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Formation of Cyclic Steps by water on Ice Surface

by

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Abstract

Cyclic steps are long-wave bedforms that are characterized by hydraulic jumps. On the upstream side of a step, the flow is subcritical, accelerating towards the downstream end to become a supercritical flow. These steps can be observed in various environments, such as rivers, ocean floors, and ice surfaces on Earth, and also on Mars's ice caps. The present study focuses on cyclic steps formed on ice surfaces. Understanding the formation and characteristics of cyclic steps on ice can enhance our knowledge of the cold region environment, such as temperature variations, flow features, and morphologies. In addition, it could help the research of space exploration to provide more options for human beings in the future. Cyclic steps on ice are formed by wind or water by heat transfer. It is difficult to know the normal flow condition of the wind, so we study the formation of cyclic steps formed by flowing water on ice.

In this study, we propose an analytical model to reproduce the formation of cyclic steps created by water on ice. We assume a steady and uniform Froude-supercritical flow over a flat ice surface, which causes the surface to become unstable and results in the spontaneous formation of a series of steps. The formation process involves both mechanical and thermodynamic processes. The water temperature is assumed higher than the ice temperature but lower than the air temperature. The model is governed by four equations: the momentum equation, the continuity equation, the heat transfer equation in water, and the energy balance equation (the Stefan equation).

We use the variables in the normal flow condition without steps as normalization. In addition, we apply the quasi-steady assumption to the governing equations since changes in the ice surface occur so slowly that the flow equations can be approximated as steady. Based on experimental observations, cyclic steps were found that migrate upstream while maintaining their form because of the temperature setting of air, water, and ice. Once this equilibrium state is reached, the average channel slope, wavelength, wave height, and wave migration speed reach constant values. The entire ice surface moves downward since the vertical degradation of ice is more active than the upslope progradation. We introduce a moving coordinate system to describe these features of cyclic steps on the ice.

After further manipulation, the governing equations are reduced to three normalized equations, and we need corresponding boundary conditions to solve them. The flow transitions from a supercritical to a subcritical regime through a hydraulic jump. We thus apply the momentum conservation before and after each hydraulic jump as one boundary condition. Furthermore, we assume the ice elevation and average water temperature are constant before and after hydraulic jumps. However, the problem cannot be closed, and wavelength cannot be determined since these

boundary conditions are relationships among variables rather than specific values. We assume the water temperature value at the downstream end of a hydraulic jump to close the problem. During the calculation, we find that the minimum value corresponds to the longest wavelength. We assume it is the most stable when the wavelength is the longest and use it in further analysis.

This two-point boundary value problem is solved using the shooting method with the Newton-Raphson scheme. The calculation is initiated from the vicinity of the Froude critical point (i.e., the point where the Froude number is unity) to avoid singularity and progresses towards upstream and downstream directions. After the calculations, we can generate variations in depth-averaged water temperature, depth-averaged velocity, and bed elevation over one step wavelength. Once we know the variables in the normal flow condition, the actual scale steps and dimensional variables can be obtained. The results calculated by our model are compared with the previous experimental results to validate our model. The results can explain the formation process of cyclic steps on the ice and suggest relationships among wavelengths and heat transfer capacity of air and water, and the Froude number in the normal flow condition. The calculated steps in the analysis are compared with experimental observations to validate our model.

Cyclic steps formed by water on ice and those formed by long-runout turbidity currents share some similarities. We refer to the previous theory of dividing the currents into two layers to capture the features of long-runout turbidity currents that can run long distances without dissipating. The high-concentrated lower layer is in an equilibrium state and interacts little with the upper layer. This configuration results in the lower layer similar to the open channel flow. We compare these two types of cyclic steps regarding formulations, formation process, and relationships among wavelengths and parameters.

Keywords: Cyclic steps; heat transfer; sediment transport; ice surface; Spiral troughs on Mars

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1 Introduction

Cyclic steps are trains of self-generated rhythmic bedforms. They form due to instability when Froude-supercritical flow passes over the interface between solid and liquid. The flow is subcritical at the upstream side of each step, accelerating to become supercritical at the downstream end, as shown in Figure 1 [1]. A hydraulic jump is presented at each step's boundary, which helps dissipate energy and facilitate the transition of the supercritical flow to the subcritical flow. The research interest in cyclic steps has increased due to their significant impact on geomorphology and sediment transport.

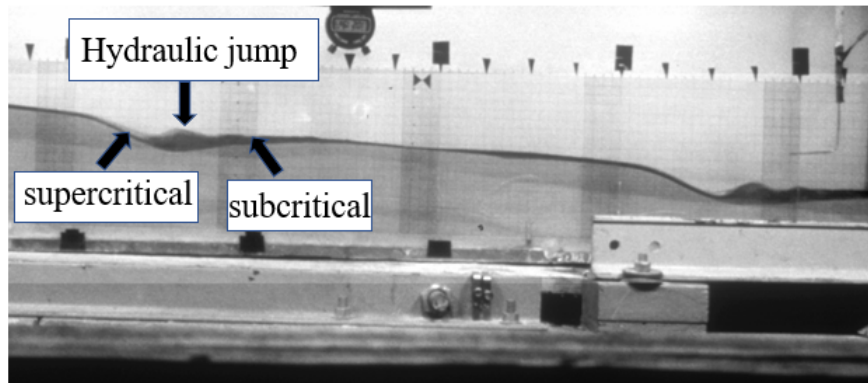


Figure 1: The experiment of cyclic steps created by flow over an erodible bed. Two steps, each terminating in a hydraulic jump, are visible. Flow is from left to right. [1]

Cyclic steps are closely tied to antidunes [2], so it is common for researchers to misidentify cyclic steps as antidunes [3]. They are both bedforms that form in response to fluid flow, typically in fluvial or marine environments. The Froude-supercritical normal flow over an erodible bed is unstable, and this instability is often manifested in the form of antidunes. Antidunes are temporary features that form, collapse, and reform. However, cyclic steps are much more stable than antidunes. This is because cyclic steps maintain their boundaries through hydraulic jumps, whereas antidunes do not necessarily have hydraulic jumps. Additionally, the flow regime over antidunes is predominantly Froude-supercritical, whereas cyclic steps exhibit an alternation of Froude-supercritical and Froude-subcritical flow. Furthermore, cyclic steps are considered long-wave bedforms, with wavelengths that are several orders of magnitude larger than the layer thickness of water, while antidunes have relatively short wavelengths. In certain instances, antidunes can

evolve to cyclic steps via nonlinear interaction between the flow and the bed [4], for example, wave breaking with density flows [5].

According to Slootman et al. [3], Jorritsma [6] was the first to observe the spontaneous formation of upward-migrating cyclic steps in a laboratory experiment. Winterwerp et al. [7] provided the first clear description of cyclic steps, and the term "cyclic steps" was coined by Parker et al. [8]. Cyclic steps have been observed in a variety of settings, including the ocean floor in response to turbidity currents (Figure 2 [9]) [10] [11], erodible beds in channels [2], the north polar ice cap surface of Mars [12], volcano flanks [13], deltas [14][15], bedrock channels [16], and in the depositional record of fluvial environments [17][18].

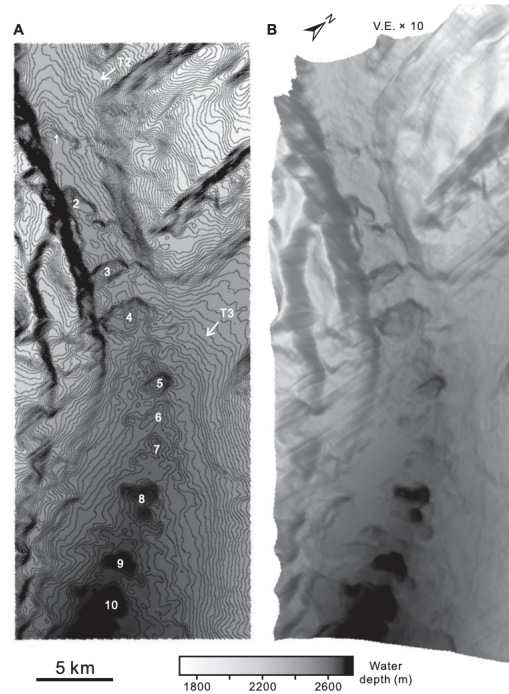


Figure 2: Cyclic steps 1-10 are found within the South Taiwan Shoal Canyon. (A) Isobaths are shown at intervals of 10 meters. (B) A three-dimensional relief bathymetric map. T2-T3 denotes tributaries. V.E. denotes vertical exaggeration. [9]

Laboratory experiments have contributed significantly to our understanding of cyclic steps. Taki and Parker [1] conducted experiments to reproduce transportation cyclic steps, as shown in Figure 1, which are associated with the suspension

and deposition of sediment. On the upstream side, slow flow causes deposition, while erosion caused by swift flow occurs on the downstream side. They also found that cyclic steps migrate upstream without changing form. Further experiments by Yokokawa et al. [19] demonstrated the formation of cyclic steps in subaerial conditions in a tilted flume. They found that the slopes should be steep enough to cause supercritical flows with Froude numbers above those needed to form antidunes.

Several models described the formation and features of cyclic steps. Parker and Izumi [2] introduced a model to analyze cyclic steps formed purely by erosion on cohesive soil in open channel flow and proposed to use threshold velocity (the velocity for the onset of bed erosion) to determine the wavelength of one step. This threshold velocity corresponds to the longest wavelength in the calculation. Sun and Parker [20] developed an analytical model for transportational cyclic steps formed in alluvium with abundant suspended sediment. They discussed the features and formation domain of cyclic steps during their analysis. Izumi et al. [16] conducted a theoretical study to explain the formation of cyclic steps in mixed bedrock-alluvial channels and obtained the conditions for forming cyclic steps in bedrock. They discovered that cyclic steps only develop on slopes with high gradients when the sediment supply is relatively small compared to the transport capacity.

Our study focuses mainly on ice environments. Boundary waves forming on the ice surface are often observed. A previous study by Carey et al. [21] reported the presence of ripples and dunes on the underside of river ice cover in the St. Croix River. Gilpin et al. [22] have performed experiments and linear stability analyses on the formation of waves at the ice-water interface. These waves are believed to be formed due to variations in heat transfer caused by the interaction between the ice surface and flow. Furthermore, Camporeale et al. [23] conducted a linear stability analysis on ice ripples formation by the water and suggested that the ice-water interface may become unstable under laminar flow conditions at large Reynolds numbers ($Re = 1 - 10^3$).

Cyclic steps are also a type of boundary wave that can be observed on ice. Kalstrom et al. [24] identified the steps formed by a superglacial meltwater stream. After the water flows through the initially flat ice surface, it becomes unstable, resulting in a sequence of steps on the ice-water interface. The spiral troughs observed on the north polar ice cap of Mars, created by the Katabatic wind, are considered to be cyclic steps (Figure 3) [12]. These steps are formed due to the solidification and sublimation of water vapor contained in the katabatic wind. A katabatic wind is a wind type that arises when cold, dense air moves downward from higher elevations and flows downhill, propelled by gravitational force. The Coriolis force primarily influences the direction, leading to a spiral flow path. Similar bedforms created by the katabatic wind have also been discovered on the Antarctic ice sheet [25] [26], and Yokokawa et al. [4] suggested that some of them

might be cyclic steps. Studying cyclic steps on ice can provide valuable insights into cold regions, including temperature variations, geomorphology changes, and flow features. Moreover, it can enhance our understanding of spiral troughs on Mars, supporting space exploration.

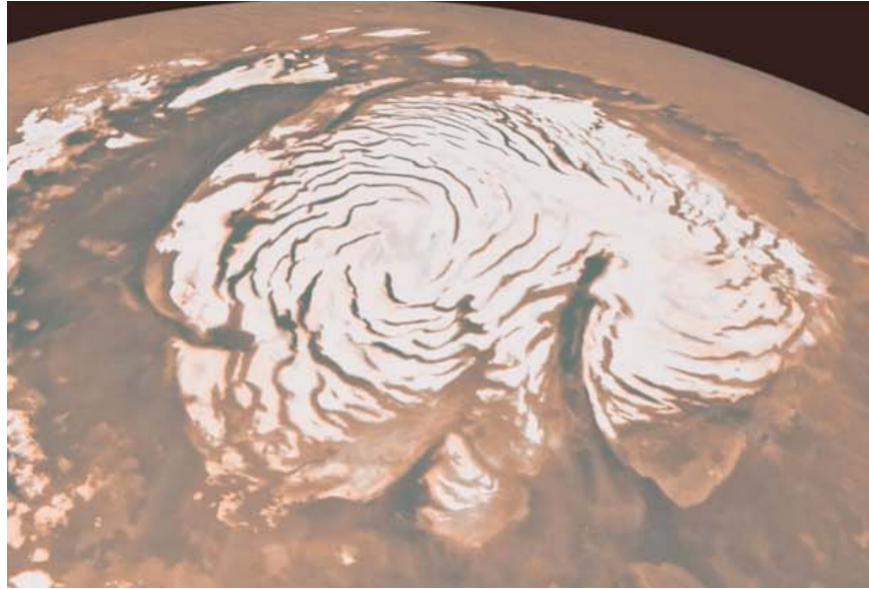


Figure 3: Spiral troughs on the North Polar Ice Cap of Mars. By NASA/JPL-Caltech/MSSS. accessed on date: 2023-03-07
https://www.nasa.gov/mission_pages/MRO/multimedia/pia13163.html

Based on our current understanding, there are few studies conducted on cyclic steps formed on ice. Naito et al. [27] conducted experiments about downstream migrating steps on the ice and proposed a model to analyze steps based on experiments. However, in general, cyclic steps migrate upstream while maintaining their shape. Yokokawa et al. [4] created cyclic steps on icy substrates in a laboratory setting (as shown in Figure 4) and presented a linear stability analysis to explain the upstream-migrating antidunes, which are the precursors of cyclic steps. They observed that the boundary between ice and water is susceptible to perturbations under Froude-supercritical flow. As for cyclic steps formed by wind, Sumida et al. [28] conducted experiments on cyclic steps formed by wind and found that longer wavelengths tend to occur on steeper slopes.

Cyclic steps formed by water on ice and long-runout turbidity currents have similarities and differences. We compare these two types later. Turbidity currents

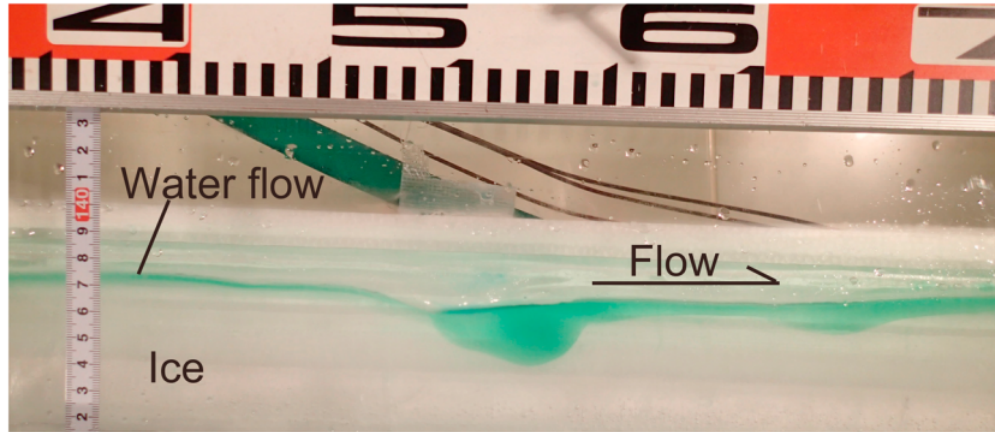


Figure 4: The experiment of cyclic steps on ice [4]. Flow is from left to right.

are sediment gravity flow in which sediment is suspended by fluid turbulence. They are the essential sediment transport processes and always occur in the lakes and oceans. Slumping, earthquakes, and other geological disturbances can trigger them. As a result of its potent ability to erode the seabed and transport sediment, it is regarded as one of the fundamental mechanisms responsible for shaping and creating submarine morphologies. People have been paying increasing attention to turbidity currents since the study of turbidity currents plays a vital role in exploiting resources and preventing disasters. Turbidity currents could bring abundant natural resources but might cause severe damage. These currents can transport turbidite with rich hydrocarbon matter, the most critical petroleum and methane hydrate source. However, reaching fast speeds, turbidity currents are capable of causing damage to artificial structures such as telecommunication cables on the sea floor.

Parker et al. [29] presented a three-equation model to describe self-accelerating turbidity currents. This model comprises the conservatives of the fluid mass equation, the sediment mass equation, and the mean flow momentum equation. If the velocity of the current is fast enough, it can pick up additional sediment from the bed, leading to an increase in velocity. Consequently, currents can accelerate and entrain more bed sediment in a self-reinforcing loop. Parker et al. [29] also proposed a four-equation model that includes the turbulent kinetic energy equation to satisfy the energy constraint. Both models account for water entrainment from the ambient water, which causes the currents to thicken and dilute sediment. Izumi et al. [31] combined the model proposed by Parker et al. [29] and the Exner equation to reproduce the formation of submarine gullies formed by turbidity currents. Dias

et al. [30] presented a model of cyclic steps created by turbidity currents and applied their model in the Monterey East channel [11] to reproduce cyclic steps.

