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Ida Volcano with an Elastic Conduit

Itaru Maeda

Division of Earth and Planetary Sciences, Graduate School of Science, Hokkaido University, Sapporo 060, Japan

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Abstract

A model for volcanic eruption is proposed, which is a modification of Ida volcano. The model constitutes of a spherical elastic magma chamber and an elastic conduit which differs from Ida volcano. Behavior of the present system was analyzed in essentially two cases, variation of magma supply being constant and bell shape. For the constant magma supply, the phase portrait and vector field of the radius of the conduit were constructed and a stable fixed point was determined. From the analysis, it is found that the system does not erupt intermittently like as Ida volcano but converges asymptotically to a steady state in which magma supply is balanced with the outflow, i.e., eruption. For the variable magma supply, the outflow is not in phase with the supply. If the initial radius of the conduit is small, the outflow lasts long after the supply had terminated. The variation of the outflow is resembled to that of real volcanic activities.

1. Introduction

Ida proposed a model for volcanic eruption (1995). The model constitutes an elastic magma chamber with a conduit surrounded by viscous fluid. Inflow of magma to the chamber is assumed to be constant. Outflow, that is eruption, is driven by buoyancy which is caused by the difference between densities of the magma and the surrounding fluid. Because the viscosity of magma will be high, Poiseuille flow is assumed for the outflow, which means that the amount of flow is proportional to (radius of the conduit$=a)^4$. This model shows intermittent eruptions though the eruption is periodic. Ida claims that this intermittence explains actual volcanic activities which are neither uniform nor steady in general.

Usually volcanic eruptions accompany earthquakes which are the elastic response to variations of the medium surrounding the magma chamber and the conduit. From physical point of view, that a conduit is a hole in viscous fluid would not be sufficiently realistic, though certain type of volcanos may be explained by this model. As a bit more realistic model, we propose to replace the viscous conduit by an elastic conduit.

Ida model may be more realistic for the process of upward movement of magma deep in the earth. There may be a magma chamber as a stage of the last but one for the process. In and out flow of magma at this stage may be described by Ida model. We can place our magma chamber above Ida magma chamber. The connection can be described by a viscous conduit given by Ida model. Then inflow of magma to our chamber is given by Ida model.

In this report, we will give a short summary of Ida model at first. Then we will give quasi-static elastic conduit model which is described by a first order nonlinear differential equation. The model can be analyzed by so-call graphical method when input flow is constant. The model will be analyzed numerically when the input flow is a function of time the shape of which imitates an eruption by Ida volcano.

2. Ida volcano

Ida model is described by four equations:

$$J=\frac{\pi g \Delta\rho}{8\eta_f}a^4 \tag{1}$$

$$p=kv \tag{2}$$

$$\dot{a}=\frac{1}{2\eta_c}ap \tag{3}$$

$$\dot{v}=I-J \tag{4}$$

Outflow J is driven by buoyancy of density difference $\Delta\rho$ in the gravitational field g. The force is opposed by viscosity of magma η_f. J is proportional to the fourth power of the radius of the conduit a. Pressure deviation p (here after we will simply call p pressure) from the lithostatic pressure at the depth of the magma chamber is proportional to the volumetric deviation v of the chamber which is considered as a spherical cavity in an infinite elastic medium with stiffness k.

Eq. (3) is the most important part of Ida theory. Change in the conduit radius a is determined by p acting on the surface of the conduit in a viscous fluid with viscosity η_c. We will replace this viscous conduit with an elastic one. Eq. (4) states simply the conservation of mass, where inflow I is assumed to be constant in Ida theory.

