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| Title | Logarithmic A-hypergeometric series II |
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| Citation | Beiträge zur Algebra und Geometrie / Contributions to Algebra and Geometry, 64(4), 1057-1086 https://doi.org/10.1007/s13366-022-00669-5 |
| Issue Date | 2022 |
| Doc URL | https://hdl.handle.net/2115/90608 |
| Rights | This version of the article has been accepted for publication, after peer review (when applicable) and is subject to Springer Nature's AM terms of use, but is not the Version of Record and does not reflect post-acceptance improvements, or any corrections. The Version of Record is available online at: http://dx.doi.org/10.1007/s13366-022-00669-5 |
| Type | journal article |
| File Information | Beitr. Algebr. Geom.pdf |

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# LOGARITHMIC $A$-HYPERGEOMETRIC SERIES II

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ABSTRACT. In this paper, following [6], we continue to develop the perturbing method of constructing logarithmic series solutions to a regular $A$-hypergeometric system.

Fixing a fake exponent of an $A$-hypergeometric system, we consider some spaces of linear partial differential operators with constant coefficients. Comparing these spaces, we construct a fundamental system of series solutions with the given exponent by the perturbing method. In addition, we give a sufficient condition for a given fake exponent to be an exponent. As important examples of the main results, we give fundamental systems of series solutions to Aomoto-Gel'fand systems and to Lauricella's $F_C$ systems with special parameter vectors, respectively.

## 1. INTRODUCTION

Let $A = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n) = (a_{ij})$ be a $d \times n$-matrix of rank $d$ with coefficients in $\mathbb{Z}$. Throughout this paper, we assume the homogeneity of $A$, i.e., we assume that all $\boldsymbol{a}_j$ belong to one hyperplane off the origin in $\mathbb{Q}^d$. Let $\mathbb{N}$ be the set of nonnegative integers. Let $I_A$ denote the toric ideal in the polynomial ring $\mathbb{C}[\partial_{\boldsymbol{x}}] = \mathbb{C}[\partial_1, \dots, \partial_n]$, i.e.,

$$I_A = \langle \partial_{\boldsymbol{x}}^{\boldsymbol{u}} - \partial_{\boldsymbol{x}}^{\boldsymbol{v}} \,|\, A\boldsymbol{u} = A\boldsymbol{v},\ \boldsymbol{u}, \boldsymbol{v} \in \mathbb{N}^n \rangle \subseteq \mathbb{C}[\partial_{\boldsymbol{x}}].$$

Here and hereafter we use the multi-index notation; for example, $\partial_{\boldsymbol{x}}^{\boldsymbol{u}}$ means $\partial_1^{u_1} \cdots \partial_n^{u_n}$ for $\boldsymbol{u} = (u_1, \dots, u_n)^T$. Given a column vector $\boldsymbol{\beta} = (\beta_1, \dots, \beta_d)^T \in \mathbb{C}^d$, let $H_A(\boldsymbol{\beta})$ denote the left ideal of the Weyl algebra

$$D = \mathbb{C}\langle \boldsymbol{x}, \partial_{\boldsymbol{x}} \rangle = \mathbb{C}\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle$$

generated by $I_A$ and

$$\sum_{j=1}^{n} a_{ij}\theta_j - \beta_i \qquad (i = 1, \dots, d),$$

2020 *Mathematics Subject Classification.* Primary: 33C70.

*Key words and phrases.* $A$-hypergeometric systems; the method of Frobenius.

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where $\theta_j = x_j\partial_j$. The quotient $M_A(\boldsymbol{\beta}) = D/H_A(\boldsymbol{\beta})$ is called the *$A$-hypergeometric system with parameter $\boldsymbol{\beta}$*, and a formal series annihilated by $H_A(\boldsymbol{\beta})$ an *$A$-hypergeometric series with parameter $\boldsymbol{\beta}$*. The homogeneity of $A$ is known to be equivalent to the regularity of $M_A(\boldsymbol{\beta})$ by Hotta [4] and Schulze, Walther [9].

Logarithm-free series solutions to $M_A(\boldsymbol{\beta})$ were constructed by Gel'fand et al. [2, 3] for a generic parameter $\boldsymbol{\beta}$, and more generally in [8].

Note that the logarithmic coefficients of $A$-hypergeometric series solutions are polynomials of $\log x^{\boldsymbol{b}}$ ($\boldsymbol{b} \in L$) [5, Proposition 5.2], where

$$L := \mathrm{Ker}_{\mathbb{Z}}(A) = \{\boldsymbol{u} \in \mathbb{Z}^n \mid A\boldsymbol{u} = \boldsymbol{0}\}.$$

To construct logarithmic series solutions, the second author [6] introduced a method of perturbation by a finite subset $B = \{\boldsymbol{b}^{(1)}, \dots, \boldsymbol{b}^{(h)}\} \subset L$, and explicitly described logarithmic series solutions for a fake exponent and a set $B$ that satisfy certain conditions [6, Theorems 5.4, 6.2 and Remarks 5.6, 6.3].

In this paper, following [6], we continue to develop the perturbing method of constructing logarithmic series solutions to a regular $A$-hypergeometric system.

Fixing a fake exponent of an $A$-hypergeometric system, we consider some spaces of linear partial differential operators with constant coefficients. Comparing these spaces, we construct a fundamental system of series solutions with the given exponent by the perturbing method. In addition, we give a sufficient condition for a given fake exponent to be an exponent. As important examples of the main results, we give fundamental systems of series solutions to Aomoto-Gel'fand systems and to Lauricella's $F_C$ systems with special parameter vectors, respectively.

This paper is organized as follows. In Section 2, we first recall a power series to perturb from [6], associated with a fake exponent $\boldsymbol{v}$ and a linearly independent subset $B$ of $L$. In particular, we discuss properties of each term $a_{\boldsymbol{u}}(\boldsymbol{s})$ appearing in the series (for the definition of $a_{\boldsymbol{u}}(\boldsymbol{s})$, see (1)), and modify the series by changing the range of the sum from $\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$ in [6] to $\mathcal{N}$ which incorporates $B$. We give a refinement of [6, Theorem 6.2] as Theorem 2.7.

In Section 3, for a fake exponent $\boldsymbol{v}$ of the $A$-hypergeometric ideal $H_A(\boldsymbol{\beta})$ with respect to a generic weight vector $\boldsymbol{w}$, we recall the structure of the ideal $Q_{\boldsymbol{v}}$ associated with the fake indicial ideal $\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))$ and that of its orthogonal complement $Q_{\boldsymbol{v}}^{\perp}$ defined in [8, Sections 2.3 and 3.6]. We introduce ideals $P_{\mathcal{N}}$ and $P_B$ of $\mathbb{C}[\boldsymbol{s}]$, and their orthogonal complements $P_{\mathcal{N}}^{\perp}$ and $P_B^{\perp}$. Then we discuss relations among these ideals. Under a certain condition, we can derive $Q_{\boldsymbol{v}}^{\perp}$ as the image of a linear map from $P_{\mathcal{N}}^{\perp}$ (Proposition 3.11, Theorem 3.14).

In Section 4, we give a sufficient condition for a fake exponent $\boldsymbol{v}$ to be an exponent (Proposition 4.1). Then we construct a fundamental system of solutions with the exponent $\boldsymbol{v}$ (Theorem 4.4) by applying Theorem 2.7 and the results of Section 3 under the condition that $B$ is a basis of $L$, which is the main theorem of this paper.

In Sections 5 and 6, we deal with the Aomoto-Gel'fand systems and Lauricella's $F_C$ systems, which are important examples of $H_A(\boldsymbol{\beta})$. We discuss a fundamental system of solutions to $H_A(\mathbf{0})$ in each system. In each case, we have a unique fake exponent $\boldsymbol{v} = \mathbf{0}$. Taking a basis $B$ of $L$, we can obtain a fundamental system of series solutions for $\boldsymbol{\beta} = \mathbf{0}$.

## 2. Refinement of [6, Theorem 6.2]

In this section, we refine [6, Theorem 6.2].

Recall that for $\boldsymbol{v} = (v_1, \dots, v_n)^T \in \mathbb{C}^n$ its support $\operatorname{supp}(\boldsymbol{v})$ and its *negative support* $\operatorname{nsupp}(\boldsymbol{v})$ are defined as

$$\operatorname{supp}(\boldsymbol{v}) := \{j \in \{1, \dots, n\} \mid v_j \neq 0\},$$
$$\operatorname{nsupp}(\boldsymbol{v}) := \{j \in \{1, \dots, n\} \mid v_j \in \mathbb{Z}_{<0}\},$$

respectively.

For $\boldsymbol{v} \in \mathbb{C}^n$ and $\boldsymbol{u} \in \mathbb{N}^n$, set

$$[\boldsymbol{v}]_{\boldsymbol{u}} := \prod_{j=1}^{n} v_j(v_j - 1) \cdots (v_j - u_j + 1).$$

Here recall that $\mathbb{N} = \{0, 1, 2, \cdots\}$.

Note that we can uniquely write $\boldsymbol{u} \in \mathbb{Z}^n$ as the sum $\boldsymbol{u} = \boldsymbol{u}_+ - \boldsymbol{u}_-$ with $\boldsymbol{u}_+, \boldsymbol{u}_- \in \mathbb{N}^n$ and $\operatorname{supp}(\boldsymbol{u}_+) \cap \operatorname{supp}(\boldsymbol{u}_-) = \emptyset$.

Let $B = \{\boldsymbol{b}^{(1)}, \dots, \boldsymbol{b}^{(h)}\} \subset L$. We write the same symbol $B$ for the $n \times h$ matrix $(\boldsymbol{b}^{(1)}, \dots, \boldsymbol{b}^{(h)})$.

Set

$$\operatorname{supp}(B) := \bigcup_{k=1}^{h} \operatorname{supp}(\boldsymbol{b}^{(k)}) \subset \{1, \dots, n\},$$

which means the set of all labels for nonzero rows in $B$.

Let $\boldsymbol{s} = (s_1, \dots, s_h)^T$ be indeterminates, and let

$$(B\boldsymbol{s})_j := \sum_{k=1}^{h} b_j^{(k)} s_k \in \mathbb{C}[\boldsymbol{s}] := \mathbb{C}[s_1, \dots, s_h]$$

for $j = 1, \dots, n$. Set

$$(B\boldsymbol{s})^J := \prod_{j \in J} (B\boldsymbol{s})_j \in \mathbb{C}[\boldsymbol{s}]$$

for $J \subset \{1, \ldots, n\}$. Note that $(B\boldsymbol{s})_j = 0$ if $j \notin \mathrm{supp}(B)$, hence we have $(B\boldsymbol{s})^J = 0$ if $J \not\subset \mathrm{supp}(B)$.

**Lemma 2.1.** *Let $B = \{\boldsymbol{b}^{(1)}, \ldots, \boldsymbol{b}^{(h)}\} \subset L$, $\boldsymbol{u}, \boldsymbol{u}' \in L$ and $\boldsymbol{v} \in \mathbb{C}^n$. Let $\boldsymbol{s} = (s_1, \ldots, s_h)^T$ be indeterminates. Then $[\boldsymbol{v} + B\boldsymbol{s} + \boldsymbol{u}]_{\boldsymbol{u}'_+} \neq 0$ if and only if* $\mathrm{nsupp}(\boldsymbol{v} + \boldsymbol{u} - \boldsymbol{u}') \subset \mathrm{supp}(B) \cup \mathrm{nsupp}(\boldsymbol{v} + \boldsymbol{u})$. *In particular, $[\boldsymbol{v} + B\boldsymbol{s} + \boldsymbol{u}]_{\boldsymbol{u}_+} \neq 0$ if and only if* $\mathrm{nsupp}(\boldsymbol{v}) \subset \mathrm{supp}(B) \cup \mathrm{nsupp}(\boldsymbol{v} + \boldsymbol{u})$.

*Proof.* Note that

$$[\boldsymbol{v} + B\boldsymbol{s} + \boldsymbol{u}]_{\boldsymbol{u}'_+} = \prod_{j; u'_j > 0} (v_j + (B\boldsymbol{s})_j + u_j) \cdots (v_j + (B\boldsymbol{s})_j + u_j - u'_j + 1).$$

Hence, $[\boldsymbol{v} + B\boldsymbol{s} + \boldsymbol{u}]_{\boldsymbol{u}'_+} = 0$ if and only if there exists $j$ such that $v_j + u_j - u'_j \in \mathbb{Z}_{<0}$, $v_j + u_j \in \mathbb{N}$, and $b_j^{(k)} = 0$ for all $k$.

Hence $[\boldsymbol{v} + B\boldsymbol{s} + \boldsymbol{u}]_{\boldsymbol{u}'_+} = 0$ if and only if $\mathrm{nsupp}(\boldsymbol{v} + \boldsymbol{u} - \boldsymbol{u}') \not\subset \mathrm{supp}(B) \cup \mathrm{nsupp}(\boldsymbol{v} + \boldsymbol{u})$. □

Let $\boldsymbol{w}$ be a generic weight. Recall that $\boldsymbol{v}$ is called a *fake exponent* of $H_A(\boldsymbol{\beta})$ with respect to $\boldsymbol{w}$ if $A\boldsymbol{v} = \boldsymbol{\beta}$ and $[\boldsymbol{v}]_{\boldsymbol{u}_+} = 0$ for all $\boldsymbol{u} \in L$ with $\boldsymbol{u}_+ \cdot \boldsymbol{w} > \boldsymbol{u}_- \cdot \boldsymbol{w}$, where $\boldsymbol{u} \cdot \boldsymbol{w} = \sum_{j=1}^n u_j w_j$.

Throughout this paper, fix a generic weight $\boldsymbol{w}$, a fake exponent $\boldsymbol{v}$ of $H_A(\boldsymbol{\beta})$ with respect to $\boldsymbol{w}$.

We abbreviate $\mathrm{nsupp}(\boldsymbol{v} + \boldsymbol{u})$ to $I_{\boldsymbol{u}}$ for $\boldsymbol{u} \in L$. In particular, $I_{\boldsymbol{0}} = \mathrm{nsupp}(\boldsymbol{v})$. Then the condition $\mathrm{nsupp}(\boldsymbol{v}) \subset \mathrm{supp}(B) \cup \mathrm{nsupp}(\boldsymbol{v} + \boldsymbol{u})$ in Lemma 2.1 can be rewritten as

$$I_{\boldsymbol{0}} \subset \mathrm{supp}(B) \cup I_{\boldsymbol{u}}.$$

For $\boldsymbol{u} \in L$ with $I_{\boldsymbol{0}} \subset \mathrm{supp}(B) \cup I_{\boldsymbol{u}}$, let

$$a_{\boldsymbol{u}}(\boldsymbol{s}) := \frac{[\boldsymbol{v} + B\boldsymbol{s}]_{\boldsymbol{u}_-}}{[\boldsymbol{v} + B\boldsymbol{s} + \boldsymbol{u}]_{\boldsymbol{u}_+}}. \tag{1}$$

Note that the denominator is nonzero by Lemma 2.1.

**Lemma 2.2.** *Let $\boldsymbol{u}, \boldsymbol{u}' \in L$. Assume that $\boldsymbol{u}$ satisfies $I_{\boldsymbol{0}} \subset \mathrm{supp}(B) \cup I_{\boldsymbol{u}}$. Then the following hold.*

(i) *$a_{\boldsymbol{u}}(\boldsymbol{s}) \neq 0$ if and only if $I_{\boldsymbol{u}} \subset \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$, if and only if $\mathrm{supp}(B) \cup I_{\boldsymbol{u}} = \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$.*

(ii) *If $I_{\boldsymbol{u}} \cup I_{\boldsymbol{u}-\boldsymbol{u}'} \not\subset \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$, then $\partial^{\boldsymbol{u}'_+}(a_{\boldsymbol{u}}(\boldsymbol{s}) x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}}) = 0$.*

*Proof.* (i) We have $[\boldsymbol{v} + B\boldsymbol{s}]_{\boldsymbol{u}_-} = 0$ if and only if there exists $j$ such that $[v_j + \sum_{k=1}^h s_k b_j^{(k)}]_{-u_j} = [v_j]_{-u_j} = 0$, namely $v_j \in \mathbb{N}, v_j + u_j \in \mathbb{Z}_{<0}$,

and $b_j^{(k)} = 0$ for all $k$. Hence it is equivalent to saying that there exists $j$ such that $j \in I_{\boldsymbol{u}} \setminus (\mathrm{supp}(B) \cup I_{\boldsymbol{0}})$, or $I_{\boldsymbol{u}} \not\subset \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$.

