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COMBINATORIALLY EQUIVALENT HYPERPLANE ARRANGEMENTS

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ABSTRACT. We study the combinatorics of hyperplane arrangements over arbitrary fields. Specifically, we determine in which situation an arrangement and its reduction modulo a prime number have isomorphic lattices via the use of minimal strong σ -Gröbner bases. Moreover, we prove that the Terao's conjecture over finite fields implies the conjecture over the rationals.

1. INTRODUCTION

Let V be a vector space of dimension l over a field K . Fix a system of coordinates (x_1, \dots, x_l) of V^* . We denote by $S = S(V^*) = K[x_1, \dots, x_l]$ the symmetric algebra of V^* . A hyperplane arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ is a finite collection of hyperplanes in V . We refer to [12] as main reference on the theory of arrangements.

The *lattice of intersections* $L(\mathcal{A})$ is a fundamental combinatorial invariant of an arrangement \mathcal{A} . In fact one of the most studied topics in the theory of arrangements is to identify which topological and algebraic invariants of an arrangement are determined by its lattice of intersections.

To pursue this type of questions, Athanasiadis ([3], [4] and [5]), inspired by [9] and [8], initiated and systematically applied the “finite field method”, i.e. the study of the combinatorics of arrangements and their reduction modulo prime numbers. See also [7] for related work. After its introduction, this method has been used by several authors ([10], [11], [2] and [13]) to solve similar problems. The purpose of this paper is to study the combinatorics of arrangements over arbitrary fields and determine in which situation an arrangement and its reduction modulo a prime have isomorphic lattices.

The paper is organized as follows. In Section 2, we recall the basic notions on hyperplane arrangements. In Section 3, we describe how to characterize when two arrangements are combinatorially equivalent. In Section 4, we use the results of Section 3 to describe the primes p for which \mathcal{A} and \mathcal{A}_p are combinatorially equivalent. In Section 5, we show that the knowledge of Terao's conjecture in finite characteristic implies the conjecture over the rationals. In Section 6, we describe a method to compute *good primes* via

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minimal strong σ -Gröbner bases. In Section 7, we show that computing the *good* and *l-lucky primes* for an arrangement is equivalent to compute all the primes that divide its *lcm-period* (as defined in [11]).

2. PRELIMINARIES

Let K be a field. A finite set of affine hyperplanes $\mathcal{A} = \{H_1, \dots, H_n\}$ in K^l is called a **hyperplane arrangement**. For each hyperplane H_i we fix a polynomial $\alpha_i \in S = K[x_1, \dots, x_l]$ such that $H_i = \alpha_i^{-1}(0)$, and let $Q(\mathcal{A}) = \prod_{i=1}^n \alpha_i$. An arrangement \mathcal{A} is called **central** if each H_i contains the origin of K^l . In this case, each α_i is a linear homogeneous polynomial, and hence $Q(\mathcal{A})$ is homogeneous of degree n .

Define the **lattice of intersections** of \mathcal{A} by

$$L(\mathcal{A}) = \left\{ \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A} \right\},$$

where if $\mathcal{B} = \emptyset$, we identify $\bigcap_{H \in \mathcal{B}} H$ with K^l . We endow $L(\mathcal{A})$ with a partial order defined by $X \leq Y$ if and only if $Y \subseteq X$, for all $X, Y \in L(\mathcal{A})$. Note that this is the reverse inclusion. Define a rank function on $L(\mathcal{A})$ by $\text{rk}(X) = \text{codim}(X)$. Moreover, we define $\text{rk}(\mathcal{A}) = \text{codim}(\bigcap_{H \in \mathcal{A}} H)$. $L(\mathcal{A})$ plays a fundamental role in the study of hyperplane arrangements, in fact it determines the combinatorics of the arrangement. Let

$$L^p(\mathcal{A}) = \{X \in L(\mathcal{A}) \mid \text{rk}(X) = p\},$$

we call \mathcal{A} **essential** if $L^1(\mathcal{A}) \neq \emptyset$.

Let $\mu: L(\mathcal{A}) \rightarrow \mathbb{Z}$ be the **Möbius function** of $L(\mathcal{A})$ defined by

$$\mu(X) = \begin{cases} 1 & \text{for } X = K^l, \\ -\sum_{Y < X} \mu(Y) & \text{if } X > K^l. \end{cases}$$

The **characteristic polynomial** of \mathcal{A} is

$$\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim(X)}.$$

Given $\mathcal{A} = \{H_1, \dots, H_n\}$ an arrangement in K^l , the operation of **coning** allows to transform \mathcal{A} into a central arrangement $c\mathcal{A} = \{\tilde{H}_1, \dots, \tilde{H}_{n+1}\}$ in K^{l+1} . The hyperplane \tilde{H}_{n+1} corresponds to the hyperplane at infinity H_∞ of \mathcal{A} . Moreover, $\bar{\mathcal{A}} = \{\bar{H}_1, \dots, \bar{H}_{n+1}\}$ denotes the projectivization of $c\mathcal{A}$, which is an arrangement induced by $c\mathcal{A}$ in the projective space $K\mathbb{P}^l$. We will say that $\bar{\mathcal{A}}$ is **essential** if $\bigcap_{i=1}^{n+1} \bar{H}_i = \emptyset$.

Associated to each hyperplane arrangement \mathcal{A} , it can be naturally defined its **Tutte polynomial**

$$T_{\mathcal{A}}(x, y) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \mathcal{B} \text{ central}}} (x-1)^{\text{rk}(\mathcal{A})-\text{rk}(\mathcal{B})} (y-1)^{|\mathcal{B}|-\text{rk}(\mathcal{B})}.$$

As shown in [2], it turns out that the Tutte polynomial and the characteristic polynomial are related by

$$\chi(\mathcal{A}, t) = (-1)^{\text{rk}(\mathcal{A})} t^{l-\text{rk}(\mathcal{A})} T_{\mathcal{A}}(1-t, 0).$$

It is sometimes useful to consider a simple transformation of the Tutte polynomial. The **coboundary polynomial** of \mathcal{A} is

$$\bar{\chi}_{\mathcal{A}}(x, y) = \sum_{\substack{\mathcal{B} \subseteq \mathcal{A} \\ \mathcal{B} \text{ central}}} x^{\text{rk}(\mathcal{A})-\text{rk}(\mathcal{B})} (y-1)^{|\mathcal{B}|}.$$

It is easy to check that

$$\bar{\chi}_{\mathcal{A}}(x, y) = (y-1)^{\text{rk}(\mathcal{A})} T_{\mathcal{A}}\left(\frac{x+y-1}{y-1}, y\right),$$

and

$$T_{\mathcal{A}}(x, y) = \frac{1}{(y-1)^{\text{rk}(\mathcal{A})}} \bar{\chi}_{\mathcal{A}}((x-1)(y-1), y).$$

3. COMBINATORIAL EQUIVALENCE

The results in this section are a generalization of certain ones from [16]. Fix a pair (l, n) with $l \geq 1$ and $n \geq 0$. Let $A_n(K^l)$ be the set of affine arrangements of n distinct linearly ordered hyperplanes in K^l . In other words, each element \mathcal{A} of $A_n(K^l)$ is a collection $\mathcal{A} = \{H_1, \dots, H_n\}$, where H_1, \dots, H_n are distinct affine hyperplanes in K^l .

