Correct bounds on the Ising lace-expansion coefficients

Akira Sakai*

February 22, 2022

Abstract

The lace expansion for the Ising two-point function was successfully derived in [25, Proposition 1.1]. It is an identity that involves an alternating series of lace-expansion coefficients. In the same paper, we claimed that the expansion coefficients obey certain diagrammatic bounds which imply faster x-space decay (as the two-point function cubed) above the critical dimension d_c ($=4$ for finite-variance models) if the spin-spin coupling is ferromagnetic, translation-invariant, summable and symmetric with respect to the underlying lattice symmetries. However, we recently found a flaw in the proof of [25, Lemma 4.2], a key lemma to the aforementioned diagrammatic bounds.

In this paper, we no longer use the problematic [25, Lemma 4.2], and prove new diagrammatic bounds on the expansion coefficients that are slightly more complicated than those in [25, Proposition 4.1] but nonetheless obey the same fast decay above the critical dimension d_c. Consequently, the lace-expansion results for the Ising and φ^4 models in the literature are all saved. The proof is based on the random-current representation and its source-switching technique of Griffiths, Hurst and Sherman, combined with a double expansion: a lace expansion for the lace-expansion coefficients.

Contents

1 Background **2**

2 Definition **4**
2.1 The Ising model . . . 4
2.2 The random-current representation . . . 4
2.3 The source-switching technique (SST) . . . 6
2.4 The lace expansion . . . 9
2.5 Problematic bounds on the expansion coefficients . . . 11

3 Main results **12**
3.1 Results for the 0th-order expansion coefficient . . . 12
3.2 Results for the 1st-order expansion coefficient . . . 13
3.3 Results for the higher-order expansion coefficients . . . 16

*Faculty of Science, Hokkaido University, Japan. https://orcid.org/0000-0003-0943-7842

4 Proofs of the main results 18
4.1 Proof of Theorem 3.1 18
4.2 Proof of Theorem 3.2 29
4.3 Proof of Theorem 3.3 31
4.4 Proof of Theorem 3.5 33

5 Application to the spread-out model 35
5.1 Bound on the 0^{th}-order expansion coefficient 37
5.2 Bound on the 1^{st}-order expansion coefficient 39
5.3 Bound on the higher-order expansion coefficient 42

1 Background

The (ferromagnetic) Ising model is a paradigmatic model in statistical physics that exhibits a phase transition and critical behavior. One of the most powerful methods to investigate those phenomena is to use the random-current representation, which is a sophisticated version of the high-temperature expansion and provides a way to translate spin correlations into connectivity of the corresponding vertices via paths of bonds with positive current. It was initiated by Griffiths, Hurst and Sherman [13] to prove the GHS inequality, and then made the most of it by Aizenman et al., in 1980s. Since then, the random-current representation has given rise to many useful results for the Ising model, such as the uniqueness of the critical point and mean-field bounds on critical exponents [2, 3], a sufficient condition, known as the bubble condition, for the mean-field behavior [1, 4, 5] and a sufficient condition for the continuity of the spontaneous magnetization [3]. Those sufficient conditions hold in dimensions above 4 and 2, respectively, if the critical two-point function obeys an infrared bound on the underlying short-range random-walk Green function, which is true for reflection-positive models [11]. However, the reflection-positivity is too restrictive and may easily be violated by slight modification of the spin-spin coupling, such as introducing relatively large next-nearest-neighbor interaction. Moreover, the reflection-positivity alone does not imply infrared asymptotics of the critical two-point function, i.e., the anomalous dimension $\eta = 0$, even in high dimensions; only a one-sided inequality is proved. To prove universal results, it is desirable to get rid of this strong symmetry condition.

The lace expansion is one of the few mathematically rigorous methods to prove mean-field critical behavior in high dimensions. Since it does not require reflection-positivity, we can deal with a wider class of spin-spin couplings. It is also applied to other models, such as percolation [17], for which it is argued that reflection-positivity does not hold. The first lace expansion was invented by Brydges and Spencer [8] for weakly self-avoiding walk. Since then, it has been extended to strictly self-avoiding walk [19], lattice trees and lattice animals [18], oriented percolation [23], the contact process [24], the Ising model [25], the $|\varphi|^4$ model [7, 26] and the random-connection model [20]; see also [27] for the development of the subject until mid 2000s. In general, the lace expansion gives rise to a recursion equation for the two-point function, which is almost identical to that for the Green function of the underlying random walk. The difference between the two is the kernel: an alternating series of the lace-expansion coefficients for the former, and the 1-step distribution for the latter. If the alternating series is absolutely convergent, then it can be treated as a 1-step distribution (after normalization) and the critical two-point function exhibits the same infrared asymptotics as the Green function. Therefore, absolute summability of the expansion coefficients (and existence of their lower-order moments) is a sufficient condition for the mean-field behavior.

To prove this sufficient condition for all dimensions above the model-dependent upper critical dimension d_c, we need correlation inequalities, such as the famous BK inequality for percolation (see [6] for the ever simplest proof), with which the expansion coefficients can be bounded by optimal diagrams consisting of two-point functions. For example, the 0^{th}-order expansion coefficient for bond percolation is the probability that there are at least two bond-disjoint paths of occupied bonds from o to x, and by the BK inequality, it is bounded by the two-point function squared: $\mathbb{P}_p(o \Rightarrow x) \leq \mathbb{P}_p(o \to x)^2$. The higher-order expansion coefficients for percolation are bounded similarly by diagrams that can be decomposed into triangles, which implies $d_c = 6$ for percolation.

For the Ising model, there was no equivalent to the BK inequality to control the expansion coefficients that are defined by using the aforementioned random-current representation. Inspired by the so-called Source-Switching Technique (SST) [13], which is a way to exchange sources between two current configurations, we came up with [25, Lemma 4.2] that was supposed to provide optimal diagrammatic bounds on the expansion coefficients. However, as explained more in detail in Section 2.5, we found a flaw in its proof, thanks to an inquiry by Duminil-Copin, and the diagrammatic bounds [25, Proposition 4.1] became no longer reliable; directly affected are the proof of the bound on the 0^{th}-order expansion coefficient in [25, pp.306–307] and [25, Lemma 4.4]; the rest of that paper is secure.

In this paper, we prove new diagrammatic bounds on the Ising lace-expansion coefficients that are slightly more complicated than those in [25, Proposition 4.1] but nonetheless obey the same x-space decay in high dimensions. As an example, we demonstrate how to derive the wanted x-space decay from the new diagrammatic bounds for sufficiently spread-out (finite-variance) models in dimensions $d > 4$; as a byproduct, we obtain better multiplicative constants in the x-space decay of the lace-expansion coefficients of order $j \geq 2$ (see Corollary 5.13 below). The proof of those diagrammatic bounds is based on the standard SST and a double expansion, i.e., a lace expansion for the expansion coefficients along the "earliest" path of odd current joining the two sources. See Section 4.1 for more details.

The rest of this paper is organized as follows. In Section 2.1, we define the model and introduce some notation. In Section 2.2, we explain the random-current representation. In Section 2.3, the aforementioned SST and its implications are summarized; one of them (Lemma 2.4) is nontrivial and its proof shares the key idea (i.e., the use of the earliest path of odd current) used in the proof of the aforementioned new diagrammatic bounds on the expansion coefficients. In Section 2.4, we briefly review the lace expansion and its results obtained by assuming bounds on the expansion coefficients. In Section 2.5, we explain why the proof of [25, Lemma 4.2], on which the previous diagrammatic bounds in [25, Proposition 4.1] rely, does not work. Then, in Section 3, we present the new diagrammatic bounds on four main building blocks of the expansion coefficients. Those bounds are proven in Sections 4.1–4.4, respectively. Finally, in Section 5, we demonstrate how to use the new diagrammatic bounds for the spread-out model in $d > 4$.

2 Definition

2.1 The Ising model

For simplicity, we consider the d-dimensional integer lattice $\mathbb{Z}^d$ as space (this assumption is not essential as long as the concerned graph is transitive, as in [25]). Let $J : \mathbb{Z}^d \to [0, \infty)$ be symmetric in such a way that $J(x) = J(y)$ as long as $|x| = |y|$. Let $\{J_{x,y}\}$ be a collection of spin-spin couplings that satisfy $J_{x,y} = J(y-x)$. We say that a subset $\Lambda \subset \mathbb{Z}^d$ is a connected domain if any pair of distinct vertices $x, y \in \Lambda$ are connected by a path of bonds in $\mathbb{B}_\Lambda = \{\{x, y\} \subset \Lambda : J_{x,y} > 0\}$, i.e., there is a sequence $\{v_j\}_{j=0}^n \subset \Lambda$ such that $v_0 = x$, $v_n = y$ and $\{v_{j-1}, v_j\} \in \mathbb{B}_\Lambda$ for all $j = 1, \dots, n$. We assume $J(o) = 0$, i.e., there are no self-bonds. Given a finite $\Lambda \subset \mathbb{Z}^d$, we define the Ising Hamiltonian as

$$H_\Lambda(\boldsymbol{\varphi}) = -\sum_{\{x,y\}\subset\Lambda} J_{x,y}\varphi_x\varphi_y, \tag{2.1}$$

where $\boldsymbol{\varphi} = \{\varphi_x\}_{x\in\Lambda} \in \{\pm1\}^\Lambda$ is a spin configuration. Then, we define the finite-volume two-point function and its infinite-volume limit at the inverse temperature $\beta \in [0, \infty)$ as

$$\langle\varphi_x\varphi_y\rangle_{\beta,\Lambda} = \frac{\displaystyle\sum_{\boldsymbol{\varphi}\in\{\pm1\}^\Lambda} \varphi_x\varphi_y e^{-\beta H_\Lambda(\boldsymbol{\varphi})}}{\displaystyle\sum_{\boldsymbol{\varphi}\in\{\pm1\}^\Lambda} e^{-\beta H_\Lambda(\boldsymbol{\varphi})}}, \qquad G_\beta(x) = \lim_{\Lambda\uparrow\mathbb{Z}^d}\langle\varphi_o\varphi_x\rangle_{\beta,\Lambda}, \tag{2.2}$$

where the limit is nonnegative and unique due to monotonicity in terms of volume-increasing limits (see Lemma 2.2 below). The summable model (i.e., $\sum_x J(x) < \infty$) is known to exhibit a phase transition at the critical point defined by

$$\beta_c = \sup\Big\{\beta \geq 0 : \sum_{x\in\mathbb{Z}^d} G(x) < \infty\Big\}. \tag{2.3}$$

2.2 The random-current representation

Let $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. Given a finite $\Lambda \subset \mathbb{Z}^d$ and a current configuration $\boldsymbol{n} = \{n_b\}_{b\in\mathbb{B}_\Lambda} \in \mathbb{Z}_+^{\mathbb{B}_\Lambda}$ and a bond set $B \subset \mathbb{B}_\Lambda$, we define the source set and the weight on B as

$$\partial\boldsymbol{n} = \Big\{x : \sum_{b\ni x} n_b \text{ is odd}\Big\}, \qquad w_B(\boldsymbol{n}) = \prod_{\{x,y\}\in B} \frac{(\beta J_{x,y})^{n_{x,y}}}{n_{x,y}!}. \tag{2.4}$$

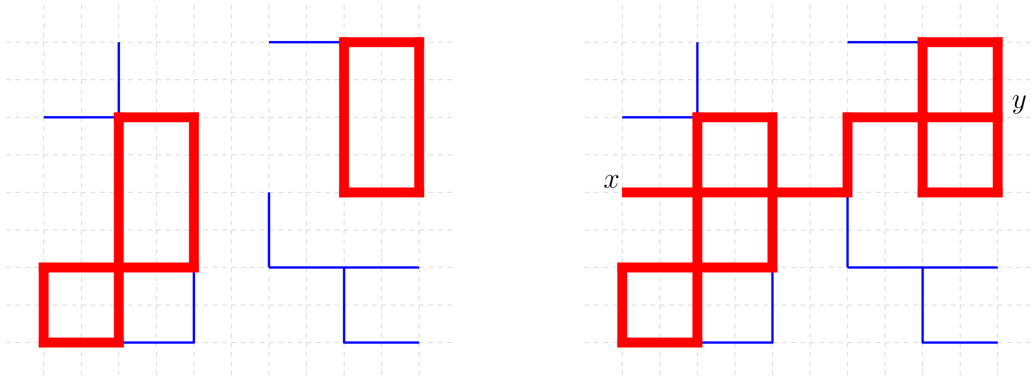


Figure 1: Current configurations satisfying the source constraint in (2.5) (left) and that in (2.6) (right). Bonds with odd current are bold (in red), while those with positive-even current are thin-solid (in blue).

Then, by simple arithmatic, we obtain the rewrite

$$\sum_{\boldsymbol{\varphi}\in\{\pm1\}^\Lambda} e^{-\beta H_\Lambda(\boldsymbol{\varphi})} = \sum_{\boldsymbol{\varphi}\in\{\pm1\}^\Lambda} \prod_{\{x,y\}\in\mathbb{B}_\Lambda} e^{J_{x,y}\varphi_x\varphi_y} = \sum_{\boldsymbol{\varphi}\in\{\pm1\}^\Lambda} \prod_{\{x,y\}\in\mathbb{B}_\Lambda} \sum_{n_{x,y}\in\mathbb{Z}_+} \frac{(\beta J_{x,y}\varphi_x\varphi_y)^{n_{x,y}}}{n_{x,y}!}$$

$$= \sum_{\boldsymbol{n}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda}} w_{\mathbb{B}_\Lambda}(\boldsymbol{n}) \prod_{x\in\Lambda} \underbrace{\sum_{\varphi_x=\pm1} \varphi_x^{\sum_{b\ni x} n_b}}_{2\times\mathbb{1}_{\{x\notin\partial\boldsymbol{n}\}}} = 2^{|\Lambda|} \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda}:\\ \partial\boldsymbol{n}=\varnothing}} w_{\mathbb{B}_\Lambda}(\boldsymbol{n}), \tag{2.5}$$

where $\mathbb{1}_{\{\cdots\}}$ is the indicator function. By this representation, we can interpret the partition function (= the denominator in the definition of the two-point function) as the sum of the weight over the current configurations in which bonds with odd current form loops (see the left of Figure 1). Similarly, we have

$$\sum_{\boldsymbol{\varphi}\in\{\pm1\}^\Lambda} \varphi_x\varphi_y e^{-\beta H_\Lambda(\boldsymbol{\varphi})} = 2^{|\Lambda|} \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda}:\\ \partial\boldsymbol{n}=x\triangle y}} w_{\mathbb{B}_\Lambda}(\boldsymbol{n}), \tag{2.6}$$

where $x \triangle y$ is an abbreviation for the heavier notation of symmetric difference $\{x\}\triangle\{y\}$. By this representation, we can interpret the numerator in the definition of the two-point function as the sum of the weight over the current configurations in which there is a path of bonds with odd current between x and y in the sea of loops with odd current (see the right of Figure 1). As a result, we obtain the random-current representation for the two-point function: for any $B \subset \mathbb{B}_\Lambda$,

$$\langle\varphi_x\varphi_y\rangle_B = \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{B}:\\ \partial\boldsymbol{n}=x\triangle y}} \frac{w_B(\boldsymbol{n})}{Z_B}, \qquad\qquad \langle\varphi_x\varphi_y\rangle_{\mathbb{B}_\Lambda} = \langle\varphi_x\varphi_y\rangle_{\beta,\Lambda}, \tag{2.7}$$

where

$$Z_B = \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^B:\\ \partial\boldsymbol{n}=\varnothing}} w_B(\boldsymbol{n}). \tag{2.8}$$

From now on, we omit the β-dependence if unnecessary, such as $G(x) = G_\beta(x)$. Similarly, we have the random-current representation for the four-point function:

$$\langle\varphi_x\varphi_y\varphi_u\varphi_v\rangle_B = \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^B:\\ \partial\boldsymbol{n}=x\triangle y\triangle u\triangle v}} \frac{w_B(\boldsymbol{n})}{Z_B}. \tag{2.9}$$

2.3 The source-switching technique (SST)

One of the advantages of the random-current representation (as compared to other similar representations, such as the high-temperature expansion) is the following source-switching technique (SST) of Griffiths, Hurst and Sherman [13]. It provides a way to exchange sources between two current configurations.

Lemma 2.1 (SST, e.g., Lemma 2.3 in [25])**.** *For any finite $B \subset B' \subset \mathbb{B}_{\mathbb{Z}^d}$, $x, y \in V(B)$ (= the set of end vertices of bonds in B) and $\boldsymbol{N} \in \mathbb{Z}_+^{B'}$,*

$$\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^B:\\ \partial\boldsymbol{n}=x\triangle y}} \prod_{b\in B} \binom{N_b}{n_b} = \mathbb{1}_{\{x \underset{\boldsymbol{N}}{\longleftrightarrow} y \text{ in } B\}} \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^B:\\ \partial\boldsymbol{n}=\varnothing}} \prod_{b\in B} \binom{N_b}{n_b}, \tag{2.10}$$

where $x \underset{\boldsymbol{N}}{\longleftrightarrow} y$ in B means either $x = y$ or there is a path from x to y consisting of bonds $b \in B$ with positive current $N_b > 0$.

Using the random-current representation and the SST, we can easily show the following consequences of Griffiths' inequalities [12] for Ising ferromagnets.

Lemma 2.2. *For every $x, y \in \mathbb{Z}^d$ and $\beta \geq 0$, the two-point function $\langle\varphi_x\varphi_y\rangle_B$, provided $x, y \in V(B)$, is nonnegative and nondecreasing in terms of the bond set B. As a result, there is a unique translation-invariant infinite-volume limit $G(y-x) = \lim_{\Lambda\uparrow\mathbb{Z}^d}\langle\varphi_x\varphi_y\rangle_{\mathbb{B}_\Lambda}$.*

Proof. Since $J_b \geq 0$ for any $b \in \mathbb{B}_{\mathbb{Z}^d}$, the weight $w_B(\boldsymbol{n})$ in (2.7) is nonnegative for any $B \subset \mathbb{B}_{\mathbb{Z}^d}$, and so are $\langle\varphi_x\varphi_y\rangle_B$ and its infinite-volume limit as $B \uparrow \mathbb{B}_{\mathbb{Z}^d}$ (if it exists). To prove monotonicity, we consider $B \subset B' \subset \mathbb{B}_{\mathbb{Z}^d}$. By the random-current representation (2.7), the difference of the two-point functions on B' and B is written as

$$\begin{aligned}\langle\varphi_x\varphi_y\rangle_{B'}-\langle\varphi_x\varphi_y\rangle_B &= \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{B'}:\\ \partial\boldsymbol{m}=x\triangle y}}\frac{w_{B'}(\boldsymbol{m})}{Z_{B'}}-\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{B}:\\ \partial\boldsymbol{n}=x\triangle y}}\frac{w_B(\boldsymbol{n})}{Z_B}\\ &= \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{B'}:\\ \partial\boldsymbol{m}=x\triangle y}}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{B}:\\ \partial\boldsymbol{n}=\varnothing}}\frac{w_{B'}(\boldsymbol{m})}{Z_{B'}}\frac{w_B(\boldsymbol{n})}{Z_B}-\sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{B'}:\\ \partial\boldsymbol{m}=\varnothing}}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{B}:\\ \partial\boldsymbol{n}=x\triangle y}}\frac{w_{B'}(\boldsymbol{m})}{Z_{B'}}\frac{w_B(\boldsymbol{n})}{Z_B}\\ &= \sum_{\substack{\boldsymbol{N}\in\mathbb{Z}_+^{B'}:\\ \partial\boldsymbol{N}=x\triangle y}}\frac{w_{B'}(\boldsymbol{N})}{Z_{B'}Z_B}\Bigg(\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{B}:\\ \partial\boldsymbol{n}=\varnothing}}\prod_{b\in B}\binom{N_b}{n_b}-\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{B}:\\ \partial\boldsymbol{n}=x\triangle y}}\prod_{b\in B}\binom{N_b}{n_b}\Bigg). \end{aligned} \tag{2.11}$$

However, by (2.10), the last line equals

$$\begin{aligned}&\sum_{\substack{\boldsymbol{N}\in\mathbb{Z}_+^{B'}:\\ \partial\boldsymbol{N}=x\triangle y}}\frac{w_{B'}(\boldsymbol{N})}{Z_{B'}Z_B}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{B}:\\ \partial\boldsymbol{n}=\varnothing}}\prod_{b\in B}\binom{N_b}{n_b}\Big(1-\mathbb{1}_{\{x\underset{\boldsymbol{N}}{\longleftrightarrow}y\text{ in }B\}}\Big)\\ &=\sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{B'}:\\ \partial\boldsymbol{m}=x\triangle y}}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{B}:\\ \partial\boldsymbol{n}=\varnothing}}\frac{w_{B'}(\boldsymbol{m})}{Z_{B'}}\frac{w_B(\boldsymbol{n})}{Z_B}\Big(1-\mathbb{1}_{\{x\underset{\boldsymbol{m}+\boldsymbol{n}}{\longleftrightarrow}y\text{ in }B\}}\Big),\end{aligned} \tag{2.12}$$

which is nonnegative. This proves monotonicity of $\langle\varphi_x\varphi_y\rangle_B$ in B and the uniqueness of the infinite-volume limit. To prove translation-invariance of the limit, we take two hypercubes $\Lambda(o) \subset \Lambda'(o) \subset \mathbb{Z}^d$, both centered at o, such that $\mathbb{B}_{\Lambda(x)} \subset B \subset \mathbb{B}_{\Lambda'(x)}$, where $\Lambda(x)$ is the translation of $\Lambda(o)$ by x. Then, by the monotonicity, we have

$$\langle\varphi_o\varphi_{y-x}\rangle_{\mathbb{B}_{\Lambda(o)}}=\langle\varphi_x\varphi_y\rangle_{\mathbb{B}_{\Lambda(x)}}\le\langle\varphi_x\varphi_y\rangle_B\le\langle\varphi_x\varphi_y\rangle_{\mathbb{B}_{\Lambda'(x)}}=\langle\varphi_o\varphi_{y-x}\rangle_{\mathbb{B}_{\Lambda'(o)}}. \tag{2.13}$$

Since both ends go to the same limit $G(y-x)$, this proves translation-invariance of the limit. ■

Other relevant results already proven in the previous work by using the random-current representation and the SST are summarized as follows:

Lemma 2.3. *(i) For any $x \neq o$,*

$$G(x) \le \tilde{G}(x) = (\tau * G)(x), \tag{2.14}$$

where $*$ represents a convolution in $\mathbb{Z}^d$: $(\tau * G)(x) = \sum_{y\in\mathbb{Z}^d}\tau(y)\,G(x-y)$.

