博士学位論文

Limit theorems on random matrices and finite free probability

（ランダム行列と有限自由確率論に関する極限定理）

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Contents

Symbols and Notations

The following list describes several symbols that will be used later in this thesis.

I. Number Sets

$\mathbb{N}$ Natural numbers with its minimum element 1

$\mathbb{Z}$ Integers

$\mathbb{Q}$ Rational numbers

$\mathbb{R}$ Real numbers

$\mathbb{C}$ Complex numbers

$\mathbb{T}$ Unit circle on $\mathbb{C}$ $\{z \in \mathbb{C} : |z| = 1\}$

$[n]$ The n point set $\{1, 2, \ldots, n\}$

II. Posets and Möbius Functions

$P = (P, \leq)$ Partially ordered set (for short, poset), page 12

μ_P Möbius function defined on a poset P, page 13

$(\mathrm{B}(n), \mu_n^{\mathrm{B}})$ Boolean poset and its Möbius function μ_n^{B}, page 12

$(\mathrm{P}(n), \mu_n^{\mathrm{P}})$ All partition and its Möbius function, page 12

$(\mathrm{NC}(n), \mu_n^{\mathrm{NC}})$ Non-crossing partition and its Möbius function, page 13

$\mathrm{Kr}(\pi)$ Kreweras complement of $\pi \in \mathrm{NC}(n)$, page 14

III. Measures and Operations

$(\Omega, \mathcal{F}, \mathbb{P})$ Probability space; the triplet of sample space Ω, σ-algebra $\mathcal{F}$ and probability $\mathbb{P}$.

$\mathbb{E}[X]$ Expectation of a random variable X

$\mathrm{M}_n(\mu)$ Moments of a probability measure μ, see equation (0.0.1), page 1

$\mathcal{D}_c(\mu)$ Dilation of a probability measure μ, page 1

$\mu \boxplus \nu$ Free additive convolution of μ and ν, page 4

$\mu \boxtimes \nu$ Free multiplicative convolution of μ and ν, page 4

$\kappa_n(\mu)$ Free cumulants of a probability measure μ, see equation (2.2.1), page 16

R_μ R-transform of μ, page 16

S_μ S-transform of μ, page 16

Σ_t Free unitary normal distribution on $\mathbb{T}$ with parameter t, page 50

Π_t Free unitary Poisson distribution on $\mathbb{T}$ with parameter t, page 51

λ_t Multiplicative free semicircle distribution on $[0,\infty)$ with parameter t, page 54

$\Phi(\mu)$ The limit measure in Tucci's limit theorem , see equation (4.6.1), page 55

IV. Polynomials and Operations

$\mathbb{C}[x]$ The set of all polynomials with complex coefficients

$\operatorname{lead} f$ The leading coefficient of a polynomial f, page 1

$\deg f$ The degree of a polynomial f, page 1

$\mu[\![p]\!]$ Empirical root distribution of a polynomial p, see equation (0.0.2), page 2

$\mathcal{D}_c(p)$ Dilation of a polynomial p, page 2

$\phi_\alpha(p)$ Root dilation of a monic polynomial p having non-negative roots, see equation (0.0.3), page 2

$p \boxplus_d q$ Finite additive convolution of p and q, see equation (0.0.4), page 2

$p \boxtimes_d q$ Finite multiplicative convolution of p and q, see equation (0.0.5), page 2

$\kappa_n^{(d)}(p)$ Finite free cumulants of p, see equation (4.1.1), page 41

$\widehat{L}_d^{(\lambda)}(x)$ Normalized Laguerre polynomial, see equation (4.1.2), page 41

$H_d(z,t)$ Unitary Hermite polynomial, see equation (4.4.1), page 49

$L_{d,m}(z)$ Unitary Laguerre polynomial, see equation (4.4.3), page 51

V. Matrices and Operations

I_N Identity matrix of size N

P_N Projection of size N with rank $N-1$

D_N Diagonal matrix of size N

U_N Haar unitary matrix of size N , page 32

Tr_N The un-normalized trace

tr_N The normalized trace $\frac{1}{N}\operatorname{Tr}_N$

VI. Miscellaneous

$\mathfrak{S}_n$ Symmetric group acting on $[n]$

$\#(g)$ The cycle number of $g \in \mathfrak{S}_n$, page 15

$|g|$ The length of $g \in \mathfrak{S}_n$, page 15

α_π The multiplicative expansion for a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\pi \in \mathrm{P}(n)$, page 2

$\operatorname{Wg}(g,N)$ Weingarten function on $\mathfrak{S}_n$ with s N, page 34

μ_n^{Wg} The main part of Weingarten function on $\mathfrak{S}_n$, page 34

Notations

Denote by $\mathcal{P}$ and $\mathcal{P}_+$ the set of all probability measures on $\mathbb{R}$ and $[0,\infty)$, respectively. Additionally, we define $\mathcal{P}_c$ and $\mathcal{P}_{+,c}$ as the set of all compactly supported probability measures on $\mathbb{R}$ and $[0,\infty)$, respectively. The notation $\xrightarrow{w}$ means the weak convergence of probability measures.

Let $L^{\infty-}(\Omega)$ be the set of random variables X on Ω such that

$$\mathbb{E}[|X|^n] < \infty$$

for all $n \in \mathbb{N}$. We always assume in this thesis that every random variable is in $L^{\infty-}(\Omega)$ unless otherwise noted. Similarly, for a probability measure μ on $\mathbb{C}$, assume

$$\int_{\mathbb{C}} |z|^n \mu(dz) < \infty, \qquad n \in \mathbb{N}$$

and define

$$\mathrm{M}_n(\mu) := \int_{\mathbb{C}} z^n \mu(dz), \qquad n \in \mathbb{N} \tag{0.0.1}$$

called the moments of μ. Define the dilation $\mathcal{D}_c(\mu)(B) := \mu(c^{-1}B)$ for a probability measure μ on $\mathbb{C}$, $c \neq 0$ and a Borel set B in $\mathbb{C}$.

For a polynomial $f \in \mathbb{C}[x]$ written as $f(x) = a_d x^d + \cdots + a_1 x + a_0$ ($a_d \neq 0$), we define $\deg f := d$ and $\mathrm{lead}(f) := a_d$. A polynomial $f \in \mathbb{C}[x]$ is said to be *monic* if $\mathrm{lead}(f) = 1$. Also, $f \in \mathbb{C}[x]$ is said to be *real-rooted* if all roots of f are in $\mathbb{R}$. The following subsets of $\mathbb{C}[x]$ are often used in this thesis:

- $\mathbb{C}[x]_0 := \{f \in \mathbb{C}[x] : f(0) = 0\}$;
- $\mathbb{P}_{\mathrm{mon}}(d) := \{p \in \mathbb{C}[x] : p \text{ is monic and } \deg p = d\}$.

We use the following notations for a polynomial p of degree d.

- The number $e_k(p)$ denotes the $(d-k)$-th coefficient of p for $k = 0, 1, \ldots, d$, and it is often useful to write
$$\widetilde{e}_k(p) := \binom{d}{k}^{-1} e_k(p), \qquad k = 0, 1, \ldots, d,$$
instead of $e_k(p)$. Then a polynomial $p(x) = \sum_{k=0}^d (-1)^k e_k(p) x^{d-k}$ can be written as
$$p(x) = \sum_{k=0}^d (-1)^k \binom{d}{k} \widetilde{e}_k(p) x^{d-k}.$$
- The empirical root distribution of p is the probability measure
$$\mu[\![p]\!] := \frac{1}{d} \sum_{\substack{z \in \mathbb{C} \\ p(z)=0}} m_p(z) \delta_z, \tag{0.0.2}$$
where $m_p(z)$ denotes the multiplicity of the root at z.
- For $c \neq 0$, $\mathcal{D}_c(p)(x) := c^d p(x/c)$ for $p \in \mathbb{P}_{\mathrm{mon}}(d)$.
- For a monic polynomial $p(x) = \prod_{k=1}^d (x - \lambda_k)$ with nonnegative roots,
$$\phi_\alpha(p)(x) := \prod_{k=1}^d (x - \lambda_k^\alpha), \qquad \alpha > 0. \tag{0.0.3}$$

For $p, q \in \mathbb{P}_{\mathrm{mon}}(d)$, one defines the *finite free additive convolution* $p \boxplus_d q$ to be

$$(p \boxplus_d q)(x) = \sum_{k=0}^{d} (-1)^k \binom{d}{k} \sum_{i+j=k} \frac{k!}{i!j!} \widetilde{e}_i(p) \widetilde{e}_j(q) x^{d-k}. \tag{0.0.4}$$

For $p \in \mathbb{P}_{\mathrm{mon}}(d)$, $p^{\boxplus_d m}$ denotes the m-th power of finite free additive convolution of p. One also defines the *finite free multiplicative convolution* $p \boxtimes_d q$ to be

$$(p \boxtimes_d q)(x) := \sum_{k=0}^{d} (-1)^k \binom{d}{k} \widetilde{e}_k(p) \widetilde{e}_k(q) x^{d-k}. \tag{0.0.5}$$

In particular, $p^{\boxtimes_d n}$ denotes the n-th power of finite free multiplicative convolution of p.

For a finite multi-set $\Lambda = \{\lambda_1, \cdots, \lambda_d\}$ of complex numbers, *the k-th elementary symmetric polynomials* $e_k(\Lambda)$ is denoted by

$$e_k(\Lambda) := \sum_{J \subset [d],\ |J|=k} \left(\prod_{j \in J} \lambda_j \right), \qquad e_0(\Lambda) := 1,$$

where $[d] = \{1, 2, \cdots, d\}$. In addition, we define

$$\widetilde{e}_k(\Lambda) := \binom{d}{k}^{-1} e_k(\Lambda)$$

for each $0 \leq k \leq d$.

Throughout this thesis, for a sequence $\{\alpha_n\}_{n \in \mathbb{N}} \subset \mathbb{C}$ and a partition π of $[n]$, we define

$$\alpha_\pi := \prod_{V \in \pi} \alpha_{|V|}.$$

Similarly, for a permutation $g \in \mathfrak{S}_k$, let $\mathrm{Tr}_g[A_1, A_2, \ldots, A_k]$ be the product of traces according to the cycle decomposition of g; for example if $g = (1,3,2,5)(4)(6,9)(7,8) \in \mathfrak{S}_9$ then $\mathrm{Tr}_g[A_1, A_2, \ldots, A_9] = \mathrm{Tr}(A_1 A_3 A_2 A_5)\,\mathrm{Tr}(A_4)\,\mathrm{Tr}(A_6 A_9)\,\mathrm{Tr}(A_7 A_8)$.

Chapter 1

Introduction

Background of the research

Free probability theory is a branch of mathematics that emerged in the 1980s from the concept of free independence introduced by Voiculescu when studying operator algebras generated by free groups. The group von Neumann algebra $L(G)$ is defined as the weak closure of group algebra $\mathbb{C}[G]$ for a discrete group G acting on $\ell^2(G)$. A naive question arises here: for different free groups $\mathbb{F}_n$ and $\mathbb{F}_m$, are the corresponding von Neumann algebras $L(\mathbb{F}_n)$ and $L(\mathbb{F}_m)$ isomorphic or not? It is known as *the free group factor isomorphism problem* and is still an open problem in operator algebra theory. Voiculescu obtained new insights into the structure of free group factors $L(\mathbb{F}_n)$ by paying attention to the property of freeness of groups and also algebras (e.g., see [MS17; VDN92]).

He also found the celebrated application to random matrix theory called *asymptotic freeness* in [Voi91]. Roughly speaking, it states that an independent family of random matrices satisfies the free relation as the dimensions tend to infinity. In other words, we can obtain information about an independent family of random matrices by using free probability if the dimensions are large enough. This discovery led to a deeper investigation of the relationship with random matrix theory [AGZ10], e.g., large deviations and free entropy [BCG03; Voi02], BBP phase transition [BBP05; Bel+17], additivity violation of minimum output entropy [BCN16; Has09], etc. Since then, many researchers have studied this fruitful topic having interesting connections to other fields of mathematics, e.g., combinatorics of noncrossing partitions [NS06], representation theory of symmetric groups [Bia98; Bia01], quantum information theory [NC00].

One of the crucial approaches to free probability theory is to pursue the analogy of classical probability theory, which we call the usual probability theory based on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to contrast the difference. There are so many probabilistic analogues between free and classical probability theories. For example, independences, convolutions, cumulants, the law of large numbers (LLN), the central limit theorem (CLT), normal distributions, Lévy–Khintchine representations, entropies, and so on.

More recently, in [Mar21; MSS22], Marcus, Spielman and Srivastava investigated a relationship between polynomial convolutions and the sum of random matrices related to free probability theory. They gave the affirmative answer to the Kadison–Singer problem and the construction of non-trivial Ramanujan graphs. Since the 2010s, this research area has been referred to as *finite free probability* because there are many analogues for free probability. Recent progress in this field includes the development of finite free cumulants by Arizmendi and Perales [AP18], which can be used to derive various results in finite free probability using a combinatorial approach.

Motivation

Free probability and random matrices

Let us briefly discuss free probability theory, which is not the main subject but is a motivation for the research addressed in this thesis. The reader is referred to [MS17; NS06] for further details.

Definition 1.0.1 (Non-commutative probability space)**.** Let $\mathcal{A}$ be a $\mathbb{C}$-algebra with the unit $1_{\mathcal{A}}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ a linear map such that $\varphi(1_{\mathcal{A}}) = 1$. We call the pair $(\mathcal{A}, \varphi)$ a *noncommutative probability space.* Elements $a \in \mathcal{A}$ are called *noncommutative random variables.*

We often impose more assumptions on $(\mathcal{A}, \varphi)$; $\mathcal{A}$ is a C^*-algebra, φ is a positive map, etc. Then, for a selfadjoint element $a \in \mathcal{A}$, there exists a corresponding probability measure μ_a such that

$$\varphi(a^k) = \int_{\mathbb{R}} x^k \mu_a(dx), \qquad k \in \mathbb{N}$$

by the continuous functional calculus.

Definition 1.0.2 (Free independence)**.** For $(\mathcal{X}_i)_{i\in I}$ a family of subsets of $\mathcal{A}$, let $\mathcal{A}_i := \mathrm{alg}(1_{\mathcal{A}}, \mathcal{X}_i)$ for $i \in I$. We call $(\mathcal{X}_i)_{i\in I}$ *freely independent* if

$$\varphi(a_1 \cdots a_n) = 0$$

whenever $n \in \mathbb{N}$, $(i_1, \ldots, i_n) \in I^n$, $i_j \neq i_{j+1}$ such that $i_1 \neq i_2, \ldots, i_{n-1} \neq i_n$ and $a_j \in \mathcal{A}_{i_j}$ such that $\varphi(a_j) = 0$ for every $j = 1, \ldots, n$.

Voiculescu introduced this concept to understand the structure of free group factors $L(\mathbb{F}_n)$ and then developed another kind of probability theory; convolutions, normal distributions, the law of large numbers, the central limit theorem, and so on. For example, we can define the free additive (resp. multiplicative) convolution $\mu \boxplus \nu$ (resp. $\mu \boxtimes \nu$) for probability measures μ, ν on $\mathbb{R}$ as the distribution of $a + b$ (resp. ab) where noncommutative random variables a, b are free independent, a has the distribution μ and b has the distribution ν, see [Voi87] and [BV93] for more details.

A hint for the application of free probability to random matrix theory can be traced back to Wigner's theorem in the 1950s. Let us take a brief look. Basically, in random matrix theory, the main interest lies in the eigenvalue distribution. For A_N an $N \times N$ hermitian matrix and $\{\lambda_i\}_{i=1}^N$ the eigenvalues of A_N, the *empirical eigenvalue distribution (EED)* of A_N is defined as

$$\mu_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}.$$

To understand the empirical eigenvalue distribution μ_N, it is basic to look at the moments

$$\mathrm{M}_k(\mu_N) = \frac{1}{N} \sum_{i=1}^{N} \lambda_i^k = \mathrm{tr}_N(A_N^k), \qquad k \in \mathbb{N}.$$

As an example, let $A_N = (a_{ij})_{i,j=1}^N$ be a GUE (Gaussian Unitary Ensemble) random matrix, i.e. $a_{ij} = x_{ij} + \sqrt{-1} y_{ij}$ is a complex Gaussian random variable normalized such that $\sqrt{N} a_{ij}$ is a standard complex Gaussian random variable ($\mathbb{E}[a_{ij}] = 1$, $\mathbb{E}[|a_{ij}|^2] = 1/N$), $a_{ij} = \overline{a_{ji}}$, and $\{x_{ij}\}_{i\geq j} \cup \{y_{ij}\}_{i>j}$ are independent. Although we can even compute precisely the density on the space of $N \times N$ hermitian matrices and the joint distribution of the eigenvalues in the case of

GUE, the moments are also directly calculated by using Wick's formula and the combinatorics of pair partitions:

$$\mathbb{E}[\mathrm{tr}_N(A_N^k)] = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ \sum_{\pi \in \mathrm{P}_2(2l)} N^{\#(\gamma_{2l}\pi)-l-1}, & \text{if } k = 2l \text{ for some } l \in \mathbb{N}, \end{cases}$$

for a precise discussion and notation here, see [MS17, Chapter 1]. The condition $\#(\gamma_{2l}\pi) = l+1$ implies $\pi \in \mathrm{NC}_2(2l)$ and we finally obtain

$$\lim_{N\to\infty} \mathbb{E}[\mathrm{tr}_N(A_N^{2l})] = \frac{1}{l+1}\binom{2l}{l} = \int_{\mathbb{R}} x^{2l} \mu_{\mathrm{sc}}(dx),$$

where $\mu_{\mathrm{sc}}(dx) = (2\pi)^{-1}\sqrt{4-x^2}\mathbb{1}_{[-2,2]}(x)dx$ is called the *(standard) semicircle distribution.* By refining the discussion above, we can obtain a stronger result: the empirical distributions of A_N weakly converge to μ_{sc} almost surely. More generally, this result holds even if the random matrix elements a_{ij} do not follow Gaussian distributions (such matrix is called a *Wigner matrix*) and is widely known as *Wigner's theorem* [Wig55]. One of Voiculescu's fundamental contributions was the discovery that the semicircle distribution plays the role of the normal distribution in free probability theory. For instance, the free central limit theorem holds: the limit of $\mathcal{D}_{1/\sqrt{n}}(\mu^{\boxplus n})$ exists and coincides with the standard semicircle distribution μ_{sc} for any probability measure $\mu \in \mathcal{P}$ with mean 0 and variance 1.

It is also well known that a GUE random matrix has the density

$$\frac{1}{Z_N} \exp\left(-\frac{1}{2}\operatorname{Tr} H^2\right)$$

on the space of hermitian matrices $H = (H_{ij})_{i,j=1}^N$, where Z_N is the normalization constant. As can be seen easily from this, it follows that a GUE random matrix exhibits unitary invariance: its distribution in the space of $N \times N$ hermitian matrices is unchanged under any unitary adjoint action.

Definition 1.0.3 (Unitarily invariant)**.** A random hermitian matrix A_N is said to be *unitarily invariant* if its distribution is invariant for any unitary adjoint action.

Definition 1.0.4 (Haar Unitary)**.** Let $\mathcal{U}_N$ be the $N \times N$ unitary group. There exists a unique left-invariant probability measure on $\mathcal{U}_N$ because $\mathcal{U}_N$ is a compact group. Let U_N be a unitary random matrix whose distribution is the left-invariant probability measure on $\mathcal{U}_N$ and then we call U_N a *Haar unitary matrix of size N.*

Let A_N be a unitarily invariant hermitian of size N whose eigenvalues are $\{\lambda_i\}_{i=1}^N$. It is known that a diagonalization $A_N = U_N D_N U_N{}^*$ exists, where $D_N = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$ and U_N is a Haar unitary random matrix of size N and independent of D_N (see [CM14, Proposition 6.1]).

While Wigner's theorem is about a single random matrix sequence, let us consider such two sequences. Then the concept of free independence naturally appears as the dimensions tend to infinity. Note that asymptotic freeness can be applied in various settings, not only in this typical case.

Proposition 1.0.5 (Asymptotic freeness)**.** Let A_N and B_N be sequences of hermitian matrices such that their EEDs converge in moments. Then A_N and $U_N B_N U_N^*$ satisfy the free independence asymptotically as the dimensions N tend to infinity.

A simple consequence of the asymptotic freeness means: let μ_1 (resp. μ_2) be the limit measure of EEDs of A_N (resp. B_N) and then the EEDs of the sum of $A_N + U_N B_N U_N^*$ converge to $\mu_1 \boxplus \mu_2$ in moments. It is a classical problem in linear algebra to describe the eigenvalues of $A_N + B_N$ for

given two hermitian matrices A_N and B_N, e.g. Weyl's inequality and Horn's conjecture [KT01]. However, if B_N is at, in some sense, probabilistically the most general position for A_N, then asymptotic freeness implies that the overall behavior of the eigenvalues of the sum $A_N + B_N$ is almost determined when the dimension is large enough.

Asymptotic representation theory and the Markov–Krein correspondence

Another interesting appearance of free probability can be seen in *asymptotic representation theory* initiated by Vershik et al. [LS77; VK77], which was pointed out by Biane in the 1990s [Bia98]. Loosely speaking, the main part of the tensor representation of symmetric groups $\mathfrak{S}_n$ can be described by free convolution as the degree n tends to infinity.

Let λ be a Young diagram of size n displayed in the Russian style. We then identify λ as a scaled piecewise linear function and equivalently as its local minimal and maximal points described by interlacing integer sequences $\{x_i\}_{i=1}^N$ and $\{y_j\}_{j=1}^{N-1}$, i.e.

$$x_1 < y_1 < x_2 < \cdots < x_{N-1} < y_N < x_N.$$

In addition, we also identify λ as a signed measure $\tau_\lambda := \sum_{i=1}^N \delta_{x_i} - \sum_{j=1}^{N-1} \delta_{y_j}$ and as a probability measure $\mathfrak{m}_\lambda$ on $\mathbb{R}$ called the *(Kerov) transition measure* of λ by the *Markov–Krein correspondence*:

$$\int_{\mathbb{R}} \frac{1}{1-zx}\, d\mathfrak{m}_\lambda(x) = \exp\left[\int_{\mathbb{R}} \log \frac{1}{1-zx}\, d\tau_\lambda(x)\right], \qquad z \in \mathbb{C} \setminus \mathbb{R}. \tag{1.0.1}$$

The Markov–Krein correspondence generally provides a bijection between the probability measures $\mathfrak{m}$ on $\mathbb{R}$ and certain Schwartz distributions τ. In many examples, τ is a signed measure, and in such a case τ is called the *Rayleigh measure* of $\mathfrak{m}$. In general, τ is the derivative (in the sense of Schwartz distribution) of a so-called Rayleigh function; see [Ker98] for further details.

Via this correspondence, we can consider, for example, a Markov chain of Young diagrams following the branching rule induced from the representation theory of symmetric groups, called the *Plancherel measure*. Then the almost surely limit of measures $\mathcal{D}_{1/\sqrt{n}}(\mathfrak{m}_\lambda)$ is known to be the standard semicircle distribution μ_{sc} and the corresponding limit shape of scaled Young diagrams is also known as the *VKLS (Vershik–Kerov–Logan–Shepp) curve*. Moreover, given two sequences of young diagrams $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$ such that their scaled transition measures converge to measures $\mathfrak{m}_1$ and $\mathfrak{m}_2$ with a few regularity conditions, then the main part of the tensor representation of two irreducible representations corresponding to $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$ is described as the free convolution $\mathcal{D}_{1/\sqrt{2}}(\mathfrak{m}_1 \boxplus \mathfrak{m}_2)$. In any case, it is essential to understand the moments and free cumulants of transition measures $\mathfrak{m}$ and Rayleigh measures τ, and also their relations between them: $\mathrm{M}_k(\mathfrak{m})$, $\mathrm{M}_k(\tau)$, $\kappa_k(\mathfrak{m})$, etc. This kind of research is known as the study of *Kerov–Olshanski algebra* [Hor16]. In this thesis, we show a relation between the moments of a Rayleigh measure τ and the free cumulants of a transition measure μ by the combinatorics of non-crossing partitions, see Theorem 2.3.1.

Kerov additionally investigated interlacing sequences appearing in different contexts using the Markov–Krein correspondence: roots of two orthogonal polynomials of large consecutive degrees [Ker93]; eigenvalues of large random matrices and those of their principal submatrices, in the case of randomly rotated real Wigner matrices [Ker93]. Then the case of Wigner and Wishart matrices (without random rotation) was studied by Bufetov [Buf13]. There are also situations where the distribution τ above appears as a probability measure: Poisson–Dirichlet processes (see [Ker98, Section 4.1] and references therein); self-decomposable distributions for monotone convolution [FHS20]; Harish-Chandra–Izykson–Zuber integral of rank one at a high temperature regime [MP22]. The reason why the same correspondence appears in different contexts is still unclear to the author.

In this thesis, we prove a concentration phenomenon analogous to those in [Buf13; Ker93] in the setting of unitarily invariant hermitian random matrices. Although it seems to be a "folklore

theorem" in random matrix theory, there is no proof, and even, only [GY23] among the literature states it explicitly as a conjecture to the author's best knowledge. The main result of Chapter 3 is Theorem 3.1.1, which answers to a conjecture announced by [GY23].

Finite free probability

Finite free probability is a remarkable recent development in free probability. This research area has attracted attention since the 2010s, when Marcus, Spielmann and Srivastava, the three founders, solved the Kadison–Singer conjecture and discovered a connection between the original method of solving it and free probability theory. The most important findings were the following.

Proposition 1.0.6. Let A and B be $N \times N$ hermitian matrices. Then

$$\mathbb{E}[\det(xI - A - UBU^*)] = \det(xI - A) \boxplus_d \det(xI - B), \tag{1.0.2}$$

where U is a Haar unitary matrix and the expectation is taken over U. A similar statement for the multiplication also holds:

$$\mathbb{E}[\det(xI - AUBU^*)] = \det(xI - A) \boxtimes_d \det(xI - B). \tag{1.0.3}$$

The operation $\boxplus_d$ (resp. $\boxtimes_d$) is called finite free additive (resp. multiplicative) convolution although these operations were already defined and studied in the 1920s by Szegö and Walsh [Sze22; Wal22]. The concept of asymptotic freeness leads to the consideration of the connection between finite and free probability. In fact, there are various finite free versions of results in free probability. Moreover, in many cases, taking the limit of them as the degree d tends to infinity gives the corresponding result in free probability. In other words, the inspiration from free probability theory makes many probabilistic analogues in this polynomial setting.

Specifically, for any polynomial $p = \prod_{i=1}^{d}(x - \lambda_i)$ with degree d, we have the law of large numbers:

$$n^{-d} p^{\boxplus_d n}(nx) \to (x - \alpha)^d \tag{1.0.4}$$

where $\alpha = \sum_{i=1}^{d} \lambda_i / d$ is the average of the roots of p. Similarly, for any polynomial p with degree d having only real roots such that $\sum_{i=1}^{d} \lambda_i / d = 0$ and $\sigma^2 = \sum_{i=1}^{d} \lambda_i^2 / d$, we have the central limit theorem:

$$n^{-\frac{d}{2}} p^{\boxplus_d n}(\sqrt{n}x) \to \mathrm{He}_d\left(x, \frac{d\sigma^2}{d-1}\right), \tag{1.0.5}$$

where $\mathrm{He}_d(x, \sigma^2)$ is the *Hermite polynomial* of degree d:

$$\mathrm{He}_d(x, \sigma^2) := \sigma^d \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} (-1)^{2k} \binom{d}{2k} \frac{(-1)^k (2(k+1))!!}{2(k+1)\sigma^{2k}} x^{d-2k}.$$

That is, Hermite polynomials play the role of normal distributions in this framework.

Hermite polynomials are well known as the orthogonal polynomials for Gaussian distributions and are utilized for the analysis of GUEs. Interestingly, the average of the characteristic polynomial of the GUE (and more generally a Wigner matrix) is a Hermite polynomial [FG06]. It is also the famous fact that Hermite polynomials have only real roots because they are orthogonal polynomials. In addition, the limit of empirical root distributions $\mu[\![\mathrm{He}_d(x, 1)]\!]$ is the standard semicircle distribution μ_{sc} [MG60]. We can easily prove this result by using finite free cumulants $\{\kappa_n^{(d)}\}_{n=1}^{d}$ because the Hermite polynomial $\mathrm{He}_d(x, 1)$ is characterized as $\kappa_1^{(d)}(\mathrm{He}_d(x, 1)) = 0$, $\kappa_2^{(d)}(\mathrm{He}_d(x, 1)) = 1$, and $\kappa_n^{(d)}(\mathrm{He}_d(x, 1)) = 0$ for $3 \le n \le d$, see also Proposition 4.1.7.

Chapter 4 of this thesis pursues this kind of probabilistic analogy in finite free probability, specifically in the case of multiplicative convolution. In contrast to the classical case, limit

theorems for multiplication are not directly derived from those for addition. It is interesting to note that even the LLN for multiplication exhibits a significant difference.

The LLN is a well-known result that a sample average of independent identically distributed random variables with finite mean concentrates on the theoretical mean when the sample size is sufficiently large. As an analogous result in free probability, the LLN for free random variables was also established [LP97]. More precisely, for any $\mu \in \mathcal{P}$ with mean α, we have $\mathcal{D}_{1/n}(\mu^{\boxplus n}) \xrightarrow{w} \delta_\alpha$ as $n \to \infty$, where $\mu^{\boxplus n}$ is the n-th power of free additive convolution of μ.

The LLN can also be applied to the multiplication of independent positive random variables, whether classically or freely. In classical probability, the LLN for multiplication can be easily formulated and investigated by using the exponential mapping of those random variables. Additionally, the multiplicative CLT can also be considered, which results in the *log normal distribution*. This distribution is widely used in various fields of science, particularly in statistics.