One remarkable feature of submarine turbidity currents is that they can travel over 1000 km without dissipating, and the models proposed by Parker et al. [29] cannot explain this feature. Luchi et al. [32] was the first to propose a solution to the physical mechanism of why turbidity currents can travel long distances. They discovered that the turbidity current could be divided into an upper layer with a low suspended sediment concentration and a lower layer with a high suspended sediment concentration. Paull et al. [10], and Pope et al. [33] conducted field observations that partially support the assumption that a turbidity current comprises two layers. The lower driving layer has an invariant suspended sediment concentration profile, velocity profile, and flow thickness, while the upper driven layer continues to entrain ambient water and becomes dilute. The two layers have a density interface with little friction, momentum exchange, and diffusion. The velocity gradient and the shear stress are almost zero at the interface, and gravity drives the upper layer and balances the shear stress at the bottom of the lower layer. The lower layer can travel long distances and interacts less with the upper layer because of the little friction between the two layers. We can assume that the highly concentrated lower layer is in equilibrium. An analysis proposed by Izumi[34] revealed three significant non-dimensional parameters, the fall velocity relative to the friction velocity, the slope of the ocean floor, and the particle Reynolds number, determine the thickness of the high-concentrated lower layer.

2 Formulation

2.1 Problem Statement

Water-driven and wind-driven boundary waves on ice are both associated with heat transfer. Since the existence of density stratification in the Katabatic wind, the cold and denser lower layer can be seen as the open channel flow. Therefore, the formation of cyclic steps on ice formed by Katabatic wind and water are similar. To understand these boundary waves, studying their formation process and the relationships among their features and surrounding environments is necessary. Nevertheless, it is difficult to represent the formation of cyclic steps generated by wind since we cannot obtain the variables in the normal flow condition of wind. In order to examine the characteristics of morphologies formed by heat transfer, we suggest employing an analytical model to reproduce cyclic steps created by water on an ice surface, as shown in Figure 5.

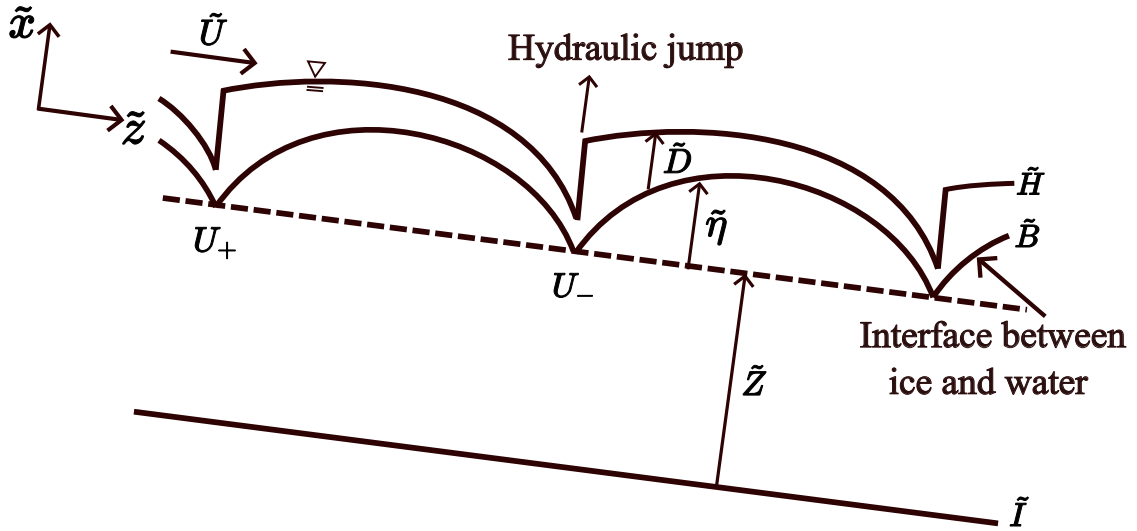


Figure 5: Conceptual sketch of cyclic steps on the ice. The variable $\tilde{H}$ is the water surface elevation, $\tilde{B}$ is the water bottom elevation, and $\tilde{I}$ is the ice bottom elevation.

The linear stability analysis, as performed by Yokokawa et al. [4], did not directly analyze the cyclic steps on the ice. Cyclic steps themselves cannot be described in terms of linear stabilities since they are a fundamentally nonlinear phenomenon, and the discontinuity at each hydraulic jump prevents the Taylor expansion on the base state. Therefore, we propose an analytical model based

on the model presented by Naito et al. [27] to reproduce the formation of cyclic steps on the ice. We employ the depth-integrated flow equations, depth-integrated heat transfer, and Stephen equations. Naito et al. [27] used simple heat conduction equations to describe heat transfer between air, water, and ice. However, this method may not accurately depict the heat flux, and thus we aim to find better descriptions in our model. We use empirical formulas to describe the heat flux in water, ice, and ambient air instead.

According to the theory proposed by Naito et al. [27], the steps' wavelengths depend on the water velocity before hydraulic jumps. However, they did not provide a method to determine wavelengths. We aim to address this issue in our study. Our analysis found that the wavelength could reach its maximum value as the temperature at the downstream end of the hydraulic jump varies. We assume that the wavelength of a single step in nature is always the longest and use this maximum wavelength for further analysis. In addition, we normalize the variables by the values in the normal flow condition before the steps are developed, enabling us to obtain the actual scale of cyclic steps. Furthermore, we validate the accuracy of our model using the experimental data presented by Yokokawa et al. [4]. Finally, we compare these steps with those formed by long-runout turbidity currents.

2.2 Governing Equations

As shown in Figure 5, we assume a steady and uniform Froude-supercritical flow moving over a flat ice surface. The surface becomes unstable, resulting in the spontaneous formation of a series of steps, which are permanently stabilized by the presence of hydraulic jumps. At the interface between water and ice, solidification and melting occur simultaneously. We assume that the fluid temperature is lower than the surrounding air but higher than the ice temperature. Similar temperature distributions can be observed on spiral troughs of Mars [12].

The linear stability analysis presented by Yokokawa et al. [4] cannot describe cyclic steps themselves. Hence, to directly analyze cyclic steps on the ice, we refer to the approach of Naito et al. [27]. The scale of cyclic steps streamwise is much larger than the scale in the upward direction, and we thus integrate the Navier-Stokes equations and the heat equation of flowing water from the ice surface to the water surface. The focus is on the variation in the streamwise direction, such as velocity, water temperature, ice elevation, and water depth.

2.2.1 Depth Integrated Flow equations

The Reynolds-averaged Navier-Stokes and continuity equations are

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{\partial \tilde{u}\tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{u}\tilde{w}}{\partial \tilde{z}} = \frac{1}{\rho_f} \left(\frac{\partial \tilde{T}_{xx}}{\partial \tilde{x}} + \frac{\partial \tilde{T}_{xz}}{\partial \tilde{z}} \right) + g \sin \alpha \quad (1)$$

$$\frac{\partial \tilde{w}}{\partial \tilde{t}} + \frac{\partial \tilde{u}\tilde{w}}{\partial \tilde{x}} + \frac{\partial \tilde{w}\tilde{w}}{\partial \tilde{z}} = \frac{1}{\rho_f} \left(\frac{\partial \tilde{T}_{xz}}{\partial \tilde{x}} + \frac{\partial \tilde{T}_{zz}}{\partial \tilde{z}} \right) - g \cos \alpha \quad (2)$$

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0 \quad (3)$$

where $\tilde{t}$ is the time, $\tilde{x}$ is the coordinate in the streamwise, $\tilde{z}$ is the coordinate in the depth direction, $\tilde{u}$ and $\tilde{w}$ are the velocity components in the $\tilde{x}$ and $\tilde{z}$ directions respectively, ρ_f is the density of water ($= 1,000 \text{ kg/m}^3$), g is the gravity acceleration ($= 9.8 \text{ m/s}^2$), α is the slope angle of the channel, and $\tilde{T}_{ij}$ ($i, j = \tilde{x}, \tilde{z}$) is the stress tensor. $\sim$ means dimensional variables and is later removed to denote dimensionless variables. The boundary layer of the flow quickly develops in the streamwise direction since the scale in the streamwise direction is much larger than that in the depth direction in the shallow water approximation. Therefore, because the boundary layer can be assumed to be in equilibrium, it is possible to ignore the effect of boundary layer development in the shallow water approximation.

The stress tensor can be written in the form

$$\tilde{T}_{xx} = -\tilde{p} + 2\nu \frac{\partial \tilde{u}}{\partial \tilde{x}} - \overline{\tilde{u}'\tilde{u}'} \quad (4)$$

$$\tilde{T}_{xz} = \nu \left(\frac{\partial \tilde{u}}{\partial \tilde{z}} + \frac{\partial \tilde{w}}{\partial \tilde{x}} \right) - \overline{\tilde{u}'\tilde{w}'} \quad (5)$$

$$\tilde{T}_{zz} = -\tilde{p} + 2\nu \frac{\partial \tilde{w}}{\partial \tilde{z}} - \overline{\tilde{w}'\tilde{w}'} \quad (6)$$

where $\tilde{p}$ is the hydrostatic pressure, ν is the viscosity, and $\prime$ denotes the turbulent fluctuation of the corresponding velocity component. The terms, including the viscosity ν are usually sufficiently small compared with the Reynolds stress terms ($= \overline{\tilde{u}'\tilde{u}'}, \overline{\tilde{u}'\tilde{w}'}, \overline{\tilde{w}'\tilde{w}'}$), we ignore the viscosity terms hereafter.

In addition, the scale in the stream direction is much larger than that in the depth direction for shallow water flow. According to this assumption, we obtain the following scale estimation

$$\frac{\partial}{\partial \tilde{x}} \ll \frac{\partial}{\partial \tilde{z}} \quad (7)$$

Applying the above estimation to the continuity equation, we obtain the following relation

$$\tilde{u} \gg \tilde{w} \quad (8)$$

Assuming the turbulence is isotropic, we have the relation

$$\overline{\tilde{u}'\tilde{u}'} \sim \overline{\tilde{u}'\tilde{w}'} \sim \overline{\tilde{w}'\tilde{w}'} \quad (9)$$

With this in mind, we rewrite Equations 1 and 2 in the form

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{\partial \tilde{u}\tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{u}\tilde{w}}{\partial \tilde{z}} = -\frac{1}{\rho_f} \frac{\partial \tilde{P}}{\partial \tilde{x}} + \frac{1}{\rho_f} \frac{\partial \tilde{T}_{xz}}{\partial \tilde{z}} + g \sin \alpha \quad (10)$$

$$0 = -\frac{1}{\rho_f} \frac{\partial \tilde{P}}{\partial \tilde{z}} - g \cos \alpha \quad (11)$$

where $\tilde{P}$ is the total pressure, including the normal component of the Reynolds stress, such that

$$\tilde{P} = \tilde{p} + \overline{\tilde{u}'\tilde{u}'} = \tilde{p} + \overline{\tilde{w}'\tilde{w}'} \quad (12)$$

In the above relation, we have assumed that the normal components of the Reynolds stress are isotropic, and $\overline{\tilde{u}'\tilde{u}'} = \overline{\tilde{w}'\tilde{w}'}$.

There is a boundary condition of vanishing the pressure at the water surface, as shown below

$$\tilde{P}(\tilde{H}) = 0 \quad (13)$$

Integrating Equation 11 with the use of the above boundary condition, we obtain

$$\tilde{P} = \rho_f (\tilde{H} - \tilde{z}) g \quad (14)$$

Integrating Equation 10 with Equation 14 substituted from the bottom ($\tilde{z} = \tilde{B}$) to the water surface ($\tilde{z} = \tilde{H}$), we obtain

$$\int_{\tilde{B}}^{\tilde{H}} \frac{\partial \tilde{u}}{\partial \tilde{t}} d\tilde{z} + \int_{\tilde{B}}^{\tilde{H}} \frac{\partial \tilde{u}\tilde{u}}{\partial \tilde{x}} d\tilde{z} + \int_{\tilde{B}}^{\tilde{H}} \frac{\partial \tilde{u}\tilde{w}}{\partial \tilde{z}} d\tilde{z} = - \int_{\tilde{B}}^{\tilde{H}} g \frac{\partial \tilde{H}}{\partial \tilde{x}} d\tilde{z} + \frac{1}{\rho_f} \int_{\tilde{B}}^{\tilde{H}} \frac{\partial \tilde{T}_{xz}}{\partial \tilde{z}} d\tilde{z} + \int_{\tilde{B}}^{\tilde{H}} g \sin \alpha d\tilde{z} \quad (15)$$

The first, second, and third terms on the left-hand side of the above equation are written in the form, of respectively

$$\int_{\tilde{B}}^{\tilde{H}} \frac{\partial \tilde{u}}{\partial \tilde{t}} d\tilde{z} = \frac{\partial}{\partial \tilde{t}} \int_{\tilde{B}}^{\tilde{H}} \tilde{u} d\tilde{z} - \frac{\partial \tilde{H}}{\partial \tilde{t}} \tilde{u}(\tilde{H}) + \frac{\partial \tilde{B}}{\partial \tilde{t}} \tilde{u}(\tilde{B}) \quad (16)$$

$$\int_{\tilde{B}}^{\tilde{H}} \frac{\partial \tilde{u}\tilde{u}}{\partial \tilde{x}} d\tilde{z} = \frac{\partial}{\partial \tilde{x}} \int_{\tilde{B}}^{\tilde{H}} \tilde{u}\tilde{u} d\tilde{z} - \frac{\partial \tilde{H}}{\partial \tilde{x}} \tilde{u}^2(\tilde{H}) + \frac{\partial \tilde{B}}{\partial \tilde{x}} \tilde{u}^2(\tilde{B}) \quad (17)$$

$$\int_{\tilde{B}}^{\tilde{H}} \frac{\partial \tilde{u} \tilde{w}}{\partial \tilde{z}} d\tilde{z} = \tilde{u}(\tilde{H}) \tilde{w}(\tilde{H}) - \tilde{u}(\tilde{B}) \tilde{w}(\tilde{B}) \quad (18)$$

The first, second, and third terms on the right-hand side of Equation 15 are written in the form of respectively

$$- \int_{\tilde{B}}^{\tilde{H}} g \frac{\partial \tilde{H}}{\partial \tilde{x}} d\tilde{z} = -g \tilde{D} \frac{\partial \tilde{H}}{\partial \tilde{x}} \quad (19)$$

$$\frac{1}{\rho} \int_{\tilde{B}}^{\tilde{H}} \frac{\partial \tilde{T}_{xz}}{\partial \tilde{z}} d\tilde{z} = -\frac{\tilde{T}_{xz}(\tilde{B})}{\rho} \quad (20)$$

$$\int_{\tilde{B}}^{\tilde{H}} g \sin \alpha d\tilde{z} = g \tilde{D} \sin \alpha \quad (21)$$

where $\tilde{D}$ is the flow depth. Substituting Equations 16–21 into Equation 15, we obtain

$$\begin{aligned} & \frac{\partial}{\partial \tilde{t}} \int_{\tilde{B}}^{\tilde{H}} \tilde{u} d\tilde{z} - \frac{\partial \tilde{H}}{\partial \tilde{t}} \tilde{u}(\tilde{H}) + \frac{\partial \tilde{B}}{\partial \tilde{t}} \tilde{u}(\tilde{B}) + \frac{\partial}{\partial \tilde{x}} \int_{\tilde{B}}^{\tilde{H}} \tilde{u} \tilde{u} d\tilde{z} - \frac{\partial \tilde{H}}{\partial \tilde{t}} \tilde{u}^2(\tilde{h}) + \frac{\partial \tilde{B}}{\partial \tilde{t}} \tilde{u}^2(\tilde{h}) \\ & + \tilde{u}(\tilde{H}) \tilde{w}(\tilde{H}) - \tilde{u}(\tilde{B}) \tilde{w}(\tilde{B}) = -g \tilde{D} \frac{\partial \tilde{H}}{\partial \tilde{x}} - \frac{\tilde{T}_{xz}(\tilde{B})}{\rho_f} + g \tilde{D} \sin \alpha \end{aligned} \quad (22)$$

The velocity in the depth direction at the water surface and the bed correspond to the total derivatives of the water surface and bed elevations, such that

$$\tilde{w}(\tilde{H}) = \frac{D\tilde{H}}{D\tilde{t}} = \frac{\partial \tilde{H}}{\partial \tilde{t}} + \tilde{u}(\tilde{H}) \frac{\partial \tilde{H}}{\partial \tilde{x}} \quad (23)$$

$$\tilde{w}(\tilde{B}) = \frac{D\tilde{B}}{D\tilde{t}} = \frac{\partial \tilde{B}}{\partial \tilde{t}} + \tilde{u}(\tilde{B}) \frac{\partial \tilde{B}}{\partial \tilde{x}} \quad (24)$$

Substituting Equations 23 and 24 into Equation 22, we obtain

$$\frac{\partial}{\partial \tilde{t}} \int_{\tilde{B}}^{\tilde{H}} \tilde{u} d\tilde{z} + \frac{\partial}{\partial \tilde{x}} \int_{\tilde{B}}^{\tilde{H}} \tilde{u} \tilde{u} d\tilde{z} = -g \tilde{D} \frac{\partial \tilde{H}}{\partial \tilde{x}} - \frac{\tilde{T}_{xz}(\tilde{B})}{\rho_f} + g \tilde{D} \sin \alpha \quad (25)$$

We introduce the depth-averaged velocity $\tilde{U}$ written in the form

$$\tilde{U} = \frac{1}{\tilde{D}} \int_{\tilde{B}}^{\tilde{H}} \tilde{u} d\tilde{z} \quad (26)$$

We also introduce the depth averaged-momentum $\tilde{U}^2$ written in the form

$$\phi \tilde{U}^2 = \frac{1}{\tilde{D}} \int_{\tilde{B}}^{\tilde{H}} \tilde{u}^2 d\tilde{z} \quad (27)$$

where ϕ is a coefficient defined by the above equation, and estimated to be close to unity. With the use of Equations 26 and 27, Equation 25 is reduced

$$\frac{\partial \tilde{U} \tilde{D}}{\partial \tilde{t}} + \frac{\partial \tilde{U}^2 \tilde{D}}{\partial \tilde{x}} = -g \tilde{D} \frac{\partial \tilde{H}}{\partial \tilde{x}} - \frac{\tilde{T}_{xz}(\tilde{B})}{\rho_f} + g \tilde{D} \sin \alpha \quad (28)$$

where ϕ has been assumed to be unity.