(ii) For any finite $B \subset \mathbb{B}_{\mathbb{Z}^d}$ with $o, x, y \in V(B)$,

$$\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{B}:\\ \partial\boldsymbol{n}=o\triangle x}}\frac{w_B(\boldsymbol{n})}{Z_B}\,\mathbb{1}_{\{o\underset{\boldsymbol{n}}{\longleftrightarrow}y\}} \le G(y)\,G(x-y). \tag{2.15}$$

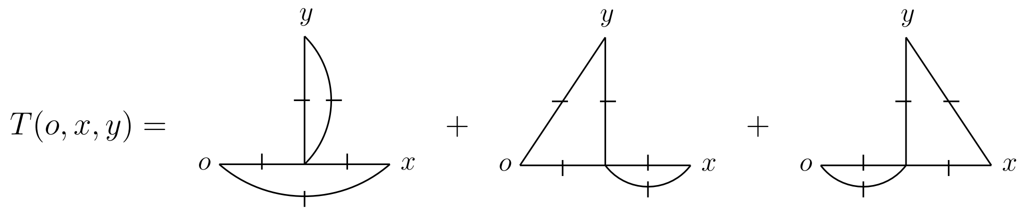


Figure 2: A schematic representation of $T(o,x,y)$. The slashed line segments represent G.

In particular, when $x = o$,

$$\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^B:\\ \partial\boldsymbol{n}=\varnothing}} \frac{w_B(\boldsymbol{n})}{Z_B}\, \mathbb{1}_{\{o \underset{\boldsymbol{n}}{\longleftrightarrow} x\}} \le G(x)^2. \tag{2.16}$$

(iii) For any finite $B, B' \subset \mathbb{B}_{\mathbb{Z}^d}$ with $o, x, y \in V(B)$,

$$\sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{B'}:\\ \partial\boldsymbol{m}=\varnothing}} \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^B:\\ \partial\boldsymbol{n}=o\triangle x}} \frac{w_{B'}(\boldsymbol{m})}{Z_{B'}} \frac{w_B(\boldsymbol{n})}{Z_B}\, \mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{n}}{\longleftrightarrow} y \text{ in } B\}} \le \sum_v G(v)\, G(x-v) \sum_{j=0}^{\infty} (\tilde{G}^2)^{*j}(y-v), \tag{2.17}$$

where $(\tilde{G}^2)^{*j}$ is the j-fold convolution of $\tilde{G}^2$ (recall the definition of $\tilde{G}$ in (2.14)).

Sketch proof. The result (i) is obtained by taking the infinite-volume limit of a finite-volume version in [9, (2.34)–(2.35)]. The result (ii) is obtained by multiplying the left-hand side of (2.15) by $1 = \sum_{\partial\boldsymbol{m}=\varnothing} w_B(\boldsymbol{m})/Z_B$, then using $\mathbb{1}_{\{o \underset{\boldsymbol{n}}{\longleftrightarrow} y\}} \le \mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{n}}{\longleftrightarrow} y\}}$, and finally applying the SST. The result (iii) is a simple extension of [25, (4.51)–(4.62)]. ■

Let (cf., Figure 2)

$$\begin{aligned} T(o,x,y) = \sum_z G(z)\, G(x-z)\, G(z-y) \\ \times \Big(G(x)\, G(z-y) + G(y)\, G(z-x) + G(z)\, G(y-x) \Big). \end{aligned} \tag{2.18}$$

We will later use the following lemma to bound the 1st- and higher-order expansion coefficients (cf., (3.15)–(3.16) and (3.22)–(3.23)).

Lemma 2.4. *For any finite $B \subset \mathbb{B}_\Lambda$ with $o, x, y \in V(B)$,*

$$\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^B:\\ \partial\boldsymbol{n}=\varnothing}} \frac{w_B(\boldsymbol{n})}{Z_B} \mathbb{1}_{\{o \underset{\boldsymbol{n}}{\longleftrightarrow} x\} \cap \{o \underset{\boldsymbol{n}}{\longleftrightarrow} y\}} \le T(o, x, y). \tag{2.19}$$

The proof of (2.19) turns out to require the same idea as in the proof of new diagrammatic bounds on the expansion coefficients, as well as Lebowitz' inequality [22], by which a four-point function is bounded by the three terms in the second line of (2.18). We prove Lemma 2.4 in Section 4.1.

2.4 The lace expansion

By heavy use of the random-current representation and the SST, we derived in [25, Section 2.2] the lace expansion (= the recursion equation (2.21) below) for the two-point function. To explain it, we first define

$$\tau(x) = \tanh\big(\beta J(x)\big), \tag{2.20}$$

which is zero for $x = o$, as $J(o) = 0$ (see above (2.1)). Then, for any $\beta \ge 0$, there exist lace-expansion coefficients $\{\pi_{\mathbb{B}_\Lambda}^{(i)}\}_{i\in\mathbb{Z}_+}$, which are nonnegative functions on Λ for ferromagnetic models, such that the following identity holds for every $j \in \mathbb{Z}_+$ [25, Proposition 1.1]:

$$\langle\varphi_o\varphi_x\rangle_{\mathbb{B}_\Lambda} = \Pi_{\mathbb{B}_\Lambda}^{(j)}(x) + \sum_{v\in\Lambda} (\Pi_{\mathbb{B}_\Lambda}^{(j)} * \tau)(o, v)\, \langle\varphi_v\varphi_x\rangle_{\mathbb{B}_\Lambda} + (-1)^{j+1} R_{\mathbb{B}_\Lambda}^{(j+1)}(x), \tag{2.21}$$

where the remainder $R_{\mathbb{B}_\Lambda}^{(j+1)}$ is nonnegative for ferromagnetic models, and

$$\Pi_{\mathbb{B}_\Lambda}^{(j)}(x) = \sum_{i=0}^{j} (-1)^i \pi_{\mathbb{B}_\Lambda}^{(i)}(x). \tag{2.22}$$

In fact, the identity (2.21) holds independently of the signs of the spin-spin couplings. However, as defined in the beginning of this section, if we restrict our attention to ferromagnetic models, then $\pi_{\mathbb{B}_\Lambda}^{(j)}(x)$ and $R_{\mathbb{B}_\Lambda}^{(j+1)}(x)$ are proven at the end of [25, Section 2.2.3] to obey the bounds

$$\pi_{\mathbb{B}_\Lambda}^{(j)}(x) \ge \delta_{j,0}\delta_{o,x}, \qquad R_{\mathbb{B}_\Lambda}^{(j+1)}(x) \le \sum_{v\in\Lambda} (\pi_{\mathbb{B}_\Lambda}^{(j)} * \tau)(o, v)\, \langle\varphi_v\varphi_x\rangle_{\mathbb{B}_\Lambda}. \tag{2.23}$$

We note that the lace expansion (2.21) looks similar to the recursion equation for the random-walk Green function S_p generated by the 1-step distribution D with fugacity p:

$$S_p(x) = \delta_{o,x} + pD * S_p(x). \tag{2.24}$$

If $D(x)$ decays faster than $|x|^{-d-\alpha}$ for some $\alpha > 2$ (hence $\sigma^2 = \sum_x |x|^2 D(x) < \infty$), then the critical Green function $S_1(x)$ exists in dimensions $d > 2$ and exhibits the asymptotic behavior $\sim \frac{a_d}{\sigma^2}|x|^{2-d}$ as $|x|\uparrow\infty$, where $a_d=\frac{d}{2}\Gamma(\frac{d-2}{2})\pi^{-d/2}$, which is obtained by the time-integral of the d-dimensional heat kernel defined by the diagonal covariance matrix with all entries $1/d$ (see, e.g., [9]).

Suppose that J is the spread-out interaction with parameter $L\in[1,\infty)$ of the form

$$J(x)=L^{-d}h(x/L) \qquad [x\neq o], \tag{2.25}$$

where $h:[-1,1]^d\to[0,\infty)$ is a bounded probability distribution, which is piecewise continuous and symmetric with respect to rotations by $\pi/2$ and reflections against cordinate hyperplanes. It has been shown (the latest reference is [10, Theorem 1.3], where we regard $\alpha=\infty$) that, if $d>2$, then

$$\theta=\sup_{x\neq o}\frac{S_1(x)}{(|x|\vee L)^{2-d}}=O(L^{-2}), \tag{2.26}$$

where S_1 is generated by the 1-step distribution $D=\tau/\|\tau\|_1$. It has also been shown (the latest reference is [9, Section 3.3], where we regard $\alpha=\infty$) that, if $d>4$, $\theta\ll 1$ and

$$\sup_{j\in\mathbb{Z}_+}|\Pi_{\mathbb{B}_\Lambda}^{(j)}(x)-\delta_{o,x}|\le O(L^{-d})\,\delta_{o,x}+\frac{O(\theta^3)}{(|x|\vee L)^{3(d-2)}}, \qquad R_{\mathbb{B}_\Lambda}^{(j)}(x)\xrightarrow[j\uparrow\infty]{}0 \tag{2.27}$$

hold uniformly in $x\in\Lambda\subset\mathbb{Z}^d$ and $\beta\le\beta_\mathrm{c}$, then the aforementioned similarity to random walk is justified and that $G_{\beta_\mathrm{c}}(x)\sim Aa_d|x|^{2-d}$ as $|x|\uparrow\infty$, where, by denoting the limit $\lim_{\beta\uparrow\beta_\mathrm{c}}\lim_{\Lambda\uparrow\mathbb{Z}^d}\lim_{j\uparrow\infty}\Pi_{\beta,\mathbb{B}_\Lambda}^{(j)}$ by Π_{β_c},

$$A=\frac{1}{\|\tau_{\beta_\mathrm{c}}\|_1}\Bigg(\sum_{x\in\mathbb{Z}^d}|x|^2\bigg(D(x)+\frac{\Pi_{\beta_\mathrm{c}}(x)}{\sum_y\Pi_{\beta_\mathrm{c}}(y)}\bigg)\Bigg)^{-1}. \tag{2.28}$$

For the nearest-neighbor model, where $J(x)=\mathbb{1}_{\{|x|=1\}}$, the same asymptotic behavior $G_{\beta_\mathrm{c}}(x)\sim Aa_d|x|^{2-d}$ as $|x|\uparrow\infty$ has been shown for $d\gg 4$ [25, Proposition 3.3(i)] provided that

$$\sum_x|\Pi_{\mathbb{B}_\Lambda}^{(j)}(x)|=1+O(\theta), \qquad \sum_x|x|^2|\Pi_{\mathbb{B}_\Lambda}^{(j)}(x)|=O(\theta), \tag{2.29}$$

$$|\Pi_{\mathbb{B}_\Lambda}^{(j)}(x)|=\frac{O(1)}{(|x|\vee 1)^{d+2}}, \qquad R_{\mathbb{B}_\Lambda}^{(j)}(x)\xrightarrow[j\uparrow\infty]{}0 \tag{2.30}$$

hold uniformly in $j\in\mathbb{Z}_+$, $x\in\Lambda\subset\mathbb{Z}^d$ and $\beta\le\beta_\mathrm{c}$, where

$$\theta=\max\Big\{\|D*S_1^{*2}\|_\infty,\ \sup_x x_1^2S_1(x)\Big\}=O\big((d-4)^{-1}\big). \tag{2.31}$$

The implicit constants in $O(\theta)$ in (2.29) and in $O((d-4)^{-1})$ in (2.31) are independent of d, but that in $O(1)$ in (2.30) may be large depending on d. Using the obtained asymptotics of $G_{\beta_\mathrm{c}}(x)$, we can improve the power exponent $d+2$ in (2.30) to the same in (2.27), i.e., $3(d-2)=d+2+4(d-4)$, and thanks to this excess power, the correction term to the leading asymptotics of $G_{\beta_\mathrm{c}}(x)$ can be evaluated (see, e.g., [15, Theorem 1.4]). Due to the extra assumption (2.29), which is to ensure convergence of the lace expansion, the nearest-neighbor model is more cumbersome than the spread-out model[1]. But, nonetheless, it is doable. So far, so good...

2.5 Problematic bounds on the expansion coefficients

To verify the above assumptions in (2.27) and (2.29)–(2.30), we want to bound the lace-expansion coefficients by diagrams consisting of two-point functions G. For example, we claimed in [25] that

$$\pi_{\mathbb{B}_\Lambda}^{(0)}(x)=\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda}:\\ \partial\boldsymbol{n}=o\triangle x}}\frac{w_{\mathbb{B}_\Lambda}(\boldsymbol{n})}{Z_{\mathbb{B}_\Lambda}}\mathbb{1}_{\{o\underset{\boldsymbol{n}}{\Longleftrightarrow}x\}}\overset{(*)}{\le}\langle\varphi_o\varphi_x\rangle_{\mathbb{B}_\Lambda}^3\le G(x)^3, \tag{2.34}$$

where $o\underset{\boldsymbol{n}}{\Longleftrightarrow}x$ means either $o=x$ or there are at least two bond-disjoint paths from o to x consisting of bonds b with positive current $n_b>0$. The last inequality is due to the monotonicity in volume mentioned below (2.2). The inequality $(*)$ is the issue to be discussed in this paper.

We also claimed in [25] that the higher-order expansion coefficients $\pi_{\mathbb{B}_\Lambda}^{(j)}(x)$, $j\ge1$, were bounded similarly, but by more involved diagrams. There are two key lemmas to show those diagrammatic bounds: [25, Lemmas 4.3 & 4.4]. The former is affirmatively obtained by repeated use of the SST. On the other hand, the latter (= [25, Lemma 4.4]) is based on the same idea used in showing the inequality $(*)$ in (2.34). As a result, we cannot justify the credibility of the diagrammatic bounds in [25, Proposition 4.1].

The common culprit is [25, Lemma 4.2], which was supposed to be an extension of the SST and to derive a similar inequality to the BK inequality for percolation. As seen in (2.10), the identity due to the SST holds if and only if o is connected to y via a path of bonds in B with positive current in the superposition $\boldsymbol{m}+\boldsymbol{n}=\{m_b+n_b\}_{b\in\mathbb{B}_\Lambda}$. Any

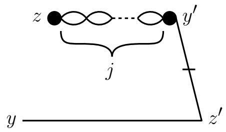
[1]The proof for the nearest-neighbor model is slightly different from the spread-out model. It goes roughly as follows. First, by assuming (2.29) and convergence of the remainder term in (2.30), we can show that, for $d\gg4$,

$$\max\Big\{\|\tau_\beta\|_1-1,\ \|D*G_\beta^{*2}\|_\infty,\ \sup_x x_1^2G_\beta(x)\Big\}=O\big((d-4)^{-1}\big), \tag{2.32}$$

uniformly in $\beta<\beta_{\mathrm{c}}$. Applying this to the desired diagrammatic bounds on the expansion coefficients, which we prove in this paper, we can show that (2.29) and convergence of the remainder term in (2.30) indeed hold for all $d\gg4$. By the standard continuity argument, the above bounds can be extended all the way to $\beta=\beta_{\mathrm{c}}$. Since $\sum_x|x|^r|\Pi_{\mathbb{B}_\Lambda}(x)|$ for $r=0,2$ is bounded, we can show that $\bar G^{(r)}$ is finite as long as $r<d-2$ and that $\bar W^{(r)}$ is finite as long as $r<d-4$ [25, Proposition 3.3(ii)], where

$$\bar G^{(r)}=\sup_x|x|^rG_{\beta_{\mathrm{c}}}(x),\qquad\qquad \bar W^{(r)}=\sup_x\sum_y|y|^rG_{\beta_{\mathrm{c}}}(y)\,G_{\beta_{\mathrm{c}}}(x-y). \tag{2.33}$$

Then, by using $\bar G^{(2)}<\infty$ and $\bar W^{(r)}<\infty$, we can show that $\sum_x|x|^{r+2}|\Pi_{\mathbb{B}_\Lambda}(x)|$ is bounded [25, Proposition 3.3(iii)]. Repeat this procedure until r reaches $(d+2)/3$ $(<d-2)$, so that $G_{\beta_{\mathrm{c}}}(x)=O(|x|^{-(d+2)/3})$. Applying this to the desired diagrammatic bounds on the expansion coefficients, we can show the point-wise bound in (2.30). For those who want to know more, please refer to [25, Section 3.2].

$U^m(y,z;y',z') = \sum_{j=0}^{m}$ 

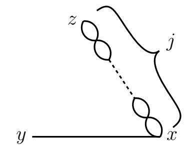
$V^m(y,z;x) = \sum_{j=1}^{m}$ 

Figure 3: Schematic representations of $U^m(y,z;y',z')$ and $V^m(y,z;x)$ for $m \geq 1$. The slashed line segment represents G (as in Figure 2), while the other unslashed ones represent $\tilde{G} = \tau * G$. In addition, the small filled discs represent $\delta + \tau^2$.

such path can be used to define a bijection[2] between two sets of pairs $(\boldsymbol{m}, \boldsymbol{n})$ with $\boldsymbol{m}+\boldsymbol{n}$ fixed: one with $\partial\boldsymbol{m} = \varnothing$, $\partial\boldsymbol{n} = o \vartriangle x$ and the other with $\partial\boldsymbol{m} = o \vartriangle y$, $\partial\boldsymbol{n} = y \vartriangle x$. To generalize this idea to deal with bond-disjoint connections, such as $o \underset{\boldsymbol{n}}{\Longleftrightarrow} x$, and exchange sources among more than two current configurations simultaneously, in the proof of [25, Lemma 4.2] we used the "earliest" path from o to x and another disjoint one to define a bijection between two sets of triples $(\boldsymbol{n}, \boldsymbol{m}_1, \boldsymbol{m}_2)$ with $\boldsymbol{n} + \boldsymbol{m}_1$ and $\boldsymbol{n} + \boldsymbol{m}_2$ fixed: one with the source constraint $\partial\boldsymbol{n} = o \vartriangle x$, $\partial\boldsymbol{m}_1 = \partial\boldsymbol{m}_2 = \varnothing$ and the other with $\partial\boldsymbol{n} = \partial\boldsymbol{m}_1 = \partial\boldsymbol{m}_2 = o \vartriangle x$. The ordering used in defining the earliest path was non-local, so as to ensure existence of another unaffected path after removal of the earliest. It turns out that this non-local rule disrupts construction of a bijection; for some cases, the image is empty because the first two earliest paths used to define the bijection are no longer the first two earliest in the image. Therefore, we have decided to abandon [25, Lemma 4.2] and to seek alternative diagrammatic bounds on the expansion coefficients.

3 Main results

3.1 Results for the 0^{th}-order expansion coefficient

First we recall (2.14):

$$\tilde{G}(x) = (\tau * G)(x). \tag{3.1}$$

To explain the new diagrammatic bounds on the expansion coefficients, we define (see Figure 3)

[2]In fact, what we do is to consider the multigraph $G_{\boldsymbol{m}+\boldsymbol{n}} = (\Lambda, \mathbb{B}_\Lambda^{\boldsymbol{m}+\boldsymbol{n}})$, where each bond $b \in \mathbb{B}_\Lambda$ is duplicated $m_b + n_b$ times. Choose any path $(o = v_0, v_1, \dots, v_n = y)$ of bonds $b_j = \{v_{j-1}, v_j\}$ with $m_{b_j} + n_{b_j} > 0$, let $e_j = \{v_{j-1}, v_j\}$ be one of those $m_{b_j} + n_{b_j}$ edges, and set $\omega = \{e_j\}_{j=1}^n$. Then, the aforementioned bijection is defined by taking the symmetric difference between ω and $E \subset \mathbb{B}_\Lambda^{\boldsymbol{m}+\boldsymbol{n}}$ with the set of odd-degree vertices $\partial E = o \vartriangle x$; the image satisfies the required condition $\partial(E \vartriangle \omega) = y \vartriangle x$.

$$U^m(y,z;y',z') = \tilde{G}(z'-y)\,G(z'-y')$$
$$\times \begin{cases} G(y'-z)^2 & (m=0), \\ \Big((\delta+\tau^2) * \sum_{j=0}^{m} (\tilde{G}^2)^{*j} * (\delta+\tau^2)\Big)(y'-z) & (m \ge 1), \end{cases} \tag{3.2}$$

$$V^m(y,z;x) = \tilde{G}(x-y) \sum_{j=1}^{m} (\tilde{G}^2)^{*j}(x-z), \tag{3.3}$$

where $(\tilde{G}^2)^{*0}(x) = \delta_{o,x}$ by convention. Also, we define

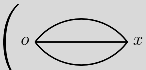
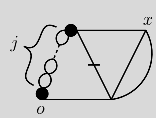
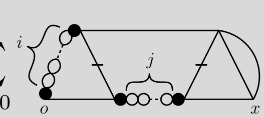
$$X^m_{o,x} = \sum_{i=0}^{\infty} \sum_{\substack{u_0,\dots,u_i,\\ v_0,\dots,v_i:\\ u_0=v_0=o}} \prod_{j=1}^{i} U^m(u_{j-1},v_{j-1};u_j,v_j)\, V^m(u_i,v_i;x), \tag{3.4}$$

where the empty product $\prod_{j=1}^{0}$ for $i=0$ is regarded as 1. To make it short, we abbreviate this to

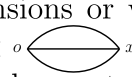
$$X^m_{o,x} = \sum_{i=0}^{\infty} \big((U^m)^{\star i} \star V^m\big)_{o,x}. \tag{3.5}$$

In the following diagrammatic bounds (in Theorems 3.1, 3.2, 3.3 and 3.5), we only use the two extremes: $m=1$ or ∞ (cf., e.g., (3.11)); U^0 is used only for quantitative estimates (cf., e.g., (5.9)).

Theorem 3.1. *Let $x \ne o$. For the ferromagnetic models defined above (2.1),*

$$\pi^{(0)}_{\mathbb{B}_\Lambda}(x) \le 2X^1_{o,x}$$
$$= 2\Bigg($$  $$+ \sum_{j=0}^{1}$$  $$+ \sum_{i,j=0}^{1}$$  $$+ \cdots \Bigg). \tag{3.6}$$

We will prove this theorem in Section 4.1.

The bound in (3.6) may look awful, but in fact, it is not so. Presumably a nonzero bubble $\tilde{G}^2$ is small in sufficiently high dimensions or with sufficiently large spread-out parameter in $d>4$, the dominant term is $2\times$  $= 2V^1(o,o;x) = 2\tilde{G}(x)^3$ and the other terms are geometrically small. We will demonstrate this for the spread-out model in $d>4$ in Section 5.1.