However, it is not easy to consider the LLN for multiplication in free probability since $e^{X+Y} \neq e^X e^Y$ for (non-commutative) random variables X and Y. For example, we can consider the two different limits of

$$(\mu^{\frac{1}{n}})^{\boxtimes n}, \tag{1.0.6}$$

$$(\mu^{\boxtimes n})^{\frac{1}{n}}, \tag{1.0.7}$$

where (i) ν^α denotes the push-forward of a measure ν by the mapping $x \mapsto x^\alpha$ for $\alpha \in \mathbb{R}$ and (ii) $\mu^{\boxtimes n}$ is the n-th power of free multiplicative convolution of $\mu \in \mathcal{P}_+$. Interestingly, their limits generally do not coincide.

As a matter of fact, Ho studied the CLT for multiplication and discovered that the distributions $(\mu^{\frac{1}{\sqrt{n}}})^{\boxtimes n}$ converge to multiplicative free semicircle distribution λ_t if $\log \mu$ has the zero mean and variance $t > 0$ [Ho11]. The limit of (1.0.6) can be derived by slightly modifying Ho's method and the limit is the delta measure concentrated at the mean of $\log \mu$.

The limit of (1.0.7) was obtained by Tucci for a probability measure μ with bounded support [Tuc10]. After that, the result was extended to include unbounded cases by Haagerup and Möller [HM13]. As a result, the limit distribution of the sequence (1.0.7) always exists but surprisingly not a delta measure except for trivial cases. For $\mu \neq \delta_0$, the limit measure $\Phi(\mu) \in \mathcal{P}_+$ is characterized by the S-transform, see Section 4.6 for details.

Therefore we call the convergence of (1.0.6) *the LLN for multiplicative free convolution* and the convergence of (1.0.7) *Tucci's limit theorem* in this thesis. We obtain finite versions of these results, see Theorems 4.5.1 and 4.6.4 in Chapter 4.

Main results and outline of this thesis

After this introduction, we prepare technical notions in Chapter 2 for the later chapters. In particular, the following two theorems are a part of our main results.

Theorem 2.3.1. Suppose that μ is a probability measure on $\mathbb{R}$ with finite moments of all orders and τ is defined by the Markov–Krein correspondence (1.0.1). Then the formula

$$\mathrm{M}_k(\tau) = \sum_{\rho \in \mathrm{NC}(k)} (k + 1 - |\rho|)\, \kappa_\rho(\mu)$$

holds for every $k \in \mathbb{N}$, where $\mathrm{NC}(k)$ is the set of non-crossing partitions of $\{1, \ldots, k\}$ and $\kappa_\rho(\mu)$ is the free cumulant of μ.

The formula above will be used to prove Theorem 3.1.1 in Chapter 3. Note that this formula is more or less known; however, we provide a different proof by observing a combinatorial structure of non-crossing partitions.

Theorem 2.4.20. Suppose that $f_i \in \mathbb{C}[x]_0$ with $\deg f_i = m_i$ for each $1 \le i \le k$, and let $M = \sum_{i=1}^k m_i$. Then we have

$$\sum_{\pi \in \mathcal{P}(n)} \prod_{i=1}^{k} \left(\sum_{V \in \pi} f_i(|V|) \right) \mu_n^{\mathrm{P}}(\pi, 1_n) = \begin{cases} (n-1)! n^{k-1} \prod_{i=1}^k m_i \operatorname{lead} f_i, & n = M - (k-1), \\ 0, & n > M - (k-1), \end{cases}$$

where $\mathrm{P}(n)$ denotes the set of all partitions of $[n]$ and $1_n := \{\{1, \dots, n\}\} \in \mathrm{P}(n)$, and μ_n^{P} is the Möbius function on $[n]$, see Chapter 2 for details.

Theorem 2.4.20 plays an important role in investigating the three limit theorems below relating finite free probability, namely Theorems 4.3.1, 4.4.2 and 4.5.2.

In Chapter 3, we show a concentration phenomenon on the empirical eigenvalue distribution (EED) of the principal submatrix in a random hermitian matrix whose distribution is invariant under unitary conjugacy. More precisely, if the EED of the whole matrix converges to some deterministic probability measure μ, then the difference of rescaled EEDs of the whole matrix and of its principal submatrix concentrates at the Rayleigh measure (in general, a Schwartz distribution) associated with μ by the Markov–Krein correspondence. The whole statements are based on [FH22].

Theorem 3.1.1. Let $\mu_N, \tau_N, \widehat{\mu}_N, \widehat{\tau}_N$ be defined in Section 3.1, μ be a probability measure on $\mathbb{R}$ and τ be related to μ by the Markov–Krein correspondence (1.0.1). Assume that

$$\sup_{N \ge 1} \mathbb{E}[\mathrm{M}_k(\mu_N)] < \infty \quad \text{and} \quad \mathrm{M}_k(\mu) < \infty, \qquad k \in 2\mathbb{N}$$

and μ_N converges in moments to μ in probability:

$$\lim_{N \to \infty} \mathbb{P}[|\mathrm{M}_k(\mu_N) - \mathrm{M}_k(\mu)| \ge \epsilon] = 0, \qquad k \in \mathbb{N},\ \epsilon > 0.$$

Then we have

$$\lim_{N \to \infty} \| \mathrm{M}_k(\widehat{\tau}_N) - \mathrm{M}_k(\tau) \|_{L^2} = 0, \qquad k \in \mathbb{N},$$

and

$$\lim_{N \to \infty} \mathbb{P}[|\mathrm{M}_k(\widehat{\mu}_N) - \mathrm{M}_k(\mu)| \ge \epsilon] = 0, \qquad k \in \mathbb{N},\ \epsilon > 0.$$

In particular, if the moment problem for $\{\mathrm{M}_k(\mu)\}_{k \ge 1}$ is determinate then $\widehat{\mu}_N$ weakly converges to μ in probability:

$$\lim_{N \to \infty} \mathbb{P}\left[\left| \int_{\mathbb{R}} f(x)\, \widehat{\mu}_N(dx) - \int_{\mathbb{R}} f(x)\, \mu(dx) \right| \ge \epsilon \right] = 0, \qquad f \in C_b(\mathbb{R}),\ \epsilon > 0.$$

Chapter 4 of this thesis mainly focuses on limit theorems in finite free probability, which is primarily based on [AFU23; FU23]. The field of finite free probability has made significant progress in limit theorems for finite free convolutions and combinatorial structures, as seen in [AGP23; AP18; Kab21; Kab22; Mar21]. This thesis aims to study further limit theorems for finite free convolutions and their connections to the free probability theory. In the proof of limit theorems, we use a series of combinatorial identities on sums over partitions proved in Chapter 2. These identities allow for new proofs of the recent results by Kabluchko [Kab21; Kab22] using purely combinatorial tools. Compared to Kabluchko's analytical approach using the saddle-point argument, our proof is more straightforward.

First, we study the limit of $\{p_d^{\boxtimes_d m}\}_{m \in \mathbb{N}}$ as $m \to \infty$ for a fixed monic polynomial p_d of degree d.

Theorem 4.2.4. Let us consider $p \in \mathbb{P}_{\mathrm{mon}}(d)$ with nonnegative roots.

(1) We have

$$\lim_{m\to\infty} p^{\boxtimes_d m}(x) = \begin{cases} x^d, & \widetilde{e_1}(p) < 1, \\ x^d - dx^{d-1}, & \widetilde{e_1}(p) = 1 \text{ and } \widetilde{e_2}(p) < 1, \\ (x-1)^d, & \widetilde{e_1}(p) = 1 \text{ and } \widetilde{e_2}(p) = 1. \end{cases}$$

The limit does not exist if $\widetilde{e_1}(p) > 1$.

(2) Assume that $\widetilde{e_1}(p) > 0$. Then

$$\lim_{m\to\infty} \mathcal{D}_{1/\widetilde{e_1}(p)^m}(p^{\boxtimes_d m})(x) = \begin{cases} x^d - dx^{d-1}, & \widetilde{e_2}(p) < \widetilde{e_1}(p)^2, \\ (x-1)^d, & \widetilde{e_2}(p) = \widetilde{e_1}(p)^2. \end{cases}$$

This is a preliminary result that describes the behavior of m-fold finite free multiplicative convolution $\boxtimes_d$ of a monic polynomial p with nonnegative roots, as m tends to infinity. It is different in flavor from the others, but independently interesting in its own right.

In Section 4.3, we show a finite free analogue of a result of Sakuma and Yoshida [SY13], which is a limit theorem related to free multiplicative and additive convolution. Let us introduce a detailed description of their result to formulate the problem. Let μ be a probability measure on $[0,\infty)$ that has the second moment and is not δ_0. Put $s := 1/m_1(\mu) > 0$ and $\alpha = \mathrm{Var}(\mu)/(m_1(\mu))^2$. Then Sakuma and Yoshida in [SY13, Theorems 9 and 11] proved that there exists a probability measure η_α on $[0,\infty)$ such that

$$\mathcal{D}_{s^m/m}((\mu^{\boxtimes m})^{\boxplus m}) \xrightarrow{w} \eta_\alpha.$$

In addition, it holds for the measure η_α that

$$\kappa_n(\eta_\alpha) = \frac{(\alpha n)^{n-1}}{n!}, \qquad n \in \mathbb{N},$$

where $\kappa_n(\rho)$ is the n-th free cumulant of a probability measure ρ on $\mathbb{R}$. Free cumulants are an important combinatorial tool to treat the free additive and multiplicative convolutions (see [NS06] for details). According to [AP18], one can also define and consider the finite free cumulants $\kappa_n^{(d)}(p)$ of $p \in \mathbb{P}_{\mathrm{mon}}(d)$ to treat the finite free additive convolution $\boxplus_d$ from a viewpoint of combinatorics. The definition of finite free cumulants and their fundamental properties are summarized in Section 4.1.

To describe the corresponding theorem, let us define

$$e_n(t,\mu) := \exp\left(-t\binom{n}{2}\kappa_2(\mu)\right),$$

for $n \in \mathbb{N}$, $t > 0$ and a probability measure μ on $\mathbb{R}$.

Theorem 4.3.1. Let us consider $p_d \in \mathbb{P}_{\mathrm{mon}}(d)$ with nonnegative roots such that $\kappa_1^{(d)}(p_d) = 1$, and let μ be a probability measure with compact support. Assume that $\mu[\![p_d]\!] \xrightarrow{w} \mu$ as $d \to \infty$. Then

(1) For $n \in \mathbb{N}$, we have

$$\lim_{\substack{m\to\infty \\ m/d\to t>0}} \kappa_n^{(d)}\left(\mathcal{D}_{1/m}\left((p_d^{\boxtimes_d m})^{\boxplus_d m}\right)\right) = \frac{(-1)^{n-1}}{t^{n-1}(n-1)!} \sum_{\pi\in \mathrm{P}(n)} e_\pi(t,\mu)\mu_n^{\mathrm{P}}(\pi, 1_n).$$

(2) For $n \in \mathbb{N}$, we have

$$\lim_{\substack{m\to\infty \\ m/d\to 0}} \kappa_n^{(d)}\left(\mathcal{D}_{1/m}\left((p_d^{\boxtimes_d m})^{\boxplus_d m}\right)\right) = \frac{(\kappa_2(\mu)n)^{n-1}}{n!},$$

where the limit coincides with the n-th free cumulant of $\eta_{\kappa_2(\mu)}$.

In Section 4.4, we give alternative proofs for the results shown by Kabluchko in [Kab21; Kab22] by using finite free cumulants and combinatorial formulas deduced from Theorem 2.4.20.

Theorem 4.4.2. (1) (Kabluchko [Kab22]) Let us define $H_d(z;t)$ as the *unitary Hermite polynomial.* Then we get

$$\mu[\![H_d(z;t)]\!] \xrightarrow{w} \Sigma_t, \qquad d \to \infty,$$

where Σ_t is the *free unitary normal distribution* (see [Bia97a] and [BV92, Lemma 6.3]).

(2) (Kabluchko [Kab21]) Let us define $L_{d,m}(z)$ as the *unitary Laguerre polynomial.* Then we get

$$\mu[\![L_{d,m}]\!] \xrightarrow{w} \Pi_t, \qquad d \to \infty,$$

where Π_t is the *free unitary Poisson distribution* (see [Kab21] and [BV92, Lemma 6.4]).

In Section 4.5, we show the central limit theorem for finite free multiplicative convolution of polynomials with nonnegative roots and investigate a connection to free probability.

Theorem 4.5.2. (1) Let $d \geq 2$. Suppose $p(x) = \prod_{k=1}^{d}(x - e^{\theta_k})$ such that $\frac{1}{d}\sum_{k=1}^{d}\theta_k = 0$ and $\frac{1}{d}\sum_{k=1}^{d}\theta_k^2 = \sigma^2$. Then we have

$$\lim_{m\to\infty} \phi_{1/\sqrt{m}}(p)^{\boxtimes_d m} = I_d\left(x; \frac{d\sigma^2}{d-1}\right),$$

and

$$I_d(x;t) := \sum_{k=0}^{d}(-1)^k\binom{d}{k}\exp\left(\frac{k(d-k)}{2d}t\right)x^{d-k}, \qquad t \geq 0.$$

(2) As $d \to \infty$, we have

$$\mu[\![I_d(x;t)]\!] \xrightarrow{w} \lambda_t,$$

where λ_t is *the multiplicative free semicircle distribution* on $[0,\infty)$.

Lastly, we prove the finite free analogue of Tucci's limit theorem in Section 4.6.

Theorem 4.6.4. Consider a monic polynomial p of degree d with non-negative real roots Λ and let $k = k(p)$ be the number of zeros in Λ. Let $\Lambda^{(n)} := \{\lambda_1^{(n)} \geq \lambda_2^{(n)} \geq \cdots \geq \lambda_d^{(n)}\}$ be the set of non-negative real roots of $p^{\boxtimes_d n}$. Then

$$\lim_{n\to\infty}(\lambda_i^{(n)})^{\frac{1}{n}} = \frac{\widetilde{e}_i(\Lambda)}{\widetilde{e}_{i-1}(\Lambda)}, \qquad 1 \leq i \leq d-k.$$

Chapter 2

Combinatorics

In this chapter, we prepare technical notions. We start from basic combinatorics of finite posets and Möbius functions defined on it. Then as concrete examples, we deal with several classes of partitions, which will be used to define various cumulants. The last two sections form a part of the main results. Formula (2.3.1) will be used to prove Theorem 3.1.1 in Section 3.3. The second one (2.4.20) will play a prominent role in proving main theorems in Chapter 4.

2.1 Partially ordered sets and Möbius inversion formula

A *partially ordered set* (for short, poset) is a set equipped with a partial order. More precisely, a pair $P = (P, \leq)$ is called a poset if P is a set and $\leq$ is a relation on P, that is, reflexive, antisymmetric and transitive. A poset P is said to be finite if the number of elements in P is finite.

We give three examples as important posets in this thesis as follows.

Example 2.1.1. (1) Define $\mathrm{B}(n)$ as the set of all subsets of $[n]$. The set $\mathrm{B}(n)$ can be equipped with the following partial order $\leq$:

$$V \leq W \overset{\text{def}}{\Longleftrightarrow} V \subset W \quad (V \text{ is a subset of } W)$$

for $V, W \in \mathrm{B}(n)$. Then $\mathrm{B}(n) = (\mathrm{B}(n), \leq)$ is a finite poset. It is easy to verify that the minimum and maximum elements of $\mathrm{B}(n)$ are $\emptyset$ and $[n]$, respectively.

(2) We call $\pi = \{V_1, \ldots, V_r\}$ a *partition* of the set $[n]$ if it satisfies that

(i) V_i is a non-empty subset of $[n]$ for all $i = 1, 2, \ldots, r$;

(ii) $V_i \cap V_j = \emptyset$ if $i \neq j$;

(iii) $V_1 \cup \cdots \cup V_r = [n]$.

Each subset V_i is called a *block* of π and $|V_i|$ denotes the number of elements in V_i, namely the *size* of V_i.

Let $\mathrm{P}(n)$ be the set of all partitions in $[n]$. The set $\mathrm{P}(n)$ can be equipped with the reversed refining order $\leq$:

$$\pi \leq \sigma \overset{\text{def}}{\Longleftrightarrow} \text{each block of } \pi \text{ is completely contained in one of the blocks of } \sigma.$$

Then $\mathrm{P}(n) = (\mathrm{P}(n), \leq)$ is a finite poset. The minimum and maximum elements of $\mathrm{P}(n)$ (with respect to $\leq$) are given by $0_n := \{\{1\}, \{2\}, \ldots, \{n\}\}$ and $1_n := \{\{1, 2, \ldots, n\}\}$, respectively.

(3) For a partition $\pi \in \mathrm{P}(n)$, we say that π is crossing if there exists a pair of different blocks $V, W \in \pi$ such that the following relation holds:

$$i_1 < j_1 < i_2 < j_2$$

for some $i_1, i_2 \in V$ and $j_1, j_2 \in W$. If π is not crossing, π is said to be *non-crossing*. Define $\mathrm{NC}(n)$ as the set of non-crossing partitions of $\mathrm{P}(n)$ with the relation $\leq$ induced from $\mathrm{P}(n)$. Then $(\mathrm{NC}(n), \leq)$ is a sub-poset of $\mathrm{P}(n)$.

Remark 2.1.2. 1. Via the following bijection, $\mathrm{B}(n)$ can be identified as a sub-poset of $\mathrm{NC}(n+1)$, which is called the set of *interval partitions* and plays an important role in *boolean probability theory*, see [SW97]. Given $V \subset [n]$, it corresponds to a partition π of $[n+1]$ such that $i, j \in [n+1]$ $(i < j)$ are in the same block of π if and only if $l \in V$ for all $i \leq l \leq j-1$.

2. The cardinalities of $\mathrm{B}(n)$, $\mathrm{P}(n)$ and $\mathrm{NC}(n)$ are known to be 2^n, B_n, and $C_n = \frac{1}{n+1}\binom{2n}{n}$, respectively, where the last two satisfy the following recursive relations

$$B_0 = B_1 = 1, \qquad B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k \qquad n \geq 1,$$

$$C_0 = C_1 = 1, \qquad C_{n+1} = \sum_{k=0}^{n} C_{n-k} C_k \qquad n \geq 1. \tag{2.1.1}$$

They are called the *Bell numbers* and the *Catalan numbers*, respectively.

Let $P = (P, \leq)$ be a finite poset. Denote $P^{(2)} := \{(\pi, \sigma) \in P \times P : \pi \leq \sigma\}$. For $F, G : P^{(2)} \to \mathbb{C}$, their convolution $F * G : P^{(2)} \to \mathbb{C}$ is defined as

$$(F * G)(\pi, \sigma) := \sum_{\substack{\rho \in P \\ \pi \leq \rho \leq \sigma}} F(\pi, \rho) G(\rho, \sigma).$$

The *zeta function* $\zeta_P : P^{(2)} \to \mathbb{C}$ of P is defined by

$$\zeta_P(\pi, \sigma) = 1, \qquad (\pi, \sigma) \in P^{(2)}.$$

The *Möbius function* μ_P of P is defined as the inverse of ζ_P with respect to the convolution $*$.

The following inversion principle is one of the most important properties of incidence algebras.

Proposition 2.1.3 (Möbius inversion formula)**.** Let P be a finite poset and $\mu_P : P^{(2)} \to \mathbb{Z}$ the Möbius function associated with P. Then, for any functions $f, g : P \to \mathbb{C}$, the relation

$$f(\pi) = \sum_{\sigma \leq \pi} g(\sigma) \qquad \pi \in P$$

is equivalent to the relation

$$g(\pi) = \sum_{\sigma \leq \pi} f(\sigma) \mu_P(\sigma, \pi) \qquad \pi \in P.$$

The following formula on the Möbius functions is often used in this thesis.

Proposition 2.1.4. Let P be a finite poset with the maximum 1_P and μ_P the Möbius function on P. Then the following identity holds:

$$\sum_{\pi \leq \sigma} \mu_P(\sigma, 1_P) = \delta_{\pi, 1_P} := \begin{cases} 1, & \pi = 1_P, \\ 0, & \pi \neq 1_P. \end{cases}$$

Example 2.1.5. (1) Let μ_n^{B} denote the Möbius function of $\mathrm{B}(n)$. It is easy to verify that $\mu_n^{\mathrm{B}}(W,V) = (-1)^{|V|-|W|}$ for $W \leq V$ in $\mathrm{B}(n)$. In particular, we have $\mu_n^{\mathrm{B}}(W,[n]) = (-1)^{n-|W|}$ for all $W \in \mathrm{B}(n)$.

(2) Let μ_n^{P} denote the Möbius function of $\mathrm{P}(n)$. The function μ_n^{P} can be explicitly computed as follows: for $\pi, \sigma \in \mathrm{P}(n)$,

$$\mu_n^{\mathrm{P}}(\pi,\sigma) = (-1)^{|\pi|-|\sigma|}(2!)^{r_3}(3!)^{r_4}\cdots((n-1)!)^{r_n},$$

where $|\pi|$ denotes the number of blocks of $\pi \in \mathrm{P}(n)$ and r_i is the number of blocks of σ that contain exactly i blocks of π. In particular, we have

$$\mu_n^{\mathrm{P}}(\pi, 1_n) = (-1)^{|\pi|-1}(|\pi|-1)!$$

and

$$\mu_n^{\mathrm{P}}(0_n, \sigma) = (-1)^{n-|\sigma|}(2!)^{t_3}(3!)^{t_4}\cdots((n-1)!)^{t_n},$$

where t_i is the number of blocks of σ of size i.

(3) Let μ_n^{NC} denote the Möbius function associated with $\mathrm{NC}(n)$. The concrete values of $\mu_n^{\mathrm{NC}}(\pi,\sigma)$ can be expressed by using the Catalan numbers (2.1.1). There are two important facts:

(a) $s_n := \mu_n^{\mathrm{NC}}(0_n, 1_n) = (-1)^{n-1}C_{n-1}$ for $n \in \mathbb{N}$;

(b) For $(\pi,\sigma) \in \mathrm{NC}(n)^{(2)}$, each chain $[\pi,\sigma] := \{\rho \in \mathrm{NC}(n) \mid \pi \leq \rho \leq \sigma\}$ has the natural decomposition $[\pi,\sigma] \cong \mathrm{NC}(1)^{t_1} \times \mathrm{NC}(2)^{t_2} \times \cdots \times \mathrm{NC}(n)^{t_n}$.

Thus, $\mu_n^{\mathrm{NC}}(\pi,\sigma) = {s_1}^{t_1}{s_2}^{t_2}\cdots{s_n}^{t_n}$. In particular, if we define $\mu_n^{\mathrm{NC}}(\sigma) := \mu_n^{\mathrm{NC}}(0_n,\sigma)$ then

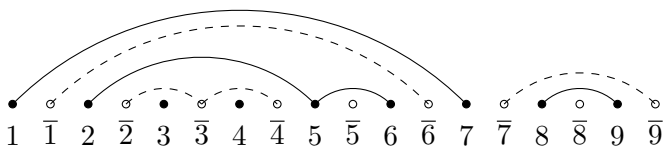
$$\mu_n^{\mathrm{NC}}(\sigma) = \prod_{1\leq i\leq k}(-1)^{|V_i|-1}C_{|V_i|-1}$$

for $\sigma = \{V_1,\ldots,V_k\} \in \mathrm{NC}(n)$.

Kreweras complement

The *Kreweras complement* of a non-crossing partition $\pi \in \mathrm{NC}(n)$ is defined as follows. Inserting additional points $[\overline{n}] := \{\overline{1},\overline{2},\ldots,\overline{n}\}$ to $[n]$, suppose that $L_n = \{1,\overline{1},2,\overline{2},\ldots,n,\overline{n}\}$ is a linearly ordered set with the order as displayed. It is clear that $\mathrm{NC}(L_n)$ is isomorphic to $\mathrm{NC}(2n)$. Take the maximal non-crossing partition σ of $[\overline{n}]$ such that $\pi \cup \sigma \in \mathrm{NC}(L_n)$. Then deleting bars over the integers, we call σ the Kreweras complement of π and denoted by $\mathrm{Kr}(\pi)$. For convenience, we sometimes keep the bars and regard $\mathrm{Kr}(\pi)$ as a non-crossing partition on $[\overline{n}]$.

Example 2.1.6. If $\pi = \{\{1,7\},\{2,5,6\},\{3\},\{4\},\{8,9\}\}$ then the following picture



shows that $\mathrm{Kr}(\pi) = \{\{1,6\},\{2,3,4\},\{5\},\{7,9\},\{8\}\}$.

Correspondence between non-crossing partitions and symmetric groups

The set of non-crossing partitions can be embedded into the symmetric group. Here we collect needed facts. For further details, the reader is referred to [Bia97c; NS06]

The length function on symmetric groups is defined as the minimal number l for which g can be written as a product of l transpositions. It has the following properties: for all $g, h \in \mathfrak{S}_n$,

$$\begin{aligned} &|hgh^{-1}| = |g|, \\ &|gh| \leq |g| + |h|, \\ &|gh| \equiv |g| + |h| \pmod 2. \end{aligned} \tag{2.1.2}$$

The number $\#(g)$ of cycles in the cycle decomposition of g is known to satisfy

$$\#(g) + |g| = n.$$

Let d be a metric on $\mathfrak{S}_n$ defined by $d(g,h) = |g^{-1}h|$. The geodesic set from the unit e to $\gamma_n := (1, \dots, n)$ is defined by

$$\mathfrak{S}_{\mathrm{NC}}(\gamma_n) = \{\, g \in \mathfrak{S}_n \mid d(e,g) + d(g,\gamma_n) = d(e,\gamma_n)\ (= n-1) \,\}.$$

For $g, h \in \mathfrak{S}_{\mathrm{NC}}(\gamma_n)$, denote by $g \leq h$ if g and h are on a common geodesic and $d(e,g) \leq d(e,h)$, namely, if $d(e,g) + d(g,h) = d(e,h)$, or equivalently, $|g| + |g^{-1}h| = |h|$.

For a partition $\pi \in \mathrm{NC}(n)$, each block $V = \{i_1, i_2, \dots, i_p\} \in \pi$ whose elements are arranged in the increasing order associates the cyclic permutation $h = (i_1, i_2, \dots, i_p)$, so that π associates the permutation $\mathcal{P}_\pi := h_1 h_2 \cdots h_l$, where $l = \#(\mathcal{P}_\pi) = |\pi|$. This embedding becomes a poset isomorphism

$$\mathcal{P} \colon \mathrm{NC}(n) \longrightarrow \mathfrak{S}_{\mathrm{NC}}(\gamma_n),$$

see [NS06, Proposition 23.23]. We need the following facts later: for $g = \mathcal{P}_\sigma, h = \mathcal{P}_\pi \in \mathfrak{S}_{\mathrm{NC}}(\gamma_n)$,

1. the relation $\sigma \leq \pi$ holds in $\mathrm{NC}(n)$ if and only if $|g| + |g^{-1}h| + |h^{-1}\gamma_n| = n - 1$,
2. $g^{-1}\gamma_n = \mathcal{P}_{\mathrm{Kr}(\sigma)}$; in particular $\#(g^{-1}\gamma_n) = |\mathrm{Kr}(\sigma)|$,
3. $\mu_n^{\mathrm{NC}}(\sigma, \pi) = \mu_n^{\mathrm{Wg}}(g^{-1}h)$,

where μ_n^{Wg} is the main part of Weingarten function, (3.1.6).

Similar results hold for $\gamma_n^{(1)}\gamma_n^{(2)} \in \mathfrak{S}_{2n}$ instead of γ_{2n}, where

$$\gamma_n^{(1)} = (1, \dots, n)(n+1) \cdots (2n) \qquad \text{and} \qquad \gamma_n^{(2)} = (1) \cdots (n)(n+1, \dots, 2n).$$

Correspondingly, let

$$(1_n^{(1)}, 1_n^{(2)}) := \{\{1, \dots, n\}, \{n+1, \dots, 2n\}\},$$

then $\mathcal{P}_{(1_n^{(1)}, 1_n^{(2)})} = \gamma_n^{(1)}\gamma_n^{(2)}$. Via the embedding

$$\mathrm{NC}(n) \times \mathrm{NC}(n) \cong [0_{2k}, (1_n^{(1)}, 1_n^{(2)})] \subset \mathrm{NC}(2n)$$

the restriction of the mapping $\mathcal{P}$ induces an isomorphism between $\mathrm{NC}(n) \times \mathrm{NC}(n)$ and

$$\mathfrak{S}_{\mathrm{NC}}(\gamma_n^{(1)}\gamma_n^{(2)}) = \{\, g \in \mathfrak{S}_{2k} \mid d(e,g) + d(g, \gamma_n^{(1)}\gamma_n^{(2)}) = d(e, \gamma_n^{(1)}\gamma_n^{(2)})\ (= 2n-2) \,\}. \tag{2.1.3}$$

For $(\pi_1, \pi_2) \in \mathrm{NC}(n) \times \mathrm{NC}(n)$ and $h = \mathcal{P}_{(\pi_1,\pi_2)} \in \mathfrak{S}_{\mathrm{NC}}(\gamma_n^{(1)}\gamma_n^{(2)})$, the element $h^{-1}\gamma_n^{(1)}\gamma_n^{(2)}$ corresponds to $(\mathrm{Kr}(\pi_1), \mathrm{Kr}(\pi_2))$ under the isomorphism $\mathcal{P}$, and in particular $\#(h^{-1}\gamma_n^{(1)}\gamma_n^{(2)}) = |\mathrm{Kr}(\pi_1)| + |\mathrm{Kr}(\pi_2)| = |\mathrm{Kr}(\pi)| + 1$, where $\pi = (\pi_1, \pi_2)$ is regarded as a partition in $\mathrm{NC}(2n)$. Note that this relation can be clearly understood in terms of the *relative Kreweras complement*; however, we will not use this technical notion since it is not directly needed in this thesis.

