}&A_1-B_1 = -e^{-i\eta_1 E},\\ &A_1 e^{i\eta_1 H_1}+B_1 e^{-i\eta_1 H_1}-A_2 e^{i\eta_2 H_2}-B_2 e^{-i\eta_2 H_2} = -e^{i\eta_1(E-H_1)},\\ &A_1 e^{i\eta_1 H_1}-B_1 e^{-i\eta_1 H_1}-A_2(\mu_2\eta_2/\mu_1\eta_1)e^{i\eta_2 H_2}\\ &\qquad +B_2(\mu_2\eta_2/\mu_1\eta_1)e^{-i\eta_2 H_2} = e^{i\eta_1(E-H_1)},\\ &A_2 e^{i\eta_2(H_1+H_2)}+B_2 e^{-i\eta_2(H_1+H_2)}-B_3 e^{-i\eta_3(H_1+H_2)} = 0,\\ &A_2 e^{i\eta_2(H_1+H_2)}-B_2 e^{-i\eta_2(H_1+H_2)}+B_3(\mu_3\eta_3/\mu_2\eta_2)e^{-i\eta_3(H_1+H_2)} = 0.\end{aligned}\right\} \quad (9)$$

Denoting the determinant constructed from the coefficients of the arbitrary constants in (9) as D, it is calculated as follows :

$$D = \begin{vmatrix} 1 & -1 & 0 & 0 & 0\\ e^{i\eta_1 H_1} & e^{-i\eta_1 H_1} & -e^{i\eta_2 H_1} & -e^{-i\eta_2 H_1} & 0\\ e^{i\eta_1 H_1} & -e^{-i\eta_1 H_1} & -(\mu_2\eta_2/\mu_1\eta_1)e^{i\eta_2 H_1} & (\mu_2\eta_2/\mu_1\eta_1)e^{-i\eta_2 H_1} & 0\\ 0 & 0 & e^{i\eta_2(H_1+H_2)} & e^{-i\eta_2(H_1+H_2)} & -e^{-i\eta_3(H_1+H_2)}\\ 0 & 0 & e^{i\eta_2(H_1+H_2)} & -e^{-i\eta_2(H_1+H_2)} & (\mu_3\eta_3/\mu_2\eta_2)e^{-i\eta_3(H_1+H_2)} \end{vmatrix}$$

$$
= e^{-i\eta_3(H_1+H_2)}
\cdot \begin{vmatrix}
e^{-i\eta_1 H_1} & -e^{i\eta_1 H_1} & 0 & 0 & 0 \\
1 & 1 & -1 & -1 & 0 \\
1 & -1 & -(\mu_2\eta_2/\mu_1\eta_1) & (\mu_2\eta_2/\mu_1\eta_1) & 0 \\
0 & 0 & e^{i\eta_2 H_2} & e^{-i\eta_2 H_2} & -1 \\
0 & 0 & e^{i\eta_2 H_2} & -e^{-i\eta_2 H_2} & (\mu_3\eta_3/\mu_2\eta_2)
\end{vmatrix}
$$

$$
= e^{-i\eta_3(H_1+H_2)} e^{i(\eta_1 H_1+\eta_2 H_2)}
\cdot \begin{vmatrix}
e^{-2i\eta_1 H_1} & -1 & 0 & 0 & 0 \\
1 & 1 & -1 & -1 & 0 \\
1 & -1 & -(\mu_2\eta_2/\mu_1\eta_1) & (\mu_2\eta_2/\mu_1\eta_1) & 0 \\
0 & 0 & 1 & e^{-2i\eta_2 H_2} & -1 \\
0 & 0 & 1 & -e^{-2i\eta_2 H_2} & (\mu_3\eta_3/\mu_2\eta_2)
\end{vmatrix}
$$

$$
= \frac{1}{4} e^{-i\eta_3(H_1+H_2)} e^{i(\eta_1 H_1+\eta_2 H_2)}
\cdot \begin{vmatrix}
e^{-2i\eta_1 H_1} & -1 & 0 & 0 & 0 \\
2 & 0 & -(1+\mu_2\eta_2/\mu_1\eta_1) & -(1-\mu_2\eta_2/\mu_1\eta_1) & 0 \\
0 & -2 & 1-\mu_2\eta_2/\mu_1\eta_1 & 1+\mu_2\eta_2/\mu_1\eta_1 & 0 \\
0 & 0 & 2 & 0 & -(1-\mu_3\eta_3/\mu_2\eta_2) \\
0 & 0 & 0 & 2e^{-2i\eta_2 H_2} & 1+\mu_3\eta_3/\mu_2\eta_2
\end{vmatrix}
$$

$$
\begin{aligned}
&= e^{-i\eta_3(H_1+H_2)} e^{i(\eta_1 H_1+\eta_2 H_2)} \left\{K_{12}\, e^{-2i\eta_1 H_1} + K_{23}\, e^{-2i(\eta_1 H_1+\eta_2 H_2)} \right. \\
&\quad \left. -(1+K_{12}K_{23}\, e^{-2i\eta_2 H_2})\right\} \left\{1+(\mu_2\eta_2/\mu_1\eta_1)\right\} \left\{1+(\mu_3\eta_3/\mu_2\eta_2)\right\}, \qquad (10)
\end{aligned}
$$

in which

$$
\left.\begin{aligned}
K_{12} &= \left\{1-(\mu_2\eta_2/\mu_1\eta_1)\right\}/\left\{1+(\mu_2\eta_2/\mu_1\eta_1)\right\}, \\
K_{23} &= \left\{1-(\mu_3\eta_3/\mu_2\eta_2)\right\}/\left\{1+(\mu_3\eta_3/\mu_2\eta_2)\right\}.
\end{aligned}\right\} \qquad (11)
$$

K_{12} and K_{23} mean reflecting coefficients of SH-waves respectively on the boundaries from the first to the second layer and from the second to the third layer.

Calculating each constant contained in (9), one has

$$
A_1 D =
\begin{vmatrix}
-e^{-i\eta_1 E} & -1 & 0 & 0 & 0 \\
-e^{i\eta_1(E-H_1)} & e^{-i\eta_1 H_1} & -e^{i\eta_2 H_1} & -e^{-i\eta_2 H_1} & 0 \\
e^{i\eta_1(E-H_1)} & -e^{-i\eta_1 H_1} & -(\mu_2\eta_2/\mu_1\eta_1)\, e^{i\eta_2 H_1} & (\mu_2\eta_2/\mu_1\eta_1)\, e^{-i\eta_2 H_1} & 0 \\
0 & 0 & e^{i\eta_2(H_1+H_2)} & e^{-i\eta_2(H_1+H_2)} & -e^{-i\eta_3(H_1+H_2)} \\
0 & 0 & e^{i\eta_2(H_1+H_2)} & -e^{-i\eta_2(H_1+H_2)} & (\mu_3\eta_3/\mu_2\eta_2) e^{-i\eta_3(H_1+H_2)}
\end{vmatrix}
$$

$$
\begin{aligned}
&= -\left(e^{i\eta_1 E}+e^{-i\eta_1 E}\right) \left\{K_{12} e^{-2i\eta_1 H_1} + K_{23}\, e^{-2i(\eta_1 H_1+\eta_2 H_2)}\right\} e^{-i\eta_3(H_1+H_2)} e^{i(\eta_1 H_1+\eta_2 H_2)} \\
&\quad \cdot \left\{1+(\mu_2\eta_2/\mu_1\eta_1)\right\} \left\{1+(\mu_3\eta_3/\mu_2\eta_2)\right\},
\end{aligned}
$$

$$B_1D = -\left[e^{i\eta_1 E}\{K_{12}e^{-2i\eta_1 H_1}+K_{23}e^{-2i(\eta_1 H_1+\eta_2 H_2)}\}+e^{-i\eta_1 E}(1+K_{12}K_{23}e^{-2i\eta_2 H_2})\right] \cdot e^{-i\eta_3(H_1+H_2)}e^{i(\eta_1 H_1+\eta_2 H_2)}\left\{1+(\mu_2\eta_2/\mu_1\eta_1)\right\}\left\{1+(\mu_3\eta_3/\mu_2\eta_2)\right\},$$

$$A_2D = -2\left(e^{i\eta_1 E}+e^{-i\eta_1 E}\right)e^{-i\eta_3(H_1+H_2)}e^{-i\eta_2(H_1+H_2)}\left\{1-(\mu_3\eta_3/\mu_2\eta_2)\right\},$$

$$B_2D = -2\left(e^{i\eta_1 E}+e^{-i\eta_1 E}\right)e^{-i\eta_3(H_1+H_2)}e^{-i\eta_2(H_1+H_2)}\left\{1+(\mu_3\eta_3/\mu_2\eta_2)\right\},$$

$$B_3D = -4\left(e^{i\eta_1 E}+e^{-i\eta_1 E}\right).$$

Putting (10) into the above expressions, one has

$$A_1 = \left(e^{i\eta_1 E}+e^{-i\eta_1 E}\right)\{K_{12}e^{-2i\eta_1 H_1}+K_{23}e^{-2i(\eta_1 H_1+\eta_2 H_2)}\}/F(\xi,\omega),$$

$$B_1 = \left[e^{i\eta_1 E}\{K_{12}e^{-2i\eta_1 H_1}+K_{23}e^{-2i(\eta_1 H_1+\eta_2 H_2)}\}+e^{-i\eta_1 E}(1+K_{12}K_{23}e^{-2i\eta_2 H_2})\right]/F(\xi,\omega),$$

$$A_2 = \left(e^{i\eta_1 E}+e^{-i\eta_1 E}\right)e^{-i(\eta_1 H_1+\eta_2 H_2)}e^{-i\eta_2(H_1+H_2)}K_{23}(1+K_{12})/F(\xi,\omega),$$

$$B_2 = \left(e^{i\eta_1 E}+e^{-i\eta_1 E}\right)e^{-i(\eta_1 H_1+\eta_2 H_2)}e^{i\eta_2(H_1+H_2)}(1+K_{12})/F(\xi,\omega),$$

$$B_3 = \left(e^{i\eta_1 E}+e^{-i\eta_1 E}\right)e^{i\eta_3(H_1+H_2)}e^{-i(\eta_1 H_1+\eta_2 H_2)}(1+K_{12})(1+K_{23})/F(\xi,\omega),$$

where

$$F(\xi,\omega) = (1+K_{12}K_{23}e^{-2i\eta_2 H_2})-\{K_{12}e^{-2i\eta_1 H_1}+K_{23}e^{-2i(\eta_1 H_1+\eta_2 H_2)}\}. \tag{12}$$