Definition 3.1. Given $\mathcal{A} \in A_n(K^l)$, define

$$\mathcal{I}(\bar{\mathcal{A}}) = \{(i_1, \dots, i_{l+1}) \in [n+1]_{<}^{l+1} \mid \bar{H}_{i_1} \cap \dots \cap \bar{H}_{i_{l+1}} \neq \emptyset\},$$

where $[n+1] = \{1, \dots, n+1\}$ and $[n+1]_{<}^{l+1} = \{(i_1, \dots, i_{l+1}) \in [n+1]^{l+1} \mid i_1 < \dots < i_{l+1}\}$.

The space $\mathcal{I}(\bar{\mathcal{A}})$ allows us to check if \mathcal{A} and $\bar{\mathcal{A}}$ are essential.

Lemma 3.2. Given $\mathcal{A} \in A_n(K^l)$, the following conditions are equivalent

- (1) \mathcal{A} is essential.
- (2) $\bar{\mathcal{A}}$ is essential.
- (3) $\mathcal{I}(\bar{\mathcal{A}}) \neq [n+1]_{<}^{l+1}$.

Proof. We start by proving that (3) is equivalent to (2). If (3) is satisfied, then there exists $(i_1, \dots, i_{l+1}) \in [n+1]_{<}^{l+1}$ such that $\bar{H}_{i_1} \cap \dots \cap \bar{H}_{i_{l+1}} = \emptyset$, and hence $\bar{\mathcal{A}}$ is essential. On the other hand, if $\bar{\mathcal{A}}$ is essential then there exist $l+1$ hyperplanes $\bar{H}_{i_1}, \dots, \bar{H}_{i_{l+1}}$ in $\bar{\mathcal{A}}$ whose intersection is empty. This shows that the conditions (2) and (3) are equivalent.

We will now prove that (1) is equivalent to (3). Condition (3) is equivalent to the existence of $(i_1, \dots, i_{l+1}) \in [n+1]_{<}^{l+1}$ such that $\bar{H}_{i_1} \cap \dots \cap \bar{H}_{i_{l+1}} = \emptyset$. This happens if and only if there exist l hyperplanes $H_{i_1}, \dots, H_{i_l} \in \mathcal{A}$ such that $\bar{H}_{i_1} \cap \dots \cap \bar{H}_{i_l} \cap \bar{H}_{n+1} = \emptyset$ if and only if there exist l hyperplanes $H_{i_1}, \dots, H_{i_l} \in \mathcal{A}$ such that $H_{i_1} \cap \dots \cap H_{i_l}$ is a point. This last fact is equivalent to (1). \square

Let K_1 and K_2 be two fields (non necessarily distinct), and consider $\mathcal{A}^{(j)} = \{H_1^{(j)}, \dots, H_n^{(j)}\} \in A_n(K_j^l)$, for $j = 1, 2$, two hyperplane arrangements.

Definition 3.3. $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are *combinatorially equivalent* if

$$\dim(H_{i_1}^{(1)} \cap \dots \cap H_{i_p}^{(1)}) = \dim(H_{i_1}^{(2)} \cap \dots \cap H_{i_p}^{(2)}),$$

for all $1 \leq p \leq n$ and $1 \leq i_1 < \dots < i_p \leq n$, where the dimension of the empty set is equal to -1 . In this case, we write $\mathcal{A}^{(1)} \sim \mathcal{A}^{(2)}$.

The following result is a generalization of [16, Proposition 3].

Theorem 3.4. Let \mathcal{A} be an essential arrangement in K^l . Then $\mathcal{I}(\bar{\mathcal{A}})$ determines $L(\mathcal{A})$, and vice versa.

Proof. Consider $(i_1, \dots, i_k) \in [n]_{<}^k$. Since \mathcal{A} is essential, then $\dim(H_{i_1} \cap \dots \cap H_{i_k}) = l - k$ if and only if there exist $1 \leq i_{k+1} < \dots < i_l \leq n$ such that $\dim(H_{i_1} \cap \dots \cap H_{i_l}) = 0$. Passing to the projectivization, this is equivalent to the existence of $1 \leq i_{k+1} < \dots < i_l \leq n$ such that $\bar{H}_{i_1} \cap \dots \cap \bar{H}_{i_l} \cap \bar{H}_\infty = \emptyset$. This fact is then equivalent to the existence of $1 \leq i_{k+1} < \dots < i_l \leq n$ such that $(i_1, \dots, i_l, n+1) \notin \mathcal{I}(\bar{\mathcal{A}})$. From the knowledge of which $(i_1, \dots, i_k) \in [n]_{<}^k$ have $\dim(H_{i_1} \cap \dots \cap H_{i_k}) = l - k$, we can easily reconstruct $L(\mathcal{A})$. This shows that $\mathcal{I}(\bar{\mathcal{A}})$ determines $L(\mathcal{A})$.

Consider $(i_1, \dots, i_{l+1}) \in [n+1]_{<}^{l+1}$. If $i_{l+1} = n+1$, then $\bar{H}_{i_{l+1}} = \bar{H}_\infty$. Moreover, $(i_1, \dots, i_{l+1}) \notin \mathcal{I}(\bar{\mathcal{A}}) \Leftrightarrow \bar{H}_{i_1} \cap \dots \cap \bar{H}_{i_l} \cap \bar{H}_\infty = \emptyset \Leftrightarrow H_{i_1} \cap \dots \cap H_{i_l}$ is a point $\Leftrightarrow \dim(H_{i_1} \cap \dots \cap H_{i_l}) = 0$. Suppose now that $i_{l+1} < n+1$ and let $\mathcal{B} = \{H_{i_1}, \dots, H_{i_{l+1}}\}$. We have $(i_1, \dots, i_{l+1}) \notin \mathcal{I}(\bar{\mathcal{A}}) \Leftrightarrow \bar{H}_{i_1} \cap \dots \cap \bar{H}_{i_{l+1}} = \emptyset \Leftrightarrow H_{i_1} \cap \dots \cap H_{i_{l+1}} = \emptyset$ and $\bar{H}_{i_1} \cap \dots \cap \bar{H}_{i_{l+1}} \cap \bar{H}_\infty = \emptyset \Leftrightarrow H_{i_1} \cap \dots \cap H_{i_{l+1}} = \emptyset$ and $\bar{\mathcal{B}}$ is essential. By Lemma 3.2, this is equivalent to $H_{i_1} \cap \dots \cap H_{i_{l+1}} = \emptyset$ and \mathcal{B} is essential. This fact is then equivalent to $\dim(H_{i_1} \cap \dots \cap H_{i_{l+1}}) = -1$ and there exist l hyperplanes in \mathcal{B} whose intersection is a point and hence it is zero dimensional. This shows that $L(\mathcal{A})$ determines $\mathcal{I}(\bar{\mathcal{A}})$. \square

