Doctoral Thesis

Associated groups of involutive quandles and their second quandle homology

(対合的カンドルの付随群とその２次カンドルホモロジー)

Aoi Hasegawa

Department of Mathematics, Graduate School of Science,
Hokkaido University

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CONTENTS

1. Introduction

1.1. **Quandles.** A *quandle* is a nonempty set Q equipped with a binary operation $\triangleleft\colon Q \times Q \to Q$, $(x,y) \mapsto x \triangleleft y$ which satisfies the following three conditions:

(1) $x \triangleleft x = x$ $(x \in Q)$,
(2) for all $y \in Q$, the map $\bullet \triangleleft y : Q \to Q, x \mapsto x \triangleleft y$ is bijective,
(3) $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$ $(x,y,z \in Q)$.

Quandles are algebraic structures whose axiomatization comes from Reidemeister moves in knot theory. Since a group G equipped with the conjugation operation $g \triangleleft h := h^{-1}gh$ is a quandle (which is called a conjugacy quandle of G and denoted by $\mathrm{Conj}(G)$), quandles are also algebraic structures obtained by axiomatizing and generalizing the conjugation operations of groups. The term "quandle" was coined by Joyce [15] in 1982. Quandle have been applied to the study of knots and links, set-theoretical solutions of the Yang-Baxter equation, symmetric spaces and so on (see, e.g., Joyce [15], Kubo-Nagashiki-Okuda-Tamaru [17], and Lebed-Vendramin [18]). For any quandle Q, one can define a group $\mathrm{As}(Q)$ called the *associated group* of Q, which is defined by the presentation

$$\mathrm{As}(Q) := \langle e_x \ (x \in Q) \mid {e_y}^{-1} e_x e_y = e_{x \triangleleft y} \ (x, y \in Q) \rangle.$$

The functor from the category of quandles to the category of groups given by $Q \mapsto \mathrm{As}(Q)$ is left adjoint to the functor from the category of groups to the category of quandles given by $G \mapsto \mathrm{Conj}(G)$. In other words, there is a natural bijection

$$\mathrm{Hom}_{\mathrm{groups}}(\mathrm{As}(Q), G) \cong \mathrm{Hom}_{\mathrm{quandles}}(Q, \mathrm{Conj}(G)).$$

In this way, associated groups play an important role in understanding quandles themselves. However, since $\mathrm{As}(Q)$ is defined by presentations, it is difficult to investigate the structures of $\mathrm{As}(Q)$.

1.2. **Quandle homologies.** In 1995, Fenn, Rourke and Sanderson introduced a word "trunks" in [10] leading to the homology theory of racks (rack is a nonempty set equipped with a binary operation satisfying (2) and (3)). A modification to quandle homology theory was given in [6, 7] to define knot invariants in a state-sum form. In 2007, Eisermann posted a preprint on a way of calculating the second quandle homology (published as [9] in 2014). However, calculations of quandle homologies are difficult in general even if formulas for calculating quandle homologies are given.

1.3. **The purpose of the thesis.** This thesis consists of two parts. In the first half, inspired by Akita-Hasegawa-Tanno [2], we will introduce a quotient group $\mathrm{As}_f(Q)$ of $\mathrm{As}(Q)$ defined by a presentation for an involutive quandle Q. We will show that our $\mathrm{As}_f(Q)$ satisfies the following remarkable properties:

(1) $\mathrm{As}_f(Q)$ is of finite order whenever Q is finite (Proposition 3.3).
(2) The projection from $\mathrm{As}(Q)$ to $\mathrm{As}_f(Q)$ yields an isomorphism from the commutator subgroup of $\mathrm{As}(Q)$ to the commutator subgroup of $\mathrm{As}_f(Q)$ (Theorem 3.8).

(3) The commutative square

$$\begin{array}{ccc} \mathrm{As}(Q) & \xrightarrow{ab_{\mathrm{As}(Q)}} & \mathrm{As}(Q)_{ab} \\ \big\downarrow \pi & & \big\downarrow \pi_{ab} \\ \mathrm{As}_f(Q) & \xrightarrow{ab_{\mathrm{As}_f(Q)}} & \mathrm{As}_f(Q)_{ab} \end{array}$$

is a pullback, where ab stands for the abelianization, and the induced square of Eilenberg-MacLane spaces

$$\begin{array}{ccc} K(\mathrm{As}(Q),1) & \longrightarrow & K(\mathrm{As}(Q)_{ab},1) \\ \big\downarrow & & \big\downarrow \\ K(\mathrm{As}_f(Q),1) & \longrightarrow & K(\mathrm{As}_f(Q)_{ab},1) \end{array}$$

is a homotopy pullback (Theorem 3.10).

(4) For a connected involutive quandle Q and $x_0 \in Q$, the second quandle homology of Q is isomorphic to the abelianization of the intersection of the stabilizer of x_0 with respect to the $\mathrm{As}_f(Q)$-action on Q and the commutator subgroup of $\mathrm{As}_f(Q)$ (Theorem 4.7).

In the second half, as applications of (4), we calculate the second quandle homology for certain Coxeter quandles (Coxeter quandles will be defined in section 2). For the Coxeter quandle Q associated with a Coxeter group W, the group $\mathrm{As}_f(Q)$ is isomorphic to W (Proposition 5.8). For certain Coxeter quandles, the following property holds:

(5) Let W be a Coxeter group whose odd Coxeter graph is a tree and Q_W be its Coxeter quandle, $x_0 \in Q_W$. Then, the second quandle homology of Q_W is isomorphic to the abelianization of the intersection of the centralizer of x_0 and the commutator subgroup of W (Theorem 5.10).

Let (W,S) be a Coxeter system whose odd Coxeter graph is a tree, and $x_0 \in S$. Then, the centralizer of x_0 is a Coxeter group $W_{\Gamma(x_0)\cup\{\pm\alpha_{x_0}\}}$ which is a subgroup of W and is obtained from the root subsystem $\Gamma(x_0)\cup\{\pm\alpha_{x_0}\}$, where α_{x_0} is a root such that x_0 is a reflection of α_{x_0} and $\Gamma(x_0)$ is the set of roots orthogonal to α_{x_0} (Theorem 5.4). Let $(W_{\Gamma(x_0)\cup\{\pm\alpha_{x_0}\}}, S' = \{s'_0, s_1, \ldots, s'_n\})$ be a Coxeter system. Then, the intersection of the centralizer of x_0 and the commutator subgroup of W is isomorphic to a group generated by $R = \{r_1, r_2, \ldots, r_n\}$ with the relators $r_i^{m(s'_0,s'_i)}$ and $(r_i^{-1}r_j)^{m(s'_i,s'_j)}$ $(1 \le i < j \le n)$ (the special case of Theorem 5.6). Therefore we can calculate the second quandle homology of Q_W. In particular, for certain types of Coxeter groups, we calculated the second quandle homologies:

(6) If W is a Coxeter group of type A_n $(n \ge 1)$, D_n $(n \ge 4)$, E_6, E_7, E_8, $I_2(n)$ $(n \ge 3$ and n is odd$)$, H_4 or H_3, then the results of the calculation of the second quandle homology $H_2(Q_W)$ are shown in the Table 1.1 (Theorem 6.1).

Type	$A_n (n \geq 1)$	$D_n (n \geq 4)$	E_6	E_7	E_8	$I_2(n) (n \geq 3)$	H_4	H_3
$H_2(Q)$	$0\ (n \leq 2)$ $\mathbb{Z}_2\ (n \geq 3)$	$\mathbb{Z}_2^3\ (n = 4)$ $\mathbb{Z}_2^2\ (n \geq 5)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0 (n is odd)	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$

TABLE 1.1. The results of the calculation of $H_2(Q)$ in case Q is a Coxeter quandle Q_W.

1.4. **Outline of this thesis.** In section 2, we define quandles and $\mathrm{As}(Q)$, and summarize properties of $\mathrm{As}(Q)$. In section 3, the definition of a group $\mathrm{As}_f(Q)$ and the proof of (1), (2) and (3) are given. In section 4, quandle homology is defined, and the proof of (4) is given. In section 5, we recall the definition of Coxeter groups and their properties. Also, we give a proof of (5). In section 6, we introduce symbols used from section 7 to section 14 and list results of the second quandle homology. From section 7 to section 14, the result of (6) are given.

1.5. **Notation.** We denote by $\mathbb{Z}$, $\mathbb{Z}_{\geq 1}$, and $\mathbb{H}$ the set of integers, the set of integers greater than or equal to 1, and the set of quaternions respectively. For $n \in \mathbb{Z}_{\geq 1}$, we denote the n-dimensional Euclidean space by $\mathbb{R}^n$. For a set X, we denote by $|X|$ the cardinality of X. For a group G, we define $G_{ab} := G/[G,G]$ as the abelianization of G and $ab_G \colon G \to G_{ab}$ as the natural projection, and write $[g] := g[G,G] \in G_{ab}$ for $g \in G$. We denote the direct sum of $|X|$ copies of a group G by $G^{\oplus X}$. For a quandle Q and $x, y \in Q$, let $(\bullet \triangleleft y)^{-1}$ be the inverse map of $\bullet \triangleleft y$. Then, $x \triangleleft^{-1} y$ is defined by $(\bullet \triangleleft y)^{-1}(x)$. Moreover, for any integer N,

$$x \triangleleft^N y := \begin{cases} (\cdots(x \underbrace{\triangleleft y) \triangleleft y \cdots) \triangleleft y}_{N} & (N > 0) \\ x & (N = 0) \\ (\cdots(x \underbrace{\triangleleft^{-1} y) \triangleleft^{-1} y \cdots) \triangleleft^{-1} y}_{N} & (N < 0). \end{cases}$$

Part 1. **Quandles**

2. Quandles and its associated groups

We recall the definition of quandles. A *quandle* is a nonempty set Q equipped with a binary operation $\triangleleft$: $Q \times Q \to Q$, $(x, y) \mapsto x \triangleleft y$ which satisfies the following three conditions:

(1) $x \triangleleft x = x$ $(x \in Q)$,
(2) for all $y \in Q$, the map $\bullet \triangleleft y : Q \to Q, x \mapsto x \triangleleft y$ is bijective,
(3) $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$ $(x, y, z \in Q)$.

A quandle Q is said to be *involutive* if $x \triangleleft^2 y = x$ for all x, $y \in Q$. A subset A of a quandle Q with a quandle operation $\triangleleft$ is called a *subquandle* (of Q) if A also forms a quandle under $\triangleleft$.

Example 2.1 (Trivial quandle)**.** For any nonempty set Q, Q is an involutive quandle with the operation $x \triangleleft y = x$ $(x, y \in Q)$ and is called the *trivial quandle*.

Example 2.2 (Dihedral quandle)**.** Let $R_n = \mathbb{Z}/m\mathbb{Z}$. By the following operation, R_n is an involutive quandle and is called a *dihedral quandle*:

$$x \triangleleft y := 2y - x \ (x, y \in R_n).$$

Example 2.3 (Alexander quandle)**.** Let $\mathbb{Z}[T^{\pm 1}]$ be the Laurent polynomial ring. Then, any $\mathbb{Z}[T^{\pm 1}]$-module M is made into a quandle by the following operation, and is called an *Alexander quandle*:

$$x \triangleleft y := y + T(x - y) \ (x, y \in Q).$$

If $T^2(x) = x$ for all $x \in M$, then M is an involutive quandle.

Example 2.4 (Generalized Alexander quandle)**.** Let let ϕ be an automorphism from a group G to itself. G is a quandle with the following operation, and is called a *generalized Alexander quandle*:

$$x \triangleleft y := \phi(xy^{-1})y \ (x, y \in G).$$

If $\phi^2 = id_G$, then G is an involutive quandle.

Example 2.5 (Conjugacy quandle)**.** For a group G, G is a quandle with the operation $x \triangleleft y := y^{-1}xy$ $(x, y \in Q)$, and is called the *conjugacy quandle* of G and is denoted by $\mathrm{Conj}(G)$.

Example 2.6 (Core quandle)**.** For a group G, G is an involutive quandle with the operation $x \triangleleft y := yx^{-1}y$ $(x, y \in Q)$, and is called the *core quandle* of G.

Example 2.7 (Dehn quandle)**.** Let G be a group and S a subset of G which generates G. Then, $D(S) := \cup_{g \in G} g^{-1}Sg$ is a subquandle of $\mathrm{Conj}(G)$ and is called a *Dehn quandle* of S. If $s^2 = 1$ holds for all $s \in S$, then $D(S)$ is involutive.

Example 2.8 (Coxeter qunadle)**.** Let (W, S) be a Coxeter system (Coxeter systems will be defined in section 5). Then, a Dehn quandle of S is called a *Coxeter quandle* associated with (W, S) and denoted by Q_W. Coxeter quandles are involutive since $s^2 = 1$ holds for all $s \in S$.

For any quandle Q, one can define a group $\mathrm{As}(Q)$ called the *associated group* of Q, which is defined by the presentation

$$\mathrm{As}(Q) \coloneqq \langle e_x \ (x \in Q) \mid {e_y}^{-1} e_x e_y = e_{x \triangleleft y} \ (x, y \in Q) \rangle.$$

For a quandle Q, the well-defined right action on Q by $\mathrm{As}(Q)$ can be defined as $x \cdot e_y \coloneqq x \triangleleft y$ $(x, y \in Q)$. We denote the set of orbits by $O(Q)$. We say that Q is *connected* if $\mathrm{As}(Q)$ acts on Q transitively, *i.e.*, $|O(Q)| = 1$.

