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Doctoral dissertation

Critical points for the spread-out models of self-avoiding walk, lattice trees and lattice animals

(有限記憶自己回避歩行，格子木，格子動物の臨界点について)

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Preface

The goal of this thesis is to study the critical point for four statistical models in upper-critical dimension: memory-τ walk, self-avoiding walk, lattice trees and lattice animals. These are known as statistical-mechanical models for branched polymers. The value of the critical point is one of interesting topics related to the critical phenomena, and it has been studied for various statistical-mechanical models. The lace expansion is useful tool for investigating the critical point in upper-crititical dimension. Also in this thesis, we rely on the lace expansion for our analysis.

First, we explain some background and basic knowledge.

Background

In physics, we refer to substances whose size is large enough for us to recognize as **macroscopic systems**. On the other hand, tiny substances like molecules or atoms, which constitute macroscopic systems are called **microscopic systems**.

The phenomenon caused by the interaction of innumerable microscopic systems is called a **cooperative phenomenon**, and studying the phenomenon is at the center of interest of statistical mechanics. The most prominent cooperative phenomenon is known as a **phase transition**, in which the properties of a substance change qualitatively at a certain value (**critical points**) when parameters such as temperature and pressure are altered. One of the most famous examples of the phase transition is the transformation of water under atmospheric pressure (1013 hPa), turning into ice at 0°C and vapor at 100°C. This transformation results from the interaction of innumerable H_2O molecules contained in water.

Phase transitions has been observed in various statistical mechanical models. Even more interesting is the fact that various physical observabls exhibit anomalous behavior, such as divergence around the critical point, which is called **critical phenomenon**. It is conjectured that the manner of such divergence around the critical point obeys a power law with an exponent called a **critical exponent**. The critical exponents are believed to depend on dimension and symmetry of the model but to be independent of details of it such as lattice structure. This property is known as **universality**. Due to universality, statistical mechanical models can be classified into classes for the value of the critical exponent, which is called the **universality class**. This makes the critical phenomena universal research objectives.

It is still challenging to give a proof for the existence of the critical exponent and its value in low dimension. However, it is proved for various models that the critical exponent in sufficiently high dimension is identical to that of the so called **mean-field model**, in which individual particles or elements uniformly influence their surrounding environment, neglecting the effects of many-body interactions, i.e., the interactions between particles are averaged and the particles behave as if they were in a homogeneous mean field. For example, the mean-field model of the **self-avoiding walk**, a model for walks without no self-intersection, is the **simple random walk**. We call the dimension above which the value of the critical exponent degenerates to the mean-field value the **critical dimension** and it is model-dependent.

The lace expansion is a most powerful tool for understanding the critical phenomena in the upper critical dimension. In fact, the lace expansion played an important role in identifying the value of the critical exponent in $d > d_c$. Brydges and Spencer [7] first introduced the lace expansion. Their lace expansion was for the **weakly self-avoiding walk** which is a model for walks in which self-intersection is allowed but imposed probabilistic penalty. Then, the lace expansion has been successfuly developed to be applied to various statistical mechanics models.

Critical point and critical exponent

The 2**-point function** is most important observable for statistical mechanics, which is defined for each models. The **susceptibility** χ_p is defined by sum of the 2-point function. **It has been known that there is a critical point p_c such that χ_p is finite if and only if $p < p_\mathrm{c}$ and diverges as $p \uparrow p_\mathrm{c}$.** Especially, for the models such that χ_p can be written with a form of infinite series, we can define the critical point p_c by radius of convergence of χ_p. For example, self-avoiding walk, memory-τ walk, lattice trees and animals, which we consider in this thesis, are such models. **The critical point p_c is model-dependent** and the precise value of it has been investigated for various statistical-mechanical models, especially above their respective critical dimension d_c. The divergence of χ_p as $p \uparrow p_c$ is the representative critical phenomena. It is conjectured that there is **the critical exponent** γ such that

$$\chi_p \sim (p_c - p)^{-\gamma},$$

where the notation $\sim$ means $\log \chi_p / \log(p_c - p)^{-\gamma}$ converges to 1 as $p \uparrow p_c$. As an example, let us consider the nearest-neighbor simple random walk of which 1-step transition probability is given by $D(x) = \frac{\mathbb{1}\{\sum_{i=1}^{d} |x_i| = 1\}}{2d}$. Let $0 \le p < 1$. The 2-point function $C_p(x)$

and susceptibility χ_p for the simple random walk are defined by

$$C_p(x) = \sum_{n=0}^{\infty} D^{*n}(x)p^n, \quad \chi_p = \sum_{x\in\mathbb{Z}^d} C_p(x) = \sum_{n=0}^{\infty} p^n = \frac{1}{1-p}. \tag{0.0.1}$$

where $D^{*n}(x)$ denote n-fold convolution of D, i.e., n-step transition probability of the simple random walk from o to x. Therefore, in the case of the simple random walk, $p_c = 1$ and $\gamma = 1$. We call these values the mean-field values for the self-avoiding walk and other models which belong to the same universality class as the self-avoiding walk. For the memory-τ walk, it is known that $\gamma = 1$ for all $d \geq 1$, which means that the universality class is same as the simple random walk. For the self-avoiding walk, it is proven (e.g.,[19, 20]) that $\gamma = 1$ for $d > 4$, however, the values of γ for $d \geq 4$ are still left open. For the lattice trees and the lattice animals, it is proven for both lattice trees and animals (e.g., [16, 35]) that $\gamma = \frac{1}{2}$ for $d > 8$, but for $d \leq 8$, it is still open.

The lace expansion

We use $G_p(x)$ to denote the 2-point function to be considered, of which definition depends on each model. Only in the simple random walk, we use $C_p(x)$ defined in (0.0.1). The starting point of the lace expansion method is a convolution equation for the 2-point function expressed in the following form:

$$G_p(x) = I_p(x) + \sum_{y\in\mathbb{Z}^d} J_p(y)G_p(x-y) \qquad (p < p_c) \tag{0.0.2}$$

where I_p and J_p are functions on $\mathbb{Z}^d$ which are translation invariant and absolutely summable. As understandable examples, we here consider the simple random walk and the self-avoiding walk. The 2-point functions for each models can be written as

$$C_p(x) = \delta_{o,x} + \sum_{y\in\mathbb{Z}^d} pD(y)C_p(x-y), \tag{0.0.3}$$

$$G_p(x) = \delta_{o,x} + \sum_{y\in\mathbb{Z}^d} pD(y)G_p(x-y) + \sum_{y\in\mathbb{Z}^d} \Pi_p(y)G_p(x-y), \tag{0.0.4}$$

where $\Pi_p(x)$ is an alternating series: $\Pi_p(x) = \sum_{n\in\mathbb{N}}(-1)^n \pi_p^{(n)}(x)$, and $\pi_p^{(n)}(x)$ $(n \in \mathbb{N})$ are nonnegative. We call $\{\pi_p^{(n)}\}_{n\in\mathbb{N}}$ the lace-expansion coefficients. As seen in (0.0.3)-(0.0.4), the term in which the lace-expansion coefficients appear is unique to the expansion for $G_p(x)$ because the lace-expansion coefficients arise due to the removal of the local self-avoidance constraint in a self-avoifding walk. If we neglect the third term in (0.0.4), $G_p(x)$ seems to exhibit similar behavior to $C_p(x)$. In $d > d_c$, it is known that the lace expansion coefficients are small enough for the series Π_p to converge to a small constant (see e.g., [37, 52] and reference therein). This can be interpreted intuitively as the self-constraint

Models	m	C	d_c
Self-avoiding walk	1	$\sum_{n=2}^{\infty} U^{*n}(o)$ —[a]	4
Contact Process		$\sum_{n=2}^{\infty} U^{*n}(o)$ —[a]	
Oriented Percolation		$\frac{1}{2}\sum_{n=2}^{\infty} U^{*2n}(o)$ —[a]	
Percolation		$U^{*2}(o) + \sum_{n=3}^{\infty} \frac{n+1}{2} U^{*n}(o)$ —[a]	6
Memory-τ walk		$\sum_{n=2}^{\infty} U^{*n}(o) - \frac{2}{d-2}\left(\frac{d}{2\pi\Sigma_U^2}\right)^{d/2} \tau^{-\frac{d-2}{2}}$ $(d>4)$ —[b]	2
Lattice trees	$1/e$	$\sum_{n=2}^{\infty} \frac{n+1}{2e} U^{*n}(o)$ —[c]	8
Lattice animals		$\sum_{n=2}^{\infty} \frac{n+1}{2e} U^{*n}(o) - \frac{1}{2e^2}\sum_{n=3}^{\infty} U^{*n}(o)$ —[c]	

表 1: The asymptotics of the ctirical point $p_c = m + CL^{-d} + O(L^{-d-1})$ for the spread-out models as $L \uparrow \infty$ in $d > d_c$ (for the memory τ walk, in $d > 4$), where m is the mean-field value and C is the model dependent constant. Here, U^{*n} is the n-fold convolution of the uniform distribution on the d-dimensional ball $\{x \in \mathbb{R}^d : \|x\| \le 1\}$ and Σ_U^2 is its variance. [a] van der Hofstad and Sakai [26], [b] Kawamoto [33], [c] Kawamoto and Sakai [34].

being weak in $d > d_c$. Therefore, in $d > d_c$, the heart of the lace expansion analysis is to show the mean-field behavior of physical quantities representing the 2-point function by bounding the lace-expansion coefficients above by constants. However, in $d \le d_c$, since the constraint has strong effect, it is difficult to show that the lace-expansion coefficients are small enough. This is the main reason why there are still many open problems in $d \le d_c$.