3.2 Results for the 1st-order expansion coefficient

Next we present a diagrammatic bound on the following $\Theta'_{o,x;A}$, which shows up in the bounds [25, Lemma 4.3] on the higher-order expansion coefficients $\pi^{(j)}_{\mathbb{B}_\Lambda}(x)$ for $j \ge 1$ (see

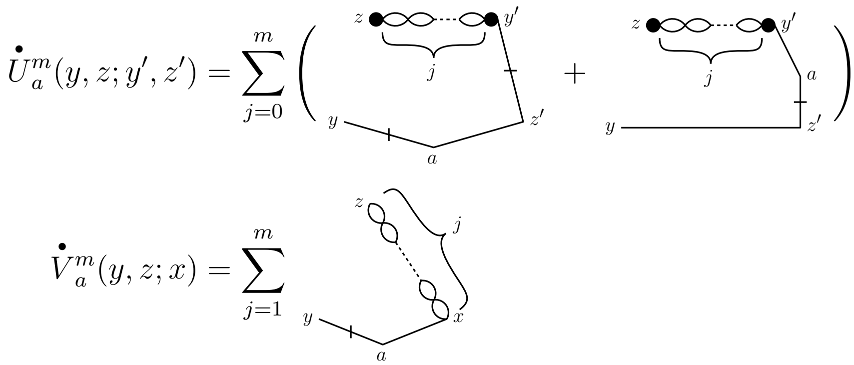


Figure 4: Schematic representations of $\dot{U}_a^m(y,z;y',z')$ and $\dot{V}_a^m(y,z;x)$ for $m \geq 1$.

(3.14) below). Given a vertex set $A \subset \Lambda$, we denote by $\mathbb{B}_{A^c}$ the set of bonds whose end vertices are both in A^c, and define

$$\Theta'_{o,x;A} = \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\mathbb{B}_{A^c}}:\\ \partial\boldsymbol{m}=\varnothing}} \frac{w_{\mathbb{B}_{A^c}}(\boldsymbol{m})}{Z_{\mathbb{B}_{A^c}}} \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda}:\\ \partial\boldsymbol{n}=o\triangle x}} \frac{w_{\mathbb{B}_\Lambda}(\boldsymbol{n})}{Z_{\mathbb{B}_\Lambda}} \mathbb{1}_{\{o \overset{A}{\underset{\boldsymbol{m}+\boldsymbol{n}}{\Longleftrightarrow}} x\}}, \tag{3.7}$$

where $o \overset{A}{\underset{\boldsymbol{m}+\boldsymbol{n}}{\Longleftrightarrow}} x$ means that $o \underset{\boldsymbol{m}+\boldsymbol{n}}{\Longleftrightarrow} x$ and that all paths from o to x with positive current in $\boldsymbol{m}+\boldsymbol{n}$ must go through the set A (i.e., every path from o to x consisting of bonds with positive current in $\boldsymbol{m}+\boldsymbol{n}$ has at least one bond with an endpoint in A). Then we define (see Figure 4)

$$\begin{aligned}\dot{U}_a^m(y,z;y',z') &= \Big(G(a-y)\,\tilde{G}(z'-a)\,G(z'-y') + \tilde{G}(z'-y)\,\tilde{G}(a-y')\,G(z'-a)\Big) \\ &\quad\times \begin{cases} G(y'-z)^2 & (m=0), \\ \Big((\delta+\tau^2) * \sum_{j=0}^m (\tilde{G}^2)^{*j} * (\delta+\tau^2)\Big)(y'-z) & (m \geq 1), \end{cases} \end{aligned} \tag{3.8}$$

$$\dot{V}_a^m(y,z;x) = G(a-y)\,\tilde{G}(x-a) \sum_{j=1}^{m} (\tilde{G}^2)^{*j}(x-z), \tag{3.9}$$

and let

$$\dot{X}^m_{o,x;a} = \sum_{i=0}^{\infty} \Big((U^m)^{\star i} \star \dot{V}_a^m\Big)_{o,x} + \sum_{i,j=0}^{\infty} \Big((U^m)^{\star i} \star \dot{U}_a^m \star (U^m)^{\star j} \star V^m\Big)_{o,x}. \tag{3.10}$$

Theorem 3.2 (cf., Lemma 4.4 of [25]). *Let $x \neq o$. For the ferromagnetic models defined above (2.1),*

$$\Theta'_{o,x;A} \leq 2 \sum_{a\in A} \Big(X^{\infty}_{o,x}\,\delta_{x,a} + \dot{X}^{\infty}_{o,x;a}\Big). \tag{3.11}$$

We will prove this theorem in Section 4.2. As discussed at the end of the previous subsection, if a nonzero bubble $\tilde{G}^2$ is small, then the dominant terms in $X^{\infty}_{o,x}$ and $\dot{X}^{\infty}_{o,x;a}$ in (3.11) are $V^1(o,o;x)$ and $\dot{V}^1_a(o,o;x)$, respectively, and the others are geometrically small. We will demonstrate this statement for the spread-out model in $d>4$ in Section 5.2.

Next we present a diagrammatic bound on $\tilde{\pi}^{(0)}_{\mathbb{B}_\Lambda;y}(x)$, which is a variant of $\pi^{(0)}_{\mathbb{B}_\Lambda}(x)$ and is defined as

$$\tilde{\pi}^{(0)}_{\mathbb{B}_\Lambda;y}(x)=\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda}:\\ \partial\boldsymbol{n}=o\triangle x}}\frac{w_{\mathbb{B}_\Lambda}(\boldsymbol{n})}{Z_{\mathbb{B}_\Lambda}}\mathbb{1}_{\{o\underset{\boldsymbol{n}}{\Longleftrightarrow}x\}\cap\{o\underset{\boldsymbol{n}}{\longleftrightarrow}y\}}. \tag{3.12}$$

This shows up in the bounds on the higher-order expansion coefficients $\pi^{(j)}_{\mathbb{B}_\Lambda}(x)$ for $j\ge 1$. For example, by using $\tilde{\mathcal{C}}^b_{\boldsymbol{n}}(o)=\{z\in\Lambda : o\underset{\boldsymbol{n}}{\longleftrightarrow}z \text{ in } \mathbb{B}_\Lambda\setminus b\}$ (here and in the rest of the paper, we abbreviate $\mathbb{B}_\Lambda\setminus\{b\}$ to $\mathbb{B}_\Lambda\setminus b$ for any $b\in\mathbb{B}_\Lambda$) and substituting [25, (4.33)] to the first line in [25, (4.37)], we can get

$$\pi^{(1)}_{\mathbb{B}_\Lambda}(x)\le\sum_{u,v}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda}:\\ \partial\boldsymbol{n}=o\triangle u}}\frac{w_{\mathbb{B}_\Lambda}(\boldsymbol{n})}{Z_{\mathbb{B}_\Lambda}}\mathbb{1}_{\{o\underset{\boldsymbol{n}}{\Longleftrightarrow}u\}}\tau(v-u)\sum_y\big(\delta_{v,y}+\tilde{G}(y-v)\big)\Theta'_{y,x;\tilde{\mathcal{C}}^{\{u,v\}}_{\boldsymbol{n}}(o)}. \tag{3.13}$$

Now, by Theorem 3.2 and $\Theta'_{x,x;A}=\mathbb{1}_{\{x\in A\}}$, we obtain

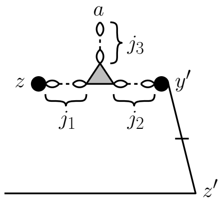
$$\pi^{(1)}_{\mathbb{B}_\Lambda}(x)\le\sum_{a,u,y}\underbrace{\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda}:\\ \partial\boldsymbol{n}=o\triangle u}}\frac{w_{\mathbb{B}_\Lambda}(\boldsymbol{n})}{Z_{\mathbb{B}_\Lambda}}\mathbb{1}_{\{o\underset{\boldsymbol{n}}{\Longleftrightarrow}u\}\cap\{o\underset{\boldsymbol{n}}{\longleftrightarrow}a\}}}_{\tilde{\pi}^{(0)}_{\mathbb{B}_\Lambda;a}(u)}\big(\tau*(\delta+\tilde{G})\big)(y-u)$$

$$\times 2\Big(\delta_{y,x}\,\delta_{x,a}+X^{\infty}_{y,x}\,\delta_{x,a}+\dot{X}^{\infty}_{y,x;a}\Big). \tag{3.14}$$

In the previous work, we claimed that $\tilde{\pi}^{(0)}_{\mathbb{B}_\Lambda;y}(x)$ obeys the diagrammatic bound [25, (4.16)], but its proof given around [25, (4.23)–(4.26)] is based on the problematic [25, Lemma 4.2]. We no longer use it and prove the following theorem instead, in which we use for $m\ge 1$ (see Figure 5)

$$\ddot{U}^m_a(y,z;y',z')=\tilde{G}(z'-y)\,G(z'-y')\sum_{\substack{u_1,u_2,u_3,\\ v_1,v_2,v_3}}(\delta+\tau^2)(z-u_1)\,(\delta+\tau^2)(y'-u_2)\,\delta_{u_3,a}$$
$$\times\prod_{i=1}^{3}\sum_{j_i=0}^{m-1}(\tilde{G}^2)^{*j_i}(u_i-v_i)\,T(v_1,v_2,v_3), \tag{3.15}$$

$$\ddot{V}^m_a(y,z;x)=\mathbb{1}_{\{z\ne x\}}\,\tilde{G}(x-y)\sum_{\substack{u_1,u_2,u_3,\\ v_1,v_2,v_3}}(\delta+\tau^2)(z-u_1)\,\delta_{u_2,x}\,\delta_{u_3,a}$$
$$\times\prod_{i=1}^{3}\sum_{j_i=0}^{m-1}(\tilde{G}^2)^{*j_i}(u_i-v_i)\,T(v_1,v_2,v_3), \tag{3.16}$$

$$\ddot{U}^m_a(y,z;y',z') = \sum_{j_1,j_2,j_3=0}^{m-1}$$ 

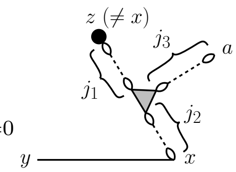
$$\ddot{V}^m_a(y,z;x) = \sum_{j_1,j_2,j_3=0}^{m-1}$$ 

Figure 5: Schematic representations of $\ddot{U}^m_a(y,z;y',z')$ and $\ddot{V}^m_a(y,z;x)$ for $m \geq 1$. The shaded triangles represent $T(v_1,v_2,v_3)$ in (3.15)–(3.16).

and

$$\ddot{X}^m_{o,x;y} = \sum_{i=0}^{\infty} \Big((U^m)^{\star i} \star \tfrac{1}{2} \ddot{V}^m_y \Big)_{o,x} + \sum_{i,j=0}^{\infty} \Big((U^m)^{\star i} \star \ddot{U}^m_y \star (U^m)^{\star j} \star V^m \Big)_{o,x}. \tag{3.17}$$

Theorem 3.3. *Let $x \neq o$. For the ferromagnetic models defined above (2.1),*

$$\tilde{\pi}^{(0)}_{\mathbb{B}_\Lambda;y}(x) \leq 2\Big(X^1_{o,x}\,\delta_{x,y} + \dot{X}^1_{o,x;y} + \ddot{X}^1_{o,x;y} \Big). \tag{3.18}$$

We will prove this theorem in Section 4.3. In addition to what we have stated below Theorem 3.2, we will also demonstrate in Section 5.2 that the dominant term in $\ddot{X}^1_{o,u;a}$ is $\ddot{V}^1_a(o,o;u)$ for the spread-out model in $d > 4$.

Applying (3.18) and $\tilde{\pi}^{(0)}_{\mathbb{B}_\Lambda;y}(o) \leq G(y)^2$ (cf., (2.16)) to (3.14) yields the following diagrammatic bound on $\pi^{(1)}_{\mathbb{B}_\Lambda}(x)$.

Corollary 3.4. *For the ferromagnetic models defined above (2.1),*

$$\pi^{(1)}_{\mathbb{B}_\Lambda}(x) \leq 4 \sum_{a,u,y} \Big(G(y)^2\,\delta_{o,u} + X^1_{o,u}\,\delta_{u,a} + \dot{X}^1_{o,u;a} + \ddot{X}^1_{o,u;a} \Big) \\ \times \big(\tau * (\delta + \tilde{G})\big)(y-u) \Big(\delta_{y,x}\,\delta_{x,a} + X^\infty_{y,x}\,\delta_{x,a} + \dot{X}^\infty_{y,x;a} \Big). \tag{3.19}$$

3.3 Results for the higher-order expansion coefficients

Finally we present a diagrammatic bound on $\Theta''_{o,x,y;A}$, which is a variant of $\Theta'_{o,x;A}$ and is defined as

$$\Theta''_{o,x,y;A} = \sum_{\substack{\boldsymbol{m} \in \mathbb{Z}_+^{\mathbb{B}_{A^c}}: \\ \partial \boldsymbol{m} = \varnothing}} \frac{w_{\mathbb{B}_{A^c}}(\boldsymbol{m})}{Z_{\mathbb{B}_{A^c}}} \sum_{\substack{\boldsymbol{n} \in \mathbb{Z}_+^{\mathbb{B}_\Lambda}: \\ \partial \boldsymbol{n} = o \triangle x}} \frac{w_{\mathbb{B}_\Lambda}(\boldsymbol{n})}{Z_{\mathbb{B}_\Lambda}} \mathbb{1}_{\{o \overset{A}{\underset{\boldsymbol{m}+\boldsymbol{n}}{\Longleftrightarrow}} x\} \cap \{o \underset{\boldsymbol{m}+\boldsymbol{n}}{\longleftrightarrow} y\}}. \tag{3.20}$$

This shows up in the bounds on the higher-order expansion coefficients $\pi^{(j)}_{\mathbb{B}_\Lambda}$ for $j \geq 2$. For example, $\pi^{(2)}_{\mathbb{B}_\Lambda}(x)$ is bounded in a similar way to (3.14) as (cf., [25, (4.38)])

$$\begin{aligned}\pi^{(2)}_{\mathbb{B}_\Lambda}(x) \leq 2 \sum_u \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda}:\\ \partial\boldsymbol{n}=o\triangle u}} \frac{w_{\mathbb{B}_\Lambda}(\boldsymbol{n})}{Z_{\mathbb{B}_\Lambda}} \mathbb{1}_{\{o \underset{\boldsymbol{n}}{\Longleftrightarrow} u\}} \\ \times \sum_{\substack{v,z,y,\\ u',a'}} \tau(v-u)\, G(z-v) \Big(\tilde{G}(y-z)\, \delta_{z,a'}\, \Theta'_{y,u';\tilde{\mathcal{C}}^{\{u,v\}}_{\boldsymbol{n}}(o)} + \delta_{y,z}\, \Theta''_{y,u',a';\tilde{\mathcal{C}}^{\{u,v\}}_{\boldsymbol{n}}(o)} \Big) \\ \times \sum_{y'} \big(\tau * (\delta + \tilde{G})\big)(y'-u') \Big(X^{\infty}_{y',x}\, \delta_{x,a'} + \dot{X}^{\infty}_{y',x;a'} \Big). \end{aligned} \tag{3.21}$$

To show a diagrammatic bound on $\Theta''_{o,x,y;A}$, we introduce the following building blocks of the diagrams: for $m \geq 1$,

$$\begin{aligned}\dddot{U}^m_{a,v}(y,z;y',z') = \Big(G(a-y)\, \tilde{G}(z'-a)\, G(z'-y') + \tilde{G}(z'-y)\, \tilde{G}(a-y')\, G(z'-a) \Big) \\ \times \sum_{\substack{u_1,u_2,u_3,\\ v_1,v_2,v_3}} (\delta+\tau^2)(z-u_1)\, (\delta+\tau^2)(y'-u_2)\, \delta_{u_3,v} \\ \times \prod_{i=1}^{3} \sum_{j_i=0}^{m-1} (\tilde{G}^2)^{*j_i}(u_i - v_i)\, T(v_1,v_2,v_3), \end{aligned} \tag{3.22}$$

$$\begin{aligned}\dddot{V}^m_{a,v}(y,z;x) = \mathbb{1}_{\{z\neq x\}}\, G(a-y)\, \tilde{G}(x-a) \sum_{\substack{u_1,u_2,u_3,\\ v_1,v_2,v_3}} (\delta+\tau^2)(z-u_1)\, \delta_{u_2,x}\, \delta_{u_3,v} \\ \times \prod_{i=1}^{3} \sum_{j_i=0}^{m-1} (\tilde{G}^2)^{*j_i}(u_i - v_i)\, T(v_1,v_2,v_3). \end{aligned} \tag{3.23}$$

Depending on which of the six terms in $\ddot{X}^m_{o,x;y}$ in (3.17) the extra vertex a is added, we define

$$\begin{aligned}\dddot{X}^m_{o,x;a,y} = \sum_{i=0}^{\infty} \Big((U^m)^{\star i} \star \tfrac{1}{2} \dddot{V}^m_{a,y} \Big)_{o,x} + \sum_{i,j=0}^{\infty} \Big((U^m)^{\star i} \star \dddot{U}^m_{a,y} \star (U^m)^{\star j} \star V^m \Big)_{o,x} \\ + \sum_{i,j=0}^{\infty} \Big((U^m)^{\star i} \star \dot{U}^m_a \star (U^m)^{\star j} \star \tfrac{1}{2} \ddot{V}^m_y \Big)_{o,x} + \sum_{i,j=0}^{\infty} \Big((U^m)^{\star i} \star \ddot{U}^m_y \star (U^m)^{\star j} \star \dot{V}^m_a \Big)_{o,x} \\ + \sum_{i,j,k=0}^{\infty} \Big((U^m)^{\star i} \star \dot{U}^m_a \star (U^m)^{\star j} \star \ddot{U}^m_y \star (U^m)^{\star k} \star V^m \Big)_{o,x} \\ + \sum_{i,j,k=0}^{\infty} \Big((U^m)^{\star i} \star \ddot{U}^m_y \star (U^m)^{\star j} \star \dot{U}^m_a \star (U^m)^{\star k} \star V^m \Big)_{o,x}. \end{aligned} \tag{3.24}$$

Theorem 3.5 (cf., Lemma 4.4 of [25])**.** *Let $x \neq o$. For the ferromagnetic models defined above (2.1),*

$$\Theta''_{o,x,y;A} \le 2\sum_{a\in A}\bigg(\ddot{X}^{\infty}_{o,x;y}\delta_{a,x} + \dddot{X}^{\infty}_{o,x;a,y} + \sum_{y'}\Big(\ddot{X}^{\infty}_{o,x;a}\delta_{y',x} + \dddot{X}^{\infty}_{o,x;y',a}\Big)\sum_{i=0}^{\infty}(\tilde{G}^2)^{*i}(y-y')\bigg). \tag{3.25}$$

We will prove this theorem in Section 4.4 and also demonstrate in Section 5.3 that the main contribution to $\dddot{X}^{m}_{o,x;a,y}$ comes from $\dddot{V}^{m}_{a,y}(o,o;x)$ for the spread-out model in $d > 4$.

Applying (3.18) and (3.25) to (3.21) and using $\Theta''_{x,x,y;A} \le \mathbb{1}_{\{x\in A\}}\sum_{j=0}^{\infty}(\tilde{G}^2)^{*j}(y)$ would yield a diagrammatic bound on $\pi^{(2)}_{\mathbb{B}_\Lambda}(x)$. We refrain from stating it explicitly. The higher-order expansion coefficients can be bounded in a similar way.

4 Proofs of the main results

In Section 4.1, we first prove Theorem 3.1 in detail, as it provides a common foundation for the other three theorems. We prove those three theorems in Sections 4.2–4.4, respectively, only focusing on differences from Theorem 3.1. In the course of the proof of Theorem 3.1 (at the end of Step 1 in Section 4.1), we also prove Lemma 2.4.

4.1 Proof of Theorem 3.1

The proof is progressed along the following five steps.

1. Rewrite $\pi^{(0)}_{\mathbb{B}_\Lambda}(x)$ for $x \neq o$ by identifying the "earliest" path from o to x with odd current.

2. A double expansion: a sort of lace expansion along the earliest path chosen above. Let $N \ge 1$ be the number of lace edges in the expansion.