The following lemma will be used repeatedly in this thesis:

Lemma 2.9 ([20], section 2.3)**.** *Let $x \in Q$, $g \in \mathrm{As}(Q)$. Then, $g^{-1} e_x g = e_{x \cdot g}$.*

The following lemma reveals the structure of the abelianization of $\mathrm{As}(Q)$:

Lemma 2.10 ([20], Lemma 2.7)**.** *Let $Q = \sqcup_{j \in O(Q)} Q_j$ be the decomposition of a quandle Q into orbits of the $\mathrm{As}(Q)$-action. For each $Q_i \in O(Q)$, define a well-defined group homomorphism $\eta_i \colon \mathrm{As}(Q) \to \mathbb{Z}$ by*

$$\eta_i(e_x) = \begin{cases} 1 & (\text{if } x \in Q_i) \\ 0 & (\text{if } x \notin Q_i). \end{cases}$$

Then, the direct sum $\oplus_{j \in O(Q)} \eta_j$ yields the abelianization $\mathrm{As}(Q)_{ab} \cong \mathbb{Z}^{\oplus O(Q)}$.

Since $ab_{\mathrm{As}(Q)} \colon \mathrm{As}(Q) \to \mathrm{As}(Q)_{ab}$ is surjective, $\mathrm{As}(Q)$ is an infinite group by Lemma 2.10.

Remark 2.11. Let Q be a quandle and R be the complete representation system of $O(Q)$. By Lemma 2.10, we have

$$\mathrm{As}(Q)_{ab} = \bigoplus_{x \in R} \langle [e_x] \rangle, \quad \langle [e_x] \rangle \cong \mathbb{Z} \ (x \in R).$$

3. Definition of $\mathrm{As}_f(Q)$ and its properties

Let Q be an involutive quandle. $\mathrm{As}_f(Q)$ is defined as the following presentation:

$$\mathrm{As}_f(Q) = \langle \overline{e}_x \ (x \in Q) \mid {\overline{e}_y}^{-1} \overline{e}_x \overline{e}_y = \overline{e}_{x \triangleleft y},\ \overline{e}_z^2 = 1 \ (x, y, z \in Q) \rangle.$$

Moreover, let $\pi : \mathrm{As}(Q) \to \mathrm{As}_f(Q)$ be a homomorphism which maps e_x to $\overline{e}_x$. Z_Q is defined by $\ker \pi$. Let x, y be elements of an involutive quandle Q. By definition of $\mathrm{As}(Q)$, $e_x^{-2} e_y e_x^2 = e_{y \triangleleft^2 x}$ holds. Since Q is involutive, $e_{y \triangleleft^2 x} = e_y$ holds. Therefore the following lemma follows:

Lemma 3.1. *Let x, y be elements of an involutive quandle Q. Then, $e_x^2 e_y = e_y e_x^2$.*

Corollary 3.2. *Let Q be an involutive quandle. Then, Z_Q is generated by $\{e_x^2 \mid x \in Q\}$, and is contained in the center of $\mathrm{As}(Q)$.*

Proof. Since Z_Q is a normal closure of $\{e_x^2 \mid x \in Q\}$, Z_Q is generated by $\{g^{-1} e_x^2 g \mid g \in \mathrm{As}(Q),\ x \in Q\}$. By Lemma 3.1, $g^{-1} e_x^2 g = e_x^2$ holds for any elements of $g \in \mathrm{As}(Q)$ and $x \in Q$. Therefore, Z_Q is generated by $\{e_x^2 \mid x \in Q\}$, and is contained in the center of $\mathrm{As}(Q)$. ∎

Since Z_Q is contained in the center of $\mathrm{As}(Q)$ by Corollary 3.2, the extension

$$1 \to Z_Q \hookrightarrow \mathrm{As}(Q) \xrightarrow{\pi} \mathrm{As}_f(Q) \to 1$$

is a central extension.

The following proposition was proved in [19, Theorem 5.2]. We prove the following proposition by a method similar to the proof of [12, Lemma 2.19].

Proposition 3.3. *If Q is a finite involutive quandle, $\mathrm{As}_f(Q)$ is a finite group.*

Proof. Put a total ordering in Q. For x, $y \in Q$, there exists some z, $w \in Q$ such that $z \geq w$, $e_x e_y = e_z e_w$. In fact, the case $x \geq y$ is obvious, and for $x < y$ we can write $e_x e_y = e_{y \triangleleft^{-1} x} e_x$. It follows by repeating the same argument a finite number of times. Suppose that $Q = \{x_1, x_2, \ldots, x_k\}$ and $x_1 \geq x_2 \geq \ldots \geq x_k$. From what we have shown in the first part, $\mathrm{As}_f(Q)$ can be written in the form $\overline{e}_{x_1}^{n_1}\, \overline{e}_{x_2}^{n_2} \cdots \overline{e}_{x_k}^{n_k}$. Furthermore, since $\overline{e}_x^2 = 1_{\mathrm{As}_f(Q)}$ for each $x \in Q$, we may assume that $n_1, n_2, \ldots, n_k$ are 0 or 1. Therefore $\mathrm{As}_f(Q)$ is a finite group. □

Remark 3.4. By the proof of Proposition 3.3, the order of $As_f(Q)$ is equal to or less than $2^{|Q|}$ if Q is a finite involutive quandle.

Let Q be an involutive quandle, R be the complete set of representatives of $O(Q)$. By Corollary 3.2, $Z_Q = \langle e_x^2, x \in R \rangle$ is satisfied. The following lemma is similar to Lemma 2.10, except that the direct sum component is a cyclic group of order 2.

Lemma 3.5. *Let Q be an involutive quandle. Then,*

$$\mathrm{As}_f(Q)_{ab} = \bigoplus_{x \in R} \langle [\overline{e}_x] \rangle,\ \langle [\overline{e}_x] \rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

Proof. The relation $\overline{e}_y \overline{e}_{x \triangleleft y} = \overline{e}_x \overline{e}_y$ $(x, y \in Q)$ in $\mathrm{As_f}(Q)$ corresponds to the relation $[\overline{e}_y][\overline{e}_{x \triangleleft y}] = [\overline{e}_x][\overline{e}_y]$, which is equivalent to $[\overline{e}_{x \triangleleft y}] = [\overline{e}_x]$ $(x, y \in Q)$. Furthermore, since $[\overline{e}_x^2] = 0$ and $[\overline{e}_x][\overline{e}_y] = [\overline{e}_y][\overline{e}_x]$ holds for x, $y \in R$, the lemma follows. □

From $\pi : \mathrm{As}(Q) \to \mathrm{As}_f(Q)$, the homomorphism $\pi_{ab} : \mathrm{As}(Q)_{ab} \to \mathrm{As}_f(Q)_{ab}$ is induced. Then, $\pi_{ab}([e_x]) = [\overline{e}_x]$ $(x \in Q)$.

Lemma 3.6. *Let Q be an involutive quandle. Then, $\ker \pi_{ab}$ is a free abelian group with a basis $\{[e_x^2] \mid x \in R\}$.*

Proof. By Lemma 2.10 and Lemma 3.5

$$\ker \pi_{ab} = \bigoplus_{x \in R} \langle [e_x^2] \rangle$$

is satisfied. Furthermore, for $x \in Q$, we have $\langle [e_x^2] \rangle \cong \mathbb{Z}$. Therefore $\ker \pi_{ab}$ has $\{[e_x^2] \mid x \in R\}$ as its basis. □

Let $ab_{\mathrm{As}(Q)} : \mathrm{As}(Q) \to \mathrm{As}(Q)_{ab}$ be the abelianization. We now turn our attention to the following commutative diagram consisting of four short exact sequences.

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & Z_Q \arrow[r, "ab_{\mathrm{As}(Q)}"] \arrow[d, hook] & \ker(\pi_{ab}) \arrow[d, hook] & \\
1 \arrow[r] & {[\mathrm{As}(Q),\mathrm{As}(Q)]} \arrow[r, hook] \arrow[d, "\pi"] & \mathrm{As}(Q) \arrow[r, "ab_{\mathrm{As}(Q)}"] \arrow[d, "\pi"] & \mathrm{As}(Q)_{ab} \arrow[r] \arrow[d, "\pi_{ab}"] & 1 \\
1 \arrow[r] & {[\mathrm{As}_f(Q),\mathrm{As}_f(Q)]} \arrow[r, hook] & \mathrm{As}_f(Q) \arrow[r, "ab_{\mathrm{As}_f(Q)}"] \arrow[d] & \mathrm{As}_f(Q)_{ab} \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

Theorem 3.7. *The restriction $ab_{\mathrm{As}(Q)} \mid_{Z_Q} : Z_Q \to \ker\pi_{ab}$ is an isomorphism. In particular, Z_Q is a free Abelian group with $\{e_x^2 \mid x \in R\}$ as its basis, and hence there is a central extension of the form*

$$0 \to \mathbb{Z}^{\oplus R} \to \mathrm{As}(Q) \to \mathrm{As}_f(Q) \to 0.$$

Proof. Since Z_Q is contained in the center of $\mathrm{As}(Q)$, Z_Q is an abelian group. Therefore, by Lemma 3.6, we can define a homomorphism $\phi : \ker\pi_{ab} \to Z_Q$ such that $\phi([e_x^2]) := e_x^2$ $(x \in R)$. Then, ϕ is the inverse map of $ab_{\mathrm{As}(Q)}$. □

Theorem 3.8. *Restriction of π to $[\mathrm{As}(Q),\mathrm{As}(Q)]$,*

$$\pi \mid_{[\mathrm{As}(Q),\mathrm{As}(Q)]} : [\mathrm{As}(Q),\mathrm{As}(Q)] \to [\mathrm{As}_f(Q),\mathrm{As}_f(Q)]$$

is an isomorphism.

Proof. Since π is surjective, $\pi \mid_{[\mathrm{As}(Q),\mathrm{As}(Q)]}$ is also surjective. We show its injectivity. Let $g \in [\mathrm{As}(Q),\mathrm{As}(Q)]$ be such that $\pi(g) = 1$. Then $g \in Z_Q$ by the lower left commutative square of the above diagram, and $[g] = 1$ holds by the upper right commutative square of the above diagram. Therefore, we have $g = 1$ from Lemma 3.7. □

We will characterize the associated group $\mathrm{As}(Q)$ as a pullback involving $As_f(Q)$ and its abelianization, and describe the Eilenberg-MacLane space $K(\mathrm{As}(Q),1)$ of $\mathrm{As}(Q)$ as a homotopy pullback. To do so, we only have to invoke the following result due to Kishimoto [16]:

Lemma 3.9 (Kishimoto [16], Lemma 3.8 and Lemma 4.9)**.** *Suppose that there is a commutative square of groups*

$$\begin{tikzcd}
G_1 \arrow[r, "f_1"] \arrow[d, "g"] & H_1 \arrow[d, "h"] \\
G_2 \arrow[r, "f_2"] & H_2
\end{tikzcd}$$

where f_1 is surjective. Then the square is a pullback if and only if the canonical map $\ker f_1 \to \ker f_2$ is an isomorphism. If the square is a pullback and f_1, f_2 are surjective, then the induced square of Eilenberg-MacLane spaces

$$\begin{array}{ccc} K(G_1,1) & \longrightarrow & K(H_1,1) \\ \downarrow & & \downarrow \\ K(G_2,1) & \longrightarrow & K(H_2,1) \end{array}$$

is a homotopy pullback.

Combining Theorem 3.8 and Lemma 3.9, we obtain the following result:

Theorem 3.10. *For any involutive quandle Q, the commutative square*

$$\begin{array}{ccc} \mathrm{As}(Q) & \xrightarrow{ab_{\mathrm{As}(Q)}} & \mathrm{As}(Q)_{ab} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{ab}} \\ \mathrm{As}_f(Q) & \xrightarrow{ab_{\mathrm{As}_f(Q)}} & \mathrm{As}_f(Q)_{ab} \end{array}$$

is a pullback, and the induced square of Eilenberg-MacLane spaces

$$\begin{array}{ccc} K(\mathrm{As}(Q),1) & \longrightarrow & K(\mathrm{As}(Q)_{ab},1) \\ \downarrow & & \downarrow \\ K(\mathrm{As}_f(Q),1) & \longrightarrow & K(\mathrm{As}_f(Q)_{ab},1) \end{array}$$

is a homotopy pullback.

By Lemma 2.10, Lemma 3.5 and Theorem 3.10, the following diagram

$$\begin{array}{ccc} K(\mathrm{As}(Q),1) & \longrightarrow & \prod_{i\in O(Q)} K(\mathbb{Z},1) \\ \downarrow & & \downarrow \\ K(\mathrm{As}_f(Q),1) & \longrightarrow & \prod_{i\in O(Q)} K(\mathbb{Z}_2,1) \end{array}$$

is a homotopy pullback. Moreover, since $K(\mathbb{Z},1)$and $K(\mathbb{Z}/2\mathbb{Z},1)$ are homotopy equivalent to the circle S^1 and the infinite dimensional real projective space $\mathbb{R}\mathrm{P}^\infty$ respectively, the following diagram

$$\begin{array}{ccc} K(\mathrm{As}(Q),1) & \longrightarrow & \prod_{i\in O(Q)} S^1 \\ \downarrow & & \downarrow \\ K(\mathrm{As}_f(Q),1) & \longrightarrow & \prod_{i\in O(Q)} \mathbb{R}\mathrm{P}^\infty \end{array}$$

is a homotopy pullback.

