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Doctoral Dissertation

Free multiarrangements and integral expressions of their derivations

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Preface

0.1 Abstract

Given a reflection group generated by reflections, we can obtain a set of fixed subspaces of the reflections which forms a reflection arrangement. The fixed point set of each reflection is called reflecting hyperplane. In [19], they developed general theory of hyperplane arrangements. Many aspects of hyperplane arrangements, e.g. combinatorial, algebraic, and topological properties, have been studied.

Studying the freeness of a central arrangement $\mathcal{A}$ is to consider its derivation module $\mathcal{D}(\mathcal{A})$. When $\mathcal{D}(\mathcal{A})$ is free, we say $\mathcal{A}$ is free. It is well known that every reflection arrangement is free. However, most arrangements are not free. On the other hand, the algebraic property of the module $\mathcal{D}(\mathcal{A})$ reflects the combinatorial property of $\mathcal{A}$. Terao conjectured the free arrangement $\mathcal{A}$ implies the freeness of the arrangement $\mathcal{B}$ with the same intersection lattices. This conjecture has not been solved even in the three-dimensional case. In a word, studying freeness of arrangements is a major topic.

In the process of studying free arrangements, many researchers have found that multiarrangements play an important role [30, 31, 32]. Many results applied to hyperplane arrangements are generalized to multiarrangements [6, 7].

Many researchers have studied the freeness of special multiarrangements [1, 12]. But it is quite difficult to construct an explicit basis for multiarrangements, even for two-dimensional cases [25, 27]. The doctoral thesis is inspired by the integral expressions of quasi-invariants [8, 10, 13, 15] and based on [14]. We construct the basis for certain multiarrangements by integral expressions.

0.2 Organization

In the first chapter, we give some definitions and classical theorems of arrangements, multiarrangements and reflection arrangements, we also introduce the monomial group. In the second chapter, we introduce the integral expressions and then construct a basis for multiarrangements defined in the first chapter. In the third chapter, we recall the primitive derivations, and observe its action on the integral expressions. In the fourth chapter, we give more applications for integral expressions.

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Contents

Chapter 1

Introduction

1.1 Hyperplane arrangements

Let V be an ℓ-dimensional vector space over a field $\mathbb{K}$ and V^* be the dual space. A hyperplane H is an affine subspace of dimension $(\ell-1)$. Fix $b,\ a_1,\ a_2, \dots, a_\ell \in \mathbb{K}$, then $H = \{(x_1,\ x_2, \dots,\ x_\ell) \in V \mid \alpha_H = a_1x_1 + a_2x_2 + \cdots + a_\ell x_\ell - b = 0\}$. A hyperplane arragement $\mathcal{A}$ is a collection of n hyperplanes in V. Let $L(\mathcal{A}) = \{H_I \mid H_I = \bigcap_{i\in I} H_i \neq \varnothing,\ \text{for } I \subset [n] = \{1,\ 2, \dots,\ n\}\}$. By convention, set $H_\varnothing = V$. We say $\mathcal{A}$ is central, if $\bigcap_{i\in[n]} H_i \neq \varnothing$. Define a partial order on L by $X \leq Y \iff Y \subseteq X$. Note that this is a reverse inclusion and the intersection poset of a central arrangement is a geometric lattice.

We usually consider the cone $\mathbf{c}\mathcal{A}$ of an arrangement $\mathcal{A}$. Define the cone $\mathbf{c}H$ of a hyperplane H by $\mathbf{c}H = \{(x_1,\ x_2, \dots,\ x_\ell,\ z) \in \mathbb{K}^{\ell+1} \mid \alpha_H = a_1x_1 + a_2x_2 + \cdots + a_\ell x_\ell = bz\}$. Then $\mathbf{c}\mathcal{A} = \{\mathbf{c}H \mid H \in \mathcal{A}\} \bigcup \{\{z = 0\}\}$.

Let S be the symmetric algebra of the dual space V^*. Then $S \cong \mathbb{K}[x_1, ..., x_\ell]$. Let $\mathrm{Der}(S)$ be the module of derivations

$$\mathrm{Der}(S) := \{\theta : S \to S \mid\ \theta \text{ is } \mathbb{K}\text{-linear},\ \theta(fg) = f\theta(g) + \theta(f)g,\ \forall f, g \in S\}.$$

It has a natural basis: $\mathrm{Der}(S) = \oplus_{i=1}^{\ell} S\partial_{x_i}$. For convenience, write $\partial_i = \partial_{x_i}$. We say a nonzero derivation $\theta = \sum_{i=1}^{\ell} f_i\partial_i \in \mathrm{Der}(S)$ is homogeneous of degree d if each f_i is zero or a homogeneous polynomial of degree d in S. For a central arrangement $\mathcal{A}$, define

$$\mathcal{D}(\mathcal{A}) := \{\theta \in \mathrm{Der}(S) \mid \theta(\alpha_H) \in \alpha_H S,\quad \text{for any } H \in \mathcal{A}\}.$$

We say $\mathcal{A}$ is **free** if $\mathcal{D}(\mathcal{A})$ is a free S-module. In fact, all 2-dimensional arrangements are free. Consider the 3-dimensional arrangement $\mathcal{A}$ defined by $xyz(ax + by + cz)$ drawn in Figure 1.1 for $abc \neq 0$. Then $\mathcal{A}$ is not free by [19]. It implies that freeness of arrangements is a very rare property.

Define the Euler vector field $\theta_E = \sum_{i\in[\ell]} x_i\partial_i$. Then $\theta_E \in \mathcal{D}(\mathcal{A})$ for any arrangements $\mathcal{A}$. For a free arrangement $\mathcal{A}$, $\mathcal{D}(\mathcal{A})$ has a homogeneous basis and the degrees of the basis form an ℓ-tuple $\exp(\mathcal{A})$, called the exponents of $\mathcal{A}$.

Let $\mathcal{A}$ be a nonempty arrangement and $H \in \mathcal{A}$. Let $\mathcal{A}' = \mathcal{A}\backslash\{H\}$, and $\mathcal{A}'' = \mathcal{A}^H = \{H' \cap H \mid H' \in \mathcal{A}',\ H' \cap H \neq \varnothing\}$. We call $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ a triple of arrangement with respect to H.

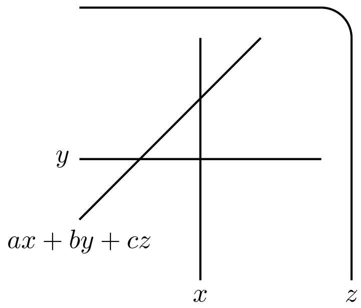

Figure 1.1: The arrangement $xyz(ax+by+cz)$

Theorem 1.1.1. *[19](Addition-Deletion Theorem) Any two of the following statements imply the third:*

$$\begin{aligned}
&\mathcal{A} \text{ is free, } \exp(\mathcal{A}) = (1, c_2, \ldots, c_{\ell-1}, c_\ell)\\
&\mathcal{A}' \text{ is free, } \exp(\mathcal{A}') = (1, c_2, \ldots, c_{\ell-1}, c_\ell - 1)\\
&\mathcal{A}'' \text{ is free, } \exp(\mathcal{A}'') = (1, c_2, \ldots, c_{\ell-1}).
\end{aligned}$$

Define the characteristic polynomial $\chi(\mathcal{A}, t)$ of an arrangement $\mathcal{A}$ by $\chi(\mathcal{A}, t) = \sum_{I\subset[\ell],\ H_I\neq\varnothing}(-1)^{|I|}t^{\dim H_I}$. The following theorem provides a method to show the non-freeness of an arrangement.

Theorem 1.1.2. *[23](Factorization Theorem) If $\mathcal{A}$ is free with exponents $(c_1, c_2, ..., c_\ell)$, then the characteristic polynomial factors $\chi(\mathcal{A}, t) = \prod_{i\in[\ell]}(t - c_i)$.*

1.2 Multiarrangements

For a central arrangement $\mathcal{A}$, let $m : \mathcal{A} \to \mathbb{Z}_{>0}$ be a multiplicity. Then the pair $(\mathcal{A},\ m)$ is called a multiarrangement. Its defining polynomial $Q(\mathcal{A},\ m) = \prod_{H\in\mathcal{A}} \alpha_H^{m(H)}$.

Definition 1.2.1. *For a central arrangement $\mathcal{A}$ and a hyperplane H. Let $(A^H,\ m^H)$ be the Ziegler restriction of $\mathcal{A}$ onto H where m^H is defined by*

$$m^H(H' \cap H) = |\{H'' \in \mathcal{A} \setminus \{H\} \mid H'' \cap H = H' \cap H\}|.$$

Define the module of vector fields by

$$\mathcal{D}(\mathcal{A},\ m) = \{\theta \in \mathrm{Der}(S) \mid \theta(\alpha_H) \in \alpha_H^{m(H)} S, \quad \text{for any } H \in \mathcal{A}\}.$$

We say $(\mathcal{A},\ m)$ is **free** if $\mathcal{D}(\mathcal{A},\ m)$ is a free S-module. Let $\exp(\mathcal{A},\ m)$ denote the exponents of $(\mathcal{A},\ m)$ consists of the degrees of elements of the homogeneous basis. Note that, when m is equal to 1, $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A},\ 1)$.

Let $\theta_1, \theta_2, \ldots, \theta_\ell \in \mathcal{D}(\mathcal{A},\ m)$ be homogeneous derivations. Their coefficient matrix which is called Saito matrix is shown below.

$$(\theta_i(x_j)) = \begin{pmatrix} \theta_1(x_1) & \theta_1(x_2) & \cdots & \theta_1(x_\ell) \\ \theta_2(x_1) & \theta_2(x_2) & \cdots & \theta_2(x_\ell) \\ \vdots & \vdots & \ddots & \vdots \\ \theta_\ell(x_1) & \theta_\ell(x_2) & \cdots & \theta_\ell(x_\ell) \end{pmatrix}.$$

Then $\det(\theta_j(x_i))\in SQ(\mathcal{A},\ m)$ by [19, 32].

The following criterion is used to show a set of vector fields forms a basis for arrangements or multiarrangements.

Theorem 1.2.2. *[19, 32] (Saito-Ziegler's Criterion) Let $\theta_1,\theta_2,\dots,\theta_\ell\in\mathcal{D}(\mathcal{A},\ m)$ be homogeneous derivations with $\deg\theta_i=d_i$ for $i\in\{1,2,\dots,\ell\}$. Then $\theta_1,\theta_2,\dots,\theta_\ell$ form a basis for $\mathcal{D}(\mathcal{A},\ m)$ if and only if they are S-independent and $\sum_{i=1}^{\ell} d_i = |m| = \sum_{H\in\mathcal{A}}\ m(H)$.*

The following theorems are the characterizations for the freeness of an arrangement $\mathcal{A}$ by its Ziegler restriction.

Theorem 1.2.3. *[31, 32] For an arrangement $\mathcal{A}$ and a hyperplane H. Assume that $\theta_1,\ \theta_2,\dots,\ \theta_{\ell-1}\in\mathcal{D}(\mathcal{A})$ satisfy $\theta_i(\alpha_H)=0$. If $\theta_1|_H,\ \theta_2|_H,\dots,\ \theta_{\ell-1}|_H$ form a basis for the Ziegler restriction $(\mathcal{A}^H,\ m^H)$ of $\mathcal{A}$ onto H, then $\theta_E,\ \theta_1,\ \theta_2,\dots,\ \theta_{\ell-1}$ form a basis for $\mathcal{A}$ with exponents $(1,\ \deg\theta_1,\dots,\ \deg\theta_{\ell-1})$.*

Theorem 1.2.4. *[30] Assume that $\ell\geq 4$. Fix a hyperplane H of an arrangement $\mathcal{A}$. Then $\mathcal{A}$ is free with exponents $(1,c_2,c_3,\dots,c_\ell)$ if and only if the following conditions are satisfied.*

(1) The Ziegler restriction $(\mathcal{A}^H,\ m^H)$ is free with exponents $(c_2,c_3,\dots,c_\ell)$.

(2) The arrangement $\mathcal{A}$ is locally free along H, i.e., $\mathcal{A}_x:=\{H'\in\mathcal{A}\mid x\in H'\}$ is free for all $x\in H\setminus\{0\}$.

1.3 Reflection arrangements

Let us recall some definitions and basic properties of reflecion groups and reflection arrangements.

Definition 1.3.1. *[19] Let $G\subset GL(V)$ be a finite group. Let $\mathrm{Fix}(g)=\{v\in V\mid gv=v\}$ be the fixed subspace of $g\in G$. The action of G on V^* is defined by $(gx)(v)=x(g^{-1}v)$ for $g\in G,\ x\in V^*$. It has similar actions on S and $\mathrm{Der}(S)$. For $g\in G$, $f\in S$ and $\theta\in\mathrm{Der}(S)$, we have $(gf)(v)=f(g^{-1}v)$ and $(g\theta)(f)=g(\theta(g^{-1}f))$.*

Definition 1.3.2. *[19] An element $g\in GL(V)$ is a reflection if it has finite order and $\mathrm{Fix}(g)$ is a hyperplane H_g. Such a hyperplane H_g is called reflecting hyperplane of g. The finite group generated by refections is called reflection group. The set $\mathcal{A}=\mathcal{A}(G)$ of reflecting hyperplanes of reflection group G is called a refletion arrangement. The reflection arrangement $\mathcal{A}(G)$ is also called a Coxeter arrangement.*

It is well known that every reflection arrangement is free [19]. Terao [24] constructed bases for Coxeter multiarrangements with constant multiplicities. Wakamiko [25, 26] constructed the explicit bases for certain 2-dimensional Coxeter arrangements. More explicit bases for multiarrangements are expected to construct.

Example 1.3.3. *The Coxeter arrangement $\mathcal{A}_{\ell-1}$ of type A is defined by $\prod_{1\leq i<j\leq\ell}(x_i-x_j)$. It is also called a braid arrangement. The Coxeter arrangement $\mathcal{B}_\ell$ of type B is defined by $\prod_{1\leq i\leq\ell}x_i\prod_{1\leq i<j\leq\ell}(x_i^2-x_j^2)$. The Coxeter arrangement $\mathcal{D}_\ell$ of type D is defined by $\prod_{1\leq i<j\leq\ell}(x_i^2-x_j^2)$.*

Under the action of the reflecion group G, we have the invariant polynomials. As shown bellow, they form an invariant ring $S^G = \{f \mid gf = f \text{ for any } g \in G\}$.