By the assumption, the inclusion $I_{\boldsymbol{u}} \subset \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$ is equivalent to the equality $\mathrm{supp}(B) \cup I_{\boldsymbol{u}} = \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$.

(ii) Suppose that $I_{\boldsymbol{u}} \cup I_{\boldsymbol{u}-\boldsymbol{u}'} \not\subset \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$. Note that

$$\partial^{\boldsymbol{u}'_+}(a_{\boldsymbol{u}}(\boldsymbol{s})x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}}) = a_{\boldsymbol{u}}(\boldsymbol{s})[\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}]_{\boldsymbol{u}'_+}x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}-\boldsymbol{u}'_+}.$$

Hence, if $I_{\boldsymbol{u}} \not\subset \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$, then $\partial^{\boldsymbol{u}'_+}(a_{\boldsymbol{u}}(\boldsymbol{s})x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}}) = 0$ by (i).

If $I_{\boldsymbol{u}} \subset \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$, that is, $I_{\boldsymbol{u}-\boldsymbol{u}'} \setminus (\mathrm{supp}(B) \cup I_{\boldsymbol{u}}) \neq \emptyset$, then we have $[\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}]_{\boldsymbol{u}'_+} = 0$ and $\partial^{\boldsymbol{u}'_+}(a_{\boldsymbol{u}}(\boldsymbol{s})x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}}) = 0$ by Lemma 2.1. □

We recall the definitions related to $\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$ for a fake exponent $\boldsymbol{v}$ from [6] and modify them.

Let

$$\mathcal{G} := \left\{ \partial^{\boldsymbol{g}^{(i)}_+} - \partial^{\boldsymbol{g}^{(i)}_-} \,\middle|\, i = 1, \ldots, m \right\}$$

denote the reduced Gröbner basis of $I_A$ with respect to $\boldsymbol{w}$ with $\partial^{\boldsymbol{g}^{(i)}_+} \in \mathrm{in}_{\boldsymbol{w}}(I_A)$ for all $i$. Note that the $\mathcal{G}$ in [6, Section 4] should be *the reduced* Gröbner basis. Set

$$C(\boldsymbol{w}) := \sum_{i=1}^{m} \mathbb{N}\boldsymbol{g}^{(i)}.$$

A collection $\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$ of negative supports $I_{\boldsymbol{u}}$ ($\boldsymbol{u} \in L$) is defined by

$$\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}) := \{ I_{\boldsymbol{u}} \,|\, \boldsymbol{u} \in L. \text{ If } I_{\boldsymbol{u}} = I_{\boldsymbol{u}'} \text{ for } \boldsymbol{u}' \in L, \text{ then } \boldsymbol{u}' \in C(\boldsymbol{w}).\} .$$

In addition, define

$$\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c := \{ I_{\boldsymbol{u}} \,|\, \boldsymbol{u} \in L \} \setminus \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}).$$

We modify the definition of $\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$. To do this, in this paper we make the following assumption.

**Assumption 2.3.** A subset $B = \{\boldsymbol{b}^{(1)}, \ldots, \boldsymbol{b}^{(h)}\} \subset L$ is linearly independent, hence $\mathrm{rank}(B) = h$, and satisfies

$$I_{\boldsymbol{0}} \subset \mathrm{supp}(B) \cup I_{\boldsymbol{u}}$$

for any $\boldsymbol{u} \in L$.

*Remark* 2.4. (1) In the proofs of Lemma 2.6 and Theorem 2.7 below, we need the condition $I_{\boldsymbol{0}} \subset I_{\boldsymbol{u}-\boldsymbol{u}'} \cup \mathrm{supp}(B)$ for any $\boldsymbol{u}, \boldsymbol{u}' \in L$, which Assumption 2.3 guarantees.

(2) If a linearly independent set $B$ satisfies $I_{\boldsymbol{0}} \subset \mathrm{supp}(B)$, then it satisfies Assumption 2.3. For example, this condition holds for each of the following cases:

(i) $\mathrm{supp}(B) = \{1, \ldots, n\}$,
(ii) $\mathrm{nsupp}(\boldsymbol{v}) = \emptyset$.

(3) If $B$ is a basis of $L$, then $B$ satisfies

$$\mathrm{supp}(B) \cup I_{\boldsymbol{u}} = \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$$

for all $\boldsymbol{u} \in L$. Indeed, since $B$ is a basis of $L$, we see that if $j \notin \mathrm{supp}(B)$ then $\boldsymbol{u}_j = 0$ for all $\boldsymbol{u} \in L$. This implies that $I_{\boldsymbol{u}} \setminus \mathrm{supp}(B) = I_{\boldsymbol{0}} \setminus \mathrm{supp}(B)$ for all $\boldsymbol{u} \in L$.

Define a subset $\mathcal{N}$ of $\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$ as a modification of $\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})$ by

$$\mathcal{N} := \{I_{\boldsymbol{u}} \,|\, \boldsymbol{u} \in L, \mathrm{supp}(B) \cup I_{\boldsymbol{u}} = \mathrm{supp}(B) \cup I_{\boldsymbol{0}}\} \cap \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}), \tag{2}$$

and set

$$\mathcal{N}^c := \{I_{\boldsymbol{u}} \,|\, \boldsymbol{u} \in L, \mathrm{supp}(B) \cup I_{\boldsymbol{u}} = \mathrm{supp}(B) \cup I_{\boldsymbol{0}}\} \cap \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c.$$

Consider the subset $L'$ of $L$ defined by

$$L' := \{\boldsymbol{u} \in L \,|\, I_{\boldsymbol{u}} \in \mathcal{N}\}. \tag{3}$$

By definition, we see that $L' \subset C(\boldsymbol{w})$.

Let

$$K_{\mathcal{N}} := \bigcap_{I \in \mathcal{N}} I,$$

and define the homogeneous ideal $P_{\mathcal{N}}$ of $\mathbb{C}[\boldsymbol{s}]$ for $\mathcal{N}$ as

$$P_{\mathcal{N}} := \left\langle (B\boldsymbol{s})^{I \cup J \setminus K_{\mathcal{N}}} \,\middle|\, I \in \mathcal{N}, J \in \mathcal{N}^c \right\rangle. \tag{4}$$

In addition, we define the orthogonal complement $P^{\perp}$ for a homogeneous ideal $P \subset \mathbb{C}[\boldsymbol{s}]$ as

$$\begin{aligned} P^{\perp} &:= \{q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}] \,|\, (q(\partial_{\boldsymbol{s}}) \bullet h(\boldsymbol{s}))|_{\boldsymbol{s}=\boldsymbol{0}} = 0 \text{ for all } h(\boldsymbol{s}) \in P\} \\ &= \{q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}] \,|\, q(\partial_{\boldsymbol{s}}) \bullet P \subset \langle s_1, \ldots, s_h \rangle\}, \end{aligned} \tag{5}$$

where $\mathbb{C}[\partial_{\boldsymbol{s}}] := \mathbb{C}[\partial_{s_1}, \ldots, \partial_{s_h}]$, and the symbol $\bullet$ denotes the natural action of $\mathbb{C}[\partial_{\boldsymbol{s}}]$ on polynomials of $\mathbb{C}[\boldsymbol{s}]$. Since $P$ and $\langle s_1, \ldots, s_h \rangle$ are both homogeneous, $P^{\perp}$ is homogeneous with respect to the usual total ordering.

**Example 2.5.** (cf. [6, Examples 3.3, 4.8, 6.4]) Let $n = 5$, $d = 3$, and

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 1 & 0 \end{bmatrix}.$$

Let $\boldsymbol{\beta} = (1,0,0)^T$ and $\boldsymbol{w} = (1,1,1,1,0)$. Then $\boldsymbol{v} = (0,0,0,0,1)^T$ is the unique exponent, and

$$\mathcal{G} = \{\underline{\partial_{x_1}\partial_{x_3}} - \partial_{x_5}^2, \underline{\partial_{x_2}\partial_{x_4}} - \partial_{x_5}^2\},$$

where the underlined terms are the leading ones. Put $\boldsymbol{g}^{(1)} := (1,0,1,0,-2)^T$ and $\boldsymbol{g}^{(2)} := (0,1,0,1,-2)^T$. Recall that

$$\begin{aligned}\mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}) &= \{\emptyset = I_{\boldsymbol{0}}, \{5\}\},\\ \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c &= \{\{1,3\},\{2,4\},\{1,3,5\},\{2,4,5\},\{1,2,3,4\}\}.\end{aligned}$$

Let $B := \{\boldsymbol{g}^{(1)}, \boldsymbol{g}^{(2)}\}$. Then we have $\mathrm{supp}(B) = \{1,2,3,4,5\}$, and

$$\begin{aligned}\mathcal{N} &= \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}),\\ \mathcal{N}^c &= \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c,\\ K_{\mathcal{N}} &= \emptyset.\end{aligned}$$

The homogeneous ideal $P_{\mathcal{N}} \subset \mathbb{C}[\boldsymbol{s}] = \mathbb{C}[s_1, s_2]$ and the vector space $P_{\mathcal{N}}^{\perp} \subset \mathbb{C}[\partial_{\boldsymbol{s}}] = \mathbb{C}[\partial_{s_1}, \partial_{s_2}]$ are given as

$$\begin{aligned}P_{\mathcal{N}} &= \langle (B\boldsymbol{s})^{\{1,3\}}, (B\boldsymbol{s})^{\{2,4\}} \rangle = \langle s_1^2, s_2^2 \rangle,\\ P_{\mathcal{N}}^{\perp} &= \{q(\partial_{s_1}, \partial_{s_2}) \in \mathbb{C}[\partial_{s_1}, \partial_{s_2}] \,|\, q(\partial_{s_1}, \partial_{s_2}) \bullet \langle s_1^2, s_2^2 \rangle \subset \langle s_1, s_2 \rangle\}\\ &= \mathbb{C}1 + \mathbb{C}\partial_{s_1} + \mathbb{C}\partial_{s_2} + \mathbb{C}\partial_{s_1}\partial_{s_2}.\end{aligned}$$

We consider another case. Let $B_1 = \{\boldsymbol{g}^{(1)}\}$. Then we have $\mathrm{supp}(B_1) = \{1,3,5\}$ and

$$\begin{aligned}\mathcal{N}_1 &= \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v}) = \{\emptyset = I_{\boldsymbol{0}}, \{5\}\},\\ \mathcal{N}_1^c &= \{\{1,3\}, \{1,3,5\}\},\\ K_{\mathcal{N}_1} &= \emptyset.\end{aligned}$$

The homogeneous ideal $P_{\mathcal{N}_1} \subset \mathbb{C}[s]$ and the vector space $P_{\mathcal{N}_1}^{\perp} \subset \mathbb{C}[\partial_s]$ are given as

$$\begin{aligned}P_{\mathcal{N}_1} &= \langle (B_1 s)^{\{1,3\}} \rangle = \langle s^2 \rangle,\\ P_{\mathcal{N}_1}^{\perp} &= \{q(\partial_s) \in \mathbb{C}[\partial_s] \,|\, q(\partial_s) \bullet \langle s^2 \rangle \subset \langle s \rangle\} = \mathbb{C}1 + \mathbb{C}\partial_s.\end{aligned}$$

Throughout this paper, put

$$m(\boldsymbol{s}) := (B\boldsymbol{s})^{I_{\boldsymbol{0}} \setminus K_{\mathcal{N}}}.$$

The following lemma guarantees that we may plug $\boldsymbol{s} = \boldsymbol{0}$ into the series appearing in Theorem 2.7.

**Lemma 2.6.** *Let $\mathcal{N}$ be the set defined by* (2)*, and let $\boldsymbol{u}, \boldsymbol{u}' \in L$. Then, under Assumption 2.3, each term of the power series for $m(\boldsymbol{s}) \cdot a_{\boldsymbol{u}}(\boldsymbol{s}) \cdot [\boldsymbol{v} + B\boldsymbol{s} + \boldsymbol{u}]_{\boldsymbol{u}'_+}$ in the indeterminates $\boldsymbol{s}$ is divided by $(B\boldsymbol{s})^{I_{\boldsymbol{u}} \cup I_{\boldsymbol{u}-\boldsymbol{u}'} \setminus K_{\mathcal{N}}}$.*

*Proof.* By [6, Lemma 6.1], there exists a formal power series $g(\boldsymbol{y})$ in the indeterminates $\boldsymbol{y} = (y_1, \ldots, y_n)$ such that

$$\begin{aligned} & a_{\boldsymbol{u}}(\boldsymbol{s}) \cdot [\boldsymbol{v} + B\boldsymbol{s} + \boldsymbol{u}]_{\boldsymbol{u}'_+} \\ &= \frac{[\boldsymbol{v} + B\boldsymbol{s}]_{\boldsymbol{u}_-}}{[\boldsymbol{v} + B\boldsymbol{s} + \boldsymbol{u}]_{\boldsymbol{u}_+}} \cdot [\boldsymbol{v} + B\boldsymbol{s} + \boldsymbol{u}]_{\boldsymbol{u}'_+} \\ &= \left( \frac{(B\boldsymbol{s})^{I_{\boldsymbol{u}} \setminus I_{\boldsymbol{0}}}}{(B\boldsymbol{s})^{I_{\boldsymbol{0}} \setminus I_{\boldsymbol{u}}}} \cdot (B\boldsymbol{s})^{I_{\boldsymbol{u}-\boldsymbol{u}'} \setminus I_{\boldsymbol{u}}} \right) \cdot g((B\boldsymbol{s})_1, \ldots, (B\boldsymbol{s})_n) \\ &= \frac{(B\boldsymbol{s})^{(I_{\boldsymbol{u}} \cup I_{\boldsymbol{u}-\boldsymbol{u}'}) \setminus I_{\boldsymbol{0}}}}{(B\boldsymbol{s})^{I_{\boldsymbol{0}} \setminus (I_{\boldsymbol{u}} \cup I_{\boldsymbol{u}-\boldsymbol{u}'})}} \cdot g((B\boldsymbol{s})_1, \ldots, (B\boldsymbol{s})_n). \end{aligned}$$

Hence we have

$$\begin{aligned} & m(\boldsymbol{s}) \cdot a_{\boldsymbol{u}}(\boldsymbol{s}) \cdot [\boldsymbol{v} + B\boldsymbol{s} + \boldsymbol{u}]_{\boldsymbol{u}'_+} \\ &= (B\boldsymbol{s})^{I_{\boldsymbol{0}} \setminus K_{\mathcal{N}}} \cdot \frac{(B\boldsymbol{s})^{(I_{\boldsymbol{u}} \cup I_{\boldsymbol{u}-\boldsymbol{u}'}) \setminus I_{\boldsymbol{0}}}}{(B\boldsymbol{s})^{I_{\boldsymbol{0}} \setminus (I_{\boldsymbol{u}} \cup I_{\boldsymbol{u}-\boldsymbol{u}'})}} \cdot g((B\boldsymbol{s})_1, \ldots, (B\boldsymbol{s})_n) \\ &= (B\boldsymbol{s})^{I_{\boldsymbol{u}} \cup I_{\boldsymbol{u}-\boldsymbol{u}'} \setminus K_{\mathcal{N}}} \cdot g((B\boldsymbol{s})_1, \ldots, (B\boldsymbol{s})_n), \end{aligned}$$

and the assertion holds. □

We can refine the main results [6, Theorem 5.4, Theorem 6.2] as follows.