The continuity equation Equation 3 is integrated from the bottom to the water surface. The result is

$$\frac{\partial}{\partial \tilde{x}} \int_{\tilde{B}}^{\tilde{H}} \tilde{u} d\tilde{z} - \frac{\partial \tilde{H}}{\partial \tilde{x}} u(\tilde{H}) + \tilde{w}(\tilde{H}) - \tilde{w}(\tilde{B}) = 0 \quad (29)$$

Again, applying Equations 23 and 24 to the above equation, we obtain

$$\frac{\partial \tilde{D}}{\partial \tilde{t}} + \frac{\partial \tilde{U} \tilde{D}}{\partial \tilde{x}} = 0 \quad (30)$$

Equation 28 is rewritten in the form

$$\frac{\partial \tilde{U}}{\partial \tilde{t}} \tilde{D} + \tilde{U} \frac{\partial \tilde{D}}{\partial \tilde{t}} + \tilde{U} \tilde{D} \frac{\partial \tilde{U}}{\partial \tilde{x}} + \tilde{U} \frac{\partial \tilde{U} \tilde{D}}{\partial \tilde{x}} = -g \tilde{D} \frac{\partial \tilde{H}}{\partial \tilde{x}} - \frac{\tilde{T}_{xz}(\tilde{B})}{\rho_f} + g \tilde{D} \sin \alpha \quad (31)$$

The first term in the continuity equation is the time rate of change of fluid mass, and the second is the streamwise change in flow discharge.

Applying Equation 30 to the above equation, we obtain

$$\frac{\partial \tilde{U}}{\partial \tilde{t}} + \tilde{U} \frac{\partial \tilde{U}}{\partial \tilde{x}} = -g \frac{\partial \tilde{H}}{\partial \tilde{x}} - \frac{\tilde{\tau}_b}{\rho_f \tilde{D}} + g \sin \alpha \quad (32)$$

where $\tilde{\tau}_b$ is the bed shear stress corresponding to $\tilde{T}_{xz}(\tilde{B})$. The above equation can be rewritten into

$$\frac{\partial \tilde{U}}{\partial \tilde{t}} + \tilde{U} \frac{\partial \tilde{U}}{\partial \tilde{x}} = -g \frac{\partial \tilde{D}}{\partial \tilde{x}} - g \frac{\partial \tilde{\eta}}{\partial \tilde{x}} - \frac{\tilde{\tau}_b}{\rho_f \tilde{D}} + g S \quad (33)$$

where $\tilde{\eta}$ denotes the ice elevation and $\sin \alpha$ can be seen as slope S . On the left-hand side of the above equation, the first term is the time rate of change of depth-integrated momentum, and the second is the net convective inflow of momentum. On the right side, the first two terms are pressure. The third term is the shear force, and the last is the gravitational force.

2.2.2 Depth integrated Heat transfer equations of flowing water

The variation of heat flux in the vertical direction is significantly greater than in the streamwise direction. Consequently, the integration method utilized in the shallow water equation is employed to derive the heat transfer equation for flowing water. The heat transfer equation for flowing water is expressed in the following form

$$\rho_f c_f \left(\frac{\partial \tilde{T}}{\partial \tilde{t}} + \tilde{u} \frac{\partial \tilde{T}}{\partial \tilde{x}} + \tilde{w} \frac{\partial \tilde{T}}{\partial \tilde{z}} \right) + \frac{\partial \tilde{F}_{\tilde{x}}}{\partial \tilde{x}} + \frac{\partial \tilde{F}_{\tilde{z}}}{\partial \tilde{z}} = 0 \quad (34)$$

where c_f is the specific heat capacity of water ($=4184 \text{ J/kgK}$), $\tilde{T}$ is the water temperature, $\tilde{F}_{\tilde{x}}$ and $\tilde{F}_{\tilde{z}}$ represent the heat flux in the $\tilde{x}$ and $\tilde{z}$ directions, respectively, within the flowing water. They are expressed as follows

$$\tilde{F}_{\tilde{x}} = -k_f \frac{\partial \tilde{T}}{\partial \tilde{x}} \quad (35)$$

$$\tilde{F}_{\tilde{z}} = -k_f \frac{\partial \tilde{T}}{\partial \tilde{z}} \quad (36)$$

where k_f denotes the thermal conductivity of the flowing water. Equation 34 can be rewritten as below

$$\rho_f c_f \left(\frac{\partial \tilde{T}}{\partial \tilde{t}} + \frac{\partial \tilde{u} \tilde{T}}{\partial \tilde{x}} + \frac{\partial \tilde{w} \tilde{T}}{\partial \tilde{z}} \right) = \frac{\partial}{\partial \tilde{z}} \left(k_f \frac{\partial \tilde{T}}{\partial \tilde{z}} \right) \quad (37)$$

Since $\partial \tilde{F}_{\tilde{x}} / \partial \tilde{x}$ is sufficiently small compared to $\partial \tilde{F}_{\tilde{z}} / \partial \tilde{z}$, it is ignored hereafter. The three terms on the left-hand side of the above equation are integrated from the bed ($\tilde{z} = \tilde{B}$) to the water surface ($\tilde{z} = \tilde{H}$) as follows, respectively

$$\int_{\tilde{B}}^{\tilde{H}} \frac{\partial \tilde{T}}{\partial \tilde{t}} d\tilde{z} = \frac{\partial}{\partial \tilde{t}} \int_{\tilde{B}}^{\tilde{H}} \tilde{T} d\tilde{z} - \frac{\partial \tilde{H}}{\partial \tilde{t}} \tilde{T}(\tilde{H}) + \frac{\partial \tilde{B}}{\partial \tilde{t}} \tilde{T}(\tilde{B}) \quad (38)$$

$$\int_{\tilde{B}}^{\tilde{H}} \frac{\partial \tilde{u} \tilde{T}}{\partial \tilde{x}} d\tilde{z} = \frac{\partial}{\partial \tilde{x}} \int_{\tilde{B}}^{\tilde{H}} \tilde{u} \tilde{T} d\tilde{z} - \frac{\partial \tilde{H}}{\partial \tilde{x}} \tilde{u}(\tilde{H}) \tilde{T}(\tilde{H}) + \frac{\partial \tilde{B}}{\partial \tilde{x}} \tilde{u}(\tilde{B}) \tilde{T}(\tilde{B}) \quad (39)$$

$$\int_{\tilde{B}}^{\tilde{H}} \frac{\partial \tilde{w} \tilde{T}}{\partial \tilde{z}} d\tilde{z} = \tilde{w}(\tilde{H}) \tilde{T}(\tilde{H}) - \tilde{w}(\tilde{B}) \tilde{T}(\tilde{B}) \quad (40)$$

The right-hand side of Equation 37 is integrated to be

$$\int_{\tilde{B}}^{\tilde{H}} \frac{\partial}{\partial \tilde{z}} \left(k_f \frac{\partial \tilde{T}}{\partial \tilde{z}} \right) d\tilde{z} = \tilde{F}_{\tilde{z}}(\tilde{B}) - \tilde{F}_{\tilde{z}}(\tilde{H}) \quad (41)$$

Substituting Equations 38–41, 23 and 24 into Equation 37, we obtain

$$\rho_f c_f \left(\frac{\partial \tilde{D} \tilde{\Theta}}{\partial \tilde{t}} + \phi_f \frac{\partial \tilde{U} \tilde{D} \tilde{\Theta}}{\partial \tilde{x}} \right) = \tilde{F}_{\tilde{z}}(\tilde{B}) - \tilde{F}_{\tilde{z}}(\tilde{H}) \quad (42)$$

where $\tilde{\Theta}$ is the depth-averaged temperature defined by

$$\tilde{\Theta} = \frac{1}{\tilde{D}} \int_{\tilde{B}}^{\tilde{H}} \tilde{T} d\tilde{z} \quad (43)$$

The coefficient ϕ_f is defined by

$$\phi_f \tilde{U} \tilde{\Theta} = \frac{1}{\tilde{D}} \int_{\tilde{B}}^{\tilde{H}} \tilde{u} \tilde{T} d\tilde{z} \quad (44)$$

and is expected to be close to unity. We assume that ϕ_f is unity. With the use of Equations 30, 43 and 44, Equation 42 is rewritten in the form

$$\rho_f c_f \left(\frac{\partial \tilde{\Theta}}{\partial \tilde{t}} + \tilde{U} \frac{\partial \tilde{\Theta}}{\partial \tilde{x}} \right) = \frac{\tilde{F}_{\tilde{z}}(\tilde{B}) - \tilde{F}_{\tilde{z}}(\tilde{H})}{\tilde{D}} \quad (45)$$

The left side of the above equation is the streamwise change and time rate change of energy due to convection.

2.2.3 Stefan Condition

The difference in heat flux plays a vital role in forming cyclic steps. The heat flux going out from the water-ice interface to the ice below and coming into the interface from the water above causes melting and solidifying. The released heat balance with the total heat absorbed. The Stefan condition describes the progression of the boundary between two phases of a material undergoing a phase transition. This equation is an energy balance that defines the position of the moving interface. The temperatures of the liquid and solid phases near the interface must equal the temperature of the interface, which is the melting point. The Stefan condition is in the following form

$$\rho_s h_s \frac{\partial \tilde{\eta}}{\partial \tilde{t}} = \tilde{F}_{\tilde{z}}(\tilde{B}) - \tilde{F}_{\tilde{x}}(\tilde{B}) \quad (46)$$

where h_s is the latent heat of melting, ρ_s is the density of ice, $\tilde{F}_{\tilde{x}}$ is the heat flux from the ice to the interface in the $\tilde{z}$ direction.

2.3 Calculation of heat flux

Equations 45 and 46 involve three terms related to heat flux, $\tilde{F}_{fz}(\tilde{B})$, $\tilde{F}_{sz}(\tilde{B})$, and $\tilde{F}_{fz}(\tilde{H})$, which are defined empirically.

A formula estimates the heat exchange flux between ice and water is shown below

$$\tilde{F}_{fz}(\tilde{B}) = -\tilde{h}_f(\tilde{\Theta} - \tilde{T}_m) \quad (47)$$

where $\tilde{T}_m$ is the temperature at the ice-water interface and kept to the melting point, $\tilde{h}_f$ is a bulk heat transfer coefficient of water above the ice surface ($= \rho_f c_f C_{fh} \tilde{U}$), c_f is the specific heat capacity of water ($= 4184 \text{ J/kgK}$), C_{fh} is a dimensionless ice-water transfer coefficient. According to Shirasawa et al. [35], the value of C_{fh} used in Resolute Passage, Northwest Territories, Canada was in the range of $3 \times 10^{-5} - 2 \times 10^{-2}$. In the study by Hamblin et al. [36], the value of C_{fh} was found to be $(0.8 \pm 0.3) \times 10^{-3}$ for the exchange of heat between a lake and an ice sheet. Li et al. [37] proposed that C_{fh} at ice-water interface in the low flow velocities (0.024 m/s-0.110 m/s) is 1.1×10^{-3} .

Heat conduction is a primary mode of heat transfer within a solid object, often described using Fourier's Law. In this context, we apply Fourier's Law to describe heat transfer within the ice, as shown below

$$\tilde{F}_x(\tilde{B}) = -k_s \frac{\tilde{T}_m - \tilde{T}_i}{\tilde{Z}} \quad (48)$$

where $\tilde{T}_i$ is the temperature at the ice bottom, k_s is the thermal conductivity of ice ($= 2.22 \text{ W/(mK)}$) and $\tilde{Z}$ is the ice thickness.

Due to a boundary layer above the water surface, a significant temperature difference exists between the air and the water surface. The heat flux from the surrounding air to the water depends on air velocity and temperature distributions. However, obtaining this information can be difficult, so we adopt a simplification proposed by Yokokawa et al. [4]. We assume that the heat flux from the surrounding air to the water surface is equivalent to that from the water surface to the flowing water. As most studies focus on heat transfer from a solid surface to a fluid rather than between gas and liquid, we use a formula commonly employed in oceanography or geophysics to calculate the heat flux, as shown below

$$\tilde{F}_{fz}(\tilde{H}) = -\tilde{h}_a(\tilde{T}_a - \tilde{\Theta}) \quad (49)$$

where $\tilde{T}_a$ is the air temperature, $\tilde{h}_a$ is a bulk heat transfer coefficient of air ($= \rho_a c_a C_{ah}$), ρ_a is the air density ($= 1.29 \text{ kg/m}^3$), c_a is the specific heat of air ($= 1005 \text{ J/kgK}$), C_{ah} is a normalized heat transfer coefficient which is influenced by the wind speed. Assuming the wind speed is from 1 to 10 m/s, it has been reported

that C_{ah} takes a range of values from 1.5×10^{-3} to 1.4×10^{-2} [38]. As a result, the value of $\tilde{h}_a$ is from 1.94 to 18.15.

Apply Equations 47–49 into Equations 45 and 46, we can obtain

$$\rho_f c_f \tilde{D} \left(\frac{\partial \tilde{\Theta}}{\partial \tilde{t}} + \tilde{U} \frac{\partial \tilde{\Theta}}{\partial \tilde{x}} \right) = -\tilde{h}_f (\tilde{\Theta} - \tilde{T}_m) + \tilde{h}_a (\tilde{T}_a - \tilde{\Theta}) \quad (50)$$

$$\rho_s h_s \frac{\partial \tilde{\eta}}{\partial \tilde{t}} = -\tilde{h}_f (\tilde{\Theta} - \tilde{T}_m) + k_s \frac{\tilde{T}_m - \tilde{T}_i}{\tilde{Z}} \quad (51)$$

2.4 Normalization

We use the following normalization

$$\tilde{U} = \tilde{U}_0 U \quad (52)$$

$$(\tilde{D}, \tilde{H}, \tilde{Z}, \tilde{\eta}) = \tilde{D}_0 (D, H, Z, \eta) \quad (53)$$

$$\tilde{x} = \frac{\tilde{D}_0}{C_D} x \quad (54)$$

$$\tilde{T} = T(\tilde{T}_a - \tilde{T}_m) + \tilde{T}_m \quad (55)$$

$$\tilde{\Theta} = \Theta(\tilde{T}_a - \tilde{T}_m) + \tilde{T}_m \quad (56)$$

$$\tilde{t} = \left[\frac{\rho_s h_s \tilde{D}_0}{\rho_f c_f C_{fh} \tilde{U}_0 (\tilde{T}_a - \tilde{T}_m)} \right] t \quad (57)$$

$$\tilde{\tau}_b = \rho_f C_D \tilde{U}^2 \quad (58)$$

where the subscript 0 denotes variables in the normal flow condition (steady, uniform) before the development of steps, and C_D is the drag coefficient. The shallow water approximation allows the above relationship between the bed shear stress and the flow velocity (Equation 58). The bed shear stress is related to the drag coefficient, and the drag coefficient is generally a weak function of relative flow depth (flow depth/roughness height). However, in this study, we assume the drag coefficient is a constant for simplicity.

We use the quasi-steady approximation in the governing equations by assuming a constant water discharge and a significantly slower evolution of the ice surface compared to the flow change. As a result, the flow can be regarded as steady and can exclude the temporal terms in the governing equations, except for the Stefan equation. By incorporating Equations 52–58, we can rewrite Equations 30, 33, 50 and 51, as follows

$$C_D \tilde{U}_0 U \frac{\partial \tilde{U}_0 U}{\partial \tilde{D}_0 x} = -g C_D \frac{\partial \tilde{D}_0 D}{\partial \tilde{D}_0 x} - g C_D \frac{\partial \tilde{D}_0 \eta}{\partial \tilde{D}_0 x} - \frac{\rho_f C_D \tilde{U}_0^2 U^2}{\rho_f \tilde{D}_0 D} + g S \quad (59)$$

$$C_D \frac{\partial \tilde{U}_0 U \tilde{D}_0 D}{\partial \tilde{D}_0 x} = 0 \quad (60)$$

$$\tilde{D}_0 D \rho_f c_f C_D \tilde{U}_0 U \frac{\partial (\Theta(\tilde{T}_a - \tilde{T}_m) + \tilde{T}_m)}{\partial \tilde{D}_0 x} = -\tilde{h}_f \Theta(\tilde{T}_a - \tilde{T}_m) + \tilde{h}_a (\tilde{T}_a - \Theta T_a (\tilde{T}_a - \tilde{T}_m)) \quad (61)$$

$$\rho_s h_s \frac{\partial \tilde{D}_0 \eta}{\partial \left[\frac{\rho_s h_s \tilde{D}_0}{\rho_f c_f C_{fh} \tilde{U}_0 (\tilde{T}_a - \tilde{T}_m)} \right] t} = -\tilde{h}_f (\Theta(\tilde{T}_a - \tilde{T}_m) + \tilde{T}_m) + k_s \frac{-T_i (\tilde{T}_a - \tilde{T}_m) + \tilde{T}_m}{\tilde{D}_0 Z} \quad (62)$$

The above equations can be reduced to the following form

$$U \frac{\partial U}{\partial x} = -F_0^{-2} \left(\frac{\partial D}{\partial x} + \frac{\partial \eta}{\partial x} \right) + 1 - \frac{U^2}{D} \quad (63)$$

$$\frac{\partial U D}{\partial x} = 0 \quad (64)$$

$$U D \frac{\partial \Theta}{\partial x} = -\lambda U \Theta + \xi (1 - \Theta) \quad (65)$$

$$\frac{\partial \eta}{\partial t} = -U \Theta - \gamma \frac{T_i}{Z} \quad (66)$$

where F_0 denotes the Froude number associated with the normal flow before step formation, λ represents the normalized coefficient of ice-water heat transfer, ξ is a non-dimensional parameter that indicates the ratio between the air-water heat transfer coefficient and the water-ice heat transfer coefficient, and γ denotes a non-dimensional parameter defined as the ratio between the thermal conductivity coefficient and the heat transfer coefficient in water. These parameters have the following expressions

$$F_0 = \frac{\tilde{U}_0}{\sqrt{g \tilde{D}_0}} \quad (67)$$

$$\lambda = \frac{C_{fh}}{C_D} \quad (68)$$

$$\xi = \frac{\tilde{h}_a}{\rho_f c_f \tilde{U}_0 C_D} \quad (69)$$

$$\gamma = \frac{k_s}{\rho_f c_f C_{fh} \tilde{U}_0 \tilde{D}_0} \quad (70)$$

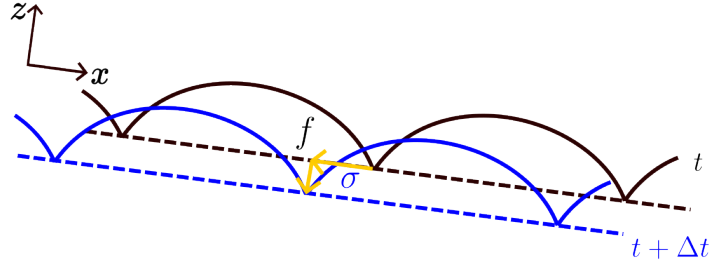


Figure 6: Cyclic steps migrate upstream and downward.