Theorem 1.3.4. *[11] (Chevalley's Theorem) Let G be a reflection group of V over a field $\mathbb{K}$ of characteristic 0. The invariant ring S^G can be identified with $\mathbb{K}[f_1, f_2, \ldots, f_\ell]$ for some homogeneous polynomials $f_1, f_2, \ldots, f_\ell \in S^G$.*

1.4 Monomial groups and corresponding arrangements

Definition 1.4.1. *[19] Let $V = \oplus_{i=1}^{\ell} \mathbb{C}e_i$. Let $r \geq 2$ be an integer, $\theta = exp(2\pi i/r)$, and $C(r)$ be the cyclic group generated by θ. Let $\sigma \in S_\ell$ and $\epsilon : [\ell] = \{1, 2, \ldots, \ell\} \to C(r)$ be any function. Define the transformation $g(\sigma, \epsilon)$ by $g(\sigma, \epsilon)e_i = \epsilon(i)e_{\sigma(i)}$. The full monomial group $G = G(r, 1, \ell)$ is defined by*

$$G = \{g(\sigma, \epsilon) \mid \sigma \in S_\ell,\ \epsilon : [\ell] \to C(r)\}.$$

The full monomial group G is generated by reflections of two types. The first type is

$$\begin{cases} g(\sigma, \epsilon)e_i = \epsilon(i)e_i, \text{ for } \epsilon(i) \neq 1, \\ g(\sigma, \epsilon)e_j = e_j, \text{ for } i \neq j. \end{cases}$$

Its fixed subspace is hyperplane $H_i = \mathrm{Ker}(x_i)$. And $g(\sigma, \epsilon)^r = 1$. The order of reflections of this type is r. The second type is

$$\begin{cases} g(\sigma, \epsilon)e_i = \zeta^{-1}e_j,\ g(\sigma, \epsilon)e_j = \zeta e_i, \text{ for } i \neq j, \\ g(\sigma, \epsilon)e_k = e_k, \text{ for } k \neq i,\ k \neq j. \end{cases}$$

Its fixed subspace is hyperplane $H_{i,j}(\zeta) = \mathrm{Ker}(x_i - \zeta x_j)$. And $g(\sigma, \epsilon)^2 = 1$. The order of reflections of this type is 2. Then the defining polynomial of its corresponding arrangement is defined by

$$\prod_{1 \leq i \leq \ell} x_i \prod_{1 \leq i < j \leq \ell} (x_i^r - x_j^r).$$

Definition 1.4.2. *[19] Let p be a divisor of $r \geq 2$. The monomial group $G(r, p, \ell)$ is a subgroup of the full monomial group consisting of all transformations $g(\sigma, \epsilon)$ where $\prod \epsilon(i)$ is a power of θ^p.*

Note that $\prod \epsilon(i) \in C_{r/p}$. If $p < r$, the corresponding arrangement is the same as the arrangement of full monomial group $G(r, 1, \ell)$. For the generators $g(\sigma, \epsilon)$ of the first type, we have $g(\sigma, \epsilon)^{r/p} = 1$. The order of reflections of this case is r/p. If $p = r$, the corresponding arrangement consists of hyperplanes $H_{i,j}(\zeta)$ where $1 \leq i \neq j \leq \ell$. Then the defining polynomial of its corresponding arrangement is defined by

$$\prod_{1 \leq i < j \leq \ell} (x_i^r - x_j^r).$$

For $r = 2$, the group $G(2, 1, \ell)$ is the Coxeter group of type B and $G(2, 2, \ell)$ is the Coxeter group of type D.

We have known some free multiplicities for monomial groups given in [17].

Theorem 1.4.3. *[17] Let $m \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}$ and integers $\overline{m_i} \in [0, r-1]$ such that $m_i = r(m+k)+1+\overline{m_i} \geq 0$ for $i \in \{1, 2, \dots, \ell\}$. Then $q := \lfloor \frac{m_i - 1}{r} \rfloor = m+k$ does not depend on i. Set $a := (\ell - 1)r$, $m' = \sum m_i$ and $c := ma + qr + 1$.*

(1) The multiarrangements $(\mathcal{A}, \mu)$ with defining polynomial

$$Q(\mathcal{A},\ \mu) = \prod_{1 \leq i \leq \ell} x_i^{m_i} \prod_{1 \leq i < j \leq \ell} (x_i^r - x_j^r)^{2m+1}$$

is free with expoents

$$\begin{aligned}\exp(\mathcal{A},\ \mu) &= (c + m' - \ell(qr+1),\ c + r,\ c + 2r, \dots,\ c + (\ell - 1)r) \\ &= (mr\ell + rk + 1 + \sum \overline{m_i}, \\ &\quad r(k+1) + mr\ell + 1,\ r(k+2) + mr\ell + 1, \dots,\ r(k + \ell - 1) + mr\ell + 1).\end{aligned}$$

(2) The multiarrangements $(\mathcal{A},\ \mu')$ with defining polynomial

$$Q(\mathcal{A},\ \mu') = \prod_{1 \leq i \leq \ell} x_i^{m_i} \prod_{1 \leq i < j \leq \ell} (x_i^r - x_j^r)^{2m}$$

is free with expoents

$$\exp(\mathcal{A},\ \mu') = (ma + m_1, \dots,\ ma + m_\ell).$$

Note that if $m_i = 0$, we have $k = -m-1$ and $\overline{m_i} = r-1$. If $m_i = r(m+k)+1$, $\overline{m_i} = 0$.

This was proved in [17] by using a flat connection for the flat systems of invariants. In chapter 2, we give an explicit basis which gives a new proof of Theorem 1.4.3.

Chapter 2

Integral expressions for derivations

2.1 Bandlow-Musiker's integral formula and its generalization

Recall that $\mathcal{A}_{\ell-1}$ is the Coxeter arrangement of type A. Here let $m : \mathcal{A}_{\ell-1} \to \mathbb{Z}_{>0}$ be a constant multiplicity $m(H) = 2m+1$. Set $\theta_0 = \partial_1 + \partial_2 + \cdots + \partial_\ell$. Take $g(t) = \prod_{1\le i\le \ell}(t - x_i)$. For $k, m \ge 0$, we define the following vector field which is called Bandlow-Musiker's integral formula

$$\eta_k^m = \sum_{i,j=1}^{\ell} \int_{x_i}^{x_j} t^k g(t)^m dt \partial_i. \tag{2.1}$$

In [22], we have $\eta_k^m(x_i - x_j) = \ell \int_{x_i}^{x_j} t^k g(t)^m dt$.

Theorem 2.1.1. *[8, 22] For $k, m \ge 0$, $\eta_k^m(x_i - x_j)$ is divisible by $(x_i - x_j)^{2m+1}$. Furthermore, the vector fields $\eta_0^m, \eta_1^m, \dots \eta_{\ell-2}^m, \theta_0$ form a basis for $\mathcal{D}(\mathcal{A}_{\ell-1},\ m)$.*

Define the linear operator $\int^{x_i} dt$ by $\int^{x_i} t^k dt = \frac{x_i^{k+1}}{k+1}$ for $k \ne -1$. When $k \ge 0$, it is equivalent to the definite integral $\int_0^{x_i} dt$. When $k < -1$, $\int^{x_i} dt = \int_\infty^{x_i} dt$.

For $r \ge 2,\ m \ge 0,\ n \in \mathbb{Z},\ 1 \ne \zeta \in C_r$, we have

$$\begin{aligned}\int^x t^{rn}(t^r - x^r)^m dt &= (1-\zeta)^{-1} \int_{\zeta x}^x t^{rn}(t^r - x^r)^m dt \\ &\doteq \left(\int_\zeta^1 t^{rn}(t^r - 1)^m dt\right) x^{r(n+m)+1}.\end{aligned}$$

Proposition 2.1.2. *Let r, m, n, ζ be as above. Then $\int_\zeta^1 t^{rn}(t^r - 1)^m dt \ne 0$.*

Proof. We have

$$\begin{aligned}\int_\zeta^1 t^{rn}(t^r - 1)^m dt &\doteq \int_\zeta^1 (t^r - 1)^m dt^{rn+1} \\ &= (t^r - 1)^m t^{rn+1}|_\zeta^1 - \int_\zeta^1 t^{rn+1} d(t^r - 1)^m \\ &\doteq \int_\zeta^1 t^{rn+r}(t^r - 1)^{m-1} dt.\end{aligned}$$

By using the integral by parts and induction on m, we can prove the integral $\int_\zeta^1 t^{rn}(t^r - 1)^m dt$ is nonzero. $\square$

The above proposition implies that $\int^x t^{rn}(t^r - x^r)^m dt$ is a nonzero multiple of $x^{r(n+m)+1}$. Hence $\deg(\int^x t^{rn}(t^r - x^r)^m dt) = r(n+m)+1$.

The Bandlow-Musiker's integral formula (2.1) can be generalized by changing the integrand. Let $\ell \geq 2$. Let $a_1, a_2, \dots, a_\ell, b \in \mathbb{Z}_{\geq 0}$. We define the vector field $\theta_{a_1,\dots,a_\ell,b}$ as

$$\theta_{a_1,\dots,a_\ell,b} := \sum_{i=1}^{\ell} \left(\int^{x_i} t^b (t-x_1)^{a_1} \dots (t-x_\ell)^{a_\ell} dt \right) \partial_i. \tag{2.2}$$

Note that $\deg \theta_{a_1,\dots,a_\ell,b} = \sum_{i=1}^{\ell} a_i + b + 1$. The vector field $\theta_{a_1,\dots,a_\ell,b}$ has very nice properties as follows.

Proposition 2.1.3. *Let $(\mathcal{A},\ m)$ be the multiarrangement defined by*

$$Q(\mathcal{A},\ m) = \prod_{i=1}^{\ell} x_i^{a_i+b+1} \cdot \prod_{1 \leq i < j \leq \ell} (x_i - x_j)^{a_i+a_j+1}.$$

Then the vector field $\theta_{a_1,\dots,a_\ell,b}$ is contained in $\mathcal{D}(\mathcal{A},\ m)$.

Proof. It is clear that $\theta_{a_1,\dots,a_\ell,b}(x_i) = \int_0^{x_i} t^b (t-x_1)^{a_1} \dots (t-x_\ell)^{a_\ell} dt$. The integrand is divisible by $t^b(t-x_i)^{a_i}$, then $\theta_{a_1,\dots,a_\ell,b}(x_i)$ is divisible by $x_i^{a_i+b+1}$. For $i < j$, we have $\theta_{a_1,\dots,a_\ell,b}(x_i - x_j) = \int_{x_j}^{x_i} t^b (t-x_1)^{a_1} \dots (t-x_\ell)^{a_\ell} dt$. After setting $t' = t - x_j$, the integral is $\int_0^{x_i - x_j} dt'$ where the integrand is divisible by $(t')^{a_j}(t' - (x_i - x_j))^{a_i}$. Then $\theta_{a_1,\dots,a_\ell,b}(x_i - x_j)$ is divisible by $(x_i - x_j)^{a_i+a_j+1}$. $\square$

We want to use the generalized integral expressions to construct a basis for certain multiarrangements. To prove a set of vector fields form a basis by Therorem 1.2.2, it is equivalent to show the sum of their degrees is equal to the sum of multiplicites and the determinant of their Saito matrix is not 0. Recall that when we prove the Vandermonde determinant is not equal to 0, we only need to show that there is a non-zero coefficient for a certain monomial. The expansion of the determinant of the following Vandermonde matrix

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{\ell-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{\ell-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_\ell & x_\ell^2 & \dots & x_\ell^{\ell-1} \end{pmatrix}$$

has a unique monomial $x_1^{\ell-1} x_2^{\ell-2} \dots x_{\ell-1}$. We introduce the lexicographic ordering among monomials. That is, $x_1^{a_1} x_2^{a_2} \dots x_\ell^{a_\ell} > x_1^{b_1} x_2^{b_2} \dots x_\ell^{b_\ell}$ if and only if $a_1 = b_1,\ a_2 = b_2,\ \dots,\ a_{i-1} = b_{i-1}$ and $a_i > b_i$ for some i. Then the unique monomial $x_1^{\ell-1} x_2^{\ell-2} \dots x_{\ell-1}$ is the leading monomial in the Vandermonde determinant.

Note that $\int_0^x t^b (t-x)^a dt = (\int_0^1 t^b (t-1)^a dt) x^{a+b+1} \neq 0$ for $a,\ b \in \mathbb{Z}_{\geq 0}$. By comparing the leading monomial of each entry of the Saito matrix, it is not difficult to show the following lemma.

Lemma 2.1.4. *Let $b_1, \dots, b_\ell \in \mathbb{Z}_{\geq 0}$ with $b_1 < b_2 < \dots < b_\ell$. Then the ℓ vector fields $\theta_{a_1,\dots,a_\ell,b_1}, \theta_{a_1,\dots,a_\ell,b_2}, \dots, \theta_{a_1,\dots,a_\ell,b_\ell}$ are linear independent over S.*

Proof. Observe the following matrix where each entry is the non-zero leading monomial of Saito matrix up to a constant,

$$M = \begin{pmatrix} x_1^{a_1+\cdots+a_\ell+b_1+1} & x_1^{a_1}x_2^{a_2+\cdots+a_\ell+b_1+1} & \cdots & x_1^{a_1}\dots x_{\ell-1}^{a_{\ell-1}}x_\ell^{a_\ell+b_1+1} \\ x_1^{a_1+\cdots+a_\ell+b_2+1} & x_1^{a_1}x_2^{a_2+\cdots+a_\ell+b_2+1} & \cdots & x_1^{a_1}\dots x_{\ell-1}^{a_{\ell-1}}x_\ell^{a_\ell+b_2+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{a_1+\cdots+a_\ell+b_\ell+1} & x_1^{a_1}x_2^{a_2+\cdots+a_\ell+b_\ell+1} & \cdots & x_1^{a_1}\dots x_{\ell-1}^{a_{\ell-1}}x_\ell^{a_\ell+b_\ell+1} \end{pmatrix}.$$

The monomial of the expansion of $\det(M)$ has the form $x_1^{c_1}x_2^{c_2}\dots x_\ell^{c_\ell}$. And c_1 is at most $\ell a_1+a_2+\cdots+a_\ell+b_\ell+1$. It requires the entry $x_1^{a_1+a_2+\cdots+a_\ell+b_\ell+1}$. Then c_2 is at most $(\ell-1)a_2+a_3+\cdots+a_\ell+b_{\ell-1}+1$. It requires the entry $x_1^{a_1}x_2^{a_2+\cdots+a_\ell+b_{\ell-1}+1}$. By similar consideration, the product of anti-diagonal entries gives the unique maximum monomial

$$x_1^{\ell a_1+a_2+\cdots+a_\ell+b_\ell+1}x_2^{(\ell-1)a_2+a_3+\cdots+a_\ell+b_{\ell-1}+1}\dots x_\ell^{a_\ell+b_1+1}.$$

Hence the determinant of the Saito matrix is nonzero, and $\theta_{a_1,\dots,a_\ell,b_1},\theta_{a_1,\dots,a_\ell,b_2},\dots,\theta_{a_1,\dots,a_\ell,b_\ell}$ are linear independent over S. $\square$

By Theorem 1.2.2, Propositive 2.1.3, and the above lemma, we can choose a set of vector fields $\theta_{a_1,\dots,a_\ell,b}$ with mutually distinct b to construct a basis for certain multiarrangements.

Proposition 2.1.5. *The vector fields $\theta_{a_1,\dots,a_\ell,b}$, $\theta_{a_1,\dots,a_\ell,b+1}$, $\dots$, $\theta_{a_1,\dots,a_\ell,b+\ell-1}$ form a basis for multiarrangements $(\mathcal{A},\ m)$ defined in Proposition 2.1.3.*

Proof. It is clear that they are contained in $\mathcal{D}(\mathcal{A},\ m)$ and linearly independent over S. Recall that $\deg\theta_{a_1,\dots,a_\ell,b}=\sum_{i=1}^{\ell}a_i+b+1$. Then $\sum_{i=0}^{\ell-1}\deg\theta_{a_1,\dots,a_\ell,b+i}=(\sum_{i=1}^{\ell}a_i)\ell+b\ell+\ell(\ell+1)/2=|m|$. Hence they form a basis for $(\mathcal{A},\ m)$. $\square$

In [4], Abe, Nuida, and Numata gave some free multiplicities on braid arrangements. As part of their results, the following multiarrangement is free. By similar consideration, we can construct a basis for it.