3. Proof for the $N = 1$ case.

4. Proof for the $N \ge 2$ case, part 1: bounds on the contributions from lace edges.

5. Proof for the $N \ge 2$ case, part 2: bound on the contribution from the earliest path.

Step 1. First we note that, due to the source constraint in the definition of $\pi^{(0)}_{\mathbb{B}_\Lambda}$, there must be a path from o to x (assumed not to be o) of bonds with odd current. To identify a unique one among those paths, we introduce a fixed (e.g., lexicographic) ordering in the set of bonds incident on each vertex. Given a pair of bonds $b_1 = \{u, v_1\}$ and $b_2 = \{u, v_2\}$ that are incident on a common vertex u, we write $b_1 \preceq b_2$ (and $v_1 \preceq v_2$) if b_1 is earlier than or equal to b_2 in that ordering. Let $\Omega(o, x)$ be the set of nonzero paths from o to

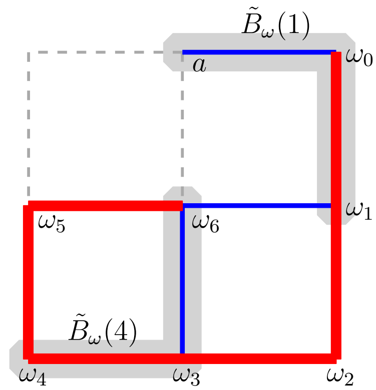


Figure 6: Suppose that Λ consists of nine vertices as depicted, and that bonds incident on each vertex x are ordered in a counter-clockwise way as $\{x, x+e_1\} \preceq \{x, x+e_2\} \preceq \{x, x-e_1\} \preceq \{x, x-e_2\}$, where $e_1 = (1,0)$ and $e_2 = (0,1)$. Given $\omega = (\omega_0, \omega_1, \dots, \omega_6)$ as depicted, we have $\tilde{B}_\omega(0) = \varnothing$, $\tilde{B}_\omega(1) = \{\{\omega_0, \omega_1\}, \{\omega_0, a\}\}$ (in grey), $\tilde{B}_\omega(2) = \{\{\omega_1, \omega_2\}, \{\omega_1, \omega_6\}\}$, $\tilde{B}_\omega(3) = \{\{\omega_2, \omega_3\}\}$, $\tilde{B}_\omega(4) = \{\{\omega_3, \omega_4\}, \{\omega_3, \omega_6\}\}$ (in grey), $\tilde{B}_\omega(5) = \{\{\omega_4, \omega_5\}\}$, $\tilde{B}_\omega(6) = \{\{\omega_5, \omega_6\}\}$, so that B_ω is the set of bold bonds (in red) and $\tilde{B}_\omega \setminus B_\omega$ is the set of thin-solid bonds (in blue).

x each of which may intersect to itself (except for the terminal x) but does not traverse any bond more than once:

$$\Omega(z,x) = \left\{ \omega = (\omega_0, \dots, \omega_{|\omega|}) : \begin{array}{c} |\omega| \geq 1,\ \omega_0 = z,\ \omega_{|\omega|} = x,\ \omega_j \neq x \text{ for } j < |\omega|, \\ \{\omega_{i-1}, \omega_i\} \neq \{\omega_{j-1}, \omega_j\} \text{ for } 1 \leq i < j \leq |\omega| \end{array} \right\}. \tag{4.1}$$

Given an $\omega \in \Omega(o,x)$, we let $B_\omega = \{\{\omega_{j-1}, \omega_j\} : j = 1, \dots, |\omega|\}$ and inductively define the bond set $\tilde{B}_\omega$ ($\supset B_\omega$) as (cf., Figure 6)

$$\tilde{B}_\omega(j) = \begin{cases} \varnothing & (j = 0), \\ \Big\{\{\omega_{j-1}, v\} \notin \bigcup_{i=0}^{j-1} \tilde{B}_\omega(i) : v \preceq \omega_j\Big\} & (j \geq 1), \end{cases} \qquad \tilde{B}_\omega = \bigcup_{j=0}^{|\omega|} \tilde{B}_\omega(j). \tag{4.2}$$

Then we can rewrite the random-current representation of $\pi^{(0)}_{\mathbb{B}_\Lambda}(x)$ in (2.34) as

$$\begin{aligned} \pi^{(0)}_{\mathbb{B}_\Lambda}(x) &= \delta_{o,x} + \sum_{\substack{\boldsymbol{n} \in \mathbb{Z}_+^{\mathbb{B}_\Lambda}: \\ \partial \boldsymbol{n} = o \triangle x}} \frac{w_{\mathbb{B}_\Lambda}(\boldsymbol{n})}{Z_{\mathbb{B}_\Lambda}} \mathbb{1}_{\{o \underset{\boldsymbol{n}}{\Longleftrightarrow} x\}} \sum_{\omega \in \Omega(o,x)} \mathbb{1}_{\{\omega \text{ the earliest odd path}\}}(\boldsymbol{n}) \\ &= \delta_{o,x} + \sum_{\substack{\boldsymbol{n} \in \mathbb{Z}_+^{\mathbb{B}_\Lambda}: \\ \partial \boldsymbol{n} = o \triangle x}} \frac{w_{\mathbb{B}_\Lambda}(\boldsymbol{n})}{Z_{\mathbb{B}_\Lambda}} \mathbb{1}_{\{o \underset{\boldsymbol{n}}{\Longleftrightarrow} x\}} \sum_{\omega \in \Omega(o,x)} \prod_{b \in B_\omega} \mathbb{1}_{\{n_b \text{ odd}\}} \prod_{b' \in \tilde{B}_\omega \setminus B_\omega} \mathbb{1}_{\{n_{b'} \text{ even}\}}. \end{aligned} \tag{4.3}$$

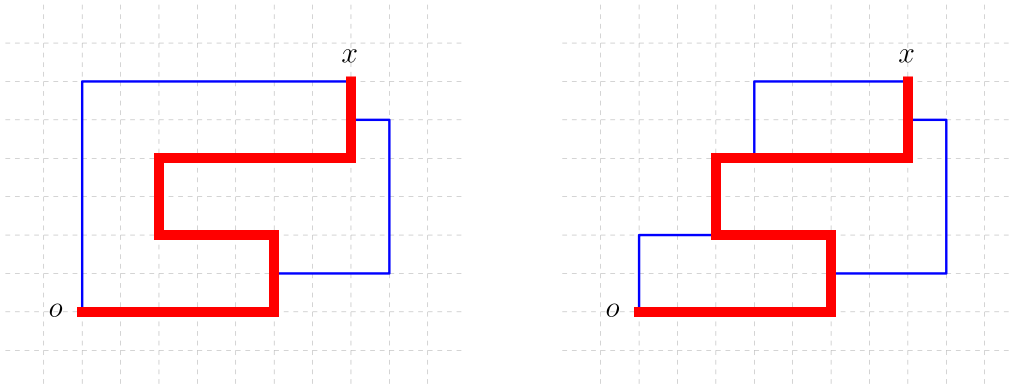


Figure 7: Current configurations for $\pi^{(0)}_{\mathbb{B}_\Lambda}(x)$ in (4.3). As in Figure 1, bonds with odd current are bold (in red), while those with positive-even current are thin-solid (in blue). On the left, o and x are still connected by a path of bonds with positive current, even after removal of the path of bonds with odd current, which is not the case on the right.

By splitting the weight as $w_{\mathbb{B}_\Lambda}(\boldsymbol{n}) = w_{\tilde{B}_\omega}(\boldsymbol{m})\, w_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}(\boldsymbol{k})$, where $\boldsymbol{m}$ and $\boldsymbol{k}$ are the restrictions of $\boldsymbol{n}$ on $\tilde{B}_\omega$ and on $\mathbb{B}_\Lambda \setminus \tilde{B}_\omega$, repectively, we have the rewrite for $x \neq o$:

$$\pi^{(0)}_{\mathbb{B}_\Lambda}(x) = \sum_{\omega\in\Omega(o,x)} \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde{B}_\omega}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde{B}_\omega\setminus B_\omega}} \frac{w_{\tilde{B}_\omega}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda}} \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}:\\ \partial\boldsymbol{k}=\varnothing}} w_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}(\boldsymbol{k})\, \mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}}{\Longleftrightarrow} x\}}. \tag{4.4}$$

Remark. Suppose $\mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}}{\Longleftrightarrow} x\}} \leq \mathbb{1}_{\{o \underset{\boldsymbol{k}}{\longleftrightarrow} x\}}$ in (4.4), i.e., the double connection implies existence of a path from o to x of positive current in the restricted region $\mathbb{B}_\Lambda \setminus \tilde{B}_\omega$ (as on the left of Figure 7, where we regard the set of bold red bonds as $\tilde{B}_\omega$). Then, by (2.16), the sum over $\boldsymbol{k}$ is bounded as

$$\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}:\\ \partial\boldsymbol{k}=\varnothing}} w_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}(\boldsymbol{k})\, \mathbb{1}_{\{o \underset{\boldsymbol{k}}{\longleftrightarrow} x\}} \leq G(x)^2 Z_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega} = G(x)^2 \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}:\\ \partial\boldsymbol{k}=\varnothing}} w_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}(\boldsymbol{k}). \tag{4.5}$$

As a result, we obtain that, for $x \neq o$,

$$\begin{aligned} &\sum_{\omega\in\Omega(o,x)} \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde{B}_\omega}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde{B}_\omega\setminus B_\omega}} \frac{w_{\tilde{B}_\omega}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda}} \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}:\\ \partial\boldsymbol{k}=\varnothing}} w_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}(\boldsymbol{k})\, \mathbb{1}_{\{o \underset{\boldsymbol{k}}{\longleftrightarrow} x\}} \\ &\overset{(4.5)}{\leq} G(x)^2 \sum_{\omega\in\Omega(o,x)} \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde{B}_\omega}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde{B}_\omega\setminus B_\omega}} \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}:\\ \partial\boldsymbol{k}=\varnothing}} \frac{w_{\tilde{B}_\omega}(\boldsymbol{m})\, w_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}(\boldsymbol{k})}{Z_{\mathbb{B}_\Lambda}} \end{aligned}$$

$$= G(x)^2 \underbrace{\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda}:\\ \partial\boldsymbol{n}=o\triangle x}} \frac{w_{\mathbb{B}_\Lambda}(\boldsymbol{n})}{Z_{\mathbb{B}_\Lambda}}}_{\langle\varphi_o\varphi_x\rangle_{\mathbb{B}_\Lambda}} \overset{(2.14)}{\leq} \tilde{G}(x)^3. \tag{4.6}$$

However, the above argument is incomplete, due to the possibility of $\mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}}{\Longleftrightarrow} x\}} > \mathbb{1}_{\{o \underset{\boldsymbol{k}}{\longleftrightarrow} x\}}$, as depicted on the right of Figure 7, i.e., o and x may no longer connected after removal of $\tilde{B}_\omega$. To overcome this problem, we will introduce a notion of lace in Step 2.

Proof of Lemma 2.4. First, by multiplying $1 = \sum_{\partial\boldsymbol{m}=\varnothing} w_B(\boldsymbol{m})/Z_B$, using the monotonicity $\{o \underset{\boldsymbol{n}}{\longleftrightarrow} x\} \subset \{o \underset{\boldsymbol{m}+\boldsymbol{n}}{\longleftrightarrow} x\}$ and then the SST, we obtain

$$\begin{aligned}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^B:\\ \partial\boldsymbol{n}=\varnothing}} \frac{w_B(\boldsymbol{n})}{Z_B}\mathbb{1}_{\{o\underset{\boldsymbol{n}}{\longleftrightarrow}x\}\cap\{o\underset{\boldsymbol{n}}{\longleftrightarrow}y\}} &\leq \sum_{\substack{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^B:\\ \partial\boldsymbol{m}=\partial\boldsymbol{n}=\varnothing}} \frac{w_B(\boldsymbol{m})}{Z_B}\frac{w_B(\boldsymbol{n})}{Z_B}\mathbb{1}_{\{o\underset{\boldsymbol{m}+\boldsymbol{n}}{\longleftrightarrow}x\}\cap\{o\underset{\boldsymbol{m}+\boldsymbol{n}}{\longleftrightarrow}y\}}\\ &\overset{\text{SST}}{=} \sum_{\substack{\boldsymbol{m},\boldsymbol{n}\in\mathbb{Z}_+^B:\\ \partial\boldsymbol{m}=\partial\boldsymbol{n}=o\triangle x}} \frac{w_B(\boldsymbol{m})}{Z_B}\frac{w_B(\boldsymbol{n})}{Z_B}\mathbb{1}_{\{o\underset{\boldsymbol{m}+\boldsymbol{n}}{\longleftrightarrow}y\}}.\end{aligned} \tag{4.7}$$

Now, as explained above in Step 1, we can identify the earliest path $\omega \in \Omega(o,x)$ of bonds $b \in B$ with odd m_b. Splitting the weight $w_B(\boldsymbol{m})$ as $w_{\tilde{B}_\omega}(\boldsymbol{k})\, w_{B\setminus\tilde{B}_\omega}(\boldsymbol{\ell})$, we obtain

$$(4.7) = \sum_{\omega\in\Omega(o,x)} \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\tilde{B}_\omega}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde{B}_\omega\setminus B_\omega}} \frac{w_{\tilde{B}_\omega}(\boldsymbol{k})}{Z_B} \sum_{\substack{\boldsymbol{\ell}\in\mathbb{Z}_+^{B\setminus\tilde{B}_\omega}:\\ \partial\boldsymbol{\ell}=\varnothing}} w_{B\setminus\tilde{B}_\omega}(\boldsymbol{\ell}) \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^B:\\ \partial\boldsymbol{n}=o\triangle x}} \frac{w_B(\boldsymbol{n})}{Z_B}\mathbb{1}_{\{o\underset{\boldsymbol{k}+\boldsymbol{\ell}+\boldsymbol{n}}{\longleftrightarrow}y\}}. \tag{4.8}$$

Notice that, if $o \underset{\boldsymbol{k}+\boldsymbol{\ell}+\boldsymbol{n}}{\longleftrightarrow} y$, then there must be a $z \in V(\tilde{B}_\omega)$ such that $o \underset{\boldsymbol{k}}{\longleftrightarrow} z$ and $z \underset{\boldsymbol{\ell}+\boldsymbol{n}}{\longleftrightarrow} y$ in $B \setminus \tilde{B}_\omega$ (zero length is allowed). Therefore, the above expression is bounded by

$$\begin{aligned}&\sum_{\omega\in\Omega(o,x)} \sum_{z\in V(\tilde{B}_\omega)} \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\tilde{B}_\omega}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde{B}_\omega\setminus B_\omega}} \frac{w_{\tilde{B}_\omega}(\boldsymbol{k})}{Z_B} \sum_{\substack{\boldsymbol{\ell}\in\mathbb{Z}_+^{B\setminus\tilde{B}_\omega}:\\ \partial\boldsymbol{\ell}=\varnothing}} w_{B\setminus\tilde{B}_\omega}(\boldsymbol{\ell}) \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^B:\\ \partial\boldsymbol{n}=o\triangle x}} \frac{w_B(\boldsymbol{n})}{Z_B}\mathbb{1}_{\{z\underset{\boldsymbol{\ell}+\boldsymbol{n}}{\longleftrightarrow}y \text{ in } B\setminus\tilde{B}_\omega\}}\\ &\overset{\text{SST}}{=} \sum_{\omega\in\Omega(o,x)} \sum_{z\in V(\tilde{B}_\omega)} \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\tilde{B}_\omega}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde{B}_\omega\setminus B_\omega}} \frac{w_{\tilde{B}_\omega}(\boldsymbol{k})}{Z_B} Z_{B\setminus\tilde{B}_\omega} \underbrace{\sum_{\substack{\boldsymbol{\ell}\in\mathbb{Z}_+^{B\setminus\tilde{B}_\omega}:\\ \partial\boldsymbol{\ell}=z\triangle y}} \frac{w_{B\setminus\tilde{B}_\omega}(\boldsymbol{\ell})}{Z_{B\setminus\tilde{B}_\omega}}}_{\langle\varphi_z\varphi_y\rangle_{B\setminus\tilde{B}_\omega}} \underbrace{\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^B:\\ \partial\boldsymbol{n}=o\triangle x\triangle z\triangle y}} \frac{w_B(\boldsymbol{n})}{Z_B}}_{\langle\varphi_o\varphi_x\varphi_z\varphi_y\rangle_B\ (\because (2.9))}.\end{aligned} \tag{4.9}$$

Furthermore, by Lebowitz' inequality [22] and Lemma 2.2,

$$\begin{aligned}\langle\varphi_o\varphi_x\varphi_z\varphi_y\rangle_B &\leq \langle\varphi_o\varphi_x\rangle_B\,\langle\varphi_z\varphi_y\rangle_B + \langle\varphi_o\varphi_y\rangle_B\,\langle\varphi_x\varphi_z\rangle_B + \langle\varphi_o\varphi_z\rangle_B\,\langle\varphi_x\varphi_y\rangle_B\\ &\leq G(x)\,G(z-y) + G(y)\,G(z-x) + G(z)\,G(y-x).\end{aligned} \tag{4.10}$$

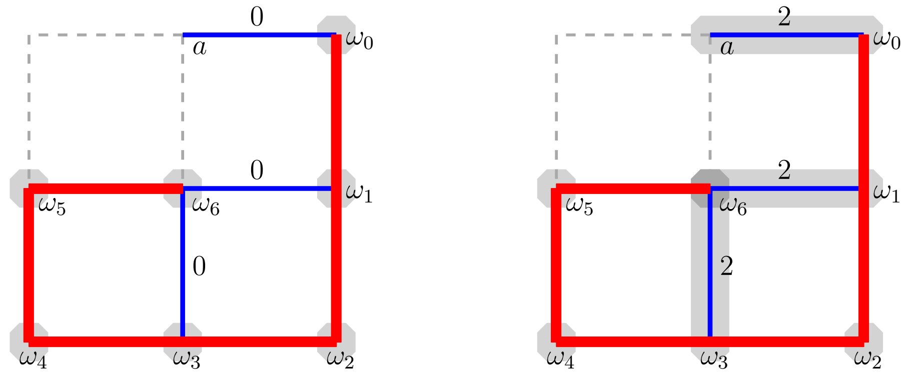


Figure 8: Consider the same setting as Figure 6 and let $\boldsymbol{m}$ be a current configuration that assigns odd numbers to the red bonds in B_ω. If $\boldsymbol{m}$ assigns zeros to the blue bonds in $\tilde{B}_\omega \setminus B_\omega$ (as depicted in the left figure), then $\tilde{V}_{\boldsymbol{m}}(j) = \{\omega_j\}$ (in grey) for all $j = 0, 1, \ldots, 6$. On the other hand, if $\boldsymbol{m}$ assigns to the blue bonds positive even numbers (say, 2, as in the right figure), then $\tilde{V}_{\boldsymbol{m}}(0) = \{\omega_0, a\}$, $\tilde{V}_{\boldsymbol{m}}(1) = \{\omega_1, \omega_6\}$ and $\tilde{V}_{\boldsymbol{m}}(3) = \{\omega_3, \omega_6\}$.

As a result, we obtain

$$
\begin{aligned}
(4.9) &\le \sum_{z} \sum_{\omega\in\Omega(o,x)} \mathbb{1}_{\{z\in V(\tilde{B}_\omega)\}} \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\tilde{B}_\omega}: \\ \text{odd on } B_\omega, \\ \text{even on } \tilde{B}_\omega\setminus B_\omega}} \frac{w_{\tilde{B}_\omega}(\boldsymbol{k})}{Z_B} Z_{B\setminus\tilde{B}_\omega} G(z-y) \\
&\qquad \times \Big(G(x)\,G(z-y) + G(y)\,G(z-x) + G(z)\,G(y-x)\Big) \\
&\le \sum_{z} \underbrace{\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{B}: \\ \partial\boldsymbol{n}=o\triangle x}} \frac{w_B(\boldsymbol{n})}{Z_B} \mathbb{1}_{\{o\underset{\boldsymbol{n}}{\longleftrightarrow} z\}}}_{\le G(z)\,G(z-x)\ \ (\because\ (2.15))} G(z-y) \\
&\qquad \times \Big(G(x)\,G(z-y) + G(y)\,G(z-x) + G(z)\,G(y-x)\Big),
\end{aligned}
\tag{4.11}
$$

as required. ■

Step 2. To overcome the problem explained below (4.6), we use $\tilde{B}_\omega$ as a time line for an expansion, similar to the lace expansion, of the sum over $\boldsymbol{k}$ in (4.4); we call this a double expansion. First, we define the vertex set $\tilde{V}_{\boldsymbol{m}}(j)$ for a current configuration $\boldsymbol{m} \in \mathbb{Z}_+^{\tilde{B}_\omega}$ satisfying the constraint in the second sum on the right-hand side of (4.4) as (cf., Figure 8)

$$
\tilde{V}_{\boldsymbol{m}}(j) = \begin{cases} \{\omega_j\} \cup \Big\{v : \{\omega_j, v\} \in \tilde{B}_\omega(j+1),\ m_{\omega_j,v} \text{ is positive-even}\Big\} & (0 \le j < |\omega|), \\ \{\omega_{|\omega|}\} & (j = |\omega|). \end{cases}
\tag{4.12}
$$

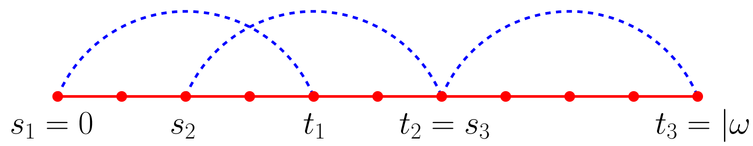


Figure 9: An example of a lace consisting of three edges s_1t_1, s_2t_2, s_3t_3. A dashed arc from s to t represents $\tilde{V}_{\boldsymbol{m}}(s) \underset{\boldsymbol{k}}{\longleftrightarrow} \tilde{V}_{\boldsymbol{m}}(t)$.

Given those vertex sets and a $\boldsymbol{k} \in \mathbb{Z}_+^{\mathbb{B}_\Lambda \setminus \tilde{B}_\omega}$ in the third sum on the right-hand side of (4.4), we uniquely define a lace $\mathsf{L}_{\boldsymbol{m},\boldsymbol{k}} = \{s_j t_j\}_{j=1}^N$ as follows (see Figure 9):

- First we define

$$s_1 = 0, \qquad\qquad t_1 = \max\big\{j : \tilde{V}_{\boldsymbol{m}}(0) \underset{\boldsymbol{k}}{\longleftrightarrow} \tilde{V}_{\boldsymbol{m}}(j)\big\}. \tag{4.13}$$

  where $\tilde{V}_{\boldsymbol{m}}(0) \underset{\boldsymbol{k}}{\longleftrightarrow} \tilde{V}_{\boldsymbol{m}}(j)$ means either $\tilde{V}_{\boldsymbol{m}}(0) \cap \tilde{V}_{\boldsymbol{m}}(j) \neq \varnothing$ or there is a nonzero path of bonds b with $k_b > 0$ from a vertex in $\tilde{V}_{\boldsymbol{m}}(0)$ to another in $\tilde{V}_{\boldsymbol{m}}(j)$. If $t_1 = |\omega|$ (as on the left of Figure 7), then it is done with $\mathsf{L}_{\boldsymbol{m},\boldsymbol{k}} = \{0|\omega|\}$ and $N = 1$.

- If $t_1 < |\omega|$, then, since $o \underset{\boldsymbol{m}+\boldsymbol{k}}{\Longleftrightarrow} x$, there must be an s_2t_2 uniquely defined as

$$t_2 = \max\big\{j : \exists i \le t_1 \text{ s.t. } \tilde{V}_{\boldsymbol{m}}(i) \underset{\boldsymbol{k}}{\longleftrightarrow} \tilde{V}_{\boldsymbol{m}}(j)\big\}, \tag{4.14}$$

$$s_2 = \min\big\{i : \tilde{V}_{\boldsymbol{m}}(i) \underset{\boldsymbol{k}}{\longleftrightarrow} \tilde{V}_{\boldsymbol{m}}(t_2)\big\}. \tag{4.15}$$

  If $t_2 = |\omega|$, then it is done with $\mathsf{L}_{\boldsymbol{m},\boldsymbol{k}} = \{0t_1, s_2|\omega|\}$ and $N = 2$.

- Repeat this procedure until it reaches $t_N = |\omega|$ with $\mathsf{L}_{\boldsymbol{m},\boldsymbol{k}} = \{s_j t_j\}_{j=1}^N$.

Let $\mathcal{L}_{[0,|\omega|]}^{(N)}$ be the set of N-edge graphs $\Gamma = \{s_j t_j\}_{j=1}^N$ satisfying

$$\begin{cases} 0 = s_1 < t_1 = |\omega| & (N = 1), \\ 0 = s_1 < s_2 \le t_1 < t_2 = |\omega| & (N = 2), \\ 0 = s_1 < s_2 \le \dots < s_n \le t_{n-1} < s_{n+1} \le t_n < \dots \le t_{N-1} < t_N = |\omega| & (N \ge 3). \end{cases} \tag{4.16}$$

Then, we can rewrite the sum over $\boldsymbol{k}$ in (4.4) to obtain that, for $x \neq o$,

$$\pi_{\mathbb{B}_\Lambda}^{(0)}(x) = \sum_{N=1}^{\infty} \sum_{\omega \in \Omega(o,x)} \sum_{\Gamma \in \mathcal{L}_{[0,|\omega|]}^{(N)}} \sum_{\substack{\boldsymbol{m} \in \mathbb{Z}_+^{\tilde{B}_\omega}: \\ \text{odd on } B_\omega, \\ \text{even on } \tilde{B}_\omega \setminus B_\omega}} \frac{w_{\tilde{B}_\omega}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda}} \times \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}_+^{\mathbb{B}_\Lambda \setminus \tilde{B}_\omega}: \\ \partial \boldsymbol{k} = \varnothing}} w_{\mathbb{B}_\Lambda \setminus \tilde{B}_\omega}(\boldsymbol{k}) \, \mathbb{1}_{\{\mathsf{L}_{\boldsymbol{m},\boldsymbol{k}} = \Gamma\}} \prod_{st \in \Gamma} \mathbb{1}_{\{\tilde{V}_{\boldsymbol{m}}(s) \underset{\boldsymbol{k}}{\longleftrightarrow} \tilde{V}_{\boldsymbol{m}}(t)\}}. \tag{4.17}$$

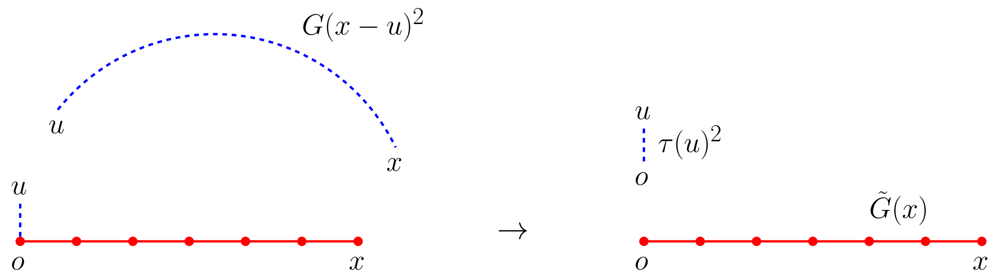
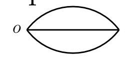


Figure 10: Explanation of how to decompose (4.18) into two-point functions. The left figure explains the extraction of $G(x-u)^2$ as in (4.20), and then the right figure explains the extraction of $\tau(u)^2$ and $\tilde G(x)$ after removal of $G(x-u)^2$ as in (4.21)–(4.24).

