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MINIMAL CYLINDERS IN THE THREE-DIMENSIONAL HEISENBERG GROUP

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Abstract. We study minimal cylinders in the three-dimensional Heisenberg group Nil_3 using the generalized Weierstrass type representation, the so-called loop group method. We characterize all non-vertical minimal cylinders in terms of pairs of two closed plane curves which have the same signed area. Moreover, as a byproduct of the construction, spacelike CMC cylinders can also be obtained.

1. Introduction and the main result

(A) The construction of surfaces with special properties is an important and standard task of classical differential geometry. A particularly important class of surfaces in the three-dimensional Heisenberg group is formed by the minimal surfaces. They have been investigated by many authors for many years, e.g., [5, 13, 14, 15, 12, 7]. In [8] the basic loop group construction of all such surfaces with contractible domain of definition has been presented. In [9] this was generalized to minimal surfaces with non-trivial topology. More generally minimal surfaces with symmetries have been investigated. As an illustration of our technique we have discussed equivariant surfaces and helicoidal surfaces in some detail. The present paper continues the discussion of special surfaces in the three-dimensional Heisenberg group Nil_3 by investigating minimal cylinders.

Before we can describe the present work we would like to recall the generalized Weierstrass type representation for non-vertical minimal surfaces in Nil_3. For details and notation we refer to [8, 9]. In this paper we consider exclusively Riemann surfaces M and their universal cover $\widetilde{M} = \mathbb{D}$, a contractible open subset of $\mathbb{C}$, since there is no minimal S^2 in Nil_3.

Step 1: Choose any holomorphic potential $\eta = A(z,\lambda)dz$ so that $A(z,\lambda) = \sum_{k=-1}^{\infty} A_k(z)\lambda^k$ is defined for $\lambda \in \mathbb{C}^*$ and all $z \in \mathbb{D}$ and the diagonal entries are even, and the off-diagonal entries are odd in λ. Moreover, assume that the $(1,2)$-entry of A_{-1} never vanishes.

Step 2: Solve the ordinary differential equation $dC = C\eta$ on $\mathbb{D}$ with initial condition identity at some base point z_0. Here we consider a holomorphic potential and therefore do not have any problems to solve the ode (Below we extend what is explained here to certain meromorphic $\mathfrak{su}_{1,1}$-potentials).

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Step 3: Consider the connected component $\mathcal{I}_{z_0}$, containing z_0, of all points in $\mathbb{D}$ such that $C(z,\lambda)$ has for all $\lambda \in S^1$ an $\mathrm{SU}_{1,1}$-Iwasawa (twisted) loop group decomposition on $\mathcal{I}_{z_0}$, that is, $C = FV_+$, where $F : \mathcal{I}_{z_0} \to \Lambda\mathrm{SU}_{1,1\sigma}$ and $V_+ : \mathcal{I}_{z_0} \to \Lambda^+\mathrm{SL}_2\mathbb{C}_\sigma$.

Then F is on $\mathbb{D}(z_0)$ the extended frame of some minimal surface in Nil_3, that is, F takes values in $\Lambda\mathrm{SU}_{1,1\sigma}$ and its Maurer-Cartan form has the particular λ-distribution characteristic for the Maurer-Cartan forms of extended frames. Also note, the leading term V_0 of V_+ is a diagonal matrix and we can assume w.l.g. that it has positive diagonal elements. In this case the decomposition $C = FV_+$ is unique.

Step 4: Next we apply Sym-type formulas to the extended frame F as follows: First we compute the spacelike CMC surface in Minkowski space $f_{\mathbb{L}_3}$ and its Gauss map $N_{\mathbb{L}_3}$ respectively by the formulas

$$f_{\mathbb{L}_3} = -i\lambda(\partial_\lambda F)F^{-1} - \frac{i}{2}\operatorname{Ad}(F)\sigma_3, \quad \text{and} \quad N_{\mathbb{L}_3} = \frac{i}{2}\operatorname{Ad}(F)\sigma_3 \tag{1.1}$$

with $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then we finally compute f^λ by the formula

$$f^\lambda := \Xi_{\mathrm{nil}} \circ \hat{f} \quad \text{with} \quad \hat{f} = (f_{\mathbb{L}_3})^o - \frac{i}{2}\lambda(\partial_\lambda f_{\mathbb{L}_3})^d, \tag{1.2}$$

where the superscripts "o" and "d" denote the off-diagonal and diagonal part, respectively. Moreover, Ξ_{nil} denotes the exponential map from $\mathfrak{nil}_3$ to Nil_3. Then $f^\lambda, \lambda \in S^1$ defines a minimal surface in Nil_3 for each $\lambda \in S^1$.

The Sym-type formula (1.2) first was stated in [6], and the form of (1.2) has been given in [8].

(B) It is clear that the choice of holomorphic potentials η is the most important task. Since the extended frames of a minimal surface in Nil_3 and a spacelike CMC surfaces in the Minkowski 3-space $\mathbb{L}_3$ are the same, see (1.1) and (1.2), the construction technique of spacelike CMC surfaces can be applied. Moreover, it has been known that there exists a natural choice of holomorphic potentials for CMC-cylinders in the Euclidean 3-space, see [10]. Note that the extended frames of a CMC surface in $\mathbb{E}^3$ and a spacelike CMC surface in $\mathbb{L}_3$ just take values in different real forms of the loop group of $\mathrm{SL}_2\mathbb{C}$. We thus follow the strategy from Section 5.2 in [10] for the construction of all non-vertical minimal cylinders in Nil_3.

By the appendix we can assume without loss of generality that one can start from a holomorphic and periodic matrix $C_0(z)$ of period $p > 0$ of the form

$$C_0(z) = \begin{pmatrix} a_0(z) & b_0(z) \\ \overline{b_0(\bar{z})} & \overline{a_0(\bar{z})} \end{pmatrix} \quad , \tag{1.3}$$

where $a_0(z), b_0(z)$ are periodic functions of period $p > 0$ satisfying $\det C_0(z) = a_0(z)\overline{a_0(\bar{z})} - b_0(z)\overline{b_0(\bar{z})} = 1$. In particular $C_0(z)$ is in $\mathrm{SU}_{1,1}$ for $z \in \mathbb{R}$. Moreover, we assume in addition $C_0(z=0) = \mathrm{id}$.

Remark 1.1. If we also want to admit the case $C_0(z+p) = -C_0(z)$, then we can change in two ways to the periodic case:

(a) We change C_0 by a gauge k, where k is a diagonal matrix with entries $\exp(i\pi z/p)$ and $\exp(-i\pi z/p)$. This does not change anything of geometric substance. However, strictly speaking it changes the original surface to another one by diagonal dressing.

(b) Or we consider the period $2p$.