Remark 3.11. Theorem 3.10 generalizes a result of Kishimoto [16, Theorem 3.9] where a similar result was proved for Coxeter quandles.

Since the group $As_f(Q)$ is defined in terms of generators and relations, an explicit description of $As_f(Q)$ is difficult in general. In the rest of this section, we will give an explicit description of $As_f(Q)$ for a certain class of Dehn quandles. The following theorem is inspired by Dhanwani-Raundal-Singh [8, Theorem 3.14] and Akita-Hasegawa-Tanno [2, Theorem 7.1].

Theorem 3.12. *Let G be a group having a presentation $G = \langle S \mid R \rangle$ and $D(S)$ be the associated Dehn quandle. Assume that $s^2 \in R$ for all $s \in S$ and other relators in R are of the form $w^{-1}swt^{-1}$ $(s,t \in S, w \in G)$. Then the surjection $\mathrm{As}(D(S)) \to G$ induces a well-defined isomorphism $\mathrm{As}_f(D(S)) \to G$, $\overline{e}_x \mapsto x$.*

Proof. To simplify the notation, we write e_x instead of $\overline{e}_x$ $(x \in D(S))$ so that the group $\mathrm{As}_f(D(S))$ is presented by

$$\mathrm{As}_f(D(S)) \coloneqq \langle e_x\ (x \in D(S)) \mid\ e_y^{-1}e_xe_y = e_{y^{-1}xy},\ e_x^2 = 1\ (x,y \in D(S))\rangle.$$

Since $s^2 = 1$ for all $s \in S$, It follows that the surjection $\mathrm{As}(D(S)) \to G$ induces a well-defined surjection $\mathrm{As}_f(D(S)) \to G$, $e_x \mapsto x$. We will construct the inverse of $\mathrm{As}_f(D(S)) \to G$. To do so, let $F(S)$ be the free group on S and define a surjective homomorphism $\psi\colon F(S) \to \mathrm{As}_f(D(S))$ by $s \mapsto e_s$ $(s \in S)$. The surjectivity follows from that $\mathrm{As}_f(D(S))$ is generated by e_s $(s \in S)$. For a relator $w^{-1}swt^{-1} \in R$, write $w = s_1^{\varepsilon_1}s_2^{\varepsilon_2}\cdots s_k^{\varepsilon_k}$ $(s_i \in S, \varepsilon_i \in \{\pm 1\})$. By the iterated use of a relation $e_y^{-\varepsilon}e_xe_y^{\varepsilon} = e_{y^{-\varepsilon}xy^{\varepsilon}}$ $(x,y \in D(S), \varepsilon \in \{\pm 1\})$, we have

$$\begin{aligned}\psi(w^{-1}sw) &= \psi((s_1^{\varepsilon_1}\cdots s_k^{\varepsilon_k})^{-1}s(s_1^{\varepsilon_1}\cdots s_k^{\varepsilon_k})) = (e_{s_1}^{\varepsilon_1}\cdots e_{s_k}^{\varepsilon_k})^{-1}e_s(e_{s_1}^{\varepsilon_1}\cdots e_{s_k}^{\varepsilon_k})\\ &= e_{(s_1^{\varepsilon_1}\cdots s_k^{\varepsilon_k})^{-1}s(s_1^{\varepsilon_1}\cdots s_k^{\varepsilon_k})} = e_{w^{-1}sw} = e_t = \psi(t).\end{aligned}$$

For $s^2 \in R$, we have

$$\psi(s^2) = e_s^2 = 1.$$

Consequently, the homomorphism ψ induces a surjective homomorphism $\bar{\psi}\colon G \to \mathrm{As}_f(D(S))$, $s \mapsto e_s$, which is the inverse of $\mathrm{As}_f(D(S)) \to G$. □

Remark 3.13. Corollary 3.2 and Theorem 3.7 are proven in case of finite quandles in Lebed-Vendramin [19, Theorem 5.2]. In Lebed-Vendramin [19, Section 5], for a finite rack Q (rack is a nonempty set equipped with a binary operation satisfying (2) and (3) in the definition of quandles), $\overline{G}_Q$ is defined as follows:

$$\overline{G}_Q \coloneqq \langle e_x\ (x \in Q) \mid {e_y}^{-1}e_xe_y = e_{x\triangleleft y},\ e_z^{D_z} = 1\ (x,y,z \in Q)\rangle,$$

where $D_z \coloneqq \max\{2,\ \mathrm{ord}(\bullet \triangleleft z)\}$ and $\mathrm{ord}(\bullet \triangleleft z) \in \mathbb{Z}_{\geq 1}$ is the order of $\bullet \triangleleft z$. If Q is a finite involutive quandle, $\overline{G}_Q$ is the same as $\mathrm{As}_f(Q)$ since $D_z = 2$ for any $z \in Q$.

Remark 3.14. In Akita-Hasegawa-Tanno [2, §3], for a quandle Q, F_Q is defined as follows:

$$F_Q \coloneqq \langle e_x\ (x \in Q) \mid {e_y}^{-1}e_xe_y = e_{x\triangleleft y},\ e_z^{\mathrm{ord}(\bullet\triangleleft z)} = 1\ (x,y,z \in Q,\ \mathrm{ord}(\bullet \triangleleft z) < \infty)\rangle.$$

If Q is an involutive quandle and $\mathrm{ord}(\bullet \triangleleft z) = 2$ for any $z \in Q$, F_Q is the same as $\mathrm{As}_f(Q)$. However, if there exists $z \in Q$ such that $\mathrm{ord}(\bullet \triangleleft z) = 1$, F_Q is different from $\mathrm{As}_f(Q)$. For instance, if Q is the trivial quandle consisting of n elements, F_Q is the trivial group and $\mathrm{As}_f(Q)$ is isomorphic to $\mathbb{Z}_2^n$. If Q is an involutive quandle which have an element z such that $\mathrm{ord}(\bullet \triangleleft z) = 1$, then the projection $\rho\colon \mathrm{As}_f(Q) \to F_Q$ is surjective but not injective. Therefore, intuitively speaking, $\mathrm{As}_f(Q)$ is a quotient group slightly larger than F_Q.

4. Quandle homology

We define quandle homology based on [14]. Let Q be a quandle. For $n \in \mathbb{Z}_{\geq 1}$, let $C_n^R(Q)$ be a free abelian group with Q^n as its basis. When $n = 0$, let $C_0^R(Q)$ be a free abelian group with $Q^0 := \{(\ )\}$ as its basis. When $n < 0$, let $C_n^R(Q)$ be the trivial group. For $n \in \mathbb{Z}_{\geq 1}$, We also define the subgroup $C_n^D(Q)$ of $C_n^R(Q)$ as follows:

$$C_n^D(Q) := \mathrm{Span}_{\mathbb{Z}}\langle (x_1, x_2, \ldots, x_n) \in Q^n \mid x_{i-1} = x_i \text{ for some } i \in \{2, 3, \ldots, n\}\rangle.$$

Let $C_n(Q) := C_n^R(Q)/C_n^D(Q)$. For $n \in \mathbb{Z}_{\geq 1}$, we also define the boundary map $\partial_n^R :$ $C_n^R(Q) \to C_{n-1}^R(Q)$ as follows:

$$\begin{aligned}\partial_n^R(x_1, x_2, \ldots, x_n) := \sum_{1 \leq i \leq n} &(-1)^i ((x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \\ &- (x_1 \triangleleft x_i, x_2 \triangleleft x_i, \ldots, x_{i-1} \triangleleft x_i, x_{i+1}, \ldots, x_n)).\end{aligned}$$

If $n \leq 0$, we define $\partial_n^R\colon C_n^R(Q) \to C_{n-1}^R(Q)$ as the zero map. Since $\partial(C_n^D(Q)) \subset C_{n-1}^D(Q)$, the boundary map $\partial_n\colon C_n(Q) \to C_{n-1}(Q)$ is induced and $(C_n(Q), \partial_n)$ is a chain complex. Then, we define $H_n(Q) := H_n((C_n(Q), \partial_n))$.

The following proposition is known, however, we present it with a proof.

Proposition 4.1. *Let Q be a quandle. Then,*

$$H_0(Q) \cong \mathbb{Z},\ H_1(Q) \cong \mathbb{Z}^{\oplus O(Q)}.$$

Proof. Since $C_0^D(Q)$ and $C_1^D(Q)$ are the trivial group by the definition of $C_n^D(Q)$, $C_0(Q) \cong C_0^R(Q)$ and $C_1(Q) \cong C_1^R(Q)$ hold. Since ∂_0^R and ∂_1^R are the zero map by the definition of ∂_n^R, $\mathrm{Ker}\partial_0 = C_0(Q)$ and $\mathrm{Ker}\partial_1 = C_1(Q)$ hold and $\mathrm{Im}\partial_1$ is the trivial group. Therefore, the following isomorphism holds:

$$H_0(Q) \cong C_0(Q) \cong C_0^R(Q) \cong \mathbb{Z}.$$

We identify $C_1(Q)$ and $C_1^R(Q)$ by the isomorphism $C_1(Q) \cong C_1^R(Q)$. Let $Q = \sqcup_{j \in O(Q)} Q_j$ be the decomposition of a quandle Q into orbits of the $\mathrm{As}(Q)$-action. Since $\partial_2^R(x_1, x_2) = x_1 - x_1 \triangleleft x_2$ holds for any $(x_1, x_2) \in Q^2$, for any $x \in O(Q)$ and $y \in Q_x$, $y + \mathrm{Im}\partial_2 = x + \mathrm{Im}\partial_2$ holds. Therefore, the following isomorphism holds:

$$H_1(Q) = C_1(Q)/\mathrm{Im}\partial_2 \cong \bigoplus_{j \in O(Q)} \mathbb{Z}(j + \mathrm{Im}\partial_2) \cong \mathbb{Z}^{\oplus O(Q)}.$$

□

The following theorem exists for calculating the second quandle homology:

Theorem 4.2 (Eisermann [9], Theorem 9.9)**.** *Let $Q = \sqcup_{j \in O(Q)} Q_j$ be the decomposition of a quandle Q into orbits of the* $\mathrm{As}(Q)$*-action. Let* $\oplus_{j \in O(Q)} \eta_j\colon \mathrm{As}(Q) \to$

$\mathbb{Z}^{\oplus O(Q)}$ *be the group homomorphism defined in Lemma 2.10. For each $i \in O(Q)$, fix $x_i \in Q_i$. Then the following isomorphism holds:*

$$H_2(Q) \cong \bigoplus_{j \in O(Q)} (\mathrm{Stab}_{\mathrm{As}(Q)}(x_j) \cap \mathrm{Ker}\eta_j)_{ab},$$

where $\mathrm{Stab}_{\mathrm{As}(Q)}(x_j)$ is a stabilizer of x_j with respect to the action defined in section 2.

Remark 4.3. Assume that Q is a connected quandle. Let $x_0 \in Q$. Then, the isomorphism in Theorem 4.2 is as follows:

$$H_2(Q) \cong (\mathrm{Stab}_{\mathrm{As}(Q)}(x_0) \cap [\mathrm{As}(Q), \mathrm{As}(Q)])_{ab}.$$

The following examples are the known results of the second quandle homology.

Example 4.4 (Alexander quandle, [20], Proposition 5.29)**.** Let M be a $\mathbb{Z}[T^{\pm 1}]$-module and regard M by an Alexander quandle. Assume that $M = (1-T)M$. Then,

$$H_2(M) \cong (M \otimes_{\mathbb{Z}} M)/(x \otimes y - Ty \otimes x)_{x,y \in M}.$$

Example 4.5 (Dihedral quandle R_n, [9], Example 1.14)**.** Let n (≥ 3) be an odd integer. Then, $H_2(R_n) = 0$ holds.

Example 4.6 (Dehn quandle of transpositions, [9], Example 1.15)**.** Let $\mathfrak{S}_n$ $(n \geq 3)$ be the symmetric group on n points and $T_n \subset \mathfrak{S}_n$ be the set of transpositions. Then,

$$H_2(D(T_n)) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & (n \geq 4) \\ 0 & (n = 3). \end{cases}$$

Let Q be an involutive quandle. Observe that there is a well-defined right $\mathrm{As}_f(Q)$-action on Q given by $y \cdot \overline{e}_x := y \triangleleft x$ $(x, y \in Q)$, and the $\mathrm{As}(Q)$-action on Q is the one induced by the $\mathrm{As}_f(Q)$-action through the projection $\pi : \mathrm{As}(Q) \to \mathrm{As}_f(Q)$. Theorem 4.2 can be reformulated using subgroups of $\mathrm{As}_f(Q)$.