4. MODULAR CASE

From now on we will assume that $\mathcal{A} = \{H_1, \dots, H_n\}$ is a central and essential arrangement in \mathbb{Q}^l . After clearing denominators, we can suppose that $\alpha_i \in \mathbb{Z}[x_1, \dots, x_l]$ for all $i = 1, \dots, n$, and hence that $Q(\mathcal{A}) = \prod_{i=1}^n \alpha_i \in \mathbb{Z}[x_1, \dots, x_l]$. Moreover, we can also assume that there exists no prime number p that divides any α_i .

Let p be a prime number, and consider the canonical homomorphism $\pi_p: \mathbb{Z}[x_1, \dots, x_l] \rightarrow \mathbb{F}_p[x_1, \dots, x_l]$. Since \mathcal{A} is central and we assume that there exists no prime number p that divides any α_i , this implies that $\pi_p(\alpha_i)$ is a non-zero linear homogeneous polynomial, for all $i = 1, \dots, n$. Since we are interested in the case when \mathcal{A} and its reduction modulo p are both arrangements with the same number of hyperplanes, we call p **good** for \mathcal{A} if $\pi_p(Q(\mathcal{A}))$ is reduced. Clearly, this is equivalent to the requirement that $\pi_p(\alpha_i)$ and $\pi_p(\alpha_j)$ are not one multiple of the other, for all $1 \leq i < j \leq n$. Notice that the number of primes p that are non-good for \mathcal{A} is finite, see [13].

Let now p be a good prime for \mathcal{A} . Consider $\mathcal{A}_p = \{(H_1)_p, \dots, (H_n)_p\}$ the arrangement in \mathbb{F}_p^l defined by $\pi_p(Q(\mathcal{A})) \in \mathbb{F}_p[x_1, \dots, x_l]$ and define $(\alpha_i)_p = \pi_p(\alpha_i)$. Hence, by construction, $\mathcal{A} \in A_n(\mathbb{Q}^l)$ and $\mathcal{A}_p \in A_n(\mathbb{F}_p^l)$. Moreover, since \mathcal{A} is central, also \mathcal{A}_p is central.

Definition 4.1. Given $\mathcal{A} = \{H_1, \dots, H_n\} \in A_n(K^l)$, define

$$\mathfrak{J}(\mathcal{A}) = \{(i_1, \dots, i_l) \in [n]_{<}^l \mid \dim(H_{i_1} \cap \dots \cap H_{i_l}) = 0\}.$$

Remark 4.2. \mathcal{A} is essential if and only if $\mathfrak{J}(\mathcal{A}) \neq \emptyset$.

Lemma 4.3. The following facts are equivalent

- (1) $\mathcal{I}(\bar{\mathcal{A}}) = \mathcal{I}(\bar{\mathcal{A}}_p)$.
- (2) $\mathfrak{J}(\mathcal{A}) = \mathfrak{J}(\mathcal{A}_p)$.

Proof. If (1) is satisfied, since \mathcal{A} is essential, then by Lemma 3.2, also \mathcal{A}_p is essential. Similarly, if (2) is satisfied, then by Remark 4.2, also \mathcal{A}_p is essential.

Since both \mathcal{A} and \mathcal{A}_p are central, then for all $(i_1, \dots, i_{l+1}) \in [n]_{<}^{l+1}$, we have that $(i_1, \dots, i_{l+1}) \in \mathcal{I}(\bar{\mathcal{A}}) \cap \mathcal{I}(\bar{\mathcal{A}}_p)$. Now $(i_1, \dots, i_l) \in \mathfrak{J}(\mathcal{A})$ if and only if $H_{i_1} \cap \dots \cap H_{i_l}$ is a point. This is equivalent to $\bar{H}_{i_1} \cap \dots \cap \bar{H}_{i_l} \cap \bar{H}_\infty = \emptyset$ and hence to $(i_1, \dots, i_l, n+1) \notin \mathcal{I}(\bar{\mathcal{A}})$. A similar proof shows that $(i_1, \dots, i_l) \in \mathfrak{J}(\mathcal{A}_p)$ if and only if $(i_1, \dots, i_l, n+1) \notin \mathcal{I}(\bar{\mathcal{A}}_p)$. Putting these three properties together we get our result. \square

Since the goal of this section is to determine in which situation an arrangement and its reduction modulo a prime number have isomorphic lattices via the use of minimal strong σ -Gröbner bases, we will now recall some properties of ideals in $\mathbb{Z}[x_1, \dots, x_l]$.

Consider I an ideal of $\mathbb{Z}[x_1, \dots, x_l]$ and σ a term ordering. Given $f \in \mathbb{Z}[x_1, \dots, x_l]$, we define the **leading term** of f as $\text{LT}_\sigma(f) = \max_\sigma\{t \in \text{Supp}(f)\}$, the **leading coefficient** of f as the coefficient multiplying the $\text{LT}_\sigma(f)$ in the writing of f and we denote it by $\text{LC}_\sigma(f)$, and the **leading monomial** of f as $\text{LM}_\sigma(f) = \text{LC}_\sigma(f) \text{LT}_\sigma(f)$.

Definition 4.4. *Let I be an ideal of $\mathbb{Z}[x_1, \dots, x_l]$, σ a term ordering and $G = \{g_1, \dots, g_t\}$ a set of non-zero polynomials in I . Then we say that G is a **minimal strong σ -Gröbner basis** for I if the following conditions hold true*

- (1) G forms a set of generators of I ;
- (2) for each $f \in I$, there exists $i \in \{1, \dots, t\}$ such that $\text{LM}_\sigma(g_i)$ divides $\text{LM}_\sigma(f)$;
- (3) if $i \neq j$, then $\text{LM}_\sigma(g_i)$ does not divide $\text{LM}_\sigma(g_j)$.