Set

$$\zeta_0(z) = C_0^{-1}\partial_z C_0 dz = \begin{pmatrix} \nu(z) & \kappa(z) \\ \kappa(\bar{z}) & -\nu(z) \end{pmatrix} dz, \tag{1.4}$$

then ν and κ are periodic functions of period $p > 0$ and $\overline{\nu(\bar{z})} = -\nu(z)$. It is easy to see that

$$\kappa(z) = \overline{a_0(\bar{z})}\partial_z b_0(z) - b_0(z)\partial_z \overline{a_0(\bar{z})} \tag{1.5}$$

and $\nu(z) = \overline{a_0(\bar{z})}\partial_z a_0(z) - b_0(z)\partial_z \overline{b_0(\bar{z})}$ hold. We take a meromorphic function h on $\mathbb{D}$, which is periodic of period p, and set

$$\zeta(z,\lambda) = \begin{pmatrix} \nu(z) & \lambda^{-1}h(z) + \lambda(\kappa(z) - h(z)) \\ \lambda^{-1}(\overline{\kappa(\bar{z})} - \overline{h(\bar{z})}) + \lambda\overline{h(\bar{z})} & -\nu(z) \end{pmatrix} dz. \tag{1.6}$$

Then $\zeta(z, \lambda = 1) = \zeta_0(z)$ is meromorphic for $z \in \mathbb{D}$ and in $\mathfrak{su}_{1,1}$ for $z \in \mathbb{R} \cap \mathbb{D}$, that is, ζ_0 is $\mathfrak{su}_{1,1}$ on $\mathbb{R} \cap \mathbb{D}$.

The potential ζ satisfies the assumption stated in Proposition 5.1 of [10] after replacing "skew-hermitian" by "$\mathfrak{su}_{1,1}$", and will be called an $\mathfrak{su}_{1,1}$-*potential* .

Using the $\mathfrak{su}_{1,1}$-potential ζ as the input data for Step 1 above, one can construct a minimal surface in Nil_3 with particularly nice properties. We will explain them in Section 2 below.

Remark 1.2. The $\mathfrak{su}_{1,1}$-potential ζ is meromorphic and its poles can be off from the real line. In general poles of ζ give special properties of the resulting minimal surface, for examples, ends, branch points or smooth points etc.

(a) Branch points: At a point z_1 where the product of the two (off-diagonal) terms of A_{-1} vanishes, considering the partial derivatives of (the Sym formula for) the minimal surface in Nil_3 immersion derived from the potential. It is easy to see that the differential of the "immersion" is 1 or 0. Thus we obtain a branch point.
(b) Ends: From [9] one infers that sometimes poles of specific structure give "ends".

(C) To state the main result of this paper, we introduce two complex functions ℓ and m defined by

$$\ell(t) = \int_0^t \left\{ a_0(s)^2(2h(s) - \kappa(s)) + b_0(s)^2(2\overline{h(s)} - \overline{\kappa(s)}) \right\} ds \tag{1.7}$$

$$m(t) = a_0(t)b_0(t), \tag{1.8}$$

for $t \in \mathbb{R} \cap \mathbb{D}$. Note that ℓ and m give plane curves in $\mathbb{C}$, and the curve m is determined by the periodic matrix C_0 only, and in particular, the curve $m(t)$ is automatically periodic and thus closed.

Now we can formulate the main result of this paper.

Theorem 1. *The following statements hold*:

(i) *Let C_0 and ζ be a periodic matrix and (after choosing a function h) the corresponding $\mathfrak{su}_{1,1}$-potential as defined in* (1.3) *and* (1.6)*, respectively, and let ℓ and m be the plane curves defined in* (1.7) *and* (1.8) *respectively. Then ζ gives, via the generalized Weierstrass*

representation explained in part (A) *above, a non-vertical minimal cylinder in* Nil_3 *with possibly singular points if the curves* ℓ *and* m *are closed and their signed enclosed areas are equal.*

(ii) *Let* $\tilde{\ell}$ *and* $\tilde{m}$ *be analytic closed plane curves whose signed enclosed areas are equal. Then there exist a periodic matrix* C_0 *and an* $\mathfrak{su}_{1,1}$*-potential* ζ *such that the plane curves* ℓ *and* m *in* (1.7) *and* (1.8) *are* $\tilde{\ell}$ *and* $\tilde{m}$ *respectively, and* ζ *gives a non-vertical minimal cylinder in* Nil_3 *via the generalized Weierstrass representation explained in part* (A) *above.*

(iii) *All non-vertical minimal cylinders in* Nil_3 *can be constructed from two analytic curves which satisfy the conditions in* (ii).

The proof will be given in Section 2.4 below.

Remark 1.3. One of the closed curves ℓ or m can be degenerate, i.e. for example in Figure 1, $m = 0$ while ℓ is a regular curve. On the one hand, in Example 2.10

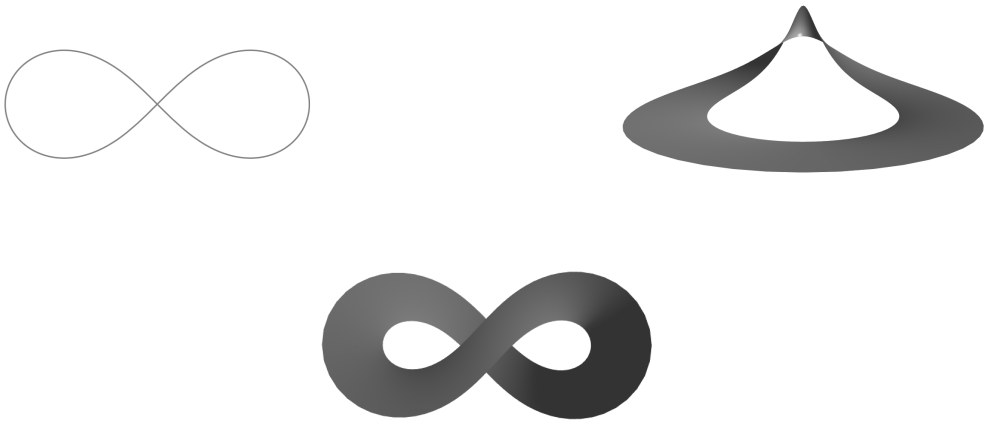


FIGURE 1. The closed plane curve ℓ (left, the lemniscate) and two views of the corresponding non-vertical minimal cylinder in Nil_3 (right, bottom) from the potential ζ stated in (2.16) with h defined by first one in (2.18) and initial frame $C_0(z) \equiv \mathrm{id}$. Note, we have $T = 0$ in this case, and the curve m is degenerate to a point 0. The figures were drawn by using [3] developed by David Brander.

2. Minimal cylinders in Nil_3

2.1. **Constructing C_0 and ζ for non-vertical minimal surfaces in Nil_3.** We show first that each non-vertical minimal cylinder in Nil_3 can be obtained from a potential ζ defined by using a matrix C_0 and a function h as discussed in the introduction.

We follow the scheme outlined in [9] and change "Mutatis Mutandis". First we recall that we have shown in Section 2.3.1 loc.cit. that after multiplication by some diagonal matrix L the extended frame F has a meromorphic extension $U = FL$. In Section 2.3.2 loc.cit. we have

discussed the extension of symmetries to the meromorphic extension. The closing condition is stated in Theorem 2.11. loc.cit. We thus have proven the results of Section 3 of [10].

2.2. **The construction of frame periodic minimal immersions in** Nil_3 **from** C_0 **and** ζ **and** h**.** We have already discussed the construction of a minimal surface in Nil_3 from the initial data C_0 and h in (B) and (C) of the introduction.