Purpose of this thesis

In this thesis, we consider the **spread-out model**, i.e., the underlying random walk is allowed to take steps only in $\Lambda = \{x \in \mathbb{Z}^d : 0 < \|x\| \le L\}$, where $\|\cdot\|$ is an arbitrary fixed norm on $\mathbb{R}^d$ and L is a given sufficiently large positive constant. The results detailed in this thesis, which the author has worked on, are divided into two subjects: **[I]** to investigate rate of convergence of the critical point for the memory-τ walk in terms of τ, and **[II]** to obtain asymptotics as $L \uparrow \infty$ of the value of the critical point for the lattice trees and the lattice animals. The analysis for both results is highly related to the lace expansion. Below is a brief overview of the two results.

[I]: Rate of convergence of the critical point for memory-τ walk [33] The memory-τ walk is a model for walks with no loops of length τ or less. The self-avoiding walk is defined as the $\tau \uparrow \infty$ limit of the memory-τ walk. We use ω to denote an simple random walk on $\mathbb{Z}^d$ taking steps in Λ. Let $\mathcal{W}^\tau(o,x)$ (resp., $\mathcal{W}^\infty(o,x)$) be the set of memory-τ walk ($\tau < \infty$) (resp., self-avoiding walk ($\tau = \infty$)) from o to x. The 2-point function is defined to be $G_p^\tau(x) = \sum_{\omega \in \mathcal{W}^\tau(o,x)} (p/|\Lambda|)^{|\omega|}$ for $\tau < \infty$ and for

$\tau = \infty$, where $|\omega|$ is the number of steps of ω. It is known that the critical point for memory-τ walk p_c^τ converge to that for self-avoiding walk p_c^∞ as $\tau \uparrow \infty$ (e.g., [37]).

In [33], the author give the estimate for rate of convergence of p_c^τ:

$$p_c^\infty - p_c^\tau = C_d \tau^{-\frac{d-2}{2}} + O(\tau^{-\frac{d-2}{2}}(\log \tau)^{-1}) \quad (\tau \uparrow \infty) \tag{0.0.5}$$

for $d > d_c = 4$ where the exact expression of a constant C_d is identified [33]. This improves the estimate obtained by Madras and Slade [37]: $p_c^\infty - p_c^\tau \le K\tau^{-(1+\delta)}$, where $\delta < (d-4)/2 \wedge 1$ and $K < \infty$ may depend on d, δ, L but not on τ. The proof is based on comparison of the lace expansions for $G_p^\tau(x)$ and $G_p^\infty(x)$.

[II]: The critical points for lattice trees and lattice animals [34] A spread-out lattice animal is defined by $A = (V_A, E_A)$, where V_A is a finite subset of $\mathbb{Z}^d$ and any pair of vertices in V_A are connected by a path of edges $E_A \subset \{\{x,y\} : x - y \in \Lambda\}$. A lattice tree T is a lattice animal with no loops. The 2-point functions $\tau_p(x)$ for lattice trees (resp., lattice animals) is defined by $\tau_p(x) = \sum_{T \in \mathcal{T}_{o,x}} (p/|\Lambda|)^{|E_T|}$, where $\mathcal{T}_{o,x}$ (resp., $\mathcal{A}_{o,x}$) is the set of T with $\{o,x\} \subset V_T$ (resp., A with $\{o,x\} \subset V_A$).

In [34], Sakai and the author show

$$p_c = 1/e + CL^{-d} + O(L^{-d-1}) \tag{0.0.6}$$

for $d > d_c = 8$, where the exact expression of the model-dependent constant C for both models are identified [34]. This improves the estimate on p_c achieved by Penrose [43]: $p_c = 1/e + O(L^{-2d/7} \log L)$ $(d \ge 1)$. The proof is based on a novel use of the lace expansion for $\tau_p(x)$.

Organization

The remainder of this thesis is organized as follows. In Section 1.1, the models are defined precisely and in Section 1.2 we summarize the previous results related to the critical points for several statistical-mechanical models, focusing on memory-τ walk, self-avoiding walk, lattice trees and lattice animals.

In Chapter 2, we introduce the lace expansion, which is key for our analysis for the main results stated in Section 1.2. Based on the derivation introduced by Brydges and Spencer[7], we define the lace-expansion coefficients for the four models defined in Chapter 1.

In Chapter 3, we concentrate on proving the results [**I**] (Theorem 1.2.1 below). First, in Section 3.1.1, we summarize the results in [29] by van der Hofstad and Slade which we heavily use in the proof of the main statement. The core of this chapter is Section 3.2 where we estimate the proportionality constant in Theorem 1.2.1. We divide the leading term which is expressed with the lace-expansion coefficients ($\hat{\Pi}^{\infty,(1)}_{p_c^\infty,>\tau}$ in (3.0.19)) further into two parts: a dominant term and the remainder, which we estimate in Section 3.2.1 and in Section 3.2.2, respectively. In Appendix, we give the proof of Lemma 3.1.5, which is used to approximate variance of D defined by (1.1.1). Finally, in Section 3.3, we discuss the possibilities regarding the extension of Theorem 1.2.1 to other models.

In Chapter 4, we provide the proof of the results [**II**] (Theorem 1.2.2 below). In Section 4.1, we show that p_c is close (up to order L^{-2d}) to p_1 that satisfies the identity $p_1 g_{p_1} = 1$, which is heavily used in the analysis in Sections 4.2 and 4.3. Here, g_p is called 1-point function which we define in Section 1.1.2. Section 4.2 is devoted to evaluating g_{p_1} for lattice trees. The 1-point function is split into two parts, G and H, which are investigated in Sections 4.2.1 and 4.2.2, respectively. Finally, in Section 4.3, we demonstrate how to evaluate the difference between lattice trees and lattice animals.

Notations

Given absolutely summable functions on $\mathbb{Z}^d$, f, g and h, the notation $*$ denotes convolutions on $\mathbb{Z}^d$

$$(f * g)(x) = \sum_{y \in \mathbb{Z}^d} f(y) g(x-y), \tag{0.0.7}$$

and the notation $\hat{}$ denotes Fourier transform and the inverse, i.e.,

$$\hat{h}(k) = \sum_{x \in \mathbb{Z}^d} h(x) e^{ik\cdot x}, \quad h(x) = \int_{[-\pi,\pi]^d} \hat{h}(k) e^{-ik\cdot x} \frac{d^d k}{(2\pi)^d}, \tag{0.0.8}$$

where $k \in [-\pi, \pi]^d$ and $k \cdot x = \sum_{j=1}^d k_j x_j$. We use $\|\cdot\|_\infty$ and $\|\cdot\|_1$ to denote the l_∞ norm and the l_1 norm respectively:

$$\|h\|_\infty = \sup_{x \in \mathbb{Z}^d} h(x), \quad \|h\|_1 = \sum_{x \in \mathbb{Z}^d} |h(x)|, \quad \left\|\hat{h}\right\|_1 = \int_{[-\pi,\pi]^d} |\hat{h}(k)| \frac{d^d k}{(2\pi)^d}. \tag{0.0.9}$$

第1章 Introduction

1.1 Models

In this section, we introduce the spread-out self-avoiding walk, the memory-τ walk (Section 1.1.1), the lattice trees and the lattice animals (Section 1.1.2). First, we here summarize the notations needed in the rest of thesis. Let $\Lambda = \{x \in \mathbb{Z}^d : 0 < \|x\| \leq L\}$, where $\|\cdot\|$ is an arbitrary fixed norm on $\mathbb{R}^d$ and L is a given positive constant. We define a uniform distribution over Λ defined by

$$D(x) = \frac{1}{|\Lambda|}\mathbb{1}_{\{x\in\Lambda\}}, \tag{1.1.1}$$

where $|\Lambda|$ denotes the cardinality of Λ and $\mathbb{1}_{\{.\}}$ is an indicator function. The function D will be used as a transition probability of the underlying random walk. Only for the self-avoiding walk and the memory-τ walk, we restrict the norm $\|\cdot\|$ to sup-norm; $\|x\|_\infty := \max_j |x_j|$. Hence, in this case,

$$D(x) = \frac{\mathbb{1}_{\{0<\|x\|_\infty\leq L\}}}{(2L+1)^d - 1}. \tag{1.1.2}$$

However, we believe that the main result (Theorem 1.2.1 below) holds for any fixed norm on $\mathbb{R}^d$.