Step 3. As a practice before considering more complicated cases, we first prove that the $N=1$ term in (4.17) is bounded by $2V^1(o,o;x)=2\tilde G(x)^3$, which is $2\times$  in (3.6). Since $\mathcal{L}^{(1)}_{[0,|\omega|]}=\{\{0|\omega|\}\}$, $\tilde V_{\boldsymbol{m}}(|\omega|)=\{x\}$ (cf., (4.12)) and $\mathbb{1}_{\{\mathsf{L}_{\boldsymbol{m},\boldsymbol{k}}=\Gamma\}}\le 1$, the $N=1$ term in (4.17) is bounded by

$$\sum_{\omega\in\Omega(o,x)}\sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde B_\omega}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde B_\omega\setminus B_\omega}}\frac{w_{\tilde B_\omega}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda}}\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\tilde B_\omega}:\\ \partial\boldsymbol{k}=\varnothing}} w_{\mathbb{B}_\Lambda\setminus\tilde B_\omega}(\boldsymbol{k})\sum_{u\in\tilde V_{\boldsymbol{m}}(0)}\mathbb{1}_{\{u\underset{\boldsymbol{k}}{\longleftrightarrow}x\}}. \tag{4.18}$$

We note that

$$\mathbb{1}_{\{u\in\tilde V_{\boldsymbol{m}}(0)\}}=\delta_{o,u}+\mathbb{1}_{\{u\in\tilde V_{\boldsymbol{m}}(0)\setminus\{o\}\}}. \tag{4.19}$$

By (4.6), the contribution from $\delta_{o,u}$ is bounded by $\tilde G(x)^3$, while the contribution from $\mathbb{1}_{\{u\in\tilde V_{\boldsymbol{m}}(0)\setminus\{o\}\}}$ is bounded as (see Figure 10)

$$\begin{aligned}
&\sum_u\sum_{\omega\in\Omega(o,x)}\sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde B_\omega}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde B_\omega\setminus B_\omega}}\frac{w_{\tilde B_\omega}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda}}\mathbb{1}_{\{u\in\tilde V_{\boldsymbol{m}}(0)\setminus\{o\}\}}\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\tilde B_\omega}:\\ \partial\boldsymbol{k}=\varnothing}} w_{\mathbb{B}_\Lambda\setminus\tilde B_\omega}(\boldsymbol{k})\,\mathbb{1}_{\{u\underset{\boldsymbol{k}}{\longleftrightarrow}x\}}\\
&\overset{(2.16)}{\le}\sum_u G(x-u)^2\sum_{\omega\in\Omega(o,x)}\sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde B_\omega}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde B_\omega\setminus B_\omega}}\frac{w_{\tilde B_\omega}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda}}Z_{\mathbb{B}_\Lambda\setminus\tilde B_\omega}\,\mathbb{1}_{\{u\in\tilde V_{\boldsymbol{m}}(0)\setminus\{o\}\}}\\
&\le\sum_u G(x-u)^2\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda}:\\ \partial\boldsymbol{n}=o\triangle x}}\frac{w_{\mathbb{B}_\Lambda}(\boldsymbol{n})}{Z_{\mathbb{B}_\Lambda}}\mathbb{1}_{\{n_{o,u}>0,\text{ even}\}},
\end{aligned}\tag{4.20}$$

where we have used $\mathbb{1}_{\{u\in\tilde{V}_{\boldsymbol{m}}(0)\setminus\{o\}\}} \le \mathbb{1}_{\{m_{o,u}>0,\text{ even}\}}$. Notice that

$$\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda}:\\ \partial\boldsymbol{n}=o\triangle x}} \frac{w_{\mathbb{B}_\Lambda}(\boldsymbol{n})}{Z_{\mathbb{B}_\Lambda}} \mathbb{1}_{\{n_{o,u}>0,\text{ even}\}} = \sum_{n_{o,u}>0,\text{ even}} \frac{(\beta J_{o,u})^{n_{o,u}}}{n_{o,u}!} \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\{o,u\}}:\\ \partial\boldsymbol{n}=o\triangle x}} \frac{w_{\mathbb{B}_\Lambda\setminus\{o,u\}}(\boldsymbol{n})}{Z_{\mathbb{B}_\Lambda}}$$

$$= \big(\cosh(\beta J_{o,u})-1\big) \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\{o,u\}}:\\ \partial\boldsymbol{n}=o\triangle x}} \frac{w_{\mathbb{B}_\Lambda\setminus\{o,u\}}(\boldsymbol{n})}{Z_{\mathbb{B}_\Lambda}}$$

$$= \big(\cosh(\beta J_{o,u})-1\big) \frac{Z_{\mathbb{B}_\Lambda\setminus\{o,u\}} \langle\varphi_o\varphi_x\rangle_{\mathbb{B}_\Lambda\setminus\{o,u\}}}{Z_{\mathbb{B}_\Lambda}}. \tag{4.21}$$

Since

$$Z_{\mathbb{B}_\Lambda} \ge Z_{\mathbb{B}_\Lambda\setminus\{o,u\}} \sum_{n_{o,u}\text{ even}} \frac{(\beta J_{o,u})^{n_{o,u}}}{n_{o,u}!} = Z_{\mathbb{B}_\Lambda\setminus\{o,u\}} \cosh(\beta J_{o,u}), \tag{4.22}$$

and since

$$\frac{\cosh(\beta J_{o,u})-1}{\cosh(\beta J_{o,u})} \le \frac{\cosh(\beta J_{o,u})-1}{\cosh(\beta J_{o,u})} \underbrace{\frac{\cosh(\beta J_{o,u})+1}{\cosh(\beta J_{o,u})}}_{\ge 1} = \frac{\sinh^2(\beta J_{o,u})}{\cosh^2(\beta J_{o,u})} = \tau(u)^2, \tag{4.23}$$

we obtain that, for $x \ne o$,

$$(4.20) \le \tilde{G}(x) \underbrace{\sum_u G(x-u)^2\, \tau(u)^2}_{\le \tilde{G}(x)^2} \le \tilde{G}(x)^3. \tag{4.24}$$

As a result, we arrive at

$$(4.18) \le 2\tilde{G}(x)^3 = 2V^1(o,o;x). \tag{4.25}$$

Step 4. Next we investigate the contribution to (4.17) from more general $N \ge 2$, which is bounded by

$$\sum_{\substack{\{u_j,v_j\}_{j=1}^N\\ \text{(no intersection)}}} \sum_{\omega\in\Omega(o,x)} \sum_{\{s_jt_j\}_{j=1}^N\in\mathcal{L}_{[0,|\omega|]}^{(N)}} \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde{B}_\omega}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde{B}_\omega\setminus B_\omega}} \frac{w_{\tilde{B}_\omega}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda}} \prod_{j=1}^N \mathbb{1}_{\{u_j\in\tilde{V}_{\boldsymbol{m}}(s_j)\}} \mathbb{1}_{\{v_j\in\tilde{V}_{\boldsymbol{m}}(t_j)\}}$$

$$\times \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}:\\ \partial\boldsymbol{k}=\varnothing}} w_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}(\boldsymbol{k})\, \mathbb{1}_{\{\mathsf{L}_{\boldsymbol{m},\boldsymbol{k}}=\{s_jt_j\}_{j=1}^N\}} \prod_{j=1}^N \mathbb{1}_{\{u_j\underset{\boldsymbol{k}}{\longleftrightarrow}v_j\}}, \tag{4.26}$$

where the first sum is over N pairs of vertices that do not intersect: $\{u_i,v_i\}\cap\{u_j,v_j\}=\varnothing$ for $i \ne j$. This constraint is due to the compatibility with the condition that $\mathsf{L}_{\boldsymbol{m},\boldsymbol{k}} =$

$\{s_jt_j\}_{j=1}^N$. For example, if the lace is given as in Figure 9, then it looks as if the coincidence $v_2 = u_3$ is possible, but it is not; if $v_2 = u_3$, then u_2 is directly connected to x by using bonds b with $k_b > 0$, which implies that the number of lace edges is two, not three as suggested in Figure 9. Therefore, the constraint in the first sum in (4.26) is unnecessary as long as we keep the indicator $\mathbb{1}_{\{\mathsf{L}_{\boldsymbol{m},\boldsymbol{k}}=\{s_jt_j\}_{j=1}^N\}}$. However, since we drop this indicator at some point in the following analysis, we keep this constraint for now.

Let $N = 2$, for example, and let $\Gamma = \{st, s't'\} = \{0t, s'|\omega|\}$. Then the sum over $\boldsymbol{k}$ is (n.b., $\tilde{V}_{\boldsymbol{m}}(|\omega|) = \{x\}$)

$$\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}:\\ \partial\boldsymbol{k}=\varnothing}} w_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}(\boldsymbol{k})\, \mathbb{1}_{\{\mathsf{L}_{\boldsymbol{m},\boldsymbol{k}}=\{0t,s'|\omega|\}\}} \mathbb{1}_{\{u_1 \underset{\boldsymbol{k}}{\longleftrightarrow} v_1\}}\, \mathbb{1}_{\{u_2 \underset{\boldsymbol{k}}{\longleftrightarrow} x\}}. \tag{4.27}$$

Notice that, under the constraint $\mathsf{L}_{\boldsymbol{m},\boldsymbol{k}} = \{0t, s'|\omega|\}$ (recall (4.13)–(4.15), where lace edges are defined by min/max of connectivity), the clusters $\mathcal{C}_{\boldsymbol{k}}(u_j) = \{y : u_j \underset{\boldsymbol{k}}{\longleftrightarrow} y\}$ are disjoint, i.e., $\mathcal{C}_{\boldsymbol{k}}(u_1) \cap \mathcal{C}_{\boldsymbol{k}}(u_2) = \varnothing$; otherwise the number N of lace edges is reduced to 1, which is a contradiction to the present situation: $N = 2$. Therefore, we can condition on $\mathcal{C}_{\boldsymbol{k}}(u_1)$, say, $\mathcal{C}_{\boldsymbol{k}}(u_1) = A$ ($\ni v_1$), split the weight $w_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}(\boldsymbol{k})$ as $w_{\mathbb{B}_{\bar{A}}}(\boldsymbol{k}_1)\, w_{\mathbb{B}_{A^{\rm c}}}(\boldsymbol{k}_2)$, where $\mathbb{B}_{\bar{A}}$ is an abbreviation for $\mathbb{B}_\Lambda \setminus \tilde{B}_\omega \setminus \mathbb{B}_{A^{\rm c}}$ (recall that $\mathbb{B}_{A^{\rm c}}$ is the set of bonds whose end vertices are both in $A^{\rm c}$, as defined above (3.7)), and then sum over $\boldsymbol{k}_2$ to obtain

$$\begin{aligned}
(4.27) \quad &\le \sum_{A\ni v_1} \sum_{\substack{\boldsymbol{k}_1\in\mathbb{Z}_+^{\mathbb{B}_{\bar{A}}}:\\ \partial\boldsymbol{k}_1=\varnothing}} w_{\mathbb{B}_{\bar{A}}}(\boldsymbol{k}_1)\, \mathbb{1}_{\{\mathcal{C}_{\boldsymbol{k}_1}(u_1)=A\}} \sum_{\substack{\boldsymbol{k}_2\in\mathbb{Z}_+^{\mathbb{B}_{A^{\rm c}}}:\\ \partial\boldsymbol{k}_2=\varnothing}} w_{\mathbb{B}_{A^{\rm c}}}(\boldsymbol{k}_2)\, \mathbb{1}_{\{u_2 \underset{\boldsymbol{k}_2}{\longleftrightarrow} x\}} \\
&\overset{(2.16)}{\le} G(x-u_2)^2 \sum_{A\ni v_1} \sum_{\substack{\boldsymbol{k}_1\in\mathbb{Z}_+^{\mathbb{B}_{\bar{A}}}:\\ \partial\boldsymbol{k}_1=\varnothing}} w_{\mathbb{B}_{\bar{A}}}(\boldsymbol{k}_1)\, Z_{\mathbb{B}_{A^{\rm c}}}\, \mathbb{1}_{\{\mathcal{C}_{\boldsymbol{k}_1}(u_1)=A\}} \\
&= G(x-u_2)^2 \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}:\\ \partial\boldsymbol{k}=\varnothing}} w_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}(\boldsymbol{k})\, \mathbb{1}_{\{u_1 \underset{\boldsymbol{k}}{\longleftrightarrow} v_1\}} \\
&\overset{(2.16)}{\le} G(x-u_2)^2\, G(v_1-u_1)^2\, Z_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}.
\end{aligned} \tag{4.28}$$

It is easy to extend the above analysis to more general $N \geq 3$. As a result, we obtain (see Figure 11)

$$\begin{aligned}
(4.26) \le \sum_{\substack{\{u_j,v_j\}_{j=1}^N \\ \text{(no intersection)}}} \prod_{i=1}^N G(v_i-u_i)^2 \sum_{\omega\in\Omega(o,x)} \sum_{\{s_jt_j\}_{j=1}^N\in\mathcal{L}_{[0,|\omega|]}^{(N)}} \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde{B}_\omega}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde{B}_\omega\setminus B_\omega}} \frac{w_{\tilde{B}_\omega}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda}} Z_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega} \\
\times \prod_{j=1}^N \mathbb{1}_{\{u_j\in\tilde{V}_{\boldsymbol{m}}(s_j)\}}\, \mathbb{1}_{\{v_j\in\tilde{V}_{\boldsymbol{m}}(t_j)\}}.
\end{aligned} \tag{4.29}$$

Step 5. We complete the proof for $N \geq 2$ by first extracting $2N-1$ factors of $\delta+\tau^2$ and then extracting $2N-1$ two-point functions from the sum over $\omega \in \Omega(o,x)$ in (4.29), just as

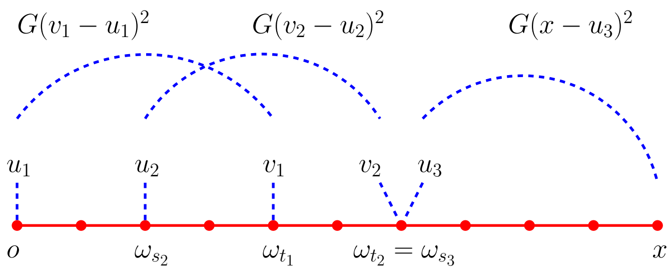


Figure 11: Explanation of Step 4: extracting the two-point functions $\prod_{i=1}^{3} G(v_i-u_i)^2$ in (4.29) from (4.26) with three lace edges (cf., Figure 9).

done for $N=1$ in (4.21)–(4.24). To do so, we first use $\mathbb{1}_{\{u\in\tilde{V}_{\boldsymbol{m}}(s)\}}=\delta_{u,\omega_s}+\mathbb{1}_{\{u\in\tilde{V}_{\boldsymbol{m}}(s)\setminus\{\omega_s\}\}}$ (cf., (4.19)) to rewrite the sum over $\omega\in\Omega(o,x)$ in (4.29) as

$$\sum_{\omega\in\Omega(o,x)}\sum_{\{s_jt_j\}_{j=1}^N\in\mathcal{L}^{(N)}_{[0,|\omega|]}}\sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde{B}_\omega}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde{B}_\omega\setminus B_\omega}}\frac{w_{\tilde{B}_\omega}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda}}Z_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}$$
$$\times\prod_{j=1}^N\Big(\delta_{u_j,\omega_{s_j}}+\mathbb{1}_{\{u_j\in\tilde{V}_{\boldsymbol{m}}(s_j)\setminus\{\omega_{s_j}\}\}}\Big)\Big(\delta_{v_j,\omega_{t_j}}+\mathbb{1}_{\{v_j\in\tilde{V}_{\boldsymbol{m}}(t_j)\setminus\{\omega_{t_j}\}\}}\Big)$$
$$=\sum_{\omega\in\Omega(o,x)}\sum_{\{s_jt_j\}_{j=1}^N\in\mathcal{L}^{(N)}_{[0,|\omega|]}}\sum_{\substack{I,J\subset[N]:\\ N\in J^{\rm c}}}\bigg(\prod_{i\in I^{\rm c}}\delta_{u_i,\omega_{s_i}}\prod_{j\in J^{\rm c}}\delta_{v_j,\omega_{t_j}}\bigg)$$
$$\times\sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde{B}_\omega}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde{B}_\omega\setminus B_\omega}}\frac{w_{\tilde{B}_\omega}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda}}Z_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}\prod_{\substack{i\in I,\\ j\in J}}\mathbb{1}_{\{u_i\in\tilde{V}_{\boldsymbol{m}}(s_i)\setminus\{\omega_{s_i}\}\}}\mathbb{1}_{\{v_j\in\tilde{V}_{\boldsymbol{m}}(t_j)\setminus\{\omega_{t_j}\}\}},\tag{4.30}$$

where $[N]=\{1,\dots,N\}$, and $J^{\rm c}$ is an abbreviation for $[N]\setminus J$. Since $\{u_j,v_j\}_{j=1}^N$ are N pairs of vertices that do not intersect, the bonds in $\tilde{B}_{I,J}=\{\{\omega_{s_i},u_i\},\{\omega_{t_j},v_j\}\}_{i\in I,j\in J}\subset\tilde{B}_\omega\setminus B_\omega$ (depicted as dashed short line segments in Figure 11) are distinct. Then, by the same analysis as in (4.21)–(4.24), the last line of (4.30) is bounded above by

$$\prod_{\substack{i\in I,\\ j\in J}}\tau(\omega_{s_i}-u_i)^2\,\tau(\omega_{t_j}-v_j)^2\sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde{B}_\omega\setminus\tilde{B}_{I,J}}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde{B}_\omega\setminus\tilde{B}_{I,J}\setminus B_\omega}}\frac{w_{\tilde{B}_\omega\setminus\tilde{B}_{I,J}}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda\setminus\tilde{B}_{I,J}}}Z_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}.\tag{4.31}$$

Therefore, by changing the order of summations, we obtain

$$(4.30)\le\sum_{\substack{y_1,\dots,y_N;\\ z_1,\dots,z_N:\\ y_1=o,\\ z_N=x}}\sum_{I\subset[N]}\bigg(\prod_{i\in I^{\rm c}}\delta_{u_i,y_i}\prod_{i'\in I}\tau(y_{i'}-u_{i'})^2\bigg)\sum_{\substack{J\subset[N]:\\ N\in J^{\rm c}}}\bigg(\prod_{j\in J^{\rm c}}\delta_{v_j,z_j}\prod_{j'\in J}\tau(z_{j'}-v_{j'})^2\bigg)$$

$$\times \sum_{\omega\in\Omega(o,x)} \sum_{\{s_jt_j\}_{j=1}^N \in \mathcal{L}^{(N)}_{[0,|\omega|]}} \prod_{j=1}^N \delta_{\omega_{s_j},y_j}\delta_{\omega_{t_j},z_j} \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde{B}_\omega\setminus\tilde{B}_{I,J}}: \\ \text{odd on } B_\omega, \\ \text{even on } \tilde{B}_\omega\setminus\tilde{B}_{I,J}\setminus B_\omega}} \frac{w_{\tilde{B}_\omega\setminus\tilde{B}_{I,J}}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda\setminus\tilde{B}_{I,J}}} Z_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}. \tag{4.32}$$

The remaining task is to extract $2N-1$ two-point functions one by one from the second line of (4.32). To do so, we first split the sum over $\omega \in \Omega(o,x)$ into $\xi \in \Omega(o, z_{N-1})$ with $\xi \not\ni x$ and $\eta \in \tilde{\Omega}_{\xi,I,J}(z_{N-1},x)$, where

$$\tilde{\Omega}_{\xi,I,J}(z,x) = \big\{\eta \in \Omega(z,x) : \forall j=1,\dots,|\eta|,\ \{\eta_{j-1},\eta_j\} \notin \tilde{B}_\xi \cup \tilde{B}_{I,J}\big\}, \tag{4.33}$$

which is the set of nonzero paths from z to x ($\neq z$, due to the definition of $\Omega(z,x)$ in (4.1)) consisting of bonds that are not yet explored by ξ or used to extract τ^2 as in (4.32). Then, by splitting the weight $w_{\tilde{B}_\omega\setminus\tilde{B}_{I,J}}(\boldsymbol{m})$ as $w_{\tilde{B}_\xi\setminus\tilde{B}_{I,J}}(\boldsymbol{k})\, w_{\tilde{B}_\eta\setminus(\tilde{B}_\xi\cup\tilde{B}_{I,J})}(\boldsymbol{m})$ and multiplying $1 = Z_{\mathbb{B}_\Lambda\setminus(\tilde{B}_\xi\cup\tilde{B}_{I,J})}/Z_{\mathbb{B}_\Lambda\setminus(\tilde{B}_\xi\cup\tilde{B}_{I,J})}$, the second line of (4.32) equals

$$\begin{aligned}
&\sum_{\substack{\xi\in\Omega(o,z_{N-1}): \\ \xi\not\ni x}} \sum_{\substack{\{s_jt_j\}_{j=1}^{N-1} \in \mathcal{L}^{(N-1)}_{[0,|\xi|]}, \\ s_N\in(t_{N-2},t_{N-1}]}} \prod_{j=1}^{N-1} \delta_{\xi_{s_j},y_j}\delta_{\xi_{t_j},z_j}\delta_{\xi_{s_N},y_N} \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\tilde{B}_\xi\setminus\tilde{B}_{I,J}}: \\ \text{odd on } B_\xi, \\ \text{even on } \tilde{B}_\xi\setminus\tilde{B}_{I,J}\setminus B_\xi}} \frac{w_{\tilde{B}_\xi\setminus\tilde{B}_{I,J}}(\boldsymbol{k})}{Z_{\mathbb{B}_\Lambda\setminus\tilde{B}_{I,J}}} Z_{\mathbb{B}_\Lambda\setminus(\tilde{B}_\xi\cup\tilde{B}_{I,J})} \\
&\times \mathbb{1}_{\{z_{N-1}\neq x\}} \sum_{\eta\in\tilde{\Omega}_{\xi,I,J}(z_{N-1},x)} \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde{B}_\eta\setminus(\tilde{B}_\xi\cup\tilde{B}_{I,J})}: \\ \text{odd on } B_\eta, \\ \text{even on } \tilde{B}_\eta\setminus(\tilde{B}_\xi\cup\tilde{B}_{I,J})\setminus B_\eta}} \frac{w_{\tilde{B}_\eta\setminus(\tilde{B}_\xi\cup\tilde{B}_{I,J})}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda\setminus(\tilde{B}_\xi\cup\tilde{B}_{I,J})}} Z_{\mathbb{B}_\Lambda\setminus(\tilde{B}_\xi\cup\tilde{B}_\eta)}.
\end{aligned} \tag{4.34}$$