2.2 Free cumulants

This section deals with the *free cumulants* κ_n associated with the non-crossing partitions $\mathrm{NC}(n)$. The reader is referred to [NS06] for further details.

For a probability measure μ on $\mathbb{R}$ with finite moments of all orders, the free cumulants $\{\kappa_n(\mu)\}_{n\in\mathbb{N}}$ of μ are determined recursively by the moment-cumulant formula

$$\mathrm{M}_\sigma(\mu) = \sum_{\substack{\pi\in\mathrm{NC}(n)\\ \pi\leq\sigma}} \kappa_\pi(\mu), \qquad \sigma\in\mathrm{NC}(n), \quad n\in\mathbb{N}. \tag{2.2.1}$$

Actually, it suffices to take $\sigma = 1_n, n = 1, 2, 3, \ldots$ to determine the free cumulants, and then the above formula can be proved for all $\sigma\in\mathrm{NC}(n), n = 1, 2, 3, \ldots$. More explicitly, free cumulants can be expressed as

$$\kappa_\pi(\mu) = \sum_{\substack{\sigma\in\mathrm{NC}(n)\\ \sigma\leq\pi}} \mathrm{M}_\sigma(\mu)\mu_n^{\mathrm{NC}}(\sigma,\pi), \qquad \pi\in\mathrm{NC}(n), \quad n\in\mathbb{N}, \tag{2.2.2}$$

where μ_n^{NC} is the Möbius function on the poset $\mathrm{NC}(n)$.

We compute the free cumulants of free unitary normal distribution Σ_t and free unitary Poisson distribution Π_t introduced in Section 4.4. A calculation strategy is the use of the *Lagrange inversion theorem* (see, e.g., [Com74, p. 148, Theorem A]).

Recall that, for a probability measure μ (on $[0,\infty)$ or $\mathbb{T}$) with nonzero first moment, we obtain

$$R_\mu(zS_\mu(z)) = z$$

on a neighborhood of 0, where $R_\mu(z)$ is the R-transform of μ and $S_\mu(z)$ is the S-transform of μ, see [BV92; BV93] for details. If $f_\mu(z) := zS_\mu(z)$ is analytic on a neighborhood of 0 and $f_\mu(0) = 0$ and also $f'_\mu(0)\neq 0$, then the Lagrange inversion theorem implies that

$$\kappa_n(\mu) = \frac{1}{n!}\lim_{z\to 0}\left(\frac{d}{dz}\right)^{n-1}\left(\frac{z}{f_\mu(z)}\right)^n.$$

Using the above strategy, the following known results are rigorously proved.

Proposition 2.2.1 (see [DGN15])**.** For $n\in\mathbb{N}$ and $t>0$, we have

$$\kappa_n(\Sigma_t) = \exp\left(-\frac{nt}{2}\right)\frac{(-nt)^{n-1}}{n!}.$$

Proof. Recall that, for any $t>0$,

$$S_{\Sigma_t}(z) = \exp\left(t\left(z+\frac{1}{2}\right)\right).$$

Then it is easy to verify that $f_{\Sigma_t}(z) := zS_{\Sigma_t}(z)$ is analytic on a neighborhood of 0 and $f_{\Sigma_t}(0) = 0$ and also $f'_{\Sigma_t}(0) = e^{t/2}\neq 0$. The Lagrange inversion theorem implies that

$$\begin{aligned}\kappa_n(\Sigma_t) &= \frac{1}{n!}\lim_{z\to 0}\left(\frac{d}{dz}\right)^{n-1}\left(\frac{z}{f_{\Sigma_t}(z)}\right)^n\\ &= \frac{1}{n!}\lim_{z\to 0}\left(\frac{d}{dz}\right)^{n-1}\exp\left(-nt\left(z+\frac{1}{2}\right)\right)\\ &= \exp\left(-\frac{nt}{2}\right)\frac{(-nt)^{n-1}}{n!}.\end{aligned}$$

□

Proposition 2.2.2. For $n \in \mathbb{N}$ and $t > 0$, we get

$$\kappa_n(\Pi_t) = (-1)^{n-1} 2^n e^{-2nt} \sum_{k=1}^{n-1} \frac{(-t)^k}{k!} (2n)^{k-1} \binom{n-2}{k-1}. \tag{2.2.3}$$

To show this, we use the Lagrange inversion theorem again. According to [Kab21], we have

$$S_{\Pi_t}(z) = \exp\left(\frac{t}{z+\frac{1}{2}}\right), \qquad t > 0.$$

One can see that, $f_{\Pi_t}(z) = zS_{\Pi_t}(z)$ is analytic on a neighborhood of 0, and $f_{\Pi_t}(0) = 0$ and also $f'_{\Pi_t}(0) = e^{2t} \neq 0$. By the Lagrange inversion theorem, we have

$$\begin{aligned} \kappa_n(\Pi_t) &= \frac{1}{n!} \lim_{z\to 0} \frac{d^{n-1}}{dz^{n-1}} \left(\frac{z}{f_{\Pi_t}(z)}\right)^n \\ &= \frac{1}{n!} \lim_{z\to 0} \frac{d^{n-1}}{dz^{n-1}} \exp\left(\frac{-nt}{z+\frac{1}{2}}\right). \end{aligned} \tag{2.2.4}$$

To compute this, we prepare the following result derived from Faá Di Bruno's formula.

Lemma 2.2.3. Let u be an analytic function on $\mathbb{C}$. Then

$$\frac{d^n}{dz^n} e^{u(z)} = \left(\sum_{\pi \in \mathrm{P}(n)} \prod_{V \in \pi} u^{(|V|)}(z) \right) e^{u(z)}.$$

Proof. According to Faá Di Bruno's formula, for analytic functions f and g, we have

$$\frac{d^n}{dz^n} f(g(z)) = \sum_{\pi \in \mathrm{P}(n)} f^{(|\pi|)}(g(z)) \cdot \prod_{V \in \pi} g^{(|V|)}(z).$$

Taking $f(z) = e^z$ and $g(z) = u(z)$, and observing that $f^{(|\pi|)}(z) = e^z$, we obtain the desired result. □

The complete computation is as follows.

Proof of Proposition 2.2.2. By (2.2.4) and Lemma 2.2.3, we obtain

$$\begin{aligned} \kappa_n(\Pi_t) &= \frac{1}{n!}(-2)^{n-1} e^{-2nt} \left(\sum_{\pi \in \mathrm{P}(n-1)} (-2nt)^{|\pi|} \prod_{V \in \pi} |V|! \right) \\ &= \frac{1}{n!}(-2)^{n-1} e^{-2nt} \left(\sum_{k=1}^{n-1} (-2nt)^k \sum_{\substack{\pi \in \mathrm{P}(n-1) \\ |\pi|=k}} \prod_{V \in \pi} |V|! \right) \\ &= (-1)^{n-1} 2^n e^{-2nt} \left(\sum_{k=1}^{n-1} (-t)^k (2n)^{k-1} \sum_{\substack{\pi \in \mathrm{P}(n-1) \\ |\pi|=k}} \frac{\prod_{V \in \pi} |V|!}{(n-1)!} \right). \end{aligned}$$

Hence, it is enough to prove

$$\sum_{\substack{\pi \in \mathrm{P}(n-1) \\ |\pi|=k}} \frac{\prod_{V \in \pi} |V|!}{(n-1)!} = \frac{1}{k!} \binom{n-2}{k-1}$$

by comparing with (2.2.3). Note that an elementary combinatorial argument shows that

$$\sum_{\substack{p_1,\dots,p_{n-1}\geq 0\\ p_1+\cdots+(n-1)p_{n-1}=n-1\\ p_1+\cdots+p_{n-1}=k}} \binom{n}{p_1,\dots,p_{n-1}} = \binom{n-2}{k-1}, \qquad 1\leq k\leq n-1. \tag{2.2.5}$$

Thus, since the number of partitions with p_1 blocks of size 1, p_2 blocks of size 2,..., p_n blocks of size n is equal to

$$\frac{n!}{p_1!p_2!\cdots p_n!(1!)^{p_1}(2!)^{p_2}\cdots(n!)^{p_n}},$$

we have

$$\begin{aligned}\sum_{\substack{\pi\in \mathrm{P}(n-1)\\ |\pi|=k}} \frac{\prod_{V\in\pi}|V|!}{(n-1)!} &= \frac{1}{k!}\sum_{\substack{p_1,\dots,p_{n-1}\geq 0\\ p_1+\cdots+(n-1)p_{n-1}=n-1\\ p_1+\cdots+p_{n-1}=k}} \binom{n}{p_1,\dots,p_{n-1}}\\ &= \frac{1}{k!}\binom{n-2}{k-1},\end{aligned}$$

where the last equality follows from (2.2.5). □

2.3 Kreweras decomposition

The goal of this section is to prove the next theorem.

Theorem 2.3.1. Suppose that μ is a probability measure on $\mathbb{R}$ with finite moments of all orders and τ is defined by the Markov–Krein correspondence (1.0.1). Then the formula

$$\mathrm{M}_k(\tau) = \sum_{\pi\in \mathrm{NC}(k)} (k+1-|\pi|)\,\kappa_\pi(\mu) \tag{2.3.1}$$

holds for every $k\in\mathbb{N}$, where $\mathrm{NC}(k)$ is the set of non-crossing partitions of $\{1,\dots,k\}$ and $\kappa_\pi(\mu)$ is the free cumulant of μ.

Remark 2.3.2. This formula gives an explicit combinatorial relation between two bases in the Kerov–Olshanski algebra: the moments of τ and free cumulants of μ. It can be easily proved by combining known formulas for complete symmetric functions as follows. The moments of τ and the free cumulants of μ can be identified with the elements $\{p_n(A)\}_{n\geq 1}$ and $\{(-1)^n e_n^*(A)\}_{n\geq 1}$ in [Las09], respectively; the latter fact is noted on page 2242 of [Las09]. Combining (4.5) and the formula right before (4.10) in [Las09] allows one to express $\{p_n^*(A)\}_{n\geq 1}$ in terms of $\{e_n(A)\}_{n\geq 1}$ as a sum over integer partitions. Applying the involution gives a formula that expresses $\{p_n(A)\}_{n\geq 1}$ in terms of $\{(-1)^n e_n^*(A)\}_{n\geq 1}$. This formula can be transformed into the sum over non-crossing partitions via [NS06, Corollary 9.12], which amounts to Theorem 2.3.1.

Let us start to prove the combinatorial formula (2.3.1) by the induction on the degree k. In this subsection, we keep the assumptions and notation in Theorem 2.3.1. To begin, the original formula for the Markov–Krein correspondence (1.0.1) implies the recursive relation

$$\mathrm{M}_k(\tau) = k\,\mathrm{M}_k(\mu) - \sum_{r=1}^{k-1} \mathrm{M}_r(\tau)\,\mathrm{M}_{k-r}(\mu), \qquad k\in\mathbb{N}, \tag{2.3.2}$$

which is exactly the relation satisfied by complete symmetric functions and Newton power sums ([Ker98, (3.2.4) and Section 3.4]).

Thanks to the moment-cumulant formula (2.2.1), the RHS of the desired formula (2.3.1) may be transformed into

$$\begin{aligned}\sum_{\pi\in\mathrm{NC}(k)}(k+1-|\pi|)\,\kappa_\pi(\mu) &= k\sum_{\pi\in\mathrm{NC}(k)}\kappa_\pi(\mu)-\sum_{\pi\in\mathrm{NC}(k)}(|\pi|-1)\,\kappa_\pi(\mu)\\ &= k\,\mathrm{M}_k(\mu)-\sum_{\pi\in\mathrm{NC}(k)}(|\pi|-1)\,\kappa_\pi(\mu).\end{aligned}$$

Hence, according to the recursive equation (2.3.2), Formula (2.3.1) is eventually equivalent to

$$\sum_{\pi\in\mathrm{NC}(k)}(|\pi|-1)\,\kappa_\pi(\mu)=\sum_{r=1}^{k-1}\mathrm{M}_r(\tau)\,\mathrm{M}_{k-r}(\mu). \tag{2.3.3}$$

By the induction hypothesis up to the degree $k-1$ and the moment-cumulant formula, the RHS of (2.3.3) can be written as

$$\sum_{r=1}^{k-1}\sum_{\substack{\overline{\pi}\in\mathrm{NC}(r)\\ \underline{\pi}\in\mathrm{NC}(k-r)}}|\,\mathrm{Kr}(\overline{\pi})|\,\kappa_{\overline{\pi}}(\mu)\,\kappa_{\underline{\pi}}(\mu). \tag{2.3.4}$$

The cardinality $|\,\mathrm{Kr}(\overline{\pi})|$ can be interpreted as the number of inserting $\underline{\pi}$ into $\overline{\pi}$ in the following way:

(P1) pick $r\in\{1,2,\ldots,k-1\}$, $\overline{\pi}\in\mathrm{NC}(r)$ and $\underline{\pi}\in\mathrm{NC}(k-r)$;

(P2) pick a block B of $\mathrm{Kr}(\overline{\pi})$, where $\mathrm{Kr}(\overline{\pi})$ is interpreted as a partition on the points $[\overline{r}]$ interlacing with $[r]$;

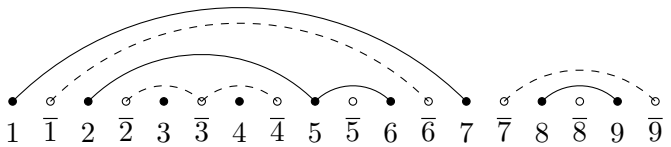
(P3) substitute the partition $\underline{\pi}$ into the last point of B.

The steps (P2) and (P3) provide a way to insert $\underline{\pi}$ into $\overline{\pi}$, which yields a non-crossing partition $\pi\in\mathrm{NC}(k)$; see also Example 2.3.3. The sum (2.3.4) can then be expressed as

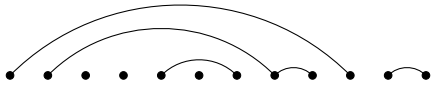
$$\sum_{\pi}\kappa_\pi(\mu), \tag{2.3.5}$$

where π runs over all the non-crossing partitions appearing as a result of (P1)–(P3). Note that the same non-crossing partition π may appear more than once, and the sum (2.3.5) needs to count the multiplicity. Actually, in order to have (2.3.3), we need to demonstrate that each $\pi\in\mathrm{NC}(k)$ appears exactly $|\pi|-1$ times. To achieve this, we introduce the notion of *Kreweras decomposition* of a non-crossing partition, which describes the relation between $\pi,\overline{\pi}$ and $\underline{\pi}$ above.

Example 2.3.3. For the non-crossing partitions $\overline{\pi}=\{\{1,7\},\{2,5,6\},\{3\},\{4\},\{8,9\}\}$ and $\underline{\pi}=\{\{1,3\},\{2\}\}$, the Kreweras complement $\mathrm{Kr}(\overline{\pi})$ is the partition described by the dashed curves below



and hence the Kreweras complement has the blocks $\{\overline{1},\overline{6}\},\{\overline{2},\overline{3},\overline{4}\},\{\overline{5}\},\{\overline{7},\overline{9}\},\{\overline{8}\}$. According to (P3) we are allowed to place $\underline{\pi}$ at any point of $\{\overline{6},\overline{4},\overline{5},\overline{8},\overline{9}\}$. For example, if we choose $\overline{4}$ then the resulting non-crossing partition π is



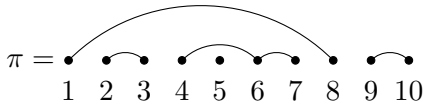
Definition 2.3.4. 1. For $\sigma \in \mathrm{NC}(r)$, a *Kreweras point* of σ is the last point of a block of the Kreweras complement $\mathrm{Kr}(\sigma)$ regarded as a partition on $[\overline{r}]$ that interlaces with $[r]$.

2. For $\pi \in \mathrm{NC}(k)$, a pair $(\overline{\pi}, \underline{\pi})$ of non-empty disjoint subsets of π such that $\overline{\pi} \cup \underline{\pi} = \pi$ and the union of all elements of $\underline{\pi}$ is an interval of $[k]$, that is, there exist some $i < j$ such that

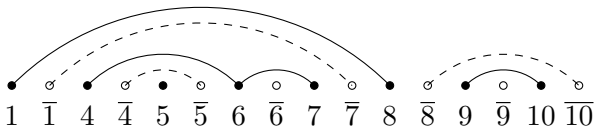
$$\bigcup_{V \in \underline{\pi}} V = [i, j] = \{i, i+1, \ldots, j\}.$$

If the position of $\underline{\pi}$ is a Kreweras point of $\overline{\pi}$, then we call $(\overline{\pi}, \underline{\pi})$ a Kreweras decomposition of π, $\overline{\pi}$ an outer partition of π and $\underline{\pi}$ an inner partition of π.

Example 2.3.5. The non-crossing partition $\pi = \{\{1,8\},\{2,3\},\{4,6,7\},\{5\},\{9,10\}\}$ can be described as



and it has the four inner partitions $\pi_1 = \{\{2,3\},\{4,6,7\},\{5\}\}, \pi_2 = \{\{4,6,7\},\{5\}\}, \pi_3 = \{\{5\}\}, \pi_4 = \{\{9,10\}\}$. Any other subsets of π are not inner partitions; for example, $\pi' = \{\{2,3\}\}$ has the support $\{2,3\}$ of interval form, but the Kreweras complement of $\pi \setminus \pi'$ is described by the dashed curves and white singletons in the picture



so that the position of the removed block $\{2,3\}$ was at the point $\overline{1}$, which was not the last point of the block $\{\overline{1}, \overline{7}\}$.

The goal is then to demonstrate that each $\pi \in \mathrm{NC}(k)$ has exactly $|\pi| - 1$ Kreweras decompositions. The proof is based on the induction, which depends on the following nesting structure of inner partitions.

Lemma 2.3.6. Suppose that $\pi \in \mathrm{NC}(k)$ and its first block which contains 1 divides $[k]$ into (non-empty) l segments $I_1, \ldots, I_l$. Then $\pi_j := \pi|_{I_j}$ is an inner partition of π for every j, and moreover, every inner partition of π_j is an inner partition of π. Conversely, any inner partition of π is some π_j or its inner partition.

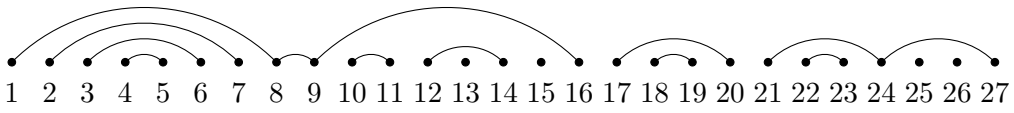
Proof. It is clear that all π_j $(j = 1, \ldots, l)$ are inner partitions of π. Then we take any inner partition $\underline{\pi_j}$ of π_j for $j = 1, \ldots, l$. Note that the Kreweras complement $\mathrm{Kr}(\pi_j)$ equals $\mathrm{Kr}(\pi)$ restricted to the interval I_j. Hence, since $\underline{\pi_j}$ is at a Kreweras point of the outer partition $\overline{\pi_j} = \pi_j \setminus \underline{\pi_j}$, $\underline{\pi_j}$ is also at a Kreweras point of $\pi \setminus \underline{\pi_j}$.

Conversely, we take any inner partition $\underline{\pi}$ of $\overline{\pi}$. By the definition of inner partitions, $\underline{\pi}$ is supported on some interval I_j. If $\underline{\pi}$ contains the first block of π_j, then $\underline{\pi}$ equals π_j. Otherwise, the support of $\underline{\pi}$ is a sub-interval of I_j which does not intersect the first block of π_j, and since $\underline{\pi}$ is at a Kreweras point of the outer partition $\overline{\pi}$, $\underline{\pi}$ is also at a Kreweras point of $\pi_j \setminus \underline{\pi}$. □

Proposition 2.3.7. Let $k \geq 2$. Each $\pi \in \mathrm{NC}(k)$ has exactly $|\pi| - 1$ Kreweras decompositions.

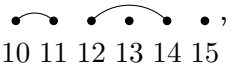
Proof. The proof runs by the induction. It is clear that the statement is true when $k = 2$. Then we assume the statement is true up to $k-1$ and take $\pi \in \mathrm{NC}(k)$ ($|\pi| > 1$). Suppose that the first block of π divides $[k]$ into l segments $I_1, \ldots, I_l$. Then all $\{\pi_j = \pi|_{I_j}\}_{j=1}^{l}$ are inner partitions of π. By Lemma 2.3.6, a subset of π is an inner partition of π if and only if it is one of $\{\pi_j\}_{j=1}^{l}$ or an inner partition of some π_j. Therefore, by the induction hypothesis, the number of inner partitions of π is $l + \sum_{j=1}^{l}(|\pi_j| - 1) = |\pi| - 1$. □

Example 2.3.8. We take $\pi \in \mathrm{NC}(27)$ to be

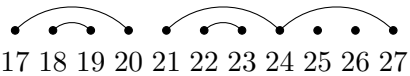


in which $|\pi| = 14$. The three non-crossing partitions

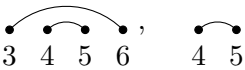
$\pi_1 =$ 

, $\pi_2 =$ 

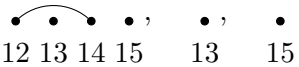
,

$\pi_3 =$ 

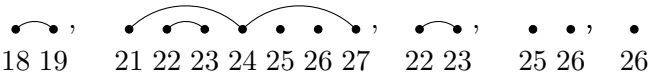
are inner partitions of π and the two inner partitions of π_1



are inner partitions of π. In the same way, the three inner partitions of π_2



and the five inner partitions of π_3



are also inner partitions of π. Thus π has 13 inner partitions: π_1, π_2, π_3 and the inner partitions of them.

2.4 Combinatorial formula related to finite free probability

In this section, we investigate the value of

$$\sum_{\pi \in \mathrm{P}(n)} \prod_{i=1}^{k} \left(\sum_{V \in \pi} f_i(|V|) \right) \mu_n^{\mathrm{P}}(\pi, 1_n) \tag{2.4.1}$$

for polynomials $f_1, \ldots, f_k$ without constant terms. These values will play an important role in considering the convergence of finite free cumulants in Chapter 4.

At first, we define useful polynomials $r_k(z)$ and $s_k(z)$ to understand the combinatorics behind Formula (2.4.1)

Definition 2.4.1. For any polynomial $f(x) \in \mathbb{C}[x]_0$, define a family of functions $\{\varphi_n(t)\}_{n\in\mathbb{N}}$ and its generating series $M(t,z)$ as follows. For $n \in \mathbb{N}$

$$\varphi_n(t) := \exp(f(n)t)$$

and

$$M(t,z) := 1 + \sum_{n=1}^{\infty} \frac{\varphi_n(t)}{n!} z^n.$$

Also, we define $\{\psi_n(t)\}_{n\in\mathbb{N}}$, $\{r_k(z)\}_{k\in\mathbb{N}}$ and $\{s_k(z)\}_{k\in\mathbb{N}}$ (see Remark 2.4.4) as characterizing the following identities:

$$M(t,z) = e^z \left(1 + \sum_{k=1}^{\infty} \frac{r_k(z)}{k!} t^k\right)$$

and

$$\log(M(t,z)) = \sum_{n=1}^{\infty} \frac{\psi_n(t)}{n!} z^n = z + \sum_{k=1}^{\infty} \frac{s_k(z)}{k!} t^k.$$

There are a lot of useful relations between them.

Proposition 2.4.2. Let $\{\varphi_n(t)\}_{n\in\mathbb{N}}$, $\{\psi_n(t)\}_{n\in\mathbb{N}}$, $\{r_k(z)\}_{k\in\mathbb{N}}$, and $\{s_k(z)\}_{k\in\mathbb{N}}$ be defined as above.

(1) For all $n, k \in \mathbb{N}$, we obtain

$$\varphi_n(t) = \sum_{\pi\in \mathrm{P}(n)} \psi_\pi(t), \qquad \text{or equivalently} \qquad \psi_n(t) = \sum_{\pi\in \mathrm{P}(n)} \varphi_\pi(t)\mu_n^{\mathrm{P}}(\pi, 1_n),$$

and

$$r_k(z) = \sum_{\pi\in \mathrm{P}(k)} s_\pi(z), \qquad \text{or equivalently} \qquad s_k(z) = \sum_{\pi\in \mathrm{P}(k)} r_\pi(z)\mu_k^{\mathrm{P}}(\pi, 1_k). \tag{2.4.2}$$

(2) For all $n, k \in \mathbb{N}$, we have

$$s_k^{(n)}(0) = \psi_n^{(k)}(0) = \sum_{\pi\in \mathrm{P}(n)} \left(\sum_{V\in\pi} f(|V|)\right)^k \mu_n^{\mathrm{P}}(\pi, 1_n). \tag{2.4.3}$$

(3) For all $n, k \in \mathbb{N}$,

$$r_k^{(n)}(0) = \sum_{l=1}^{n} \binom{n}{l} (-1)^{n-l} f(l)^k. \tag{2.4.4}$$

Proof. The statement (1) follows from the moment-cumulant formula. Since $\varphi_\pi(0) = 1$ and

$$\varphi_\pi^{(k)}(t) = \left(\sum_{V\in\pi} f(|V|)\right)^k \varphi_\pi(t)$$

for $k \in \mathbb{N}$ and $\pi \in \mathrm{P}(n)$, we obtain

$$\psi_n^{(k)}(0) = \sum_{\pi\in \mathrm{P}(n)} \varphi_\pi^{(k)}(0)\mu_n^{\mathrm{P}}(\pi, 1_n) = \sum_{\pi\in \mathrm{P}(n)} \left(\sum_{V\in\pi} f(|V|)\right)^k \mu_n^{\mathrm{P}}(\pi, 1_n).$$

Because we just exchanged the order of sum, we have $s_k^{(n)}(0) = \psi_n^{(k)}(0)$ as the desired result in (2). By the definitions of $M(t,z)$ and $r_k(z)$,

$$r_k^{(n)}(0) = \sum_{l=1}^{n} \binom{n}{l} (-1)^{n-l} \varphi_l^{(k)}(0).$$

It is easy to verify that $\varphi_l^{(k)}(0) = f(l)^k$ as desired. □

Example 2.4.3. Here, as the simplest polynomial, take $f(x) = x$. Then it immediately follows that $\varphi_n(t) = \exp(nt)$, $\psi_1(t) = e^t$ and $\psi_n(t) = 0$ for $n \geq 2$. Hence, $s_k(z) = z$ and

$$r_k(z) = \sum_{\pi \in \mathrm{P}(k)} z^{|\pi|} \tag{2.4.5}$$

for every $k \in \mathbb{N}$.

Remark 2.4.4. An easy consequence of (2.4.4) and (2.4.5) is

$$\sum_{l=1}^{n} \binom{n}{l} (-1)^{n-l} l^k = 0 \tag{2.4.6}$$

for $k \in \mathbb{N}$ if $n > k$. Hence, for any polynomial $f \in \mathbb{C}[x]_0$, corresponding $r_k(f)$ is a polynomial because $r_k^{(n)}(0) = 0$ if $n > k \deg(f)$ from (2.4.6). Also, $s_k(f)$ is a polynomial due to the moment-cumulant formula (2.4.2).

After this, to consider the cases of various polynomials f, we will emphasize the corresponding polynomials $r_k(f)$, $s_k(f)$ just as appeared in the remark above, and the variables t and z will not be denoted when they are unimportant.

Lemma 2.4.5. Let f, g be polynomials in $\mathbb{C}[x]_0$, $\alpha, \beta \in \mathbb{C}$ and $k \in \mathbb{N}$. Then

(1) $r_1(\alpha f + \beta g) = \alpha r_1(f) + \beta r_1(g)$.

(2) $r_k(f) = r_1(f^k)$.

Proof. Both are derived directly from (2.4.4). □

Next, we will generalize the definition of polynomials r_k and s_k as determined by polynomials $\{f_i\}_{i=1}^k$ and satisfying multi-linearity. According to Lemma 2.4.5, the following can be understood as a natural extension.

Definition 2.4.6. Let us consider $f_1, \ldots, f_k \in \mathbb{C}[x]_0$. We define

$$r_k(f_1, \ldots, f_k) := r_1(f_1 \cdots f_k). \tag{2.4.7}$$

Likewise, for $\pi \in \mathrm{P}(k)$,

$$r_\pi[f_1, \ldots, f_k] := \prod_{V = \{i_1, \ldots, i_{|V|}\} \in \pi} r_{|V|}(f_{i_1}, \ldots, f_{i_{|V|}}).$$

Moreover, we define

$$s_\pi[f_1, \ldots, f_k] := \sum_{\substack{\sigma \in \mathrm{P}(k) \\ \sigma \leq \pi}} r_\sigma[f_1, \ldots, f_k] \mu_k^{\mathrm{P}}(\sigma, \pi) \tag{2.4.8}$$

for $\pi \in \mathrm{P}(k)$. In particular, $s_k(f_1, \ldots, f_k)$ denotes $s_{1_k}[f_1, \ldots, f_k]$.

Clearly, $s_k(f,\dots,f) = s_k(f)$ for all polynomial $f \in \mathbb{C}[x]_0$, and hence this is a generalization. The benefits of this generalization are the subsequent properties.

Proposition 2.4.7. For $k \in \mathbb{N}$ and $f_1, \dots, f_k \in \mathbb{C}[x]_0$, we have

$$r_k(f_1,\dots,f_k) = \sum_{\pi\in \mathrm{P}(k)} s_\pi[f_1,\dots,f_k].$$

Likewise, we have

$$s_\pi[f_1,\dots,f_k] = \prod_{V=\{i_1,\dots,i_{|V|}\}\in\pi} s_{|V|}(f_{i_1},\dots,f_{i_{|V|}})$$

for $\pi \in \mathrm{P}(k)$.