The heat-kernel bound is important for our analysis (see e.g., [26, (1.10)] and [29]):

$$\|D^{*n}\|_\infty \leq O(\beta)n^{-\frac{d}{2}} \quad (n \geq 1), \qquad \left\|\hat{D}^{n+2}\right\|_1 \leq O(\beta)(n+1)^{-\frac{d}{2}} \quad (n \geq 0). \tag{1.1.3}$$

By (1.1.3), we obtain

$$\sum_{n=1}^\infty D^{*n}(x) \overset{d\geq 2}{=} O(\beta), \qquad \sum_{n=1}^\infty nD^{*n}(x) \overset{d\geq 4}{=} O(\beta), \qquad \sum_{n=1}^\infty n^2 D^{*n}(x) \overset{d\geq 6}{=} O(\beta). \tag{1.1.4}$$

Moreover, it is also known that

$$1 - \hat{D}(k) \geq \frac{|k|^2}{2\pi^2 d} \quad (k \in [-\pi,\pi]^d). \tag{1.1.5}$$

For (1.1.5), see e.g., [52, Appendix A] and references therein.

We use U^{*n} to denote the n-fold convolution of the uniform distribution on the d-dimensional box $\{x \in \mathbb{R}^d : \|x\| \le 1\}$. For example, if $\|x\| = \|x\|_\infty := \max_j |x_j|$, then, for all $n \in \mathbb{N}$,

$$U(x) = \frac{\mathbb{1}_{\{\|x\|_\infty \le 1\}}}{2^d}, \qquad U^{*(n+1)}(x) = \int_{\mathbb{R}^d} U^{*n}(y)\, U(x-y)\, \mathrm{d}^d y, \tag{1.1.6}$$

$$\Sigma_U^2 := \int U(x)|x|^2 d^d x < \infty. \tag{1.1.7}$$

These quantities (1.1.6)-(1.1.7) are the spread-out limit of the underlying random walk generated by D [26, Section 4]. For example, for $d > 4$,

$$\sum_{n=2}^{\infty} \frac{n+1}{2} D^{*n}(o) = L^{-d} \sum_{n=2}^{\infty} \frac{n+1}{2} U^{*n}(o) + O(L^{-d-1}), \tag{1.1.8}$$

where the error term $O(L^{-d-1})$ is due to Riemann-sum approximation.

1.1.1 The self-avoiding walk and the memory-τ walk

The memory-τ walk is a model for walks with no loops of length τ or less. The self-avoiding walk is defined as the $\tau \uparrow \infty$ limit of the memory-τ walk, i.e., an ensemble of walks with no self-intersections. The memory-τ walk was used to analyze the original lace expansion and a lot of results related to the self-avoiding walk were proved via the memory-τ walk (see, e.g., [7, 50]).

For $n \ge 0$, an n-step spread-out simple random walk on $\mathbb{Z}^d$ taking steps in Λ is defined as a sequence $\omega = (\omega_0, \omega_1, \cdots, \omega_n)$ of sites in $\mathbb{Z}^d$ with $\omega_{i+1} - \omega_i \in \Lambda$ for $i = 1, \cdots, n$. We difine the weight function of ω by

$$W_p(\omega) = \prod_{i=1}^{|\omega|} pD(\omega_i - \omega_{i-1}), \tag{1.1.9}$$

where $|\omega|$ denotes the number of steps of ω. By convention, the empty product in (1.1.9) that arises when $|\omega| = 0$ is defined to be 1. Let $\mathcal{W}_n(x, y)$ be the set of the n-step walks with $\omega_0 = x$ and $\omega_n = y$. In particular, $\mathcal{W}_0(x, y)$ consists of the 0-step walk if $x = y$ and the empty set otherwise. To consider walks which remember self-avoidance constraint only for a time span τ, we difine for $1 \le \tau \le \infty$

$$\mathcal{U}_{st} = -\delta_{\omega_s, \omega_t}, \quad K_\tau[a, b] = \prod_{\substack{a \le s < t \le b \\ |s-t| \le \tau}} (1 + \mathcal{U}_{st}) \tag{1.1.10}$$

and

$$C_{p,n}^{\tau}(x) = \sum_{\omega \in \mathcal{W}_n(o,x)} W_p(\omega) K_\tau[0, n]. \tag{1.1.11}$$

This defines the self-avoiding walk ($\tau = \infty$) and the memory-τ walk ($\tau < \infty$) all at once. Since $K_1[0, |\omega|] = 1$ from the definition of (1.1.1), the case of $\tau = 1$ is identical to the simple random walk. The two-point function is defined to be

$$G_p^\tau(x) = \sum_{n=0}^{\infty} C_{p,n}^\tau(x) = \sum_{\omega \in \mathcal{W}(o,x)} W_p(\omega) K_\tau[0, |\omega|], \tag{1.1.12}$$

where $\mathcal{W}(o,x) = \cup_{n=0}^{\infty} \mathcal{W}_n(o,x)$. The susceptibility χ_p is the sum of the 2-point function

$$\chi_p^\tau = \sum_{x \in \mathbb{Z}^d} G_p^\tau(x) = \sum_{n=0}^{\infty} p^n \hat{C}_{1,n}^\tau, \tag{1.1.13}$$

where $\hat{C}_{1,n}^\tau = \sum_{x \in \mathbb{Z}^d} C_{1,n}^\tau(x)$. The submultiplicative property $\hat{C}_{1,n+m}^\tau \leq \hat{C}_{1,n}^\tau \hat{C}_{1,m}^\tau$ implies the existence of the connective constant

$$\mu_\tau = \lim_{n \to \infty} (\hat{C}_{1,n}^\tau)^{1/n} = \inf_{n \geq 1} (\hat{C}_{1,n}^\tau)^{1/n}. \tag{1.1.14}$$

The susceptibility has the radius of convergence

$$p_c^\tau = \frac{1}{\mu_\tau}. \tag{1.1.15}$$

For example, simple calculation shows $p_c^1 = 1$. This p_c^τ is called the critical points.

1.1.2 The lattice trees and the lattice animals

We consider spread-out lattice animals $A = (V_A, E_A)$, where the vertex set V_A is a finite subset of $\mathbb{Z}^d$ and any pair of vertices in V_A are connected by a path of spread-out edges $E_A \subset \{\{x, y\} : x - y \in \Lambda\}$. A lattice tree is a lattice animal with no loops. We define the weight function for a tree T as

$$W_p(T) = \prod_{\{x,y\} \in E_T} pD(x - y) = \left(\frac{p}{|\Lambda|}\right)^{|E_T|}, \tag{1.1.16}$$

and similarly for a lattice animal A as $W_p(A)$. For a finite set $X \subset \mathbb{Z}^d$, we denote by $\mathcal{T}_X$ (resp., $\mathcal{A}_X$) the set of lattice trees T with $X \subset V_T$ (resp., lattice animals A with $X \subset V_A$); if X consists of a vertex or two, we simply write, e.g., $\mathcal{T}_o$ (for $X = \{o\}$; see Figure 1.1) or $\mathcal{T}_{o,x}$ (for $X = \{o, x\}$; see Figure 1.2).

The generating functions we want to investigate are the 1-point and 2-point functions, defined respectively as

$$g_p = \sum_{T \in \mathcal{T}_o} W_p(T), \qquad \tau_p(x) = \sum_{T \in \mathcal{T}_{o,x}} W_p(T), \tag{1.1.17}$$

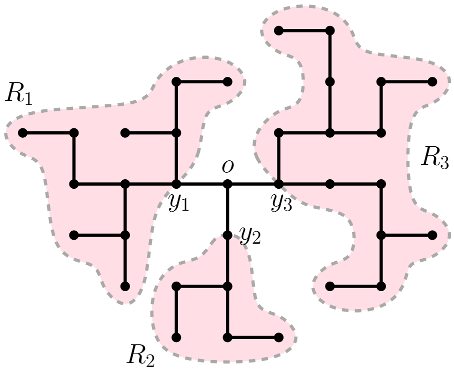


図 1.1: A sample T of $\mathcal{T}_o$. Removal of all edges $\{o, y_j\} \in E_T$ leaves disjoint subtrees R_j rooted at y_j: $V_T \setminus \{o\} = \bigcup_j V_{R_j}$ and $E_T \setminus \bigcup_j \{\{o, y_j\}\} = \bigcup_j E_{R_j}$ (e.g., [34]).

for lattice trees, and similarly defined for lattice animals. The susceptibility χ_p^{LT} is defined by

$$\chi_p^{\text{LT}} = \sum_{x \in \mathbb{Z}^d} \tau_p(x) = \sum_{x \in \mathbb{Z}^d} \sum_{T \in \mathcal{T}_o} \mathbb{1}_{\{x \in V_T\}} W_p(T) = \sum_{T \in \mathcal{T}_o} |V_T| \, W_p(T), \tag{1.1.18}$$

for lattice trees, and similarly for lattice animals (replacing LT and $T \in \mathcal{T}_o$ by LA and $A \in \mathcal{A}_o$). The susceptibility χ_p^{LT} also has a series expansion form:

$$\chi_p^{\text{LT}} = \sum_{n=1}^{\infty} n t_n p^{n-1}, \qquad t_n = \sum_{T \in \mathcal{T}_o} \mathbb{1}_{\{|V_T| = n\}} W_1(T). \tag{1.1.19}$$

As in (1.1.14), $\lim_{n\uparrow\infty} t_n^{1/n}$ exists and the critical point is defined by

$$p_c^{\text{LT}} = \lim_{n \to \infty} (n t_n)^{-1/n}. \tag{1.1.20}$$

1.2 Main results

In this section, we first summarize the estimates of the critical points of various models obtained so far, with particular focus on self-avoiding walk, memory-τ walk, lattice trees and lattice animals, and then introduce our main theorems (Section 1.2.1 and Section 1.2.2). From now on, we drop subscripts from $p_c^{\cdot}$ to simplify the notation and use p_c to denote the critical point of the respective models, except when we have to emphasize the model dependence.