Yokokawa et al. [4] reported the observation of cyclic steps on ice that migrate upstream without changing shape. This phenomenon of the steps that migrate upstream is mainly due to the water temperature being higher than the ice but lower than the air temperature. We have the same temperature setting, and thus cyclic steps are assumed to migrate upstream in our study. In addition, in the experiments conducted by Yokokawa et al. [4], the entire ice surface moved downward since the heat flux from the water was not necessarily balanced with that of the ice. To describe these features of upstream migrating cyclic steps as shown in Figure 6, we referred to the study of cyclic steps over an erodible bed by Sun and Parker [20] and introduced a moving coordinate system as shown below

$$\hat{x} = x + ft \quad (71)$$

$$\hat{\eta} = \eta + \sigma \hat{t} \quad (72)$$

$$\hat{t} = t \quad (73)$$

where f is the normalized migration speed in the streamwise direction, and σ is the normalized degradation speed of the ice surface in the depth direction. Applying this transformation to the governing equations (Equations 63–66), we obtain

$$U \frac{\partial U}{\partial \hat{x}} = -F_0^{-2} \left(\frac{\partial D}{\partial \hat{x}} + \frac{\partial \eta}{\partial \hat{x}} \right) + 1 - \frac{U^2}{D} \quad (74)$$

$$\frac{\partial UD}{\partial \hat{x}} = 0 \quad (75)$$

$$UD \frac{\partial \Theta}{\partial \hat{x}} = -\lambda U \Theta + \xi(1 - \Theta) \quad (76)$$

$$\frac{\partial \eta}{\partial \hat{t}} + f \frac{d\eta}{d\hat{x}} = (-U\Theta + \zeta) \quad (77)$$

where ζ is used to present the sum of the heat flux from the ice surface to the ice bottom and the migration speed of the ice surface in the depth direction for simplicity. Its form is shown below

$$\zeta = -\gamma \frac{T_i}{Z} - \sigma \quad (78)$$

Our analysis is focused on a self-preserving quasi-steady step shape that moves upstream while maintaining its shape. Therefore, we drop the time derivative in Equation 77 to maintain this solution. For simplicity, we drop the hat in Equations 74–77 and then obtain a new version of the governing equations by replacing partial derivatives with total derivatives, as shown below

$$U \frac{dU}{dx} = -F_0^{-2} \left(\frac{dD}{dx} + \frac{d\eta}{dx} \right) + 1 - \frac{U^2}{D} \quad (79)$$

$$\frac{dUD}{dx} = 0 \quad (80)$$

$$UD \frac{d\Theta}{dx} = -\lambda U \Theta + \xi(1 - \Theta) \quad (81)$$

$$f \frac{d\eta}{dx} = -U\Theta + \zeta \quad (82)$$

In the normal flow condition, Equation 33 can be reduced and the gravitational force is in balance with the shear stress, as shown here

$$\frac{\tilde{\tau}_b}{\rho_f \tilde{D}_0} = gS \quad (83)$$

so the bed shear stress can be rewritten to

$$\tilde{\tau}_b = \rho_f g S \tilde{D}_0 \quad (84)$$

By using Equation 58 and 84, we can derive

$$F_0 = \sqrt{S/C_D} \quad (85)$$

Equation 79 can be reduced with Equations 80 and 82 to obtain the following form

$$\frac{dU}{dx} = \frac{-F_0^{-2} f^{-1} (-U\Theta + \xi) + 1 - U^3}{U - F_0^{-2} U^{-2}} \quad (86)$$

With the aid of Equation 80, the heat transfer equation in water takes the form

$$\frac{d\Theta}{dx} = -\lambda U\Theta + \xi(1 - \Theta) \quad (87)$$

2.5 Boundary conditions

The flow accelerates from subcritical to supercritical and then transitions through a hydraulic jump, as depicted in Figure 5. Momentum conservation is maintained between the upstream and downstream sides of the hydraulic jump, as shown below

$$\rho_f \tilde{U}_-^2 \tilde{D}_- + \frac{1}{2} \rho_f g \tilde{D}_-^2 = \rho_f \tilde{U}_+^2 \tilde{D}_+ + \frac{1}{2} \rho_f g \tilde{D}_+^2 \quad (88)$$

where subscripts - and + denote the upstream and downstream sides of the hydraulic jump, respectively. The normalized boundary condition is written to

$$U_-^2 D_- + \frac{1}{2} D_-^2 F_0^{-2} = U_+^2 D_+ + \frac{1}{2} D_+^2 F_0^{-2} \quad (89)$$

With the aid of continuity equation (Equation 74), the above equation can be reduced to

$$U_- = U_+ \frac{1 + \sqrt{1 + 8F_0^2 U_+^3}}{4F_0^2 U_+^3} \quad (90)$$

The origin of the coordinate system is located at the downstream side of the hydraulic jump, as illustrated in Figure 5. The ice elevation is assumed to be

zero both before and after the hydraulic jump. The boundary condition of the ice elevation is

$$\eta_- = \eta_+ \quad (91)$$

As shown below, we assume that the water temperature remains continuous before and after the hydraulic jump

$$\Theta_+ = \Theta_- \quad (92)$$

3 Solution

3.1 Solution Method

The method used to solve this problem is based on the approach of Sun and Parker [20] to some extent. The Froude number is essential in forming cyclic steps since the critical Froude number distinguishes between supercritical and subcritical flows. At the Froude critical point, the Froude number is

$$F_r = \frac{\tilde{U}_c}{\sqrt{g\tilde{D}_c}} = 1 \quad (93)$$

where the subscript c denotes variables at the Froude critical point (i.e. $F_r = 1$). With the aid of normalization Equations 52 and 53, the above equation can be rewritten to

$$\frac{\tilde{U}_0^2 U_c^2}{g\tilde{D}_0^2 D_c^2} = 1 \quad (94)$$

With the aid of Equation 67, the above equation is in the following form

$$F_0 \frac{U_c^2}{D_c^2} = 1 \quad (95)$$

Thus, the velocity at the Froude critical point, U_c can be calculated as

$$U_c = F_0^{-\frac{2}{3}} \quad (96)$$

At the Froude critical point, the denominator of Equation 86 becomes zero, leading to singularity. To avoid this singularity, the numerator must also become zero simultaneously as shown here

$$-F_0^{-2} f^{-1} (-U_c \Theta_c + \zeta) + 1 - U_c^3 = 0 \quad (97)$$

With these conditions met, we can determine the migration speed, f , as shown following

$$f = \frac{-F_0^{-2/3} \Theta_c + \zeta}{F_0^2 - 1} \quad (98)$$

The governing equations 82, 86, 87 with the boundary conditions 90-92 form a two-point boundary value problem. The three boundary conditions are all the relationships between the variables before and after hydraulic jumps rather than specific values of variables. Therefore, these weak boundary conditions cannot

determine one step's wavelength. The wavelength is defined as the distance between two adjacent hydraulic jumps. One more boundary condition is necessary to close the problem.

In the study of cyclic steps created by flow over a cohesive bed [2], the threshold velocity, which denotes the onset of bed erosion, identifies the wavelength. Several factors, such as water content, ionic content of pure water, etc., influence the threshold velocity. However, no threshold velocity is associated with the formation of cyclic steps on the ice. To overcome the deficiency that the wavelength of cyclic steps on ice is difficult to determine, we assume the value of Θ_+ as one more boundary condition.

We solve this two-point boundary value problem by the shooting method with the Newton-Raphson scheme. The calculation starts from the vicinity of the Froude critical point to avoid singularity. The temperature gradient $d\Theta/dx$ and the ice elevation gradient $d\eta/dx$ at the Froude critical point x_c are expressed in following forms with the aids of Equation 87 and Equation 82, respectively

$$\left. \frac{d\Theta}{dx} \right|_{x=x_c} = -\lambda F_0^{-\frac{2}{3}} \Theta_c + \xi(1 - \Theta_c) \quad (99)$$

$$\left. \frac{d\eta}{dx} \right|_{x=x_c} = \frac{1}{f} \left(-F_0^{-\frac{2}{3}} \Theta_c + \zeta \right) \quad (100)$$

We cannot calculate the velocity gradient, dU/dx , at the Froude critical point by taking the limit $U \rightarrow F_0^{-2/3}$. To ensure a smooth transition across the condition $Fr = 1$, we apply L'Hopital's rule to Equation 86. The gradient dU/dx evaluated at the Froude critical point must satisfy the following relation

$$3U_c'^2 + 3U_c'F_0^{-\frac{4}{3}} + F_0^{-2}f^{-1}[-U_c'\Theta_c + F_0^{-\frac{4}{3}}\lambda\Theta_c - F_0^{-\frac{2}{3}}\xi(1 - \Theta_c)] = 0 \quad (101)$$

where U_c' denotes dU/dx at the Froude critical point. Since the flow transitions from subcritical to supercritical regime at the Froude critical point, the solution for dU/dx at the Froude critical point should be real and positive to ensure a physically reasonable solution.

We calculate dU/dx , $d\eta/dx$ and $d\Theta/dx$ from the vicinity of the Froude critical point and guess the integration's initial values of Θ_c and ζ . Then U_c is determined from Equation 96, and η_c is assumed as zero. We calculate down in the downstream direction and up in the upstream direction until Θ_+ equals the value we assume. The hydraulic jump occurs at these points, so Equation 90 must be satisfied. We adjust the guessed value of Θ_c and ζ using the Newton-Raphson scheme with the Jacobian matrix. Once Equation 90 and 91 are satisfied, we obtain the correct Θ_c and ζ .

3.2 Calculations of Parameters

The drag coefficient C_D can be obtained using Equation 85, which relates to the Froude number. The drag coefficient is expressed as

$$C_D = s/F_0^2 \quad (102)$$

We provide an alternative method for calculating C_D , as shown below. In the normal flow condition, the momentum equation for the laminar flow can be rewritten into

$$\frac{d}{d\tilde{z}}\left(\nu \frac{d\tilde{u}_0}{d\tilde{z}}\right) + gs = 0 \quad (103)$$

where $\tilde{u}_0$ is the water velocity in the normal flow condition, and ν is the kinematic viscosity ($1.7 \times 10^{-6} m^2/s$). We integrate the above equation in $\tilde{z}$ direction to obtain

$$\nu \frac{d\tilde{u}_0}{d\tilde{z}} = -gs\tilde{z} + K \quad (104)$$

where K is a constant. When $\tilde{z} = \tilde{D}_0$, the shear stress does not exist so that we can get

$$K = gs\tilde{D}_0 \quad (105)$$

With the aid of Equation 105, Equation 104 can be integrated to

$$\tilde{u}_0 = \frac{gs}{\nu} \left(\tilde{D}_0\tilde{z} - \frac{\tilde{z}^2}{2} \right) + K' \quad (106)$$

where K' is a constant. We find $K' = 0$ when $\tilde{D}_0 = 0$ and $\tilde{z} = 0$. The average water velocity in the normal flow condition can be expressed as

$$\tilde{U}_0 = \frac{1}{\tilde{D}_0} \int_0^{\tilde{D}_0} \tilde{u}_0 d\tilde{z} = \frac{1}{\tilde{D}_0} \int_0^{\tilde{D}_0} \frac{gs}{\nu} \left(\tilde{D}_0\tilde{z} - \frac{\tilde{z}^2}{2} \right) d\tilde{z} = \frac{gs\tilde{D}_0^2}{3\nu} \quad (107)$$

The friction velocity $\tilde{u}_*$ can be expressed as

$$\tilde{u}_* = \sqrt{g\tilde{D}_0 s} = \tilde{U}_0 \sqrt{C_D} \quad (108)$$

The drag coefficient can be rewritten with the help of Equation 107, as shown below

$$C_D = \frac{\tau_b}{\rho \tilde{U}_0^2} = \frac{\rho g \tilde{D}_0 s}{\rho \tilde{U}_0^2} = \frac{g \tilde{D}_0 s}{\left(\frac{gs\tilde{D}_0^2}{3\nu}\right)^2} = \frac{9\nu^2}{gs\tilde{D}_0^3} \quad (109)$$

With the aid of Equation 108, the above equation is in the following form

$$C_D = \frac{9\nu^2}{\tilde{u}_*^2 \tilde{D}_0^2} = \frac{9}{\tilde{u}_*^2 \tilde{D}_0^2 C_D} \quad (110)$$

Equation 110 can be reduced with the help of Reynolds number as shown in the following form

$$C_D = \frac{3}{\text{Re}} \quad (111)$$

where Reynolds number Re is expressed as

$$\text{Re} = \tilde{U}_0 \tilde{D}_0 / \nu \quad (112)$$

If the flow is laminar, C_D can be determined using the Reynolds number Re with Equation 111. Our analytical model is compared to the experimental results of Yokokawa et al. [4] later, and the Reynolds numbers measured in the experiments range from 228 to 2190, indicating that flows cannot be all regarded as laminar. Therefore, we use Equation 102 to calculate C_D .

According to Yokokawa et al. [4], the ice-water interface becomes unstable only when the flow is in the supercritical regime. Therefore, F_0 should be larger than unity. We assume the Froude number in the normal flow condition ranges from 1.01 to 10. The channel slope is assumed to be between 0.05 and 0.5. As a result, the drag coefficient ranges from 5×10^{-4} to 0.49.

The parameter λ is a normalized coefficient representing the ability to transfer heat flux from water to ice. It is calculated as the ratio of the water-ice heat transfer coefficient (C_{fh}) to the drag coefficient (C_D). As previously mentioned, the value of C_{fh} ranges from 3×10^{-5} to 2×10^{-2} , resulting in a range of λ from 6.12×10^{-5} to 40. The parameter ξ represents the ratio of the air-water heat transfer coefficient to the water-ice heat transfer coefficient and is given by $\tilde{h}_a / (\rho_f c_f \tilde{U}_0 C_D)$. The value of $\tilde{h}_a$ was previously discussed to be from 1.94 to 18.15, and $\tilde{U}_0$ is assumed to be between 0.1 and 5 m/s. The parameter ξ thus ranges from 1.89×10^{-7} to 8.68×10^{-2} . After the calculation, the variation of the water velocity, the water temperature, and the ice elevation can be obtained.

4 Results and Discussion

4.1 Determine the wavelength

During the analysis, we find that the temperature at the downstream of the hydraulic jump Θ_+ cannot be arbitrarily selected. We identify specific ranges for Θ_+ , as shown in Figure 7, which must be met for a solution existing.