Recall, if ζ is the potential in (1.6) and let $C = C(z,\lambda)$ denote the solution to $dC = C\zeta$, $C(0,\lambda) = \mathrm{id}$. Then it is easy to see that C takes values in $\Lambda\mathrm{SU}_{1,1\sigma}$ for $z \in \mathbb{R}$, since ζ takes values in $\Lambda\mathfrak{su}_{1,1\sigma}$ for $z \in \mathbb{R}$. Our construction implies $C(z,1) = C_0(z)$, whence the first closing condition in Theorem 2.11 of [9] is satisfied, that is,

$$M(\lambda) \in \Lambda\mathrm{SU}_{1,1\sigma} \quad \text{and} \quad M(\lambda = 1) = \pm\,\mathrm{id} \tag{2.1}$$

for the monodromy matrix M defined by $C(z+p,\lambda) = M(\lambda)C(z,\lambda)$. Moreover, $C(z,\lambda)$ is the extended frame along $z \in \mathbb{R}$ up to a U_1 factor, that is $F = Ck$ for some $k \in \mathrm{U}_1$ on $z \in \mathbb{R}$. Then the extended frame F of a minimal surface in Nil_3 satisfies $F(z+p,\lambda) = M(\lambda)F(z,\lambda)k(z)$ and (2.1). Such a surface will be called a *frame periodic minimal immersion in* Nil_3.

The following theorem is an exact analogue of Theorem 5.4 in [10] for $\mathfrak{su}_{1,1}$-potentials and the proof is as well.

Theorem 2.1. *Let C_0 be given by* (1.3)*. Set $\zeta_0 = C_0^{-1}dC_0$ and define a $\mathfrak{su}_{1,1}$-potential ζ by* (1.6)*. Then ζ defines a frame periodic minimal immersion into* Nil_3*. Moreover, every frame periodic minimal immersion in* Nil_3 *can be obtained this way.*

2.3. **The remaining two closing conditions for minimal cylinders in** Nil_3**.** From Theorem 2.11 of [9] we derive that in addition to the condition of frame periodic we also need to satisfy two more closing conditions to obtain a minimal "cylinder" in Nil_3:

$$X^o(\lambda = 1) = O \quad \text{and} \quad Y^d(\lambda = 1) = O, \tag{2.2}$$

where $X = -i\lambda(\partial_\lambda M)M^{-1}$ and $Y = -\frac{1}{2}\lambda\partial_\lambda(\lambda(\partial_\lambda M)M^{-1})$, considered as functions of λ, and where the superscript "d" and "o" means "diagonal part" and "off-diagonal part" respectively. It is natural to call $M(\lambda) = \pm\,\mathrm{id}$ the *first* closing condition. Accordingly, the condition $X^o(\lambda = 1) = O$ and the condition $Y^d(\lambda = 1) = O$ will be called the *second* and the *third* closing condition, respectively.

To consider the second and third closing conditions, we next prove [16, Theorem 5.1.2] in our setting.

Proposition 2.2. *Let C be the solution of $dC = C\zeta$ with $C(0,\lambda) = \mathrm{id}$ where the 1-form ζ is given as before by* (1.6)*. Then the following formula holds*:

$$(\partial_\lambda C(z))C(z)^{-1} = \int_0^z C(t,\lambda)(\partial_\lambda \zeta(t,\lambda))C(t,\lambda)^{-1}$$

for all real z and integration along the real axis.

Proof. Consider the z-derivative of $(\partial_\lambda C)C^{-1}$:

$$\begin{aligned}\partial_z((\partial_\lambda C)C^{-1})dz &= (\partial_\lambda(\partial_z C))C^{-1}dz - (\partial_\lambda C)C^{-1}(\partial_z C)C^{-1}dz\\ &= (\partial_\lambda(C\zeta))C^{-1} - (\partial_\lambda C)C^{-1}(C\zeta)C^{-1}\\ &= C(\partial_\lambda\zeta)C^{-1}.\end{aligned}$$

Now an integration from 0 to z along the real axis yields

$$(\partial_\lambda C(z,\lambda))C(z,\lambda)^{-1} = \int_0^z C(t,\lambda)(\partial_\lambda\zeta(t,\lambda))C(t,\lambda)^{-1} + A(\lambda).$$

But since $C(0,\lambda) = \mathrm{id}$, substituting $z = 0$ above yields $A(\lambda) = 0$ for all λ. This completes the proof. □

Since $C(z+p,\lambda) = M(\lambda)C(z,\lambda)$ implies $M(\lambda) = C(p,\lambda)$, we infer

Corollary 2.3. *Let $X(\lambda) = -i\lambda(\partial_\lambda M(\lambda))M(\lambda)^{-1}$ and $Y(\lambda) = -\frac{1}{2}\lambda\partial_\lambda(\lambda(\partial_\lambda M(\lambda))M(\lambda)^{-1})$ respectively. Then the following formulas hold:*

$$X(\lambda) = -i\int_0^p C(t,\lambda)(\lambda\partial_\lambda\zeta(t,\lambda))C(t,\lambda)^{-1}, \tag{2.3}$$

$$Y(\lambda) = -\frac{i}{2}\lambda\partial_\lambda X(\lambda) = -\frac{1}{2}\lambda\partial_\lambda\left(\int_0^p C(t,\lambda)(\lambda\partial_\lambda\zeta(t,\lambda))C(t,\lambda)^{-1}\right). \tag{2.4}$$

Using Proposition 2.2 and Corollary 2.3, it is straightforward to rewrite the second and third closing conditions for $\lambda = 1$ in a simple way.

Proposition 2.4. *Let ζ be as before a $\mathfrak{su}_{1,1}$-potential as given in* (1.6)*. Moreover, let α be the integrand in* (1.7)*,*

$$\alpha(s) = a_0(s)^2(2h(s) - \kappa(s)) + b_0(s)^2(2\overline{h(s)} - \overline{\kappa(s)}). \tag{2.5}$$

Then the second closing condition $X^o = 0$ is for $\lambda = 1$ equivalent to

$$\int_0^p \alpha(t)dt = 0. \tag{2.6}$$

Moreover, the third closing condition $Y^d = 0$ is for $\lambda = 1$ equivalent to

$$\int_0^p \mathrm{Im}\left(\alpha(t)\left\{\int_0^t \overline{\alpha(s)}ds\right\}\right)dt = \int_0^p \mathrm{Im}\left(a_0(t)\overline{b_0(t)}\kappa(t)\right)dt. \tag{2.7}$$

Proof. Equation (2.6) follows from Corollary 2.3, since the integrand of $X(\lambda = 1)$ is

$$C_0(t)\left\{\lambda\partial_\lambda\zeta(t,\lambda)\right\}|_{\lambda=1}C_0(t)^{-1}$$

with $(1,2)$-entry equal to $-i\alpha(t)$. To evaluate the third (and last) closing condition we need to differentiate first of all $X(\lambda)$ for λ, and then evaluate at $\lambda = 1$.