The critical point p_c is model-dependent and much work have been done for investigating the precise value of the critical point for various models in statistical mechanics, especially above their respective upper-critical dimension d_c (see [37, 52] and references therein). As

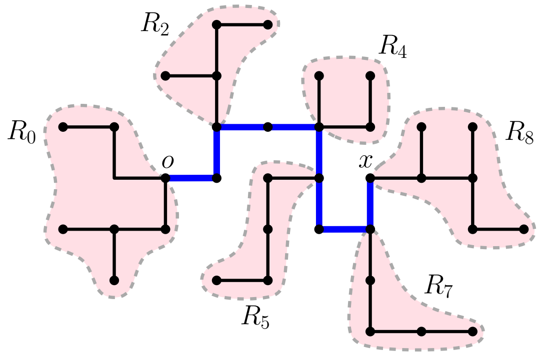


図 1.2: A sample tree in $\mathcal{T}_{o,x}$. Removal of the backbone edges (in blue) yields disjoint subtrees $\{R_j\}$, called ribs. In this example, R_1, R_3 and R_6 are single-vertex trees (e.g., [34]).

we mentioned in Preface, χ_p is finite if and only if $p < p_{\rm c}$ and diverges as $p \uparrow p_{\rm c}$. In particular, it has already been known that in $d > d_c$, χ_p obey the power law $(p_c - p)^{-\gamma}$ with the respective mean-field value of the critical exponent γ.

In [26], van der Hofstad and Sakai applied the lace expansion to the spread-out models (defined by D in (1.1.1)) of self-avoiding walk, percolation, oriented percolation and the contact process, and showed that, for all d bigger than the respective critical dimension $d_{\rm c}$,

$$p_{\rm c} = 1 + CL^{-d} + O(L^{-d-1}), \tag{1.2.1}$$

as $L \uparrow \infty$, where 1 is the mean-field value, and the model-dependent constant C has the following random-walk representation:

$$C = \begin{cases} \displaystyle\sum_{n=2}^{\infty} U^{*n}(o) & \text{[self-avoiding walk, the contact process]}, \\ \displaystyle\frac{1}{2}\sum_{n=2}^{\infty} U^{*2n}(o) & \text{[oriented percolation]}, \\ \displaystyle U^{*2}(o) + \sum_{n=3}^{\infty} \frac{n+1}{2} U^{*n}(o) & \text{[percolation]}. \end{cases} \tag{1.2.2}$$

We want to achieve a similar result for memory-τ walk, lattice trees and lattice animals, i.e., we want to obtain a random-walk representation for the difference between $p_{\rm c}$ and the respective mean-field value 1 and $1/e$, and see how the model-dependence arises in it.

1.2.1 The critical point of self-avoiding walk and memory-τ walk

Since $\hat{C}^{\tau}_{1,n} = \sum_x C^{\tau}_{1,n}(x)$ is decreasing in τ, μ_τ is decreasing in τ (see (1.1.14) and (1.1.15)). Hence, we can conclude

$$p_c^\tau \nearrow p_c^\infty \quad (\tau \uparrow \infty) \tag{1.2.3}$$

(see [37, Lemma 1.2.3]). Various approaches have been known for the estimate of μ_τ for the nearest-neighbor model. The exact value of μ_τ is obtained for small τ (see, e.g., [11, 42]), which gives an upper bound on μ_∞. Our main concern of this thesis is the speed of convergence of (1.2.3). For the nearest-neighbor model $\Lambda = \{x \in \mathbb{Z}^d : \sum_{i=1}^d |x_i| = 1\}$, Kesten showed in [32, Theorem 1] that for each even integer τ and $d \geq \max\{13\tau - 13, 5\}$,

$$\mu_\tau - \mu_\infty = O(|\Lambda|^{-\frac{\tau+2}{2}}), \quad \text{i.e.,} \quad p_c^\infty - p_c^\tau = O(|\Lambda|^{-\frac{\tau+2}{2}}) \quad (d \uparrow \infty) \tag{1.2.4}$$

with $|\Lambda| = 2d$. His approach to show (1.2.4) is to remove loops from a memory-τ self-avoiding path and construct a self-avoiding path. He combined (1.2.4) with the value of μ_4 and obtained

$$\mu_\infty = 1 - |\Lambda|^{-1} - |\Lambda|^{-2} + O(|\Lambda|^{-3}). \tag{1.2.5}$$

Furthermore, in [22], Hara and Slade showed that there is an asymptotic expansion of μ_∞ for the nearest-neighbor model and that all coefficients in expansion are integers by using the lace expansion for the memory-τ walk. Because of Kesten's result (1.2.4), it was sufficient for them to prove the existence of such $|\Lambda|^{-1}$- expansion of μ_τ. They also obtained more improved estimate compared to (1.2.5):

$$\mu_\infty = 1 - |\Lambda|^{-1} - |\Lambda|^{-2} - 3|\Lambda|^{-3} - 16|\Lambda|^{-4} - 102|\Lambda|^{-5} + O(|\Lambda|^{-6}). \tag{1.2.6}$$

The proof is based on the lace expansion for the self-avoiding walk.

For the spread-out model, Madras and Slade proved in [37, Lemma 6.8.6] that for $d > 4$ and sufficiently large L, there is a constant $K < \infty$ such that

$$p_c^\infty - p_c^\tau \leq K\tau^{-(1+\delta)} \tag{1.2.7}$$

where $\delta < (d-4)/2 \wedge 1$ and K may depend on d, δ, L but not on τ. They also showed (1.2.7) for the nearest-neighbor model with sufficiently large d. Their proof of (1.2.7) is based on the lace expansion for the memory-τ walk. They used the estimate (1.2.7) to show that there exists a constant $B < \infty$ such that $\sup_x C^\infty_{p_c^\infty,n}(x) \leq Bn^{-d/2}$ for sufficiently spread-out self-avoiding walk, which is the result about the self-avoiding walk proven via the memory-τ walk. However, in [29], van der Hofstad and Slade succeeded in getting $\sup_x C^\infty_{p_c^\infty,n}(x) \leq O(L^{-d})n^{-d/2}$ without making use of the memory-τ walk.

Our main theorem is the following, which is proved in Chapter 3:

Theorem 1.2.1 ([33]). *Let $d > 4$ and $L \gg 1$. There is a constant C_d such that as $\tau \uparrow \infty$,*

$$p_c^\infty - p_c^\tau = C_d \tau^{-\frac{d-2}{2}} + O(\tau^{-\frac{d-2}{2}}(\log \tau)^{-1}). \tag{1.2.8}$$

Moreover, as $L \uparrow \infty$

$$C_d = \frac{2}{d-2}\left(\frac{d}{2\pi\Sigma_U^2}\right)^{\frac{d}{2}} L^{-d} + O(L^{-d-1}) \tag{1.2.9}$$

where Σ_U^2 is defined by (1.1.7).

From Theorem 1.2.1, we can conclude that the speed of convergence of p_c^τ to p_c^∞ is of order $\tau^{-(d-2)/2}$, which is better than the known result (1.2.7). Moreover, we can identify the exact expression for the proportionality constant up to an error in terms of L^{-d} of order L^{-d-1} and an error in terms of τ of oreder $\tau^{-(d-2)/2}(\log \tau)^{-1}$.

The proof of Theorem 1.2.1 is based on comparison of the lace expansions for the self-avoiding walk and the memory-τ walk. We will introduce the derivation of the lace expansion in Section 2.1, Chapter 2.

