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Polymer brush inspired by ribosomal RNA transcription

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Abstract

Pre-ribosomal RNAs are synthesized during the transcription by RNA polymerase I molecules localized at the surfaces of a nucleolus subcompartment. Inspired by the ribosomal RNA transcription, we here develop a scaling theory of a brush of polymers, where monomers are added to their grafted ends in the steady state. Our theory predicts that monomers newly added to the polymers stay at the vicinity of the surface due to the slow dynamics of the polymers and thus the polymer volume fraction increases with increasing the polymerization rate. The excluded volume interaction between polymers and reactant monomers suppresses the diffusion of reactant monomers and thus decreases the polymerization rate. The extent of the suppression of monomer diffusion increases with increasing the polymerization rate because the diffusion length decreases, rather than the condensation of polymers due to their slow dynamics.

Keywords: Polymer brush, Polymer dynamics, Polymerization, Transcription, Nascent ribosomal RNA

1 Introduction

Nucleolus is a condensate of ribosomal RNAs and proteins and is a factory of ribosomes. In higher organisms, nucleolus forms a multi-phase structure,

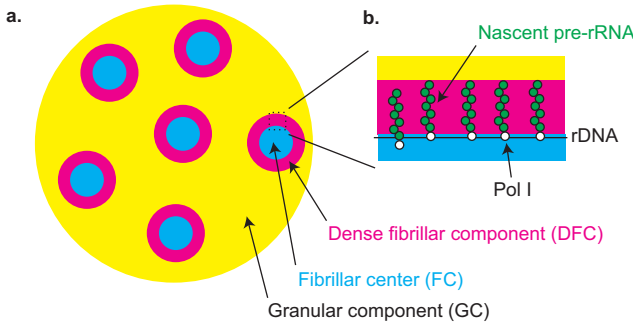
2 *Polymer brush inspired by ribosomal RNA transcription*

Fig. 1 Multi-phase structure of nucleolus: **a.** A nucleolus is composed of three phases – fibrillar centers (FCs), granular component (GC), and dense fibrillar component (DFC). Multiple FCs are dispersed in a GC and there is a layer of DFC at the surface of each FC. **b.** Ribosomal DNAs (rDNAs) are localized at the surfaces of FCs and is transcribed by RNA polymerase Is (Pol Is) entrapped in FCs. Nascent pre-rRNAs are end-grafted to the surfaces of FCs via Pol Is. These nascent pre-rRNAs are modeled as a polymer brush.

where multiple subcompartments, called fibrillar centers (FCs), are dispersed in another subcompartment, called a granular component (GC) [1], see fig. 1a. Indeed, distinct phases, called dense fibrillar components (DFCs), are assembled between each FC and GC. Nucleoli and their subcompartments have been thought to be assembled by liquid-liquid phase separation (LLPS) because of the fact that the major protein components of each phase show phase separation in test tube experiments [2]. However, condensates assembled by LLPS show coarsening and coalescence to minimize the surface energy. It is in contrast to the situation of nucleolus, in which multiple FCs are dispersed.

Indeed, if RNA polymerase I (Pol I), which transcribes precursor ribosomal RNAs (pre-rRNAs), is depleted, FCs show fusion to be one large condensate, much like condensates assembled by LLPS [3]. This implies that the transcription of pre-rRNAs somehow suppresses the fusion of FCs. The transcription of pre-rRNAs happens at the interface between FCs and DFCs [4, 5]. Multiple copies of pre-rRNAs are tandemly repeated in the DNA region localized at the interfaces (only 2 - 3 active copies per FC in an estimate [4], but can be more) and many Pol Is are loaded to each active copy of length ~ 15 kbps for transcription [1]. One interesting hypothesis is that nascent pre-rRNAs, which are end-grafted to the surfaces of FCs via Pol Is, act as polymer brushes that suppress the fusion of FCs [6], much like the situation that polymer brushes at the surfaces of colloids suppress the flocculation as predicted by Witten and Pincus [7], see fig. 1b.

Single-stranded RNAs are relatively flexible polymers with the aspect ratio of its Kuhn unit in the order of unity. In the physiological salt concentration, the electrostatic interaction between RNAs is screened by salt ions. Nascent pre-rRNAs end-grafted at a surface can be thus treated as a brush of electrically neutral flexible polymers. The simplest model of a neutral flexible polymer brush is the Alexander model, with which the monomer concentration is assumed to be uniform in the brush [8]. By using the Flory-type argument,

Alexander predicted that the height of the brush is proportional to the number of monomers in the chains, implying that the chains in the brush are stretched due to the excluded volume interaction with the neighboring chains. The scaling argument by de Gennes augments the structures in the length scales shorter than the brush height by using the fact that the correlation length of chains in the brush is set by the distance between the grafting points [9]. The self-consistent field theory predicts that the monomer concentration is a parabolic function of the distance from the surface in the classical limit [10–12].

In general salt conditions, RNAs should be treated as polyelectrolytes. In the low salt concentration, the charges of polyelectrolytes in the brush are mainly neutralized by counterions. In his pioneering work, Pincus used the scaling theory to predict that if the linear charge density and the grafting density of polyelectrolytes in the brush are large enough, counterions are entrapped in the brush region due to the strong electrostatic interaction between the polyelectrolytes and counterions, and the polyelectrolytes are swollen due to the osmotic pressure of the counterions (osmotic regime) [13]. If the free energy due to the translational entropy of counterions dominates the free energy due to the electrostatic interaction between polyelectrolytes and counterions, the brush collapses because counterions escape from the brush (Pincus brush regime). Polyelectrolyte brushes experience the crossover between the osmotic and Pincus brush regimes by applying electric fields because polyelectrolytes collapse to neutralize the charges induced by the applied electric fields [14]. In the high salt concentration, polyelectrolyte charges are mainly neutralized by salt ions (salty brush regime). In this regime, a polyelectrolyte brush behaves as a brush of electrically neutral polymers because the monomer-monomer electrostatic interactions are screened by salt ions and are reduced to the second virial coefficient [13].

Chemically similar to a polymer brush of RNAs is a polymer brush of DNA. DNA brushes are useful systems to study how the transcription dynamics depends on the local packing density of DNA [15–17]. In the low salt concentration, DNAs in a brush are swollen by the osmotic pressure of counterions due to the polyelectrolyte nature of DNA [18]. Double-stranded DNA is a semiflexible polymer with the aspect ratio of its Kuhn unit is in the order of 100 [19]. Schaefer, Joanny, and Pincus predicted that semiflexible polymers have the marginal solvent regime, in which the mean-field theory is effective although the two-body interaction between the Kuhn segments dominates the three-body interaction between them, because the excluded volume of segments with high aspect ratio is small [20]. In the physiological salt concentration, DNA brush behaves as a brush of electrically neutral semiflexible polymers in the marginal solvent regime [18]. RNA brushes probably do not have the marginal solvent regime because the aspect ratio of their Kuhn segments is relatively small.

What is the difference between the nascent pre-rRNA brushes at the surfaces of nucleolus subcompartments and the above-mentioned concepts of polymer brushes? First, RNA-binding proteins (RBPs) are bound to some RNAs and the multi-valent interaction between these RBPs drives

LLPS [21, 22]. Indeed, nuclear bodies, such as paraspeckles, are assembled by the multi-valent interaction between RBPs bound to architectural RNAs [22–25]. Fibrillarin RNA-binding proteins are a major component of the DFC layers at the interface between FCs and GC and are the RBPs of nascent pre-rRNAs [2, 4]. Fibrillarin shows phase separation in the physiological concentration [2]. In our recent theory, we have predicted that the DFCs at the interface between FCs and GC are polymer brushes of nascent pre-rRNAs that are swollen to accommodate RBPs in the brush [6]. There are analogous systems with DNA – DNA brushes with DNA-binding proteins, such as histones [26, 27], transcription factors [28], and Structural Maintenance of Chromosome proteins [29–32]. Second, nascent pre-rRNAs are end-grafted to the surface via Pol Is that add nucleoside triphosphates (which are monomers of RNAs) to the nascent pre-rRNAs. Because of the slow dynamics of polymers, RNA monomers newly added to the nascent pre-rRNAs stay at the vicinity of the surface for a relatively long time. The high concentration of RNA monomers at the surface may suppress the diffusion of reactant RNA monomers and limit the kinetics of transcription. This situation is very different from cases in which monomers are added to the free ends of polymers in a brush, such as the free-radical polymerization initiated from a surface [33, 34].

Inspired by the dynamical nature of the nascent pre-rRNA brush, we here use the scaling theory to predict the structure of polymers in a brush, where monomers are added to the grafted-ends with a constant rate (sec. 2) and the polymerization rate, which is due to the diffusion of reactant monomers (sec. 3). We do not take into account the transcriptional termination and co-transcriptional processing to focus on the polymer dynamics driven by the polymerization reaction. **In this paper, we treat cases in which the heights of end-grafted RNAs are smaller than the radius of the curvature of the surface to avoid the complication arising from the spherical geometry, see fig. 1.** The nascent pre-rRNAs during transcription is a type of living polymers, which are defined as non-equilibrium polymers due to the polymerization and depolymerization kinetics [35]. Indeed, our theory predicts that the elongation of Pol I transcription is too slow for the polymer dynamics to be relevant for the case of pre-rRNA transcription. However, we anticipate that this theory can be extended to polymerization reaction in other biological contexts and may motivate polymer chemists to design polymer brushes by taking advantage of the slow relaxation dynamics of polymers.