Eq.s (1)-(4) can be reduced to a system of two dimensional first order nonlinear differential equations. They are further simplified by non-dimensionalization:

$$\dot{\alpha}=\alpha\beta, \tag{5}$$
$$\dot{\beta}=\gamma-\alpha^4 \tag{6}$$

where

$$a=a_0\alpha,\ v=v_0\beta,\ t=t_0\tau,$$

and

$$a_0=1,\ t_0=\frac{1}{4}\sqrt{\frac{\pi gk\Delta\rho}{\eta_t\eta_c}},\ v_0=\frac{1}{2}\sqrt{\frac{\pi g\Delta\rho\eta_c}{k\eta_f}},\ \gamma=\frac{8\eta_f}{\pi g\Delta\rho}I.$$

The dot over α and β means differentiation with respect to τ.

Ida showed that his volcano could erupt intermittently even if the input flow was constant. This is the important consequence of his theory. It can be explained by the difference in response time of the conduit (viscous) and chamber (elastic). The eruption is perfectly periodic and the variation of the outflow for each eruption has a bell shape. The period depends on initial values of α, the smaller the α is, the longer it becomes.

3. A model with an elastic conduit

We replace the viscous conduit by an elastic one. We will approximate our conduit by a hole with infinite length in an infinite elastic medium. We take a cylindrical coordinate system $(r,\ \theta,\ z)$ whose z-axis is along the conduit. The displacement field is a function of r only. Governing equation of the displacement in r-direction u is

$$\frac{1}{\gamma}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)-\frac{u}{r^2}=\frac{1}{V_p^2}\frac{\partial^2 u}{\partial t^2}. \tag{7}$$

The boundary condition is

$$\sigma_{rr}(a,t)=\left[(\lambda+2\mu)\frac{\partial u}{\partial r}+\lambda\frac{u}{r}\right]_{r=a}=-p(t). \tag{8}$$

In Eq.s (7) and (8), V_p is P-wave velocity, λ and μ are the elastic constants.

We assume that the variation of the system is sufficiently slow so that the inertial term can be neglected. In this quasi-static approximation, the right-hand side (RHS for short) of Eq. (7) equals zero. There are two independent solutions of Eq. (7) with zero RHS. One of the solutions $u=cr$ may be physically rejected. Then the solution is

$$u(t)=\frac{c(t)}{r}, \tag{9}$$

where the integration constant $c(t)$ can be a function of time but its variation should be sufficiently slow.

Substitution of Eq. (9) into Eq. (8) gives

$$2\mu\frac{c}{a^2(t)}=p(t). \tag{10}$$

Because the velocity of the displacement at a equals to da/dt, eliminating c from Eq.s (9) and (10), we obtain

$$\frac{da}{dt}=\frac{d}{dt}\left(\frac{pa}{2\mu}\right),$$

which immediately gives

$$a(t)=\frac{1}{2\mu}p(t)a(t)+d. \tag{11}$$

The integration constant d has an important meaning. Eq. (11) is rewritten by Eq. (2) as

$$a=\frac{k}{2\mu}av+d \tag{12}$$

Using Eq.s (1), (4), and (12), we obtain, under the assumption of $d\neq 0$,

$$\frac{da}{dt}=\frac{k}{2\mu d}Ia^2-\frac{\pi g\Delta\rho k}{16\eta_f\mu d}a^6. \tag{13}$$

Now we will non-dimensionalize Eq. (13) ; put a and t as

$$a=a_0\alpha,\ t=t_0\tau,$$

and substitute them into Eq. (13) then determine a_0 and t_0 so that all the coefficients in Eq. (13) are equal to 1. We obtain

$$a_0=\left(\frac{8\eta_f}{\pi g\Delta\rho}\right)^{\frac{1}{4}},\ t_0=\frac{2\mu d}{k}\left(\frac{8\eta_f}{\pi g\Delta\rho}\right)^{-\frac{1}{4}}.$$

With there constants, Eq. (13) becomes

$$\dot{\alpha}=I\alpha^2-\alpha^6, \tag{14}$$

where the dot over α means differentiation with respect to τ.