Theorem 4.7. *For a connected involutive quandle Q and $x_0 \in Q$, we have*

$$H_2(Q) \cong (\mathrm{Stab}_{\mathrm{As}_f(Q)}(x_0) \cap [\mathrm{As}_f(Q), \mathrm{As}_f(Q)])_{ab}.$$

Proof. The map π induces a surjective homomorphism

$$\mathrm{Stab}_{\mathrm{As}(Q)}(x_0) \to \mathrm{Stab}_{\mathrm{As}_f(Q)}(x_0),$$

where $\mathrm{Stab}_{\mathrm{As}_f(Q)}(x_0)$ is the stabilizer of x_0 with respect to the $\mathrm{As}_f(Q)$-action on Q. Together with Theorem 3.8, we conclude that the projection $\mathrm{As}(Q) \to \mathrm{As}_f(Q)$ induces an isomorphism

$$\mathrm{Stab}_{\mathrm{As}(Q)}(x_0) \cap [\mathrm{As}(Q), \mathrm{As}(Q)] \cong \mathrm{Stab}_{\mathrm{As}_f(Q)}(x_0) \cap [\mathrm{As}_f(Q), \mathrm{As}_f(Q)].$$

Combining with Theorem 4.2, the following isomorphism holds:

$$\begin{aligned} H_2(Q) &\cong (\mathrm{Stab}_{\mathrm{As}(Q)}(x_0) \cap [\mathrm{As}(Q), \mathrm{As}(Q)])_{ab} \\ &\cong (\mathrm{Stab}_{\mathrm{As}_f(Q)}(x_0) \cap [\mathrm{As}_f(Q), \mathrm{As}_f(Q)])_{ab}. \end{aligned}$$

□

Part 2. Second Quandle homologies

5. Coxeter Groups

In this section, we recall the definition of Coxeter group and some properties. Let (S,m) be a pair consisting of a finite set S and a map $m : S\times S \to \mathbb{N}\cup\{\infty\}$ satisfying the following two conditions:

(1) for all $s \in S$, $m(s,s) = 1$,
(2) for any distinct $s,t \in S$, $2 \le m(s,t) = m(t,s) \le \infty$.

Then the group defined by the presentation

$$W := \langle s\ (s \in S) \mid (st)^{m(s,t)} = 1\ (s,t \in S, m(s,t) < m(s,t) < \infty)\rangle \tag{5.1}$$

is called a *Coxeter group* and (W,S) is called a *Coxeter system*. A *Coxeter graph* of (W,S) (simply, a *Coxeter graph* of W) is defined by a partially labeled and undirected finite graph whose vertex set is S and whose edge set consists of the unordered pairs $\{s,t\} \subset S$ such that $m(s,t) \ge 3$ and whose edges with $m(s,t) \ge 4$ are labeled with $m(s,t)$. Moreover, a Coxeter group is *irreducible* if its Coxeter graph is connected and is *reducible* if it is not irreducible.

It is easy to verify that the defining relations in (5.1) are equivalent to the following two types of relations

$$(st)_{m(s,t)} = (ts)_{m(s,t)}\ (s,t \in S, s \neq t, m(s,t) < \infty),\ s^2 = 1\ (s \in S), \tag{5.2}$$

where the symbol $(st)_{m(s,t)}$ is defined by

$$(st)_{m(s,t)} := \underbrace{ststs\cdots}_{m(s,t)}.$$

For example, $(st)_2 = st, (st)_3 = sts, (st)_4 = stst$, and so on. Observe that a relation $(st)_{m(s,t)} = (ts)_{m(s,t)}$ $(m(s,t) < \infty)$ in (5.2) can be rewritten in terms of conjugations as

$$(st)^{-1}_{m(s,t)-1}t(st)_{m(s,t)-1} = \begin{cases} s & \text{if } m(s,t) \text{ odd,} \\ t & \text{if } m(s,t) \text{ even.}\end{cases} \tag{5.3}$$

It is known that the order of $s \in W$ is 2 for all $s \in S$.

We recall reflections, reflection groups, root systems and simple root systems. Let V be a real inner product space, and let $\langle\lambda,\mu\rangle$ be an inner product of two elements λ and μ of V. Then, for an element α of V, s_α is defined as follows and called a *reflection* of α:

$$s_\alpha(\lambda) := \lambda - \frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha.$$

A *finite reflection group* is a finite group generated by reflections. A finite subset Φ of nonzero vectors in V is said to be a *root system* if it satisfies the following two conditions:

(1) for all α in Φ, $\Phi\cap\mathbb{R}\alpha = \{\pm\alpha\}$,
(2) for all α in Φ, $s_\alpha\Phi = \Phi$.

Let Φ be a root system. Then, a subset Δ of Φ is a *simple root system* if it satisfies the following two conditions :

(1) Δ is a vector space basis for the $\mathbb{R}$-span of Φ,
(2) for all α in Φ, α is a linear combination of Δ with coefficients all of the same sign (all nonnegative or all nonpositive).

For any root system Φ, there is a finite reflection group $W(\Phi)$ and a simple system $\Delta \subset \Phi$ such that $W(\Phi)$ is generated by $\{s_\alpha \mid s_\alpha \text{ is a reflection of } \alpha \in \Delta\}$. For any finite reflection group W, there is a root system Φ and a simple system $\Delta \subset \Phi$ such that $W = W(\Phi)$. The following theorem claims on a relationship between finite Coxeter groups and finite reflection groups.

Theorem 5.1 ([13], Theorem 6.4)**.** *The following conditions on the Coxeter group W are equivalent:*

(1) W is finite.
(2) W is a finite reflection group.

By Theorem 5.1, for a finite Coxeter group W, there is a root system Φ and a simple system Δ such that W is generated by $\{s_\alpha \mid s_\alpha \text{ is a reflection of } \alpha \in \Delta\}$.
It is known that irreducible finite Coxeter group is classified into 10 types.

Theorem 5.2 ([3], Chapter VI, §4.1, Theorem 1)**.** *An irreducible finite Coxeter group is a Coxeter group whose Coxeter graph is one of the ten types in Figure 5.1, where the subscript of each type denotes the number of vertices.*

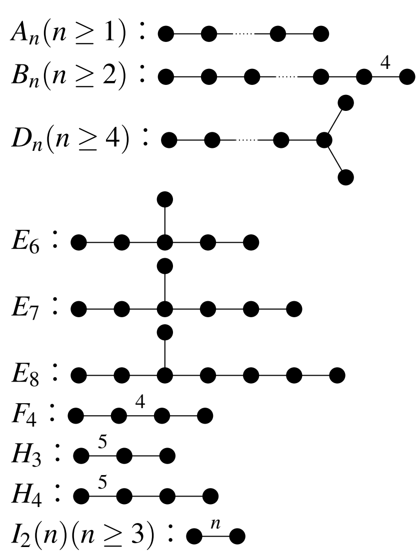

FIGURE 5.1. Coxeter graphs of irreducible finite Coxeter groups

If W is an irreducible finite Coxeter group whose Coxeter graph is type X, then we say that W is a Coxeter group of type X and denote W by $W(X)$. The following theorem claims that a Coxeter group is isomorphic to the direct product of irreducible Coxeter groups.

Theorem 5.3 ([13], Proposition 6.1)**.** *Let (W,S) be a Coxeter system and S is finite. If the connected components of the Coxeter graph of W is $\Gamma_1,\Gamma_2,\ldots,\Gamma_n$, let $S_1,S_2,\ldots,S_n$ be the corresponding subsets of S and W_{S_i} be the subgroup of W generated by S_i. Then, W is the direct product of irreducible Coxeter groups $W_{S_1},W_{S_2},\ldots,W_{S_n}$.*

Suppose that W is a reducible finite Coxeter group and the direct product of $W(X_1),W(X_2),\ldots,W(X_n)$. Then, we say that W is a Coxeter group of type $X_1\sqcup X_2\sqcup\cdots\sqcup X_n$ and denote W by $W(X_1\sqcup X_2\sqcup\cdots\sqcup X_n)$.

Let (W,S) be a Coxeter system such that W is finite and Φ be a root system of W. For $\Gamma\subset\Phi$, we define W_Γ to be the subgroup of W generated by $\{s_\gamma\in W\mid\gamma\in\Gamma\}$. For $s\in S$, let α_s be a root in Φ such that s is a reflection of α_s and define $\Gamma(s)$ the set of roots γ with $\langle\gamma,\alpha_s\rangle=0$. Moreover, for $s\in S$, let Y_s is the set of $u\in W$ with $\alpha_s=u(\alpha_s)$ and $N(u)\cap\Gamma(s)=\emptyset$. Then Y_s is a subgroup of W (see Brink [4] for details).

We also define the *odd Coxeter graph* of W to be the graph with vertex set S and edge set $\{\{s,t\}\mid s,t\in S \text{ such that } s\neq t \text{ and } m(s,t) \text{ is odd}\}$. Theorem 5.4 and Theorem 5.6 are tools for calculating the second quandle homology on and after section 7.

Theorem 5.4 (Brink [4], p.466, Theorem)**.** *Let (W,S) be a Coxeter system such that W is finite. For $s\in S$, α_s be a root such that s is a reflection of α_s and $Z_W(s)$ be the centralizer of s in W. Then, $Z_W(s)$ is the semidirect product of $W_{\Gamma(s)\cup\{\pm\alpha_s\}}$ by Y_s : that is,*

$$Z_W(s)=W_{\Gamma(s)\cup\{\pm\alpha_s\}}\rtimes Y_s.$$

Moreover, $W_{\Gamma(s)\cup\{\pm\alpha_s\}}$ is a Coxeter group with root system $\Gamma(s)\cup\{\pm\alpha_s\}$, and Y_s is free of rank $e(s)-n(s)+1$, where $e(s)$ denotes the number of edges and $n(s)$ the number of vertices of the connected component of the odd Coxeter graph of W containing s.

If Coxeter graph of a Coxeter group W is a forest, then the connected component of the odd Coxeter graph containing s is a tree. Therefore $e(s)-n(s)+1=0$ and Y_s is the trivial group. If $W_{\Gamma(s)}$ is a Coxeter group of type X, then $W_{\Gamma(s)\cup\{\pm\alpha_s\}}$ is a Coxeter group of type $X\sqcup A_1$ since $s_\gamma s=ss_\gamma$ holds for $\gamma\in\Gamma(s)$.

Summarizing the above discussion, the following corollary is obtained:

Corollary 5.5. *Let W be a Coxeter group whose Coxeter graph is a forest. Then, Y_s in Theorem 5.4 is the trivial group. Moreover, if $W_{\Gamma(s)}$ is a Coxeter group of type X, $W_{\Gamma(s)\cup\{\pm\alpha_s\}}$ is a Coxeter group of type $X\sqcup A_1$.*

The following theorem is provides a special representation for the alternating subgroup of Coxeter groups.

Theorem 5.6 (Brenti-Reiner-Roichman [5], Proposition 2.1.1)**.** *Let (W,S) be a Coxeter system, $S=\{s_0,s_1,\ldots,s_n\}$, $R=\{r_1,r_2,\ldots,r_n\}$, $\varepsilon\colon W\to\{\pm1\}$ be the homomorphism uniquely defined by $\varepsilon(s)=-1$ for all $s\in S$ and $f\colon R\to W$ be a map* *defined by* $f(r_i) = s_0 s_i$ *for all* $i \in \{1, 2, \ldots, n\}$*. Then,* f *gives the following isomorphism:*

$$\mathrm{Ker}\varepsilon \cong \langle R \mid r_i^{m(s_0,s_i)} = (r_i^{-1} r_j)^{m(s_i,s_j)} = 1 (1 \le i < j \le n) \rangle.$$

Remark 5.7. If the odd Coxeter graph of W is connected, $\mathrm{Ker}\varepsilon$ is equal to the commutator subgroup of W since ts is equal to a commutator $[t, (st)_{m(s,t)-1}^{-1}]$ for any $s, t \in S$ by (5.3).

By Example 2.8, a Coxeter quandle Q_W is an involutive Dehn quandle $D(S)$. Observe that a relation $(st)_{m(s,t)} = (ts)_{m(s,t)}$ $(m(s,t) < \infty)$ in (5.2) can be rewritten in terms of conjugations as

$$(st)_{m(s,t)-1}^{-1} t (st)_{m(s,t)-1} = \begin{cases} s & \text{if } m(s,t) \text{ odd}, \\ t & \text{if } m(s,t) \text{ even}. \end{cases}$$

Therefore, by Theorem 3.12, the following proposition holds:

Proposition 5.8. *Let* (W,S) *be a Coxeter system and* $\phi \colon \mathrm{As}(Q_W) \to W$ *be a homomorphism such that* $\phi(e_s) = s (s \in S)$*. Then* ϕ *induces a well-defined isomorphism* $\overline{\phi} \colon \mathrm{As}_f(Q_W) \to W$*,* $\overline{e}_x \mapsto x$*.*

By restricting the domain of the above isomorphism $\overline{\phi}$ to the commutator subgroup of $\mathrm{As}_f(Q_W)$, the following theorem is obtained.

Theorem 5.9. *Let* W *be a Coxeter group and* $\overline{\phi}$ *be the isomorphism in Proposition 5.8. Then the isomorphism obtained from* $\overline{\phi}$ *by restricting its domain to* $[\mathrm{As}_f(Q_W), \mathrm{As}_f(Q_W)]$ *yields the the following isomorphism:*

$$[\mathrm{As}_f(Q_W), \mathrm{As}_f(Q_W)] \cong [W, W].$$

By Proposition 5.8, the definition of $\mathrm{As}_f(Q_W)$-action on Q_W and the definition of Coxeter quandle operation, $\mathrm{Stab}_{\mathrm{As}_f(Q)}(x_0)$ is isomorphic to the centralizer of x_o. Together with Theorem 5.9, the following theorem holds:

Theorem 5.10. *Let* W *be a Coxeter group whose odd Coxeter graph is a tree and* Q_W *be its Coxeter quandle,* $x_0 \in Q_W$*. Then,*

$$H_2(Q_W) \cong (\mathrm{Z}_W(x_0) \cap [W, W])_{ab},$$

where $\mathrm{Z}_W(x_0)$ *is a centralizer of* x_0*.*

Let (W,S) be a Coxeter system whose odd Coxeter graph is a tree, and $s_0 \in S$. By Theorem 5.4, $Z_W(s_0) = W_{\Gamma(s_0) \cup \{\pm\alpha_{s_0}\}}$ holds and $W_{\Gamma(s_0) \cup \{\pm\alpha_{s_0}\}}$ is a Coxeter group. Furthermore, by Theorem 5.6 and remark 5.7, the following equality holds:

$$\mathrm{Z}_W(s_0) \cap [W, W] = \mathrm{Ker}\varepsilon \mid_{W_{\Gamma(s_0) \cup \{\pm\alpha_{s_0}\}}} .$$

Therefore we can calculate the presentation of $(\mathrm{Z}_W(s_0) \cap [W, W])_{ab}$ by Theorem 5.6.