2 Polymer brush inspired by pre-rRNA transcription

2.1 Mushroom regime

We first treat a single polymer (such as nascent pre-rRNA) end-grafted to a planar surface via a catalyst (such as Pol I) that adds monomers to its grafted end with a constant rate k_p . **We consider the steady state, where**

the structure of the end-grafted polymer does not change as time elapses. The Kuhn length of the polymer is b . For simplicity, we treat the dynamics of polymers in the brush by using the Rouse model, with which the hydrodynamic interaction between monomers is neglected [36, 37]. This model predicts that there is a number $g(t)$ of monomers (at time t), where subchains composed of smaller number of monomers have relaxed and thus assume the equilibrium conformation, while subchains composed of larger number of monomers have not yet relaxed. Without the hydrodynamic interaction, the diffusion constant of a subchain composed of g monomers is $D_g (= D_1/g)$, where D_1 is the diffusion constant of a monomer. The relaxation time of the subchain is $\xi^2/D_g (= \tau_1 g^{11/5})$ with the size of the subchain, $\xi = bg^{3/5}$, in an athermal solvent with the Flory exponent and the relaxation time of a monomer $\tau_1 (= b^2/D_1)$. The number $g(t)$ of monomers in a subchain that has relaxed in time t is derived as

$$g(t) = \left(\frac{t}{\tau_1} \right)^{5/11} \quad (1)$$

by using $\xi^2/D_g \approx t$. **Eq. (1) implies that the number $g(t)$ of monomers in relaxed subchains increases as time elapses.**

The number of monomers newly added to the chain in time t is $k_p t$. The relaxation is faster than the polymerization for $k_p t < g(t)$. The polymerization kinetics becomes as fast as the relaxation at a crossover time $\tau_c (= k_p^{-11/6} \tau_1^{-5/6})$. The number of monomers in a subchain relaxed in the time τ_c is

$$g_c = g(\tau_c) = (k_p \tau_1)^{-5/6}. \quad (2)$$

The size of the subchain is

$$\xi_c = bg_c^{3/5} = \frac{b}{\sqrt{k_p \tau_1}}. \quad (3)$$

The polymer volume fraction at this length scale is

$$\phi_c = \frac{b^3 g_c}{\xi_c^3} = (k_p \tau_1)^{2/3}. \quad (4)$$

Now we consider the structure of the chain in the length scale larger than ξ_c , see fig. 2a. ξ_c is the minimum length scale of the relaxed region in the chain. **Eq. (1) predicts that the subchain composed of $g(t - t_0)$ monomers around the monomer polymerized at time t_0 has already relaxed at time t ($t - t_0 > \tau_c$), but subchains in the larger length scales have not. We define the subchain composed of $g(t - t_0)$, which is at the border between the relaxed and not-yet-relaxed states, as a blob (which we call a relaxed blob). Because the size of a relaxed blob depends on**

the time t_0 at which it is polymerized, the polymer is treated as a chain of relaxed blobs of varying sizes, see fig. 2a. In the limit of large polymerization rate, $k_p \rightarrow \infty$, newly added monomers stay at the position where they are added to the chain and are then pushed out by monomers added next due to the excluded volume interaction between monomers. This process fills the space at the vicinity of the catalyst with monomers. In a finite polymerization rate, relaxed blobs (instead of monomers) fill the volume around the catalyst, see fig. 2a. The volume r^3 occupied by a subchain polymerized in time $t(> \tau_c)$ has the form

$$\begin{aligned} r^3 &= \xi_c^3 + \int_0^{t-\tau_c} k_p dt_0 \frac{\xi^3(t-t_0)}{g(t-t_0)} \\ &\approx \xi_c^3 \left(\frac{t}{\tau_c} \right)^{15/11}, \end{aligned} \quad (5)$$

where the first term of the first equation is the volume of the smallest blob and the second term of the first equation is the volume filled by larger blobs. The subchain filling the volume r^3 is composed of n ($= k_p t$) monomers and the size of the subchain thus has the form

$$r \approx \xi_c \left(\frac{n}{g_c} \right)^{5/11}, \quad (6)$$

see fig. 2b. This approach is an extension of the Daoud-Cotton scaling theory [38] to the relaxed blobs filled in the geometry of a single chain (and a planar brush): in our case, the size of blobs increases as a function of the distance r from the grafted point because longer time has elapsed after monomers are added to the chain for blobs at larger distance r . The local polymer volume fraction for $r > \xi_c$ thus has the form

$$\phi_p(r) = \frac{b^3 g}{\xi^3} = \phi_c \left(\frac{r}{\xi_c} \right)^{-4/5}, \quad (7)$$

see fig. 2c. Eq. (7) is derived by using $\xi = b g^{3/5}$ and then use eq. (1), $n = k_p t$, and eq. (6) to represent it as a function of r . Eq. (7) indeed has the same form as the average polymer volume fraction $b^3 n / r^3$. The polymer volume fraction ϕ_p thus is larger than an end-grafted polymer at the equilibrium for $r > \xi_c$ due to the slow relaxation dynamics of polymers, compare the solid and broken lines in fig. 2c.

2.2 Brush regime

Next, we treat a brush of polymers that are end-grafted to a surface via catalysts that add monomers to their grafted-ends with a constant rate k_p , see fig.

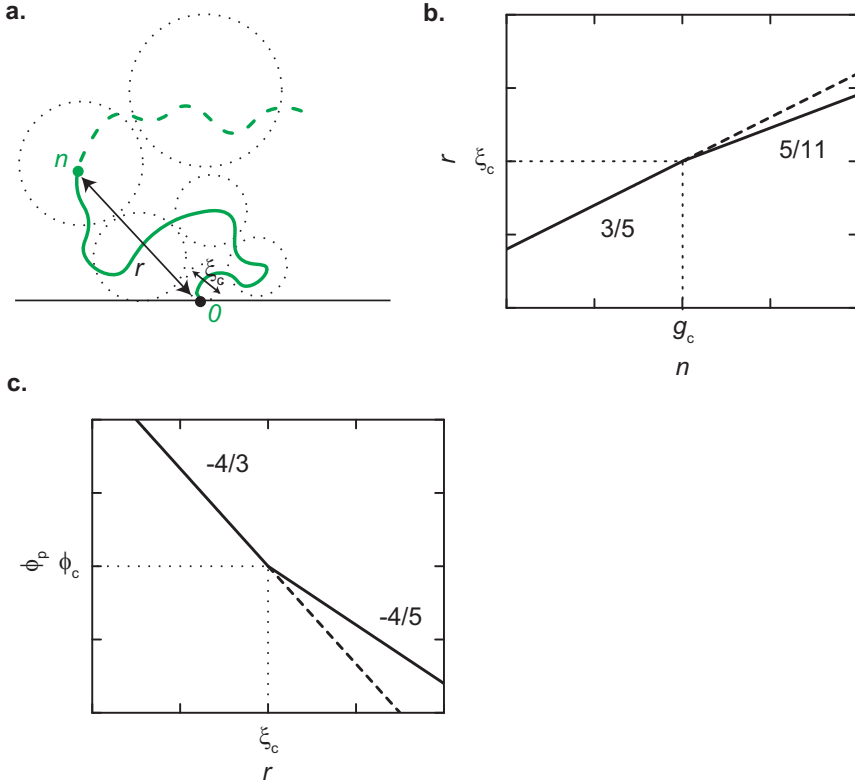


Fig. 2 Mushroom regime. **a.** A polymer (nascent pre-rRNA) synthesized by a catalyst (Pol I) at a surface. In general, there can be multiple catalysts at the surface, but they are separated so that the interaction between end-grafted polymers is negligible (mushroom regime). Monomers are added to the grafted-end of the polymer with a constant rate. The polymer is considered as a chain of blobs, where the subchain inside of each blob is already relaxed. The smallest blob has the length ξ_c and is composed of g_c monomers, see eqs. (2) and (3). These blobs are space-filling. **b.** The size r of a subchain is shown as a function of the number n of monomers of the subchain (double-logarithm plot). **c.** The polymer volume fraction ϕ_p is shown as a function of the distance r from the grafting point (double-logarithm plot). The broken lines in **b** and **c** are the case of an end-grafted polymer in the thermodynamic equilibrium.