Putting these values into (7) and employing the same integral as that in (8), one gains the displacements of SH-waves expressed as follows:

$$\psi_0+\psi_1 = 2\int_{-\infty}^{\infty}e^{i(\omega t-\xi x)}\cos\eta_1 z\left[e^{i\eta_1 E}\{K_{12}e^{-2i\eta_1 H_1}+K_{23}e^{-2i(\eta_1 H_1+\eta_2 H_2)}\} +e^{-i\eta_1 E}(1+K_{12}K_{23}e^{-2i\eta_2 H_2})\right]\{F(\xi,\omega)\}^{-1}(d\xi/\eta_1),\quad 0\leqq z\leqq E, \tag{13}$$

$$\psi_0+\psi_1 = 2\int_{-\infty}^{\infty}e^{i(\omega t-\xi x)}\cos\eta_1 E\left[e^{i\eta_1 z}\{K_{12}e^{-2i\eta_1 H_1}+K_{23}e^{-2i(\eta_1 H_1+\eta_2 H_2)}\} +e^{-i\eta_1 z}(1+K_{12}K_{23}e^{-2i\eta_2 H_2})\right]\{F(\xi,\omega)\}^{-1}(d\xi/\eta_1),\quad E\leqq z\leqq H_1, \tag{14}$$

$$\psi_2 = 2\int_{-\infty}^{\infty}e^{i(\omega t-\xi x)}\cos\eta_1 E\cdot e^{-i(\eta_1 H_1+\eta_2 H_2)}(1+K_{12})\left[K_{23}e^{i\eta_2\{z-(H_1+H_2)\}} +e^{-i\eta_2\{z-(H_1+H_2)\}}\right]\{F(\xi,\omega)\}^{-1}(d\xi/\eta_1), \tag{15}$$

$$\psi_3 = 2\int_{-\infty}^{\infty}e^{i(\omega t-\xi x)}\cos\eta_1 E\cdot e^{-i\eta_3\{z-(H_1+H_2)\}}e^{-i(\eta_1 H_1+\eta_2 H_2)}(1+K_{12})(1+K_{23}) \cdot\{F(\xi,\omega)\}^{-1}(d\xi/\eta_1). \tag{16}$$

4. Characteristic equation

If the next equation,

$$F(\xi,\omega) = 0, \tag{17}$$

is satisfied with ξ, this ξ is a ploe of the integrals from (13) to (16).

By now, ω and ξ are taken as any complexes respectively expressed in (5) and (6). If $\hat{\omega}\neq 0$ in (5), the amplitude of the wave will be damped with increase of time. As the present problem does not consider such a case,

$$\hat{\omega} = 0 \tag{18}$$

will be assumed in (5).

Combining (18) with (4), (5) and (6), the next conditions will also be obtained:

$$\bar{\xi}\hat{\xi} = \bar{\eta}_j \hat{\eta}_j = 0 . \tag{19}$$

If $\hat{\xi} \neq 0$ in (19), $\bar{\xi}$ must be zero and the wave cannot progress in x-direction. If this case will be disregarded, $\bar{\xi}$ must not be zero, namely

$$\hat{\xi} = 0 . \tag{20}$$

Equation (17) cannot be the characteristic equation of LOVE-waves until it meets the conditions of (18) and (20).

Using (12) and (17), the characteristic equation of LOVE-waves in three layers can be written as follows:

$$F(\xi, \omega) = (1 + K_{12} K_{23} e^{-2i\eta_2 H_2}) - \{K_{12} e^{-2i\eta_1 H_1} + K_{23} e^{-2i(\eta_1 H_1 + \eta_2 H_2)}\} = 0 \tag{21}$$

or

$$1 - \frac{e^{-2i\eta_1 H_1} (K_{12} + K_{23} e^{-2i\eta_2 H_2})}{1 + K_{12} K_{23} e^{-2i\eta_2 H_2}} = 0 , \tag{22}$$

which can be rewritten by use of (11)

$$\tan \eta_1 H_1 = \left(i \frac{\mu_2 \eta_2}{\mu_1 \eta_1}\right) \frac{i \{\tan \eta_2 H_2 - (i\mu_3 \eta_3 / \mu_2 \eta_2)\}}{1 + (i\mu_3 \eta_3 / \mu_2 \eta_2) \tan \eta_2 H_2} . \tag{23}$$

Denoting here

$$\tan \delta_{12} = i \frac{\mu_2 \eta_2}{\mu_1 \eta_1} \quad \text{and} \quad \tan \delta_{23} = i \frac{\mu_3 \eta_3}{\mu_2 \eta_2} , \tag{24}$$

(23) will become

$$\tan \eta_1 H_1 = i \tan \delta_{12} \tan \left\{ \eta_1 H_1 \left(\frac{\eta_2 H_2}{\eta_1 H_1} \right) - \delta_{23} \right\} . \tag{25}$$

Recalling

$$\frac{\eta_2}{\eta_1} = \frac{v_1}{v_2} \left\{ \frac{1 - (v_2/v_1)^2 (v_1/c)^2}{1 - (v_1/c)^2} \right\}^{1/2} \quad \text{and} \quad \frac{\eta_3}{\eta_2} = \frac{v_2}{v_3} \left\{ \frac{1 - (v_3/v_1)^2 (v_1/c)^2}{1 - (v_2/v_1)^2 (v_1/c)^2} \right\}^{1/2} ,$$

one sees the right hand side of (25) to consist of functions of $\eta_1 H_1$ alone, if the conditions of underground and v_1/c are given.

Therefore $\eta_1 H_1$ corresponding to any v_1/c will be found from intersecting points of the two curves respectively obtained from the left and the right hand sides of (25), $\eta_1 H_1$ being a parameter. Because

$$\eta_1 H_1 = \frac{\omega H_1}{v_1} \left\{ 1 - \left(\frac{v_1}{c} \right)^2 \right\}^{1/2} ,$$

$\omega H_1/v_1$ can be easily calculated if v_1/c and $\eta_1 H_1$ are known. Thus by the calculation of $\omega H_1/v_1$ for various v_1/c, the dispersion curve may be drawn.

5. Classification in the type of the amplitude distribution

According to (19), each η_1, η_2 and η_3 is real or purely imaginary. If η_3 in (23) is assumed to be real, neither η_1 nor η_2 can be real or purely imaginary. If η_3 is purely imaginary, on the contrary, each η_1 and η_2 can be either real or purely imaginary. Therefore, from (19), the following combinations alone will be possible:

(i)		(ii)		(iii)	
$\bar{\eta}_1$	S	$\bar{\eta}_1$	S	$-i\hat{\eta}_1$	E
$-i\hat{\eta}_2$	E	$\bar{\eta}_2$	S	$\bar{\eta}_2$	S
$-i\hat{\eta}_3$	E ,	$-i\hat{\eta}_3$	E ,	$-i\hat{\eta}_3$	E ,

(26)

in which notations of S and E are correspondent to those used by SATO[3].

Putting (26) into (24) and (25), one has each characteristic equation corresponding to the above three types of the amplitude distribution.

$$
\left.\begin{aligned}
&\text{(i)}\quad \tan\bar{\delta}_{12} = (\mu_2\hat{\eta}_2)/(\mu_1\bar{\eta}_1),\ \tanh\hat{\delta}_{23} = (\mu_3\hat{\eta}_3)/(\mu_2\hat{\eta}_2),\ \delta_{23} = i\hat{\delta}_{23}\,,\\
&\qquad \tan\bar{\eta}_1 H_1 = \tan\bar{\delta}_{12}\ \tanh\left\{\bar{\eta}_1 H_1\left(\frac{\hat{\eta}_2 H_2}{\bar{\eta}_1 H_1}\right) + \hat{\delta}_{23}\right\},\\
&\text{(ii)}\quad \tanh\hat{\delta}_{12} = (\mu_2\bar{\eta}_2)/(\mu_1\bar{\eta}_1),\ \tan\bar{\delta}_{23} = (\mu_3\hat{\eta}_3)/(\mu_2\bar{\eta}_2)\,,\ \delta_{12} = i\hat{\delta}_{12}\,,\\
&\qquad \tan\bar{\eta}_1 H_1 = -\tanh\hat{\delta}_{12}\ \tan\left\{\bar{\eta}_1 H_1\left(\frac{\bar{\eta}_2 H_2}{\bar{\eta}_1 H_1}\right) - \bar{\delta}_{23}\right\},\\
&\text{(iii)}\quad \tan\bar{\delta}_{12} = -(\mu_2\bar{\eta}_2)/(\mu_1\hat{\eta}_1),\ \tan\bar{\delta}_{23} = (\mu_3\hat{\eta}_3)/(\mu_2\bar{\eta}_2)\,,\\
&\qquad \tanh\hat{\eta}_1 H_1 = \tan\bar{\delta}_{12}\ \tan\left\{\hat{\eta}_1 H_1\left(\frac{\bar{\eta}_2 H_2}{\hat{\eta}_1 H_1}\right) - \bar{\delta}_{23}\right\},
\end{aligned}\right\} \tag{27}
$$

where

$$
\bar{\eta}_j = \bar{\xi}\left\{(c/v_j)^2 - 1\right\}^{1/2} \quad\text{and}\quad \hat{\eta}_j = \bar{\xi}\left\{1-(c/v_j)^2\right\}^{1/2}. \tag{28}
$$