4. Analysis

We first analyze Eq. (14) for the case of constant inflow, i.e., I=constant, by an analytical or so-called geometrical method. The fixed points of the phase portrait for α are $-I^{1/4}$, 0, and $I^{1/4}$. The point zero is unstable and the others are stable. One dimensional vector field of α is depicted by horizontal arrows in Fig. 1 in which a vertical arrow indicates the fixed point $I^{1/4}$. The gray area in the figure has no physical meaning, so that we will consider the fixed point $I^{1/4}$ only. From this figure, we can construct the solution of Eq. (14) without numerical calculations as follows.

When initial value of α, $\alpha(0)$, is less than α_1, "velocity" of α is positive and increases until α reaches α_1, so that the curve of α as a function of time is downward convex. Having passed α_1, the velocity decreases, so that the curve becomes upward convex. If $\alpha_1 < \alpha(0) < I^{1/4}$, the velocity simply decreases and the curve of α is upward convex from the start. If $\alpha(0)$ is greater than $I^{1/4}$, the velocity is negative and the absolute value of it decreases. If the initial values of α happen to hit the fixed point, it keeps the value $I^{1/4}$ forever. For the first three cases, the curves converge asymptotically to the fourth one. All these features of the solution are given in Fig. 2.

We have assumed that $d \neq 0$ in Eq. (12) to get Eq. (14). Now the meaning of the assumption is clear. If d is zero in Eq. (12), $v = 2\mu/k$ that is constant, so that $I = J$ in Eq. (4). This is equivalent to $I = \alpha^4$, the fixed point. The physical situation at the fixed point is as follows; our system is in perfect equilibrium elastically and steady for magma flow, the inflow being equal to the outflow. This situation is not of interesting and never reached if the inflow varies. We will consider cases of variable inflow next.

As stated above, original Ida model may be adequate for deeper magma systems and the outflow from Ida magma could be the inflow to our system.

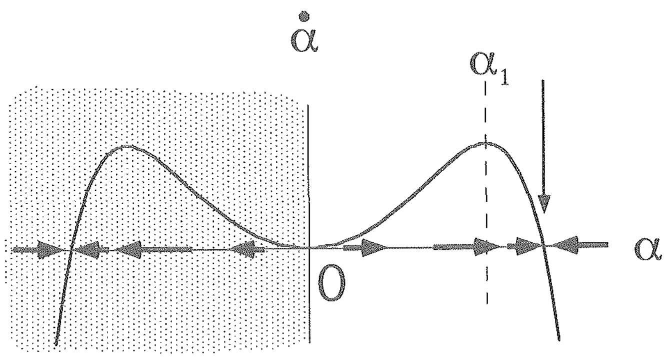

Fig. 1. Phase portrait and vector field of α. Vertical arrow indicates the position of stable fixed point having physical meaning.

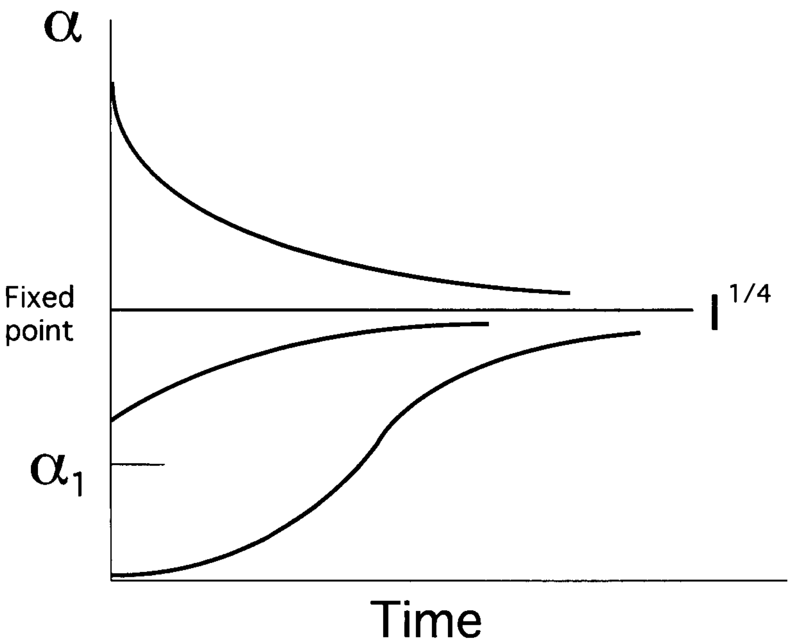

Fig. 2. Solution curves of Eq. (14) constructed from Fig. 1 with four different initial conditions.