1.2.2 The critical point of lattice trees and lattice animals

The best estimate so far on p_c for the spread-out lattice trees and lattice animals was achieved by Penrose [43]. He investigated $(|\Lambda|^{n-1}t_n)/n$ (t_n is defined by (1.1.19)) and showed that the nth root of it is asymptotically $e|\Lambda| + O(|\Lambda|^{5/7}\log|\Lambda|)$ as $|\Lambda| \uparrow \infty$ [43]. Therefore, for large $|\Lambda|$,

$$p_\mathrm{c} \overset{(1.1.20)}{=} \lim_{n\uparrow\infty}(nt_n)^{-1/n} = \frac{|\Lambda|}{e|\Lambda| + O(|\Lambda|^{5/7}\log|\Lambda|)} = \frac{1}{e} + O(|\Lambda|^{-2/7}\log|\Lambda|), \tag{1.2.10}$$

which is true for all dimensions $d \geq 1$. Penrose also claimed in [43, Section 3.1] that p_c for lattice animals obeys the same bound, due to the result of Klarner [30].

A weaker estimate, $p_\mathrm{c} = 1/e + o(1)$ as $L \uparrow \infty$ for all d bigger than the critical dimension $d_\mathrm{c} = 8$, was obtained by Miranda and Slade [39]. In fact, their main concern was to obtain $1/d$ expansions of p_c for the nearest-neighbor models. In [38, 40], they showed that,

$$p_\mathrm{c} = \frac{1}{e} + \frac{3}{2e}|\Lambda|^{-1} + \begin{cases} \dfrac{115}{24e}|\Lambda|^{-2} + o(|\Lambda|^{-2}) & \text{[lattice trees]}, \\ \left(\dfrac{115}{24e} - \dfrac{1}{2e^2}\right)|\Lambda|^{-2} + o(|\Lambda|^{-2}) & \text{[lattice animals]}, \end{cases} \tag{1.2.11}$$

as $|\Lambda| = 2d \uparrow \infty$. The proof is based on the lace expansion for the 2-point function $\tau_p(x)$ and an expansion for the 1-point function g_p based on inclusion-exclusion. Notice that the model-dependence appears only from the $O(|\Lambda|^{-2})$ term, which is due to unit squares contained in g_p for lattice animals, but not in g_p for lattice trees.

Our main theorem is the following, which is proved in Chapter 4:

Theorem 1.2.2 ([34]). *For both models with $d > 8$ and $L \uparrow \infty$,*

$$p_{\mathrm{c}} = \frac{1}{e} + CL^{-d} + O(L^{-d-1}), \tag{1.2.12}$$

where the model-dependent constant C has the following random-walk representation:

$$C_{\mathrm{LT}} = \sum_{n=2}^{\infty} \frac{n+1}{2e} U^{*n}(o), \qquad C_{\mathrm{LA}} = C_{\mathrm{LT}} - \frac{1}{2e^2} \sum_{n=3}^{\infty} U^{*n}(o). \tag{1.2.13}$$

The difference in p_{c} already shows up in the first error term of order L^{-d} for the spread-out models, while it appears in (1.2.11) from the second error term of order d^{-2} for the nearest-neighbor models, as mentioned earlier. This is due to closed loops of length bigger than 2 in g_p for lattice animals. The smallest among such loops for the spread-out model is of length 3 and of order L^{-d}, while that for the nearest-neighbor model is of length 4 and of order d^{-2} (see Lemma 4.3.1 below). Identifying coefficients of the higher order terms for the spread-out models may need more work since they are absorbed in the error term $O(L^{-d-1})$ in (1.12), which is inherent in Riemann-sum approximation, just as mentioned below (1.1.8).

The proof of the above theorem is based on the lace expansion for the 2-point function and detailed analysis of the 1-point function, similarly to the previous work by Miranda and Slade [40]. We will introduce the lace expansion for lattice trees and lattice animals in Section 2.2, Chapter 2. However, the use of the lace expansion is kept to be a minimum in the proof of Theorem 1.2.2, so you do not necessarily have to understand Section 2.2 in order to read the proof of Theorem 1.2.2.

第2章 The lace expansion

The lace expansion was first introduced by Brydges and Spencer [7]. In [7], they proved that the variance of the end point of a T-step weakly self-avoiding walk in $d > 4$ is of order T and the scaling limit behave as Gaussian, where the weakly self-avoiding walk is a model for walks in which self-intersection is allowed but imposed probability penalty. Then, the lace expansion has been successfuly developed to be applied to various statistical mechanics models and has played an important role in showing mean-field critical behavior in high dimensions. For example, the models are self-avoiding walk for $d > 4$ (e.g., [7, 16, 20]), lattice trees and lattice animals for $d > 8$ (e.g., [16, 18, 21]), percolation for $d > 6$ (e.g., [16, 17]), oriented percolation and the contact process for the spatial dimension $d > 4$ (e.g., [41, 46]), and the Ising and φ^4 models for $d > 4$ (e.g., [6, 47, 48, 49]).

In this chapter, we provide the original derivation of the lace expansion by Brydges and Spencer [7] which is the so-called algebraic lace expansion, for memory-τ walk, lattice trees, and lattice animals. The algebraic lace expansion was derived through an expansion and resummation procedure, inspired by the cluster expansion of statistical mechanics. Another approach to the lace expansion is the inclusion-exclusion approach, developed by Hara and Slade (e.g., [51, 22]). The inclusion-exclusion approach is useful for gaining intuition in the derivation, while the algebraic approach is more systematic and considered in a general context. For derivations of the lace expansion for other models, see, e.g., [7, 37, 52].

2.1 The lace expansion for memory-τ walk

Given an interval $[a, b]$ with nonnegative integers a, b, an open interval (s, t) for $a \le s < t \le b$ is abbreviated to st, which we call an edge. A set of edges is called graph, which is denoted by Γ. Then we obtain a rewrite of $K_\tau[a, b]$ for $1 \le \tau \le \infty$ in (1.1.10) as

$$K_\tau[a, b] = \prod_{\substack{a \le s < t \le b \\ |s-t| \le \tau}} (1 + \mathcal{U}_{st}) = \sum_{\Gamma \in \mathcal{B}_\tau[a,b]} \prod_{st \in \Gamma} \mathcal{U}_{st}, \tag{2.1.1}$$

where $\mathcal{U}_{st} = -\delta_{\omega(s),\omega(t)}$ and $\mathcal{B}_\tau[a, b]$ is used to denote the set of all graphs on $[a, b]$, of which the length of any edges is τ or less than τ. From now on, we use the subscript τ for $1 \le \tau \le \infty$ in the notation of the set of graph to indicate that all graph in the set consist of edges of length τ or less. Also, we abbreviate the notation ∞, i.e., for example, $\mathcal{G}_\infty[a,b] = \mathcal{G}[a,b]$. A graph Γ is said to be connected if both a and b are endpoints of edges in Γ, i.e., for any $c \in (a,b)$, there are $s,t \in [a,b]$ such that $s < c < t$ and $st \in \Gamma$. In other words, Γ is connected if $\bigcup_{st\in\Gamma}(s,t) = (a,b)$. The set of connected graph is denoted by $\mathcal{G}_\tau[a,b]$. Let

$$J_\tau[a,b] = \sum_{\Gamma\in\mathcal{G}_\tau[a,b]} \prod_{st\in\Gamma} \mathcal{U}_{st}. \tag{2.1.2}$$

The following lemma is a starting point of the lace expansion for the 2-point function.

Lemma 2.1.1 ([51]). *For any $a < b$,*

$$K_\tau[a,b] = K_\tau[a+1,b] + \sum_{j=a+1}^{b} J_\tau[a,j]K_\tau[j,b]. \tag{2.1.3}$$

(2.1.3) holds for both $\tau < \infty$ and $\tau = \infty$.