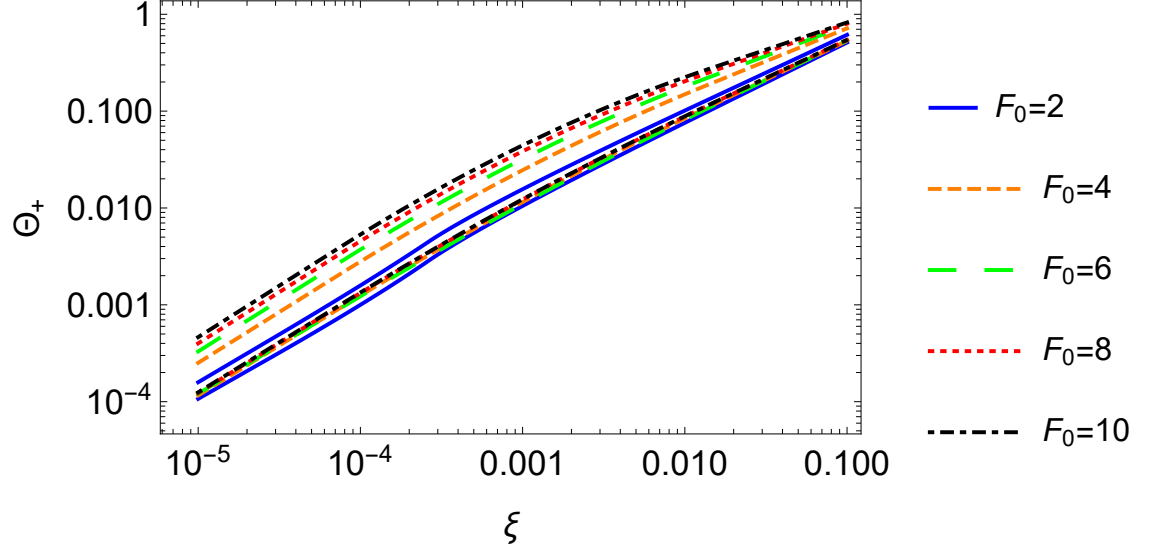
An explanation exists regarding the maximum value of Θ_+ . Our calculations reveal that a negative value for the migration speed parameter f results in no solution. Furthermore, our solutions indicate that the temperature variation over one step always exhibits a distinct pattern, wherein Θ_c is greater than Θ_+ and the minimum water temperature occurs at the point $x = x_+$. By using Equation 98 and establishing the relationship between Θ_c and Θ_+ , we obtain

$$\Theta_+ < \Theta_c < \zeta F_0^{2/3} \quad (113)$$

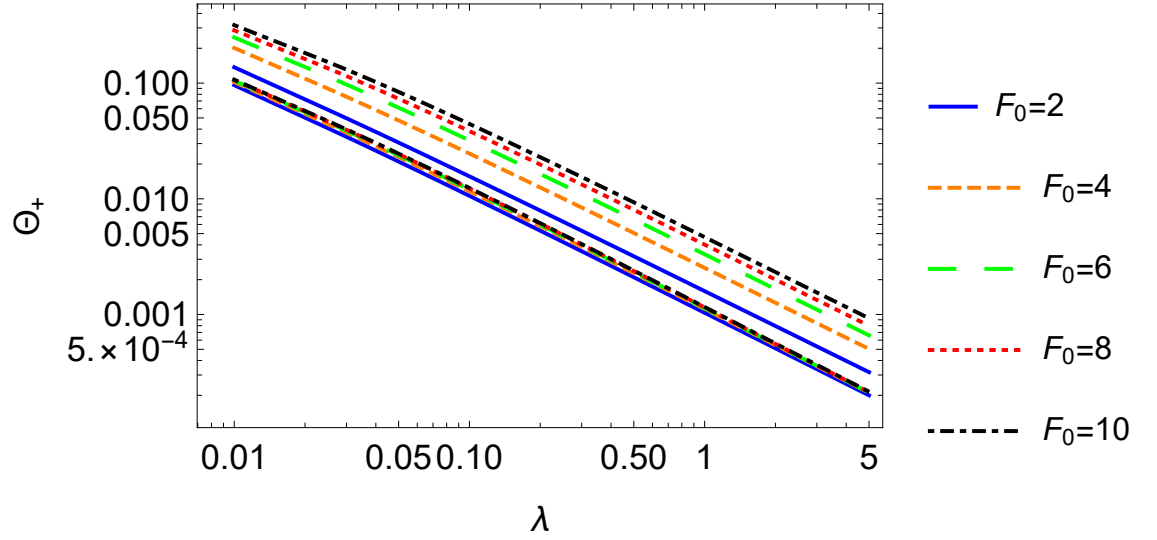
Thus, exceeding the upper limit for Θ_+ leads to a negative migration speed f value and no solution for the analytical model.

Moreover, there exists a lower limit for Θ_+ . The average water velocity continues to increase until reaching the upstream of the hydraulic jump. As a result, the numerator of Equation 86 should not be zero except at the Froude critical point. However, if Θ_+ falls below its lower limit, the numerator of Equation 86 becomes zero at a location other than the Froude critical point.

The value of Θ_+ plays a critical role in determining the wavelength. Figure 8 suggests that the wavelength increases with increasing Θ_+ and approaches the maximum value as the lower bound in Θ_+ is approached. It is said that, with specified ξ , λ , and F_0 , the longest wavelength occurs when Θ_+ is at its lowest. Parker and Izumi [2], in their study of purely erosional cyclic steps, assumed that the stability of these steps is achieved when the velocity at the downstream end of hydraulic jumps equals the minimum possible value. They considered this minimum to be the threshold velocity at the onset of bed erosion. Sun and Parker [20] performed an analysis employing the same assumption that the velocity at the downstream end of hydraulic jumps is equal to the minimum possible value and found it to be in reasonable agreement with their experiments [1]. In their cases, this minimum corresponds to the threshold velocity at the onset of the entrainment of sand on the bed into flowing water. The longest wavelength appears when the velocity at the downstream end of hydraulic jumps takes the minimum possible value. Their assumption that the smallest possible velocity appears at the downstream end of hydraulic jumps is synonymous with the assumption that the longest possible wavelength appears. In the case of cyclic steps on the ice, there is no minimum velocity at the downstream end of hydraulic jumps. We, therefore,



(a)



(b)

Figure 7: Possible range of Θ_+ . The upper and lower lines with the same color denote the largest and smallest possible value for Θ_+ , respectively. (a) in which $\lambda = 0.1$. (b) in which $\xi=0.001$.

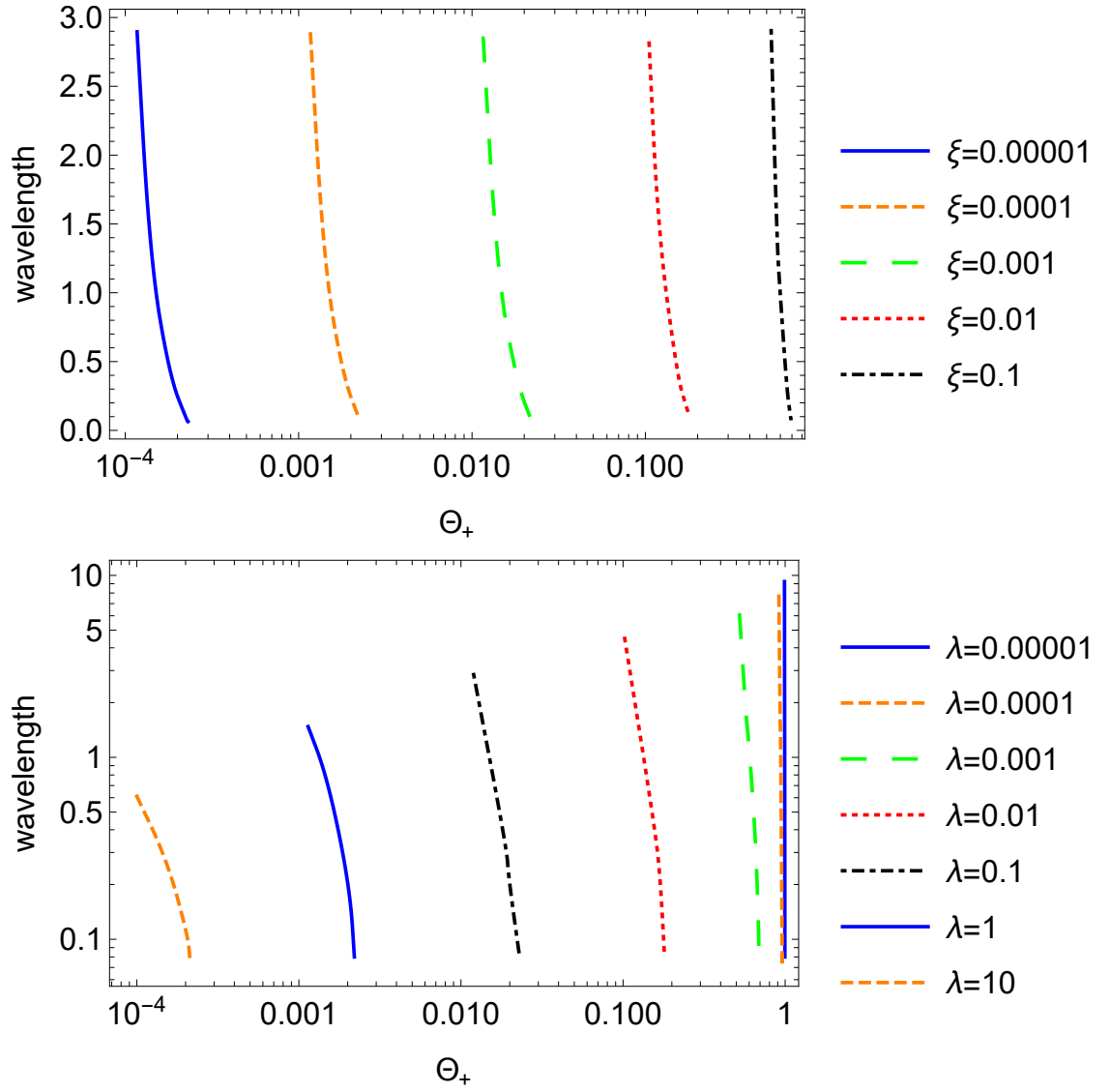


Figure 8: The relationships between wavelength and Θ_+ . (a) $F_0 = 4$, $\lambda = 0.1$. (b) $F_0 = 4$, $\xi = 0.001$

assume that the longest wavelength, which is probably the most stable, appears in our analysis.

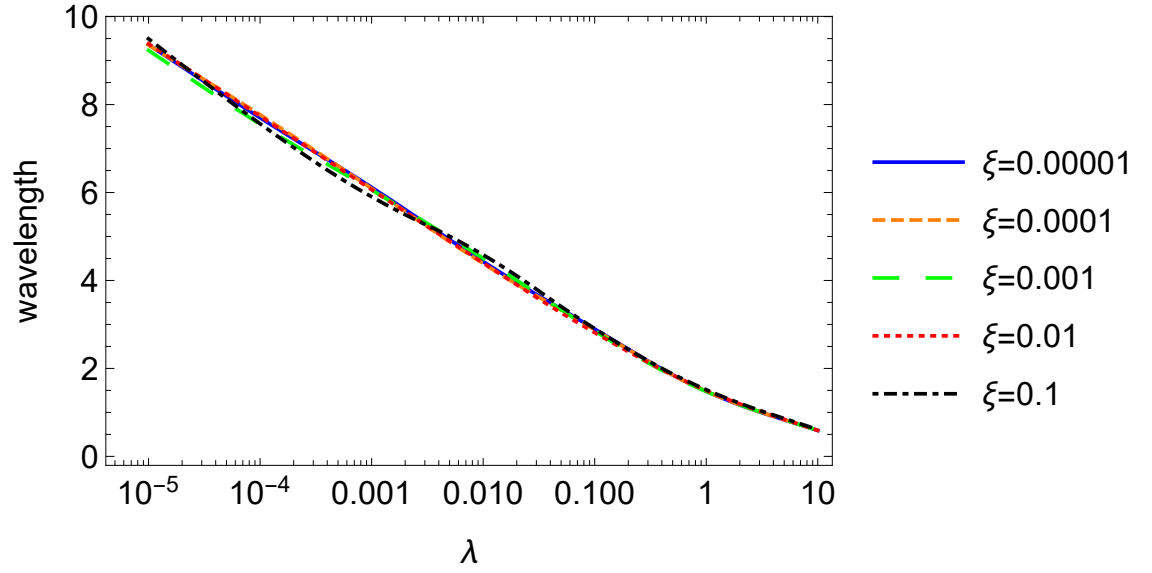
4.2 The Relationships among Parameters

The relationships among parameters and wavelengths are shown in Figure 9. Figure 9a shows that the wavelength of cyclic steps on the ice increases with decreasing λ , but is nearly invariable to changes in ξ . The former may be due to the following reasons. The amount of heat transfer across the water–ice interface per unit length reduces when the heat transfer capacity is small. In order to maintain equilibrium, a longer distance, i.e., a longer wavelength, is required to release the amount of heat transfer coming from the air–water interface. Meanwhile, the reason for the latter is not known yet, but it can be stated as follows. Keeping the water–ice heat transfer coefficient a constant, ξ is a function only of the air–water heat transfer coefficient, denoting the air–water heat transfer capacity. According to Figure 9a, ξ does not influence wavelength. It is found that the wavelength is more sensitive to the water–ice heat transfer capacity than the air–water heat transfer capacity.

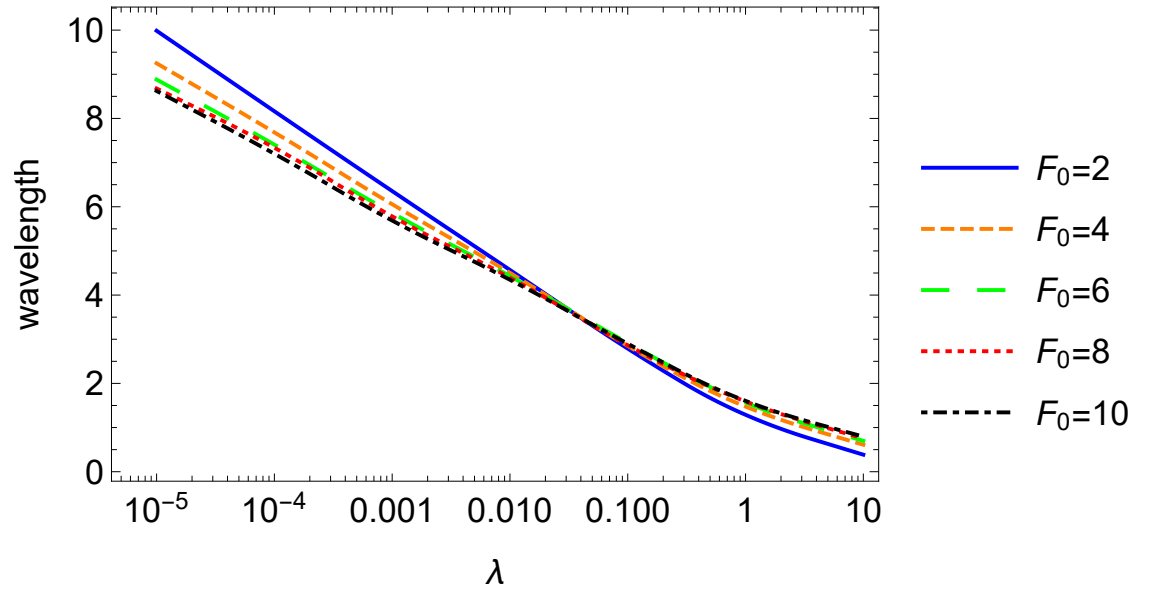
Figure 9b suggests that the wavelength increases with F_0 when λ is larger than 0.034, while the wavelength decreases with increasing F_0 when λ is smaller than 0.034. Generally, larger values of F_0 correspond to steeper average channel slopes because F_0 is defined by Equation 67. Therefore, the wavelength decreases with increasing channel slope when the water–ice heat transfer capacity is small, while the opposite trend is observed if the capacity is large. Regarding purely erosional cyclic steps, Parker and Izumi [2] found that the wavelength increases with decreasing Froude number under normal flow conditions. Our study also shows the same trend when the water–ice heat transfer capacity is small. This suggests that mechanical processes also dominate in the case of ice steps when the water–ice heat transfer capacity is small, and that the same trend is observed as in the case of purely erosional and transportational cyclic steps. When the water–ice heat transfer capacity is large, on the other hand, thermodynamic processes become important, and the relationships between wavelength and F_0 show the opposite trend. Sumida et al. [28] observed an increase in the wavelength with the Froude number. Though the detailed mechanism of an increase in the wavelength with the Froude number is not yet well understood, the water–ice heat transfer capacity is also expected to be large in Sumida et al.’s experiments.

4.3 Comparison with observed and predicted results

We compare the results calculated by our model with the experimental results of Yokokawa et al. [4]. They conducted eight experiments to replicate the formation



(a)



(b)

Figure 9: The relationships among wavelength and parameters. (a) $F_0 = 4$ (b) $\xi = 0.001$ (Cyclic steps on ice)

of cyclic steps on the ice. Figure 10 displays an experimental result, and Figure 11 depicts the step development process during the experiment. In one case where F_0 was less than unity, no bedforms were observed. This case is not considered in our study.

We obtain typical parameter values from the experiments [4], presented in Table 1. As mentioned previously, the drag coefficient can be determined by the Froude number ($C_D = S/F_0^2$). The water temperature corresponds to $\tilde{h}_a$ and C_{fh} . By comparing the predicted and observed water temperatures, we select suitable values for $\tilde{h}_a$ and C_{fh} . We assume C_{fh} as 6×10^{-5} in our analysis. Except for the experiment CSIM120913A, the values of $\tilde{h}_a$ were smaller than 18.15. Since $\tilde{h}_a$ is influenced by wind velocity, There is a possibility that the wind velocity was relatively faster in case CSIM120913A. The parameters λ and ξ can be calculated using Equations 68 and Equation 69, respectively. We present the normalized variations of averaged water velocity, water temperature, ice surface, and water surface for the experiment CSIM120910A in Figure 12.

Run Number	s	$\tilde{T}_a$ ($^{\circ}C$)	$\tilde{\Theta}_f$ ($^{\circ}C$)	F_0	$\tilde{D}_0$ ($\times 10^{-3}m$)	$\tilde{U}_0$ (m/s)	ξ ($\times 10^{-4}$)	λ ($\times 10^{-3}$)	C_D ($\times 10^{-2}$)
CSIM120908A	0.0875	5.5	0.7	1.52	2.05	0.22	2.26	1.58	3.79
CSIM120910A	0.0875	9.3	1.9	2.42	1.45	0.29	9.54	4.02	1.49
CSIM120910B	0.0875	5.7	0.7	3.17	1.65	0.40	8.59	6.89	0.87
CSIM120911A	0.3640	5.5	0.5	4.31	2.12	0.62	2.69	3.06	1.96
CSIM120911B	0.0875	5.2	0.2	2.89	5.70	0.68	2.10	5.73	1.05
CSIM120913A	0.0875	5.3	2.2	1.11	2.29	0.17	5.82	0.84	7.10
CSIM120914A	0.3640	5.4	0.5	4.37	1.25	0.48	2.85	3.15	1.91

Table 1: Input parameters for the model calculation.

The equations are normalized using variables in the normal flow condition, which allows us to obtain dimensional results once the variables in the normal flow condition are known. Figure 13 shows the dimensional results of experiment CSIM120910A. At $x = 0$, the flow reaches the Froude-critical point. Subsequently, the subcritical flow transitions to supercritical flow and accelerates until the end of one step. Energy dissipation through hydraulic jumps occurs in a relatively short zone, and the flow returns to the subcritical regime after the hydraulic jump. The water temperature increases upstream of one step due to less heat leaving the water than from the air. The acceleration of water results in more heat release, and the water temperature decreases when the heat flux from water to ice exceeds the heat flux from the air. The wavelength is several orders of magnitude larger than the water depth, confirming the satisfaction of the long wave assumption. The cyclic

steps on the ice reproduced by our models are consistent with the observed results.