Since

$$\big(\tilde{B}_\eta \setminus (\tilde{B}_\xi \cup \tilde{B}_{I,J})\big) \cap \big(\mathbb{B}_\Lambda \setminus (\tilde{B}_\xi \cup \tilde{B}_\eta)\big) = \varnothing, \tag{4.35}$$

$$\big(\tilde{B}_\eta \setminus (\tilde{B}_\xi \cup \tilde{B}_{I,J})\big) \cup \big(\mathbb{B}_\Lambda \setminus (\tilde{B}_\xi \cup \tilde{B}_\eta)\big) = \mathbb{B}_\Lambda \setminus (\tilde{B}_\xi \cup \tilde{B}_{I,J}), \tag{4.36}$$

and since η is the earliest path of odd current in the restricted region $\mathbb{B}_\Lambda \setminus (\tilde{B}_\xi \cup \tilde{B}_{I,J})$, the last line of (4.34) is exactly equal to $\langle\varphi_{z_{N-1}}\varphi_x\rangle_{\mathbb{B}_\Lambda\setminus\tilde{B}_{I,J}\setminus\tilde{B}_\xi}$, which is further bounded by $\tilde{G}(x-z_{N-1})$ for $z_{N-1}\neq x$. Repeating the above analysis to extract two-point functions one by one and using $\mathbb{1}_{\{\xi\not\ni x\}} \le \mathbb{1}_{\{\xi_{s_N}\neq x\}}$, we obtain

$$(4.34) \le \tilde{G}(y_2) \prod_{i=1}^{N-1} G(z_i - y_{i+1}) \prod_{j=2}^{N-1} \tilde{G}(y_{j+1}-z_{j-1})\, \tilde{G}(x-z_{N-1})\, \mathbb{1}_{\{y_N\neq x\}}, \tag{4.37}$$

where we have used (2.14) to gain $\tilde{G}$ instead of G for $y_2 \neq o$ and $y_{j+1} \neq z_{j-1}$ for all $j = 2,\dots,N-1$, due to the construction (4.16) of lace edges: each $\tilde{G}$ (resp., G) in (4.37) corresponds to a strict inequality (resp., an inequality) in (4.16). The empty product

$\prod_{j=2}^{0}$ is regarded as 1 by convention, as always. As a result, we obtain

$$(4.32) \le \sum_{\substack{y_1,\dots,y_N,\\ z_0,\dots,z_N:\\ y_1=z_0=o,\\ z_N=x}} \prod_{j=1}^{N-1} G(z_j-y_{j+1})\,\tilde G(y_{j+1}-z_{j-1})\,\tilde G(x-z_{N-1})\,\mathbb{1}_{\{y_N\ne x\}}$$
$$\times \underbrace{\sum_{I\subset[N]}\bigg(\prod_{i\in I^{\rm c}}\delta_{u_i,y_i}\prod_{i'\in I}\tau(y_{i'}-u_{i'})^2\bigg)}_{\prod_{j=1}^{N}(\delta+\tau^2)(y_j-u_j)}\underbrace{\sum_{\substack{J\subset[N]:\\ N\in J^{\rm c}}}\bigg(\prod_{j\in J^{\rm c}}\delta_{v_j,z_j}\prod_{j'\in J}\tau(z_{j'}-v_{j'})^2\bigg)}_{\prod_{j=1}^{N-1}(\delta+\tau^2)(z_j-v_j)\,\delta_{v_N,x}}, \tag{4.38}$$

hence

$$(4.29) \le \sum_{\substack{y_1,\dots,y_N,\\ z_0,\dots,z_{N-1}:\\ y_1=z_0=o}} \prod_{j=1}^{N-1} \underbrace{\Big((\delta+\tau^2)*G^2*(\delta+\tau^2)\Big)(z_j-y_j)\,G(z_j-y_{j+1})\,\tilde G(y_{j+1}-z_{j-1})}_{\le U^1(z_{j-1},y_j;z_j,y_{j+1})\ (\because (2.14))}$$
$$\times \underbrace{\tilde G(x-z_{N-1})\,\Big((\delta+\tau^2)*G^2\Big)(x-y_N)\,\mathbb{1}_{\{y_N\ne x\}}}_{\le 2V^1(z_{N-1},y_N;x)\ (\because (4.25))}$$
$$\le 2\Big((U^1)^{\star(N-1)}\star V^1\Big)_{o,x}. \tag{4.39}$$

Combining this for $N\ge 2$ with (4.25) for $N=1$ and recalling the definition (3.5) of $X^1_{o,x}$, we complete the proof of Theorem 3.1. ■

4.2 Proof of Theorem 3.2

Recall the definition (3.7) of $\Theta'_{o,x;A}$:

$$\Theta'_{o,x;A} = \sum_{\substack{\boldsymbol{\ell}\in\mathbb{Z}_+^{\mathbb{B}_{A^{\rm c}}}:\\ \partial\boldsymbol{\ell}=\varnothing}} \frac{w_{\mathbb{B}_{A^{\rm c}}}(\boldsymbol{\ell})}{Z_{\mathbb{B}_{A^{\rm c}}}} \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda}:\\ \partial\boldsymbol{n}=o\triangle x}} \frac{w_{\mathbb{B}_\Lambda}(\boldsymbol{n})}{Z_{\mathbb{B}_\Lambda}}\,\mathbb{1}_{\{o\underset{\boldsymbol{n}+\boldsymbol{\ell}}{\overset{A}{\Longleftrightarrow}}x\}}. \tag{4.40}$$

Since it is similar to $\pi^{(0)}_{\mathbb{B}_\Lambda}(x)$, we can follow the same line of proof as explained in the previous subsection, by taking note of the following two differences:

(i) All paths from o to x with positive current in the superposition of two current configurations must go through the vertex set A, so that the earliest path $\omega\in\Omega(o,x)$ of odd current also contains a vertex in A.

(ii) A double connection from o to x is achieved by the superposition of two current configurations, not by a single current configuration as in $\pi^{(0)}_{\mathbb{B}_\Lambda}(x)$, and one of them is defined in the restricted region $\mathbb{B}_{A^{\rm c}}$.

Now we begin the proof of Theorem 3.2. First, by identifying the earliest path $\omega\in\Omega(o,x)$ of bonds b with odd n_b (as done in Step 1 of the previous subsection) and then relaxing the through-A condition to $\omega \cap A \neq \varnothing$, we obtain the following inequality similar to (4.4):

$$\Theta'_{o,x;A} \le \sum_{\substack{\omega\in\Omega(o,x):\\ \omega\cap A\neq\varnothing}} \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde{B}_\omega}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde{B}_\omega\setminus B_\omega}} \frac{w_{\tilde{B}_\omega}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda}} \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}:\\ \partial\boldsymbol{k}=\varnothing}} w_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}(\boldsymbol{k}) \sum_{\substack{\boldsymbol{\ell}\in\mathbb{Z}_+^{\mathbb{B}_{A^{\rm c}}}:\\ \partial\boldsymbol{\ell}=\varnothing}} \frac{w_{\mathbb{B}_{A^{\rm c}}}(\boldsymbol{\ell})}{Z_{\mathbb{B}_{A^{\rm c}}}} \mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}+\boldsymbol{\ell}}{\Longleftrightarrow} x\}}. \tag{4.41}$$

Then, by using the double expansion with a lace $\mathsf{L}_{\boldsymbol{m},\boldsymbol{k}+\boldsymbol{\ell}}$ (as done in Step 2 of the previous subsection), we obtain the following inequality similar to (4.17):

$$\begin{aligned}
\Theta'_{o,x;A} &\le \sum_{N=1}^{\infty} \sum_{\substack{\omega\in\Omega(o,x):\\ \omega\cap A\neq\varnothing}} \sum_{\Gamma\in\mathcal{L}^{(N)}_{[0,|\omega|]}} \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde{B}_\omega}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde{B}_\omega\setminus B_\omega}} \frac{w_{\tilde{B}_\omega}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda}} \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}:\\ \partial\boldsymbol{k}=\varnothing}} w_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}(\boldsymbol{k}) \sum_{\substack{\boldsymbol{\ell}\in\mathbb{Z}_+^{\mathbb{B}_{A^{\rm c}}}:\\ \partial\boldsymbol{\ell}=\varnothing}} \frac{w_{\mathbb{B}_{A^{\rm c}}}(\boldsymbol{\ell})}{Z_{\mathbb{B}_{A^{\rm c}}}} \\
&\qquad \times \mathbb{1}_{\{\mathsf{L}_{\boldsymbol{m},\boldsymbol{k}+\boldsymbol{\ell}}=\Gamma\}} \prod_{st\in\Gamma} \mathbb{1}_{\{\tilde{V}_{\boldsymbol{m}}(s) \underset{\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} \tilde{V}_{\boldsymbol{m}}(t) \text{ in } \mathbb{B}_\Lambda\setminus\tilde{B}_\omega\}} \\
&\le \sum_{N=1}^{\infty} \sum_{\substack{\{u_j,v_j\}_{j=1}^N\\ \text{(no intersection)}}} \sum_{\substack{\omega\in\Omega(o,x):\\ \omega\cap A\neq\varnothing}} \sum_{\{s_jt_j\}_{j=1}^N\in\mathcal{L}^{(N)}_{[0,|\omega|]}} \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde{B}_\omega}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde{B}_\omega\setminus B_\omega}} \frac{w_{\tilde{B}_\omega}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda}} \prod_{j=1}^N \mathbb{1}_{\{u_j\in\tilde{V}_{\boldsymbol{m}}(s_j)\}} \mathbb{1}_{\{v_j\in\tilde{V}_{\boldsymbol{m}}(t_j)\}} \\
&\qquad \times \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}:\\ \partial\boldsymbol{k}=\varnothing}} w_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}(\boldsymbol{k}) \sum_{\substack{\boldsymbol{\ell}\in\mathbb{Z}_+^{\mathbb{B}_{A^{\rm c}}}:\\ \partial\boldsymbol{\ell}=\varnothing}} \frac{w_{\mathbb{B}_{A^{\rm c}}}(\boldsymbol{\ell})}{Z_{\mathbb{B}_{A^{\rm c}}}} \mathbb{1}_{\{\mathsf{L}_{\boldsymbol{m},\boldsymbol{k}+\boldsymbol{\ell}}=\{s_jt_j\}_{j=1}^N\}} \prod_{j=1}^N \mathbb{1}_{\{u_j \underset{\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} v_j \text{ in } \mathbb{B}_\Lambda\setminus\tilde{B}_\omega\}}.
\end{aligned} \tag{4.42}$$

However, we cannot use (2.16) here to extract $Z_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}\prod_{j=1}^N G(v_j-u_j)^2$ from the double sum over $\boldsymbol{k},\boldsymbol{\ell}$ (as done in Step 3 and Step 4 in the previous subsection), due to the difference (ii) mentioned above. Instead, as described in (2.17), the last line of (4.42) is bounded by chains of nonzero bubbles $Z_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}\prod_{j=1}^N\sum_{i=0}^\infty(\tilde{G}^2)^{*i}(v_j-u_j)$. Then, we obtain the following inequality similar to (4.29):

$$\begin{aligned}
\Theta'_{o,x;A} &\le \sum_{N=1}^{\infty} \sum_{\substack{\{u_j,v_j\}_{j=1}^N\\ \text{(no intersection)}}} \prod_{j=1}^N \sum_{i_j=0}^{\infty} (\tilde{G}^2)^{*i_j}(v_j-u_j) \sum_{\substack{\omega\in\Omega(o,x):\\ \omega\cap A\neq\varnothing}} \sum_{\{s_jt_j\}_{j=1}^N\in\mathcal{L}^{(N)}_{[0,|\omega|]}} \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde{B}_\omega}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde{B}_\omega\setminus B_\omega}} \frac{w_{\tilde{B}_\omega}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda}} \\
&\qquad \times Z_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega} \prod_{j=1}^N \mathbb{1}_{\{u_j\in\tilde{V}_{\boldsymbol{m}}(s_j)\}} \mathbb{1}_{\{v_j\in\tilde{V}_{\boldsymbol{m}}(t_j)\}}.
\end{aligned} \tag{4.43}$$

The remaining task is to extract two-point functions and factors of $\delta+\tau^2$ from the above sum over ω, as done in Step 5 of the previous subsection. However, since $\omega\cap A\neq\varnothing$, among $2N-1$ segments $\{\omega_{[0,s_2)},\omega_{[s_2,t_1)},\omega_{[t_1,s_3)},\dots,\omega_{[s_N,t_{N-1})},\omega_{[t_{N-1},|\omega|]}\}$, where $\omega_{[s,t)}=(\omega_s,\omega_{s+1},\dots,\omega_{t-1})$ and $\omega_{[t_{N-1},|\omega|]}=\omega_{[t_{N-1},|\omega|)}\cup\{\omega_{|\omega|}\}$, there is always a segment that contains a vertex $a \in A$. Therefore, to bound (4.43), we replace the product of $2N-1$ two-point functions in (4.38), i.e.,

$$\prod_{j=1}^{N-1} G(z_j - y_{j+1})\,\tilde{G}(y_{j+1} - z_{j-1})\,\tilde{G}(x - z_{N-1}), \qquad \text{where} \quad z_0 = o, \tag{4.44}$$

by

$$\begin{aligned}&\sum_{a\in A}\Bigg(\prod_{j=1}^{N-1} G(z_j - y_{j+1})\,\tilde{G}(y_{j+1} - z_{j-1})\Big(G(a - z_{N-1})\,\tilde{G}(x-a) + G(x - z_{N-1})\,\delta_{a,x}\Big)\\ &\quad+\sum_{i=1}^{N-1}\Big(G(a-y_{i+1})\,\tilde{G}(z_i - a)\,\tilde{G}(y_{i+1} - z_{i-1}) + G(z_i - y_{i+1})\,G(a - z_{i-1})\,\tilde{G}(y_{i+1} - a)\Big)\\ &\quad\times\prod_{\substack{j=1\\(j\neq i)}}^{N-1} G(z_j - y_{j+1})\,\tilde{G}(y_{j+1} - z_{j-1})\,\tilde{G}(x - z_{N-1})\Bigg),\end{aligned} \tag{4.45}$$

which is obtained by applying (2.15) to each segment. Assembling all the above estimates yields the wanted bound (3.11), just as done in (4.39) for $\pi^{(0)}_{\mathbb{B}_\Lambda}(x)$. ■

4.3 Proof of Theorem 3.3

Recall the definition (3.12) of $\tilde{\pi}^{(0)}_{\mathbb{B}_\Lambda;y}(x)$:

$$\tilde{\pi}^{(0)}_{\mathbb{B}_\Lambda;y}(x) = \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda}:\\ \partial\boldsymbol{n}=o\triangle x}} \frac{w_{\mathbb{B}_\Lambda}(\boldsymbol{n})}{Z_{\mathbb{B}_\Lambda}}\,\mathbb{1}_{\{o \underset{\boldsymbol{n}}{\Longleftrightarrow} x\}\cap\{o \underset{\boldsymbol{n}}{\longleftrightarrow} y\}}. \tag{4.46}$$

This looks simpler than $\Theta'_{o,x;A}$, as we only need to control one current configuration, not two. It turns out to be a little more involved, due to the extra $\mathbb{1}_{\{o \underset{\boldsymbol{n}}{\longleftrightarrow} y\}}$, as explained now.

First, by identifying the earliest path $\omega \in \Omega(o,x)$ of bonds b with odd n_b (as done in Step 1 of Section 4.1), we can rewrite $\tilde{\pi}^{(0)}_{\mathbb{B}_\Lambda;y}(x)$ as (cf., (4.4))

$$\tilde{\pi}^{(0)}_{\mathbb{B}_\Lambda;y}(x) = \sum_{\omega\in\Omega(o,x)} \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde{B}_\omega}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde{B}_\omega\setminus B_\omega}} \frac{w_{\tilde{B}_\omega}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda}} \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}:\\ \partial\boldsymbol{k}=\varnothing}} w_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}(\boldsymbol{k})\,\mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}}{\Longleftrightarrow} x\}\cap\{o \underset{\boldsymbol{m}+\boldsymbol{k}}{\longleftrightarrow} y\}}. \tag{4.47}$$

Then, by the double expansion as in Step 2 of Section 4.1, we obtain (see (4.17) for the equality below and then (4.26) for the inequality)

$$
\begin{aligned}
(4.47) &= \sum_{N=1}^{\infty} \sum_{\omega \in \Omega(o,x)} \sum_{\Gamma \in \mathcal{L}^{(N)}_{[0,|\omega|]}} \sum_{\substack{\boldsymbol{m} \in \mathbb{Z}_+^{\tilde{B}_\omega}: \\ \text{odd on } B_\omega, \\ \text{even on } \tilde{B}_\omega \setminus B_\omega}} \frac{w_{\tilde{B}_\omega}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda}} \\
&\quad \times \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}_+^{\mathbb{B}_\Lambda \setminus \tilde{B}_\omega}: \\ \partial \boldsymbol{k} = \varnothing}} w_{\mathbb{B}_\Lambda \setminus \tilde{B}_\omega}(\boldsymbol{k}) \, \mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}}{\longleftrightarrow} y\}} \, \mathbb{1}_{\{\mathsf{L}_{\boldsymbol{m},\boldsymbol{k}} = \Gamma\}} \prod_{st \in \Gamma} \mathbb{1}_{\{\tilde{V}_{\boldsymbol{m}}(s) \underset{\boldsymbol{k}}{\longleftrightarrow} \tilde{V}_{\boldsymbol{m}}(t)\}} \\
&\le \sum_{N=1}^{\infty} \sum_{\substack{\{u_j,v_j\}_{j=1}^N \\ \text{(no intersection)}}} \sum_{\omega \in \Omega(o,x)} \sum_{\{s_j t_j\}_{j=1}^N \in \mathcal{L}^{(N)}_{[0,|\omega|]}} \sum_{\substack{\boldsymbol{m} \in \mathbb{Z}_+^{\tilde{B}_\omega}: \\ \text{odd on } B_\omega, \\ \text{even on } \tilde{B}_\omega \setminus B_\omega}} \frac{w_{\tilde{B}_\omega}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda}} \prod_{j=1}^{N} \mathbb{1}_{\{u_j \in \tilde{V}_{\boldsymbol{m}}(s_j)\}} \, \mathbb{1}_{\{v_j \in \tilde{V}_{\boldsymbol{m}}(t_j)\}} \\
&\quad \times \sum_{\substack{\boldsymbol{k} \in \mathbb{Z}_+^{\mathbb{B}_\Lambda \setminus \tilde{B}_\omega}: \\ \partial \boldsymbol{k} = \varnothing}} w_{\mathbb{B}_\Lambda \setminus \tilde{B}_\omega}(\boldsymbol{k}) \, \mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}}{\longleftrightarrow} y\}} \, \mathbb{1}_{\{\mathsf{L}_{\boldsymbol{m},\boldsymbol{k}} = \{s_j t_j\}_{j=1}^N\}} \prod_{st \in \Gamma} \mathbb{1}_{\{u_j \underset{\boldsymbol{k}}{\longleftrightarrow} v_j\}}. \qquad (4.48)
\end{aligned}
$$

Next we investigate the effect of the indicator $\mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}}{\longleftrightarrow} y\}}$. Since $\{\mathcal{C}_{\boldsymbol{k}}(u_j)\}_{j=1}^N$ are disjoint, i.e., $\mathcal{C}_{\boldsymbol{k}}(u_i) \cap \mathcal{C}_{\boldsymbol{k}}(u_j) = \varnothing$ for $i \neq j$, we have the rewrite

$$
\mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}}{\longleftrightarrow} y\}} = \sum_{i=1}^{N} \mathbb{1}_{\{u_i \underset{\boldsymbol{k}}{\longleftrightarrow} y\}} + \mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}}{\longleftrightarrow} y\} \setminus \bigcup_{i=1}^N \{u_i \underset{\boldsymbol{k}}{\longleftrightarrow} y\}}. \qquad (4.49)
$$

By conditioning on clusters (as done in Steps 3 & 4 in Section 4.1) and using Lemma 2.4, the contribution to (4.48) from $\mathbb{1}_{\{u_N \underset{\boldsymbol{k}}{\longleftrightarrow} y\}}$ is bounded by (cf., (4.39))

$$
\big((U^1)^{\star(N-1)} \star \ddot{V}^1_y\big)_{o,x} = \sum_{\substack{y_1,\dots,y_N, \\ z_0,\dots,z_{N-1}: \\ y_1 = z_0 = o}} \prod_{j=1}^{N-1} U^1(z_{j-1}, y_j; z_j, y_{j+1}) \, \ddot{V}^1_y(z_{N-1}, y_N; x), \qquad (4.50)
$$

while the contribution from each $\mathbb{1}_{\{u_i \underset{\boldsymbol{k}}{\longleftrightarrow} y\}}$ with $i < N$ is bounded by

$$
2\Big((U^1)^{\star(i-1)} \star \ddot{U}^1_y \star (U^1)^{\star(N-1-i)} \star V^1\Big)_{o,x}, \qquad (4.51)
$$

where $\ddot{U}^1_y$ and $\ddot{V}^1_y$ are defined in (3.15)–(3.16).

