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Complex Roots in the Characteristic Equation for LOVE-type Waves in a Layer over a Half Space

Kyozi TAZIME

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Abstract

In order to investigate surface waves thoroughly, it is necessary to try to find complex roots besides real ones in the characteristic equation. In the present paper, for simplicity, SH-waves are treated. Two types of waves have been found. One resembles somewhat the ordinary LOVE waves and the other the normal modes in a plate.

1. Introduction

It is well known that LOVE waves exist when the velocity of SH-waves in a superficial layer is smaller than that in a lower half space. When the half space is absent, on the other hand, normal modes of SH-waves can again exist. In this case the superficial layer may be taken as a plate. SH-waves feel no resistance to liquid. In other words, liquid is nothing but a vacuum for SH-waves.

From the viewpoint of rigidity ratio μ_2/μ_1, the above description will be arranged as follows, assuming that densities ρ_1 and ρ_2 are equal to each layer.

(a) When $\mu_2/\mu_1 > 1$, there exist LOVE waves which are a kind of normal modes.

(b) When $\mu_2/\mu_1 = 0$, normal modes can again exist.

In both cases no energy may run off the superficial layer. However it may be physically expected that some waves akin to (b) will be observed, if μ_2/μ_1 is not exactly zero but nearly zero. It can not be recognized physically that the waves will appear quite suddenly at $\mu_2/\mu_1 = 0$.

Admitting the above consideration, normal modes of SH-waves must exist everywhere between $1 > \mu_2/\mu_1 \geqq 0$, although some relative difficulty of existence may be unavoidable.

2. General considerations on complex wave numbers

When displacement of SH-waves is written as

$$\psi_j\,(t, x, z) \qquad \text{where} \quad j = 1 \text{ or } 2, \tag{2.1}$$

this must satisfy the next equation,

$$\partial^2 \psi_j / \partial t^2 = v_j^2 \nabla^2 \psi_j . \tag{2.2}$$

Taking space coordinates as those illustrated in Fig. 1,

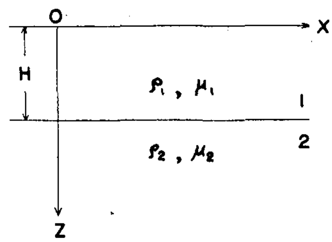


Fig. 1. The space coordinates under consideration.

(2.2) can have the following particular solution:

$$\psi_j = e^{i(\omega t - \xi x)} (C_j e^{i\eta_j z} + D_j e^{-i\eta_j z}) \tag{2.3}$$

in which

$$\xi^2 + \eta_j^2 = k_j^2 \qquad \text{and} \qquad k_j = \omega / v_j . \tag{2.4}$$

If ξ and k_j are complex quantities,

$$\xi = \bar{\xi} + i \hat{\xi} \qquad \text{and} \qquad k_j = \bar{k}_j + i \hat{k}_j , \tag{2.5}$$

η_j must also be complex,

$$\eta_j = \bar{\eta}_j + i \hat{\eta}_j \tag{2.6}$$

where $\bar{\xi}$, $\hat{\xi}$, ..., $\hat{\eta}_j$ are real quantities.

Putting (2.5) and (2.6) into (2.4), one gets

$$\eta_j^2 = \bar{k}_j^2 - \hat{k}_j^2 - (\bar{\xi}^2 - \hat{\xi}^2) + 2 i (\bar{k}_j \hat{k}_j - \bar{\xi} \hat{\xi}) \tag{2.7}$$

and finds that

$$\bar{\eta}_j \hat{\eta}_j = \bar{k}_j \hat{k}_j - \bar{\xi} \hat{\xi} . \tag{2.8}$$

The sign of $\bar{\eta}_j \hat{\eta}_j$ on the lower half of ξ-plane is illustrated in Fig. 2 where the suffix j is omitted. The white circle indicates the branch point where $\eta_j = 0$.

The signs of $\bar{\eta}_j$ and $\hat{\eta}_j$ themselves have not been decided. In spite of various combinations of the signs of $\bar{\eta}_j$ and $\hat{\eta}_j$ in (2.6), either C_2 or D_2 in (2.3) must be zero, because the second layer, that is the lowest layer in the present problem, has no reflecting surface under itself. Physically, the next condition may be considered:

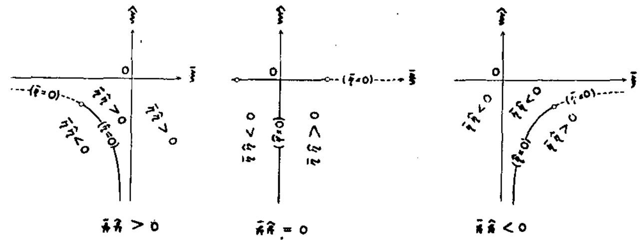


Fig. 2. The sign of $\bar{\eta}_j \hat{\eta}_j$.

$$\psi_2 \text{ should neither increase its amplitude nor regress in } x\text{- and } z\text{-directions below the source.} \qquad (2.9)$$

The condition in x-direction can be satisfied well in the fourth quadrant of ξ-plane. Yet the condition in z-direction is too severe to find complex roots in the characteristic equation. If C_2 may be abandoned, for example, the region where $\bar{\eta}_2 > 0$ and $\hat{\eta}_2 < 0$ can satisfy the condition completely. But the region where $\bar{\eta}_2 < 0$ and $\hat{\eta}_2 < 0$ or the region where $\bar{\eta}_2 > 0$ and $\hat{\eta}_2 > 0$ may satisfy the condition only incompletely.

In the former region, ψ_2 does not increase its amplitude but regresses in z-direction. In the latter region, on the contrary, ψ_2 does not regress but increases its amplitude in z-direction. Some compromise will be proposed to this question.

Now several boundary conditions are given in the present case as follows:

stress must be zero at $z = 0$

and (2.10)

displacements as well as stresses must be continuous at $z = H$.

Because (2.3) must satisfy (2.10) and also $C_2 = 0$, the next relation will be obtained,

$$\tan \eta_1 H = i\,(\mu_2/\mu_1)(\eta_2/\eta_1) \qquad (2.11)$$

which is the characteristic equation for LOVE-type waves.