6. THE SECOND QUANDLE HOMOLOGY OF COXETER QUANDLES

In this section, we introduce and recall symbols used from section 7 to section 14 and list results of the second quandle homology. In section 7 onwards, we calculate the second quandle homology of Coxeter quandle Q_W if W is an irreducible finite Coxeter group whose odd Coxeter graph is a tree.

For $n \in \mathbb{Z}_{\geq 1}$, we denote by $e_1, e_2, \ldots, e_n$ the coordinate vectors of $\mathbb{R}^n$. Moreover, for $i, j \in \mathbb{Z}$, we denote by $\varepsilon_{i,j}$ Kronecker delta. For $1 \leq i < j \leq n$, $f_{i,j}$ is a linear map from $\mathbb{R}^n$ to itself that reverses the signs of the i-th and j-th components respectively and does not change any other sign, and $g_{i,j}$ is a linear map from $\mathbb{R}^n$ to itself that permutes the i-th and j-th components and maps the other components to itself.

Let (W,S) be a Coxeter system such that W is finite and Φ be a root system of W. For $s \in S$, let α_s be a root in Φ such that s is a reflection of α_s. We defined $\Gamma(s)$ as the set of roots γ with $\langle \gamma, \alpha_s \rangle = 0$ in section 5.

Suppose that an inner product of $\mathbb{R}^n$ is the standard inner product of $\mathbb{R}^n$. Let $\lambda = x + y\boldsymbol{i} + z\boldsymbol{j} + w\boldsymbol{k}$ (1, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ is the standard basis of $\mathbb{H}$) be an element of $\mathbb{H}$. Then, we denote $x - y\boldsymbol{i} - z\boldsymbol{j} - w\boldsymbol{k}$ by $\bar{\lambda}$. For λ, $\mu \in \mathbb{H}$, We define an inner product $\langle \lambda, \mu \rangle$ of two elements λ and μ of $\mathbb{H}$ by

$$\langle \lambda, \mu \rangle = \frac{\lambda \bar{\mu} + \mu \bar{\lambda}}{2}.$$

Root systems and simple systems of type A_n, D_n, E_6, E_7 and E_8 are cited from [3]. Root systems and simple systems of type H_3 and H_4 are cited from [13].

Theorem 6.1. *Let $W(X)$ be a Coxeter group of type X. Then, the results of the calculation of $H_2(Q_{W(X)})$ are shown in the Table 6.1.*

Type X	$A_n (n \geq 1)$	$D_n (n \geq 4)$	E_6	E_7	E_8	$I_2(n) (n \geq 3)$	H_4	H_3
$H_2(Q)$	0 $(n \leq 2)$ $\mathbb{Z}_2$ $(n \geq 3)$	$\mathbb{Z}_2^3$ $(n = 4)$ $\mathbb{Z}_2^2$ $(n \geq 5)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0 *(n is odd)*	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$

TABLE 6.1. The results of the calculation of $H_2(Q)$ in case Q is a Coxeter quandle $Q_{W(X)}$

7. TYPE A_n

Let $n \geq 1$. Then, a root system $\Phi(A_n)$ and a simple root system $\Delta(A_n)$ of $W(A_n)$ are respectively as follows:

$$\Phi(A_n) = \{\pm(e_i - e_j) \in \mathbb{R}^{n+1} \mid 1 \leq i < j \leq n+1,\ i \neq j\},$$
$$\Delta(A_n) = \{e_i - e_{i+1} \mid 1 \leq i \leq n\}.$$

The reflection corresponding to $e_n - e_{n+1}$ is $g_{n,n+1}$.

Lemma 7.1. *Let $s = g_{n,n+1}$. Then,*

$$\Gamma(s) = \begin{cases} \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n-1\} & (n \geq 3) \\ \emptyset & (n \leq 2). \end{cases}$$

Proof. Let $1 \leq i < j \leq n+1$. Then,

$$\begin{aligned}\langle e_n - e_{n+1}, e_i - e_j \rangle &= \langle e_n, e_i \rangle - \langle e_n, e_j \rangle - \langle e_{n+1}, e_i \rangle + \langle e_{n+1}, e_j \rangle \\ &= \delta_{n,i} - \delta_{n,j} - \delta_{n+1,i} + \delta_{n+1,j}.\end{aligned}$$

If $(i,j) = (n,n+1)$, $\langle e_n - e_{n+1}, e_i - e_j \rangle$ is not 0. Suppose that $(i,j) \neq (n,n+1)$. Then, $n \geq 2$.

(1) If $j < n$, then $n \geq 3$ and $\delta_{n,i} - \delta_{n,j} - \delta_{n+1,i} + \delta_{n+1,j} = 0-0-0+0 = 0$.
(2) If $j = n$, since $i \leq n-1$, $\delta_{n,i} - \delta_{n,j} - \delta_{n+1,i} + \delta_{n+1,j} = 0-1-0+0 = -1$.
(3) If $j = n+1$, since $i \leq n-1$, $\delta_{n,i} - \delta_{n,j} - \delta_{n+1,i} + \delta_{n+1,j} = 0-0-0+1 = 1$.

□

Lemma 7.2. *Let $s = g_{n,n+1}$. Then,*

$$Z_{W(A_n)}(s) = \begin{cases} W(A_{n-2} \sqcup A_1) & (n \geq 3) \\ W(A_1) & (n \leq 2). \end{cases}$$

Proof. By Lemma 7.1, the following holds:

$$W_{\Gamma(s) \cup \{\pm e_n - e_{n+1}\}} = \begin{cases} W(A_{n-2} \sqcup A_1) & (n \geq 3) \\ W(A_1) & (n \leq 2). \end{cases}$$

Therefore, the above statement holds by Theorem 5.4. □

Theorem 7.3.

$$H_2(Q_{W(A_n)}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & (n \geq 3) \\ 0 & (n \leq 2). \end{cases}$$

Proof. If $n \leq 2$, $\mathrm{Ker}\varepsilon\ |_{W(A_1)} = 0$ holds by Theorem 5.6. Hence $H_2(Q_{W(A_n)}) = 0$ holds by Theorem 5.10.

Suppose that $n \geq 3$. Let $s_0 = g_{n,n+1}$ and $s_i = s_{\alpha_{i,i+1}} (i = 1,2,\ldots,n-2)$. By Theorem 5.6, $\mathrm{Ker}\varepsilon\ |_{W(A_{n-2} \sqcup A_1)}$ is generated by $\{r_i \mid i = 1,2,\ldots,n-2\}$ and have the following relations:

$$r_i^2 = (r_i^{-1} r_{i+1})^3 = (r_j^{-1} r_k)^2 = 1 \ (i,j,k = 1,2,\ldots,n-2 \text{ and } |j-k| \geq 2).$$

The relation $(r_i^{-1} r_{i+1})^3 = 1$ implies $[r_i] = [r_{i+1}] \in (\mathrm{Ker}\varepsilon\ |_{W(A_{n-2} \sqcup A_1)})_{ab}$. Therefore,

$$\mathrm{Ker}\varepsilon\ |_{W(A_{n-2} \sqcup A_1)})_{ab} = \langle r_1 \mid r_1^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}$$

holds. By Theorem 5.10, $H_2(Q_{W(A_n)}) \cong \mathbb{Z}/2\mathbb{Z}$ holds. □

Remark 7.4. The results of $H_2(Q_{W(A_n)})$ is the same as that that calculated in Eisermann [9].

8. TYPE D_n

Let $n \geq 4$. Then, a root system $\Phi(D_n)$ and a simple root system $\Delta(D_n)$ of $W(D_n)$ are respectively as follows:

$$\Phi(D_n) = \{\pm(e_i - e_j), \pm(e_i + e_j) \in \mathbb{R}^n \mid 1 \leq i < j \leq n\},$$
$$\Delta(D_n) = \{e_i - e_{i+1}, e_{n-1} + e_n \in \mathbb{R}^n \mid 1 \leq i \leq n-1\}.$$

The reflection corresponding to $e_{n-1} + e_n$ is $g_{n-1,n} \circ f_{n-1,n}$.

Lemma 8.1. *Let $s = g_{n-1,n} \circ f_{n-1,n}$. Then,*

$$\Gamma(s) = \{\pm(e_i - e_j), \pm(e_i + e_j) \mid 1 \leq i < j \leq n-2\} \cup \{\pm(e_{n-1} - e_n)\}.$$

Proof. Let $1 \leq i < j \leq n$. Then,

$$\langle e_i - e_j, e_{n-1} + e_n \rangle = \langle e_i, e_{n-1} \rangle + \langle e_i, e_n \rangle - \langle e_j, e_{n-1} \rangle - \langle e_j, e_n \rangle. \tag{8.1}$$

If $j < n-1$,

$$(\text{left side of equation (8.1)}) = 0 + 0 - 0 - 0 = 0.$$

If $j = n-1$,

$$(\text{left side of equation (8.1)}) = 0 + 0 - 1 - 0 = -1.$$

If $j = n$,

$$(\text{left side of equation (8.1)}) = \delta_{i,n-1} + 0 - 0 - 1 = \begin{cases} 0 & (i = n-1) \\ -1 & (i < n-1). \end{cases}$$

Since

$$\langle e_i + e_j, e_{n-1} + e_n \rangle = \langle e_i, e_{n-1} \rangle + \langle e_i, e_n \rangle + \langle e_j, e_{n-1} \rangle + \langle e_j, e_n \rangle$$

holds, $\langle e_i + e_j, e_{n-1} + e_n \rangle$ is 0 if $1 \leq i < j \leq n-2$, and is not 0 if $i \geq n-1$. $\square$

Lemma 8.2. *Let $n \geq 4$, $s = g_{n-1,n} \circ f_{n-1,n}$. Then,*

$$Z_{W(D_n)}(s) = \begin{cases} W(A_1 \sqcup A_1 \sqcup A_1 \sqcup A_1) & (n = 4) \\ W(A_3 \sqcup A_1 \sqcup A_1) & (n = 5) \\ W(D_{n-2} \sqcup A_1 \sqcup A_1) & (n \geq 6). \end{cases}$$

Proof. If $n = 4$, $W_{\Gamma(s) \cup \{\pm(e_3+e_4)\}}$ is the Coxeter group of type $A_1 \sqcup A_1 \sqcup A_1 \sqcup A_1$ since $e_1 - e_2$, $e_1 + e_2$, $e_3 - e_4$ and $e_3 + e_4$ are mutually orthogonal. If $n = 5$, $W_{\Gamma(s) \cup \{\pm(e_4+e_5)\}}$ is the Coxeter group of type $A_3 \sqcup A_1 \sqcup A_1$ since its Coxeter graph is shown in Figure 8.1.

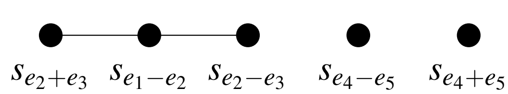

FIGURE 8.1. A Coxeter graph of $W_{\Gamma(s) \cup \{\pm(e_4+e_5)\}}$

If $n \geq 6$, $W_{\Gamma(s) \cup \{\pm(e_{n-1}+e_n)\}}$ is the Coxeter group of type $D_{n-2} \sqcup A_1$. Therefore the above statement holds by Theorem 5.4. $\square$

Theorem 8.3.

$$H_2(Q_{W(D_n)}) \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^3 & (n=4) \\ (\mathbb{Z}/2\mathbb{Z})^2 & (n \geq 5). \end{cases}$$

Proof. Let $s_0 = g_{n-1,n} \circ f_{n-1,n}$ and $s_1 = s_{e_{n-1}-e_n}$.

Suppose that $n=4$. Let $s_2 = s_{e_1-e_2}$ and $s_3 = s_{e_1+e_2}$. By Theorem 5.6, the following isomorphism holds:

$$\mathrm{Ker}\varepsilon\mid_{W(A_1 \sqcup A_1 \sqcup A_1 \sqcup A_1)} \cong (\mathbb{Z}/2\mathbb{Z})^3.$$

Therefore, its abelianization is also isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$.

Suppose that $n=5$. Let $s_2 = s_{e_2+e_3}$, $s_3 = s_{e_1-e_2}$ and $s_4 = s_{e_2-e_3}$. $\mathrm{Ker}\varepsilon\mid_{W(A_3 \sqcup A_1 \sqcup A_1)}$ is generated by $\{r_i \mid i = 1,2,3,4\}$ and have the following relations:

$$\begin{aligned}
&r_i^2 = 1 \ (i = 1,2,3,4) \\
&(r_1^{-1}r_j)^2 = (r_2^{-1}r_4)^2 = 1 \ (j = 2,3,4) \\
&(r_2^{-1}r_3)^3 = (r_3^{-1}r_4)^3 = 1.
\end{aligned}$$

The relations $(r_2^{-1}r_3)^3 = (r_3^{-1}r_4)^3 = 1$ imply $[r_2] = [r_3] = [r_4] \in (\mathrm{Ker}\varepsilon\mid_{W(A_3 \sqcup A_1 \sqcup A_1)})_{ab}$. Therefore,

$$(\mathrm{Ker}\varepsilon\mid_{W(A_3 \sqcup A_1 \sqcup A_1)})_{ab} = \langle r_1, r_2 \mid r_1^2 = r_2^2 = 1, r_1r_2 = r_2r_1 \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$$

holds. By Theorem 5.10, $H_2(Q_{W(D_5)}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ holds.

