Proceedings of 49th Sapporo Symposium on Partial Differential Equations

Edited by
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H. Kuroda, S. Masaki, T. Ozawa, T. Sakajo, and K. Tsutaya

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H. Kuroda, S. Masaki, T. Ozawa, T. Sakajo, and K. Tsutaya

Sapporo, 2024

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Preface

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 19 through August 21 in 2024 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school. Late Professor Taira Shirota started the symposium almost 50 years ago. Late Professor Kôji Kubota and late Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

S.-I. Ei (Josai University)
Y. Giga (The University of Tokyo)
N. Hamamuki (Hokkaido University)
S. Jimbo (Hokkaido University)
H. Kubo (Hokkaido University)
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S. Masaki (Hokkaido University)
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T. Sakajo (Kyoto University)
K. Tsutaya (Hirosaki University)

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Program Committee	Shin-Ichiro Ei, Yoshikazu Giga, Nao Hamamuki, Shuichi Jimbo, Hideo Kubo, Hirotoshi Kuroda, Satoshi Masaki, Tohru Ozawa, Takashi Sakajo, Kimitoshi Tsutaya
URL	https://www.math.sci.hokudai.ac.jp/sympo/sapporo/program240819.html

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On the stability of bifurcating periodic patterns of the compressible Navier-Stokes equations

Yoshiyuki Kagei
Tokyo Institute of Technology

1 Introdunction

This talk is concerned with a bifurcation and stability of periodic patterns in the compressible Navier-Stokes systems. We will consider a stationary bifurcation problem and stability of bifurcating solutions. This kind of problems are classical for the incompressile Navier-Stokes systems which are roughly speaking, classified in semilinear parabolic systems. Indeed, One can apply the Crandall-Rabinowitz theory ([2, 3] to see the bifurcation of periodic patterns and their stability. On the other hand, the compressible Navier-Stokes system is classified in quasilinear hyperbolic-parabolic system, so the classical bifurcation theory is not directly applicable. The first result on the bifurcation in the compressible Navier-Stokes systems was given by Nishida-Padula-Teramoto ([13]) where the bifurcation of periodic patterns was shown in a compressible thermal convection problem between two pararell plates. The incorempressible limit of the bifurcation pattens were also considered in [13]; it was indeed proved in [13] that the compressible bifurcating patterns converge to the incompressible ones when the Oberbeck-Boussinesq parameter tends to zero. In [8] it was proved that the bifurcation of traveling periodic patterns bifurcate from the plane Poiseuille flows when the Mach number is around 1.5 for the Reynolds number about 10.5. In this talk we mainly consider the stability of the bifurcating patterns for the Taylor problem in viscous compressible fluids between two concentric infinitely long cylinders.

2 Taylor problem

Let us consider a viscous fluid between two concentric cylinders. The inner cylinder rotates with a uniform speed ω and the outer one is at rest. Oen could then imagine that a rotating laminar flow would occur accompanying with the rotation of the inner cylinder. Indeed, if the rotating speed ω is sufficiently small, then the laminar flow, the so called *Couette flow* is stably observed. When ω increases, beyond a certain critical value of ω, say, ω_0, the Couette flow becomes unstable and a vortex pattern called *the Taylor vortex* appears; the pattern is periodic along the axis of the cylinder. This phenomenon is mathematically formulated as a bifurcation and stability problem for fluid dynamical equations.

In the case of incomrepssible fluid, The bifurcation of the Taylor vortices and their stability under periodic perturbations were studied by Iudovich [5], Velte [15] and Kirchgäsner-Sorger [11] and so on. The Crandall-Rabinowitz theory is also applicable. See a book [1] by Chossat-Iooss [1] for details. One can also consider the stability of the Taylor vortices under the localized perturbations, i.e., perturbations that are not periodic but decays at infinity. As for this problem Schneider ([14]) proved the stability of the Eckhaus stable Taylor vortices under the localized perturbations and also showed that the perturbations behave like a heat kernel with coefficient of a periodic function as time goes to infinity.

In this talk we will consider the same problem for the compressible Navier-Stokes system when the Mach number is sufficiently small. The governing system of equations is written in a nondimensional form as

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \boldsymbol{v}) = 0, \\ \rho(\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}) - \dfrac{1}{\mathcal{R}} \Delta \boldsymbol{v} - \dfrac{1}{3\mathcal{R}} \nabla \operatorname{div} \boldsymbol{v} + \dfrac{1}{\varepsilon^2} \nabla p(\rho) = \mathbf{0}. \end{cases} \tag{2.1}$$

Here ρ and $\boldsymbol{v}$ are unknowns; ρ is the density and $\boldsymbol{v}$ is the velocity field; $\mathcal{R} > 0$ and $\varepsilon > 0$ are nondimensional numbers; $\mathcal{R}$ is the Reynolds number which is proportional to μ/ω with the viscosity coefficient μ; $\varepsilon > 0$ is the Mach number; $p(\rho)$ denotes the pressure which is assumed to be a smooth function of ρ. In this talk p is normalized as, due to the nondimensionalization, $p'(1) = 1$. We will consider the situation where ρ is sufficiently close to the constant state 1.

The underlying domain Ω_α is given, by taking the axis of the cylinders as z-axis, in the following way:

$$\Omega_\alpha = \{(r, \theta, z) : r_1 < r < r_2,\ \theta \in \mathbb{T}_{2\pi},\ z \in \mathbb{T}_{\frac{2\pi}{\alpha}}\}.$$

Here (r, θ, z) denotes the cylindrical coordinates; r_1, r_2 and α are nondimensional positive numbers; $\mathbb{T}_\beta = \mathbb{R}/\beta\mathbb{Z}$.

We impose the periodic boundary condition in z variable with period $2\pi/\alpha$ on ρ and $\boldsymbol{v}$, which is due to the expected Taylor vortex is periodic in z. On the lateral boundaries, we impose the nonslip boundary condition

$$v^\theta|_{r=r_1} = 1,\ v^\theta|_{r=r_2} = 0,\ v^r|_{r=r_j} = v^z|_{r=r_j} = 0\ (j = 1, 2) \tag{2.2}$$

Here $(v^r,\ v^\theta,\ v^z)$ denotes the (r, θ, z) component of the velocity field in the cylindrical coordinates, namely, $\boldsymbol{v} = v^r\boldsymbol{e}_r + v^\theta\boldsymbol{e}_\theta + v^z\boldsymbol{e}_z$, where $\boldsymbol{e}_r = {}^\top(\cos\theta, \sin\theta, 0)$, $\boldsymbol{e}_\theta = {}^\top(-\sin\theta, \cos\theta, 0)$ and $\boldsymbol{e}_z = {}^\top(0, 0, 1)$. Here and inwhat follows ${}^\top\cdot$ stands for the transposition.

The problem (2.1)–(2.2) has the following stationary solution $u^\varepsilon_C = {}^\top(\rho^\varepsilon_C, \boldsymbol{v}_C)$ corresponding to the compressible Couette flow:

$$\rho^\varepsilon_C = \rho^\varepsilon_C(r) = 1 + O(\varepsilon^2), \quad \boldsymbol{v}_C = v^\theta_C(r)\boldsymbol{e}_\theta.$$

Here $\rho^\varepsilon_C(r)$ are $v^\theta_C(r)$ are smooth funcions of r. Note that the velocity field $\boldsymbol{v}_C$ satisfies the incompressible Navier-Stokes system

$$\left\{\begin{array}{l} \operatorname{div}\boldsymbol{v} = 0, \\ \partial_t\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} - \nu\Delta\boldsymbol{v} + \nabla p = \boldsymbol{0} \end{array}\right. \tag{2.3}$$

with a suitable pressure p.

If $0 < \mathcal{R} \ll 1$ and $0 < \varepsilon \ll 1$, then one can see that the Couette flow u^ε_C is asymptotically stable. The question here is what happens in the stability of the Couette flow u^ε_C when the Ryenolds number increases.

2.1 Perturbation equations

To consider the stability of the Couette flow u^ε_C, we rewrite the system (2.1) in the system of equations for the perturbation of the Couette u^ε_C. The perturbation is denoted by $u = {}^\top(\phi, \boldsymbol{w}) = {}^\top(\varepsilon^{-2}(\rho - \rho^\varepsilon_C), \boldsymbol{v} - \boldsymbol{v}_C)$.

We impose the following symmetries. Since the Taylor vortices are expected to be axisymmetric, we assume that the perturbation $u = {}^\top(\phi, \boldsymbol{w})$ is axi ymmetric:

$$\phi = \phi(r, z, t),\ \boldsymbol{w} = w^r(r, z, t)\boldsymbol{e}_r + w^\theta(r, z, t)\boldsymbol{e}_\theta + w^z(r, z)\boldsymbol{e}_z.$$

(i.e., ϕ, w^j $(j = r, \theta, z)$ are independent of θ.)

Due to the axisymmetric assumption, one can see that $\operatorname{div}(\phi \boldsymbol{v}_C) = 0$, so the equations for the perturbation u are written as follows.

$$\left\{\begin{array}{l} \partial_t \phi + \frac{1}{\varepsilon^2}\operatorname{div}(\rho_C^\varepsilon \boldsymbol{w}) = -\operatorname{div}(\phi \boldsymbol{w}), \\ \partial_t \boldsymbol{w} - \dfrac{1}{\mathcal{R}\rho_C^\varepsilon}\Delta \boldsymbol{w} - \dfrac{1}{3\mathcal{R}\rho_C^\varepsilon}\nabla \operatorname{div} \boldsymbol{w} + \nabla\left(\dfrac{p'(\rho_C^\varepsilon)}{\rho_C^\varepsilon}\phi\right) \\ \qquad + \boldsymbol{v}_C \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_C = \boldsymbol{g}(\varepsilon, \mathcal{R}; u, \partial_x u, \partial_x^2 \boldsymbol{w}). \end{array}\right. \tag{2.4}$$

Here $\boldsymbol{g} = -\boldsymbol{w} \cdot \nabla \boldsymbol{w} + \varepsilon^2 \tilde{\boldsymbol{g}}(\varepsilon, \mathcal{R}; u, \partial_x u, \partial_x^2 \boldsymbol{w})$ denotes the nonlinearity.

On the lateal boundaries $r = r_1, r_2$, the boundary condition is written as

$$\boldsymbol{w}|_{r=r_1,r_2} = \boldsymbol{0}. \tag{2.5}$$

The periodic boundary condition is put into the definition of the domain Ω_α.

Furthermore, we impose on the density perturbation ϕ the following zero mean value condition

$$\int_{\Omega_\alpha} \phi \, dx = 0. \tag{2.6}$$

This naturally follows from the conservation of mass.

Remark 2.1 In this talk, we consider the situation where the Mach number ε is sufficiently small, $0 < \varepsilon \ll 1$. If $0 < \varepsilon \ll 1$, then compressible flow can be regarded as a perturbation of incompressible flow; in (2.4), if $\varepsilon \to 0$, then one could formally obtain

$$\left\{\begin{array}{l} \operatorname{div} \boldsymbol{w} = 0, \\ \partial_t \boldsymbol{w} - \dfrac{1}{\mathcal{R}}\Delta \boldsymbol{w} + \nabla \phi + \boldsymbol{v}_C \cdot \nabla \boldsymbol{w} + \boldsymbol{w} \cdot \nabla \boldsymbol{v}_C = -\boldsymbol{w} \cdot \nabla \boldsymbol{w}, \end{array}\right. \tag{2.7}$$

which is the system for the perturbation of the incomrepssible Couette flow. This limiting process $\varepsilon \to 0$ is a singular one from a quasilinear hyperbolic-parabolic system to a semilinear parabolic system.

3 Bifurcation of Taylor vortices

We introduce notation. $H^m(\Omega_\alpha)$ denotes the m th order L^2 Sobolev space on Ω_α; its norm is denoted by $\|\cdot\|_{H^m}$.

We define $L^2_*(\Omega_\alpha)$ and $H^k_*(\Omega_\alpha)$ by

$$L^2_*(\Omega_\alpha) = \{f \in L^2(\Omega_\alpha);\ \textstyle\int_{\Omega_\alpha} f(x)dx = 0\},$$

$$H^k_*(\Omega_\alpha) = H^k(\Omega_\alpha) \cap L^2_*(\Omega_\alpha) \ \ (k \geq 1).$$

We also define $L^2_\sigma(\Omega_\alpha)$ by

$$L^2_\sigma(\Omega_\alpha) = \{\boldsymbol{v} \in L^2(\Omega_\alpha)^3;\ \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega_\alpha,\ \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega_\alpha} = 0\}.$$

Here $\boldsymbol{n}$ denotes the unit outer normal to $\partial\Omega_\alpha$. As is well know, the Helmholtz decomposition holds:

$$L^2(\Omega_\alpha)^3 = L^2_\sigma(\Omega_\alpha) \oplus G^2(\Omega_\alpha).$$

Here $G^2(\Omega_\alpha) = \{\nabla p;\ p \in H^1_*(\Omega)\}$ is the orthogonal complement to $L^2_\sigma(\Omega_\alpha)$ in $L^2(\Omega)^3$. The orthogonal projection to $L^2_\sigma(\Omega_\alpha)$ is denoted by $\mathbb{P}$.

Let X be a function space consisting of functions $u = {}^\top(\phi, \boldsymbol{w})$ on Ω_α. Here ϕ and $\boldsymbol{w}$ are a scalar field and a vector field on Ω_α. X_{sym} denote a function space of functions $u = {}^\top(\phi, \boldsymbol{w})$ in X with the following symmetries.

- axisymmetry: $\phi = \phi(r, z)$, $\boldsymbol{w} = w^r(r, z)\boldsymbol{e}_r + w^\theta(r, z)\boldsymbol{e}_\theta + w^z(r, z)\boldsymbol{e}_z$,
- reflection symmetry about $z = 0$:

  $\phi(r, -z) = \phi(r, z)$, $w^j(r, -z) = w^j(r, z)$ $(j = r, \theta)$, $w^z(r, -z) = -w^z(r, z)$.

3.1 Incompressible Taylor problem

We introduce the linearized operator around the incomressible Couette flow $\boldsymbol{v}_C$ for the incompressible problem (2.7) with (2.5). The operator $\mathbb{L}_\mathcal{R}$: $L^2_{\sigma,sym}(\Omega_\alpha) \to L^2_{\sigma,sym}(\Omega_\alpha)$ is defined by

$$D(\mathbb{L}_\mathcal{R}) = \left[H^2(\Omega_\alpha) \cap H^1_0(\Omega_\alpha)\right]^3 \cap L^2_{\sigma,sym}(\Omega_\alpha),$$

$$\mathbb{L}_\mathcal{R}\boldsymbol{w} = -\frac{1}{\mathcal{R}}\mathbb{P}\Delta\boldsymbol{w} + \mathbb{P}(\boldsymbol{v}_C\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{v}_C), \quad \boldsymbol{w} \in D(\mathbb{L}_\mathcal{R}).$$

In this talk we assume the following conditions for the spectrum of $\mathbb{L}_\mathcal{R}$.

Assumption (A) There is a smooth curve $\mathcal{R} = \mathcal{R}_0(\alpha)$ $(\alpha > 0)$ satisfying $\mathcal{R}_0''(\alpha) > 0$ and $\lim_{\alpha\to 0}\mathcal{R}_0(\alpha) = \lim_{\alpha\to\infty}\mathcal{R}_0(\alpha) = \infty$ with minimum $\mathcal{R}_c > 0$

at $\alpha = \alpha_c$. (The curve $\mathcal{R} = \mathcal{R}_0(\alpha)$ is called the *neutral curve*; $\mathcal{R}_c$ and α_c are called the *critical Reynolds number* and *critical wave number*, resupectively.)

Furthermore, for each $\alpha > 0$ there are constants $a_0 > 0$ and $\Lambda_0 > 0$, $\tilde{\mathcal{R}}_0 > \mathcal{R}_0$ such that it holds

$$\sigma(-\mathbb{L}_{\mathcal{R}}) \subset \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \le -b_0|\operatorname{Im}\lambda|^2 - \Lambda_0\} \cup \{\lambda_{\mathcal{R}}\}$$

for $0 < \mathcal{R} \le \tilde{\mathcal{R}}_0$, where $\lambda_{\mathcal{R}} \in \mathbb{R}$ is a simple eigenvalue satisfying

$$\lambda_{\mathcal{R}_0} = 0, \quad \left.\frac{d\lambda_{\mathcal{R}}}{d\mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_0} > 0.$$

Remark 3.1 (i) Velte ([15]) and Iudovich ([5]) showed that $\lambda_{\mathcal{R}_0} = 0$ and $\frac{d\lambda_{\mathcal{R}}}{d\mathcal{R}}\big|_{\mathcal{R}=\mathcal{R}_0} > 0$ (a.e. $\alpha > 0$).

(ii) Experiments and numerical simulations support Assumption (A) for a physically relevant range of α. See, e.g., [1, 11].

By Assumption (A), the Couette flow becomes unstable when $\mathcal{R} > \mathcal{R}_0$. By a bifurcation theory (e.g., [2]), one can see that there occurs the bifurcation of a Taylor vortex in the incompressible problem (2.7) with (2.5) under Assumption (A) as follows.

Applying the Helmholtz projection $\mathbb{P}$ to the second equation of (2.7), we have the stationary problem

$$\mathbb{L}_\nu \boldsymbol{w} = -\mathbb{P}(\boldsymbol{w} \cdot \nabla \boldsymbol{w}), \quad \boldsymbol{w} \in D(\mathbb{L}_\nu). \tag{3.1}$$

Here the trivial solution $\boldsymbol{w} = \boldsymbol{0}$ of (3.1) corresponds to the Couette flow $\boldsymbol{v}_C$.

Proposition 3.2 ([5, 15, 11]) *For $\mathcal{R} > \mathcal{R}_0$, $\mathcal{R} \sim \mathcal{R}_0$, the incompressible problem* (3.1) *has a nontrivial solution $\boldsymbol{w}_\delta \in [H^2(\Omega_\alpha) \cap H^1_0(\Omega_\alpha)]^3 \cap L^2_{\sigma,sym}(\Omega_\alpha)$* (*incompressible Taylor vortex*)*, where*

$$\boldsymbol{w}_\delta = \delta \boldsymbol{w}^{(0)} + \delta^2 \boldsymbol{W}_\delta, \quad \delta = c_0\sqrt{\mathcal{R} - \mathcal{R}_0}.$$

Here $\boldsymbol{w}_\delta$ is a C^∞ function of δ with values in $H^2(\Omega_\alpha)^3$; $\|\boldsymbol{W}_\delta\|_{H^2} = O(1)$ as $\delta \to 0$; c_0 is a positive constant; and $\boldsymbol{w}^{(0)}$ is an eigenfunction of $-\mathbb{L}_{\mathcal{R}_0}$ for eigenvalue 0.

3.2 Instability of the compressible Couette flow

Let us consider the instability of the compressible Couette flow u_C^ε. We introduce the linearized operator around u_C^ε for the compressible problem (2.4)–(2.6). We define the linearized operator $L_{\mathcal{R}}^\varepsilon : H^1_{*,sym}(\Omega_\alpha)\times L^2_{sym}(\Omega_\alpha)^3 \to H^1_{*,sym}(\Omega_\alpha) \times L^2_{sym}(\Omega_\alpha)^3$ by

$$
\begin{aligned}
D(L_{\mathcal{R}}^\varepsilon) &= H^1_{*,sym}(\Omega_\alpha) \times \left[H^2_{sym}(\Omega_\alpha) \cap H^1_{0,sym}(\Omega_\alpha)\right]^3, \\
L_{\mathcal{R}}^\varepsilon u &= \begin{pmatrix} 0 & \frac{1}{\varepsilon^2}\mathrm{div}\,(\rho_C^\varepsilon\,\cdot\,) \\ \nabla\left(\frac{p'(\rho_C^\varepsilon)}{\rho_C^\varepsilon}\,\cdot\right) & -\frac{1}{\mathcal{R}\rho_C^\varepsilon}\Delta - \frac{1}{3\mathcal{R}\rho_C^\varepsilon}\nabla\mathrm{div}\, + \boldsymbol{v}_C\cdot\nabla + {}^\top(\nabla \boldsymbol{v}_C)\cdot \end{pmatrix}\begin{pmatrix}\phi \\ \boldsymbol{w}\end{pmatrix}, \\
u &= {}^\top(\phi, \boldsymbol{w}) \in D(L_{\mathcal{R}}^\varepsilon).
\end{aligned}
$$

If the Mach number ε is sufficiently small, we have the following about the spectrum of $L_{\mathcal{R}}^\varepsilon$.

Theorem 3.3 ([9]) *There are constants $\varepsilon_1 > 0$, $\eta_1 > 0$, $\Lambda_1 > 0$ and $\tilde{\alpha}_c > 0$ such that for each $0 < \varepsilon \le \varepsilon_1$ and α with $|\alpha - \alpha_c| \le \tilde{\alpha}_c$, there exists $\mathcal{R}_0^\varepsilon(\alpha) = \mathcal{R}_0(\alpha) + O(\varepsilon^2)$ with minimum $\mathcal{R}_c^\varepsilon$ at $\alpha = \alpha_c^\varepsilon (= \alpha_c + O(\varepsilon^2))$ such that for $|\mathcal{R} - \mathcal{R}_c| \le \eta_1$, it holds that*

$$\sigma(-L_{\mathcal{R}}^\varepsilon) \subset \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \le -\Lambda_1\} \cup \{\lambda_{\mathcal{R}}^\varepsilon\}.$$

where $\lambda_{\mathcal{R}}^\varepsilon \in \mathbb{R}$ is a simple eigenvalue satisfying

$$\lambda_{\mathcal{R}_0^\varepsilon}^\varepsilon = 0, \qquad \left.\frac{d\lambda_{\mathcal{R}}^\varepsilon}{d\mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_0^\varepsilon} > 0.$$

Remark 3.4 We give one remark about the axisymmetric assumption. As for the incompressible problem we do not need the axisymmetric assumption to show the bifurcation of the Taylor vortices. On the other hand, in the compressible problem, we assume that perturbations of the Couette flow are axisymmetric. This is due to the following technical reason. When ε is sufficiently small, one can show that the spectrum near the imaginary part is decomposed into three parts; one is near the origin; and the remaining two are in a region with $|\mathrm{Im}\,\lambda| = O(\varepsilon^{-1})$. The part near the origin is shown to be a perturbation of the corresponding part of the spectrum of $-\mathbb{L}_{\mathcal{R}}$, while the part with $|\mathrm{Im}\,\lambda| = O(\varepsilon^{-1})$ arises from acoustic modes. To show the part with $|\mathrm{Im}\,\lambda| = O(\varepsilon^{-1})$ lies in the left half plane, we here use the axisymmetric assumption.

By Theorem 3.3, we see that the compressible Couette flow becomes unstable when $\mathcal{R} > \mathcal{R}_0^\varepsilon$ and one could expect a stationary bifurcation from the Couette flow. However, the classical theory available for the incompressible problem cannot be directly applied to the compressible problem. The is due to the fact that the nonlinear term $-\mathrm{div}\,(\phi \boldsymbol{w})$ causes the derivative loss, i.e., it is not Fréchet differentiable in the classical setting of the bifurcation analysis.

3.3 Bifurcation of the compressible Taylor vortex

We consider the stationary problem for the compressible problem

$$L_{\mathcal{R}}^\varepsilon u = F(\varepsilon, \mathcal{R}, u), \quad u = {}^\top(\phi, \boldsymbol{w}) \in D(L_{\mathcal{R}}^\varepsilon). \tag{3.2}$$

Here $F(\varepsilon, \mathcal{R}, u)$ denotes the nonlinearity

$$F(\varepsilon, \mathcal{R}, u) = \begin{pmatrix} -\mathrm{div}\,(\phi \boldsymbol{w}) \\ \boldsymbol{g}(\varepsilon, \mathcal{R}; u, \partial_x u, \partial_x^2 \boldsymbol{w}) \end{pmatrix}$$

By using the method of [8] we have the following bifurcation of the compressible Taylor vortices.

Theorem 3.5 *Let $\alpha \sim \alpha_c$. For $\mathcal{R} > \mathcal{R}_0^\varepsilon$, $\mathcal{R} \sim \mathcal{R}_0^\varepsilon$, the compressible problem (3.2) has a nontrivial solution $u_\delta^\varepsilon \in H_*^4(\Omega_\alpha) \times [H^5(\Omega_\alpha) \cap H_0^1(\Omega_\alpha)]^3$ (compressible Taylor vortex), where*

$$u_\delta^\varepsilon = \delta u^{(0),\varepsilon} + \delta^2 U_\delta^\varepsilon, \quad \delta = c_0^\varepsilon \sqrt{\mathcal{R} - \mathcal{R}_0^\varepsilon}.$$

Here u_δ^ε is a C^ℓ function of δ with values in $H^{3-\ell} \times H^{4-\ell}$ ($\ell = 0, 1, 2$); $\|U_\delta^\varepsilon\|_{H^4 \times H^5} = O(1)$ as $\delta \to 0$ c_0^ε is a positive constant; and $u^{(0),\varepsilon}$ is an eigenfunction of $-L_{\mathcal{R}_0^\varepsilon}^\varepsilon$ for eigenvalue 0.

Remark 3.6 (i) In Theorem 3.5 the density component ϕ_δ^ε of the bifurcating solution has the mean value zero in Ω_α. One can also construct the bifurcating solution $u_{\delta,m}^\varepsilon = {}^\top(\phi_{\delta,m}^\varepsilon, \boldsymbol{w}_{\delta.m}^\varepsilon)$ with non vanishing mean value condition $\int_{\Omega_\alpha} \phi_{\delta,m}^\varepsilon \, dx = m$ if $|m|$ is sufficiently small. This will be need to consider the stability of the bifurcating Taylor vortex u_δ^ε under the localized perturbations.

(ii) One can prove the bifurcation of the Taylor vortex u_δ^ε in $H_*^2(\Omega_\alpha) \times [H^3(\Omega_\alpha) \cap H_0^1(\Omega_\alpha)]^3$. However, to have enough information for the stability analysis of u_δ^ε we need to have some smoothness with respect to δ which follows from the higher order regularity of u_δ^ε.

4 Stability of the compressible Taylor votex undet periodic perturbations

We next consider the stability of the Taylor vortex u^ε_δ under axisymmetric perturbations in Ω_α. Here we do not impose the reflection symmetry.

We introduce the linearized operator around the Taylor vortex $u^\varepsilon_{T,\delta} = u^\varepsilon_C + u^\varepsilon_\delta$. We define the operator $\tilde{L}^\varepsilon_\delta : H^1_{*,axi}(\Omega_\alpha) \times (L^2(\Omega_\alpha))^3_{axi} \to H^1_{*,axi}(\Omega_\alpha) \times (L^2(\Omega_\alpha))^3_{axi}$ by

$$\begin{aligned} D(\tilde{L}^\varepsilon_\delta) = \{ & u \in H^1_{*,axi}(\Omega_\alpha) \times (L^2(\Omega_\alpha))^3_{axi}; \\ & \boldsymbol{w} \in H^1_0(\Omega_\alpha)^3, \tilde{L}^\varepsilon_\delta u \in H^1_{*,axi}(\Omega_\alpha) \times (L^2(\Omega_\alpha))^3_{axi}\}, \\ \tilde{L}^\varepsilon_\delta u = {} & L^\varepsilon_{\mathcal{R}^\varepsilon_\delta} u + M(u^\varepsilon_\delta)u + G_u(u^\varepsilon_\delta)u, \quad u \in D(\tilde{L}^\varepsilon_\delta). \end{aligned}$$

Here the subscript $\cdot_{axi}$ stands for the axisymmetricity; $\mathcal{R}^\varepsilon_\delta$ denotes the Reynolds number $\mathcal{R}$ defined by $\delta = c^\varepsilon_0\sqrt{\mathcal{R} - \mathcal{R}^\varepsilon_0}$ in Theorem 3.5; $M(u^\varepsilon_\delta)u = N(u^\varepsilon_\delta, u) + N(u, u^\varepsilon_\delta)$. Since $u^\varepsilon_\delta = O(\delta)$, we have$\tilde{L}^\varepsilon_\delta = L^\varepsilon_{\mathcal{R}^\varepsilon_0} + O(\delta)$, but $M(u^\varepsilon_\delta)u$ is a singular perturbation term as $\delta \to 0$.

Theorem 4.1 ([7]) *Let $\alpha \sim \alpha_c$. There are constants $\varepsilon_2 > 0$, $\delta_2 > 0$ and $\Lambda_2 > 0$ such that if $0 < \varepsilon \le \varepsilon_2$, $|\delta| \le \delta_2$, then*

$$\sigma(-\tilde{L}^\varepsilon_\delta) \subset \{\lambda \in \mathbb{C}; \mathrm{Re}\,\lambda \le -\Lambda_2\} \cup \{0, \lambda^\varepsilon_\delta\}.$$

Here 0 is a semisimple eigenvalue with multiplicity 2; and $\lambda^\varepsilon_\delta$ is a simple eigenvalue and $\lambda^\varepsilon_\delta$ is a C^2 function of δ satisfying $\lambda^\varepsilon_\delta = -a^\varepsilon\delta^2 + O(\delta^3)$ with some constant $a^\varepsilon > 0$. The kernel of $-\tilde{L}^\varepsilon_\delta$ is spanned by $\partial_z u^\varepsilon_\delta$ and $\partial_m u^\varepsilon_{\delta,m}|_{m=0}$.

Remark 4.2 The stability under nonaxisymmetric perturbations has not yet been solved, This is due to the fact that the stability of the Couette flow under nonaxisymmetric perturbations has not yet been known around the bifurcation point; if this could be done, then then one could prove Theorem 4.1 under nonaxisymmetric perturbations.

5 Stability of the compressible Taylor vortices under localized perturbations

We briefly consider the stability of the compressible Taylor vortices under localized perturbations. Theorem 4.1 shows that for a fixed α, the Taylor vortex is stable under the axisymmetric periodic perturbations of period $2\pi/\alpha$ in z. If one considers the stability under localized perturbation (nonperiodic but decays at $z = \pm\infty$, the stability of the Taylor vortices depends on the value of $(\alpha, \mathcal{R})$; this is called the Eckhaus stability criterion ([4]) which is obtained by the linearized stability analysis. In particular, the Taylor vortex with $\alpha = \alpha_c$ is stable for all $\mathcal{R} > \mathcal{R}_c$, $\mathcal{R} \sim \mathcal{R}_c$.

As for the incompressible Taylor problem, Schneider ([14]) proved the Eckhaus stability criterion. (See also [10, 12].) Furthermore, it was proved in [14] that the perturbation $\boldsymbol{w}(t)$ of the Taylor vortex $\boldsymbol{v}_T$ with $\alpha = \alpha_c$ behaves in large time as

$$\boldsymbol{w}(t) \ \sim \ \partial_z \boldsymbol{w}_\delta(x) t^{-\frac{1}{2}} g_\kappa(\tfrac{z}{\sqrt{t}}) \ \text{ as } t \to \infty, \quad g_\kappa(z) = (4\pi\kappa)^{-\frac{1}{2}} e^{-\frac{|z|^2}{4\kappa}} \ (\kappa > 0).$$

The appearance of coefficient $\partial_z \boldsymbol{w}_\delta(x)$ is due to the translation invariance in z.

As for the compressible problem we can also show the Eckhaus stability and instability under the axisymmetric localized perturbations. We also have the following result on large time behavior of the perturbation of the compressible Taylor vortex with $\alpha = \alpha_c^\varepsilon$.

Theorem 5.1 ([7]) *Let $u_{T,\delta}^\varepsilon$ be the Taylor vortex with $\alpha = \alpha_c^\varepsilon$. Then $u_{T,\delta}^\varepsilon$ is nonlinearly stable; if $\|u_0\|_{H^2 \cap L^1} \ll 1$, then $u \in C([0,\infty); H^2_{axi} \times H^2_{axi})$ and*

$$u(t) \sim \sigma_1(z,t) f_{1,\delta}^\varepsilon + \sigma_2(z,t) f_{2,\delta}^\varepsilon + O(t^{-\frac{3}{4}+\kappa}) \ \text{ in } \ L^2 \ \ (\kappa > 0)$$

as $t \to \infty$. Here $f_{j,\delta}^\varepsilon(x', z)$ is $\frac{2\pi}{\alpha}$-periodic in z; and $\sigma_j = \sigma_j(z,t)$ satisfy

$$\begin{cases} \partial_t \sigma_1 - c\varepsilon^{-2}\mathcal{R}\, \partial_z^2 \sigma_1 + c_1 \partial_z(\sigma_1^2) = 0, \\ \partial_t \sigma_2 - \gamma_\delta^\varepsilon \partial_z^2 \sigma_2 + c_2 \partial_z(\sigma_1^2) = 0. \end{cases}$$

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Continuity of a spatial derivative for certain very singular parabolic equations

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1 Introduction

This abstract briefly reports the speaker's recent works [21, 22], concerned with regularity results for the parabolic $(1,\,p)$-Laplace equation. We consider a real-valued weak solution $u = u(x,\,t)$ to

$$\partial_t u - \Delta_1 u - \Delta_p u = 0 \quad \text{in} \quad \Omega_T \coloneqq \Omega \times (0,\,T), \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with $n \geq 2$, and $T \in (0,\,\infty)$ is a fixed constant. We aim to prove that a spatial gradient $\nabla u = (\partial_{x_j} u)_{j=1,\,\dots,\,n}$ is continuous.

The mathematical analysis concerning (1) at least goes back to the classical textbook by Duvaut–Lions [11], especially for $p = 2$. There, elliptic or parabolic $(1,\,2)$-Laplace equations are used to model the motion of a Bingham fluid, a non-Newtonian fluid with both aspects of plasticity and viscosity. Also, the $(1,\,3)$-Laplace operator often appears as the first variation of crystal surface energy (see e.g., [17]).

For the parabolic p-Laplace equation with $p \in (1,\,\infty)$, the Hölder gradient continuity is well-established (see [6] as a fundamental monograph). In the borderline case $p = 1$, this regularity result is generally violated. For $(1,\,p)$-Laplace problems, the smoothness of a spatial gradient appears not completely understood because of the anisotropic diffusivity of the one-Laplace operator. At least for the gradient continuity, this kind of smoothness is shown in a qualitative approach.

1.1 Definition of weak solutions

We introduce and fix some notations to define weak solutions to (1). The symbol $p_{\mathrm{c}} \coloneqq 2n/(n+2) \in (1,\,2)$ denotes the critical exponent for $n \geq 3$. For $n \geq 3$ and $p \in (1,\,p_{\mathrm{c}})$, we set $s_{\mathrm{c}} \coloneqq n(2-p)/p \geq 2$ as a borderline exponent related to the integrability of u. Following [13, 16], we introduce a standard norm space

$$(V_0,\,\|\cdot\|_{V_0}) \coloneqq \begin{cases} \left(W_0^{1,\,p}(\Omega),\,\|\nabla\cdot\|_{L^p(\Omega)}\right) & (p_{\mathrm{c}} < p < \infty), \\ \left(W_0^{1,\,p}(\Omega) \cap L^2(\Omega),\,\|\nabla\cdot\|_{L^p(\Omega)} + \|\cdot\|_{L^2(\Omega)}\right) & (1 < p \leq p_{\mathrm{c}}). \end{cases}$$

Then, the continuous dual V_0' is given by $V_0' = W^{-1,\,p'}(\Omega)$ for $p \in (p_{\mathrm{c}},\,\infty)$ and $V_0' = W^{-1,\,p'}(\Omega) + L^2(\Omega)$, where $p' \coloneqq p/(p-1) \in (1,\,\infty)$ denotes the Hölder conjugate exponent of p. We define the parabolic function space

$$X_0^p(0,\,T;\,\Omega) \coloneqq \left\{ u \in L^p(0,\,T;\,V_0) \;\middle|\; \partial_t u \in L^{p'}(0,\,T;\,V_0') \right\} \subset C([0,\,T];\,L^2(\Omega)),$$

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where the last inclusion follows from the Lions–Magenes lemma [16, Chapter III, Proposition 1.2], since the Gelfand triple $V_0 \hookrightarrow L^2(\Omega) \hookrightarrow V_0'$ holds. Similarly, we define

$$X^p(0,\,T;\,\Omega) \coloneqq \left\{ u \in L^p(0,\,T;\,V) \,\middle|\, \partial_t u \in L^{p'}(0,\,T;\,V_0') \right\},$$

where $V \coloneqq W^{1,\,p}(\Omega)$ for $p \in (p_{\mathrm{c}},\,\infty)$ and $V \coloneqq W^{1,\,p}(\Omega) \cap L^2(\Omega)$ for $p \in (1,\,p_{\mathrm{c}}]$.

Definition 1 A function $u \in X^p(0,\,T;\,\Omega) \cap C([0,\,T];\,L^2(\Omega))$ is called a weak solution to (1) when there exists $Z \in L^\infty(\Omega_T;\,\mathbb{R}^n)$ satisfying the following;

1. For a.e. $(x,\,t) \in \Omega_T$, the inclusion $Z(x,\,t) \in \partial|\cdot|(\nabla u(x,\,t))$ holds, where $\partial|\cdot|$ denotes the subdifferential operator of the Euclidean norm $|\cdot|$.
2. There holds $\partial_t u - \operatorname{div}\left(Z + |\nabla u|^{p-2}\nabla u\right) = 0$ in $L^{p'}(0,\,T;\,V_0')$.

The main result is Theorem 2 ([21, Theorem 1.2], [22, Theorem 1.2]).