**Theorem 2.7.** *Let $\mathcal{N}$ be the set defined by* (2)*. Set*

$$F_{\mathcal{N}}(\boldsymbol{x}, \boldsymbol{s}) := \sum_{\boldsymbol{u} \in L'} a_{\boldsymbol{u}}(\boldsymbol{s}) x^{\boldsymbol{v} + B\boldsymbol{s} + \boldsymbol{u}},$$

*and*

$$\widetilde{F}_{\mathcal{N}}(\boldsymbol{x}, \boldsymbol{s}) := m(\boldsymbol{s}) F_{\mathcal{N}}(\boldsymbol{x}, \boldsymbol{s}),$$

*where $L'$ is defined by* (3)*.*

*Then $(q(\partial_s) \bullet \widetilde{F}_{\mathcal{N}}(\boldsymbol{x}, \boldsymbol{s}))_{|\boldsymbol{s}=\boldsymbol{0}}$ are solutions to $M_A(\boldsymbol{\beta})$ for any $q(\partial_{\boldsymbol{s}}) \in P_{\mathcal{N}}^{\perp}$.*

*Proof.* Let $\boldsymbol{u}' \in L$ and $\boldsymbol{u} \in L'$. If $\partial^{\boldsymbol{u}'_+}(a_{\boldsymbol{u}}(\boldsymbol{s}) x^{\boldsymbol{v} + B\boldsymbol{s} + \boldsymbol{u}}) \neq 0$, then we have $I_{\boldsymbol{u}} \cup I_{\boldsymbol{u}-\boldsymbol{u}'} \subset \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$ by Lemma 2.2, and hence $\mathrm{supp}(B) \cup I_{\boldsymbol{u}-\boldsymbol{u}'} = \mathrm{supp}(B) \cup I_{\boldsymbol{0}}$ by Assumption 2.3. Thus $I_{\boldsymbol{u}-\boldsymbol{u}'} \notin \mathcal{N}$ implies $I_{\boldsymbol{u}-\boldsymbol{u}'} \in \mathcal{N}^c$.

Similar to the arguments in the proofs of [6, Theorem 5.4, Theorem 6.2], we see that

$$\begin{aligned}(\partial^{\boldsymbol{u}'_+} - \partial^{\boldsymbol{u}'_-}) \bullet \widetilde{F}_{\mathcal{N}}(\boldsymbol{x}, \boldsymbol{s}) \\ = \sum_{\boldsymbol{u}\in L', I_{\boldsymbol{u}-\boldsymbol{u}'}\in\mathcal{N}^c} m(\boldsymbol{s})\partial^{\boldsymbol{u}'+}(a_{\boldsymbol{u}}(\boldsymbol{s})x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}}) \\ - \sum_{\boldsymbol{u}\in L', I_{\boldsymbol{u}+\boldsymbol{u}'}\in\mathcal{N}^c} m(\boldsymbol{s})\partial^{(-\boldsymbol{u})'+}(a_{\boldsymbol{u}}(\boldsymbol{s})x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}}).\end{aligned}$$

Let $q(\partial_{\boldsymbol{s}}) \in P_{\mathcal{N}}^{\perp}$. Then the series $(q(\partial_s) \bullet \widetilde{F}_{\mathcal{N}}(\boldsymbol{x}, \boldsymbol{s}))_{|\boldsymbol{s}=\boldsymbol{0}}$ is a solution to $M_A(\boldsymbol{\beta})$ if

$$\left(q(\partial_s) \bullet \left(m(\boldsymbol{s})\partial^{\boldsymbol{u}'+}a_{\boldsymbol{u}}(\boldsymbol{s})x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}}\right)\right)_{|\boldsymbol{s}=\boldsymbol{0}} = 0$$

for any $\boldsymbol{u} \in L'$ and $\boldsymbol{u}' \in L$ with $I_{\boldsymbol{u}-\boldsymbol{u}'} \in \mathcal{N}^c$.

By Lemma 2.6, each coefficient of

$$\begin{aligned}m(\boldsymbol{s})(\partial^{\boldsymbol{u}'+}a_{\boldsymbol{u}}(\boldsymbol{s})x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}}) \\ = m(\boldsymbol{s})a_{\boldsymbol{u}}(\boldsymbol{s})[\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}]_{\boldsymbol{u}'_+}x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}-\boldsymbol{u}'_+}\end{aligned}$$

in the indeterminates $\boldsymbol{s}$ is divided by $(B\boldsymbol{s})^{I_{\boldsymbol{u}}\cup I_{\boldsymbol{u}-\boldsymbol{u}'}\setminus K_{\mathcal{N}}}$, hence belongs to $P_{\mathcal{N}}$. By the definition of $P_{\mathcal{N}}^{\perp}$, the assertion holds. □

## 3. Relations between $P_{\mathcal{N}}^{\perp}$ and $Q_{\boldsymbol{v}}^{\perp}$

In this section, we recall $Q_{\boldsymbol{v}}$ and its orthogonal complement $Q_{\boldsymbol{v}}^{\perp}$ defined in [8, Section 2.3], and discuss relations between $P_{\mathcal{N}}^{\perp}$ and $Q_{\boldsymbol{v}}^{\perp}$. For the definitions of $P_{\mathcal{N}}$ and $P_{\mathcal{N}}^{\perp}$, see (4) and (5).

Consider the fake indicial ideal $\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))$ of $H_A(\boldsymbol{\beta})$ with respect to $\boldsymbol{w}$:

$$\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta})) := \langle A\theta_{\boldsymbol{x}} - \boldsymbol{\beta}\rangle + \widetilde{\mathrm{in}}_{\boldsymbol{w}}(I_A) \subset \mathbb{C}[\theta_{\boldsymbol{x}}] := \mathbb{C}[\theta_1, \ldots, \theta_n].$$

Here $\widetilde{\mathrm{in}}_{\boldsymbol{w}}(I_A)$ is the distraction of the initial ideal $\mathrm{in}_{\boldsymbol{w}}(I_A)$ with respect to $\boldsymbol{w}$ (cf. [8, Section 3.1]). Related to the reduced Gröbner basis $\mathcal{G} = \{\partial^{\boldsymbol{g}_+^{(i)}} - \partial^{\boldsymbol{g}_-^{(i)}} \mid i = 1, \ldots, m\}$ of $I_A$ with respect to $\boldsymbol{w}$ with $\partial^{\boldsymbol{g}_+^{(i)}} \in \mathrm{in}_{\boldsymbol{w}}(I_A)$ for all $i$, define

$$G^{(i)} := I_{-\boldsymbol{g}^{(i)}} \setminus I_{\boldsymbol{0}} = \{j \in \{1, \ldots, n\} \mid v_j \in \mathbb{N}, g_j^{(i)} - v_j > 0\}$$

for $i = 1, \ldots, m$. Since

$$\widetilde{\mathrm{in}}_{\boldsymbol{w}}(I_A) = \left\langle [\theta_{\boldsymbol{x}}]_{\boldsymbol{g}_+^{(i)}} := \prod_{j;\, g_j^{(i)}>0} \prod_{\nu=0}^{g_j^{(i)}-1} (\theta_j - \nu) \,\middle|\, i = 1, \ldots, m \right\rangle$$

by [8, Theorem 3.2.2], we see that its primary component at a fake exponent $\boldsymbol{v}$ is

$$\widetilde{\mathrm{in}}_{\boldsymbol{w}}(I_A)_{\boldsymbol{v}} = \left\langle ([\theta]_{\boldsymbol{g}_+^{(i)}})_{\boldsymbol{v}} := \prod_{j\in G^{(i)}} (\theta_j - v_j) \,\middle|\, i = 1,\dots,m \right\rangle. \tag{6}$$

We obtain the homogeneous ideal $Q_{\boldsymbol{v}}$ of $\mathbb{C}[\theta_{\boldsymbol{x}}]$ from $\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))_{\boldsymbol{v}}$ by replacing $\theta_j \mapsto \theta_j + v_j$ for $j = 1,\dots,n$ (cf. [8, Section 2.3]). Namely,

$$Q_{\boldsymbol{v}} = \langle\, A\theta_{\boldsymbol{x}} \,\rangle + \left\langle \prod_{j\in G^{(i)}} \theta_j \,\middle|\, i = 1,\dots,m \right\rangle. \tag{7}$$

The orthogonal complement $Q_{\boldsymbol{v}}^{\perp}$ of $Q_{\boldsymbol{v}}$ is defined by

$$Q_{\boldsymbol{v}}^{\perp} := \{f \in \mathbb{C}[\boldsymbol{x}] \,|\, \phi(\partial_{\boldsymbol{x}})(f) = 0 \text{ for all } \phi = \phi(\theta_{\boldsymbol{x}}) \in Q_{\boldsymbol{v}}\}.$$

Note that $Q_{\boldsymbol{v}}^{\perp}$ is a graded $\mathbb{C}$-vector space with the usual grading.

**Proposition 3.1.** *Let $f(\boldsymbol{x})$ be a polynomial. Then $x^{\boldsymbol{v}} f(\log \boldsymbol{x})$ is a solution to $\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))$ if and only if $f(\boldsymbol{x})$ satisfies the following conditions:*

(i) $f(\boldsymbol{x}) \in \mathbb{C}[\boldsymbol{x}G] := \mathbb{C}[\boldsymbol{x}\boldsymbol{g}^{(1)},\dots,\boldsymbol{x}\boldsymbol{g}^{(m)}]$.
(ii) $\partial_{\boldsymbol{x}}^{G^{(i)}} \bullet f(\boldsymbol{x}) = 0$ *for all* $i = 1,\dots,m$.

*Here*

$$\boldsymbol{x}G := (\boldsymbol{x}\boldsymbol{g}^{(1)},\dots,\boldsymbol{x}\boldsymbol{g}^{(m)}) = \left(\sum_{j=1}^{n} g_j^{(1)} x_j, \dots, \sum_{j=1}^{n} g_j^{(m)} x_j\right)$$

*for $\boldsymbol{x} = (x_1,\dots,x_n)$.*

*Proof.* By [8, Theorem 2.3.11], the function $x^{\boldsymbol{v}} f(\log \boldsymbol{x})$ is a solution to $\mathrm{find}_{\boldsymbol{w}}(H_A(\beta))$ if and only if $f(\boldsymbol{x}) \in Q_{\boldsymbol{v}}^{\perp}$. From $f(\boldsymbol{x}) \in \langle A\partial_{\boldsymbol{x}}\rangle^{\perp}$, we see (i) [5, Lemma 5.1]. (ii) follows from Equation (6). □

**Example 3.2** (Continuation of Example 2.5)**.** Note that $\boldsymbol{v} - \boldsymbol{g}^{(1)} = (-1,0,-1,0,3)^T$ and $\boldsymbol{v} - \boldsymbol{g}^{(2)} = (0,-1,0,-1,3)^T$. Thus we see that

$$G^{(1)} = I_{-\boldsymbol{g}^{(1)}} \setminus I_{\boldsymbol{0}} = \mathrm{nsupp}(\boldsymbol{v} - \boldsymbol{g}^{(1)}) \setminus \mathrm{nsupp}(\boldsymbol{v}) = \{1,3\},$$
$$G^{(2)} = I_{-\boldsymbol{g}^{(2)}} \setminus I_{\boldsymbol{0}} = \mathrm{nsupp}(\boldsymbol{v} - \boldsymbol{g}^{(1)}) \setminus \mathrm{nsupp}(\boldsymbol{v}) = \{2,4\}.$$

The ideal $Q_{\boldsymbol{v}} \subset \mathbb{C}[\theta_{\boldsymbol{x}}] = \mathbb{C}[\theta_1,\theta_2,\theta_3,\theta_4,\theta_5]$ is given as

$$Q_{\boldsymbol{v}} = \langle \theta_1+\theta_2+\theta_3+\theta_4+\theta_5, -\theta_1+\theta_2+\theta_3-\theta_4, -\theta_1-\theta_2+\theta_3+\theta_4, \theta_1\theta_3, \theta_2\theta_4\rangle.$$

In addition, we see that

$$Q_{\boldsymbol{v}}^{\perp} = \mathbb{C}\cdot 1 + \mathbb{C}\cdot \boldsymbol{x}\boldsymbol{g}^{(1)} + \mathbb{C}\cdot \boldsymbol{x}\boldsymbol{g}^{(2)} + \mathbb{C}\cdot (\boldsymbol{x}\boldsymbol{g}^{(1)})\cdot(\boldsymbol{x}\boldsymbol{g}^{(2)}).$$

To compare $Q_{\boldsymbol{v}}$ with $P_{\mathcal{N}}$, we consider the graded ring homomorphism $\Phi_B : \mathbb{C}[\theta_{\boldsymbol{x}}] \to \mathbb{C}[\boldsymbol{s}]$ defined by $\theta_j \mapsto (B\boldsymbol{s})_j$ for $j = 1, \ldots, n$. By the linear independence of $B$, we see that $\Phi_B$ is surjective. Define $P_B := \Phi_B(Q_{\boldsymbol{v}})$. By the ring isomorphism theorem, $\Phi_B$ induces the ring isomorphism

$$\widetilde{\Phi}_B : \mathbb{C}[\theta_{\boldsymbol{x}}]/\Phi_B^{-1}(P_B) \simeq \mathbb{C}[\boldsymbol{s}]/P_B.$$

Since $\langle A\theta_{\boldsymbol{x}} \rangle$ is vanished by $\Phi_B$, we have

$$P_B = \left\langle (B\boldsymbol{s})^{G^{(i)}} \,\middle|\, i = 1, \ldots, m \right\rangle. \tag{8}$$

**Proposition 3.3.** *Let $J \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c$. Then $G^{(i)} \subset J \setminus I_{\boldsymbol{0}}$ for some $i$.*

*Proof.* By definition and [6, Lemma 4.2], we see that there exists $\boldsymbol{u} \in L \setminus C(\boldsymbol{w})$ such that $J = I_{\boldsymbol{u}}$ and $\partial^{\boldsymbol{u}_+} \notin \mathrm{in}_{\boldsymbol{w}}(I_A)$. Hence $\partial^{\boldsymbol{u}_-} = \mathrm{in}_{\boldsymbol{w}}(\partial^{\boldsymbol{u}_-} - \partial^{\boldsymbol{u}_+})$ is divided by some $\partial^{\boldsymbol{g}_+^{(i)}}$. Let $j \in G^{(i)} = I_{-\boldsymbol{g}_+^{(i)}} \setminus I_{\boldsymbol{0}}$. Then $v_j \in \mathbb{N}$ and $v_j - g_j^{(i)} \in \mathbb{Z}_{<0}$. Since $g_j^{(i)} \in \mathbb{Z}_{>0}$, we see that $g_j^{(i)} \leq -u_j$ and $v_j + u_j \leq v_j - g_j^{(i)} < 0$. Thus we have $j \in I_{\boldsymbol{u}} \setminus I_{\boldsymbol{0}} = J \setminus I_{\boldsymbol{0}}$. □

Three ideals $Q_{\boldsymbol{v}}$, $P_{\mathcal{N}}$, and $P_B$ are related as follows.

**Proposition 3.4.** *Let $Q_{\boldsymbol{v}}$, $P_{\mathcal{N}}$, and $P_B$ be the ones in* (7), (4), *and* (8), *respectively. Then, the following hold.*

(i) *$m(\boldsymbol{s}) \cdot P_B \subset P_{\mathcal{N}} \subset P_B$. In particular, if $K_{\mathcal{N}} = I_0$, then $P_{\mathcal{N}} = P_B$.*
(ii) *If $B$ is a basis of $L$, then $\Phi_B^{-1}(P_B) = Q_{\boldsymbol{v}}$.*

*Proof.* (i) Let $I \in \mathcal{N}$ and $J \in \mathcal{N}^c$. Since $J \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c$ and $K_{\mathcal{N}} \subset I_{\boldsymbol{0}}$, $I \cup J \setminus K_{\mathcal{N}}$ contains some $G^{(i)}$ by Proposition 3.3. Hence the inclusion $P_{\mathcal{N}} \subset P_B$ holds.

For any $i = 1, \ldots, m$, since $-\boldsymbol{g}^{(i)} \notin C(\boldsymbol{w})$ we see that $I_{-\boldsymbol{g}^{(i)}} \in \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c$. If $I_{-\boldsymbol{g}^{(i)}} \notin \mathcal{N}^c$, then

$$\mathrm{supp}(B) \cup I_{-\boldsymbol{g}^{(i)}} \neq \mathrm{supp}(B) \cup I_{\boldsymbol{0}}. \tag{9}$$

By Assumption 2.3, (9) implies that $G^{(i)} = I_{-\boldsymbol{g}^{(i)}} \setminus I_{\boldsymbol{0}} \not\subset \mathrm{supp}(B)$. Hence we have $(B\boldsymbol{s})^{G^{(i)}} = 0$. If $I_{-\boldsymbol{g}^{(i)}} \in \mathcal{N}^c$, then

$$m(\boldsymbol{s})(B\boldsymbol{s})^{G^{(i)}} = (B\boldsymbol{s})^{I_{\boldsymbol{0}} \cup I_{-\boldsymbol{g}^{(i)}} \setminus K_{\mathcal{N}}} \in P_{\mathcal{N}}.$$

Hence we have $m(\boldsymbol{s}) \cdot P_B \subset P_{\mathcal{N}}$.