A little attention must be paid to (i) among (27), because $\hat{\delta}_{23}$ cannot have any solution if $(\mu_3\hat{\eta}_3)/(\mu_2\hat{\eta}_2) > 1$. In this case $\tanh\hat{\delta}_{23}$ must be replaced by $\coth\hat{\delta}_{23}$, that is

$$
\text{(i}'\text{)}\quad \coth\hat{\delta}_{23}{}' = (\mu_3\hat{\eta}_3)/(\mu_2\hat{\eta}_2),\ \text{namely}\ \cot\delta_{23}{}' = -\tan\delta_{23},\quad \delta_{23}{}' = i\hat{\delta}_{23}{}'. \tag{29}
$$

Due to this replacement another equation in (i) among (27) must also change its form to

$$
\tan\bar{\eta}_1 H_1 = \tan\bar{\delta}_{12}\ \coth\left\{\bar{\eta}_1 H_1\left(\frac{\hat{\eta}_2 H_2}{\bar{\eta}_1 H_1}\right) + \hat{\delta}_{22}{}'\right\}.
$$

In (ii) among (27) $\hat{\delta}_{12}$ cannot have any solution, if $(\mu_2\bar{\eta}_2)/(\mu_1\bar{\eta}_1)>1$. The similar replacement to the above must be adopted for (ii) among (27). Thus one has

$$\left.\begin{aligned}&\text{(ii')}\quad \coth\hat{\delta}'_{12}=(\mu_2\bar{\eta}_2)/(\mu_1\bar{\eta}_1),\ \text{namely}\ \cot\delta_{12}'=-\tan\delta_{12},\ \delta'_{12}=i\hat{\delta}'_{12},\\ &\qquad \tan\bar{\eta}_1H_1=-\coth\hat{\delta}_{12}'\tan\left\{\bar{\eta}_1H_1\left(\frac{\bar{\eta}_2H_2}{\bar{\eta}_1H_1}\right)-\hat{\delta}_{23}\right\}.\end{aligned}\right\}\qquad(29)$$

It will be worth while to see that δ' in (29) is connected to δ in (27) by

$$\delta=\delta'-\pi/2. \qquad (30)$$

It must be noticed, in the characteristic equations among (27) and (29), that $\tan\bar{\eta}_1H_1$ is a periodic function, though $\tanh\bar{\eta}_1H_1$ is not one. Owing to the periodicity of $\tan\bar{\eta}_1H_1$, the dispersion curve will have many branches in higher orders.

6. Displacements of LOVE-waves

LOVE-waves are to be calculated from the pole of the integrand from (13) to (16), expressions of general SH-waves. Thus LOVE-waves in each layer may be expressed as follows:

$$\left.\begin{aligned}[\psi_0+\psi_1]_{\text{LOVE}}&=\pi i\times(\text{residue})\\ &=4\pi i\,e^{i(\omega t-\xi x)}\cos\eta_1E\cos\eta_1z(1+K_{12}K_{23}{}^{-2i\eta_2H_2})\\ &\qquad\qquad /\{\eta_1F_\xi(\xi,\omega)\},\\ [\psi_2]_{\text{LOVE}}&=2\pi i\,e^{i(\omega t-\xi x)}\cos\eta_1E\cdot e^{-i(\eta_1H_1+\eta_2H_2)}(1+K_{12})\{K_{23}e^{i\eta_2(z-H_1-H_2)}\\ &\qquad +e^{-i\eta_2(z-H_1-H_2)}\}/\{\eta_1F_\xi(\xi,\omega)\},\\ [\psi_3]_{\text{LOVE}}&=2\pi i\,e^{i(\omega t-\xi x)}\cos\eta_1E\cdot e^{-i\eta_3(z-H_1-H_2)}e^{-i(\eta_1H_1+\eta_2H_2)}(1+K_{12})\\ &\qquad\qquad\cdot(1+K_{23})/\{\eta_1F_\xi(\xi,\omega)\}.\end{aligned}\right\}\qquad(31)$$

On the other hand, one sees in (21)

$$U=d\omega/d\xi=-F_\xi(\xi,\omega)/F_\omega(\xi,\omega), \qquad (32)$$

in which U means group velocity; it is possible to obtain the next relation from (32);

$$\left.\begin{aligned}&\omega\left(\frac{1}{U}-\frac{1}{c}\right)=-\frac{1}{F_\xi(\xi,\omega)}\left(\omega\frac{\partial}{\partial\omega}+\xi\frac{\partial}{\partial\xi}\right)F(\xi,\omega),\\ &\text{where}\\ &\left(\omega\frac{\partial}{\partial\omega}+\xi\frac{\partial}{\partial\xi}\right)\left(\frac{\eta_2}{\eta_1},\frac{\eta_3}{\eta_2}\right)=0\ \text{and}\ \left(\omega\frac{\partial}{\partial\omega}+\xi\frac{\partial}{\partial\xi}\right)\eta_j=\eta_j.\end{aligned}\right\}\qquad(33)$$

Performing the operation of (33) on (21), one has

$$\omega\left(\frac{1}{U}-\frac{1}{c}\right)=\frac{2i}{F_\xi(\xi,\omega)}\{\eta_2H_2K_{12}K_{23}\,e^{-2i\eta_2H_2}-\eta_1H_1K_{12}\,e^{-2i\eta_1H_1}$$
$$-(\eta_1H_1+\eta_2H_2)\,K_{23}\,e^{-2i(\eta_1H_1+\eta_2H_2)}\} \tag{34}$$

or

$$\left(\frac{v_1}{U}-\frac{v_1}{c}\right)\left\{1-\left(\frac{v_1}{c}\right)^2\right\}^{-1/2}=-\frac{2\,i}{\eta_1F_\xi\,(\xi,\omega)}\,\{\eta_1H_1\,(1+K_{12}K_{23}\,e^{-2i\eta_2H_2})$$
$$+\eta_2H_2\,(1-K_{12}\,e^{-2i\eta_1H_1})\}\,. \tag{35}$$

By means of (35), (31) will be rewritten as

$$\left.\begin{aligned}
[\psi_0+\psi_1]_{\text{LOVE}}&=-e^{i(\omega t-\xi x)}\cos\,\eta_1E\,\cos\,\eta_1z\cdot 2\pi\,(v_1/U-v_1/c)\\
&\qquad\cdot\{1-(v_1/c)^2\}^{-1/2}\cdot p\,,\\
[\psi_2]_{\text{LOVE}}&=-\,e^{i(\omega t-\xi x)}\cos\,\eta_1E\cdot 2\pi(v_1/U-v_1/c)\,\{1-(v_1/c)^2\}^{-1/2}\\
&\qquad\cdot\{q\,e^{i\eta_2(z-H_1)}+r\,e^{-i\eta_2(z-H_1)}\}/2\,,\\
[\psi_3]_{\text{LOVE}}&=-e^{i(\omega t-\xi x)}\cos\,\eta_1E\cdot e^{-i\eta_3(z-H_1-H_2)}\cdot 2\,\pi\,(v_1/U-v_1/c)\\
&\qquad\cdot\{1-(v_1/c)^2\}^{-1/2}\cdot s/2\,,
\end{aligned}\right\} \tag{36}$$

in which

$$\left.\begin{aligned}
p&=\frac{1+K_{12}K_{23}\,e^{-2i\eta_2H_2}}{\eta_1H_1\,(1+K_{12}K_{23}\,e^{-2i\eta_2H_2})+\eta_2H_2\,(1-K_{12}\,e^{-2i\eta_1H_1})}\,,\\
q&=\frac{e^{-i\eta_1H_1}(1+K_{12})K_{23}\,e^{-2i\eta_2H_2}}{\eta_1H_1\,(1+K_{12}K_{23}\,e^{-2i\eta_2H_2})+\eta_2H_2\,(1-K_{12}\,e^{-2i\eta_1H_1})}\,,\\
r&=\frac{e^{-i\eta_1H_1}(1+K_{12})}{\eta_1H_1\,(1+K_{12}K_{23}\,e^{-2i\eta_2H_2})+\eta_2H_2\,(1-K_{12}\,e^{-2i\eta_1H_1})}\,,\\
s&=\frac{e^{-i(\eta_1H_1+\eta_2H_2)}\,(1+K_{12})\,(1+K_{23})}{\eta_1H_1\,(1+K_{12}K_{23}\,e^{-2i\eta_2H_2})+\eta_2H_2\,(1-K_{12}\,e^{-2i\eta_1H_1})}\,.
\end{aligned}\right\} \tag{37}$$

As in (36), calculations containing complex values will be necessary. To avoid this difficulty, more considerations must be continued.