Theorem 2 *Assume that $u \in X^p(0,\,T;\,\Omega) \cap C([0,\,T];\,L^2(\Omega))$ is a weak solution to (1). In addition, when $n \geq 3$ and $p \in (1,\,p_{\mathrm{c}}]$, let $u \in L^s_{\mathrm{loc}}(\Omega_T)$ be in force with $s > s_{\mathrm{c}} \geq 2$. Then, $\nabla u \in C^0(\Omega_T;\,\mathbb{R}^n)$.*

It should be noted that Theorem 2 matches the parabolic p-Laplace regularity results found in [4, 8, 9]. In particular, we cannot remove the higher integrability assumption when $p \in (1,\,p_{\mathrm{c}}]$ (see [10] as a related item).

1.2 Truncated gradients

The main difficulty of parabolic $(1,\,p)$-Laplace equations arises in the degenerate region of ∇u, often called the facet of u. More precisely, formally differentiating (1) by the space variable x_j, we find out that a spatial derivative $v_j \coloneqq \partial_{x_j} u$ satisfies

$$\partial_t v_j - \operatorname{div}\left(\nabla^2 E(\nabla u)\nabla v_j\right) = 0, \quad \text{where} \quad E(z) \coloneqq b|z| + |z|^p/p\,(z \in \mathbb{R}^n). \tag{2}$$

The main problem is that the coefficient matrix $\nabla^2 E(\nabla u)$ is not everywhere uniformly elliptic. To explain this, we would like to measure uniform ellipticity by computing the ellipticity ratio of $\nabla^2 E(\nabla u)$, which is defined as the maximum eigenvalue divided by the minimum one. For (2), the eigenvalues of $\nabla^2 E(\nabla u)$ are either $\Lambda(\nabla u) \coloneqq |\nabla u|^{-1} + |\nabla u|^{p-2}$ or $\lambda(\nabla u) \coloneqq (p-1)|\nabla u|^{p-2}$, provided $\nabla u \neq 0$. These eigenvalues are not compatible with each other. Indeed, we have $\Lambda(\nabla u)/\lambda(\nabla u) \simeq 1 + |\nabla u|^{1-p} \to \infty$ as $\nabla u \to 0$. In other words, $\nabla^2 E(\nabla u)$ becomes no longer uniformly elliptic as $\nabla u \to 0$. This result reflects the well-known fact that the ellipticity of the one-Laplace operator Δ_1 always degenerates in the gradient direction, while it becomes singular in other directions. The anisotropic property of Δ_1, which differs from the isotropic one of Δ_p, appears hard to deal with in the existing regularity theory. In particular, it appears that we cannot apply some standard regularity theories, including the De Giorgi–Nash–Moser theory and the Schauder theory, to the facet $\{\nabla u = 0\}$. Thanks to this difficulty, no quantitative continuity estimate of v_j across the facet of u is known.

However, we can also observe that the ellipticity ratio $\Lambda(\nabla u)/\lambda(\nabla u)$ is bounded if $|\nabla u| \geq \delta > 0$, depending on the positive parameter δ. In other words, we may regard the equation (1) as uniformly elliptic on $\{|\nabla u| \geq \delta > 0\}$. From this viewpoint, we would like to show that the truncated gradient, defined as

$$\mathcal{G}_\delta(\nabla u) \coloneqq (|\nabla u| - \delta)_+ \frac{\nabla u}{|\nabla u|}$$

is locally Hölder continuous, depending on the truncation parameter $\delta \in (0,\,1)$. Since the mapping $\mathcal{G}_\delta \colon \mathbb{R}^n \to \mathbb{R}^n$ converges uniformly to the identity mapping, this implies that the continuity of $\mathcal{G}_\delta(\nabla u)$ yields that of ∇u. Although the Hölder bound estimates of $\mathcal{G}_\delta(\nabla u)$ may blow up as $\delta \to 0$, we are able to prove the continuity of $\mathcal{G}_0(\nabla u) = \nabla u$ in a qualitative approach. It is highly worth noting that similar regularity results are found in the very degenerate elliptic problem of the form

$$-\operatorname{div}\left(\nabla E^*(\nabla v)\right) = 0, \tag{3}$$

where $E^*(\zeta) = (|\zeta|-1)_+^{p'}/p'$ ($\zeta \in \mathbb{R}^n$) denotes the Fenchel dual of E. In the three papers [1, 5, 15], it is proved that a solution v to (3) satisfies $\mathcal{G}_1(\nabla v) \in C^0$, by showing $\mathcal{G}_{1+\delta}(\nabla v) \in C^0$ for each fixed $\delta > 0$. Among them, the author is highly inspired by the recent work [1]. There, local Hölder bounds of $\mathcal{G}_{1+\delta}(\nabla v)$ are shown by standard regularity methods including the comparison with harmonic functions and De Giorgi's truncation. The same regularity result has been proved for the very degenerate parabolic problem [2]. The gradient continuity results of $(1,\,p)$-Laplace-type very singular problems ([12, 19, 20] for elliptic cases and [21, 22] for parabolic cases) are counterparts of those of degenerate ones [1, 2].

2 Approximate problems

To give a rigorous proof of local Hölder bounds of $\mathcal{G}_\delta(\nabla u)$, we have to appeal to approximation arguments. This is mainly because it seems difficult to deal with (2) in the sense of $L^2(0,\,T;\,W^{-1,\,2}(\Omega))$, since the standard difference quotient method may not work directly to (1). For this reason, instead of (1), we consider

$$\partial_t u_\varepsilon - \operatorname{div}\left(\nabla E_\varepsilon(\nabla u_\varepsilon)\right) = 0, \tag{4}$$

where $\varepsilon \in (0,\,1)$ denotes an approximation parameter, and E_ε is a regularized energy density that suitably approximates $E(z) = |z| + |z|^p/p$. For each $\varepsilon \in (0,\,1)$, we introduce $\rho_\varepsilon \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ by $\rho_\varepsilon(y) \coloneqq \varepsilon^{-n}\rho(\varepsilon^{-1}y)$ for $y \in \mathbb{R}^n$. Here $\rho \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$ is a sphere symmetric, non-negative function that is supported in a closed unit ball, and satisfies $\|\rho\|_{L^1(\mathbb{R}^n)} = 1$. For each fixed $\varepsilon \in (0,\,1)$, we define

$$E_\varepsilon(z) \coloneqq \int_{\mathbb{R}^n} \rho_\varepsilon(y) E(z-y)\,\mathrm{d}y = (\rho_\varepsilon * E)(z) \quad \text{for} \quad z \in \mathbb{R}^n$$

In other words, we straightforwardly approximate E by convoluting with the standard mollifier ρ_ε (see [19, §2]). The relaxed density $E_\varepsilon \in C^\infty(\mathbb{R}^n)$ satisfies

$$c\lambda_\varepsilon(z)|w|^2 \leq \nabla^2 E_\varepsilon(z) w \cdot w \leq C\Lambda(z)|w|^2$$

for all $z,\,w \in \mathbb{R}^n$, where $c \in (0,\,1)$ and $C \in (1,\,\infty)$ are constant, and $\lambda_\varepsilon(z) \coloneqq \left(\varepsilon^2 + |z|^2\right)^{p/2-1}$ and $\Lambda_\varepsilon(z) \coloneqq \left(\varepsilon^2 + |z|^2\right)^{p/2-1} + \left(\varepsilon^2 + |z|^2\right)^{-1/2}$.

We consider a weak solution to (4) under a suitable Dirichlet boundary condition. More precisely, for given $u_\star \in X^p(0,\,T;\,\Omega) \cap C([0,\,T];\,L^2(\Omega))$, we consider the weak solution of

$$\left\{ \begin{array}{rcll} \partial_t u_\varepsilon - \operatorname{div}(\nabla E_\varepsilon(\nabla u_\varepsilon)) & = & 0 & \text{in} \quad \Omega_T, \\ u_\varepsilon & = & u_\star & \text{on} \quad \partial_{\mathrm{p}}\Omega_T. \end{array} \right. \tag{5}$$

In other words, we consider $u_\varepsilon \in u_\star + X_0^p(0,\,T;\,\Omega)$ satisfying $u_\varepsilon|_{t=0} = u_\star|_{t=0}$ in $L^2(\Omega)$, and $\partial_t u_\varepsilon - \operatorname{div}(\nabla E_\varepsilon(\nabla u_\varepsilon)) = 0$ in $L^{p'}(0,\,T;\,V_0')$. We would like to prove that the solution u_ε uniquely converges to u_0, the weak solution of

$$\left\{ \begin{array}{rcll} \partial_t u_0 - \operatorname{div}(\nabla E(\nabla u_0)) & = & 0 & \text{in} \quad \Omega_T, \\ u_0 & = & u_\star & \text{on} \quad \partial_{\mathrm{p}}\Omega_T. \end{array} \right. \tag{6}$$

in the sense of Definition 3 below.

Definition 3 For given $u_\star \in X^p(0,\,T;\,\Omega)\cap C([0,\,T];\,L^2(\Omega))$, a function $u_0 \in u_\star + X_0^p(0,\,T;\,\Omega) \subset C([0,\,T];\,L^2(\Omega))$ is called the weak solution of (6), if $u_0|_{t=0} = u_\star|_{t=0}$ holds in $L^2(\Omega)$, and if u_0 is a weak solution to (1) in the sense of Definition 1.

For parabolic p-Laplace equations, unique existence theories are established in the monographs [13, Chapitre 2], [16, Chapter III], which rely on the monotone operator theory and the Galerkin method. By following standard arguments therein, the unique existence of (5) is shown. However, the methods therein might not work for (6), since the energy density E lacks C^1-regularity at the origin. We note that, the uniqueness of (6) follows from the vector inequality

$$\left(|z_1|^{p-2}z_1 - |z_2|^{p-2}z_2\right)\cdot(z_1-z_2) \geq c_p \begin{cases} |z_1-z_2|^p & (p\geq 2),\\ (|z_1|^2+|z_2|^2)^{(p-2)/2}|z_1-z_2|^2 & (1<p<2), \end{cases}$$

often called the strong monotonicity of the p-Laplace operator. Combining this property with the monotonicity of the subdifferential operator $\partial|\cdot|$, we straightforwardly prove the uniqueness of (6). These monotonicity properties are valid for E_ε. Indeed, the inequality

$$(\nabla E_\varepsilon(z_1) - \nabla E_\varepsilon(z_2))\cdot(z_1-z_2) \geq c \begin{cases} |z_1-z_2|^p & (p\geq 2),\\ (\varepsilon^2+|z_1|^2+|z_2|^2)^{(p-2)/2}|z_1-z_2|^2 & (1<p<2), \end{cases} \tag{7}$$

holds for all $z_1,\,z_2\in\mathbb{R}^n$, where the constant $c\in(0,\,1)$ is independent of $\varepsilon\in(0,\,1)$. The strong monotonicity (7) enables us to show Proposition 4 (see [21, Proposition 2.4], [22, Proposition 3.3] for the details), which states that the weak solution of (6) is constructed as the limit of (5).

Proposition 4 *Let u_ε be a weak solution of (5). Then, for each fixed $\tau\in(0,\,T)$, we have $\nabla u_\varepsilon \to \nabla u_0$ in $L^p(\Omega_{T-\tau})$ and a.e. in $\Omega_{T-\tau}$, up to a subsequence. Here u_0 is the unique weak solution of (6) with Ω_T replaced by $\Omega_{T-\tau}$.*

3 Weak formulations and local a priori estimates

By Proposition 4, the regularity problems of (1) are reduced to those of (4). We consider a weak solution to (4) in a parabolic cylinder $\mathcal{Q}$. In other words, $\mathcal{Q}$ is of the form $\mathcal{Q} = Q_{R_0}(x_\star,\,t_\star) := B_{R_0}(x_\star)\times(t_\star - R_0^2,\,t_\star] \Subset \Omega_T$ with $B_R(x_\star)\Subset\Omega$ denoting the open ball whose center and radius are $x_\star$ and R_0. Then, it is not restrictive to assume

$$\nabla^2 u_\varepsilon \in L^2_{\mathrm{loc}}(\mathcal{Q};\,\mathbb{R}^{n\times n}) \quad\text{and}\quad \nabla u_\varepsilon\in L^\infty_{\mathrm{loc}}(\mathcal{Q};\,\mathbb{R}^{n\times n}), \tag{8}$$

whose estimates may depend on ε. Indeed, the ratio $\Lambda_\varepsilon(\nabla u_\varepsilon)/\lambda_\varepsilon(\nabla u_\varepsilon) = 1+\left(\varepsilon^2+|\nabla u_\varepsilon|^2\right)^{(1-p)/2}$ is everywhere bounded by $1+\varepsilon^{1-p}$. This fact enables us to carry out standard parabolic arguments, including the difference quotient method, Moser's iteration, and De Girogi's truncation (see [6, Chapter VIII]). From the improved regularity (8), we give a priori estimates for u_ε. Here we have to note that the ellipticity ratio is controlled by $V_\varepsilon := \sqrt{\varepsilon^2+|\nabla u_\varepsilon|^2}$, not by $|\nabla u_\varepsilon|$. Hence, we introduce another truncated gradient, defined as

$$\mathcal{G}_{\delta,\,\varepsilon}(\nabla u_\varepsilon) := (V_\varepsilon-\delta)_+\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|},$$

which makes sense as long as $0<\varepsilon<\delta$.

By (8), for each index $j\in\{\,1,\,\dots\,,\,n\,\}$, we may treat

$$\partial_t\partial_{x_j}u_\varepsilon - \operatorname{div}\left(\nabla^2 E_\varepsilon(\nabla u_\varepsilon)\nabla\partial_{x_j}u_\varepsilon\right) = 0 \tag{9}$$

in a weak sense (e.g., in $L^2_{\mathrm{loc}}(W^{-1,\,2}_{\mathrm{loc}})$). We systematically deduce weak formulations of V_ε.

Lemma 5 *Let u_ε be a weak solution to (4) with (8). Choose an arbitrary function $\psi\colon [0,\infty)\to[0,\infty)$ that is both Lipschitz and non-decreasing. For this ψ, we define*

$$\Psi(\sigma) \coloneqq \int_0^\sigma \tau\psi(\tau)\,\mathrm{d}\tau \quad \textit{for} \quad \sigma\in[0,\infty).$$

Then, for any $\zeta\in C_{\mathrm{c}}^1(\mathcal{Q})$, we have

$$\begin{aligned}
&-\iint_{\mathcal{Q}} \Psi(V_\varepsilon)\partial_t\zeta\,\mathrm{d}x\mathrm{d}t + \iint_{\mathcal{Q}} \nabla^2E_\varepsilon(\nabla u_\varepsilon)\nabla\left[\Psi(V_\varepsilon)\right]\cdot\nabla\zeta\,\mathrm{d}x\mathrm{d}t\\
&+\iint_{\mathcal{Q}} \left(\nabla^2E_\varepsilon(\nabla u_\varepsilon)\nabla V_\varepsilon\cdot\nabla V_\varepsilon\right)\psi'(V_\varepsilon)V_\varepsilon\zeta\,\mathrm{d}x\mathrm{d}t\\
&+\iint_{\mathcal{Q}} \sum_{j=1}^n \left(\nabla^2E_\varepsilon(\nabla u_\varepsilon)\nabla\partial_{x_j}u_\varepsilon\cdot\nabla\partial_{x_j}u_\varepsilon\right)\psi(V_\varepsilon)\zeta\,\mathrm{d}x\mathrm{d}t = 0.
\end{aligned}\tag{10}$$

The identity (10) is deduced by testing $\zeta\psi(V_\varepsilon)\partial_{x_j}u_\varepsilon$ into (9), and taking a summation for the index $j\in\{1,\dots,n\}$ (see also [21, Lemma 3.1]). This kind of weak formulation is found in parabolic p-Laplace equations [6, Chapter VIII]. It is worth mentioning that for a classical solution to the heat equation $\partial_t w-\Delta w=0$, we easily deduce

$$\partial_t|\nabla w|^2-\Delta|\nabla w|^2+2|\nabla^2w|^2 = 2\nabla w\cdot\nabla(\partial_t-\Delta)w=0,$$

by carrying out similar procedures. Lemma 5, as well as [6, Chapter VIII, Proposition 3.2], generalizes this classical computation.

The weak formulation (10) informs us of two basic results. Firstly, the convex composite function $\Psi(V_\varepsilon)$ is a subsolution to a certain parabolic equation. Secondly, local L^2-energy estimates for ∇^2u_ε can be deduced by choosing suitable ψ. These viewpoints are fully used in showing the improved regularity of ∇u_ε or $\mathcal{G}_{\delta,\varepsilon}(\nabla u_\varepsilon)$.

3.1 Local gradient bounds

Before showing Hölder bounds of truncated gradients, we would like to show local uniform gradient bounds. More precisely, for a fixed parabolic cylinder $\mathcal{Q}'\Subset\mathcal{Q}$, we aim to find a constant μ_0 satisfying

$$\|V_\varepsilon\|_{L^\infty(\mathcal{Q}')}\le\mu_0, \tag{11}$$

independent of $\varepsilon\in(0,1)$. By Moser's iteration, we have Proposition 6 ([21, Theorem 2.7], [22, Proposition 5.3]).

Proposition 6 *Let u_ε be a weak solution to (4) in $\mathcal{Q}$ with (8). For $p\in(p_{\mathrm{c}},\infty)$, we have*

$$\operatorname*{ess\,sup}_{Q_R} V_\varepsilon \le C(n,p)\left[\fint\!\!\fint_{Q_{2R}} (1+V_\varepsilon^p)\,\mathrm{d}x\mathrm{d}t\right]^{d/p}$$

for any fixed $Q_{2R}(x_0,t_0)\Subset\mathcal{Q}$, where $d\coloneqq p/2$ for $p\ge2$, and $d\coloneqq 2p/[p(n+2)-2n]$. For $p\in(1,p_{\mathrm{c}}]$, we choose an exponent $q\in[2,\infty)$ satisfying $q>q_{\mathrm{c}}\coloneqq n(2-p)/2$. Then, we have

$$\operatorname*{ess\,sup}_{Q_R} V_\varepsilon \le C(n,p,q)\left[\fint\!\!\fint_{Q_{2R}} (1+V_\varepsilon^q)\,\mathrm{d}x\mathrm{d}t\right]^{1/(q-q\mathrm{c})}$$

for any fixed $Q_{2R}(x_0,t_0)\Subset\mathcal{Q}$.

For p-harmonic flows with $p \in (p_{\mathrm{c}}, \infty)$, [3] shows local gradient bounds by Moser's iteration. In showing Proposition 6, we have to carefully choose test functions whose support is contained in $\{Q_{2R} \mid |\nabla u_{\varepsilon}| \geq 1\}$, which is substantially different from [3]. In particular, instead of V_{ε}, we introduce

$$w_{\varepsilon,\, j} := (\partial_{x_j} u_{\varepsilon} - 1)_{+} - (-\partial_{x_j} u_{\varepsilon} - 1)_{+}, \quad W_{\varepsilon} := \sqrt{1 + \sum_{j=1}^{n} w_{\varepsilon,\, j}^2}.$$

The proof of Proposition 6 is completed by showing local L^{∞}-bounds of W_{ε}.

The proof of (11) differs depending on whether $p \in (p_{\mathrm{c}}, \infty)$ or $p \in (1, p_{\mathrm{c}}]$. By Proposition 6, (11) is straightforwardly deduced when $p \in (p_{\mathrm{c}}, \infty)$. For $p \in (1, p_{\mathrm{c}})$, in addition to Proposition 6, we have to prove uniform L^q-estimates of ∇u_{ε} for any fixed $q \in (p, \infty)$. This is verified when the local boundedness of u_{ε} is guaranteed, as stated in Proposition 7 [22, Proposition 5.2].

Proposition 7 *Let u_{ε} be a weak solution to (4) in $\mathcal{Q}$ with (8) and $p \in (1, p_{\mathrm{c}}]$. Moreover, assume that there exists a positive constant M_0 such that*

$$\|u_{\varepsilon}\|_{L^{\infty}(\mathcal{Q})} \leq M_0. \tag{12}$$

Then, for each fixed $q \in (p, \infty)$, we have

$$\iint_{Q_R} V_{\varepsilon}^{q} \, \mathrm{d}x\mathrm{d}t \leq C(n,\, p,\, q,\, M_0,\, R) \iint_{Q_{2R}} (V_{\varepsilon}^{p} + 1) \, \mathrm{d}x\mathrm{d}t$$

for any $Q_{2R} = Q_{2R}(x_0,\, t_0) \Subset \mathcal{Q}$.

Proposition 7 boils down to the claim

$$\iint_{Q_{r_1}} V_{\varepsilon}^{\alpha+2} \, \mathrm{d}x\mathrm{d}t \leq \frac{C(n,\, p,\, \alpha,\, M_0)}{(r_2 - r_1)^2} \iint_{Q_{r_2}} \left(V_{\varepsilon}^{\alpha+p} + 1\right) \mathrm{d}x\mathrm{d}t$$

for $\alpha \in [0, \infty)$, $R \leq r_1 < r_2 \leq 2R$, since finitely many iterations yield the desired estimate. This claim is shown by integration by parts, (12), and local L^2-energy estimates of $\nabla^2 u_{\varepsilon}$ from Lemma 5.

Hence, to deduce (11), it suffices to verify (12) by two steps. Firstly, we apply Moser's iteration to prove $u \in L^{\infty}(\mathcal{Q})$ [22, Proposition 4.1], where the higher integrability assumption $u \in L_{\mathrm{loc}}^{s}$ $(s > s_{\mathrm{c}})$ is essentially used (see [4, 10] and [7, Appendix A]). Secondly, we prove that u_{ε}, a weak solution of

$$\begin{cases} \partial_t u_{\varepsilon} - \operatorname{div}(\nabla E_{\varepsilon}(\nabla u_{\varepsilon})) &= 0 \quad \text{in} \quad \mathcal{Q}, \\ u_{\varepsilon} &= u \quad \text{on} \quad \partial_{\mathrm{p}} \mathcal{Q}, \end{cases}$$

satisfies $\|u_{\varepsilon}\|_{L^{\infty}(\mathcal{Q})} \leq \|u\|_{L^{\infty}(\mathcal{Q})}$ [22, Corollary 4.4]. In other words, the equation (4) satisfies a weak maximum principle (see the monograph [14]), which is an easy consequence of (7). Combining these two results, we find a uniform constant M_0 satisfying (12).

3.2 Local Hölder bounds of truncated gradients

Theorem 8 plays an important role in the proof of Theorem 2 [21, Theorem 2.8].

Theorem 8 *Fix $\delta \in (0, 1)$. Let $\varepsilon \in (0, \delta/8)$, and u_{ε} be a weak solution to (4) in $\mathcal{Q}$ with (8). Assume that there exists a constant μ_0, which is independent of ε, such that (11) holds for $\mathcal{Q}' \Subset \mathcal{Q}$. Then, there exists a sufficiently small number $\alpha \in (0, 1)$, which may depend on δ and μ_0 but is independent of ε, such that*

$$\mathcal{G}_{2\delta,\, \varepsilon}(\nabla u_{\varepsilon}) := \left(\sqrt{\varepsilon^2 + |\nabla u_{\varepsilon}|^2} - 2\delta\right)_{+} \frac{\nabla u_{\varepsilon}}{|\nabla u_{\varepsilon}|}$$

is locally Hölder continuous in $\mathcal{Q}'$.

The proof of Theorem 8 is carried out by a careful truncation trick. To explain this, we consider a small parabolic cylinder $Q_r(x_*,\, t_*) \Subset \mathcal{Q}'$. Assume that $\mu \in [0,\, \mu_0 - \delta]$ satisfy

$$\operatorname*{ess\,sup}_{Q_r(x_*,\, t_*)} V_\varepsilon \leq \mu + \delta.$$

In other words, μ stands for the local bound parameter of V_ε or $|\mathcal{G}_{\delta,\,\varepsilon}(\nabla u_\varepsilon)|$. Throughout the proof of Theorem 8, we often assume

$$\delta < \mu, \tag{13}$$

since otherwise we just find the trivial case $\mathcal{G}_{2\delta,\,\varepsilon}(\nabla u_\varepsilon) \equiv 0$ in $Q_r(x_*,\, t_*)$. By this truncation trick and the good condition (13), we can avoid some delicate cases where a gradient could vanish. We emphasize that our analysis does not rely on intrinsic scaling methods, which are often useful in the p-Laplace regularity theory (see [6, 7]).

Theorem 8 itself is based on classical methods from the Schauder theory or the De Giorgi–Nash–Moser theory. Roughly speaking, we appeal to the comparison with heat equations if the super level set of V_ε is sufficiently large [21, Proposition 2.10], and otherwise, we apply De Giorgi's truncation [21, Proposition 2.9]. In the former case, we need to verify the integral average of ∇u_ε never degenerates, which is shown by non-trivial energy estimates of $\nabla^2 u_\varepsilon$. In the latter case, we prove an oscillation lemma by fully making use of the fact that $(V_\varepsilon - \delta)_+^2 = |\mathcal{G}_{\delta,\,\varepsilon}(\nabla u_\varepsilon)|^2$ is a subsolution to a uniformly parabolic equation. This is easy to check by choosing $\psi(\sigma) \coloneqq 2(1 - \delta/\sigma)_+$ in Lemma 5. In showing [21, Propositions 2.9–2.10], we carefully use (13). In any possible case, $\mathcal{G}_{2\delta,\,\varepsilon}(\nabla u_\varepsilon)$ locally satisfies Campanato-type integral growth estimates.

Finally, we give a brief sketch of the proof of Theorem 2. Fix $\mathcal{Q}' \Subset \mathcal{Q} \Subset \Omega_{T-\tau} \Subset \Omega_T$ arbitrarily, and we would like to show $\mathcal{G}_{2\delta}(\nabla u) \in C^0(\mathcal{Q}';\, \mathbb{R}^n)$. For $p \in (p_{\mathrm{c}},\, \infty)$, we consider u_ε, a weak solution of (5) with $u_\star = u$. Proposition 4 implies that $\nabla u_\varepsilon \to \nabla u$ strongly in $L^p(\Omega_{T-\tau};\, \mathbb{R}^n)$ and a.e. in $\Omega_{T-\tau}$. In particular, we may let $\|\nabla u_\varepsilon\|_{L^p(\Omega_{T-\tau})}$ be bounded with respect to ε. Combining this with Proposition 6, we can find a constant μ_0 satisfying (11). By Theorem 8 and the Arzelá–Ascoli theorem, we may let $\mathcal{G}_{2\delta,\,\varepsilon}(\nabla u_\varepsilon)$ converge to a continuous mapping $v_{2\delta}$ uniformly in $\mathcal{Q}'$. Since $\nabla u_\varepsilon \to \nabla u$ a.e., $\mathcal{G}_{2\delta,\,\varepsilon}(\nabla u_\varepsilon)$ also converges to $\mathcal{G}_{2\delta}(\nabla u)$ a.e. in $\mathcal{Q}'$. These two convergence results imply that $\mathcal{G}_{2\delta}(\nabla u)$ has a continuous representative $v_{2\delta}$. Finally, letting $\delta \to 0$, we conclude $\nabla u \in C^0(\mathcal{Q}';\, \mathbb{R}^n)$, which completes the proof. The remaining case $p \in (1,\, p_{\mathrm{c}}]$ is also completed by carrying out local approximation arguments and using Propositions 6–7 to deduce (11), where $u \in L^s_{\mathrm{loc}}(\Omega_T)$ $(s > s_{\mathrm{c}})$ is needed.

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POSITIVITY-PROPAGATION FOR DOUBLY NONLINEAR PARABOLIC TYPE EQUATIONS AND ITS APPLICATION TO REGULARITY †

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Abstract We shall consider the second-ordered partial differential equations of parabolic type, which have the p-Laplacian operator and the time-derivative of power-nonlinearity. We call the equations the doubly nonlinear parabolic type equations. Some interactions of fast and slow diffusions may appear, and they make effect of regularity of solutions. Our aim is to study the regularity of sign-changing weak solutions of them. It is generally necessary to treat sign-changing solutions, since the equations considered here are not translation-invariant on unknown functions and this property is also true even for the porous medium equations. We modify the so-called expansion of positivity and apply it for obtaining the decay of local oscillation of sigh-changing solutions. Our proof of regularity is based on the nowadays classical De Giorgi's measure theoretical alternative approach.

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1 Introduction

Let Ω be an open set in $\mathbb{R}^n$, $n \geq 1$, and $T > 0$. We shall consider quasi-linear, parabolic partial differential equations of the form

$$\partial_t(|u|^{q-1}u) - \operatorname{div} A(x,t,u,Du) = 0 \quad \text{in } \Omega_T. \tag{1.1}$$

Here $q > 0$ is a given number, $\Omega_T := \Omega \times (0,T)$, the unknown function $u = u(x,t)$ is a real-valued function defined for $(x,t) \in \Omega_T$, and, as usual, $\partial_t := \partial/\partial t$ and $\partial_\alpha = \partial/\partial x_\alpha$, $\alpha = 1,\dots,n$, are the partial derivatives on time and space, respectively, $D := (\partial_1,\dots,\partial_n)$ the gradient on space. The vector-valued function $A : \Omega_T \times \mathbb{R}^{n+1} \ni (x,t,u,\zeta) \mapsto A(x,t,u,\zeta) \in \mathbb{R}^n$ is measurable on $(x,t) \in \Omega_T$ for any $(u,\zeta) \in \mathbb{R}^{n+1}$, continuous on $(u,\zeta) \in \mathbb{R}^{n+1}$ for almost every $(x,t) \in \Omega_T$ and fulfills the following structure conditions

$$\begin{cases} A(x,t,u,\zeta)\cdot\zeta \geq C_0|\zeta|^p \\ |A(x,t,u,\zeta)| \leq C_1|\zeta|^{p-1} \end{cases} \quad \text{almost everywhere } (x,t) \in \Omega_T,\ \forall u \in \mathbb{R},\ \forall \zeta \in \mathbb{R}^n, \tag{1.2}$$

where $p > 1$ is a given number and C_i, $i = 0,1$, are given positive constants with $C_0 \leq C_1$. The prototype of (1.1) is

$$\partial_t(|u|^{q-1}u) - \Delta_p u = 0 \quad \text{in } \Omega_T := \Omega \times (0,T), \tag{1.3}$$

which contains the porous medium operator for $p = 2$ and $p-$Laplacian one for $q = 1$, $\Delta_p u := \operatorname{div}\left(|Du|^{p-2}Du\right)$, and thus, is called doubly nonlinear parabolic equation.

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Let Γ be the parabolic boundary, defined as $\Gamma = \Omega \times \{t = 0\} \bigcup \partial\Omega \times (0, T)$. For a compact set $\mathcal{K} \subset \Omega_T$, the parabolic distance from $\mathcal{K}$ to Γ is defined by

$$d_{\mathrm{s}}(\mathcal{K}, \Gamma) := \inf_{(x,t)\in\mathcal{K},\,(x',t')\in\Gamma} \left\{ |x - x'| + \omega_0^{\frac{p-(q+1)}{p}} |t - t'|^{\frac{1}{p}} \right\} \quad \text{if } 0 < q \leq p - 1,$$
$$d_{\mathrm{f}}(\mathcal{K}, \Gamma) := \inf_{(x,t)\in\mathcal{K},\,(x',t')\in\Gamma} \left\{ \omega_0^{\frac{q+1-p}{p}} |x - x'| + |t - t'|^{\frac{1}{p}} \right\} \quad \text{if } q \geq p - 1.$$

The main result in this report is the following Hölder regularity of weak solutions to (1.1), of which the definition is in Definition 1. The result is on the Hölder regularity of weak solutions, those may be sign-changing, in the *fast-fast* duffusion and homogeneous cases, that is $q \geq p - 1$ and $1 < p \leq 2$.

Theorem 1 (*fast-fast diffusion and homogeneous cases*) *Let $q \geq p - 1$ and $1 < p \leq 2$. Suppose that u is a bounded weak solution to (1.1) in Ω_T. Then, u is locally Hölder continuous in Ω_T. Precisely, there exist a positive constants $C = C(C_0, C_1, n, p, q) > 1$ and $\gamma = \gamma(C_0, C_1, n, p, q) < 1$ such that, the local oscillation estimate*

$$|u(x_1, t_1) - u(x_2, t_2)| \leq C\,\omega_0 \left(\frac{\omega_0^{\frac{q+1-p}{p}} |x_1 - x_2| + |t_1 - t_2|^{\frac{1}{p}}}{d_{\mathrm{f}}(\mathcal{K}, \Gamma)} \right)^{\gamma}$$

holds ture for any compact set $\mathcal{K} \subset \Omega_T$ and all pairs of points (x_1, t_1), $(x_2, t_2) \in \mathcal{K}$, where ω_0 is the local essential oscillation of u in $\mathcal{K}$, $\omega_0 = \operatorname{ess\,osc}_{\mathcal{K}} u$.

The regularity theory for linear parabolic equations has already established and the method of proofs is well-known in the regularity circle. For instance, we shall refer to [17] for linear parabolic equations and [7, 13] for linear elliptic ones. The regularity issues for doubly nonlinear parabolic type equations have been recently studied by many authors. The Hölder regularity is shown by [4] in the homogeneous case that $q = p - 1$, which treats a sign-changing weak solution and shows some technique for regularity based on DeGiorgi's measure theoretical approach. In advance there were the Hölder regularity results for nonnegative weak solutions obtained in [15] for $q = p - 1 > 2$ and in [16] for $q = p = 1 < 2$, where the Harnack inequality is used (see [14, 23]). Here we shall note that the power nonlinearity of solution itself makes the equation *not translation invariant* on solution-variable and thus, we should distinguish between the constant-sign and sign-chainging of solutions in the oscillation estimate for Hölder regularity. Such algebraic difficulty has already been appeared in the porous medium equations (refer to [2, 8, 18]). The nonhomogeneity $q \neq p - 1$ of doubly nonlinear parabolic equations can be controlled by the use of intrinsic scaling method originally developed by DiBenedetto (see [10, Sect. III, p. 41; Sect. IV, p. 77] and also refer to [9, 6, 22] for the degenerate and singular parabolic $p-$Laplace equations. The Hölder regularity has also recently proven in the non-homogeneous cases that $p > 2$ and $0 < q < p - 1$ (see [5]), and that $1 < p < 2$ and $0 < p - 1 < q$ (see [19]). Although the above results surely offer some techniques well-worked for the double nonlinearity of the equation, the regularity issues have been remained to be settled for doubly nonlinear equations. The expansion of positivity is reminiscent of a weak Harnack type inequality and may play an essential role in the Hölder regularity estimates for doubly nonlinear parabolic type equations. Refer to [3, 20] for the existence of a weak solution to (1.1).

From the expansion of positivity for nonnegative weak solutions, we obtain an alternative proof of the Hölder regularirty for the evolutionary $p-$Laplace equation (see [21]), which was originally established due to DiBenedetto [10, Theorem 1.1, pp. 41-42; pp.77-78] (also refer to [9, 6]).

Let Γ be the parabolic boundary, defined as $\Gamma = \Omega \times \{t=0\} \bigcup \partial\Omega \times (0,T)$. For a compact set $\mathcal{K} \subset \Omega_T$, the parabolic distance from $\mathcal{K}$ to Γ is defined by

$$d_{\mathrm{S}}(\mathcal{K}, \Gamma) := \inf_{(x,t)\in\mathcal{K},\,(x',t')\in\Gamma} \left\{ |x-x'| + \omega_0^{\frac{p-2}{p}} |t-t'|^{\frac{1}{p}} \right\} \quad \text{if } p \geq 2,$$
$$d_{\mathrm{f}}(\mathcal{K}, \Gamma) := \inf_{(x,t)\in\mathcal{K},\,(x',t')\in\Gamma} \left\{ \omega_0^{\frac{2-p}{p}} |x-x'| + |t-t'|^{\frac{1}{p}} \right\} \quad \text{if } 1 < p \leq 2.$$

Theorem 2 (Evolutionary $p-$Laplace equation) *Let $q = 1$ and $p > 1$. Suppose that u is a bounded weak solution to (1.1) in Ω_T. Then, u is locally Hölder continuous in Ω_T. Precisely, there exist positive constants $C = C(C_0, C_1, n, p) > 1$ and $\gamma = \gamma(C_0, C_1, n, p) < 1$ such that, the local oscillation estimates hold true as follows: If $p \geq 2$, then,*

$$|u(x_1, t_1) - u(x_2, t_2)| \leq C\,\omega_0 \left(\frac{|x_1 - x_2| + \omega_0^{\frac{p-2}{p}} |t_1 - t_2|^{\frac{1}{p}}}{d_{\mathrm{S}}(\mathcal{K}, \Gamma)} \right)^{\gamma} \tag{1.4}$$

and, if $1 < p \leq 2$, then,

$$|u(x_1, t_1) - u(x_2, t_2)| \leq C\,\omega_0 \left(\frac{\omega_0^{\frac{2-p}{p}} |x_1 - x_2| + |t_1 - t_2|^{\frac{1}{p}}}{d_{\mathrm{f}}(\mathcal{K}, \Gamma)} \right)^{\gamma} \tag{1.5}$$

hold ture for any compact set $\mathcal{K} \subset \Omega_T$ and all pairs of points (x_1, t_1), $(x_2, t_2) \in \mathcal{K}$, where ω_0 is the local essential oscillation of u in $\mathcal{K}$, $\omega_0 = \mathrm{ess\ osc}_{\mathcal{K}} u$.

2 Preliminaries

In this section, first we shall fix some notation and we present the definition of a weak solution to (1.1). We also gather some technical tools.