(ii) Since $B$ is a basis of $L$, we have $\mathrm{Ker}(\Phi_B) = \langle A\theta_{\boldsymbol{x}} \rangle$. Thus the assertion ${\Phi_B}^{-1}(P_B) = Q_{\boldsymbol{v}}$ holds from (7). □

**Example 3.5** (Continuation of Example 2.5 and 3.2)**.** Consider the case where $B = \{\boldsymbol{g}^{(1)}, \boldsymbol{g}^{(2)}\}$. Then we have $m(\boldsymbol{s}) = (B\boldsymbol{s})^{\emptyset} = 1$, and

$$P_B = \langle (B\boldsymbol{s})^{G^{(1)}}, (B\boldsymbol{s})^{G^{(2)}} \rangle = \langle s_1^2, s_2^2 \rangle = P_{\mathcal{N}}.$$

Furthermore, since $B$ is a basis of $L$, we see that

$$\Phi_B^{-1}(P_B) = \Phi_B^{-1}(\langle (B\boldsymbol{s})^{G^{(1)}}, (B\boldsymbol{s})^{G^{(2)}} \rangle) = \langle \theta_1\theta_3, \theta_2\theta_4 \rangle + \langle A\theta_{\boldsymbol{x}} \rangle = Q_{\boldsymbol{v}}.$$

Consider the other case where $B_1 = \{\boldsymbol{g}^{(1)}\}$. Then we have $m(s) = (B_1 s)^{\emptyset} = 1$, and

$$P_{B_1} = \langle (B_1 s)^{G^{(1)}} \rangle = \langle s^2 \rangle = P_{\mathcal{N}_1}.$$

We see that $B_1$ does not span $L$ and that

$$\Phi_B^{-1}(P_{B_1}) = \Phi_{B_1}^{-1}(\langle (B_1 s)^{G^{(1)}} \rangle) = \langle \theta_1\theta_3 \rangle + \langle A\theta_{\boldsymbol{x}} \rangle \subsetneq Q_{\boldsymbol{v}}.$$

We consider relations between $P_{\mathcal{N}}^{\perp}$ and $P_B^{\perp}$, and between $P_B^{\perp}$ and $Q_{\boldsymbol{v}}^{\perp}$. Recall the construction of a basis of orthogonal complements in [8, Section 2.3].

Let $P$ be a homogeneous ideal of $\mathbb{C}[\boldsymbol{s}]$. Fix any term order $\prec$ on $\mathbb{C}[\boldsymbol{s}]$, and let $\mathcal{H} \subset \mathbb{C}[\boldsymbol{s}]$ be the reduced Gröbner basis of $P$ with respect to $\prec$. For any $\boldsymbol{\mu} \in \mathbb{N}^h$ with $\boldsymbol{s}^{\boldsymbol{\mu}} \in \mathrm{in}_{\prec}(P)$, there exist unique $c_{\boldsymbol{\mu},\boldsymbol{\nu}} \in \mathbb{C}$ for $\boldsymbol{\nu} \in \mathbb{N}^h$ with $|\boldsymbol{\nu}| = |\boldsymbol{\mu}|$ and $\boldsymbol{s}^{\boldsymbol{\nu}} \notin \mathrm{in}_{\prec}(P)$ such that

$$p_{\boldsymbol{\mu}}(\boldsymbol{s}) := \boldsymbol{s}^{\boldsymbol{\mu}} - \sum_{\substack{\boldsymbol{\nu} \in \mathbb{N}^h;\, |\boldsymbol{\nu}| = |\boldsymbol{\mu}|, \\ \boldsymbol{s}^{\boldsymbol{\nu}} \notin \mathrm{in}_{\prec}(P)}} c_{\boldsymbol{\mu},\boldsymbol{\nu}} \boldsymbol{s}^{\boldsymbol{\nu}} \in P.$$

We obtain $p_{\boldsymbol{\mu}}(\boldsymbol{s})$ by taking the normal form modulo $\mathcal{H}$ for the monomial $\boldsymbol{s}^{\boldsymbol{\mu}}$. For $\boldsymbol{\nu} \in \mathbb{N}^h$ with $\boldsymbol{s}^{\boldsymbol{\nu}} \notin \mathrm{in}_{\prec}(P)$, define the homogeneous polynomial $q_{\boldsymbol{\nu}}(\partial_{\boldsymbol{s}})$ of degree $|\boldsymbol{\nu}|$ by

$$q_{\boldsymbol{\nu}}(\partial_{\boldsymbol{s}}) := \frac{1}{\boldsymbol{\nu}!} \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}} + \sum_{\substack{\boldsymbol{\mu} \in \mathbb{N}^h;\, |\boldsymbol{\mu}| = |\boldsymbol{\nu}|, \\ \boldsymbol{s}^{\boldsymbol{\mu}} \in \mathrm{in}_{\prec}(P)}} \frac{c_{\boldsymbol{\mu},\boldsymbol{\nu}}}{\boldsymbol{\mu}!} \partial_{\boldsymbol{s}}^{\boldsymbol{\mu}} \in \mathbb{C}[\partial_{\boldsymbol{s}}].$$

**Lemma 3.6.** *Let $P$ be a homogeneous ideal of $\mathbb{C}[\boldsymbol{s}]$. Fix any term order $\prec$ on $\mathbb{C}[\boldsymbol{s}]$, and let $\mathcal{H} \subset \mathbb{C}[\boldsymbol{s}]$ be the reduced Gröbner basis of $P$ with respect to $\prec$. Then*

$$\{p_{\boldsymbol{\mu}}(\boldsymbol{s}) \mid \boldsymbol{s}^{\boldsymbol{\mu}} \in \mathrm{in}_{\prec}(P)\}$$

*and*

$$\{q_{\boldsymbol{\nu}}(\partial_{\boldsymbol{s}}) \mid \boldsymbol{\nu} \in \mathbb{N}^h \text{ with } \boldsymbol{s}^{\boldsymbol{\nu}} \notin \mathrm{in}_{\prec}(P)\}$$

*form $\mathbb{C}$-bases of $P$ and $P^{\perp}$, respectively.*

*Proof.* This is similar to [8, Proposition 2.3.13]. □

**Lemma 3.7.** *Let $P$ and $\widetilde{P}$ be homogeneous ideals of $\mathbb{C}[\boldsymbol{s}]$. Then $P \subset \widetilde{P}$ if and only if $P^\perp \supset \widetilde{P}^\perp$.*

*Proof.* Assume that $P \subset \widetilde{P}$. By definition, $P^\perp \supset \widetilde{P}^\perp$ is clear.

Conversely, assume that $P^\perp \supset \widetilde{P}^\perp$. Fix any term order $\prec$ on $\mathbb{C}[\boldsymbol{s}]$, and let $\widetilde{\mathcal{H}}$ be the reduced Gröbner basis of $\widetilde{P}$.

Let $\{\widetilde{q}_{\boldsymbol{\nu}}(\partial_{\boldsymbol{s}}) \,|\, \boldsymbol{\nu} \in \mathbb{N}^h \text{ with } \boldsymbol{s}^{\boldsymbol{\nu}} \notin \mathrm{in}_\prec(\widetilde{P})\}$ be the $\mathbb{C}$-basis of $\widetilde{P}^\perp$ as in Lemma 3.6. Let $p \in P$. Applying the division algorithm with respect to $\widetilde{\mathcal{H}}$ to $p$, we can express $p$ as

$$p = \widetilde{p} + \sum_{\boldsymbol{\lambda}; \boldsymbol{s}^{\boldsymbol{\lambda}} \notin \mathrm{in}_\prec(\widetilde{P})} d_{\boldsymbol{\lambda}} s^{\boldsymbol{\lambda}}$$

with some $\widetilde{p} \in \widetilde{P}$ and $d_{\boldsymbol{\lambda}} \in \mathbb{C}$. For each $\boldsymbol{\nu} \in \mathbb{N}^h$ with $\boldsymbol{s}^{\boldsymbol{\nu}} \notin \mathrm{in}_\prec(\widetilde{P})$, since $[\partial_{\boldsymbol{s}}^{\boldsymbol{\mu}} \bullet \boldsymbol{s}^{\boldsymbol{\lambda}}]_{|\boldsymbol{s}=\boldsymbol{0}} = \boldsymbol{\mu}! \delta_{\boldsymbol{\mu},\boldsymbol{\lambda}}$ for any $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$, where $\delta_{\boldsymbol{\mu},\boldsymbol{\lambda}}$ denotes the Kronecker delta, we have

$$\begin{aligned}
&[\widetilde{q}_{\boldsymbol{\nu}}(\partial_{\boldsymbol{s}}) \bullet p]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&\quad = \left[ \widetilde{q}_{\boldsymbol{\nu}}(\partial_{\boldsymbol{s}}) \bullet \left( \widetilde{p} + \sum_{\boldsymbol{\lambda}; \boldsymbol{s}^{\boldsymbol{\lambda}} \notin \mathrm{in}_\prec(\widetilde{P})} d_{\boldsymbol{\lambda}} s^{\boldsymbol{\lambda}} \right) \right]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&\quad = \sum_{\boldsymbol{\lambda}; \boldsymbol{s}^{\boldsymbol{\lambda}} \notin \mathrm{in}_\prec(\widetilde{P})} d_{\boldsymbol{\lambda}} \left[ \widetilde{q}_{\boldsymbol{\nu}}(\partial_{\boldsymbol{s}}) \bullet \boldsymbol{s}^{\boldsymbol{\lambda}} \right]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&\quad = \sum_{\boldsymbol{\lambda}; \boldsymbol{s}^{\boldsymbol{\lambda}} \notin \mathrm{in}_\prec(\widetilde{P})} d_{\boldsymbol{\lambda}} \left[ \left( \frac{1}{\boldsymbol{\nu}!} \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}} + \sum_{\substack{\boldsymbol{\mu} \in \mathbb{N}^h; |\boldsymbol{\mu}| = |\boldsymbol{\nu}|, \\ \boldsymbol{s}^{\boldsymbol{\mu}} \in \mathrm{in}_\prec(\widetilde{P})}} \frac{c_{\boldsymbol{\mu},\boldsymbol{\nu}}}{\boldsymbol{\mu}!} \partial_{\boldsymbol{s}}^{\boldsymbol{\mu}} \right) \bullet \boldsymbol{s}^{\boldsymbol{\lambda}} \right]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&\quad = d_{\boldsymbol{\nu}}.
\end{aligned}$$

It follows from the assumption $P^\perp \supset \widetilde{P}^\perp$ that $d_{\boldsymbol{\nu}} = 0$ for all $\boldsymbol{\nu} \in \mathbb{N}^h$ with $\boldsymbol{s}^{\boldsymbol{\nu}} \notin \mathrm{in}_\prec(\widetilde{P})$. Hence, we have $p \in \widetilde{P}$. □

Let $\mathbb{C}[\partial_{\boldsymbol{z}}] := \mathbb{C}[\partial_{z_1}, \ldots, \partial_{z_h}]$ be the ring of partial differential operators with constant coefficients in indeterminates $\boldsymbol{z} = (z_1, \ldots, z_h)$. To describe relations between $P_{\mathcal{N}}^\perp$ and $P_B^\perp$, and between $P_B^\perp$ and $Q_{\boldsymbol{v}}^\perp$, we define an action of $\mathbb{C}[\partial_{\boldsymbol{z}}]$ on $\mathbb{C}[\partial_{\boldsymbol{s}}]$ and a ring homomorphism $\Psi_B$ from $\mathbb{C}[\partial_{\boldsymbol{s}}]$ to $\mathbb{C}[\boldsymbol{x}]$.

For $U(\partial_{\boldsymbol{z}}) \in \mathbb{C}[\partial_{\boldsymbol{z}}]$ and $q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$, we define a $\mathbb{C}$-linear operation $U(\partial_{z_1}, \ldots, \partial_{z_h}) \star q(\partial_{\boldsymbol{s}})$ by

$$U(\partial_{z_1}, \ldots, \partial_{z_h}) \star q(\partial_{\boldsymbol{s}}) := (U(\partial_{\boldsymbol{z}}) \bullet q(\boldsymbol{z}))|_{\boldsymbol{z}=\partial_{\boldsymbol{s}}} \in \mathbb{C}[\partial_{\boldsymbol{s}}].$$

**Lemma 3.8.** *The following hold for the $\star$-operation.*

(i) *Let $k = 1, \ldots, h$ and $q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$. Then*

$$\partial_{z_k} \star q(\partial_{\boldsymbol{s}}) = q(\partial_{\boldsymbol{s}}) s_k - s_k q(\partial_{\boldsymbol{s}}) \in \mathbb{C}\langle \boldsymbol{s}, \partial_{\boldsymbol{s}} \rangle.$$

(ii) *Let $U(\partial_z), U'(\partial_z) \in \mathbb{C}[\partial_{\boldsymbol{z}}]$, and $q(\partial_{\boldsymbol{z}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$. Then*

$$U(\partial_{\boldsymbol{z}}) \star (U'(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}})) = (U(\partial_{\boldsymbol{z}}) U'(\partial_{\boldsymbol{z}})) \star q(\partial_s).$$

(iii) *Let $U(\partial_{\boldsymbol{z}}) = \prod_{\nu=1}^{N} l_\nu(\partial_{\boldsymbol{z}}) \in \mathbb{C}[\partial_{\boldsymbol{z}}]$ be the product of non-zero linear homogeneous polynomials $l_\nu(\partial_{\boldsymbol{z}})$, and let $q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$. Then there exists $r(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$ such that $U(\partial_{\boldsymbol{z}}) \star r(\partial_{\boldsymbol{s}}) = q(\partial_{\boldsymbol{s}})$.*

*Proof.* (i) For any $k = 1, \ldots, h$, and $\boldsymbol{\mu} \in \mathbb{N}^h$, we have

$$\partial_{z_k} \star \partial_{\boldsymbol{s}}^{\boldsymbol{\mu}} = \mu_k \partial_{\boldsymbol{s}}^{\boldsymbol{\mu} - \boldsymbol{e}_k} = \partial_{\boldsymbol{s}}^{\boldsymbol{\mu}} s_k - s_k \partial_{\boldsymbol{s}}^{\boldsymbol{\mu}}.$$

(ii) It suffices to show that the equality holds for $U(\partial_{\boldsymbol{z}}) = \partial_{\boldsymbol{z}}^{\boldsymbol{\lambda}}$, $U'(\partial_{\boldsymbol{z}}) = \partial_{\boldsymbol{z}}^{\boldsymbol{\mu}}$, and $q(\partial_{\boldsymbol{s}}) = \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}}$ with $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbb{N}^h$. We see that

$$\begin{aligned}
\partial_{\boldsymbol{z}}^{\boldsymbol{\lambda}} \star (\partial_{\boldsymbol{z}}^{\boldsymbol{\mu}} \star \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}}) &= \partial_{\boldsymbol{z}}^{\boldsymbol{\lambda}} \star ((\partial_{\boldsymbol{z}}^{\boldsymbol{\mu}} \bullet \boldsymbol{z}^{\boldsymbol{\nu}})|_{|\boldsymbol{z}=\partial_{\boldsymbol{s}}}) \\
&= [\boldsymbol{\nu}]_{\boldsymbol{\mu}} \partial_{\boldsymbol{z}}^{\boldsymbol{\lambda}} \star \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}-\boldsymbol{\mu}} \\
&= [\boldsymbol{\nu}]_{\boldsymbol{\mu}} (\partial_{\boldsymbol{z}}^{\boldsymbol{\lambda}} \bullet \boldsymbol{z}^{\boldsymbol{\nu}-\boldsymbol{\mu}})_{|\boldsymbol{z}=\partial_{\boldsymbol{s}}} \\
&= [\boldsymbol{\nu}]_{\boldsymbol{\mu}} [\boldsymbol{\nu} - \boldsymbol{\mu}]_{\boldsymbol{\lambda}} \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}-\boldsymbol{\mu}-\boldsymbol{\lambda}} \\
&= [\boldsymbol{\nu}]_{\boldsymbol{\lambda}+\boldsymbol{\mu}} \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}-(\boldsymbol{\lambda}+\boldsymbol{\mu})} \\
&= (\partial_{\boldsymbol{z}}^{\boldsymbol{\lambda}+\boldsymbol{\mu}} \bullet \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}})_{|\boldsymbol{z}=\partial_{\boldsymbol{s}}} \\
&= \partial_{\boldsymbol{z}}^{\boldsymbol{\lambda}+\boldsymbol{\mu}} \star \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}},
\end{aligned}$$

and hence the assertion holds.