3a. These catalysts are distributed to the surface with the surface density σ . The correlation length of the brush is the distance ξ_b ($= \sigma^{-1/2}$) between the grafting points [9]. In the length scale smaller than ξ_b , the polymer conformation follows the same scaling relationship with the case of mushroom regime because there is no interaction between polymers, see the region of $z < \xi_b$ in fig. 3b. By using eq. (6), the number of monomers in $r < \xi_b$ is derived as

$$g_b = k_p \tau_b = g_{eq} \left(\frac{k_p \tau_r}{g_{eq}} \right)^{4/15}, \quad (8)$$

where τ_b is the time necessary to polymerize g_b monomers. In eq. (8), we used the number g_{eq} of monomers in the correlation volume ξ_b^3 in the thermodynamic equilibrium and the relaxation time τ_r of the subchain composed of g_{eq} monomers:

$$g_{eq} = \frac{1}{(\sigma b^2)^{5/6}} \quad (9)$$

$$\tau_r = \frac{\xi_b^2}{D_1/g_{eq}} = \tau_1(\sigma b^2)^{-11/6}. \quad (10)$$

The number g_b of monomers in the volume ξ_b^3 returns to that g_{eq} in the equilibrium for $k_p^{-1}g_{eq} = \tau_r$, see eq. (8). The number of monomers in the region $r < \xi_b$ increases with increasing the polymerization rate k_p . The polymer volume fraction at the length scale of ξ_b is $\phi_b = b^3 g_b / \xi_b^3$, see eq. (8).

In the length scale larger than ξ_b , the relaxed blobs are confined in the area ξ_b^2 due to the interaction between end-grafted polymers and are stacked towards the z -direction. The height z of n -th monomer from the grafted-end is

$$\begin{aligned} z &= \sigma \left(\xi_c^3 + \int_0^{t-\tau_c} k_p dt_0 \frac{\xi^3(t-t_0)}{g(t-t_0)} \right) \\ &\approx \sigma \xi_c^3 \left(\frac{n}{k_p \tau_c} \right)^{15/11}, \end{aligned} \quad (11)$$

where we used $n = k_p t$ to derive the last form. **The position z of a blob is determined by the fact that the volume z/σ is filled with the smallest blob (the first term of the first equation of eq. (11)) and larger blobs (the second term of the first equation of eq. (11)), where these blobs are produced during time t ($= n/k_p$) after the blob at z is polymerized.** Note that $z = \xi_b$ for $n = g_b$ and thus eq. (11) smoothly crossovers with eq. (6) at $n = g_b$.

The size of the relaxed blob $\xi(t)$ becomes the size of the blob ξ_b of the equilibrium brush in the time $t = \tau_r$. The distance with which a monomer travels in the relaxation time τ_r is derived by substituting $n = g_r$ ($= k_p \tau_r$) into eq. (11) as

$$\lambda = \xi_b \frac{k_p \tau_r}{g_{eq}}, \quad (12)$$

see fig. 3b. It is interesting to note that eq. (12) has the form of the brush height of polymers composed of g_r ($= k_p \tau_r$) monomers in the thermodynamic equilibrium. This implies that the effect of the slow dynamics of polymers is not dramatic. Eq. (12) returns to $\lambda = \xi_b$ for $k_p^{-1}g_{eq} = \tau_r$.

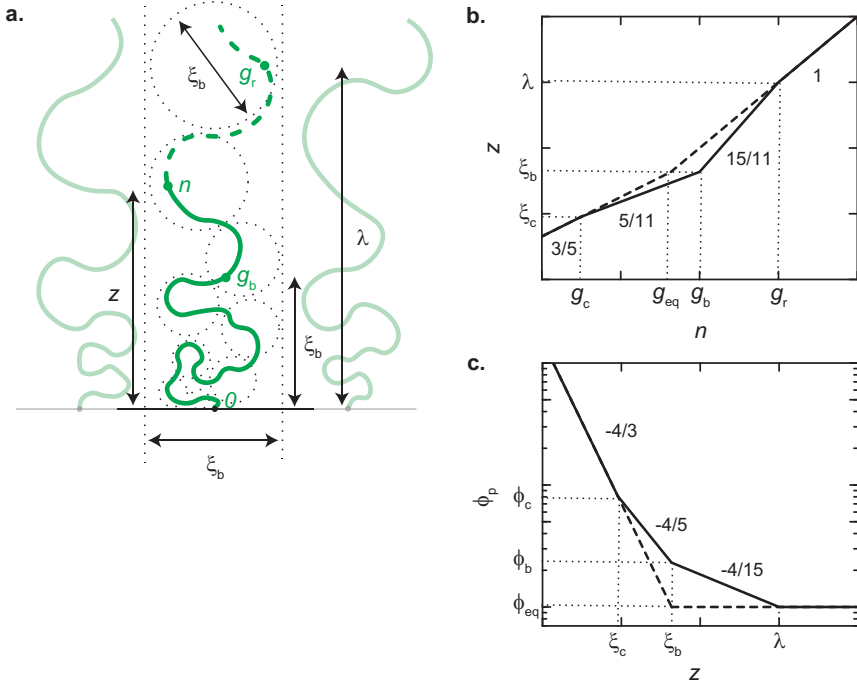


Fig. 3 Brush regime: **a.** Polymers (pre-rRNAs) synthesized by catalysts (Pol Is) at a surface. The grafting density σ is relatively large so that the interaction between the end-grafted polymers is significant. Monomers are added to the grafted-ends of the polymers with a constant rate (brush regime). An end-grafted polymer in the brush is considered as a chain of blobs, where the subchain in each blob is already relaxed. These blobs are space-filling and are confined in the area ξ_b^2 ($= \sigma^{-1}$) due to the interaction between end-grafted polymers. **b.** The height z of a subchain is shown as a function of the number n of monomers of the subchain (double-logarithm). **c.** The polymer volume fraction ϕ_p is shown as a function of the distance z from the surface (double-logarithm). The broken lines in **b** and **c** are the case of an end-grafted polymer in the thermodynamic equilibrium.

The local polymer volume fraction for $\xi_b < z < \lambda$ has the form

$$\phi_p(z) = \frac{b^3 g}{\xi^3} = \phi_{\text{eq}} \left(\frac{z}{\lambda} \right)^{-4/15}, \quad (13)$$

see fig. 3c. Eq. (13) is derived by using $\xi = bg^{3/5}$ and then use eq. (1), $n = k_p t$, and eq. (11) to represent it as a function of z . Eq. (13) has the same form as the average polymer volume fraction $b^3 \sigma n / z$. In eq. (13), we used the polymer volume fraction ϕ_{eq} of a blob of the equilibrium brush

$$\phi_{\text{eq}} = \frac{b^3 g_{\text{eq}}}{\xi_b^3} = (\sigma b^2)^{2/3}. \quad (14)$$

The polymer volume fraction $\phi_p(z)$ for $z > \lambda$ is ϕ_{eq} , see fig. 3c.

2.3 Brush height

For the case of the transcription of nascent pre-rRNAs on the surfaces of FCs, these pre-rRNAs experience the co-transcriptional processing and are released when Pol I reach the terminator sequence. To focus on the roles played by the polymerization kinetics in the structure of the polymer brush, we neglect these complex processes and consider cases in which the end-grafted polymers are randomly scissored by other catalysts (such as ribonucleases). In this case, the number N of monomers in each end-grafted polymer in the brush follows the kinetic equation

$$\frac{dN}{dt} = k_p - \frac{N}{\tau_d}, \quad (15)$$

where τ_d is the average time of polymer degradation. In the steady state, the number of monomers in each end-grafted chain is $N = k_p \tau_d$.

i) The case of $\tau_d > \tau_r$: In the limit of small polymerization rate k_p , the number N of monomers in each end-grafted polymer is too small and the interaction between the polymers is negligible (mushroom regime). Each end-grafted polymer is relaxed in the entire length scale for $k_p \tau_r / g_{eq} < 1$ (quasi-equilibrium mushroom regime) and thus the size of the end-grafted polymer is

$$h = b(k_p \tau_d)^{3/5} = \xi_b \left(\frac{\tau_d k_p \tau_r}{\tau_r g_{eq}} \right)^{3/5}, \quad (16)$$

see the cyan line in fig. 4. It is not the true equilibrium because of the polymerization and degradation. **The size does not depend on the distance ξ_b between the grafting points in the mushroom regime, but is expressed in terms of ξ_b in the last form of eq. (16) to show the crossover.** For $h > \xi_b$, the interaction between end-grafted polymers becomes significant and the end-grafted polymers thus become a brush (quasi-equilibrium brush regime). The brush height has the form

$$h = \xi_b \frac{\tau_d k_p \tau_r}{\tau_r g_{eq}}, \quad (17)$$

see the cyan line in fig. 4. For the case of $k_p \tau_r / g_{eq} < 1$, the polymers in the brush are relaxed in the entire length scale. In contrast, the polymer volume fraction becomes larger than that at the equilibrium in the region $\xi_c < z < \lambda$, see fig. 3. The brush height is given by eq. (17) both for $k_p \tau_r / g_{eq} < 1$ and $k_p \tau_r / g_{eq} > 1$. This implies that although the polymer volume fraction increases due to the slow polymer dynamics in $\xi_c < z < \lambda$, its effect on the brush height is only within the numerical factor of order unity, see also fig. 3b.

ii) The case of $\tau_d < \tau_r$: In the limit of small polymerization rate k_p , the system is in the quasi-equilibrium mushroom regime, see eq. (16) and the magenta line in fig. 4. For $k_p \tau_r / g_{eq} > (\tau_r / \tau_d)^{6/11}$, the end-grafted polymers