The weak solution to (1.1) is defined as follows:

Definition 1 *A measurable function u defined on Ω_T is said a weak supersolution (subsolution) of (1.1), provided that the following conditions are satisfied:*

(D1) $u \in L^p_{\mathrm{loc}}\left(0, T; W^{1,p}_{\mathrm{loc}}(\Omega)\right) \cap C_{\mathrm{loc}}\left(0, T; L^{q+1}_{\mathrm{loc}}(\Omega)\right)$

(D2) *The integral identity*

$$\int_{\mathcal{K}} \left(|u|^{q-1}u\right)(t)\varphi(t)\,dx \Bigg|_{t=t_1}^{t=t_2} + \int_{\mathcal{K}\times(t_1,t_2)} \left(-|u|^{q-1}u\,\partial_t\varphi + A(x,t,u,Du)\cdot D\varphi\right) dxdt \geq (\leq) 0$$

holds for any compact subset $\mathcal{K} \subset \Omega$, any sub-interval $[t_1, t_2] \subset (0, T]$ and for all nonnegative test functions $\varphi \in L^p_{\mathrm{loc}}\left(0, T; W^{1,p}_0(\Omega)\right) \cap W^{1,q+1}_{\mathrm{loc}}\left(0, T; L^{q+1}(\Omega)\right)$ satisfying $\varphi = 0$ almost everywhere $(\Omega \setminus \mathcal{K}) \times (0, T]$

A measurable function u defined on Ω_T is called a weak solution to (1.1) if u is simultaneously a weak super and subsolution.

For any $k, v \in \mathrm{I\!R}$, let

$$(v-k)_+ = \max\{v-k,\, 0\}, \qquad (k-v)_+ = \max\{k-v,\, 0\},$$

those are crucialy used in the local energy estimates of De Girogi's type. The following auxiliary functions are used for appropriately dealing with the time-derivative terms in the local energy estimates.

$$\mathcal{A}_-(v,k) = q\int_v^k |\tau|^{q-1}(k-\tau)_+ d\tau, \qquad \mathcal{A}_+(v,k) = q\int_k^v |\tau|^{q-1}(\tau-k)_+ d\tau, \tag{2.1}$$

those are clearly nonnegative. We also notice the simple relation between two quantities in (2.1).

Lemma 3 *There holds for any $v, k \in \mathrm{I\!R}$ that*

$$\mathcal{A}_+(v,k) = \mathcal{A}_-(-v,-k). \tag{2.2}$$

Some algebraic inequalities are needed for estimating the integral stemming from the time-derivative and the $p-$Laplace terms. The proof is referred in [1, Lemma 2.2] and [12, inequality (24)].

Lemma 4 *For any $q > 0$ there exists a positive constant $C = C(q)$ such that, for all $\xi, \eta \in \mathrm{I\!R}$,*

$$\frac{1}{C}\left||\xi|^{q-1}\xi - |\eta|^{q-1}\eta\right| \le (|\xi| + |\eta|)^{q-1}|\xi - \eta| \le C\left||\xi|^{q-1}\xi - |\eta|^{q-1}\eta\right|. \tag{2.3}$$

From (2.3) in Lemma 4 we obtain the following inequality.

Lemma 5 *For any $q > 0$, there exits a positive constant $C = C(q)$ such that, for all $v, k \in \mathrm{I\!R}$,*

$$\frac{1}{C}(|v| + |k|)^{q-1}(k-v)_+^2 \le \mathcal{A}_-(v,k) \le C(|v| + |k|)^{q-1}(k-v)_+^2; \tag{2.4}$$

$$\frac{1}{C}(|v| + |k|)^{q-1}(v-k)_+^2 \le \mathcal{A}_+(v,k) \le C(|v| + |k|)^{q-1}(v-k)_+^2. \tag{2.5}$$

3 Local energy estimates

Now we present local energy estimates available for weak supersolutions and subsolutions, those are fundamental in the Hölder regularity estimates. We use the scaling transformation intrinsic to the doubly nonlinear parabolic type equations to obtain the local energy estimates. After the intrinsic scaling transformation, all of local energy estimates are performed on the "unit" space-time cylinders. Let $B := B(1)(0)$ be the unit ball with center of origin, and $I := (0,1)$, $J := (-1,0)$ be the unit intervals. Put the unit space-time cylinders $P := B \times I$ and $Q := B \times J$.

First we shall note the following relation between a weak supersolution and a weak subsolution.

Lemma 6 (relation between weak subsolution and supersolution) *If v is a weak subsolution (supersolution) in P, then $w := -v$ satisfies the integral inequality*

$$\begin{aligned}&\int_B (|w|^{q-1} w\,\phi)(s)\,dy\Big|_{s=s_1}^{s=s_2}\\ &+\int_P \Big(-|w|^{q-1} w\,\partial_s\phi + (-A)(y,s,-w,-D_y w)\cdot D_y\phi\Big)\,dyds \geq (\leq) 0\end{aligned} \tag{3.1}$$

for any sub-interval $(s_1,\, s_2) \subset I$ and all nonnegative test functions $\phi \in L^p(I; W_0^{1,p}(B)) \cap W^{1,q+1}(I; L^{q+1}(B))$.

Lemma 7 *Let v be a weak supersolution in P. For any bounded domain $K \subset B$, any interval $(s_1, s_2) \subset I$, all $k \in \mathbb{R}$ and all non-negative, Lipschitz cut-off function ζ vanishing on the lateral boundary $\partial K \times (s_1, s_2)$, there holds, with a positive constant $C = C(p, q, C_0, C_1)$*

$$\begin{aligned}&\int_K \zeta^p\,\mathcal{A}_-(v(s),k)dy\Big|_{s=s_1}^{s=s_2} + \int_{K\times(s_1,\,s_2)} \zeta^p\ |D(k-v)_+|^p\,dyds\\ &\leq C \int_{K\times(s_1,\,s_2)} \big((k-v)_+^p|D\zeta|^p + \mathcal{A}_-(v,k)|\partial_t\zeta^p|\big)\,dyds.\end{aligned} \tag{3.2}$$

If v be a weak subsolution of (1.1) in P, then it holds that for all $k \in \mathbb{R}$ and all ζ as above

$$\begin{aligned}&\int_K \zeta^p\,\mathcal{A}_+(v(s),k)dy\Big|_{s=s_1}^{s=s_2} + \int_{K\times(s_1,\,s_2)} \zeta^p\ |D(v-k)_+|^p\,dyds\\ &\leq C \int_{K\times(s_1,\,s_2)} \big((v-k)_+^p|D\zeta|^p + \mathcal{A}_+(v,k)|\partial_t\zeta^p|\big)\,dyds\end{aligned} \tag{3.3}$$

If v be a weak supersolution (subsolution) of (1.1) in Q, then, the formulas (3.2) and (3.3) holds, with P and I replaced by Q and J.

4 Local infimum and supremum estimates

Let M, ρ, c_2 be positive numbers and $(x_0,\, t_0) \in \Omega_T := \Omega \times (0,\, T)$. Put

$$\mathcal{B}_0 := \begin{cases} B(8\rho)(x_0) & \text{if } q \geq p-1 \\ B(16\rho)(x_0) & \text{if } q > p-1 \end{cases}, \quad \mathcal{P}_0 := \mathcal{B}_0 \times \big(t_0,\, t_0 + c_2 M^{q+1-p}\rho^p\big).$$

Let ω be a positive number and put some quantities as

$$M = \frac{1}{4}\omega, \quad \mu^- = \operatorname*{ess\,inf}_{\mathcal{P}_0} u, \quad \mu^+ = \operatorname*{ess\,sup}_{\mathcal{P}_0} u. \tag{4.1}$$

Throughout this section we assume the followings:

$$\mathcal{P}_0 \subset \Omega_T, \quad \omega \geq \mu^+ - \mu^-. \tag{4.2}$$

4.1 Sign-changing weak solutions

We shall consider a behavior of a *sigh-changing* weak solution of (1.1) in a local in space-time region. We observe that, if the amplitude of solution is small in a local in space-time region, the solution is lower-bounded in a smaller region.

To state the results we use some notations: For positive numbers τ, δ, ε and a positive integer j_* we define the positive number γ_0, c_2, $\widetilde{c}_2$ as

$$\gamma_0 := \begin{cases} \left(\dfrac{p-(q+1)}{p}\right)^{\frac{1}{q}} \left(\dfrac{\varepsilon}{2^{j_*}}\right)^{\frac{p-1}{q}} \delta\, e^{\frac{\tau+\frac{4^p 2}{\delta}\left(\frac{\varepsilon}{2^{j_*}}\right)^{q+1-p}}{q+1-p}} & \text{if } q<p-1 \\ \dfrac{\varepsilon}{2^{j_*+1}} & \text{if } q=p-1 \\ \dfrac{\varepsilon}{2^{j_*+3}} e^{-\frac{(4^p+2)\varepsilon^{p-(q+1)} 2^{(j_*+1)(q+1-p)}}{c_2'(q+1-p)}} & \text{if } q>p-1 \end{cases}$$

$$c_2 := \begin{cases} e^{\tau} \delta e^{\frac{4^p}{\delta}\left(\frac{\varepsilon}{2^{j_*}}\right)^{q+1-p}} & \text{if } q<p-1 \\ \delta & \text{if } q \geq p-1 \end{cases}$$

$$\widetilde{c}_2 := \begin{cases} e^{\tau} \delta e^{\frac{4^p-2^p}{\delta}\left(\frac{\varepsilon}{2^{j_*}}\right)^{q+1-p}} & \text{if } q<p-1 \\ \delta - \frac{1}{2^p} & \text{if } q=p-1 \\ \delta - c_2' \gamma_0^{q+1-p} & \text{if } q>p-1 \end{cases}$$

For positive numbers M, ρ, t_0 and $x_0 \in \mathbb{R}^n$ put the local parabolic cylinders

$$\mathcal{P} = B(2\rho)(x_0) \times \left(t_0 + M^{q+1-p}\rho^p \widetilde{c}_2,\ \ t_0 + M^{q+1-p}\rho^p c_2\right).$$

Lemma 8 (local infimum estimate) *Suppose (4.1) and (4.2) to be valid. Let u be a weak supersolution of (1.1) in $\mathcal{P}_0$. For any $\alpha \in (0,1)$, there exist positive constants $\delta_0 = \delta_0(\alpha, C_0, C_1, n, p, q) < 1$, $\varepsilon_0 = \varepsilon_0(\alpha, n, p, q) < 1$ and a positive integer $j_0^* = j_0^*(\alpha, C_0, C_1, n, p, q)$ such that the following assertion holds true for all positive numbers $\delta \leq \delta_0$, $\varepsilon \leq \varepsilon_0$, any positive integer $j_* \geq j_0^*$ and any positive τ: Assume that $-\xi M \leq \mu^- \leq \xi M$ for a positive number $\xi \leq \gamma_0$ and that the condition*

$$\left|B(\rho)(x_0) \cap \left\{u(t_0) \geq \mu^- + M\right\}\right| \geq \alpha\, |B(\rho)|$$

is valid for positive numbers α and M. Then, there holds that

$$u \geq \mu^- + \gamma_0 M \quad \text{almost everywhere } \mathcal{P}.$$

Lemma 9 (local supremum estimate) *Suppose (4.1) and (4.2) to be valid. Let u be a weak subsolution of (1.1) in $\mathcal{P}_0$. For any $\alpha \in (0,1)$, there exist positive constants $\delta_0 = \delta_0(\alpha, C_0, C_1, n, p, q) < 1$, $\varepsilon_0 = \varepsilon_0(\alpha, n, p, q) < 1$ and a positive integer $j_0^* = j_0^*(\alpha, C_0, C_1, n, p, q)$ such that the following assertion holds true for all positive numbers $\delta \leq \delta_0$, $\varepsilon \leq \varepsilon_0$, any positive integer $j_* \geq j_0^*$ and any positive τ: Assume that $-\xi M \leq \mu^+ \leq \xi M$ for a positive number $\xi \leq \gamma_0$ and that the condition*

$$\left|B(\rho)(x_0) \cap \left\{u(t_0) \leq \mu^+ - M\right\}\right| \geq \alpha\, |B(\rho)|$$

is valid for positive numbers α and M. Then, there holds that

$$u \leq \mu^+ - \gamma_0 M \quad \text{almost everywhere } \mathcal{P}.$$

4.2 Reduction to the $p-$Laplace equation

We shall continue the study of a behavior of a sign-changing weak solution of (1.1) in a local space-time region. In Sect. 4.1, we assume the smallness of amplitude of a solution. Here, we assume the local essential infimum (supremum) to be lower and upper bounded by negative (positive) numbers and thus, to be of negative (positive) constant-sign. This assumption is given by the small number ξ and is consistent with the smallness of amplitude in Sect. 4.1. Under such assumption the behavior of solution is restricted near the local infimum (supremum) by truncation at an appropriate level and then, the truncated solution is subtracted by the local infimum (supremum). In this way the truncated and translated solution becomes a nonnegative weak supersolution (subsolution) of the evolutionary $p-$Laplace type equation. Therefore we can apply the expansion of positivity to derive a decay of a local infimum (supremum) in smaller region. This method is orginally introduced in [5, Lemmata 3.3 and 3.5]. The notation (4.2) and (4.1) is also used here. To state the result we set some notations: For positive numbers τ, δ and $j_* \geq 1$, let γ, c_2, $\widetilde{c}_2$ be the positive numbeers as

$$\gamma := \frac{\varepsilon}{2^{j_*+2}} \times \begin{cases} e^{\frac{\tau}{2-p}} e^{\frac{4^p}{\delta(2-p)}\left(\frac{\varepsilon}{2^{j_*}}\right)^{2-p}} & \text{if } p > 2 \\ 2 & \text{if } p = 2 \\ e^{-\frac{2^{(j_*+1)(2-p)}4}{c_2'(2-p)}} & \text{if } 1 < p < 2 \end{cases}$$

$$c_2 := \begin{cases} e^{\tau}\delta e^{\frac{4^p}{\delta}\left(\frac{\varepsilon}{2^{j_*}}\right)^{2-p}} & \text{if } p > 2 \\ \delta & \text{if } 1 < p \leq 2 \end{cases}$$

$$\widetilde{c}_2 := \begin{cases} e^{\tau}\delta e^{\frac{4^p-2^p}{\delta}\left(\frac{\varepsilon}{2^{j_*}}\right)^{2-p}} & \text{if } p > 2 \\ \delta - \dfrac{1}{2^p} & \text{if } p = 2 \\ \delta - c_2'\gamma^{2-p} & \text{if } 1 < p < 2 \end{cases}$$

Further, for a positive number ξ let $C' = C'(q,p) > 0$ and

$$\gamma' := \gamma\frac{\xi}{4} \times \begin{cases} \xi^{q-1} & \text{if } q \geq 1 \\ 1 & \text{if } 0 < q < 1 \end{cases}$$

$$c_2' := C'\, c_2 \times \begin{cases} 1 & \text{if } q \geq 1 \\ \xi^{(1-q)(2-p)} & \text{if } 0 < q < 1 \end{cases}$$

$$\widetilde{c}_2' := C'\, \widetilde{c}_2 \times \begin{cases} 1 & \text{if } q \geq 1 \\ \xi^{(1-q)(2-p)} & \text{if } 0 < q < 1 \end{cases}$$

and put the local in space-time region $\mathcal{P}'$ as

$$\mathcal{P}' := \left(t_0 + \rho^p\left(\frac{\xi M}{4}\right)^{q+1-p}\widetilde{c}_2',\ t_0 + \rho^p\left(\frac{\xi M}{4}\right)^{q+1-p} c_2'\right).$$

Lemma 10 (local infimum estimate) *Suppose (4.1) and (4.2) to be valid. Let u be a weak supersolution of (1.1) in $\mathcal{P}_0$. For any numbers $\xi, \alpha \in (0,1)$, there exist positive constants $\delta_0 = \delta_0(\xi, \alpha, C_0, C_1, n, p, q) < 1$, $\varepsilon_0 = \varepsilon_0(\xi, \alpha, n, p, q) < 1$ and a positive integer $j_0^* = j_0^*(\xi, \alpha, C_0, C_1, n, p, q)$ such that $\varepsilon_0, \delta_0 \searrow 0$ and $j_0^* \nearrow \infty$ as $\xi \searrow 0$ and the following assertion holds true for all positive numbers $\delta \leq \delta_0$, $\varepsilon \leq \varepsilon_0$, any positive integer $j_* \geq j_0^*$ and any positive number τ : Suppose that the following assumptions to be valid : $-(\xi+4)M \leq \mu^- \leq -\xi M$ and*

$$\left|B(\rho)(x_0) \cap \left\{u(t_0) \geq \mu^- + \frac{\xi}{4}M\right\}\right| \geq \alpha|B(\rho)|.$$

Then, there holds the following estimation :

$$u \geq \mu^- + \gamma_1 M \quad \textit{almost everywhere } \mathcal{P}',$$

where $\gamma_1 = C\gamma'$ *with* $C = C(q) > 0$.

Lemma 11 (local supremum estimate) *Suppose (4.1) and (4.2) to be valid. Let* u *be a weak subsolution of (1.1) in* $\mathcal{P}_0$*. For any numbers* $\xi, \alpha \in (0,1)$*, there exist positive constants* $\delta_0 = \delta_0(\xi, \alpha, C_0, C_1, n, p, q) < 1$*,* $\varepsilon_0 = \varepsilon_0(\xi, \alpha, n, p, q) < 1$ *and a positive integer* $j_0^* = j_0^*(\xi, \alpha, C_0, C_1, n, p, q)$ *such that* $\varepsilon_0, \delta_0 \searrow 0$ *and* $j_0^* \nearrow \infty$ *as* $\xi \searrow 0$ *and the following assertion holds true for all positive numbers* $\delta \leq \delta_0$*,* $\varepsilon \leq \varepsilon_0$*, any positive integer* $j_* \geq j_0^*$ *and any positive number* τ *: Suppose the following assumptions to be valid :* $\xi M \leq \mu^+ \leq (\xi + 4)M$ *and*

$$\left| B(\rho)(x_0) \cap \left\{ u(t_0) \leq \mu^+ - \frac{\xi}{4} M \right\} \right| \geq \alpha |B(\rho)|.$$

Then, there holds following estimation :

$$u \leq \mu^+ - \gamma_1 M \qquad \text{almost everywhere } \mathcal{P}',$$

where $\gamma_1 = C\gamma'$ *with* $C = C(q) > 0$.

5 Oscillation estimates

In this section we shall present the oscillation estimates for the proof of Theorem 1. Let $q \geq p - 1$ and $p \geq 2$. Suppsoe that u is a weak solution of (1.1) and bounded,

$$D_0 := \operatorname*{ess\,sup}_{\Omega_T} |u| < \infty.$$

Set some notations: Let ω be a positive number and $M = \frac{1}{4}\omega$. For a point $x_0 \in \Omega$, positive numbers t_0, ρ and c, let the local parabolic cylinders as

$$\mathcal{Q}(\rho, c; x_0, t_0) = B(\rho)(x_0) \times \left(t_0 - c\, M^{q+1-p}\, \rho^p,\ t_0 \right),$$

Let c be a positive number and

$$\mathcal{Q} := \mathcal{Q}(16\rho, c;\, x_0, t_0), \quad \mu^- := \operatorname*{ess\,inf}_{\mathcal{Q}} u, \quad \mu^+ := \operatorname*{ess\,sup}_{\mathcal{Q}} u.$$

We also assume the following relation to be satisified:

$$\mathcal{Q} \subset \Omega_T, \quad \mu^+ - \mu^- \leq \omega. \tag{5.1}$$

Take a subregion of $\mathcal{Q}$ with the vertex (x_0, t_0), $\mathcal{Q}' := \mathcal{Q}(2\rho, c;\, x_0, t_0)$.

Lemma 12 *There exist positive constants* $\eta = \eta(C_0, C_1, n, p) < 1$ *and* $c = c(C_0, C_1, n, p) < 1$ *such that the altrenative assertion holds true: Suppose that the conditions (5.1) to be valid. Assume that* $\mu^+ - \mu^- \geq 2M$*. If it is veified that*

$$\left| \mathcal{Q} \cap \{ u \leq \mu^- + M \} \right| \leq \frac{1}{2} \left| \mathcal{Q} \right|,$$

then, there holds that

$$u \geq \mu^- + \eta M \quad \textit{almost everywhere } \mathcal{Q}'$$

If it is veified that

$$|\mathcal{Q} \cap \{u \leq \mu^- + M\}| > \frac{1}{2} |\mathcal{Q}|,$$

then, there holds that

$$u \leq \mu^+ - \eta M \quad \textit{almost everywhere } \mathcal{Q}'$$

The assertion in Lemma 12 follows from Lemmata 8, 9, 10 and 11 and the postive constants c and η can be naturally chosen by using the positive numbers γ, c_2, $\widetilde{c}_2$, γ', c_2' and $\widetilde{c}_2'$ in them.

From Lemma 12 we obtain the decay of local oscillation. Let

$$\mu'_- := \operatorname*{ess\,inf}_{\mathcal{Q}'} u, \qquad \mu'_+ := \operatorname*{ess\,sup}_{\mathcal{Q}'} u.$$

Lemma 13 *Under the setting as Lemma 12 there holds that*

$$\mu'_+ - \mu'_- \leq (1 - \eta)(\mu^+ - \mu^-).$$

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Asymptotic modelling of the current flow system described by the $p(x)$-Laplacian

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Abstract

Starting from the $p(x)$-Laplace thermistor model, we rigorously derive the effective model describing the current flow in a thin domain. We assume that the exponent $p(x)$ is constant by parts and exhibiting the oscillating interface. Depending on the parameter describing these oscillations, we investigate the asymptotic behavior of the solution by using the adaptation of the periodic unfolding method. Our work is motivated by the industrial applications consisting of thin organic light-emitting diodes.

Key words. $p(x)$-Laplacian, thin domain, oscillating interface, unfolding method.
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1 Introduction

The technology of large-area light-emitting diodes made of organic semiconductor materials (briefly, OLEDs) has gained much attention in the modern lightening industry due to its sustainability. Recently, a stationary thermistor problem was analyzed in [10, 11, 14] in order to study the electrothermal effects in OLEDs, such as self-heating and inhomogeneous current distributions. The model treated by the authors, takes the following form:

$$\begin{cases} -\text{div }(\sigma(x,T,|\nabla u|)\nabla u) = 0 \quad \text{in} \quad \Omega, \\ -\text{div }(\lambda(x)\nabla T) = (1-\eta)\sigma(x,T,|\nabla u|)|\nabla u|^2 \quad \text{in} \quad \Omega, \\ u = u_D \quad \text{on} \quad \Gamma_D, \\ \sigma(x,T,|\nabla u|)\nabla u\nu = 0 \quad \text{on} \quad \Gamma_N, \\ \lambda(x)\nabla T = k(x)(T-T_a) \quad \text{on} \quad \partial\Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain, σ and λ are the electrical and thermal conductivities, respectively, and $\eta \in [0,1]$ is the efficiency of the light outcoupling. As usual in the thermistor problems, the electrical conductivity σ depends on the temperature T and the electric field $E = -\nabla u$, with u denoting the electrostatic potential. Moreover, the electrical conductivity is given by

$$\sigma(x,T,|\nabla u|) = \sigma_0(x)F(x,T)|\nabla u|^{p(x)-2},$$

where σ_0 is an effective conductivity, F is the Arrhenius temperature factor and $p(x)$ is a power-law exponent depending on the spacial coordinates (see [14]). In particular,

$$p(x) = p_i \quad \text{in} \quad \Omega_i,$$

with $\cup_{i=1}^n \overline{\Omega}_i = \overline{\Omega}$ and $\{\Omega_i\}$ is a disjoint family of subdomains. In [10], the authors addressed the well-posedness of the above problem, whereas in [11, 14] proposed the lower-dimensional asymptotic models in

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a thin-domain settings which naturally appear in such applications. In this case, an important issue arises when treating the equation the temperature equation. Namely, it is not obvious how to obtain the boundedness of the solution (in an appropriate function space) by a constant independent of the domain. This represents the essential issue when one performs the asymptotic analysis of the PDE systems posed in singularly perturbed domains.

Problem settings. Motivated by the above discussion, in order to be able to rigorously investigate the asymptotic behaviour of the corresponding solution, we shall decouple the thermistor system, by taking $\eta = 1$. In view of that, we focus on the part of the system satisfied by the electrostatic potential and consider the following problem:

$$\text{(P)} \qquad \begin{cases} -\text{div}\,(\mu(T^\varepsilon)|\nabla u^\varepsilon|^{p_\varepsilon(x,y)-2}\nabla u^\varepsilon) = f^\varepsilon \quad \text{in} \quad R^\varepsilon = (0,1)\times(-\varepsilon,\varepsilon), \\ |\nabla u^\varepsilon|^{p_\varepsilon(x,y)-2}\nabla u^\varepsilon \nu^\varepsilon = 0 \quad \text{on} \quad \Gamma_N^\varepsilon = (0,1)\times\{-\varepsilon,\varepsilon\}, \\ u^\varepsilon = 0 \quad \text{on} \quad \Gamma_D^\varepsilon = \partial R^\varepsilon \setminus \Gamma_N^\varepsilon, \end{cases}$$

where ν^ε is the unit outward normal to Γ_N^ε. Note that we add the forcing term f^ε on the right-hand side in the current-flow equation. We make the following assumptions:

Hypothesis H_1.

H_{11} *The function $g : \mathbb{R} \to \mathbb{R}$, defining the interface*

$$\Gamma^\varepsilon = \left\{ \left(x, \varepsilon g\left(\frac{x}{\varepsilon^\alpha}\right)\right) : x \in (0,1) \right\}, \quad \alpha > 0,$$

is L-periodic, Lipschitz with

$$-1 < g_2 \le g(y_1) \le g_1 < 1, \quad g_2 \ne g_1;$$

H_{12} *The exponent $p_\varepsilon : R^\varepsilon \to \mathbb{R}_+^*$ satisfies (see Figure 1)*

$$p_\varepsilon = p_\varepsilon(x,y) = \begin{cases} p_1 & in \quad R_1^\varepsilon = \left\{(x,y) : x \in [0,1],\, \varepsilon g\left(\frac{x}{\varepsilon}\right) \le y < \varepsilon\right\} \\ p_2 & in \quad R_2^\varepsilon = \left\{(x,y) : x \in (0,1),\, -\varepsilon < y < \varepsilon g\left(\frac{x}{\varepsilon}\right)\right\} \end{cases}$$

and without loss of generality, we suppose $1 \le p_1 < p_2$.

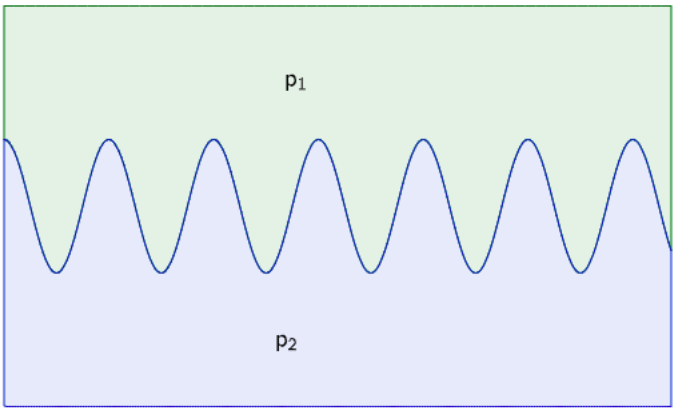

Figure 1: The exponent p_ε.

Next, we impose the following hypothesis on the given terms appearing in the equation $(\text{P})_1$:

Hypothesis H_2.

H_{21} *The temperature $T^\varepsilon \in L^\infty(R^\varepsilon)$ and there exists $T \in L^\infty(0,1)$ such that*

$$\|T^\varepsilon - T\|_{L^\infty(R^\varepsilon)} \to 0, \quad as \quad \varepsilon \to 0;$$

H_{22} *The electrical conductivity $\mu : \mathbb{R} \to \mathbb{R}$ is positive, locally Lipschitz and bounded from bellow by a strictly positive constant;*

H_{23} $f^\varepsilon \in L^{p_\varepsilon(x,y)}(R^\varepsilon)$ *with* $\mathcal{T}_\varepsilon f^\varepsilon \rightharpoonup \hat{f}$ *weakly in* $L^{p(\hat{y})}((0,1)\times(0,L)\times(-1,1))$, *where* $\mathcal{T}_\varepsilon$ *is the unfolding operator (see the next section for its rigorous definition), where* $\mathcal{T}_\varepsilon$ *is the unfolding operator (see the next section for its rigorous definition), where*

$$p(y_1, y_2) = p(\hat{y}) = \begin{cases} p_1, & \text{if} \quad \hat{y} \in Y_1^* = \{\hat{y} : y_1 \in (0,L),\, g(y_1) \le y_2 \le 1\} \\ p_2, & \text{if} \quad \hat{y} \in Y_2^* = \{\hat{y} : y_1 \in (0,L),\, -1 < y_2 < g(y_1)\} \end{cases}$$

We remark that hypothesis H_{21} holds if one considers a stationary Poisson equation with sufficiently regular data. Furthermore, H_{23} allows us to deal with the forcing term and determine the limit problem.

The variational formulation of (P) reads as follows:

$$\int_{R^\varepsilon} \mu(T^\varepsilon)\, |\nabla u^\varepsilon|^{p_\varepsilon(x,y)-2}\, \nabla u^\varepsilon \nabla\varphi dxdy = \int_{R^\varepsilon} f^\varepsilon \varphi dxdy,$$

for all $\varphi \in W_D^{1,p_\varepsilon}(R^\varepsilon)$, where $W_D^{1,p_\varepsilon}(R^\varepsilon)$ denotes the Sobolev space $W^{1,p_\varepsilon}(R^\varepsilon)$ satisfying Dirichlet condition on Γ_D^ε. It is straightforward to establish that the above problem has an unique solution, using Minty–Browder's Theorem.

For the purpose of the analysis in the sequel, since we have the discontinuous variable exponent depending on R^ε, we rewrite the above variational formulation to obtain:

$$\int_{R_1^\varepsilon} \mu(T^\varepsilon)|\nabla u_1^\varepsilon|^{p_1-2}\nabla u_1^\varepsilon \nabla\varphi_1 dxdy + \int_{R_2^\varepsilon} \mu(T^\varepsilon)|\nabla u_2^\varepsilon|^{p_2-2}\nabla u_2^\varepsilon \nabla\varphi_2 dxdy = \sum_{i=1}^{2}\int_{R_i^\varepsilon} f^\varepsilon \varphi_i dxdy,$$

for all $(\varphi_1, \varphi_2) \in W_D^{1,p_\varepsilon}(R_1^\varepsilon) \times W_D^{1,p_\varepsilon}(R_2^\varepsilon)$ with $\varphi_1 = \varphi_2$ on $\partial R_1^\varepsilon \cap \partial R_2^\varepsilon$.

Statement of the effective problems. We propose the following:

$$\begin{cases} -\left[\mu(T)\left(q_1\, |u_x|^{p_1-2}\, u_x + q_2\, |u_x|^{p_2-2}\, u_x\right)\right]_x = f \quad \text{in} \quad (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1.1}$$

Depending on the value of the parameter α determining the interface oscillations, we deduce:

- If $\alpha = 1$ (critical case), then

$$q_i = \int_{Y_i^*} |(1,0) + \nabla X|^{p_i-2}\, ((1,0)+\nabla X)\,(1,0) d\hat{y}, \quad i = 1,2,$$

  and X is the weak solution of the auxiliary problem

$$\begin{cases} -\text{div}\, \left(|(1,0)+\nabla X|^{p(\hat{y})-2}((1,0)+\nabla X)\right) = 0 \quad \text{in} \quad Y = (0,L)\times(-1,1) \\ |(1,0)+\nabla X|^{p(\hat{y})-2}((1,0)+\nabla X)\eta = 0 \quad \text{on} \quad (0,L)\times\{-1,1\} \\ L - \text{periodic in} \quad y_1 \quad \text{with} \quad \displaystyle\int_{Y^*} X d\hat{y} = 0. \end{cases}$$

- If $\alpha > 1$, then

$$q_i = g_i, \quad i = 1,2.$$

- If $\alpha < 1$, the limit problem reads:

$$\begin{cases} -\left[\mu(T)B(u_x)\right]_x = f \quad \text{in} \quad (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.2}$$

  where

$$B(\lambda) = \int_0^L \sum_{i=1}^{2} |X_\lambda|^{p_i-2}\, X_\lambda dy_1 \quad \text{and} \quad f = \int_Y \hat{f} f \hat{y}$$

with X_λ being the solution of

$$\left\{ \begin{array}{l} \displaystyle\sum_{i=1}^{2} \left\{ |\lambda + (X_\lambda)_{y_1}(y_1)|^{p_i-2} \left(\lambda + (X_\lambda)_{y_1}(y_1)\right) [1 + (-1)^i g(y_1)] \right\}_{y_1} = 0 \quad \text{in} \quad (0, L) \\ X_\lambda(0) = X_\lambda(L). \end{array} \right.$$

Note that, in the above problem, we are not in position to explicitly determine X_λ. This is due to the simple fact that we are summing a p_1-Laplacian and a p_2-Laplacian ($p_1 \neq p_2$). Hence, although we obtain a limit problem, it is not a p-Laplacian equation anymore. On the other hand, the monotonicity properties and the uniqueness of solution still remain.

Literature overview. Now, we make a short review of the literature on the subject. From pioneering works [12, 20], where the authors studied the Laplacian operator, to more recent ones [19], where the authors studied the p-Laplacian operator, thin domains were considered due to its practical importance. We must also mention more recent works where the authors studied the effects of oscillating boundaries in thin domains, from linear problems, [1, 5], to nonlinear ones, [2, 3, 4, 15, 16, 17, 18]. We also refer the reader to [8, 9], where the authors treated monotone problems in domains with oscillating boundaries.

As mentioned earlier, our work is directly inspired by very interesting results brought in [10, 11, 14]. It is important to emphasize that the $p(x)$-Laplacian considered by the authors has the exponent $p(x)$ being constant by parts and each of these regions are also thin with a flat interface between the different constant values of $p(x)$. The main novelty of our work is that we allow the interface to oscillate, as one can see in Figure 1. For the presentation's sake, we have simplified the exponent for only two different regions. Nevertheless, one can easily extend the analysis presented here to a case for multiple regions with the same order of oscillations ε^α and periodicity L given above. Furthermore, one can also consider higher dimensional situations. It is worth to mention that such oscillating interfaces make perfect sense from the point of view of the applications, since materials (organic or not) hardly posses a perfectly flat surfaces (see e.g. [21, 22]).

Since the interface that separates the different constant values given by $p_\varepsilon(x, y)$ is oscillating, we have that the two regions R_1^ε and R_2^ε are, in fact, thin domains with oscillating boundaries. This enables us to make use of the tools developed in the previous works (see e.g. [4, 5]). For that reason, we adapt the so-called periodic unfolding method (see [6] and the references therein) and determine the asymptotic behaviour of the solution, as $\varepsilon \to 0$ (see Theorem 2.2). Notice that, differently from the previous works, the variable exponent plays an important role when one defines the function spaces and, for this reason, we must revisit an important convergence result. We believe that our results will gain attention in the engineering community as well, since we propose the effective lower-dimensional model that can be used for numerical simulations.

2 The convergence results

We recall, for the sake of convenience, the definition of the unfolding operator:

Definition 2.1. *Let $\varphi \in L^{p_\varepsilon(x,y)}(R^\varepsilon)$. We define the unfolding operator $\mathcal{T}_\varepsilon : L^{p_\varepsilon(x,y)}(R^\varepsilon) \to L^{p(\hat{y})}((0,1) \times Y)$ by*

$$\mathcal{T}_\varepsilon \varphi(x, \hat{y}) = \varphi\left(\varepsilon^\alpha \left[\frac{x}{\varepsilon^\alpha}\right] L + \varepsilon y_1, \varepsilon y_2\right) \quad \textit{for} \quad (x, y_1, y_2) = (x, \hat{y}) \in (0,1) \times Y.$$

The convergence results:

Theorem 2.2. *Let u^ε be the weak solution of* (P) *satisfying Hypothesis H_1 and H_2.*
IF $\alpha = 1$, there exist $(u, u_1, u_2) \in W_0^{1,p_1}(0,1) \times L^{p_1}((0,1); W^{1,p_1}(Y_1^*)) \times L^{p_2}((0,1); W^{1,p_2}(Y_2^*))$ such that

$$\mathcal{T}_\varepsilon^i u^\varepsilon \to u \quad \textit{strongly in} \quad L^{p_i}\left((0,1); W^{1,p_i}(Y_i^*)\right),$$
$$\mathcal{T}_\varepsilon^i \nabla u^\varepsilon \rightharpoonup (u_x, 0) + \nabla_{\hat{y}} u_1^i \quad \textit{weakly in} \quad L^{p_i}\left((0,1) \times Y_i^*\right),$$

for $i = 1, 2$. Moreover, u is the weak solution of (1.1).

If $\alpha < 1$, then there exist $(u, u_1^1, u_1^2) \in W^{1,p_2}(0,1) \times L^{p_1}\left((0,1); W^{1,p_1}(Y_1^*)\right) \times L^{p_2}\left((0,1); W^{1,p_2}(Y_2^*)\right)$ with $\frac{\partial u_1^i}{\partial y_2} = 0$ and $u_1^1 = u_1^2 = u_1(x, y_1)$ such that

$$\begin{aligned}
&\mathcal{T}_\varepsilon^i u_i^\varepsilon \to u \quad \textit{strongly in} \quad L^{p_i}\left((0,1); W^{1,p_i}(Y_i^*)\right), \quad i = 1, 2, \\
&\mathcal{T}_\varepsilon^i \nabla u_i^\varepsilon \rightharpoonup \frac{\partial u}{\partial x} + \frac{\partial u_1}{\partial y_1} \quad \textit{weakly in} \quad L^{p_i}\left((0,1) \times Y_i^*\right), \quad i = 1, 2,
\end{aligned}$$

with u being the solution of (1.2).