Proof. It follows from the standard discussion using the Möbius inversion formula, see [NS06, Lectures 9–11]. □

Example 2.4.8. As an interesting example, we take the polynomials $g_m(x) = x^m$ for $m \in \mathbb{N}$, then consider $s_k(g_{m_1},\dots,g_{m_k})$ for positive integers $\{m_i\}_{i=1}^k$ and let $M = \sum_{i=1}^k m_i$. First, note that $g_1(x) = x$ and also

$$\begin{aligned} r_k(g_{m_1},\dots,g_{m_k}) &= r_1(g_M) \\ &= r_M(g_1) \\ &= \sum_{\pi\in\mathrm{P}(M)} z^{|\pi|} \end{aligned}$$

due to Lemma 2.4.5 and Equation (2.4.5).

The map $\mathrm{P}(k) \to \mathrm{P}(M)$, $\sigma \mapsto \widehat{\sigma}$, is defined as each point $\{i\}$ expanding m_i-interval, e.g., $\widehat{0}_k = \{\{1,\dots,m_1\},\{m_1+1,\dots,m_1+m_2\},\dots,\{M-m_k+1,\dots,M\}\}$; in particular, $\mathrm{P}(k)$ and $[\widehat{0}_k, 1_M]$ are poset isomorphism via this map. Then we have

$$\begin{aligned} s_k(g_{m_1},\dots,g_{m_k}) &= \sum_{\sigma\in\mathrm{P}(k)} r_\sigma[g_{m_1},\dots,g_{m_k}]\mu_k^{\mathrm{P}}(\sigma,1_k) \\ &= \sum_{\sigma\in\mathrm{P}(k)} r_{\widehat{\sigma}}[g_1,\dots,g_1]\mu_M^{\mathrm{P}}(\widehat{\sigma},1_M) \\ &= \sum_{\sigma\in\mathrm{P}(k)} \sum_{\substack{\pi\in\mathrm{P}(M)\\ \pi\le\widehat{\sigma}}} z^{|\pi|}\mu_M^{\mathrm{P}}(\widehat{\sigma},1_M) \\ &= \sum_{\pi\in\mathrm{P}(M)} z^{|\pi|} \sum_{\substack{\sigma\in\mathrm{P}(k)\\ \pi\le\widehat{\sigma}}} \mu_M^{\mathrm{P}}(\widehat{\sigma},1_M) \\ &= \sum_{\pi\in\mathrm{P}(M)} z^{|\pi|} \sum_{\pi\vee\widehat{0}_k\le\rho} \mu_M^{\mathrm{P}}(\rho,1_M) \\ &= \sum_{\substack{\pi\in\mathrm{P}(M)\\ \pi\vee\widehat{0}_k=1_M}} z^{|\pi|}, \end{aligned}$$

where we used the result of Example 2.4.3 on the second line and Proposition 2.1.4 on the fifth line. Thus, $\deg s_k(g_{m_1},\dots,g_{m_k}) = M-(k-1)$ and $\operatorname{lead} s_k(g_{m_1},\dots,g_{m_k}) = \#\{\pi \in \mathrm{P}(M) : \pi\vee\widehat{0}_k = 1_M, |\pi| = M-(k-1)\}$.

Let V be a vector space over $\mathbb{C}$. A multi-linear map $\Phi : V^k \to \mathbb{C}$ is said to be *symmetric* if for any permutation ι of $[k]$, we have

$$\Phi(x_1,\dots,x_k) = \Phi(x_{\iota(1)},\dots,x_{\iota(k)}), \qquad (x_1,\dots,x_k) \in V^k.$$

Lemma 2.4.9. The both r_k and s_k are symmetric multi-linear maps from $\mathbb{C}[x]_0^k$ to $\mathbb{C}[z]_0$.

Proof. By the definition (2.4.7) and Lemma 2.4.5, it is clear that r_k is a symmetric multi-linear map. For s_k, the multi-linearity and the invariance of symmetric action are derived from those of r_k by the moment-cumulant formula (2.4.8). □

We obtain a generalized result of Formula (2.4.3) as follows.

Proposition 2.4.10. For any $k, n \in \mathbb{N}$ and $f_1, \dots, f_k \in \mathbb{C}[x]_0$, we get

$$s_k^{(n)}(f_1, \dots, f_k)(0) = \sum_{\pi \in \mathrm{P}(n)} \prod_{i=1}^{k} \left(\sum_{V \in \pi} f_i(|V|) \right) \mu_n^{\mathrm{P}}(\pi, 1_n)$$

The key to proving Proposition 2.4.10 is the following lemma.

Lemma 2.4.11 (see [Tho14])**.** Let V be a vector space over $\mathbb{C}$, (generally, a field). If $\Phi : V^k \to \mathbb{C}$ is a symmetric multi-linear map, then it is written as

$$\Phi(x_1, \dots, x_k) = \frac{1}{k!} \sum_{l=1}^{k} (-1)^{k-l} \sum_{J:\ |J|=l} \widetilde{\Phi} \left(\sum_{j \in J} x_j \right),$$

where $\widetilde{\Phi}(x) := \Phi(x, \dots, x)$ for $x \in V$.

Proof of Proposition 2.4.10. Note that a map $(f_1, \dots, f_k) \mapsto s_k^{(n)}(f_1, \dots, f_k)(0)$ from $\mathbb{C}[x]_0^k$ to $\mathbb{C}$ is symmetric multi-linear by Lemma 2.4.9. Also, define a symmetric multi-linear map $\Phi_{k,n} : \mathbb{C}[x]_0^k \to \mathbb{C}$ by

$$\Phi_{k,n}(f_1, \dots, f_k) := \sum_{\pi \in \mathrm{P}(n)} \prod_{i=1}^{k} \left(\sum_{V \in \pi} f_i(|V|) \right) \mu_n^{\mathrm{P}}(\pi, 1_n).$$

One can see that, for any $k, n \in \mathbb{N}$ and $f \in \mathbb{C}[x]_0$,

$$s_k^{(n)}(f, \dots, f)(0) = \Phi_{k,n}(f, \dots, f),$$

due to Formula (2.4.3). Thus, Lemma 2.4.11 implies the desired result. □

By Proposition 2.4.10, the problem boils down to finding the degree and leading coefficient of polynomials $s_k(f_1, \dots, f_k)$. As a special family of polynomials, let us take

$$c_m(x) := \binom{x}{m},$$

for $m \in \mathbb{N}$. Then the general cases are induced from them by multi-linearity of s_k; see Theorem 2.4.20.

Consider $r_k(c_{m_1}, \dots, c_{m_k})$ for positive integers $\{m_i\}_{i=1}^k$ then its coefficient is

$$\begin{aligned} r_k^{(n)}(c_{m_1}, \dots, c_{m_k})(0) &= r_1^{(n)}(c_{m_1} \cdots c_{m_k})(0) \\ &= \sum_{l=1}^{n} \binom{n}{l} (-1)^{n-l} c_{m_1}(l) \cdots c_{m_k}(l) \\ &= \sum_{l=1}^{n} \binom{n}{l} (-1)^{n-l} \binom{l}{m_1} \cdots \binom{l}{m_k} \end{aligned} \tag{2.4.9}$$

by (2.4.4). This value has a combinatorial meaning as follows.

Proposition 2.4.12. It holds that

$$r_k^{(n)}(c_{m_1},\dots,c_{m_k})(0) = \#R^{(n)}_{(m_1,\dots,m_k)},$$

where $R^{(n)}_{(m_1,\dots,m_k)} := \{(W_1,\dots,W_k)\in \mathrm{B}(n)^k : |W_i| = m_i,\ \bigcup_{i=1}^k W_i = [n]\}$. In particular,

(1) $\deg r_k(c_{m_1},\dots,c_{m_k}) = \sum_{i=1}^k m_i =: M$,

(2) $\operatorname{lead} r_k(c_{m_1},\dots,c_{m_k}) = \dfrac{1}{\left(\sum_{i=1}^k m_i\right)!}\cdot r_k^{(M)}(c_{m_1},\dots,c_{m_k}) = \dfrac{1}{\prod_{i=1}^k m_i!}$.

Proof. Equation (2.4.9) implies

$$r_k^{(n)}(c_{m_1},\dots,c_{m_k})(0) = \sum_{l=1}^{n}\sum_{\substack{V\in\mathrm{B}(n)\\|V|=l}}\sum_{\substack{W_1,\dots,W_k\subset V\\|W_1|=m_1,\dots,|W_k|=m_k}} (-1)^{n-l}.$$

Define $W = \bigcup_{i=1}^k W_i$. Exchanging the order of summations shows

$$\begin{aligned} r_k^{(n)}(c_{m_1},\dots,c_{m_k})(0) &= \sum_{\substack{W_1,\dots,W_k\in\mathrm{B}(n)\\|W_1|=m_1,\dots,|W_k|=m_k}} \sum_{l=1}^{n}\left(\sum_{\substack{W\subset V\\|V|=l}}(-1)^{n-l}\right)\\ &= \sum_{\substack{W_1,\dots,W_k\in\mathrm{B}(n)\\|W_1|=m_1,\dots,|W_k|=m_k}} \sum_{W\subset V}(-1)^{n-|V|}\\ &= \sum_{\substack{W_1,\dots,W_k\in\mathrm{B}(n)\\|W_1|=m_1,\dots,|W_k|=m_k}} \delta_{W,[n]}\\ &= \#R^{(n)}_{(m_1,\dots,m_k)}, \end{aligned}$$

where the third equation follows from Proposition 2.1.4 and Example 2.1.5 (1). The properties (1) and (2) follow easily from the definition of the set $R^{(n)}_{(m_1,\dots,m_k)}$. □

Similar to the polynomial $r_k(c_{m_1},\dots,c_{m_k})$, we can give a combinatorial interpretation to the coefficients of $s_k(c_{m_1},\dots,c_{m_k})$ which reflects the intrinsic decomposition of $R^{(n)}_{(m_1,\dots,m_k)}$. Let us prepare a few concepts to explain it.

- Define the natural map $\tau : \mathrm{B}(n)\setminus\emptyset \to \mathrm{P}(n)$ as

  $$\tau(W) = \{W\}\cup\{\{j\} : j\notin W\}$$

  for all non-empty subsets $W\subset[n]$.

- An *ordered partition* of $[n]$ is a tuple $\mathbf{V} = (V_1,\dots,V_l)$ such that $\{V_1,\dots,V_l\}\in\mathrm{P}(n)$.

- There is a canonical map that makes a partition $\pi = \{V_1,\dots,V_l\}\in\mathrm{P}(n)$ correspond to an ordered partition $\overrightarrow{\pi} = (V_1,\dots,V_l)$ such that $1\in V_1$ and V_i contains the minimum number in $[n]\setminus(V_1\cup\cdots\cup V_{i-1})$ for $i\geq 2$. We call $\overrightarrow{\pi}$ the *natural* ordered partition of $[n]$ associated with $\pi\in\mathrm{P}(n)$. Clearly, this map is one-to-one correspondence.

In particular, the following concept plays an important role in interpreting the coefficients of $s_k(c_{m_1},\dots,c_{m_k})$ by combinatorics.

Definition 2.4.13. Let $W = (W_1, \ldots, W_k) \in R^{(n)}_{(m_1,\ldots,m_k)}$.

(1) W is said to be *separable* when $\vee_{i=1}^{k} \tau(W_i) \neq 1_n \in \mathrm{P}(n)$.

(2) W is said to be *essential* if W is not separable.

Let $S^{(n)}_{(m_1,\ldots,m_k)}$ denote the set of essential tuples in $R^{(n)}_{(m_1,\ldots,m_k)}$.

The set $R^{(n)}_{(m_1,\ldots,m_k)}$ can be decomposed into its components which consist of essential tuples. One can see that, for $(W_1, \ldots, W_k) \in R^{(n)}_{(m_1,\ldots,m_k)}$, there are uniquely a natural ordered partition $\overrightarrow{\pi} = (V_1, \ldots, V_l)$ of $[k]$ and an ordered partition $\mathbf{U} = (U_1, \ldots, U_l)$ of $[n]$ with $\bigvee_{r\in V_j} \tau(W_r) = \tau(U_j)$ for each $1 \leq j \leq l$.

Example 2.4.14. Take $(W_1, W_2, W_3) \in R^{(5)}_{(2,2,2)}$, where

$$W_1 = \{1, 4\}, \quad W_2 = \{2, 4\} \quad \text{and} \quad W_3 = \{3, 5\}.$$

Then $\overrightarrow{\pi} = (V_1, V_2) = (\{1, 2\}, \{3\})$ and $\mathbf{U} = (U_1, U_2) = (\{1, 2, 4\}, \{3, 5\})$. It is clear that $\mathbf{U} = \left(\bigcup_{r\in V_1} W_r,\ \bigcup_{r\in V_2} W_r\right)$. Moreover, $\tau(W_1) \vee \tau(W_2) = \{\{1, 2, 4\}, \{3\}, \{5\}\} = \tau(U_1)$ and $\tau(W_3) = \tau(U_2)$.

Define

$$S^{\overrightarrow{\pi},\mathbf{U}}_{(m_1,\ldots,m_k)} = \left\{ (W_1, \ldots, W_k) \in R^{(n)}_{(m_1,\ldots,m_k)} : \bigcup_{r\in V_j} W_r = U_j,\ \bigvee_{r\in V_j} \tau(W_r) = \tau(U_j) \right\}$$

for a natural ordered partition $\overrightarrow{\pi} = (V_1, \ldots, V_l)$ of $[k]$, an ordered partition $\mathbf{U} = (U_1, \ldots, U_l)$ of $[n]$ and positive integers $\{m_i\}_{i=1}^{k}$. Note that $S^{\overrightarrow{\pi},\mathbf{U}}_{(m_1,\ldots,m_k)}$ is isomorphic to $S^{(|U_1|)}_{V_1} \times \cdots \times S^{(|U_l|)}_{V_l}$ where $S^{(i)}_{V} := S^{(i)}_{(m_{r_1},\ldots,m_{r_t})}$ when $V = \{r_1, \ldots, r_t\}$.

A consequence of the above one-to-one correspondence is that

$$\begin{aligned} R^{(n)}_{(m_1,\ldots,m_k)} &= \bigcup_{\overrightarrow{\pi}=(V_1,\ldots,V_l)} \bigcup_{\mathbf{U}=(U_1,\ldots,U_l)} S^{\overrightarrow{\pi},\mathbf{U}}_{(m_1,\ldots,m_k)} \\ &= \bigcup_{\overrightarrow{\pi}=(V_1,\ldots,V_l)} \bigcup_{\substack{i_1,\ldots,i_l\geq 1 \\ i_1+\cdots+i_l=n}} \bigcup_{\substack{\mathbf{U}=(U_1,\ldots,U_l) \\ |U_j|=i_j}} S^{\overrightarrow{\pi},\mathbf{U}}_{(m_1,\ldots,m_k)} \end{aligned}$$

and therefore

$$\#R^{(n)}_{(m_1,\ldots,m_k)} = \sum_{\pi=\{V_1,\ldots,V_l\}\in\mathrm{P}(k)} \sum_{\substack{i_1,\ldots,i_l\geq 1 \\ i_1+\cdots+i_l=n}} \binom{n}{i_1, \ldots, i_l} \prod_{j=1}^{l} \#S^{(i_j)}_{V_j}. \tag{2.4.10}$$

The value $s^{(n)}_k(c_{m_1}, \ldots, c_{m_k})(0)$ can be computed by counting the number of $S^{(n)}_{(m_1,\ldots,m_k)}$.

Proposition 2.4.15. It holds that

$$s^{(n)}_k(c_{m_1}, \ldots, c_{m_k})(0) = \#S^{(n)}_{(m_1,\ldots,m_k)}.$$

In particular, $s_k(c_{m_1}, \ldots, c_{m_k})$ is a polynomial of degree $\sum_{i=1}^{k} m_i - (k-1)$.

Proof. By using the induction, it is not difficult to see that there are no essential tuples in $R^{(n)}_{(m_1,\dots,m_k)}$ if $n > \sum_{i=1}^k m_i - (k-1)$, which means $\#S^{(n)}_{(m_1,\dots,m_k)} = 0$.

Define

$$\tilde{s}_k(m_1,\dots,m_k)(z) = \sum_{n=1}^{\infty} \frac{\#S^{(n)}_{(m_1,\dots,m_k)}}{n!} z^n.$$

For the conclusion, it suffices to show that $s_k(c_{m_1},\dots,c_{m_k}) = \tilde{s}_k(m_1,\dots,m_k)$, which is equivalent to

$$r_k(c_{m_1},\dots,c_{m_k})(z) = \sum_{\pi\in \mathrm{P}(k)} \tilde{s}_\pi(m_1,\dots,m_k)(z) \tag{2.4.11}$$

because of the Möbius inversion formula. By Proposition 2.4.12, Equation (2.4.11) is equivalent to

$$\#R^{(n)}_{(m_1,\dots,m_k)} = \sum_{\pi=\{V_1,\dots,V_l\}\in \mathrm{P}(k)} \sum_{\substack{i_1,\dots,i_l\geq 1\\ i_1+\cdots+i_l=n}} \binom{n}{i_1,\dots,i_l} \prod_{j=1}^{l} \#S^{(i_j)}_{V_j}$$

for $n\in\mathbb{N}$, which is exactly the same as (2.4.10). □

Now, the last problem is to determine the leading coefficient of $s_k(c_{m_1},\dots,c_{m_k})$, i.e., to count $S^{(M-(k-1))}_{(m_1,\dots,m_k)}$ where $M = \sum_{i=1}^k m_i$. The main strategy is to use the mathematical induction, which requires a slight modification of $S^{(M-(k-1))}_{(m_1,\dots,m_k)}$.

Definition 2.4.16. Let $\{m_i\}_{i=1}^k$ and $\{l_i\}_{i=1}^{M-(k-1)}$ be sequences of positive integers and $L := l_1+\cdots+l_{M-(k-1)}$. Define

$$T^{(l_1,\dots,l_{M-(k-1)})}_{(m_1,\dots,m_k)} := \left\{ (W_1,\dots,W_k)\in \mathrm{B}(L)^k : |W_i| = m_i,\ \bigvee_{i=1}^k \tau(W_i)\vee \widehat{0}_{M-(k-1)} = 1_L \right\},$$

where $\widehat{0}_{M-(k-1)} := \{\{1,\dots,l_1\},\{l_1+1,\dots,l_1+l_2\},\dots,\{\sum_{i=1}^{M-(k-1)-1} l_i+1,\dots,\sum_{i=1}^{M-(k-1)} l_i\}\}$.

It is clear from Definitions 2.4.13 and 2.4.16 that $S^{(M-(k-1))}_{(m_1,\dots,m_k)}$ is a specific case of $T^{(l_1,\dots,l_{M-(k-1)})}_{(m_1,\dots,m_k)}$. That is,

$$S^{(M-(k-1))}_{(m_1,\dots,m_k)} = T^{(1,\dots,1)}_{(m_1,\dots,m_k)},$$

where $(1,\dots,1)$ is a $(M-(k-1))$-tuple which consists only of 1.

Example 2.4.17. Let us look at examples for small k.

- For any positive integers m_1 and $l_1,\dots,l_{m_1}$, one has
$$\#T^{(l_1,\dots,l_{m_1})}_{(m_1)} = l_1\cdots l_{m_1}.$$
- For any positive integers m_1,m_2 and $l_1,\dots,l_{m_1+m_2-1}$, one has
$$\#T^{(l_1,\dots,l_{m_1+m_2-1})}_{(m_1,m_2)} = \sum_{\substack{I\subset[m_1+m_2-1]\\ |I|=m_1}} \left(\prod_{i\in I} l_i\right) \#T^{(l'_1,\dots,l'_{m_2})}_{(m_2)},$$
where $l'_1 = \sum_{i\in I} l_i$ and $\{l'_2,\dots,l'_{m_2}\} = \{l_i \mid i\in[m_1+m_2-1]\setminus I\}$. Thus,
$$\begin{aligned}\#T^{(l_1,\dots,l_{m_1+m_2-1})}_{(m_1,m_2)} &= \sum_{\substack{I\subset[m_1+m_2-1]\\ |I|=m_1}} \left(\prod_{i=1}^{m_1+m_2-1} l_i\right) \sum_{i\in I} l_i \\ &= \left(\prod_{i=1}^{m_1+m_2-1} l_i\right)\binom{m_1+m_2-2}{m_1-1} \sum_{i=1}^{m_1+m_2-1} l_i.\end{aligned}$$

Applying the counting technique used in the above example to the general case yields the following results.

Proposition 2.4.18. Let $\{m_i\}_{i=1}^{k}$ and $\{l_j\}_{j=1}^{M-(k-1)}$ be sequences of positive integers, where $M = \sum_{i=1}^{k} m_i$. Then

$$\#T_{(m_1,\dots,m_k)}^{(l_1,\dots,l_{M-(k-1)})} = \left(\prod_{i=1}^{M-(k-1)} l_i\right) \binom{M-k}{m_1-1,\dots,m_k-1} \left(\sum_{i=1}^{M-(k-1)} l_i\right)^{k-1}. \tag{2.4.12}$$

Proof. We use the induction for k as follows. Formula (2.4.12) holds for $k = 1$ because we mentioned in Example 2.4.17. Assume Formula (2.4.12) holds up to $k-1$. Note that

$$\#T_{(m_1,\dots,m_k)}^{(l_1,\dots,l_{M-(k-1)})} = \sum_{\substack{I\subset[M-(k-1)]\\ |I|=m_1}} \left(\prod_{i\in I} l_i\right) \#T_{(m_2,\dots,m_k)}^{(l'_1,\dots,l'_{m_2+\cdots+m_k-(k-1)})},$$

where $l'_1 = \sum_{i\in I} l_i$ and $\{l'_2,\dots,l'_{m_2+\cdots+m_k-(k-1)}\} = \{l_i \mid i \in [M-(k-1)] \setminus I\}$. Thus, by the induction hypothesis (2.4.12),

$$\begin{aligned}
&\#T_{(m_1,\dots,m_k)}^{(l_1,\dots,l_{M-(k-1)})}\\
&= \sum_{\substack{I\subset[M-(k-1)]\\ |I|=m_1}} \left(\prod_{i=1}^{M-(k-1)} l_i\right) \left(\sum_{i\in I} l_i\right) \binom{m_2+\cdots+m_k-(k-1)}{m_2-1,\dots,m_k-1} \left(\sum_{i=1}^{M-(k-1)} l_i\right)^{k-2}\\
&= \left(\prod_{i=1}^{M-(k-1)} l_i\right) \binom{m_2+\cdots+m_k-(k-1)}{m_2-1,\dots,m_k-1} \left(\sum_{i=1}^{M-(k-1)} l_i\right)^{k-2} \sum_{\substack{I\subset[M-(k-1)]\\ |I|=m_1}} \left(\sum_{i\in I} l_i\right)\\
&= \left(\prod_{i=1}^{M-(k-1)} l_i\right) \binom{m_2+\cdots+m_k-(k-1)}{m_2-1,\dots,m_k-1} \left(\sum_{i=1}^{M-(k-1)} l_i\right)^{k-2} \binom{M-k}{m_1-1} \left(\sum_{i=1}^{M-(k-1)} l_i\right)\\
&= \left(\prod_{i=1}^{M-(k-1)} l_i\right) \binom{M-k}{m_1-1,\dots,m_k-1} \left(\sum_{i=1}^{M-(k-1)} l_i\right)^{k-1}.
\end{aligned}$$

□

Thus, we can get the leading coefficient of $s_k(c_{m_1},\dots,c_{m_k})$ as a corollary.

Corollary 2.4.19. Let $\{m_i\}_{i=1}^{k}$ be a sequence of positive integers and $M = \sum_{i=1}^{k} m_i$. Then

$$\operatorname{lead} s_k(c_{m_1},\dots,c_{m_k}) = \frac{(M-(k-1))^{k-2}}{\prod_{i=1}^{k}(m_i-1)!}.$$

Proof. Recall that $\deg s_k(c_{m_1},\cdots,c_{m_k}) = M-(k-1)$ by Proposition 2.4.15. Then Propositions 2.4.15 and 2.4.18 imply that

$$\begin{aligned}
s_k^{(M-(k-1))}(c_{m_1},\dots,c_{m_k})(0) &= \#S_{(m_1,\dots,m_k)}^{(M-(k-1))}\\
&= \#T_{(m_1,\dots,m_k)}^{(1,\dots,1)}\\
&= \binom{M-k}{m_1-1,\dots,m_k-1}(M-(k-1))^{k-1}\\
&= \frac{(M-(k-1))!}{\prod_{i=1}^{k}(m_i-1)!}\cdot(M-(k-1))^{k-2},
\end{aligned}$$

as desired. □

Combining the above propositions, we show the main result of this section.

Theorem 2.4.20. Suppose that $f_i \in \mathbb{C}[x]_0$ with $\deg f_i = m_i$ for each $1 \leq i \leq k$, and let $M = \sum_{i=1}^k m_i$. Then we have

$$\sum_{\pi \in \mathrm{P}(n)} \prod_{i=1}^k \left(\sum_{V \in \pi} f_i(|V|) \right) \mu_n^{\mathrm{P}}(\pi, 1_n) = \begin{cases} (n-1)! n^{k-1} \prod_{i=1}^k m_i \operatorname{lead} f_i, & n = M - (k-1), \\ 0, & n > M - (k-1). \end{cases}$$

Proof. By Proposition 2.4.10, the statement is equivalent to the following:

- $\deg s_k(f_1, \ldots, f_k) = M - (k-1)$;
- $\operatorname{lead} s_k(f_1, \ldots, f_k) = (M-(k-1))^{k-2} \cdot \prod_{i=1}^k (m_i \operatorname{lead} f_i)$.

Because the family of polynomials $\{c_m(x)\}_{m \in \mathbb{N}}$ is a basis of $\mathbb{C}[x]_0$, the polynomials $\{f_i\}_{i=1}^k$ can be uniquely expressed as linear combinations of $\{c_m(x)\}_{m \in \mathbb{N}}$:

$$f_i(x) = \sum_{j=1}^{m_i} a_j^{(i)} c_j(x)$$

for $1 \leq i \leq k$. Here, note that $c_m(x)$ is a polynomial of degree m and the leading coefficient $1/m!$ and hence $a_{m_i}^{(i)} = m! \operatorname{lead} f_i$. Next, by Proposition 2.4.9,

$$s_k(f_1, \ldots, f_k) = \sum_{j_1=1}^{m_1} \cdots \sum_{j_k=1}^{m_k} a_{j_1}^{(1)} \cdots a_{j_k}^{(k)} s_k(c_{j_1}, \ldots, c_{j_k}).$$

Then, by Proposition 2.4.15,

$$\begin{aligned} \deg s_k(f_1, \ldots, f_k) &= \deg s_k(c_{m_1}, \ldots, c_{m_k}) \\ &= M - (k-1). \end{aligned}$$

Hence, by Corollary 2.4.19,

$$\begin{aligned} \operatorname{lead} s_k(f_1, \ldots, f_k) &= a_{m_1}^{(1)} \cdots a_{m_k}^{(k)} \operatorname{lead} s_k(c_{m_1}, \ldots, c_{m_k}) \\ &= \operatorname{lead} s_k(c_{m_1}, \ldots, c_{m_k}) \prod_{i=1}^k m_i! \operatorname{lead} f_i \\ &= \frac{(M-(k-1))^{k-2}}{\prod_{i=1}^k (m_i - 1)!} \prod_{i=1}^k m_i! \operatorname{lead} f_i \\ &= (M-(k-1))^{k-2} \prod_{i=1}^k m_i \operatorname{lead} f_i. \end{aligned}$$

□

We give a few specific cases of Theorem 2.4.20.

Corollary 2.4.21.

$$\sum_{\pi \in \mathrm{P}(n)} \left(\sum_{V \in \pi} \binom{|V|}{2} \right)^k \mu_n^{\mathrm{P}}(\pi, 1_n) = \begin{cases} (n-1)! n^{k-1}, & n = k+1, \\ 0, & n > k+1, \end{cases} \tag{2.4.13}$$

and

$$\sum_{\pi \in \mathrm{P}(n)} \left(\sum_{V \in \pi} |V|^2 \right)^k \mu_n^{\mathrm{P}}(\pi, 1_n) = \begin{cases} 2^k (n-1)! n^{k-1}, & n = k+1, \\ 0, & n > k+1. \end{cases} \tag{2.4.14}$$

These formulas will be used in Chapter 4.

Finally, we conclude this section by noting that the combination of Example 2.4.8 and Theorem 2.4.20 leads to the following result as a byproduct.

Corollary 2.4.22. For positive integers $\{m_i\}_{i=1}^k$, define $M = \sum_{i=1}^k m_i$ and $\widehat{0}_k = \{\{1, \ldots, m_1\}, \{m_1 + 1, \ldots, m_1 + m_2\}, \ldots, \{M - m_k + 1, \ldots, M\}\}$. Then

$$\#\{\sigma \in \mathrm{P}(M) : \sigma \vee \widehat{0}_k = 1_M, |\sigma| = M - (k-1)\} = (M - (k-1))^{k-2} \prod_{i=1}^k m_i.$$

Chapter 3

Random Matrices

We prove a concentration phenomenon on the empirical eigenvalue distribution (EED) of the principal submatrix in a random hermitian matrix whose distribution is invariant under unitary conjugacy; for example, this class includes GUE (Gaussian Unitary Ensemble) and Wishart matrices. More precisely, if the EED of the whole matrix converges to some deterministic probability measure μ, then the difference of rescaled EEDs of the whole matrix and of its principal submatrix concentrates at the Rayleigh measure (in general, a Schwartz distribution) associated with μ by the Markov–Krein correspondence. For the proof, we use the moment method with Weingarten calculus and free probability.