Proposition 2.1.6. *Let $(\mathcal{A},\ m)$ be the multiarrangement defined by*

$$Q(\mathcal{A},\ m)=\prod_{1\le i<j\le\ell}(x_i-x_j)^{a_i+a_j+1}.$$

Then the vector fields $\theta_{a_1,\dots,a_\ell,b}$ for $b=0,\dots,\ell-2$ together with $\theta_0=\sum_{i=1}^{\ell}\partial_i$ form a basis for $(\mathcal{A},\ m)$.

Proof. It is clear that they are in $\mathcal{D}(\mathcal{A},\ m)$ and linearly independent over S. We have $\deg\theta_0+\sum_{i=0}^{\ell-2}\deg\theta_{a_1,\dots,a_\ell,i}=(\sum_{i=1}^{\ell}a_i)(\ell-1)+\ell(\ell-1)/2=|m|$. Hence they form a basis for $(\mathcal{A},\ m)$. $\square$

2.2 Integral expressions for $(\mathcal{A},\ \mu)$

By integral expressions, we can construct a basis for free multiarrangements defined in Theorem 1.4.3. In this section, we construct a basis for Theorem

1.4.3 (1). Let $\lambda(t) = \prod_{i=1}^{\ell}(t^r - x_i^r)$ for $r \geq 2$. For $m \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{Z}$, define the vector field η_s^m by

$$\eta_s^m = \sum_{i=1}^{\ell} \left(\int^{x_i} t^{rs} \lambda(t)^m dt \right) \partial_i.$$

It is easy to see that if s is small enough, η_s^m is not a polynomial vector field. By simple caculation, we have the following.

Proposition 2.2.1. *(1) If $s \geq -m$, then η_s^m is a polynomial vector field.*

(2) Let $m,\ k,\ r,\ \overline{m_i}$ be as shown in Theorem 1.4.3. The vector field $\left(\prod_{i=1}^{\ell} x_i^{\overline{m_i}}\right) \eta_k^m$ is a polynomial vector field

Proof. (1) We have

$$\begin{aligned}
\eta_s^m(x_i) &= \int^{x_i} t^{rs} \lambda(t)^m dt \\
&= \sum_{a=0}^{m(\ell-1)} \sum_{b=0}^{m} f_{a,b} t^{r(s+a+b)+1} x_i^{r(m-b)} |_{t=x_i} \\
&= \sum_{a=0}^{m(\ell-1)} \sum_{b=0}^{m} f_{a,b} x_i^{r(m+s+a)+1} \\
&\in x_i^{r(m+s)+1} S,
\end{aligned}$$

where $f_{a,b} \in \mathbb{C}[x_1, \dots, \hat{x_i}, \dots, x_\ell]$ for $s \geq -m$. Hence $\eta_s^m(x_i)$ is a polynomial. And deg $\eta_s^m = mr\ell + rs + 1$.

(2) By similar argument, we have $\left(\prod_{i=1}^{\ell} x_i^{\overline{m_i}}\right) \eta_k^m(x_i) \in x_i^{r(m+k)+1+\overline{m_i}} S = x^{m_i} S$. Hence it is a polynomial. And deg $\left(\prod_{i=1}^{\ell} x_i^{\overline{m_i}}\right) \eta_k^m = mr\ell + rk + 1 + \sum_{i=1}^{\ell} \overline{m_i}$. ☐

Proposition 2.2.2. *Let $m,\ k,\ r,\ \overline{m_i}$ be as shown in Theorem 1.4.3. Then the vector fields $\left(\prod_{i=1}^{\ell} x_i^{\overline{m_i}}\right) \eta_k^m,\ \eta_s^m$ are in $\mathcal{D}(\mathcal{A},\ \mu)$ for $s \geq k+1$.*

Proof. By the Proposition 2.2.1, they are polynomial vector fields, $\eta_s^m(x_i) \in x_i^{r(m+s)+1} S$, and $\left(\prod_{i=1}^{\ell} x_i^{\overline{m_i}}\right) \eta_k^m(x_i) \in x_i^{r(m+k)+1+\overline{m_i}} S = x^{m_i} S$. If $s \geq k+1$, $\eta_s^m(x_i) \in x_i^{m_i} S = x_i^{r(m+k)+\overline{m_i}+1}$.

For $\zeta \in C_r$ and $i \neq j$, we want to show $\eta_s^m(x_i - \zeta x_j) \in (x_i - \zeta x_j)^{2m+1} S$. Note that $\zeta^r = 1$. We have

$$\begin{aligned}
\eta_s^m(x_i - \zeta x_j) &= \int^{x_i} t^{rs} \lambda(t)^m dt - \zeta \int^{x_j} t^{rs} \lambda(t)^m dt \\
&= \int^{x_i} t^{ru} \lambda(t)^m dt - \zeta \int^{\zeta x_j} (t')^{ru} \lambda(t')^m \zeta^{-1} dt' \\
&= \int_{\zeta x_j}^{x_i} t^{ru} \lambda(t)^m dt.
\end{aligned}$$

Since the integrand is divisible by $(t-x_i)^m(t-\zeta x_j)^m$, and $\eta_s^m(x_i-\zeta x_j)$ is a polynomial, it follows that $\eta_u^m(x_i-\zeta x_j)$ is divisible by $(x_i-\zeta x_j)^{2m+1}$. Similarly $\left(\prod_{i=1}^{\ell} x_i^{\overline{m}_i}\right)\eta_k^m(x_i-\zeta x_j)\in(x_i-\zeta x_j)^{2m+1}S$ □

Theorem 2.2.3. *Let $m,\ k,\ r,\ \overline{m_i}$ be as shown in Theorem 1.4.3. Then the vector fields*

$$\left(\prod_{i=1}^{\ell} x_i^{\overline{m}_i}\right)\eta_k^m,\quad \eta_{k+1}^m,\quad \eta_{k+2}^m,\quad \ldots,\quad \eta_{k+\ell-1}^m$$

form a basis of $\mathcal{D}(\mathcal{A},\ \mu)$.

Proof. By the similar discussion in Lemma 2.1.4, it is easily seen that they are linearly independent over S. Note that (deg $\left(\prod_{i=1}^{\ell} x_i^{\overline{m}_i}\right)\eta_k^m$, deg $\eta_{k+1}^m,\ldots,$ deg $\eta_{k+\ell-1}^m) =$ $\exp(\mathcal{A},\mu)$. By the Theorem 1.2.2, and the Proposition 2.2.1, 2.2.2, it is clear that they form a basis.

□

Example 2.2.4. *Let $r=2$, $m\geq 0$, and $m_i=2m+1$ for $i\in[\ell]$. Then the multiarrangement $(\mathcal{A},\ \mu)$ is $(\mathcal{B}_\ell,\ 2m+1)$ defined by*

$$\prod_{1\leq i\leq \ell} x_i^{2m+1}\cdot\prod_{1\leq i<j\leq\ell}(x_i^2-x_j^2)^{2m+1}.$$

The vector fields

$$\eta_s^m=\sum_{j=1}^{\ell}\left(\int^{x_j} t^{2s}\prod_{i=1}^{\ell}(t^2-x_i^2)^m dt\right)\partial_j$$

$(0\leq s\leq \ell-1)$ *form a basis for $\mathcal{D}(\mathcal{B}_\ell,\ 2m+1)$.*

Example 2.2.5. *Let $m\geq 0$, and $m_i=rm+1$ for $i\in[\ell]$. Then the multiarrangement $(\mathcal{A},\ \mu)$ is defined by*

$$\prod_{1\leq i\leq \ell} x_i^{rm+1}\cdot\prod_{1\leq i<j\leq\ell}(x_i^r-x_j^r)^{2m+1}.$$

The vector fields

$$\eta_s^m=\sum_{j=1}^{\ell}\left(\int^{x_j} t^{rs}\prod_{i=1}^{\ell}(t^r-x_i^r)^m dt\right)\partial_j$$

$(0\leq s\leq \ell-1)$ *form a basis for $\mathcal{D}(\mathcal{A},\ \mu)$.*

Example 2.2.6. *Let $m_i=0$, $k=-m-1$, and $\overline{m_i}=r-1$. Then the multiarrangement $(\mathcal{A},\ \mu)$ is defined by*

$$\prod_{1\leq i<j\leq\ell}(x_i^r-x_j^r)^{2m+1}.$$

The vector fields

$$\left(\prod_{i=1}^{\ell} x_i^{r-1}\right)\eta_{-m-1}^m = \left(\prod_{i=1}^{\ell} x_i^{r-1}\right)\sum_{j=1}^{\ell}\left(\int^{x_j} t^{-r(m+1)}\prod_{i=1}^{\ell}(t^r - x_i^r)^m dt\right)\partial_j$$

and

$$\eta_s^m = \sum_{i=1}^{\ell}\left(\int^{x_i} t^{rs}\prod_{j=1}^{\ell}(t^r - x_j^r)^m dt\right)\partial_i,$$

$(-m \le s \le -m+\ell-2)$ *form a basis for* $\mathcal{D}(\mathcal{A},\ \mu)$.

2.3 Integral expressions for $(\mathcal{A},\ \mu')$

This section, we construct a basis for Theorem 1.4.3 (2) by integral expressions. Let $\lambda_i(t) = \frac{\lambda(t)}{t^r - x_i^r}$.

For $m \in \mathbb{Z}_{>0}$ and $1 \le i \le \ell$, define the vector field σ_i^m by

$$\sigma_i^m = x_i^{\overline{m_i}}\sum_{j=1}^{\ell}\left(\int^{x_j} t^{r(k+1)}\lambda(t)^{m-1}\lambda_i(t)dt\right)\partial_j.$$

Proposition 2.3.1. *Let* $m,\ k,\ r,\ \overline{m_i}$ *be as shown in Theorem 1.4.3. Then* σ_i^m *is a polynomial vector field.*

Proof. It is similar to the proof of Proposition 2.2.1. For $i \ne j$, we have $\sigma_i^m(x_j) \in x_j^{r(m+k+1)+1}S$. And $\sigma_i^m(x_i) \in x_i^{r(m+k)+1+\overline{m_i}}S$. It is clear that $\deg \sigma_i^m = mr\ell + rk + 1 + \overline{m_i} = m(\ell-1)r + m_i$. □

Proposition 2.3.2. *Let* $m,\ k,\ r,\ \overline{m_i}$ *be as shown in Theorem 1.4.3. Then the vector field* $\sigma_i^m \in \mathcal{D}(\mathcal{A},\mu')$.

Proof. From the above proposition, we only need to show $\sigma_i^m(x_p - \zeta x_q) = x_i^{\overline{m_i}}\int_{\zeta x_q}^{x_p} t^{r(k+1)}\lambda(t)^{m-1}\lambda_i(t)dt \in (x_p - \zeta x_q)^{2m}S$ for $\zeta \in C_r$ and $p \ne q$.

If $i \notin \{p,q\}$, since the integrand is divisible by $(t^r - x_p^r)^m(t^r - x_q^r)^m$, we have $\eta_i(x_p - \zeta x_q) \in (x_p - \zeta x_q)^{2m+1}S$.

If $i \in \{p,q\}$, since the integrand is divisible by $(t^r - x_p^r)^{m-1}(t^r - x_q^r)^m$ or $(t^r - x_p^r)^m(t^r - x_q^r)^{m-1}$, we have $\eta_i(x_p - \zeta x_q) \in (x_p - \zeta x_q)^{2m}S$. □

Theorem 2.3.3. *Let* $m,\ k,\ r,\ \overline{m_i}$ *be as shown in Theorem 1.4.3. Then the vector fields*

$$\sigma_1^m,\ \sigma_2^m,\ \ldots,\ \sigma_\ell^m$$

form a basis of $\mathcal{D}(\mathcal{A},\ \mu')$.

Proof. Note that $(\deg\ \sigma_1^m,\ \deg\ \sigma_2^m, \ldots,\ \deg\ \sigma_\ell^m) = \exp(\mathcal{A},\mu')$. By the Theorem 1.2.2, the Proposition 2.3.1, 2.3.2, it remains to show $\sigma_1^m,\ \sigma_2^m, \ldots,\ \sigma_\ell^m$ are linearly independent over S. Observe the following matrix where each entry is the non-zero leading monomial of Saito matrix up to a constant,

$$M = \begin{pmatrix} x_1^{rk+\ell rm+1} & x_1^{r(m-1)}x_2^{r(k+1)+rm(\ell-1)+1} & x_1^{r(m-1)}x_2^{rm}x_3^{r(k+1)+rm(\ell-2)+1} & \cdots \\ x_1^{rk+\ell rm+1} & x_1^{rm}x_2^{rk+rm(\ell-1)+1} & x_1^{rm}x_2^{r(m-1)}x_3^{r(k+1)+rm(\ell-2)+1} & \cdots \\ x_1^{rk+\ell rm+1} & x_1^{rm}x_2^{rk+rm(\ell-1)+1} & x_1^{rm}x_2^{rm}x_3^{rk+rm(\ell-2)+1} & \cdots \\ x_1^{rk+\ell rm+1} & x_1^{rm}x_2^{rk+rm(\ell-1)+1} & x_1^{rm}x_2^{rm}x_3^{rk+rm(\ell-2)+1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that each diagonal monomial $m_{i,i}$ is the maximum monomial in each column, and $m_{i,i} > m_{j,i}$ for $i > j$. Then the product of diagonal entries gives the unique maximum monomial

$$x_1^{r(k-m)+2rm\ell+1} x_2^{r(k-3m)+2rm\ell+1} \dots x_\ell^{r(k+m)+1}.$$

Hence $\sigma_1^m,\ \sigma_2^m, \dots,\ \sigma_\ell^m$ are linearly independent over S. $\square$

Example 2.3.4. *Let $m_i = rm$ for $i = 1, 2, \dots, \ell$. Then $k = -1$ and $\overline{m_i} = r-1$. The multiarrangement $(\mathcal{A},\ \mu')$ is defined by*

$$\prod_{1\le i\le \ell} x_i^{rm} \prod_{1\le i<j\le \ell} (x_i^r - x_j^r)^{2m}.$$

Then the vector fields

$$\sigma_i^m = x_i^{r-1} \sum_{j=1}^{\ell} \left(\int^{x_j} \lambda(t)^{m-1} \lambda_i(t) dt \right) \partial_j$$

$(i = 1, 2, \dots, \ell)$ form a basis for $\mathcal{D}(\mathcal{A},\ \mu')$.

Example 2.3.5. *Let $m_i = 0$ for $i = 1, 2, \dots, \ell$. Then $k = -m-1$ and $\overline{m_i} = r-1$. The multiarrangement $(\mathcal{A},\ \mu')$ is defined by*

$$\prod_{1\le i<j\le \ell} (x_i^r - x_j^r)^{2m}.$$

Then the vector fields

$$\sigma_i^m = x_i^{r-1} \sum_{j=1}^{\ell} \left(\int^{x_j} t^{-rm} \lambda(t)^{m-1} \lambda_i(t) dt \right) \partial_j$$

$(i = 1, 2, \dots, \ell)$ form a basis for $\mathcal{D}(\mathcal{A},\ \mu')$.