If $\alpha > 1$, there exists $u \in W^{1,p_2}(0,1)$, such that

$$\begin{aligned}
&|||u^\varepsilon - u|||_{L^{p_2}(R^\varepsilon)} \to 0, \\
&\mathcal{T}_\varepsilon^{12}\left(\mu(T^\varepsilon)\, |\nabla u_{12}^\varepsilon|^{p(\hat{y})-2}\, \nabla u_{12}^\varepsilon \cdot (1,0)\right) \rightharpoonup 0 \quad \textit{weakly in} \quad L^{p(\hat{y})'}\left((0,1) \times Y_{12}\right),
\end{aligned}$$

with u the solution of (1.1).

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MODIFIED SCATTERING FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH LINEAR LONG-RANGE POTENTIALS

HARUYA MIZUTANI

1. Introduction

In this talk, we present recent progress [8] (joint work with Masaki Kawamoto) on a modified scattering theory for the following nonlinear Schrödinger equation with a linear potential V:

$$i\partial_t u - Hu = F(u), \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.1}$$

where $n = 1, 2, 3$, $u = u(t, x)$ is a $\mathbb{C}$-valued unknown function, $F(u) = \nu|u|^{\frac{2}{n}}u$ with $\nu \in \mathbb{R}$,

$$H = H_0 + V, \quad H_0 = -\frac{1}{2}\Delta = -\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2},$$

and $V = V(x)$ is a real-valued function converging to 0 as $|x| \to \infty$. A typical example of V we have in mind is the repulsive inverse power potential of the form

$$V(x) = a|x|^{-\rho}, \quad a > 0, \quad n/4 < \rho \le 1,$$

in particular the repulsive Coulomb potential $a|x|^{-1}$ in three space dimensions $n = 3$.

We are interested in the *final state problem* for (1.1). Namely, for a given function $u_+(x)$ (called the *scattering datum*), we seek to construct a time-dependent function $v(t, x)$ (call the *asymptotic profile*) and a global solution $u(t, x)$ to (1.1) scattering to $v(t)$:

$$\|u(t) - v(t)\|_{L^2} \to 0, \quad t \to \infty.$$

In this regard, the nonlinearity $F(u)$ is known to be critical. Indeed, no non-trivial solution scatter to a free solution ([1, 9]), but the asymptotic profile must have an additional correction term in general:

Theorem 1.1 ([11] for $n = 1$ and [4] for $n = 2, 3$)**.** *Suppose $n = 1, 2, 3$, $\widehat{u_+} \in H^2$ and in addition $\widehat{u_+} \in L^1$ if $n = 3$. If $\|\widehat{u_+}\|_{L^\infty}$ is small enough, then there exists a unique global (mild) solution $u \in C(\mathbb{R}; L^2(\mathbb{R}^n))$ to* (1.1) *with $V \equiv 0$ such that, for any $1/2 < \alpha < 1$,*

$$\|u(t) - v_{\mathrm{O}}(t)\|_{L^2} \lesssim t^{-\alpha}, \quad t \to \infty, \tag{1.2}$$

where

$$v_{\mathrm{O}}(t, x) = (it)^{-n/2} e^{i|x|^2/(2t)} e^{-i\nu|\widehat{u_+}(x/t)|^{2/n}\log|t|}\widehat{u_+}(x/t).$$

The above type statement is called the *modified scattering* as v_{O} has a phase modification $e^{-i\lambda|\widehat{u_+}(x/t)|^{2/n}\log|t|}$ compared with $(it)^{-n/2}e^{i|x|^2/(2t)}\widehat{u_+}(x/t)$ which is the leading term of $e^{-itH_0}u_+$ as $t \to \infty$. We refer to [6, 7] for further development on the modified scattering for the final state problem and [5] on the modified scattering for the associated Cauchy problem (with $V \equiv 0$).

For the linear Schrödinger equation (1.1) with $\nu = 0$ and long-range potential $V(x) = O(\langle x\rangle^{-\rho})$ with $0 < \rho \le 1$, the situation is similar: no non-trivial solution scatter to the free solution, but the modified wave operator exists and is asymptotically complete (see e.g. textbook [2]). In particular, the following theorem is known:

Theorem 1.2 ([12]). *Let $n \geq 1$, $0 < \rho \leq 1$ and $V \in C^2(\mathbb{R}^n;\mathbb{R})$ satisfy*
$$|\partial_x^\alpha V(x)| \leq C_\alpha \langle x\rangle^{-\rho-|\alpha|}$$
on $\mathbb{R}^n$ for all $|\alpha| \leq 2$. Then, for any $u_+ \in L^2(\mathbb{R}^n)$ there exists $u_0 \in L^2(\mathbb{R}^n)$ such that
$$\|e^{-itH}u_0 - u_{\mathrm{Y}}(t)\|_{L^2} \to 0, \quad t \to \infty, \tag{1.3}$$
where
$$\upsilon_{\mathrm{Y}}(t) = (it)^{-n/2} e^{i\Psi_0(t,x)}\widehat{u_+}(x/t),$$
and Ψ_0 is an approximate solution to the Hamilton–Jacobi equation
$$-\partial_t \Psi_0 = \frac{1}{2}|\nabla_x \Psi_0|^2 + V(x)$$
satisfying $\nabla_x \Psi_0(t,x) = x/t + o(1)$ as $t \to \infty$. In particular, the modified wave operator W_{Y}^+ : $L^2(\mathbb{R}^n) \ni u_+ \mapsto u_0 = \lim_{t\to\infty} e^{itH}\upsilon_{\mathrm{Y}}(t) \in L^2(\mathbb{R}^n)$ exists as a partial isometry on $L^2(\mathbb{R}^n)$.

We refer to [3] for the asymptotic completeness of Yafaev's modified wave operator.

While the modified scattering theory has been extensively studied for both linear and nonlinear Schrödinger equations, the literature is much more sparse both for the final state and Cauchy problems in the mixed case, namely (1.1) with $\nu \neq 0$ and $V \not\equiv 0$. If $n = 1$ and the potential V is of very short-range, say $\langle x\rangle V \in L^1(\mathbb{R})$, then the modified scattering for the Cauchy problem has been proved by several works (see e.g. [10]), where the asymptotic profile is essentially the same as υ_{O} and there is no (linear) phase modification as for Yafaev's profile υ_{Y}. In particular, to the best of our knowledge, there seems to be no known positive results on the modified scattering for (1.1) if V is of long-range type, which is the main target of the paper [8].

2. Main theorem

To state the main result, we introduce several assumptions and notation. Let us first introduce the assumption on the potential.

Assumption A. V is real-valued and satisfies the following two conditions (A1), (A2):

(A1) $V = V^{\mathrm{S}} + V^{\mathrm{L}} + V^{\mathrm{sing}}$ with some real-valued $V^{\mathrm{S}}, V^{\mathrm{L}}, V^{\mathrm{sing}}$ on $\mathbb{R}^n$ such that:
- V^{S} satisfies $V^{\mathrm{S}} \in L^\infty(\mathbb{R}^n)$ and there exists $\rho_{\mathrm{S}} > 1 + \frac{n}{4}$ such that
$$|V^{\mathrm{S}}(x)| \lesssim \langle x\rangle^{-\rho_{\mathrm{S}}}, \quad x \in \mathbb{R}^n. \tag{2.1}$$
- $V^{\mathrm{L}} \in C^2(\mathbb{R}^n)$ and there exists $\frac{n}{4} < \rho_{\mathrm{L}} \leq 1 + \frac{n}{4}$ such that for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq 2$ there exists $C_\alpha > 0$ such that
$$|\partial_x^\alpha V^{\mathrm{L}}(x)| \leq C_\alpha \langle x\rangle^{-\rho_{\mathrm{L}}-|\alpha|}, \quad x \in \mathbb{R}^n. \tag{2.2}$$
- V^{sing} is relatively H_0-form compact, namely $|V^{\mathrm{sing}}|^{\frac{1}{2}}(H_0+1)^{-\frac{1}{2}}$ is a compact operator on $L^2(\mathbb{R}^n)$. Moreover, V^{sing} is compactly supported.

(A2) The unitary group e^{-itH} satisfies the global-in-time Strichartz estimates:
$$\left\|e^{-itH}f\right\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)}, \tag{2.3}$$
$$\left\|\int_0^t e^{-i(t-s)H}F(s)ds\right\|_{L^q(\mathbb{R};L^r(\mathbb{R}^n))} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R};L^{\tilde{r}'}(\mathbb{R}^n))}, \tag{2.4}$$
for any admissible pairs (q,r) and $(\tilde{q},\tilde{r})$, where (q,r) is said to be admissible if $q,r \geq 2$, $2/q = n(1/2-1/r)$ and $(q,r) \neq (2,\infty)$.

There are several concrete examples of V satisfying Assumption A:

Example 2.1. V satisfies Assumption A if *one* of the following (E1)–(E4) hold:

(E1) *Very short-range potential.* Let $n = 1, 2, 3$ and $V = V^{\mathrm{S}} + V^{\mathrm{sing}}$, where
- $V^{\mathrm{S}} \in L^\infty(\mathbb{R}^n)$ satisfies (2.1) with some $\rho_{\mathrm{S}} > 2$.
- $V^{\mathrm{sing}} \equiv 0$ if $n = 1, 2$ and $V^{\mathrm{sing}} \in L^{\frac{3}{2}}_{\mathrm{compact}}(\mathbb{R}^3)$ if $n = 3$.

Moreover, the negative part $V_- = \max\{0, -V\}$ of V satisfies $V_- \equiv 0$ if $n = 1, 2$ and $\|V_-\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 3 \cdot 2^{-\frac{4}{3}}\pi^{\frac{4}{3}}$ if $n = 3$.

(E2) *Smooth slowly decaying potential.* Let $n = 2, 3$, $V \in C^\infty(\mathbb{R}^n)$ and there exists $\frac{n}{4} < \rho < 2$ such that the following properties (H1)–(H3) hold:

(H1) For all $\alpha \in \mathbb{Z}^n_+$, there exists $C_\alpha > 0$ such that
$$|\partial_x^\alpha V(x)| \le C_\alpha \langle x\rangle^{-\rho-|\alpha|}, \quad x \in \mathbb{R}^n.$$

(H2) There exists $C_1 > 0$ such that
$$V(x) \ge C_1\langle x\rangle^{-\rho}, \quad x \in \mathbb{R}^n.$$

(H3) There exists $R_0, C_2 > 0$ such that
$$-x \cdot \nabla V(x) \ge C_2\langle x\rangle^{-\rho}, \quad |x| \ge R_0.$$

(E3) *Inverse power potential.* Let $n = 2, 3$ and $V = Z|x|^{-\rho} + \varepsilon W$, where $\frac{n}{4} < \rho < 2$ and $Z > 0$. Moreover, $\varepsilon > 0$ is a sufficiently small constant and $W \in C^\infty(\mathbb{R}^n)$ satisfies
$$|\partial_x^\alpha W(x)| \le C_\alpha\langle x\rangle^{-1-\rho-|\alpha|}.$$

(E4) *Small perturbation of (E1)–(E3).* Let $n = 3$ and $V = V_1 + V_2$, where V_1 satisfies one of (E1)–(E3) and $\|V_2\|_{L^{3/2}(\mathbb{R}^3)}$ is small enough.

In particular, if $n = 2, 3$, $a > 0$ and $\rho \in (\frac{n}{4}, 2)$ then $a\langle x\rangle^{-\rho}$ and $a|x|^{-\rho}$, satisfy Assumption A. For the scattering data, we impose:

Assumption B. $\langle x\rangle^\gamma u_+ \in L^2(\mathbb{R}^n)$ and there exists $c_0 > 0$ such that $\operatorname{supp}\widehat{u_+} \subset \{|\xi| \ge c_0\}$, where γ is specified in Theorem 2.2 below.

Given a u_+ satisfying Assumption B, we then define the asymptotic profile v as follows. We fix a cut-off function $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \chi \le 1$, $\chi(x) = 1$ for $|x| \le c_0/4$ and $\chi(x) = 0$ for $|x| \ge c_0/3$, where c_0 is given in Assumption B. With the long-range part V^{L} of V and $T_1 \ge 1$, a time dependent effective potential $V_{T_1}(t, x)$ is defined by
$$V_{T_1}(t,x) = V^{\mathrm{L}}(x)\left\{1 - \chi\left(\frac{2x}{t+T_1}\right)\right\}.$$
Note that $V_{T_1} \equiv V^{\mathrm{L}}$ for $t \ge 0$ and $|x| \ge c_0(t + T_1)/4$ and
$$|\partial_x^\alpha V_{T_1}(t,x)| \le C_\alpha\langle t\rangle^{-\rho_{\mathrm{L}}-|\alpha|}, \quad t \ge 0, \quad x \in \mathbb{R}^n,$$
where C_α is independent of T_1. Thanks to this decay property, if $T_1 \ge 1$ is large enough, one can construct a (real-valued) solution $\Psi \in C^1([1,\infty)\times\mathbb{R}^n)\cap C([1,\infty); C^2(\mathbb{R}^n))$ to the Hamilton-Jacobi equation
$$-\partial_t\Psi(t,x) = \frac{1}{2}|\nabla\Psi(t,x)|^2 + V_{T_1}(t,x)$$

such that, for $t \ge 1$,

$$\left\|\nabla\Psi(t,x)-\frac{x}{t}\right\|_{L^\infty(\mathbb{R}^n)}+t\left\|\Delta\Psi(t,x)-\frac{n}{t}\right\|_{L^\infty(\mathbb{R}^n)}\lesssim \max\{t^{-\rho_{\mathrm{L}}},t^{-1}\log t\}$$

Then the asymptotic profile v is defined by

$$v(t,x):=(it)^{-n/2}e^{i\Psi(t,x)}e^{-i\nu|\widehat{u_+}(x/t)|^{2/n}\log t}\widehat{u_+}(x/t).$$

v is exactly a mixture of Ozawa's profile v_{O} and Yafaev's profile v_{Y}, but there is no interaction between linear and nonlinear phase correction terms.

Now we state the main result.

Theorem 2.2 ([8]). *Let $1 \le n \le 3$, $1 \le \gamma \le 2$ if $n = 1$, $\frac{n}{2} < \gamma < 1 + \frac{2}{n}$ if $n = 2, 3$ and $c_0 > 0$. Suppose V, u_+ satisfy Assumptions A and B and $\|\widehat{u_+}\|_{L^\infty}$ be small enough. Let*

$$\frac{n}{4}<b<\min\left\{\frac{\gamma}{2},\ \rho_{\mathrm{L}},\ \rho_{\mathrm{S}}-1,1\right\}.$$

Then there exists a unique solution $u \in C(\mathbb{R}; L^2(\mathbb{R}^n))$ to (1.1) satisfying

$$\|u(t)-v(t)\|_{L^2}+\|u-v\|_{L^q([t,\infty);L^r(\mathbb{R}^n))}\lesssim t^{-b},\quad t\to+\infty.$$

for any admissible pair (q, r).

3. Idea of the proof

We here explain briefly the idea of the proof of Theorem 2.2. Let $V^{\mathrm{S}} \equiv V^{\mathrm{sing}} \equiv 0$ and $\rho_{\mathrm{L}} < 1$ for simplicity. Thus, we focus on how to deal with the long-range part V^{L}.

3.1. **Integral equation.** Recall the Dollard (often also called MDFM) decomposition:

$$e^{-itH_0}=\mathcal{M}(t)\mathcal{D}(t)\mathcal{F}\mathcal{M}(t),$$

where $\mathcal{M}(t)f(x) = e^{\frac{i|x|^2}{2t}}f(x)$, $\mathcal{D}(t)f(x) = (it)^{-\frac{n}{2}}f(t/x)$ and $\mathcal{F}f = \widehat{f}$ denotes the Fourier trans form with respect to the x-variable. Define a modified free evolution $U_\Psi(t)$ by

$$U_\Psi(t):=\mathcal{M}_\Psi(t)\mathcal{D}(t)\mathcal{F}\mathcal{M}(t),\quad \mathcal{M}_\Psi(t)f(x):=e^{i\Psi(t,x)}f(x)$$

We also let $w(t,x) = e^{-i\nu|\widehat{u_+}(x)|^{2/n}\log t}\widehat{u_+}(x)$ such that the profile function v is of the form

$$v=\mathcal{M}_\Psi(t)\mathcal{D}(t)w=U_\Psi(t)\mathcal{F}^{-1}w-\mathcal{M}_\Psi(t)\mathcal{D}(t)\mathcal{R}(t)w,$$

where $\mathcal{R}(t) = \mathcal{F}(\mathcal{M}-1)\mathcal{F}^{-1}$. To utilize the support condition of u_+, we split v into the low-velocity and high-velocity parts as

$$v=\chi_t(x)v+(1-\chi_t(x))v,$$

where $\chi_t(x) = \chi(x/t)$ with χ given in the definition of the effective potential V_{T_1}. Then,

$$v=\chi_t\mathcal{M}_\Psi\mathcal{D}w+(1-\chi_t)\mathcal{M}_\Psi\mathcal{D}w=(1-\chi_t)U_\Psi\mathcal{F}^{-1}w-(1-\chi_t)\mathcal{M}_\Psi\mathcal{D}\mathcal{R}w,\tag{3.1}$$

where the low-velocity part $\chi_t\mathcal{M}_\Psi\mathcal{D}w$ vanishes identically since

$$\chi_t\mathcal{M}_\Psi\mathcal{D}w=\mathcal{M}_\Psi\mathcal{D}\chi w$$

and $\operatorname{supp}\chi \cap \operatorname{supp}\widehat{u_+} = \emptyset$. With the expression (3.1) at hand, calculating

$$e^{-itH}i\partial_te^{itH}[u-v-(1-\chi_t)\mathcal{M}_\Psi\mathcal{D}\mathcal{R}w],$$

one can rewrite (1.1) as

$$e^{-itH}i\partial_te^{itH}(u-v-E_1)=F(u)-F(v)+E_2+E_3,$$

where

$$\begin{aligned} E_1(t) &= \mathcal{M}_\Psi(t)\mathcal{D}(t)(1-\chi)\mathcal{R}(t)w(t) \\ E_2(t) &= -t^{-1}\mathcal{M}_\Psi(t)\mathcal{D}(t)(1-\chi)\mathcal{R}(t)F(w(t)), \\ E_3(t) &= -e^{-itH}[i\partial_t, e^{itH}(1-\chi_t)U_\Psi(t)\mathcal{F}^{-1}]w(t) \end{aligned}$$

and $[A,B]=AB-BA$ denotes the commutator of A and B. Equation (1.1) thus is reformulated into the following integral equation:

$$u(t) = v(t) + E_1(t) + i\int_t^\infty e^{-i(t-s)H}\left\{F(u(s)) - F(v(s)) + E_2(s) + E_3(s)\right\} ds. \tag{3.2}$$

3.2. **Energy estimate.** The ultimate goal then is to show that the map $u \mapsto$ RHS of (3.2) is a contraction in an appropriate energy space equipped with the norm

$$\sup_{t\ge T} t^b \|u(t)\|_{L^2} + \sup_{t\ge T} t^b \|u\|_{L^q([t,\infty);L^r)}$$

with some T large enough and admissible pair (q,r).

All the terms in (3.2) except for E_3 can be dealt by completely the same argument as in [11, 4, 6] based on Strichartz estimates for e^{-itH}, the nonlinear estimates (see e.g. [6])

$$\begin{aligned} \|w(t)\|_{H^\gamma} &\lesssim (1+|\log t|^{\lceil\gamma\rceil}\|\widehat{u_+}\|_{H^\gamma}^{\frac{2\lceil\gamma\rceil}{n}})\|\widehat{u_+}\|_{H^\gamma}, \\ \|F(w(t))\|_{H^\gamma} &\lesssim (1+|\log t|^{\lceil\gamma\rceil}\|\widehat{u_+}\|_{H^\gamma}^{\frac{2\lceil\gamma\rceil}{n}})\|\widehat{u_+}\|_{H^\gamma}\|\widehat{u_+}\|_{L^\infty}^{\frac{2}{n}}, \end{aligned}$$

and the decay estimates for $\mathcal{D}(t)$ and $\mathcal{M}(t)-1$ (recall that $\mathcal{R}=\mathcal{F}(\mathcal{M}-1)\mathcal{F}^{-1}$):

$$\|\mathcal{D}(t)f\|_{L^r} = |t|^{-n(\frac12-\frac1r)}\|f\|_{L^r}, \quad \|(\mathcal{M}(t)-1)f\|_{L^r} \lesssim |t|^{-\delta}\||x|^{2\delta}f\|_{L^r}$$

where $1\le r\le\infty$, $0\le\delta\le1$ and $\lceil\gamma\rceil = \min\{m\in\mathbb{Z} \mid m\ge\gamma\}$ denotes the smallest integer greater than or equal to γ. In particular, no specific property of Ψ is used in this step, but only the fact that Ψ is real-valued is sufficient.

Calculating the commutator shows the term E_3 is of the form $E_3 = I_1 + I_2 + I_3$ with

$$\begin{aligned} I_1 &= -(1-\chi_t)\mathcal{M}_\Psi\mathcal{A}_\Psi\mathcal{D}\mathcal{F}\mathcal{M}\mathcal{F}^{-1}w, \\ I_2 &= \left\{\frac{i}{t}\left(\nabla\Psi - \frac{x}{t}\right)\cdot(\nabla\chi)_t + \frac{1}{2t^2}(\Delta\chi)_t\right\}U_\Psi\mathcal{F}^{-1}w, \\ I_3 &= -\frac{1}{t^2}\mathcal{M}_\Psi\mathcal{D}\nabla\chi\cdot\mathcal{F}\mathcal{M}\mathcal{F}^{-1}\nabla w, \end{aligned}$$

where

$$\mathcal{A}(t) = -\partial_t\Psi - \frac12|\nabla\Psi|^2 - V_{T_1} + i\left(\nabla\Psi - \frac{x}{t}\right)\cdot\nabla + i\left(\Delta\Psi - \frac{n}{t}\right).$$

By the properties of Ψ above, one can obtain an energy estimate for $E_3(t)$ which is sufficient to close the argument.

3.3. **Construction of Ψ.** It remains to construct Ψ for which we follow closely the argument in Dereziński-Gérard [2, Sections 1.5, 1.8 and A.3] based on the method of characteristics. A rough strategy is as follows:

- We first construct a unique Hamilton flow $(X(t,\xi),\Xi(t,\xi))$ associated with the classical energy $\frac12|\xi|^2 + V_{T_1}(t,x)$ with the initial position $X(0,\xi)=0$ and the final momentum $\lim_{t\to\infty}\Xi(t,\xi)=\xi$.

- As an action integral associated with this Hamilton flow, we next obtain the solution S to the (momentum space) Hamilton–Jacobi equation
$$\partial_t S(t,\xi) = \frac{1}{2}|\xi|^2 + V_{T_1}(t, \nabla_\xi S(t,\xi))$$
with the asymptotics $t^{-1}\nabla_\xi S(t,\xi) = \xi + O(t^{-\rho_{\mathrm{L}}})$ as $t \to \infty$.
- Finally, Ψ can be constructed as the Legendre transform of S.

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GRADIENT PROFILE FOR THE RECONNECTION OF VORTEX LINES WITH THE BOUNDARY IN TYPE-II SUPERCONDUCTORS

YI C. HUANG AND HATEM ZAAG

Abstract. In a recent work, Duong, Ghoul and Zaag determined the gradient profile for blowup solutions of standard semilinear heat equation with power nonlinearities in the (supposed to be) generic case. Their method refines the constructive techniques introduced by Bricmont and Kupiainen and further developed by Merle and Zaag. In this paper, we extend their refinement to the problem about the reconnection of vortex lines with the boundary in a type-II superconductor under planar approximation, a physical model derived by Chapman, Hunton and Ockendon featuring the finite time quenching for the nonlinear heat equation

$$\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} + e^{-h} - \frac{1}{h^\beta}, \quad \beta > 0$$

subject to initial boundary value conditions

$$h(\cdot, 0) = h_0 > 0, \quad h(\pm 1, t) = 1.$$

We derive the intermediate extinction profile with refined asymptotics, and with extinction time T and extinction point 0, the gradient profile behaves as $x \to 0$ like

$$\lim_{t\to T} (\nabla h)(x,t) \quad \sim \quad \frac{1}{\sqrt{2\beta}} \frac{x}{|x|} \frac{1}{\sqrt{|\log|x||}} \left[\frac{(\beta+1)^2}{8\beta} \frac{|x|^2}{|\log|x||} \right]^{\frac{1}{\beta+1}-\frac{1}{2}},$$

agreeing with the gradient of the extinction profile previously derived by Merle and Zaag. Our result holds with general boundary conditions and in higher dimensions.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 1$. Let $\beta > 0$. In this paper, we are interested in the quenching problem modelled on the nonlinear heat equation on Ω

$$\frac{\partial h}{\partial t} = \Delta h - F(h), \tag{1.1}$$

where

$$F = F_\beta = \frac{1}{h^\beta} + \widetilde{F} \in C^\infty(\mathbb{R}_+), \quad \text{with } \mathbb{R}_+ = (0, \infty).$$

We shall assume that $\widetilde{F}$ satisfies

$$\widetilde{F}(h) = o\left(\frac{1}{h^\beta}\right) \quad \text{and} \quad \widetilde{F}'(h) = o\left(\frac{1}{h^{\beta+1}}\right), \quad \text{as } h \to 0, \tag{1.2}$$

and h is subject to initial data $h_0 = h(\cdot, 0) > 0$ and the Dirichlet boundary condition $h \equiv 1$ on $\partial\Omega$. Finite time *quenching* or *extinction* of a solution h to the Cauchy

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problem for (1.1) at $x_0 \in \Omega$, means that for some $T \in \mathbb{R}_+$, h has limit value 0 at (x_0, T). We remark that the perturbative assumption (1.2) is typically satisfied by

$$\widetilde{F} \equiv 0, \quad \widetilde{F}(h) = e^{-h}, \quad \text{or} \quad \widetilde{F}(h) = \frac{1}{h^{\beta'}} \ \text{ for some } \beta' < \beta.$$

Moreover, since our main concern in this paper is in the quenching scenario $h \to 0$, apart from the smoothness $F \in C^\infty(\mathbb{R}_+)$ there is no need (and no gain) to prescribe the behaviour of $F(h)$ for $h \geq 1$. Equation (1.1) can also be considered on the whole space $\mathbb{R}^N$, if F further satisfies the decay assumption

$$|F(h)| + |F'(h)| \leq Ce^{-h} \quad \text{as} \quad h \to +\infty, \tag{1.3}$$

and if the boundary condition is replaced by a growth condition, say,

$$h(x,t) \sim a_1 |x| \quad \text{as} \quad |x| \to +\infty \tag{1.4}$$

for some $a_1 > 0$. We remark that (1.3)-(1.4) are again not within our main concern.

Now, by introducing the following family of transformations

$$u(x,t) = u(\alpha, x, t) = \frac{\alpha^{\frac{\alpha}{\beta+1}}}{h(x,t)^\alpha} \quad (\alpha > 0) \tag{1.5}$$

for (1.1) we are then led to study the blowup problem of the following nonlinear heat equation with power-type nonlinearity and a particular perturbation term

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u - a\frac{|\nabla u|^2}{u} + f(u), \\ &f(u) = \alpha^{\frac{\beta}{\beta+1}} u^{1+\frac{1}{\alpha}} F(\alpha^{\frac{1}{\beta+1}} u^{-\frac{1}{\alpha}}) = u^p + \widetilde{f}(u), \end{aligned} \tag{1.6}$$

where $u > 0$, $u \equiv 1$ on $\partial\Omega$, (a, p) is computed from (α, β) by

$$1 < a = 1 + \frac{1}{\alpha} < p = \frac{1 + \alpha + \beta}{\alpha}, \tag{1.7}$$

and $\widetilde{f} \in C^\infty(\mathbb{R}_+)$ satisfies

$$\widetilde{f}(u) = o(u^p) \quad \text{and} \quad \widetilde{f}'(u) = o(u^{p-1}) \quad \text{as} \quad u \to +\infty. \tag{1.8}$$

If $\Omega = \mathbb{R}^N$ we further assume that f satisfies

$$|f(u)| + |f'(u)| \leq Cu^{1+\frac{1}{\alpha}} \exp(-\alpha^{\frac{1}{\beta+1}} u^{-\frac{1}{\alpha}}) \quad \text{as} \quad u \to 0, \tag{1.9}$$

and the solution u is subject to the decay condition

$$u(x,t) \sim \frac{\widetilde{a}_1}{|x|} \quad \text{as} \quad |x| \to +\infty \tag{1.10}$$

for some $\widetilde{a}_1 > 0$. Conversely, given $1 < a < p < \infty$ for (1.6), we recover by $a = 1 + \frac{1}{\alpha}$ and $p = \frac{1+\alpha+\beta}{\alpha}$ the exponent β for (1.1) and the transformation index α in (1.5). If we use, instead of β, the p and a notations to denote objects for h, we mean $\alpha = 1$.

In the non-perturbed case of (1.6) ($a = \widetilde{f} = 0$), namely, for the equation

$$\frac{\partial u}{\partial t} = \Delta u + |u|^{p-1} u, \tag{1.11}$$

where $u(\cdot, t) : \mathbb{R}^N \to \mathbb{R}$, $p > 1$ and $p < \frac{N+2}{N-2}$ if $N \geq 3$, Fujita [Fuj66], Levine [Lev73], Ball [Bal77] and Kavian [Kav87] obtained obstructions to global existence in time, using monotonicity properties and the maximum principle. Another method has been followed by Merle and Zaag [MZ97b] (see also the earlier papers by Giga and Kohn [GK85, GK87, GK89], Berger and Kohn [BK88], Herrero and Velázquez

[HV92a, HV92b, HV93], Bricmont and Kupiainen [BK94] and Zaag [Zaa98]). Once an asymptotic profile (that is a function that $u(t)$ approaches to, after a time-dependent rescaling, as $t \to T < +\infty$) is derived formally, the existence of a solution $u(t)$ which blows up at time T with the suggested profile is then proved rigorously, using analysis of (1.11) near the given profile and reduction of the existence problem to a finite-dimensional one. More precisely, in [MZ97b] the following asymptotic behaviour

$$\left\| (T-t)^{\frac{1}{p-1}} u(\cdot,t) - \Phi_0\left(\frac{\cdot - x_0}{\sqrt{(T-t)|\log(T-t)|}}\right)\right\|_{L^\infty(\Omega)} \longrightarrow 0 \quad \text{as} \quad t \to T, \tag{1.12}$$

is justified, where x_0 is the only blowup point, T is the blowup time, and

$$\Phi_0(z) = \frac{1}{\left(p-1+\frac{(p-1)^2}{4p}|z|^2\right)^{\frac{1}{p-1}}}. \tag{1.13}$$

Let $\kappa = \Phi_0(0) = (p-1)^{-\frac{1}{p-1}}$. The so-called final blowup profile for (1.11)

$$\lim_{t\to T} u(x,t) \quad \sim \quad \left[\frac{(p-1)^2}{8p}\frac{|x-x_0|^2}{|\log|x-x_0||}\right]^{-\frac{1}{p-1}} \quad \text{as} \quad x \to x_0,$$

was later derived by Zaag in [Zaa98].

Note that the constructive method in [MZ97b] is robust and in fact has been extended by Zaag and collaborators to a considerably large class of parabolic problems such as: Merle and Zaag [MZ97a] for the quenching problems (see also Duong and Zaag [DZ19] for MEMS devices and Duong, Kavallaris and Zaag [DKZ21] for Gierer-Meinhardt system), Masmoudi and Zaag [MZ08], Nouaili and Zaag [NZ18], Duong, Nouaili and Zaag [DNZ19b, DNZ22] for the complex Ginzburg-Landau equation, Ebde and Zaag [EZ11], Nguyen and Zaag [NZ16], Tayachi and Zaag [TZ19], Duong, Nguyen and Zaag [DNZ19a], Ghoul, Nguyen and Zaag [GNZ17] and Abdelhedi and Zaag [AZ21a, AZ21b] for perturbed nonlinear source terms (with or without gradient perturbation), Nouaili and Zaag [NZ15] for a non-variational complex valued heat equation, and Ghoul, Nguyen and Zaag [GNZ18b, GNZ18a] for semilinear parabolic systems where the nonlinearities have no gradient structure.

In a broader context, this method has also proved to be efficient for different PDEs of different type, and no list can be exhaustive (see Merle, Raphaël and Szeftel [MRS20] for anisotropic type I blowup for heat equation, del Pino, Musso and Wei [DPMW20], Harada [Har20], Collot, Merle and Raphaël [CMR20] for type II blowup for heat equation, Raphaël and Schweyer [RS14], Collot, Ghoul, Masmoudi and Nguyen [CGMN22] for Keller-Segel system, Krieger, Schlag and Tataru [KST09], Côte and Zaag [CZ13], Donninger and Schörkhuber [DS14] for wave equation, Martel [Mar05] for generalized Korteweg-de Vries equation, Elgindi, Ghoul and Masmoudi [EGM21] for incompressible Euler, Buckmaster, Shkoller and Vicol [BSV22] for 2D isentropic compressible Euler, Merle, Raphaël, Rodnianski and Szeftel [MRRS22a, MRRS22b] for Schrödinger equation and 3D compressible fluids, etc.).

Back to Equation (1.1) (with $0 < h < 1$ and $F(h) = \frac{1}{h^\beta}$), few results are known except the case on $(-1,1) \subset \mathbb{R}$ with Dirichlet boundary conditions, where some criteria of quenching are obtained and, upon assuming that the initial data h_0 satisfies

$$\frac{\partial^2 h_0}{\partial x^2} - \frac{1}{h_0^\beta} \leq 0,$$

the quenching rates and quenching profiles are obtained, see Levine [Lev89], Guo [Guo91] (also Fila, Hulshof and Quittner [FHQ92]), Filippas and Guo [FG93] and the references therein. By adapting the techniques of [BK94] and [MZ97b], Merle and Zaag [MZ97a] succeeded in constructing a stable quenching solution.

Recently, Duong, Ghoul and Zaag [DGZ21] refined the constructive techniques of [MZ97b] and obtained the gradient profile of blowup solutions. Our aim here is to determine the gradient profile of the quenching solution constructed in [MZ97a]. To recall the known results and also state ours, let us introduce some definitions.

Definition 1.1 (Intermediate profile)**.** Introduce the intermediate profile

$$\widehat{\Phi}(z) = \left(\beta + 1 + \frac{(\beta+1)^2}{4\beta}|z|^2\right)^{\frac{1}{\beta+1}}, \quad z \in \mathbb{R}^N, \tag{1.14}$$

with the intermediate gradient profile being

$$\nabla\widehat{\Phi}(z) = \frac{\beta+1}{2\beta} z \left(\beta + 1 + \frac{(\beta+1)^2}{4\beta}|z|^2\right)^{-\frac{\beta}{\beta+1}}, \quad z \in \mathbb{R}^N. \tag{1.15}$$

Here z will be a time-dependent rescaling of x as in (1.12)-(1.13) above.

Now we prescribe the final extinction profile (near extinction point) that is expected in our construction. For later purpose we also prescribe its global behaviour.

Definition 1.2 (Final profile)**.** Let $x_0 \in \Omega \subset \mathbb{R}^N$. Define $H^*_{x_0} \in C^\infty(\Omega \backslash \{x_0\})$ by

(i) If $\Omega = \mathbb{R}^N$, then

$$H^*_{x_0}(x) = H^*(x - x_0), \tag{1.16}$$

where $H^* \in C^\infty(\mathbb{R}^N \backslash \{0\})$ is defined as follows:

$$H^*(x) = \left[\frac{(\beta+1)^2}{8\beta} \frac{|x|^2}{|\log|x||}\right]^{\frac{1}{\beta+1}}, \quad 0 < |x| \leq \min\left\{C(a_1, \beta), \frac{1}{2}\right\}, \tag{1.17}$$

$$H^*(x) = a_1|x|, \quad |x| \geq 1, \tag{1.18}$$

with a_1 given in (1.4) and with the additional requirements that

$$\forall\ x \neq 0, \quad H^*(x) > 0 \quad \text{and} \quad |\nabla H^*(x)| > 0.$$

(ii) If Ω is bounded, then with $\rho_0 = \operatorname{dist}(x_0, \partial\Omega)$,

$$H^*_{x_0}(x) = \left[\frac{(\beta+1)^2}{8\beta} \frac{|x - x_0|^2}{|\log|x - x_0||}\right]^{\frac{1}{\beta+1}}, \quad 0 < |x - x_0| \leq \min\left\{C(\beta), \frac{\rho_0}{4}\right\}, \tag{1.19}$$

$$H^*_{x_0}(x) = 1, \quad |x - x_0| \geq \frac{\rho_0}{2},$$

with the additional requirements that

$$\forall\ x \neq x_0, \quad H^*_{x_0}(x) > 0 \quad \text{and} \quad |\nabla H^*_{x_0}(x)| > 0.$$

Remark 1.3. Since our analysis will be local, we shall not precise

$$H^*(x) \quad \text{when} \quad \min\left\{C(a_1,\beta), \frac{1}{2}\right\} \le |x| \le 1;$$

$$H^*_{x_0}(x) \quad \text{when} \quad \min\left\{C(\beta), \frac{\rho_0}{4}\right\} \le |x-x_0| \le \frac{\rho_0}{2}.$$

Similarly, (1.18) is imposed only in accordance with (1.4), itself introduced in accordance with the vortex line model considered in [CHO98]. In view of this, it is more reasonable to assume $H^*(x) = a_\beta |x|^{\frac{2}{\beta+1}}$ for $|x| \ge 1$ instead of (1.18). These different growth functions will not make any difference in the *local Cauchy theory*. Thus we relieve ourselves from further precision on the profile beyond (1.17) and (1.19). For the same reason and when Ω is bounded, other types of boundary behaviours instead of $H^*_{x_0}(x) = 1$ for $|x-x_0| \ge \frac{\rho_0}{2}$ can be considered.