Proof of Lemma 2.1.1. We abbreviate the notation τ here.

$$\begin{aligned} K[a,b] &= \sum_{\Gamma\in\mathcal{B}[a,b]} \prod_{st\in\Gamma} \mathcal{U}_{st} \\ &= \underbrace{\sum_{\Gamma\in\mathcal{B}[a+1,b]} \prod_{st\in\Gamma} \mathcal{U}_{st}}_{=K[a+1,b]} + \sum_{\Gamma\in\mathcal{B}[a,b]\setminus\mathcal{B}[a+1,b]} \prod_{st\in\Gamma} \mathcal{U}_{st}. \end{aligned} \tag{2.1.4}$$

We define the largest value of j such that the set of edges of Γ with both ends in $[a,j]$ forms a connected graph on $[a,j]$ by $j(\Gamma)$. Since there is at least one edge containing a in $\Gamma \in \mathcal{B}[a,b]\setminus\mathcal{B}[a+1,b]$, $j(\Gamma)$ always exists and is bigger than $a+1$. Thus, all edges in $\Gamma \in \mathcal{B}[a,b]\setminus\mathcal{B}[a+1,b]$ can be separated into 2 graphs: $\mathcal{G}[a,j]$ and $\mathcal{B}[j,b]$. By summing over $j(\Gamma)$, the second term can be written as

$$\begin{aligned} \sum_{\Gamma\in\mathcal{B}[a,b]\setminus\mathcal{B}[a+1,b]} \prod_{st\in\Gamma} \mathcal{U}_{st} &= \sum_{j=a+1}^{b} \sum_{\substack{\Gamma\in\mathcal{B}[a,b]\setminus\mathcal{B}[a+1,b] \\ j(\Gamma)=j}} \prod_{st\in\Gamma} \mathcal{U}_{st} \\ &= \sum_{j=a+1}^{b} \underbrace{\sum_{\Gamma\in\mathcal{G}[a,j]} \prod_{st\in\Gamma} \mathcal{U}_{st}}_{=J[a,j]} \underbrace{\sum_{\Gamma'\in\mathcal{B}[j,b]} \prod_{s't'\in\Gamma} \mathcal{U}_{s't'}}_{=K[j,b]} \end{aligned} \tag{2.1.5}$$

which completes the proof of (2.1.3). ∎

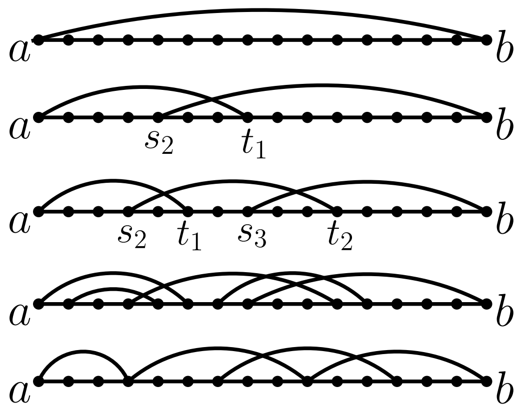


図 2.1: The 3 figures from the top are examples of $L \in \mathcal{L}_\tau^{(N)}[a,b]$ for $N = 1, 2, 3$. An arc represents an edge. The length of each edge is less than or equal to τ. If $b - a$ is strictly bigger than τ, then obviously $\mathcal{L}_\tau^{(1)}[a,b] = \varnothing$. If we apply the procedure to the second figure from the bottom, which is an connected graph, we get the lace for $N = 3$ of the third figure from the top. The bottom figure is an example of a connected graph for lattice trees and lattice animals while this graph is not said to be connected in context of memory-τ or self-avoiding walk (e.g., [33]).

A connected graph L is said to be a lace, if L is a minimally connected graph, i.e., a connected graph for which the removal of any edge results in a disconnected graph. For $N \geq 1$, a lace L on $[a,b]$ is identical to be a set of N edges $L = \{s_1t_1, s_2t_2, \cdots, s_Nt_N\}$ such that $s_1 = a < s_2$, $s_{i+1} < t_i \leq s_{i+2}$ for $1, \cdots, N-2$ and $s_N < t_{N-1} < t_N = b$ (see Figure 2.1). We use $\mathcal{L}_\tau^{(N)}[a,b]$ to denote a set of laces on $[a,b]$ each of which consist of N edges.

Given a connected graph Γ, we define a lace $\mathsf{L}_\Gamma = \{s_1t_1, s_2t_2, \cdots\}$ associated to Γ by determing $t_1, s_1, t_2, s_2, \cdots$ in the following way:

1. $t_1 = \max\{t : at \in \Gamma\}, \quad s_1 = a,$
2. $t_{i+1} = \max\{t : \exists s < t_i \text{ such that } st \in \Gamma\}, \quad s_{i+1} = \min\{s : st_{i+1} \in \Gamma\}.$

We iterate this procedure until $t_{i+1} = b$. Given a lace L, an edge $st \notin L$ such that $\mathsf{L}_{L\cup\{st\}} = L$ is called a compatible edge with L. We use $\mathcal{C}_\tau(L)$ to denote a set of compatible edges with L of length τ or less. For example, see Figure 2.1. Then we can rewrite $J_\tau[a,b]$ in (2.1.2) as

$$J_\tau[a,b] = \sum_{N=1}^{\infty} (-1)^N J_\tau^{(N)}[a,b], \tag{2.1.6}$$

$$J_\tau^{(N)}[a,b] = \sum_{L\in\mathcal{L}_\tau^{(N)}[a,b]} \prod_{st\in L} (-\mathcal{U}_{st}) \prod_{s't'\in\mathcal{C}_\tau(L)} (1 + \mathcal{U}_{s't'}). \tag{2.1.7}$$

Using (2.1.3) and (2.1.6) for (1.1.11) yields the recursion equation

$$C^{\tau}_{p,n+1}(x) = \sum_{y\in\mathbb{Z}^d} pD(y)C^{\tau}_{p,n}(x-y) + \sum_{y\in\mathbb{Z}^d}\sum_{m=2}^{n+1} \pi^{\tau}_{p,m}(y)C^{\tau}_{p,n+1-m}(x-y), \tag{2.1.8}$$

where

$$\pi^{\tau}_{p,n}(x) = \sum_{N=1}^{\infty} (-1)^N \pi^{\tau,(N)}_{p,n}(x), \tag{2.1.9}$$

$$\pi^{\tau,(N)}_{p,n}(x) = \sum_{\omega\in\mathcal{W}_n(o,x)} W_p(\omega)J^{(N)}_{\tau}[0,n]. \tag{2.1.10}$$

By summing both sides of (2.1.8) over $n \geq 0$, we obtain an expansion for the 2-point function:

$$G^{\tau}_p(x) = \sum_{y\in\mathbb{Z}^d} pD(y)G^{\tau}_p(x-y) + \sum_{y\in\mathbb{Z}^d} \Pi^{\tau}_p(y)G^{\tau}_p(x-y) \tag{2.1.11}$$

where

$$\Pi^{\tau}_p(x) = \sum_{m=2}^{\infty} \pi^{\tau}_{p,m}(x) = \sum_{N=1}^{\infty} (-1)^N \Pi^{\tau,(N)}_p(x), \quad \Pi^{\tau,(N)}_p(x) = \sum_{m=2}^{\infty} \pi^{\tau,(N)}_{p,m}(x). \tag{2.1.12}$$

2.2 The lace expansion for lattice trees and lattice animals

We begin with the lattice trees. Since a tree $T \in \mathcal{T}_{o,x}$ can be divided into a unique path $\omega = (\omega_0, \omega_1, \ldots, \omega_{|\omega|})$ from $\omega_0 = o$ to $\omega_{|\omega|} = x$, called a backbone, and disjoint subtrees $R_j \in \mathcal{T}_{\omega_j}$, called ribs (see Figure 1.2), we can rewrite the the 2-point function for lattice tree as

$$\begin{aligned}
\tau_p(x) &\overset{(1.1.17)}{=} \sum_{T\in\mathcal{T}_{o,x}} W_p(T) \\
&= g_p\delta_{o,x} + \sum_{\omega\in\mathcal{W}(o,x)} \left(\frac{p}{|\Lambda|}\right)^{|\omega|} \prod_{j=0}^{|\omega|} \sum_{R_j\in\mathcal{T}_{\omega_j}} W_p(R_j) \prod_{0\le s<t\le|\omega|} \mathbb{1}_{\{V_{R_s}\cap V_{R_t}=\varnothing\}} \\
&= g_p\delta_{o,x} + \sum_{\omega\in\mathcal{W}(o,x)} \left(\frac{p}{|\Lambda|}\right)^{|\omega|} \prod_{j=0}^{|\omega|} \sum_{R_j\in\mathcal{T}_{\omega_j}} W_p(R_j) \underbrace{\prod_{0\le s<t\le|\omega|} (1+\mathcal{U}_{st})}_{:=K[0,|\omega|]}
\end{aligned} \tag{2.2.1}$$

where

$$\mathcal{U}_{st} = -\mathbb{1}_{\{V_{R_s}\cap V_{R_t}\neq\varnothing\}}. \tag{2.2.2}$$

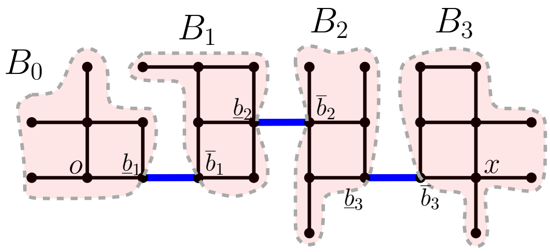


図 2.2: A sample animal in $\mathcal{A}_{o,x}$. Removal of the backbone edges (in blue) yields disjoint subanimals $\{B_j\}$. Each animal B_j contains a double connection between $\overline{b}_j$ and $\underline{b}_{j+1}$, i.e., $B_j \in \mathcal{D}_{\overline{b}_j,\underline{b}_{j+1}}$.