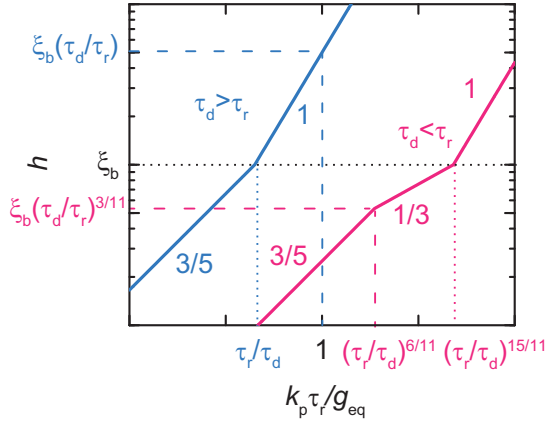


Fig. 4 Brush height: The brush height h is shown as a function of rescaled polymerization rate $k_p \tau_r / g_{eq}$ for $\tau_d > \tau_r$ (cyan) and $\tau_d < \tau_r$ (magenta). τ_d is the degradation time of polymers and τ_r is the relaxation time, see eqs. (10) and (15). The end-grafted polymers form mushrooms for $h < \xi_b$ and form a brush for $h > \xi_b$, see the black dotted line. The crossovers between the mushroom and brush regimes are shown by the cyan and magenta dotted lines. The cyan and magenta broken lines are the crossover between the quasi-equilibrium and non-equilibrium regimes.

are relaxed in the length scales smaller than ξ_c (more precisely, the size of a relaxed blob), but not in the larger length scale (non-equilibrium mushroom regime). The size of the non-equilibrium mushroom is

$$h = \xi_c \left(\frac{k_p \tau_d}{g_c} \right)^{5/11} = \xi_b \left(\frac{\tau_d}{\tau_r} \right)^{5/11} \left(\frac{k_p \tau_r}{g_{eq}} \right)^{1/3}, \quad (18)$$

see the magenta line in fig. 4. Eq. (18) is derived by substituting $n = N$ into eq. (6). **The size does not depend on the distance ξ_b between the grafting point in the mushroom regime, but is expressed in terms of ξ_b in the last form of eq. (18) to show the crossover.** For $h > \xi_b$, the interaction between end-grafted polymers becomes significant (non-equilibrium brush regime). The brush height is

$$h = \xi_b \left(\frac{\tau_d}{\tau_r} \right)^{15/11} \frac{k_p \tau_r}{g_{eq}}, \quad (19)$$

see the magenta line in fig. 4. Eq. (19) is derived by substituting $n = N$ in eq. (11). In the non-equilibrium brush regime, the end-grafted polymers are relaxed in the length scale smaller than ξ_c (more precisely, the size of a relaxed blob), but not in the larger length scale.

3 Polymerization kinetics

3.1 Dynamics of reactant monomers

In the previous section, we derived the polymer volume fraction and the brush height by treating the polymerization rate k_p as a control parameter. The polymerization rate k_p depends on the volume fraction of (unreacted) reactant monomers at the position of the catalysts and the flux of these monomers, where both of them are affected by the polymer volume fraction in the brush. In this section, we theoretically predict how the polymer dynamics driven by the polymerization affects the kinetics of polymerization. For simplicity, we treat cases in which the polymerization reaction is diffusion-limited.

The flux of reactant monomers has the form

$$\mathbf{J}_m = -D_m \phi_m \nabla \left(\frac{\mu_m}{k_B T} \right), \quad (20)$$

where ϕ_m is the volume fraction of reactant monomers and μ_m is the chemical potential of reactant monomers. D_m is the diffusion constant of reactant monomers. k_B is the Boltzmann constant and T is the absolute temperature. The chemical potential of reactant monomers has the form

$$\frac{\mu_m}{k_B T} = \log \phi_m + \phi_p. \quad (21)$$

The volume fraction ϕ_p of polymers is given by eqs. (7) and (13). The flux $\mathbf{J}_m$ is defined as the volume of reactant monomers passing a unit area in a unit time and thus has the unit of m/s. Eq. (21) is derived by assuming that the solvent is athermal both for reactant monomers and monomers in polymers because of the same chemical structure, see also the Appendix A for the derivation.

We use the coordinate z to represent the positions in the brush for $z > \xi_b$ and use the radial coordinate r to represent the position for $r < \xi_b$. The catalyst acts as a sink of reactant monomers. In the steady state, the flux of reactant monomers in the z -direction is

$$J_m = -\sigma k_p b^3 \quad (22)$$

for $z > \xi_b$. For $r < \xi_b$, we use the flux in the radial direction

$$r^2 J_m = -k_p b^3 \quad (23)$$

to ensure the continuity of the flux at $r = \xi_b$. **Eqs. (22) and (23) result from the fact that the catalysts act as sinks of reactant monomers, each absorbs k_p reactant monomers per unit time, and it is equal to the number of reactant monomers passing through cross-sections.**

These equations are formulated with the picture that the flux has the spherical symmetry in the length scale of the reaction radius and is approximated as the planar symmetry in the length scale larger than ξ_b because the length scale ξ_b is much larger than the reaction radius r_c (much like the fact that electromagnetic waves are assumed to be planar at far away from a point source [39]). We thus used the same notation J_m for the radical component of the flux in $r < \xi_b$ and the z -component of the flux in $z > \xi_b$. **The monomer volume fraction ϕ_m is derived as eqs. (B8) and (B13) by solving eq. (20) with eqs. (22) and (23) for the linear and spherical geometries, respectively, see also Appendix B.**

3.2 Boundary conditions

The top of the brush is in equilibrium with the external solution that acts as the reservoir of reactant monomers. Rigorously speaking, there is no steady state in the solution of the diffusion equation for the planar geometry. We thus assume that the polymer brush is on a spherical surface, but the surface can be approximated as a planar surface if its curvature is relatively small. The monomer volume fraction at the top of the brush, $z = h$, is

$$\phi_m(h) = \left(\phi_{m\infty} - \frac{b^3 k_p}{D_m} \sigma r_{\text{ex}} \right) e^{-\phi_p(h)} \quad (24)$$

due to the local equilibrium at the brush top. **r_{ex} is the distance between the top of the brush and the center of the curvature of the spherical surface and $\phi_{m\infty}$ is the volume fraction of reactant monomers at $r \rightarrow \infty$.** The second term in the round bracket of eq. (24) is derived by assuming that the monomer volume fraction at the exterior of the brush follows the diffusion equation, **see Appendix B.**

The length scale with which reactant monomers can diffuse in the reaction time k_p^{-1} is $l_d (= \sqrt{D_m/k_p})$. The diffusion constant of reactant monomers is approximately equal to the diffusion constant of monomers in the polymers, $D_m \approx D_1$, because both of the monomers have the same size. The diffusion length l_d is thus equal to the length scale ξ_c , see eq. (3). This implies that the distribution of reactant monomers is at the local equilibrium for $r < \xi_c$. **For the diffusion-limited polymerization, the polymerization is faster than the diffusion of reactant monomers. The boundary condition is thus**

$$\phi_m(\xi_c) = 0 \quad (25)$$

for cases in which the length scale ξ_c is larger than the reaction radius r_c because reactant monomers in $r < \xi_c$ have already reacted.

3.3 Perturbation analysis

The polymerization rate k_p is derived by using the relationship

$$\frac{\phi_{m\infty} D_m}{b^3 k_p} = \frac{1}{\xi_b^2} \int_{\xi_b}^h dz' e^{\phi_p(z')} + \int_{\xi_c}^{\xi_b} dr' \frac{e^{\phi_p(r')}}{r'^2} + \frac{r_{ex}}{\xi_b^2} \quad (26)$$

in the brush regime. **Eq. (26) is derived by imposing the boundary conditions, eqs. (24) and (25), to eqs. (B8) and (B13), which are the general solutions of eq. (20).** The corresponding relationship for the mushroom regime is derived by replacing ξ_b to h . The subchain in a relaxed blob is swollen due to the two-body excluded volume interaction between the monomers. The polymer volume fraction ϕ_p is small in the length scale larger than ξ_c and eq. (26) can be expanded in a power series of ϕ_p . We therefore treat the polymer volume fraction ϕ_p as perturbation and derive the polymerization rate k_p in the form

$$k_p = k_{p0} + k_{p1}, \quad (27)$$

where k_{p0} is the polymerization rate for the case of $\phi_p = 0$ and k_{p1} is an order linear to ϕ_p .