Since

$$\tan(\bar{\eta}_1 + i\,\hat{\eta}_1)H = (\tan \bar{\eta}_1 H \operatorname{sech}^2 \hat{\eta}_1 H + i \tanh \hat{\eta}_1 H \sec^2 \bar{\eta}_1 H) / (1 + \tan^2 \bar{\eta}_1 H \tanh^2 \hat{\eta}_1 H)$$

and (2.12)

$$i\,(\bar{\eta}_2 + i\hat{\eta}_2)/(\bar{\eta}_1 + i\hat{\eta}_1) = \{\hat{\eta}_1\bar{\eta}_2 - \bar{\eta}_1\hat{\eta}_2 + i\,(\bar{\eta}_1\bar{\eta}_2 + \hat{\eta}_1\hat{\eta}_2)\}/(\bar{\eta}_1^2 + \hat{\eta}_1^2)\,,$$

the characteristic equation can have no complex root in the region $\bar{\eta}_2\hat{\eta}_2 > 0$ if

$\bar{\eta}_2$ and $\hat{\eta}_2$ are negative.

On the other hand, expressions (2.3) and (2.11) are never disturbed by changing the sign from η_1 to $-\eta_1$. From now on, it will be assumed without failure of generality that

$\bar{\eta}_j$ is positive in every region
and
$\hat{\eta}_j$ is negative in the region $\bar{\eta}_j\hat{\eta}_j < 0$ but positive in the region $\bar{\eta}_j\hat{\eta}_j > 0$. (2.13)

If $\hat{\eta}_j$ is positive, however, ψ_2 seems to increase its amplitude infinitely in z-direction. It must be ascertained whether this is merely apparent or actual.

3. Ray theoretical investigations on complex wave numbers

Introducing incident angles, (2.4) will be expressed by

$$\xi = k_j \sin \theta_j \qquad \text{and} \qquad \eta_j = k_j \cos \theta_j \,. \tag{3.1}$$

Putting

$$\theta_j = \bar{\theta}_j - i\,\hat{\theta}_j \tag{3.2}$$

where $\bar{\theta}_j$ and $\hat{\theta}_j$ are real and positive, the next relations will be obtained:

$$\left.\begin{aligned} \bar{\xi} &= k_j \sin \bar{\theta}_j \cosh \hat{\theta}_j \,, \quad \hat{\xi} = -k_j \cos \bar{\theta}_j \sinh \hat{\theta}_j \\ \bar{\eta}_j &= k_j \cos \bar{\theta}_j \cosh \hat{\theta}_j \,, \quad \hat{\eta}_j = k_j \sin \bar{\theta}_j \sinh \hat{\theta}_j \,, \end{aligned}\right\} \tag{3.3}$$

assuming that k_j is real.

Then the phase lag of ψ_2 will be calculated,

$$\begin{aligned} &-i\,\xi x - i\,\eta_1 H - i\,\eta_2(z-H) \\ &\quad = -i\left\{\bar{\xi}x + \bar{\eta}_1 H + \bar{\eta}_2\,(z-H)\right\} + \left\{\hat{\xi}x + \hat{\eta}_1 H + \hat{\eta}_2\,(z-H)\right\} . \end{aligned} \tag{3.4}$$

Putting here

$$r_1^2 = x_0^2 + H^2 \qquad \text{and} \quad r_2^2 = (x-x_0)^2 + (z-H)^2 \,,$$

the first brace on the right hand side in (3.4) will be

$$r_1 k_1 \cosh \hat{\theta}_1 + r_2 k_2 \cosh \hat{\theta}_2 \,. \tag{3.5}$$

The second brace in (3.4) becomes

$$\begin{aligned} &k_1 \sinh \hat{\theta}_1 (-x_0 \cos \bar{\theta}_1 + H \sin \bar{\theta}_1) \\ &\qquad + k_2 \sinh \hat{\theta}_2 \left\{-(x-x_0) \cos \bar{\theta}_2 + (z-H) \sin \bar{\theta}_2\right\} . \end{aligned} \tag{3.6}$$

If $\bar{\theta}_1$ and $\bar{\theta}_2$ are taken as an incident angle and a refracting one respectively, $r_1 = PQ$ and $r_2 = QR$ as shown in Fig. 3. The value of (3.5) means the phase lag of ψ_2 at z from $z=0$.

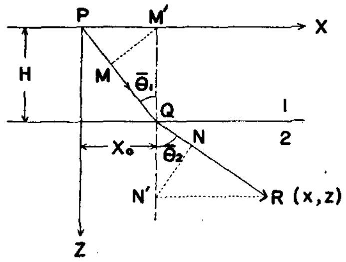


Fig. 3. Ray paths in the layers.

One sees in Fig. 3,

$$MM' = x_0 \cos \bar{\theta}_1 = H \sin \bar{\theta}_1$$

and

$$NN' = (x-x_0) \cos \bar{\theta}_2 = (z-H) \sin \bar{\theta}_2 \,.$$

Therefore (3.6) must be zero, namely

$$\begin{aligned} \psi_2 &\propto \exp\{-i\,\xi x - i\,\eta_1 H - i\,\eta_2(z-H)\} \\ &= \exp(-i\,k_1 r_1 \cosh \hat{\theta}_1 - i\,k_2 r_2 \cosh \hat{\theta}_2) \,. \end{aligned} \tag{3.7}$$

In spite of the existence of both $\hat{\xi}_1$ and $\hat{\eta}_2$, (3.7) shows that

ψ_2 will neither regress in every direction nor increase its amplitude in its progressing direction. (3.8)

Looking back to the previous section, one sees that (2.9) ought to be modified by (3.8).

4. A course for finding complex roots

In order to find complex roots satisfying (2.11), phase velocity c will be introduced,

$$c = \omega/\xi = \bar{c} + i\,\hat{c} \tag{4.1}$$

in which $\bar{c}$ and $\hat{c}$ are real and positive.