Chapter 3

Primitive derivations

3.1 Invariant rings for monomial groups

Let $\mathbb{K} = \mathbb{C}$ and $G = G(r,p,\ell)$. For $1 \leq j \leq \ell$, define the elementary symmetric polynomials $e_j(x_1, x_2, \ldots, x_\ell)$ by

$$e_j(x_1, x_2, \ldots, x_\ell) = \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq \ell} x_{i_1} x_{i_2} \ldots x_{i_j}.$$

Set

$$\begin{cases} P_j = (-1)^j e_j(x_1^r, x_2^r, \ldots, x_\ell^r), \text{ for } 1 \leq j \leq \ell - 1, \\ P_\ell = (x_1 x_2 \ldots x_\ell)^{r/p}. \end{cases}$$

We have $P_\ell^p = e_\ell(x_1^r, x_2^r, \ldots, x_\ell^r) = (x_1 x_2 \ldots x_\ell)^r$. Then $P_1, P_2, \ldots, P_\ell$ are the basic invariant polynomials of G by [9]. We obtain

$$\mathbb{C}[x_1, x_2, \ldots, x_\ell]^G = \mathbb{C}[P_1, P_2, \ldots, P_\ell].$$

Recall that the Jacobian of the basic invariant polynomials is

$$\det\left(\frac{\partial P_i}{\partial x_j}\right) \doteq (x_1 x_2 \ldots x_\ell)^{r/p-1} \prod_{1 \leq i < j \leq \ell} (x_i^r - x_j^r).$$

Moreover, $P_1, P_2, \ldots, P_\ell$ can be thought of as the coordinate system of the quotient space $\mathbb{C}^\ell/G$. Hence, we can define the vector field $\frac{\partial}{\partial P_1}, \frac{\partial}{\partial P_2}, \ldots, \frac{\partial}{\partial P_\ell}$ by

$$\begin{cases} \frac{\partial}{\partial P_i}(P_j) = 1 \text{ for } i = j, \\ \frac{\partial}{\partial P_i}(P_j) = 0 \text{ otherwise.} \end{cases}$$

Each $\frac{\partial}{\partial P_i}$ has the following form.

$$\frac{\partial}{\partial P_i} = (-1)^{\ell-i} \frac{1}{Q} \det \begin{pmatrix} \frac{\partial P_1}{\partial x_1} & \frac{\partial P_2}{\partial x_1} & \cdots & \frac{\partial P_{i-1}}{\partial x_1} & \frac{\partial P_{i+1}}{\partial x_1} & \cdots & \frac{\partial P_\ell}{\partial x_1} & \frac{\partial}{\partial x_1} \\ \frac{\partial P_1}{\partial x_2} & \frac{\partial P_2}{\partial x_2} & \cdots & \frac{\partial P_{i-1}}{\partial x_2} & \frac{\partial P_{i+1}}{\partial x_2} & \cdots & \frac{\partial P_\ell}{\partial x_2} & \frac{\partial}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial P_1}{\partial x_\ell} & \frac{\partial P_2}{\partial x_\ell} & \cdots & \frac{\partial P_{i-1}}{\partial x_\ell} & \frac{\partial P_{i+1}}{\partial x_\ell} & \cdots & \frac{\partial P_\ell}{\partial x_\ell} & \frac{\partial}{\partial x_\ell} \end{pmatrix},$$

where Q is the Jacobian as shown above.

3.2 Primitive derivations

For vector fields θ_1, θ_2 with $\theta_2 = \sum_{i=1}^{\ell} f_i \partial_i$, we define an affine connection $\nabla : \mathrm{Der}(S) \times \mathrm{Der}(S) \to \mathrm{Der}(S)$ by

$$\nabla_{\theta_1} \theta_2 = \sum_{i=1}^{\ell} (\theta_1(f_i)) \partial_i.$$

For $G(r, p, \ell)$, recall the section 1.4 that we have the order of the reflection corresponding to the reflection hyperplane. Define the order multiplicity ω by assigning the order of the reflection to each reflecting hyperplane. If $p < r$, the reflection arrangement $\mathcal{A}(G(r, p, \ell))$ consists of H_i and $H_{i,j}(\zeta)$ for $1 \leq i \leq \ell$, $1 \leq i < j \leq \ell$, and $\zeta \in C(r)$. We have the order multiplicity

$$\begin{cases} \omega(H_i) = \frac{r}{p}, \\ \omega(H_{i,j}(\zeta)) = 2. \end{cases}$$

If $p = r$, the reflection arrangement $\mathcal{A}(G(r, r, \ell))$ consists of $H_{i,j}(\zeta)$ for $1 \leq i < j \leq \ell$, and $\zeta \in C(r)$. We have the order multiplicity

$$\omega(H_{i,j}(\zeta)) = 2.$$

We say a complex reflection group is well-generated if it can be generated by ℓ reflections. For an irreducible well-generated unitary reflection group W, let $F_1, F_2, \ldots, F_\ell$ in $\mathbb{C}[x_1, x_2, \ldots, x_\ell]$ be the homogeneous basic invariant polynomials for W. Among $F_1, F_2, \ldots, F_\ell$, we have the unique polynomial F_ℓ with the largest degree. Similarly, define the vector field $D = \frac{\partial}{\partial F_\ell}$. Call D the primitive derivation.

It is well-known that the group $G(r, p, \ell)$ is well-generated if and only if $p \in \{1, r\}$. In this case, we have the following primitive derivation

$$D = \begin{cases} \frac{\partial}{\partial P_\ell} \text{ for } p = 1, \\ \frac{\partial}{\partial P_{\ell-1}} \text{ for } p = r. \end{cases}$$

In [17], Hoge, Mano, Röhrle, and Stump used the primitive derivation to give some free multiplicities on the reflection arrangement $\mathcal{A}$ related to W by constructing a basis for these free multiarrangements. They generalized Yoshinaga's results and gave the following theorem. They generalized the results in [29] to W.

Theorem 3.2.1. *[17] Let W be an irreducible, well-generated unitary refection group with reflection arrangement $\mathcal{A}$. Define the order multiplicity $\omega : \mathcal{A} \to \mathbb{Z}_{>0}$ by assigning the order of reflection to each reflecting hyperplane. Suppose that $\tau : \mathcal{A} \to \{0, 1\}$ such that $\mathcal{D}(\mathcal{A},\ \tau)$ is free with a homogeneous basis $\theta_1, \ldots, \theta_\ell$. Then $\mathcal{D}(\mathcal{A},\ m\omega + \tau)$ is free with a basis $\nabla_{\theta_1} \nabla_D^{-m} \theta_E, \ldots, \nabla_{\theta_\ell} \nabla_D^{-m} \theta_E$, for $m \geq 0$.*

Given a basis for $\mathcal{D}(\mathcal{A})$, we can use the above theorem to construct a basis for $\mathcal{D}(\mathcal{A},\ m\omega + 1)$. The affine connection ∇_D also induces an isomorphism of T-modules

$$\nabla_D : D(\mathcal{A},\ (m+1)\omega + \tau)^W \xrightarrow{\simeq} D(\mathcal{A},\ m\omega + \tau)^W,$$

where $T = \{F \in \mathbb{C}[x_1, x_2, \ldots, x_\ell]^W \mid D(F) = 0\}$. For $m \geq 0$, we have known that $D(\mathcal{A},\ m\omega + \tau)^W$ is a free module over $\mathbb{C}[x_1, x_2, \ldots, x_\ell]^W$. The basis of $\mathcal{D}(\mathcal{A},\ m\omega + \tau)^W$ also spans $\mathcal{D}(\mathcal{A},\ m\omega + \tau)$ as S-modules. Hence ∇_D sends a basis of $\mathcal{D}(\mathcal{A},\ (m+1)\omega + \tau)^W$ to $\mathcal{D}(\mathcal{A},\ m\omega + \tau)^W$, which forms a basis of $\mathcal{D}(\mathcal{A},\ m\omega + \tau)$. In general, it is hard to give an explicit description of $\nabla_D^{-1}\theta$ for $\theta \in \mathcal{D}(\mathcal{A},\ m\omega + \tau)$. However, the integral expressions can apply the affine connection ∇_D very well.

Recall that

$$\lambda(t) = \prod_{1 \leq i \leq \ell} (t^r - x_i^r) = t^{r\ell} + P_1 t^{r(\ell-1)} + \cdots + P_{\ell-1} t^r + (-1)^\ell P_\ell^p.$$

We have

$$\frac{\partial \lambda(t)}{\partial P_i} \doteq \begin{cases} t^{r(\ell-i)} \text{ for } 1 \leq i \leq \ell - 1, \\ P_\ell^{p-1} \text{ for } i = \ell. \end{cases}$$

Note that the vector field η_s^m defined in Section 2.2

$$\begin{aligned} \eta_s^m &= \sum_{i=1}^{\ell} \left(\int^{x_i} t^{rs} \lambda(t)^m dt \right) \partial_i \\ &= \sum_{i=1}^{\ell} \left(\int^{x_i} t^{rs} (t^{r\ell} + P_1 t^{r(\ell-1)} + \cdots + P_{\ell-1} t^r + (-1)^\ell P_\ell^p)^m dt \right) \partial_i. \end{aligned}$$

If $D = \frac{\partial}{\partial P_\ell}$ and $p = 1$, we have

$$D(\lambda(t)^m) = (-1)^\ell m \lambda(t)^{m-1},$$

and

$$\nabla_D \eta_s^m = (-1)^\ell m \eta_s^{m-1}.$$

If $D = \frac{\partial}{\partial P_{\ell-1}}$, we have

$$D(\lambda(t)^m) = m t^r \lambda(t)^{m-1},$$

and

$$\nabla_D \eta_s^m = m \eta_{s+1}^{m-1}.$$

Example 3.2.2. *Consider the multiarrangement $(\mathcal{A},\ \mu)$ as shown in Example 2.2.5. It corresponds to the group $G(r, 1, \ell)$. We have $D = \frac{\partial}{\partial P_\ell}$. When $m = 0$, $\eta_s^0 \doteq \sum_{i=1}^{\ell} x_i^{rs+1} \partial_i$. It is known in [19] that the vector fields $\eta_0^0, \eta_1^0, \ldots, \eta_{\ell-1}^0$ form a basis for the arrangement $\mathcal{A}$ defined by*

$$\prod_{1 \leq i \leq \ell} x_i \cdot \prod_{1 \leq i < j \leq \ell} (x_i^r - x_j^r).$$

For $m > 0$, we have $\nabla_D^m \eta_s^m = (-1)^{m\ell} m!\ \eta_s^0$. Clearly,

$$\eta_s^m \doteq \nabla_D^{-m} \eta_s^0$$

$(0 \leq s \leq \ell - 1)$ form a basis for $(\mathcal{A},\ \mu)$.

Example 3.2.3. *Consider the multiarrangement $(\mathcal{A},\ \mu')$ as shown in Example 2.3.4. It corresponds to the group $G(r,1,\ell)$. We have $D = \frac{\partial}{\partial P_\ell}$. Note that*

$$\eta_0^0 = \sum_{j=1}^{\ell} \int^{x_j} dt\partial_j = \theta_E.$$

And $\nabla_D \eta_s^m = (-1)^\ell m \eta_s^{m-1}$. Then $\nabla_D^{-m}\eta_0^0 \doteq \eta_0^m$. For $1 \le i \le \ell$, we have

$$\nabla_{\partial_i}\eta_0^m = -rmx_i^{r-1}\sum_{j=1}^{\ell}\int^{x_j}\lambda(t)^m/(t^r - x_i^r)dt\partial_j \doteq \sigma_i^m,$$

where σ_i^m is defined in Example 2.3.4. Clearly,

$$\sigma_i^m \doteq \nabla_{\partial_i}\nabla_D^{-m}\theta_E$$

$(1 \le i \le \ell)$ form a basis for $(\mathcal{A},\ \mu')$.

Example 3.2.4. *Consider the multiarrangement $(\mathcal{A},\ \mu) = (\mathcal{A},\ m\omega + 1)$ as shown in Example 2.2.6. It corresponds to the group $G(r,r,\ell)$. We have $D = \frac{\partial}{\partial P_{\ell-1}}$. As shown in Example 2.2.6, we have the vector fields*

$$\left(\prod_{i=1}^{\ell} x_i^{r-1}\right)\eta_{-1}^0 = \left(\prod_{i=1}^{\ell} x_i^{r-1}\right)\sum_{j=1}^{\ell}\left(\int^{x_j} t^{-r}dt\right)\partial_j$$

and

$$\eta_s^0 = \sum_{i=1}^{\ell}\left(\int^{x_i} t^{rs}dt\right)\partial_i,$$

$(0 \le s \le \ell-2)$ form a basis for the arrangement $\mathcal{A}$ defined by

$$\prod_{1\le i<j\le \ell}(x_i^r - x_j^r).$$

Note that $\nabla_D \eta_s^m \doteq \eta_{s+1}^{m-1}$. We have

$$\nabla_D^{-m}\left(\prod_{i=1}^{\ell} x_i^{r-1}\right)\eta_{-1}^0 \doteq \left(\prod_{i=1}^{\ell} x_i^{r-1}\right)\eta_{-m-1}^m$$

and

$$\nabla_D^{-m}\eta_s^0 \doteq \eta_{-m+s}^m \qquad \text{for } s \ge 0$$

Clearly, the vector fields

$$\left(\prod_{i=1}^{\ell} x_i^{r-1}\right)\eta_{-m-1}^m \text{ and } \eta_s^m\ (-m \le s \le -m+\ell-2)$$

form a basis for $(\mathcal{A},\ m\omega+1)$.

Example 3.2.5. *Consider the multiarrangement* $(\mathcal{A},\ \mu')$ *as shown in Example 2.3.5. It corresponds to the group* $G(r, r, \ell)$*. We have* $D = \frac{\partial}{\partial P_{\ell-1}}$*. We have known* $\eta_0^0 = \theta_E$ *and* $\nabla_D \eta_s^m \doteq \eta_{s+1}^{m-1}$*. Then*

$$\nabla_D^{-m} \eta_0^0 \doteq \eta_{-m}^m.$$

For $1 \leq i \leq \ell$*, we have*

$$\nabla_{\partial_i} \eta_{-m}^m = -rmx_i^{r-1} \sum_{j=1}^{\ell} \int^{x_j} t^{-rm} \lambda(t)^m / (t^r - x_i^r) dt \partial_j \doteq \sigma_i^m,$$

where σ_i^m *is defined in Example 2.3.5. Clearly,*

$$\sigma_i^m \doteq \nabla_{\partial_i} \nabla_D^{-m} \theta_E$$

$(1 \leq i \leq \ell)$ *form a basis for* $(\mathcal{A},\ \mu')$*.*

Chapter 4

More applications for integral expressions

4.1 Integral expressions of multiarrangements of three lines

In this section, we consider the multiarrangement $(\mathcal{A},\ m)$ in dimension 2, in particular, the case $\mathcal{A}$ is 3 lines. Wakamiko [25] constructed a basis for $(\mathcal{A},\ m)$ by using the generalized binomial coefficients. We construct a basis for $(\mathcal{A},\ m)$ according to the situation. It is trivial, if there exists a hyperplane $H \in \mathcal{A}$ such that $m(H) = 0$.