Figure 14 compares the wavelength and wave steepness (wave height divided by wavelength). Wave height is the distance between the highest point of a crest and the lowest point. The predicted wavelength is sensitive to changes in λ . Since λ is defined by Equation 68, wavelengths are influenced by the dimensionless ice-water heat transfer coefficient, C_{fh} . This coefficient is strongly affected by factors like ice roughness and water velocity [36][37]. However, a constant value of C_{fh} is used for all experimental cases, meaning the comparisons must be considered approximate at best. The water velocity in the CSIM120913A case is slower than in other cases, as shown in Table 1, so there is a possibility that the chosen value of C_{fh} is too small in this case. Despite these limitations, the agreement between observed and predicted values is acceptable, except for CSIM120913A.

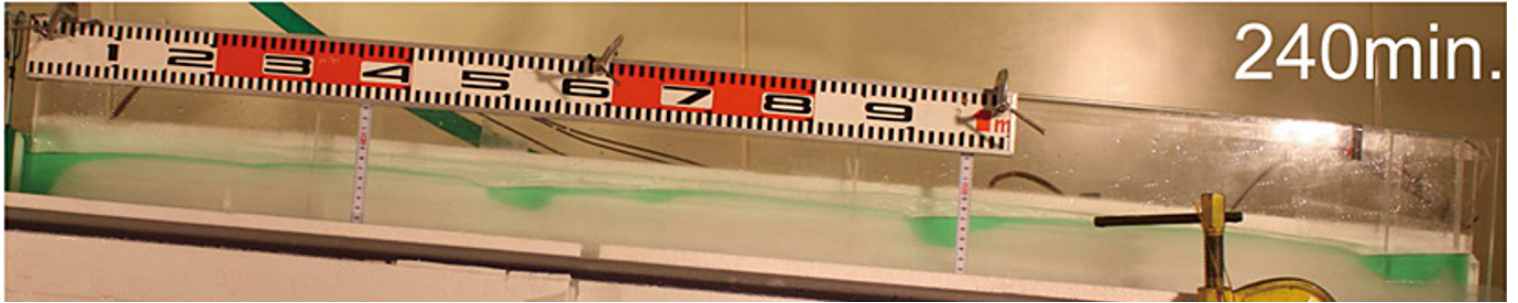


Figure 10: The experiment of cyclic steps on ice [4]. 240min subsequent to run initiation (Run number CSIM120908A) . Flow is left to right.

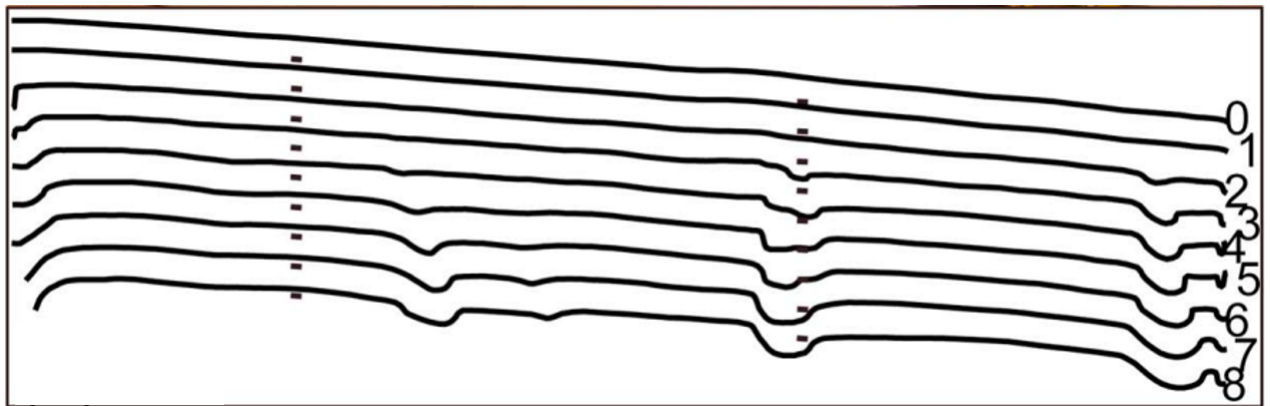
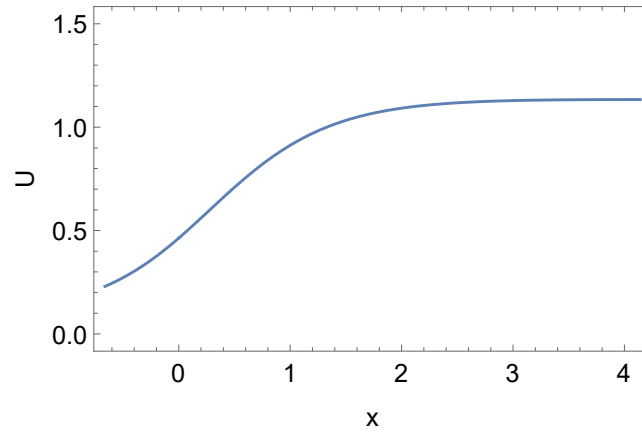
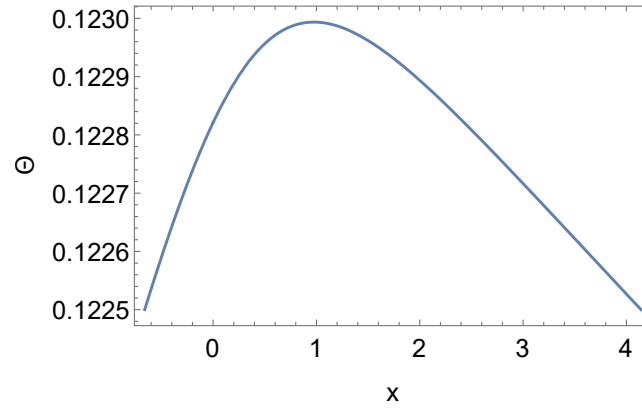


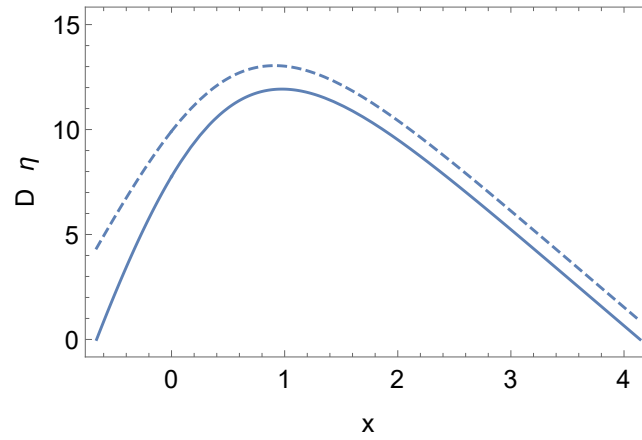
Figure 11: Development process of steps [4]. (Run number CSIM120908A). (1) 30 min, (2) 60 min, (3) 90 min, (4) 120 min, (5) 150 min, (6) 180 min, (7) 210 min, (8) 234 min subsequent to run initiation. Flow is left to right. (9) Line drawings of ice surface for every 30 min.



(a)

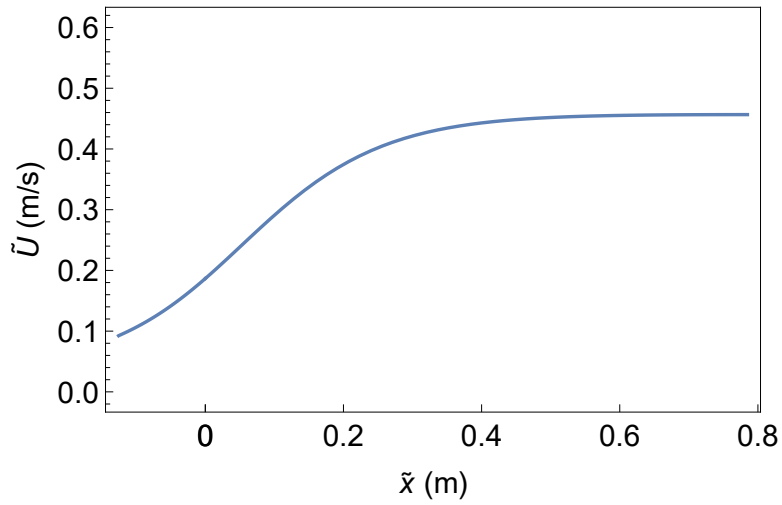


(b)

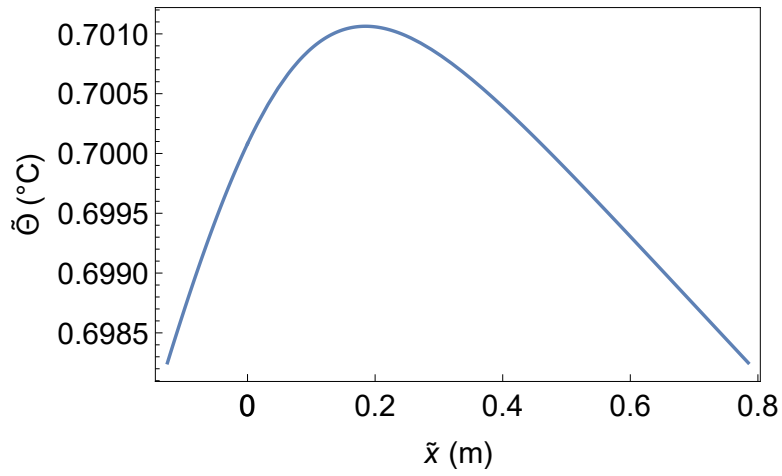


(c)

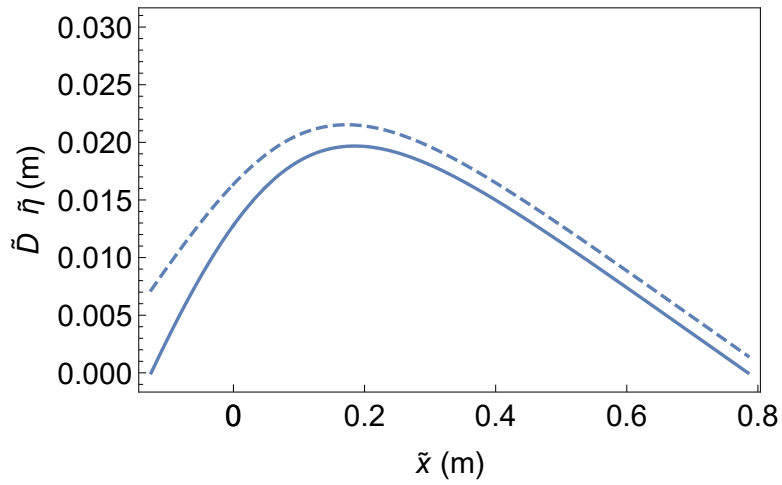
Figure 12: The variations of normalized variables over one step wavelength in CSIM120910b. (a) the variation of non-normalized velocity. (b) the variation of non-normalized average water temperature. (c) the variation of non-normalized water surface (dashed line) and ice surface (solid line).



(a)

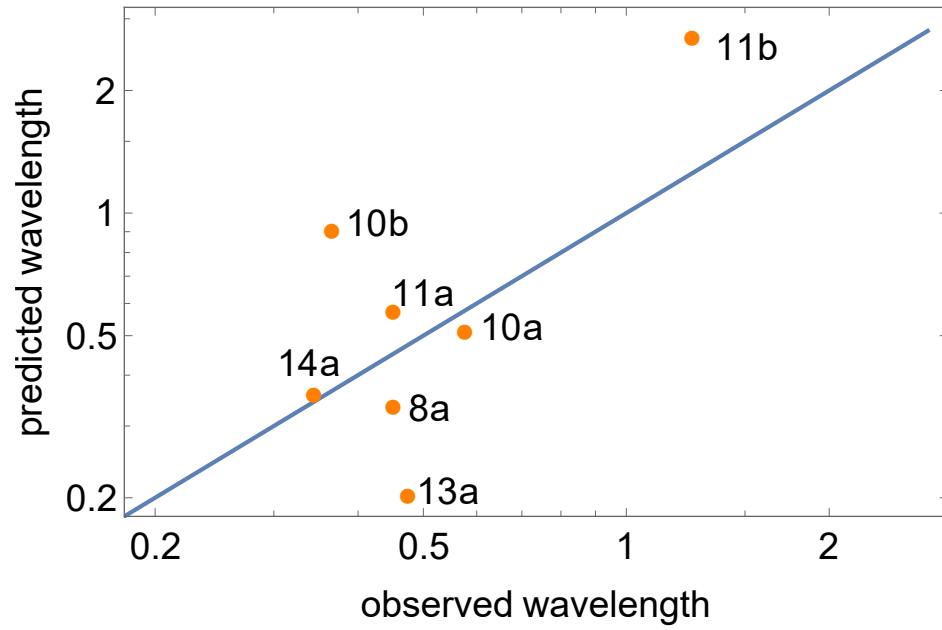


(b)

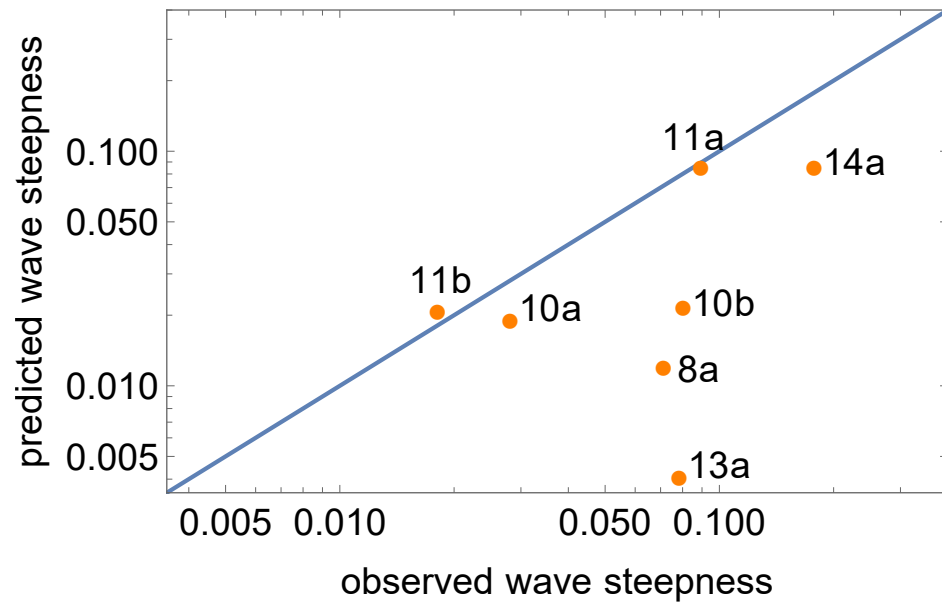


(c)

Figure 13: The variations of non-normalized variables over one step wavelength in CSIM120910B. (a) the variation of non-normalized velocity. (b) the variation of non-normalized average water temperature. (c) the variation of non-normalized water surface (dashed line) and ice surface (solid line).



(a)



(b)

Figure 14: Comparison with observed and predicted results, (a) wavelength (b) wave steepness

4.4 Comparisons with submarine cyclic steps formed by long-runout turbidity currents

4.4.1 Formulations

Cyclic steps formed on ice exhibit similarities and differences to the transportational cyclic steps created by long-runout turbidity currents on the ocean floor. In the case of submarine cyclic steps, an initially flat bed becomes unstable and transforms into a surface covered with a sequence of steps, as illustrated in Figure 14. We compare these two types of cyclic steps regarding their formulations, formation processes, and the relationships between wavelengths and parameters.

Previous research on submarine cyclic steps did not consider the characteristics of long-runout turbidity currents, which can travel over long distances (e.g.[30]). To address this issue, we propose an analytical model to reproduce the formation of submarine cyclic steps based on the turbidity currents model of Parker et al. [29]. Furthermore, we apply the theory of Luchi et al. [32] that currents can be divided into two layers to ensure that our model captures the features of long-runout turbidity currents (Figure 154). The upper layer continues to mix with the surrounding water and becomes increasingly diluted. The lower layer propels the flow and maintains a consistent suspended sediment concentration, flow thickness, and velocity profile. In this configuration, there exists a distinct density interface between the two layers. The velocity gradient and shear stress are almost zero at this distinct interface. The gravitational force propels the upper layer and balances the shear stress at the bottom of the lower layer. The high-concentrated lower layer can be assumed as in an equilibrium state under these conditions. The long-runout turbidity currents is analogous to the open channel flow.

We integrate the governing equations over the lower layer with a high concentration. After this manipulation, the layer-averaged momentum equation takes the same form as the one obtained in Parker et al.'s [29] model, as shown below

$$\tilde{U} \frac{\partial \tilde{U}}{\partial \tilde{x}} = -\frac{Rg}{2} \tilde{h} \frac{\partial \tilde{C}}{\partial \tilde{x}} - Rg\tilde{C} \frac{\partial \tilde{h}}{\partial \tilde{x}} - Rg\tilde{C} \frac{\partial \tilde{\eta}}{\partial \tilde{x}} + Rg\tilde{C}S - \frac{\tilde{\tau}_b}{\rho\tilde{h}} \quad (114)$$

where $\tilde{C}$ is the layer-averaged suspended sediment concentration, $\tilde{h}$ is the high-concentrated layer thickness of turbidity currents, R is the submerged specific gravity, $\tilde{\eta}$ is the deviation from the bed elevation without steps, S is the average slope, ρ is the density of water. $\tilde{\tau}_b$ is the bed shear stress related to the bed friction coefficient C_D ($\tilde{\tau}_b = \rho C_D \tilde{U}^2$). In addition, we denote dimensional variables by the tilde and drop it to denote non-dimensional or normalized equivalents.