One has from (4.1)

$$\bar{\xi}/k_j = (v_j/\bar{c})\,\{1 + (\hat{c}/\bar{c})^2\}^{-1} \quad \text{and} \quad \hat{\xi}/\bar{\xi} = -\hat{c}/\bar{c} \tag{4.2}$$

and obtains

$$(\bar{\xi}/k_j - v_j/2\,\bar{c})^2 + (\hat{\xi}/k_j)^2 = (v_j/2\,\bar{c})^2 \tag{4.3}$$

in which k_j is assumed to be real.

Therefore $\bar{c}/v_j$ will rest upon the circle on ξ-plane as shown in Fig. 4. Locations of $\hat{c}/\bar{c}$ on ξ-plane are very easy, owing to the second relation of (4.2). The chain line in Fig. 4 will be explained later.

Substituting (4.1) into (2.6), η_j in the region $\bar{\eta}_j \hat{\eta}_j > 0$ can be expressed with the assumption of (2.13) as follows;

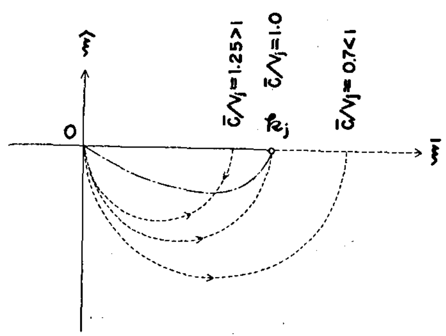


Fig. 4. Locations of $\bar{c}/v_j$ and $\hat{c}/\bar{c}$ on ξ-plane. Arrows indicate the increasing direction of ε_j.

$$\eta_j = \bar{\eta}_j + i\hat{\eta}_j = |\eta_j| \exp(i\varepsilon_j), \quad 0 \leqq \varepsilon_j \leqq \pi/2 \tag{4.4}$$

in which

$$\left.\begin{aligned} \tan 2\varepsilon_j &= 2(\hat{c}/\bar{c})/[(\bar{c}/v_j)^2\{1+(\hat{c}/\bar{c})^2\}^2+(\hat{c}/\bar{c})^2-1] \\ &\text{and} \\ |\eta_j| &= (\omega/\bar{c})R_j \end{aligned}\right\} \tag{4.5}$$

where

$$R_j = \left\{1+\left(\frac{\hat{c}}{\bar{c}}\right)^2\right\}^{-1}\left[\left\{\left(\frac{\bar{c}}{v_j}\right)^2\left(1+\frac{\hat{c}^2}{\bar{c}^2}\right)^2+\left(\frac{\hat{c}}{\bar{c}}\right)^2-1\right\}^2+\left(2\frac{\hat{c}}{\bar{c}}\right)^2\right]^{1/4}. \tag{4.6}$$

Using these expressions, one will get

$$i(\bar{\eta}_2+i\hat{\eta}_2)/(\bar{\eta}_1+i\hat{\eta}_1) = (R_2/R_1)\{\sin(\varepsilon_1-\varepsilon_2)+i\cos(\varepsilon_1-\varepsilon_2)\}. \tag{4.7}$$

Putting here

$$\left.\begin{aligned} P &= (\mu_2/\mu_1)(R_2/R_1)\sin(\varepsilon_1-\varepsilon_2) \\ &\text{and} \\ Q &= (\mu_2/\mu_1)(R_2/R_1)\cos(\varepsilon_1-\varepsilon_2), \end{aligned}\right\} \tag{4.8}$$

and using the relation (2.12), (2.11) will become

$$\left.\begin{aligned} \tan 2\bar{\eta}_1 H &= 2P/(1-P^2-Q^2) \\ &\text{and} \\ \tanh 2\hat{\eta}_1 H &= 2Q/(1+P^2+Q^2). \end{aligned}\right\} \tag{4.9}$$

Then eliminating ω from (4.9), one has

$$\tan\varepsilon_1 = \hat{\eta}_1/\bar{\eta}_1 = \tanh^{-1}\{2Q/(1+P^2+Q^2)\}/\tan^{-1}\{2P/(1-P^2-Q^2)\} \tag{4.10}$$

which means the relation between $\bar{c}/v_1$ and $\hat{c}/\bar{c}$, the parameter being v_2/v_1.

If any $\hat{c}/\bar{c}$ against $\bar{c}/v_1$ may be found with (4.10), the corresponding ω will be decided by (4.9), namely

$$\frac{\omega H}{v_1}=\frac{\bar{c}}{2v_1}\cdot\frac{\tanh^{-1}\{2Q/(1+P^2+Q^2)\}}{R_1\sin\varepsilon_1}$$
$$=\frac{\bar{c}}{2v_1}\cdot\frac{\tan^{-1}\{2P/(1-P^2-Q^2)\}}{R_1\cos\varepsilon_1}. \tag{4.11}$$

5. Preliminary investigations before the numerical solution

The relation between ε_j and $\hat{c}/\bar{c}$ is calculated by (4.5) and is illustrated in Fig. 5 where the parameter is $\bar{c}/v_j$ but the suffix j is omitted. The relation between R_j and $\hat{c}/\bar{c}$ and that between $\tan\varepsilon_j$ and $\hat{c}/\bar{c}$ are also illustrated in Figs. 6 and 7.

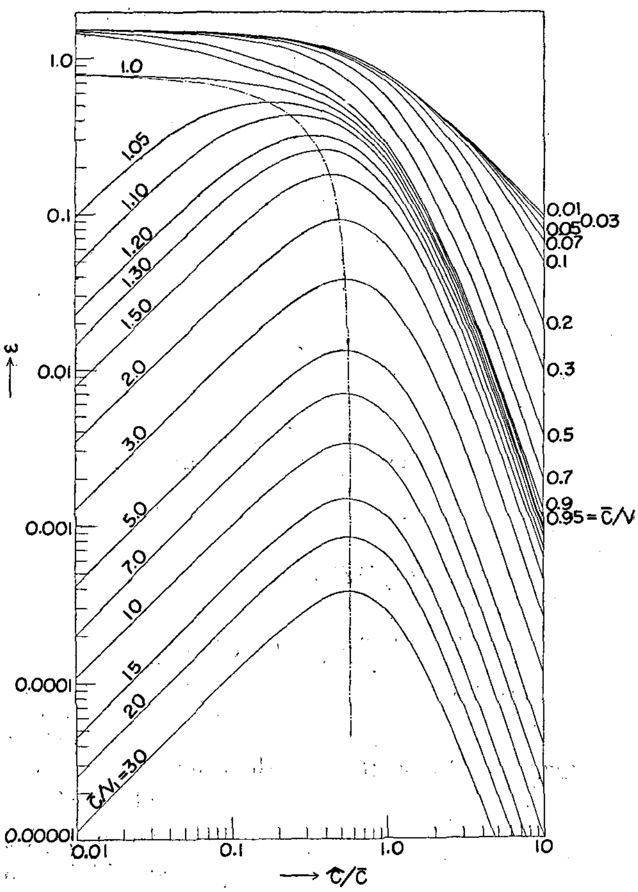


Fig. 5. The relation between ε_j and $\hat{c}/\bar{c}$.