Suppose that $n \geq 6$. Let $s_i = s_{e_{i-1}-e_i}$ $(i = 2,3,\ldots,n-2)$ and $s_{n-1} = s_{e_{n-3}+e_{n-2}}$. $\mathrm{Ker}\varepsilon\mid_{W(D_{n-2} \sqcup A_1)}$ is generated by $\{r_i \mid i = 1,2,\ldots,n-1\}$ and have the following relations:

$$\begin{aligned}
&r_i^2 = 1 \ (i = 1,2,\ldots,n-1) \\
&(r_1^{-1}r_j)^2 = (r_2^{-1}r_4)^2 = 1 \ (j = 2,3,\ldots,n-1) \\
&(r_k^{-1}r_{k+1})^3 = (r_{n-3}^{-1}r_{n-1})^3 = 1 \ (k = 2,3,\ldots,n-3) \\
&(r_{n-1}^{-1}r_l)^2 = 1 \ (l = 2,3,\ldots,n-4,n-2) \\
&(r_p^{-1}r_q)^2 = 1 \ (p,q = 2,\ldots,n-2 \text{ and } |p-q| \geq 2).
\end{aligned}$$

The relations $(r_k^{-1}r_{k+1})^3 = (r_{n-3}^{-1}r_{n-1})^3 = 1 (k = 2,3,\ldots,n-3)$ imply $[r_i] = [r_{i+1}] \in (\mathrm{Ker}\varepsilon\mid_{W(D_{n-2} \sqcup A_1)})_{ab}$ for $i = 2,3,\ldots,n-2$. Therefore,

$$(\mathrm{Ker}\varepsilon\mid_{W(D_{n-2} \sqcup A_1)})_{ab} = \langle r_1, r_2 \mid r_1^2 = r_2^2 = 1, r_1r_2 = r_2r_1 \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$$

holds. By Theorem 5.10, $H_2(Q_{W(D_n)}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ holds. $\square$

9. Type E_6

A root system $\Phi(E_6)$ and a simple root system $\Delta(E_6)$ of $W(E_6)$ are respectively as follows:

$$\begin{aligned}\Phi(E_6) =&\{\pm(e_i-e_j),\pm(e_i+e_j)\in\mathbb{R}^8 \mid 1\le i<j\le 5\}\\ &\cup\{\pm\frac{1}{2}(\sum_{i=1}^{5}\varepsilon_i e_i-e_6-e_7+e_8)\in\mathbb{R}^8 \mid \varepsilon_i=\pm1(i=1,2,\ldots,5),\prod_{i=1}^{5}\varepsilon_i=1\},\\ \Delta(E_6) =&\{\frac{1}{2}(e_1-e_2-e_3-e_4-e_5-e_6-e_7+e_8)\}\\ &\cup\{e_1+e_2,e_{i+1}-e_i\in\mathbb{R}^8 \mid 1\le i\le 4\}.\end{aligned}$$

Lemma 9.1. *Let s be a reflection corresponding to $\alpha=2^{-1}(e_1-e_2-e_3-e_4-e_5-e_6-e_7+e_8)$ which is an element of $\Phi(E_6)$. Then,*

$$\Gamma(s)=\{\pm(e_1+e_i)\mid 2\le i\le 5\}\cup\{\pm(e_i-e_j)\mid 2\le i<j\le 5\}\cup\{\pm\alpha_i\mid 1\le i\le 5\},$$

where α_i $(1\le i\le 5)$ are defined as follows:

$$\begin{aligned}\alpha_1&=\frac{1}{2}(e_1+e_2+e_3+e_4+e_5-e_6-e_7+e_8)\\ \alpha_2&=\frac{1}{2}(-e_1-e_2+e_3+e_4+e_5-e_6-e_7+e_8)\\ \alpha_3&=\frac{1}{2}(-e_1+e_2-e_3+e_4+e_5-e_6-e_7+e_8)\\ \alpha_4&=\frac{1}{2}(-e_1+e_2+e_3-e_4+e_5-e_6-e_7+e_8)\\ \alpha_5&=\frac{1}{2}(-e_1+e_2+e_3+e_4-e_5-e_6-e_7+e_8).\end{aligned}$$

Proof. Let $1\le i<j\le 5$. By direct calculations, the following holds:

$$\langle e_i-e_j,2\alpha\rangle=\langle e_i,e_1\rangle-\langle e_j,e_1\rangle+\sum_{k=2}^{5}(-\langle e_i,e_k\rangle+\langle e_j,e_k\rangle)=\begin{cases}2\ (i=1)\\0\ (i>1),\end{cases}$$

$$\langle e_i+e_j,2\alpha\rangle=\langle e_i,e_1\rangle+\langle e_j,e_1\rangle-\sum_{k=2}^{5}((\langle e_i,e_k\rangle+\langle e_j,e_k\rangle)=\begin{cases}0\quad (i=1)\\-2\ (i>1),\end{cases}$$

$$\langle\sum_{i=1}^{5}\varepsilon_i e_i-e_6-e_7+e_8,2\alpha\rangle=\varepsilon_1-\sum_{k=2}^{5}\varepsilon_k+3.$$

Suppose that $\varepsilon_1=1$. Since $\prod_{k=2}^{5}\varepsilon_k=1$, $\sum_{k=2}^{5}\varepsilon_k=\pm4$ or 0 holds. Therefore, $\langle\sum_{i=1}^{5}\varepsilon_i e_i-e_6-e_7+e_8,2\alpha\rangle=0$ holds only when $\sum_{k=2}^{5}\varepsilon_k=4$. The only root of the form $\sum_{i=1}^{5}\varepsilon e_i-e_6-e_7+e_8$ that satisfies $\varepsilon_1=1$ and $\sum_{k=2}^{5}\varepsilon_k=4$ is α_1.

Suppose that $\varepsilon_1=-1$. Since $\prod_{k=2}^{5}\varepsilon_k=-1$, $\sum_{k=2}^{5}\varepsilon_k=\pm2$ or 0 holds. Therefore, $\langle\sum_{i=1}^{5}\varepsilon_i e_i-e_6-e_7+e_8,2\alpha\rangle=0$ holds only when $\sum_{k=2}^{5}\varepsilon_k=2$. The only roots of the form $\sum_{i=1}^{5}\varepsilon e_i-e_6-e_7+e_8$ that satisfy $\varepsilon_1=-1$ and $\sum_{k=2}^{5}\varepsilon_k=2$ are $\alpha_i(2\le i\le 5)$. □

Lemma 9.2. *Let s be a reflection corresponding to $\alpha = 2^{-1}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$. Then,*

$$Z_{W(E_6)}(s) = W(A_5 \sqcup A_1).$$

Proof. By Theorem 5.4, $Z_{W(E_6)}(s) = W_{\Gamma(s)\cup\{\pm\alpha\}}$ holds. By Lemma 9.1, $W_{\Gamma(s)\cup\{\pm\alpha\}}$ is the Coxeter group of type $A_5 \sqcup A_1$ since its Coxeter graph is shown in Figure 9.1, where $\beta = 2^{-1}(e_1 - e_2 - e_3 - e_4 + e_5 + e_6 + e_7 - e_8)$. $\square$

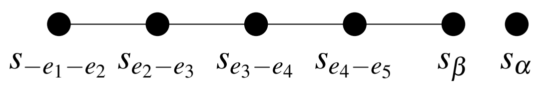

FIGURE 9.1. A Coxeter graph of $W_{\Gamma(s)\cup\{\pm\alpha\}}$

Theorem 9.3.

$$H_2(Q_{W(E_6)}) \cong \mathbb{Z}/2\mathbb{Z}.$$

Proof. Let $\alpha = 2^{-1}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$ and $\beta = 2^{-1}(e_1 - e_2 - e_3 - e_4 + e_5 + e_6 + e_7 - e_8)$. Let $s_0 = s_\alpha$, $s_1 = s_{-e_1-e_2}$, $s_i = s_{e_i - e_{i+1}} (i = 2,3,4)$ and $s_5 = s_\beta$. By Theorem 5.6, $\mathrm{Ker}\varepsilon\,|_{W(A_5\sqcup A_1)}$ is generated by $\{r_i \mid i = 1,2,3,4,5\}$ and have the following relations:

$$r_i^2 = (r_j^{-1} r_k)^2 = 1 \ (i,j,k = 1,2,3,4,5 \text{ and } |j-k| \geq 2)$$
$$(r_l^{-1} r_{l+1})^3 = 1 \ (l = 1,2,3,4).$$

The relations $(r_l^{-1} r_{l+1})^3 = 1$ $(l = 1,2,3,4)$ imply $[r_1] = [r_2] = [r_3] = [r_4] = [r_5] \in (\mathrm{Ker}\varepsilon\,|_{W(A_5\sqcup A_1)})_{ab}$. Therefore,

$$(\mathrm{Ker}\varepsilon\,|_{W(A_5\sqcup A_1)})_{ab} = \langle r_1 \mid r_1^2 = 1\rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

holds. By Theorem 5.10, $H_2(Q_{W(E_6)}) \cong \mathbb{Z}/2\mathbb{Z}$ holds. $\square$

10. TYPE E_7

A root system $\Phi(E_7)$ and a simple root system $\Delta(E_7)$ of $W(E_7)$ are respectively as follows:

$$\begin{aligned}
\Phi(E_7) =& \{\pm(e_i - e_j), \pm(e_i + e_j) \in \mathbb{R}^8 \mid 1 \leq i < j \leq 6\} \cup \{\pm(e_7 - e_8)\} \\
& \cup \{\pm\frac{1}{2}(\sum_{i=1}^{6} \varepsilon_i e_i + e_7 - e_8) \in \mathbb{R}^8 \mid \varepsilon_i = \pm 1 (i = 1,2,\ldots,6), \prod_{i=1}^{6} \varepsilon_i = -1\}, \\
\Delta(E_7) =& \{\frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)\} \\
& \cup \{e_1 + e_2, e_{i+1} - e_i \in \mathbb{R}^8 \mid 1 \leq i \leq 5\}.
\end{aligned}$$

Lemma 10.1. *Let s be a reflection corresponding to $\alpha = 2^{-1}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$ which is an element of $\Delta(E_7)$. Then,*

$$\begin{aligned}\Gamma(s) =& \{\pm(e_1+e_i) \mid 2 \le i \le 6\} \cup \{\pm(e_i - e_j) \mid 2 \le i < j \le 6\} \\ & \cup \{\pm\frac{1}{2}(e_1 + \sum_{i=2}^{6} \varepsilon_i e_i + e_7 - e_8) \in \mathbb{R}^8 \mid \varepsilon_i = \pm 1 (i = 2,\dots,6), \sum_{i=2}^{6} \varepsilon_i = 3\} \\ & \cup \{\pm\frac{1}{2}(-e_1 + \sum_{i=2}^{6} \varepsilon_i e_i + e_7 - e_8) \in \mathbb{R}^8 \mid \varepsilon_i = \pm 1 (i = 2,\dots,6), \sum_{i=2}^{6} \varepsilon_i = 1\}.\end{aligned}$$

Proof. Let $1 \le i < j \le 6$. By direct calculations, the following holds:

$$\begin{aligned}&\langle e_7 - e_8, 2\alpha\rangle = -\langle e_7, e_7\rangle - \langle e_8, e_8\rangle = -2,\\ &\langle e_i - e_j, 2\alpha\rangle = \langle e_i, e_1\rangle - \langle e_j, e_1\rangle + \sum_{k=2}^{6}(-\langle e_i, e_k\rangle + \langle e_j, e_k\rangle) = \begin{cases} 2 \ (i=1) \\ 0 \ (i>1),\end{cases}\\ &\langle e_i + e_j, 2\alpha\rangle = \langle e_i, e_1\rangle + \langle e_j, e_1\rangle - \sum_{k=2}^{6}(\langle e_i, e_k\rangle + \langle e_j, e_k\rangle) = \begin{cases} 0 \quad (i=1) \\ -2 \ (i>1),\end{cases}\\ &\langle \sum_{i=1}^{6} \varepsilon_i e_i - e_7 + e_8, 2\alpha\rangle = \varepsilon_1 - \sum_{k=2}^{6} \varepsilon_k + 2.\end{aligned}$$

Suppose that $\varepsilon_1 = 1$. Since $\prod_{k=2}^{5} \varepsilon_k = 1$, $\sum_{k=2}^{6} \varepsilon_k = -5$ or -1 or 3 holds. Therefore, $\langle \sum_{i=1}^{5} \varepsilon_i e_i - e_6 - e_7 + e_8, 2\alpha\rangle = 0$ holds if and only if $\sum_{k=2}^{6} \varepsilon_k = 3$.