Remark 4.5 (c.f. [1], Lemma 4.5.8). *The reduced σ -Gröbner basis of an ideal I of $\mathbb{Z}[x_1, \dots, x_l]$ is also a minimal strong σ -Gröbner basis of I . Moreover, every minimal strong σ -Gröbner basis of I is also a σ -Gröbner basis.*

Proposition 4.6 ([1], Exercise 4.5.9). *Let I be a non-zero ideal of $\mathbb{Z}[x_1, \dots, x_l]$ and σ a term ordering. Then there always exists a minimal strong σ -Gröbner basis of I*

Lemma 4.7 ([13], Lemma 5.9). *Let I be an ideal of the polynomial ring $\mathbb{Z}[x_1, \dots, x_l]$, and σ a term ordering. Let G_1 and G_2 be two minimal strong σ -Gröbner bases of I . Then $\{\text{LM}_\sigma(g) \mid g \in G_1\} = \{\text{LM}_\sigma(g) \mid g \in G_2\}$. Consequently we have $|G_1| = |G_2|$ and $\{\text{LC}_\sigma(g) \mid g \in G_1\} = \{\text{LC}_\sigma(g) \mid g \in G_2\}$.*

Remark 4.8. *The previous lemma implies that $\{\text{LM}_\sigma(g) \mid g \in G\}$ generates the monomial ideal $\text{LM}_\sigma(I)$, for G any minimal strong σ -Gröbner basis of I .*

By Lemma 4.7, we can introduce the following definition. See [13] and [14], for more details.

Definition 4.9. *Let I be an ideal of $\mathbb{Z}[x_1, \dots, x_l]$, and σ be a term ordering. If a prime number p does not divide the leading coefficient of any polynomial in a minimal strong σ -Gröbner basis for I , then we will say p is **σ -lucky** for I .*

In other words, p is σ -lucky for I if and only if it is a non-zero divisor in $\mathbb{Z}[x_1, \dots, x_l] / \text{LM}_\sigma(I)$.

Remark 4.10. *Given I an ideal of $\mathbb{Z}[x_1, \dots, x_l]$ and σ a term ordering, since a minimal strong σ -Gröbner basis is finite, then the number of primes that are not σ -lucky for I is finite.*

Now that we have all the tools to work with minimal strong σ -Gröbner basis, we can use them to study the combinatorics of arrangements.

Proposition 4.11. *Consider $(i_1, \dots, i_l) \in [n]_{<}^l$ and p a good prime for \mathcal{A} that is σ -lucky for the ideal $I_{\mathbb{Z}} = \langle \alpha_{i_1}, \dots, \alpha_{i_l} \rangle_{\mathbb{Z}} \subseteq \mathbb{Z}[x_1, \dots, x_l]$. Then the following fact are equivalent*

- (1) $(i_1, \dots, i_l) \in \mathfrak{I}(\mathcal{A})$.
- (2) $(i_1, \dots, i_l) \in \mathfrak{I}(\mathcal{A}_p)$.

Proof. Consider the ideal $I = \langle \alpha_{i_1}, \dots, \alpha_{i_l} \rangle_{\mathbb{Q}} \subseteq \mathbb{Q}[x_1, \dots, x_l]$ and the ideal $I_p = \langle (\alpha_{i_1})_p, \dots, (\alpha_{i_l})_p \rangle \subseteq \mathbb{F}_p[x_1, \dots, x_l]$.

If $(i_1, \dots, i_l) \in \mathfrak{I}(\mathcal{A}_p)$, then $(H_{i_1})_p \cap \dots \cap (H_{i_l})_p$ is the origin, and hence $I_p = \langle x_1, \dots, x_l \rangle$. This implies that for each $i = 1, \dots, l$, there exists $f_i \in \mathbb{Z}[x_1, \dots, x_l]$ of degree 1 such that $x_i + pf_i \in I$. Since I is an ideal in $\mathbb{Q}[x_1, \dots, x_l]$, we can transform the f_i in such way that $f_i \in \mathbb{Q}[x_{i+1}, \dots, x_l]$. This gives us that $\langle x_1, \dots, x_l \rangle \subseteq I$. Since \mathcal{A} is central, then I is a homogenous ideal such that $I \subsetneq \mathbb{Q}[x_1, \dots, x_l]$. This shows that $\langle x_1, \dots, x_l \rangle = I$ and hence $(i_1, \dots, i_l) \in \mathfrak{I}(\mathcal{A})$.

To show the opposite inclusion, assume that $(i_1, \dots, i_l) \in \mathfrak{I}(\mathcal{A})$. This implies that $H_{i_1} \cap \dots \cap H_{i_l}$ is the origin, and hence I is zero dimensional and $I = \langle x_1, \dots, x_l \rangle$. Since I_p is a homogenous ideal generated in degree 1, $I_p \subseteq \langle x_1, \dots, x_l \rangle$. Consider now $\{g_1, \dots, g_l\}$ a minimal strong σ -Gröbner basis for $I_{\mathbb{Z}}$. Since I is zero-dimensional, then $\{\text{LM}_{\sigma}(g_1), \dots, \text{LM}_{\sigma}(g_l)\} = \{\lambda_1 x_1, \dots, \lambda_l x_l\}$, where $\lambda_i \in \mathbb{Z}_{>0}$. Since we have that $g_j = \sum_{k=1}^l h_{kj} \alpha_{i_k}$, for some $h_{kj} \in \mathbb{Z}[x_1, \dots, x_l]$, then $\pi_p(g_j) \in I_p$. Moreover, since p is σ -lucky for $I_{\mathbb{Z}}$, then $\pi_p(g_j) \neq 0$ and $\text{LM}_{\sigma}(\pi_p(g_j)) = \pi_p(\text{LM}_{\sigma}(g_j)) \neq 0$. This implies that for each $i = 1, \dots, l$, there exists $f_i \in I_p$ such that $\text{LT}_{\sigma}(f_i) = x_i$. This shows that $\langle x_1, \dots, x_l \rangle \subseteq I_p$ and hence $I_p = \langle x_1, \dots, x_l \rangle$. This implies that $(i_1, \dots, i_l) \in \mathfrak{I}(\mathcal{A}_p)$. \square

As described in Proposition 4.11, we are interested in σ -lucky primes for certain ideals over the integers. This fact motivates the following definition.

Definition 4.12. *A prime number p is called l -lucky for \mathcal{A} , if it is σ -lucky for all the ideals of the form $\langle \alpha_{i_1}, \dots, \alpha_{i_l} \rangle_{\mathbb{Z}}$ for $(i_1, \dots, i_l) \in \mathfrak{I}(\mathcal{A})$.*

Theorem 4.13. *Let \mathcal{A} be a central and essential arrangement in \mathbb{Q}^l . The following facts are equivalent*

- (1) p is a good and l -lucky prime number for \mathcal{A} .
- (2) $\mathcal{A} \rightsquigarrow \mathcal{A}_p$, i.e. \mathcal{A} and \mathcal{A}_p are combinatorially equivalent.