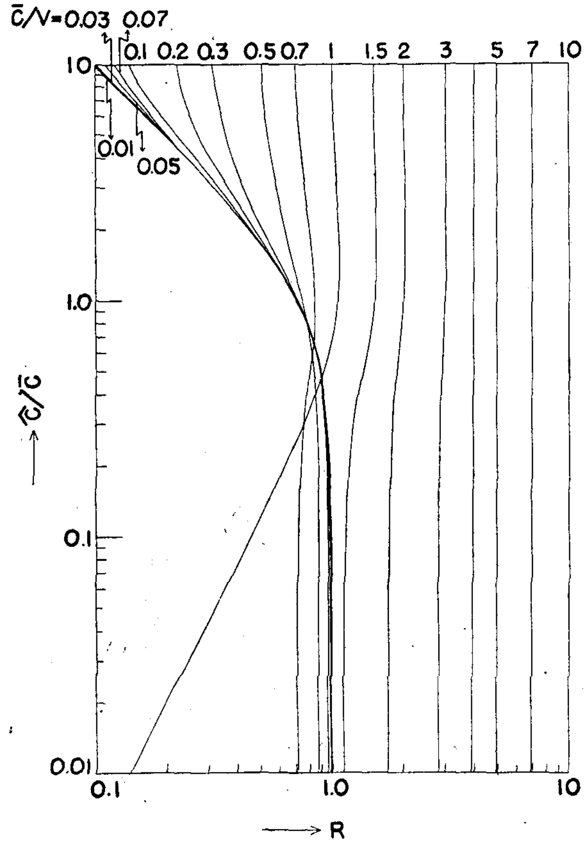


Fig. 6. The relation between R_j and $\hat{c}/\bar{c}$.

The chain lines in Figs. 4, 5 and 7 correspond to the relation

$$d\,\varepsilon_j/d\,(\hat{c}/\bar{c}) = 0\,,$$

that is

$$\hat{c}/\bar{c} = (1/3)^{1/2}\{1-(v_j/\bar{c})^2\}^{1/2}\,. \tag{5.1}$$

Table 1. P means positive and N negative.

case	(a) $v_2/v_1>1$	(b) $v_2/v_1<1$
$\varepsilon_1-\varepsilon_2$	N	P
$\tan\eta_1 H$	N	P

It is seen in Fig. 5 that the smaller the $\bar{c}/v_j$, the larger ε_j for the same value of $\hat{c}/\bar{c}$. Thus Table 1 will be obtained.

Then the multiplicity of $2\,\bar{\eta}_1 H$ in (4.9) should be somewhat restricted by the sign of $\tan\bar{\eta}_1 H\,\tan 2\,\bar{\eta}_1 H$.

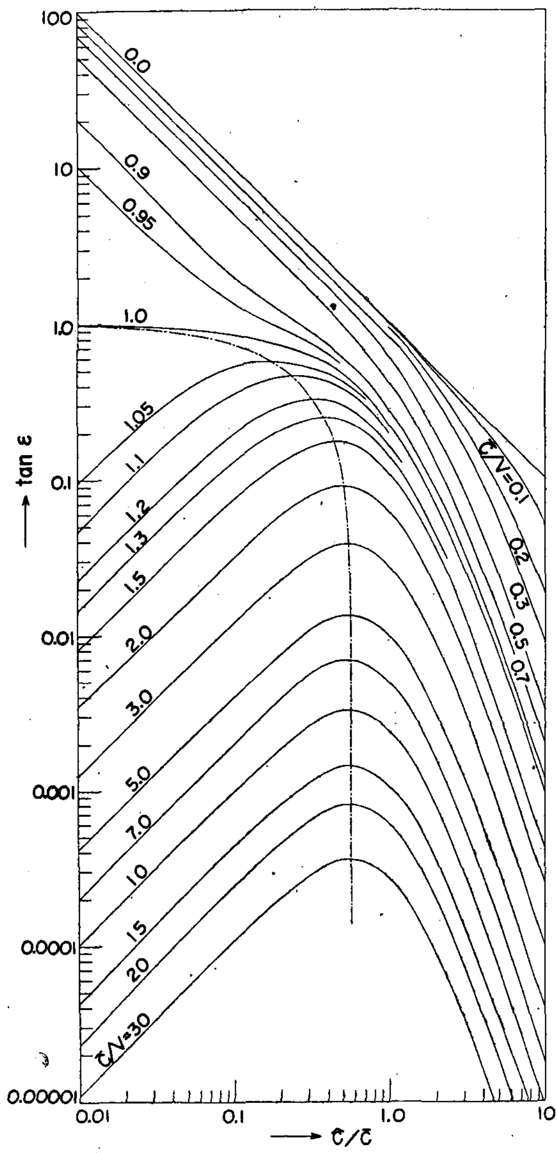


Fig. 7. The relation between tan ε_j and $\hat{c}/\bar{c}$.