3.1 Main result

Let X_N be an hermitian random matrix of size N whose distribution is invariant under conjugacy by unitary matrices and let $\Lambda_N = (\lambda_1^{(N)} \leq \cdots \leq \lambda_N^{(N)})$ be its eigenvalues. It is known that a diagonalization $X_N = U_N D_N U_N{}^*$ exists, where $D_N = \mathrm{diag}(\lambda_1^{(N)}, \lambda_2^{(N)}, \ldots, \lambda_N^{(N)})$ and U_N is a Haar unitary random matrix of size N and independent of D_N (see [CM14, Proposition 6.1]).

For the principal submatrix $\tilde{X}_N$ made by removing the last row and column of X_N, Cauchy's interlacing law says that the eigenvalues $\tilde{\Lambda}_N = (\tilde{\lambda}_1^{(N)} \leq \cdots \leq \tilde{\lambda}_{N-1}^{(N)})$ of $\tilde{X}_N$ interlace with Λ_N (see [Tao12, Exercise 1.3.14]):

$$\lambda_1^{(N)} \leq \tilde{\lambda}_1^{(N)} \leq \lambda_2^{(N)} \leq \tilde{\lambda}_2^{(N)} \leq \cdots \leq \lambda_{N-1}^{(N)} \leq \tilde{\lambda}_{N-1}^{(N)} \leq \lambda_N^{(N)}.$$

In many examples, the empirical eigenvalue distribution $\mu_N = (1/N)\sum_{i=1}^N \delta_{\lambda_i^{(N)}}$ of the random matrix X_N converges, as $N \to \infty$, to a non-random probability measure, and we do assume so. Then it is not hard to see (at least with a mild assumption) that the empirical eigenvalue distribution $\tilde{\mu}_N$ of $\tilde{X}_N$ also converges to the same limit. Our main result roughly says that the Rayleigh measure

$$\widehat{\tau}_N := N\mu_N - (N-1)\tilde{\mu}_N = \sum_{i=1}^N \delta_{\lambda_i^{(N)}} - \sum_{j=1}^{N-1} \delta_{\tilde{\lambda}_j^{(N)}}$$

is close to the Rayleigh measure τ_N linked to the transition measure μ_N by the Markov–Krein correspondence. Note that τ_N is of the form

$$\tau_N = \sum_{i=1}^N \delta_{\lambda_i^{(N)}} - \sum_{j=1}^{N-1} \delta_{\eta_j^{(N)}}$$

where $(\eta_1^{(N)} \leq \cdots \leq \eta_{N-1}^{(N)})$ is a sequence also interlacing with Λ_N (see [Ker98, Eq. (2)]).

Since our arguments are based on the moment method, we denote by $\mathrm{M}_k(\zeta)$ for simplicity the k-th moment of a measure or Schwartz distribution ζ when it is well defined. It should be

noted here that if a probability measure μ has finite moments of all orders, then τ defined via (1.0.1) also has finite moments of all orders (see [AD57, Theorem A (d)] and [Ker98, Section 3.4]). Furthermore, for convenience of statements, let $\widehat{\mu}_N$ be the transition measure associated with the Rayleigh measure $\widehat{\tau}_N$; then the main result can alternatively be phrased that $\widehat{\mu}_N$ is close to μ_N.

Theorem 3.1.1. Let $\mu_N, \tau_N, \widehat{\mu}_N, \widehat{\tau}_N$ be as above, μ be a probability measure on $\mathbb{R}$ and τ be related to μ by the Markov–Krein correspondence (1.0.1). Assume that

$$\sup_{N\geq 1} \mathbb{E}[\mathrm{M}_k(\mu_N)] < \infty \quad \text{and} \quad \mathrm{M}_k(\mu) < \infty, \qquad k \in 2\mathbb{N} \tag{3.1.1}$$

and μ_N converges in moments to μ in probability:

$$\lim_{N\to\infty} \mathbb{P}[|\mathrm{M}_k(\mu_N) - \mathrm{M}_k(\mu)| \geq \epsilon] = 0, \qquad k \in \mathbb{N},\ \epsilon > 0. \tag{3.1.2}$$

Then we have

$$\lim_{N\to\infty} \| \mathrm{M}_k(\widehat{\tau}_N) - \mathrm{M}_k(\tau)\|_{L^2} = 0, \qquad k \in \mathbb{N},$$

and

$$\lim_{N\to\infty} \mathbb{P}[|\mathrm{M}_k(\widehat{\mu}_N) - \mathrm{M}_k(\mu)| \geq \epsilon] = 0, \qquad k \in \mathbb{N},\ \epsilon > 0.$$

In particular, if the moment problem for $\{\mathrm{M}_k(\mu)\}_{k\geq 1}$ is determinate then $\widehat{\mu}_N$ weakly converges to μ in probability:

$$\lim_{N\to\infty} \mathbb{P}\left[\left|\int_{\mathbb{R}} f(x)\,\widehat{\mu}_N(dx) - \int_{\mathbb{R}} f(x)\,\mu(dx)\right| \geq \epsilon\right] = 0, \qquad f \in C_b(\mathbb{R}),\ \epsilon > 0.$$

Remark 3.1.2. (i) Since the relation between the moments $\{\mathrm{M}_n(\mu)\}_{n\in\mathbb{N}}$ (resp. $\{\mathrm{M}_n(\mu_N)\}_{n\in\mathbb{N}}$) and $\{\mathrm{M}_k(\tau)\}_{k\in\mathbb{N}}$ (resp. $\{\mathrm{M}_k(\tau_N)\}_{k\in\mathbb{N}}$) is the same as that between complete symmetric functions and Newton power sums (see (2.3.2) below), the convergence (3.1.2) holds if and only if τ_N converges in moments to τ in probability.

(ii) The combination of (3.1.1) and (3.1.2) implies the convergence of moments in L^p norm for every $p \in [1, \infty)$; see Proposition 3.2.3.

(iii) The assumptions (3.1.1) and (3.1.2) are satisfied by appropriately normalized Gaussian Unitary Ensemble (GUE) [HP00, Theorem 4.1.5], where μ is the standard semicircle distribution $(1/(2\pi))\sqrt{4-x^2}\,dx$. For GUE (actually, more general Wigner matrices), a finer result on the fluctuation of $\widehat{\tau}_N$ from τ is also known in [ES18] stated in the language of rectangular Young diagrams; see also [Sod17].

The proof is based on Weingarten calculus and free probability which allow us to compute the moments of the principal submatrix:

$$\mathbb{E} \circ \mathrm{Tr}[(\tilde{X}_N)^k] = \mathbb{E} \circ \mathrm{Tr}[D_N U_N P_N U_N{}^* D_N U_N P_N U_N{}^* \cdots D_N U_N P_N U_N{}^*], \tag{3.1.3}$$

where $P_N = \mathrm{diag}(1, 1, \ldots, 1, 0)$.

In fact, the joint distribution of $(\tilde{\lambda}_1^{(N)} \leq \cdots \leq \tilde{\lambda}_{N-1}^{(N)})$ is explicit under the condition that $(\lambda_1^{(N)} \leq \cdots \leq \lambda_N^{(N)})$ is a constant sequence; it is proportional to the Vandermonde determinant [Bar01, Proposition 4.2] (see also the expository paper [Far15]). Using this explicit formula might be an alternative approach for computing (3.1.3) and hence for the proof of Theorem 3.1.1; however, the author is not sure whether this direction is promising.

Weingarten calculus

The computation of mixed moments of Haar unitary random matrices U_N and deterministic matrices is called Weingarten calculus. Recall that for $g \in \mathfrak{S}_k$, let $\mathrm{Tr}_g[A_1, A_2, \ldots, A_k]$ be the product of traces according to the cycle decomposition of g; for example if $g = (1,3,2,5)(4)(6,9)(7,8)$ then $\mathrm{Tr}_g[A_1, A_2, \ldots, A_9] = \mathrm{Tr}(A_1A_3A_2A_5)\,\mathrm{Tr}(A_4)\,\mathrm{Tr}(A_6A_9)\,\mathrm{Tr}(A_7A_8)$. Similarly, for a sequence $\{\alpha_n\}_{n\geq 1} \subset \mathbb{C}$ we define α_g to be the product of α_n's according to the sizes of cycles; in the above example, $\alpha_g = \alpha_4\alpha_1{\alpha_2}^2$.

Let A_i, B_i $(i = 1, \ldots, k)$ be $N \times N$ matrices. Then

$$\begin{aligned}&\mathbb{E} \circ \mathrm{Tr}_g[A_1U_NB_1U_N{}^*, \ldots, A_kU_NB_kU_N{}^*] \\ &\quad = \sum_{\substack{g_1,g_2,g_3 \in \mathfrak{S}_k \\ g_1g_2g_3 = g}} \mathrm{Tr}_{g_1}[A_1, \ldots, A_k]\,\mathrm{Tr}_{g_2}[B_1, \ldots, B_k]\,\mathrm{Wg}(g_3, N)\end{aligned}$$

for all $g \in \mathfrak{S}_k$. In particular, in the case of $g = \gamma_k = (1, 2, \ldots, k)$ the formula above specializes to

$$\begin{aligned}&\mathbb{E} \circ \mathrm{Tr}[(A_1U_NB_1U_N{}^*) \cdots (A_kU_NB_kU_N{}^*)] \\ &\quad = \sum_{g,h \in \mathfrak{S}_k} \mathrm{Tr}_g[A_1, \ldots, A_k]\,\mathrm{Tr}_h[B_1, \ldots, B_k]\,\mathrm{Wg}(h^{-1}g^{-1}\gamma_k, N),\end{aligned} \tag{3.1.4}$$

see [CŚ06, Proposition 2.3]. The coefficients $\mathrm{Wg}(g, N)$ are called the Weingarten function. Its asymptotic behavior for large N is known in the form

$$N^{k+|g|}\,\mathrm{Wg}(g, N) = \mu_k^{\mathrm{Wg}}(g) + O\left(N^{-2}\right), \qquad g \in \mathfrak{S}_k. \tag{3.1.5}$$

The number $|g|$, called the length function, is the minimal number l for which g can be written as a product of l transpositions, and the value $\mu_k^{\mathrm{Wg}}(g)$ above is expressed in terms of the Catalan numbers $C_n = (2n)!/(n!(n+1)!)$ as

$$\mu_k^{\mathrm{Wg}}(g) = \prod_{1 \leq j \leq l} (-1)^{|h_j|} C_{|h_j|} \tag{3.1.6}$$

where $g = h_1 \cdots h_l$ is the cycle decomposition of g; see [CM17, Theorem 2.7].

3.2 Moment convergence

Some results on the moment method for random measures are collected below. The proofs are basic. Let $\mathfrak{p}, \mathfrak{p}_n, n \in \mathbb{N}$, be random probability measures on $\mathbb{R}$ with an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ below.

Proposition 3.2.1. Suppose that $\mathfrak{p}_n, \mathfrak{p}, n \in \mathbb{N}$ have finite moments of all orders almost surely, and the moment problem for $\{\mathrm{M}_k(\mathfrak{p})\}_{k\geq 1}$ is determinate almost surely. If

$$\lim_{n\to\infty} \mathbb{P}[|\mathrm{M}_k(\mathfrak{p}_n) - \mathrm{M}_k(\mathfrak{p})| \geq \epsilon] = 0, \qquad k \in \mathbb{N},\ \epsilon > 0, \tag{3.2.1}$$

then $\mathfrak{p}_n$ weakly converges to $\mathfrak{p}$ in probability:

$$\lim_{n\to\infty} \mathbb{P}\left[\left|\int_{\mathbb{R}} f(x)\,d\mathfrak{p}_n(x) - \int_{\mathbb{R}} f(x)\,d\mathfrak{p}(x)\right| \geq \epsilon\right] = 0, \qquad f \in C_b(\mathbb{R}),\ \epsilon > 0. \tag{3.2.2}$$

Proof. For later use, we first verify the existence of a subsequence of $\{\mathfrak{p}_n\}_{n\geq 1}$ which weakly converges to $\mathfrak{p}$ almost surely. Let $\Omega_0 \in \mathcal{F}$ be such that $\mathbb{P}[\Omega_0] = 1$ and the moment problem for $\{\mathrm{M}_k(\mathfrak{p}^\omega)\}_{k\geq 1}$ is determinate for all $\omega \in \Omega_0$. For $k = 1$, there exists a subsequence $\{n(1,\ell)\}_{\ell=1}^{\infty}$

of $\mathbb{N}$ and $\Omega_1 \subset \Omega_0$ such that $\Omega_1 \in \mathcal{F}, \mathbb{P}[\Omega_1] = 1$ and $\mathrm{M}_1(\mathfrak{p}^\omega_{n(1,\ell)})$ converges to $\mathrm{M}_1(\mathfrak{p}^\omega)$ for all $\omega \in \Omega_1$. For $k = 2$, there exists a subsequence $\{n(2,\ell)\}_{\ell=1}^\infty$ of $\{n(1,\ell)\}_{\ell=1}^\infty$ and $\Omega_2 \subset \Omega_1$ such that $\Omega_2 \in \mathcal{F}, \mathbb{P}[\Omega_2] = 1$ and $\mathrm{M}_2(\mathfrak{p}^\omega_{n(2,\ell)})$ converges to $\mathrm{M}_2(\mathfrak{p}^\omega)$ for all $\omega \in \Omega_2$. In this way we obtain subsequences $\{n(k,\ell)\}_{\ell=1}^\infty$ and decreasing subsets Ω_k of probability one for $k \geq 1$. Define $\tilde{\Omega} := \cap_{k\geq 1}\Omega_k$ and $n(\ell) := n(\ell,\ell)$; then $\mathrm{M}_k(\mathfrak{p}^\omega_{n(\ell)})$ converges to $\mathrm{M}_k(\mathfrak{p}^\omega)$ as $\ell \to \infty$ for all $\omega \in \tilde{\Omega}$ and all $k \in \mathbb{N}$. Since the moment problem for the limit sequence is determinate, we conclude by [Chu68, Theorem 4.5.5] that $\mathfrak{p}^\omega_{n(\ell)}$ weakly converges to $\mathfrak{p}^\omega$ as $\ell \to \infty$ for all $\omega \in \tilde{\Omega}$.

To finish the proof, suppose to the contrary that the desired conclusion (3.2.2) is false: there exist $f \in C_b(\mathbb{R})$, $\epsilon, \delta > 0$ and a subsequence of $\{\mathfrak{p}_n\}_{n\geq 1}$, denoted by $\{\mathfrak{p}_{n'}\}$, such that for all n'

$$\mathbb{P}\left[\left|\int_{\mathbb{R}} f(x)\, d\mathfrak{p}_{n'}(x) - \int_{\mathbb{R}} f(x)\, d\mathfrak{p}(x)\right| \geq \epsilon\right] \geq \delta. \tag{3.2.3}$$

However, we can extract a further subsequence of $\{\mathfrak{p}_{n'}\}$ which weakly converges to $\mathfrak{p}$ almost surely as we discussed. For this subsequence, the LHS of (3.2.3) must tend to zero, a contradiction. □

Remark 3.2.2. A similar result and proof are found in [Gre63, p. 178–180].

Proposition 3.2.3. Suppose that

$$\sup_{n\geq 1} \mathbb{E}[\mathrm{M}_k(\mathfrak{p}_n)] < \infty \quad \text{and} \quad \mathbb{E}[\mathrm{M}_k(\mathfrak{p})] < \infty, \qquad k \in 2\mathbb{N}. \tag{3.2.4}$$

Then the condition (3.2.1) is equivalent to

$$\|\,\mathrm{M}_k(\mathfrak{p}_n) - \mathrm{M}_k(\mathfrak{p})\|_{L^p} \to 0, \qquad p \in [1,\infty),\ k \in \mathbb{N}. \tag{3.2.5}$$

Proof. It suffices to prove that (3.2.1) implies (3.2.5); the other direction is well known.

For $p \in [1,\infty)$ choose $\ell \in 2\mathbb{N}$ such that $\ell \geq p$. The Hölder inequality implies that $|\,\mathrm{M}_k(\mathfrak{p}_n)|^\ell \leq \mathrm{M}_{k\ell}(\mathfrak{p}_n)$ and hence

$$\|\,\mathrm{M}_k(\mathfrak{p}_n)\|_{L^p} \leq \|\,\mathrm{M}_k(\mathfrak{p}_n)\|_{L^\ell} \leq (\mathbb{E}[\mathrm{M}_{k\ell}(\mathfrak{p}_n)])^{\frac{1}{\ell}}. \tag{3.2.6}$$

Combining the above and (3.2.4), as well as similar inequalities for $\mathrm{M}_k(\mathfrak{p})$, yields that

$$\sup_{n\in\mathbb{N}} \|\,\mathrm{M}_k(\mathfrak{p}_n)\|_{L^p} < \infty \quad \text{and} \quad \|\,\mathrm{M}_k(\mathfrak{p})\|_{L^p} < \infty, \qquad k \in \mathbb{N},\ p \in [1,\infty). \tag{3.2.7}$$

By standard arguments we obtain

$$\begin{aligned}
\|\,\mathrm{M}_k(\mathfrak{p}_n) - \mathrm{M}_k(\mathfrak{p})\|_{L^p}^p &= \mathbb{E}[|\mathrm{M}_k(\mathfrak{p}_n) - \mathrm{M}_k(\mathfrak{p})|^p 1_{\{|\mathrm{M}_k(\mathfrak{p}_n)-\mathrm{M}_k(\mathfrak{p})|\geq\epsilon\}}] \\
&\quad + \mathbb{E}[|\mathrm{M}_k(\mathfrak{p}_n) - \mathrm{M}_k(\mathfrak{p})|^p 1_{\{|\mathrm{M}_k(\mathfrak{p}_n)-\mathrm{M}_k(\mathfrak{p})|<\epsilon\}}] \\
&\leq (\mathbb{E}[|\mathrm{M}_k(\mathfrak{p}_n) - \mathrm{M}_k(\mathfrak{p})|^{2p}])^{1/2} (\mathbb{P}[|\mathrm{M}_k(\mathfrak{p}_n) - \mathrm{M}_k(\mathfrak{p})| \geq \epsilon])^{1/2} + \epsilon^p \\
&\leq (\|\,\mathrm{M}_k(\mathfrak{p}_n)\|_{L^{2p}} + \|\,\mathrm{M}_k(\mathfrak{p})\|_{L^{2p}})^p (\mathbb{P}[|\mathrm{M}_k(\mathfrak{p}_n) - \mathrm{M}_k(\mathfrak{p})| \geq \epsilon])^{1/2} + \epsilon^p.
\end{aligned}$$

Applying (3.2.7) and (3.2.1) to the above finishes the proof. □

Proposition 3.2.4. Suppose that (3.2.1) and (3.2.4) hold. Then

$$\lim_{n\to\infty} \mathbb{E}[P(\mathrm{M}_1(\mathfrak{p}_n), \mathrm{M}_2(\mathfrak{p}_n), \ldots, \mathrm{M}_k(\mathfrak{p}_n))] = \mathbb{E}[P(\mathrm{M}_1(\mathfrak{p}), \mathrm{M}_2(\mathfrak{p}), \ldots, \mathrm{M}_k(\mathfrak{p}))]$$

for every $k \in \mathbb{N}$ and every polynomial $P \in \mathbb{C}[x_1, x_2, \ldots, x_k]$.

Proof. This is a consequence of Proposition 3.2.3 and the following standard fact: if random variables $Y, Z, Y_n, Z_n, n \in \mathbb{N}$ satisfy $Y_n \to Y$ in L^p and $Z_n \to Z$ in L^p for all $p \in [1,\infty)$, then $Y_n Z_n \to YZ$ in L^p for all $p \in [1,\infty)$. □

3.3 Proof of the main result

In this section, we follow the notation in Theorem 3.1.1. The index N is omitted for readability when no confusion occurs. The main part of the proof of Theorem 3.1.1 is the following.

Theorem 3.3.1. Assume that

$$\sup_{N\geq 1}\mathbb{E}[\mathrm{M}_k(\mu_N)]<\infty, \qquad k\in 2\mathbb{N}. \tag{3.3.1}$$

Then, for every $k\in\mathbb{N}$ and $\ell\in\{1,2\}$, it holds that

$$\mathbb{E}[\mathrm{M}_k(\widehat{\tau}_N)^\ell]=\mathbb{E}[\mathrm{M}_k(\tau_N)^\ell]+O\left(N^{-1}\right).$$

Remark 3.3.2. Whether the above result holds for $\ell\geq 3$ is unknown.

Proof. Note first that the assumption (3.3.1) implies that

$$\sup_{N\geq 1}\mathbb{E}[|\mathrm{M}_g(\mu_N)|]<\infty, \qquad g\in\mathfrak{S}_k$$

for every $k\in\mathbb{N}$, thanks to the iterative use of Schwarz inequality and (3.2.6).

(i) $\ell=1$**.** A key of the proof is the calculations of

$$\sum_{j=1}^{N-1}\mathbb{E}[\tilde{\lambda}_j{}^k]=\mathbb{E}\circ\mathrm{Tr}[(PU^*DUP)^k], \tag{3.3.2}$$

where $P=\mathrm{diag}(1,\dots,1,0)$. The RHS of (3.3.2) is calculated into

$$\begin{aligned}\mathbb{E}\circ\mathrm{Tr}[(PU^*DUP)^k]&=\mathbb{E}\circ\mathrm{Tr}[(DUPU^*)^k]\\&=\sum_{g,h\in\mathfrak{S}_k}\mathbb{E}\circ\mathrm{Tr}_g[D,\dots,D]\,\mathrm{Tr}_h[P,\dots,P]\,\mathrm{Wg}(h^{-1}g^{-1}\gamma_k) \qquad (3.3.3)\\&=\sum_{g,h\in\mathfrak{S}_k}\mathbb{E}\circ\mathrm{Tr}_g[D,\dots,D]\,\mathrm{Tr}_{h^{-1}\gamma_k}[P,\dots,P]\,\mathrm{Wg}(g^{-1}h)\\&=\sum_{g,h\in\mathfrak{S}_k}N^{\#(g)}\mathbb{E}\circ\mathrm{tr}_g[D,\dots,D](N-1)^{\#(h^{-1}\gamma_k)}\,\mathrm{Wg}(g^{-1}h), \qquad (3.3.4)\end{aligned}$$

where (3.1.4) was used on the second line and the change of variables $h\mapsto h^{-1}\gamma_k$ was employed on the third line.

On the other hand, if the projection P is replaced by the identity I in (3.3.3), then the same calculations lead to

$$\begin{aligned}\sum_{i=1}^{N}\mathbb{E}[\lambda_i{}^k]=\mathbb{E}\circ\mathrm{Tr}[D^k]&=\mathbb{E}\circ\mathrm{Tr}[(DUIU^*)^k]\\&=\sum_{g,h\in\mathfrak{S}_k}N^{\#(g)}\mathbb{E}\circ\mathrm{tr}_g[D,\dots,D]N^{\#(h^{-1}\gamma_k)}\,\mathrm{Wg}(g^{-1}h). \qquad (3.3.5)\end{aligned}$$

Taking the difference of (3.3.4) and (3.3.5) provides

$$\begin{aligned}\mathbb{E}[\mathrm{M}_k(\widehat{\tau}_N)]&=\sum_{i=1}^{N}\mathbb{E}[\lambda_i{}^k]-\sum_{j=1}^{N-1}\mathbb{E}[\tilde{\lambda}_j{}^k]\\&=\sum_{g,h\in\mathfrak{S}_k}N^{\#(g)}\mathbb{E}\circ\mathrm{tr}_g[D,\dots,D]\#(h^{-1}\gamma_k)N^{\#(h^{-1}\gamma_k)-1}\left(1+O(N^{-1})\right)\mathrm{Wg}(g^{-1}h).\end{aligned}$$

Here we use the asymptotic expansion (3.1.5) of the Weingarten functions to get

$$\mathbb{E}[\mathrm{M}_k(\widehat{\tau}_N)] = \sum_{|g|+|g^{-1}h|+|h^{-1}\gamma_k|=k-1} \#(h^{-1}\gamma_k)\mathbb{E}[\mathrm{M}_g(\mu_N)]\mu_k^{\mathrm{Wg}}(g^{-1}h) + O\left(N^{-1}\right). \tag{3.3.6}$$

Using the isomorphism explained in Section 2.1, we may rewrite (3.3.6) in terms of non-crossing partitions:

$$\begin{aligned}\mathbb{E}[\mathrm{M}_k(\widehat{\tau}_N)] &= \sum_{\sigma\le\pi\in\mathrm{NC}(k)} |\mathrm{Kr}(\pi)|\,\mathbb{E}[\mathrm{M}_\sigma(\mu_N)]\mu_k^{\mathrm{NC}}(\sigma,\pi) + O\left(N^{-1}\right)\\ &= \sum_{\pi\in\mathrm{NC}(k)} |\mathrm{Kr}(\pi)|\,\mathbb{E}[\kappa_\pi(\mu_N)] + O\left(N^{-1}\right),\end{aligned} \tag{3.3.7}$$

where the cumulant–moment formula (2.2.2) was used in the last line. Combining (3.3.7) and Theorem 2.3.1 implies the desired conclusion.

(ii) $\ell = 2$**.** Taking the expectation of $\mathrm{M}_k(\widehat{\tau}_N)^2 = (\mathrm{Tr}[D^k] - \mathrm{Tr}[(DUPU^*)^k])^2$ with Weingarten calculus yields

$$\begin{aligned}&\mathbb{E}[\mathrm{M}_k(\widehat{\tau}_N)^2]\\ &= \mathbb{E}\circ\mathrm{Tr}_{\gamma_k^{(1)}\gamma_k^{(2)}}[(DUIU^*)^k,(DUIU^*)^k] - \mathbb{E}\circ\mathrm{Tr}_{\gamma_k^{(1)}\gamma_k^{(2)}}[(DUIU^*)^k,(DUPU^*)^k]\\ &\quad - \mathbb{E}\circ\mathrm{Tr}_{\gamma_k^{(1)}\gamma_k^{(2)}}[(DUPU^*)^k,(DUIU^*)^k] + \mathbb{E}\circ\mathrm{Tr}_{\gamma_k^{(1)}\gamma_k^{(2)}}[(DUPU^*)^k,(DUPU^*)^k]\\ &= \sum_{g,h\in\mathfrak{S}_{2k}} N^{\#(g)}\mathbb{E}[\mathrm{M}_g(\mu_N)]\mathcal{T}_k(h)\,\mathrm{Wg}(g^{-1}h)\end{aligned} \tag{3.3.8}$$

where

$$\mathcal{T}_k(h) := \mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[I^k,I^k] - \mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[I^k,P^k] - \mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[P^k,I^k] + \mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[P^k,P^k].$$

Note that, for readability, we use the abbreviation $\mathrm{Tr}_g[A^k,B^k] = \mathrm{Tr}_g[A_1,\dots,A_k,B_1,\dots,B_k]$ when $A = A_1 = \cdots = A_k$ and $B = B_1 = \cdots = B_k$.