$$\begin{aligned} 2Y(\lambda) &= -i\lambda\partial_\lambda X(\lambda) \\ &= -\int_0^p \{\lambda\partial_\lambda C(t,\lambda)\}\{\lambda\partial_\lambda\zeta(t,\lambda)\}C(t,\lambda)^{-1} \\ &\quad -\int_0^p C(t,\lambda)[\lambda\partial_\lambda\{\lambda\partial_\lambda\zeta(t,\lambda)\}]C(t,\lambda)^{-1} \\ &\quad +\int_0^p C(t,\lambda)\{\lambda\partial_\lambda\zeta(t,\lambda)\}C(t,\lambda)^{-1}\{\lambda\partial_\lambda C(t,\lambda)\}C(t,\lambda)^{-1} \end{aligned}$$

A straightforward computation shows

$$\lambda\partial_\lambda\left(\lambda\partial_\lambda\zeta(t,\lambda)\right)|_{\lambda=1} = \begin{pmatrix} 0 & \kappa(t) \\ \overline{\kappa(t)} & 0 \end{pmatrix} dt.$$

Recall that $C(t,\lambda=1) = C_0(t) = \begin{pmatrix} \frac{a_0(t)}{b_0(t)} & \frac{b_0(t)}{a_0(t)} \end{pmatrix}$ holds. Therefore the $(1,1)$-entry of the second summand can be computed at $\lambda = 1$ as

$$\int_0^p \left(a_0(t)\overline{b_0(t)}\kappa(t) - \overline{a_0(t)}b_0(t)\overline{\kappa(t)}\right) dt. \tag{2.8}$$

Moreover, the first and third summands can be computed at $\lambda = 1$ by using Proposition 2.2 as

$$\begin{aligned} &-\int_0^p \left\{\int_0^t C_0(s)(\partial_\lambda\zeta(s,\lambda))|_{\lambda=1}C_0(s)^{-1}\right\}\left\{C_0(t)(\lambda\partial_\lambda\zeta(t,\lambda))|_{\lambda=1}C_0(t)^{-1}\right\} \\ &+\int_0^p \left\{C_0(t)(\lambda\partial_\lambda\zeta(t,\lambda))|_{\lambda=1}C_0(t)^{-1}\right\}\left\{\int_0^t C_0(s)(\partial_\lambda\zeta(s,\lambda))|_{\lambda=1}C_0(s)^{-1}\right\}. \end{aligned}$$

Putting $A(t) = C_0(t)(\partial_\lambda\zeta(t,\lambda))|_{\lambda=1}C_0(t)^{-1}$ this is equivalent to

$$\int_0^p \left[A(t), \int_0^t A(s)\right]. \tag{2.9}$$

where $[P,Q] = PQ - QP$ denotes the matrix commutator. It is easy to compute the $(1,1)$-entry of (2.9) as

$$-\int_0^p \left(\alpha(t)\int_0^t \overline{\alpha(s)}ds - \overline{\alpha(t)}\int_0^t \alpha(s)ds\right) dt. \tag{2.10}$$

Putting together (2.8) and (2.10), the condition (2.7) for $Y(\lambda = 1)^d = 0$ follows. This completes the proof. □

Remark 2.5. In general the closing conditions in (2.2) can be relaxed as follows:

$$\tilde{X}^o(\lambda = 1) = 0 \quad \text{and} \quad \tilde{Y}^d(\lambda = 1) = 0, \tag{2.11}$$

where $\tilde{X} = -i\lambda(\partial_\lambda\tilde{M})\tilde{M}^{-1}$ and $\tilde{X} = -\frac{1}{2}\partial_\lambda(\lambda(\partial_\lambda\tilde{M})\tilde{M}^{-1})$ with $\tilde{M} = \{M(\lambda)\}^n$ for some $n \in \mathbb{N}$. Then the second and the third closing condition in (2.6) and (2.7) respectively can be stated by replacing the period "p" by "np", for some $n \in \mathbb{N}$.

2.4. **The proof of Theorem 1.**

Proof. We assume that the two curves l and m are closed and have the same signed area. We want to show that the closing conditions for a minimal cylinder in Nil_3 are satisfied. Recall that the first closing condition is satisfied by construction.

(i) We look at the closing conditions (2.6) and (2.7) more closely. It is evident that the second closing condition, i.e., (2.6) is equivalent to that ℓ is a closed curve with period $p > 0$. It thus remains to consider the third closing condition. But by using the relation (1.5) between κ, a_0 and b_0 and integration by parts, the third closing condition, (2.7), is easily seen to be equivalent to

$$\int_0^p \mathrm{Im}\left(\overline{\ell(t)}\ell'(t)\right) dt = \int_0^p \mathrm{Im}\left(\overline{m(t)}m'(t)\right) dt.$$

On the left-hand side let $\ell(t) = x(t) + iy(t)$ be written in terms of its real and its imaginary part. Then a straightforward computation shows that

$$\int_0^p \overline{\ell(t)}\ell'(t)dt = \frac{1}{2}\int_0^p (x(t)^2 + y(t)^2)'dt - i\int_0^p (x(t)y(t))'dt + 2i\int_0^p x(t)y'(t)dt$$

showing that the original integral expresses (up to the factor $2i$) the signed area of ℓ. The analogous argument applies to the curve m. Since we have assumed that the signed areas of ℓ and m are equal, the two integrals above are equal and thus the third closing condition is satisfied.

(ii) First we define periodic functions a_0 and b_0 by the equations:

$$\tilde{m}(z) = a_0(z)b_0(z), \quad a_0(z)\overline{a_0(\bar{z})} - b_0(z)\overline{b_0(\bar{z})} = 1. \tag{2.12}$$

Note, we first solve for functions defined on $\mathbb{R}$ and extend holomorpically, since our original functions were assumed to be real-analytic. Note that the functions a_0 and b_0 are determined up to some phase factors, i.e., $a_0e^{i\theta}$ and $b_0e^{-i\theta}$ for some real function θ. Then the periodic matrix C_0 is defined by

$$C_0(z) = \begin{pmatrix} a_0(z) & b_0(z) \\ \overline{b_0(\bar{z})} & \overline{a_0(\bar{z})} \end{pmatrix} k, \quad k = \begin{pmatrix} e^{i\theta(z)} & 0 \\ 0 & e^{-i\theta(z)} \end{pmatrix}.$$

The diagonal matrix k is just a gauge and thus the functions a_0 and b_0 are uniquely determined by (2.12). Actually, different C_0 may yield the same cylinder: If we have $\tilde{C_0} = C_0k$, with k diagonal unitary, then the Iwasawa decomposition yields a new F, namely Fk. But the Sym-formula ignores k.

Next the plane curve $\tilde{\ell}$ is real analytic and also $\tilde{\ell}'(t)$ is. Thus $\tilde{\ell}'(t)$ can be extended holomorphically around $t \in \mathbb{R}$ and we denote this curve by α. Then we define a function μ by

$$\mu(z) = \frac{1}{a_0(z)\overline{a_0(\bar{z})} + b_0(z)\overline{b_0(\bar{z})}}\left(\overline{a_0(\bar{z})}^2\alpha(z) - b_0(z)^2\overline{\alpha(\bar{z})}\right), \tag{2.13}$$

where a_0, b_0 are entries of C_0. From the construction it is easy to see that

$$\alpha(z) = a_0(z)^2\mu(z) + b_0(z)^2\overline{\mu(\bar{z})}$$

holds. Then we define h to be

$$h(z) = \frac{\kappa(z) + \mu(z)}{2},$$

where the function $\kappa(z) = \overline{a_0(\bar{z})}\partial_z b_0(z) - \partial_z \overline{a_0(\bar{z})} b_0(z)$ is given by (1.5), and define the potential ζ in (1.6) by the data C_0 and h. Then the plane curve $\tilde{\ell}$ is exactly ℓ in (1.7) and ζ clearly produces a non-vertical minimal cylinder in Nil_3.