To bound the contribution to (4.48) from $\mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}}{\longleftrightarrow} y\} \setminus \bigcup_{j=1}^N \{u_j \underset{\boldsymbol{k}}{\longleftrightarrow} y\}}$ in (4.49) is not much difficult, as we can follow the same line up to (4.32), with its second line replaced by

$$\sum_{\omega\in\Omega(o,x)}\sum_{\{s_jt_j\}_{j=1}^N\in\mathcal{L}^{(N)}_{[0,|\omega|]}}\prod_{j=1}^N\delta_{\omega_{s_j},y_j}\delta_{\omega_{t_j},z_j}\sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde{B}_\omega\setminus\tilde{B}_{I,J}}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde{B}_\omega\setminus\tilde{B}_{I,J}\setminus B_\omega}}\frac{w_{\tilde{B}_\omega\setminus\tilde{B}_{I,J}}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda\setminus\tilde{B}_{I,J}}}$$
$$\times\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}:\\ \partial\boldsymbol{k}=\varnothing}}w_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}(\boldsymbol{k})\sum_{i=1}^{2N-1}\mathbb{1}_{\{\omega_{I_i}\underset{\boldsymbol{m}+\boldsymbol{k}}{\longleftrightarrow}y\}\setminus\bigcup_{j>i}\{\omega_{I_j}\underset{\boldsymbol{m}+\boldsymbol{k}}{\longleftrightarrow}y\}},\tag{4.52}$$

where $\omega_I=(\omega_s,\dots,\omega_{t-1})$ for $I=[s,t)$ (cf., below (4.43)) and

$$(I_1,I_2,I_3,\dots,I_{2N-2},I_{2N-1})=([0,s_2),[s_2,t_1),[t_1,s_3),\dots,[s_N,t_{N-1}),[t_{N-1},|\omega|]).\tag{4.53}$$

Then, by repeated applications of conditioning on clusters to extract two-point functions one by one (as done in showing (4.37)) and using (2.15) to deal with the indicator $\mathbb{1}_{\{\omega_{I_i}\underset{\boldsymbol{m}+\boldsymbol{k}}{\longleftrightarrow}y\}}$, we can bound (4.52) by (4.45) with A replaced by a singleton $\{y\}$. Therefore, the contribution to (4.48) from $\mathbb{1}_{\{o\underset{\boldsymbol{m}+\boldsymbol{k}}{\longleftrightarrow}y\}\setminus\bigcup_{j=1}^N\{u_j\underset{\boldsymbol{k}}{\longleftrightarrow}y\}}$ in (4.49) obeys the same bound as $\Theta'_{o,x;\{y\}}$, with $U^\infty,\dot{U}^\infty_a,V^\infty,\dot{V}^\infty_a$ replaced by $U^1,\dot{U}^1_y,V^1,\dot{V}^1_y$, respectively. Combining this with (4.50)–(4.51), we complete the proof of Theorem 3.3. ■

4.4 Proof of Theorem 3.5

Recall the definition (3.20) of $\Theta''_{o,x,y;A}$:

$$\Theta''_{o,x,y;A}=\sum_{\substack{\boldsymbol{\ell}\in\mathbb{Z}_+^{\mathbb{B}_{A^c}}:\\ \partial\boldsymbol{\ell}=\varnothing}}\frac{w_{\mathbb{B}_{A^c}}(\boldsymbol{\ell})}{Z_{\mathbb{B}_{A^c}}}\sum_{\substack{\boldsymbol{n}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda}:\\ \partial\boldsymbol{n}=o\triangle x}}\frac{w_{\mathbb{B}_\Lambda}(\boldsymbol{n})}{Z_{\mathbb{B}_\Lambda}}\mathbb{1}_{\{o\underset{\boldsymbol{n}+\boldsymbol{\ell}}{\overset{A}{\Longleftrightarrow}}x\}\cap\{o\underset{\boldsymbol{n}+\boldsymbol{\ell}}{\longleftrightarrow}y\}}.\tag{4.54}$$

First, by identifying the earliest path $\omega\in\Omega(o,x)$ of bonds b with odd n_b (as done in Step 1 of Section 4.1; cf., (4.41) and (4.47)), we can rewrite $\Theta''_{o,x,y;A}$ as

$$\Theta''_{o,x,y;A}=\sum_{\omega\in\Omega(o,x)}\sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde{B}_\omega}:\\ \text{odd on } B_\omega,\\ \text{even on } \tilde{B}_\omega\setminus B_\omega}}\frac{w_{\tilde{B}_\omega}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda}}\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}:\\ \partial\boldsymbol{k}=\varnothing}}w_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}(\boldsymbol{k})$$
$$\times\sum_{\substack{\boldsymbol{\ell}\in\mathbb{Z}_+^{\mathbb{B}_{A^c}}:\\ \partial\boldsymbol{\ell}=\varnothing}}\frac{w_{\mathbb{B}_{A^c}}(\boldsymbol{\ell})}{Z_{\mathbb{B}_{A^c}}}\mathbb{1}_{\{o\underset{\boldsymbol{m}+\boldsymbol{k}+\boldsymbol{\ell}}{\overset{A}{\Longleftrightarrow}}x\}\cap\{o\underset{\boldsymbol{m}+\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow}y\}}.\tag{4.55}$$

Then, by the double expansion (as done in Step 2 of Section 4.1), we obtain the following inequality that is a mixture of (4.42) and (4.48):

$$\Theta''_{o,x,y;A} \le \sum_{N=1}^{\infty} \sum_{\substack{\{u_j,v_j\}_{j=1}^N \\ \text{(no intersection)}}} \sum_{\omega\in\Omega(o,x)} \sum_{\{s_jt_j\}_{j=1}^N \in \mathcal{L}^{(N)}_{[0,|\omega|]}} \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}_+^{\tilde{B}_\omega}: \\ \text{odd on } B_\omega, \\ \text{even on } \tilde{B}_\omega\setminus B_\omega}} \frac{w_{\tilde{B}_\omega}(\boldsymbol{m})}{Z_{\mathbb{B}_\Lambda}} \prod_{j=1}^{N} \mathbb{1}_{\{u_j\in\tilde{V}_{\boldsymbol{m}}(s_j)\}} \mathbb{1}_{\{v_j\in\tilde{V}_{\boldsymbol{m}}(t_j)\}}$$

$$\times \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}_+^{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}: \\ \partial\boldsymbol{k}=\varnothing}} w_{\mathbb{B}_\Lambda\setminus\tilde{B}_\omega}(\boldsymbol{k}) \sum_{\substack{\boldsymbol{\ell}\in\mathbb{Z}_+^{\mathbb{B}_{A^{\rm c}}}: \\ \partial\boldsymbol{\ell}=\varnothing}} \frac{w_{\mathbb{B}_{A^{\rm c}}}(\boldsymbol{\ell})}{Z_{\mathbb{B}_{A^{\rm c}}}} \mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}+\boldsymbol{\ell}}{\overset{A}{\longleftrightarrow}} x\}} \mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} y\}}$$

$$\times \mathbb{1}_{\{\mathsf{L}_{\boldsymbol{m},\boldsymbol{k}+\boldsymbol{\ell}}=\{s_jt_j\}_{j=1}^N\}} \prod_{j=1}^{N} \mathbb{1}_{\{u_j \underset{\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} v_j \text{ in } \mathbb{B}_\Lambda\setminus\tilde{B}_\omega\}}. \tag{4.56}$$

Then we rewrite $\mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} y\}}$, by using (4.49) with $\boldsymbol{k}$ replaced by $\boldsymbol{k}+\boldsymbol{\ell}$, as

$$\mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} y\}} = \sum_{i=1}^{N} \mathbb{1}_{\{u_i \underset{\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} y\}} + \mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} y\}\setminus\bigcup_{i=1}^N\{u_i \underset{\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} y\}}. \tag{4.57}$$

For the contribution from $\sum_{i=1}^N \mathbb{1}_{\{u_i \underset{\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} y\}}$, we ignore $\mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}+\boldsymbol{\ell}}{\overset{A}{\longleftrightarrow}} x\}}$ and impose the through-A condition only on $\boldsymbol{m}$ by replacing the sum over ω by $\sum_{\omega:\omega\cap A\neq\varnothing}$, as done in (4.41). Then the contribution from $\sum_{i=1}^N \mathbb{1}_{\{u_i \underset{\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} y\}}$ is bounded by the right-hand side of (4.43) with $\prod_{j=1}^N \sum_{i_j=0}^\infty (\tilde{G}^2)^{*i_j}(v_j-u_j)$ replaced by

$$\sum_{i=1}^{N} \prod_{\substack{j=1 \\ (j\neq i)}}^{N} \sum_{t_j=0}^{\infty} (\tilde{G}^2)^{*t_j}(v_j-u_j) \sum_{\substack{z_1,z_2,z_3, \\ z_1',z_2',z_3'}} (\delta+\tau^2)(u_i-z_1)\,(\delta+\tau^2)(v_i-z_2)\,\delta_{z_3,y}$$

$$\times \prod_{k=1}^{3} \sum_{t_k=0}^{\infty} (\tilde{G}^2)^{*t_k}(z_i'-z_i)\,T(z_1',z_2',z_3'). \tag{4.58}$$

The rest is the same as described in the last paragraph of Section 4.2. Consequently, the contribution from $\sum_{i=1}^N \mathbb{1}_{\{u_i \underset{\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} y\}}$ is bounded by

$$2\sum_{a\in A}\Bigg(\ddot{X}^{\infty}_{o,x;y}\delta_{a,x} + \sum_{i=0}^{\infty}\Big((U^\infty)^{\star i}\star \tfrac{1}{2}\dddot{V}^{\infty}_{a,y}\Big)_{o,x} + \sum_{i,j=0}^{\infty}\Big((U^\infty)^{\star i}\star \dddot{U}^{\infty}_{a,y}\star (U^\infty)^{\star j}\star V^\infty\Big)_{o,x}$$

$$+ \sum_{i,j=0}^{\infty}\Big((U^\infty)^{\star i}\star \dot{U}^{\infty}_{a}\star (U^\infty)^{\star j}\star \tfrac{1}{2}\ddot{V}^{\infty}_{y}\Big)_{o,x} + \sum_{i,j=0}^{\infty}\Big((U^\infty)^{\star i}\star \ddot{U}^{\infty}_{y}\star (U^\infty)^{\star j}\star \dot{V}^{\infty}_{a}\Big)_{o,x}$$

$$+ \sum_{i,j,k=0}^{\infty}\Big((U^\infty)^{\star i}\star \dot{U}^{\infty}_{a}\star (U^\infty)^{\star j}\star \ddot{U}^{\infty}_{y}\star (U^\infty)^{\star k}\star V^\infty\Big)_{o,x}$$

$$+ \sum_{i,j,k=0}^{\infty}\Big((U^\infty)^{\star i}\star \ddot{U}^{\infty}_{y}\star (U^\infty)^{\star j}\star \dot{U}^{\infty}_{a}\star (U^\infty)^{\star k}\star V^\infty\Big)_{o,x}\Bigg). \tag{4.59}$$

For the contribution from $\mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} y\}\setminus\bigcup_{i=1}^N\{u_i \underset{\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} y\}}$ in (4.57), on the other hand, we again ignore the indicator $\mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}+\boldsymbol{\ell}}{\overset{A}{\longleftrightarrow}} x\}}$, but impose the through-A condition on $\boldsymbol{k}+\boldsymbol{\ell}$ by replacing $\prod_{j=1}^N \mathbb{1}_{\{u_j \underset{\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} v_j \text{ in } \mathbb{B}_\Lambda\setminus\tilde{B}_\omega\}}$ in (4.56) by

$$\sum_{a\in A}\sum_{i=1}^N \prod_{\substack{j=1\\(j\neq i)}}^N \mathbb{1}_{\{u_j \underset{\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} v_j \text{ in } \mathbb{B}_\Lambda\setminus\tilde{B}_\omega\}} \mathbb{1}_{\{u_i \underset{\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} v_i \text{ in } \mathbb{B}_\Lambda\setminus\tilde{B}_\omega\}\cap\{u_i \underset{\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} a \text{ in } \mathbb{B}_\Lambda\setminus\tilde{B}_\omega\}}, \tag{4.60}$$

which yields (4.58) with $\delta_{z_3,y}$ replaced by $\delta_{z_3,a}$ and summed over $a \in A$. Then the rest is almost the same as described in the last paragraph of Section 4.3, except for two things: there are three current configurations involved, instead of two as in (4.52), and one of them is restricted on $\mathbb{B}_{A^c}$. Taking them into account, we can conclude that the contribution from $\mathbb{1}_{\{o \underset{\boldsymbol{m}+\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} y\}\setminus\bigcup_{i=1}^N\{u_i \underset{\boldsymbol{k}+\boldsymbol{\ell}}{\longleftrightarrow} y\}}$ in (4.57) is bounded by (4.59) with a and y swapped, y replaced by y' and then multiplied by $\sum_{i=0}^\infty (\tilde{G}^2)^{*i}(y-y')$. This completes the proof of Theorem 3.5. ■

5 Application to the spread-out model

In this section, we demonstrate how to use the diagrammatic bounds proven in the previous section to derive the wanted x-space decay (see (5.11), (5.37) and (5.43) below) for the spread-out model with $L \gg 1$ in dimensions $d > 4$. To do so, we repeatedly use the following convolution bound [9, Lemma 3.2(i)], which is an improved version of [16, Proposition 1.7].

Lemma 5.1 (Convolution bound for the spread-out model [9])**.** *Let*

$$|||x|||_L = |x| \vee L. \tag{5.1}$$

For any $a \ge b > 0$ with $a+b > d$, there is an L-independent constant $C = C(a,b,d) < \infty$ such that

$$\sum_{y\in\mathbb{Z}^d} |||x-y|||_L^{-a}\, |||y|||_L^{-b} \le \begin{cases} CL^{d-a}|||x|||_L^{-b} & (a>d),\\ C|||x|||_L^{d-a-b} & (a<d). \end{cases} \tag{5.2}$$

Throughout this section, we assume the following bound on $\|\tau\|_1$ and G.

Assumption 5.2. *Let $J_{o,x}$ be the spread-out interaction (2.25) and define $\theta = O(L^{-2})$ as in (2.26). We assume*

$$\|\tau\|_1 \vee \sup_{x\neq o} \frac{G(x)}{\theta|||x|||_L^{2-d}} \leq 2. \tag{5.3}$$

As explained earlier, if $d > 4$, $\theta \ll 1$ and (2.27) holds uniformly in $x \in \Lambda \subset \mathbb{Z}^d$ and $\beta \leq \beta_c$, then $G_{\beta_c}(x) \sim A' S_1(x)$ as $|x| \uparrow \infty$, where $A' \overset{(2.28)}{=} A\sigma^2 = (1+O(L^{-2}))/\|\tau_{\beta_c}\|_1$ (the latest reference is [10, (1.41)] with $\alpha = \infty$) and $\|\tau_{\beta_c}\|_1 = \|\Pi_{\beta_c}\|_1^{-1} = 1 + O(L^{-d})$ (due to (2.27)). As a result, the assumption (5.3) indeed holds at β_c. In fact, we can show that (5.3) holds uniformly in $\beta_{\rm MF} \leq \beta < \beta_c$, where $\beta_{\rm MF}$ is the mean-field critical point (i.e., $\|\tau_{\beta_{\rm MF}}\|_1 = 1$), and the result at β_c is obtained by the continuity in β of the left-hand side of (5.3). For more details, see, e.g., [9, Theorem 3.3] with $\alpha = \infty$.

Now we are left to show the inequality (2.27) for each $\beta < \beta_c$, under Assumption 5.2. To do so, we will frequently use the following bound and notation:

Lemma 5.3. *Under Assumption 5.2, we have*

$$\sup_x \frac{\tilde{G}(x)}{|||x|||_L^{2-d}} \leq O(\theta), \tag{5.4}$$

where the implicit constant in $O(\theta)$ is independent of L. This means that $\tilde{G}$ obeys the same x-space bound on G, modulo L-independent constant multiplication. We denote this by

$$\tilde{G}(x) \lesssim G(x). \tag{5.5}$$

Notice that, by repeated use of (5.5), we have

$$(\tau^{*2} * \tilde{G})(x) \lesssim (\tau^{*2} * G)(x) = (\tau * \tilde{G})(x) \lesssim (\tau * G)(x) = \tilde{G}(x). \tag{5.6}$$

We will use this relation to bound the diagrammatic bounds on the expansion coefficients.

Proof of Lemma 5.3. Since h in (2.25) is bounded and supported on $[-1,1]^d$,

$$\tau(x) = O(L^{-d})\,\mathbb{1}_{\{\|x\|_\infty \leq L\}} \leq \frac{O(L^2)}{|||x|||_L^{d+2}}. \tag{5.7}$$

By (5.2) and (5.3), we obtain

$$\tilde{G}(x) = \tau(x) + \sum_{y\neq o} \tau(x-y)\,G(y) \leq \frac{O(\theta)}{|||x|||_L^{d-2}}. \tag{5.8}$$

In particular, $\tilde{G}(o) = O(L^{-d})$, while $G(o) = 1$. This implies the relation (5.5). ∎

5.1 Bound on the 0^{th}-order expansion coefficient

First we recall Theorem 3.1. The following proposition provides a bound on $\pi^{(0)}_{\mathbb{B}_\Lambda}(x)$ for $x \neq o$.

Proposition 5.4. *Under Assumption 5.2, if $d > 4$ and $\theta \ll 1$, then, for any $m \geq 1$,*

$$U^m(y, z; y', z') \lesssim U^0(y, z; y', z'), \qquad V^m(y, z; x) \lesssim V^1(y, z; x), \tag{5.9}$$

i.e., U^m and V^m obey the same x-space bounds on U^0 and V^1, respectively, modulo L-independent constant multiplication. As a result, for $x \neq o$ and $m \geq 1$,

$$X^m_{o,x} \lesssim V^1(o, o; x) = \tilde{G}(x)^3. \tag{5.10}$$

The following is an immediate consequence of (5.10) and Theorem 3.1.

Corollary 5.5 (cf., (3.3) of [25]). *Under Assumption 5.2, if $d > 4$ and $\theta \ll 1$, then*

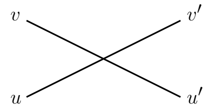
$$\delta_{o,x} \leq \pi^{(0)}_{\mathbb{B}_\Lambda}(x) \leq \delta_{o,x} + \frac{O(\theta^3)}{|||x|||_L^{3(d-2)}}, \tag{5.11}$$

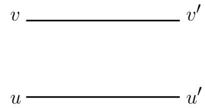
where the implicit constant in $O(\theta^3)$ may depend on d, but not on L.

Proof of Proposition 5.4. First we prove (5.9). By the convolution bound (5.2), degree-4 vertices can be eliminated one by one when $d > 4$, as follows. Since $|u - x| \vee |x - u'| \geq |u - u'|/2$ and $|v - x| \vee |x - v'| \geq |v - v'|/2$, we have

$$\begin{aligned}
&\sum_x \tilde{G}(u - x)\, \tilde{G}(x - u')\, \tilde{G}(v - x)\, \tilde{G}(x - v') \\
&\overset{(5.4)}{\leq} \sum_x \frac{O(\theta)}{|||u - x|||_L^{d-2}} \frac{O(\theta)}{|||x - u'|||_L^{d-2}} \frac{O(\theta)}{|||v - x|||_L^{d-2}} \frac{O(\theta)}{|||x - v'|||_L^{d-2}} \\
&\leq \frac{O(\theta)^4}{|||u - u'|||_L^{d-2} |||v - v'|||_L^{d-2}} \Bigg(\underbrace{\sum_x \frac{\mathbb{1}_{\{|u-x| \leq |x-u'|\}}\, \mathbb{1}_{\{|v-x| \leq |x-v'|\}}}{|||u - x|||_L^{d-2} |||v - x|||_L^{d-2}}}_{\leq C|||u-v|||_L^{4-d} \leq CL^{4-d} \ (\because (5.2)\ \&\ d>4)} + [3 \text{ others}] \Bigg) \\
&\leq \frac{O(\theta)}{|||u - u'|||_L^{d-2}} \frac{O(\theta)}{|||v - v'|||_L^{d-2}}\, O(\theta^2 L^{4-d}),
\end{aligned} \tag{5.12}$$

where [3 others] is the sum of the contributions from $\mathbb{1}_{\{|u-x| \leq |x-u'|\}} \mathbb{1}_{\{|v-x| > |x-v'|\}}$, from $\mathbb{1}_{\{|u-x| > |x-u'|\}} \mathbb{1}_{\{|v-x| \leq |x-v'|\}}$ and from $\mathbb{1}_{\{|u-x| > |x-u'|\}} \mathbb{1}_{\{|v-x| > |x-v'|\}}$. Since $\theta = O(L^{-2})$, this may be depicted as



$\lesssim$


$\times\, L^{-d}$. (5.13)

By taking $u = v$ and $u' = v'$, we obtain $\tilde{G}^2 * \tilde{G}^2 \lesssim L^{-d}\,\tilde{G}^2$. By repeated use of this relation, we obtain

$$\sum_{j=1}^{m}(\tilde{G}^2)^{*j} \lesssim \tilde{G}^2 \sum_{j=1}^{\infty} L^{-d(j-1)} \lesssim \tilde{G}^2, \tag{5.14}$$

which proves the second relation in (5.9). Similarly, $(\delta+\tau^2) * \sum_{j=0}^{m}(\tilde{G}^2)^{*j} * (\delta+\tau^2)$ in (3.2) can be evaluated, by using $\tau \le \tilde{G}$, (5.14) and then (5.5), as

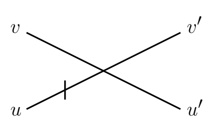
$$\begin{aligned}
(\delta+\tau^2)^{*2} * \sum_{j=0}^{m}(\tilde{G}^2)^{*j} \quad &\overset{\tau\le\tilde{G}}{\le} \quad (\delta+\tilde{G}^2)^{*2} * \Bigg(\delta + \sum_{j=1}^{m}(\tilde{G}^2)^{*j}\Bigg) \\
&\overset{(5.14)}{\lesssim} \quad (\delta+\tilde{G}^2)^{*3} \\
&\overset{(5.13)}{\lesssim} \quad \delta+\tilde{G}^2 \\
&\le \quad (\delta+\tilde{G})^2 \\
&\overset{(5.5)}{\lesssim} \quad G^2.
\end{aligned} \tag{5.15}$$

This implies that the sum of the bubble chain in Figure 3 (including the black disks at both ends of the chain) can be replaced by G^2, at the cost of L-independent constant multiplication. This proves the first relation in (5.9).