At first K_{12} and K_{23} in (37) will, by means of (11) and (24), be expressed respectively by δ_{12} and by δ_{23} as follows:

$$K_{12}=\frac{1+i\,\tan\,\delta_{12}}{1-i\,\tan\,\delta_{12}}=e^{2i\delta_{12}}\quad\text{and}\quad K_{23}=e^{2i\delta_{23}}\,. \tag{38}$$

Then one will have, from (25), the next useful relation:

$$\frac{e^{2i(\delta_{12}-\eta_1H_1)}}{1+e^{2i\delta_{12}}\,e^{-2i(\eta_2H_2-\delta_{23})}}=\frac{1}{1+e^{-2i\delta_{12}}\,e^{-2i(\eta_2H_2-\delta_{23})}}\,. \tag{39}$$

Therefore, from (38) and (39), one gets

$$\frac{1-K_{12}\,e^{-2i\eta_1H_1}}{1+K_{12}K_{23}\,e^{-2i\eta_2H_2}}=\frac{1-e^{2i(\delta_{12}-\eta_1H_1)}}{1+e^{2i\delta_{12}}\,e^{-2i(\eta_2H_2-\delta_{23})}}=\frac{1}{1+e^{2i\delta_{12}}\,e^{-2i(\eta_2H_2-\delta_{23})}}$$
$$-\frac{1}{1+e^{-2i\delta_{12}}\,e^{-2i(\eta_2H_2-\delta_{23})}}=-\,i\,\frac{\sin 2\,\delta_{12}}{\cos 2\,\delta_{12}+\cos 2\,(\eta_2H_2-\delta_{23})}\,, \tag{40}$$

$$\frac{e^{-i\eta_1 H_1}(1+K_{12})}{1+K_{12}K_{23}e^{-2i\eta_2 H_2}} = \frac{e^{-i\eta_1 H_1}(1+e^{2i\delta_{12}})}{1+e^{2i\delta_{12}}e^{-2i(\eta_2 H_2-\delta_{23})}} = e^{i(\eta_2 H_2-\delta_{23})}\frac{\cos\eta_1 H_1}{\cos(\eta_2 H_2-\delta_{23})}, \tag{41}$$

and

$$\frac{e^{-i(\eta_1 H_1+\eta_2 H_2)}(1+K_{12})(1+K_{23})}{1+K_{12}K_{23}e^{-2i\eta_2 H_2}} = \frac{e^{-i(\eta_1 H_1+\eta_2 H_2)}(1+e^{2i\delta_{12}})(1+e^{2i\delta_{23}})}{1+e^{2i\delta_{12}}e^{-2i(\eta_2 H_2-\delta_{23})}}$$
$$= 4\cos\delta_{12}\cos\delta_{23}\,e^{-i(\eta_2 H_2-\delta_{23})}\frac{e^{i(\delta_{12}-\eta_1 H_1)}}{1+e^{2i\delta_{12}}e^{-2i(\eta_2 H_2-\delta_{23})}} = 2\frac{\cos\eta_1 H_1\cos\delta_{23}}{\cos(\eta_2 H_2-\delta_{23})}. \tag{42}$$

Thus (37) can be transformed to

$$\left.\begin{aligned}
p &= \frac{1}{\eta_1 H_1}\left\{1-i\frac{\eta_2 H_2}{\eta_1 H_1}\cdot\frac{\sin 2\delta_{12}}{\cos 2\delta_{12}+\cos 2(\eta_2 H_2-\delta_{23})}\right\}^{-1},\\
r &= p\cdot e^{i(\eta_2 H_2-\delta_{23})}\frac{\cos\eta_1 H_1}{\cos(\eta_2 H_2-\delta_{23})},\\
s &= 2p\cdot\frac{\cos\eta_1 H_1\cos\delta_{23}}{\cos(\eta_2 H_2-\delta_{23})},\\
\text{and}\quad & \\
q\,e^{i\eta_2(z-H_1)}+r\,e^{-i\eta_2(z-H_1)} &= p\,\{e^{i\eta_2(z-H_1)}e^{-2i(\eta_2 H_2-\delta_{23})}\\
&\quad +e^{-i\eta_2(z-H_1)}\}\,e^{i(\eta_2 H_2-\delta_{23})}\frac{\cos\eta_1 H_1}{\cos(\eta_2 H_2-\delta_{23})}\\
&= 2p\cos\eta_1 H_1\frac{\cos\{\eta_2(z-H_1)-(\eta_2 H_2-\delta_{23})\}}{\cos(\eta_2 H_2-\delta_{23})}.
\end{aligned}\right\} \tag{43}$$

At last, using (43), (36) arrives at a more convenient form:

$$\left.\begin{aligned}
[\psi_0+\psi_1]_{\text{LOVE}} &= -2\pi A(\xi)\cdot e^{i(\omega t-\xi x)}\cos\eta_1 E\cos\eta_1 z,\\
[\psi_2]_{\text{LOVE}} &= -2\pi A(\xi)\,e^{i(\omega t-\xi x)}\\
&\quad\cdot\cos\eta_1 E_1\cos\eta_1 H_1\cos\{\eta_2(z-H_1-H_2)+\delta_{23}\}/\cos(\eta_2 H_2-\delta_{23}),\\
[\psi_3]_{\text{LOVE}} &= -2\pi A(\xi)\,e^{i(\omega t-\xi x)}e^{-i\eta_3(z-H_1-H_2)}\\
&\quad\cdot\cos\eta_1 E\cos\eta_1 H_1\cos\delta_{23}/\cos(\eta_2 H_2-\delta_{23}),
\end{aligned}\right\} \tag{44}$$

where

$$2\pi A(\xi) = \frac{Tv_1}{H_1}\left(\frac{v_1}{U}-\frac{v_1}{c}\right)\left\{1-\left(\frac{v_1}{c}\right)^2\right\}^{-1}\left\{1-i\frac{\eta_2 H_2}{\eta_1 H_1}\cdot\frac{\sin 2\delta_{12}}{\cos 2\delta_{12}+\cos 2(\eta_2 H_2-\delta_{23})}\right\}^{-1} \tag{45}$$

may be called the amplitude function of LOVE-waves in three layers, because it has some analogies to that in two layers.

7. Practical expressions of LOVE-waves

Now, δ_{12} and δ_{23} may be real or purely imaginary, as was seen in (27) and

(29), so (38) must be classified, in practice, as follows in the similar manner to (26):

$$\left.\begin{array}{lllll} \text{(i)} & K_{12}=e^{2i\delta_{12}} & \text{and} & K_{23}=e^{-2\hat{\delta}_{23}}, \\ \text{(i')} & K_{12}=e^{2i\delta_{12}} & \text{and} & K_{23}=-e^{-2\hat{\delta}'_{23}}, \\ \text{(ii)} & K_{12}=e^{-2\hat{\delta}_{12}} & \text{and} & K_{23}=e^{2i\delta_{23}} \\ \text{(ii')} & K_{12}=-e^{-2\hat{\delta}'_{12}} & \text{and} & K_{23}=e^{2i\delta_{23}}, \\ \text{(iii)} & K_{12}=e^{2i\delta_{12}} & \text{and} & K_{23}=e^{2i\delta_{23}}. \end{array}\right\} \quad (46)$$

Corresponding to (46), the above expressions (44) and (45) must be classified in practice to the various cases;

(i)

$$[\psi_0+\psi_1]_{\text{Love}} = -2\pi A(\xi)\cos\bar{\eta}_1 E\cos\bar{\eta}_1 z,$$

$$[\psi_2]_{\text{Love}} = -2\pi A(\xi)\cos\bar{\eta}_1 E\cos\bar{\eta}_1 H_1 \cdot\cosh\{-\hat{\eta}_2(z-H_1-H_2)+\hat{\delta}_{23}\}/\cosh(\hat{\eta}_2 H_2+\hat{\delta}_{23}),$$

$$[\psi_3]_{\text{Love}} = -2\pi A(\xi)\cos\bar{\eta}_1 E\cos\bar{\eta}_1 H_1\, e^{-\hat{\eta}_3(z-H_1-H_2)} \cdot\cosh\hat{\delta}_{23}/\cosh(\hat{\eta}_2 H_2+\hat{\delta}_{23}),$$

where

$$2\pi A(\xi) = \frac{Tv_1}{H_1}\left(\frac{v_1}{U}-\frac{v_1}{c}\right)\left\{1-\left(\frac{v_1}{c}\right)^2\right\}^{-1}\left\{1-\frac{\hat{\eta}_2 H_2}{\bar{\eta}_1 H_1}\cdot\frac{\sin 2\bar{\delta}_{12}}{\cos 2\bar{\delta}_{12}+\cosh(\hat{\eta}_2 H_2+\hat{\delta}_{23})}\right\}^{-1}.$$

(i')

$$\cos\{\eta_2(z-H_1-H_2)+\delta_{23}\}/\cos(\eta_2 H_2-\delta_{23}) = \sinh\{-\hat{\eta}_2(z-H_1-H_2)+\hat{\delta}_{23}'\}/\sinh(\hat{\eta}_2 H_2+\hat{\delta}_{23}'),$$

$$\cos\delta_{23}/\cos(\eta_2 H_2-\delta_{23}) = \sinh\hat{\delta}_{23}'/\sinh(\hat{\eta}_2 H_2+\hat{\delta}_{23}'),$$

$$i\frac{\eta_2 H_2}{\eta_1 H_1}\cdot\frac{\sin 2\delta_{12}}{\cos 2\delta_{12}+\cos 2(\eta_2 H_2-\delta_{23})} = \frac{\hat{\eta}_2 H_2}{\bar{\eta}_1 H_1}\cdot\frac{\sin 2\bar{\delta}_{12}}{\cos 2\bar{\delta}_{12}-\cosh 2(\hat{\eta}_2 H_2+\hat{\delta}_{23}')}.$$

(ii)

$$[\psi_0+\psi_1]_{\text{Love}} = -2\pi A(\xi)\cos\bar{\eta}_1 E\cos\bar{\eta}_1 z,$$

$$[\psi_2]_{\text{Love}} = -2\pi A(\xi)\cos\bar{\eta}_1 E\cos\bar{\eta}_1 H_1 \cdot\cos\{\bar{\eta}_2(z-H_1-H_2)+\bar{\delta}_{23}\}/\cos(\bar{\eta}_2 H_2-\bar{\delta}_{23}),$$

$$[\psi_3]_{\text{Love}} = -2\pi A(\xi)\cos\bar{\eta}_1 E\cos\bar{\eta}_1 H_1\, e^{-\hat{\eta}_3(z-H_1-H_2)}\cos\bar{\delta}_{23}/\cos(\bar{\eta}_2 H_2-\bar{\delta}_{23}),$$

where

$$2\pi A(\xi) = \frac{Tv_1}{H}\left(\frac{v_1}{U}-\frac{v_1}{c}\right)\left\{1-\left(\frac{v_1}{c}\right)^2\right\}^{-1}\left\{1+\frac{\bar{\eta}_2 H_2}{\bar{\eta}_1 H_1}\cdot\frac{\sinh 2\hat{\delta}_{12}}{\cosh 2\hat{\delta}_{12}+\cos 2(\bar{\eta}_2 H_2-\bar{\delta}_{23})}\right\}^{-1}.$$