$$2\bar{\eta}_1 H = \mathrm{Tan}^{-1}\{2P/(1-P^2-Q^2)\} + (2n-1)\pi \quad \text{for} \quad \tan\bar{\eta}_1 H \tan 2\bar{\eta}_1 H < 0 \qquad (5.2)$$

and

$$2\bar{\eta}_1 H = \mathrm{Tan}^{-1}\{2P/(1-P^2-Q^2)\} + 2n\pi \qquad \text{for} \quad \tan\bar{\eta}_1 H \tan 2\bar{\eta}_1 H > 0$$

in which n means any integer

If each $\bar{c}/v_j$ is different from unity,

$$(\mu_2/\mu_1)(R_2/R_1) \lesseqgtr 1 \qquad \text{for} \quad v_2/v_1 \lesseqgtr 1, \tag{5.3}$$

resulting in Table 2

Table 2 P means positive and N negative

case	(a)	(b)
P	N	P
$1-P^2-Q^2$	N	P
$\tan 2\eta_1 H$	P	P

If $\bar{c}/v_1 \gg 1$, one sees in Figs. 5 and 6 that $\cos \mathcal{E}_1 = 1$ and $R_1 = \bar{c}/v_1$. In this condition both $\tanh^{-1}\{2Q/(1+P^2+Q^2)\}$ and $\tan^{-1}\{2P/(1-P^2-Q^2)\}$ have considerably smaller values than unity and (4 11) will be reduced to

$$\omega H/v_1 = -\pi/2 + n\pi \qquad \text{for case (a)}$$

and

$$\omega H/v_1 = n\pi \qquad \text{for case (b)}, \tag{5.4}$$

although v_2/v_1 reaches neither ∞ nor 0

6. Complex roots found numerically in the characteristic equation

(4 10) has been solved numerically for case (a) The result is illustrated in Fig 8 where $v_2/v_1 = 4$ and $\rho_2/\rho_1 = 1$ are assumed Solutions cannot be easily determined for lower orders on the left side of the chain line on which

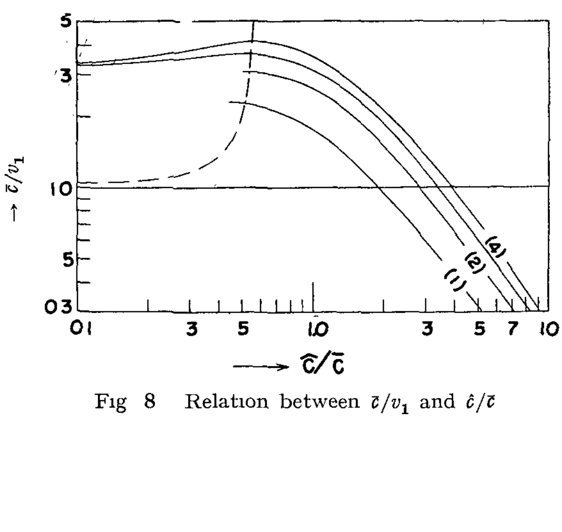


Fig 8 Relation between $\bar{c}/v_1$ and $\hat{c}/\bar{c}$

$\bar{c}/v_1$ becomes maximum.

By making use of (4.2) and (4.3), the locus of the complex root can be drawn on ξ-plane as shown with full lines in Fig. 9 where the number in parenthesis means the order of the root. In these figures chain lines correspond to (5.1).

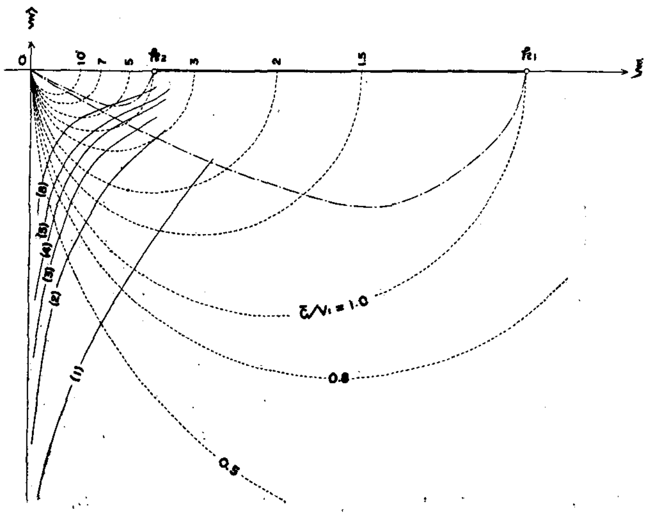


Fig. 9. Locus of the complex root on ξ-plane.

Similarly to case (a), several relations have been obtained for case (b). These results are illustrated in Figs. 10 to 12 where higher orders than the second are omitted. The relation between Tv_1/H and $\bar{c}/v_1$ is also shown in Fig. 13 where broken lines are dispersion curves for $v_2/v_1=0$ and dotted lines indicate $\hat{c}/v_1$.

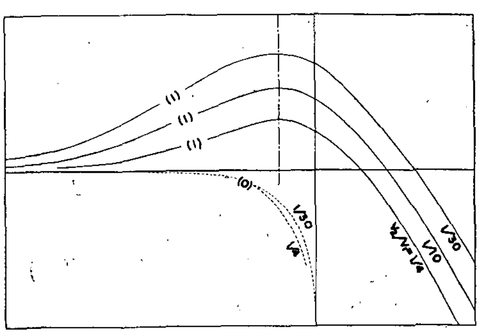


Fig. 10. Relation between $\bar{c}/v_1$ and $\hat{c}/\bar{c}$ for zeroth and first orders.

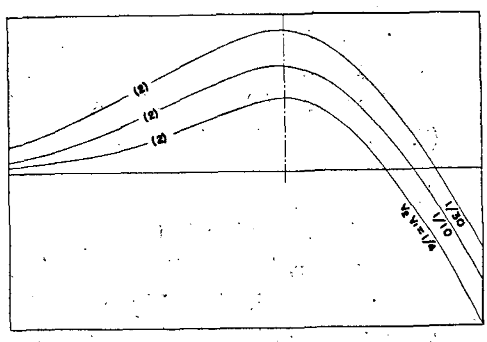


Fig. 11. Relation between $\bar{c}/v_1$ and $\hat{c}/\bar{c}$ for the second order.