For the case of $\phi_p \rightarrow 0$, eq. (26) is rewritten as

$$\phi_{m\infty} - \frac{b^3 k_{p0}}{D_m} \left(\frac{h_0}{\xi_b^2} + \frac{1}{\xi_{c0}} - \frac{2}{\xi_b} + \frac{r_{ex}}{\xi_b^2} \right) = 0, \quad (28)$$

where h_0 and ξ_{c0} are the brush height h and length scale ξ_c with the polymerization rate k_{p0} . In the brush regime, k_{p1} is derived as

$$\frac{k_{p1}}{k_{p0}} = -\frac{1}{\Delta} \left[\frac{1}{\xi_b^2} \int_{\xi_b}^{h_0} dz' \phi_{p0}(z') + \int_{\xi_{c0}}^{\xi_b} dr' \frac{\phi_{p0}(r')}{r'^2} \right] \quad (29)$$

with

$$\Delta = \frac{2h_0}{\xi_b^2} + \frac{3}{2} \frac{1}{\xi_{c0}} - \frac{2}{\xi_b} + \frac{r_{ex}}{\xi_b^2}, \quad (30)$$

where ϕ_{p0} is the polymer volume fraction ϕ_p with the polymerization rate k_{p0} . Eq. (29) is derived by expanding eq. (26) in a power series of k_{p1} and ϕ_p and omitting the higher order terms. The corresponding relationship for the mushroom regime is derived by replacing ξ_b to h (more precisely, Δ has an additional term $-2/(3h_0)$ in the mushroom regime, but it is smaller than other terms and is omitted). Because r_{ex} is the largest length scale in eq. (30), we use $\Delta \approx r_{ex}/\xi_b^2$ in the following.

i) The case of $\tau_d > \tau_r$: The end-grafted polymers form a brush at the crossover between the quasi-equilibrium and non-equilibrium regimes,

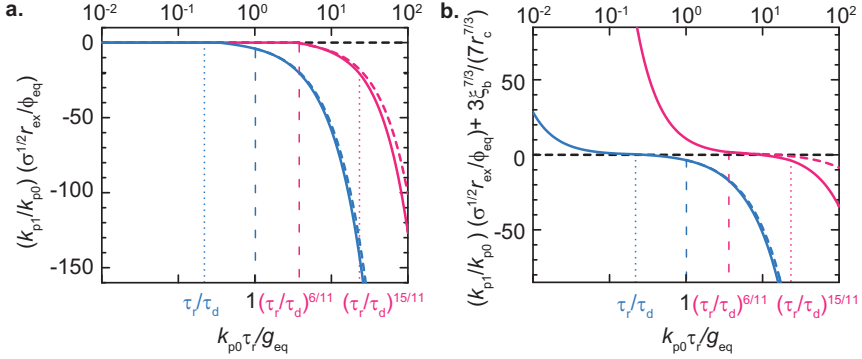


Fig. 5 Perturbation analysis: The deviation k_{p1} of the polymerization rate due to the polymer volume fraction ϕ_p in the brush is shown as a function of the polymerization rate k_{p0} without the perturbation of ϕ_p for $\tau_d > \tau_r$ (cyan) and $\tau_d < \tau_r$ (magenta). **a** and **b** are the results of the calculations by using eq. (25) and $\phi_m(r_c) = 0$, respectively. The length scale ξ_c depends on the polymerization rate k_p and the reaction radius r_c is a constant. The polymer volume fraction with the polymer dynamics (the solid line in fig. 3c) is used to derive the cyan and magenta solid lines. The cyan and magenta broken lines are derived by using the polymer volume fraction at the thermodynamic equilibrium (the broken line in fig. 3c). The vertical axis of **b** is shifted from **a** by a constant term $3\xi_b^{7/3}/(7r_c^{7/3})$ (which is indeed the sole term that depends on r_c), but **a** and **b** are shown in the same scale. The horizontal axis is logarithmic and the vertical axis is linear. We used $\tau_d/\tau_r = 5.0$ (cyan) and 0.1 (magenta) for the calculation.

$k_{p0}\tau_r/g_{eq} = 1$, see the cyan line in fig. 4. For $k_{p0}\tau_r/g_{eq} < (\tau_r/\tau_d)^{2/3}$, the distribution of reactant monomers in the entire region of the brush (or the mushrooms) is at the thermodynamic equilibrium, $l_d > h_0$, and thus the polymer volume fraction in the brush (or mushroom) does not affect the polymerization rate, $k_{p1} = 0$, see the cyan solid line in fig. 5a. For $(\tau_r/\tau_d)^{2/3} < k_{p0}\tau_r/g_{eq} < 1$, the distribution of reactant monomers in the brush (or the mushrooms) become out of the equilibrium, $l_d < h_0$, but the polymers are relaxed in the entire length scales. The form of the deviation k_{p1} of the polymerization rate due to the polymer volume fraction for $k_{p0}\tau_r/g_{eq} > 1$ is given in the Appendix C. In this regime, the deviation has an asymptotic form

$$\frac{k_{p1}}{k_{p0}} \approx -\frac{\phi_{eq}\xi_b}{r_{ex}} \left[\left(\frac{k_{p0}\tau_r}{g_{eq}} \right)^{7/6} + \frac{\tau_d}{\tau_r} \frac{k_{p0}\tau_r}{g_{eq}} \right] \quad (31)$$

for large polymerization rate $k_{p0}\tau_r/g_{eq}$ and ratio τ_d/τ_r , see the cyan solid line in fig. 5a. The polymerization rate k_{p0} changes not only the polymer volume fraction (compare the solid and broken lines in fig. 3c), but also the brush height (see fig. 4) and the diffusion length l_d . To analyze the contribution of the polymer dynamics, we compared the deviation k_{p1} derived by using eqs. (7) and (13) and the deviation derived by using the polymer volume fraction of the equilibrium polymer brush. This calculation reveals that the condensation of polymers due to their slow relaxation dynamics decreases the

polymerization rate, but only slightly, compare the solid and broken lines in fig. 5a. Indeed, the polymer dynamics contributes to the first term of eq. (31) only by the numerical factor of order unity, see Appendix C. The diffusion length l_d ($= \xi_{c0}$) decreases and the brush height h_0 increases with increasing the polymerization rate k_{p0} . The absolute value of the deviation k_{p1} increases with the polymerization rate k_{p0} due to the fact that reactant monomers have to travel deeper into the region of large polymer volume fraction in the small length scale (the first term of eq. (31)) and have to travel longer distance (the second term of eq. (31)) with increasing the polymerization rate k_{p0} .

ii) The case of $\tau_d < \tau_r$: The end-grafted polymers are still mushrooms at the crossover between the quasi-equilibrium and non-equilibrium regimes, $k_{p0}\tau_r/g_{eq} = (\tau_r/\tau_d)^{6/11}$, see the magenta line in fig. 4. For $k_{p0}\tau_r/g_{eq} < (\tau_r/\tau_d)^{6/11}$, the distribution of reactant monomers in the entire region of the mushrooms is at the thermodynamic equilibrium, $l_d > h_0$, and thus the polymer volume fraction in the mushrooms does not affect the polymerization rate, $k_{p1} = 0$, see the magenta solid line in fig. 5a. For $(\tau_r/\tau_d)^{6/11} < k_{p0}\tau_r/g_{eq} < (\tau_r/\tau_d)^{15/11}$, the polymer volume fraction ϕ_p in the mushrooms due to the polymer dynamics and the distribution of reactant monomers becomes out of the equilibrium. The deviation k_{p1} of polymerization rate due to the polymer volume fraction is negative and its absolute value increases gradually with increasing $k_p\tau_r/g_{eq}$ through the crossover between the mushroom and the brush regimes at $k_{p0}\tau_r/g_{eq} = (\tau_r/\tau_d)^{15/11}$, see the solid magenta line in fig. 5a. The deviation k_{p1} of polymerization rate has an asymptotic form

$$\frac{k_{p1}}{k_{p0}} \approx -\frac{\phi_{eq}\xi_b}{r_{ex}} \left(\frac{k_{p0}\tau_r}{g_{eq}} \right)^{7/6} \quad (32)$$

for large polymerization rate $k_{p0}\tau_r/g_{eq}$ and small time scale ratio τ_d/τ_r . The condensation of polymers due to their slow relaxation dynamics slightly decreases the polymerization rate k_p , compare the solid and broken magenta lines in fig. 5a. The contribution of the polymer dynamics to eq. (32) is only the numerical factor of order unity, see Appendix C. For $\tau_d < \tau_r$, the fact that the diffusion length l_d decreases with increasing the polymerization rate k_{p0} indeed makes the dominant contribution to the suppression of the diffusion of reactant monomers, see also Appendix C. Indeed, if the boundary condition $\phi_m(r_c) = 0$ is used instead of eq. (25), the condensation of polymers due to their slow dynamics contributes to the deviation k_{p1} more significantly than the case in which eq. (25) is used, compare the magenta solid and broken lines in fig. 5b. This implies that the suppression of polymerization rate, eq. (32), is mainly due to the fact that the diffusion length l_d decreases with increasing the polymerization rate k_{p0} . Because of the small height of the brush, the polymer volume fraction in the larger length scales, $z > \lambda_b$, does not contribute to the polymerization rate significantly even in the brush regime.

Parameter	meaning	Value	Ref.
b	Kuhn length	4 nm (= 12 b)	[40]
D_1	Monomer diffusion constant	$5 \times 10^{-11} \text{ m}^2/\text{s}$	
v_e	Elongation rate	6 kb/min	[41, 42]

Table 1 Estimate of parameters involved in ribosomal RNA transcription: The Kuhn length of a pre-rRNA is estimated by using the value of single-stranded DNA. The monomer diffusion constant is estimated by $D_1 \approx \frac{k_B T}{6\pi\eta_w b}$, where $k_B T$ ($\approx 4 \times 10^{-21}$ J) is the thermal energy and η_w ($\approx 1 \text{ mPa} \cdot \text{s}$) is the viscosity of water.