(ii′) $$i\frac{\eta_2 H_2}{\eta_1 H_1}\cdot\frac{\sin 2\delta_{12}}{\cos 2\delta_{12}+\cos 2(\eta_2 H_2-\delta_{23})}=\frac{\bar{\eta}_2 H_2}{\bar{\eta}_1 H_1}\cdot\frac{\sinh 2\hat{\delta}_{12}'}{-\cosh 2\hat{\delta}_{12}+\cos 2(\bar{\eta}_2 H_2-\bar{\delta}_{23})}.$$

(iii) $$[\psi_0+\psi_1]_{\text{LOVE}}=-2\pi A(\xi)\cosh\hat{\eta}_1 E\cosh\hat{\eta}_1 z,$$

$$[\psi_2]_{\text{LOVE}}=-2\pi A(\xi)\cosh\hat{\eta}_1 E\cosh\hat{\eta}_1 H_1\cdot\cos\{\bar{\eta}_2(z-H_1-H_2)+\bar{\delta}_{23}\}/\cos(\bar{\eta}_2 H_2-\bar{\delta}_{23}),$$

$$[\psi_3]_{\text{LOVE}}=-2\pi A(\xi)\cosh\hat{\eta}_1 E\cosh\hat{\eta}_1 H_1 e^{-\hat{\eta}_3(z-H_1-H_2)}\cos\bar{\delta}_{23}/\cos(\bar{\eta}_2 H_2-\bar{\delta}_{23}),$$

where

$$2\pi A(\xi)=\frac{Tv_1}{H_1}\left(\frac{v_1}{U}-\frac{v_1}{c}\right)\left\{\left(\frac{v_1}{c}\right)^2-1\right\}^{-1}\left\{1+\frac{\bar{\eta}_2 H_2}{\hat{\eta}_1 H_1}\cdot\frac{\sin 2\bar{\delta}_{12}}{\cos 2\bar{\delta}_{12}+\cos 2(\bar{\eta}_2 H_2-\bar{\delta}_{23})}\right\}^{-1}.$$

In the above expression of LOVE-waves, the common coefficient of $exp\{i(\omega t-\xi x)\}$ is omitted.

8. Some special cases

When $c=v_2$, η_2 is zero and the right hand side of (25) becomes indeterminate, due to the next relation :

$$\eta_j H_j=(\omega H_j/v_j)\{1-(v_j/c)^2\}^{-1/2}. \tag{47}$$

One must calculate the limitting value of it. For this purpose it is more convenient to use (23) in place of (25). At the limit of $\eta_2=0$, (23) will be transformed to

$$\cot\eta_1 H_1=\frac{\mu_1\eta_1}{\mu_3\eta_3}+\frac{\mu_1}{\mu_2}\cdot\frac{H_2}{H_1}\cdot\eta_1 H_1. \tag{48}$$

It is very easy to find $\eta_1 H_1$ which is satisfied by (48), because the right hand side of (48) is a straight line, contrary to the trigonometric function as that was in (25), with respect to $\eta_1 H_1$.

Also in (45),

$$\beta=-i\frac{\eta_2 H_2}{\eta_1 H_1}\cdot\frac{\sin 2\delta_{12}}{\cos 2\delta_{12}+\cos 2(\eta_2 H_2-\delta_{23})}=\frac{\eta_2 H_2}{\eta_1 H_1}\cdot\frac{1-K_{12}e^{-2i\eta_1 H_1}}{1+K_{12}K_{23}e^{-2i\eta_2 H_2}}$$

$$=\frac{H_2(1-K_{12}e^{-2i\eta_1 H_1})}{H_1\eta_1}\cdot\frac{\eta_2}{1+K_{12}K_{23}e^{-2i\eta_2 H_2}}$$

is indeterminate, if $\eta_2=0$ is put in directly. However,

$$\lim_{\eta_2\to 0}\beta=\left(\frac{H_2}{H_1}\cdot\frac{1-e^{-2i\eta_1 H_1}}{\eta_1}\right)\left\{2i\frac{\mu_2}{\mu_1\eta_1}\left(\frac{\mu_1\eta_1}{\mu_3\hat{\eta}_3}+\frac{\mu_1\eta_1}{\mu_2}H_2-i\right)\right\}^{-1}$$

$$= \frac{H_2}{H_1} \cdot \frac{\mu_1}{\mu_2} \cdot \frac{1-e^{-2i\eta_1 H_1}}{2i} \cdot \frac{1}{\cot \eta_1 H_1 - i},$$

by means of (48). Thus one has, at last,

$$\lim_{\eta_2 \to 0} \beta = \frac{H_2}{H_1} \cdot \frac{\mu_1}{\mu_2} \sin^2 \eta_1 H_1 \tag{49}$$

in which $(\eta_1 H_1)_{c=v_2}$ was already obtained from (48).

It is very easy to find the next values from (i) and (ii) in (27),

$$\left. \begin{aligned} &Tv_1/H_1 = 0 && \text{for } c = v_1 \\ \text{and} \quad &Tv_1/H_1 = (2/n)\left\{1-(v_1/v_3)^2\right\}^{1/2} && \text{for } c = v_3 \end{aligned} \right\} \tag{50}$$

where n means the order of LOVE-waves.

When $c=v_1$, in the case of (iii), η_1 is zero and (21) is reduced to

$$1-K_{23}\, e^{-2i\eta_2 H_2} = 0$$

or

$$\tan \eta_2 H_2 = i\,(\mu_3\eta_3)/(\mu_2\eta_2)\,. \tag{51}$$

It is very easy to find the value of $(Tv_1/H_1)_{c=v_1}$, getting $\eta_2 H_2$ satisfied by (51).

Also in (45);

$$2\pi A\,(\xi) = \omega\left(\frac{1}{U} - \frac{1}{c}\right) \cdot \beta',$$

$$\beta' = \frac{1}{\eta_1} \cdot \frac{1+K_{12}K_{23}\, e^{-2i\eta_2 H_2}}{\eta_1 H_1\,(1+K_{12}K_{23}\, e^{-2i\eta_2 H_2}) + \eta_2 H_2\,(1-K_{12}\, e^{-2i\eta_1 H_1})}$$

is indeterminate, if $\eta_1=0$ is put in directly. However,

$$\lim_{\eta_1 \to 0} \beta' = \lim_{\eta_1 \to 0} \frac{1}{\eta_1} \cdot \frac{1+K_{12}K_{23}\, e^{-2i\eta_2 H_2}}{\eta_2 H_2\,(1-K_{12}\, e^{-2i\eta_1 H_1})} = \frac{\mu_1}{\mu_2} \cdot \frac{1}{\eta_2 H_2} \cdot \frac{1}{\eta_2}$$
$$= \frac{\mu_1}{\mu_2} \cdot \frac{1}{\eta_2 H_2} \cdot \frac{v_2}{\omega}\left\{1-\left(\frac{v_2}{v_1}\right)^2\right\}^{-1/2},$$

Thus one has, at last,

$$2\pi A\,(\xi) = \left(\frac{1}{U} - \frac{1}{c}\right) \frac{\mu_1}{\mu_2}\left\{1-\left(\frac{v_2}{v_1}\right)^2\right\}^{-1/2}\left(\frac{v_2}{\eta_2 H_2}\right) \tag{52}$$

in which $(\eta_2 H_2)_{c=v_1}$ was already obtained from (51).

It is very easy to find the next values from (iii) and (ii) in (27),

$$\left. \begin{aligned} &Tv_1/H_1 = 0 && \text{for } c = v_2 \\ \text{and} \quad &Tv_1/H_1 = (2/n)\left\{1-(v_1/v_3)^2\right\}^{1/2} && \text{for } c = v_3\,. \end{aligned} \right\} \tag{53}$$

If $H_2/H_1=0$, it being that $H_2=0$, (21) will be reduced to

$$F(\xi, \omega) = 1 - \frac{K_{12}+K_{23}}{1+K_{12}K_{23}} e^{-2i\eta_1 H_1} = 1 - K_{13} e^{-2i\eta_1 H_1} = 0\,, \tag{54}$$

in which $K_{13} = \{1-(\mu_3\eta_3)/(\mu_1\eta_1)\}/\{1+(\mu_3\eta_3)/(\mu_1\eta_1)\}$

is considered to be the reflecting coefficient between the first and the third layers. The equation of (54) is nothing but the characteristic equation in two layers, the first and the third.

If $H_2/H_1=\infty$, it being that $H_1=0$, (21) will be reduced to

$$1-K_{23}\,e^{-2i\eta_2 H_2} = 0\,. \tag{55}$$

This is also nothing but the characterisitc equation in two layers, the second and the third.

9. Numerical calculations of the dispersion curve

If the densities are equal in every layer, namely

$$\rho_1 = \rho_2 = \rho_3\,, \tag{56}$$

the following four combinations of rigidities in the underground are to be considered for the existence of LOVE-waves;

$$\left.\begin{array}{lll} \text{(A)} & \mu_1<\mu_2<\mu_3\,; & \mu_2/\mu_1>1 \text{ and } \mu_3/\mu_2>1, \\ \text{(B)} & \mu_2<\mu_1<\mu_3\,; & \mu_2/\mu_1<1 \text{ and } \mu_3/\mu_2>1, \\ \text{(C)} & \mu_2<\mu_3<\mu_1\,; & \mu_2/\mu_1<1 \text{ and } \mu_3/\mu_2>1, \\ \text{(D)} & \mu_1<\mu_3<\mu_2\,; & \mu_2/\mu_1>1 \text{ and } \mu_3/\mu_2<1, \end{array}\right\} \tag{57}$$

among which the former two cases alone will be calculated here.