Proof. Assume that p is a good and l -lucky prime number for \mathcal{A} . Since p is l -lucky for \mathcal{A} , by Proposition 4.11, $\mathfrak{J}(\mathcal{A}) = \mathfrak{J}(\mathcal{A}_p)$. By Lemma 4.3, this implies that $\mathcal{I}(\bar{\mathcal{A}}) = \mathcal{I}(\bar{\mathcal{A}}_p)$. We can then conclude that $\mathcal{A} \sim \mathcal{A}_p$ by Theorem 3.4.

Vice versa, assume now that $\mathcal{A} \sim \mathcal{A}_p$. This clearly implies that \mathcal{A} and \mathcal{A}_p are both (simple) arrangements with $|\mathcal{A}| = |\mathcal{A}_p|$. This then forces p to be good for \mathcal{A} . Suppose that p is not l -lucky for \mathcal{A} . This implies that there exists $\{i_1, \dots, i_l\} \in \mathfrak{J}(\mathcal{A})$ such that p divides a leading coefficient in a minimal strong σ -Gröbner basis of $I_{\mathbb{Z}} = \langle \alpha_{i_1}, \dots, \alpha_{i_l} \rangle_{\mathbb{Z}}$. Since $\{i_1, \dots, i_l\} \in \mathfrak{J}(\mathcal{A})$, we can consider $\{g_1, \dots, g_l\}$ a minimal strong σ -Gröbner basis for $I_{\mathbb{Z}}$ such that $\text{LM}_{\sigma}(g_i) = \lambda_i x_{i_i}$, where $\lambda_i \in \mathbb{Z}_{>0}$ for all $i = 1, \dots, l$. Consider $r = \min\{j \in [l] \mid p \text{ divides } \lambda_j\}$. Since $\mathcal{A} \sim \mathcal{A}_p$ and $\{i_1, \dots, i_l\} \in \mathfrak{J}(\mathcal{A})$, then $\{i_1, \dots, i_l\} \in \mathfrak{J}(\mathcal{A}_p)$ and hence $I_p = \langle (\alpha_{i_1})_p, \dots, (\alpha_{i_l})_p \rangle = \langle x_1, \dots, x_l \rangle$. In particular, $x_r \in I_p$, and hence there exists $g \in \mathbb{Z}[x_1, \dots, x_l]$ such that $f_r = x_r + pg \in I_{\mathbb{Z}}$. Since p does not divide λ_i with $i < r$, there exist $\gamma_1, \dots, \gamma_{r-1} \in \mathbb{Z}$ such that $\tilde{f}_r = f_r + \sum_{j=1}^{r-1} p\gamma_j g_j \in I_{\mathbb{Z}}$ with $\text{LM}_{\sigma}(\tilde{f}_r) = (1 + p\beta)x_r$ for some $\beta \in \mathbb{Z}$. Clearly, p does not divide $1 + p\beta$ and hence $\lambda_r x_r$ does not divide $\text{LM}_{\sigma}(\tilde{f}_r)$ but this is impossible since $\{g_1, \dots, g_l\}$ is a minimal strong σ -Gröbner basis for $I_{\mathbb{Z}}$. \square

By the discussion at the beginning of Section 4 and Remark 4.10, the set of prime numbers that are good and l -lucky for \mathcal{A} is infinite. This implies that Theorem 4.13 is a generalization of [15, Proposition 3.11.9], since our result describes explicitly how to compute the prime numbers for which \mathcal{A} and \mathcal{A}_p are not combinatorially equivalent.

Since the characteristic polynomial of an arrangement is determined by its lattice of intersections, we have the following

Corollary 4.14. *Let \mathcal{A} be a central and essential arrangement in \mathbb{Q}^l , and p a good and l -lucky prime number for \mathcal{A} . Then $\chi(\mathcal{A}, t) = \chi(\mathcal{A}_p, t)$.*

Remark 4.15. *Let q be a power of a prime p and $\mathcal{A}_{\mathbb{F}_q}$ the arrangement in \mathbb{F}_q^l defined by the class of $Q(\mathcal{A})$ in $\mathbb{F}_q[x_1, \dots, x_l]$. Then the same argument of Theorem 4.13 shows that if p is good and l -lucky for \mathcal{A} , then $\mathcal{A} \sim \mathcal{A}_{\mathbb{F}_q}$.*

In [2], Ardila described a finite field method to compute the coboundary polynomial, and hence the Tutte polynomial, of a given arrangement. His result involved the use of powers of large enough primes to make sure that \mathcal{A} and $\mathcal{A}_{\mathbb{F}_q}$ are combinatorially equivalent. Thanks to Theorem 4.13, we can rewrite his result as follows.

Theorem 4.16. *Let \mathcal{A} be a central and essential arrangement in \mathbb{Q}^l , and p a good and l -lucky prime number for \mathcal{A} . Then*

$$\bar{\chi}_{\mathcal{A}}(q, t) = \sum_{P \in \mathbb{F}_q^l} t^{h(P)},$$

where $h(P)$ denotes the number of hyperplanes of $\mathcal{A}_{\mathbb{F}_q}$ that contain P .

5. ON TERAO'S CONJECTURE

We first recall the basic notions and properties of free hyperplane arrangements.

We denote by $\text{Der}_{K^l} = \{\sum_{i=1}^l f_i \partial_{x_i} \mid f_i \in S\}$ the S -module of **polynomial vector fields** on K^l (or S -derivations). Let $\delta = \sum_{i=1}^l f_i \partial_{x_i} \in \text{Der}_{K^l}$. Then δ is said to be **homogeneous of polynomial degree d** if f_1, \dots, f_l are homogeneous polynomials of degree d in S . In this case, we write $\text{pdeg}(\delta) = d$.

Let \mathcal{A} be a central arrangement in K^l . Define the **module of vector fields logarithmic tangent** to \mathcal{A} (or logarithmic vector fields) by

$$D(\mathcal{A}) = \{\delta \in \text{Der}_{K^l} \mid \delta(\alpha_i) \in \langle \alpha_i \rangle S, \forall i\}.$$

The module $D(\mathcal{A})$ is obviously a graded S -module and we have

$$D(\mathcal{A}) = \{\delta \in \text{Der}_{K^l} \mid \delta(Q(\mathcal{A})) \in \langle Q(\mathcal{A}) \rangle S\}.$$

Definition 5.1. *A central arrangement \mathcal{A} in K^l is said to be **free with exponents** (e_1, \dots, e_l) if and only if $D(\mathcal{A})$ is a free S -module and there exists a basis $\delta_1, \dots, \delta_l$ of $D(\mathcal{A})$ such that $\text{pdeg}(\delta_i) = e_i$, or equivalently $D(\mathcal{A}) \cong \bigoplus_{i=1}^l S(-e_i)$.*

A lot it is known about free arrangements, however there is still some mystery around the notion of freeness. See [12], [17] and [6] for more details on freeness. For example, Teramo's conjecture asserting the dependence of freeness only on the combinatorics is the longstanding open problem in this area.