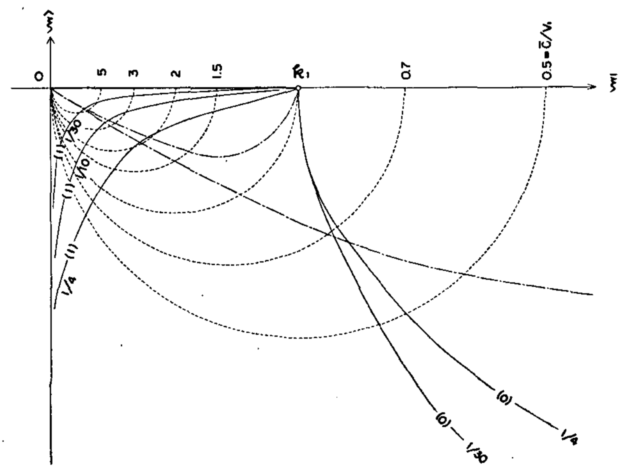


Fig. 12. Locus of the complex root on ξ-plane.

Indeed complex roots have been found in both cases (a) and (b), but it is still questionable whether these complex roots will produce practical waves or not. This question must be discussed in the following sections.

7. Ray theoretical interpretations on complex roots

Using reflecting coefficient,

$$K = \{1-(\mu_2\eta_2)/(\mu_1\eta_1)\}/\{1+(\mu_2\eta_2)/(\mu_1\eta_1)\}\,, \tag{7.1}$$

one can express the characteristic equation by

$$1 - K \exp(-2\,i\,\eta_1 H) = 0 \tag{7.2}$$

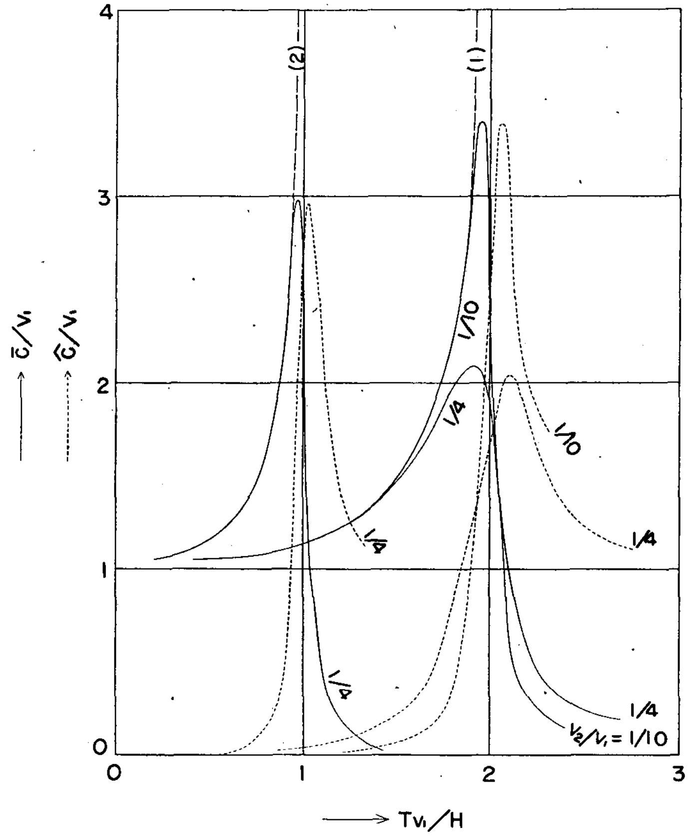
which is the same equation as (2.11).

Taking ξ as real, K is illustrated in Figs. 14(a) and (b) where $\rho_2/\rho_1 = 1$ is assumed.

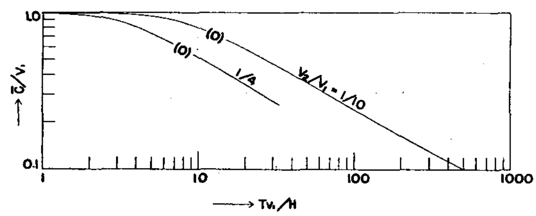
For case (a). If v_2/v_1 approaches to ∞, K may coincide with -1 and (7.2) will be reduced at the same time to

$$\cos\eta_1 H = 0\,. \tag{7.3}$$

This is nothing but the characteristic equation for LOVE waves at the limit when $v_2/v_1 = \infty$; it is equivalent to the upper equation of (5.2) or (5.4). It must be noticed in Fig. 14 that K can become near -1 within the region $v_2 < c < \infty$ if v_2/v_1 is very large, although the phase velocity of LOVE waves should be restricted within $v_1 < c < v_2$.



(a)



(b)

Fig. 13. Relations between $\bar{c}/v_1$, $\hat{c}/v_1$ and Tv_1/H.

It may be considered, from a practical viewpoint, that (7.3) can be satisfied approximately even if v_2/v_1 does not become exactly ∞. This consideration suggests that there may exist in layers where $1 < v_2/v_1 < \infty$ some waves

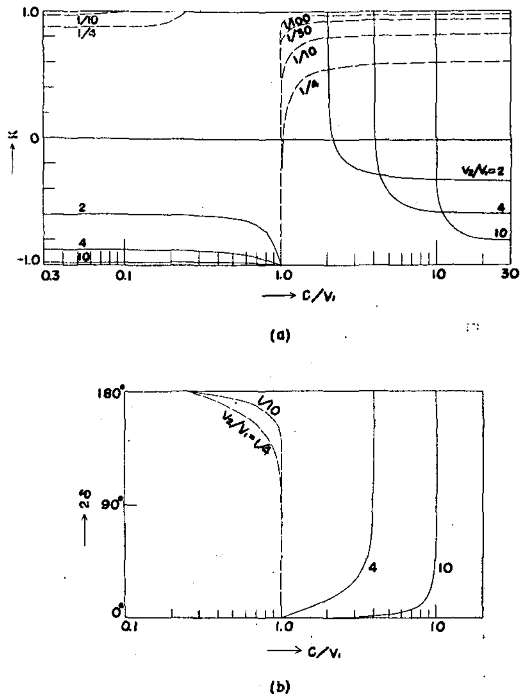


Fig. 14. Reflecting coefficient.

approximately characterized by (7.3). They are different from LOVE waves but have larger phase velocity than v_2.