(iii) To prove the final statement, we consider a non-vertical minimal cylinder in Nil_3. Then without loss of generality, there exists the extended frame $F \in \Lambda\mathrm{SU}_{1,1\sigma}$ such that F is periodic on $\mathbb{R}$ with period $p > 0$ and $F(z+p,\lambda) = M(\lambda)F(z)$ and the monodromy $M(\lambda)$ satisfies all closing conditions, that is, $M|_{\lambda=1} = \pm\,\mathrm{id}$, $X^o|_{\lambda=1} = O$ and $Y^d|_{\lambda=1} = O$. We have shown in Section 2.3.1 in [9] that there exists some diagonal matrix L such that $U = FL$ has a meromorphic extension, see e.g., Theorem 3.2 in [10]. Then the Maurer-Cartan form of U is a potential $\zeta = U^{-1}dU$ of the desired type, and the two plane curves ℓ and m given by C_0 and ζ satisfy the properties in (ii). □

2.5. **Examples of non-vertical minimal cylinders in** Nil_3 **constructed from matrices** C_0**.** In this subsection, we illustrate the theory presented in the previous section by constructing non-vertical minimal cylinders in terms of simple specific periodic matrices C_0 and specific functions h.

Thus we start by considering a diagonal matrix

$$C_0(z) = \begin{pmatrix} e^{icz} & 0 \\ 0 & e^{-icz} \end{pmatrix}, \quad c \in \mathbb{R}. \tag{2.14}$$

Note that $a_0(z) = e^{icz}$ and $b_0(z) = 0$ in $C_0(z)$. If $c \neq 0$, then we have the minimum period p of $C_0(z)$ given by $\pi/|c| > 0$, and if $c = 0$, then we have any period. We compute

$$\zeta_0(z) = C_0(z)^{-1}\partial_z C_0(z)dz = \begin{pmatrix} ic & 0 \\ 0 & -ic \end{pmatrix} dz. \tag{2.15}$$

We note that in our case we have $\kappa(z) \equiv 0$ (the off-diagonal part of ζ_0 is identically zero). For the construction of ζ we still need to choose a periodic function $h(z)$ of period p, which must be a positive integer multiple of the minimum period of C_0. Then we obtain

$$\zeta(z,\lambda) = \begin{pmatrix} ic & \lambda^{-1}h(z) - \lambda\overline{h(\bar{z})} \\ -\lambda^{-1}\overline{h(\bar{z})} + \lambda h(z) & -ic \end{pmatrix} dz. \tag{2.16}$$

From Corollary 2.3 we derive that $X(\lambda = 1)$ is off-diagonal and the integrals of both entries are complex conjugate. Since the first closing condition is always satisfied by our construction we only need to consider the second and the third closing condition. For this we compute the functions α and m in (2.5) and (1.8). We obtain

$$\alpha(t) = 2e^{2ict}h(t) \quad \text{and} \quad m(t) = 0. \tag{2.17}$$

Example 2.6 (The case $c = 0$). We construct non-vertical minimal cylinders from the constant frame $C_0(z) \equiv \mathrm{id}$. For this we only need to choose a periodic holomorphic function h.

We present two examples, by choosing

$$h(z) = \frac{1 + i\sin z}{(i + \sin z)^2}, \quad \text{or} \quad h(z) = \cos(z) - i\sin(3z). \tag{2.18}$$

Note that we choose the period p for both choices of h as $p = 2\pi$. For our first choice of h the curve $\ell(t) = \int_0^t 2h(s)ds$ is the lemniscate, that is,

$$\ell(t) = -\frac{2\cos z}{i + \sin z},$$

and it is clear that the signed area of ℓ is zero. Moreover, the resulting non-vertical minimal cylinder does not have any branch point on $\mathbb{R}$, since $h^2(z)$+"positive function" is the conformal factor of the metric and h in (2.18) does vanish on $\mathbb{R}$. This example is shown in Figure 1.

For our second choice of h the curve $\ell(t)$ can be computed as

$$\ell(t) = \frac{2i}{3}\cos 3t + 2\sin t,$$

and the signed area of ℓ is zero. The plane curve ℓ and the minimal cylinder given by our second choice of h as stated in (2.18) are shown in Figure 2.



FIGURE 2. A closed plane curve ℓ (left) and the corresponding non-vertical minimal cylinder in Nil_3 (right) from the $\mathfrak{su}_{1,1}$-potential ζ in (2.16) with the second choice of h in (2.18) and identity initial condition. The signed area enclosed by ℓ is zero in this case. The figure is constructed by [3].

Example 2.7 (The case $c = 1$). We construct a on-vertical minimal cylinder in Nil_3 from the simple periodic frame $C_0(z)$ as stated in (2.14) for $c = 1$. We choose the periodic holomorphic function h

$$h(z) = \exp(-i\pi/4) + \sqrt{6}\cos(4z). \tag{2.19}$$

Note that the minimal period is $\frac{\pi}{2}$, but we choose the period $p = \pi$, and the plane curve ℓ is given by

$$\ell(t) = \frac{1}{6}e^{-2it}\left(3\sqrt{6}i - \sqrt{6}ie^{8it} - 6\sqrt{i}e^{4it}\right).$$

It is easy to see that ℓ is closed curve whose signed area is zero for the period p. Therefore the resulting minimal surface in Nil_3 is a cylinder. Moreover, the resulting non-vertical minimal cylinder does not have any branch point on $\mathbb{R}$, since $h^2(z)$+"positive function" is the conformal factor of the metric and h in (2.19) does vanish on $\mathbb{R}$. The corresponding curve ℓ and the corresponding surface are shown in Figure 3.

Remark 2.8. The non-vertical minimal cylinders in Examples 2.6 and 2.7 have a branch point at $z_0 \in \mathbb{C} \setminus \mathbb{R}$ such that $h(z_0) = 0$, where h is a function given in (2.18).



FIGURE 3. The closed plane curve ℓ (left) and the corresponding non-vertical minimal cylinder in Nil_3 (right) constructed from the $\mathfrak{su}_{1,1}$-potential ζ stated in (2.16) with h as defined in (2.19) and $C_0(z)$ defined as in (2.14) for $c = 1$. The signed area enclosed by ℓ is zero in this case. The right figure is constructed by [3].