4 Discussion

We have used the scaling theory to predict the polymer volume fraction and height in a polymer brush inspired by ribosomal RNA transcription. This polymer brush is grown by adding monomers to the grafted-ends of polymers. Our theory predicts that the polymerization kinetics increases the polymer volume fraction in the length scales $\xi_c < z < \lambda$, see eqs. (3) and (12) for the definition of ξ_c and λ . The polymerization rate decreases as the polymer volume fraction in the brush increases because it suppresses the diffusion of reactant monomers. The deviation k_{p1} of polymerization rate due to the polymer volume fraction in the brush is a negative value and its absolute value increases with increasing the polymerization rate k_{p0} without the perturbation due to the polymer volume fraction. It is mainly because of the fact that diffusion length l_d decreases or the brush height increases with increasing the polymerization rate k_{p0} and the condensation of polymers due to the slow polymer dynamics plays only a minor role, see eqs. (31) and (32).

For simplicity, we have assumed that the polymerization rate is the diffusion-limited reaction. For the general case, the polymerization rate is calculated by using the rate equation $k_p = k_0 \phi_m(\xi_c) e^{\phi_p(\xi_c) - \phi_p(r_c)}$ for the boundary condition at $r = \xi_c$, instead of using eq. (25), where k_0 is the rate constant. For the case of the rate-limited reaction, the polymerization rate is trivially $k_p = k_0 \phi_{m\infty} e^{-\phi_p(r_c)}$ and the polymer dynamics is not involved as long as $r_c < \xi_c$.

Because the polymer brush is inspired by ribosomal RNA transcription, it may be of interest to estimate how much extent the polymerization dynamics affects the structure of nascent pre-rRNAs. The relaxation time of monomers is estimated as $\tau_1 = b^2/D_1 \approx 0.3 \mu\text{s}$ and the elongation rate (per monomer) is 8 s^{-1} , see table 1. The number g_c of monomers in the smallest relaxed blob is thus 5×10^4 , corresponding to 600 kb. The putative region, which is localized at the DFC, is 200 b and is much smaller than g_c . Even the total length of pre-rRNA, $\approx 15 \text{ kb}$, is smaller than g_c . This implies that the transcription of pre-rRNAs by Pol I is too slow for the polymer dynamics to be relevant and thus is safe to neglect this effect. This justifies the approach used in our previous theory that neglects the polymer dynamics to predict the multi-phase structure of nucleolus [6].

We have treated cases in which the height of the brush is smaller than the radius of the curvature of the surface to avoid complications

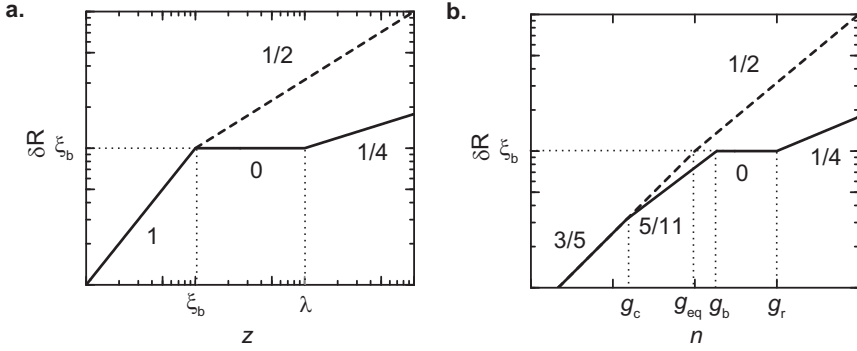


Fig. 6 Magnitude of lateral fluctuation: The magnitude δR of the lateral fluctuation is shown as a function of the distance z from the surface (a) and the number n of monomers from the grafted ends (b) of polymers in a non-equilibrium polymer brush (solid) and an equilibrium brush (broken).

arising from the geometry. In the context of rRNA transcription, the thickness of DFC layer is as large as the radius of FC. The geometry of the system can be taken into account in the spirit of the Daoud-Cotton theory [38]. We expect that our analysis of the planar geometry may be useful more in engineering contexts.

Our theory predicts that the longest length scale λ in which the polymer volume fraction increases due to the polymer dynamics is determined by the relaxation time τ_r of a subchain in the equilibrium blob, which is much smaller than the relaxation time of the whole chain. One may wonder why the relaxation of a polymer in a brush is faster than the relaxation of a free polymer in a solution. Indeed, it is not the case. In the thermodynamic equilibrium, a polymer brush is viewed as a melt of equilibrium blobs [9]. The equilibrium blobs in a chain show the lateral fluctuation with the magnitude $\delta R_{\text{eq}} = \xi_b (N/g_{\text{eq}})^{1/2}$. The relaxation time of the whole chain is the time scale of the lateral excursion of equilibrium blobs. We thus expect that the magnitude of the lateral fluctuation is ξ_b for the length scales $\xi_b < z < \lambda$ and increases with the distance z for $z > \lambda$ because the distance z reflects the time t after the addition of monomers in the blobs, see fig. 6a. Because of the connectivity of blobs, the blobs show the Rouse dynamics. The magnitude of the fluctuation for $z > \lambda$ is thus $\delta R = \xi_b (t/\tau_r)^{1/4} (= \xi_b (z/\lambda)^{1/4})$, see fig. 6a (or b represented as a function of the number n of monomers from the grafted ends). This implies that polymers in a dynamic brush show smaller lateral excursion than polymers in an equilibrium brush. The relaxation time of the whole chain does not affect the polymer volume fraction ϕ_p in the brush.

Inspired by the transcription of pre-rRNAs, we have proposed a brush of polymers, where monomers are added to their grafted-ends with a constant rate. This brush is different from a brush of polymers, where monomers are added to their free-ends [33, 34], such as the case of free radical polymerization from initiators grafted at a surface. However, the difference is only significant when the polymerization kinetics is fast or polymers in the brush are very long.

As such, one does not have to worry about the slow relaxation dynamics of polymers in the context of the transcription of pre-rRNAs. We anticipate that our theory may be extended to polymerization reactions in other biological contexts, where the slow relaxation dynamics of polymers is relevant, and/or may motivate polymer chemists to design a polymer brush by taking advantage of their slow relaxation dynamics and polymerization kinetics.

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Appendix A Chemical potential of reactant monomers

The mean field free energy of the subchain (composed of g monomers) in a relaxed blob (of the size of ξ) has the form

$$\begin{aligned} \frac{F}{k_B T} &= \frac{3}{2} \frac{\xi^2}{b^2 g} + \frac{\xi^3}{b^3} \left[\phi_m \log \phi_m + (1 - \phi_p - \phi_m) \log(1 - \phi_p - \phi_m) - \frac{\mu_m}{k_B T} \phi_m \right] \\ &\simeq \frac{3}{2} \frac{\xi^2}{b^2 g} + \frac{1}{2} \frac{b^3 g^2}{\xi^3} + \frac{\xi^3}{b^3} \left(\phi_m \log \phi_m - \phi_m + \phi_p \phi_m - \frac{\mu_m}{k_B T} \phi_m \right), \quad (\text{A1}) \end{aligned}$$

where ϕ_m and μ_m are the volume fraction and chemical potential of reactant monomers. k_B is the Boltzmann constant and T is the absolute temperature. In the first form of eq. (A1), we take into account the free energy due to the entropic elasticity of the subchain (the first term) and the mixing entropy of solvent molecules and reactant monomers (the second term). We assumed that the solvent is athermal both for polymers and reactant monomers because they have the same chemistry. The last form of eq. (A1) is derived by expanding the free energy in a power series of ϕ_m and ϕ_p and neglected the higher order terms.

The minimization of eq. (A1) with respect to ϕ_m leads to the form of the chemical potential

$$\frac{\mu_m}{k_B T} = \log \phi_m + \phi_p. \quad (\text{A2})$$

The minimization of the free energy with respect to the size ξ leads to the form

$$-\frac{3\xi}{b^2 g} + \frac{3}{2} \frac{b^3 g^2}{\xi^4} + \frac{3\xi^2}{b^3} \phi_m = 0. \quad (\text{A3})$$

Eq. (A3) represents the balance of the elastic force of the subchain (first term), the force due to the excluded volume interaction between monomers in the subchain (second term), and the force due to the osmotic pressure of reactant monomers (third term). For simplicity, we here treat cases in which the osmotic

pressure of reactant monomers is small and thus does not affect the size of relaxed blobs. In such cases, the size of a relaxed blob is $\xi \approx bg^{3/5}$.