(iii) We show the statement by induction on $N$. First, let $U(\partial_{\boldsymbol{z}})$ be a non-zero linear homogeneous polynomial, and let $q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$. By changing coordinates, we may assume that $U(\partial_{\boldsymbol{z}}) = \partial_{z_1}$. Put

$$q(\partial_{\boldsymbol{s}}) = \sum_{\boldsymbol{\nu} \in \mathbb{N}^h} d_{\boldsymbol{\nu}} \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}}.$$

Then

$$r(\partial_{\boldsymbol{s}}) = \sum_{\boldsymbol{\nu} \in \mathbb{N}^h} \frac{d_{\boldsymbol{\nu}}}{\nu_1 + 1} \partial_{\boldsymbol{s}}^{\boldsymbol{\nu} + \boldsymbol{e}_1}$$

satisfies $U(\partial_{\boldsymbol{z}}) \star r(\partial_{\boldsymbol{s}}) = q(\partial_{\boldsymbol{s}})$.

Next, fix $N > 1$, and let $U(\partial_{\boldsymbol{z}}) = \prod_{\nu=1}^{N} l_\nu(\partial_{\boldsymbol{z}})$ such that $l_\nu(\partial_{\boldsymbol{z}})$ are non-zero linear homogeneous polynomials. Assume that the assertion holds for any product of non-zero linear homogeneous polynomial of degree less than $N$. By the induction hypothesis, there exist

$r(\partial_{\boldsymbol{s}}), \widetilde{r}(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$ such that

$$l_1(\partial_{\boldsymbol{z}}) \star \widetilde{r}(\partial_{\boldsymbol{s}}) = q(\partial_{\boldsymbol{s}}), \qquad \left(\prod_{\nu=2}^{N} l_\nu(\partial_{\boldsymbol{z}})\right) \star r(\partial_{\boldsymbol{s}}) = \widetilde{r}(\partial_{\boldsymbol{s}}).$$

By (ii), we have

$$\begin{aligned} U(\partial_{\boldsymbol{z}}) \star r(\partial_{\boldsymbol{s}}) &= \left(l_1(\partial_{\boldsymbol{z}}) \left(\prod_{\nu=2}^{N} l_\nu(\partial_{\boldsymbol{z}})\right)\right) \star r(\partial_{\boldsymbol{s}}) \\ &= l_1(\partial_{\boldsymbol{z}}) \star \left(\left(\prod_{\nu=2}^{N} l_\nu(\partial_{\boldsymbol{z}})\right) \star r(\partial_{\boldsymbol{s}})\right) \\ &= l_1(\partial_{\boldsymbol{z}}) \star \widetilde{r}(\partial_{\boldsymbol{s}}) \\ &= q(\partial_{\boldsymbol{s}}), \end{aligned}$$

and hence the assertion holds. □

**Lemma 3.9.** *Let $U(\partial_{\boldsymbol{z}}) \in \mathbb{C}[\partial_{\boldsymbol{z}}]$, $q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$, and $f(\boldsymbol{s}) \in \mathbb{C}[[\boldsymbol{s}]]$. Then*

$$[q(\partial_{\boldsymbol{s}}) \bullet (U(\boldsymbol{s}) f(\boldsymbol{s}))]_{|\boldsymbol{s}=\boldsymbol{0}} = [(U(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}})) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}}.$$

*Proof.* We show that the statement holds for any monomial operator $U(\partial_{\boldsymbol{z}}) = \partial_{\boldsymbol{z}}^{\boldsymbol{\mu}}$ by induction on $|\boldsymbol{\mu}|$. Firstly, the assertion is clear for $\boldsymbol{\mu} = \boldsymbol{0}$.

Secondly, assume that $\boldsymbol{\mu} = \boldsymbol{e}_k$ for $k = 1, \ldots, k$, hence $U(\partial_{\boldsymbol{z}}) = \partial_{\boldsymbol{z}}^{\boldsymbol{\mu}} = \partial_{z_k}$. Note that $[s_k q(\partial_{\boldsymbol{s}}) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}} = 0$ for any $k = 1 \ldots, h$, $q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$, and $f(\boldsymbol{s}) \in \mathbb{C}[[\boldsymbol{s}]]$. Thus, by Lemma 3.8, we see that

$$\begin{aligned} &[q(\partial_{\boldsymbol{s}}) \bullet (U(\boldsymbol{s}) f(\boldsymbol{s}))]_{|\boldsymbol{s}=\boldsymbol{0}} \\ &\quad = [(q(\partial_{\boldsymbol{s}}) s_k) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}} \\ &\quad = [(s_k q(\partial_{\boldsymbol{s}}) + \partial_{z_k} \star q(\partial_{\boldsymbol{s}})) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}} \\ &\quad = [s_k (q(\partial_{\boldsymbol{s}}) \bullet f(\boldsymbol{s}))]_{|\boldsymbol{s}=\boldsymbol{0}} + [(\partial_{z_k} \star q(\partial_{\boldsymbol{s}})) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}} \\ &\quad = [(\partial_{z_k} \star q(\partial_{\boldsymbol{s}})) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}}. \end{aligned}$$

Hence the assertion holds for $|\boldsymbol{\mu}| = 1$.

Finally, fix $\boldsymbol{\mu} \in \mathbb{N}^h$ with $|\boldsymbol{\mu}| > 1$. Let $U(\partial_{\boldsymbol{z}}) = \partial_{\boldsymbol{z}}^{\boldsymbol{\mu}}$, $q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$, and $f(\boldsymbol{s}) \in \mathbb{C}[[\boldsymbol{s}]]$. Assume that the assertion holds for any $\widetilde{U}(\partial_{\boldsymbol{z}}) = \partial_{\boldsymbol{z}}^{\widetilde{\boldsymbol{\mu}}}$ with $|\widetilde{\boldsymbol{\mu}}| < |\boldsymbol{\mu}|$. Then there exists $k$ such that $\boldsymbol{\mu}_k > 0$. Applying the induction hypothesis to the operators $\partial_{z_k}$ and $\partial_{\boldsymbol{z}}^{\boldsymbol{\mu}-\boldsymbol{e}_k}$, respectively, we

see from Lemma 3.8 (ii) that

$$\begin{aligned}
&[q(\partial_{\boldsymbol{s}}) \bullet (U(\boldsymbol{s}) f(\boldsymbol{s}))]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&\quad = \left[q(\partial_{\boldsymbol{s}}) \bullet \left(s_k \cdot \boldsymbol{s}^{\boldsymbol{\mu}-\boldsymbol{e}_k} f(\boldsymbol{s})\right)\right]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&\quad = \left[(\partial_{z_k} \star q(\partial_{\boldsymbol{s}})) \bullet \left(\boldsymbol{s}^{\boldsymbol{\mu}-\boldsymbol{e}_k} f(\boldsymbol{s})\right)\right]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&\quad = \left[\left(\partial_{\boldsymbol{z}}^{\boldsymbol{\mu}-\boldsymbol{e}_k} \star (\partial_{z_k} \star q(\partial_{\boldsymbol{s}}))\right) \bullet f(\boldsymbol{s})\right]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&\quad = [(U(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}})) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}} \cdot
\end{aligned}$$

Hence the assertion holds. □

We define a ring homomorphism $\Psi_B : \mathbb{C}[\partial_{\boldsymbol{s}}] \to \mathbb{C}[\boldsymbol{x}]$ as

$$\Psi_B(q(\partial_{\boldsymbol{s}}))(\boldsymbol{x}) := q(\boldsymbol{x}B) = q\left(\sum_{j=1}^{n} b_j^{(1)} x_j, \ldots, \sum_{j=1}^{n} b_j^{(h)} x_j\right) \tag{10}$$

for $q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$. Note that $\Psi_B$ is injective by the linear independence of $B$.

**Proposition 3.10.** *Let $q(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$. Then*

$$\left[q(\partial_{\boldsymbol{s}}) \bullet \left(m(\boldsymbol{s}) x^{\boldsymbol{v}+B\boldsymbol{s}}\right)\right]_{|\boldsymbol{s}=\boldsymbol{0}} = x^{\boldsymbol{v}} \Psi_B\left(m(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}})\right)(\log \boldsymbol{x}),$$

*where* $\log \boldsymbol{x} := (\log x_1, \ldots, \log x_n)$.

*Proof.* Note that we can regard $x^{\boldsymbol{v}+B\boldsymbol{s}}$ as the formal series $x^{\boldsymbol{v}} e^{(\log \boldsymbol{x})B\boldsymbol{s}}$ in $\boldsymbol{s}$, where

$$(\log \boldsymbol{x})B\boldsymbol{s} := \sum_{j=1}^{n} \sum_{k=1}^{h} (\log x_j) b_j^{(k)} s_k.$$

Put $r(\partial_{\boldsymbol{s}}) := m(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}})$. Then, by Lemma 3.9,

$$\begin{aligned}
\left[q(\partial_{\boldsymbol{s}}) \bullet \left(m(\boldsymbol{s}) x^{\boldsymbol{v}+B\boldsymbol{s}}\right)\right]_{|\boldsymbol{s}=\boldsymbol{0}} &= \left[(m(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}})) \bullet x^{\boldsymbol{v}+B\boldsymbol{s}}\right]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&= \left[r(\partial_{\boldsymbol{s}}) \bullet x^{\boldsymbol{v}+B\boldsymbol{s}}\right]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&= [r((\log \boldsymbol{x})B) x^{\boldsymbol{v}+B\boldsymbol{s}}]_{|\boldsymbol{s}=\boldsymbol{0}} \\
&= x^{\boldsymbol{v}} \Psi_B(r(\partial_{\boldsymbol{s}}))(\log \boldsymbol{x}).
\end{aligned}$$

□

**Proposition 3.11.** *The following hold.*

(i) $m(\partial_{\boldsymbol{z}}) \star P_{\mathcal{N}}^{\perp} \subset P_B^{\perp} \subset P_{\mathcal{N}}^{\perp}$. *In particular, if $K_{\mathcal{N}} = I_{\boldsymbol{0}}$, then $P_{\mathcal{N}}^{\perp} = P_B^{\perp}$.*

(ii) *$m(\boldsymbol{s}) \in P_{\mathcal{N}}$ if and only if $m(\partial_{\boldsymbol{z}}) \star P_{\mathcal{N}}^{\perp} = \{0\}$.*

(iii) *If $P_{\mathcal{N}} = m(\boldsymbol{s}) \cdot P_B$, then $m(\partial_{\boldsymbol{z}}) \star P_{\mathcal{N}}^{\perp} = P_B^{\perp}$.*

*Proof.* (i) $P_B^\perp \subset P_{\mathcal{N}}^\perp$ is clear by Lemma 3.4 (i) and Lemma 3.7. Let $q(\partial_{\boldsymbol{s}}) \in P_{\mathcal{N}}^\perp$. Then, for any $f(\boldsymbol{s}) \in P_B$, Lemma 3.9 shows that

$$[(m(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}})) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}} = [q(\partial_{\boldsymbol{s}}) \bullet (m(\boldsymbol{s}) f(\boldsymbol{s}))]_{|\boldsymbol{s}=\boldsymbol{0}}\,.$$

It follows from Lemma 3.4 (i) that the right hand side is 0. Hence we have $m(\partial_{\boldsymbol{z}}) \star P_{\mathcal{N}}^\perp \subset P_B^\perp$.

(ii) Assume that $m(\boldsymbol{s}) \in P_{\mathcal{N}}$. Let $q(\partial_{\boldsymbol{s}}) \in P_{\mathcal{N}}^\perp$. Put $m(\partial_{\boldsymbol{z}}) \star q(\partial_s) = \sum_{\boldsymbol{\nu}} a_{\boldsymbol{\nu}} \partial_{\boldsymbol{s}}^{\boldsymbol{\nu}}$, where $a_{\boldsymbol{\nu}} \in \mathbb{C}$. Then, by Lemma 3.9, we have

$$\begin{aligned} \boldsymbol{\nu}! a_{\boldsymbol{\nu}} &= [(m(\partial_{\boldsymbol{z}}) \star q(\partial_s)) \bullet \boldsymbol{s}^{\boldsymbol{\nu}}]_{|\boldsymbol{s}=\boldsymbol{0}} \\ &= [q(\partial_{\boldsymbol{s}}) \bullet (m(\boldsymbol{s}) \boldsymbol{s}^{\boldsymbol{\nu}})]_{|\boldsymbol{s}=\boldsymbol{0}} = 0 \end{aligned}$$

for any $\boldsymbol{\nu} \in \mathbb{N}^h$. Hence, we have $m(\partial_{\boldsymbol{z}}) \star q(\partial_s) = 0$.

Conversely, assume that $m(\partial_{\boldsymbol{z}}) \star P_{\mathcal{N}}^\perp = \{0\}$. Let $q(\partial_{\boldsymbol{s}}) \in P_{\mathcal{N}}^\perp$. Then, Lemma 3.9 shows that

$$[q(\partial_{\boldsymbol{s}}) \bullet (m(\boldsymbol{s}) f(\boldsymbol{s}))]_{|\boldsymbol{s}=\boldsymbol{0}} = [(m(\partial_{\boldsymbol{z}}) \star q(\partial_s)) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}} = 0$$

for any $f(\boldsymbol{s}) \in \mathbb{C}[\boldsymbol{s}]$. Thus we have $q(\partial_{\boldsymbol{s}}) \in \langle m(\boldsymbol{s}) \rangle^\perp$, that is, $P_{\mathcal{N}}^\perp \subset \langle m(\boldsymbol{s}) \rangle^\perp$. By Lemma 3.7, $m(\boldsymbol{s}) \in P_{\mathcal{N}}$.