Case (A) from (57) may have the types (i) and (ii) in (26). On the other

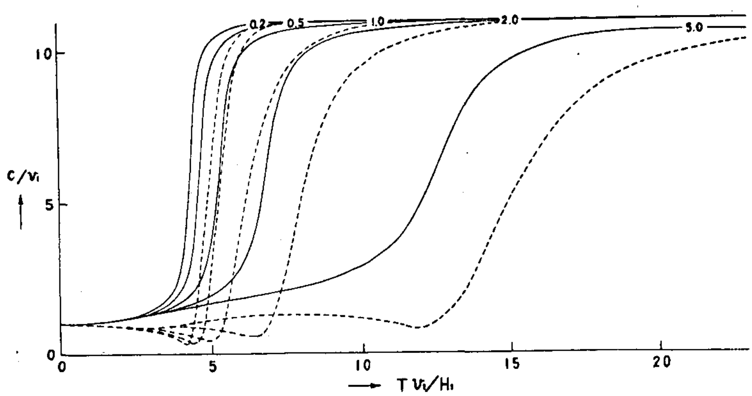

Fig. 2. (a).

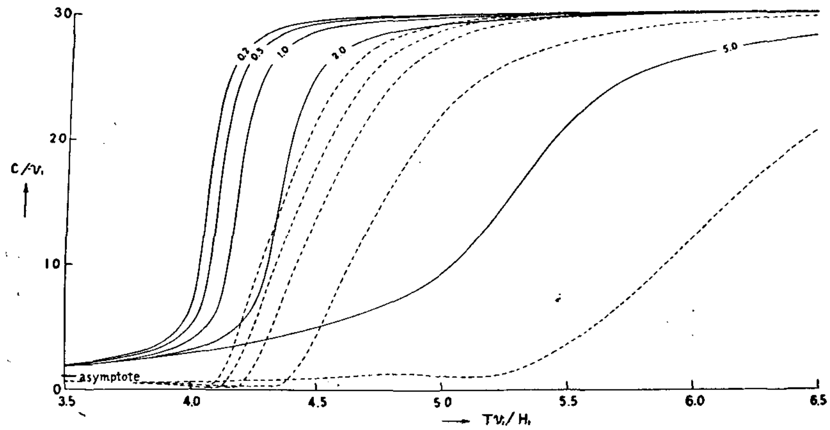

Fig. 2. (b).

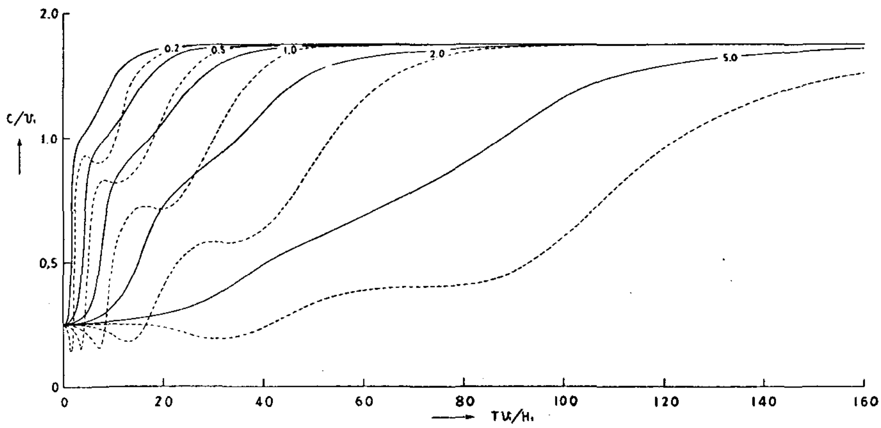

Fig. 2. (c).

hand, case (B) may have the types (iii) and (ii). Dispersion curves have been calculated by any proper formula selected from among (27) and (29).

Four examples are shown in Fig. 2, the parameter being H_2/H_1.

Group velocities, obtained graphically by phase velocities, are also shown in Fig. 2 by broken lines. Group velocity may have two minima; one is near $Tv_1/H_1=4$ and another is near $T(v_1/H_1+v_2/H_2)=4$. The former corresponds to that in two layers previously investigated[1]. The latter has appeared for the first time and is illustrated clearly in Fig. 3 where μ_3/μ_2 is kept 30.

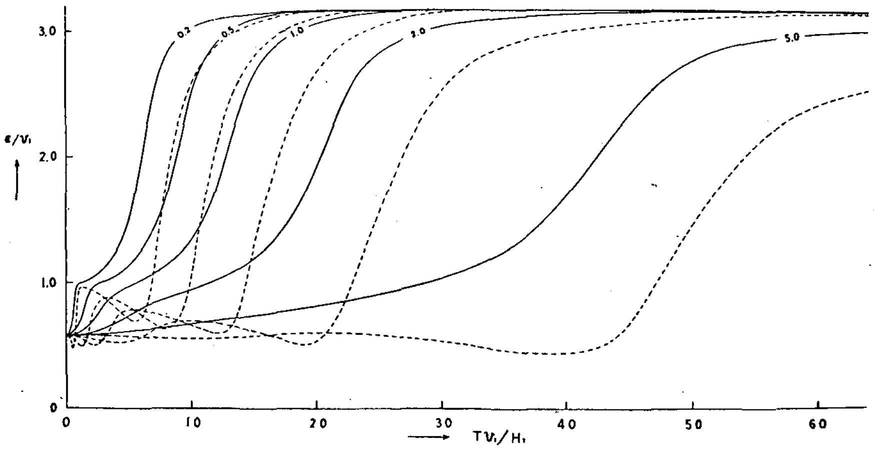

Fig. 2. (d).

Fig. 2. Dispersion curves in three stratified layers. (a) $\mu_2/\mu_1=4$ and $\mu_3/\mu_2=30$, (b) $\mu_2/\mu_1=30$ and $\mu_3/\mu_2=30$, (c) $\mu_2/\mu_1=1/16$ and $\mu_3/\mu_2=30$, (d) $\mu_2/\mu_1=1/3$ and $\mu_3/\mu_2=30$.

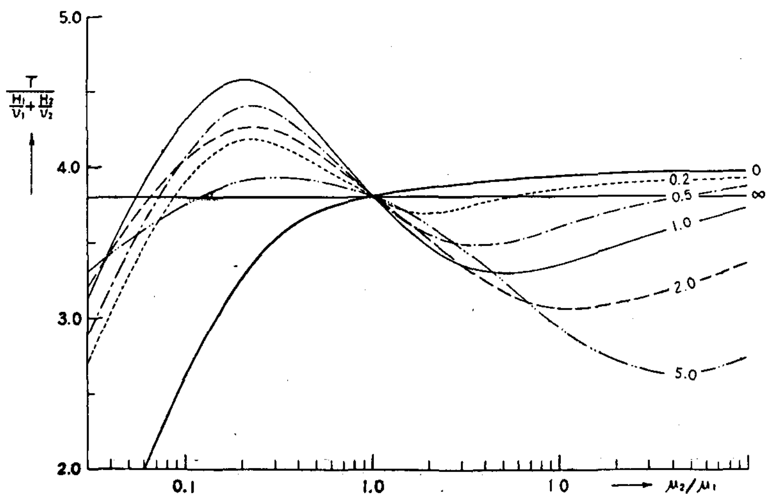

Fig. 3. The period corresponding to the second minimum group velocity. $\mu_3/\mu_2=30$. Parameter ; H_2/H_1.

10. Numerical calculation of the amplitude function

By using Fig. 2 and eq. (45), one can easily calculate the amplitude function as illustrated in Fig. 4.

Each curve in Fig. 4 may have two maxima, corresponding respectively to the two minimum group velocities. The first maximum of the amplitude function in three layers seems to have some correspondence to the maximum of the amplitude function in two layers. The second maximum of it has appeared for the first time in the present case. The period corresponding to

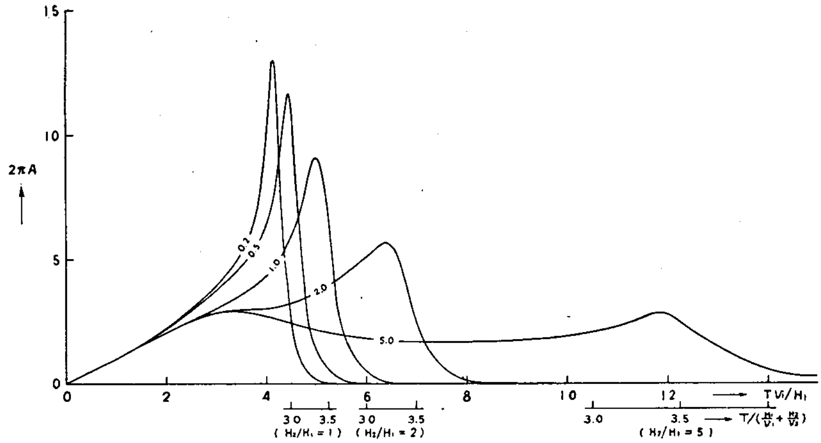

Fig. 4. (a).

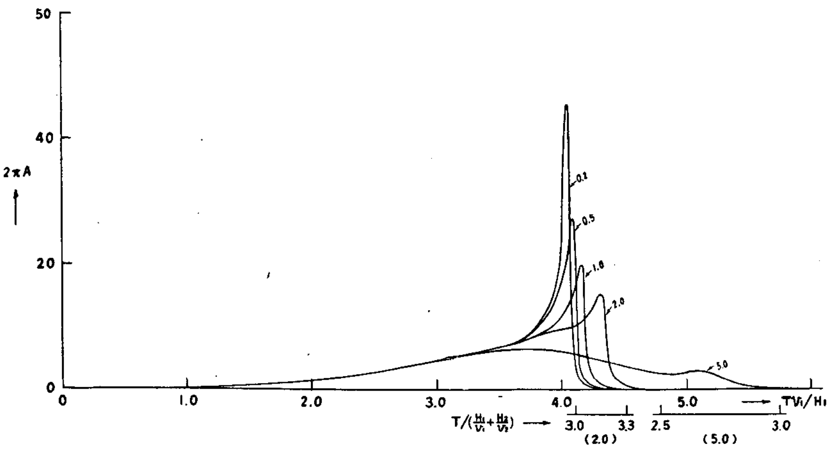

Fig. 4. (b).