Remark 1.4. For $\beta = 1$, the (vortex line) profile $H^*(x)$ in (1.17)-(1.18) is nearly straight (up to a logarithm $\log|x|$). For $\beta > 1$, the profile forms a cusp at $x = 0$.

Remark 1.5 (Gradient profile)**.** The gradient of H^* via (1.17) satisfies that as $x \to 0$,

$$\nabla H^*(x) \quad \sim \quad \frac{1}{\sqrt{2\beta}} \frac{x}{|x|} \frac{1}{\sqrt{|\log|x||}} \left[\frac{(\beta+1)^2}{8\beta} \frac{|x|^2}{|\log|x||}\right]^{\frac{1}{\beta+1}-\frac{1}{2}}. \tag{1.20}$$

When $|x| \to 0$, $|\nabla H^*(x)|$ blows up if $\beta > 1$ and extinguishes if $0 < \beta \le 1$.

As the containing space of initial data h_0 and the evolution time-slices $h(\cdot,t)$ $(0 \le t < T)$, h being a solution to (1.1), we shall need the following notion.

Definition 1.6 (Trajectory space for quenching problem)**.** Define $\mathbf{H}$ as follows:

(i) If $\Omega = \mathbb{R}^N$, then for some $x_0 \in \mathbb{R}^N$, we set

$$\mathbf{H} = \mathbf{H}_{\psi_{x_0}} = \left\{\mathbf{h} : (\mathbf{h} - \psi_{x_0}) \in W^{1,\infty}(\mathbb{R}^N),\ \frac{1}{\mathbf{h}} \in L^\infty(\mathbb{R}^N)\right\}, \tag{1.21}$$

where $\psi_{x_0} \in C^\infty(\mathbb{R}^N)$ satisfies

$$\psi_{x_0} \equiv 0, \text{ if } |x-x_0| \le 1;$$

$$\psi_{x_0}(x) = a_1|x-x_0|, \text{ if } |x-x_0| \ge 2.$$

Here, a_1 is the constant given in (1.4).

(ii) If Ω is a bounded domain in $\mathbb{R}^N$, then we set

$$\mathbf{H} = \left\{\mathbf{h} \in W^{1,\infty}(\Omega) : \frac{1}{\mathbf{h}} \in L^\infty(\Omega)\right\}. \tag{1.22}$$

When no ambiguity arises in the context, we shall no longer precise the reference point x_0 in the definition of $\mathbf{H}$. The following was proved in [MZ97a].

Theorem A (Existence and stability of vortex reconnection with the boundary or finite time quenching obeying prescribed behaviours given in Definitions 1.1 and 1.2, Merle and Zaag)**.** *Assume that either Ω is bounded, or $\Omega = \mathbb{R}^N$ and* (1.3)-(1.4) *hold.*

(1) (Existence) For all $x_0 \in \Omega$*, there exists a positive* $h_0 \in \mathbf{H}$ *such that for a* $T > 0$*, equation* (1.1) *with initial data* h_0 *has a unique solution* $h(\cdot, t)$ *on* $[0, T)$ *satisfying*

$$\lim_{t \to T} h(x_0, t) = 0.$$

Furthermore,
(i) (Intermediate extinction profile)

$$\lim_{t \to T} \left\| \frac{(T-t)^{\frac{1}{\beta+1}}}{h(\cdot, t)} - \frac{1}{\widehat{\Phi}(z_{x_0}(\cdot, t))} \right\|_{L^\infty(\Omega)} = 0, \tag{1.23}$$

where $\widehat{\Phi}$ *is given in* (1.14) *and*

$$z_{x_0}(x, t) = \frac{x - x_0}{\sqrt{(T-t)|\log(T-t)|}}.$$

(ii) (Final extinction profile) $h^*(x) := \lim_{t \to T} h(x, t)$ *exists for all* $x \in \Omega$ *and*

$$h^*(x) \sim H^*_{x_0}(x) \quad \textit{as} \quad x \to x_0,$$

where $H^*_{x_0}$ *is given precisely in* (1.16), (1.17) *and* (1.19).

(2) (Stability). For every $\varepsilon > 0$*, there exists a neighbourhood* $\mathcal{V}_0$ *of* h_0 *in* $\mathbf{H}$ *with the following property:*

For each $\widetilde{h}_0 \in \mathcal{V}_0$*, there exists* $\widetilde{x}_0 \in \Omega$ *and* $\widetilde{T} > 0$ *satisfying*

$$|T - \widetilde{T}| + |x_0 - \widetilde{x}_0| \le \varepsilon$$

such that (1.1) *with initial data* $\widetilde{h}_0$ *has a unique solution* $\widetilde{h}(\cdot, t)$ *on* $[0, \widetilde{T})$ *satisfying*

$$\lim_{t \to \widetilde{T}} \widetilde{h}(\widetilde{x}_0, t) = 0.$$

Furthermore,
(i) (Intermediate extinction profile)

$$\lim_{t \to \widetilde{T}} \left\| \frac{(\widetilde{T}-t)^{\frac{1}{\beta+1}}}{\widetilde{h}(\cdot, t)} - \frac{1}{\widehat{\Phi}(\widetilde{z}_{x_0}(\cdot, t))} \right\|_{L^\infty(\Omega)} = 0,$$

where $\widehat{\Phi}$ *is given in* (1.14) *and*

$$\widetilde{z}_{x_0}(x, t) = \frac{x - \widetilde{x}_0}{\sqrt{(\widetilde{T}-t)|\log(\widetilde{T}-t)|}}.$$

(ii) (Final extinction profile) $\widetilde{h}^*(x) := \lim_{t \to \widetilde{T}} \widetilde{h}(x, t)$ *exists for all* $x \in \Omega$ *and*

$$\widetilde{h}^*(x) \sim H^*_{\widetilde{x}_0}(x) \quad \textit{as} \quad x \to \widetilde{x}_0,$$

where $H^*_{\widetilde{x}_0}$ *is given precisely in* (1.16), (1.17) *and* (1.19).

2. Statement of the main result

It turns out that the final profile is stable under differentiation:

$$(\nabla h)^*(x) \sim \nabla h^*(x)$$

$$\left(\lim_{t\to T} \nabla h(x,t) \sim \nabla \lim_{t\to T} h(x,t)\right)$$

as $x \to x_0$. More precisely, we can prove the following result.

Theorem 2.1 (Construction of finite time extinction solutions with refined asymptotics and determination of the final extinction profile and the final gradient profile)**.** *Assume that either Ω is bounded, or $\Omega = \mathbb{R}^N$ and (1.3)-(1.4) hold.*

For all $x_0 \in \Omega$, there exists a positive $h_0 \in \mathbf{H}$ such that for a $T = T(h_0) \in (0, e^{-1})$, equation (1.1) with initial data h_0 has a unique solution $h(\cdot,t)$ on $[0,T)$ satisfying

$$\lim_{t\to T} h(x_0,t) = 0.$$

Furthermore, for this h and for all $t \in [0,T)$, we have:

(i) (Intermediate extinction profile and refined asymptotics)

$$\left\| \frac{(T-t)^{\frac{1}{\beta+1}}}{h(\cdot,t)} - \frac{1}{\widehat{\Phi}(z_{x_0}(\cdot,t))} \right\|_{L^\infty(\Omega)} \le C \frac{\log(|\log(T-t)|)}{|\log(T-t)|} \tag{2.1}$$

and for each $K > 0$ and $\Omega_{t,K} = \left\{x \in \Omega : |x - x_0| \le K\sqrt{(T-t)|\log(T-t)|}\right\}$,

$$\left\| (T-t)^{-\frac{1}{\beta+1}+\frac{1}{2}} \nabla h(\cdot,t) - \frac{(\nabla\widehat{\Phi})(z_{x_0}(\cdot,t))}{|\log(T-t)|^{\frac{1}{2}}} \right\|_{L^\infty(\Omega_{t,K})} \le C(K) \frac{\log(|\log(T-t)|)}{|\log(T-t)|}. \tag{2.2}$$

Here, $\widehat{\Phi}$ is given in (1.14) and $z_{x_0}(x,t)$ is given in Theorem A.

(ii) (Final extinction profile and final gradient profile)

(ii-1) $h^*(x) := \lim_{t\to T} h(x,t)$ exists for all $x \in \Omega$, and

$$h^*(x) \sim H^*_{x_0}(x) \quad \textit{as} \quad x \to x_0, \tag{2.3}$$

where $H^*_{x_0}$ is given precisely in (1.16), (1.17) and (1.19).

(ii-2) $(\nabla h)^*(x) := \lim_{t\to T} \nabla h(x,t)$ exists for all $x \in \Omega\backslash\{x_0\}$, and

$$(\nabla h)^*(x) \sim \nabla H^*_{x_0}(x) \quad \textit{as} \quad x \to x_0. \tag{2.4}$$

See [HZ24] for the full proof.

Remark 2.2. The main contribution of the theorem is to discover precise error estimates (2.1)-(2.2) and their consequence (2.4). Note that (1.23) suffices for the derivation of (2.3), which is same as in Theorem A. The error estimate in (2.1) refines $|\log(T-t)|^{-\frac{1}{2}}$ which is implicit in (1.23) (see [MZ97a]). As for (2.2), the error term one obtains from the estimates of [MZ97a] would be $C|\log(T-t)|^{-\frac{1}{2}}$, which is of the same size as $\frac{\nabla\widehat{\Phi}}{\sqrt{|\log(T-t)|}}$, preventing us from obtaining an equivalent for ∇h. Thus, the improved error estimate we get in (2.2) thanks to our strategy is crucial.

Remark 2.3. We obtained indeed an estimate stronger than (2.2): $\forall\, t \in [0, T)$,

$$\begin{aligned}\left\| \frac{(T-t)^{\frac{1}{\beta+1}+\frac{1}{2}} \nabla h(\cdot,t)}{h^2(\cdot,t)} - \frac{(\nabla \widehat{\Phi})(z_{x_0}(\cdot,t))}{|\log(T-t)|^{\frac{1}{2}} \widehat{\Phi}^2(z_{x_0}(\cdot,t))} \right\|_{L^\infty(\Omega)} \\ \leq C \frac{\log(|\log(T-t)|)}{|\log(T-t)|}.\end{aligned} \tag{2.5}$$

Remark 2.4 (Stability of the gradient profile)**.** We use the method which was first introduced by Bressan in [Bre90, Bre92] for the heat equation with an exponential source, then for (1.11) in Bricmont and Kupiainen [BK94] and Merle and Zaag [MZ97b] (also Fermanian, Merle and Zaag [FKMZ00]). This consists in a formal approach where the profile is obtained through an inner/outer expansion with matched asymptotics, followed by a rigorous proof where the PDE is linearized around the profile candidate. Then, the negative part of the spectrum (which is infinite dimensional) is controlled thanks to the decaying properties of the Laplacian, whereas the nonnegative part (which is finite dimensional) is controlled thanks to the degree theory. Via this finite dimensional reduction and the interpretation of the parameters of the finite-dimensional problem in terms of the blowup time and the blowup point, the refined estimates (2.1)-(2.2) and the final profiles (2.3)-(2.4) are stable under perturbations (in the same form) of initial data.

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Roughness-induced effects on a reaction-diffusion equation defined in a thin domain

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Key words. convection-diffusion-reaction equation; Robin boundary condition; thin domain; rough boundary; unfolding method.

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Remark. This note is based on the references [13] and [14].

1 Introduction

Reaction-diffusion problems and flows posed in thin domains (domains whose longitudinal dimension is much larger than the transverse one) are of great interest due to their practical importance. In real-life applications, the boundary of such domains are usually not perfectly smooth, i.e. they usually have some small rugosities, dents, etc. In solid mechanics, the typical examples of such structures would be thin rods, plates or shells. Lubrication devices and blood circulatory system are the obvious examples associated to fluid mechanics (see for instance [6, 12, 17]). No matter the context is, introducing the small parameter as the perturbation quantity in the domain boundary makes the analysis very challenging from the mathematical point of view.

Here, we discuss the main ideas of our approach, in a bidimensional reaction-diffusion problem with a convective term and Robin-type boundary condition. We suppose the upper boundary of our thin domain has an oscillating behavior. We set a purely periodic thin domain modeling roughness and thickness of the same order. Namely, the considered domain reads:

$$R^\varepsilon = \left\{(x,y)\in\mathbb{R}^2 : 0<x<1, 0<y<\varepsilon h\left(\frac{x}{\varepsilon}\right)\right\}, \quad 0<\varepsilon\ll 1.$$

We address the following elliptic boundary-value problem:

$$\begin{cases} -\kappa\Delta u^\varepsilon + Q^\varepsilon(y)\partial_x u^\varepsilon + cu^\varepsilon = f^\varepsilon \text{ in } R^\varepsilon, \\ \kappa\dfrac{\partial u^\varepsilon}{\partial\nu^\varepsilon} = \varepsilon^\alpha(g(x)-u^\varepsilon) \text{ on } \Gamma^\varepsilon = \left\{(x,y)\in\mathbb{R}^2 : 0<x<1, y=\varepsilon h\left(\dfrac{x}{\varepsilon}\right)\right\}, \\ \dfrac{\partial u^\varepsilon}{\partial\nu^\varepsilon} = 0 \text{ on } \partial R^\varepsilon\backslash\Gamma^\varepsilon. \end{cases} \tag{1}$$

Here $\kappa, c = \text{const.} > 0$, the vector $\nu^\epsilon = (\nu_1^\epsilon, \nu_2^\epsilon)$ is the unit outward normal to ∂R^ϵ and $\frac{\partial}{\partial\nu^\epsilon}$ is the outside normal derivative. For the function Q^ε, we assume

$$Q^\varepsilon(y) = Q\left(\frac{y}{\varepsilon}\right),$$

where $Q\in L^\infty(0,h_1)$ is a non-negative function, $h_1 = \max_{x\in\mathbb{R}} h(x)$. This assumption is reasonable from the point of view of the applications, since we are tackling the process in a thin domain and Q^ε can be interpreted as the entering (unidirectional) velocity in e.g. the solute transport problem (see [11, 12]). The boundary perturbation function h satisfies the usual assumptions

($\mathbf{H_h}$) *$h:\mathbb{R}\to\mathbb{R}$ is a strictly positive, Lipschitz and L-periodic with $h'\in L^\infty(\mathbb{R})$. Also, if $h_0 = \min_{x\in\mathbb{R}} h(x)$ and $h_1 = \max_{x\in\mathbb{R}} h(x)$, we have $0<h_0\le h(x)\le h_1$ for all $x\in\mathbb{R}$.*

Finally, we suppose $g\in L^2(0,1)$. As you can see, the governing equation is endowed with the Robin-type boundary condition which models the reaction catalyzed by the upper wall. By taking the reaction coefficient in the form ε^α, $\alpha\in\mathbb{R}$ (see $(1)_2$), our aim is to address different order of magnitudes of the prescribed reaction mechanism. Such type of elliptic boundary-value problems describes many processes naturally arising in chemical engineering, in particular related to microfluidic applications (see e.g. [17]). Our goal is to study the asymptotic behavior of the described problem, as $\varepsilon\to 0$.

We employ the homogenization technique based on the unfolding method proposed in [8, 9] (see also [10]). Due to its ability to elegantly treat the surface integrals, the unfolding method has been extensively used for derivation of lower-dimensional approximations in the last period. We refer the reader e.g. to [1, 3, 6, 16]. In this work, we adapt the variant of this method introduced by Arrieta and Villanueva-Pesqueira [4, 5] for thin domains. As a result, we obtain three different asymptotic models, depending on the value of the coefficient α. More precisely, for $\alpha<1$, the process turns out to be dominated by the function g from the Robin boundary condition, with $g\in H^1(0,1)$ (see Theorem 3.2). For $\alpha>1$, the effective model does not depend on g (see Theorem

3.3), meaning that the reaction mechanism does not affect the process. Between those two cases, we identify the critical (and the most interesting) case $\alpha = 1$ capturing the effects of the domain's geometry and all the physical processes relevant to the problem as well (see Theorem 3.1). We firmly believe that the results presented here could prove useful in numerical simulations of the convection-diffusion-reaction problems in thin domains with irregularities.

Finally, we provide more bibliographic remarks on the subject. In [7], the Neumann problem for the Laplace equation posed in a domain (of thickness $\mathcal{O}(1)$) with highly oscillating boundary has been considered via asymptotic expansion method. Using rigorous analysis in appropriate functional setting, a thin-domain situation has been addressed in [2]. It should be emphasized that, in both papers, a homogeneous Neumann boundary condition has been imposed and that the transition to a Robin-type boundary condition cannot be considered straightforward whatsoever. The present work can be viewed as the continuation of our recent work [15] in which a thin domain without boundary oscillations has been studied. Notice that introducing boundary irregularities to the problem forced us to completely change the approach.

2 Basic facts

First we notice that the variational formulation of (1) reads:

$$\begin{aligned}\int_{R^\varepsilon} \kappa \nabla u^\varepsilon \nabla \varphi + Q^\varepsilon(y)\partial_x u^\varepsilon \varphi + c u^\varepsilon \varphi dxdy + \varepsilon^\alpha \int_{\Gamma^\varepsilon} u^\varepsilon \varphi dS \\ = \int_{R^\varepsilon} f^\varepsilon \varphi dxdy + \varepsilon^\alpha \int_{\Gamma^\varepsilon} g\varphi dS\,, \qquad \forall \varphi \in H^1(R^\varepsilon).\end{aligned} \tag{2}$$

Remark 2.1. *The existence and uniqueness in $H^1(R^\varepsilon)$ follows as in [15, Lemma 3.4] under condition $c > ||Q||^2_{L^\infty(0,h_1)}/4\kappa$. In fact, it can be proved that the bilinear form $a_\varepsilon : H^1(R^\varepsilon) \times H^1(R^\varepsilon) \mapsto \mathbb{R}$ set by*

$$a_\varepsilon(u,v) = \int_{R^\varepsilon} \kappa \nabla u \nabla v + Q^\varepsilon(y)\partial_x uv + cuvdxdy$$

is continuous and uniformly coercive.

Thus, we have a family of solutions $\{u^\varepsilon\}_{\varepsilon>0}$ given by (2) and we are concerned here about the asymptotic behavior at $\varepsilon = 0$.

The unfolding operator

In order to study the convergence of the solutions u^ε, we apply the unfolding method firstly introduced in [8, 9] (see also [10]) for oscillating coefficients and perforated domains. Here, we just give some notations and recall the main results concerning this method in the thin domain situation. The proofs and all the details can be found in [4, 5].

Throughout this note, we use the following notations. We call Y^* the representative cell of the thin domain R^ε which is given by

$$Y^* = \{(y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 < L \text{ and } 0 < y_2 < h(y_1)\}.$$

The average of $\varphi \in L^1_{loc}(\mathbb{R}^2)$ on a measure set $\mathcal{O} \subset \mathbb{R}^2$ is denoted by $\langle \varphi \rangle_{\mathcal{O}} := \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} \varphi(x)\,dx$, where $|\mathcal{O}|$ sets the Lebesgue measure of any measure set $\mathcal{O}$. We will also need the following functional spaces which are defined by periodic functions in the variable $y_1 \in (0, L)$

$$\begin{aligned} L^2_\#(Y^*) &= \{\varphi \in L^2(Y^*) \,:\, \varphi(y_1,y_2) \text{ is } L\text{-periodic in } y_1\,\}, \\ L^2_\#((0,1)\times Y^*) &= \{\varphi \in L^2((0,1)\times Y^*) \,:\, \varphi(x,y_1,y_2) \text{ is } L\text{-periodic in } y_1\,\}, \\ H^1_\#(Y^*) &= \{\varphi \in H^1(Y^*) \,:\, \varphi|_{\partial_{left}Y^*} = \varphi|_{\partial_{right}Y^*}\}.\end{aligned}$$

If we denote by $[a]_L$ the unique integer number such that $a = [a]_L L + \{a\}_L$ where $\{a\}_L \in [0, L)$, then for each $\varepsilon > 0$ and any $x \in \mathbb{R}$, we have

$$x = \varepsilon \left[\frac{x}{\varepsilon}\right]_L L + \varepsilon \left\{\frac{x}{\varepsilon}\right\}_L \quad \text{where } \left\{\frac{x}{\varepsilon}\right\}_L \in [0, L).$$

Let us also denote

$$I_\varepsilon = \text{ Int } \left(\bigcup_{k=0}^{N_\varepsilon} [kL\varepsilon, (k+1)L\varepsilon] \right),$$

where N_ε is the largest integer such that $\varepsilon L(N_\varepsilon + 1) \leq 1$. We also set

$$\Lambda_\varepsilon = (0,1)\backslash I_\varepsilon = [\varepsilon L(N_\varepsilon + 1), 1), \quad R_0^\varepsilon = \left\{(x,y) \in \mathbb{R}^2 : x \in I_\varepsilon, 0 < y < \varepsilon h\left(\frac{x}{\varepsilon}\right)\right\}$$
$$\text{and} \quad R_1^\varepsilon = \left\{(x,y) \in \mathbb{R}^2 : x \in \Lambda_\varepsilon, 0 < y < \varepsilon h\left(\frac{x}{\varepsilon}\right)\right\}.$$

Notice that we have $\Lambda_\varepsilon = \emptyset$ if $\varepsilon L(N_\varepsilon + 1) = 1$. In this case $R_0^\varepsilon = R^\varepsilon$ and $R_1^\varepsilon = \emptyset$.

Definition 2.1. *Let φ be a Lebesgue-measurable function in R^ε. The unfolding operator $\mathcal{T}_\varepsilon$ acting on φ is defined as the following function in $(0,1) \times Y^*$:*

$$\mathcal{T}_\varepsilon \varphi(x, y_1, y_2) = \begin{cases} \varphi\left(\varepsilon \left[\frac{x}{\varepsilon}\right]_L L + \varepsilon y_1, \varepsilon y_2\right) & a.e.\ (x, y_1, y_2) \in I_\varepsilon \times Y^*, \\ 0 \ \ a.e.\ (x, y_1, y_2) \in \Lambda_\varepsilon \times Y^*. \end{cases}$$

Proposition 2.2. *The unfolding operator satisfies the following properties:*

1. *$\mathcal{T}_\varepsilon$ is linear and satisfies $\mathcal{T}_\varepsilon(\varphi\psi) = \mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\psi)$;*
2. *Let φ a Lebesgue function in Y^* extended periodically in the first variable. Then, $\varphi^\varepsilon(x,y) = \varphi\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$ is mesurable in R^ε and $\mathcal{T}_\varepsilon(\varphi^\varepsilon)(x, y_1, y_2) = \varphi(y_1, y_2)$. Moreover, if $\varphi \in L^2(Y^*)$, then $\varphi^\varepsilon \in L^2(R^\varepsilon)$;*
3. *For all $\varphi^\varepsilon \in L^1(R^\varepsilon)$, we have*
$$\frac{1}{L} \int_{(0,1)\times Y^*} \mathcal{T}_\varepsilon(\varphi)(x, y_1, y_2) dx dy_1 dy_2 = \frac{1}{\varepsilon} \int_{R^\varepsilon} \varphi(x,y) dx dy - \frac{1}{\varepsilon} \int_{R_1^\varepsilon} \varphi(x,y) dx dy;$$
4. *$T_\varepsilon(\varphi) \in L^2((0,1) \times Y^*)$ for all $\varphi \in L^2(R^\varepsilon)$ with*
$$||\mathcal{T}_\varepsilon(\varphi)||_{L^2((0,1)\times Y^*)} = \left(\frac{L}{\varepsilon}\right)^{\frac{1}{2}} ||\varphi||_{L^2(R_0^\varepsilon)} \leq \left(\frac{L}{\varepsilon}\right)^{\frac{1}{2}} ||\varphi||_{L^2(R^\varepsilon)}.$$
5. *$\partial_{y_1} T_\varepsilon(\varphi) = \varepsilon T_\varepsilon(\partial_x \varphi)$ and $\partial_{y_2} T_\varepsilon(\varphi) = \varepsilon T_\varepsilon(\partial_y \varphi)$ a.e. $(0,1) \times Y^*$ for all $H^1(R^\varepsilon)$;*
6. *If $\varphi \in H^1(R^\varepsilon)$, then $\mathcal{T}_\varepsilon(\varphi) \in L^2\left((0,1); H^1(Y^*)\right)$ with*
$$||\partial_{y_1} \mathcal{T}_\varepsilon(\varphi)||_{L^2((0,1)\times Y^*)} = \varepsilon \left(\frac{L}{\varepsilon}\right)^{\frac{1}{2}} ||\partial_x \varphi||_{L^2(R_0^\varepsilon)} \leq \varepsilon \left(\frac{L}{\varepsilon}\right)^{\frac{1}{2}} ||\partial_x \varphi||_{L^2(R^\varepsilon)},$$
$$||\partial_{y_2} \mathcal{T}_\varepsilon(\varphi)||_{L^2((0,1)\times Y^*)} = \varepsilon \left(\frac{L}{\varepsilon}\right)^{\frac{1}{2}} ||\partial_y \varphi||_{L^2(R_0^\varepsilon)} \leq \varepsilon \left(\frac{L}{\varepsilon}\right)^{\frac{1}{2}} ||\partial_y \varphi||_{L^2(R^\varepsilon)}.$$

Here, we use the following rescaled norms in the thin open sets

$$|||\varphi|||_{L^2(R^\varepsilon)} = \varepsilon^{-1/2} ||\varphi||_{L^2(R^\varepsilon)}, \forall \varphi \in L^2(R^\varepsilon),$$
$$|||\varphi|||_{H^1(R^\varepsilon)} = \varepsilon^{-1/2} ||\varphi||_{H^1(R^\varepsilon)} \forall \varphi \in H^1(R^\varepsilon).$$

Definition 2.3. *A sequence (φ_ε) satisfies the unfolding criterion for integrals (u.c.i) if*

$$\frac{1}{\varepsilon} \int_{R_1^\varepsilon} |\varphi_\varepsilon| dx dy \to 0.$$

It is known that any sequence $(\varphi_\varepsilon) \subset L^2(R^\varepsilon)$ with norm $|||\cdot|||_{L^2(R^\varepsilon)}$ uniformly bounded satisfies the (u.c.i). Moreover, if we have (ψ_ε) set as $\psi_\varepsilon(x,y) = \psi\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$ for any $\psi \in L^2(Y^*)$, then $(\varphi_\varepsilon \psi_\varepsilon)$ also satisfies (u.c.i).

In the sequel, we recall some convergence properties of the unfolding operator.

Proposition 2.4. *1. Let $\varphi \in L^2(0,1)$. Then, $\mathcal{T}_\varepsilon \varphi \to \varphi$ strongly in $L^2\left((0,1)\times Y^*\right)$.*

2. Let (φ_ε) be a sequence in $L^2(0,1)$ such that $\varphi_\varepsilon \to \varphi$ strongly in $L^2(0,1)$. Then, $\mathcal{T}_\varepsilon \varphi_\varepsilon \to \varphi$ strongly in $L^2\left((0,1)\times Y^*\right)$.

Next, let us set the operator V_ε getting other convergence results:

$$V_\varepsilon(x) := \frac{1}{\varepsilon h_0}\int_0^{\varepsilon h_0} \varphi_\varepsilon(x,s)\, ds \qquad \text{a.e. } x \in (0,1). \tag{3}$$

Proposition 2.5. *Let $\varphi_\varepsilon \in H^1(R^\varepsilon)$ with $|||\varphi_\varepsilon|||_{H^1(R^\varepsilon)}$ uniformly bounded and $V_\varepsilon(x)$ defined as in (3). Then, there exists a function $\varphi \in H^1(0,1)$ such that, up to subsequences*

$$\begin{aligned}
&V_\varepsilon \rightharpoonup \varphi \text{ weakly in } H^1(0,1) \text{ and strongly in } L^2(0,1),\\
&\mathcal{T}_\varepsilon V_\varepsilon \to \varphi \text{ strongly in } L^2\left((0,1)\times Y^*\right),\\
&|||\varphi_\varepsilon - V_\varepsilon|||_{L^2(R^\varepsilon)} \to 0,\\
&|||\varphi_\varepsilon - \varphi|||_{L^2(R^\varepsilon)} \to 0,\\
&\mathcal{T}_\varepsilon \varphi_\varepsilon \to \varphi \text{ strongly in } L^2\left((0,1); H^1(Y^*)\right).
\end{aligned}$$

Finally, we recall a compactness result proved in [5, Theorem 3.1]. It allows us to pass to the limit in the gradient sequences.

Theorem 2.6. *Let $\varphi_\varepsilon \in H^1(R^\varepsilon)$ with $|||\varphi_\varepsilon|||_{H^1(R^\varepsilon)}$ uniformly bounded. Then, there exist $\varphi \in H^1(0,1)$ and $\varphi_1 \in L^2((0,1); H^1_\#(Y^*))$ such that (up to a subsequence)*

$$\begin{aligned}
&\mathcal{T}_\varepsilon \varphi_\varepsilon \to \varphi \text{ strongly in } L^2\left((0,1); H^1(Y^*)\right),\\
&\mathcal{T}_\varepsilon \partial_x \varphi_\varepsilon \rightharpoonup \partial_x \varphi + \partial_{y_1}\varphi_1 \text{ weakly in } L^2\left((0,1)\times Y^*\right),\\
&\mathcal{T}_\varepsilon \partial_y \varphi_\varepsilon \rightharpoonup \partial_{y_2}\varphi_1 \text{ weakly in } L^2\left((0,1)\times Y^*\right).
\end{aligned}$$

Boundary unfolding

Next we set the unfolding operator on the oscillating upper boundary. We have adapted the one set in [8, 9] yielding the appropriated results to our case. Notice that under assumptions $(\mathbf{H_h})$ we have that Γ^ε is Lipschitz.

Definition 2.7. *Let ϕ be a measurable function on Γ_ε. The boundary unfolding operator $\mathcal{T}^b_\varepsilon$ is defined by*

$$\mathcal{T}^b_\varepsilon \phi(x,y) = \begin{cases} \phi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]L + \varepsilon y\right) & a.e \quad I_\varepsilon \times \partial_u Y^*,\\ 0 \quad a.e \quad \Lambda_\varepsilon \times \partial_u Y^* \end{cases}$$

where $\partial_u Y^*$ is the upper boundary of the representative cell Y^* given by

$$\partial_u Y^* = \{(y_1,y_2) \in \mathbb{R}^2 \ : \ y_1 \in (0,L), \text{ and } y_2 = h(y_1)\}.$$

Proposition 2.8. *The boundary unfolding satisfies the following properties:*

1. *$\mathcal{T}^b_\varepsilon$ is linear;*
2. *$\mathcal{T}^b_\varepsilon(\varphi\psi) = \mathcal{T}^b_\varepsilon(\varphi)\mathcal{T}^b_\varepsilon(\psi)$, for all φ, ψ Lebesgue measurable in Γ^ε;*
3. *For any $\varphi \in L^1(R^\varepsilon)$,*

$$\frac{1}{L}\int_{(0,1)\times\partial_u Y^*} \mathcal{T}^b_\varepsilon \varphi(x,y)\, dx d\sigma(y) = \int_{\Gamma_0^\varepsilon} \varphi dS = \int_{\Gamma^\varepsilon} \varphi dS - \int_{\Gamma_1^\varepsilon} \varphi dS,$$

   where Γ_i^ε is the upper boundary of R_i^ε for $i = 0, 1$.

4. *Suppose that* $\varphi \in L^2(\Gamma^\varepsilon)$. *Then,*

$$||\mathcal{T}^b_\varepsilon \varphi||_{L^2((0,1)\times\partial_u Y^*)} \leq \frac{1}{L}||\varphi||_{L^2(\Gamma^\varepsilon)}.$$

5. *(Unfolding criterion for integrals) Suppose that* $\varphi_\varepsilon \in L^2(\Gamma_\varepsilon)$ *is such that* $||\varphi_\varepsilon||_{L^2(\Gamma^\varepsilon)} \leq c$, *with* c *independent on* ε. *Then,*

$$\int_{\Gamma^\varepsilon_1} |\varphi| dS \to 0.$$

6. *Let* $\psi_\varepsilon \in H^1(R^\varepsilon)$ *such that* $\mathcal{T}_\varepsilon\psi_\varepsilon \rightharpoonup \widehat{\psi}$ *in* $L^2((0,1); H^1(Y^*))$ *with* $\widehat{\psi} \in H^1(0,1)$. *Then,*

$$\mathcal{T}^b_\varepsilon \psi_\varepsilon \rightharpoonup \widehat{\psi} \quad in \quad L^2((0,1); H^{\frac{1}{2}}(\partial_u Y^*)).$$

Proof. See [13, Proposition 2.8]. □

3 Main results

As emphasized, the asymptotic behavior of the considered problem greatly depends on the value of the coefficient α appearing in the Robin boundary condition $(1)_2$. First we mention the theorem obtained for the critical case $\alpha = 1$. After that, we address two remaining characteristic cases $\alpha < 1$ and $\alpha > 1$, respectively.

Case $\alpha = 1$

Theorem 3.1. *Let* u^ε *be the solution of the problem* (1) *with* $\alpha = 1$, $f^\varepsilon \in L^2(R^\varepsilon)$ *and* $|||f^\varepsilon|||_{L^2(R^\varepsilon)}$ *uniformly bounded. Also, assume there exists* $\hat{f} \in L^2((0,1)\times Y^*)$ *such that*

$$\mathcal{T}_\varepsilon f^\varepsilon \rightharpoonup \hat{f} \quad weakly\ in\ L^2((0,1)\times Y^*).$$

Then, there exist $u \in H^1(0,1)$ *and* $u_1 \in L^2((0,1); H^1_\#(Y^*))$ *such that*

$$\begin{gathered}\mathcal{T}_\varepsilon u_\varepsilon \to u \ strongly\ in\ L^2\left((0,1); H^1(Y^*)\right),\\ \mathcal{T}_\varepsilon \partial_x u_\varepsilon \rightharpoonup \partial_x u + \partial_{y_1} u_1 \ weakly\ in\ L^2((0,1)\times Y^*),\\ \mathcal{T}_\varepsilon \partial_y u_\varepsilon \rightharpoonup \partial_{y_2} u_1 \ weakly\ in\ L^2((0,1)\times Y^*).\end{gathered}$$

Moreover, we have that u *is the solution of*

$$\begin{cases} -\kappa q u_{xx} + p u_x + \left(c + \dfrac{|\partial_u Y^*|}{|Y^*|}\right) u = \bar{f} + \dfrac{|\partial_u Y^*|}{|Y^*|} g & in\ (0,1),\\ u_x(0) = u_x(1) = 0, \end{cases}$$

where the homogenized coefficients q *and* p *are given by*

$$q = \frac{1}{|Y^*|}\int_{Y^*}(1-\partial_{y_1}X)\,dy_1dy_2 \quad and \quad p = \frac{1}{|Y^*|}\int_{Y^*} Q\,(1-\partial_{y_1}X)\,dy_1dy_2$$

and $X \in H^1_\#(Y^*)$ *with* $\int_{Y^*} X dy_1 dy_2 = 0$ *is the unique solution of*

$$\int_{Y^*} \nabla X \nabla\varphi dy_1 dy_2 = \int_{Y^*} \partial_{y_1}\varphi dy_1 dy_2 \qquad \forall \varphi \in H^1_\#(Y^*).$$

Also, the forcing term $\bar{f}$ *is given by* $\bar{f} = \dfrac{1}{|Y^*|}\int_{Y^*} \hat{f} dy_1 dy_2$.

Proof. See [13, Theorem 3.1]. □

Remark 3.1. *Notice the effect of the oscillating behavior on the homogenized coefficients* q *and* p *given by the auxiliary function* X. *Also, we emphasize the effect of the Lebesgue measure of the sets* Y^* *and* $\partial_u Y^*$ *at the limit equation, as well as, the flux condition on the border sets by* g. *All these ingredients can be seen at the homogenized equation.*

Remark 3.2. *See [5] for a discussing on some properties of the homogenized coefficient* q. *In particular, they show that* $0 < q < 1$.

Cases $\alpha < 1$ and $\alpha > 1$

Next, we mention the results concerning to the sub and super critical assumptions on the roughness parameter α. The proof can be found respectively in [13, Theorem 3.2] and [13, Theorem 3.3].