By using the evident decomposition $I = P + Q$ with $Q = \mathrm{diag}(0,\dots,0,1)$, we have the following expansion

$$\begin{aligned}&\mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[I^k,I^k]\\ &= \mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[I^k,I^{k-1},P+Q]\\ &= \mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[I^k,I^{k-1},P] + \mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[I^k,I^{k-1},Q]\\ &= \mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[I^k,I^{k-2},P+Q,P] + \mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[I^k,I^{k-1},Q]\\ &= \mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[I^k,I^{k-2},P,P] + \mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[I^k,I^{k-2},Q,P] + \mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[I^k,I^{k-1},Q]\\ &\cdots\\ &= \mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[I^k,P^k] + \sum_{j=1}^{k}\mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[I^k,I^{k-j},Q,P^{j-1}].\end{aligned} \tag{3.3.9}$$

In the same way,

$$\mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[P^k,I^k] = \mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[P^k,P^k] + \sum_{j=1}^{k}\mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[P^k,I^{k-j},Q,P^{j-1}]. \tag{3.3.10}$$

Combining (3.3.9) and (3.3.10) together we get

$$\mathcal{T}_k(h) = \sum_{j=1}^{k} \left(\mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[I^k, I^{k-j}, Q, P^{j-1}] - \mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[P^k, I^{k-j}, Q, P^{j-1}] \right).$$

Again, a similar argument yields

$$\mathcal{T}_k(h) = \sum_{i=1}^{k}\sum_{j=1}^{k} \mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[I^{k-i}, Q, P^{i-1}, I^{k-j}, Q, P^{j-1}].$$

When we decompose $h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}$ into cycles, the contribution of the cycle which contains Q is at most 1 in $\mathrm{Tr}_{h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}}[I^{k-i}, Q, P^{i-1}, I^{k-j}, Q, P^{j-1}]$. Therefore, we get the upper bound

$$\mathcal{T}_k(h) = O\left(N^{\#(h^{-1}\gamma_k^{(1)}\gamma_k^{(2)})-1}\right) = O\left(N^{2k-|h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}|-1}\right). \tag{3.3.11}$$

Here we also use the asymptotic expansion (3.1.5) of the Weingarten functions and two elementary facts about the length function in symmetric groups $\mathfrak{S}_{2k}$: $|g|+|g^{-1}h|+|h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}| \geq 2k-2$ and $|g| + |g^{-1}h| + |h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}| \neq 2k-1$ since $|g| + |g^{-1}h| + |h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}| \equiv |\gamma_k^{(1)}\gamma_k^{(2)}| \equiv 2k-2 \pmod 2$ by using the length property (2.1.2). Applying those facts and (3.3.11) to (3.3.8) reveals that

$$\mathbb{E}[\mathrm{M}_k(\widehat{\tau}_N)^2] = \sum_{\substack{g,h\in\mathfrak{S}_{2k} \\ |g|+|g^{-1}h|+|h^{-1}\gamma_k^{(1)}\gamma_k^{(2)}|=2k-2}} N^{-|g|-|g^{-1}h|}\mathbb{E}[\mathrm{M}_g(\mu_N)]\mathcal{T}_k(h)\mu_{2k}^{\mathrm{Wg}}(g^{-1}h) + O\left(N^{-1}\right).$$

By using the isomorphism (2.1.3), the last expression can be rewritten in terms of non-crossing partitions:

$$\begin{aligned}
\mathbb{E}[\mathrm{M}_k(\widehat{\tau}_N)^2] &= \sum_{\substack{\sigma,\pi\in\mathrm{NC}(2k) \\ \sigma\leq\pi\leq(1_k{}^{(1)},1_k{}^{(2)})}} \mathbb{E}[\mathrm{M}_\sigma(\mu_N)]\mu_{2k}^{\mathrm{NC}}(\sigma,\pi)\frac{\mathcal{T}_k(\mathcal{P}_\pi)}{N^{|\mathrm{Kr}(\pi)|-1}} + O\left(N^{-1}\right) \\
&= \sum_{\substack{\pi\in\mathrm{NC}(2k) \\ \pi\leq(1_k{}^{(1)},1_k{}^{(2)})}} \mathbb{E}[\kappa_\pi(\mu_N)]\frac{\mathcal{T}_k(\mathcal{P}_\pi)}{N^{|\mathrm{Kr}(\pi)|-1}} + O\left(N^{-1}\right) \\
&= \sum_{\pi_1,\pi_2\in\mathrm{NC}(k)} \mathbb{E}[\kappa_{\pi_1}(\mu_N)\,\kappa_{\pi_2}(\mu_N)]\frac{\mathcal{T}_k(\mathcal{P}_{(\pi_1,\pi_2)})}{N^{|\mathrm{Kr}(\pi_1)|+|\mathrm{Kr}(\pi_2)|-2}} + O\left(N^{-1}\right).
\end{aligned}$$

Note that

$$\begin{aligned}
\mathcal{T}_k(\mathcal{P}_{(\pi_1,\pi_2)}) &= \sum_{i=1}^{k}\sum_{j=1}^{k} \mathrm{Tr}_{(\mathrm{Kr}(\pi_1),\mathrm{Kr}(\pi_2))}[I^{k-i}, Q, P^{i-1}, I^{k-j}, Q, P^{j-1}] \\
&= N^{|\mathrm{Kr}(\pi_1)|+|\mathrm{Kr}(\pi_2)|-2}|\mathrm{Kr}(\pi_1)||\mathrm{Kr}(\pi_2)| + O\left(N^{|\mathrm{Kr}(\pi_1)|+|\mathrm{Kr}(\pi_2)|-3}\right).
\end{aligned} \tag{3.3.12}$$

This is because the contribution of a cycle is 0 if it contains both P and Q, and is 1 if it contains Q and no P; from those observations, the main contributions appear when both Q's are at Kreweras points of π_1 and π_2, respectively, and so (3.3.12) follows. Hence we arrive at the formula

$$\mathbb{E}[\mathrm{M}_k(\widehat{\tau}_N)^2] = \sum_{\pi_1,\pi_2\in\mathrm{NC}(k)} |\mathrm{Kr}(\pi_1)||\mathrm{Kr}(\pi_2)|\,\mathbb{E}[\kappa_{\pi_1}(\mu_N)\kappa_{\pi_2}(\mu_N)] + O\left(N^{-1}\right).$$

Applying Theorem 2.3.1 to the RHS finishes the proof. □

Remark 3.3.3. Note that the calculations for $\ell = 1$ are similar to those in [NS06, pp.379-393] where asymptotic freeness is proved for matrices rotated by independent Haar unitaries.

Proof of Theorem 3.1.1. According to Theorem 2.3.1 , $\mathrm{M}_k(\tau_N)^\ell$ is a polynomial on $\{\mathrm{M}_n(\mu_N)\}_{n\in\mathbb{N}}$, so that Proposition 3.2.4 allows us to pass to the limit:

$$\lim_{N\to\infty} \mathbb{E}[\mathrm{M}_k(\tau_N)] = \mathrm{M}_k(\tau) \quad \text{and} \quad \lim_{N\to\infty} \mathbb{E}[\mathrm{M}_k(\tau_N)^2] = \mathrm{M}_k(\tau)^2, \qquad k \in \mathbb{N}.$$

Combining the above and Theorem 3.3.1 yields that

$$\lim_{N\to\infty} \mathbb{E}[\mathrm{M}_k(\widehat{\tau}_N)] = \mathrm{M}_k(\tau) \quad \text{and} \quad \lim_{N\to\infty} \mathbb{E}[\mathrm{M}_k(\widehat{\tau}_N)^2] = \mathrm{M}_k(\tau)^2, \qquad k \in \mathbb{N},$$

which readily implies $\| \mathrm{M}_k(\widehat{\tau}_N) - \mathrm{M}_k(\tau)\|_{L^2} \to 0$. In particular, $\mathrm{M}_k(\widehat{\tau}_N)$ converges to $\mathrm{M}_k(\tau)$ in probability for every $k \in \mathbb{N}$. Since $\mathrm{M}_k(\widehat{\mu}_N)$ and $\mathrm{M}_k(\mu)$ are respectively expressed by a common polynomial evaluated at $\{\mathrm{M}_k(\widehat{\tau}_N)\}_{k\geq 1}$ and $\{\mathrm{M}_k(\tau)\}_{k\geq 1}$, it follows that $\mathrm{M}_k(\widehat{\mu}_N)$ converges to $\mathrm{M}_k(\mu)$ in probability. Finally, if the moment problem for $\{\mathrm{M}_k(\mu)\}_{k\geq 1}$ is determinate then we conclude that $\widehat{\mu}_N$ weakly converges to μ in probability by Proposition 3.2.1. □

Chapter 4

Finite Free Probability

In this chapter, we provide various limit theorems in finite free probability. To begin with, we introduce basic notions and results in finite free probability. The most essential concepts in this theory are finite free additive and multiplicative convolutions of polynomials. In particular, we pay more attention to multiplicative convolution in this thesis.

First, we study the limit of $\{p_d^{\boxtimes_d m}\}_{m\in\mathbb{N}}$ as $m \to \infty$ for a fixed monic polynomial p_d of degree d. In the second, we give the finite free analogue of Sakuma and Yoshida's limit theorem, that is, the limit of $\{\mathcal{D}_{1/m}((p_d^{\boxtimes_d m})^{\boxplus_d m})\}_{m\in\mathbb{N}}$ as $m \to \infty$ in two separate regimes; (i) $m/d \to t$ for some $t > 0$, and (ii) $m/d \to 0$. As the third result, we give alternative proofs of Kabluchko's limit theorems for the unitary Hermite polynomial and the unitary Laguerre polynomials via combinatorial identities. The fourth result is the central limit theorem for finite free multiplicative convolution and a discovery related to the multiplicative free semicircle distributions. Lastly, we provide a finite free analogue of Tucci's limit theorem in Section 4.6. Moreover, we study the empirical root distributions of the limit polynomials when their degree tends to infinity.

4.1 Preliminaries

In this section, we introduce some concepts and preliminary results on finite free probability that are used in the remainder of this thesis; see [AGP23; Mar21; MSS22] for more details on finite free probability.

For any $p, q \in \mathbb{P}_{\mathrm{mon}}(d)$, one defines the finite free additive convolution

$$(p \boxplus_d q)(x) = \sum_{k=0}^{d} (-1)^k \binom{d}{k} \sum_{i+j=k} \frac{k!}{i!j!} \widetilde{e}_i(p)\widetilde{e}_j(q) x^{d-k}.$$

For $p \in \mathbb{P}_{\mathrm{mon}}(d)$, a polynomial $p^{\boxplus_d m}$ denotes the m-th power of p with respect to finite free additive convolution. Note that, if $p, q \in \mathbb{P}_{\mathrm{mon}}(d)$ are real-rooted, then so is $p \boxplus_d q \in \mathbb{P}_{\mathrm{mon}}(d)$ (see [MSS22, Theorem 1.3]). The finite free additive convolution plays an important role in studying the characteristic polynomials of the sum of (random) matrices. For a $d \times d$ real symmetric matrix A, χ_A denotes the characteristic polynomial of A. Then we obtain

$$(\chi_A \boxplus_d \chi_B)(x) = \mathbb{E}_Q \det[xI_d - A - QBQ^*],$$

where the expectation is taken over unitary matrices Q distributed uniformly on the unitary group of degree d (see [MSS22, Theorem 1.2]). Moreover, the finite additive convolution is closely related to the free additive convolution $\boxplus$ which describes the law of sum of freely independent non-commutative random variables (see [BV93; Maa92; Voi86] for detailed information on free additive convolution). For example, Marcus [Mar21] obtained typical limit theorems (LLN, CLT and Poisson's law of small numbers, etc.) for finite free additive convolution. According

to the evidence above, we can treat finite free probability as a discrete approximation of free probability.

Proposition 4.1.1. (see [AP18, Corollary 5.5]) Suppose that $p_d, q_d \in \mathbb{P}_{\mathrm{mon}}(d)$ are real-rooted and μ, ν are probability measures on $\mathbb{R}$ with compact support. If $\mu[\![p_d]\!] \xrightarrow{w} \mu$ and $\mu[\![q_d]\!] \xrightarrow{w} \nu$ as $d \to \infty$, respectively, then $\mu[\![p_d \boxplus_d q_d]\!] \xrightarrow{w} \mu \boxplus \nu$ as $d \to \infty$.

Similarly, for $p, q \in \mathbb{P}_{\mathrm{mon}}(d)$, the *finite free multiplicative convolution* $p \boxtimes_d q$ is defined by

$$(p \boxtimes_d q)(x) = \sum_{i=0}^{d} (-1)^i \binom{d}{i} \widetilde{e}_i(p) \widetilde{e}_i(q) x^{d-i}.$$

For $p \in \mathbb{P}_{\mathrm{mon}}(d)$, we denote by $p^{\boxtimes_d m}$ the m-th power of p for finite free multiplicative convolution. Note that, if p has only nonnegative roots and q is real-rooted, then $p \boxtimes_d q$ has only real roots (see, e.g., [Mar66, Section 16, Exercise 2]). If p, q have only roots located on $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, then so is $p \boxtimes_d q$ (see, e.g., [Sze22, Satz 3]). According to [MSS22, Theorem 1.5], the finite free multiplicative convolution describes the characteristic polynomial of the product of positive definite matrices. More precisely, if A and B are $d \times d$ positive definite matrices, then

$$(\chi_A \boxtimes_d \chi_B)(x) = \mathbb{E}_Q \det[xI_d - AQBQ^*].$$

Furthermore, it is also known that $\boxtimes_d$ is closely related to free multiplicative convolution $\boxtimes$ which describes the law of multiplication of freely independent random variables (see [BV93] for further details on free multiplicative convolution).

Proposition 4.1.2. (see [AGP23, Theorem 1.4]) Let us consider $p_d, q_d \in \mathbb{P}_{\mathrm{mon}}(d)$ in which p_d has only nonnegative roots and q_d is real-rooted. Further, consider probability measures μ, ν on $\mathbb{R}$ with compact support, in which μ is supported on $[0, \infty)$. If $\mu[\![p_d]\!] \xrightarrow{w} \mu$ and $\mu[\![q_d]\!] \xrightarrow{w} \nu$ as $d \to \infty$, respectively, then $\mu[\![p_d \boxtimes_d q_d]\!] \xrightarrow{w} \mu \boxtimes \nu$ as $d \to \infty$.

Also, according to [Kab21, Proposition 2.9], the same statement holds when p_d, q_d have only roots located on $\mathbb{T}$ and μ, ν are probability measures on $\mathbb{T}$.

There is a useful concept to understand finite free additive and multiplicative convolutions by combinatorics. For $p \in \mathbb{P}_{\mathrm{mon}}(d)$, the *finite free cumulant* of p is defined by

$$\kappa_n^{(d)}(p) := \frac{(-d)^{n-1}}{(n-1)!} \sum_{\pi \in \mathrm{P}(n)} \widetilde{e}_\pi(p) \mu_n^{\mathrm{P}}(\pi, 1_n), \tag{4.1.1}$$

for $n = 1, 2, \ldots, d$ (see [AP18, Proposition 3.4] for details).

Example 4.1.3. (1) Let us set $p(x) = x^d - dx^{d-1}$. Since $\widetilde{e_1}(p) = 1$ and $\widetilde{e}_i(p) = 0$ for all $i = 2, \ldots, d$, it is easy to see that $\kappa_n^{(d)}(p) = d^{n-1}$ for $n = 1, 2, \ldots, d$.

(2) Consider $\lambda > 0$. We define the *normalized Laguerre polynomial*

$$\widehat{L}_d^{(\lambda)}(x) = \sum_{i=0}^{d} (-1)^i \binom{d}{i} \frac{(d\lambda)_i}{d^i} x^{d-i}, \tag{4.1.2}$$

where $(\alpha)_i := \alpha(\alpha - 1) \cdots (\alpha - i + 1)$. Then the finite free cumulants of $\hat{L}_d^{(\lambda)}$ are given by $\kappa_n^{(d)}(\widehat{L}_d^{(\lambda)}) = \lambda$ for $n = 1, 2, \ldots, d$ (see [AGP23]).

According to [AP18, Proposition 3.6], the finite free cumulant linearizes the finite free additive convolution:

$$\kappa_n^{(d)}(p \boxplus_d q) = \kappa_n^{(d)}(p) + \kappa_n^{(d)}(q)$$

for $p, q \in \mathbb{P}_{\mathrm{mon}}(d)$. In particular, we have

$$\kappa_n^{(d)}(p^{\boxplus_d m}) = m\kappa_n^{(d)}(p), \qquad m \in \mathbb{N}.$$

In the following, we give a formula for finite free cumulants of $p^{\boxtimes_d m}$ for $p \in \mathbb{P}_{\mathrm{mon}}(d)$ and $m \in \mathbb{N}$. First, the following is directly derived from the definition of finite free multiplicative convolution.

Lemma 4.1.4. For a family $\{p_i\}_{i=1}^m \subset \mathbb{P}_{\mathrm{mon}}(d)$, one has

$$\widetilde{e}_j(p_1 \boxtimes_d \cdots \boxtimes_d p_m) = \prod_{i=1}^m \widetilde{e}_j(p_i) \qquad (j = 1, \ldots, d).$$

In particular, if all $p_i = p$ for some $p \in \mathbb{P}_{\mathrm{mon}}(d)$, then

$$\widetilde{e}_j(p^{\boxtimes_d m}) = \widetilde{e}_j(p)^m.$$

Second, it is also known that

$$\kappa_n^{(d)}(p \boxtimes_d q) = \frac{(-d)^{n-1}}{(n-1)!} \sum_{\substack{\sigma,\tau \in \mathrm{P}(n) \\ \sigma \vee \tau = 1_n}} d^{|\sigma|+|\tau|-2n} \mu_n^{\mathrm{P}}(0_n, \sigma)\mu_n^{\mathrm{P}}(0_n, \tau)\kappa_\sigma^{(d)}(p)\kappa_\tau^{(d)}(q)$$

by [AGP23, Theorem 1.1]. In particular, if $p = q$, then

$$\kappa_n^{(d)}(p^{\boxtimes_d 2}) = \frac{(-d)^{n-1}}{(n-1)!} \sum_{\substack{\sigma_1,\sigma_2 \in \mathrm{P}(n) \\ \sigma_1 \vee \sigma_2 = 1_n}} \left(\prod_{i=1}^2 d^{|\sigma_i|-n} \mu_n^{\mathrm{P}}(0_n, \sigma_i)\kappa_{\sigma_i}^{(d)}(p) \right).$$

In general, the following holds.

Proposition 4.1.5. For a family $\{p_i\}_{i=1}^m \subset \mathbb{P}_{\mathrm{mon}}(d)$, we have

$$\begin{aligned}
\kappa_n^{(d)}(p_1 \boxtimes_d \cdots \boxtimes_d p_m) &= \frac{(-d)^{n-1}}{(n-1)!} \sum_{\pi \in \mathrm{P}(n)} \prod_{i=1}^m \sum_{\sigma_i \le \pi} d^{|\sigma_i|-n} \mu_n^{\mathrm{P}}(0_n, \sigma_i)\kappa_{\sigma_i}^{(d)}(p_i)\mu_n^{\mathrm{P}}(\pi, 1_n) \\
&= \frac{(-d)^{n-1}}{(n-1)!} \sum_{\substack{\sigma_1,\ldots,\sigma_m \in \mathrm{P}(n) \\ \sigma_1 \vee \cdots \vee \sigma_m = 1_n}} \left(\prod_{i=1}^m d^{|\sigma_i|-n} \mu_n^{\mathrm{P}}(0_n, \sigma_i)\kappa_{\sigma_i}^{(d)}(p_i) \right).
\end{aligned}$$

In particular, if all $p_i = p$ for some $p \in \mathbb{P}_{\mathrm{mon}}(d)$, then

$$\begin{aligned}
\kappa_n^{(d)}(p^{\boxtimes_d m}) &= \frac{(-d)^{n-1}}{(n-1)!} \sum_{\pi \in \mathrm{P}(n)} \left(\sum_{\sigma \le \pi} d^{|\sigma|-n} \mu_n^{\mathrm{P}}(0_n, \sigma)\kappa_\sigma^{(d)}(p) \right)^m \mu_n^{\mathrm{P}}(\pi, 1_n) \\
&= \frac{(-d)^{n-1}}{(n-1)!} \sum_{\substack{\sigma_1,\ldots,\sigma_m \in \mathrm{P}(n) \\ \sigma_1 \vee \cdots \vee \sigma_m = 1_n}} \left(\prod_{i=1}^m d^{|\sigma_i|-n} \mu_n^{\mathrm{P}}(0_n, \sigma_i)\kappa_{\sigma_i}^{(d)}(p) \right).
\end{aligned}$$

Proof. By the definition of finite free cumulant (4.1.1) and Lemma 4.1.4, we obtain

$$\begin{aligned}
\kappa_n^{(d)}(p_1 \boxtimes_d \cdots \boxtimes_d p_m) &= \frac{(-d)^{n-1}}{(n-1)!} \sum_{\pi \in \mathrm{P}(n)} \widetilde{e}_\pi(p_1 \boxtimes_d \cdots \boxtimes_d p_m)\mu_n^{\mathrm{P}}(\pi, 1_n) \\
&= \frac{(-d)^{n-1}}{(n-1)!} \sum_{\pi \in \mathrm{P}(n)} \prod_{i=1}^m \widetilde{e}_\pi(p_i)\mu_n^{\mathrm{P}}(\pi, 1_n). \qquad (4.1.3)
\end{aligned}$$

According to [AP18, Proposition 3.4], since

$$\widetilde{e}_\pi(p) = \sum_{\sigma \le \pi} d^{|\sigma|-n} \mu_n^{\mathrm{P}}(0_n, \sigma) \kappa_\sigma^{(d)}(p),$$

for any $p \in \mathbb{P}_{\mathrm{mon}}(d)$, the first equality holds by (4.1.3).

The second equality is proved as follows:

$$\begin{aligned}
&\sum_{\pi \in \mathrm{P}(n)} \prod_{i=1}^{m} \sum_{\sigma_i \le \pi} d^{|\sigma_i|-n} \mu_n^{\mathrm{P}}(0_n, \sigma_i) \kappa_{\sigma_i}^{(d)}(p_i) \mu_n^{\mathrm{P}}(\pi, 1_n) \\
&\quad = \sum_{\pi \in \mathrm{P}(n)} \left(\sum_{\sigma_1, \dots, \sigma_m \le \pi} \prod_{i=1}^{m} d^{|\sigma_i|-n} \mu_n^{\mathrm{P}}(0_n, \sigma_i) \kappa_{\sigma_i}^{(d)}(p_i) \right) \mu_n^{\mathrm{P}}(\pi, 1_n) \\
&\quad = \sum_{\sigma_1, \dots, \sigma_m \in \mathrm{P}(n)} \prod_{i=1}^{m} d^{|\sigma_i|-n} \mu_n^{\mathrm{P}}(0_n, \sigma_i) \kappa_{\sigma_i}^{(d)}(p_i) \left(\sum_{\sigma_1 \vee \cdots \vee \sigma_m \le \pi} \mu_n^{\mathrm{P}}(\pi, 1_n) \right) \\
&\quad = \sum_{\substack{\sigma_1, \dots, \sigma_m \in \mathrm{P}(n) \\ \sigma_1 \vee \cdots \vee \sigma_m = 1_n}} \left(\prod_{i=1}^{m} d^{|\sigma_i|-n} \mu_n^{\mathrm{P}}(0_n, \sigma_i) \kappa_{\sigma_i}^{(d)}(p_i) \right),
\end{aligned}$$

where we used Proposition 2.1.4 on the third line. □

Example 4.1.6. By using the first equation in Proposition 4.1.5, we get

$$\kappa_n^{(d)} \left((\widehat{L}_d^{(1)})^{\boxtimes_d m} \right) = \frac{(-d)^{n-1}}{(n-1)!} \sum_{\pi \in \mathrm{P}(n)} \left(\sum_{\sigma \le \pi} d^{|\sigma|-n} \mu_n^{\mathrm{P}}(0_n, \sigma) \right)^m \mu_n^{\mathrm{P}}(\pi, 1_n)$$

because $\kappa_n^{(d)}(\widehat{L}_d^{(1)}) = 1$ for all n. The formula implies that

$$\kappa_2^{(d)} \left((\widehat{L}_d^{(1)})^{\boxtimes_d m} \right) = d \left\{ 1 - \left(1 - \frac{1}{d} \right)^m \right\}$$

and

$$\kappa_3^{(d)} \left((\widehat{L}_d^{(1)})^{\boxtimes_d m} \right) = d^2 \left\{ 1 - \frac{3}{2} \left(1 - \frac{1}{d} \right)^m + \frac{1}{2} \left(1 - \frac{1}{d} \right)^m \left(1 - \frac{2}{d} \right)^m \right\}.$$

According to [AP18, Theorem 5.4], it is known that the finite free cumulants approach free cumulants introduced by Speicher. A consequence of this is the following criteria for convergence in distribution.

Proposition 4.1.7. Let us consider $p_d \in \mathbb{P}_{\mathrm{mon}}(d)$ and a probability measure μ with compact support. The following assertions are equivalent.

(1) $\mu[\![p_d]\!] \xrightarrow{w} \mu$ as $d \to \infty$.

(2) For all $n \in \mathbb{N}$, $\lim_{d \to \infty} \kappa_n^{(d)}(p_d) = \kappa_n(\mu)$.

4.2 Simple limits for $p^{\boxtimes_d m}$ as $m \to \infty$

In this section, we investigate the relatively simple limits of $p^{\boxtimes_d m}$ for $p \in \mathbb{P}_{\mathrm{mon}}(d)$ having nonnegative roots. In order to study it, we give some properties of a sequence $\{\widetilde{e}_i(p)\}_{i=0}^d$. First, let us recall Newton's inequality and Maclaurin's inequality (see, e.g., [HLP52, Section 2.22]).

Proposition 4.2.1 (Newton's inequality)**.** Let $p \in \mathbb{P}_{\mathrm{mon}}(d)$ be a monic polynomial with real roots. Then

$$\widetilde{e}_{i+1}(p)\widetilde{e}_{i-1}(p) \leq \widetilde{e}_i(p)^2, \qquad i = 1, 2, \ldots, d-1.$$

The equality holds if and only if its roots are the same α in which case $\widetilde{e}_i(p) = \alpha^i$ for $i = 1, 2, \ldots, d$.

Proposition 4.2.2 (Maclaurin's inequality)**.** Let $p \in \mathbb{P}_{\mathrm{mon}}(d)$ be a monic polynomial with positive roots. Then

$$\widetilde{e}_1(p) \geq \widetilde{e}_2(p)^{\frac{1}{2}} \geq \cdots \geq \widetilde{e}_d(p)^{\frac{1}{d}} \tag{4.2.1}$$

with equality if and only if its roots are the same α.

Remark 4.2.3. The inequality (4.2.1) itself holds even when p has zero roots. More precisely, if p has exactly k zero roots then $\widetilde{e}_1(p) > \cdots > \widetilde{e}_{d-k}(p)^{\frac{1}{d-k}}$ and $\widetilde{e}_{d-k+1}(p) = \cdots = \widetilde{e}_d(p) = 0$.

Recall that, for $p \in \mathbb{P}_{\mathrm{mon}}(d)$ and $c \neq 0$,

$$\mathcal{D}_c(p)(x) = c^d p(x/c),$$

and hence the definition of finite free cumulants implies

$$\kappa_n^{(d)}(\mathcal{D}_c(p)) = c^n \kappa_n^{(d)}(p). \tag{4.2.2}$$

Theorem 4.2.4. Let us consider $p \in \mathbb{P}_{\mathrm{mon}}(d)$ with nonnegative roots.

(1) We have

$$\lim_{m\to\infty} p^{\boxtimes_d m}(x) = \begin{cases} x^d, & \widetilde{e_1}(p) < 1, \\ x^d - dx^{d-1}, & \widetilde{e_1}(p) = 1 \text{ and } \widetilde{e_2}(p) < 1, \\ (x-1)^d, & \widetilde{e_1}(p) = 1 \text{ and } \widetilde{e_2}(p) = 1. \end{cases}$$

The limit does not exist if $\widetilde{e_1}(p) > 1$.

(2) Assume that $\widetilde{e_1}(p) > 0$. Then

$$\lim_{m\to\infty} \mathcal{D}_{1/\widetilde{e}_1(p)^m}(p^{\boxtimes_d m})(x) = \begin{cases} x^d - dx^{d-1}, & \widetilde{e}_2(p) < \widetilde{e}_1(p)^2, \\ (x-1)^d, & \widetilde{e}_2(p) = \widetilde{e}_1(p)^2. \end{cases}$$

Proof. For each i, we get

$$\widetilde{e}_i(p^{\boxtimes_d m}) = \widetilde{e}_i(p)^m \xrightarrow{m\to\infty} \begin{cases} 0, & \widetilde{e}_i(p) < 1, \\ 1, & \widetilde{e}_i(p) = 1, \\ \infty, & \widetilde{e}_i(p) > 1. \end{cases}$$

Accordingly, it does not converge in the case $\widetilde{e}_1(p) > 1$.

Next, we consider the case $\widetilde{e}_1(p) \leq 1$.

- If $\widetilde{e}_1(p) < 1$, then $\widetilde{e}_i(p) < 1$ for $i = 2, \ldots, d$ by Maclaurin's inequality. Then we get

$$p^{\boxtimes_d m}(x) = \sum_{i=0}^{d} (-1)^i \binom{d}{i} \widetilde{e}_i(p)^m x^{d-i} \xrightarrow{m\to\infty} x^d.$$

- If $\widetilde{e}_1(p) = 1$, then $\widetilde{e}_2(p) \leq 1$ by Maclaurin's inequality. There are two possible cases.

(i) If $\widetilde{e}_2(p) < 1$, then $\widetilde{e}_i(p) < 1$ for $i = 2, \dots, d$, and therefore

$$p^{\boxtimes_d m}(x) = \sum_{i=0}^{d} (-1)^i \binom{d}{i} \widetilde{e}_i(p)^m x^{d-i} \xrightarrow{m\to\infty} x^d - dx^{d-1}.$$

(ii) If $\widetilde{e}_2(p) = 1$, then $\widetilde{e}_i(p) = 1$ for $i = 2, \dots, d$ by Proposition 4.2.1. It means $p(x) = (x-1)^d$. Thus we have

$$p^{\boxtimes_d m}(x) = (x-1)^d \xrightarrow{m\to\infty} (x-1)^d.$$

For the latter part, assume $\widetilde{e}_1(p) > 0$. The condition $\widetilde{e}_2(p) < \widetilde{e}_1(p)^2$ means that $\widetilde{e}_i(p) < \widetilde{e}_1(p)^i$ for all $i = 2, \dots, d$ by Maclaurin's inequality. Then

$$\begin{aligned}
\mathcal{D}_{1/\widetilde{e}_1(p)^m}(p^{\boxtimes_d m})(x) &= \frac{1}{\widetilde{e}_1(p)^{md}} \cdot p^{\boxtimes_d m}(\widetilde{e}_1(p)^m x) \\
&= x^d + \sum_{i=1}^{d} (-1)^i \binom{d}{i} \widetilde{e}_i(p)^m \cdot \widetilde{e}_1(p)^{-im} x^{d-i} \\
&= x^d - dx^{d-1} + \sum_{i=2}^{d} (-1)^i \binom{d}{i} \left(\frac{\widetilde{e}_i(p)}{\widetilde{e}_1(p)^i} \right)^m x^{d-i} \\
&\to x^d - dx^{d-1},
\end{aligned}$$

as $m \to \infty$ because $\widetilde{e}_i(p) < \widetilde{e}_1(p)^i$ for all $i = 2, \dots, d$.

If $\widetilde{e}_2(p) = \widetilde{e}_1(p)^2$, then $\widetilde{e}_i(p) = \alpha^i$ for some α by Proposition 4.2.1. Then a similar way to the above computation shows

$$\mathcal{D}_{1/\widetilde{e}_1(p)^m}(p^{\boxtimes_d m})(x) = (x-1)^d.$$

□

4.3 Finite free analogue of Sakuma–Yoshida's limit theorem

In this section, we study the finite free analogue of the limit theorem by Sakuma and Yoshida [SY13] as already mentioned in Introduction. More precisely, our purpose in this section is to investigate the limit behavior of the sequence of finite free cumulants of

$$\mathcal{D}_{1/m}((p_d^{\boxtimes_d m})^{\boxplus_d m}),$$

as $m \to \infty$ for $p_d \in \mathbb{P}_{\mathrm{mon}}(d)$ with nonnegative roots, such that $\kappa_1^{(d)}(p_d) = 1$. Recall that, Sakuma and Yoshida investigated the asymptotic expansion of S-transform, in contrast to that, Arizmendi and Vargas [AV12] gave another proof by focusing on the combinatorial structure of the non-crossing partitions. Here, we will take the latter approach, i.e., the convergence of finite free cumulants.