Appendix B Volume fraction of reactant monomers

B.1 Brush region

For $z > \xi_b$, eq. (20) is rewritten as

$$-\sigma k_p b^3 = -D_m \left[\frac{\partial}{\partial z} \phi_m(z) + \phi_m(z) \frac{\partial}{\partial z} \phi_p(z) \right], \quad (\text{B4})$$

by using eqs. (21) and (22). We solve eq. (B4) in the form of

$$\phi_m(z) = A_m(z) e^{-\phi_p(z)}. \quad (\text{B5})$$

Substituting eq. (B5) into eq. (B4) leads to the form

$$\frac{\partial}{\partial z} A_m(z) = \frac{\sigma k_p b^3}{D_m} e^{\phi_p(z)}. \quad (\text{B6})$$

By using eq. (B6), the function $A_m(z)$ is derived as

$$A_m(z) = A_m(h) + \frac{\sigma k_p b^3}{D_m} \int_h^z dz' e^{\phi_p(z')}. \quad (\text{B7})$$

By substituting eq. (B7) into eq. (B5), the monomer volume fraction $\phi_m(z)$ for $z > \xi_b$ is derived as

$$\phi_m(z) = \phi_m(h) e^{\phi_p(h) - \phi_p(z)} - \frac{b^3 \sigma k_p}{D_m} \int_z^h dz' e^{\phi_p(z') - \phi_p(z)}, \quad (\text{B8})$$

where we represented $A_m(h)$ in terms of $\phi_m(h)$ by using eq. (B5) to derive the first term.

For $r < \xi_b$, eq. (20) is rewritten as

$$-\frac{k_p b^3}{r^2} = -D_m \left[\frac{\partial}{\partial r} \phi_m(r) + \phi_m(r) \frac{\partial}{\partial r} \phi_p(r) \right] \quad (\text{B9})$$

by using eqs. (21) and (23). We solve eq. (B9) in the form

$$\phi_m(r) = B_m(r) e^{-\phi_p(r)}. \quad (\text{B10})$$

Substituting eq. (B10) into eq. (B9) leads to the form

$$\frac{\partial}{\partial r} B_m(r) = \frac{k_p b^3}{D_m} \frac{e^{\phi_p(r)}}{r^2} \quad (\text{B11})$$

By using eq. (B11), the function $B_m(r)$ is derived as

$$B_m(r) = B_m(\xi_b) + \frac{k_p b^3}{D_m} \int_{\xi_b}^r dr' \frac{e^{\phi_p(r')}}{r'^2}. \quad (\text{B12})$$

By substituting eq. (B12) into eq. (B10), the monomer volume fraction $\phi_m(r)$ for $r < \xi_b$ is derived as

$$\phi_m(r) = \phi_m(\xi_b) e^{\phi_p(\xi_b) - \phi_p(r)} - \frac{b^3 k_p}{D_m} \int_r^{\xi_b} dr' \frac{e^{\phi_p(r') - \phi_p(r)}}{r'^2}, \quad (\text{B13})$$

where we represented $B_m(\xi_b)$ in terms of $\phi_m(\xi_b)$ by using eq. (B10) to derive the first term. Because the polymer volume fraction ϕ_p is continuous at $z = \xi_b$, the monomer volume fraction ϕ_m is also continuous at this position. In the mushroom regime, the monomer volume fraction $\phi_m(r)$ is derived by replacing ξ_b to h .

B.2 Exterior

For simplicity, we assume that at the exterior of the brush region, the volume fraction ϕ_m of reactant monomers follows the diffusion equation

$$D_m \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \left(\frac{\partial}{\partial r} \phi_m(r) \right) \right) = 0 \quad (\text{B14})$$

with the boundary conditions $\phi_m(r) = \phi_{m\infty}$ for $r \rightarrow \infty$ and

$$-D_m \frac{\partial}{\partial r} \phi_m(r) \Big|_{r=r_{\text{ex}}} = -b^3 \sigma k_p \quad (\text{B15})$$

By using eqs. (B14) and (B15), the volume fraction $\phi_m(r)$ has the form

$$\phi_m(r) = \phi_{m\infty} - \frac{b^3 k_p}{D_m} \frac{\sigma r_{\text{ex}}^2}{r}. \quad (\text{B16})$$

This leads to the volume fraction ϕ_m at $r = r_{\text{ex}}$

$$\phi_m(r_{\text{ex}}) = \phi_{m\infty} - \frac{b^3 k_p}{D_m} \sigma r_{\text{ex}}. \quad (\text{B17})$$

The chemical potential μ_{m} is continuous at the top of the brush, see also eq. (21). This leads to eq. (24).

Appendix C Deviation of polymerization rate by polymer volume fraction

We here summarize the deviation k_{p1} of polymerization rate by the polymers in the brush as a function of p ($= k_{\text{p0}}\tau_{\text{r}}/g_{\text{eq}}$):

i) The case of $\tau_{\text{d}} > \tau_{\text{r}}$.

1) Brush regime - with polymer dynamics:

$$\frac{k_{\text{p1}}}{k_{\text{p0}}} = -\frac{\phi_{\text{eq}}\xi_{\text{b}}}{r_{\text{ex}}} \left[\frac{5}{9}p^{7/6}(1 - p^{-9/10}) + \frac{15}{11}p(1 - p^{-11/15}) + p \left(\frac{\tau_{\text{d}}}{\tau_{\text{r}}} - 1 \right) \right] \quad (\text{C18})$$

- without polymer dynamics:

$$\frac{k_{\text{p1}}}{k_{\text{p0}}} = -\frac{\phi_{\text{eq}}\xi_{\text{b}}}{r_{\text{ex}}} \left[\frac{3}{7}(p^{7/6} - 1) + \frac{\tau_{\text{d}}}{\tau_{\text{r}}}p - 1 \right]. \quad (\text{C19})$$

ii) The case of $\tau_{\text{d}} < \tau_{\text{r}}$.

2) The mushroom regime - with polymer dynamics:

$$\frac{k_{\text{p1}}}{k_{\text{p0}}} = -\frac{\phi_{\text{eq}}\xi_{\text{b}}}{r_{\text{ex}}} \frac{5}{9}p^{7/6} \left(1 - \left(\frac{\tau_{\text{r}}}{\tau_{\text{d}}} \right)^{9/11} p^{-3/2} \right). \quad (\text{C20})$$

- without polymer dynamics:

$$\frac{k_{\text{p1}}}{k_{\text{p0}}} = -\frac{\phi_{\text{eq}}\xi_{\text{b}}}{r_{\text{ex}}} \frac{3}{7} \left(p^{7/6} - \left(\frac{\tau_{\text{r}}}{\tau_{\text{d}}} \right)^{7/5} p^{-7/5} \right). \quad (\text{C21})$$

3) Brush regime - with polymer dynamics

$$\frac{k_{\text{p1}}}{k_{\text{p0}}} = -\frac{\phi_{\text{eq}}\xi_{\text{b}}}{r_{\text{ex}}} \left[\frac{5}{9}p^{7/6} (1 - p^{-9/10}) + \frac{15}{11}p^{4/15} \left(\frac{\tau_{\text{d}}}{\tau_{\text{r}}} p^{11/15} - 1 \right) \right]. \quad (\text{C22})$$

-without polymer dynamics

$$\frac{k_{\text{p1}}}{k_{\text{p0}}} = -\frac{\phi_{\text{eq}}\xi_{\text{b}}}{r_{\text{ex}}} \left[\frac{3}{7} (p^{7/6} - 1) + \frac{\tau_{\text{d}}}{\tau_{\text{r}}}p - 1 \right]. \quad (\text{C23})$$

If the boundary condition $\phi_{\text{m}}(r_{\text{c}}) = 0$ is used (instead of eq. (25)), the lower boundary ξ_{c0} of the integral in the second term of eq. (29) should be

replaced to r_c . In this case, the deviation k_{p1} is calculated by using

$$\int_{r_c}^{\xi_b} dr' \frac{e^{\phi_p(r')}}{r'^2} = \int_{\xi_c}^{\xi_b} dr' \frac{e^{\phi_p(r')}}{r'^2} + \int_{r_c}^{\xi_c} dr' \frac{e^{\phi_p(r')}}{r'^2}. \quad (C24)$$

The deviation k_{p1} including the first term of eq. (C24) has been already derived in eqs. (C18) - (C23). The deviation δk_{p1} due to the second term of eq. (C24) is

$$\begin{aligned} \frac{\delta k_{p1}}{k_{p0}} &= -\frac{\xi_b^2}{r_{ex}} \int_{r_c}^{\xi_{c0}} dr' \frac{\phi_p(r')}{r'^2} \\ &= -\frac{\phi_{eq} \xi_b}{r_{ex}} \frac{3}{7} \left[\left(\frac{\xi_b}{r_c} \right)^{7/3} - p^{7/6} \right]. \end{aligned} \quad (C25)$$

The deviation for the new boundary condition is $k_{p1} + \delta k_{p1}$. We note that the polymer dynamics does not affect the polymer volume fraction ϕ_p in the length scales $r_c < r < \xi_{c0}$. We thus used the polymer volume fraction for the equilibrium to derive eq. (C25).

Data Availability Statement: No Data associated with the manuscript.