For case (b). If v_2/v_1 approaches to zero, K may coincide with 1 and (7.2) will be reduced at the same time to

$$\sin \eta_1 H = 0 . \tag{7.4}$$

This is nothing but the characteristic equation for the normal modes of SH-waves in a plate bounded by air and is equivalent to the lower equation of (5.2) or (5.4). It must be noticed again in Fig. 14 that K can become near 1 within the region $v_1 < c < \infty$ if v_2/v_1 is very small. It may be considered, from a practical point of view, that (7.4) can be satisfied approximately even if v_2/v_1 does not become exactly zero. This consideration suggests that some waves approximately characterized by (6.4) may exist in layers where $0 < v_2/v_1 < 1$.

Because all waves which might be derived from complex roots in the characteristic equation must have wave fronts oblique to the boundary surface and cannot decrease their amplitudes in z-direction in the lowest half space, they will escape the usual definition of "surface waves".

8. Group velocity and amplitude function

When the characteristic equation is satisfied, the root will introduce group velocity. If the root is complex, group velocity U will also be complex.

$$U=\frac{\mathrm{d}\,\omega}{\mathrm{d}\,\xi}=\left\{\left(\frac{\mathrm{d}\,\bar{\xi}}{\mathrm{d}\,\omega}\right)^2+\left(\frac{\mathrm{d}\,\hat{\xi}}{\mathrm{d}\,\omega}\right)^2\right\}^{-1}\left(\frac{\mathrm{d}\,\bar{\xi}}{\mathrm{d}\,\omega}-i\frac{\mathrm{d}\,\hat{\xi}}{\mathrm{d}\,\omega}\right). \tag{8.1}$$

On the other hand, it was seen in Fig. 10 or 13 that

$$\mathrm{d}\,\bar{c}/\mathrm{d}\,T \lesseqgtr 0 \qquad \text{for} \quad \mathrm{d}\,\bar{c}/\mathrm{d}\,(\hat{c}/\bar{c}) \lesseqgtr 0\,, \tag{8.2}$$

in connection with Fig. 8 or 11. This means that $\mathrm{d}\bar{\xi}/\mathrm{d}\omega$ as well as $\mathrm{d}\hat{\xi}/\mathrm{d}\omega$ is always positive, excepting the zeroth order in case (b), on the locus of the complex root, as shown in Fig. 15 where arrows indicate the increasing direction of ω.

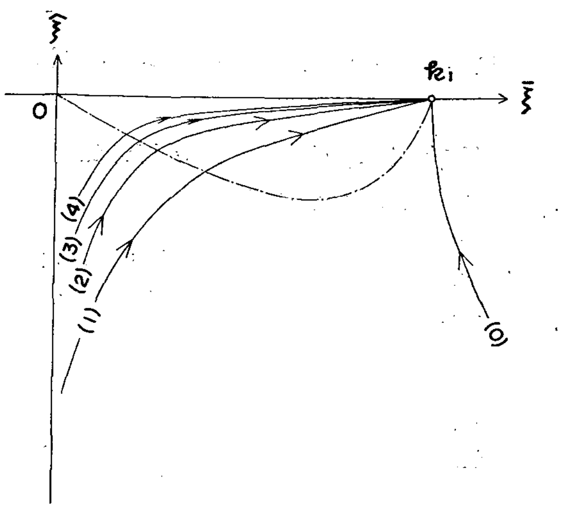


Fig. 15. Analytical relation between $\bar{\xi}$, $\hat{\xi}$ and ω. $v_2/v_1=1/4$.

Therefore $\mathrm{Re}\,U$ is positive but $\mathrm{Im}\,U$ is negative on the whole locus of the complex root except that of the zeroth order. (8.2) gives almost no restriction for practical exsistence of the complex root, at least as to group velocity.

Amplitude function of LOVE waves in a layer over a half space has been expressed by

$$2\pi\,A(\xi)=\frac{T\,v_1}{H}\left\{1-\left(\frac{v_1}{c}\right)^2\right\}^{-1}\left(\frac{v_1}{U}-\frac{v_1}{c}\right). \qquad (8.3)^{1)}$$

It must be kept in mind that amplitude function of normal modes in general is proportional to $(1/U-1/c)^{2)}$.

However,

$$\frac{1}{U}=\frac{\mathrm{d}\xi}{\mathrm{d}\omega}=\frac{1}{c}+\frac{T}{c^2}\frac{\mathrm{d}c}{\mathrm{d}T}\,. \tag{8.4}$$

Therefore (8.3) can be rewritten as

$$2\pi A(\xi)=\left(\frac{T v_1}{H}\right)^2\left\{\left(\frac{c}{v_1}\right)^2-1\right\}^{-1}\frac{\mathrm{d}\,(c/v_1)}{\mathrm{d}\,(T v_1/H)}\,. \tag{8.5}$$

Because of (4.1) and (5.2), the higher the order of the solution, the smaller the period. Owing to (8.5), this will be said in other words, the higher the order, the smaller the amplitude.

Using (4.6), one has

$$\eta_1/\xi=R_1\exp i\,(\delta+\varepsilon_1) \qquad \text{in which} \quad \tan\delta=\hat{c}/\bar{c}\,. \tag{8.6}$$

Therefore

$$(c/v_1)^2-1=(\eta_1/\xi)^2=R_1{}^2\exp 2\,i\,(\delta+\varepsilon_1)\,. \tag{8.7}$$

On the other hand,

$$\mathrm{d}\,c/\mathrm{d}\,T=\sec\delta'(\mathrm{d}\,\bar{c}/\mathrm{d}\,T)\exp\,(i\,\delta') \qquad \text{in which} \quad \tan\delta'=\mathrm{d}\,\hat{c}/\mathrm{d}\,\bar{c}\,. \tag{8.8}$$