Next we consider lattice animals. Given $A \in \mathcal{A}_{o,x}$, we define a backbone of A by the set of pivotal edges for the event that o is connected to x. In other words, removal of any edges of a backbone is fatal for the connection between o and x (see Figure 2.2). Let $\mathcal{B}$ be an arbitrary sequence of directed edges $b_1, \ldots, b_{|\mathcal{B}|}$:

$$\mathcal{B} = \{(\underline{b}_1, \overline{b}_1), \ldots, (\underline{b}_{|\mathcal{B}|}, \overline{b}_{|\mathcal{B}|})\}. \tag{2.2.3}$$

We use $\mathcal{D}_{o,x}$ to denote the set of all animals such that there is at least one pair of edge-disjoint paths from o to x, where we say "o is doubly connected to x" and use $o \Longleftrightarrow x$ to denote such situation. By convention, we define $\mathcal{D}_{x,x}$ by $\mathcal{A}_x$. Let $\overline{b}_0 = o$ and $\underline{b}_{|\mathcal{B}|+1} = x$. Then the 2-point function for lattice animals can be rewritten as

$$\tau_p(x) \overset{(1.1.17)}{=} \sum_{A \in \mathcal{A}_{o,x}} W_p(A) = g_p \delta_{o,x} + \sum_{\substack{\mathcal{B} \\ |\mathcal{B}| \geq 1}} \left(\frac{p}{|\Lambda|}\right)^{|\mathcal{B}|} \prod_{j=0}^{|\mathcal{B}|} \sum_{B_j \in \mathcal{D}_{\overline{b}_j,\underline{b}_{j+1}}} W_p(B_j) K[0, |\mathcal{B}|] \tag{2.2.4}$$

where in the definition of $K[0, |\mathcal{B}|]$ in (2.2.1)

$$\mathcal{U}_{st} = -\mathbb{1}_{\{V_{B_s} \cap V_{B_t} \neq \varnothing\}}. \tag{2.2.5}$$

To obtain an expansion of the 2-point function for both lattice trees and lattice animals, we use the same terminology introduced in Section 2.1, with a change in the definition of connected graph. A graph Γ is said to be connected if both a and b are endpoints of edges in Γ, i.e., for any $c \in (a, b)$, there are $s, t \in [a, b]$ such that $s < c < t$ with either $st \in \Gamma$ or, $sc \in \Gamma$ and $ct \in \Gamma$. In other words, Γ is connected if $\bigcup_{st \in \Gamma}[s, t] = [a, b]$ (see Figure 2.1). This definition of connected graph for lattice trees and lattice animals is more suitable for dealing with the interaction between ribs. The set of all connected graph is denoted by $\mathcal{G}[a, b]$. From the definiton, $\mathcal{G}[a, b]$ encompasses that for memory-τ walk and self-avoiding walk. In a same way introduced in Section 2.1, we can obtain a lace and compatible edges associated to an connected graph defined in this context. Let

$$J[a,b] = \sum_{N=1}^{\infty}(-1)^N J^{(N)}[a,b], \tag{2.2.6}$$

$$J^{(N)}[a,b] = \sum_{L\in\mathcal{L}^{(N)}[a,b]} \prod_{st\in L}(-\mathcal{U}_{st}) \prod_{s't'\in\mathcal{C}(L)} (1+\mathcal{U}_{s't'}). \tag{2.2.7}$$

By following the proof of Lemma 2.1.1 with the modified definition of connected graph, we obtain the following identity, where we omit the proof.

Lemma 2.2.1 ([51]). *For any $b \geq 1$,*

$$K[0,b] = K[1,b] + \sum_{j=1}^{b-1} J[0,j]K[j+1,b] + J[0,b] \tag{2.2.8}$$

Then, by substituting (2.2.8) to (2.2.1) and (2.2.4) respectively, we obtain an expansion for the 2-point function for both lattice trees and lattice animals:

$$\begin{aligned}\tau_p(x) &= g_p\delta_{o,x} + h_p(x) + \pi_p(x)\\ &\quad + \sum_{u,v}\Big(g_p\delta_{o,u} + h_p(u) + \pi_p(u)\Big)\, pD(v-u)\,\tau_p(x-v),\end{aligned} \tag{2.2.9}$$

where

$$\pi_p(x) = \sum_{n\in\mathbb{N}}(-1)^n\pi_p^{(n)}(x), \tag{2.2.10}$$

$$\pi_p^{(n)}(x) = \begin{cases} \displaystyle\sum_{\substack{\omega\in\mathcal{W}(o,x)\\|\omega|\geq 1}} \left(\frac{p}{|\Lambda|}\right)^{|\omega|} \prod_{j=0}^{|\omega|}\sum_{R_j\in\mathcal{T}_{\omega_j}} W_p(R_j)J^{(n)}[0,|\omega|] & \text{[lattice trees]},\\ \displaystyle\sum_{\substack{\mathcal{B}\\|\mathcal{B}|\geq 1}} \left(\frac{p}{|\Lambda|}\right)^{|\omega|} \prod_{j=0}^{|\mathcal{B}|}\sum_{B_j\in\mathcal{D}_{\overline{b}_j,\underline{b}_{j+1}}} W_p(B_j)J^{(n)}[0,|\mathcal{B}|] & \text{[lattice animals]},\end{cases} \tag{2.2.11}$$

and

$$h_p(x) = \begin{cases} 0 & \text{[lattice trees]},\\ (1-\delta_{o,x})\displaystyle\sum_{A\in\mathcal{D}_{o,x}} W_p(A) & \text{[lattice animals]}.\end{cases} \tag{2.2.12}$$

第3章 Rate of convergence of the critical point for the memory-τ walk

In the rest of paper, we will frequently use

$$\beta = L^{-d}, \tag{3.0.1}$$

which is a small parameter as L increases. Recall our main theorem stated in Section 1.2.1.

Theorem 3.0.1. *Let $d > 4$ and $L \gg 1$. There is a constant C_d such that as $\tau \uparrow \infty$,*

$$p_c^\infty - p_c^\tau = C_d \tau^{-\frac{d-2}{2}} + O(\tau^{-\frac{d-2}{2}}(\log \tau)^{-1}). \tag{3.0.2}$$

Moreover, as $L \uparrow \infty$

$$C_d = \frac{2}{d-2}\left(\frac{d}{2\pi\Sigma_U^2}\right)^{\frac{d}{2}} \beta + O(\beta L^{-1}) \tag{3.0.3}$$

where Σ_U^2 is defined by (1.1.7).

In the rest of Chapter 3, we prove Theorem 3.0.1, based on [33]. All items in this chapter that do not specify a source are from [33]. First, we rewrite $p_c^\infty - p_c^\tau$ with the lace-expansion coefficient and see where the leading term of (3.0.2) arises.

By summing both sides of (2.1.11) over $x \in \mathbb{Z}^d$, and then by solving the equation for χ_p^τ, we formally obtain

$$\chi_p^\tau = \frac{1}{1 - p - \hat{\Pi}_p^\tau}, \tag{3.0.4}$$

where

$$\hat{\Pi}_p^\tau = \sum_{x\in\mathbb{Z}^d} \Pi_p^\tau(x) = \sum_{N=1}^\infty (-1)^N \hat{\Pi}_p^{\tau,(N)}, \quad \hat{\Pi}_p^{\tau,(N)} = \sum_{x\in\mathbb{Z}^d} \Pi_p^{\tau,(N)}(x). \tag{3.0.5}$$

Since χ_p^τ diverges as $p \uparrow p_c^\tau$, we obtain the identity

$$p_c^\tau = 1 - \hat{\Pi}^\tau_{p_c^\tau}. \tag{3.0.6}$$

To prove Theorem 3.0.1, we first rewrite $p_c^\infty - p_c^\tau$ as

$$\begin{aligned} p_c^\infty - p_c^\tau &= (1 - \hat{\Pi}^\infty_{p_c^\infty}) - (1 - \hat{\Pi}^\tau_{p_c^\tau}) \\ &= -(p_c^\infty - p_c^\tau) \frac{\hat{\Pi}^\infty_{p_c^\infty} - \hat{\Pi}^\infty_{p_c^\tau}}{p_c^\infty - p_c^\tau} - \left(\hat{\Pi}^\infty_{p_c^\tau} - \hat{\Pi}^\tau_{p_c^\tau}\right). \end{aligned} \tag{3.0.7}$$

By the mean-value theorem, there is a $p \in (p_c^\tau, p_c^\infty)$ such that $\frac{\hat{\Pi}^\infty_{p_c^\infty} - \hat{\Pi}^\infty_{p_c^\tau}}{p_c^\infty - p_c^\tau} = \partial_p \hat{\Pi}^\infty_p$.