The continuity equation is

$$\frac{\partial \tilde{U} \tilde{h}}{\partial \tilde{x}} = 0 \quad (115)$$

The high-concentrated layer of turbidity currents is in an equilibrium state and thus can be considered an open channel flow. Consequently, models for cyclic steps on the ice and those for cyclic steps formed by long-runout turbidity currents share similar momentum and continuity equations. However, the momentum equation for submarine cyclic steps takes into account the suspended sediment concentration as the primary driving force of turbidity currents.

$$\frac{\partial \tilde{U} \tilde{C} \tilde{h}}{\partial \tilde{x}} = \tilde{v}_s (\tilde{E} - \tilde{D}_e) \quad (116)$$

45

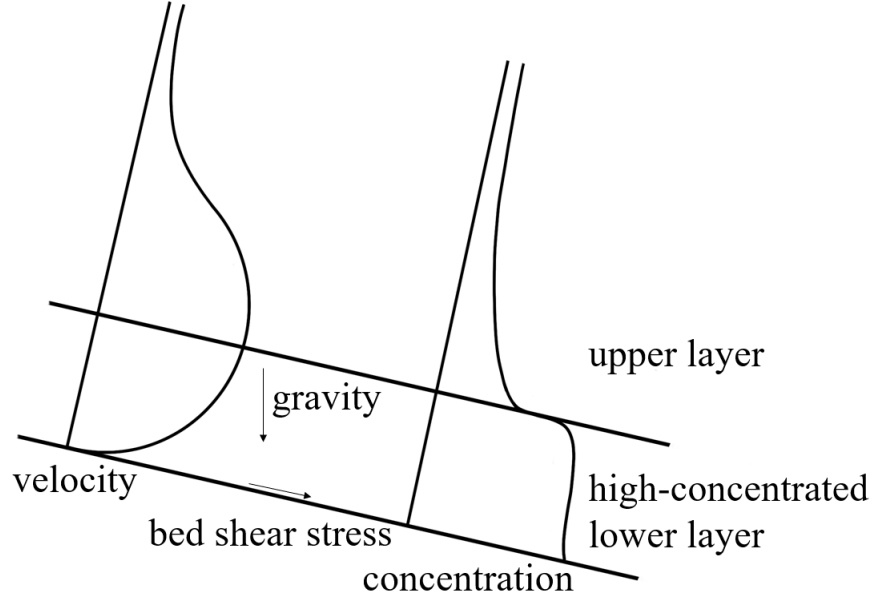


Figure 16: Sketch of the model.

(Equation 50).

One more governing equation in the model of submarine cyclic steps is the Exner equation, and it is expressed as

$$(1 - \lambda_p) \frac{\partial \tilde{\eta}}{\partial \tilde{t}} = \tilde{v}_s (\tilde{D}_e - \tilde{E}) \quad (117)$$

where λ_p is the porosity of the ocean floor. The Exner equation governs the balance of sediment between erosion and deposition on the bed. The variations in sediment erosion and deposition rates lead to bed elevation variations. In contrast, the formation of cyclic steps on ice involves melting and freezing caused by heat transfer. The Stefan equation (Equation 51) governs the energy balance and determines the position of the moving interface.

Like the cyclic steps on the ice, we use the variables in the normal flow condition before developing steps as normalization. Since cyclic steps move upstream without changing form, we apply the moving coordinate system into the governing equations (Equations 114–117) as in the study of cyclic steps on the ice. After further manipulation, the Exner equation 117 is then simplified to

$$f \frac{d\eta}{dx} = C - E \quad (118)$$

The parameter f in the above equation represents the migration speed.

We finally obtain the normalized governing equations in the flowing form

$$\frac{dU}{dx} = \frac{\left(\frac{C}{f} - \frac{1}{2U\beta}\right)(E - C) + \sigma C - \frac{U^3}{\text{Ri}_n}}{\frac{U}{\text{Ri}_n} - \frac{C}{U^2}} \quad (119)$$

$$\frac{dC}{dx} = \frac{E - C}{\beta} \quad (120)$$

where β is a non-dimensional parameter representing the suspendability of sediment, σ is the normalized slope, and Ri_n is the Richardson number in the normal flow condition in the absence of steps, which read, respectively

$$\beta = \frac{c_D \tilde{U}_n \tilde{C}_n}{\tilde{v}_s \tilde{D}_n} \quad (121)$$

$$\sigma = \frac{S}{C_D} \quad (122)$$

$$\text{Ri}_n = \frac{Rg \tilde{H}_n \tilde{C}_n}{\tilde{U}_n^2} \quad (123)$$

Here the subscripts n denotes variables for the case of normal flow in the absence of steps.

The Richardson number Ri is the key to the emergence of submarine cyclic steps. The Richardson critical point denotes the location where the Richardson number equals unity, dividing the flow into supercritical and subcritical regimes. Generally, the Froude number Fr is used to describe free-surface fluids, and thus it is used in the model of cyclic steps on the ice. The Richardson number is typically used to describe the stability and flow characteristics of flows that contain particles or sediment, so we apply it in the model of submarine cyclic steps.

4.4.2 Formation Process

Typically, the suspended load is the primary mode of transport in forming cyclic steps created by long-runout turbidity currents on the ocean floor. They are characterized by a series of repeating erosional and depositional features. The deposition prevails on the upstream side of a single step where the flow is in the subcritical regime. Erosion dominates on the downstream side, where the flow is faster. If the flow velocity fails to decrease to the threshold value (e.i. U_{th} , is the non-dimensional threshold velocity for entrainment), erosion of the bed will persist, and periodic cyclic steps will not form.

The formation of cyclic steps on the ice is a mechanics and thermodynamics process. The evolution of the ice surface through melting and freezing is primarily due to the difference in heat flux, as shown in Figure 17. We tend to analyze the heat flux variation and apply the data of experiment case CSIM120910B as an example. The first term of the right-hand side of Equation 87 means the heat flux from water to the interface, and its variation over one step wavelength is shown in Figure 18. The second term is the heat flux from air to water, and the variation is in Figure 19. Equation 87 describes the variation of average water temperature. The water temperature must be greater than Θ_+ to allow sufficient heat flux into the interface to match that going out, resulting in the melting of ice. The water temperature increases when less heat flux transfers from ice than into the water from the air. As water flow accelerates, more heat goes out from the water, and the water temperature begins to decrease. The water temperature reduces to Θ_+ to allow the formation of periodic cyclic steps.

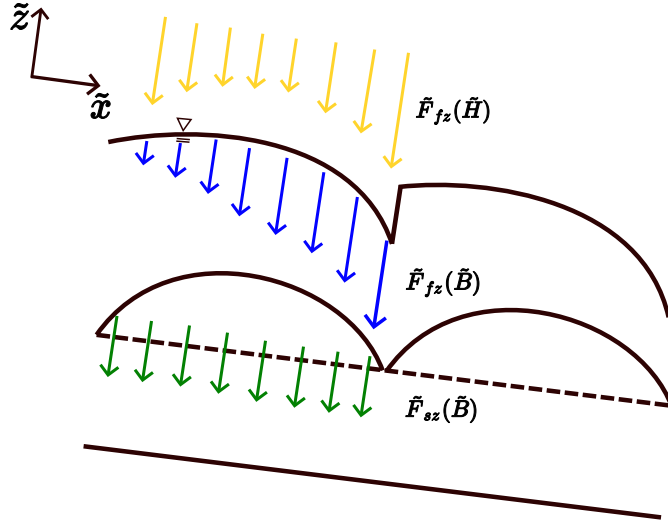


Figure 17: Conceptual sketch of heat flux. Longer arrows denote more heat flux, and shorter arrows represent less heat flux. $\tilde{F}_{fz}(\tilde{H})$ is the heat flux from air to water, $\tilde{F}_{fz}(\tilde{B})$ is the heat flux coming into the interface from the water above, $\tilde{F}_{sz}(\tilde{B})$ is the heat flux going out from the interface to the ice bottom.

On the right-hand side of Equation 82, the first term denotes the heat flux from

water to the interface, and the variation can be seen in Figure 20. The second term is the heat flux from the interface to the ice bottom, and the value is 0.11. Solidification begins after the hydraulic jump because more heat is transferred from the interface than into it. As the flow accelerates downstream, more heat transfers to the ice surface. When more heat enters the interface than exits, the ice elevation decreases due to melting.

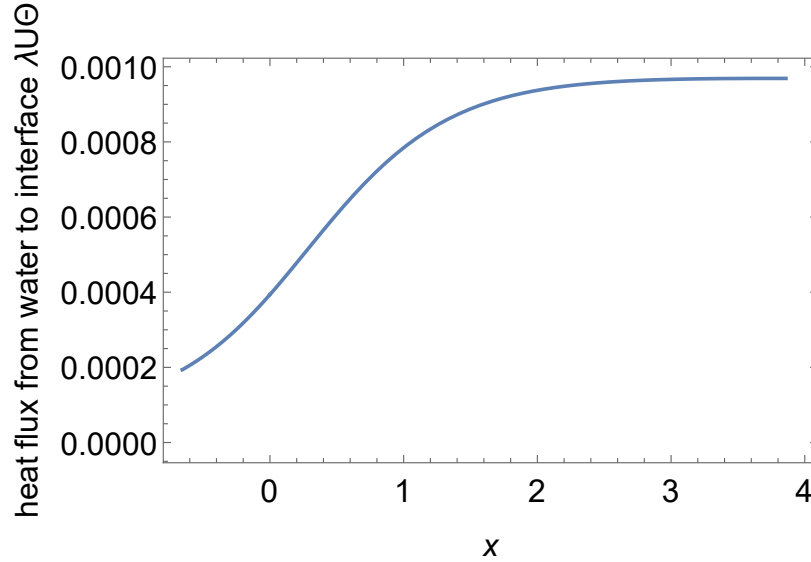


Figure 18: The variation of heat flux from water to the ice surface over one step wavelength. The setting of parameters corresponds to the experiment case CSIM120910B.

Most cyclic steps are observed migrating upstream. The reason that cyclic steps formed by long-runout turbidity currents migrating upstream is explained following. The amount of deposition from the Froude-subcritical flow exceeds that from the Froude-supercritical flow. Similarly, the Froude-supercritical flow causes greater erosion than the erosion caused by the Froude-subcritical flow on the stoss side.

However, in the case of cyclic steps on ice, they are observed migrating upstream or downstream in different settings. The phenomenon of the steps on ice that migrate upstream (e.g.[4]) is mainly due to the water temperature being higher than the ice but lower than the air temperature. The shallow supercritical flow significantly impacts the ice bed under high atmospheric temperatures more than the deeper regime. Hence, the melting dominates in the downstream regime, and solidifying

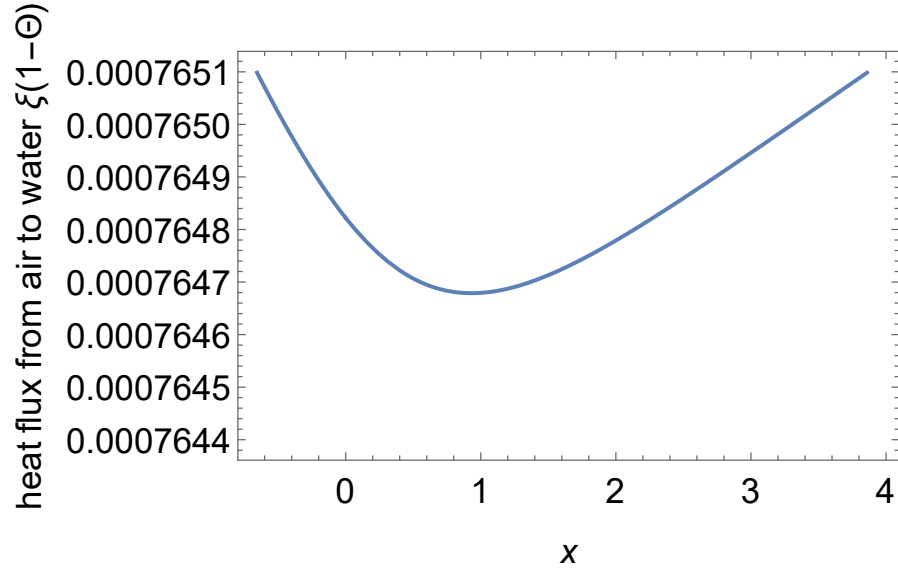


Figure 19: The variation of heat flux from air to the water over one step wavelength. The setting of parameters corresponds to the experiment case CSIM120910B.

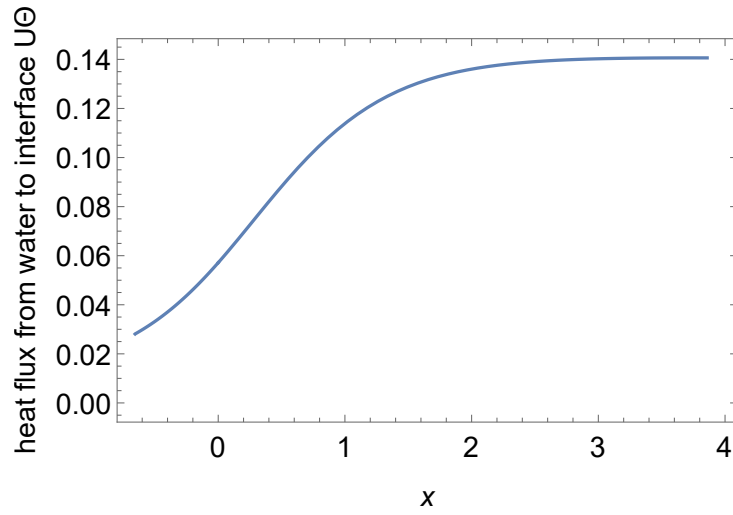


Figure 20: The variation of heat flux from water to the ice surface over one step wavelength. The setting of parameters corresponds to the experiment case CSIM120910B.

is main in the upstream. In contrast, Naito et al. [27] observed cyclic steps migrating downstream as the air temperature was lower than the melting point. The aggradation occurs mainly in the downstream part of the step due to the low ambient temperature being more easily affected by the shallow water depth.

4.4.3 Relationships among wavelengths and parameters

The relationships among wavelengths and parameters in the cases of submarine cyclic steps and cyclic steps on ice have different patterns. In the study of submarine cyclic steps, the wavelength of cyclic steps varies with the Richardson number in the normal flow condition Ri_n and the suspendability parameter β as shown in Figure 21. The wavelength is defined as the distance between two adjacent hydraulic jumps.

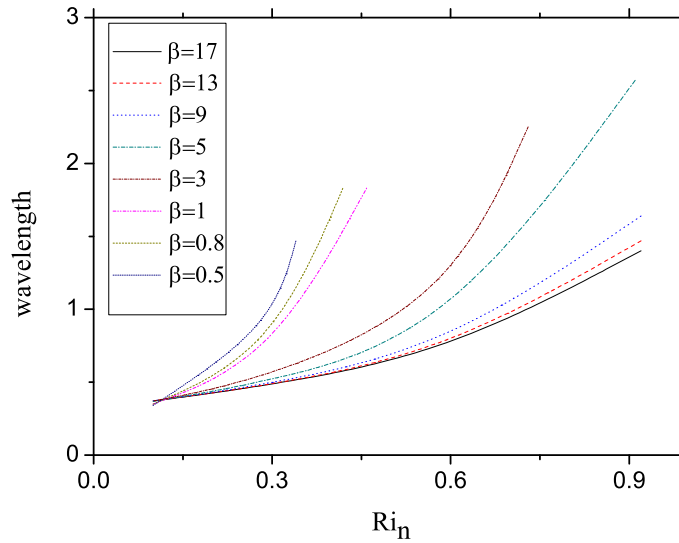


Figure 21: The relationships among typical parameters (Cyclic steps formed by long-runout turbidity currents).

During the calculation, if Ri_n exceeds a specific value or β falls below a certain threshold, there is no solution. When β is smaller than 0.12, the proposed model has no solution, indicating that cyclic steps do not form if the suspendability is insufficient. It is because the large suspendability and the generation of abundant suspended sediment are essential for cyclic steps formation. Moreover, we observe that, as the suspendability decreases, steps will only form if the slope is considerably

steep. If the suspendability is sufficiently large, cyclic steps can appear even on mild slopes. Additionally, we find that large suspendability leads to short wavelengths. The reason may be that when the suspendability is more prominent, fewer suspended particles settle out of the fluid to form cyclic steps. Overall, our analysis suggests that the formation of cyclic steps depends on the bed's slope and the sediment's suspendability.

If the value of drag coefficient is specified, Ri_n corresponds to the average slope according to the expression of $Ri_n (=C_D/S)$. To compare with the cyclic steps on the ice, we use the Froude number in the normal flow condition F_n rather than the Richardson number. Figure 21 is replotted to Figure 22, showing that cyclic steps with longer wavelengths tend to form for gentle slopes, while shorter wavelength steps occur on steeper slopes. This tendency may be because steep slopes generate more rapid flow variations, leading to shorter intervals between hydraulic jumps. In the South Taiwan Shoal Canyon, Zhong et al. [9] observed that longer wavelengths appear on the relatively milder slope while shorter wavelengths are on steeper slopes, as shown in Figure 23. They reported that wave height ranges from 15 m to 81 m. Wavelengths vary between 1800 m to 8414 m in the upper segment and 4078 m to 10452 m in the lower segment. Using the present model, we obtain the bed profiles over a one-step wavelength in both the upper and lower segments within the South Taiwan Shoal Canyon as shown in Figure 24, and the same trends are gained.

To see the relationships between wavelengths and Froude number in the normal flow condition F_0 in the case of cyclic steps on the ice, we replot Figure 9b to Figure 25. It is obvious that wavelengths increase with the decrease of F_0 when the water-ice heat transfer capacity λ is small, and it is the same as in the analysis of submarine cyclic steps. However, when the water-ice heat transfer capacity is large, the thermodynamic process is the primary formation process, and the opposite trend between wavelengths and F_0 appears.