Suppose that $\varepsilon_1 = -1$. Since $\prod_{k=2}^{5} \varepsilon_k = -1$, $\sum_{k=2}^{5} \varepsilon_k = -3$ or 1 or 5 holds. Therefore, $\langle \sum_{i=1}^{5} \varepsilon_i e_i - e_6 - e_7 + e_8, 2\alpha\rangle = 0$ holds if and only if $\sum_{k=2}^{6} \varepsilon_k = 1$. □

Lemma 10.2. *Let s be a reflection corresponding to $\alpha = 2^{-1}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$. Then,*

$$Z_{W(E_7)}(s) = W(D_6 \sqcup A_1).$$

Proof. $Z_{W(E_7)}(s) = W_{\Gamma(s)\cup\{\pm\alpha\}}$ holds by Theorem 5.4. By Lemma 10.1, $W_{\Gamma(s)\cup\{\pm\alpha\}}$ is the Coxeter group of type $D_6 \sqcup A_1$ since its Coxeter graph is shown in Figure 10.1, where $\beta = 2^{-1}(e_1 - e_2 - e_3 - e_4 + e_5 + e_6 + e_7 - e_8)$. □

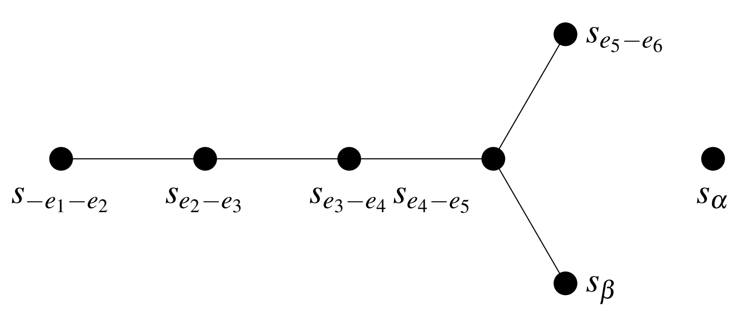

FIGURE 10.1. A Coxeter graph of $W_{\Gamma(s)\cup\{\pm\alpha\}}$

Theorem 10.3.

$$H_2(Q_{W(E_7)}) \cong \mathbb{Z}/2\mathbb{Z}.$$

Proof. Let $\alpha = 2^{-1}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$ and $\beta = 2^{-1}(e_1 - e_2 - e_3 - e_4 + e_5 + e_6 + e_7 - e_8)$. Let $s_0 = s_\alpha$, $s_1 = s_{-e_1-e_2}$, $s_i = s_{e_i - e_{i+1}} (i = 2,3,4,5)$ and $s_6 = s_\beta$.

By Theorem 5.6, $\mathrm{Ker}\varepsilon\mid_{W(D_6 \sqcup A_1)}$ is generated by $\{r_i \mid i = 1,2,\ldots,6\}$ and have the following relations:

$$\begin{aligned}
&r_i^2 = 1 \ (i,j,k = 1,2,\ldots,6)\\
&(r_i^{-1}r_{i+1})^3 = (r_4^{-1}r_6)^3 = 1 \ (i = 1,2,3,4).
\end{aligned}$$

The relations $(r_i^{-1}r_{i+1})^3 = (r_4^{-1}r_6)^3 = 1$ $(i = 1,2,3,4)$ imply $[r_1] = [r_2] = [r_3] = [r_4] = [r_5] = [r_6] \in (\mathrm{Ker}\varepsilon\mid_{W(D_6 \sqcup A_1)})_{ab}$. Therefore,

$$(\mathrm{Ker}\varepsilon\mid_{W(D_6 \sqcup A_1)})_{ab} = \langle r_1 \mid r_1^2 = 1\rangle \cong \mathbb{Z}/2\mathbb{Z}$$

holds. By Theorem 5.10, $H_2(Q_{W(E_7)}) \cong \mathbb{Z}/2\mathbb{Z}$ holds. □

11. Type E_8

A root system $\Phi(E_8)$ and a simple root system $\Delta(E_8)$ of $W(E_8)$ are respectively as follows:

$$\begin{aligned}
\Phi(E_8) =&\{\pm(e_i - e_j), \pm(e_i + e_j) \in \mathbb{R}^8 \mid 1 \le i < j \le 8\}\\
&\cup \{\frac{1}{2}\sum_{i=1}^{8} \varepsilon_i e_i \in \mathbb{R}^8 \mid \varepsilon_i = \pm 1 (i = 1,2,\ldots,8), \prod_{i=1}^{8} \varepsilon_i = 1\},\\
\Delta(E_8) =&\{\frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)\}\\
&\cup \{e_1 + e_2, e_{i+1} - e_i \in \mathbb{R}^8 \mid 1 \le i \le 6\}.
\end{aligned}$$

Lemma 11.1. *Let $s = g_{1,2} \circ f_{1,2}$. Then,*

$$\begin{aligned}
\Gamma(s) =&\{\pm(e_i + e_j) \mid 3 \le i < j \le 8\} \cup \{\pm(e_i - e_8), \pm(e_1 - e_2) \mid 3 \le i < j \le 7\}\\
&\cup \{\pm\frac{1}{2}(e_1 - e_2 + \sum_{i=3}^{8} \varepsilon_i e_i) \mid \varepsilon_i = \pm 1 (i = 3,4,\ldots,8), \prod_{i=3}^{8} \varepsilon_i = -1\}.
\end{aligned}$$

Proof. Let $1 \leq i < j \leq 8$. By direct calculations, the following holds:

$$\langle e_i - e_j, e_1 + e_2 \rangle = \langle e_i, e_1 \rangle + \langle e_i, e_2 \rangle - \langle e_j, e_1 \rangle - \langle e_j, e_2 \rangle = \begin{cases} 0 \ (i = 1 \text{ and } j = 2) \\ 1 \ (i = 1 \text{ and } 3 \leq j) \\ 1 \ (i = 2) \\ 0 \ (3 \leq i \text{ and } j = 8), \end{cases}$$

$$\langle e_i + e_j, e_1 + e_2 \rangle = \langle e_i, e_1 \rangle + \langle e_i, e_2 \rangle + \langle e_j, e_1 \rangle + \langle e_j, e_2 \rangle = \begin{cases} 2 \ (i = 1 \text{ and } j = 2) \\ 1 \ (i = 2) \\ 0 \ (3 \leq i), \end{cases}$$

$$\langle \sum_{i=1}^{8} \varepsilon_i e_i, e_1 + e_2 \rangle = \varepsilon_1 + \varepsilon_2.$$

Since ε_1 and ε_2 are ± 1, $\varepsilon_1 + \varepsilon_2 = 0$ or ± 2 holds. Moreover, since $\prod_{k=1}^{8} \varepsilon_k = 1$, $\prod_{k=3}^{8} \varepsilon_k = -1$ holds if and only if $\varepsilon_1 + \varepsilon_2 = 0$. Therefore, $\langle \sum_{i=1}^{8} \varepsilon_i e_i, e_1 + e_2 \rangle = 0$ holds if and only if $\prod_{k=3}^{8} \varepsilon_k = -1$. □

Corollary 11.2. *Let $s = g_{1,2} \circ f_{1,2}$. Then,*

$$Z_{W(E_8)}(s) = W(E_7 \sqcup A_1).$$

Proof. By Theorem 5.4 and Lemma 11.1, $Z_{W(E_8)}(s) = W_{\Gamma(s) \cup \{\pm(e_1+e_2)\}}$ holds and $W_{\Gamma(s) \cup \{\pm(e_1+e_2)\}}$ is the Coxeter group of type $E_7 \sqcup A_1$ since its Coxeter graph is shown in Figure 11.1, where $\beta = 2^{-1}(e_1 + e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8)$. □

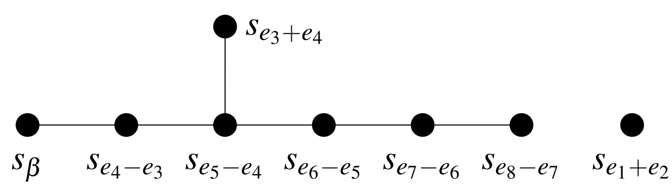

FIGURE 11.1. A Coxeter graph of $W_{\Gamma(s) \cup \{\pm(e_1+e_2)\}}$

Theorem 11.3.

$$H_2(Q_{W(E_8)}) \cong \mathbb{Z}/2\mathbb{Z}.$$

Proof. Let $\beta = 2^{-1}(e_1 + e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8)$. Let $s_0 = s_{e_1+e_2}$, $s_1 = s_\alpha$, $s_2 = s_{e_3+e_4}$, $s_i = s_{e_i - e_{i-1}} (i = 4,5,6,7,8)$. By Theorem 5.6, $\mathrm{Ker}\varepsilon \mid_{W(E_7 \sqcup A_1)}$ is generated by $\{r_i \mid i = 1,2,\ldots,7\}$ and have the following relations:

$$r_i^2 = 1 \ (i = 1,2,\ldots,6)$$
$$(r_j^{-1} r_{j+1})^3 = (r_3^{-1} r_7)^3 = 1 \ (j = 1,2,\ldots,6)$$
$$(r_k^{-1} r_l)^2 = 1 \ (k,l = 1,2,\ldots,6 \text{ and } |k-l| \geq 2 \text{ and } (k,l) \neq (3,7),(7,3)).$$

The relations $(r_j^{-1}r_{j+1})^3 = (r_3^{-1}r_7)^3 = 1$ $(j = 1,2,\ldots,6)$ imply $[r_1] = [r_2] = [r_3] = [r_4] = [r_5] = [r_6] = [r_7] \in (\mathrm{Ker}\varepsilon\,|_{W(E_7 \sqcup A_1)})_{ab}$. Therefore,

$$(\mathrm{Ker}\varepsilon\,|_{W(E_7 \sqcup A_1)})_{ab} = \langle r_1 \mid r_1^2 = 1\rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

holds. By Theorem 5.10, $H_2(Q_{W(E_8)}) \cong \mathbb{Z}/2\mathbb{Z}$ holds. □

12. Type $I_2(n)$

Let $n \geq 3$. A root system $\Phi(I_2(n))$ and a simple system $\Delta(I_2(n))$ are respectively as follows:

$$\Phi(I_2(n)) = \{\cos(k\pi/n)e_1 + \sin(k\pi/n)e_2 \in \mathbb{R}^2 \mid k = 0,1,\ldots,2n-1\}$$
$$\Delta(I_2(n)) = \{e_1, \cos((n-1)\pi/n)e_1 + \sin((n-1)\pi/n)e_2\}.$$

The root corresponding to the reflection about the y-axis is e_1.

Lemma 12.1. *Let s be a reflection about y-axis. Then,*

$$\Gamma(s) = \begin{cases} \{\pm e_2\} & (n \text{ is even}) \\ \emptyset & (n \text{ is odd}). \end{cases}$$

Proof. Let $k = 0,1,\ldots,2n-1$. Then,

$$\langle e_1, \cos(k\pi/n)e_1 + \sin(k\pi/n)e_2\rangle = \cos(k\pi/n).$$

Since $0 \leq k < 2n$, $0 \leq k\pi/n < 2\pi$ holds. Therefore, when $\cos(k\pi/n) = 0$,

$$k\pi/n = \pi/2, 3\pi/2 \quad (\text{i.e. } k = n/2, 3n/2).$$

If n is even, the only roots satisfying $\cos(k\pi/n) = 0$ are $\pm e_2$. If n is odd, there is no root satisfying $\cos(k\pi/n) = 0$. □

Lemma 12.2. *Let $n(\geq 3)$ be an odd integer, s be a reflection about the y-axis. Then,*

$$Z_{W(I_2(n))}(s) = W(A_1).$$

Proof. By Theorem 5.4 and Lemma 12.1, it follows immediately that

$$Z_{W(I_2(n))}(s) = W_{\{\pm e_1\}} = W(A_1).$$

□

Theorem 12.3. *Let $n(\geq 3)$ be an odd integer. Then,*

$$H_2(Q_{W(I_2(n))}) \cong 0.$$

Proof. Since $\mathrm{Ker}\varepsilon\,|_{W(A_1)}$ is the trivial group by Theorem 5.6, its abelianization is also the trivial group. By Theorem 5.10, $H_2(Q_{W(I_2(n))}) \cong 0$ holds. □

Remark 12.4. The results of $H_2(Q_{W(I_2(n))})$ is the same as that calculated in Eisermann [9].

13. Type H_4

Let $a = \cos(\pi/5) = (1+\sqrt{5})/4$ and $b = \cos(2\pi/5) = (-1+\sqrt{5})/4$. A root system $\Phi(H_4)$ consists of the unit vectors in $\mathbb{H}$ obtained from 1, $2^{-1}(1+\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})$ and $a+2^{-1}\boldsymbol{i}+b\boldsymbol{j}$ by even permutations of coordinates and arbitrary sign changes, and a simple system $\Delta(H_4)$ is $\{a-2^{-1}\boldsymbol{i}+b\boldsymbol{j}, -a+2^{-1}\boldsymbol{i}+b\boldsymbol{j}, 2^{-1}+b\boldsymbol{i}-a\boldsymbol{j}, -2^{-1}-a\boldsymbol{i}+b\boldsymbol{k}\}$.