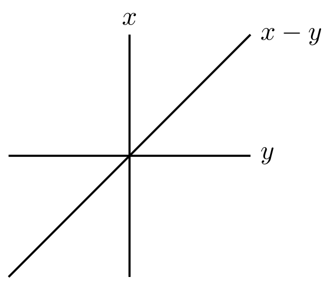

Figure 4.1: The 3-lines arrangement $xy(x-y)$

We say $(\mathcal{A},\ m)$ is balanced, if $|m| \geq 2m(H)$ for any $H \in \mathcal{A}$. The balanced line multiarrangements have been studied in [2, 3]. If $\mathcal{A}$ is not balanced, there exists a hyperplane $H \in \mathcal{A}$ such that $2m(H) > |m|$. Without loss of generality, we may assume that $H = \{x = 0\}$, $(\mathcal{A},\ m)$ is defined by $x^p y^q (x-y)^r$, and $p > q + r$ for $p, q, r \in \mathbb{Z}_{>0}$. Then $\theta_1 = y^q (x-y)^r \partial_y \in \mathcal{D}(\mathcal{A},\ m)$ is a nonzero derivation of the smallest degree, together with a vector field $\theta_2 = x^p \partial_x + (\sum_{i=0}^{r} \binom{p}{i} (x-y)^i y^{p-i}) \partial_y$ form a basis for $(\mathcal{A},\ m)$. If $(\mathcal{A},\ m)$ is balanced, we may assume that $p \leq q + r$ and $p \geq \max\{q, r\}$. We use integral expressions to construct a basis. Define the vector field $\theta_{a,b,c}$ as

$$\theta_{a,b,c} = \left(\int^x t^c (t-x)^a (t-y)^b dt \right) \partial_x + \left(\int^y t^c (t-x)^a (t-y)^b dt \right) \partial_y.$$

We can prove the following theorem.

Theorem 4.1.1. *Let $(\mathcal{A},\ m)$ be a multiarrangement defined as above. Then*

(1) Suppose $p+q+r$ is odd. Let

$$(a,\ b,\ c) = \left(\frac{p-q+r-1}{2},\ \frac{-p+q+r-1}{2},\ \frac{p+q-r-1}{2}\right).$$

Then $\theta_{a,b,c}$ and $\theta_{a,b,c+1}$ form a basis for $\mathcal{D}(\mathcal{A},\ m)$.

(2) Suppose $p+q+r$ is even. Let

$$(a,\ b,\ c) = \left(\frac{p-q+r-2}{2},\ \frac{-p+q+r}{2},\ \frac{p+q-r}{2}\right),$$

and

$$(a',\ b',\ c') = \left(\frac{p-q+r}{2},\ \frac{-p+q+r}{2},\ \frac{p+q-r-2}{2}\right).$$

Then $\theta_{a,b,c}$ and $\theta_{a',b',c'}$ form a basis for $\mathcal{D}(\mathcal{A},\ m)$.

Proof. (1) Since $p \geq \max\{q,\ r\}$, $p \leq q+r$, and $p+q+r$ is odd, we have $a, b, c \in Z_{\geq 0}$.

By Proposition 2.1.3 and Lemma 2.1.4, it is clear that $\theta_{a,b,c}$ and $\theta_{a,b,c+1}$ are contained in $D(\mathcal{A},\ m)$ and linearly independent over S. Note that $\deg \theta_{a,b,c} + \deg \theta_{a,b,c+1} = p+q+r = |m|$. By Theorem 1.2.2, $\theta_{a,b,c}$ and $\theta_{a,b,c+1}$ form a basis for $\mathcal{D}(\mathcal{A},\ m)$.

(2) Since $p \geq \max\{q,\ r\}$, $p+q+r$ is even, and $p, q, r \in \mathbb{Z}_{>0}$, we have $p+q-r-2 \geq 0$ and $p-q+r-2 \geq 0$.

By Proposition 2.1.3, it is clear that $\theta_{a,b,c}$ and $\theta_{a',b',c'}$ are contained in $D(\mathcal{A},\ m)$. By simple calculation, we have $\deg \theta_{a,b,c} + \deg \theta_{a,b,c+1} = p+q+r = |m|$. By Theorem 1.2.2, it suffices to show that $\theta_{a,b,c}$ and $\theta_{a',b',c'}$ are linearly independent over S. Note that $(a',\ b',\ c') = (a+1,\ b,\ c-1)$. Observe the following matrix where each entry is the non-zero leading monomial of Saito matrix up to a constant,

$$\begin{pmatrix} x^{a+b+c+1} & x^a y^{b+c+1} \\ x^{a+b+c+1} & x^{a+1} y^{b+c} \end{pmatrix}.$$

Then the product of diagonal entries gives the unique maximum monomial

$$x^{2a+b+c+2} y^{b+c}.$$

Hence $\theta_{a,b,c}$ and $\theta_{a',b',c'}$ form a basis for $\mathcal{D}(\mathcal{A},\ m)$. □

The following examples are corresponding to the above cases.

Example 4.1.2. *(1) Let $(\mathcal{A},\ m)$ be the multiarrangement defined by $x^{1025} y^{916} (x-y)^{488}$. Then the following vector fields*

$$\left(\int^x t^{726}(t-x)^{298}(t-y)^{189} dt\right) \partial_x + \left(\int^y t^{726}(t-x)^{298}(t-y)^{189} dt\right) \partial_y,$$
$$\left(\int^x t^{727}(t-x)^{298}(t-y)^{189} dt\right) \partial_x + \left(\int^y t^{727}(t-x)^{298}(t-y)^{189} dt\right) \partial_y$$

form a basis for $\mathcal{D}(\mathcal{A},\ m)$ with exponents $(1214,\ 1215)$.

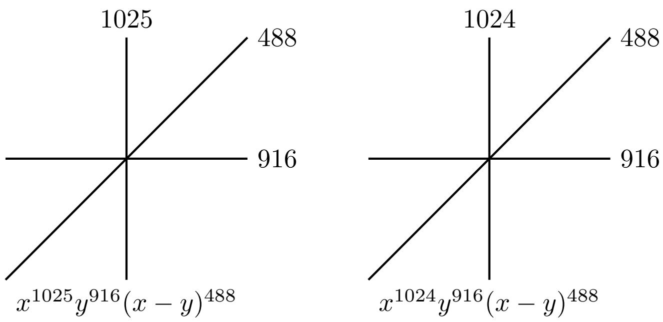

Figure 4.2: Two 3-lines multiarrangements

(2) Let $(\mathcal{A},\ m)$ be the multiarrangement defined by by $x^{1024}y^{916}(x-y)^{488}$. Then the following vector fields

$$\left(\int^x t^{726}(t-x)^{297}(t-y)^{190}dt\right)\partial_x + \left(\int^y t^{726}(t-x)^{297}(t-y)^{190}dt\right)\partial_y,$$
$$\left(\int^x t^{725}(t-x)^{298}(t-y)^{190}dt\right)\partial_x + \left(\int^y t^{725}(t-x)^{298}(t-y)^{190}dt\right)\partial_y$$

form a basis for $\mathcal{D}(\mathcal{A},\ m)$ with exponents $(1214,\ 1214)$.

4.2 Deformations for Catalan arrangements

Suyama-Terao [20] used the Bernoulli polynomials to construct the basis for the Shi arrangements of type A. Suyama [21] constructed the basis for the Shi arrangements of type B or type C by using new Bernoulli-like polynomials. Gao-Pei-Terao [16] constructed the basis for the Shi arrangements of type D by using Bernoulli polynomials and their relatives. Abe-Suyama [5] constructed the basis for the extended Catalan and Shi arrangements of type A_2. Suyama-Yoshinaga also [22] constructed the explicit basis for Catalan and Shi arrangements of type A by deforming the basis shown in Theorem 2.1.1. They used the discrete analogue of integral expressions to construct the basis. And Theorem 1.2.3 played an important role. In this section, we consider the Catalan arrangements of type B.

Definition 4.2.1. *For $\ell \geq 2$ and $m \geq 0$, define the m-Catalan arrangement $\mathrm{Cat}_m(\mathcal{B}_\ell)$ of type B by*

$$\prod_{\substack{1\leq i\leq \ell\\ k\in\{-m,-m+1,\dots,m\}}}(x_i-k)\cdot\prod_{\substack{1\leq i<j\leq \ell\\ k\in\{-m,-m+1,\dots,m\}}}(x_i-x_j-k)(x_i+x_j-k).$$

Then the cone $c\,\mathrm{Cat}_m(\mathcal{B}_\ell)$ of $\mathrm{Cat}_m(\mathcal{B}_\ell)$ is defined by the homogeneous polynomial

$$z\cdot\prod_{\substack{1\leq i\leq \ell\\ k\in\{-m,-m+1,\dots,m\}}}(x_i-kz)\cdot\prod_{\substack{1\leq i<j\leq \ell\\ k\in\{-m,-m+1,\dots,m\}}}(x_i-x_j-kz)(x_i+x_j-kz).$$

Its Ziegler restriction $(c\,\mathrm{Cat}_m(\mathcal{B}_\ell)^{H_0},\ m^{H_0}) = (\mathcal{B}_\ell,\ 2m+1)$ with respect to

$H_0 = \{z = 0\}$ is defined by

$$\prod_{1 \le i \le \ell} x_i^{2m+1} \cdot \prod_{1 \le i < j \le \ell} (x_i^2 - x_j^2)^{2m+1}.$$

It is well known that $c\,\mathrm{Cat}_m(\mathcal{B}_\ell)$ is free [30]. We also have a basis for $(\mathcal{B}_\ell,\ 2m+1)$ as shown in Example 2.2.4. It seems that we can use Theorem 1.2.3 to give the basis for $c\,\mathrm{Cat}_m(\mathcal{B}_\ell)$ by deforming the basis of $(\mathcal{B}_\ell,\ 2m+1)$. But it is hard to construct the basis even for the case $\ell = 2$.

From now, we consider the case $\mathcal{B}_2$. Let $\lambda(t) = (t^2 - x^2)(t^2 - y^2)$. For $i, m \in \mathbb{Z}_{\ge 0}$, define η_i^m as follows

$$\begin{aligned}
\eta_i^m &= \int^x t^{2i}\lambda^m(t)dt\partial_x + \int^y t^{2i}\lambda^m(t)dt\partial_y \\
&= \int^x t^{2i}(t^2 - x^2)^m(t^2 - y^2)^m dt\partial_x + \int^y t^{2i}(t^2 - x^2)^m(t^2 - y^2)^m dt\partial_y \\
&= \int^x \sum_{j,k=0}^{m} \binom{m}{j}\binom{m}{k} t^{2i+2(m-j)+2(m-k)}(-x^2)^j(-y^2)^k dt\partial_x \\
&\quad + \int^y \sum_{k,j=0}^{m} \binom{m}{k}\binom{m}{j} t^{2i+2(m-k)+2(m-j)}(-x^2)^k(-y^2)^j dt\partial_y \\
&= \sum_{k=0}^{m} C_k x^{4m+1+2i-2k}y^{2k}\partial_x + \sum_{k=0}^{m} C_k y^{4m+1+2i-2k}x^{2k}\partial_y.
\end{aligned}$$

Let $f_i^m(x, y) = \sum_{k=0}^m C_k x^{4m+1+2i-2k}y^{2k}$. Then $\eta_i^m = f_i^m(x, y)\partial_x + f_i^m(y, x)\partial_y$. By Example 2.2.4, η_0^m and η_1^m form a basis for $\mathcal{D}(\mathcal{B}_\ell,\ 2m+1)$.

The method of [22] for type A dose not work for type B. We need other methods for the deformation. Define the falling factorial as the polynomial with respect to one variable x

$$(x + a)^{\underline{n}} = (x + a)(x + a - 1)\cdots(x + a - n + 1),$$

and the rising factorial as the polynomial with respect to one variable x

$$(x + a)^{\overline{n}} = (x + a)(x + a + 1)\cdots(x + a + n - 1).$$

By convention, set $(x + a)^{\underline{0}} = (x + a)^{\overline{0}} = 1$.

We have the following deformations.

Definition 4.2.2. *Let $f_i^m(x, y)$ and η_i^m be as above for $i \ge 0$. Define the deformation of $f_i(x, y)$ by*

$$\widetilde{f_i^m}(x, y) = \sum_{k=0}^{m} C_k(x + 2m + i - k)^{\underline{4m+1+2i-2k}}(y + m + i)^{\underline{k}}(y - m - i)^{\overline{k}},$$

and the deformation of η_i^m by

$$\widetilde{\eta_i^m} = \widetilde{f_i^m}(x, y)\partial_x + \widetilde{f_i^m}(y, x)\partial_y.$$

Take a vector field $\theta = \sum_{i=1}^{\ell} \prod_{k=1,2,\cdots,n} \alpha_{ik}\partial_i$, where each $\alpha_{ik} = \sum_{j=1}^{\ell} c_{ikj}x_j + c_{ik}$ is of degree 1. The homogeneous form of θ is obtained by homogenizing each α_{ik} to $\sum_{j=1}^{\ell} c_{ikj}x_j + c_{ik}z$. It is easy to check the following examples.

Example 4.2.3. *(1) The case* $m = 1$.
$\eta_0^1 = (x^5 - 5x^3y^2)\partial_x + (y^5 - 5y^3x^2)\partial_y$,
$\eta_1^1 = (3x^7 - 7x^5y^2)\partial_x + (3y^7 - 7y^5x^2)\partial_y$.
The homogeneous forms of the following deformations of η_0^1, η_1^1 *and the Euler derivation* θ_E *form a basis for the cone of* $\mathrm{Cat}_1(\mathcal{B}_2)$.
$\widetilde{\eta_0^1} = ((x+2)^{\underline{5}} - 5(x+1)^{\underline{3}}(y+1)(y-1))\partial_x + ((y+2)^{\underline{5}} - 5(y+1)^{\underline{3}}(x+1)(x-1))\partial_y$,
$\widetilde{\eta_1^1} = (3(x+3)^{\underline{7}} - 7(x+2)^{\underline{5}}(y+2)(y-2))\partial_x + (3(y+3)^{\underline{7}} - 7(y+2)^{\underline{5}}(x+2)(x-2))\partial_y$.

(2) The case $m = 2$.
$\eta_0^1 = (x^9 - 6x^7y^2 + 21x^5y^4)\partial_x + (y^9 - 6y^7x^2 + 21y^5x^4)\partial_y$,
$\eta_1^1 = (5x^{11} - 22x^9y^2 + 33x^7y^4)\partial_x + (5y^{11} - 22y^9x^2 + 33y^7x^4)\partial_y$.
The homogeneous forms of the following deformations of η_0^1, η_1^1 *and the Euler derivation* θ_E *form a basis for the cone of* $\mathrm{Cat}_2(\mathcal{B}_2)$.
$\widetilde{\eta_0^1} = ((x+4)^{\underline{9}} - 6(x+3)^{\underline{7}}(y+2)(y-2) + 21(x+2)^{\underline{5}}(y+2)^{\underline{2}}(y-2)^{\overline{2}})\partial_x + ((y+4)^{\underline{9}} - 6(y+3)^{\underline{7}}(x+2)(x-2) + 21(y+2)^{\underline{5}}(x+2)^{\underline{2}}(x-2)^{\overline{2}})\partial_y$,
$\widetilde{\eta_1^1} = (5(x+5)^{\underline{11}} - 22(x+4)^{\underline{9}}(y+3)(y-3) + 33(x+3)^{\underline{7}}(y+3)^{\underline{2}}(y-3)^{\overline{2}})\partial_x + (5(y+5)^{\underline{11}} - 22(y+4)^{\underline{9}}(x+3)(x-3) + 33(y+3)^{\underline{7}}(x+3)^{\underline{2}}(x-3)^{\overline{2}})\partial_y$.