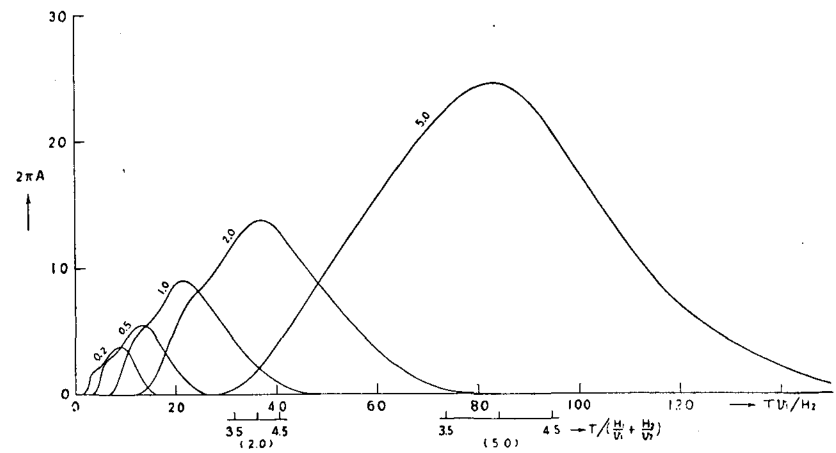

Fig. 4. (c).

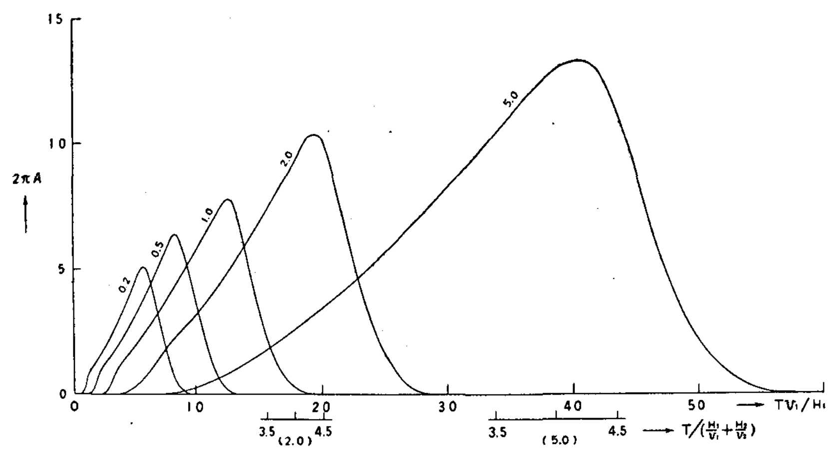

Fig. 4, (d).

Fig. 4. The amplitude function in three stratified layers. Parameter; H_2/H_1

the second maximum of the amplitude function is very similar to that in Fig. 3. Changing the abscissa from μ_2/μ_1 to H_2/H_1, the relation is shown by Fig. 5.

11. Numerical calculation of the amplitude distribution

Using the relation between period and phase velocity and that between

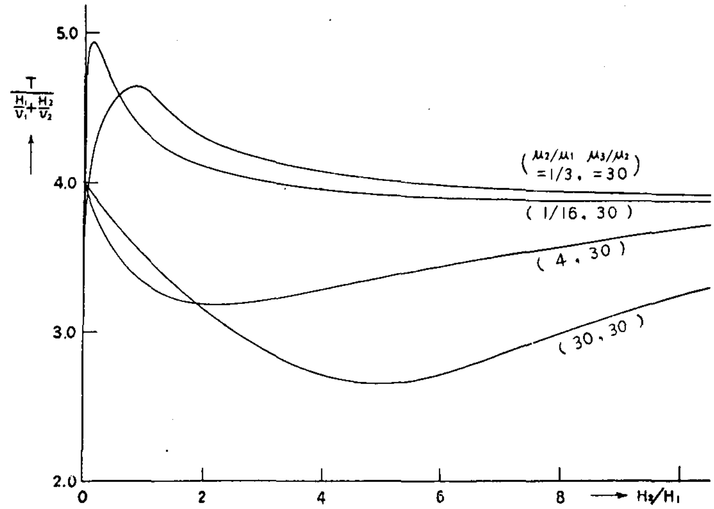

Fig. 5. The period corresponding to the second maximum of the amplitude function.

period and the amplitude function, one can calculate amplitude distribution within the layers directly from (44). Fig. 6 shows that distribution for several periods.

12. Remarks

From the more general view point, "the apparent reflecting coefficient" by ABELES is expressed as follows[4]:

$$\mathcal{R}_{j,j+1}{}^{(n-j)} = \frac{R_{j,j+1} + \mathcal{R}_{j+1,j+2}{}^{(n-j-1)}\, e^{-2i\eta_{j+1}H_{j+1}}}{1 + R_{j,j+1}\, \mathcal{R}_{j+1,j+2}{}^{(n-j-1)}\, e^{-2i\eta_{j+1}H_{j+1}}} \tag{58}$$

and

$$\mathcal{R}_{n,n+1}{}^{(0)} = R_{n,n+1}\,, \tag{59}$$

where

- n ; total number of superficial layers,
- j ; number of layers counted from the surface of the earth,
- n-j ; number of layers between the j th and the $(n-1)$ th layers, the latter being semi-infinite,
- R ; ordinary reflecting coefficient at the boundary of two layers extending semi-infinitely.

There are two superficial layers in the present problem, so $n=2$ in (58) and (59) ;

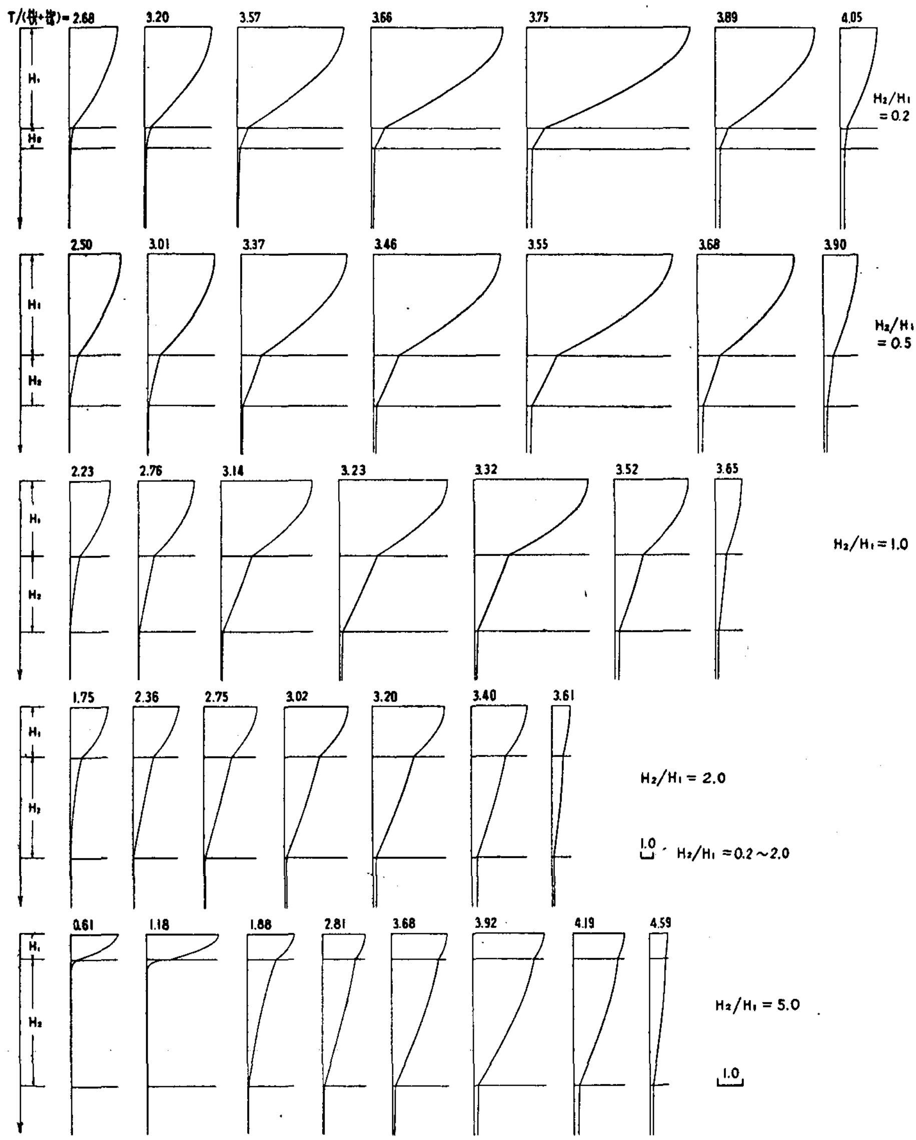

Fig. 6. (a). $\mu_2/\mu_1=4$, $\mu_3/\mu_2=30$.