Time variation of an outflow from Ida magma has a bell shape. We assume that the inflow to our magma chamber varies as the outflow of Ida magma and we approximate it by a function of

$$I(t)=\frac{I_0}{(t-s)^n+1}. \tag{15}$$

The power n may be 2 or 4 or larger but, because results are almost independent from the value selected, we will present only the case with $n=2$. We have two parameters $\alpha(0)$ and I_0. Fig. 3 shows cases with three I_0's and variable $\alpha(0)$'s. In each plot, the horizontal axis is time and the vertical one is the amount of magma flow. Dashed curves are inflows given by Eq. (15) with fixed $s=20$, so that their shapes are the same in normalized plots. The real curve in each plot represents the outflow of magma which equals to α^4. Initial value $\alpha(0)$ is attached to each plot.

It is clearly seen from the figure that the inflow and outflow are not in phase and the phase lag increases as $\alpha(0)$ decreases. Each peak of the outflow is on the curve of corresponding inflow, which is proved analytically. The time of

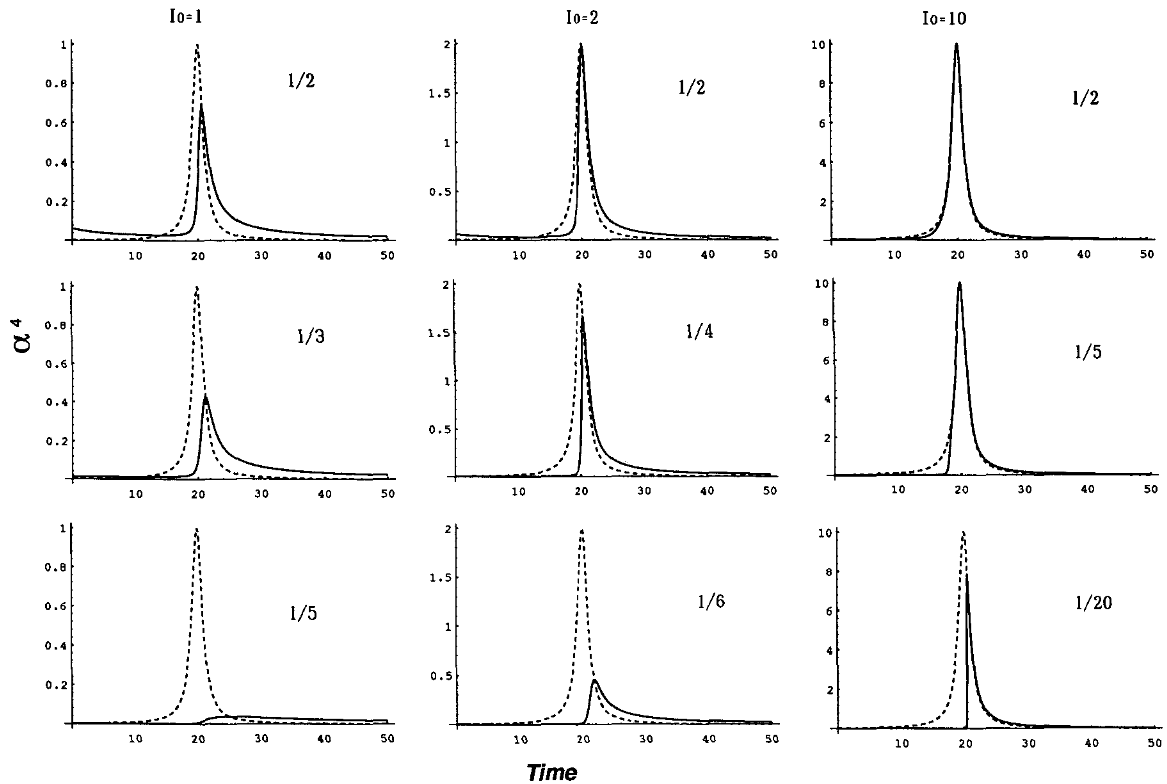

Fig. 3. Variations of inflow (dashed line) and outflow (real line) of magma as a function of intensity of the inflow and the initial radius of conduit.