Theorem 3.2. *Let u^ε be the solution of the problem (1) with $\alpha < 1$, $f^\varepsilon \in L^2(R^\varepsilon)$ and $|||f^\varepsilon|||_{L^2(R^\varepsilon)}$ uniformly bounded. Also, assume $g \in H^1(0,1)$. Then, as $\varepsilon \to 0$,*

$$\mathcal{T}_\varepsilon u^\varepsilon \to g \quad \text{strongly in } L^2\left((0,1); H^1(Y^*)\right)$$
$$\text{in such way that}$$
$$|||u^\varepsilon - g|||_{L^2(R^\varepsilon)} \to 0.$$

Theorem 3.3. *Let u^ε be the solution of the problem (1) with $\alpha > 1$, $f^\varepsilon \in L^2(R^\varepsilon)$ and $|||f^\varepsilon|||_{L^2(R^\varepsilon)}$ uniformly bounded. Assume that there exists $\hat{f} \in L^2((0,1)\times Y^*)$ such that*

$$\mathcal{T}_\varepsilon f^\varepsilon \rightharpoonup \hat{f} \quad \text{weakly in } L^2\left((0,1)\times Y^*\right).$$

Then, there exist $u \in H^1(0,1)$ and $u_1 \in L^2((0,1); H^1_\#(Y^*))$ such that

$$\begin{aligned}
\mathcal{T}_\varepsilon u_\varepsilon &\to u \text{ strongly in } L^2\left((0,1); H^1(Y^*)\right),\\
\mathcal{T}_\varepsilon \partial_x u_\varepsilon &\rightharpoonup \partial_x u + \partial_{y_1} u_1 \text{ weakly in } L^2\left((0,1)\times Y^*\right),\\
\mathcal{T}_\varepsilon \partial_y u_\varepsilon &\rightharpoonup \partial_{y_2} u_1 \text{ weakly in } L^2\left((0,1)\times Y^*\right).
\end{aligned}$$

Moreover, u is the solution of

$$\begin{cases} -\kappa q u_{xx} + p u_x + c u = \bar{f} \text{ in } (0,1) \\ u_x(0) = u_x(1) = 0, \end{cases}$$

where the homogenized coefficients q and p are given by

$$q = \frac{1}{|Y^*|}\int_{Y^*} (1 - \partial_{y_1} X)\, dy_1 dy_2 \qquad \text{and} \qquad p = \frac{1}{|Y^*|}\int_{Y^*} Q\,(1 - \partial_{y_1} X)\, dy_1 dy_2$$

and $X \in H^1_\#(Y^*)$ with $\int_{Y^*} X dy_1 dy_2 = 0$ is the unique solution of

$$\int_{Y^*} \nabla X \nabla \varphi dy_1 dy_2 = \int_{Y^*} \partial_{y_1} \varphi dy_1 dy_2 \qquad \forall \varphi \in H^1_\#(Y^*).$$

Also, the forcing term $\bar{f}$ is given by $\bar{f} = \dfrac{1}{|Y^*|}\int_{Y^*} \hat{f} dy_1 dy_2$.

Proof's idea

First let us rewrite and rescale the problem as

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{R^\varepsilon} \kappa \nabla u^\varepsilon \nabla \varphi + Q^\varepsilon(y)\partial_x u^\varepsilon \varphi + c u^\varepsilon \varphi dx dy - \frac{1}{\varepsilon}\int_{R^\varepsilon} f^\varepsilon \varphi dx dy& \\
= \varepsilon^{\alpha-1}\int_{\Gamma^\varepsilon} (g - u^\varepsilon)\varphi dS \qquad \forall \varphi \in H^1(R^\varepsilon).&
\end{aligned}$$

Next, let us assume the solutions are uniformly bounded in the norm

$$|||\cdot|||_{H^1} = \frac{1}{\sqrt{\varepsilon}}\|\cdot\|_{H^1}.$$

Then,

- if $\alpha > 1$, and $\varepsilon^{\alpha-1} \to 0$, then the homogenized equation does not depend on g.

- If $\alpha < 1$, $\varepsilon^{\alpha-1} \to \infty$, we get

$$\int_{\Gamma^\varepsilon} (g - u^\varepsilon)\varphi dS \to 0 \quad \text{ as } \varepsilon \to 0.$$

  Hence, the solutions will converge to g and the condition on the border is dominant.

- If $\alpha = 1$, $\varepsilon^{\alpha-1} = 1$ and we have a homogenized equation depending on g coexisting with all the others effects produced by the thin domain situation.

We use the unfolding operators to pass to the limit in the equation

$$\begin{aligned}
&\int_{(0,1)\times Y^*} k\mathcal{T}_\varepsilon \nabla u^\varepsilon \mathcal{T}_\varepsilon \nabla\varphi + \mathcal{T}_\varepsilon Q^\varepsilon \mathcal{T}_\varepsilon \partial_x u^\varepsilon \mathcal{T}_\varepsilon \varphi + c\mathcal{T}_\varepsilon u^\varepsilon \varphi dxdy_1dy_2 \\
&\qquad + \varepsilon^{\alpha-1} \int_{(0,1)\times\partial_u Y^*} \mathcal{T}^b_\varepsilon u^\varepsilon \mathcal{T}^b_\varepsilon \varphi d\sigma(y) \\
&+ \frac{L}{\varepsilon}\int_{R_1^\varepsilon} \kappa \nabla u^\varepsilon \nabla \varphi + Q^\varepsilon(y)\partial_x u^\varepsilon \varphi + cu^\varepsilon \varphi dxdy + L\varepsilon^{\alpha-1}\int_{\Gamma_1^\varepsilon} u^\varepsilon \varphi dS \\
&= \int_{(0,1)\times Y^*} \mathcal{T}_\varepsilon f^\varepsilon \mathcal{T}_\varepsilon \varphi dxdy_1dy_2 + \varepsilon^{\alpha-1}\int_{(0,1)\times\partial_u Y^*} \mathcal{T}^b_\varepsilon g \mathcal{T}^b_\varepsilon \varphi dxd\sigma(y) \\
&\qquad + \frac{L}{\varepsilon}\int_{R_1^\varepsilon} f^\varepsilon \varphi dxdy + L\varepsilon^{\alpha-1}\int_{\Gamma_1^\varepsilon} g\varphi dS.
\end{aligned}$$

For instance, let us see the case $\alpha = 1$. There exist $u \in H^1(0,1)$ and $u_1 \in L^2((0,1); H^1_\#(Y^*))$ such that

$$\begin{aligned}
\mathcal{T}_\varepsilon u_\varepsilon &\to u \text{ strongly in } L^2\left((0,1); H^1(Y^*)\right), \\
\mathcal{T}_\varepsilon \partial_x u_\varepsilon &\rightharpoonup \partial_x u + \partial_{y_1} u_1 \text{ weakly in } L^2\left((0,1)\times Y^*\right), \\
\mathcal{T}_\varepsilon \partial_y u_\varepsilon &\rightharpoonup \partial_{y_2} u_1 \text{ weakly in } L^2\left((0,1)\times Y^*\right).
\end{aligned}$$

Also,

$$\mathcal{T}^b_\varepsilon u_\varepsilon \to u \quad \text{in} \quad L^2((0,1); H^{\frac{1}{2}}(\partial_u Y^*)).$$

We can pass to the limit yielding

$$\begin{aligned}
&\int_{(0,1)\times Y^*} \kappa\left(\partial_x u + \partial_{y_1} u_1\right)\partial_x \varphi + Q\left(\partial_x u + \partial_{y_1} u_1\right)\varphi + cu\varphi dxdy_1dy_2 \\
+&\int_{(0,1)\times\partial_u Y^*} u\varphi dxd\sigma(y) = \int_{(0,1)\times Y^*} \hat{f}\varphi dxdy_1dy_2 + \int_{(0,1)\times\partial_u Y^*} g\varphi dxd\sigma(y).
\end{aligned}$$

To obtain the relation between u_1 and the auxiliar solution X we consider

$$v^\varepsilon(x,y) = \varepsilon\phi(x)\psi\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right), \quad (x,y) \in R^\varepsilon,$$

where $\phi \in C_0^\infty(0,1)$ and $\psi \in H^1_\#(Y^*)$. Then,

$$\begin{aligned}
\mathcal{T}_\varepsilon v^\varepsilon &\to 0, \quad \text{strongly in } L^2\left((0,1)\times Y^*\right), \\
\mathcal{T}_\varepsilon \partial_x v^\varepsilon &\to \phi\partial_{y_1}\psi, \quad \text{strongly in } L^2\left((0,1)\times Y^*\right), \\
\mathcal{T}_\varepsilon \partial_y v^\varepsilon &\to \phi\partial_{y_2}\psi, \quad \text{strongly in } L^2\left((0,1)\times Y^*\right).
\end{aligned}$$

We take v^ε as a test function getting

$$\int_{(0,1)\times Y^*} \left(\partial_x u + \partial_{y_1} u_1, \partial_{y_2} u_1\right)\phi\nabla_y \psi dxdy_1dy_2 = 0.$$

Hence, from the density of $C_0^\infty(0,1)\otimes H^1_\#(Y^*)$ in $L^2((0,1); H^1_\#(Y^*))$ we have

$$\int_{(0,1)\times Y^*} \left(\partial_x u + \partial_{y_1} u_1, \partial_{y_2} u_1\right)\nabla_y \psi dxdy_1dy_2 = 0, \quad \forall \psi \in L^2((0,1); H^1_\#(Y^*)).$$

Thus, by uniqueness, we conclude that

$$u_1(x, y_1, y_2) = -\partial_x u(x) X(y_1, y_2).$$

Now we are in position to rewrite our limit expression as

$$\int_{(0,1)\times Y^*} \kappa\left(\partial_x u - \partial_x u \partial_{y_1} X\right)\partial_x \varphi + Q\left(\partial_x u - \partial_x u \partial_{y_1} X\right)\varphi + cu\varphi dx dy_1 dy_2$$
$$+ \int_{(0,1)\times\partial_u Y^*} u\varphi dx d\sigma(y) = \int_{(0,1)\times Y^*} \hat{f}\varphi dx dy_1 dy_2 + \int_{(0,1)\times\partial_u Y^*} g\varphi dx d\sigma(y).$$

Since u and φ are independent on y_1 and y_2, we get

$$\int_0^1 \kappa \partial_x u \left[\int_{Y^*} \left(1 - \partial_{y_1} X\right) dy_1 dy_2\right] \partial_x \varphi + \left[\int_{Y^*} Q\left(1 - \partial_{y_1} X\right) dy_1 dy_2\right] \partial_x u \varphi dx + |Y^*| c \int_0^1 u\varphi dx$$
$$+ |\partial_u Y^*| \int_0^1 u\varphi dx = \int_0^1 \left[\int_{Y^*} \hat{f} dy_1 dy_2\right] \varphi dx dy_1 dy_2 + |\partial_u Y^*| \int_0^1 g\varphi dx.$$

Dividing both sides by $|Y^*|$ gives

$$\int_0^1 \left[\kappa\, q\, \partial_x u \partial_x \varphi + p\, \partial_x u \varphi + c\, u\varphi\right] dx + \frac{|\partial_u Y^*|}{|Y^*|} \int_0^1 u\varphi dx = \int_0^1 \bar{f}\varphi dx + \frac{|\partial_u Y^*|}{|Y^*|} \int_0^1 g\varphi dx,$$

for all $\varphi \in H^1(0,1)$.

Remark 3.3. *The coefficients q and p are strictly positive, and then, the limit equation is a well posed problem and u^ε is a convergent sequence.*

Recently

In [14], we have analyzed quasilinear equations

$$\begin{cases} -\operatorname{div} a\left(x, \frac{x}{\varepsilon^\beta}, \frac{y}{\varepsilon}, \nabla u^\varepsilon\right) = f^\varepsilon \quad \text{in} \quad R^\varepsilon, \\ a\left(x, \frac{x}{\varepsilon^\beta}, \frac{y}{\varepsilon}, \nabla u^\varepsilon\right)\eta^\varepsilon + \varepsilon^\alpha b\left(x, \frac{x}{\varepsilon^\beta}, \frac{y}{\varepsilon}, u^\varepsilon\right) = \varepsilon^\alpha H^\varepsilon \quad \text{in} \quad \Gamma^\varepsilon, \\ a\left(x, \frac{x}{\varepsilon^\beta}, \frac{y}{\varepsilon}, \nabla u^\varepsilon\right)\eta^\varepsilon = 0 \quad \text{on} \quad \partial R^\varepsilon \setminus \left(\Gamma^\varepsilon \cup \partial_l R^\varepsilon\right), \\ u^\varepsilon = 0 \quad \text{in} \quad \partial_l R^\varepsilon, \end{cases}$$

where
$$R^\varepsilon = \left\{(x,y) \in \mathbb{R}^{N+1} : x \in \omega, 0 < y < \varepsilon h\left(\frac{x}{\varepsilon^\beta}\right)\right\} \quad \text{for} \quad \varepsilon > 0$$

- a and b are Carathéodory functions that satisfy monotone and usual p-growth conditions.
- η^ε is the outward unit normal to ∂R^ε.
- $f^\varepsilon \in L^{p'}(R^\varepsilon)$, $H^\varepsilon \in L^{p'}(\Gamma^\varepsilon)$ and $p^{-1} + (p')^{-1} = 1$.
- $\omega \subset \mathbb{R}^N$, $N \geq 1$, is an open, bounded, connected and regular set.
- $h \in W^{1,\infty}(\mathbb{R}^N)$ is strictly positive and L-periodic for some $L \in \mathbb{R}^N$.
- The parameter $\beta > 0$ establishes different roughness on the boundary.
- $\partial_l R^\varepsilon$ denotes the lateral boundary of R^ε and Γ^ε its upper boundary.
- R^ε is a purely periodic in $\mathbb{R}^{N+1}$ whose the representative cell is

$$Y^* = \left\{(y_1, y_2) \in \mathbb{R}^N \times \mathbb{R} \ : \ y_1 \in Y, \ 0 < y_2 < h(y_1)\right\}$$
$$\text{with } \ Y = \prod_{i=1}^N (0, L_i).$$

Analogous resultas are obtained. If $\alpha < 1$ and $H^\varepsilon = g$, for any $\gamma > 0$,

$$\frac{1}{\varepsilon}\|u^\varepsilon - g\|^p_{L^p(R^\varepsilon)} \leq \varepsilon^p c + c \cdot \begin{cases} \varepsilon^{1-\alpha}, & p \geq 2,\ \beta > 0 \\ \left(\varepsilon^{1-\beta-\frac{2\gamma}{p}} + \varepsilon^{\frac{2\gamma}{2-p}}\right) & 1 < p < 2,\ \beta \leq 1, \\ \left(\varepsilon^{1-\alpha-\frac{2\gamma}{p}} + \varepsilon^{\frac{2\gamma}{2-p}+1-\beta}\right) & 1 < p < 2,\ \beta > 1. \end{cases}$$

Thus, for some appropriate combinations of β and p with $\alpha < 1$, we obtain

$$\frac{1}{\varepsilon}\|u^\varepsilon - g\|^p_{L^p(R^\varepsilon)} \to 0, \qquad \text{as } \varepsilon \to 0.$$

In particular, it holds if $\beta > 0$ and $p \geq 2$.

If $\alpha \geq 1$, let us first consider this simple case

$$\begin{cases} -\Delta u^\varepsilon = f(x) \text{ in } R^\varepsilon \\ \dfrac{\partial u^\varepsilon}{\partial \eta^\varepsilon} + \varepsilon^\alpha u^\varepsilon = \varepsilon^\alpha g \text{ on } \Gamma^\varepsilon \\ \dfrac{\partial u^\varepsilon}{\partial \eta^\varepsilon} = 0 \text{ on } \partial R^\varepsilon \setminus (\Gamma^\varepsilon \cup \partial_l R^\varepsilon) \\ u^\varepsilon = 0 \text{ on } \partial_l R^\varepsilon. \end{cases}$$

Then, the family of solutions u^ε will converge to the unique solution of

$$\begin{cases} -\text{div}\,(A\nabla u) + v(\alpha)\dfrac{|\partial_u Y^*|}{|Y^*|} u = f + v(\alpha)\dfrac{|\partial_u Y^*|}{|Y^*|} g \text{ in } \omega \\ u = 0 \quad \text{on } \partial\omega \end{cases}$$

with $v(1) = 1$ and zero otherwise. $A = (a_{ij})_{n\times n}$ is a constant matrix with

$$\begin{aligned} a_{ii} &= \frac{1}{|Y^*|}\int_{Y^*}\left(1 + \frac{\partial X_i}{\partial {y_1}_i}\right) dy_1 dy_2 \text{ and } a_{ij} = \frac{\partial X_i}{\partial {y_1}_j},\ i \neq j,\ \text{if } \beta = 1, \\ a_{ii} &= \frac{1}{\langle h\rangle_Y \langle 1/h\rangle_Y} \text{ and } a_{ij} = 0,\ i \neq j,\ \text{if } \beta < 1, \\ a_{ii} &= \frac{h_0}{\langle h\rangle_Y} \text{ and } a_{ij} = 0,\ i \neq j,\ \text{if } \beta > 1; \end{aligned}$$

$\frac{\partial}{\partial {y_1}_i}$ denotes the i-th partial derivative for $y_1 \in \mathbb{R}^N$ and $\langle\varphi\rangle_Y = \frac{1}{|Y|}\int_Y \varphi\, dy$.

The auxiliary function X_i is the unique solution of

$$\int_{Y^*}(e_i + \nabla_{y_1 y_2} X_i)\,\nabla_{y_1 y_2}\psi dy_1 dy_2 = 0 \text{ with } \int_{Y^*} X_i dy_1 dy_2 = 0, \quad \forall \psi \in H^1_\#(Y^*).$$

Remark 3.4. *Notice that now we have N auxiliary solutions.*

Another important example is the well-known p-Laplacian

$$\begin{cases} -\Delta_p u^\varepsilon = f(x) \quad \text{in} \quad R^\varepsilon \\ |\nabla u^\varepsilon|^{p-2}\dfrac{\partial u^\varepsilon}{\partial \eta^\varepsilon} + \varepsilon^\alpha |u^\varepsilon|^{p-2} u^\varepsilon = \varepsilon^\alpha |g|^{p-2} g \quad \text{on} \quad \Gamma^\varepsilon \\ |\nabla u^\varepsilon|^{p-2}\dfrac{\partial u^\varepsilon}{\partial \eta^\varepsilon} = 0 \quad \text{on} \quad \partial R^\varepsilon \setminus (\Gamma^\varepsilon \cup \partial_l R^\varepsilon) \\ u^\varepsilon = 0 \quad \text{on} \quad \partial_l R^\varepsilon. \end{cases}$$

The forcing terms are the same and the operator A is now nonlinear

$$\frac{1}{|Y^*|}A(\xi) = \begin{cases} \dfrac{1}{|Y^*|}\displaystyle\int_{Y^*} |\nabla_{y_1 y_2} X_\xi|^{p-2}\nabla_{y_1} X_\xi dy_1 dy_2, \quad \beta = 1, \\ \dfrac{|\xi|^{p-2}\,\xi}{\langle h\rangle_Y \langle h^{1-p'}\rangle^{p-1}}, \quad \beta < 1, \\ \dfrac{h_0}{\langle h\rangle_Y} |\xi|^{p-2}\,\xi, \quad \beta > 1, \end{cases}$$

where, for each $(\xi_1, \ldots, \xi_N, 0) \in \mathbb{R}^{N+1}$, X_ξ is the unique solution of

$$\textstyle\int_{Y^*} |(\xi,0) + \nabla_{y_1 y_2} X_\xi|^{p-2}((\xi,0) + \nabla_{y_1 y_2} X_\xi)\nabla_{y_1 y_2}\psi dy_1 dy_2 = 0,$$

for all $\psi \in W^{1,p}_\#(Y^*)$ with $\int_{Y^*} X_\xi dy_1 dy_2 = 0$. $A(\cdot)$ still satisfies the monotonicity properties, but is not a p-Laplacian equation as is the case with $N = 1$ (see also Donato and Moscariello (1990)).

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Remarks on the non-local q-eigenvalue problems for the p-Laplacian

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1 Introduction

First, let us recall the best constant $\lambda_{p,q}$ of the Sobolev–Poincaré inequality

$$\lambda_{p,q}\left(\int_\Omega |u|^q\,dx\right)^{p/q} \le \int_\Omega |\nabla u|^p\,dx \quad \text{for } u \in W_0^{1,p}(\Omega).$$

Here we assume that Ω is a bounded domain in $\mathbb{R}^N$, $1 < p < \infty$ and $1 < q < p^*$ (where $p^* := Np/(N-p)$ if $N > p$, $p^* := \infty$ if $N \le p$). This inequality plays an important role in analyzing the p-Laplace equation under the Dirichlet boundary condition. Moreover, $\lambda_{p,q}$ is described as the infimum of the Rayleigh quotient as follows:

$$\lambda_{p,q} = \inf\left\{ \frac{\int_\Omega |\nabla u|^p\,dx}{\left(\int_\Omega |u|^q\,dx\right)^{p/q}} \;:\; u \in W_0^{1,p}(\Omega)\setminus\{0\} \right\}. \tag{1}$$

Thanks to the compactness of the embedding under the subcritical condition ($1 < q < p^*$), the infimum in (1) is attained by some positive (eigen)function and the minimizer satisfies the p-Laplace equation with $\lambda = \lambda_{p,q}$

$$-\Delta_p u = \lambda \|u\|_q^{p-q}|u|^{q-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where Δ_p stands for the p-Laplace operator defined by $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $\|\cdot\|_q$ denotes the standard Lebesgue norm of $L^q(\Omega)$.

The equation (2) has "*a non-local term* $\|u\|_q^{p-q}$" if $p \ne q$, but it coincides with *the standard eigenvalue equation* for p-Laplacian in case $p = q$. In this talk, we call the equation (2) *a non-local q-eigenvalue problem* for p-Laplacian as called in [9]. Moreover, we say that λ is *q-eigenvalue* (of p-Laplacian) if (2) has a nontrivial solution. In this sense, we can understand that the best constant $\lambda_{p,q}$ is *the first q-eigenvalue* of a non-local q-eigenvalue problem (2).

Recently, there are many results on non-local eigenvalue problems (e.g. [2], [4], [10], [7], [13], [14], [15]). In particular, I'd like to mention three results related to our q-eigenvalue problem (2). In [8], Franzina and Lamberti studied the first q-eigenvalue $\lambda_{p,q}$ and constructed a sequence of variational q-eigenvalues. In [3], Brasco and Lindgren showed the simplicity of $\lambda_{p,q}$ when $2 < p < q$ and q is close to p. In [9], Ercole proved that there is no eigenvalue having "*nodal (sign-changing)*" eigenfunctions near $\lambda_{p,q}$ in p-superlinear case ($p < q < p^*$). However, as far as I know, there is no result on the second q-eigenvalue (for $q \ne p$). In particular, Ercole's result motivated this research because the second (standard) p-eigenvalue is the least one having a nodal p-eigenfunction.

In this talk, I'd like to focus on the least q-eigenvalue which has a "*nodal eigenfunction*". The talk is based on the results in [11] (section 3) and [17] (section 2).

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2 Non-local Eigenvalue problem with weight

First, we assume that $w \in L^r(\Omega)$ with

$$r \in [1, \infty] \quad (\text{if } N < p) \quad \text{and} \quad \frac{Np}{Np - Nq + pq} < r \leq \infty \quad (\text{if } N \geq p), \tag{3}$$

satisfies $w^+ \not\equiv 0$, where $w^\pm(x) := \max\{\pm w(x), 0\}$, that is, $w = w^+ - w^-$. We remark that (3) is equivarent to $qr' < p^*$ if $N \geq p$, where r' denotes the Hölder conjugate of r. For the weight function w above, we define

$$|u|_{q,w} := \operatorname{sign}\left(\int_\Omega w(x)|u|^q\,dx\right)\left|\int_\Omega w(x)|u|^q\,dx\right|^{1/q} \qquad \text{for } u \in L^{r'q}(\Omega),$$

where $\operatorname{sign}(0) = 0$, $\operatorname{sign}(t) = t/|t|$ if $t \neq 0$, In case $w \equiv 1$, $|\cdot|_{q,w}$ coincides with the standard Lebesgue norm $\|\cdot\|_q$ of $L^q(\Omega)$. Throughout this talk, we often omit p in the notations because it is fixed.

Now, let us consider the following non-local q-eigenvalue problem with weight w:

$$(EV;\lambda) \qquad \begin{cases} |u|_{q,w} \geq 0 \\ -\Delta_p u = \lambda w(x)\, |u|_{q,w}^{p-q}\, |u|^{q-2}u & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda \in \mathbb{R}$ is an eigenvalue parameter. Here we naturally understand that our equation becomes

$$-\Delta_p u = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

if $|u|_{q,w} = 0$ even in case $p < q$ because any solutions satisfy $\|\nabla u\|_p^p = \lambda |u|_{q,w}^p$ if $|u|_{q,w} > 0$. Since we admit an indefinite weight w in general, we can consider solutions of $(EV;\lambda)$ such that $|u|_{q,w} < 0$. However, we handle only solutions satisfying $|u|_{q,w} \geq 0$ to simplify the problem.

Definition 1. We say that $u \in W_0^{1,p}(\Omega)$ is a *(weak) solution* of $(EV;\lambda)$ if $|u|_{q,w} \geq 0$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi\,dx = \lambda\,|u|_{q,w}^{p-q}\int_\Omega w|u|^{q-2}u\varphi\,dx$$

for any $\varphi \in W_0^{1,p}(\Omega)$. As in the standard nonlinear eigenvalue problems, if $(EV;\lambda)$ has a non-trivial solution u, then λ is called a *q-eigenvalue* for the p-Laplacian, and u is a *q-eigenfunction* corresponding to λ. Here, we say that u is *nodal* (sign-changing) if $\|\nabla u^+\|_p > 0$ and $\|\nabla u^-\|_p > 0$.

Definition 2. We define *the first q-eigenvalue* λ_q^w by

$$\lambda_q^w = \inf\left\{\, R_q^w(u) \,:\, u \in W_0^{1,p}(\Omega),\ |u|_{q,w} > 0 \,\right\}, \quad R_q^w(u) := \frac{\int_\Omega |\nabla u|^p\,dx}{\left(\int_\Omega w|u|^q\,dx\right)^{p/q}}.$$

From the definition, it follows that λ_q^w is the best constant of the following Sobolev-Poincaré type inequality

$$\lambda_q^w |u|_{q,w}^p \leq \int_\Omega |\nabla u|^p\,dx \qquad \text{for } u \in W_0^{1,p}(\Omega).$$

Moreover, we can prove the following elementary results on λ_q^w as like the standard first p-eigenvalue in the standard way (see Appendix in [17] and refer to [8] also).

Proposition 1. The following assertions hold:

(i) $\lambda_q^w > 0$;

(ii) $(EV;\lambda)$ has no non-trivial solutions provided $\lambda < \lambda_q^w$;

(iii) There exists a "*non-negative*" minimizer $\varphi_q^w \in C^{0,\theta}_{\mathrm{loc}}(\Omega)$ (some $\theta \in (0,1)$) of λ_q^w and it is a solution of $(EV;\lambda_q^w)$;

(iv) $\varphi_q^w > 0$ in Ω if $1 < q < p$ and $w \geq 0$ a.e. in Ω, or if $p \leq q < p^*$;

(v) λ_q^w is simple if $1 < q \leq p$, that is, if φ is a solution of $(EV;\lambda_q^w)$, then $\varphi = t\varphi_q^w$ for some $t \in \mathbb{R}$;

(vi) λ_q^w is continuous w.r.t. q and w in the following sense: if $q_n \to q$ and $w_n \to w$ strongly in L^r, then $\lambda_{q_n}^{w_n} \to \lambda_q^w$ as $n \to \infty$;

(v) Let $\{\Omega_n\}_n$ be a sequence of domains such that $\Omega_n \subset \Omega$ for all n and $|\Omega \setminus \Omega_n| \to 0$ as $n \to \infty$. Then $\lambda_q^w(\Omega_n) \geq \lambda_q^w(\Omega)$ for all n and $\lambda_q^w(\Omega_n) \to \lambda_q^w(\Omega)$ as $n \to \infty$.

In p-superlinear case ($p < q < p^*$), it may be impossible to show the simplicity of the first q-eigenvalue generally as stated in (v), but it is recently proven in [3] provided $w \equiv 1$, $2 < p < q$ and q is close to p.

Definition 3. Define *the least nodal q-eigenvalue* as follows:

$$\mu_q^w := \inf\{\lambda > 0 \,:\, (EV;\lambda) \text{ has a nodal solution}\}.$$

In the standard eigenvalue problem (case $p = q$), it is known (see [6, Corollary 5.1]) that the second p-eigenvalue $\lambda_{p,2nd}^w$ is the least nodal p-eigenvalue, namely,

$$\mu_p^w = \lambda_{p,2nd}^w := \inf\left\{\lambda > \lambda_q^w \,:\, (EV;\lambda) \text{ has a nontrivial solution (with } p = q)\right\}.$$

So the main motivation is the following:

$\boxed{\text{Q}}$ Is μ_q^w the second q-eigenvalue?

Remark 1. By taking $u^\pm$ as test function in $(EV;\lambda)$, we see that any nodal solution u of $(EV;\lambda)$ satisfies

$$0 < \|\nabla u^\pm\|_p^p = \lambda |u|_{q,w}^{p-q}\, |u^\pm|_{q,w}^q, \quad \text{and so} \quad R_q^w(u^\pm) = \lambda\left(\frac{|u|_{q,w}}{|u^\pm|_{q,w}}\right)^{p-q}.$$

Hence, nodal solutions belong to

$$\mathcal{A}_q^w := \left\{\, u = u^+ - u^- \in W_0^{1,p}(\Omega),\ |u^\pm|_{q,w} > 0 \,\right\}.$$

Let us remark the second p-eigenvalue (in case $p = q$) without weight. It was proven in [1, Proposition 4.2.] that the second p-eigenvalue $\lambda_{p,2nd}$ with $w \equiv 1$ is characterized by using Rayleigh quotient $R_p := R_p^{w\equiv 1}$ as follows:

$$\begin{aligned}\lambda_{p,2nd} &:= \inf\{\lambda > \lambda_p \,;\, (2) \text{ has a nontrivial solution}\}\ (= \mu_p^{w\equiv 1}) \\ &= \min\left\{\, \max\{R_p(u^+), R_p(u^-)\} \,:\, u \in \mathcal{A}_p^{w\equiv 1} \,\right\},\end{aligned}$$

where $\lambda_p := \lambda_{p,p}$ is the first eigenvalue in (1) with $p = q$.

$\boxed{\text{Q2}}$ Can we describe μ_q^w by using the Rayleigh quotient R_q^w?

2.1 Main results

First, let us see p-sublinear case. In case $w \equiv 1$ the following result is shown in [12, Lemma 3.1] and [8, Theorem 4.1.].

Proposition 2. Let $1 < q \le p$ and assume that $w \ge 0$ a.e. in Ω. Then, $(EV;\lambda)$ has no constant sign and nontrivial solutions provided $\lambda \ne \lambda_q^w$, that is, only the first q-eigenvalue λ_q^w has constant sign q-eigenfunctions.

Thanks to the above result, we can see that μ_q^w becomes the second q-eigenvalue.

Theorem 1. Let $1 < q \le p$ and $w \ge 0$ a.e. in Ω. Assume that Ω has $C^{1,1}$-regularity. Then $\mu_q^w > \lambda_q^w$, and it is attained and the second q-eigenvalue, that is,

$$\mu_q^w = \min\{\lambda > \lambda_q^w \ : \ (EV;\lambda) \text{ has a nontrivial solution }\} > \lambda_q^w.$$

Moreover, μ_q^w is continuous w.r.t. q and w in the following sense: if $q_n(\in (1,p]) \to q$ and $0 \le w_n \to w$ in L^r as $n \to \infty$, then $\mu_{q_n}^{w_n} \to \mu_q^w$ as $n \to \infty$.

In the standard p-eigenvalue problem, the first and second eigenvalues can be described by using the Rayleigh quotient R_p^w. This is difficult for non-local second q-eigenvalues. Instead, we get the following estimate.

Proposition 3. Let $1 < q \le p$ and $w \ge 0$ a.e. in Ω. Assume that Ω has $C^{1,1}$-regularity. Then it holds:

$$\eta_q^w \ \le \ \mu_q^w \ \le \ \min\left\{ \max\{R_q^w(u^+), R_q^w(u^-)\} \ : \ u \in \mathcal{A}_q^w \right\},$$

where

$$\eta_q^w := \inf\left\{ \max\left\{ \left(\frac{|u^+|_{q,w}}{|u|_{q,w}}\right)^{p-q} R_q^w(u^+), \left(\frac{|u^-|_{q,w}}{|u|_{q,w}}\right)^{p-q} R_q^w(u^-) \right\} \ : \ u \in \mathcal{A}_q^w \right\}.$$

Remark that I don't know whether η_q^w is attained in p-sublinear case.

In p-superlinear case, we have the following result.

Theorem 2. Let $p < q < p^*$. Then, $\mu_q^w > \lambda_q^w$, and it is attained and characterized as follows:

$$\mu_q^w = \min\left\{ \left(R_q^w(u^+)^{q/(q-p)} + R_q^w(u^-)^{q/(q-p)} \right)^{(q-p)/q} \ : \ u \in \mathcal{A}_q^w \right\}.$$

Moreover, μ_q^w is continuous w.r.t. q and w in the following sense: if $q_n \to q \in (p,p^*)$ and $w_n \to w$ in L^r as $n \to \infty$, then $\lim_{n\to\infty} \mu_{q_n}^{w_n} = \mu_q^w$.

Remark 2. In p-superlinear case, we don't know generally whether μ_q^w is the second q-eigenvalue, but in the special case (e,g, $w \equiv 1$, Ω is ball) we know that it is the second.

3 One dimensional q-eigenvalue problems without weight

In this section, I'd like to introduce the results in the joint work with Kajikiya and Tanaka ([11]).

Let us consider the n-th q-eigenvalue $\lambda_n = \lambda_n(p,q,L)$ for the following one-dimensional p-Laplace equation:

$$(|u'|^{p-2}u')' + \lambda\|u\|_q^{p-q}|u|^{q-2}u = 0 \quad \text{in } (0,L), \qquad u(0) = u(L) = 0, \tag{4}$$

where $p, q > 1$, $L > 0$ and $\|u\|_q = (\int_0^L |u|^q\,dx)^{1/q}$.

The exact formula of the first q-eigenvalue is given as follows:

Theorem 3 ([11, Theorem 2.1.])**.** The first q-eigenvalue is represented as

$$\lambda_1(p,q,L) = 2^p p^{-1}(p-1)q^{1-p}(1+q-q/p)^{(p-q)/q}L^{-p+(q-p)/q}B(1-1/p,1/q)^p, \tag{5}$$

where $B(\cdot,\cdot)$ denotes the beta function defined by

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt \quad \text{for } x, y > 0.$$

The q-eigenfunction $\varphi_{p,q}$ corresponding to $\lambda_1(p,q,L)$ is written as

$$\varphi_{p,q}(x) = c\sin_{pq}(\pi_{pq}x/L) \quad \text{with } c \neq 0, \tag{6}$$

where $\sin_{pq}(x)$ is the unique solution of

$$(|v'|^{p-2}v')' + \frac{(p-1)q}{p}|v|^{q-2}v = 0 \quad \text{in } (0,\pi_{p,q}/2), \quad v(0) = 0, \quad v'(0) = 1,$$

(see [16] for the details) and π_{pq} is defined by

$$\pi_{pq} := 2\int_0^1 \frac{dt}{(1-t^q)^{1/p}}.$$

In case $p = q$, by replacing L with L/n simply we get the n-th p-eigenvalue of the p-Laplacian, that is, $\lambda_n(p,p,L) = \lambda_1(p,p,L/n)$. However, when $p \neq q$, $\lambda_1(p,q,L/n)$ is not the n-th q-eigenvalue because our q-eigenvalue equation (4) has a non-local term $\|u\|_q^{p-q}$. Actually, we can get the following result.

Theorem 4 ([11, Theorem 2.3.])**.** Let $p, q > 1$, $L > 0$, $n \in \mathbb{N}$ and $\varphi_{p,q}$ be the first q-eigenfunction written in (6). Then, the n-th q-eigenspace is spanned by $\varphi_{p,q}(n\,\cdot)$, that is, each q-eigenvalue is simple. Moreover, $\lambda_n(p,q,L) = n^p\lambda_1(p,q,L)$ holds.

Remark 3. Thanks to Theorem 4, we know that $\lambda_2(p,q,L)$ is the least nodal q-eigenvalue, that is, $\mu_q^{w\equiv 1}$ $(N = 1)$ is the second q-eigenvalue.

Since the one dimensional q-eigenvalues of (4) are exactly written as like (5), we can observe the asymptotic behavior of $\lambda_n(p,q,L)$ as p or q tends to 1 or ∞, and the monotonicity w.r.t. p or q (see [11]).

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UNIQUENESS OF GROUND STATES FOR 2D-NONLINEAR SCHRÖDINGER EQUATIONS WITH POINT INTERACTION

NORIYOSHI FUKAYA

1. Introduction

We consider the following stationary nonlinear Schrödinger equation

$$(-\Delta_\alpha + \lambda)u - |u|^{p-1}u = 0, \quad x \in \mathbb{R}^2, \tag{1}$$

where $p > 1$, $\lambda > 0$, and $-\Delta_\alpha$ is the Schrödinger operator with a point interaction with strength $\alpha \in \mathbb{R}$, which is realized as a self-adjoint extention of the operator

$$\begin{cases} D(H_0) = C_c^\infty(\mathbb{R}^2 \setminus \{0\}), \\ H_0 = -\Delta. \end{cases}$$

The operator $-\Delta_\alpha$ is formally expressed as

$$-\Delta_\alpha v = -\Delta v + C(\alpha, v)\delta, \quad v \in D(-\Delta_\alpha)$$

for some constant $C(\alpha, v) \in \mathbb{C}$, so it can be regarded as the Schrödinger operator with a delta potential δ (see (3) for the precise definition). The equation (1) appears when we consider the standing wave solution $\psi(t, x) = e^{i\lambda t}u(x)$ for the nonlinear Schrödinger equation with a point interaction

$$i\partial_t\psi = -\Delta_\alpha\psi - |\psi|^{p-1}\psi, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2. \tag{2}$$

The stability of ground-state standing waves for (2) are studied in [1, 3] (see (4) for the definition of ground states). The uniqueness of ground states is important to investigate its stability. Although the uniqueness for large frequencies λ is obtained in [3], that for all frequencies was not obtained. The aim of this talk is to establish the uniqueness of ground states for (1) for all frequencies.