2.6. **An example of non-vertical minimal cylinders in Nil_3 constructed from matrices C_0 with full entries.**

Example 2.9. We consider a periodic matrix C_0 with full entries:

$$C_0(z) = \begin{pmatrix} \cosh(\sin z) & \sinh(\sin z) \\ \sinh(\sin z) & \cosh(\sin z) \end{pmatrix}. \tag{2.20}$$

The minimum period p of $C_0(z)$ is obviously $p = 2\pi$. Then we compute

$$\zeta_0(z) = C_0(z)^{-1}\partial_z C_0(z)dz = \begin{pmatrix} 0 & \cos z \\ \cos z & 0 \end{pmatrix} dz, \tag{2.21}$$

thus $\kappa(z) = \cos z$. It is also easy to see that the signed are of m in (1.7) vanishes since a_0 and b_0 are real on $\mathbb{R}$. Let us choose the complex function h defined by

$$h(z) = \frac{1}{2}\cos z + (\cos z)\operatorname{sech}(2\sin z) - i\sin(3z), \tag{2.22}$$

and define an $\mathfrak{su}_{1,1}$-potential ζ as

$$\zeta(z,\lambda) = \begin{pmatrix} 0 & \lambda^{-1}h(z) + \lambda(\kappa(z) - h(z)) \\ \lambda^{-1}(\overline{\kappa(\bar{z})} - \overline{h(\bar{z})}) + \lambda\overline{h(\bar{z})} & 0 \end{pmatrix} dz. \tag{2.23}$$

Since the first closing condition is always satisfied by our construction we only need to consider the second and the third closing condition. For this we compute the function α and the signed area of m in (2.5) and (1.8). It is easy to see that the signed area of m vanishes, since m is real on $\mathbb{R}$. On the one hand, the function α in (2.5) can be computed as

$$\alpha(t) = 2(\cos t - i\sin 3t), \tag{2.24}$$

which is the same function in Example 2.6. Thus the second and third closing conditions can be satisfied.

Finally we will give an example of non-vertical minimal cylinders in Nil_3 constructed from the non-diagonal matrix C_0.

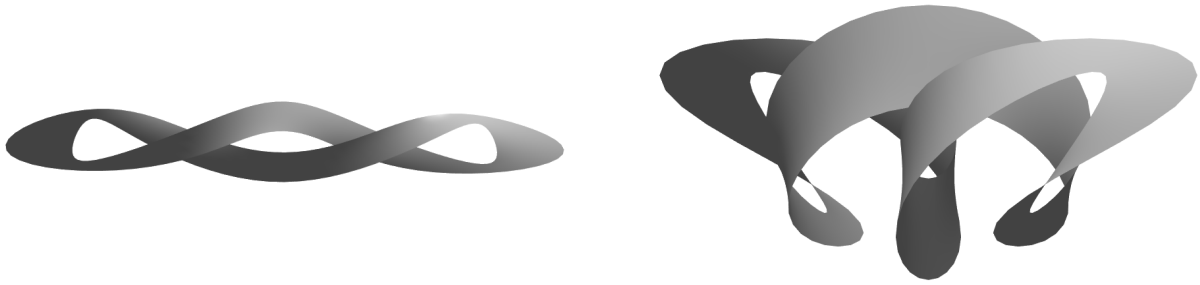


FIGURE 4. Two views of the same non-vertical minimal cylinders in Nil_3 by the $\mathfrak{su}_{1,1}$-potential in (2.23) with h in (2.22) and identity initial condition. The figures are constructed by [3].

Example 2.10. We consider a periodic matrix C_0 with full entries:

$$C_0(z) = \begin{pmatrix} e^{-iz}\cosh(\sin z) & \sinh(\sin z) \\ \sinh(\sin z) & e^{iz}\cosh(\sin z) \end{pmatrix}. \tag{2.25}$$

The minimum period p of $C_0(z)$ is obviously $p = 2\pi$. Then we compute

$$\begin{aligned} \zeta_0(z) &= C_0(z)^{-1}\partial_z C_0(z)dz \\ &= \begin{pmatrix} -i\left\{\cosh(\sin z)\right\}^2 & e^{iz}\left\{\cos z - \frac{i}{2}\sinh(2\sin z)\right\} \\ e^{-iz}\left\{\cos z + \frac{i}{2}\sinh(2\sin z)\right\} & i\left\{\cosh(\sin z)\right\}^2 \end{pmatrix} dz, \end{aligned}$$

thus

$$\nu(z) = -i\left\{\cosh(\sin z)\right\}^2, \quad \kappa(z) = e^{iz}\left\{\cos z - \frac{i}{2}\sinh(2\sin z)\right\}.$$

It is also easy to see that the signed area T of $m(t)$ in (1.8) can be computed as

$$T = -\frac{1}{8}\pi(I_0(4) - 1) \fallingdotseq -4.04556,$$

where $I_\nu(x)$ is the modified Bessel function of the first kind, see Chapter III [2] for definition. Let us choose the complex function h defined by

$$h(z) = -\frac{i}{4}\left\{2c_1 + 2i\cos z + \sinh(2\sin z)\right\}, \quad c_1 = \sqrt{|T|/\pi} \tag{2.26}$$

and define an $\mathfrak{su}_{1,1}$-potential ζ as

$$\zeta(z,\lambda) = \begin{pmatrix} \dfrac{\nu(z)}{\lambda^{-1}(\overline{\kappa(\bar{z})} - \overline{h(\bar{z})}) + \lambda\overline{h(\bar{z})}} & \lambda^{-1}h(z) + \lambda(\kappa(z) - h(z)) \\ & -\nu(z) \end{pmatrix} dz. \tag{2.27}$$

Since the first closing condition is always satisfied by our construction we only need to consider the second and the third closing condition. For this we compute the function α (2.5). As we have shown that the signed are of m is zero. On the one hand, the function α in (2.5) can be computed as

$$\alpha(t) = -ic_1 e^{-it}, \tag{2.28}$$

which is the derivative of the round circle about origin $\ell(t) = c_1 e^{-it}$ with radius c_1. Then the signed area of ℓ and m are the same, and thus the second and third conditions can be satisfied.

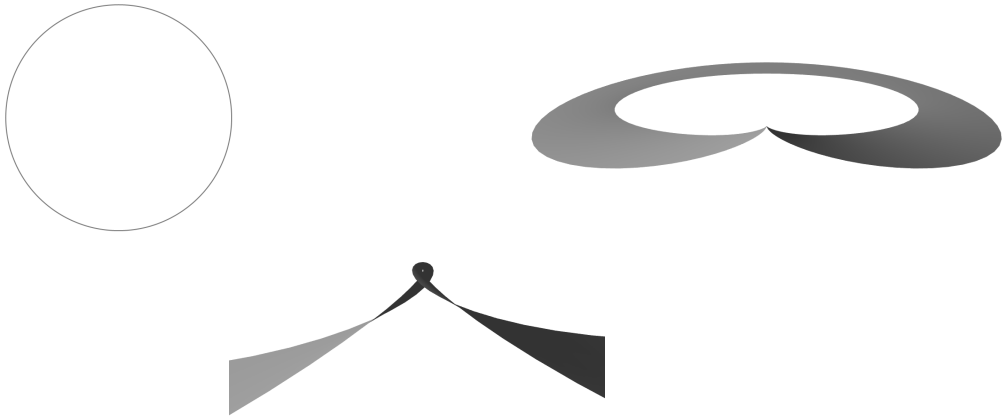


FIGURE 5. The plane curve ℓ (left, the round circle) and the corresponding non-vertical minimal cylinder in Nil_3 (right) by the $\mathfrak{su}_{1,1}$-potential in (2.27) with h in (2.26) and identity initial condition. The bottom figure shows a close-up view of the right one. The figures are constructed by [3].