Lemma 13.1. *Let s be a reflection corresponding to $\alpha = a-2^{-1}\boldsymbol{i}+b\boldsymbol{j}$. Then,*

$$\begin{aligned}\Gamma(s) = \{&\pm\boldsymbol{k}, \pm 2^{-1}(1+\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}), \pm 2^{-1}(-1-\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}), \pm(b+a\boldsymbol{i}+2^{-1}\boldsymbol{j}),\\ &\pm(2^{-1}\boldsymbol{i}+a\boldsymbol{j}+b\boldsymbol{k}), \pm(b+2^{-1}\boldsymbol{i}+a\boldsymbol{k}), \pm(2^{-1}+a\boldsymbol{i}+b\boldsymbol{k}),\\ &\pm(b\boldsymbol{i}+2^{-1}\boldsymbol{j}+a\boldsymbol{k}), \pm(2^{-1}+b\boldsymbol{i}-a\boldsymbol{j}), \pm(b+2^{-1}\boldsymbol{i}-a\boldsymbol{k}),\\ &\pm(b-a\boldsymbol{j}+2^{-1}\boldsymbol{k}), \pm(b\boldsymbol{i}+2^{-1}\boldsymbol{j}-a\boldsymbol{k}), \pm(2^{-1}\boldsymbol{i}+a\boldsymbol{j}-b\boldsymbol{k}),\\ &\pm(2^{-1}+a\boldsymbol{i}-b\boldsymbol{k}), \pm(-b+a\boldsymbol{j}+2^{-1}\boldsymbol{k})\}.\end{aligned}$$

Proof. By direct calculations of inner products, we determine $\Gamma(s)$. Let $x = 1$ or $\boldsymbol{i}$ or $\boldsymbol{j}$ or $\boldsymbol{k}$. Then,

$$\langle x, \alpha\rangle = \begin{cases} a & (x=1)\\ 2^{-1} & (x=\boldsymbol{i})\\ b & (x=\boldsymbol{j})\\ 0 & (x=\boldsymbol{k}).\end{cases}$$

Let y be a root obtained from $2^{-1}(1+\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})$ by arbitrary sign changes. Then,

$$\langle y, \alpha\rangle = \begin{cases} b & (y=2^{-1}(1+\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}))\\ -2^{-1} & (y=2^{-1}(-1+\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}))\\ b+2^{-1} & (y=2^{-1}(1-\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}))\\ 0 & (y=2^{-1}(1+\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}))\\ b & (y=2^{-1}(1+\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}))\\ 0 & (y=2^{-1}(-1-\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}))\\ -b-2^{-1} & (y=2^{-1}(-1+\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}))\\ -2^{-1} & (y=2^{-1}(-1+\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k})).\end{cases}$$

Let z be a root obtained from $a+2^{-1}\boldsymbol{i}+b\boldsymbol{j}$ by even permutations of coordinates and arbitrary sign changes. Then,

$$\langle z,\alpha\rangle=\begin{cases}2^{-1} & (y=a+2^{-1}\boldsymbol{i}+b\boldsymbol{j})\\ 0 & (y=b+a\boldsymbol{i}+2^{-1}\boldsymbol{j})\\ -b & (y=a\boldsymbol{i}+b\boldsymbol{j}+2^{-1}\boldsymbol{k})\\ 2^{-1} & (y=2^{-1}+b\boldsymbol{i}+a\boldsymbol{j})\\ 0 & (y=2^{-1}\boldsymbol{i}+a\boldsymbol{j}+b\boldsymbol{k})\\ 2^{-1} & (y=2^{-1}+b\boldsymbol{j}+a\boldsymbol{k})\\ 0 & (y=b+2^{-1}\boldsymbol{i}+a\boldsymbol{k})\\ a & (y=a+2^{-1}\boldsymbol{j}+b\boldsymbol{k})\\ 2^{-1} & (y=a+b\boldsymbol{i}+2^{-1}\boldsymbol{k})\\ 0 & (y=2^{-1}+a\boldsymbol{i}+b\boldsymbol{k})\\ 2^{-1} & (y=b+a\boldsymbol{j}+2^{-1}\boldsymbol{k})\\ 0 & (y=b\boldsymbol{i}+2^{-1}\boldsymbol{j}+a\boldsymbol{k}),\end{cases}$$

$$\langle z,\alpha\rangle=\begin{cases}-a & (y=-a+2^{-1}\boldsymbol{i}+b\boldsymbol{j})\\ a & (y=b-a\boldsymbol{i}+2^{-1}\boldsymbol{j})\\ 2^{-1} & (y=-a\boldsymbol{i}+b\boldsymbol{j}+2^{-1}\boldsymbol{k})\\ 0 & (y=2^{-1}+b\boldsymbol{i}-a\boldsymbol{j})\\ -2^{-1} & (y=2^{-1}\boldsymbol{i}-a\boldsymbol{j}+b\boldsymbol{k})\\ 2^{-1} & (y=2^{-1}+b\boldsymbol{j}-a\boldsymbol{k})\\ 0 & (y=b+2^{-1}\boldsymbol{i}-a\boldsymbol{k})\\ -2^{-1} & (y=-a+2^{-1}\boldsymbol{j}+b\boldsymbol{k})\\ -a & (y=-a+b\boldsymbol{i}+2^{-1}\boldsymbol{k})\\ a & (y=2^{-1}-a\boldsymbol{i}+b\boldsymbol{k})\\ 0 & (y=b-a\boldsymbol{j}+2^{-1}\boldsymbol{k})\\ 0 & (y=b\boldsymbol{i}+2^{-1}\boldsymbol{j}-a\boldsymbol{k}),\end{cases}$$

$$\langle z,\alpha\rangle=\begin{cases}1 & (y=a-2^{-1}\boldsymbol{i}+b\boldsymbol{j})\\ -b & (y=b+a\boldsymbol{i}-2^{-1}\boldsymbol{j})\\ -b & (y=a\boldsymbol{i}+b\boldsymbol{j}-2^{-1}\boldsymbol{k})\\ -b & (y=-2^{-1}+b\boldsymbol{i}+a\boldsymbol{j})\\ 2^{-1} & (y=-2^{-1}\boldsymbol{i}+a\boldsymbol{j}+b\boldsymbol{k})\\ -b & (y=-2^{-1}+b\boldsymbol{j}+a\boldsymbol{k})\\ 2^{-1} & (y=b-2^{-1}\boldsymbol{i}+a\boldsymbol{k})\\ 2^{-1} & (y=a-2^{-1}\boldsymbol{j}+b\boldsymbol{k})\\ 2^{-1} & (y=a+b\boldsymbol{i}-2^{-1}\boldsymbol{k})\\ -a & (y=-2^{-1}+a\boldsymbol{i}+b\boldsymbol{k})\\ 2^{-1} & (y=b+a\boldsymbol{j}-2^{-1}\boldsymbol{k})\\ -b & (y=b\boldsymbol{i}-2^{-1}\boldsymbol{j}+a\boldsymbol{k}),\end{cases}$$

$$\langle z,\alpha\rangle=\begin{cases}b & (y=a+2^{-1}\boldsymbol{i}-b\boldsymbol{j})\\ -2^{-1} & (y=-b+a\boldsymbol{i}+2^{-1}\boldsymbol{j})\\ -2^{-1} & (y=a\boldsymbol{i}-b\boldsymbol{j}+2^{-1}\boldsymbol{k})\\ a & (y=2^{-1}-b\boldsymbol{i}+a\boldsymbol{j})\\ 0 & (y=2^{-1}\boldsymbol{i}+a\boldsymbol{j}-b\boldsymbol{k})\\ b & (y=2^{-1}-b\boldsymbol{j}+a\boldsymbol{k})\\ -2^{-1} & (y=-b+2^{-1}\boldsymbol{i}+a\boldsymbol{k})\\ a & (y=a+2^{-1}\boldsymbol{j}-b\boldsymbol{k})\\ a & (y=a-b\boldsymbol{i}+2^{-1}\boldsymbol{k})\\ 0 & (y=2^{-1}+a\boldsymbol{i}-b\boldsymbol{k})\\ 0 & (y=-b+a\boldsymbol{j}+2^{-1}\boldsymbol{k})\\ b & (y=-b\boldsymbol{i}+2^{-1}\boldsymbol{j}+a\boldsymbol{k}).\end{cases}$$

□

Lemma 13.2. *Let s be a reflection corresponding to $\alpha_0=a-2^{-1}\boldsymbol{i}+b\boldsymbol{j}$. Then,*

$$Z_{W(H_4)}(s)=W(H_3\sqcup A_1).$$

Proof. By Theorem 5.4, $Z_{W(H_4)}(s)=W_{\Gamma(s)\cup\{\pm\alpha_0\}}$ holds. By Lemma 13.1, $Z_{W(H_4)}(s)$ is the Coxeter group of type $H_3\sqcup A_1$ since its Coxeter graph is shown in Figure 13.1, where $\alpha_1=-b+a\boldsymbol{j}+2^{-1}\boldsymbol{k}$, $\alpha_2=2^{-1}+b\boldsymbol{i}-a\boldsymbol{j}$ and $\alpha_3=-2^{-1}-a\boldsymbol{i}-b\boldsymbol{k}$. □

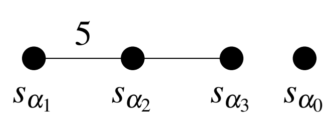

FIGURE 13.1. A Coxeter graph of $W_{\Gamma(s)\cup\{\pm\alpha_0\}}$

Theorem 13.3.

$$H_2(Q_{W(H_4)}) \cong \mathbb{Z}/2\mathbb{Z}.$$

Proof. Let $\alpha_0 = a - 2^{-1}\boldsymbol{i} + b\boldsymbol{j}$, $\alpha_1 = -b + a\boldsymbol{j} + 2^{-1}\boldsymbol{k}$, $\alpha_2 = 2^{-1} + b\boldsymbol{i} - a\boldsymbol{j}$ and $\alpha_3 = -2^{-1} - a\boldsymbol{i} - b\boldsymbol{k}$. Let $s_i = s_{\alpha_i}$ $(i = 0,1,2,3)$. By Theorem 5.6, $\mathrm{Ker}\varepsilon\,|_{W(H_3 \sqcup A_1)}$ is generated by $\{r_1, r_2, r_3\}$ and have the following relations:

$$\begin{aligned} &r_i^2 = 1 \ (i = 1,2,3) \\ &(r_1^{-1}r_3)^3 = (r_2^{-1}r_3)^3 = 1 \\ &(r_1^{-1}r_3)^2 = 1. \end{aligned}$$

The relations $(r_1^{-1}r_3)^3 = (r_2^{-1}r_3)^3 = 1$ imply $[r_1] = [r_2] = [r_3] \in (\mathrm{Ker}\varepsilon\,|_{W(H_3 \sqcup A_1)})_{ab}$. Therefore,

$$(\mathrm{Ker}\varepsilon\,|_{W(H_3 \sqcup A_1)})_{ab} = \langle r_1 \mid r_1^2 = 1\rangle \cong \mathbb{Z}/2\mathbb{Z}.$$

holds. By Theorem 5.10, $H_2(Q_{W(H_4)}) \cong \mathbb{Z}/2\mathbb{Z}$ holds. □

14. Type H_3

Let $a = \cos(\pi/5) = (1+\sqrt{5})/4$ and $b = \cos(2\pi/5) = (-1+\sqrt{5})/4$. A root system $\Phi(H_3)$ is the subset of the root system of $W(H_4)$ whose elements are orthogonal to $\boldsymbol{k}$. A simple system $\Delta(H_3)$ is $\{a - 2^{-1}\boldsymbol{i} + b\boldsymbol{j}, -a + 2^{-1}\boldsymbol{i} + b\boldsymbol{j}, 2^{-1} + b\boldsymbol{i} - a\boldsymbol{j}\}$.

Lemma 14.1. *Let s be a reflection corresponding to $a - 2^{-1}\boldsymbol{i} + b\boldsymbol{j}$. Then, $\Gamma(s) = \{\pm(2^{-1} + b\boldsymbol{i} - a\boldsymbol{j}), \pm(b + a\boldsymbol{i} + 2^{-1}\boldsymbol{j})\}$.*

Proof. By Lemma 13.1, roots of $W(H_3)$ orthogonal to $a - 2^{-1}\boldsymbol{i} + b\boldsymbol{j}$ are only $\pm(2^{-1} + b\boldsymbol{i} - a\boldsymbol{j}), \pm(b + a\boldsymbol{i} + 2^{-1}\boldsymbol{j})$. □

Lemma 14.2. *Let s be a reflection corresponding to $a - 2^{-1}\boldsymbol{i} + b\boldsymbol{j}$. Then, we have*

$$Z_{W(H_3)}(s) = W(A_1 \sqcup A_1 \sqcup A_1).$$

Proof. By Theorem 5.4, $Z_{W(H_3)}(s) = W_{\Gamma(s) \cup \{\pm(a - 2^{-1}\boldsymbol{i} + b\boldsymbol{j})\}}$ holds. Since $2^{-1} + b\boldsymbol{i} - a\boldsymbol{j}$, $b + a\boldsymbol{i} + 2^{-1}\boldsymbol{j}$ and $a - 2^{-1}\boldsymbol{i} + b\boldsymbol{j}$ are mutually orthogonal, together with Lemma 14.1, $W_{\Gamma(s) \cup \{\pm(a - 2^{-1}\boldsymbol{i} + b\boldsymbol{j})\}}$ is a Coxeter group of type $A_1 \sqcup A_1 \sqcup A_1$. □

Theorem 14.3.

$$H_2(Q_{W(H_3)}) \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

Proof. Let $s_0 = s_{a - 2^{-1}\boldsymbol{i} + b\boldsymbol{j}}$, $s_1 = s_{2^{-1} + b\boldsymbol{i} - a\boldsymbol{j}}$ and $s_2 = s_{b + a\boldsymbol{i} + 2^{-1}\boldsymbol{j}}$. By Theorem 5.6, $\mathrm{Ker}\varepsilon\,|_{W(A_1 \sqcup A_1 \sqcup A_1)}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Therefore, its abelianization is also isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. By Theorem 5.10, $H_2(Q_{W(H_3)}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ holds. □

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