References

- [1] Lafontaine, D.L.J., Riback, J.A., Bascetin, R., Brangwynne, C.P.: The nucleolus as a multiphase liquid condensate. *Nat. Rev. Mol. Cell Biol.* **22**, 165–182 (2021)
- [2] Feric, M., Vaidya, N., Harmon, T.S., Mitrea, D.M., Zhu, L., Richardson, T.M., Kriwacki, R.W., Pappu, R.V., Brangwynne, C.P.: Coexisting liquid phases underlie nucleolar subcompartments. *Cell* **165**, 1686–1697 (2016)
- [3] Ide, S., Imai, R., Ochi, H., Maeshima, K.: Transcriptional suppression of ribosomal dna with phase separation. *Sci. Adv.* **6**, 5953 (2020)
- [4] Yao, R.W., Xu, G., Wang, Y., Shan, L., Luan, P.F., Wang, Y., Wu, M., Yang, L.Z., Xing, Y.H., Yang, L., Chen, L.L.: Nascent pre-rRNA sorting via phase separation drives the assembly of dense fibrillar components in the human nucleolus. *Mol. Cell* **76**, 767–783 (2019)
- [5] Maiser, A., Dillinger, S., Längst, G., Schermelleh, L., Leonhardt, H., Németh, A.: Super-resolution in situ analysis of active ribosomal dna chromatin organization in the nucleolus. *Sci. Rep.* **10**, 7462 (2020)
- [6] Yamamoto, T., Yamazaki, T., Ninomiya, K., Hirose, T.: Lateral surface pressure generated by nascent ribosomal rna suppresses growth of fibrillar centers in the nucleolus. *bioRxiv*, <https://doi.org/10.1101/2021.09.09.459702> (2022)

- [7] Witten, T.A., Pincus, P.A.: Colloid stabilization by long grafted polymers. *Macromolecules* **19**, 2509–2513 (1986)
- [8] Alexander, S.: Adsorption of chain molecules with a polar head a scaling description. *J. Phys. France* **38**, 983–987 (1977)
- [9] de Gennes, P.G.: Conformations of polymers attached to an interface. *Macromolecules* **13**, 1069–1075 (1980)
- [10] Milner, S.T., Witten, T.A., Cates, M.E.: Theory of the grafted polymer brush. *Macromolecules* **21**, 2610–2619 (1988)
- [11] Skvortsov, A.M., Pavlushkov, I.V., Gorbunov, A.A., Zhulina, Y.B., Borisov, O.V., Pryamitsyn, V.A.: Structure of densely grafted polymeric monolayers. *Pol. Sci. USSR* **30**, 1706–1715 (1988)
- [12] Zhulina, Y.B., Pryamitsyn, V.A., Borisov, O.V.: Structure and conformational transitions in grafted polymer chain layers. a new theory. *Pol. Sci. USSR* **31**, 205–216 (1989)
- [13] Pincus, P.A.: Colloid stabilization with grafted polyelectrolytes. *Macromolecules* **24**, 2912–2919 (1991)
- [14] Yamamoto, T., Pincus, P.A.: Collapse of polyelectrolyte brushes in electric fields. *Europhys. Lett.* **95**, 48003 (2011)
- [15] Buxboim, A., Daube, S.S., Bar-Ziv, R.: Synthetic gene brushes: a structure–function relationship. *Mol. Sys. Biol.* **4**, 181 (2008)
- [16] Daube, S.S., Bracha, D., Buxboim, A., Bar-Ziv, R.H.: Compartmentalization by directional gene expression. *Proc. Nat. Aca. Sci. USA.* **107**, 2836–2841 (2010)
- [17] Yamamoto, T., Safran, S.A.: Transcription rates in dna brushes. *Soft Matter* **11**, 3017–3021 (2015)
- [18] Bracha, D., Karzbrun, E., Shemer, G., Pincus, P.A., Bar-Ziv, R.H.: Entropy-driven collective interactions in dna brushes on a biochip. *Proc. Nat. Acad. Sci. USA.* **110**, 4534–4538 (2013)
- [19] Marko, J.F., Siggia, E.D.: Stretching dna. *Macromolecules* **28**, 8759–8770 (1995)
- [20] Schaefer, D.W., Joanny, J.F., Pincus, P.: Dynamics of semiflexible polymers in solution. *Macromolecules* **13**, 1280–1289 (1980)
- [21] Yamamoto, T., Yamazaki, T., Hirose, T.: Phase separation driven by production of architectural rna transcripts. *Soft Matter* **16**, 4692–4698

(2020)

- [22] Chujo, T., Yamazaki, T., Hirose, T.: Architectural rnas (arcnas): A class of long noncoding rnas that function as the scaffold of nuclear bodies. *Biochim. Biophys. Acta* **1859**, 139–146 (2016)
- [23] Yamazaki, T., Souquere, S., Chujo, T., Kobelke, S., Chong, Y.S., Fox, A.H., Bond, C.S., Nakagawa, S., Pierron, G., Hirose, T.: Functional domains of neat1 architectural lncrna induce paraspeckle assembly through phase separation. *Mol. Cell* **70**, 1038–1053 (2018)
- [24] Yamazaki, T., Yamamoto, T., Yoshino, H., Souquere, S., Nakagawa, S., Pierron, G., Hirose, T.: Paraspeckles are constructed as block copolymer micelles. *EMBO J.* **40**, 107270 (2021)
- [25] Yamamoto, T., Yamazaki, T., Hirose, T.: Triblock copolymer micelle model of spherical paraspeckles. *Front. Mol. Biosci.* **9**, 925058 (2022)
- [26] Yamamoto, T., Schiessel, H.: Transcription driven phase separation in chromatin brush. *Langmuir* **32**, 3036–3044 (2016)
- [27] Yamamoto, T., Schiessel, H.: Transcription dynamics stabilizes nucleus-like layer structure in chromatin brush. *Soft Matter* **13**, 5307–5316 (2017)
- [28] Adame-Arana, O., Bajpai, G., Lorber, D., Volk, T., Safran, S.A.: Regulation of chromatin microphase separation by adsorbed protein complexes. *bioRxiv*, doi: <https://doi.org/10.1101/2022.09.29.510124> (2022)
- [29] Yamamoto, T., Sakaue, T., Schiessel, H.: Loop extrusion drives very different dynamics for rouse chains in bulk solutions and at interfaces. *Europhys. Lett.* **127**, 38002 (2019)
- [30] Yamamoto, T., Schiessel, H.: Dilution of contact frequency between superenhancers by loop extrusion at interfaces. *Soft Matter* **15**, 7635–7643 (2019)
- [31] Leidescher, S., Ribisel, J., Ullrich, S., Feodorova, Y., Hildebrand, E., Galitsyna, A., Bultmann, S., Link, S., Thanisch, K., Mulholland, C., Dekker, J., Leonhardt, H., Mirny, L., Solovei, I.: Spatial organization of transcribed eukaryotic genes. *Nat. Cell Biol.* **24**, 327–339 (2022)
- [32] Yamamoto, T., Kinoshita, K., Hirano, T.: Elasticity control of entangled chromosomes: crosstalk between condensin complexes and nucleosomes. *bioRxiv*, doi: <https://doi.org/10.1101/2022.11.09.515745> (2022)
- [33] Liu, H., Li, M., Lu, Z.Y., Zhang, Z.G., Sun, C.C.: Influence of surface-initiated polymerization rate and initiator density on the properties of

- polysmer brushes. *Macromolecules* **42**, 2863–2872 (2009)
- [34] Zoppe, J.O., Ataman, N.C., Mocny, P., Wang, J., Moraes, J., Klok, H.A.: Surface-initiated controlled radical polymerization: State-of-the-art, opportunities, and challenges in surface and interface engineering with polymer brushes. *Chem. Rev.* **117**, 1105–1318 (2017)
 - [35] Szwarc, M.: ‘living’ polymers. *Nature* **178**, 1168–1169 (1956)
 - [36] Doi, M., Edwards, S.F.: *The Theory of Polymer Dynamics*. Oxford Univ. Press, Oxford, UK (1986)
 - [37] Rubinstein, M., Colby, R.H.: *Polymer Physics*. Oxford Univ. Press, New York, NY (2003)
 - [38] Daoud, M., Cotton, J.P.: Star shaped polymers : a model for the conformation and its concentration dependence. *J. Phys. France.* **43**, 531–538 (1982)
 - [39] Griffiths, D.J.: *Introduction to Electrodynamics*, 4th Ed. Cambridge Univ. Press., Cambridge, UK (2017)
 - [40] Roth, E., Azaria, A.G., Girshevitz, O., Bitler, A., Garini, Y.: Measuring the conformation and persistence length of single-strand dna using origami structure. *Nano Lett.* **18**, 6703–6709 (2018)
 - [41] Dunder, M., Hoffmann-Rohrer, U., Hu, Q., Grummt, I., Rothblum, L.I., Phair, R.D., Misteli, T.: Mammalian rna polymerase in vivo. *Science* **298**, 1623–1626 (2002)
 - [42] Stefanovsky, V., Langlois, F., Gagnon-Kugler, T., Rothblum, L.I., Moss, T.: Growth factor signaling regulates elongation of rna polymerase i transcription in mammals via ubf phosphorylation and r-chromatin remodeling. *Mol. Cell* **21**, 629–639 (2006)