Remark 2.11. The Abresch-Rosenberg differential of these examples are $B(z) = h(z)\overline{h(\bar{z})}dz^2$, and $h(z)$ is given in (2.18), (2.19) or (2.22).

3. Spacelike CMC cylinders in $\mathbb{L}_3$

The loop group technique used in [8], [9] and in this paper for the construction of minimal surfaces in Nil_3 uses the close relationship with spacelike CMC surfaces in Minkowski three-space $\mathbb{L}_3$ as manifestly expressed by the closely related, but different, Sym formulas.

Spacelike CMC surfaces in $\mathbb{L}_3$ have been investigated in some detail in [4]. We trust that the present reader will be able to translate without problems notation and results from there into our setting. We therefore do not include any special notation nor any special results about this surface class into this note.

In Theorem 2.1, we have shown that the $\mathfrak{su}_{1,1}$-potentials considered there produce frame periodic minimal immersions in Nil_3 and vice versa. It is obvious that the periodic matrix C_0 stated in (1.3) also produces a spacelike CMC-immersion in $\mathbb{L}_3$ by the formula defining $f_{\mathbb{L}_3}$ in (1.1) inserting the extended frame F, and thus we have the following.

Theorem 3.1. *Let C_0 be given by* (1.3)*. Set $\zeta_0 = C_0^{-1}dC_0$ and define a $\mathfrak{su}_{1,1}$-potential ζ by* (1.6)*, where h is an arbitrary holomorphic function along $\mathbb{R} \cap \mathbb{D}$. Then ζ defines a frame periodic spacelike CMC-immersion into $\mathbb{L}_3$. Moreover, every frame periodic CMC-immersion in $\mathbb{L}_3$ can be obtained this way.*

It is known (see e.g. [4]) that the closing conditions for $f_{\mathbb{L}_3}$ at $\lambda = 1$ can be phrased as follows:

$$M(\lambda = 1) = \pm \mathrm{id} \quad \text{and} \quad (\partial_\lambda M)(\lambda = 1) = 0. \tag{3.1}$$

Since $X(\lambda)$ is defined by $-i\lambda(\partial_\lambda M)M^{-1}$, the above closing conditions can be rephrased as

$$M(\lambda = 1) = \pm \mathrm{id} \quad \text{and} \quad X(\lambda = 1) = 0. \tag{3.2}$$

Since for the frame periodic spacelike CMC-immersions we already have $M(\lambda) \in \Lambda \mathrm{SU}_{1,1\sigma}$ and $M(\lambda = 1) = \pm \mathrm{id}$, and thus only the second closing condition $X(\lambda = 1) = 0$ remains. Therefore we have the following corollary.

Corollary 3.2. *Let ζ be an $\mathfrak{su}_{1,1}$-potential as stated in* (1.6)*. Define functions α and β by the equations*

$$\alpha(t) = a_0(t)^2(2h(t) - \kappa(t)) + b_0(t)^2(2\overline{h(t)} - \overline{k(t)}), \tag{3.3}$$

$$\beta(t) = \mathrm{Im}\left\{a_0(t)\overline{b_0(t)}(2h(t) - \kappa(t))\right\}, \tag{3.4}$$

where a_0, b_0, h and κ are the functions occurring in (1.3) *and* (1.5)*, respectively. Then ζ defines a spacelike CMC-cylinder in $\mathbb{L}_3$ if and only if*

$$\int_0^p \alpha(t)dt = \int_0^p \beta(t)dt = 0 \tag{3.5}$$

hold.

Remark 3.3. Note that the function α stated in (3.3) above is the same function as defined in 2.5.

It is easy to see that for the potentials ζ occurring in Examples 2.6 and 2.7, the closing conditions stated in (3.5) are satisfied.

Let us rephrase the closing conditions in (3.5) more geometrically. First let us consider the plane curve ℓ in (1.7) as before: then $\ell(t) = \int_0^t \alpha(s)ds$ and

$$2h(z) - \kappa(z) = \frac{1}{a_0(z)\overline{a_0(\bar{z})} + b_0(z)\overline{b_0(\bar{z})}} \left(\overline{a_0(\bar{z})}^2\alpha(z) - b_0(z)^2\overline{\alpha(\bar{z})}\right),$$

hold. We now simplify the condition $\int_0^p \beta(t)dt = 0$. After plugging $2h - \kappa$ into β a straightforward computation using the relation $a_0(z)\overline{a_0(\bar{z})} - b_0(z)\overline{b_0(\bar{z})} = 1$, yields

$$\beta(z) = \mathrm{Im}\left(\overline{m(\bar{z})}\alpha(z)\right),$$

where $m(z) = a_0(z)b_0(z)$. Therefore $\int_0^p \beta(t)dt = 0$ is equivalent to that

$$\mathrm{Im}\langle \ell', m\rangle = 0,$$

i.e., the imaginary part of the L^2-inner product of m with α vanishes. Similar to Theorem 1, we have the following.

Theorem 3.4. *Let ζ be an $\mathfrak{su}_{1,1}$-potential as stated in* (1.6)*, and let ℓ and m be the curves in $\mathbb{C}$ as defined by* (1.7) *and $m = a_0b_0$ with the entries of C_0 in* (1.3)*, respectively. Then ζ, via the generalized Weierstrass representation, gives a spacelike CMC-cylinder in $\mathbb{L}_3$ if ℓ is closed and the imaginary part of the L^2-inner product of ℓ' and m vanishes.*

Conversely for closed analytic plane curves $\tilde{\ell}$ and $\tilde{m}$ such that the imaginary part of the L^2-inner product of $\tilde{\ell}'$ and $\tilde{m}$ vanishes, there exist a periodic matrix C_0 and an $\mathfrak{su}_{1,1}$-potential ζ such that the curves ℓ in (1.7) *and $m = a_0 b_0$ are $\tilde{\ell}$ and $\tilde{m}$, respectively and the generalized Weierstrass representation yields a spacelike CMC-cylinder in $\mathbb{L}_3$. Moreover, all spacelike CMC-cylinders in $\mathbb{L}_3$ can be constructed in this way.*

Example 3.5 (Two spacelike CMC cylinders -Two minimal cylinders in Nil_3 revisited). We will illustrate our results for spacelike CMC surfaces by reusing the data we had chosen in the Nil_3 case, namely two potentials for a diagonal periodic matrix C_0, with diagonal entries e^{icz} and e^{-icz}, and discuss the cases $c = 1$ and $c = 0$.

Let ζ be an $\mathfrak{su}_{1,1}$-potential as stated in (2.16) with C_0 as just above. As we have already seen that in Section 2.5, from Corollary 2.3, $X(\lambda = 1)$ is off-diagonal, and thus the second condition $\int_0^p \beta(t)dt = 0$ in (3.5) holds anyway. As before we choose a periodic holomorphic function h as

$$h(z) = \cos(z) - i\sin(3z) \quad \text{for} \quad c = 0, \tag{3.6}$$

$$h(z) = \exp(-i\pi/4) + \sqrt{6}\cos(4z) \quad \text{for} \quad c = 1. \tag{3.7}$$

Then the first condition $\int_0^p \alpha(t)dt = 0$ in (3.5) also holds since we have shown them in Examples 2.6 and 2.7.