Conjecture 5.2 (Teramo). *The freeness of a hyperplane arrangement depends only on its lattice of intersections.*

In [13], we characterized the prime numbers p for which the freeness of \mathcal{A} implies the freeness of \mathcal{A}_p and, vice versa, the ones for which the freeness of \mathcal{A}_p implies the freeness of \mathcal{A} . Specifically, we proved the following two results.

Theorem 5.3 ([13, Theorem 4.3]). *If \mathcal{A} is a free arrangement in \mathbb{Q}^l with exponents (e_1, \dots, e_l) , then \mathcal{A}_p is free in \mathbb{F}_p^l with exponents (e_1, \dots, e_l) , for all good primes except possibly a finite number of them.*

Theorem 5.4 ([13, Theorem 6.1]). *Let p be a good prime number for \mathcal{A} that is σ -lucky for $J(\mathcal{A})_{\mathbb{Z}}$, for some term ordering σ , where $J(\mathcal{A})_{\mathbb{Z}}$ denotes the Jacobian ideal of \mathcal{A} as ideal of $\mathbb{Z}[x_1, \dots, x_l]$. If \mathcal{A}_p is free in \mathbb{F}_p^l with exponents (e_1, \dots, e_l) , then \mathcal{A} is free in \mathbb{Q}^l with exponents (e_1, \dots, e_l) .*

Putting together Theorems 4.13, 5.3 and 5.4, we can now show that the knowledge of Terao's conjecture in finite characteristic implies the conjecture over the rationals.

Theorem 5.5. *If Terao's conjecture is true over all \mathbb{F}_p , then it is true over \mathbb{Q} .*

Proof. Let $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ be two central arrangements in \mathbb{Q}^l such that $\mathcal{A}^{(1)} \smile \mathcal{A}^{(2)}$, and assume that $\mathcal{A}^{(1)}$ is free with exponents (e_1, \dots, e_l) .

Consider \mathcal{P} the set of prime numbers that are good and l -lucky for $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$, and that are σ -lucky for $J(\mathcal{A}^{(2)})_{\mathbb{Z}}$. By the discussion at the beginning of Section 4 and Remark 4.10, \mathcal{P} is infinite. For every $p \in \mathcal{P}$, Theorem 4.13 gives us $(\mathcal{A}^{(1)})_p \smile \mathcal{A}^{(1)} \smile \mathcal{A}^{(2)} \smile (\mathcal{A}^{(2)})_p$. On the other hand, by Theorem 5.3, we can chose $p \in \mathcal{P}$ in such way that $(\mathcal{A}^{(1)})_p$ is free with exponents (e_1, \dots, e_l) . If Terao's conjecture is true over \mathbb{F}_p , then $(\mathcal{A}^{(2)})_p$ is free with exponents (e_1, \dots, e_l) . Finally by definition of \mathcal{P} and Theorem 5.4, $\mathcal{A}^{(2)}$ is free with exponents (e_1, \dots, e_l) . \square

It is a natural question to ask if, under the hypothesis of Theorem 5.4, \mathcal{A} and \mathcal{A}_p are combinatorially equivalent. In all the examples we considered so far, we obtained a positive answer. This is because in all considered examples, if p is σ -lucky for $J(\mathcal{A})_{\mathbb{Z}}$, then it is l -lucky for \mathcal{A} . However in general, the converse is not true.

Example 5.6. *Consider the arrangement \mathcal{A} in \mathbb{Q}^3 with defining polynomial $Q(\mathcal{A}) = xyz(x+y)(x+2y+z)$. Now 2 is the only prime that is not l -lucky for \mathcal{A} . On the other hand a direct computation shows that 2, 3 and 5 are not σ -lucky for $J(\mathcal{A})_{\mathbb{Z}}$.*

6. HOW TO COMPUTE GOOD PRIMES VIA GRÖBNER BASES

We will now describe a method to compute good primes for an arrangement using minimal strong σ -Gröbner bases.

Lemma 6.1. *Let $1 \leq i < j \leq n$. If $(\alpha_i)_p = \beta(\alpha_j)_p$ for some $\beta \in \mathbb{F}_p \setminus \{0\}$, then p is not σ -lucky for the ideal $\langle \alpha_i, \alpha_j \rangle_{\mathbb{Z}} \subseteq \mathbb{Z}[x_1, \dots, x_l]$.*

Proof. By construction α_i and α_j are distinct homogenous polynomials of degree 1, that are not one multiple of the other. This implies that there exist $g_1, g_2 \in \mathbb{Z}[x_1, \dots, x_l]$ two homogenous polynomials of degree 1 that form

a minimal strong σ -Gröbner basis for $\langle \alpha_i, \alpha_j \rangle_{\mathbb{Z}}$. Notice that in this situation $\text{LM}_{\sigma}(g_k) = \lambda_k x_{i_k}$ for $k = 1, 2$ with $x_{i_1} \neq x_{i_2}$.

Assume by absurd that $(\alpha_i)_p = \beta(\alpha_j)_p$ for some $\beta \in \mathbb{F}_p \setminus \{0\}$, but p is σ -lucky for $\langle \alpha_i, \alpha_j \rangle_{\mathbb{Z}}$. In this situation $\text{LT}_{\sigma}(\langle (\alpha_i)_p, (\alpha_j)_p \rangle) = \langle x_r \rangle$ for some $1 \leq r \leq l$. On the other hand, since p is σ -lucky for $\langle \alpha_i, \alpha_j \rangle_{\mathbb{Z}}$, we have $\text{LM}_{\sigma}(\pi_p(g_k)) = \pi_p(\text{LM}_{\sigma}(g_k)) = \pi_p(\lambda_k)x_{i_k} \neq 0$, for $k = 1, 2$. This implies that $0 \neq \pi_p(g_k) \in \langle (\alpha_i)_p, (\alpha_j)_p \rangle$ for $k = 1, 2$, and hence that $x_{i_1}, x_{i_2} \in \text{LT}_{\sigma}(\langle (\alpha_i)_p, (\alpha_j)_p \rangle) = \langle x_r \rangle$. However this is impossible. \square

In general, the converse of Lemma 6.1 does not hold.