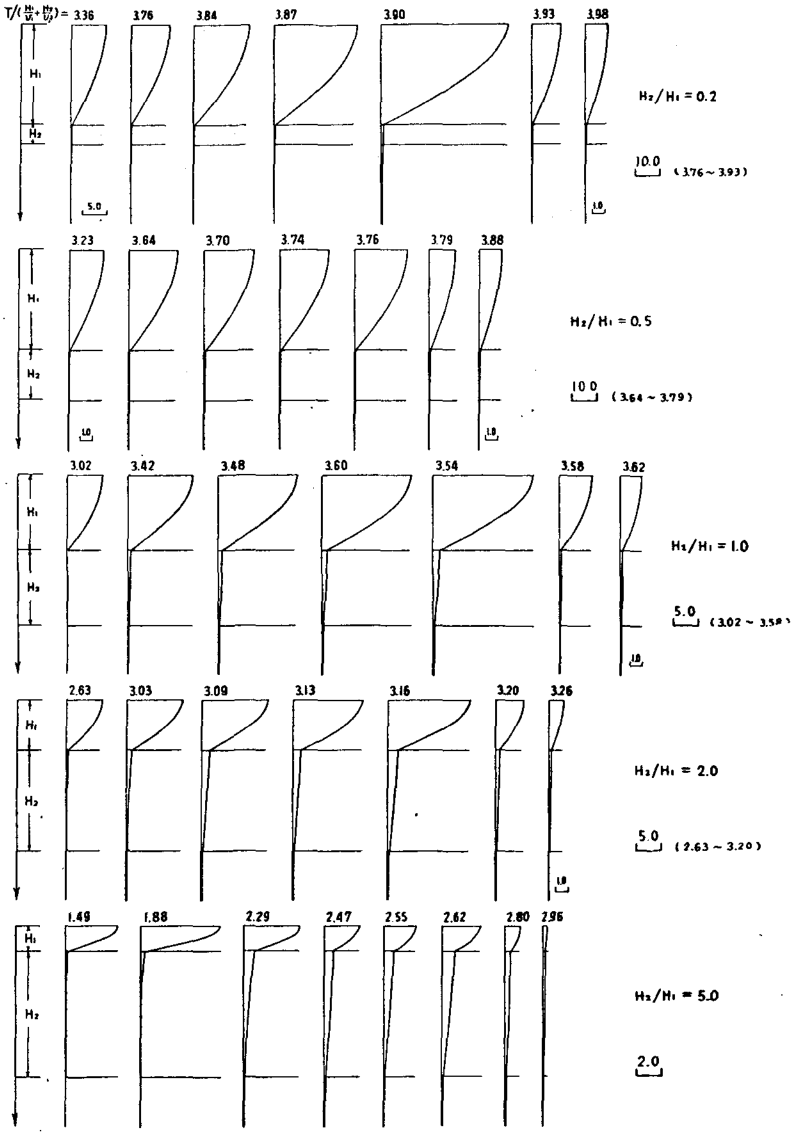

Fig. 6. (b). $\mu_2/\mu_1=30$, $\mu_3/\mu_2=30$.

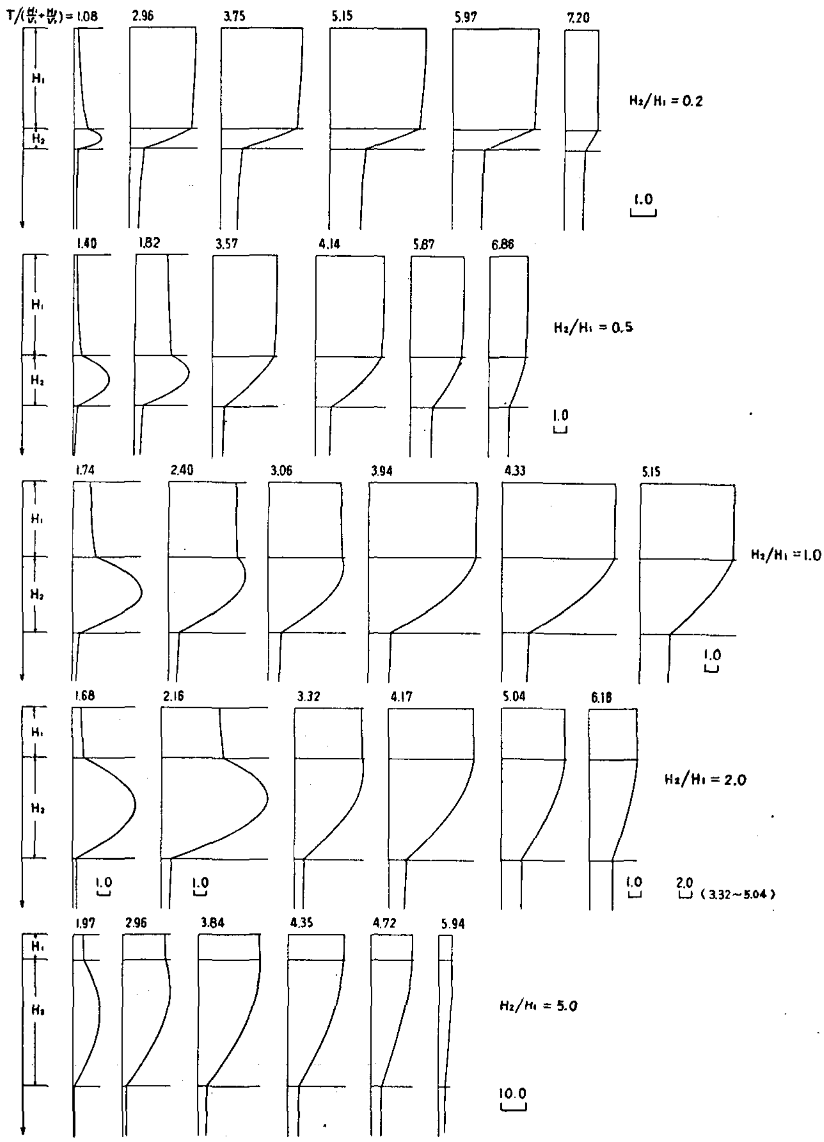

Fig. 6. (c). $\mu_2/\mu_1=1/16$, $\mu_3/\mu_2=30$.

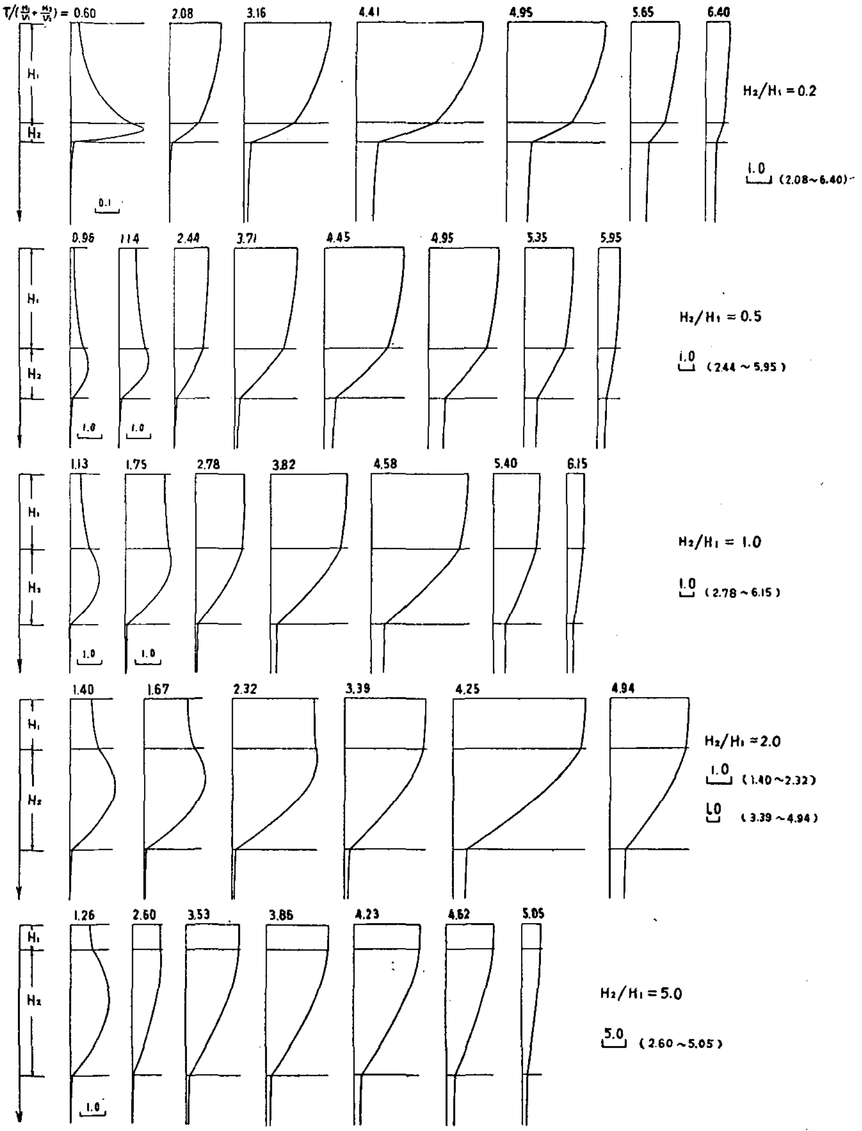

Fig. 6. (d). $\mu_2/\mu_1=1/3$, $\mu_3/\mu_2=30$.

Fig. 6. Amplitude distribution in three layers.

$$\mathcal{R}_{1,2}{}^{(1)} = \frac{R_{1,2} + \mathcal{R}_{2,3}{}^{(0)} e^{-2i\eta_2 H_2}}{1 + R_{1,2}\, \mathcal{R}_{2,3}{}^{(0)} e^{-2i\eta_2 H_2}} = \frac{R_{1,2} + R_{2,3}\, e^{-2i\eta_2 H_2}}{1 + R_{1,2}\, R_{2,3}\, e^{-2i\eta_2 H_2}}\,. \tag{60}$$

On the other hand, the characterisitc equation of LOVE-waves in two stratified layers ($n=1$) was already known;

$$K_{12} = e^{2i\eta_1 H_1}. \tag{61}$$

Using the apparent reflecting coefficient $R_{1,2}{}^{(1)}$ in (60) in place of the ordinary reflecting coefficient K_{12} in (61), the layer below the second will be taken into account by itself. Thus one has

$$\mathcal{R}_{1,2}{}^{(1)} = e^{2i\eta_1 H_1}. \tag{62}$$

Putting (62) into (60) and rewriting $R_{1,2}$ and $R_{2,3}$ respectively as K_{12} and K_{23}, one has

$$1 - \frac{e^{-2i\eta_1 H_1}(K_{12} + K_{23}\, e^{-2i\eta_2 H_2})}{1 + K_{12} K_{23}\, e^{-2i\eta_2 H_2}} = 0 \tag{63}$$

which is the same as (22).

In order to obtain the characteristic equation alone, the just above method must be more convenient than that by which (22) was arrived at. However the expression of the amplitude cannot be gotten by these simple replacements[5].

In the present paper, the zeroth order of LOVE-waves alone has been treated. Other calculations will be shown in the next paper where physical considerations as to results of the calculations will also be given.

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