Thus (8.5) will take the form

$$2\pi A(\xi)=\left(\frac{T v_1}{H}\right)^2\frac{\sec\delta'}{R_1{}^2}\frac{\mathrm{d}\,\bar{c}}{\mathrm{d}\,T}\exp i\,\{\delta'-2\,(\delta+\varepsilon_1)\} \tag{8.9}$$

which has the condition

$$\sec\delta'\,(\mathrm{d}\,\bar{c}/\mathrm{d}\,T)\geqq 0\,,$$

because

$$\frac{\mathrm{d}\,c}{\mathrm{d}\,T}=\left\{\left(\frac{\mathrm{d}\,\bar{c}}{\mathrm{d}\,T}\right)^2+\left(\frac{\mathrm{d}\,\hat{c}}{\mathrm{d}\,T}\right)^2\right\}^{1/2}\exp\,(i\,\delta')\,.$$

9. RIEMANN sheets

If a line source

$$\psi_0=\pi\,\mathrm{H}_0{}^{(2)}(kr)\exp\,(i\,\omega\,t) \tag{9.1}$$

is supposed at $z=E$ in the superficial layer, the displacement in that layer will be expressed by[1)]

$$\psi_1=2\int_{-\infty}^{\infty}e^{i(wt-\xi x)}\,\Psi_1\,(\eta_1,\eta_2)\,\mathrm{d}\xi \tag{9.2}$$

in which

$$\Psi_1(\eta_1, \eta_2) = \cos \eta_1 z \frac{e^{-i\eta_1 E} + e^{i\eta_1 E} K e^{-2i\eta_1 H}}{1 - K e^{-2i\eta_1 H}} \cdot \frac{1}{\eta_1} \quad \text{for} \quad z < E. \tag{9.3}$$

Then ψ_1 in (9.2) is customarily estimated by the next calculation

$$\psi_1 = \int_L \mathrm{d}\,\xi + \Sigma\,\mathrm{Res.}, \tag{9.4}$$

in which the first term of the right hand side means line integrals including those along branch lines and the second term is derived from poles on ξ-plane.

$k_1=0$ gives effectively no branch point, because Ψ_1 keeps the same value even if η_1 in (9.3) changes its sign. On the other hand, $k_2=0$ gives branch points, because Ψ_1 will have a different value if η_2 changes its sign.

i) When a part of $\bar{\eta}_2\hat{\eta}_2=0$ is taken as a branch line, two types of RIEMANN sheets will be constructed as shown in Fig. 16. As already examined above in section 2, there is no pole in the fourth quadrant on the upper sheet of Fig. 16(a). The lower sheet of this figure cannot be a perimissible sheet[3], because

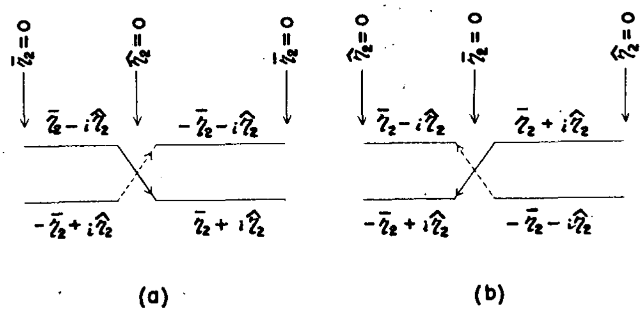


Fig. 16. RIEMANN sheets having a branch line $\hat{\eta}_2$ or $\bar{\eta}_2=0$.

ψ_2 must regress in z-direction on the left side of it. For the same reason as this, the lower sheet of Fig. 16(b) cannot also be a permissible sheet.

Looking back upon (3.8), the upper sheet of Fig. 16(b) can be a permissible RIEMANN sheet and it has poles, as has been seen in section 6. On this sheet residues of (9.4) will be obtained from the whole loci of

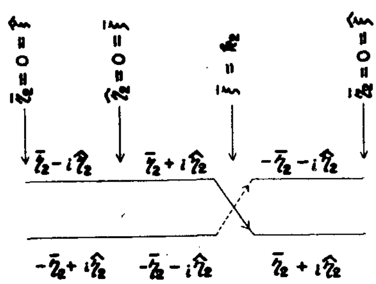


Fig. 17. RIEMANN sheets having a branch line $\xi=k_2$.

the complex root on ξ-plane, for instance Fig. 9 or Fig. 12. It will be difficult, however, to calculate the integral along the branch line $\bar{\eta}_2=0$.

ii) When $\xi=k_2$ is taken as a branch line, as LAMB did, RIEMANN sheets will be those shown by Fig. 17. On the upper sheet of this figure complex roots must exist in a part $\eta_2=\bar{\eta}_2+i\,\hat{\eta}_2$.

Looking at Fig. 9 or 12, one sees that poles will survive between $\bar{\xi}=0$ and $\bar{\xi}=k_2$ in the fourth quadrant.

iii) When the chain line starting from k_2 and $\bar{\xi}=0$ is taken as a branch line in place of $\bar{\xi}=k_2$ in Fig. 9 or 12, poles will survive only in the region surrounded by the chain line and $\hat{\xi}=0$. Again calculations along this branch line will be difficult.

iv) When the method of the steepest descent is applied in (9.4) by developping the denominator of (9.3), there exists no pole on ξ-plane.

Various values of $\Sigma\,Res.$ in (9.4) will be obtained by different selections of the branch line. This variety of $\Sigma\,Res.$ should be adjusted by $\int_L d\xi$, because ψ_1 of (9.4) must be kept constant in whatever location of the branch line.

10. Conclusions

If $\xi\,x\gg1$ will be assumed for the calculation of $\int_L d\xi$ in (9.4), as usual in cases of i), ii) and iii), the coefficient exp $(\hat{\xi}\,x)$ in (9.2) must become very small, otherwise $\hat{\xi}/\bar{\xi}$ must be neary zero. This means that almost all waves derived from complex roots will be trivial, particularly for case (a) in which $v_2/v_1>1$, since powerful LOVE waves should coexist with them. For case (b) in which $0<v_2/v_1<1$ there is no wave derived from a real root. Therefore the waves derived from complex roots may be "closeuped" when $\bar{\xi}x$ is not much larger than unity and $|\hat{\xi}/\bar{\xi}|$ is considerably smaller than unity.

It must be noted that the zeroth order of the solution can not make large amplitudes if it does exist, because dc/dT in (8.9) must be very small as shown in Fig. 13 (b).

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