Example 6.2. Consider $\alpha_1 = x + y$ and $\alpha_2 = x + 3y + z$. Then a direct computation shows $\{x + y, 2y + z\}$ is a minimal strong σ -Gröbner basis for the ideal $\langle \alpha_1, \alpha_2 \rangle_{\mathbb{Z}}$, and hence $p = 2$ is not a σ -lucky prime. However, $(\alpha_1)_2 = x + y$ and $(\alpha_2)_2 = x + y + z$ are not one multiple of the other.

After Lemma 6.1, we have the following

Definition 6.3. A prime number p is called **2-lucky** for \mathcal{A} , if it is σ -lucky for all the ideals of the form $\langle \alpha_i, \alpha_j \rangle_{\mathbb{Z}}$ for $1 \leq i < j \leq n$.

If $l = 2$, the definitions of 2-lucky (Definition 6.3) and l -lucky (Definition 4.12) coincide.

Theorem 6.4. If p is 2-lucky for \mathcal{A} , then p is good for \mathcal{A} .

Proof. By definition, if p is 2-lucky for \mathcal{A} , then p is σ -lucky for all the ideals of the form $\langle \alpha_i, \alpha_j \rangle_{\mathbb{Z}}$ for all pairs $1 \leq i < j \leq n$. By Lemma 6.1, $(\alpha_i)_p$ and $(\alpha_j)_p$ are not one multiple of the other for all pairs $1 \leq i < j \leq n$. Hence, $\pi_p(Q(\mathcal{A}))$ is reduced. \square

In general the statement of Theorem 6.4 is not an equivalence.

Example 6.5. Consider the arrangement \mathcal{A} in \mathbb{Q}^3 with defining polynomial $Q(\mathcal{A}) = xy(x + y)(x + 3y + z)$. Then a direct computation shows that $p = 2$ and $p = 3$ are not 2-lucky for \mathcal{A} . However, all prime numbers are good for \mathcal{A} .

7. ON THE PERIOD OF ARRANGEMENTS

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central and essential arrangement in \mathbb{Q}^l , with $\alpha_i \in \mathbb{Z}[x_1, \dots, x_l]$ for all $i = 1, \dots, n$. Moreover, assume that there exists no prime number p that divides any α_i . We can associate to \mathcal{A} a $l \times n$ integer matrix

$$C = (c_1, \dots, c_n) \in \text{Mat}_{l \times n}(\mathbb{Z})$$

consisting of column vectors $c_i = (c_{1i}, \dots, c_{li})^T \in \mathbb{Z}^l$, for $i = 1, \dots, n$, such that

$$\alpha_i = \sum_{k=1}^l c_{ki} x_k.$$

Similarly, for each non-empty $J = \{i_1, \dots, i_k\} \subseteq [n]$, we consider the $l \times k$ integer matrix

$$C_J = (c_{i_1}, \dots, c_{i_k}) \in \text{Mat}_{l \times k}(\mathbb{Z}).$$

For each prime number p , we can consider $(C)_p$ and $(C_J)_p$ the reductions of C and C_J , respectively, modulo p . Notice that $(C)_p$ is the matrix associated to the arrangement \mathcal{A}_p .

Since each C_J is an integer matrix, we can consider its Smith normal form. In particular, there exist two unimodular matrices $S_J \in \text{Mat}_{l \times l}(\mathbb{Z})$ and $T_J \in \text{Mat}_{k \times k}(\mathbb{Z})$ such that

$$S_J C_J T_J = \begin{pmatrix} E_J & O \\ O & O \end{pmatrix},$$

where E_J is the diagonal matrix $\text{diag}(e_{J,1}, \dots, e_{J,r})$, with $e_{J,1}, \dots, e_{J,r} \in \mathbb{Z}_{>0}$, $e_{J,1} | e_{J,2} | \dots | e_{J,r}$ and $r = \text{rk}(C_J)$. Denote $e_{J,r}$ simply by $e(J)$, and let the **lcm-period** of \mathcal{A} be

$$\rho_0 = \text{lcm}\{e(J) \mid J \subseteq [n], 1 \leq |J| \leq l\}.$$

In [11, Theorem 2.4], the authors proved the following result.

Theorem 7.1. *The function $|M(\mathcal{A}_q)| = |\mathbb{Z}_q^l \setminus \bigcup_{H \in \mathcal{A}_q} H|$ is a monic quasi-polynomial in $q \in \mathbb{Z}_{>0}$ of degree l with a period ρ_0 , where \mathcal{A}_q is the reduction of \mathcal{A} modulo q .*

In [11], the authors also defined

$$q_0 = \max_{\emptyset \neq J \subseteq [n]} \min_{S_J} \max\{|u| \mid u \text{ is an entry of } S_J C_J \text{ or } C_J\}$$

and obtained the following result in Corollary 3.3

Theorem 7.2. *The lattice of intersections $L_q = L(\mathcal{A}_q)$ is periodic in $q > q_0$ with period ρ_0 . In other words,*

$$L_{q+s\rho_0} \simeq L_q,$$

for all $q > q_0$ and $s \in \mathbb{Z}_{\geq 0}$.

As noted in [2], if p is a large prime number, then \mathcal{A} and \mathcal{A}_p are combinatorially equivalent. Putting together this fact and Theorem 7.2, we get the following result.

Corollary 7.3. *Let p be a prime number such that $p > q_0$ and p is coprime with ρ_0 . Then \mathcal{A} and \mathcal{A}_p are combinatorially equivalent.*

The rest of this section is devoted to show that the hypothesis $p > q_0$ is not necessary. Specifically, we will show that computing the good and l -lucky primes for \mathcal{A} is equivalent to computing all the prime numbers that divide ρ_0 .

Proposition 7.4. *If p is non-good for \mathcal{A} , then p divides ρ_0 .*

Proof. Assume p is non-good for \mathcal{A} . This implies that there exist a pair of indices $1 \leq i < j \leq n$ such that

$$(1) \quad (\alpha_i)_p = \beta(\alpha_j)_p,$$

for some $\beta \in \mathbb{F}_p \setminus \{0\}$. Consider now $J = \{i, j\} \subseteq [n]$. Since \mathcal{A} is central, then (1) is equivalent to the fact that C_J has rank 2 but $(C_J)_p$ has rank 1. In particular, we have that the Smith normal form of C_J is of the form

$$E_{i,j} = \begin{pmatrix} E_J \\ O \end{pmatrix},$$

where $E_J = \text{diag}(e_1, e(J))$. By definition, a matrix and its Smith normal form have the same rank. On the other hand the Smith normal form of $(C_J)_p$, up to transforming the elements on the main diagonal to 1, is $(E_{i,j})_p$ the reduction modulo p of $E_{i,j}$. This implies that $\text{rk}((E_{i,j})_p) = \text{rk}((C_J)_p) = 1$. As a consequence, p divides $e(J)$ and hence ρ_0 . \square