FIGURE 6. A spacelike CMC-cylinders in $\mathbb{L}_3$ from the $\mathfrak{su}_{1,1}$-potentials ζ in (2.16) with h in (3.6) (the left figure) or (3.6) (the right figure) with identity initial condition. The figure is constructed by [3].

APPENDIX A. BASIC RESULTS

In this appendix we will collect first basic definitions and will then present some results enabling us to use holomorphic potentials in place of meromorphic potentials.

A.1. **Notation and definitions.** We first define the twisted $\mathrm{SL}_2\mathbb{C}$ loop group as a space of continuous maps from $\mathbb{S}^1$ to the Lie group $\mathrm{SL}_2\mathbb{C}$, that is, $\Lambda\mathrm{SL}_2\mathbb{C}_\sigma = \{g : \mathbb{S}^1 \to \mathrm{SL}_2\mathbb{C} \mid g(-\lambda) = \sigma g(\lambda)\}$, where $\sigma = \mathrm{Ad}(\sigma_3)$. We restrict our attention to loops in $\Lambda\mathrm{SL}_2\mathbb{C}_\sigma$ such that the associate Fourier series of the loops are absolutely convergent. Such loops determine a Banach algebra, the so-called *Wiener algebra*, and it induces a topology on $\Lambda\mathrm{SL}_2\mathbb{C}_\sigma$, the so-called *Wiener topology*. From now on, we consider only $\Lambda\mathrm{SL}_2\mathbb{C}_\sigma$ equipped with the Wiener topology.

Let $D^{\pm}$ denote respective the inside of unit disk and the union of outside of the unit disk and infinity. We define *plus* and *minus* loop subgroups of $\Lambda\mathrm{SL}_2\mathbb{C}_\sigma$; $\Lambda^{\pm}\mathrm{SL}_2\mathbb{C}_\sigma = \{g \in \Lambda\mathrm{SL}_2\mathbb{C}_\sigma \mid g \text{ can be extended holomorphically to } D^{\pm}\}$. By $\Lambda^{+}_{*}\mathrm{SL}_2\mathbb{C}_\sigma$ we denote the subgroup of elements of $\Lambda^{+}\mathrm{SL}_2\mathbb{C}_\sigma$ which take the value identity at zero. Similarly, by $\Lambda^{-}_{*}\mathrm{SL}_2\mathbb{C}_\sigma$ we denote the subgroup of elements of $\Lambda^{-}\mathrm{SL}_2\mathbb{C}_\sigma$ which take the value identity at infinity.

We also define the $\mathrm{SU}_{1,1}$-loop group as follows:

$$\Lambda\mathrm{SU}_{1,1\sigma} = \left\{ g \in \Lambda\mathrm{SL}_2\mathbb{C}_\sigma \mid \sigma_3 \overline{g(1/\bar{\lambda})}^{t-1} \sigma_3 = g(\lambda) \right\}.$$

It will be convenient to use $\tau(g)(\lambda) = \sigma_3 \overline{g(1/\bar{\lambda})}^{t-1} \sigma_3$ for $\lambda \in S^1$. For all geometric quantities, we can assume $\lambda \in \mathbb{C}^*$.

For potentials $\zeta = \hat{\zeta} dz$ with Lie algebra elements $\hat{\zeta} \in \Lambda\mathfrak{sl}_2\mathbb{C}_\sigma$ we say that ζ is *an* $\mathfrak{su}_{1,1}$*-potential* if $\tau(\hat{\zeta})(\lambda) = -\hat{\zeta}(\lambda)$ holds.

A.2. **Holomorphic $C(z,\lambda)$.** We should probably be a bit more careful about where from we choose our variables: at one hand it is convenient to choose $\mathbb{D}$ as complex plane or unit disk, since they are invariant under complex conjugation. But since we consider already very early periodic functions with real period, we actually are interested in strips $\mathbb{S}$. There are three types of strips: the whole complex plane, the upper half-plane and a strip of finite width. In the first and the last case we can assume w.l.g. that the strip contains the real line "in the middle". Meaning the real line for the case of $\mathbb{S} = \mathbb{C}$, and the real line for a strip of type $-c_0 < \mathrm{Im}(z) < c_0$. For the third case, the upper half-plane $\mathbb{H} = \{z \mid \mathrm{Im}(z) > 0\}$, it is perhaps best to let $\mathbb{Z}$ translate parallel to the real axis and to choose the base point to be i, instead of moving the upper-half plane down so that the new domain covers the actual real line.

Proposition A.1. *Let $f : \mathcal{C} \to \mathrm{Nil}_3$ be a minimal (immersed) cylinder in Nil_3 with universal cover $\mathbb{S}$. Then there exists a maximal minimal (immersed) cylinder in Nil_3 prolonging $\mathbb{S}$.*

For the notion of "prolongable" see [17, p.207].

Proof. Clearly, if $\mathbb{S} = \mathbb{C}$, no extension is possible and the given cylinder already is maximal. If $\mathbb{S}$ is a finite strip with $\mathbb{R}$ in the middle, we consider all extensions, realized by finite strips with $\mathbb{R}$ in the middle. Then by Zorn's Lemma there exists a maximal object. If $\mathbb{S}$ is the upper half plane, then any extension has a universal cover which can be realized as a "shifted down" upper half plane. Now a procedure as in the last case yields a maximal element. □

A.3. **Holomorphic $\mathfrak{su}_{1,1}$-potentials.** For this we recall that the extended coordinate frame has a meromorphic extension. Let's have a closer look: The right upper corner of the Maurer-Cartan form of the coordinate frame F is of the form $-\lambda^{-1} e^{u/2} - \lambda \bar{B} e^{-u/2}$, where u is the metric exponent and B the Hopf differential (see [8]). Since F has a meromorphic extension to $\mathbb{D} \times \mathbb{D}$, also the expression above has such an extension. Since F is real-analytic, also $e^{u/2}$ is real analytic. Therefore there exists an open subset $\mathfrak{D}_0$ of $\mathbb{D} \times \mathbb{D}$ containing $\mathbb{D}^\sharp = \{(z, \bar{z}); z \in \mathbb{D}\}$ to which F extends holomorphically. We thus obtain:

Proposition A.2. *There exists an open subset $\mathfrak{D}_0$ of $\mathbb{D}\times\mathbb{D}$ containing $\mathbb{D}^\sharp=\{(z,\bar z);z\in\mathbb{D}\}$ to which F extends holomorphically in (z,w). In particular, also ζ extends holomorphically in (z,w) to $\mathfrak{D}_0$.*

Remark A.3.

(1) If one starts from some potential like ζ, then the maximal minimally immersed cylinder in Nil_3 is defined on the largest strip on which the $\mathrm{SU}_{1,1}$-Iwasawa decomposition is real analytic.
(2) Singularities along the boundary of such a maximal strip indicate several different geometric features of the minimal cylinder.

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