(iii) In (i), we have seen $m(\partial_{\boldsymbol{z}}) \star P_{\mathcal{N}}^\perp \subset P_B^\perp$. We show its reverse inclusion. Let $q(\partial_{\boldsymbol{s}}) \in P_B^\perp$. By Lemma 3.8 (iii), there exists $r(\partial_{\boldsymbol{s}}) \in \mathbb{C}[\partial_{\boldsymbol{s}}]$ such that $q(\partial_{\boldsymbol{s}}) = m(\partial_{\boldsymbol{z}}) \star r(\partial_{\boldsymbol{s}})$. It suffices to show that $r(\partial_{\boldsymbol{s}}) \in P_{\mathcal{N}}^\perp$. Let $f(\boldsymbol{s}) \in P_{\mathcal{N}}$. By the assumption, we have $f(\boldsymbol{s}) = m(\boldsymbol{s}) g(\boldsymbol{s})$ for some $g(\boldsymbol{s}) \in P_B$. By Lemma 3.9, we see that

$$\begin{aligned} [r(\partial_{\boldsymbol{s}}) \bullet f(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}} &= [r(\partial_{\boldsymbol{s}}) \bullet (m(\boldsymbol{s}) g(\boldsymbol{s}))]_{|\boldsymbol{s}=\boldsymbol{0}} \\ &= [(m(\partial_{\boldsymbol{z}}) \star r(\partial_s)) \bullet g(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}} \\ &= [q(\partial_s) \bullet g(\boldsymbol{s})]_{|\boldsymbol{s}=\boldsymbol{0}} = 0. \end{aligned}$$

Hence we have the assertion. □

**Example 3.12.** (cf. [6, Examples 3.2 and 4.7]) Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}$ and let $\boldsymbol{w} = (3, 1, 0, 0)$. Then the reduced Gröbner basis of $I_A$ is

$$\mathcal{G} = \{\underline{\partial_{x_1} \partial_{x_3}^2} - \partial_{x_2}^2 \partial_{x_4}, \underline{\partial_{x_2} \partial_{x_4}^2} - \partial_{x_3}^3, \underline{\partial_{x_1}^2 \partial_{x_3}} - \partial_{x_2}^3, \underline{\partial_{x_1} \partial_{x_4}} - \partial_{x_2} \partial_{x_3}\}.$$

Here underlined terms are the leading ones. Thus we have

$$\mathrm{in}_{\boldsymbol{w}}(I_A) = \langle \partial_{x_1} \partial_{x_3}^2, \partial_{x_2} \partial_{x_4}^2, \partial_{x_1}^2 \partial_{x_3}, \partial_{x_1} \partial_{x_4} \rangle.$$

Put

$$\boldsymbol{g}^{(1)} = (1, -2, 2, -1)^T, \quad \boldsymbol{g}^{(2)} = (0, 1, -3, 2)^T,$$

$$\boldsymbol{g}^{(3)} = (2, -3, 1, 0)^T, \qquad \boldsymbol{g}^{(4)} = (1, -1, -1, 1)^T.$$

Let $\boldsymbol{\beta} = (-2, -1)^T$, and let

$$B = (\boldsymbol{g}^{(1)}, \boldsymbol{g}^{(2)}) = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 2 & -3 \\ -1 & 2 \end{bmatrix}.$$

Note that $\mathrm{supp}(B) = \{1, 2, 3, 4\}$. Take $\boldsymbol{v} = (0, -2, -1, 1)^T$ as a fake exponent. Then we have

$$\begin{aligned} \mathcal{N} &= \{\{2\}, \{3\}, \{2, 3\} = I_{\mathbf{0}}\}, \\ \mathcal{N}^c &= \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}, \{1, 3, 4\}\}, \\ K_{\mathcal{N}} &= \emptyset. \end{aligned}$$

Furthermore, we have

$$G^{(1)} = I_{-\boldsymbol{g}^{(1)}} \setminus I_{\mathbf{0}} = \{1\}, \quad G^{(2)} = I_{-\boldsymbol{g}^{(2)}} \setminus I_{\mathbf{0}} = \{4\},$$

$$G^{(3)} = I_{-\boldsymbol{g}^{(3)}} \setminus I_{\mathbf{0}} = \{1\}, \quad G^{(4)} = I_{-\boldsymbol{g}^{(4)}} \setminus I_{\mathbf{0}} = \{1\}.$$

Thus the ideals $P_{\mathcal{N}}$ and $P_B$ are

$$\begin{aligned} P_{\mathcal{N}} &= \langle (B\boldsymbol{s})^{\{1,2\}}, (B\boldsymbol{s})^{\{1,3\}}, (B\boldsymbol{s})^{\{2,4\}} \rangle \\ &= \langle s_1(-2s_1 + s_2), s_1(2s_1 - 3s_2), (-2s_1 + s_2)(-s_1 + 2s_2) \rangle \\ &= \langle s_1^2, s_1 s_2, s_2^2 \rangle \end{aligned}$$

and

$$P_B = \langle (B\boldsymbol{s})^{\{1\}}, (B\boldsymbol{s})^{\{4\}} \rangle = \langle s_1, -s_1 + 2s_2 \rangle = \langle s_1, s_2 \rangle,$$

respectively. The orthogonal complements $P_{\mathcal{N}}^{\perp}$ and $P_B^{\perp}$ are

$$P_{\mathcal{N}}^{\perp} = \mathbb{C}1 + \mathbb{C}\partial_{s_1} + \mathbb{C}\partial_{s_2}$$

and

$$P_B^{\perp} = \mathbb{C}1,$$

respectively. In this case, note that

$$m(\boldsymbol{s}) = (B\boldsymbol{s})^{I_{\mathbf{0}} \setminus K_{\mathcal{N}}} = (B\boldsymbol{s})^{\{2,3\}} = (-2s_1 + s_2)(2s_1 - 3s_2) \in P_{\mathcal{N}}.$$

Hence, by Proposition 3.11, we have

$$m(\partial_{\boldsymbol{z}}) \star P_{\mathcal{N}}^{\perp} = \{0\}.$$

**Lemma 3.13.** *Let $q(\boldsymbol{z}) \in \mathbb{C}[\boldsymbol{z}] := \mathbb{C}[z_1, \ldots, z_h]$ be a homogeneous polynomial in indeterminates $\boldsymbol{z}$ of degree $r$. Then*

$$q(\partial_{\boldsymbol{s}}) \bullet \left( \frac{1}{r!} (\boldsymbol{x} B \boldsymbol{s})^r \right) = q(\boldsymbol{x} B) = \Psi_B(q(\partial_{\boldsymbol{s}}))(\boldsymbol{x}).$$

*Here,*

$$\boldsymbol{x}B\boldsymbol{s} := \sum_{j=1}^{n}\sum_{k=1}^{h} x_j b_j^{(k)} s_k$$

*denotes the quadratic form associated with $B$.*

*Proof.* We show the assertion by induction on $r$. In the case of $r = 1$, the assertion is clear. Fix $r > 1$ and assume that the assertion holds for any homogeneous polynomial of degree less than $r$. Let $\boldsymbol{\mu} \in \mathbb{N}^h$ with $|\boldsymbol{\mu}| = r$ and $\mu_k > 0$. Then, by the chain rule and the induction hypothesis, we see that

$$\begin{aligned}
\partial_s{}^{\boldsymbol{\mu}} \bullet \left(\frac{1}{r!}(\boldsymbol{x}B\boldsymbol{s})^r\right) &= \partial_s{}^{\boldsymbol{\mu}-\boldsymbol{e}_k} \bullet \left(\partial_{s_k} \bullet \left(\frac{1}{r!}(\boldsymbol{x}B\boldsymbol{s})^r\right)\right) \\
&= (\boldsymbol{x}B)_k \cdot \partial_{\boldsymbol{s}}{}^{\boldsymbol{\mu}-\boldsymbol{e}_k} \bullet \left(\frac{1}{(r-1)!}(\boldsymbol{x}B\boldsymbol{s})^{r-1}\right) \\
&= (\boldsymbol{x}B)_k (\boldsymbol{x}B)^{\boldsymbol{\mu}-\boldsymbol{e}_k} = (\boldsymbol{x}B)^{\boldsymbol{\mu}} = \Psi_B(\partial_{\boldsymbol{s}}^{\boldsymbol{\mu}})(\boldsymbol{x}).
\end{aligned}$$

□

Two vector spaces $Q_{\boldsymbol{v}}^{\perp}$ and $P_B^{\perp}$ are related as follows.

**Theorem 3.14.** *Let $\Psi_B$ be the homomorphism in* (10)*. Then, $P_B^{\perp} = \Psi_B^{-1}(Q_{\boldsymbol{v}}^{\perp})$ and $\dim_{\mathbb{C}}(P_B^{\perp}) \leq \dim_{\mathbb{C}}(Q_{\boldsymbol{v}}^{\perp})$. Furthermore, if $B$ is a basis of $L$, then $\Psi_B(P_B^{\perp}) = Q_{\boldsymbol{v}}^{\perp}$ and $\dim_{\mathbb{C}}(Q_{\boldsymbol{v}}^{\perp}) = \dim_{\mathbb{C}}(P_B^{\perp})$.*

*Proof.* Let $\deg(q(\boldsymbol{z})) = r$. Then, it follows from Lemma 3.13 that

$$\begin{aligned}
\partial_{\boldsymbol{x}}^{G^{(i)}} \bullet q(\boldsymbol{x}B) &= \partial_{\boldsymbol{x}}^{G^{(i)}} \bullet \left\{ q(\partial_{\boldsymbol{s}}) \bullet \left( \frac{1}{r!} (\boldsymbol{x}B\boldsymbol{s})^r \right) \right\} \\
&= q(\partial_{\boldsymbol{s}}) \bullet \left\{ \partial_{\boldsymbol{x}}^{G^{(i)}} \bullet \left( \frac{1}{r!} (\boldsymbol{x}B\boldsymbol{s})^r \right) \right\} \\
&= q(\partial_{\boldsymbol{s}}) \bullet \left\{ \partial_{\boldsymbol{x}}^{G^{(i)}} \bullet \left( \frac{1}{r!} \sum_{\boldsymbol{\mu} \in \mathbb{N}^n; |\boldsymbol{\mu}| = r} \frac{r!}{\boldsymbol{\mu}!} \boldsymbol{x}^{\boldsymbol{\mu}} (B\boldsymbol{s})^{\boldsymbol{\mu}} \right) \right\} \\
&= q(\partial_{\boldsymbol{s}}) \bullet \left\{ \sum_{\boldsymbol{\mu} \in \mathbb{N}^n; |\boldsymbol{\mu}| = r} \frac{\partial_{\boldsymbol{x}}^{G^{(i)}} \bullet \boldsymbol{x}^{\boldsymbol{\mu}}}{\boldsymbol{\mu}!} (B\boldsymbol{s})^{\boldsymbol{\mu}} \right\} \\
&= q(\partial_{\boldsymbol{s}}) \bullet \left\{ \sum_{\boldsymbol{\mu} \in \mathbb{N}^n; |\boldsymbol{\mu}| = r, \mathrm{supp}(\boldsymbol{\mu}) \supset G^{(i)}} \frac{\boldsymbol{x}^{\boldsymbol{\mu} - \boldsymbol{e}_{G^{(i)}}}}{(\boldsymbol{\mu} - \boldsymbol{e}_{G^{(i)}})!} (B\boldsymbol{s})^{\boldsymbol{\mu}} \right\} \\
&= \sum_{\boldsymbol{\mu} \in \mathbb{N}^n; |\boldsymbol{\mu}| = r, \mathrm{supp}(\boldsymbol{\mu}) \supset G^{(i)}} \frac{\boldsymbol{x}^{\boldsymbol{\mu} - \boldsymbol{e}_{G^{(i)}}}}{(\boldsymbol{\mu} - \boldsymbol{e}_{G^{(i)}})!} \left\{ q(\partial_{\boldsymbol{s}}) \bullet (B\boldsymbol{s})^{\boldsymbol{\mu}} \right\}
\end{aligned}$$

for any $i$. Here $\boldsymbol{e}_{G^{(i)}} := \sum_{j \in G^{(i)}} \boldsymbol{e}_j$ denotes the indicator vector of $G^{(i)}$. Since $\partial_{\boldsymbol{s}}^{\boldsymbol{p}} \bullet \boldsymbol{s}^{\boldsymbol{q}} = \boldsymbol{p}! \delta_{\boldsymbol{p},\boldsymbol{q}}$ for any $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{N}^h$ with $|\boldsymbol{p}| = |\boldsymbol{q}|$, by Proposition 3.1 we have

$$\begin{aligned}
& q(\boldsymbol{x}B) \in Q_{\boldsymbol{v}}^{\perp} \iff \partial_{\boldsymbol{x}}^{G^{(i)}} \bullet q(\boldsymbol{x}B) = 0 \text{ for all } i = 1, \ldots, m \\
\iff & q(\partial_{\boldsymbol{s}}) \bullet (B\boldsymbol{s})^{\boldsymbol{\mu}} = 0 \\
& \qquad \text{for all } i \text{ and all } \boldsymbol{\mu} \in \mathbb{N}^n \text{ with } |\boldsymbol{\mu}| = r, \mathrm{supp}(\boldsymbol{\mu}) \supset G^{(i)} \\
\iff & q(\partial_{\boldsymbol{s}}) \in P_B^{\perp}.
\end{aligned}$$

Here, the second equivalence follows from the linear independence of the monomials $\boldsymbol{x}^{\boldsymbol{\mu}}$. The third equivalence follows from the linear independence of $B$, because it yields that the ideal $P_B$ is spanned as a vector space by the polynomials whose terms are of the form $(B\boldsymbol{s})^{\boldsymbol{\mu}}$ with $\mathrm{supp}(\boldsymbol{\mu}) \supset G^{(i)}$ for some $i$. Thus we have $P_B^{\perp} = \Psi_B^{-1}(Q_{\boldsymbol{v}}^{\perp})$. Moreover, we have the inequality $\dim_{\mathbb{C}}(P_B^{\perp}) \leq \dim_{\mathbb{C}}(Q_{\boldsymbol{v}}^{\perp})$ because $\Psi_B({P_B}^{\perp}) = \Psi_B\left(\Psi_B^{-1}(Q_{\boldsymbol{v}}^{\perp})\right) \subset Q_{\boldsymbol{v}}^{\perp}$ and $\Psi_B$ is injective.

Assume that $B$ is a basis of $L$. Note that each $\boldsymbol{x}\boldsymbol{g}^{(k)} = \sum_{j=1}^n g_j^{(k)} x_j$ can be represented by a linear combination of $(\boldsymbol{x}B)_1, \ldots, (\boldsymbol{x}B)_h$. Let $f(\boldsymbol{x}) \in Q_{\boldsymbol{v}}^{\perp}$. Then, by Proposition 3.1 and the above result, there exists $q(\partial_{\boldsymbol{s}}) \in P_B^{\perp}$ such that $f(\boldsymbol{x}) = q(\boldsymbol{x}B)$. Thus we have $f(\boldsymbol{x}) = \Psi_B(q(\partial_{\boldsymbol{s}}))(\boldsymbol{x})$. □

## 4. Fundamental systems of solutions

In this section, we construct a fundamental system of series solutions with a given exponent to $M_A(\boldsymbol{\beta})$. We recall that the homogeneity of $A$ yields the regular holonomicity of $M_A(\boldsymbol{\beta})$. This means that, for a fixed generic weight $\boldsymbol{w}$, the solution space to $M_A(\boldsymbol{\beta})$ has a basis consisting of canonical series with starting monomial $x^{\boldsymbol{v}}(\log \boldsymbol{x})^{\boldsymbol{b}}$ for some exponent $\boldsymbol{v}$ and $\boldsymbol{b} \in \mathbb{N}^n$. Note that each $x^{\boldsymbol{v}}(\log \boldsymbol{x})^{\boldsymbol{b}}$ is derived as the initial monomial of a solution to the indicial ideal $\mathrm{ind}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))_{\boldsymbol{v}}$, or of an element of $Q_{\boldsymbol{v}}^{\perp}$. For the detail, see [8, Sections 2.3, 2.4, and 2.5] and Proposition 3.1.

Throughout this section, we assume that $B$ is a basis of $L$. Since $B$ satisfies Assumption 2.3 (see Remark 2.4), we have the following homomorphisms by Propositions 3.1, 3.10, and Theorem 3.14:

$$\begin{array}{ccccc} P_{\mathcal{N}}^{\perp} & \to & P_B^{\perp} & \simeq & \mathrm{Sol}(\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))_{\boldsymbol{v}}), \\ q(\partial_{\boldsymbol{s}}) & \mapsto & m(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}}) & \leftrightarrow & x^{\boldsymbol{v}} \Psi_B(m(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}}))(\log \boldsymbol{x}). \end{array} \tag{11}$$

Here, $\mathrm{Sol}(\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))_{\boldsymbol{v}})$ denotes the solution space of the fake indicial ideal $\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))_{\boldsymbol{v}}$.