It is obvious that (*the homogeneous form of* $\widetilde{\eta_i^m}$) $|_{z=0} = \eta_i^m$ for $i \geq 0$. From Theorem 1.2.3, if the following conditions are satisfied:

$$\widetilde{\eta_i^m}(x) \text{ is divisible by } \prod_{k=-m}^{m}(x-k), \text{ and}$$
$$\widetilde{\eta_i^m}(x \pm y) \text{ is divisible by } \prod_{k=-m}^{m}(x \pm y - k),$$

then the homogeneous forms of the deformations $\widetilde{\eta_0^m}$ and $\widetilde{\eta_0^m}$ together with the Euler derivation θ_E form a basis for the cone of m-Catalan arrangement $\mathrm{Cat}_m(\mathcal{B}_2)$. From the definition of $\widetilde{f_i^m}(x,y)$, $\widetilde{\eta_i^m}(x) = \widetilde{f_i^m}(x,y)$ is divisible by $\prod_{k=-m}^{m}(x-k)$. We propose the following conjecture.

Conjecture 4.2.4. *For* $i,\ m \geq 0$, $\widetilde{\eta_i^m}(x \pm y) = \widetilde{f_i^m}(x,y) \pm \widetilde{f_i^m}(y,x)$ *is divisible by* $\prod_{k=-m}^{m}(x \pm y - k)$.

Remark 4.2.5. Recently, Kawanoue [18] announced the proof of Conjecture 4.2.4.

4.3 Applications for higher-dimensional analogue

Let the arrangement $\mathcal{A} = \mathcal{A}_{\ell-1}$ be a braid arrangement.

Definition 4.3.1. *For* $\ell \geq 3$, $1 < i, j < \ell$, *define the multiarrangement* $(\mathcal{A},\ \alpha(i,j))$ *by*

$$Q(\mathcal{A},\ \alpha(i,j)) = \prod_{2 \leq k \leq i}(x_1 - x_k)^2 \cdot \prod_{i+1 \leq k \leq \ell}(x_1 - x_k) \cdot \prod_{2 \leq s < t \leq \ell-1}(x_s - x_t)^2$$
$$\cdot \prod_{j \leq k \leq \ell-1}(x_k - x_\ell)^2 \cdot \prod_{2 \leq k \leq j-1}(x_k - x_\ell).$$

Definition 4.3.2. *For $\ell \geq 3$, $1 < i, j < \ell$, $j \leq i+1$, and $m \geq 0$, define the multiarrangement $(\mathcal{A},\ \mathbf{m})$ by*

$$Q(\mathcal{A},\ \alpha(i,j)) \cdot \prod_{1 \leq s < t \leq \ell} (x_s - x_t)^{2m}.$$

The defining polynomial

$$Q(\mathcal{A},\ \mathbf{m}) = \frac{\prod_{1 \leq s < t \leq \ell}(x_s - x_t)^{2m+2}}{\prod_{i+1 \leq k \leq \ell}(x_1 - x_k) \cdot \prod_{2 \leq k \leq j-1}(x_k - x_\ell)}.$$

Then $\deg Q(\mathcal{A},\ \mathbf{m}) = \ell(\ell-1)(2m+2)/2 - (\ell - i + j - 2)$. We can use a simple graph $G = ([\ell],\ E)$ to describe the multiarrangement $(\mathcal{A},\ \mathbf{m})$. If the hyperplane $\{x_s - x_t = 0\}$ has a multiplicity $2m+2$, the edge $\{s,\ t\} \in E$. Otherwise, the edge $\{s,\ t\} \notin E$. We use a straight line between the vertices s, t $(s < t)$ to indicate a complete subgraph induced by the vertex set $\{s, s+1, \ldots, t\}$. The multiarrangement $(\mathcal{A},\ \mathbf{m})$ can be described by the following graph G (Figure 4.3) where G is the union of three complete subgraphs induced by $\{1, 2, \ldots, i\}$, $\{2, 3, \ldots, \ell-1\}$, $\{j, j+1, \ldots, \ell\}$. In this case, the multiarrangement $(\mathcal{A},\ \mathbf{m})$ is defined by

$$Q(\mathcal{A},\ \mathbf{m}) = \prod_{\{i,j\} \in G} (x_i - x_j)^{2m+2} \cdot \prod_{\{i,j\} \notin G} (x_i - x_j)^{2m+1}.$$

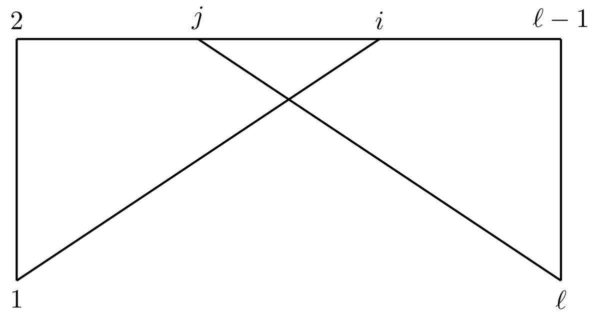

Figure 4.3: The graph G corresponding to the multiarrangement $(\mathcal{A},\ \mathbf{m})$

The multiarrangement $(\mathcal{A},\ \mathbf{m})$ is free by [4]. In [28], it also implies $(\mathcal{A},\ \mathbf{m})$ is free for $m = 0$. Let $\lambda(t) = \prod_{i=1}^{\ell}(t - x_i)$, $\lambda_k(t) = (t-x_1)(t-x_2)\ldots\widehat{(t-x_k)}\ldots(t-x_{\ell-1})$, $\lambda'_s(t) = (t-x_2)(t-x_3)\ldots\widehat{(t-x_s)}\ldots(t-x_\ell)$, and $\lambda''_h(t) = (t-x_1)(t-x_2)\ldots\widehat{(t-x_h)}\ldots(t-x_\ell)$. We first construct a basis for the multiarrangement $(\mathcal{A},\ \mathbf{m})$ where $j = i+1$.

Theorem 4.3.3. *The vector fields $\theta_k^m = \sum_{u=1}^{\ell} \int^{x_u} \lambda(t)^m \lambda_k(t) dt \partial_u$ $(1 \leq k \leq i)$ and $\theta_s^m = \sum_{u=1}^{\ell} \int^{x_u} \lambda(t)^m \lambda'_s(t) dt \partial_u$ $(i+1 \leq s \leq \ell-1)$ together with $\theta_0 = \sum_{i=1}^{\ell} \partial_i$ form a basis for the multiarrangement $(\mathcal{A},\ \mathbf{m})$ defiend by*

$$Q(\mathcal{A},\ \mathbf{m}) = \frac{\prod_{1 \leq s < t \leq \ell}(x_s - x_t)^{2m+2}}{\prod_{i+1 \leq k \leq \ell}(x_1 - x_k) \cdot \prod_{2 \leq k \leq i}(x_k - x_\ell)}.$$

The corresponding graph is shown below (Figure 4.4).

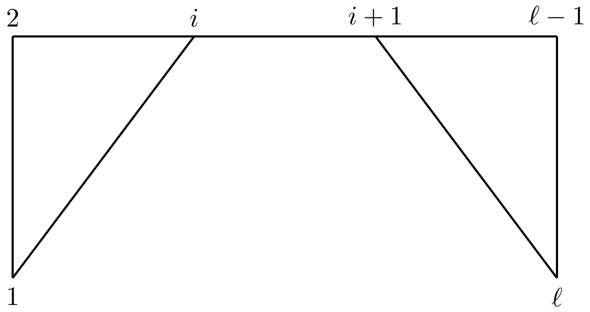

Figure 4.4: The graph corresponding to the multiarrangement $(\mathcal{A},\ \mathbf{m})$

Proof. Obviously, these vector fields are polynomial vector fields. By Proposition 2.1.3, each vector field is contained in $\mathcal{D}(\mathcal{A},\ \mathbf{m})$. Note that $\deg \theta_k^m = m\ell+\ell-1$ for $1 \leq k \leq \ell-1$. Then $\sum_{k=1}^{\ell-1} \deg \theta_k^m + \deg \theta_0 = (m\ell+\ell-1)(\ell-1) = \deg Q(\mathcal{A},\ \mathbf{m})$. The Saito matrix is shown below,

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \theta_2^m(x_1) & \theta_2^m(x_2) & \cdots & \theta_2^m(x_\ell) \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{\ell-1}^m(x_1) & \theta_{\ell-1}^m(x_2) & \cdots & \theta_{\ell-1}^m(x_\ell) \\ \theta_1^m(x_1) & \theta_1^m(x_2) & \cdots & \theta_1^m(x_\ell) \end{pmatrix}.$$

After elementary transformations, we have

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & \theta_2^m(x_2-x_1) & \cdots & \theta_2^m(x_\ell-x_1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \theta_{\ell-1}^m(x_2-x_1) & \cdots & \theta_{\ell-1}^m(x_\ell-x_1) \\ 0 & \theta_1^m(x_2-x_1) & \cdots & \theta_1^m(x_\ell-x_1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & M \end{pmatrix}.$$

Each θ_p^m has the form $\sum_{u=1}^{\ell} \int^{x_u} \prod_{q=1}^{\ell}(t-x_q)^{a_q} dt$ for $1 \leq p \leq \ell-1$. Then $\theta_p^m(x_j - x_1) = \int_{x_1}^{x_j} \prod_{q=1}^{\ell}(t-x_q)^{a_q} dt$ for $j \neq 1$. After the changes of variables $t' = t - x_1$, we have $\theta_p^m(x_j - x_1) = \int^{x_j - x_1} \prod_{q=1}^{\ell}(t' - (x_q - x_1))^{a_q} dt'$. Take the variables $y_1 = x_1$ and $y_q = x_q - x_1$ for $q \geq 2$, we have $\theta_p^m(x_j - x_1) = \int^{y_j} (t')^{a_1} \cdot \prod_{q=2}^{\ell}(t'-y_q)^{a_q} dt'$. Introduce the lexicographic ordering among monomials with respect to variables $y_1, y_2, \ldots, y_\ell$.

For $2 \le k \le \ell$, consider the $(k-1)$-th column

$$M_{k-1} = \begin{pmatrix} \theta_2^m(x_k - x_1) = \int^{y_k} (t')^{m+1} \cdot \frac{\prod_{q=2}^{\ell}(t'-y_q)^{m+1}}{(t'-y_2)(t'-y_\ell)} dt' \\ \theta_3^m(x_k - x_1) = \int^{y_k} (t')^{m+1} \cdot \frac{\prod_{q=2}^{\ell}(t'-y_q)^{m+1}}{(t'-y_3)(t'-y_\ell)} dt' \\ \vdots \\ \theta_i^m(x_k - x_1) = \int^{y_k} (t')^{m+1} \cdot \frac{\prod_{q=2}^{\ell}(t'-y_q)^{m+1}}{(t'-y_i)(t'-y_\ell)} dt' \\ \theta_{i+1}^m(x_k - x_1) = \int^{y_k} (t')^{m} \cdot \frac{\prod_{q=2}^{\ell}(t'-y_q)^{m+1}}{(t'-y_{i+1})} dt' \\ \vdots \\ \theta_1^m(x_k - x_1) = \int^{y_k} (t')^{m} \cdot \frac{\prod_{q=2}^{\ell}(t'-y_q)^{m+1}}{(t'-y_\ell)} dt' \end{pmatrix}$$

of the matrix M. If $k \le i$, observe the following column vector where each entry $m_{s,k-1}$ is the nonzero leading monomial of entries of M_{k-1},

$$\begin{pmatrix} m_{1,k-1} = \prod_{j=2}^{k-1} y_j^{m+1} \cdot y_k^{(m+1)(\ell-k+2)}/y_2 \\ m_{2,k-1} = \prod_{j=2}^{k-1} y_j^{m+1} \cdot y_k^{(m+1)(\ell-k+2)}/y_3 \\ \vdots \\ m_{k-2,k-1} = \prod_{j=2}^{k-1} y_j^{m+1} \cdot y_k^{(m+1)(\ell-k+2)}/y_{k-1} \\ m_{k-1,k-1} = \prod_{j=2}^{k-1} y_j^{m+1} \cdot y_k^{(m+1)(\ell-k+2)-1} \\ m_{k,k-1} = \prod_{j=2}^{k-1} y_j^{m+1} \cdot y_k^{(m+1)(\ell-k+2)-1} \\ \vdots \\ m_{i-1,k-1} = \prod_{j=2}^{k-1} y_j^{m+1} \cdot y_k^{(m+1)(\ell-k+2)-1} \\ m_{i,k-1} = \prod_{j=2}^{k-1} y_j^{m+1} \cdot y_k^{(m+1)(\ell-k+2)-1} \\ m_{i+1,k-1} = \prod_{j=2}^{k-1} y_j^{m+1} \cdot y_k^{(m+1)(\ell-k+2)-1} \\ \vdots \\ m_{\ell-1,k-1} = \prod_{j=2}^{k-1} y_j^{m+1} \cdot y_k^{(m+1)(\ell-k+2)-1} \end{pmatrix}.$$

If $k > i$, observe the following column vector where each entry $m_{s,k-1}$ is the nonzero leading monomial of entries of M_{k-1},

$$\begin{pmatrix} m_{1,k-1} = \prod_{j=2}^{k-1} y_j^{m+1} \cdot y_k^{(m+1)(\ell-k+2)}/y_2 \\ m_{2,k-1} = \prod_{j=2}^{k-1} y_j^{m+1} \cdot y_k^{(m+1)(\ell-k+2)}/y_3 \\ \vdots \\ m_{i-1,k-1} = \prod_{j=2}^{k-1} y_j^{m+1} \cdot y_k^{(m+1)(\ell-k+2)}/y_i \\ m_{i,k-1} = \prod_{j=2}^{k-1} y_j^{m+1} \cdot y_k^{(m+1)(\ell-k+2)}/y_{i+1} \\ m_{i+1,k-1} = \prod_{j=2}^{k-1} y_j^{m+1} \cdot y_k^{(m+1)(\ell-k+2)}/y_{i+2} \\ \vdots \\ m_{k-2,k-1} = \prod_{j=2}^{k-1} y_j^{m+1} \cdot y_k^{(m+1)(\ell-k+2)}/y_{k-1} \\ m_{k-1,k-1} = \prod_{j=2}^{k-1} y_j^{m+1} \cdot y_k^{(m+1)(\ell-k+2)-1} \\ m_{k,k-1} = \prod_{j=2}^{k-1} y_j^{m+1} \cdot y_k^{(m+1)(\ell-k+2)-1} \\ \vdots \\ m_{\ell-1,k-1} = \prod_{j=2}^{k-1} y_j^{m+1} \cdot y_k^{(m+1)(\ell-k+2)-1} \end{pmatrix}.$$

In the $(k-1)$-th column, the monomial increases strictly up to the $(k-1)$-th entry. The entry $m_{k-1,k-1} = (y_2y_3\dots y_{k-1})^{m+1}y_k^{(m+1)(\ell-k+2)-1}$ and $m_{k-1,k-1} = m_{j,k-1}$ for $j \geq k-1$. Hence the product of diagonal entries gives the unique maximum monomial. And the determinant of the Saito matrix is nonzero. It is clear that the vector fields θ_k^m $(1 \leq k \leq \ell-1)$ and θ_0 are linearly independent over S. $\square$

Then we construct a basis for the multiarrangement $(\mathcal{A},\ \mathbf{m})$ where $j \leq i+1$.