Suppose first that degree d is fixed. According to (4.2.2) and Theorem 4.2.4, we have

$$\kappa_n^{(d)}\left(\mathcal{D}_{1/m}((p_d^{\boxtimes_d m})^{\boxplus_d m})\right) = \frac{1}{m^{n-1}} \kappa_n^{(d)}(p_d^{\boxtimes_d m}) \xrightarrow{m\to\infty} 0$$

for $n = 2, \dots, d$. In this case, we get $\mathcal{D}_{1/m}((p_d^{\boxtimes_d m})^{\boxplus_d m})(x) \to (x-1)^d$ as $m \to \infty$; hence this is not an interesting result. In order to obtain a non-trivial limit of finite free cumulant, we consider the following two situations of $m \to \infty$ with (i) $m/d \to t$ for some $t > 0$, or (ii) $m/d \to 0$.

In the later discussions, we consider $p_d \in \mathbb{P}_{\mathrm{mon}}(d)$ with nonnegative roots. Additionally, we assume that

(A-1) $\kappa_1^{(d)}(p_d) = 1$, that is, $\widetilde{e}_1(p_d) = 1$;

(A-2) there exists a probability measure μ with compact support such that $\mu[\![p_d]\!] \xrightarrow{w} \mu$ as $d \to \infty$.

We define

$$e_n(t, \mu) := \exp\left(-t\binom{n}{2}\kappa_2(\mu)\right), \qquad t > 0$$

for $n \in \mathbb{N}$. Note that, for $\pi \in \mathrm{P}(n)$,

$$e_\pi(t, \mu) = \prod_{V \in \pi} \exp\left(-t\binom{|V|}{2}\kappa_2(\mu)\right) = \exp\left(-t\sum_{V \in \pi}\binom{|V|}{2}\kappa_2(\mu)\right).$$

We summarize our results as follows.

Theorem 4.3.1. Let us consider $p_d \in \mathbb{P}_{\mathrm{mon}}(d)$ with nonnegative roots such that $\kappa_1^{(d)}(p_d) = 1$, and let μ be a probability measure with compact support. Assume that $\mu[\![p_d]\!] \xrightarrow{w} \mu$ as $d \to \infty$. Then

(1) For $n \in \mathbb{N}$, we have

$$\lim_{\substack{m\to\infty \\ m/d\to t>0}} \kappa_n^{(d)}\left(\mathcal{D}_{1/m}\left((p_d^{\boxtimes_d m})^{\boxplus_d m}\right)\right) = \frac{(-1)^{n-1}}{t^{n-1}(n-1)!}\sum_{\pi\in\mathrm{P}(n)} e_\pi(t,\mu)\mu_n^{\mathrm{P}}(\pi, 1_n).$$

(2) For $n \in \mathbb{N}$, we have

$$\lim_{\substack{m\to\infty \\ m/d\to 0}} \kappa_n^{(d)}\left(\mathcal{D}_{1/m}\left((p_d^{\boxtimes_d m})^{\boxplus_d m}\right)\right) = \frac{(\kappa_2(\mu)n)^{n-1}}{n!},$$

where the limit coincides with the n-th free cumulant of $\eta_{\kappa_2(\mu)}$.

Case of $m/d \to t$ for some $t > 0$

We firstly consider the case when a ratio of a number m of finite free multiplicative convolution and degree d of polynomials converges to some $t > 0$ as $m \to \infty$ (and hence $d \to \infty$), that is, $m/d \to t$ as $m \to \infty$.

Proposition 4.3.2. Let us consider $p_d \in \mathbb{P}_{\mathrm{mon}}(d)$ satisfying (A-1) and (A-2). For $n \in \mathbb{N}$,

$$\kappa_n^{(d)}\left(\mathcal{D}_{1/m}((p_d^{\boxtimes_d m})^{\boxplus_d m})\right) \to \frac{(-1)^{n-1}}{t^{n-1}(n-1)!}\sum_{\pi\in\mathrm{P}(n)} e_\pi(t,\mu)\mu_n^{\mathrm{P}}(\pi, 1_n),$$

as $m \to \infty$ with $m/d \to t$ for some $t > 0$.

Proof. By Lemma 4.1.4, we have

$$\widetilde{e}_n(p_d^{\boxtimes_d m}) = \widetilde{e}_n(p_d)^m = \left(\kappa_1^{(d)}(p_d)^n - \binom{n}{2}\frac{\kappa_1^{(d)}(p_d)^{n-2}\kappa_2^{(d)}(p_d)}{d} + O(d^{-2})\right)^m.$$

Since $\kappa_1^{(d)}(p_d) = 1$ by (A-1) and $\kappa_2^{(d)}(p_d) \to \kappa_2(\mu)$ as $d \to \infty$ by (A-2), we then get

$$\begin{aligned}
\widetilde{e}_n(p_d^{\boxtimes_d m}) &= \left(1 - \binom{n}{2}\frac{\kappa_2^{(d)}(p_d)}{d} + O(d^{-2})\right)^m \\
&= \left\{\left(1 - \binom{n}{2}\frac{\kappa_2^{(d)}(p_d)}{d} + O(d^{-2})\right)^d\right\}^{\frac{m}{d}} \qquad (4.3.1) \\
&\to \exp\left(-t\binom{n}{2}\kappa_2(\mu)\right),
\end{aligned}$$

as $m \to \infty$ with $m/d \to t$. □

Let us denote by $\{\kappa_n(t,\mu)\}_{n\geq 1}$ the above limit of finite free cumulants.

Proposition 4.3.3. For $n \in \mathbb{N}$, we have

$$\lim_{t\to 0^+} \kappa_n(t,\mu) = \frac{(\kappa_2(\mu)n)^{n-1}}{n!}.$$

Proof. It is easy to see that $\lim_{t\to 0^+} e_\pi(t,\mu) = 1$. Note that

$$\sum_{\pi\in \mathrm{P}(n)} \mu_n^{\mathrm{P}}(\pi, 1_n) = 0$$

and

$$\frac{\partial^k}{\partial t^k} e_\pi(t,\mu) = \left(-\sum_{V\in\pi} \binom{|V|}{2} \kappa_2(\mu)\right)^k e_\pi(t,\mu).$$

Due to L'Hôpital's theorem, Proposition 4.3.2 and Equation (2.4.13), the following result is obtained:

$$\begin{aligned}
\lim_{t\to 0^+} \kappa_n(t,\mu) &= \frac{(-1)^{n-1}}{(n-1)!} \sum_{\pi\in \mathrm{P}(n)} \left\{ \lim_{t\to 0^+} \frac{\left(-\sum_{V\in\pi} \binom{|V|}{2}\kappa_2(\mu)\right)^{n-1} e_\pi(t,\mu)}{(n-1)!} \right\} \mu_n^{\mathrm{P}}(\pi, 1_n) \\
&= \frac{\kappa_2(\mu)^{n-1}}{((n-1)!)^2} \sum_{\pi\in \mathrm{P}(n)} \left(\sum_{V\in\pi} \binom{|V|}{2}\right)^{n-1} \mu_n^{\mathrm{P}}(\pi, 1_n) \\
&= \frac{(\kappa_2(\mu)n)^{n-1}}{n!}.
\end{aligned}$$

□

Example 4.3.4. For simplicity, we assume that $\kappa_2^{(d)}(p_d) = 1$. Then it also satisfies that $\kappa_2(\mu) = 1$. Then the first four cumulants are computed as follows.

- $\kappa_1(t,\mu) = 1$ and $\lim_{t\to 0^+} \kappa_1(t,\mu) = 1$.
- $\kappa_2(t,\mu) = \dfrac{e^t - 1}{t}$ and $\lim_{t\to 0^+} \kappa_2(t,\mu) = 1$.
- $\kappa_3(t,\mu) = \dfrac{e^{3t} - 3e^t + 2}{2t^2}$ and $\lim_{t\to 0^+} \kappa_3(t,\mu) = \dfrac{3}{2}$.
- $\kappa_4(t,\mu) = \dfrac{12e^t - 3e^{2t} - 4e^{3t} + e^{6t} - 6}{6t^3}$ and $\lim_{t\to 0^+} \kappa_4(t,\mu) = \dfrac{8}{3}$.

Case of $m/d \to 0$ as $m \to \infty$

Our goal is to show the next one.

Proposition 4.3.5. Let us consider $p_d \in \mathbb{P}_{\mathrm{mon}}(d)$ satisfying (A-1) and (A-2). It satisfies that

$$\kappa_n^{(d)}\left(\mathcal{D}_{1/m}((p_d^{\boxtimes_d m})^{\boxplus_d m})\right) \to \frac{(\kappa_2(\mu)n)^{n-1}}{n!}, \quad n \in \mathbb{N}$$

as $m \to \infty$ with $m/d \to 0$.

Proof. By Proposition 4.1.5,

$$\begin{aligned}
\kappa_n^{(d)}(p_d^{\boxtimes_d m}) &= \frac{(-d)^{n-1}}{(n-1)!}\sum_{\pi\in \mathrm{P}(n)} \widetilde{e}_\pi(p_d)^m \mu_n^{\mathrm{P}}(\pi, 1_n) \\
&= \frac{(-d)^{n-1}}{(n-1)!}\sum_{\substack{\sigma_1,\dots,\sigma_m\in \mathrm{P}(n)\\ \sigma_1\vee\cdots\vee\sigma_m=1_n}} \left(\prod_{i=1}^m d^{|\sigma_i|-n}\mu_n^{\mathrm{P}}(0_n,\sigma_i)\kappa_{\sigma_i}^{(d)}(p_d)\right) \\
&=: \frac{(-d)^{n-1}}{(n-1)!}\sum_{l=0}^{\infty}\frac{q_l(m)}{d^l},
\end{aligned}$$

where, on the third line, we define the polynomials $\{q_l(m)\}_{l=0}^{\infty}$ as the coefficients of the expansion with respect to d^{-l}. We immediately know that $q_l(m) = 0$ $(l = 0, \dots, n-2)$ from the second line. One can verify that the degree of $q_l(m)$ is less than or equal to l.

Look at the first line and (4.3.1), and then consider the coefficients of the expansion of $\widetilde{e}_n(p_d)^m$ $(n \in \mathbb{N})$ with respect to d^{-1}. Then it follows that its degree is less than or equal to the degree of its denominator d^l. Also, it is true about $\widetilde{e}_\pi(p_d)^m$ for $\pi \in \mathrm{P}(n)$.

Thus, what to prove is that the leading coefficient of $q_{n-1}(m)$ equals $(-\kappa_2(p_d))^{n-1}n^{n-2}$. The coefficient polynomial of d^{n-1} in the expansion of

$$\widetilde{e}_\pi(p_d)^m = \prod_{V\in\pi}\left(1 - \binom{|V|}{2}\frac{\kappa_2^{(d)}(p_d)}{d} + O(d^{-2})\right)^m$$

is computed as

$$\sum_{\substack{l_1,\dots,l_{|\pi|}\geq 0\\ l_1+\cdots+l_{|\pi|}=n-1}} \binom{m}{l_1}\cdots\binom{m}{l_{|\pi|}}\left(-\kappa_2^{(d)}(p_d)\binom{|V_1|}{2}\right)^{l_1}\cdots\left(-\kappa_2^{(d)}(p_d)\binom{|V_{|\pi|}|}{2}\right)^{l_{|\pi|}} + O(m^{n-2}),$$

so its leading coefficient is

$$\sum_{\substack{l_1,\dots,l_{|\pi|}\geq 0\\ l_1+\cdots+l_{|\pi|}=n-1}} \frac{1}{l_1!\cdots l_{|\pi|}!}\left(-\kappa_2^{(d)}(p_d)\binom{|V_1|}{2}\right)^{l_1}\cdots\left(-\kappa_2^{(d)}(p_d)\binom{|V_{|\pi|}|}{2}\right)^{l_{|\pi|}}.$$

Hence the leading coefficient of $q_{n-1}(m)$ equals to

$$\begin{aligned}
&(-\kappa_2^{(d)}(p_d))^{n-1}\sum_{\substack{\pi\in\mathrm{P}(n)\\ \pi=\{V_1,\dots,V_r\}}}\left\{\sum_{\substack{l_1,\dots,l_r\geq 0\\ l_1+\cdots+l_r=n-1}}\frac{1}{l_1!\cdots l_r!}\binom{|V_1|}{2}^{l_1}\cdots\binom{|V_r|}{2}^{l_r}\right\}\mu_n^{\mathrm{P}}(\pi,1_n) \\
&= \frac{(-\kappa_2^{(d)}(p_d))^{n-1}}{(n-1)!}\sum_{\pi\in\mathrm{P}(n)}\left(\sum_{V\in\pi}\binom{|V|}{2}\right)^{n-1}\mu_n^{\mathrm{P}}(\pi,1_n) \\
&= (-\kappa_2^{(d)}(p_d))^{n-1}n^{n-2},
\end{aligned}$$

where the last equality holds due to Equation (2.4.13). Finally, we obtain

$$\begin{aligned}
\kappa_n^{(d)}\left(\mathcal{D}_{1/m}((p_d^{\boxtimes_d m})^{\boxplus_d m})\right) &= \frac{(-d)^{n-1}}{m^{n-1}(n-1)!}\sum_{l=0}^{\infty}\frac{q_l(m)}{d^l} \\
&= \frac{(\kappa_2^{(d)}(p_d)n)^{n-1}}{n!} + O\left(\frac{m}{d}\right),
\end{aligned}$$

as desired. □

4.4 Alternative proof of Kabluchko's limit theorems

In this section, we give alternative proofs for Kabluchko's two limit theorems by using the combinatorial formulas in Chapter 2.

Kabluchko's limit theorem for unitary Hermite polynomial

Let us define $H_d(z;t)$ as a polynomial on $\mathbb{C}$ by setting

$$H_d(z;t) := \sum_{k=0}^{d} (-1)^k \binom{d}{k} \exp\left(-\frac{tk(d-k)}{2d}\right) z^{d-k}, \qquad z \in \mathbb{C},\ t > 0. \tag{4.4.1}$$

The polynomial $H_d(z;t)$ is called the *unitary Hermite polynomial* with parameter $t > 0$. By [Kab22, Lemma 2.1], all zeroes of the polynomial $H_d(z;t)$ are located on the unit circle $\mathbb{T}$. It is known as the limit polynomial of the Central Limit Theorem (CLT) for finite free multiplicative convolution of polynomials with roots located on $\mathbb{T}$ by Mirabelli [Mir21, Theorem 3.16]. For the reader's convenience, we prove this result directly from the definition of finite free multiplicative convolution.

Proposition 4.4.1. Let $d \geq 2$. Suppose $p(z) = \prod_{k=1}^{d}(z - e^{i\theta_k})$ such that

$$\frac{1}{d}\sum_{k=1}^{d} \theta_k = 0 \quad \text{and} \quad \frac{1}{d}\sum_{k=1}^{d} \theta_k^2 = \sigma^2.$$

Then

$$\lim_{m\to\infty} \phi_{1/\sqrt{m}}(p)^{\boxtimes_d m}(z) = H_d\left(z; \frac{d\sigma^2}{d-1}\right).$$

Proof. First, note that

$$0 = \left(\sum_{k=1}^{d} \theta_k\right)^2 = d\sigma^2 + 2 \sum_{1\leq j_1 < j_2 \leq d} \theta_{j_1}\theta_{j_2} \tag{4.4.2}$$

by the assumptions. It follows that

$$\begin{aligned}
\widetilde{e}_k(\phi_{1/\sqrt{m}}(p)) &= \binom{d}{k}^{-1} \sum_{1\leq j_1 < \cdots < j_k \leq d} \exp\left(\frac{i}{\sqrt{m}}(\theta_{j_1} + \cdots + \theta_{j_k})\right) \\
&= \binom{d}{k}^{-1} \left\{ \binom{d}{k} + \frac{i}{\sqrt{m}} \sum_{1\leq j_1 < \cdots < j_k \leq d} (\theta_{j_1} + \cdots + \theta_{j_k}) \right. \\
&\qquad \left. -\frac{1}{2m} \sum_{1\leq j_1 < \cdots < j_k \leq d} (\theta_{j_1} + \cdots + \theta_{j_k})^2 + O(m^{-\frac{3}{2}}) \right\} \\
&= \binom{d}{k}^{-1} \left\{ \binom{d}{k} + \frac{i}{\sqrt{m}} \binom{d-1}{k-1} \sum_{k=1}^{d} \theta_k \right. \\
&\qquad \left. -\frac{1}{2m} \left(\binom{d-1}{k-1} \sum_{k=1}^{d} \theta_k^2 + 2\binom{d-2}{k-2} \sum_{1\leq j_1 < j_2 \leq d} \theta_{j_1}\theta_{j_2} \right) + O(m^{-\frac{3}{2}}) \right\} \\
&= \binom{d}{k}^{-1} \left\{ \binom{d}{k} - \frac{d\sigma^2}{2m} \left(\binom{d-1}{k-1} - \binom{d-2}{k-2} \right) + O(m^{-\frac{3}{2}}) \right\} \\
&= 1 - \frac{k(d-k)}{2(d-1)m}\sigma^2 + O(m^{-\frac{3}{2}}),
\end{aligned}$$

where we used the assumptions and (4.4.2) on the third line. Thus, $\widetilde{e}_k(\phi_{1/\sqrt{m}}(p))^m$ goes to $\exp(-k(d-k)\sigma^2/2(d-1))$ as $m \to \infty$ for $k = 0, \ldots, d$. It means $\lim_{m\to\infty} \phi_{1/\sqrt{m}}(p)^{\boxtimes_d m}(z) = H_d(z; d\sigma^2/(d-1))$. □

Kabluchko [Kab22, Theorem 2.2] proved that the empirical root distribution of $H_d(z;t)$ converges weakly on the unit circle to the free normal distribution Σ_t on $\mathbb{T}$ with parameter $t > 0$ as $d \to \infty$, where Σ_t was introduced by [BV92] and studied by [Bia97a; Bia97b; Zho14; Zho15]. Moreover, a matricial model in which the moments of (unitary) matrix-valued Brownian motion at time t converge to ones of Σ_t as the matrix size goes to infinity, was constructed by [Céb16]. Further, the free cumulant of Σ_t was known (see [DGN15]). We already computed the free cumulant of Σ_t in Section 2.2.

We give another proof of this theorem by showing that the finite free cumulant of $H_d(z;t)$ converges to the free cumulant of Σ_t, in which the combinatorial formula (2.4.14) is essential.

Theorem 4.4.2. Consider $t > 0$. As $d \to \infty$, we have

$$\mu[\![H_d(z;t)]\!] \xrightarrow{w} \Sigma_t,$$

that is,

$$\kappa_n^{(d)}(H_d(z;t)) \to \exp\left(-\frac{nt}{2}\right)\frac{(-nt)^{n-1}}{n!} = \kappa_n(\Sigma_t).$$

Proof. By the definition of $H_d(z;t)$, we have

$$\widetilde{e}_k(H_d(z;t)) = \exp\left(-\frac{tk(d-k)}{2d}\right), \qquad k = 0, 1, \ldots, d,$$

and therefore for $\pi \in \mathrm{P}(n)$,

$$\begin{aligned}
\widetilde{e}_\pi(H_d(z;t)) &= \prod_{V\in\pi} \exp\left(-\frac{t|V|(d-|V|)}{2d}\right) \\
&= \exp\left\{-\frac{t}{2d}\left(\sum_{V\in\pi} d|V| - \sum_{V\in\pi}|V|^2\right)\right\} \\
&= \exp\left(-\frac{tn}{2}\right)\exp\left(\frac{t}{2d}\sum_{V\in\pi}|V|^2\right).
\end{aligned}$$

Due to (2.4.14), a straightforward computation shows that

$$\begin{aligned}
&\sum_{\pi\in\mathrm{P}(n)} \widetilde{e}_\pi(H_d(z;t))\mu_n^{\mathrm{P}}(\pi, 1_n) \\
&\quad= \exp\left(-\frac{tn}{2}\right)\sum_{\pi\in\mathrm{P}(n)} \exp\left(\frac{t}{2d}\sum_{V\in\pi}|V|^2\right)\mu_n^{\mathrm{P}}(\pi, 1_n) \\
&\quad= \exp\left(-\frac{tn}{2}\right)\sum_{\pi\in\mathrm{P}(n)} \left\{\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{t}{2d}\sum_{V\in\pi}|V|^2\right)^k\right\}\mu_n^{\mathrm{P}}(\pi, 1_n) \\
&\quad= \exp\left(-\frac{tn}{2}\right)\frac{1}{(n-1)!}\left(\frac{t}{2d}\right)^{n-1}\left\{\sum_{\pi\in\mathrm{P}(n)}\left(\sum_{V\in\pi}|V|^2\right)^{n-1}\mu_n^{\mathrm{P}}(\pi, 1_n)\right\} + O\left(\frac{1}{d^n}\right) \\
&\quad= \exp\left(-\frac{tn}{2}\right)\frac{1}{(n-1)!}\left(\frac{t}{2d}\right)^{n-1}\cdot 2^{n-1}(n-1)!n^{n-2} + O\left(\frac{1}{d^n}\right) \\
&\quad= \exp\left(-\frac{tn}{2}\right)\left(\frac{t}{d}\right)^{n-1} n^{n-2} + O\left(\frac{1}{d^n}\right).
\end{aligned}$$

Therefore we have

$$\begin{aligned}\kappa_n^{(d)}(H_d(z;t)) &= \frac{(-d)^{n-1}}{(n-1)!}\sum_{\pi\in \mathrm{P}(n)}\widetilde{e}_\pi(H_d(z;t))\mu_n^{\mathrm{P}}(\pi,1_n)\\ &= \exp\left(-\frac{nt}{2}\right)\frac{(-nt)^{n-1}}{n!}+O\left(\frac{1}{d}\right),\end{aligned}$$

which implies

$$\lim_{d\to\infty}\kappa_n^{(d)}(H_d(z;t)) = \exp\left(-\frac{nt}{2}\right)\frac{(-nt)^{n-1}}{n!}.$$

Combining with Proposition 2.2.1, we obtain the desired result. □

Kabluchko's limit theorem for unitary Laguerre polynomial

Let $L_{d,m}$ be the *unitary Laguerre polynomial*, that is,

$$L_{d,m}(z) = \sum_{k=0}^{d}(-1)^k\binom{d}{k}\left(1-\frac{2k}{d}\right)^m z^{d-k}. \tag{4.4.3}$$

In [Kab21, Theorem 2.7], if $m/d\to t>0$ for some $t>0$ as $d\to\infty$, then the empirical root distribution of $L_{d,m}(z)$ converges weakly on $\mathbb{T}$ to the free unitary Poisson distribution Π_t with parameter t, see [BV92; Kab21] for details on Π_t. We give a strict statement and an alternative proof of Kabluchko's limit theorem for free unitary Poisson distribution as follows.

Theorem 4.4.3. As $d\to\infty$ with $m/d\to t$ for some $t>0$, we get

$$\mu[\![L_{d,m}]\!]\xrightarrow{w}\Pi_t,$$

equivalently

$$\kappa_n^{(d)}(L_{d,m})\to(-1)^{n-1}2^n e^{-2nt}\sum_{k=1}^{n-1}\frac{(-t)^k}{k!}(2n)^{k-1}\binom{n-2}{k-1} = \kappa_n(\Pi_t),$$

where we understand $\kappa_1^{(d)}(L_{d,m}) = e^{-2t}$.

Proof. Consider $m/d\to t$ for some $t>0$ as $d\to\infty$. By the definition of the finite free cumulants, we have

$$\kappa_n^{(d)}(L_{d,m}) = \frac{(-d)^{n-1}}{(n-1)!}\sum_{\pi\in\mathrm{P}(n)}\prod_{V\in\pi}\left(1-\frac{2|V|}{d}\right)^m\mu_n^{\mathrm{P}}(\pi,1_n).$$

As $d\to\infty$, we get

$$\begin{aligned}\prod_{V\in\pi}\left(1-\frac{2|V|}{d}\right)^m &= \exp\left(m\sum_{V\in\pi}\log\left(1-\frac{2|V|}{d}\right)\right)\\ &= \exp\left(-\frac{m}{d}\cdot d\sum_{V\in\pi}\sum_{l=1}^{\infty}\frac{1}{l}\frac{(2|V|)^l}{d^\ell}\right)\\ &= e^{-2nt}\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{m}{d}\sum_{V\in\pi}\sum_{l=2}^{\infty}\frac{1}{l}\frac{(2|V|)^l}{d^{l-1}}\right)^k\\ &= e^{-2nt}\sum_{k=0}^{n-1}\frac{1}{k!}\left(-\frac{m}{d}\sum_{V\in\pi}\sum_{l=2}^{\infty}\frac{1}{l}\frac{(2|V|)^l}{d^{l-1}}\right)^k+O(d^{-n}).\end{aligned}$$

Let $t_d := m/d$ then $t_d \to t$ as $d \to \infty$ by the assumption. Then

$$\begin{aligned}\kappa_n^{(d)}(L_{d,m}) &= \frac{(-d)^{n-1}}{(n-1)!}e^{-2nt}\sum_{\pi\in \mathrm{P}(n)}\sum_{k=1}^{n-1}\frac{(-t_d)^k}{k!}\left(\sum_{V\in\pi}\sum_{l=2}^{\infty}\frac{1}{l}\frac{(2|V|)^l}{d^{l-1}}\right)^k \mu_n^{\mathrm{P}}(\pi, 1_n) + O(d^{-1}) \\ &= \frac{(-d)^{n-1}}{(n-1)!}e^{-2nt}\underbrace{\sum_{\pi\in \mathrm{P}(n)}\sum_{k=1}^{n-1}\frac{(-t_d)^k}{k!}\left(\sum_{V\in\pi}\sum_{l=1}^{\infty}\frac{1}{l+1}\frac{(2|V|)^{l+1}}{d^l}\right)^k \mu_n^{\mathrm{P}}(\pi, 1_n)}_{=:M_n(d)} + O(d^{-1}).\end{aligned}$$

For short, we write $f_l(x) := (2x)^l/l$, and then

$$\begin{aligned}M_n(d) &= \sum_{\pi\in \mathrm{P}(n)}\sum_{k=1}^{n-1}\frac{(-t_d)^k}{k!}\left(\sum_{V\in\pi}\sum_{l=1}^{\infty}\frac{f_{l+1}(|V|)}{d^l}\right)^k \mu_n^{\mathrm{P}}(\pi, 1_n) \\ &= \sum_{\pi\in \mathrm{P}(n)}\sum_{k=1}^{n-1}\frac{(-t_d)^k}{k!}\prod_{j=1}^{k}\left(\sum_{V\in\pi}\sum_{l_j=1}^{\infty}\frac{f_{l_j+1}(|V|)}{d^{l_j}}\right)\mu_n^{\mathrm{P}}(\pi, 1_n) \\ &= \sum_{k=1}^{n-1}\frac{(-t_d)^k}{k!}\sum_{l_1,\dots,l_k=1}^{\infty}\frac{1}{d^{l_1+\cdots+l_k}}\sum_{\pi\in \mathrm{P}(n)}\left(\sum_{V\in\pi}f_{l_1+1}(|V|)\right)\cdots\left(\sum_{V\in\pi}f_{l_k+1}(|V|)\right)\mu_n^{\mathrm{P}}(\pi, 1_n).\end{aligned}$$

Recall that, for each $n \in \mathbb{N}$ and $l_1, \cdots, l_k \in \mathbb{N}$,

$$s_k^{(n)}(f_{l_1+1}, \dots, f_{l_k+1})(0) = \sum_{\pi\in \mathrm{P}(n)}\left(\sum_{V\in\pi}f_{l_1+1}(|V|)\right)\cdots\left(\sum_{V\in\pi}f_{l_k+1}(|V|)\right)\mu_n^{\mathrm{P}}(\pi, 1_n).$$

By Theorem 2.4.20, if $\sum_{i=1}^k \deg f_{l_i+1} - k < n-1$, that is, $\sum_{i=1}^k l_i < n-1$ then

$$s_k^{(n)}(f_{l_1+1}, \dots, f_{l_k+1})(0) = 0,$$

also if $\sum_{i=1}^k l_i = n-1$ then

$$\begin{aligned}s_k^{(n)}(f_{l_1+1}, \dots, f_{l_k+1})(0) &= (n-1)!n^{k-1}\left(\prod_{i=1}^{k}(l_i+1)\cdot\frac{2^{l_i+1}}{l_i+1}\right) \\ &= (n-1)!2^n(2n)^{k-1}.\end{aligned}$$

Thus, we get

$$\begin{aligned}\kappa_n^{(d)}(L_{d,m}) &= \frac{(-1)^{n-1}}{(n-1)!}e^{-2nt}\sum_{k=1}^{n-1}\frac{(-t_d)^k}{k!}\sum_{\substack{l_1,\dots,l_k=1\\ l_1+\cdots+l_k=n-1}}^{\infty} s_k^{(n)}(f_{l_1+1}, \dots, f_{l_k+1})(0) + O(d^{-1}) \\ &= (-1)^{n-1}2^n e^{-2nt}\sum_{k=1}^{n-1}\frac{(-t_d)^k}{k!}(2n)^{k-1}\#\left\{(l_1, \dots, l_k) \in \mathbb{N}^k : \sum_{i=1}^k l_i = n-1\right\} + O(d^{-1}) \\ &= (-1)^{n-1}2^n e^{-2nt}\sum_{k=1}^{n-1}\frac{(-t_d)^k}{k!}(2n)^{k-1}\binom{n-2}{k-1} + O(d^{-1}).\end{aligned}$$

Combining with Proposition 2.2.2, we have obtained the result as $d \to \infty$. □

4.5 LLN and CLT for finite free multiplicative convolution on $\mathcal{P}_+$

In this section, we prove the Law of Large Numbers (LLN) and Central Limit Theorems (CLT) for finite free multiplicative convolution of polynomials with nonnegative roots. Moreover, we investigate a relation between the limit polynomial from the CLT and the multiplicative free semicircle distribution on $[0, \infty)$.