Let us give a formal derivation of $(-\Delta_\alpha)_{\alpha\in(-\infty,\infty]}$ (see [2] for more rigoras derivation). The adjoint operator of the closure $\overline{H_0}$ is given by

$$\begin{cases} D(\overline{H_0}^*) = \{v = f + cG_\lambda \,|\, f \in H^2(\mathbb{R}^2),\ c \in \mathbb{C}\} \\ (\overline{H_0}^* + \lambda)v = (-\Delta + \lambda)f. \end{cases}$$

where G_λ is a fundamental solution of the operator $-\Delta + \lambda$, that is G_λ satisfies

$$(-\Delta + \lambda)G_\lambda = \delta.$$

Note that the definition of $\overline{H_0}^*$ is independent of λ. Indeed, for $\lambda, \mu > 0$ with $\lambda \neq \mu$, since $G_\lambda - G_\mu \in H^2(\mathbb{R}^2)$, we have the independence of the domain. Let $v = f + cG_\lambda = h + dG_\mu$ for $f, h \in H^2(\mathbb{R}^2)$ and $c, d \in \mathbb{C}$. Then we have $c = d$ since $G_\lambda \notin H^2(\mathbb{R}^2)$. Moreover, noting that $(-\Delta+\lambda)(G_\lambda - G_\mu) = (\mu - \lambda)G_\mu$, we can check the relation

$$(-\Delta + \lambda)f - \lambda v = (-\Delta + \mu)h - \mu v.$$

This means the independence of the action.

Now by restricting the domain $D(\overline{H_0}^*)$ we give a family of self-adjoint operators $(-\Delta_\alpha)_{\alpha\in(-\infty,\infty]}$. For $v = f + cG_\lambda, \psi = \varphi + dG_\lambda \in D(\overline{H_0}^*)$ a direct calculation gives

$$\langle(\overline{H_0}^* + \lambda)v, \psi\rangle = \langle v, (\overline{H_0}^* + \lambda)\psi\rangle - c\overline{\varphi(0)} + f(0)\overline{d}.$$

Therefore, taking $c = C_0 f(0)$ and $d = C_0\varphi(0)$ for some $C_0 \in \mathbb{R}$, we have $\langle(\overline{H_0}^* + \lambda)v, \psi\rangle = \langle v, (\overline{H_0}^* + \lambda)\psi\rangle$. Thus, we can obtain a family of self-adjoint operators

$$\begin{cases} D(-\Delta_{C_0,\lambda}) = \left\{v = f + C_0 f(0)G_\lambda \,|\, f \in H^2(\mathbb{R}^2)\right\}, \\ (-\Delta_{C_0,\lambda} + \lambda)v = (-\Delta + \lambda)f. \end{cases}$$

This seems to depend on the two parameters C_0 and λ, but if we parametrize

$$C_0 = \frac{1}{\alpha + \beta(\lambda)}$$

with α and λ, where

$$\beta(\lambda) := \frac{1}{2\pi}\Big(\gamma + \log\frac{\sqrt{\lambda}}{2}\Big)$$

and $\gamma > 0$ is the Euler–Mascheroni constant, we see that the set

$$\left\{ v = f + \frac{f(0)}{\alpha + \beta(\lambda)}G_\lambda \,\middle|\, f \in H^2(\mathbb{R}^2) \right\}$$

is independent of the choice of λ. Therefore, we obtain the one-parameter family $(-\Delta_\alpha)_{\alpha\in(-\infty,\infty]}$ of self-adjoint operators defined by

$$\begin{cases} D(-\Delta_\alpha) = \left\{ v = f + \dfrac{f(0)}{\alpha+\beta(\lambda)} G_\lambda \,\middle|\, f \in H^2(\mathbb{R}^2) \right\}, \\ (-\Delta_\alpha + \lambda) v = (-\Delta + \lambda) f \end{cases} \tag{3}$$

for $\alpha \in \mathbb{R}$ and

$$D(-\Delta_\infty) = H^2(\mathbb{R}^2), \quad -\Delta_\infty f = -\Delta f.$$

The spectral properties of $-\Delta_\alpha$ have already been known (see [2]). For $\alpha \in \mathbb{R}$, the essential spectram is given by

$$\sigma_{\mathrm{ess}}(-\Delta_\alpha) = [0, \infty).$$

Moreover, $-\Delta_\alpha$ has a simple negative eigenvalue as

$$\sigma_{\mathrm{p}}(-\Delta_\alpha) = \{e_\alpha\}, \quad e_\alpha := -4e^{-4\pi\alpha - 2\gamma} < 0.$$

The form domain $H^1_\alpha(\mathbb{R}^2) := D[-\Delta_\alpha]$ is given by

$$H^1_\alpha(\mathbb{R}^2) = \begin{cases} \{ f + cG_\lambda \mid f \in H^1(\mathbb{R}^2),\ c \in \mathbb{R} \} & \text{if } \alpha \in \mathbb{R}, \\ H^1(\mathbb{R}^2) & \text{if } \alpha = \infty, \end{cases}$$

and we can define the quadratic form on $H^1_\alpha(\mathbb{R}^2)$ by

$$\langle (-\Delta_\alpha + \lambda) v, v \rangle = \|\nabla f\|^2_{L^2} + \lambda \|f\|^2_{L^2} + (\alpha + \beta(\lambda))|c|^2$$

for $v = f + cG_\lambda \in H^1_\alpha(\mathbb{R}^2)$.

We define the action for (1) by

$$S_\lambda(v) := \frac{1}{2}\langle (-\Delta_\alpha + \lambda) v, v \rangle - \frac{1}{p+1}\|v\|^{p+1}_{L^{p+1}}, \quad v \in H^1_\alpha(\mathbb{R}^2).$$

Then $u \in H^1_\alpha(\mathbb{R}^2)$ solves (1) if and only if $S'_\lambda(u) = 0$. We define the set of *ground states*

$$\mathcal{G}_\lambda := \{ u \in \mathcal{A}_\lambda \mid S_\lambda(u) \le S_\lambda(v),\ \forall v \in \mathcal{A}_\lambda \}, \tag{4}$$

where $\mathcal{A}_\lambda$ is the set of all nontrivial solutions for (1):

$$\mathcal{A}_\lambda := \{ u \in H^1_\alpha(\mathbb{R}^2) \mid u \neq 0,\ S'_\lambda(u) = 0 \}.$$

From the embedding $H^1_\alpha(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$ for all $q \ge 2$, we see that the $\mathcal{A}_\lambda \subset D(-\Delta_\alpha)$.

The following is known.

Proposition 1 ([1, 3])**.** *Let $p > 1$ and $\alpha \in \mathbb{R}$. The following holds.*

- *If $\lambda > |e_\alpha|$, then $\mathcal{G}_\lambda \neq \emptyset$.*
- *If $\lambda > |e_\alpha|$ and $u \in \mathcal{G}_\lambda$, then there exists $\theta \in \mathbb{R}$ such that $e^{i\theta}u$ is positive, radial, and decreasing function.*

By a elliptic regularity argument, we see that $\mathcal{A}_\lambda \subset C^2(\mathbb{R}^2 \setminus \{0\})$ for $\lambda > |e_\alpha|$.

To state our main results, we derive the boundary value problems of ordinary differential equations satisfied by the ground states. For $u = f + (\alpha + \beta(\lambda))^{-1} f(0) G_\lambda \in D(-\Delta_\alpha)$, the action of $-\Delta_\alpha + \lambda$ can be written as

$$\begin{aligned}(-\Delta_\alpha + \lambda)u = (-\Delta + \lambda)f &= (-\Delta + \lambda)(u - \frac{f(0)}{\alpha + \beta(\lambda)} G_\lambda) \\ &= (-\Delta + \lambda)u - \frac{f(0)}{\alpha + \beta(\lambda)} \delta.\end{aligned}$$

In particular,

$$(-\Delta_\alpha + \lambda)u = (-\Delta + \lambda)u \quad \text{for } x \neq 0. \tag{5}$$

It is known that

$$G_\lambda(x) = G_0(x) - \beta(\lambda) + o_{|x| \to 0}(1),$$

where

$$G_0(x) := -\frac{1}{2\pi} \log |x|$$

is the fundamental solution of the Laplacian $-\Delta$. Therefore, the boundary condition at $x = 0$ for u can be written as

$$\begin{aligned}u(x) &= f(x) + \frac{f(0)}{\alpha + \beta(\lambda)} G_\lambda(x) \\ &= \frac{f(0)}{\alpha + \beta(\lambda)} (G_0(x) + \alpha) + o_{|x| \to 0}(1). \end{aligned} \tag{6}$$

Therefore, if $u \in \mathcal{A}_\lambda$ is positive and radial, the relations (5) and (6) imply that $u(r)$ satisfies

$$\begin{cases} -u'' - \dfrac{1}{r}u' + \lambda u - u^p = 0, & r > 0, \\ u(r) = y(G_0(r) + \alpha) + o_{r \downarrow 0}(1) & \text{for some } y > 0. \end{cases} \tag{7}$$

The following is our main result.

Theorem 1. *For $p > 1$, $\alpha \in \mathbb{R}$ and $\lambda > |e_\alpha|$, the positive and decaying solutions of* (7) *are unique.*

As a corollary we have the uniqueness of ground states.

Corollary 1. *For $p > 1$, $\alpha \in \mathbb{R}$ and $\lambda > |e_\alpha|$, there exists a positive, radial, and decreasing function $\phi_\lambda \in \mathcal{G}_\lambda$ such that*

$$\mathcal{G}_\lambda = \{e^{i\theta}\phi_\lambda \,|\, \theta \in \mathbb{R}\}.$$

For the case $\alpha = \infty$, i.e., the case $u(0) = y > 0$, the same results are obtained by [4].

2. Proof

For the proof of Theorem 1, we use the following generalized Pohozaev identity derived by Shioji and Watanabe [5, 6] and follow thier arguments.

Proposition 2 ([5, 6])**.** *Let u be a solution of* (7)*, and put*

$$\begin{aligned} J[u](r) := \frac{1}{2}a(r)u'(r)^2 + b(r)u'(r)u(r) + \frac{1}{2}c(r)u(r)^2 \\ - \frac{\lambda}{2}a(r)u(r)^2 + \frac{1}{p+1}a(r)u(r)^{p+1}, \end{aligned}$$

where

$$\begin{aligned} a(r) &= r^q, \\ b(r) &= \frac{2}{p+3}r^{q-1} \\ c(r) &= \frac{8}{(p+3)^2}r^{q-2}, \\ q &= \frac{2(p+1)}{p+3}, \end{aligned}$$

Then

$$J[u]'(r) = \frac{1}{2}C(r)u(r)^2 \tag{8}$$

with

$$C(r) = \frac{1}{(p+3)^3}r^{q-3}(-32 - 2\lambda(p-1)(p+3)^2r^2).$$

Now let u and v be two positive decaying solutions for (7) with $y = y_u$ and $y = y_v$ respectively.

The following lemma is important in our proof.

Lemma 1. *$J[u](r) \geq 0$ for all $r > 0$.*

Proof. Since u decays exponentially, we see that $J[u](r) \to 0$ as $r \to \infty$. Moreover, by (8) and $C(r) < 0$, we have the conclusion. □

We put

$$w(r) := \frac{v(r)}{u(r)}$$

and

$$X(r) := w(r)^2 J[u](r) - J[v](r).$$

Then we have

$$X'(r) = 2w(r)w'(r)J[u](r).$$

Moreover, we can show the following lemmas.

Lemma 2. $\lim_{r\downarrow 0} X(r) = 0$.

Lemma 3. *If $y_v > y_u$, then $w'(r) < 0$ for all $r > 0$. Moreover,* $\lim_{r\to\infty} X(r) = 0$.

If $y_v > y_u$, Lemmas 2 and 3 derive a contradiction. Thus, we have $y_u = y_v$. In this case we have $u = v$ by the Gronwall inequality. This completes the proof of Theorem 1.

In the talk, we give more details about the proof of Lemma 2 and 3. Moreover, we mention about the open problems in the 3d-cases.

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A Minimizing Movement Scheme for Mean Curvature Flow: Addressing Contact Angles in Regular Domains

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Abstract

In this talk, we introduce an advanced numerical scheme for the mean curvature flow (MCF) with prescribed contact angle conditions, extending Chambolle's scheme. The primary contributions are twofold: (1) the development of a capillary Chambolle-type scheme tailored for MCF with contact angle conditions, and (2) the establishment of the scheme's convergence to the unique viscosity solution of the level-set equation for MCF with an oblique derivative boundary condition. This talk is based on [8] and [9].

1 Introduction

Mean curvature flow (MCF) is a geometric evolution equation that describes the motion of hypersurfaces driven by their mean curvature. It has significant applications in material science, image processing, and differential geometry. The traditional numerical approaches for MCF, such as the Almgren–Taylor–Wang (ATW) scheme and the Luckhaus–Sturzenhecker scheme, face challenges when extended to include contact angle conditions. MCF with contact angle conditions is governed by the following system of equations:

$$\begin{cases} V = -\operatorname{div}_{\Gamma(t)} \nu & \text{on} \quad \Gamma(t), \\ \angle(\nu_{\Gamma(t)}, \nu_\Omega) = \theta, & \text{on} \quad \partial\Omega \cap \partial\Gamma(t), \\ \Gamma(0) = \Gamma_0. \end{cases} \tag{1.1}$$

Here, ν denotes a unit vector field of $\Gamma(t)$, and V denotes the normal velocity of each point on $\Gamma(t)$ in the direction of ν; $\operatorname{div}_{\Gamma(t)}$ denotes the surface divergence of ν on $\Gamma(t)$ so that $-\operatorname{div}_{\Gamma(t)} \nu$ equals the ($(d-1)$-times) mean curvature of $\Gamma(t)$ in the direction of ν. Thus, the first equation of (1.1) is nothing but the mean curvature flow equation. The second equation of (1.1) is the boundary condition. The symbol $\angle(\nu, \nu_\Omega)$ denotes the angle between ν and ν_Ω, where ν_Ω denotes the outward unit normal vector field of $\partial\Omega$. The function $\theta : \partial\Omega \to (0, \pi)$ is given and prescribes the contact angle $\angle(\nu, \nu_\Omega)$.

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2 Extension of Chambolle's Scheme

2.1 Capillary ATW-Type Scheme

Bellettini and Kholmatov [5] introduced the capillary ATW energy functional defined by:

$$\mathcal{A}_\beta(F, F_0, \lambda) := \mathcal{C}_\beta(F) + \lambda \int_{F\Delta F_0} \mathrm{dist}(\cdot, \partial F_0)\, d\mathcal{L}^d,$$

where $\mathcal{C}_\beta(F)$ is the capillary functional, incorporating both the surface integral and boundary integral terms:

$$\mathcal{C}_\beta(F) := \int_{\mathbb{R}^d_+} |\nabla \chi_F| + \int_{\partial\mathbb{R}^d_+} \beta\gamma\chi_F \, d\mathcal{H}^{d-1}.$$

Here, $\gamma : BV(\mathbb{R}^d_+) \to L^1(\partial\mathbb{R}^d_+)$ is the trace operator, and β is a function that represents the prescribed contact angle on $\partial\mathbb{R}^d_+$, and χ_F denotes the characteristic function of F, say $\chi_F(x) = 1$ if $x \in F$ and $\chi_F(x) = 0$ if $x \notin F$. The functional $\mathcal{A}_\beta$ extends the traditional ATW functional [1] to include contact angle conditions. They proved a conditional convergence result of minimizers $T_\lambda(F_0)$ of $\mathcal{A}_\beta(\cdot, F_0, \lambda)$ as $\lambda \downarrow 0$ up to a subsequence. One of the drawbacks of this scheme is that we cannot have the uniqueness of minimizing sequence.

2.2 Capillary Chambolle-Type Scheme

Following the works of Chambolle [6] and Bellettini and Kholmatov [5], we introduce a capillary Chambolle-type scheme for MCF with contact angle conditions in a bounded regular domain $\Omega \subset \mathbb{R}^d$. To this end, we define the capillary Chambolle-type functional $E^\beta_h(u)$ as:

$$E^\beta_h(u) := C_\beta(u) + \frac{1}{2h}\int_\Omega (u - d_{\Omega,E_0})^2 \, d\mathcal{L}^d \qquad \text{for} \quad u \in L^2(\Omega) \cap BV(\Omega),$$

where d_{Ω,E_0} denotes the geodesic signed distance function to $E_0 \subset \Omega$, and $C_\beta(u)$ is the capillary functional defined by

$$C_\beta(u) := \int_\Omega |\nabla u| + \int_{\partial\Omega} \beta\gamma u \, d\mathcal{H}^{d-1}.$$

The scheme is iteratively applied to find a sequence of sets which approximates the solution of the MCF with contact angle conditions. Existence of a minimizer of $E^\beta_h(u)$ should be established by the convexity and the lower semicontinuity of the energy. Though the lower semicontinuity of the capillary functional $C_\beta(u)$ was already established by Modica [14], we give another proof using a tubular domain which generates a coordinate system of $\partial\Omega$, and this methodology is different from that by Modica. The precise statement is as follows:

Theorem 2.1. *The capillary functional $C_\beta(u)$ is lower semicontinuous with respect to $L^1(\Omega)$-topology.*

Thanks to the L^2 term, the energy $E^\beta_h(u)$ is strictly convex, and hence it has a unique minimizer. For $E_0 \subset \Omega$, let $w^h_{E_0}$ be the unique minimizer of $E^\beta_h(u)$. Then, we set $T_h(E_0) := \{w^h_{E_0} \le 0\}$.

3 Properties of the Capillary Chambolle-Type Scheme

The capillary Chambolle-type scheme turns out to be monotonous by a similar argument in Chambolle [6] ([9, Lemma 3.1]). Likewise, we can show that the capillary Chambolle-type scheme is consistent with the capillary ATW scheme whenever the prescribed contact angle β satisfies certain conditions.

Theorem 3.1. *Let Ω be a bounded C^2 domain in $\mathbb{R}^d$. Assume that $\beta \in L^\infty(\partial\Omega)$ satisfies $\|\beta\|_\infty \leq 1$ and $\int_{\partial\Omega} \beta d\mathcal{H}^{d-1} = 0$. Then, for any $E_0 \subset \Omega$, $T_h(E_0)$ is a minimizer of $\mathcal{A}_{-\beta}(\cdot, E_0, 1/h)$.*

It is important to characterize the subdifferential $\partial C_\beta(u)$ of the capillary functional $C_\beta(u)$. Indeed, to construct a numerical scheme, Chambolle [6] alternavely considered the duality problem of the Euler–Lagrange equation of the minimizing problem for $E_h^\beta(u)$ when the interface $\Gamma(t)$ does not meet the boundary of the domain Ω and $\beta \equiv 0$. We have the following characterization of $\partial C_\beta(u)$:

Theorem 3.2. *Let Ω be a bounded C^2 domain in $\mathbb{R}^d$. For any $(u,p) \in L^2(\Omega) \times L^2(\Omega)$, p belongs to $\partial C_\beta(u)$ if and only if there exists $z \in L^\infty(\Omega;\mathbb{R}^d)$ with $\operatorname{div} z \in L^2(\Omega)$ such that*

$$\begin{cases} p = -\operatorname{div} z & \text{in} \quad \mathcal{D}'(\Omega), \\ \beta = -[z \cdot \nu_\Omega], & \mathcal{H}^{d-1}\text{-a.e.} \quad \text{on} \quad \partial\Omega, \\ \int_\Omega (z, Du) = \int_\Omega |\nabla u| & \text{with} \quad \|z\|_\infty \leq 1, \end{cases}$$

where (z, Du) denotes the Anzellotti pair (see [2]).

We note that Theorem 3.2 is crucial to prove a convergence result which will be discussed in Section 5.

4 Numerical computation

In contrast to Chambolle [6], we employ the split Bregman method to compute the variational problem $\min_u E_h^\beta(u)$ numerically. The split Bregman method decomposes the problem into sub-problems involving only an L^1 term and an L^2 term respectively, and these problems are easier to handle computationally. To ensure that β satisfies the assumption of Theorem 3.1 on β for each time step, we set $\beta = -\cos\theta$ in a neighborhood of the contact line $\partial\Omega \cap \partial\Gamma(t)$ and assume that β equals some constant value in the remaining region of $\partial\Omega$ so that $\int_{\partial\Omega} \beta \, d\mathcal{H}^{d-1} = 0$ holds. For the results of the numerical computation, we refer the reader to [8, §6].

5 Convergence Analysis

In this section, we prove a convergence result for the capillary Chambolle-type scheme which has been developed so far. To this end, we consider the level-set equation for MCF with an oblique derivative boundary condition:

$$\begin{cases} u_t = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) & \text{in} \quad \Omega \times (0,T), \\ \langle \nabla u, \nu_\Omega \rangle + \beta |\nabla u| = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(\cdot, 0) = u_0 & \text{in} \quad \overline{\Omega}, \end{cases} \tag{5.1}$$

where $T > 0$ is a time horizon, and u_0 is a given initial function whose zero level set represents the initial interface Γ_0. We observe that if a smooth solution u of Eq. (5.1) exists, then the zero level set of $u(\cdot, t)$ coincides with the interface $\Gamma(t)$ of the MCF with contact angle conditions on setting $\beta = -\cos\theta$. Existence and uniqueness of a viscosity solution to Eq. (5.1) without the boundary condition was developed by Chen, Giga and Goto [7] and Evans and Spruck [12], independently. We refer the reader to a good instruction in the book by Giga [13] and the references therein for the theory of the level-set method.

As in [10, 11], for each $u \in UC(\overline{\Omega})$ and a time step $h > 0$, we define a function operator S_h by

$$S_h u(x) := \sup\{\lambda \in \mathbb{R} \mid x \in T_h(\{u \geq \lambda\})\}, \tag{5.2}$$

where $UC(\overline{\Omega})$ denotes the space of all uniformly continuous functions in $\overline{\Omega}$. In terms of S_h, an approximate solution $u^h : [0, T] \times \overline{\Omega} \to \mathbb{R}$ to (5.1) is defined by

$$u^h(t, x) := S_h^{\lfloor \frac{t}{h} \rfloor} u(x),$$

where $\lfloor k \rfloor$ denotes the largest integer which does not exceed $k \in (0, \infty)$.

Then, our main result is stated as follows:

Theorem 5.1. *Assume that Ω is a bounded convex set in $\mathbb{R}^d$ whose boundary is sufficiently regular so that the comparison principle holds for Eq. (5.1). Suppose that $\beta \in C^1(\partial\Omega)$ and $\|\beta\|_\infty < 1$. Assume that $|\partial_{\partial\Omega}\beta(x)| \leq k(x)$ for all $x \in \partial\Omega$. Here, $k(x)$ denotes the minimal non-negative principal (inward) curvature of $\partial\Omega$ at the point x. Then, u^h uniformly converges to the unique viscosity solution to Eq. (5.1) as $h \to 0$.*

A key step of the proof for Theorem 5.1 is to confirm that the function operator S_h fulfills the following properties:

[**Monotonicity**]

$$S_h u \leq S_h v \quad \text{if} \quad u \leq v. \tag{5.3}$$

[**Translation invariance**]

$$\begin{aligned} &S_h(u + c) = S_h u + c \quad \text{for all} \quad c \in \mathbb{R}, \\ &S_h(0) = 0. \end{aligned} \tag{5.4}$$

[**Consistency**]
For every $\varphi \in C^2(\overline{\Omega})$, and $z \in \Omega$ either $\nabla\varphi(z) \neq \mathbf{0}$ or $\nabla\varphi(z) = \mathbf{0}$ and $\nabla^2\varphi(z) = O$, and $z \in \partial\Omega$ with $\langle \nabla\varphi(z), \nu_\Omega(z) \rangle + \beta(z)|\nabla\varphi(z)| > 0$, it holds that

$$\operatorname*{limsup^*}_{h \to 0} \frac{S_h\varphi(z) - \varphi(z)}{h} \leq -F_*(\nabla\varphi(z), \nabla^2\varphi(z)). \tag{5.5}$$

Moreover, for every $\varphi \in C^2(\overline{\Omega})$, and $z \in \Omega$ either $\nabla\varphi(z) \neq \mathbf{0}$ or $\nabla\varphi(z) = \mathbf{0}$ and $\nabla^2\varphi(z) = O$, and $z \in \partial\Omega$ with $\langle \nabla\varphi(z), \nu_\Omega(z) \rangle + \beta(z)|\nabla\varphi(z)| < 0$, it holds that

$$\operatorname{liminf}_{*h \to 0} \frac{S_h\varphi(z) - \varphi(z)}{h} \geq -F^*(\nabla\varphi(z), \nabla^2\varphi(z)). \tag{5.6}$$

Here, for a function F_h on $\overline{\Omega}$ which is parametrized by $h > 0$, we have used the notation that for $x \in \overline{\Omega}$,

$$\begin{aligned}\limsup_{h\to 0}{}^{*} F_h(x) &:= \lim_{h\to 0} \sup\{F_h(y) \mid |x-y| < \delta,\, 0 < \delta < h\},\\ \mathrm{liminf}_{*h\to 0} F_h(x) &:= \lim_{h\to 0} \inf\{F_h(y) \mid |x-y| < \delta,\, 0 < \delta < h\}.\end{aligned}$$

Moreover, we define the upper(resp, lower) semi-continuous envelope F^* (resp, F_*) of F by

$$\begin{aligned}F^*(p,X) &:= \lim_{\varepsilon\to 0} \sup\{F(q,Y) \mid |p-q| < \varepsilon,\ \|X-Y\|_2 < \varepsilon\},\\ F_*(p,X) &:= \lim_{\varepsilon\to 0} \inf\{F(q,Y) \mid |p-q| < \varepsilon,\ \|X-Y\|_2 < \varepsilon\},\end{aligned}$$

where for a matrix $X = (x_{ij})_{1\le i,j\le d} \in \mathbb{R}^{d\times d}$, $\|X\|_2$ denotes the Hilbert–Schmidt norm of X, i.e., $\|X\|_2 := \sqrt{\sum_{i,j=1}^{d} x_{ij}^2}$.

If the limit equation (5.1) has a comparison principle and S_h satisfies the conditions (5.3)–(5.6), a general theory for the monotone scheme [4, Theorem 2.1] yields the desired result. In our case, we know (5.1) has a comparison principle for viscosity solutions ([9, Theorem 2.2] and [3, Theorem 3.1]), so the main task is to prove (5.3)–(5.6) for S_h [9, Theorem 4.1]. It is not difficult to prove (5.3) thanks to the monotonicity of T_h. The property (5.4) is easy to confirm from the definition of S_h. Main efforts for the convergence result of our scheme are devoted to prove (5.5) and (5.6).

To this end, we first establish a relation between S_h and T_h [9, Lemma 4.1] to interpret the formulae (5.5) and (5.6) in that for T_h. More explicitly, we show

$$\{S_h u \ge \lambda\} = T_h(\{u \ge \lambda\}) \qquad \text{for all} \quad \lambda \in \mathbb{R}. \tag{5.7}$$

The relation (5.7) is expected to hold because of the definition of S_h. One of important criteria to ensure (5.7) is the continuity of T_h [9, Lemma 3.6] which is defined by

$$\bigcap_{n=1}^{\infty} T_h(E_n) = T_h(E) \qquad \text{as} \quad n \to \infty, \tag{5.8}$$

where $\{E_n\}_{n\in\mathbb{N}}$ is an arbitrary non-increasing sequence of sets in $\overline{\Omega}$ with $E = \bigcap_{n=1}^{\infty} E_n$. Thanks to the monotonicity of T_h, we easily see that the left-hand side of (5.8) includes the right-hand side. To show the converse inclusion, we seek a subsequence $\{w_{E_n}^h\}_n$ which uniformly converges to a function w in $\overline{\Omega}$, and observe that $w = w_E^h$ in $\overline{\Omega}$. Since $w_{E_n}^h$ is uniformly bounded with respect to n by a maximum principle, the existence of such a subsequence can be proved by the Ascoli–Arzelà theorem provided that $w_{E_n}^h$ is equi-continuous with respect to $n \in \mathbb{N}$. We eventually know that the limit function w must equal w_E^h by the Lipschitz continuity of the map $L^2(\Omega) \ni g \mapsto w_g^h \in L^2(\Omega)$ [9, Proposition 3.1] and the uniform convergence of d_{Ω,E_n} to $d_{\Omega,E}$.

To derive the equi-continuity of $w_{E_n}^h$, we show that the gradient $\nabla w_{E_n}^h$ is bounded by a constant which is independent of $n \in \mathbb{N}$. For this, we adopt Bernstein's method. Namely, we are led to show that $\frac{1}{2}|\nabla w_{E_n}^h|^2$ is a subsolution to the elliptic problem with an oblique derivative boundary condition:

$$\left\{\begin{array}{rcl} w + h\,\mathrm{div}\,\nabla\phi(\nabla w) &=& g \ \text{ in } \ \Omega,\\ B(\cdot,\nabla w) &=& 0 \ \text{ on } \ \partial\Omega, \end{array}\right. \tag{5.9}$$

where $\phi(p) := |p|$, and $B(x,p) := \langle p, \nu_\Omega(x)\rangle + \beta(x)|p|$ for $x \in \partial\Omega$ and $p \in \mathbb{R}^d$. Then, we apply a comparison principle [9, Lemma 3.2] which is available for this problem. This procedure involving Bernstein's method requires the convexity of Ω and the assumption that the contact angle function β is continuously differentiable on $\partial\Omega$ and its gradient is bounded by the minimal nonnegative principal (inward) curvature of $\partial\Omega$ at each point.

Using the relation between S_h and T_h that we have obtained so far, we represent the formulae (5.5) and (5.6) in terms of the super level sets $E^\varphi_\mu := \{\varphi \geq \mu\}$ with $\mu = \varphi(z)$, and intend to prove that the following locally uniform limit in $\overline{\Omega}$ [9, Proposition 4.2]:

$$\left| \frac{w^h_{E^\varphi_\mu} - d_{\Omega, E^\varphi_\mu}}{h} + \kappa_{E^\varphi_\mu} \right| \to 0 \qquad \text{as} \quad h \to 0. \tag{5.10}$$

For the case when $z \in \Omega$, (5.10) indicates that $w^h_{E^\varphi_\mu}$ approximately solves the inclusion:

$$\frac{w - d_{\Omega, E^\varphi_\mu}}{h} + \partial C_0(w) \ni 0 \qquad \text{in} \quad \Omega,$$

which is nothing but a discretization of the first equation of (5.1). Meanwhile, if $z \in \partial\Omega$, we invoke the assumption that $\|\beta\|_\infty < 1$ to construct a viscosity super(resp, sub)solution to (5.9) which approximates the following inclusion:

$$\frac{w - d_{\Omega, E^\varphi_\mu}}{h} + \partial C_\beta(w) \ni 0 \qquad \text{in} \quad \overline{\Omega}. \tag{5.11}$$

In fact, we deduce from the characterization of $\partial C_\beta(u)$ (see [8, Theorem 2]) that (5.11) is equivalent to that w solves (5.9). In [10, 11], a viscosity super(resp, sub)solutions were constructed by a ball $B(0,R)$ which is expected to approximate E^φ_μ nearby z. This is because Chambolle's scheme yields another ball $B(0,\widetilde{R})$, and this $\widetilde{R}$ can be explicitly computed. Due to the oblique derivative boundary condition, we require another type of subsets in $\overline{\Omega}$ to approximate E^φ_μ. Here, we notice that the characterization of $\partial C_\beta(u)$ again gives rigorous solutions to (5.9) in a short time when β is constant [9, Lemma 4.2]. These rigorous solutions are so-called translating solitons in the literature. In this research, we compare E^φ_μ with these solitons instead of balls. By geometry, this soliton is available not only for $z \in \Omega$ but also for $z \in \partial\Omega$ whenever $\|\beta\|_\infty < 1$. This is the basic idea to prove the main theorem.

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A cell-cell adhesion model and chemotaxis systems

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1 Introduction

The cell sorting phenomenon is a phenomenon in which cells spontaneously migrate to appropriate positions and cooperate with other cells to create structures and functions such as tissues and organs. Understanding the cell sorting phenomenon is an extremely important research topic in the field of life sciences. In particular, during the developmental stage of an organism, cell sorting due to cell adhesion and cell sorting via chemicals are intricately intertwined.

Cells such as neurons communicate with other cells by directly touching them with their long axons, etc., and migrate appropriately. In addition, sensory epithelial cells adopt appropriate positions based on the affinity of adhesive forces between cells that are in direct contact with them. These are taxis in response to contact stimuli, and this property is called haptotaxis. The cell sorting phenomenon derived from cell adhesion can also be called the cell sorting phenomenon due to haptotaxis. In contrast, the property of moving by sensing the concentration gradient of chemicals is called chemotaxis. In connection with these, we consider aggregation/repulsion-diffusion problems involving haptotaxis or chemotaxis in one spatial dimension.

Carrillo et al. [1] proposed the following mathematical model to describe the cell-cell adhesion phenomenon:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(u\frac{\partial}{\partial x}\chi(u)) - \frac{\partial}{\partial x}(g(u)K(u)), \tag{1}$$

where $u = u(x,t)$ is the population density of cells at position x and time t. The first term of the right hand side of (1) describes the movement of cells due to pressure, where $\chi(u)$ denotes the pressure. The second term of the right hand side of (1) represents the movement of cells due to adhesion. Here, the velocity is defined by

$$K(u)(x) = \int_0^1 (u(x+r) - u(x-r))\,\omega(r)\,dr,$$

where the function ω describes how the force is dependent on the distance from x. This controls cell adhesion and repulsion. Accordingly, $K(u)$ implies that each cell counts its surrounding within the sensing radius, that is rescaled to be one, to determine the direction of movement. A typical example of g is $g(u) = u(1-u)$, that indicates that the magnitude of the total force of adhesion and/or repulsion decreases as the density locally at the cell position increases. This term restricts the solution to lie between 0 and 1. This effect is called the density saturation effect or volume filling effect.

This model has been extended to a multi-component version that reproduces the cell sorting phenomenon due to haptotaxis [1]. Moreover, it has been applied to solving real-world problems in life sciences [2, 3, 7].

Shifting the discussion to chemotaxis, the movement of cell populations due to chemotaxis has been studied using the Keller–Segel model. Here, we consider the following problem, where the fundamental movement of cells is described by the aforementioned pressure-dependent diffusion, and chemotaxis is represented by a Keller–Segel type system with a density saturation effect.

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(u\dfrac{\partial}{\partial x}\chi(u)\right) - \dfrac{\partial}{\partial x}(g(u)\dfrac{\partial}{\partial x}(av)), \\ \dfrac{\partial v}{\partial t} = d\dfrac{\partial^2 v}{\partial x^2} - v + u, \end{cases} \tag{2}$$

where $u = u(x,t)$ and $v = v(x,t)$ are the population density of cells and the concentration of the chemical substance, respectively. The positive constant d represents the diffusion coefficient of the chemical. The parameter $a \in \mathbb{R}$ represents the strength of chemotaxis, with $a > 0$ indicating that the chemical acts as an attractant, and $a < 0$ indicating that the chemical acts as a repellent. The second equation in (2) implies that the chemical diffuses, degrades, and is produced by the cells.

Both the haptotaxis model (1) and the chemotaxis model (2) represent cell dispersion and aggregation/repulsion, but there are also significant differences. When $\chi(u) = \frac{\gamma}{\gamma-1}u^{\gamma-1}$ ($\gamma > 1$), the first term on the right-hand side of both (1) and (2) is a porous medium type nonlinear diffusion, which has the property of finite propagation of the solution. This means that if the support of the initial value is compact, the support of the solution also remains compact. Furthermore, the aggregation term in (1) only involves interactions within the sensing radius. As a result, cell populations following (1) can form multiple colonies. On the other hand, in the case of system (2) with $a > 0$, even if multiple colonies are temporarily formed, the long-range interactions mediated by the linearly diffusing chemical eventually lead to the aggregation into a single colony. Despite these significant differences, their behaviors are similar in the short time.

Investigating these differences and similarities is important for understanding cell aggregation/repulsion, cell-cell adhesion, and cell sorting. In this study, we examine the relationship between the nonlocal aggregation-nonlinear diffusion equation and Keller–Segel type local chemotaxis system.

In this work, we consider problems in a bounded domain $\Omega = (-L, L]$ ($L > 0$) with the periodic boundary condition. Therefore, Ω coincides with $\mathbb{R}/(2L\mathbb{Z})$. Defining $\chi(u) = \frac{\gamma}{\gamma-1}u^{\gamma-1}$ ($\gamma > 1$) and an interaction potential $W(x) = W(|x|)$ such that $\frac{\partial W}{\partial x}(x) = \omega(|x|)\frac{x}{|x|}$, (1) can be rewritten as follows:

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} u^\gamma - \frac{\partial}{\partial x}\left(g(u) \frac{\partial}{\partial x} W * u \right) \qquad \text{in } Q_T := \Omega \times (0, T]. \tag{3}$$

Here, $*$ denotes the convolution of two periodic functions in the space variable, namely,

$$(W * u)(x,t) = \int_\Omega W(x-y)u(y,t)dy.$$

As usual, we impose the initial condition $u(0) = u_0$.

Ninomiya, Tanaka, and Yamamoto [5, 6] studied a method to approximate solutions of reaction-diffusion equations with nonlocal reactions using solutions of a FitzHugh–Nagumo type system composed only of local terms. In their study, they have shown that any potential W can be approximated by a linear combination of fundamental solutions of certain equations. Motivated by this result, we consider the following $(M+1)$-component Keller–Segel type system and examine its relationship with (3).

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2}{\partial x^2} u^\gamma - \dfrac{\partial}{\partial x}\left(g(u) \dfrac{\partial}{\partial x} \displaystyle\sum_{j=1}^M a_j v_j \right), \\ \xi \dfrac{\partial v_j}{\partial t} = d_j \dfrac{\partial^2 v_j}{\partial x^2} - v_j + u \quad (j = 1, \cdots, M) \end{cases} \qquad \text{in } Q_T. \tag{4}$$

In this model, the cells produce multiple diffusive and degradable signals, which in turn drive the cells themselves. Here ξ is a positive parameter, $d_j > 0$ is the diffusion coefficient, and $a_j \in \mathbb{R}$ implies the sensitivity of the cells to the j-th chemical. We impose the periodic boundary condition and the initial condition $(u(0), v_1(0), \ldots, v_M(0)) = (u_0, v_{01}, \ldots, v_{0M})$.