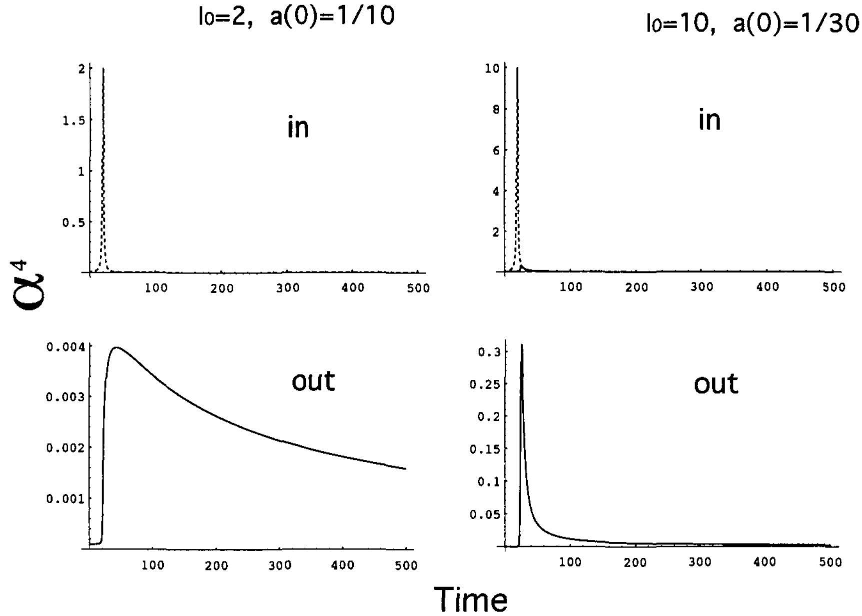

Fig. 4. Same as Fig. 3 but with a bit more extreme parameter values.

peak T satisfies $d\alpha(T)^4/dt=0$. At time T, the left-hand side of Eq. (14) is zero, so that $I(T)=\alpha(T)^4$. Because this relation holds for any functions irrelevant to Eq. (15), it is the intrinsic character of our system.

The onset of the outflow, i.e., eruption, is relatively sharp when the inflow is large comparing with cases of low inflow. The outflow becomes long lasting as the initial radius of conduit decreases. This can be seen in Fig. 4 which gives two cases with relatively small initial α, $\alpha(0)$. Plotted time span is ten times longer than that of Fig. 3. In this time span, inflow (dotted curve) is nearly a pulse like.

5. Concluding remarks

We constructed a model for volcanos. The model has an elastic conduit but retains the other features the same as original Ida model. In the followings, we will speak of the characteristics of our volcano obtained above in the original quantities. Our volcano does not erupt intermittently when supply of magma to the magma chamber is constant. This characteristic is completely different from Ida volcano. This difference may be explained as follows. In our system, both the chamber and conduit are elastic, so that the response times are the same but, in Ida model, the viscous conduit may show slow response and can over shoot which may cause the intermittence.

We, therefore, examined cases of variable supply of magma. The variation of the supply was assumed to have a bell shape. Our volcano erupts suddenly when supply of magma is large and the initial radius of the conduit is small. The variation of eruption is not the same as that of magma supply. The eruption is out of phase with the magma supply and the lag increases as the initial radius reduces. If the radius is sufficiently small, the eruption lasts long after the magma supply has stopped. The variation seems to resemble with some type of volcanic activity.

References

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