Theorem 4.5.1. Let $p(x) = \prod_{k=1}^{d}(x - e^{\theta_k})$. Assume that

$$\frac{1}{d}\sum_{k=1}^{d}\theta_k = \alpha.$$

Then

$$\lim_{m\to\infty} \phi_{1/m}(p)^{\boxtimes_d m}(x) = (x - e^{\alpha})^d.$$

Proof. A simple computation shows that

$$\begin{aligned}
\widetilde{e}_k(\phi_{1/m}(p)) &= \binom{d}{k}^{-1} \sum_{1\le j_1<\cdots<j_k\le d} \exp\left(\frac{1}{m}(\theta_{j_1} + \cdots \theta_{j_k})\right) \\
&= \binom{d}{k}^{-1} \left\{ \binom{d}{k} + \frac{1}{m} \sum_{1\le j_1<\cdots<j_k\le d} (\theta_{j_1} + \cdots \theta_{j_k}) + O(m^{-2}) \right\} \\
&= \binom{d}{k}^{-1} \left\{ \binom{d}{k} + \frac{1}{m}\binom{d-1}{k-1} \sum_{i=1}^{d} \theta_i + O(m^{-2}) \right\} \\
&= 1 + \frac{1}{m}\cdot \alpha k + O(m^{-2}).
\end{aligned}$$

This implies that

$$\lim_{m\to\infty} \widetilde{e}_k(\phi_{1/m}(p))^m = e^{\alpha k},$$

and therefore

$$\lim_{m\to\infty} \phi_{1/m}(p)^{\boxtimes_d m}(x) = \sum_{k=0}^{d}(-1)^k \binom{d}{k} e^{\alpha k} x^{d-k} = (x - e^{\alpha})^d.$$

□

We then investigate the central limit theorem for polynomials with non-negative roots and their limits as the degree goes to infinity. As a remarkable notice, their proofs are parallel to those in Section 4.4.

Theorem 4.5.2. (1) Let $d \ge 2$. Suppose $p(x) = \prod_{k=1}^{d}(x - e^{\theta_k})$ such that $\frac{1}{d}\sum_{k=1}^{d}\theta_k = 0$ and $\frac{1}{d}\sum_{k=1}^{d}\theta_k^2 = \sigma^2$. Then we have

$$\lim_{m\to\infty} \phi_{1/\sqrt{m}}(p)^{\boxtimes_d m} = I_d\left(x; \frac{d\sigma^2}{d-1}\right),$$

and

$$I_d(x;t) := \sum_{k=0}^{d}(-1)^k \binom{d}{k} \exp\left(\frac{k(d-k)}{2d}t\right) x^{d-k}, \qquad t > 0.$$

(2) As $d \to \infty$, we have

$$\mu[\![I_d(x;t)]\!] \xrightarrow{w} \lambda_t,$$

where λ_t is the multiplicative free semicircle distribution on $[0, \infty)$.

Proof. A similar argument to the proof of Proposition 4.4.1 shows that

$$\widetilde{e}_k(\phi_{1/\sqrt{m}}(p)) = 1 + \frac{k(d-k)}{2(d-1)m}\sigma^2 + O(m^{-\frac{3}{2}}).$$

Thus, $\widetilde{e}_k(\phi_{1/\sqrt{m}}(p))^m$ goes to $\exp(k(d-k)\sigma^2/2(d-1))$ as $m \to \infty$ for $k = 0, \ldots, d$. It means $\lim_{m\to\infty} \phi_{1/\sqrt{m}}(p)^{\boxtimes_d m}(z) = I_d(z; d\sigma^2/(d-1))$.

For the latter part, a similar argument to proof of Theorem 4.4.2 also shows

$$\kappa_n^{(d)}(I_d(x;t)) = \exp\left(\frac{nt}{2}\right)\frac{(nt)^{n-1}}{n!} + O\left(d^{-1}\right).$$

This implies that

$$\lim_{d\to\infty} \kappa_n^{(d)}(I_d(x;t)) = \exp\left(\frac{nt}{2}\right)\frac{(nt)^{n-1}}{n!} = \kappa_n(\mathcal{D}_{e^{t/2}}(\eta_t)),$$

as desired. □

Remark 4.5.3. The multiplicative free semicircle distribution λ_t on $[0,\infty)$, introduced in [Bia97a, Proposition 5]. Note that λ_t coincides with $\mathcal{D}_{e^{t/2}}(\eta_t)$, where η_t is the probability measure on $[0,\infty)$ appeared in [SY13]. Indeed, the n-th moment $m_n(\lambda_t)$ is given by

$$\exp\left(\frac{nt}{2}\right)\sum_{k=0}^{n-1}\frac{n^{k-1}}{k!}\binom{n}{k+1}t^k, \qquad t > 0.$$

A similar argument to the proof of Proposition 2.2.1 shows that

$$\kappa_n(\lambda_t) = \exp\left(\frac{nt}{2}\right)\frac{(nt)^{n-1}}{n!} = \kappa_n(\mathcal{D}_{e^{t/2}}(\eta_t))$$

for all $n \in \mathbb{N}$, and therefore $\lambda_t = \mathcal{D}_{e^{t/2}}(\eta_t)$ for $t > 0$.

4.6 Finite free analogue of Tucci's limit theorem

Tucci's limit theorems for Free multiplicative convolution

In this section, we introduce free multiplicative convolution and its characterization via the S-transform (see [BV93] for more details). For a probability measure $\mu \neq \delta_0$ on $[0,\infty)$, we define

$$\psi_\mu(z) := \int_0^\infty \frac{tz}{1-tz}\,\mu(dt), \qquad z \in \mathbb{C}\setminus[0,\infty).$$

It is known that its inverse function ψ_μ^{-1} exists in a neighborhood of $(\mu(\{0\})-1, 0)$, and then we define the *S-transform* of μ as

$$S_\mu(z) := \frac{z+1}{z}\psi_\mu^{-1}(z), \qquad z \in (\mu(\{0\})-1, 0).$$

According to [BV93], for probability measures $\mu, \nu \neq \delta_0$ on $[0,\infty)$, the free multiplicative convolution $\mu \boxtimes \nu$ is characterized by

$$S_{\mu\boxtimes\nu}(z) = S_\mu(z)S_\nu(z),$$

for all z in the common interval where all three S-transforms are defined. Note that the common interval is not empty since $(\mu\boxtimes\nu)(\{0\}) = \max\{\mu(\{0\}), \nu(\{0\})\}$ (see [BV93, Lemma 6.9]).

Some kind of limit theorem for free multiplicative convolution of a probability measure on $[0,\infty)$ was obtained by Tucci [Tuc10] and Haagerup and Möller [HM13].

Proposition 4.6.1. Let us consider $\mu \in \mathcal{P}_+$. As $n \to \infty$, the sequence of $(\mu^{\boxtimes n})^{\frac{1}{n}}$ converges weakly to the measure $\Phi(\mu) \in \mathcal{P}_+$ characterized by

$$\begin{aligned} &\Phi(\mu)\left(\left[0, \frac{1}{S_\mu(t-1)}\right]\right) = t, \qquad t \in (\mu(\{0\}), 1) \\ &\Phi(\mu)(\{0\}) = \mu(\{0\}). \end{aligned} \tag{4.6.1}$$

Moreover, the support of the measure $\Phi(\mu)$ is the closure of the interval

$$\left(\left(\int_0^\infty t^{-1}\mu(dt)\right)^{-1}, \int_0^\infty t\,\mu(dt)\right) \subset [0, \infty).$$

Example 4.6.2. (1) Let **MP** be the (standard) *Marchenko-Pastur distribution* which is defined as

$$\mathbf{MP}(dt) = \frac{\sqrt{t(4-t)}}{2\pi t}\mathbf{1}_{(0,4)}(t)\,dt.$$

Then $\Phi(\mathbf{MP})$ is the uniform distribution $\mathbf{U}(0,1)$ on the open interval $(0,1)$, see [Ued21].

(2) Consider $\mu = \frac{1}{2}(\delta_0 + \delta_1)$. Then we have

$$\Phi(\mu) = \frac{1}{2}\delta_0 + \frac{1}{2(1-t)^2}\mathbf{1}_{(0,1/2)}(t)dt,$$

since $S_\mu(t) = (2+2t)/(1+2t)$, $t \in (-1/2, 0)$ implies that $\Phi(\mu)([0,t]) = 2^{-1}(1-t)^{-1}$ for all $1/2 < t < 1$.

Finite free analogue of Tucci's limit theorem

In this section, we provide the finite free analogue of Tucci's limit theorems. First, we calculate the n-th power of finite free multiplicative convolution of polynomials that have only non-negative real roots. Let $\Lambda^{(n)}$ be the multi-set of roots of $p^{\boxtimes_d n}$ for $n \geq 1$ and a monic polynomial p of degree d with non-negative real roots. We put $\Lambda := \Lambda^{(1)}$, for short.

Lemma 4.6.3. Let p be a monic polynomial of degree d with non-negative real roots Λ. Then we have

$$\widetilde{e}_i(\Lambda^{(n)}) = \widetilde{e}_i(\Lambda)^n, \qquad 0 \leq i \leq d. \tag{4.6.2}$$

In particular, p and $p^{\boxtimes_d n}$ have the same multiplicity of zeros.

Proof. Note that

$$p(x) = \sum_{i=0}^{d}(-1)^i\binom{d}{i}\widetilde{e}_i(\Lambda)x^{d-i},$$

and then by the definition

$$p^{\boxtimes_d n}(x) = \sum_{i=0}^{d}(-1)^i\binom{d}{i}\widetilde{e}_i(\Lambda)^n x^{d-i}.$$

This is equivalent to (4.6.2). The last assertion holds because the number of zeros in Λ is equal to k if and only if

$$e_{d-k}(\Lambda) > 0 \quad \text{and} \quad e_{d-k+1}(\Lambda) = 0,$$

where we understand $e_{d+1}(\Lambda) = 0$. □

Due to the relation (4.6.2), we obtain the following limit for roots of finite free multiplicative convolution of polynomials.

Theorem 4.6.4. Consider a monic polynomial p of degree d with non-negative real roots Λ and let $k = k(p)$ be the number of zeros in Λ. Let $\Lambda^{(n)} := \{\lambda_1^{(n)} \geq \lambda_2^{(n)} \geq \cdots \geq \lambda_d^{(n)}\}$ be the multiset of non-negative real roots of $p^{\boxtimes_d n}$. Then

$$\lim_{n\to\infty} (\lambda_i^{(n)})^{\frac{1}{n}} = \frac{\widetilde{e}_i(\Lambda)}{\widetilde{e}_{i-1}(\Lambda)}, \qquad 1 \leq i \leq d-k.$$

Remark 4.6.5. Note that $\lambda_i^{(n)} = 0$ for $d-k+1 \leq i \leq d$ by Lemma 4.6.3.

Proof. For $1 \leq i \leq d-k$, Equation (4.6.2) implies that

$$\frac{\widetilde{e}_i(\Lambda^{(n)})}{\widetilde{e}_{i-1}(\Lambda^{(n)})} = \left(\frac{\widetilde{e}_i(\Lambda)}{\widetilde{e}_{i-1}(\Lambda)}\right)^n. \tag{4.6.3}$$

For $i = 1$, Equation (4.6.3) implies that

$$\frac{\lambda_1^{(n)} + \cdots + \lambda_d^{(n)}}{d} = \widetilde{e}_1(\Lambda^{(n)}) = \widetilde{e}_1(\Lambda)^n.$$

Since

$$\frac{\lambda_1^{(n)}}{d} \leq \frac{\lambda_1^{(n)} + \cdots + \lambda_d^{(n)}}{d} \leq \lambda_1^{(n)},$$

we obtain

$$\widetilde{e}_1(\Lambda) \leq (\lambda_1^{(n)})^{\frac{1}{n}} \leq d^{\frac{1}{n}} \widetilde{e}_1(\Lambda),$$

and therefore $(\lambda_1^{(n)})^{\frac{1}{n}} \to \widetilde{e}_1(\Lambda)$ as $n \to \infty$.

For $2 \leq i \leq d-k$, we have

$$\begin{aligned}
\frac{e_i(\Lambda^{(n)})}{e_{i-1}(\Lambda^{(n)})} &= \frac{\sum_{J\subset[d],\ |J|=i} \left(\prod_{j\in J} \lambda_j^{(n)}\right)}{\sum_{J\subset[d],\ |J|=i-1} \left(\prod_{j\in J} \lambda_j^{(n)}\right)} \\
&\geq \frac{\sum_{J\subset[d],\ |J|=i} \left(\prod_{j\in J} \lambda_j^{(n)}\right)}{\binom{d}{i-1} \lambda_1^{(n)} \lambda_2^{(n)} \cdots \lambda_{i-1}^{(n)}} \\
&\geq \frac{\lambda_1^{(n)} \lambda_2^{(n)} \cdots \lambda_i^{(n)}}{\binom{d}{i-1} \lambda_1^{(n)} \lambda_2^{(n)} \cdots \lambda_{i-1}^{(n)}} = \binom{d}{i-1}^{-1} \lambda_i^{(n)}.
\end{aligned}$$

Similar to the estimation above, we obtain

$$\frac{e_i(\Lambda^{(n)})}{e_{i-1}(\Lambda^{(n)})} \leq \frac{\binom{d}{i} \lambda_1^{(n)} \lambda_2^{(n)} \cdots \lambda_i^{(n)}}{\lambda_1^{(n)} \lambda_2^{(n)} \cdots \lambda_{i-1}^{(n)}} = \binom{d}{i} \lambda_i^{(n)}.$$

Consequently, we have

$$\binom{d}{i}^{-1} \frac{e_i(\Lambda^{(n)})}{e_{i-1}(\Lambda^{(n)})} \leq \lambda_i^{(n)} \leq \binom{d}{i-1} \frac{e_i(\Lambda^{(n)})}{e_{i-1}(\Lambda^{(n)})},$$

and therefore, by using (4.6.3),

$$\binom{d}{i-1}^{-1} \left(\frac{\widetilde{e}_i(\Lambda)}{\widetilde{e}_{i-1}(\Lambda)}\right)^n \leq \lambda_i^{(n)} \leq \binom{d}{i} \left(\frac{\widetilde{e}_i(\Lambda)}{\widetilde{e}_{i-1}(\Lambda)}\right)^n.$$

Taking the n-th root of each value in the above inequality, we get

$$\binom{d}{i-1}^{-\frac{1}{n}} \frac{\widetilde{e}_i(\Lambda)}{\widetilde{e}_{i-1}(\Lambda)} \leq (\lambda_i^{(n)})^{\frac{1}{n}} \leq \binom{d}{i}^{\frac{1}{n}} \frac{\widetilde{e}_i(\Lambda)}{\widetilde{e}_{i-1}(\Lambda)}.$$

Hence, we obtain $(\lambda_i^{(n)})^{\frac{1}{n}} \to \widetilde{e}_i(\Lambda)/\widetilde{e}_{i-1}(\Lambda)$ as $n \to \infty$. □

For a positive number $\alpha > 0$ and a monic polynomial $p(x) = \prod_{i=1}^{d}(x - \lambda_i)$ with non-negative real roots, we define

$$p^{(\alpha)}(x) := \prod_{i=1}^{d}(x - \lambda_i^{\alpha}).$$

Remark 4.6.6. According to Theorem 4.6.4, if p is a monic polynomial of degree d with non-negative real roots Λ and k is the number of zeros in Λ, then

$$\lim_{n\to\infty} (p^{\boxtimes_d n})^{(\frac{1}{n})}(x) = x^k \prod_{i=1}^{d-k} \left(x - \frac{\widetilde{e}_i(\Lambda)}{\widetilde{e}_{i-1}(\Lambda)} \right).$$

Thus, this is the finite free analogue for Tucci's limit theorem.

Remark 4.6.7. Let p be a monic polynomial of degree d with non-negative real roots $\Lambda = \{\lambda_1 \geq \cdots \geq \lambda_d\}$, k the number of zeros in Λ, and $\lambda_i^{(n)}$ the i-th real root of $p^{\boxtimes_d n}$ for $1 \leq i \leq d$. By Newton's inequality (see, e.g., [HLP52]), we have

$$\frac{\widetilde{e}_i(\Lambda)}{\widetilde{e}_{i-1}(\Lambda)} \geq \frac{\widetilde{e}_{i+1}(\Lambda)}{\widetilde{e}_i(\Lambda)}, \qquad 1 \leq i \leq d-k-1, \tag{4.6.4}$$

where the equality holds if and only if $\lambda_1 = \cdots = \lambda_d$. However, the inequality (4.6.4) can be directly proven by Theorem 4.6.4 due to $\lambda_i^{(n)} \geq \lambda_{i+1}^{(n)}$.

Consequently, we find the following remarkable phenomenon; except for trivial cases, the limit roots of $(p^{\boxtimes_d n})^{(\frac{1}{n})}$, not being zero, are all distinct.

In particular, we apply Theorem 4.6.4 to the (normalized) Laguerre polynomial and a polynomial with two real roots.

Example 4.6.8 (Case of the normalized Laguerre polynomial)**.** Consider $d \geq 1$. Recall the normalized Laguerre polynomial (4.1.2):

$$\widehat{L}_d^{(\lambda)}(x) = \sum_{i=0}^{d} (-1)^i \binom{d}{i} \frac{(d\lambda)_i}{d^i} x^{d-i}.$$

Let Λ be the set of positive real roots of $p = \widehat{L}_d^{(1)}$. Note that

$$\widetilde{e}_i(\Lambda) = \frac{d}{d} \cdot \frac{d-1}{d} \cdots \frac{d-i+1}{d}, \qquad 1 \leq i \leq d$$

and hence p has no zero roots since $p(0) = \widetilde{e}_d(\Lambda) \neq 0$.

Suppose that $\lambda_1^{(n)} \geq \cdots \geq \lambda_d^{(n)}$ are non-negative real roots of $p^{\boxtimes_d n}$. By Theorem 4.6.4, we obtain

$$\lim_{n\to\infty} (\lambda_i^{(n)})^{\frac{1}{n}} = \frac{\widetilde{e}_i(\Lambda)}{\widetilde{e}_{i-1}(\Lambda)} = \frac{\frac{d}{d} \cdot \frac{d-1}{d} \cdots \frac{d-i+1}{d}}{\frac{d}{d} \cdot \frac{d-1}{d} \cdots \frac{d-i+2}{d}} = \frac{d-i+1}{d}$$

for $1 \leq i \leq d$, where note that $\widetilde{e}_0(\Lambda) = 1$.

Example 4.6.9 (Case of a polynomial with two roots)**.** Given $d \geq 1$, consider the following monic polynomial p of $2d$ degree:

$$p(x) = x^d(x-1)^d, \qquad d \geq 1,$$

and put $\Lambda = \{\underbrace{1, \cdots, 1}_{d \text{ times}}, \underbrace{0, \cdots, 0}_{d \text{ times}}\}$ as the set of roots of p. Then we get

$$\widetilde{e}_j(\Lambda) = \frac{\binom{d}{j}}{\binom{2d}{j}}, \qquad 0 \leq j \leq d,$$

and $\widetilde{e}_j(\Lambda) = 0$ for $d+1 \leq j \leq 2d$. Suppose that $\lambda_1^{(n)} \geq \cdots \geq \lambda_d^{(n)}$ are positive real roots of $p^{\boxtimes_d n}$. By Theorem 4.6.4, we have

$$\lim_{n\to\infty} (\lambda_i^{(n)})^{\frac{1}{n}} = \frac{d-i+1}{2d-i+1},$$

for all $1 \leq i \leq d$.

Remark 4.6.10 (Rate of Convergence)**.** According to the proof of Theorem 4.6.4, it is easy to see that

$$\log(\lambda_i^{(n)})^{\frac{1}{n}} = \log\left(\frac{\widetilde{e}_i(\Lambda)}{\widetilde{e}_{i-1}(\Lambda)}\right) + O\left(\frac{1}{n}\right),$$

as $n \to \infty$.

We demonstrate an example in which the rate of convergence is of order $1/n$ and it is optimal. Consider $d = 2$ in Example 4.6.8, that is, $p(x) = x^2 - 2x + 2^{-1}$. As a consequent result of a proof of Lemma 4.6.3, we obtain $p^{\boxtimes_2 n}(x) = x^2 - 2x + 2^{-n}$ for $n \in \mathbb{N}$. Hence (positive) real roots of $p^{\boxtimes_2 n}$ are given by

$$\lambda_1^{(n)} = 1 + \sqrt{1 - 2^{-n}}, \qquad \lambda_2^{(n)} = 1 - \sqrt{1 - 2^{-n}}.$$

It is easy to see that $\lim_{n\to\infty}(\lambda_1^{(n)})^{\frac{1}{n}} = 1$, $\lim_{n\to\infty}(\lambda_2^{(n)})^{\frac{1}{n}} = 1/2$ and also

$$n\left(\log(\lambda_1^{(n)})^{\frac{1}{n}} - \log 1\right) \to \log 2, \qquad n\left(\log(\lambda_2^{(n)})^{\frac{1}{n}} - \log\frac{1}{2}\right) \to -\log 2,$$

as $n \to \infty$. Consequently, the order $1/n$ is optimal.

Further, we investigate how the empirical root distributions of the limit polynomial obtained by Theorem 4.6.4 (or Remark 4.6.6) converge weakly as $d \to \infty$. For the reason above, we henceforth emphasize their degree as follows.

Let p_d be a monic polynomial of degree d with non-negative real roots $\Lambda_d = \{\lambda_{1,d} \geq \cdots \geq \lambda_{d,d}\}$ and let k_d be the number of zeros in Λ_d. Denote by $R_i(\Lambda_d)$ the limit roots of $(p_d^{\boxtimes_d n})^{(\frac{1}{n})}$ as $n \to \infty$ for $1 \leq i \leq d$, that is,

$$R_i(\Lambda_d) = \frac{\widetilde{e}_i(\Lambda_d)}{\widetilde{e}_{i-1}(\Lambda_d)},$$

as provided in Theorem 4.6.4. In the following, we investigate relationships between the empirical root distributions:

$$\mu_d := \frac{1}{d}\sum_{i=1}^{d} \delta_{\lambda_{i,d}}, \qquad \text{and} \qquad \nu_d := \frac{1}{d}\sum_{i=1}^{d} \delta_{R_i(\Lambda_d)}.$$

Lemma 4.6.11. Let p_d be a monic polynomial of degree d with non-negative real roots Λ_d. Assume that $k_d = 0$ (equivalently, $\lambda_{d,d} > 0$ or $R_d(\Lambda_d) > 0$). Then we have

$$\int_0^\infty (\log t)\, \mu_d(dt) = \int_0^\infty (\log t)\, \nu_d(dt).$$

Proof. Note that the integrals $\int_0^\infty (\log t)\,\mu_d(dt)$ and $\int_0^\infty (\log t)\,\nu_d(dt)$ are finite since $k_d = 0$. A direct computation shows that

$$\begin{aligned}
\int_0^\infty (\log t)\,\nu_d(dt) &= \frac{1}{d}\sum_{i=1}^{d} \log R_i(\Lambda_d) \\
&= \frac{1}{d}\sum_{i=1}^{d} \log \frac{\widetilde{e}_i(\Lambda_d)}{\widetilde{e}_{i-1}(\Lambda_d)} \qquad \text{(by Theorem 4.6.4)} \\
&= \frac{1}{d}\log \widetilde{e}_d(\Lambda_d) \qquad (\text{since } \widetilde{e}_0(\Lambda_d) = 1) \\
&= \frac{1}{d}\log \prod_{i=1}^{d} \lambda_{i,d} \\
&= \frac{1}{d}\sum_{i=1}^{d} \log \lambda_{i,d} = \int_0^\infty (\log t)\,\mu_d(dt).
\end{aligned}$$

□

We study how the empirical root distributions $\nu_d = \frac{1}{d}\sum_{i=1}^{d} \delta_{R_i(\Lambda_d)}$ behave as $d \to \infty$ when $\mu_d = \frac{1}{d}\sum_{i=1}^{d} \delta_{\lambda_{i,d}}$ converges weakly to some probability measure on $[0,\infty)$.

Proposition 4.6.12. Let p_d be a monic polynomial of degree d with non-negative real roots Λ_d. Assume that there exist $\mu \in \mathcal{P}_{+,c}$ and a compact set K in $[0,\infty)$, such that the measures μ_d and μ are supported on K for all $d \geq 1$, and such that $\mu_d \xrightarrow{w} \mu$ as $d \to \infty$. Then we obtain

$$R_1(\Lambda_d) \to \int_0^\infty t\,\mu(dt)$$

as $d \to \infty$. In addition, if $0 \notin K$, then it satisfies that

$$R_d(\Lambda_d) \to \left(\int_0^\infty t^{-1}\mu(dt)\right)^{-1} \qquad \text{and} \qquad \int_0^\infty (\log t)\nu_d(dt) \to \int_0^\infty (\log t)\Phi(\mu)(dt)$$

as $d \to \infty$, where $\Phi(\mu)$ is defined in Proposition 4.6.1.

Proof. By the assumptions and Theorem 4.6.4, we get

$$R_1(\Lambda_d) = \widetilde{e}_1(\Lambda_d) = \frac{1}{d}\sum_{i=1}^{d} \lambda_{i,d} = \int_0^\infty t\,\mu_d(dt) \to \int_0^\infty t\,\mu(dt),$$

as $d \to \infty$.

Moreover, we assume that $0 \notin K$ in the following statement. Note that the functions $t \mapsto t^{-1}$ and $t \mapsto \log t$ are bounded and continuous on K. We then obtain

$$R_d(\Lambda_d) = \frac{\widetilde{e}_d(\Lambda_d)}{\widetilde{e}_{d-1}(\Lambda_d)} = \left(\frac{1}{d}\sum_{i=1}^{d} \lambda_{i,d}^{-1}\right)^{-1} = \left(\int_0^\infty t^{-1}\mu_d(dt)\right)^{-1} \to \left(\int_0^\infty t^{-1}\mu(dt)\right)^{-1}$$

as $d \to \infty$. It follows that $k_d = 0$ from $0 \notin K$. By Lemma 4.6.11, we obtain

$$\int_0^\infty (\log t)\nu_d(dt) = \int_0^\infty (\log t)\mu_d(dt) \to \int_0^\infty (\log t)\mu(dt).$$

According to [HM13, Proposition 1], the last integral equals to $\int_0^\infty (\log t)\Phi(\mu)(dt)$, and therefore we get the convergence. □

We give examples of the weak limit laws of empirical root distributions $\frac{1}{d}\sum_{i=1}^{d}\delta_{R_i(\Lambda_d)}$ as $d\to\infty$.

Example 4.6.13. (1) In Example 4.6.8, it was shown that $R_i(\Lambda_d)=\frac{d-i+1}{d}$ when we consider $p_d=\widehat{L}_d^{(1)}$. with non-negative real roots Λ_d for each $d\geq 1$. It is easy to see that

$$\frac{1}{d}\sum_{i=1}^{d}\delta_{\frac{d-i+1}{d}}\xrightarrow{w}\mathbf{U}(0,1)=\Phi(\mathbf{MP}),$$

as $d\to\infty$, where the last equality holds due to Example 4.6.2 (1).

(2) In Example 4.6.9, we obtained that $R_i(\Lambda_{2d})=\frac{d-i+1}{2d-i+1}$ for $1\leq i\leq d$ when $p_d(x)=x^d(x-1)^d$ with non-negative real roots Λ_{2d}. For any bounded continuous functions f on $[0,\infty)$, we get

$$\begin{aligned}\int_0^\infty f(t)\left[\frac{1}{2}\delta_0+\frac{1}{2d}\sum_{i=1}^{d}\delta_{\frac{d-i+1}{2d-i+1}}\right](dt)&=\frac{1}{2}f(0)+\frac{1}{2d}\sum_{i=1}^{d}f\left(\frac{d-i+1}{2d-i+1}\right)\\&=\frac{1}{2}f(0)+\frac{1}{2d}\sum_{\ell=1}^{d}f\left(\frac{\ell}{d+\ell}\right)\\&=\frac{1}{2}f(0)+\frac{1}{2d}\sum_{\ell=1}^{d}f\left(\frac{\frac{\ell}{d}}{1+\frac{\ell}{d}}\right)\\&\to\frac{1}{2}f(0)+\frac{1}{2}\int_0^1 f\left(\frac{t}{1+t}\right)dt\\&=\frac{1}{2}f(0)+\frac{1}{2}\int_0^{1/2}\frac{f(u)}{(1-u)^2}du,\end{aligned}$$

where the last equality holds by changing variable to $u=t/(1+t)$, and therefore

$$\frac{1}{2}\delta_0+\frac{1}{2d}\sum_{i=1}^{d}\delta_{\frac{d-i+1}{2d-i+1}}\xrightarrow{w}\frac{1}{2}\delta_0+\frac{1}{2(1-t)^2}\mathbf{1}_{(0,1/2)}(t)dt=\Phi\left(\frac{1}{2}(\delta_0+\delta_1)\right),$$

as $d\to\infty$, where the last equality holds due to Example 4.6.2 (2).

According to Proposition 4.6.12 and Example 4.6.13, it is natural to conjecture as follows.

Conjecture 4.6.14. Let p_d be a monic polynomial of degree d with non-negative real roots $\Lambda_d=\{\lambda_{1,d}\geq\cdots\geq\lambda_{d,d}\}$. Let us further consider $\mu\in\mathcal{P}_{+,c}$. Assume that the empirical root distributions of p_d, that is, $\frac{1}{d}\sum_{i=1}^{d}\delta_{\lambda_{i,d}}$ converge weakly to μ as $d\to\infty$. Then we obtain

$$\frac{1}{d}\sum_{i=1}^{d}\delta_{R_i(\Lambda_d)}\xrightarrow{w}\Phi(\mu)$$

as $d\to\infty$.

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