The authors [4] examined the relationship between (3) and (4) in the case where the diffusion is linear ($\gamma = 1$) and there is no density saturation effect ($g(u) = u$). We demonstrated that the solutions of these two equations are close if $M \in \mathbb{N}$, $\{a_j\}_{j=1}^M$, $\{d_j\}_{j=1}^M$ and $\xi > 0$ are chosen appropriately. Furthermore, using this Keller-Segel type approximation, we rigorously showed that the destabilization of solutions near equilibrium points of the nonlocal Fokker-Planck equation is very similar to diffusion-driven instability. In this work, we analyze the cases with nonlinear diffusion and the density saturation effect in (3) and (4).

2 Assumptions and main result

In this work, we impose the following assumptions.

(H1) $\gamma \geq 1$.

(H2) $g \in C^2(\mathbb{R})$ satisfies $g(s) = 0$ if $s < 0$ or $s > 1$, and there exists a positive constant L_g such that $|g(s)| \leq L_g |s|$ $(s \in \mathbb{R})$.

(H3) W is a sufficiently smooth, even and $2L$-periodic function satisfying

$$\lim_{x \to 1+0} \frac{d^n}{dx^n} W\left(L - \cosh^{-1}(x)\right) < \infty \quad \text{for all} \quad n \in \mathbb{N}.$$

(H4) $0 \leq u_0 \leq 1$ a.e. in Ω and $v_{0j} \in H^1(\Omega)$ $(j = 1, 2, \dots, M)$.

Problems (3) and (4) are understood in the following weak sense.

Definition 2.1. A function $u \in L^\infty(Q_T) \cap H^1(0, T; H^1(\Omega)^*)$ is said to be a weak solution of initial boundary problem of (3) if it fulfils

$$\begin{aligned} & u^\gamma \in L^2(0, T; H^1(\Omega)) \\ & \int_0^T \left\langle \frac{\partial u}{\partial t}, \varphi \right\rangle + \int_0^T \left\langle \frac{\partial}{\partial x} u^\gamma, \frac{\partial \varphi}{\partial x} \right\rangle - \int_0^T \left\langle g(u) \frac{\partial}{\partial x} W * u, \frac{\partial \varphi}{\partial x} \right\rangle = 0, \\ & \qquad\qquad \text{for all function } \varphi \in L^2(0, T; H^1(\Omega)), \\ & u(0) = u_0. \end{aligned}$$

Here and hereafter, We denote by $\langle \cdot, \cdot \rangle$ both the inner product in $L^2(\Omega)$ and the duality pairing between $H^1(\Omega)^*$ and $H_0^1(\Omega)$.

Definition 2.2. A set $(u, v_1, \dots, v_M)$ is said to be a weak solution of initial boundary problem of (4) if it fulfils

$$\begin{aligned} & u \in L^\infty(Q_T) \cap H^1(0, T; H^1(\Omega)^*) \\ & u^\gamma \in L^2(0, T; H^1(\Omega)) \\ & v_j \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)), \\ & \int_0^T \left\langle \frac{\partial u}{\partial t}, \varphi \right\rangle + \int_0^T \left\langle \frac{\partial}{\partial x} u^\gamma, \frac{\partial \varphi}{\partial x} \right\rangle - \int_0^T \left\langle g(u) \frac{\partial}{\partial x} \sum_{j=1}^M a_j v_j, \frac{\partial \varphi}{\partial x} \right\rangle = 0, \\ & \xi \int_0^T \left\langle \frac{\partial v_j}{\partial t}, \psi_j \right\rangle + d_j \int_0^T \left\langle \frac{\partial v_j}{\partial x}, \frac{\partial \psi_j}{\partial x} \right\rangle + \int_0^T \langle v_j, \psi_j \rangle = \int_0^T \langle u, \psi_j \rangle, \\ & \qquad \text{for all functions } \varphi, \psi_j \in L^2(0, T; H^1(\Omega)) \text{ and for } j = 1, 2, \dots, M, \\ & (u(0), v_1(0), \dots, v_M(0)) = (u_0, v_{01}, \dots, v_{0M}). \end{aligned}$$

We consider the fundamental solutions k_j to the following problem:

$$\begin{cases} -d_j \dfrac{\partial^2 v}{\partial x^2} + v = \delta, \\ \qquad v(-L) = v(L), \\ \quad \dfrac{\partial v}{\partial x}(-L) = \dfrac{\partial v}{\partial x}(L). \end{cases} \tag{5}$$

Here, δ denotes the Dirac delta function.

One of a key lemma of this work is as follows:

Lemma 2.3 ([4])**.** *For any smooth even interaction potential W and $\varepsilon > 0$, there exist $M \in \mathbb{N}$, $\{a_j\}_{j=1}^M$ and $\{d_j\}_{j=1}^M$ such that*

$$\left\| W - \sum_{j=1}^M a_j k_j \right\|_{C^1(\Omega)} < \varepsilon.$$

Here, k_j is the fundamental solution of (5) *with the diffusion coefficient d_j.*

The calculation methods for M, $\{a_j\}_{j=1}^M$ and $\{d_j\}_{j=1}^M$ can be found in [4].

If W_m is an approximation to W in $W^{1,1}(\Omega)$, the weak solution of (3) with W_m as the potential approximates the weak solution of (3) with W as the potential in the sense of $L^2(Q_T)$. An arbitrary W can be approximated by the linear combination of k_j as in Lemma 2.3. For these M, $\{a_j\}_{j=1}^M$ and $\{d_j\}_{j=1}^M$, we set $W_M = \sum_{j=1}^M a_j k_j$. Then, (3) with W_M as the potential coincides with the following parabolic-elliptic Keller–Segel type system:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2}{\partial x^2} u^\gamma - \dfrac{\partial}{\partial x}\left(g(u) \dfrac{\partial}{\partial x} \displaystyle\sum_{j=1}^M a_j v_j \right), \\ 0 = d_j \dfrac{\partial^2 v_j}{\partial x^2} - v_j + u \quad (j = 1, \cdots, M) \end{cases} \quad \text{in } Q_T. \tag{6}$$

Moreover, the weak solutions of (4) and (6) are close if ξ is small enough. By providing proofs for each of these steps, we conclude with the following main result.

Theorem 2.4. *Assume that* (H1)–(H4) *are satisfied. Then, for any $\varepsilon > 0$ and $T > 0$, there exists $M \in \mathbb{N}$, $\{a_j\}_{j=1}^M$, $\{d_j\}_{j=1}^M$ and $\xi > 0$ such that*

$$\|u - u_{M,\xi}\|_{L^2(Q_T)} < \varepsilon.$$

Here, u is the weak solutions of (3) *with an initial datum u_0 and $u_{M,\xi}$ is the first component of the weak solutions of* (4) *with an initial datum $(u_0, k_1 * u_0, \ldots, k_M * u_0)$.*

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THE EFFECT OF PERTURBATIONS ON THE MULTIPLICITY OF EIGENVALUES FOR THE FRACTIONAL LAPLACIAN ON BOUNDED DOMAINS

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ABSTRACT. We consider the Dirichlet eigenvalues of the fractional Laplacian $(-\Delta)^s$, with $s \in (0,1)$, related to a smooth bounded domain Ω. We prove that there exists an arbitrarily small perturbation $\tilde{\Omega} = (I+\psi)(\Omega)$ of the original domain such that all Dirichlet eigenvalues of the fractional Laplacian associated to $\tilde{\Omega}$ are simple. In addition, we prove that for a generic choice of coefficients all the eigenvalues of some non-local operators are also simple. Finally, we give a condition for which a perturbation of the coefficient or of the domain preserves the multiplicity of a given eigenvalue.

1. Introduction and statement of the result

In the last decade, there has been a great deal of interest in using the fractional Laplacian to model diverse physical phenomena. We refer the readers to Di Nezza, Palatucci and Valdinoci's survey paper [3] for a detailed exposition of the function spaces involved in the analysis of the operator, and to the recent Ros-Oton's expository paper [13] for a list of results on Dirichlet problems on bounded domains.

In this paper we will focus on the eigenvalue problem

$$\begin{cases} (-\Delta)^s \varphi_s = \lambda \varphi_s & \text{in } \Omega \\ \varphi_s = 0 & \text{in } \Omega^c = \mathbb{R}^n \setminus \Omega \end{cases}. \tag{1.1}$$

Here Ω is a bounded $C^{1,1}$ domain in $\mathbb{R}^n$ with $n \geq 1$ and $(-\Delta)^s$ with $s \in (0,1)$ is the fractional Laplacian defined, for $u \in C^2_c(\mathbb{R}^n)$, as

$$(-\Delta)^s u = C_{n,s} \mathtt{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dx = C_{n,s} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dx,$$

where $C_{n,s} := s4^s \frac{\Gamma(s+n/2)}{\pi^{n/2}\Gamma(1-s)}$ is a renormalization constant and $B_\varepsilon(x)$ is the ball of radius ε centered in x.

To avoid a priori regularity assumptions, we consider the eigenvalue problem in a weak sense. We consider the space

$$\mathcal{H}^s_0(\Omega) := \{u \in H^s(\mathbb{R}^n) \ : \ u \equiv 0 \text{ on } \Omega^c\},$$

where

$$H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) \ : \ \frac{u(x)-u(y)}{|x-y|^{\frac{n}{2}+s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \right\}.$$

Work in collaboration with M.M. Fall, A.M. Micheletti, A. Pistoia.

On $\mathcal{H}_0^s(\Omega)$ we consider the quadratic form

$$(u,v) \mapsto \mathcal{E}_s^\Omega(u,v) := \frac{C_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}dxdy.$$

Then, we call $\varphi_s \in \mathcal{H}_0^s(\Omega)$ an eigenfunction corresponding to the eigenvalue λ if

$$\mathcal{E}_s^\Omega(\varphi_s, v) = \lambda \int_{\mathbb{R}^n} \varphi_s v dx \quad \forall v \in \mathcal{H}_0^s(\Omega).$$

In the following, to simplify notation, we will omit the renormalization constant $C_{n,s}$.

It is well known (see e.g. [1] and the reference therein for an exhaustive introduction about these topics) that (1.2) admits an ordered sequence of eigenvalues

$$0 < \lambda_{1,s} < \lambda_{2,s} \le \lambda_{3,s} \le \cdots \le \lambda_{1,s} \le \cdots \to +\infty.$$

Since the first eigenvalue is strictly positive, we can endow $\mathcal{H}_0^s(\Omega)$ with the norm

$$\|u\|^2_{\mathcal{H}_0^s(\Omega)} = \mathcal{E}_s^\Omega(u,u).$$

In the local case, i.e. $s = 1$, it is well known (see [11,12]) that all the eigenvalues are simple for *generic* domains Ω.

It is natural to ask if the same results hold true in the non-local case, i.e. $s \in (0,1)$. To study domain perturbations we will consider the space

$$C^1(\mathbb{R}^n, \mathbb{R}^n) := \{\psi : \mathbb{R}^n \to \mathbb{R}^n \ : \ \psi,\ D\psi \text{ continuous and bounded}\}$$

endowed with the norm

$$\|\psi\|_1 = \sup_{x\in\mathbb{R}^n} \left(|\psi(x)| + |D\psi(x)|\right).$$

The first question is: if $\bar\lambda$ is an eigenvalue of multiplicity $\nu > 1$ of the operator $(-\Delta)^s_\Omega$ associated with the domain Ω with Dirichlet boundary condition, and U is an interval such that the intersection of the spectrum of $(-\Delta)^s_\Omega$ with U consist of the only number $\bar\lambda$, there exists a perturbation $\Omega_\psi = (I+\psi)(\Omega)$ of the domain Ω such that the intersection of the spectrum of $(-\Delta)^s_{\Omega_\psi}$ with the interval U consists exactly of ν simple eigenvalues of $(-\Delta)^s_{\Omega_\psi}$? Consequently, a second question arises: do there exist any perturbed domains $\Omega_\psi = (I+\psi)(\Omega)$ such that *all* the eigenvalues of $(-\Delta)^s_{\Omega_\psi}$ are simple?

The answer is affirmative and our main result reads as follows.

Theorem 1. *Let $s \in (0,1)$. Let Ω be a smooth bounded domain with $C^{1,1}$ boundary. Then for any $\varepsilon > 0$ there exists $\psi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, with $\|\psi\|_{C^1} < \varepsilon$, such that all the eigenvalues of the problem*

$$(-\Delta)^s\varphi = \lambda\varphi \text{ in } \Omega_\psi = (I+\psi)(\Omega), \qquad \varphi = 0 \text{ in } \mathbb{R}^n \setminus \Omega_\psi$$

are simple.

In other words, it can be said that all the eigenvalues of the problem (1.1) are simple for *generic* domains Ω, where with generic we mean that, given a domain Ω, there exists at least an arbitrarily close domain $\tilde\Omega = (I+\psi)\Omega$ for which all eigenvalues of (1.1) are simple. As a consequence of Theorem 1, we obtain the simplicity of eigenvalues of the fractional laplacian on intervals.

Corollary 2. *Let $s \in (0,1)$. Then all eigenvalues of the eigenvalue problem*

$$(-\Delta)^s\varphi = \lambda\varphi \quad \text{in } (-1,1), \qquad \varphi = 0 \quad \text{in } \mathbb{R}\setminus(-1,1)$$

are simple.

In the spirit of Theorem 1, we obtain a similar result considering Dirichlet eigenvalue fractional problem with nonconstant coefficients of the type

$$(-\Delta)^s\varphi + a(x)\varphi = \lambda\varphi \text{ in } \Omega, \qquad \varphi = 0 \text{ in } \mathbb{R}^n \setminus \Omega \tag{1.2}$$

and

$$(-\Delta)^s\varphi = \lambda\alpha(x)\varphi \text{ in } \Omega, \qquad \varphi = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \tag{1.3}$$

where $a, \alpha \in C^0(\mathbb{R}^n)$.

In the local case, simplicity of the eigenvalues with respect to a perturbation of the coefficients where proved in [14] and we are able to show the nonlocal counterpart of this result. In particular, we prove that all the eigenvalues of (1.2) and (1.3) are simple for *generic* functions a and α, respectively, in this two results.

Theorem 3. *Let $a \in C^0(\mathbb{R}^n)$ such that $\min_{\overline{\Omega}} a > 0$ or $\|a\|_{C^0(\Omega)}$ is small enough. For any $\varepsilon > 0$ there exists $b \in C^0(\mathbb{R}^n)$, with $\|b\|_{C^0} < \varepsilon$, such that all the eigenvalues of the problem*

$$(-\Delta)^s\varphi + (a(x) + b(x))\,\varphi = \lambda\varphi \text{ in } \Omega, \qquad \varphi = 0 \text{ in } \mathbb{R}^n \setminus \Omega$$

are simple.

Theorem 4. *Let $\alpha \in C^0(\mathbb{R}^n)$ such that $\min_{\overline{\Omega}} \alpha > 0$. For any $\varepsilon > 0$ there exists $\beta \in C^0(\mathbb{R}^n)$, with $\|\beta\|_{C^0} < \varepsilon$, such that all the eigenvalues of the problem*

$$(-\Delta)^s\varphi = \lambda\,(\alpha(x) + \beta(x))\,\varphi \text{ in } \Omega, \qquad \varphi = 0 \text{ in } \mathbb{R}^n \setminus \Omega$$

are simple.

It is also interesting to know if the set of perturbations of the coefficients or of the domain which *preserve* the multiplicity of the eigenvalue has some structure. In this case the results are the following.

Theorem 5. *Assume Ω be an open bounded domain of class $C^{1,1}$. Let λ_0 be an eigenvalue for Problem (1.1) with multiplicity $\nu > 1$, and let $\varphi_1, \dots, \varphi_\nu$ be an L^2-orthonormal basis for the eigenspace relative to λ_0. Let $\psi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and consider the functionals*

$$\psi \mapsto \gamma_{ij}(\psi) := \int_{\partial\Omega} \frac{\varphi_i}{\delta^s}\frac{\varphi_j}{\delta^s}\psi \cdot N, \quad i, j = 1, \dots, \nu \tag{1.4}$$

where $\delta(x) = \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega)$ and N is the exterior normal of $\partial\Omega$

Then the set $\mathscr{I}$ of the ψ's close to 0 in $C^1(\mathbb{R}^n, \mathbb{R}^n)$ such that the problem

$$(-\Delta)^s\varphi + \varphi = \lambda\varphi \text{ in } \Omega_\psi, \qquad \varphi = 0 \text{ in } \Omega_\psi^c$$

in the perturbed domain $\Omega_\psi = (I + \psi)\Omega$ admits an eigenvalue λ_ψ close to λ_0 of the same multiplicity ν is a subset of

$$\mathscr{H} := \left\{\psi \in C^1(\mathbb{R}^n, \mathbb{R}^n) \ : \ \gamma_{ij}(\psi) = 0 \text{ for } i \neq j, \ \gamma_{11}(\psi) = \gamma_{22}(\psi) = \cdots = \gamma_{\nu\nu}(\psi)\right\}.$$

In addition, if the map

$$\begin{aligned} G :& C^1(\mathbb{R}^n, \mathbb{R}^n) \to L(\mathbb{R}^\nu, \mathbb{R}^\nu) \\ G(\psi) &= (\gamma_{ij}(\psi))_{ij} \end{aligned}$$

is such that the span of $G(\psi)$ and the Identity map gives all the $\nu \times \nu$ symmetric matrices, then the set $\mathscr{I}$ is a manifold in $C^1(\mathbb{R}^\nu, \mathbb{R}^\nu)$ of codimension $\frac{\nu(\nu+1)}{2} - 1$.

Theorem 6. *Let λ_0 be an eigenvalue for Problem (1.2) (respectively Problem (1.3)) with multiplicity $\nu > 1$, and let $\varphi_1, \dots, \varphi_\nu$ be an L^2-orthonormal basis for the eigenspace relative to λ_0. Assume that $a \in C^1(\mathbb{R}^n)$ and $\min_{\bar\Omega} a > 0$, or $\|a\|_{C^1(\mathbb{R}^n)}$ small (resp. assume that $a \in C^1(\mathbb{R}^n)$ and $\min_{\bar\Omega} a > 0$). Let $b \in C^1(\mathbb{R}^n)$ and consider the functionals*

$$b \mapsto \gamma_{ij}(b) := \int_\Omega b\varphi_i\varphi_j, \quad i,j = 1,\dots,\nu \tag{1.5}$$

Then the set $\mathscr{I}$ of the b's close to 0 in $C^1(\mathbb{R}^n)$ such that the perturbed problem

$$(-\Delta)^s\varphi + (a(x) + b(x))\,\varphi = \lambda\varphi \text{ in } \Omega, \qquad \varphi = 0 \text{ in } \Omega^c$$

(respectively $(-\Delta)^s\varphi + \varphi = \lambda\,(a(x) + b(x))\,\varphi$ in $\Omega, \varphi = 0$ in Ω^c) admits an eigenvalue λ_b close to λ_0 of the same multiplicity ν is a subset of

$$\mathscr{H} := \left\{ b \in C^1(\mathbb{R}^n) \ : \ \gamma_{ij}(b) = 0 \text{ for } i \neq j,\ \gamma_{11}(b) = \gamma_{22}(b) = \dots = \gamma_{\nu\nu}(b) \right\}.$$

In addition, if the map

$$\begin{aligned} G :& C^1(\mathbb{R}^n) \to L(\mathbb{R}^\nu, \mathbb{R}^\nu) \\ G(b) &= (\gamma_{ij}(b))_{ij} \end{aligned}$$

is such that the span of $G(b)$ and the Identity map gives all the $\nu \times \nu$ symmetric matrices, then the set $\mathscr{I}$ is a manifold in $C^1(\mathbb{R}^n)$ of codimension $\frac{\nu(\nu+1)}{2} - 1$.
In particular the last claim holds if λ_0 is an eigenvalue of multiplicity $\nu = 2$.

The strategy of the proofs of the above theorems relies on an abstract result which is presented hereafter in Section 2. In particular, Theorem 10 provides us a so called *splitting condition*, which is crucial to find the perturbation term ψ (or b,β) for which all eigenvalues are simple as claimed in Theorem 1 This condition gives also the structure for the space of perturbations which leave an eigenvalues with the same multiplicity,.

Since the domain needs some technical preparation to well define the space in which the abstract Theorem applies, in this abstract we just sketch the proof of Thm. 4, while for the detailed proof we refer to [4, 7]

2. An abstract result

We recall a series of abstract results which holds in general in a Hilbert space X endowed with scalar product $< \cdot, \cdot >_X$. Later, in the paper, we will apply these abstract results to derive a splitting condition for multiple eigenvalues. The proof of these results, are contained in [12, Section 2].

Let

$$F_{ij} := \{A \in L(X,X) \ : \ \text{codim Im}A = i \text{ and } \dim\ker A = j\}$$

be the set of Fredholm operator with indices i and j in the Banach space $L(X,X) := \{A : X \to X \ : \ A \text{ linear and continuous}\}$.

We show first that F_{ij} is a smooth submanifold of codimension ij in $L(X,X)$. It is well known that if $A \in F_{ij}$, there exist closed subspaces $V, W \subset X$ such that

$$X = \ker A \oplus V \text{ and } X = W \oplus \text{Im}A.$$

Let us call $P, Q, \bar P$ and $\bar Q$ the projector on $\ker A, V, W, \text{Im}A$, respectively. It holds

Lemma 7. *We have*

$$L(X,X) = L \oplus \mathcal{V},$$

where

$$\begin{aligned}\mathcal{V} &:= \{T \in L(X,X) \ : \ T(\ker A) \subset \mathrm{Im} A\}\\ L &:= \left\{\bar{P}HP \in L(X,X) \text{ with } H \in L(X,X)\right\}.\end{aligned}$$

Proof. The claim can be showed immediately noticing that $T = \bar{P}TP + \bar{Q}TQ + \bar{P}TQ + \bar{Q}TP$ and that $\bar{Q}TQ + \bar{P}TQ + \bar{Q}TP \in \mathcal{V}$. □

Lemma 8. *We have that F_{ij} is an analytic submanifold of $L(X,X)$. In addition, for any $A \in L(X,X)$, the tangent space in A to F_{ij}, $T_A F_{ij} = \mathcal{V}$.*

We limit ourselves to give the main idea of the proof. Given $A_0 \in F_{ij}$, and given H such that $A_0 + H$ still belongs to F_{ij}, it is possible to write $H = \bar{P}HP + f(V)$ where $V \in \mathcal{V}$ and f is an analytic function. Then F_{ij} near A_0 is a smooth graph on $\mathcal{V}$.

Lemma 9. *Let $A \in F_{ij}$ such that $\ker A \not\subset \mathrm{Im} A$. Then*

$$M = \{A + H + \lambda I \in L(X,X) \ : \ \lambda \in \mathbb{R}, A + H \in F_{ij} \text{ and } H \text{ suff. small}\}$$

is an analytic manifold at $A + \lambda I$, and $T_{A+\lambda I} M = \mathcal{V} \oplus \mathrm{Span} < I >$ where $T_{A+\lambda I} M$ is the tangent space in $A + \lambda I$ to M.

Proof. By definition of $\mathcal{V}$, we have that $I \in \mathcal{V}$ if and only if $\ker A \subset \mathrm{Im} A$, which is not possible by our hypothesis on A. Thus, by Lemma 8 we have that M is a ruled manifold and the thesis follows immediately. □

We can recast the previous result considering $T : X \to X$ a selfadjoint compact operator with an eigenvalue $\bar{\lambda}$ with multiplicity ν. By Riesz theorem we have that $T - \bar{\lambda} I \in F_{\nu\nu}$ and that $\ker(T - \bar{\lambda} I) \cap \mathrm{Im}(T - \bar{\lambda} I) = \{0\}$. Moreover by Lemma 9 if U is a suitable neighborhood of $T - \bar{\lambda} I$ we have that

$$\tilde{M} = \left\{\tilde{T} + \lambda I \in L(X,X) \ : \ \lambda \in \mathbb{R} \text{ and } \tilde{T} \in F_{\nu\nu} \cap U\right\}$$

is a smooth manifold and $T_{T-\bar{\lambda} I} \tilde{M} = \tilde{\mathcal{V}} \oplus \mathrm{Span} < I >$ where

$$\tilde{\mathcal{V}} = \left\{H \in L(X,X) \ : \ H(\ker(T - \bar{\lambda} I)) \subset \mathrm{Im}(T - \bar{\lambda} I)\right\}. \tag{2.1}$$

At this point we are in position to enunciate the main result of this section.

Theorem 10. *Let $T_b : X \to X$ be a selfadjoint compact operator which depends smoothly on a parameter b belonging to a real Banach space B. Let $T_0 = T$ and let T_b be Frechet differentiable in $b = 0$. Let $x_1^0, \ldots, x_\nu^0$ be an orthonormal basis for the eigenspace relative to the eigenvalue $\bar{\lambda}$ of T. If $T_b \in \tilde{M}$ for all b with $\|b\|_{C^0}$ small, then for all b there exist a $\rho = \rho(b) \in \mathbb{R}$ such that*

$$\left\langle T'(0)[b] x_j^0, x_i^0 \right\rangle_X = \rho \delta_{ij} \text{ for } i,j = 1, \ldots, \nu. \tag{2.2}$$

Proof. By Lemma 9 we have that, if $T_b \in \tilde{M}$ for all b, then

$$T'(0)[b] \in \tilde{\mathcal{V}} \oplus \mathrm{Span} < I > .$$

So, by (2.1), for all b, there exists $\bar{\lambda}(b) \in \mathbb{R}$, such that

$$\left[T'(0)[b] - \bar{\lambda}(b) I\right] (\ker(T - \bar{\lambda} I)) \subset \mathrm{Im}(T - \bar{\lambda} I),$$

that is

$$\left\langle \left[T'(0)[b] - \bar{\lambda}(b)I\right] x_j^0, x_i^0 \right\rangle_X = 0$$

for all $i, j = 1, \dots, \nu$, which implies (2.2). □

This theorem says that if condition (2.2) is fulfilled, then the eigenvalue $\bar{\lambda}(b)$ has still multiplicity ν in a neighborhood of $b = 0$.

3. Proof of Theorem 3

Let us call

$$\mathcal{B}^a(u, v) = \mathcal{E}(u, v) + \int_{\mathbb{R}^n} au^2 dx.$$

and, since $\min_\Omega a > 0$ or $\|a\|_{C^0(\Omega)}$ is small enough, we can endow $\mathcal{H}_0^s(\Omega)$ with the norm

$$\|u\|^2_{\mathcal{H}_0^s(\Omega)} = \mathcal{B}^a(u, u) = \mathcal{E}(u, u) + \int_{\mathbb{R}^n} au^2 dx.$$

We call $\varphi^a \in \mathcal{H}_0^s(\Omega)$ an eigenfunction of $((-\Delta)^s + a)$ corresponding to the eigenvalue λ^a. Given the embedding $i : \mathcal{H}_0^s(\Omega) \to L^2(\Omega)$ we consider its adjoint operator, with respect to the scalar product $\mathcal{B}^a$,

$$i^* : L^2(\Omega) \to \mathcal{H}_0^s(\Omega).$$

It holds

$$\mathcal{B}^a\left((i^* \circ i)_a u, v\right) = \mathcal{E}\left((i^* \circ i)_a u, v\right) + \int_\Omega au(i^* \circ i)_a v = \int_\Omega uv, \tag{3.1}$$

and, if $\varphi_k^a \in \mathcal{H}_0^s(\Omega)$ is an eigenfunction of the fractional Laplacian with eigenvalue λ_k^a, then φ_k^a is an eigenfunction of $(i^* \circ i)_a$ with eigenvalue $\mu_k^a := 1/\lambda_k^a$.

In addiction (1.2) admits an ordered sequence of eigenvalues

$$0 < \lambda_1^a < \lambda_2^a \le \lambda_3^a \le \dots \le \lambda_k^a \le \dots \to +\infty$$

and all the eigenvalues λ_k^a depends continuously on a.

In the following, for $b \in C^0(\Omega)$ with $\|b\|_{L^\infty}$ small enough we consider $\mathcal{B}^{a+b}$ and $(i^* \circ i)_{a+b}$ and we put

$$B_b := \mathcal{B}^{a+b} \text{ and } E_b := (i^* \circ i)_{a+b}. \tag{3.2}$$

We have the following lemma.

Lemma 11. *The maps $b \mapsto B_b$ and $b \mapsto E_b$ are differentiable at 0 and it holds*

$$(B'(0)[b]u, v) = \int_\Omega buv,$$

$$0 = \left(B'(0)[b]E_0 u, v\right) + B_0\left(E'(0)[b]u, v\right). \tag{3.3}$$

for all $u, v \in \mathcal{H}_0^s(\Omega)$.

Remark 12. Notice that, by Lemma 11 and by (3.3), it holds

$$-B_0\left(E'(0)[b]u, v\right) = \left(B'(0)[b]E_0 u, v\right) = \int_\Omega b(E_0 u)v = \int_\Omega b\left[(i^* \circ i)_a u\right] v.$$

Remark 13. If $\mu^a = \mu$ is an eigenvalue of the map $E_0 = (i^* \circ i)_a$ with multiplicity $\nu > 1$, and $\varphi_1^a, \dots, \varphi_\nu^a$ are orthonormal eigenvectors associated to μ, then, by the previous remark we have

$$\left(B'(0)[b]E_0\varphi_i^a, \varphi_j^a\right) = \int_\Omega bE_0(\varphi_i^a)\varphi_j^a = -\mu \int_\Omega b\varphi_i^a\varphi_j^a,$$

for all $i, j = 1, \dots, \nu$.

At this point we have all the necessary architecture to apply condition (2.2) to the operator B_b and consequently to prove the splitting property for a chosen multiple eigenvalue. Indeed we have

Proposition 14. *Let $a \in C^0(\mathbb{R}^n)$ be positive on Ω or with $\|a\|_{C^0(\Omega)}$ sufficiently small. Let $\bar{\lambda}$ an eigenvalue of the operator $(-\Delta)^s_\Omega + aI$ on $\mathcal{H}^s_0$ with Dirichlet boundary condition with multiplicity $\nu > 1$. Let U be an open bounded interval such that*

$$\bar{U} \cap \sigma\left((-\Delta)^s_\Omega + aI\right) = \left\{\bar{\lambda}\right\},$$

where $\sigma\left((-\Delta)^s_\Omega + aI\right)$ is the spectrum of $(-\Delta)^s_\Omega + aI$.

Then, there exists $b \in C^0(\mathbb{R}^n)$ such that for

$$\bar{U} \cap \sigma\left((-\Delta)^s_\Omega + (a+b)I\right) = \left\{\lambda^b_1, \dots, \lambda^b_k\right\},$$

where λ^b_i is an eigenvalue of the operator $(-\Delta)^s_\Omega + (a+b)I$. Here $k > 1$ and the multiplicity of λ^b_i is ν_i with $\sum_{i=1}^k \nu_i = \nu$.

Proof. We apply Theorem 10 to the operator $E_b = (i^* \circ i)_{a+b}$ introduced in (3.2).

If μ^{a+b} is an eigenvalue of E_b which has multiplicity ν at $b = 0$ and at any b with $\|b\|_{C^0}$ small, then by condition (2.2) of Theorem 10 we have

$$B_0(E'(0)[b]\varphi_i, \varphi_j) = \rho\delta_{ij} \text{ for some } \rho \in \mathbb{R},$$

where $\{\varphi_i\}_{i=1,\dots,\nu}$ is an L^2-orthonormal basis for the eigenspace relative to μ^a. Then, in light of Remark 13, we should have that for any $b \in C^0$ small, there exists $\rho = \rho(b)$ such that

$$\mu^a \int_\Omega b\varphi_i\varphi_j = \rho(b)\delta_{ij}.$$

Then, in particular, we deduce that

$$\int_\Omega b\varphi_1\varphi_2 = 0 \text{ and } \int_\Omega b\varphi_1^2 = \int_\Omega b\varphi_2^2 \text{ for all } b \in C^0.$$

Thus $\varphi_1\varphi_2 \equiv 0$ and $\varphi_1^2 \equiv \varphi_2^2$ almost everywhere in Ω. Thus $\varphi_1 \equiv \varphi_2 \equiv 0$ a.e. in Ω, which leads us to a contradiction. Then there exists $b \in C^0$ small such that the multiplicity of μ^{a+b} is smaller that ν. Since the eigenvalue μ^{a+b} depends continuosly on b, given a neighborhood U of μ^a, for $\|b\|_{C^0}$ small we have that $\bar{U} \cap \sigma(E_b) = \left\{\mu^{a+b}_1, \dots, \mu^{a+b}_k\right\}$ with ν_i the multiplicity of μ^{a+b}_i, and where $\sum_{i=1}^k \nu_i = \nu$, and $k > 1$. Remebering the definition of E_b and that $\mu^{a+b} = 1/\lambda^{a+b}$ we have the claim. □

The next corollary follows from the previous proposition, after composing a finite number of perturbations.

Corollary 15. *There exists $b \in C^0(\mathbb{R}^n)$ with $\|b\|_{C^0}$ small enough such that*

$$\bar{U} \cap \sigma\left((-\Delta)^s_\Omega + (a+b)I\right) = \left\{\lambda^b_1, \dots, \lambda^b_\nu\right\},$$

where λ^b_i is a simple eigenvalue of the operator $(-\Delta)^s_\Omega + (a+b)I$ with Dirichlet boundary condition.

Now, iterating a numerable times we can prove Theorem 3

Lemma 16. *Given $a \in C^0(\mathbb{R}^n)$ as in the hypotesis of Theorem 3, and a sequence $\{\sigma_l\}$ of positive real numbers there esists*

- *a sequence of functions $\{b_l\} \in C^0(\mathbb{R}^n)$ with $\|b_l\|_{C^0} \le \sigma_l$*

- *a sequence of increasing integer numbers* $\{q_l\}$ *with* $q_l \nearrow +\infty$
- *a sequence of open bounded intervals* $\{U_t\}_{t=1,\dots,q_l}$ *with* $\bar U_i \cap \bar U_j = \emptyset$ *for* $i \neq j$

such that the eigenvalues $\lambda_i^{a+\sum_{j=i}^l b_j}$ *of the operator* $(-\Delta)_\Omega^s + (a + \sum_{j=i}^l b_j)I$ *are simple for* $i = 1,\dots,q_l$ *and* $\lambda_i^{a+\sum_{j=i}^l b_j} \in U_i$ *for all* $i = 1,\dots,q_l$.

Proof. Take $q \in \mathbb{N}$ such that that $\lambda_1^a,\dots,\lambda_q^a$ are simple eigenvalues for $(-\Delta)_\Omega^s + aI$ and that λ_{q+1}^a is the first eigenvalue with multiplicity ν_{q+1}. For $t = 1,\dots,q$ let $\{U_t\}$ open intervals such that $\bar U_i \cap \bar U_j = \emptyset$ for $i \neq j$ and $\lambda_t^a \in U_t$. Let us take W an open interval such that $\bar W \cap \bar U_t = \emptyset$ for all $t = 1,\dots,q$ and $\bar W \cap \sigma((-\Delta)_\Omega^s + aI) = \left\{\lambda_{q+1}^a\right\}$. At this point, by Corollary 15 we can choose $\bar b$ such that $\bar W \cap \sigma((-\Delta)_\Omega^s + (a + \bar b)I$ contains exactly ν_{q+1} simple eigenvalues. Also, we can choose a number σ_{q+1} sufficiently small, with $\|b_{q+1}\|_{C^0} \leq \sigma_{q+1}$ so that $\lambda_t^{a+\bar b} \in U_t$ for all $t = 1,\dots,q$, since the eigenvalues depends continuosly on b. At this point, by iterating this procedure a finite number of times we get the proof. □

At this point we can conclude.

Proof of Theorem 3. Let us take a sequence $\{\sigma_l\}$ with $0 < \sigma_l < \frac{1}{2^l}$, and a sequence b_l associated to σ_l as in the previous theorem. By the choice of σ_l, we have that $\sum_l b_l$ converge to some function b in $C^0(\mathbb{R}^n)$. We claim that all the eigenvalues $(-\Delta)_\Omega^s + (a+b)I$ are simple. By contradiction, suppose that there exists a $\bar q$ such that $\lambda_{\bar q}^{a+b}$ is the first multiple eigenvalue. By Theorem 16 we have that $(-\Delta)_\Omega^s + (a + \sum_{l=1}^{\bar q+1} b_l)I$ has the first $\bar q + 1$ eigenvalues simple, and that there exists $U_1,\dots,U_{\bar q+1}$ open intervals, with disjoint closure, such that $\lambda_t^{a+\sum_{l=1}^{\bar q+1} b_l} \in U_t$ for $t = 1,\dots,\bar q + 1$. On the one hand, $\lambda_{\bar q}^{a+\sum_{l=1}^N b_l} \to \lambda_{\bar q}^{a+b}$ as well as $\lambda_{\bar q+1}^{a+\sum_{l=1}^N b_l} \to \lambda_{\bar q}^{a+b}$ when $N \to \infty$ by continuity of the eigenvalues. On the other and, $\lambda_{\bar q}^{a+\sum_{l=1}^N b_l} \in U_{\bar q}$ and $\lambda_{\bar q+1}^{a+\sum_{l=1}^N b_l} \in U_{\bar q+1}$ for all N, by Theorem 16. So $\lambda_{\bar q}^{a+b} = \lambda_{\bar q+1}^{a+b} \in \bar U_{\bar q} \cap \bar U_{\bar q+1}$ which lead as to a contradiction, and the theorem is proved. □

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