**Proposition 4.1.** *If $m(\boldsymbol{s}) \notin P_{\mathcal{N}}$, then $\boldsymbol{v}$ is an exponent.*

*Proof.* Assume that $m(\boldsymbol{s}) \notin P_{\mathcal{N}}$. Then, by Proposition 3.11 (ii), there exists $q(\partial_{\boldsymbol{s}}) \in P_{\mathcal{N}}^{\perp}$ such that $m(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}}) \neq 0$. We see from Theorem 2.7 that

$$\begin{aligned} &(q(\partial_{\boldsymbol{s}}) \bullet \widetilde{F}_{\mathcal{N}}(\boldsymbol{x}, \boldsymbol{s}))_{|\boldsymbol{s}=\boldsymbol{0}} \\ &\quad = \sum_{\boldsymbol{u} \in L'} \left(q(\partial_{\boldsymbol{s}}) \bullet (m(\boldsymbol{s}) a_{\boldsymbol{u}}(\boldsymbol{s}) x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}})\right)_{|\boldsymbol{s}=\boldsymbol{0}} \\ &\quad = (q(\partial_{\boldsymbol{s}}) \bullet (m(\boldsymbol{s}) x^{\boldsymbol{v}+B\boldsymbol{s}})_{|\boldsymbol{s}=\boldsymbol{0}} \\ &\qquad\qquad + \sum_{\boldsymbol{u} \in L' \setminus \{0\}} \left(q(\partial_{\boldsymbol{s}}) \bullet (m(\boldsymbol{s}) a_{\boldsymbol{u}}(\boldsymbol{s}) x^{\boldsymbol{v}+B\boldsymbol{s}+\boldsymbol{u}})\right)_{|\boldsymbol{s}=\boldsymbol{0}} \end{aligned}$$

is a solution to $M_A(\boldsymbol{\beta})$. By Proposition 3.10, we have

$$(q(\partial_{\boldsymbol{s}}) \bullet (m(\boldsymbol{s}) x^{\boldsymbol{v}+B\boldsymbol{s}})_{|\boldsymbol{s}=\boldsymbol{0}} = x^{\boldsymbol{v}} \Psi_B(m(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}}))(\log \boldsymbol{x}),$$

hence this solution has a *non-zero* starting term. Hence $\boldsymbol{v}$ is an exponent. □

**Example 4.2** (Continuation of Example 3.12)**.** Let $A$ and $\boldsymbol{v}$ be the ones in Example 3.12. Recall that $m(\boldsymbol{s}) \in P_{\mathcal{N}}$, which is a necessary condition for the fake exponent $\boldsymbol{v}$ not to be an exponent. We see that $\boldsymbol{v}$ is not an exponent from the following calculation. Note that

$$\theta_1 - 2\theta_3 - 3\theta_4 + 1 \in \langle A\theta_{\boldsymbol{x}} - \boldsymbol{\beta} \rangle.$$

Then we have

$$\begin{aligned}
0 &\equiv \theta_1\theta_3(\theta_1 - 2\theta_3 - 3\theta_4 + 1) \\
&= \theta_1^2\theta_3 - 2\theta_1\theta_3^2 - 3\theta_1\theta_3\theta_4 + \theta_1\theta_3 \\
&= \theta_1(\theta_1 - 1)\theta_3 - 2\theta_1\theta_3(\theta_3 - 1) - 3\theta_1\theta_3\theta_4 \\
&= x_1^2x_3\partial_{x_1}^2\partial_{x_3} - 2x_1x_3^2\partial_{x_1}\partial_{x_3}^2 - 3x_1x_3x_4\partial_{x_1}\partial_{x_3}\partial_{x_4} \\
&\equiv x_1^2x_3\partial_{x_2}^3 - 2x_1x_3^2\partial_{x_2}^2\partial_{x_4} - 3x_1x_3x_4\partial_{x_2}\partial_{x_3}^2
\end{aligned}$$

modulo $H_A(\boldsymbol{\beta})$. Hence

$$x_1^2x_3\partial_{x_2}^3 - 2x_1x_3^2\partial_{x_2}^2\partial_{x_4} - 3x_1x_3x_4\partial_{x_2}\partial_{x_3}^2 \in H_A(\boldsymbol{\beta}),$$

and

$$x_1x_3^2\partial_{x_2}^2\partial_{x_4} \in \mathrm{in}_{(-\boldsymbol{w},\boldsymbol{w})}(H_A(\beta)).$$

Since

$$x_1x_3^2\partial_{x_2}^2\partial_{x_4} \bullet x^{\boldsymbol{v}} = x_1x_3^2\partial_{x_2}^2\partial_{x_4} \bullet x_2^{-2}x_3^{-1}x_4 \neq 0,$$

$\boldsymbol{v}$ is not an exponent.

**Corollary 4.3.** *Assume that $B$ is a basis of $L$. If $|I \cup J| > |I_{\mathbf{0}}|$ for any $I \in \mathcal{N}$ and $J \in \mathcal{N}^c$, then $\boldsymbol{v}$ is an exponent.*

*Proof.* For any $I \in \mathcal{N}$ and $J \in \mathcal{N}^c$, we see that

$$|I \cup J \setminus K_{\mathcal{N}}| = |I \cup J| - |K_{\mathcal{N}}| > |I_{\mathbf{0}}| - |K_{\mathcal{N}}| = |I_{\mathbf{0}} \setminus K_{\mathcal{N}}|$$

because both of $I$ and $I_{\mathbf{0}}$ contain $K_{\mathcal{N}}$. Since the degree of $m(\boldsymbol{s}) = (B\boldsymbol{s})^{I_{\mathbf{0}}\setminus K_{\mathcal{N}}}$ is less than that of any $(B\boldsymbol{s})^{I\cup J\setminus\mathcal{N}}$, $m(\boldsymbol{s})$ cannot belong to $P_{\mathcal{N}}$. By Proposition 4.1, $\boldsymbol{v}$ is an exponent. □

**Theorem 4.4.** *Assume that $B$ is a basis of $L$, and that $P_{\mathcal{N}} = m(\boldsymbol{s})\cdot P_B$. Then $\boldsymbol{v}$ is an exponent, and the set*

$$\{(q(\partial_{\boldsymbol{s}}) \bullet \widetilde{F_{\mathcal{N}}}(x,\boldsymbol{s}))_{|\boldsymbol{s}=\mathbf{0}} \,|\, q(\partial_{\boldsymbol{s}}) \in P_{\mathcal{N}}^{\perp}\}.$$

*spans the space of series solutions in the direction of $\boldsymbol{w}$ to $M_A(\boldsymbol{\beta})$ with exponent $\boldsymbol{v}$. In particular, for $q(\partial_{\boldsymbol{s}}) \in P_{\mathcal{N}}^{\perp}$, the solution $(q(\partial_{\boldsymbol{s}}) \bullet \widetilde{F}_{\mathcal{N}}(x,\boldsymbol{s}))_{|\boldsymbol{s}=\mathbf{0}}$ has the starting term $x^{\boldsymbol{v}}\Psi_B(m(\partial_{\boldsymbol{z}}) \star q(\partial_{\boldsymbol{s}}))(\log \boldsymbol{x})$.*

*Proof.* By definition, $P_B \neq \mathbb{C}[\boldsymbol{s}]$. It follows from the assumption that $m(\boldsymbol{s}) \notin P_{\mathcal{N}}$. Hence, by Proposition 4.1, $\boldsymbol{v}$ is an exponent. Moreover, by Proposition 3.11, the homomorphism (11) is surjective. Hence we have

$$\begin{aligned}
\dim_{\mathbb{C}}(P_{\mathcal{N}}^{\perp}) &\geq \dim_{\mathbb{C}}(P_B^{\perp}) = \dim_{\mathbb{C}}(Q_{\boldsymbol{v}}^{\perp}) = \dim_{\mathbb{C}}(\mathrm{Sol}(\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))_{\boldsymbol{v}})) \\
&\geq \dim_{\mathbb{C}}(\mathrm{Sol}(\mathrm{ind}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))_{\boldsymbol{v}})).
\end{aligned}$$

By [8, Proposition 2.3.6, Theorem 2.5.1, and Corollary 2.5.11], the regularity of $M_A(\boldsymbol{\beta})$ indicates that the dimension of the space of series solutions with the exponent $\boldsymbol{v}$ coincides with $\dim_{\mathbb{C}}(\mathrm{Sol}(\mathrm{ind}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))_{\boldsymbol{v}}))$. Hence, by Theorem 2.7, we have the former half of the assertion.

The latter half of the assertion follows from Proposition 3.10. □

**Example 4.5** (Continuation of Examples 2.5, 3.2, and 3.5)**.** Consider the case where $B = \{\boldsymbol{g}^{(1)}, \boldsymbol{g}^{(2)}\}$. Then, $B$ satisfies the assumption in Theorem 4.4. Recall that $m(\boldsymbol{s}) = (B\boldsymbol{s})^{\emptyset} = 1$, and

$$P_B = \langle (B\boldsymbol{s})^{G^{(1)}}, (B\boldsymbol{s})^{G^{(2)}} \rangle = \langle s_1^2, s_2^2 \rangle = P_{\mathcal{N}}.$$

Hence, we see by Proposition 4.1 that $\boldsymbol{v}$ is an exponent, and that

$$\{1, \partial_{s_1}, \partial_{s_2}, \partial_{s_1}\partial_{s_2}\}$$

is a basis of $P_B^{\perp}$. Hence $x^{\boldsymbol{v}} f(\log \boldsymbol{x})$ is a solution to $\mathrm{find}_{\boldsymbol{w}}(H_A(\boldsymbol{\beta}))_{\boldsymbol{v}}$ if and only if

$$f \in \langle 1, \boldsymbol{x}\boldsymbol{g}^{(1)}, \boldsymbol{x}\boldsymbol{g}^{(2)}, (\boldsymbol{x}\boldsymbol{g}^{(1)}) \cdot (\boldsymbol{x}\boldsymbol{g}^{(2)}) \rangle_{\mathbb{C}}.$$

By the uniqueness of an exponent, the above space coincides with the space of solutions to $M_A(\boldsymbol{\beta})$. Note that the holonomic rank of $M_A(\boldsymbol{\beta})$ is four (cf. [8, Example 3.5.2]).

**Example 4.6.** [8, Examples 3.6.3, 3.6.11, 3.6.16] Let $d = 3$, $n = 9$ and

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \end{bmatrix}.$$

Let $\boldsymbol{w} = (2, 0, 0, 0, -1, 0, 0, 0, 2)$, $\boldsymbol{\beta} = (1, 1, 1)^T = \boldsymbol{a}_5$. Consider an exponent $\boldsymbol{v} = (0, 0, 0, 0, 1, 0, 0, 0, 0)^T$. Then $K_{\mathcal{N}_B} = I_{\mathbf{0}} = \emptyset$. The reduced Gröbner basis consists of the following twenty binomials

$$\{\underline{\partial^{\boldsymbol{g}_+^{(i)}}} - \partial^{\boldsymbol{g}_-^{(i)}} \mid i = 1, 2, \ldots, 20\},$$

where

$$\begin{aligned} &\boldsymbol{g}^{(1)} := (0, 1, -1, 0, -1, 1, 0, 0, 0)^T, \boldsymbol{g}^{(2)} := (0, 0, 0, 1, -1, 0, -1, 1, 0)^T, \\ &\boldsymbol{g}^{(3)} := (0, 0, 0, 1, -2, 1, 0, 0, 0)^T, \boldsymbol{g}^{(4)} := (0, 1, 0, 0, -2, 0, 0, 1, 0)^T, \\ &\boldsymbol{g}^{(5)} := (1, -1, 0, -1, 1, 0, 0, 0, 0), \boldsymbol{g}^{(6)} := (0, 0, 0, 0, 1, -1, 0, -1, 1), \\ &\boldsymbol{g}^{(7)}, \ldots, \boldsymbol{g}^{(20)}. \end{aligned}$$

Hence we have

$$\begin{aligned}
&\{G^{(i)} = I_{-\boldsymbol{g}^{(i)}} \setminus I_{\mathbf{0}} \,|\, i = 1, \ldots, 20\} \\
&\quad = \{G^{(1)} = \{2,6\}, G^{(2)} = \{4,8\}, G^{(3)} = \{4,6\}, G^{(4)} = \{2,8\}, \\
&\qquad G^{(5)} = \{1\}, G^{(6)} = \{9\}, \{1,8\}, \{2,7\}, \{1,3\}, \{1,6\}, \{3,4\}, \\
&\qquad \{1,9\}, \{3,7\}, \{4,9\}, \{6,7\}, \{7,9\}, \{2,9\}, \{3,8\}, \{3,9\}, \{1,7\}\}.
\end{aligned}$$

Let

$$B = [\boldsymbol{g}^{(1)}, \boldsymbol{g}^{(2)}, \boldsymbol{g}^{(3)}, \boldsymbol{g}^{(4)}, \boldsymbol{g}^{(5)}, \boldsymbol{g}^{(6)}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ -1 & -1 & -2 & -2 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned}
P_{\mathcal{N}} &= P_B \\
&= \langle (s_1 + s_4 - s_5)(s_1 + s_3 - s_6), (s_2 + s_3 - s_5)(s_2 + s_4 - s_6), \\
&\qquad (s_2 + s_3 - s_5)(s_1 + s_3 - s_6), (s_1 + s_4 - s_5)(s_2 + s_4 - s_6), \\
&\qquad s_5, s_6, s_2(s_1 + s_4 - s_5), s_1(s_2 + s_3 - s_5), \\
&\qquad s_1 s_2, s_2(s_1 + s_3 - s_6), s_1(s_2 + s_4 - s_6) \rangle \\
&= \langle s_1 s_2, s_1 s_3, s_1 s_4, s_2 s_3, s_2 s_4, s_3^2, s_4^2, s_1^2 + s_3 s_4, s_2^2 + s_3 s_4, s_5, s_6 \rangle,
\end{aligned}$$

where the last generator set gives the reduced Gröbner basis with respect to the lexicographic order $<$ with $s_1 > s_2 > s_3 > s_4 > s_5 > s_6$. We see that

$$\{\boldsymbol{\nu} \in \mathbb{N}^6 \,|\, \boldsymbol{s}^{\boldsymbol{\nu}} \notin \mathrm{in}_<(P_B)\} = \{\mathbf{0}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_3 + \boldsymbol{e}_4\},$$

and hence we immediately have

$$q_{\mathbf{0}} = 1, q_{\boldsymbol{e}_1} = \partial_{s_1}, q_{\boldsymbol{e}_2} = \partial_{s_2}, q_{\boldsymbol{e}_3} = \partial_{s_3}, q_{\boldsymbol{e}_4} = \partial_{s_4}.$$

As to the last generator $q_{\boldsymbol{e}_3+\boldsymbol{e}_4}$, since

$$c_{\boldsymbol{\mu}, \boldsymbol{e}_3+\boldsymbol{e}_4} = \begin{cases} -1 & (\text{if } \boldsymbol{\mu} = 2\boldsymbol{e}_1, 2\boldsymbol{e}_2) \\ 0 & (\text{otherwise}) \end{cases},$$

we have

$$q_{\boldsymbol{e}_3+\boldsymbol{e}_4} = \partial_{s_3}\partial_{s_4} + \sum_{\boldsymbol{\mu}; |\boldsymbol{\mu}|=2} c_{\boldsymbol{\mu}, \boldsymbol{e}_3+\boldsymbol{e}_4} \frac{1}{\boldsymbol{\nu}!} \partial_{\boldsymbol{s}}^{\boldsymbol{\mu}} = \partial_{s_3}\partial_{s_4} - \frac{1}{2}\partial_{s_1}^2 - \frac{1}{2}\partial_{s_2}^2.$$

Hence a $\mathbb{C}$-basis of $Q_{\boldsymbol{v}}^{\perp}$ is given by

$$\{1, \boldsymbol{xg}^{(1)}, \boldsymbol{xg}^{(2)}, \boldsymbol{xg}^{(3)}, \boldsymbol{xg}^{(4)}, (\boldsymbol{xg}^{(3)})\cdot(\boldsymbol{xg}^{(4)}) - \frac{1}{2}(\boldsymbol{xg}^{(1)})^2 - \frac{1}{2}(\boldsymbol{xg}^{(2)})^2\}.$$

## 5. Aomoto-Gel'fand systems

In this section, let

$$A = \{\boldsymbol{a}_{i,j} \,|\, 1 \le i \le m,\, m+1 \le j \le m+l\},$$

where $\boldsymbol{a}_{i,j} = \boldsymbol{e}_i + \boldsymbol{e}_j$, and $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{l+m}\}$ is the standard basis of $\mathbb{Z}^{l+m}$.

Then $\mathbb{Z}A = \{\boldsymbol{a} \in \mathbb{Z}^{l+m} \mid \sum_{i=1}^{m} a_i = \sum_{j=m+1}^{m+l} a_j\}$, $\operatorname{rank}(A) = m+l-1$, and $\operatorname{rank}(L) = ml - (m+l-1) = (m-1)(l-1)$, where

$$L = \{[c_{ij}]_{1\le i\le m,\, m+1\le j\le m+l} \in M_{m\times l}(\mathbb{Z}) \mid \sum_{i,j} c_{ij}\boldsymbol{a}_{ij} = \boldsymbol{0}\}.$$

Since $A$ is normal, (that is, $\mathbb{N}A = \mathbb{Z}A \cap \mathbb{R}_{\ge 0}A$), $I_A$ is a Cohen-Macaulay ideal and hence $\operatorname{rank}(M_A(\boldsymbol{\beta})) = \operatorname{vol}(A)$ for any $\boldsymbol{\beta}$ (see [3]).