As a result, we have

$$X^m_{o,x} \lesssim \sum_{i=0}^{\infty}\big((U^0)^{\star i} \star V^1\big)_{o,x}. \tag{5.16}$$

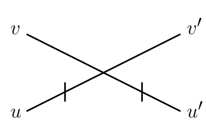
To prove (5.10), we repeatedly use the convolution bound (5.2) to eliminate all diagram vertices of degree 4 one by one. For example, if one of the four line segments in (5.13), say, between u and x, is slashed, then we use (2.14) to replace $G(u-x)$ by $\delta_{u,x}+\tilde{G}(u-x)$. The contribution from $\tilde{G}(u-x)$ is identical to (5.13). The contribution from $\delta_{u,x}$ is equal to $\tilde{G}(u-u')\,\tilde{G}(v-u)\,\tilde{G}(u-v')$. However, since $|v-u| \vee |u-v'| \ge |v-v'|/2$, we have $\tilde{G}(v-u)\,\tilde{G}(u-v') \lesssim L^{-d}\,\tilde{G}(v-v')$. Therefore,

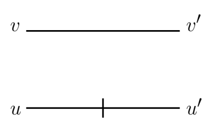


$\lesssim$

$\times\, L^{-d}$. (5.17)

Similarly, we obtain



$\lesssim$


$\times\, L^{-d}$, (5.18)

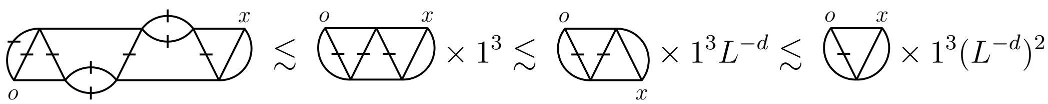
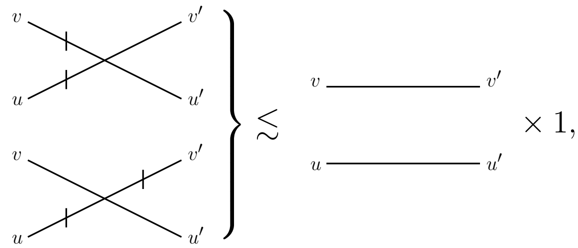
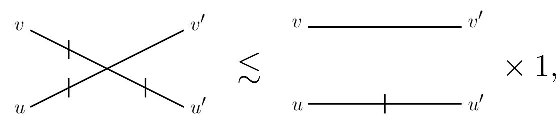


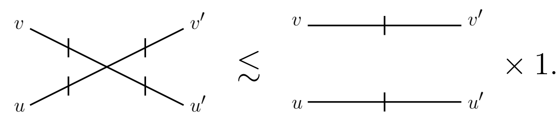
Figure 12: Reduction of the simplified version of the $n = 3$ term in (3.5) to even simpler diagrams, by using (5.20) three times and then using (5.17) twice. Using (5.17) once more yields $V^1(o, o; x) = \tilde{G}(x)^3$ multiplied by a factor of $(L^{-d})^3$.



(5.19)



(5.20)



(5.21)

As a rule of thumb, the factor of L^{-d} arises when at least one of those two removed line segments is unslashed.

To evaluate the i^{th} term in (5.16), we first use (5.20) with $u = v$ to eliminate all bubbles ($= G^2$) and then use (5.17) to eliminate all degree-4 vertices (see Figure 12). As a result, the i^{th} term in (5.16) is reduced to the simplest diagram $V^1(o, o; x) = \tilde{G}(x)^3$ multiplied by a factor of L^{-di}, which is summable in i if $L \gg 1$. This completes the proof of (5.10). ∎

5.2 Bound on the 1st-order expansion coefficient

Recall (3.14) and (3.18), where $X, \dot{X}, \ddot{X}$ are involved. We have already shown that $X^m_{o,x} \lesssim V^1(o, o; x) = \tilde{G}(x)^3$ if $d > 4$ and $\theta \ll 1$ under Assumption 5.2. It remains to investigate $\dot{X}$ and $\ddot{X}$.

First we investigate $\dot{X}$. By the same reason as in Proposition 5.4, $\dot{U}^m_a$ and $\dot{V}^m_a$ obey the same x-space bounds on $\dot{U}^0_a$ and $\dot{V}^1_a$, respectively, modulo L-independent constant multiplication, if $d > 4$ and $L \gg 1$ under Assumption 5.2. Then, by repeated use of the convolution bound (5.2) (as in Figure 12), we can show that $(U^m)^{\star i} \star \dot{V}^m_a$ and $(U^m)^{\star i} \star \dot{U}^m_a \star (U^m)^{\star j} \star V^m$ in (3.10) obey the same x-space bound on $\dot{V}^1_a(o, o; x)$ multiplied by factors of L^{-di} and $L^{-d(i+j)}$, respectively, which are summable if $L \gg 1$. As a result, we obtain the following:

Proposition 5.6. *Under Assumption 5.2, if $d > 4$ and $\theta \ll 1$, then, for any $m \geq 1$,*

$$\dot{U}_a^m(y, z; y', z') \lesssim \dot{U}_a^0(y, z; y', z'), \qquad \dot{V}_a^m(y, z; x) \lesssim \dot{V}_a^1(y, z; x). \tag{5.22}$$

As a result, for $x \neq o$,

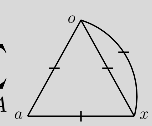
$$\dot{X}_{o,x;a}^m \lesssim \dot{V}_a^1(o, o; x) = \tilde{G}(x)^2\, G(a)\, \tilde{G}(x - a). \tag{5.23}$$

Recall (3.11), (5.10) and (5.23). Since, for $x \neq o$,

$$\left.\begin{array}{l} V^1(o, o; x)\delta_{x,a} = \tilde{G}(x)^3\delta_{x,a} \\ \dot{V}_a^1(o, o; x) = \tilde{G}(x)^2\, G(a)\, \tilde{G}(x - a) \end{array}\right\} \overset{(5.5)}{\lesssim} \tilde{G}(x)^2\, G(a)\, G(x - a), \tag{5.24}$$

and since $\Theta'_{x,x;A} = \mathbb{1}_{\{x \in A\}}$, we obtain the following:

Corollary 5.7. *Under Assumption 5.2, if $d > 4$ and $\theta \ll 1$, then*

$$\Theta'_{o,x;A} \lesssim \sum_{a \in A}$$ 

$$= \sum_{a \in A} G(x)^2\, G(a)\, G(x - a). \tag{5.25}$$

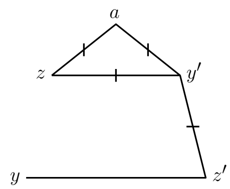
Next we investigate $\ddot{X}$. Again, by repeated use of the convolution bound (5.2), we can show that, if $d > 4$ and $L \gg 1$ under Assumption 5.2, $\ddot{U}_a^m$ and $\ddot{V}_a^m$ obey the same x-space bounds on $\ddot{U}_a^1$ and $\ddot{V}_a^1$, respectively, where

$$\ddot{U}_a^1(y, z; y', z') = \tilde{G}(z' - y)\, G(z' - y') \sum_{v,v'} (\delta + \tau^2)(z - v)\, (\delta + \tau^2)(y' - v')\, T(v, v', a), \tag{5.26}$$

$$\ddot{V}_a^1(y, z; x) = \mathbb{1}_{\{z \neq x\}}\, \tilde{G}(x - y) \sum_v (\delta + \tau^2)(z - v)\, T(v, x, a). \tag{5.27}$$

Moreover, by using (5.21) once, we have

$$T(v, x, a) \lesssim G(x - v)\, G(a - v)\, G(x - a). \tag{5.28}$$

$$\ddot{U}^0_a(y,z;y',z') =$$ 

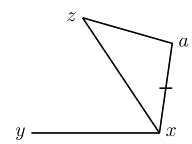
$$\ddot{V}^0_a(y,z;x) =$$ 

Figure 13: Schematic representations of $\ddot{U}^0_a(y,z;y',z')$ and $\ddot{V}^0_a(y,z;x)$ in (5.32)–(5.33).

Plugging this back to (5.27) yields

$$\ddot{V}^1_a(y,z;x) \lesssim \tilde{G}(x-y)\bigg(\mathbb{1}_{\{z\neq x\}}\underbrace{G(x-z)}_{\leq \tilde{G}(x-z)}\underbrace{G(a-z)\,G(x-a)}_{\leq 2\tilde{G}(a-z)\,G(x-a)} + \underbrace{\sum_v \tau(z-v)^2\,G(x-v)\,G(a-v)}_{\leq \tilde{G}(x-z)\,\tilde{G}(a-z)}\;G(x-a)\bigg), \tag{5.29}$$

where we have used

$$\begin{aligned} G(a-z)\,G(x-a) &\leq \big(\delta_{z,a}+\tilde{G}(a-z)\big)\,G(x-a) \\ &= G(x-z)+\tilde{G}(a-z)\,G(x-a) \\ &\overset{z\neq x}{\leq} \tilde{G}(x-z)+\tilde{G}(a-z)\,G(x-a) \\ &\overset{\delta\leq G}{\leq} 2\tilde{G}(a-z)\,G(x-a). \end{aligned} \tag{5.30}$$

Similarly, we can show (cf., (5.15))

$$\ddot{U}^1_a(y,z;y',z') \lesssim \tilde{G}(z'-y)\,G(z'-y')\,G(y'-z)\,G(a-z)\,G(y'-a). \tag{5.31}$$

Let (see Figure 13)

$$\ddot{U}^0_a(y,z;y',z') = \tilde{G}(z'-y)\,G(z'-y')\,G(y'-z)\,G(a-z)\,G(y'-a), \tag{5.32}$$

$$\ddot{V}^0_a(y,z;x) = \tilde{G}(x-y)\,\tilde{G}(x-z)\,\tilde{G}(a-z)\,G(x-a). \tag{5.33}$$

Then, $\ddot{X}^m_{o,x;y}$ in (3.17) obeys the same x-space bound on $\ddot{X}^0_{o,x;y}$. Repeated applications of the convolution bound (5.2) to $\ddot{X}^0_{o,x;y}$ (as in Figure 12), we can show that $\ddot{X}^0_{o,x;y}$ obeys the same x-space bound on $\ddot{V}^0_y(o,o;x)$, if $d>4$ and $L\gg 1$ under Assumption 5.2. As a result, we obtain the following:

Proposition 5.8. *Under Assumption 5.2, if $d > 4$ and $\theta \ll 1$, then, for any $m \geq 1$,*

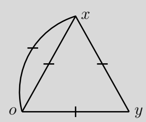
$$\ddot{U}_a^m(y,z;y',z') \lesssim \ddot{U}_a^0(y,z;y',z'), \qquad \ddot{V}_a^m(y,z;x) \lesssim \ddot{V}_a^0(y,z;x). \tag{5.34}$$

As a result, for $x \neq o$,

$$\ddot{X}_{o,x;y}^m \lesssim \ddot{V}_y^0(o,o;x) = \tilde{G}(x)^2\, \tilde{G}(y)\, G(x-y). \tag{5.35}$$

Recall (3.18), (5.10) and (5.23) (also (5.24)). Since $\tilde{\pi}_{\mathbb{B}_\Lambda;y}^{(0)}(o) \leq G(y)^2$ (cf., (2.16)), we readily obtain the following:

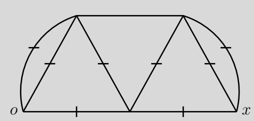
Corollary 5.9. *Under Assumption 5.2, if $d > 4$ and $\theta \ll 1$, then*

$$\tilde{\pi}_{\mathbb{B}_\Lambda;y}^{(0)}(x) \lesssim$$


$$= G(x)^2\, G(y)\, G(x-y). \tag{5.36}$$

Substituting this back into (3.14) and using (5.20) and (5.21), we can conclude the following wanted x-space decay of $\pi_{\mathbb{B}_\Lambda}^{(1)}$:

Corollary 5.10 (cf., (3.3) of [25])**.** *Under Assumption 5.2, if $d > 4$ and $\theta \ll 1$, then*

$$\pi_{\mathbb{B}_\Lambda}^{(1)}(x) \lesssim$$


$$\leq O(L^{-d})\, \delta_{o,x} + \frac{O(\theta^3)}{|||x|||_L^{3(d-2)}}, \tag{5.37}$$

where the implicit constants in $O(L^{-d})$ and $O(\theta^3)$ may depend on d, but not on L.

5.3 Bound on the higher-order expansion coefficient

Recall (3.21) and (3.25), where $X, \dot{X}, \ddot{X}, \dddot{X}$ are involved. The first three obey the bounds in (5.10), (5.23) and (5.35). It remains to investigate $\dddot{X}$ in (3.24). By Propositions 5.4, 5.6 and 5.8, we can reduce $U^m, V^m, \dot{U}^m, \dot{V}^m, \ddot{U}^m$ and $\ddot{V}^m$ in (3.24) to simpler $U^0, V^1, \dot{U}^0, \dot{V}^1, \ddot{U}^0$ and $\ddot{V}^0$, respectively, if $d > 4$ and $L \gg 1$ under Assumption 5.2. Similarly, we can reduce

$\dddot{U}^m$ and $\dddot{V}^m$ to $\dddot{U}^0$ and $\dddot{V}^0$, respectively, where

$$\dddot{U}^0_{a,v}(y,z;y',z') = \Big(G(a-y)\,\tilde{G}(z'-a)\,G(z'-y') + \tilde{G}(z'-y)\,\tilde{G}(a-y')\,G(z'-a)\Big) \times G(y'-z)\,G(v-z)\,G(y'-v), \tag{5.38}$$

$$\dddot{V}^0_{a,v}(y,z;x) = G(a-y)\,\tilde{G}(x-a)\,\tilde{G}(x-z)\,\tilde{G}(v-z)\,G(x-v). \tag{5.39}$$

Moreover, due to the observation below (5.21), the sums over i,j,k are convergent if $d>4$ and $L \gg 1$, and the dominant terms come from the $i=j=k=0$ case. Among those six terms, the largest (modulo L-independent constant multilpication) is $\dddot{V}^0_{a,y}(o,o;x)$. This is summarised as follows:

Proposition 5.11. *Under Assumption 5.2, if $d>4$ and $\theta \ll 1$, then, for any $m \geq 1$,*

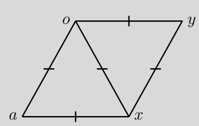
$$\dddot{U}^m_{a,v}(y,z;y',z') \lesssim \dddot{U}^0_{a,v}(y,z;y',z'), \qquad \dddot{V}^m_{a,v}(y,z;x) \lesssim \dddot{V}^0_{a,v}(y,z;x). \tag{5.40}$$

As a result, for $x \neq o$,

$$\dddot{X}^m_{o,x;a,y} \lesssim \dddot{V}^0_{a,y}(o,o;x) = G(a)\,\tilde{G}(x-a)\,\tilde{G}(x)\,\tilde{G}(y)\,G(x-y). \tag{5.41}$$

Since $\Theta''_{x,x,y;A} \leq \mathbb{1}_{\{x\in A\}} \sum_{j=0}^{\infty} (\tilde{G}^2)^{*j}(y) \lesssim \mathbb{1}_{\{x\in A\}}\, G(y)^2$, we readily obtain the following:

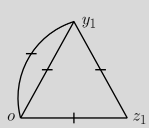
Corollary 5.12. *Under Assumption 5.2, if $d>4$ and $\theta \ll 1$, then*

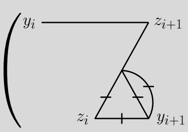
$$\Theta''_{o,x,y;A} \lesssim \sum_{a\in A}$$


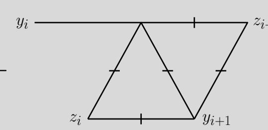
$$= \sum_{a\in A} G(a)\,G(x-a)\,G(x)\,G(y)\,G(x-y). \tag{5.42}$$

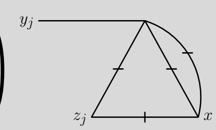
Substituting this back into (3.21), generalizing it to $\pi^{(j)}_{\mathbb{B}_\Lambda}(x)$ for $j \geq 2$, and repeatedly using the convolution bounds (5.18) and (5.20), we can conclude the following:

Corollary 5.13. *Under Assumption 5.2, if $d>4$ and $\theta \ll 1$, then, for $j \geq 2$,*

$$\pi^{(j)}_{\mathbb{B}_\Lambda}(x) \lesssim \sum_{\substack{y_1,\dots,y_j,\\ z_1,\dots,z_j}}$$


$$\prod_{i=1}^{j-1} \Bigg($$


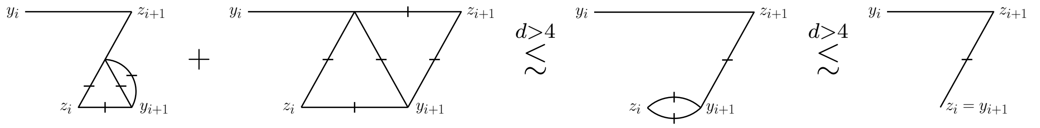
$$+$$


$$\Bigg)$$


$$\leq O(L^{-jd})\,\delta_{o,x} + \frac{O(L^{-d(j-2)}\theta^3)}{|||x|||_L^{3(d-2)}}, \tag{5.43}$$

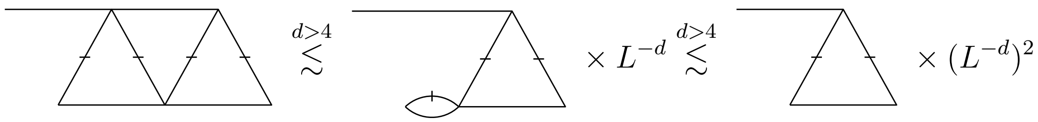
where the implicit constants in $O(L^{-jd})$ and $O(L^{-d(j-2)}\theta^3)$ may depend on d, but not on L. This is an improved version of [25, (3.3)] (see also [9, (3.4)] and [10, (3.22)].

Proof of the last line of (5.43). First we note that, by (5.20),



(5.44)

Moreover, by using (5.18) twice, we have



(5.45)

hence the recurrence formula $\pi_{\mathbb{B}_\Lambda}^{(j)}(x) \lesssim (L^{-d})^2\, \pi_{\mathbb{B}_\Lambda}^{(j-2)}(x)$ for $j \geq 4$. However, by repeated use of (5.20), we obtain $\pi_{\mathbb{B}_\Lambda}^{(2)}(x) \lesssim \tilde{G}(x)^2\, G(x)$ and $\pi_{\mathbb{B}_\Lambda}^{(3)}(x) \lesssim L^{-d}\, \tilde{G}(x)^2\, G(x)$. Therefore,

$$\pi_{\mathbb{B}_\Lambda}^{(j)}(x) \lesssim (L^{-d})^{j-2}\, \tilde{G}(x)^2\, G(x) \overset{(2.14)}{\leq} (L^{-d})^{j-2}\Big(\underbrace{\tilde{G}(o)^2}_{\lesssim L^{-2d}}\delta_{o,x} + \tilde{G}(x)^3\Big), \tag{5.46}$$

as required. ■

Acknowledgements

This work was supported by JSPS KAKENHI Grant Number 18K03406. I would like to thank Hugo Duminil-Copin for drawing my atention to the issue discussed in this paper. I would also like to thank Satoshi Handa for working together in the early stage of this project.

References

[1] M. Aizenman. Geometric analysis of ϕ^4 fields and Ising models. *Comm. Math. Phys.*, **86** (1982): 1–48.

[2] M. Aizenman, D.J. Barsky and R. Fernández. The phase transition in a general class of Ising-type models is sharp. *J. Stat. Phys.*, **47** (1987): 343–374.

[3] M. Aizenman, H. Duminil-Copin and V. Sidoravicius. Random currents and continuity of Ising model's spontaneous magnetization. *Comm. Math. Phys.*, **334** (2015): 719–742.

[4] M. Aizenman and R. Fernández. On the critical behavior of the magnetization in high-dimensional Ising models. *J. Stat. Phys.*, **44** (1986): 393–454.

[5] M. Aizenman and R. Graham. On the renormalized coupling constant and the susceptibility in ϕ^4_4 field theory and the Ising model in four dimensions. *Nucl. Phys. B*, **225** (1983): 261–288.

[6] B. Bollobás and O. Riordan. *Percolation* (Cambridge University Press, 2006).

[7] D.C. Brydges, T. Helmuth and M. Holmes. The continuous-time lace expansion. *Comm. Pure Appl. Math.*, **74** (2021): 2251–2309.

[8] D.C. Brydges and T. Spencer. Self-avoiding walk in 5 or more dimensions. *Comm. Math. Phys.*, **97** (1985): 125–148.

[9] L.-C. Chen and A. Sakai. Critical two-point functions for long-range statistical-mechanical models in high dimensions. *Ann. Probab.*, **43** (2015): 639–681.

[10] L.-C. Chen and A. Sakai. Critical two-point functions for long-range models with power-law couplings: The marginal case for $d \geq d_c$. *Comm. Math. Phys.*, **372** (2019): 543–572.

[11] J. Fröhlich, B. Simon and T. Spencer. Infrared bounds, phase transitions and continuous symmetry breaking. *Comm. Math. Phys.*, **50** (1976): 79–95.

[12] R.B. Griffiths. Correlations in Ising ferromagnets I. *J. Math. Phys.*, **8** (1967): 478–483.

[13] R.B. Griffiths, C.A. Hurst and S. Sherman. Concavity of magnetization of an Ising ferromagnet in a positive external field. *J. Math. Phys.*, **11** (1970): 790–795.

[14] G.R. Grimmett. *Percolation* (2nd ed., Springer, Berlin, 1999).

[15] T. Hara. Decay of correlations in nearest-neighbour self-avoiding walk, percolation, lattice trees and animals. *Ann. Probab.*, **36** (2008): 530–593.

[16] T. Hara, R. van der Hofstad and G. Slade. Critical two-point functions and the lace expansion for spread-out high-dimensional percolation and related models. *Ann. Probab.*, **31** (2003): 349–408.

[17] T. Hara and G. Slade. Mean-field critical behaviour for percolation in high dimensions. *Comm. Math. Phys.*, **128** (1990): 333–391.

[18] T. Hara and G. Slade. On the upper critical dimension of lattice trees and lattice animals. *J. Stat. Phys.*, **59** (1990): 1469–1510.

[19] T. Hara and G. Slade. Self-avoiding walk in five or more dimensions. I. The critical behaviour. *Comm. Math. Phys.*, **147** (1992): 101–136.

[20] M. Heydenreich, R. van der Hofstad, G. Last and K. Matzke. Lace expansion and mean-field behavior for the random connection model. Preprint (2019): arXiv:1908.11356.

[21] M. Heydenreich, R. van der Hofstad and A. Sakai. Mean-field behavior for long- and finite range Ising model, percolation and self-avoiding walk. *J. Stat. Phys.*, **132** (2008): 1001–1049.

[22] J.L. Lebowitz. GHS and other inequalities. *Comm. Math. Phys.*, **35** (1974): 87–92.

[23] B.G. Nguyen and W-S. Yang. Triangle condition for oriented percolation in high dimensions. *Ann. Prob.* **21** (1993): 1809–1844.

[24] A. Sakai. Mean-field critical behavior for the contact process. *J. Stat. Phys.* **104** (2001): 111–143.

[25] A. Sakai. Lace expansion for the Ising model. *Comm. Math. Phys.*, **272** (2007): 283–344.

[26] A. Sakai. Application of the lace expansion to the φ^4 model. *Comm. Math. Phys.*, **336** (2015): 619–648.

[27] G. Slade. The lace expansion and its applications. *Lecture Notes in Mathematics* **1879** (2006).