Theorem 4.3.4. *The vector fields*

$$\begin{aligned}
\theta_0 &= \sum_{i=1}^{\ell} \partial_i, \\
\vartheta_k^m &= \sum_{u=1}^{\ell} \int^{x_u} \lambda(t)^m \lambda_k(t) dt \partial_u \qquad (1 \leq k \leq j-1), \\
\vartheta_h^m &= (x_h - x_\ell)\theta_h^m = (x_h - x_\ell) \sum_{u=1}^{\ell} \int^{x_u} \lambda(t)^m \lambda_h(t) dt \partial_u \qquad (j \leq h \leq i), \\
\vartheta_s^m &= \sum_{u=1}^{\ell} \int^{x_u} \lambda(t)^m \lambda'_s(t) dt \partial_u \qquad (i+1 \leq s \leq \ell-1),
\end{aligned}$$

form a basis for the multiarrangement $(\mathcal{A},\ \mathbf{m})$ defined in Definition 4.3.2.

Proof. By Proposition 2.1.3, each vector field is contained in $\mathcal{D}(\mathcal{A},\ \mathbf{m})$. Clearly, $\sum_{k=1}^{\ell-1} \deg \vartheta_k^m + \deg \theta_0 = \sum_{k=1}^{\ell-1} \deg \theta_k^m + \deg \theta_0 + i - j + 1 = (m\ell + \ell - 1)(\ell - 1) + i - j + 1 = \deg Q(\mathcal{A},\ \mathbf{m})$. Since the vector fields θ_k^m $(1 \leq k \leq \ell-1)$ and θ_0 are linearly independent over S by Theorem 4.3.3, the vector fields ϑ_k^m $(1 \leq k \leq \ell-1)$ and θ_0 are linearly independent over S. $\square$

Let us consider the cases $3 \leq \ell \leq 5$.

Example 4.3.5. *Let the multiarrangement $(\mathcal{A},\ \mathbf{m})$ be described by the below graph (Figure 4.5).*

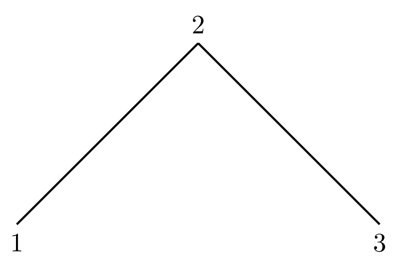

Figure 4.5: The graph of a 3-multiarrangement $\frac{\prod_{1\leq s<t\leq 3}(x_s-x_t)^{2m+2}}{x_1-x_3}$.

The vector fields

$$
\begin{aligned}
\theta_0 &= \sum_{i=1}^{3} \partial_i, \\
\vartheta_1^m &= \sum_{u=1}^{3} \int^{x_u} \lambda(t)^m \lambda_1(t) dt \partial_u = \sum_{u=1}^{3} \int^{x_u} (t-x_1)^m (t-x_2)^{m+1} (t-x_3)^m dt \partial_u, \\
\vartheta_2^m &= (x_2 - x_3) \sum_{u=1}^{3} \int^{x_u} \lambda(t)^m \lambda_2(t) dt \partial_u \\
&= (x_2 - x_3) \sum_{u=1}^{3} \int^{x_u} (t-x_1)^{m+1} (t-x_2)^m (t-x_3)^m dt \partial_u
\end{aligned}
$$

form a basis for the multiarrangement $(\mathcal{A},\ \mathbf{m})$.

Proposition 4.3.6. *The vector fields*

$$
\begin{aligned}
\theta_0 &= \sum_{i=1}^{3} \partial_i, \\
\varpi_1^m &= \sum_{u=1}^{3} \int^{x_u} \lambda(t)^m \lambda_1(t) dt \partial_u = \sum_{u=1}^{3} \int^{x_u} (t-x_1)^m (t-x_2)^{m+1} (t-x_3)^m dt \partial_u, \\
\varpi_2^m &= \sum_{u=1}^{3} \int^{x_u} \lambda(t)^m \lambda_2''(t) dt \partial_u = \sum_{u=1}^{3} \int^{x_u} (t-x_1)^{m+1} (t-x_2)^m (t-x_3)^{m+1} dt \partial_u
\end{aligned}
$$

form a basis for the above multiarrangement $(\mathcal{A},\ \mathbf{m})$ *described in Figure 4.5.*

Proof. By Proposition 2.1.3, each vector field is contained in $\mathcal{D}(\mathcal{A},\ \mathbf{m})$. Note that $\deg \varpi_1^m = 3m+2$ and $\deg \varpi_2^m = 3m+3$. Then $\sum_{k=1}^{3} \deg \varpi_k^m + \deg \theta_0 = 6m+5 = \deg Q(\mathcal{A},\ \mathbf{m})$. After elementary transformations of the Saito matrix, it is sufficient to show the determinant of following martix is nonzero.

$$
M = \begin{pmatrix} \varpi_1^m (x_2 - x_1) & \varpi_1^m (x_3 - x_1) \\ \varpi_2^m (x_2 - x_1) & \varpi_2^m (x_3 - x_1) \end{pmatrix}.
$$

Take the similar changes of variables in the proof of Theorem 4.3.3. Observe the following matrix where each entry is the nonzero leading monomial of entries of M up to a constant,

$$
\begin{pmatrix} y_2^{3m+2} & y_2^{m+1} y_3^{2m+1} \\ y_2^{3m+3} & y_2^m y_3^{2m+3} \end{pmatrix}.
$$

Then the product of the anti-diagonal entries gives the unique maximum monomial $y_2^{4m+4} y_3^{2m+1}$. Thus the determinant of M is non-zero. It is clear that the vector fields θ_0, ϖ_1^m, and ϖ_2^m are linearly independent over S. □

Example 4.3.7. *Let the multiarrangement* $(\mathcal{A},\ \mathbf{m})$ *be described by the following graph (Figure 4.6).*

Figure 4.6: The graph of a 4-multiarrangement $\frac{\prod_{1\le s<t\le 4}(x_s-x_t)^{2m+2}}{(x_1-x_3)(x_1-x_4)(x_2-x_4)}$.

The vector fields

$$\begin{aligned}
\theta_0 &= \sum_{i=1}^{4} \partial_i, \\
\theta_1^m &= \sum_{u=1}^{4} \int^{x_u} \lambda(t)^m \lambda_1(t) dt \partial_u \\
&= \sum_{u=1}^{4} \int^{x_u} (t-x_1)^m (t-x_2)^{m+1} (t-x_3)^{m+1} (t-x_4)^m dt \partial_u, \\
\theta_2^m &= \sum_{u=1}^{4} \int^{x_u} \lambda(t)^m \lambda_2(t) dt \partial_u \\
&= \sum_{u=1}^{4} \int^{x_u} (t-x_1)^{m+1} (t-x_2)^m (t-x_3)^{m+1} (t-x_4)^m dt \partial_u, \\
\theta_3^m &= \sum_{u=1}^{4} \int^{x_u} \lambda(t)^m \lambda_3'(t) dt \partial_u \\
&= \sum_{u=1}^{4} \int^{x_u} (t-x_1)^m (t-x_2)^{m+1} (t-x_3)^m (t-x_4)^{m+1} dt \partial_u
\end{aligned}$$

form a basis for the multiarrangement $(\mathcal{A},\ \mathbf{m})$.

Example 4.3.8. *Let the multiarrangement* $(\mathcal{A},\ \mathbf{m})$ *be described by the following graph (Figure 4.7).*

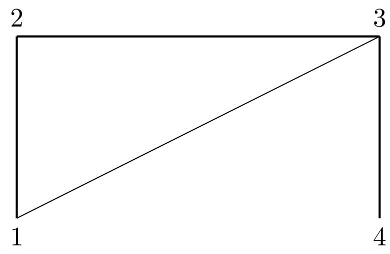

Figure 4.7: The graph of a 4-multiarrangement $\frac{\prod_{1\le s<t\le 4}(x_s-x_t)^{2m+2}}{(x_1-x_4)(x_2-x_4)}$.

The vector fields

$$\begin{aligned}
\theta_0 &= \sum_{i=1}^{4} \partial_i, \\
\vartheta_1^m &= \sum_{u=1}^{4} \int^{x_u} \lambda(t)^m \lambda_1(t) dt \partial_u \\
&= \sum_{u=1}^{4} \int^{x_u} (t-x_1)^m (t-x_2)^{m+1} (t-x_3)^{m+1} (t-x_4)^m dt \partial_u, \\
\vartheta_2^m &= \sum_{u=1}^{4} \int^{x_u} \lambda(t)^m \lambda_2(t) dt \partial_u \\
&= \sum_{u=1}^{4} \int^{x_u} (t-x_1)^{m+1} (t-x_2)^{m} (t-x_3)^{m+1} (t-x_4)^m dt \partial_u, \\
\vartheta_3^m &= (x_3 - x_4) \sum_{u=1}^{4} \int^{x_u} \lambda(t)^m \lambda_3(t) dt \partial_u \\
&= (x_3 - x_4) \sum_{u=1}^{4} \int^{x_u} (t-x_1)^{m+1} (t-x_2)^{m+1} (t-x_3)^{m} (t-x_4)^m dt \partial_u
\end{aligned}$$

form a basis for the multiarrangement $(\mathcal{A},\ \mathbf{m})$.

Proposition 4.3.9. *The vector fields*

$$\begin{aligned}
\theta_0 &= \sum_{i=1}^{4} \partial_i, \\
\varpi_1^m &= \sum_{u=1}^{4} \int^{x_u} \lambda(t)^m \lambda_1(t) dt \partial_u \\
&= \sum_{u=1}^{4} \int^{x_u} (t-x_1)^m (t-x_2)^{m+1} (t-x_3)^{m+1} (t-x_4)^m dt \partial_u, \\
\varpi_2^m &= \sum_{u=1}^{4} \int^{x_u} \lambda(t)^m \lambda_2(t) dt \partial_u \\
&= \sum_{u=1}^{4} \int^{x_u} (t-x_1)^{m+1} (t-x_2)^{m} (t-x_3)^{m+1} (t-x_4)^m dt \partial_u, \\
\varpi_3^m &= \sum_{u=1}^{4} \int^{x_u} \lambda(t)^m \lambda_3''(t) dt \partial_u \\
&= \sum_{u=1}^{4} \int^{x_u} (t-x_1)^{m+1} (t-x_2)^{m+1} (t-x_3)^{m} (t-x_4)^{m+1} dt \partial_u
\end{aligned}$$

form a basis for the above multiarrangement $(\mathcal{A},\ \mathbf{m})$ *described in Figure 4.7.*

Proof. By Proposition 2.1.3, each vector field is contained in $\mathcal{D}(\mathcal{A},\ \mathbf{m})$. Note that $\deg \varpi_k^m = 4m+3$ for $1 \le k \le 2$ and $\deg \varpi_3^m = 4m+4$. Then $\sum_{k=1}^{3} \deg \varpi_k^m +$

$\deg \theta_0 = 12m + 10 = \deg Q(\mathcal{A}, \mathbf{m})$. After elementary transformations of the Saito matrix, it is sufficient to show the determinant of following martix is nonzero.

$$M = \begin{pmatrix} \varpi_1^m(x_2 - x_1) & \varpi_1^m(x_3 - x_1) & \varpi_1^m(x_4 - x_1) \\ \varpi_2^m(x_2 - x_1) & \varpi_2^m(x_3 - x_1) & \varpi_2^m(x_4 - x_1) \\ \varpi_3^m(x_2 - x_1) & \varpi_3^m(x_3 - x_1) & \varpi_3^m(x_4 - x_1) \end{pmatrix}.$$

Take the similar changes of variables in the proof of Theorem 4.3.3. Observe the following matrix where each entry is the non-zero leading monomial of entries of M up to a constant,

$$\begin{pmatrix} y_2^{4m+3} & y_2^{m+1} y_3^{3m+2} & y_2^{m+1} y_3^{m+1} y_4^{2m+1} \\ y_2^{4m+3} & y_2^m y_3^{3m+3} & y_2^m y_3^{m+1} y_4^{2m+2} \\ y_2^{4m+4} & y_2^{m+1} y_3^{3m+3} & y_2^{m+1} y_3^m y_4^{2m+3} \end{pmatrix}.$$

Then

$$m_{1,3} \cdot \begin{vmatrix} (\int_0^1 t^{3m+2}(t-1)^m dt) y_2^{4m+3} & ((-1)^m \int_0^1 t^{2m+1}(t-1)^{m+1} dt) y_2^m y_3^{3m+3} \\ (\int_0^1 t^{3m+2}(t-1)^{m+1} dt) y_2^{4m+4} & ((-1)^{m+1} \int_0^1 t^{2m+2}(t-1)^m dt) y_2^{m+1} y_3^{3m+3} \end{vmatrix}$$
$$= m_{1,3} \cdot |M'|$$

gives the maximum monomial $y_2^{6m+5} y_3^{4m+4} y_4^{2m+1}$, where $m_{1,3} = y_2^{m+1} y_3^{m+1} y_4^{2m+1}$. Consider the matrix M'' of the coefficients of the entries of the matrix M'.

$$M'' = \begin{pmatrix} \int_0^1 t^{3m+2}(t-1)^m dt & (-1)^m \int_0^1 t^{2m+1}(t-1)^{m+1} dt \\ \int_0^1 t^{3m+2}(t-1)^{m+1} dt & (-1)^{m+1} \int_0^1 t^{2m+2}(t-1)^m dt \end{pmatrix}.$$

If m is even, then the sign of the martrix M'' is

$$\begin{pmatrix} + & - \\ - & - \end{pmatrix}.$$

If m is odd, then the sign of the martrix M'' is

$$\begin{pmatrix} - & - \\ + & - \end{pmatrix}.$$

Thus the determinant of M'' is nonzero. The maximum monomial $y_2^{6m+5} y_3^{4m+4} y_4^{2m+1}$ has a non-zero coefficient. It is clear that the vector fields ϖ_k^m $(1 \leq k \leq 3)$ and θ_0 are linearly independent over S. $\square$

Example 4.3.10. *Let the multiarrangement* $(\mathcal{A}, \mathbf{m})$ *be described by the following graph (Figure 4.8).*

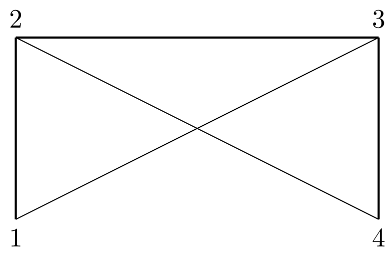

Figure 4.8: The graph of a 4-multiarrangement $\frac{\prod_{1\le s<t\le 4}(x_s-x_t)^{2m+2}}{(x_1-x_4)}$.