Take a weight vector $\boldsymbol{w}$ satisfying $w_{i,j} > w_{p,q}$ whenever $(i,j) \ne (p,q)$, $i \le p$, and $j \le q$.

Then the reduced Gröbner basis of $I_A$ with respect to $\boldsymbol{w}$ equals

$$\mathcal{G} = \{\underline{\partial^{(\boldsymbol{g}^{(i,j)}_{(p,q)})_+}} - \partial^{(\boldsymbol{g}^{(i,j)}_{(p,q)})_-} \,|\, i < p,\, j < q\},$$

and

$$\operatorname{in}_{\boldsymbol{w}}(I_A) = \langle \partial_{i,j}\partial_{p,q} \,|\, i < p,\, j < q\rangle,$$

where $\boldsymbol{g}^{(i,j)}_{(p,q)} := E_{i,j} + E_{p,q} - E_{i,q} - E_{p,j} \in L$ and $E_{i,j}$ are matrix units.

The weight $\boldsymbol{w}$ induces a staircase regular triangulation, which is unimodular (cf. [10, Example 8.12]); for example, let $m = 2, l = 4$, the standard pairs are

$$\begin{bmatrix} * & * & * & * \\ * & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & * & * & * \\ * & * & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & * & * \\ * & * & * & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & * \\ * & * & * & * \end{bmatrix},$$

where let the row-numbers be $1, \dots, m$ and the column-numbers $m+1, \dots, m+l$.

For general $l, m$, the standard pairs correspond to the paths from the southwest corner to the northeast corner going only northward or eastward. In this way, we see

$$\operatorname{vol}(A) = \binom{m+l-2}{m-1}.$$

Let

$$B := \{\boldsymbol{b}^{(i,j)} := \boldsymbol{g}^{(i,j)}_{(i+1,j+1)} \,|\, 1 \le i < m,\, m+1 \le j < m+l\}.$$

Then $B$ is a basis of $L$ and $\mathrm{supp}(B) = \{1, \ldots, m+l\}$, hence $B$ satisfies Assumption 2.3. Let $\boldsymbol{s} = (s_{(i,j)})_{1 \le i < m, m+1 \le j < m+l}$ be indeterminates such that $s_{(i,j)}$ corresponds to $\boldsymbol{b}^{(i,j)}$. For convenience, set $s_{(i,j)} := 0$ unless $(i,j) \in \{1, \ldots, m-1\} \times \{m+1, \ldots, m+l-1\}$. Then

$$\begin{aligned}(B\boldsymbol{s})_{(\mu,\nu)} &= \sum_{\substack{1 \le i < m \\ m+1 \le j < m+l}} s_{(i,j)} (\boldsymbol{g}^{(i,j)}_{(i+1,j+1)})_{(\mu,\nu)} \\ &= s_{(\mu,\nu)} - s_{(\mu,\nu-1)} - s_{(\mu-1,\nu)} + s_{(\mu-1,\nu-1)}.\end{aligned}$$

**Lemma 5.1.** *Let $\boldsymbol{\beta} = \boldsymbol{0}$. Then $\boldsymbol{v} = \boldsymbol{0}$ is a unique exponent.*

*Proof.* Since $\mathrm{in}_{\boldsymbol{w}}(I_A)$ is square-free, every fake exponent is an exponent by Theorem 3.6.6 in [8].

Let $\boldsymbol{v}$ be an exponent. Then there exists a standard pair $(\boldsymbol{a}, \sigma) = (\boldsymbol{0}, \sigma)$ corresponding to $\boldsymbol{v}$ such that

$$v_j = 0 \quad (j \notin \sigma), \qquad A\boldsymbol{v} = \boldsymbol{\beta} = \boldsymbol{0}.$$

Since the submatrix $A_\sigma = (\boldsymbol{a}_j)_{j \in \sigma}$ is invertible and satisfies $A_\sigma \boldsymbol{v}_\sigma = \boldsymbol{0}$, we have $\boldsymbol{v} = \boldsymbol{0}$. □

From now on, let $\boldsymbol{\beta} = \boldsymbol{0}$ and $\boldsymbol{v} = \boldsymbol{0}$. Hence $I_{\boldsymbol{0}} = \emptyset = K_{\mathcal{N}_B}$.

**Lemma 5.2.** (i) $\{(ij), (pq)\} \in \mathcal{N}^c = \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c$ *for* $i < p$, $j < q$.

(ii)) *Let* $J \in \mathcal{N}^c = \mathrm{NS}_{\boldsymbol{w}}(\boldsymbol{v})^c$. *Then there exist* $i < p$, $j < q$ *such that* $J \supset \{(ij), (pq)\}$.

*Proof.* (i) This follows from $G^{(ij)}_{(pq)} = \mathrm{nsupp}(\boldsymbol{v} - \boldsymbol{g}^{(ij)}_{(pq)}) = \mathrm{nsupp}(-\boldsymbol{g}^{(ij)}_{(pq)}) = \{(ij), (pq)\}$.

(ii) This is immediate from Proposition 3.3. □

**Proposition 5.3.**

$$P_B = \langle (B\boldsymbol{s})^{\{(i,j),(p,q)\}} \,|\, i < p, j < q \rangle = \langle s_{(i,j)} s_{(p,q)} \,|\, i \le p,\ j \le q \rangle.$$

*Proof.* The first equality follows from Lemma 5.2. We show the second equality. Since, for any $i$, $j$, $p$, $q$ with $i < p$ and $j < q$,

$$\begin{aligned}(B\boldsymbol{s})^{\{(i,j),(p,q)\}} &= (s_{(i,j)} - s_{(i,j-1)} - s_{(i-1,j)} + s_{(i-1,j-1)}) \\ &\quad \times (s_{(p,q)} - s_{(p,q-1)} - s_{(p-1,q)} + s_{(p-1,q-1)}) \in \mathrm{RHS},\end{aligned}$$

we only need to show the reverse inclusion. We introduce the total order $<$ on $\mathcal{X} := \{((i,j),(p,q)) \,|\, 1 \le i \le p < m, m+1 \le j \le q < m+l\}$

defined by

$$((i,j),(p,q)) < ((i',j'),(p',q')) \iff \begin{cases} i < i', \\ \text{or } [i = i' \text{ and } j < j'], \\ \text{or } [(i,j) = (i',j') \text{ and } p > p'], \\ \text{or } [(i,j,p) = (i',j',p') \text{ and } q > q']. \end{cases}$$

We prove by induction on the totally ordered set $(\mathcal{X}, <)$.

First, for the minimum element $((1, m+1), (m-1, m+l-1))$, we have

$$s_{(1,m+1)} s_{(m-1,m+l-1)} = (B\boldsymbol{s})^{\{(1,m+1),(m,m+l)\}} \in \text{LHS}.$$

Next, fix $((i,j),(p,q)) \neq ((1,m+1),(m-1,m+l-1))$ with $1 \leq i \leq p < m$ and $m+1 \leq j \leq q < m+l$. Suppose that $s_{(i',j')} s_{(p',q')} \in \text{LHS}$ for all $((i',j'),(p',q')) < ((i,j),(p,q))$. Then the leading monomial in

$$\begin{aligned} (\text{LHS} \ni)\ (B\boldsymbol{s})^{\{(i,j),(p+1,q+1)\}} &= (s_{(i,j)} - s_{(i,j-1)} - s_{(i-1,j)} + s_{(i-1,j-1)}) \\ &\quad \times (s_{(p+1,q+1)} - s_{(p+1,q)} - s_{(p,q+1)} + s_{(p,q)}) \end{aligned}$$

is $s_{(i,j)} s_{(p,q)}$. It follows from the induction hypothesis that $s_{(i,j)} s_{(p,q)} \in \text{LHS}$. Hence we have the assertion. □

**Corollary 5.4.**

$$\begin{aligned} P_B^{\perp} &= \left\langle \partial_{\boldsymbol{s}}^{\boldsymbol{p}} \,|\, \boldsymbol{s}^{\boldsymbol{p}} \notin \langle s_{(i,j)} s_{(p,q)} \,|\, i \leq p,\, j \leq q \rangle \right\rangle_{\mathbb{C}} \\ &= \langle \partial_{s_{(i_1,j_1)}} \cdots \partial_{s_{(i_r,j_r)}} \,|\, i_1 < \cdots < i_r, j_1 > \cdots > j_r \rangle_{\mathbb{C}}. \end{aligned}$$

*Furthermore,*

$$\dim_{\mathbb{C}}(P_B^{\perp}) = \sum_{r=0}^{\min\{l-1,m-1\}} \binom{l-1}{r} \binom{m-1}{r} = \binom{l+m-2}{m-1}.$$

*Proof.* The first part follows from Proposition 5.3. For the second part, compare the coefficients of $z^{m-1}$ in the following:

$$\begin{aligned} \sum_{k=0}^{l+m-2} \binom{l+m-2}{k} z^k &= (1+z)^{l+m-2} = (1+z)^{l-1}(1+z)^{m-1} \\ &= \Big(\sum_{p=0}^{l-1} \binom{l-1}{p} z^p\Big)\Big(\sum_{q=0}^{m-1} \binom{m-1}{q} z^{m-1-q}\Big) \\ &= \sum_{p,q} \binom{l-1}{p} \binom{m-1}{q} z^{m-1-q+p}. \end{aligned}$$

□

**Corollary 5.5.**

$$\{((\partial_{s_{(i_1,j_1)}} \cdots \partial_{s_{(i_r,j_r)}}) \bullet F_{\mathcal{N}}(\boldsymbol{x},\boldsymbol{s}))_{|\boldsymbol{s}=\boldsymbol{0}} \,|\, i_1 < \cdots < i_r, j_1 > \cdots > j_r\}$$

*forms a fundamental system of solutions to* $M_A(\boldsymbol{0})$.

**Proposition 5.6.**

$$Q_{\boldsymbol{0}}^{\perp} = \left\langle \prod_{k=1}^{r} (\boldsymbol{x}\boldsymbol{g}_{(i_k+1,j_k+1)}^{(i_k,j_k)}) \;\middle|\; \begin{array}{l} i_1 < i_2 < \cdots < i_r \\ j_1 > j_2 > \cdots > j_r \end{array} \right\rangle_{\mathbb{C}}.$$

*Proof.* This is immediate from Theorem 3.14 and Corollary 5.4. □

**Example 5.7.** Let $m = 2$. This case corresponds to the Lauricella's $F_D$ (e.g. see [1, §3.1.3]). Then $\mathrm{vol}(A) = \binom{l+m-2}{m-1} = l$, and

$$P_B^{\perp} = \langle 1, \partial_{s_{(1,3)}}, \partial_{s_{(1,4)}}, \ldots, \partial_{s_{(1,l+1)}} \rangle_{\mathbb{C}}.$$

**Example 5.8.** Let $l = m = 3$. Then $\mathrm{vol}(A) = \binom{l+m-2}{m-1} = \binom{4}{2} = 6$, and

$$P_B^{\perp} = \langle 1, \partial_{s_{(1,4)}}, \partial_{s_{(1,5)}}, \partial_{s_{(2,4)}}, \partial_{s_{(2,5)}}, \partial_{s_{(1,5)}}\partial_{s_{(2,4)}} \rangle_{\mathbb{C}}.$$

## 6. Lauricella's $F_C$

Let

$$A := \{\boldsymbol{a}_i := \boldsymbol{e}_0 + \boldsymbol{e}_i,\ \boldsymbol{a}_{-i} := \boldsymbol{e}_0 - \boldsymbol{e}_i \,|\, i = 1, 2, \ldots, m\}.$$

In this case, $A$ is normal, and the $A$-hypergeometric systems correspond to Lauricella's $F_C$ [7]. Then

$$\mathbb{Z}A = \{\boldsymbol{a} \in \mathbb{Z}^{m+1} \,|\, \sum_{i=0}^{m} a_i \in 2\mathbb{Z}\},$$

and

$$L = \{\boldsymbol{l} \in \mathbb{Z}^{2m} \,|\, \sum_{i=\pm 1,\ldots,\pm m} l_i = 0,\ l_i - l_{-i} = 0 \ \ (1 \le i \le m)\}.$$

We have $\mathrm{rank}(L) = 2m - (m+1) = m - 1$.

Take a weight $\boldsymbol{w}$ so that

$$w_1 + w_{-1} > w_2 + w_{-2} > \cdots > w_m + w_{-m}.$$

Then

$$\mathrm{in}_{\boldsymbol{w}}(I_A) = \langle \partial_1\partial_{-1}, \partial_2\partial_{-2}, \ldots, \partial_{m-1}\partial_{-(m-1)} \rangle,$$

and the reduced Gröbner basis $\mathcal{G}$ is given by

$$\mathcal{G} = \{\underline{\partial^{\boldsymbol{g}_+^{(i)}}} - \partial^{\boldsymbol{g}_-^{(i)}} \,|\, i = 1, 2, \ldots, m-1\},$$

where $\boldsymbol{g}^{(i)} = \boldsymbol{e}_i + \boldsymbol{e}_{-i} - \boldsymbol{e}_m - \boldsymbol{e}_{-m}$. Set $B := \{\boldsymbol{g}^{(i)} \mid i = 1, 2, \ldots, m-1\}$. Since $B$ is a basis of the free $\mathbb{Z}$-module $L$, it satisfies Assumption 2.3. Note that $\mathrm{supp}(B) = \{\pm 1, \ldots, \pm m\}$. Let $\boldsymbol{s} = (s_i)_{1 \le i \le m-1}$ be indeterminates such that $s_i$ corresponds to $\boldsymbol{b}^{(i)} := \boldsymbol{g}^{(i)}$. The standard pairs are pairs of $*$-place $\{\epsilon(i)i \mid i \in [1, m-1]\} \cup \{\pm m\}$ ($\epsilon : [1, m-1] \to \{\pm 1\}$) and 0-place its complement.

Hence $\mathrm{vol}(A) = 2^{m-1}$.

**Lemma 6.1.** *Let $\boldsymbol{\beta} = \boldsymbol{0}$. Then $\boldsymbol{v} = \boldsymbol{0}$ is a unique exponent.*

*Proof.* The proof is similar to Lemma 5.1. □

**Proposition 6.2.**

$$P_B = \langle (B\boldsymbol{s})^{\{\pm i\}} \mid i = 1, \ldots, m-1 \rangle = \langle s_i^2 \mid 1 \le i \le m-1 \rangle.$$

*Proof.* We have $G^{(i)} = I_{-\boldsymbol{g}^{(i)}} \setminus I_{\boldsymbol{0}} = \{\pm i\}$ and

$$(B\boldsymbol{s})^{\{\pm i\}} = \left( \sum_{\nu=1}^{m-1} s_\nu \boldsymbol{g}_{+i}^{(\nu)} \right) \left( \sum_{\nu=1}^{m-1} s_\nu \boldsymbol{g}_{-i}^{(\nu)} \right) = s_i^2.$$

□

**Proposition 6.3.**

$$\begin{aligned} P_B^\perp &= \left\langle \partial_{\boldsymbol{s}}^{\boldsymbol{p}} \mid \boldsymbol{s}^{\boldsymbol{p}} \notin \langle s_i^2 \mid 1 \le i \le m-1 \rangle \right\rangle_{\mathbb{C}} \\ &= \langle \partial_{\boldsymbol{s}}^I = \prod_{i \in I} \partial_{s_i} \mid I \subset \{1, \ldots, m-1\} \rangle_{\mathbb{C}}, \end{aligned}$$

*and*

$$Q_{\boldsymbol{0}}^\perp = \langle \prod_{i \in I} (\boldsymbol{x}\boldsymbol{g}^{(i)}) \mid I \subset \{1, \ldots, m-1\} \rangle_{\mathbb{C}}.$$

*Furthermore,*

$$\dim_{\mathbb{C}}(Q_{\boldsymbol{0}}^\perp) = 2^{m-1}.$$

*Proof.* This is immediate from Lemma 3.6, Theorem 3.14, and Proposition 6.2. □

**Corollary 6.4.** $\{((\prod_{i \in I} \partial_{s_i}) \bullet F_{\mathcal{N}}(\boldsymbol{x}, \boldsymbol{s}))_{|\boldsymbol{s}=\boldsymbol{0}} \mid I \subset [1, m-1]\}$ *forms a basis of solutions of $M_A(\boldsymbol{0})$.*

## Acknowledgment

The authors thank the referee for careful reading and helpful comments.

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