Proposition 7.5. *If p is not l -lucky for \mathcal{A} , then p divides ρ_0 .*

Proof. Let p be a non l -lucky prime number for \mathcal{A} . This implies that there exists $J = \{i_1, \dots, i_l\} \in \mathfrak{J}(\mathcal{A})$ such that p divides a leading coefficient in a minimal strong σ -Gröbner basis of the ideal $\langle \alpha_{i_1}, \dots, \alpha_{i_l} \rangle_{\mathbb{Z}}$. Since $J \in \mathfrak{J}(\mathcal{A})$, we have that C_J is a $l \times l$ integer matrix of rank l . Using the same strategy as when computing the Smith normal form of C_J , there exists a unimodular $l \times l$ matrix T_J such that $C_J T_J$ is lower triangular. Since $\text{rk}(C_J T_J) = \text{rk}(C_J) = l$, $C_J T_J$ has only non-zero elements on the main diagonal. Seeing that multiplying C_J on the right by T_J is equivalent to perform only column operations on C_J , we have that the columns of $C_J T_J$ represent a minimal strong σ -Gröbner basis of $\langle \alpha_{i_1}, \dots, \alpha_{i_l} \rangle_{\mathbb{Z}}$. This implies that p divides one of the elements on the main diagonal of $C_J T_J$, and hence its determinant. On the other hand, by construction, C_J and $C_J T_J$ have the same Smith normal form E_J . This implies that the determinants of $C_J T_J$ and of E_J coincide up to a sign. However since p divides the determinant of $C_J T_J$, p divides the determinant of E_J and hence $e(J)$. Finally, by definition of ρ_0 , this implies that p divides ρ_0 . \square

Theorem 7.6. *Let p be a prime number. Then the following fact are equivalent*

- (1) p is non-good or not a l -lucky prime number for \mathcal{A} .

(2) p divides ρ_0 .

Proof. By Propositions 7.4 and 7.5, (1) implies (2).

On the other hand, assume there exists p a prime number that divides the period ρ_0 , but p is good and l -lucky for \mathcal{A} . This implies that there exists $J = \{i_1, \dots, i_k\} \subseteq [n]$ such that $e(J)$ is divisible by p . In particular, since the Smith normal form of $(C_J)_p$, up to transforming the elements on the main diagonal to 1, is the reduction modulo p of the Smith normal form of C_J , this implies that $\text{rk}(C_J) > \text{rk}((C_J)_p)$ and hence that $\dim(H_{i_1} \cap \dots \cap H_{i_k}) > \dim((H_{i_1})_p \cap \dots \cap (H_{i_k})_p)$. However, this implies that \mathcal{A} and \mathcal{A}_p are not combinatorially equivalent, contradicting Theorem 4.13. \square

Corollary 7.7. *If ρ_0 is a square free integer, then it is the product of all prime numbers that are non-good or not l -lucky for \mathcal{A} .*

In general, ρ_0 is not a square free integer.

Example 7.8. *Consider \mathcal{A} the arrangement in \mathbb{Q}^3 with defining polynomial $Q(\mathcal{A}) = z(4x + z)(2x + y)(6x + y + 3z)(8x + 2y + 5z)$. In this situation, $p = 2$ is the only non-good prime number for \mathcal{A} . Moreover, all prime numbers $p \neq 2$ are l -lucky for \mathcal{A} . A direct computation shows that $\rho_0 = 16$.*

Putting together Theorems 4.13 and 7.6, we obtain the following result that generalizes Corollary 7.3.

Corollary 7.9. *Let \mathcal{A} be a central and essential arrangement in \mathbb{Q}^l . The following facts are equivalent*

- (1) p is coprime with ρ_0 .
- (2) $\mathcal{A} \sim \mathcal{A}_p$, i.e. \mathcal{A} and \mathcal{A}_p are combinatorially equivalent.

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REFERENCES

- [1] W. W. Adams and P. Loustanaunau. *An introduction to Gröbner bases*. Number 3. American Mathematical Soc., 1994.
- [2] F. Ardila. Computing the Tutte polynomial of a hyperplane arrangement. *Pacific Journal of Mathematics*, 230(1):1–26, 2007.
- [3] C. Athanasiadis. Characteristic polynomials of subspace arrangements and finite fields. *Advances in Mathematics*, 122(2):193–233, 1996.
- [4] C. Athanasiadis. Extended linial hyperplane arrangements for root systems and a conjecture of Postnikov and Stanley. *Journal of Algebraic Combinatorics*, 10(3):207–225, 1999.

- [5] C. Athanasiadis. Generalized Catalan numbers, Weyl groups and arrangements of hyperplanes. *Bulletin of the London Mathematical Society*, 36(3):294–302, 2004.
- [6] A. Bigatti, E. Palezzato, and M. Torielli. New characterizations of freeness for hyperplane arrangements. *Journal of Algebraic Combinatorics*, 51(2):297–315, 2020.
- [7] A. Björner and T. Ekedahl. Subspace arrangements over finite fields: cohomological and enumerative aspects. *Advances in Mathematics*, 129(2):159–187, 1997.
- [8] A. Blass and B. Sagan. Characteristic and Ehrhart polynomials. *Journal of Algebraic Combinatorics*, 7(2):115–126, 1998.
- [9] H. Crapo and G.-C. Rota. *On the foundations of combinatorial theory: Combinatorial geometries*. MIT press Cambridge, Mass., 1970.
- [10] H. Kamiya, P. Orlik, A. Takemura, and H. Terao. Arrangements and ranking patterns. *Annals of Combinatorics*, 10(2):219–235, 2006.
- [11] H. Kamiya, A. Takemura, and H. Terao. Periodicity of hyperplane arrangements with integral coefficients modulo positive integers. *Journal of Algebraic Combinatorics*, 27(3):317–330, 2008.
- [12] P. Orlik and H. Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.
- [13] E. Palezzato and M. Torielli. Free hyperplane arrangements over arbitrary fields. *To appear in the Journal of Algebraic Combinatorics*, 2019.
- [14] F. Pauer. On lucky ideals for Gröbner basis computations. *Journal of Symbolic Computation*, 14(5):471–482, 1992.
- [15] R.P. Stanley. *Enumerative Combinatorics, Vol. 1*. reprinted by Cambridge University Press, Cambridge, 1998.
- [16] H. Terao. Moduli space of combinatorially equivalent arrangements of hyperplanes and logarithmic Gauss–Manin connections. *Topology and its Applications*, 118(1-2):255–274, 2002.
- [17] M. Yoshinaga. Freeness of hyperplane arrangements and related topics. In *Annales de la Faculté des sciences de Toulouse: Mathématiques*, volume 23, pages 483–512, 2014.

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