The vector fields

$$
\begin{aligned}
\theta_0 &= \sum_{i=1}^{4} \partial_i,\\
\vartheta_1^m &= \sum_{u=1}^{4} \int^{x_u} \lambda(t)^m \lambda_1(t) dt \partial_u\\
&= \sum_{u=1}^{4} \int^{x_u} (t-x_1)^m (t-x_2)^{m+1} (t-x_3)^{m+1} (t-x_4)^m dt \partial_u,\\
\vartheta_2^m &= (x_2-x_4) \sum_{u=1}^{4} \int^{x_u} \lambda(t)^m \lambda_2(t) dt \partial_u\\
&= (x_2-x_4) \sum_{u=1}^{4} \int^{x_u} (t-x_1)^{m+1} (t-x_2)^{m} (t-x_3)^{m+1} (t-x_4)^m dt \partial_u,\\
\vartheta_3^m &= (x_3-x_4) \sum_{u=1}^{4} \int^{x_u} \lambda(t)^m \lambda_3(t) dt \partial_u\\
&= (x_3-x_4) \sum_{u=1}^{4} \int^{x_u} (t-x_1)^{m+1} (t-x_2)^{m+1} (t-x_3)^{m} (t-x_4)^m dt \partial_u
\end{aligned}
$$

form a basis for the multiarrangement $(\mathcal{A},\ \mathbf{m})$.

Proposition 4.3.11. *The vector fields*

$$\begin{aligned}
\theta_0 &= \sum_{i=1}^{4} \partial_i, \\
\varpi_1^m &= \sum_{u=1}^{4} \int^{x_u} \lambda(t)^m \lambda_1(t) dt \partial_u \\
&= \sum_{u=1}^{4} \int^{x_u} (t-x_1)^m (t-x_2)^{m+1} (t-x_3)^{m+1} (t-x_4)^m dt \partial_u, \\
\varpi_2^m &= \sum_{u=1}^{4} \int^{x_u} \lambda(t)^m \lambda_2''(t) dt \partial_u \\
&= \sum_{u=1}^{4} \int^{x_u} (t-x_1)^{m+1} (t-x_2)^m (t-x_3)^{m+1} (t-x_4)^{m+1} dt \partial_u, \\
\varpi_3^m &= \sum_{u=1}^{4} \int^{x_u} \lambda(t)^m \lambda_3''(t) dt \partial_u \\
&= \sum_{u=1}^{4} \int^{x_u} (t-x_1)^{m+1} (t-x_2)^{m+1} (t-x_3)^m (t-x_4)^{m+1} dt \partial_u
\end{aligned}$$

form a basis for the above multiarrangement $(\mathcal{A}, \mathbf{m})$ *described in Figure 4.8.*

Proof. By Proposition 2.1.3, each vector field is contained in $\mathcal{D}(\mathcal{A}, \mathbf{m})$. Note that $\deg \varpi_1^m = 4m{+}3$ and $\deg \varpi_k^m = 4m{+}4$ for $2 \le k \le 3$. Then $\sum_{k=1}^{3} \deg \varpi_k^m + \deg \theta_0 = 12m + 11 = \deg Q(\mathcal{A}, \mathbf{m})$. After elementary transformations of the Saito matrix, it is sufficient to show the determinant of following martix is non-zero.

$$M = \begin{pmatrix} \varpi_1^m(x_2 - x_1) & \varpi_1^m(x_3 - x_1) & \varpi_1^m(x_4 - x_1) \\ \varpi_2^m(x_2 - x_1) & \varpi_2^m(x_3 - x_1) & \varpi_2^m(x_4 - x_1) \\ \varpi_3^m(x_2 - x_1) & \varpi_3^m(x_3 - x_1) & \varpi_3^m(x_4 - x_1) \end{pmatrix}.$$

Take the similar changes of variables in the proof of Theorem 4.3.3. Observe the following matrix where each entry is the nonzero leading monomial of entries of M up to a constant,

$$\begin{pmatrix} y_2^{4m+3} & y_2^{m+1} y_3^{3m+2} & y_2^{m+1} y_3^{m+1} y_4^{2m+1} \\ y_2^{4m+4} & y_2^{m} y_3^{3m+4} & y_2^{m} y_3^{m+1} y_4^{2m+3} \\ y_2^{4m+4} & y_2^{m+1} y_3^{3m+3} & y_2^{m+1} y_3^{m} y_4^{2m+3} \end{pmatrix}.$$

Then $m_{2,1} \cdot m_{3,2} \cdot m_{1,3}$ gives the unique maximum monomial $y_2^{6m+6} y_3^{4m+4} y_4^{2m+1}$. Thus the determinant of M is nonzero. It is clear that the vector fields ϖ_k^m $(1 \le k \le 3)$ and θ_0 are linearly independent over S. □

Example 4.3.12. *Let the multiarrangement* $(\mathcal{A}, \mathbf{m})$ *be described by the following graph (Figure 4.9).*

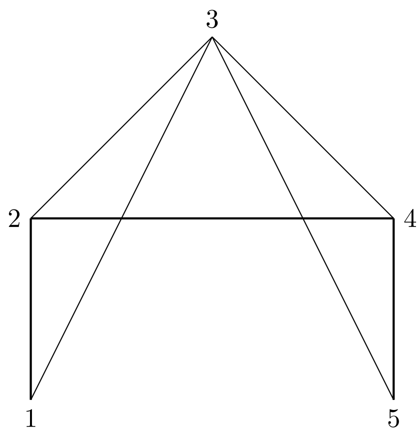

Figure 4.9: The graph of a 5-multiarrangement $\frac{\prod_{1\leq s<t\leq 5}(x_s-x_t)^{2m+2}}{(x_1-x_4)(x_1-x_5)(x_2-x_5)}$.

The vector fields

$$\begin{aligned}
\theta_0 &= \sum_{i=1}^{5} \partial_i, \\
\vartheta_1^m &= \sum_{u=1}^{5} \int^{x_u} \lambda(t)^m \lambda_1(t) dt \partial_u \\
&= \sum_{u=1}^{5} \int^{x_u} (t-x_1)^m (t-x_2)^{m+1} (t-x_3)^{m+1} (t-x_4)^{m+1} (t-x_5)^m dt \partial_u, \\
\vartheta_2^m &= \sum_{u=1}^{5} \int^{x_u} \lambda(t)^m \lambda_2(t) dt \partial_u \\
&= \sum_{u=1}^{5} \int^{x_u} (t-x_1)^{m+1} (t-x_2)^m (t-x_3)^{m+1} (t-x_4)^{m+1} (t-x_5)^m dt \partial_u, \\
\vartheta_3^m &= (x_3-x_5) \sum_{u=1}^{5} \int^{x_u} \lambda(t)^m \lambda_3(t) dt \partial_u \\
&= (x_3-x_5) \sum_{u=1}^{5} \int^{x_u} (t-x_1)^{m+1} (t-x_2)^{m+1} (t-x_3)^m (t-x_4)^{m+1} (t-x_5)^m dt \partial_u, \\
\vartheta_4^m &= \sum_{u=1}^{5} \int^{x_u} \lambda(t)^m \lambda_4'(t) dt \partial_u \\
&= \sum_{u=1}^{5} \int^{x_u} (t-x_1)^m (t-x_2)^{m+1} (t-x_3)^{m+1} (t-x_4)^m (t-x_5)^{m+1} dt \partial_u
\end{aligned}$$

form a basis for the multiarrangement $(\mathcal{A},\ \mathbf{m})$.

Proposition 4.3.13. *The vector fields*

$$
\begin{aligned}
\theta_0 &= \sum_{i=1}^{5} \partial_i,\\
\varpi_1^m &= \sum_{u=1}^{5} \int^{x_u} \lambda(t)^m \lambda_1(t) dt \partial_u\\
&= \sum_{u=1}^{5} \int^{x_u} (t-x_1)^m (t-x_2)^{m+1} (t-x_3)^{m+1} (t-x_4)^{m+1} (t-x_5)^m dt \partial_u,\\
\varpi_2^m &= \sum_{u=1}^{5} \int^{x_u} \lambda(t)^m \lambda_2(t) dt \partial_u\\
&= \sum_{u=1}^{5} \int^{x_u} (t-x_1)^{m+1} (t-x_2)^{m} (t-x_3)^{m+1} (t-x_4)^{m+1} (t-x_5)^m dt \partial_u,\\
\varpi_3^m &= \sum_{u=1}^{5} \int^{x_u} \lambda(t)^m \lambda_3''(t) dt \partial_u\\
&= \sum_{u=1}^{5} \int^{x_u} (t-x_1)^{m+1} (t-x_2)^{m+1} (t-x_3)^{m} (t-x_4)^{m+1} (t-x_5)^{m+1} dt \partial_u,\\
\varpi_4^m &= \sum_{u=1}^{5} \int^{x_u} \lambda(t)^m \lambda_4'(t) dt \partial_u\\
&= \sum_{u=1}^{5} \int^{x_u} (t-x_1)^{m} (t-x_2)^{m+1} (t-x_3)^{m+1} (t-x_4)^{m} (t-x_5)^{m+1} dt \partial_u
\end{aligned}
$$

form a basis for the above multiarrangement $(\mathcal{A},\ \mathbf{m})$ described in Figure 4.9.

Proof. By similar consideration, it is sufficient to show the determinant of following martix is nonzero.

$$
M = \begin{pmatrix}
\varpi_1^m(x_2-x_1) & \varpi_1^m(x_3-x_1) & \varpi_1^m(x_4-x_1) & \varpi_1^m(x_5-x_1)\\
\varpi_2^m(x_2-x_1) & \varpi_2^m(x_3-x_1) & \varpi_2^m(x_4-x_1) & \varpi_2^m(x_5-x_1)\\
\varpi_3^m(x_2-x_1) & \varpi_3^m(x_3-x_1) & \varpi_3^m(x_4-x_1) & \varpi_3^m(x_5-x_1)\\
\varpi_4^m(x_2-x_1) & \varpi_4^m(x_3-x_1) & \varpi_4^m(x_4-x_1) & \varpi_4^m(x_5-x_1)
\end{pmatrix}.
$$

Take the similar changes of variables in the proof of Theorem 4.3.3. Observe the following matrix where each entry is the nonzero leading monomial of entries of M up to a constant,

$$
\begin{pmatrix}
y_2^{5m+4} & y_2^{m+1}y_3^{4m+3} & y_2^{m+1}y_3^{m+1}y_4^{3m+2} & y_2^{m+1}y_3^{m+1}y_4^{m+1}y_5^{2m+1}\\
y_2^{5m+4} & y_2^{m}y_3^{4m+4} & y_2^{m}y_3^{m+1}y_4^{3m+3} & y_2^{m}y_3^{m+1}y_4^{m+1}y_5^{2m+2}\\
y_2^{5m+5} & y_2^{m+1}y_3^{4m+4} & y_2^{m+1}y_3^{m}y_4^{3m+4} & y_2^{m+1}y_3^{m}y_4^{m+1}y_5^{2m+3}\\
y_2^{5m+4} & y_2^{m+1}y_3^{4m+3} & y_2^{m+1}y_3^{m+1}y_4^{3m+2} & y_2^{m+1}y_3^{m+1}y_4^{m}y_5^{2m+2}
\end{pmatrix}.
$$

Then

$$
\begin{aligned}
&m_{4,3}\cdot m_{1,4}\cdot \begin{vmatrix}
(\int_0^1 t^{4m+3}(t-1)^m dt) y_2^{5m+4} & ((-1)^m \int_0^1 t^{3m+2}(t-1)^{m+1} dt) y_2^m y_3^{4m+4}\\
(\int_0^1 t^{4m+3}(t-1)^{m+1} dt) y_2^{5m+5} & ((-1)^{m+1} \int_0^1 t^{3m+3}(t-1)^{m} dt) y_2^{m+1} y_3^{4m+4}
\end{vmatrix}\\
&= y_2^{2m+2} y_3^{2m+2} y_4^{4m+3} y_5^{2m+1} \cdot |M'|
\end{aligned}
$$

gives the maximum monomial $y_2^{8m+7}y_3^{6m+6}y_4^{4m+3}y_5^{2m+1}$. Consider the matrix M'' of the coefficients of the entries of the matrix M'.

$$M'' = \begin{pmatrix} \int_0^1 t^{4m+3}(t-1)^m dt & (-1)^m \int_0^1 t^{3m+2}(t-1)^{m+1} dt \\ \int_0^1 t^{4m+3}(t-1)^{m+1} dt & (-1)^{m+1} \int_0^1 t^{3m+3}(t-1)^m dt \end{pmatrix}.$$

If m is even, then the sign of the martrix M'' is

$$\begin{pmatrix} + & - \\ - & - \end{pmatrix}.$$

If m is odd, then the sign of the martrix M'' is

$$\begin{pmatrix} - & - \\ + & - \end{pmatrix}.$$

Thus the determinant of M'' is nonzero. The maximum monomial $y_2^{8m+7}y_3^{6m+6}y_4^{4m+3}y_5^{2m+1}$ has a nonzero coefficient. It is clear that the vector fields ϖ_k^m $(1 \leq k \leq 4)$ and θ_0 are linearly independent over S. $\square$

We propose a conjecture for the linear independence of the vector fields.

Conjecture 4.3.14. *The vector fields*

$$\begin{aligned}
\theta_0 &= \sum_{i=1}^{\ell} \partial_i, \\
\varpi_k^m &= \sum_{u=1}^{\ell} \int^{x_u} \lambda(t)^m \lambda_k(t) dt \partial_u \qquad (1 \leq k \leq j-1), \\
\varpi_h^m &= \sum_{u=1}^{\ell} \int^{x_u} \lambda(t)^m \lambda_h''(t) dt \partial_u \qquad (j \leq h \leq i), \\
\varpi_s^m &= \sum_{u=1}^{\ell} \int^{x_u} \lambda(t)^m \lambda_s'(t) dt \partial_u \qquad (i+1 \leq s \leq \ell-1),
\end{aligned}$$

are linearly independent over S.

For $\ell \geq 2$, recall that $\theta_{a_1,\ldots,a_\ell,b} = \sum_{i=1}^{\ell} \left(\int^{x_i} t^b (t-x_1)^{a_1} \ldots (t-x_\ell)^{a_\ell} dt\right) \partial_i$. Let

$$\Theta = \{\theta_{a_1,\ldots,a_\ell,b} \mid a_i, b \in \mathbb{Z}_{\geq 0}\} \cup \{\theta_0\}.$$

It is natural to have the following conjecture.

Conjecture 4.3.15. *Any ℓ elements in Θ are linearly independent over S.*

If Conjecture 4.3.15 is ture, it is clear that Conjecture 4.3.14 is ture. Note that by Proposition 2.1.3, each vector field in Conjecture 4.3.14 is contained in $\mathcal{D}(\mathcal{A}, \mathbf{m})$ where $(\mathcal{A}, \mathbf{m})$ is defined in Definition 4.3.2. And $\deg \varpi_k^m = m\ell+\ell-1$ for $1 \leq k \leq j-1$ and $i+1 \leq k \leq \ell-1$, $\deg \varpi_k^m = m\ell+\ell$ for $j \leq k \leq i$. We have $\sum_{k=1}^{\ell} \deg \varpi_k^m + \deg \theta_0 = (m\ell+\ell-1)(\ell-1)+i-j+1 = \deg Q(\mathcal{A}, \mathbf{m})$. Then the vector fields θ_0 and ϖ_k^m $(1 \leq k \leq \ell-1)$ form a basis for the multiarrangement $(\mathcal{A}, \mathbf{m})$ defined in Definition 4.3.2.

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