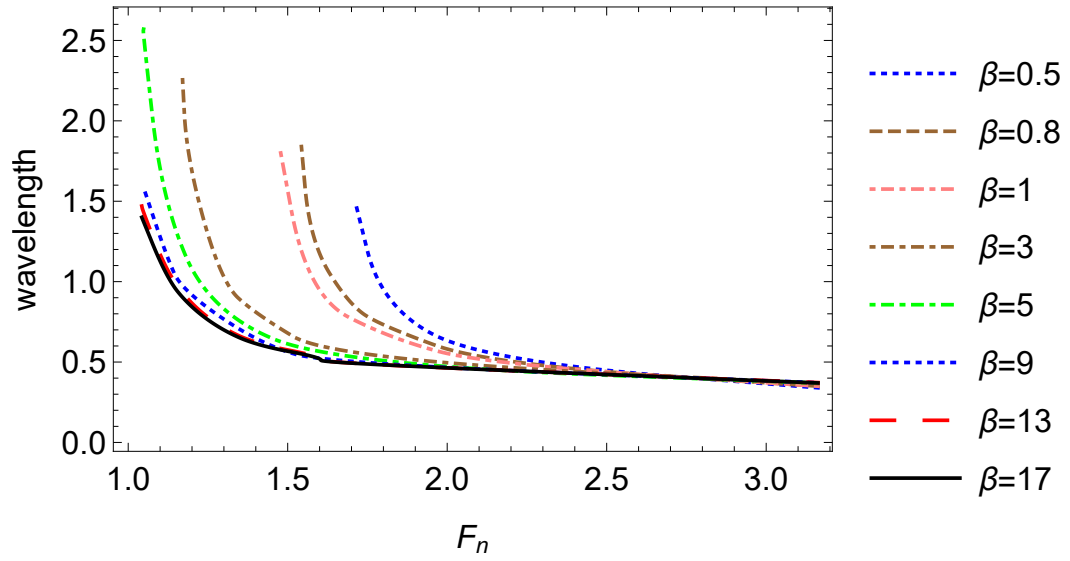


Figure 22: The relationships among typical parameters and wavelengths (Cyclic steps formed by long-runout turbidity currents).

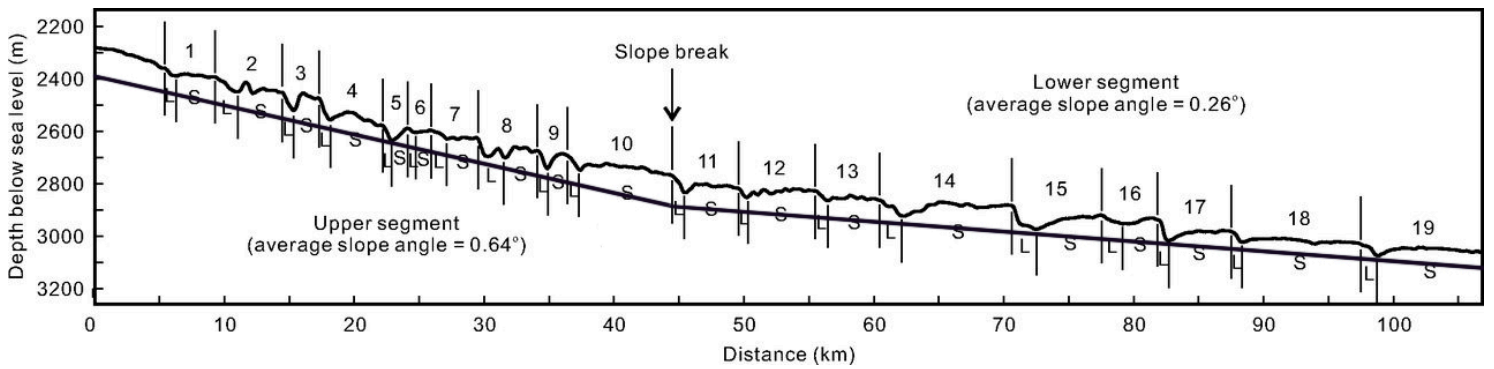
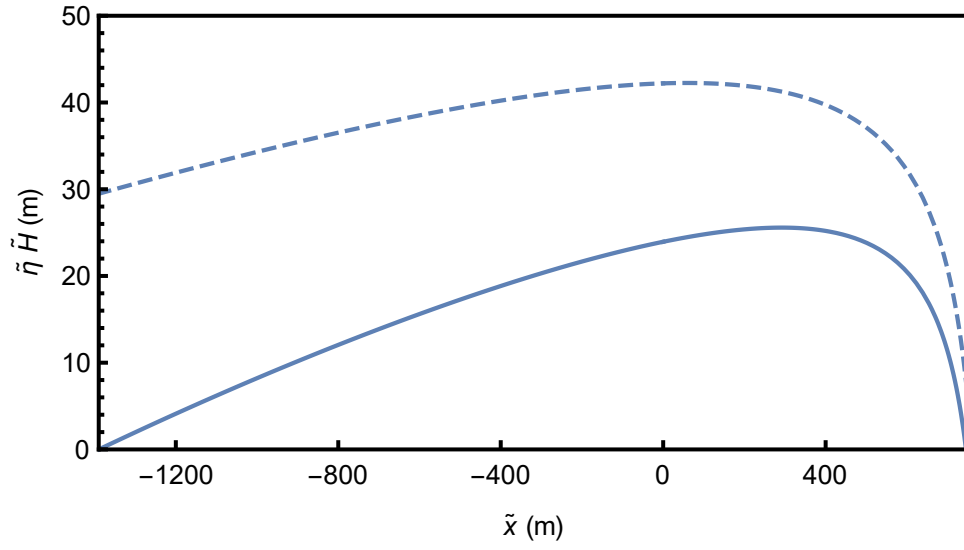
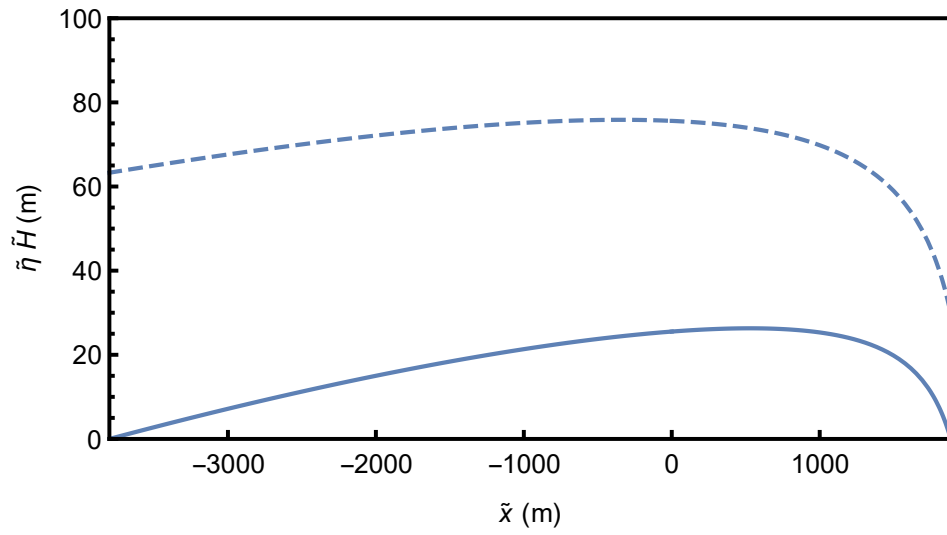


Figure 23: Cyclic steps within the South Taiwan Shoal canyon, from Zhong et al. [9]. 1-10 cyclic steps are in the upper segment and 11-19 cyclic steps are in the lower segment; *L* indicates the lee sides and *S* indicates the stoss sides.

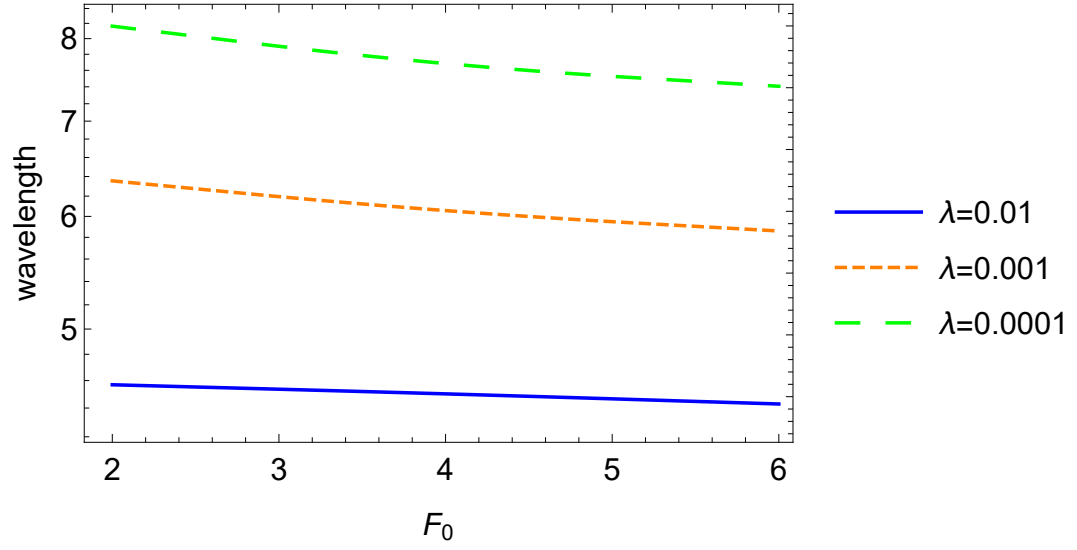


(a)

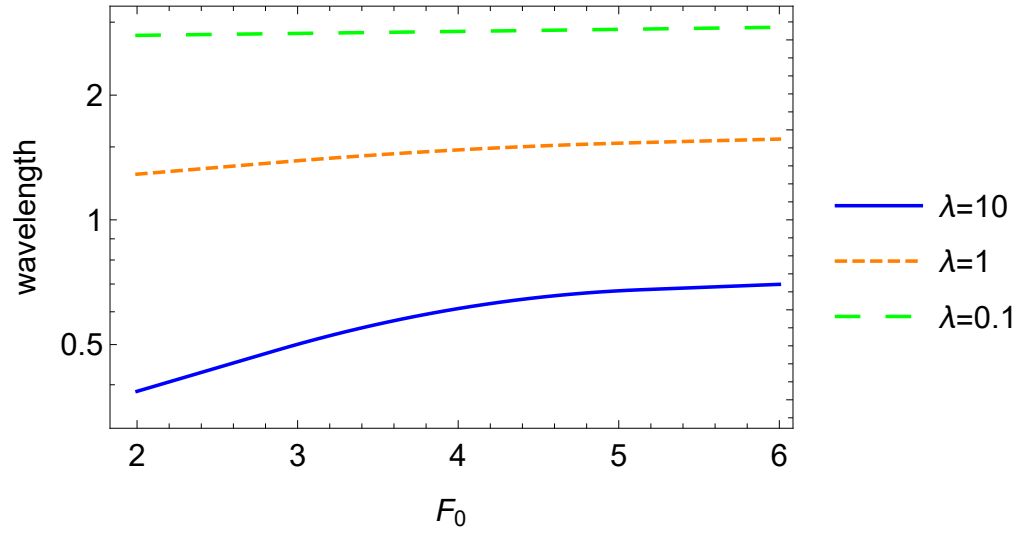


(b)

Figure 24: The distributions of non-normalized bed elevation and water surface over one step wavelength, in the upper segment.



(a)



(b)

Figure 25: The relationships among typical parameters and wavelengths (Cyclic steps on ice) (a) with relatively small water-ice heat transfer capacity λ (b) with relatively large water-ice heat transfer capacity λ

5 Conclusion

Our study presents an analytical model for reproducing cyclic steps on the ice. Solving this model can generate dimensional variables over one wavelength of the cyclic steps. Our results show that cyclic steps have periodic solutions of permanent form that migrate upstream at a constant migration speed. To address previous studies' limitations that could not directly analyze cyclic steps or determine their wavelengths. We assume the value of the water temperature at the downstream of the hydraulic jump to close the problem. We then choose the longest wavelength for further analysis, assuming it is the most stable.

This theory allows us to predict the migration speed, water temperature, average velocity, water surface profile, and ice elevation of cyclic steps formed by water on ice. The governing equations in the proposed model are normalized using variables in the normal flow condition, enabling the calculation of variations in real-scale depth-averaged water temperature, depth-averaged velocity, and ice elevation over one-step wavelength. We obtain dimensional variables by applying experimental data from Yokokawa et al. [4], and the predicted results from our theory agree reasonably well with the observed results with some limitations.

We compare cyclic steps formed on ice and long-runout turbidity currents in terms of formulation, formation process, and the factors that influence wavelengths. Similar momentum and continuity equations are used in both models. However, the Exner equation is used to govern the sediment balance in the case of submarine cyclic steps, while we use the Stefan equation to describe the energy balance in the model of cyclic steps on the ice. The erosion and deposition mainly cause the formation of submarine cyclic steps. Nevertheless, solidification and melting result in cyclic steps on the ice. Moreover, in the case of cyclic steps formed by long-runout turbidity currents, longer wavelengths appear on the relatively milder slopes, and steeper slopes facilitate shorter wavelengths. For cyclic steps on the ice, the theory predicts that longer wavelengths occur when the water-ice heat transfer coefficient is larger. In contrast, the length of one step is not associated with the air-water heat transfer coefficient. Furthermore, larger F_0 leads to longer wavelengths when the water-ice heat transfer capacity is relatively weak, but smaller F_0 results in longer wavelengths with the strong water-ice heat transfer capacity.

Notation

B : the water bottom elevation.
 C suspended sediment concentration
 C_D : the drag coefficient.
 c_a : the specific heat of air ($=1005\text{J/kgK}$).
 C_{ah} : a normalized heat transfer coefficient which is influenced by the wind speed.
 c_f : the specific heat capacity of water ($=4184\text{J/kgK}$).
 C_{fh} : a dimensionless ice-water transfer coefficient.
 D : the flow depth.
 D_e : deposition rate of suspended sediment
 E entrainment rate of suspended sediment
 e_w entrainment of water
 f : the migration speed of the steps in the streamwise direction.
 F_{fz} : the heat flux in flowing water in the depth direction $\tilde{z}$.
 F_{sz} : the the heat flux with the ice in the $\tilde{z}$ direction.
 F_0 : Froude number associated with the normal flow in the absence of steps.
 F_r : Froude number.
 g : the gravity acceleration ($= 9.8 \text{ m/s}^2$).
 h high-concentrated layer thickness of turbidity currents
 h_s : the latent heat of melting.
 h_a : bulk heat transfer coefficient of air
 h_f : the bulk heat transfer coefficient of water above ice surface ($= \rho_f c_f C_{fh} \tilde{U}$).
 H : the water surface elevation.
 I : ice bottom.
 k_s : the thermal conductivity of ice ($= 2.22\text{W/(mK)}$).
 F_{fx} : the heat flux in flowing water in $\tilde{x}$ direction.
 K : a constant.
 K' : a constant.
 k_f : the thermal conductivity of water.
 $\tilde{p}$: the hydrostatic pressure.
 $\tilde{P}$: the total pressure including the normal component of the Reynolds stress.
 R submerged specific gravity
 R_i Richardson number (–)
 s : the average slope.
 t : the time.
 T : the water temperature;
 T_m : the temperature at the ice-water interface and kept to the melting point.
 T_i : the temperature at the ice bottom.
 T_a : the air temperature.

$\tilde{T}_{ij}$ ($i, j = \tilde{x}, \tilde{z}$): the stress tensor.
 U : the depth-averaged velocity component in the $\tilde{x}$ direction.
 U_{th} threshold velocity below which erosion does not occur (–)
 $\tilde{u}$: the velocity component in the $\tilde{x}$ direction.
 $\tilde{u}_n^*$ friction velocity (–)
 $\tilde{v}_s$ = falling velocity (–)
 $\tilde{w}$: the velocity component in the $\tilde{z}$ direction.
 $\tilde{x}$: the coordinate in the streamwise.
 Z : the ice thickness.
 z : the coordinate in the depth direction.
 α : the slope angle of the channel.
 β a nondimensional parameter representing the suspendability of the turbidity currents (–)
 τ_b : the bed shear stress.
 ρ_f : the density of water ($=1 \times 10^3 \text{ kg/m}^3$).
 ρ_s : the density of ice.
 ρ_a : the air density ($=1.29 \text{ kg/m}^3$).
 λ : the normalized ice-water heat transfer coefficient ($=C_{fh}/C_D$).
 λ_p porosity of sediment (–)
 ξ : a non-dimensional parameter representing the ratio of the air-water heat transfer coefficient to the water-ice heat transfer coefficient ($=\tilde{h}_a/(\rho_f c_f \tilde{U}_0 C_D)$).
 γ : a non-dimensional parameter defined as the ratio of the thermal conductivity coefficient and heat transfer coefficient in water ($=k_s/(\rho_f c_f C_{fh} \tilde{U}_0 \tilde{D}_0)$).
 ϕ : a constant and is assumed as unity;
 ϕ_f : a constant and is assumed as unity;
 ζ : the sum of the heat flux from the ice surface to the ice bottom and the migration speed of the ice surface in the depth direction ($=-\gamma \frac{T_i}{Z} + \sigma$).
 η : the bed (ice) elevation referenced to the average bed slope (–)
 ν is the viscosity.
 Θ : the depth-averaged water temperature.
 $()_0$: for the case of normal flow in the absence of steps (cyclic steps on ice).
 $()_c$: the variables at the Froude critical point (i.e. $F_r = 1$).
 $()_n$ for the case of normal flow in the absence of steps (cyclic steps formed by long-runout turbidity currents).
 $\tilde{()}$:dimensional variables.
 ${}'$: the turbulent fluctuation of the corresponding velocity component.

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