In [28], van der Hofstad and Slade proved that for $d > 4$ and $L \gg 1$,

$$\sum_{x \in \mathbb{Z}^d} |x|^q \pi^{\infty,(N)}_{p,n}(x) \le O(\beta^N) \sigma^q N^q n^{-\frac{d-q}{2}} \quad (p \le p_c^\infty,\ N \ge 1,\ n \ge 2;\ q = 0, 2, 4). \tag{3.0.8}$$

Thus,

$$\begin{aligned} \partial_p \hat{\Pi}^\infty_p = \partial_p \sum_{N=1}^\infty (-1)^N \sum_{m=2}^\infty \sum_{x \in \mathbb{Z}^d} \pi^{\infty,(N)}_{p,m}(x) &= p^{-1} \sum_{N=1}^\infty (-1)^N \sum_{m=2}^\infty m \sum_{x \in \mathbb{Z}^d} \pi^{\infty,(N)}_{p,m}(x) \\ &\overset{(3.0.8)}{=} O(\beta). \end{aligned} \tag{3.0.9}$$

By solving (3.0.7) in terms of $p_c^\infty - p_c^\tau$, we obtain

$$\begin{aligned} p_c^\infty - p_c^\tau &= -\left(\hat{\Pi}^\infty_{p_c^\tau} - \hat{\Pi}^\tau_{p_c^\tau}\right) \frac{1}{1 + \partial_p \hat{\Pi}^\infty_p} \\ &= \left[\left(\hat{\Pi}^{\infty,(1)}_{p_c^\tau} - \hat{\Pi}^{\tau,(1)}_{p_c^\tau}\right) - \sum_{N=2}^\infty (-1)^N \left(\hat{\Pi}^{\infty,(N)}_{p_c^\tau} - \hat{\Pi}^{\tau,(N)}_{p_c^\tau}\right)\right] \frac{1}{1 + \partial_p \hat{\Pi}^\infty_p}. \end{aligned} \tag{3.0.10}$$

Since the length of any edge in $[0, n]$ for $n \le \tau$ is less than τ, we have $J^{(N)}_\infty[0, n] = J^{(N)}_\tau[0, n]$ for $N \ge 1$ and $n \le \tau$. Hence,

$$\pi^{\infty,(N)}_{p,n}(x) = \pi^{\tau,(N)}_{p,n}(x) \quad (N \ge 1,\ n \le \tau). \tag{3.0.11}$$

Moreover, since $\mathcal{L}^{(1)}_\tau[0, n] = \varnothing$ for $n \ge \tau + 1$ (see Figure 2.1), we have

$$J^{(1)}_\tau[0, n] = 0, \quad \text{i.e.,} \quad \hat{\Pi}^{\tau,(1)}_{p_c^\tau, >\tau} = 0 \quad (n \ge \tau + 1) \tag{3.0.12}$$

where for $t \ge 0$

$$\hat{\Pi}^{\tau,(N)}_{p,>t} = \sum_{x \in \mathbb{Z}^d} \sum_{n=t+1}^\infty \pi^{\tau,(N)}_{p,n}(x). \tag{3.0.13}$$

Therefore the first term of (3.0.10) can be decomposed as

$$\begin{aligned}\hat{\Pi}^{\infty,(1)}_{p_c^\tau} - \hat{\Pi}^{\tau,(1)}_{p_c^\tau} &= \hat{\Pi}^{\infty,(1)}_{p_c^\tau,>\tau} \\ &= \hat{\Pi}^{\infty,(1)}_{p_c^\infty,>\tau} - (p_c^\infty - p_c^\tau)\partial_p \hat{\Pi}^{\infty,(1)}_{p',>\tau} \end{aligned} \tag{3.0.14}$$

where the mean-value theorem yields $p' \in (p_c^\tau, p_c^\infty)$ such that $\frac{\hat{\Pi}^{\infty,(1)}_{p_c^\infty,>\tau} - \hat{\Pi}^{\infty,(N)}_{p_c^\tau,>\tau}}{p_c^\infty - p_c^\tau} = \partial_p \hat{\Pi}^{\infty,(1)}_{p',>\tau}$. Also, (3.0.8) immediately implies

$$\hat{\Pi}^{\infty,(1)}_{p_c^\infty,>\tau} = O(\beta)\tau^{-\frac{d-2}{2}}, \quad \partial_p \hat{\Pi}^{\infty,(1)}_{p',>\tau} = O(\beta)\tau^{-\frac{d-4}{2}}. \tag{3.0.15}$$

The following Proposition is important for our analysis.

Proposition 3.0.2. *Let $d > 4$ and $L \gg 1$. As $\tau \uparrow \infty$,*

$$\sum_{N=2}^{\infty} (-1)^N \left(\hat{\Pi}^{\infty,(N)}_{p_c^\tau} - \hat{\Pi}^{\tau,(N)}_{p_c^\tau}\right) = O(\beta^2)\tau^{-\frac{d-2}{2}}. \tag{3.0.16}$$

Proof of Proposition 3.0.2. By (3.0.11),

$$\hat{\Pi}^{\infty,(N)}_{p_c^\tau} - \hat{\Pi}^{\tau,(N)}_{p_c^\tau} = \hat{\Pi}^{\infty,(N)}_{p_c^\tau,>\tau} - \hat{\Pi}^{\tau,(N)}_{p_c^\tau,>\tau} \quad (N \geq 2). \tag{3.0.17}$$

Hence, the bound (3.0.8) immediately implies that the first term of (3.0.16) is $O(\beta^2)\tau^{-(d-2)/2}$. In [28], van der Hofstad and Slade proved (3.0.8) by checking if Assumption G in [29], under which the solutuion to a recursion relation satisfying other assumptions (Assumptions S,D,E in the paper) behaves as a Gaussian, holds for the self-avoifding walk. Since we can easily verify that Assumptions S,D and E holds for the memory-τ walk, to obtain (3.0.8) for $\sum_{x\in\mathbb{Z}^d} \pi^{\tau,(N)}_{p,m}(x)$, it suffices to check if Assumption G also holds for the memory-τ walk. By using the same method as [28], we can confirm that Assumption G is satisfied for the memory-τ walk (see, [28, Proposition 5.1]), which implies the second term of (3.0.17) is also $O(\beta^2)\tau^{-(d-2)/2}$. We will mention this again in Section 3.1.1. ∎

By substituting (3.0.14) and (3.0.16) into (3.0.10), we obtain

$$p_c^\infty - p_c^\tau = \hat{\Pi}^{\infty,(1)}_{p_c^\infty,>\tau} \frac{1}{1+\partial_p \hat{\Pi}^\infty_p} - (p_c^\infty - p_c^\tau)\frac{\partial_p \hat{\Pi}^{\infty,(1)}_{p',>\tau}}{1+\partial_p \hat{\Pi}^\infty_p} + O(\beta^2)\tau^{-\frac{d-2}{2}} \tag{3.0.18}$$

and then, by solving (3.0.18) for $p_c^\infty - p_c^\tau$, we obtain

$$\begin{aligned} p_c^\infty - p_c^\tau &= \hat{\Pi}^{\infty,(1)}_{p_c^\infty,>\tau} \frac{1}{1+\partial_p \hat{\Pi}^{\infty,(1)}_{p',>\tau} + \partial_p \hat{\Pi}^\infty_p} + O(\beta^2)\tau^{-\frac{d-2}{2}} \\ &= \hat{\Pi}^{\infty,(1)}_{p_c^\infty,>\tau} - \underbrace{\hat{\Pi}^{\infty,(1)}_{p_c^\infty,>\tau}}_{\overset{(3.0.15)}{=} O(\beta)\tau^{-\frac{d-2}{2}}} \underbrace{\frac{\partial_p \hat{\Pi}^{\infty,(1)}_{p',>\tau} + \partial_p \hat{\Pi}^\infty_p}{1+\partial_p \hat{\Pi}^{\infty,(1)}_{p',>\tau} + \partial_p \hat{\Pi}^\infty_p}}_{\overset{(3.0.9)(3.0.15)}{=} O(\beta)} + O(\beta^2)\tau^{-\frac{d-2}{2}}. \end{aligned} \tag{3.0.19}$$

This proves that $p_c^\infty - p_c^\tau = O(\beta)\tau^{-\frac{d-2}{2}}$. To identify the multiplicativity constant to β in the $O(\beta)$ term, we further investigate the first term on the RHS of (3.0.19).

3.1 Preliminaries

In this section, we summarize some known results which are used in Chapter 3.

3.1.1 The results by the inductive method

We will heavily use the result achieved by van der Hofstad and Slade in [29]. They gave the conditions for the solution to the recursion relation derived from the lace expansion:

$$f_{n+1}(k;p) = \sum_{m=1}^{n+1} g_m(k;p) f_{n+1-m}(k;p) + e_{n+1}(k;p) \quad (n \geq 0) \tag{3.1.1}$$

to behave as a Gaussian both in Fourier space and in terms of a local central limit theorem. Here, the functions g_m and e_m are given and the solution of (3.1.1) is $f_n(k;p)$, where $k \in [-\pi,\pi]^d$ and $p \geq 0$ are parameters. As for the self-avoiding walk and the memory-τ walk, the Fourier transform of (2.1.8):

$$\hat{C}^\tau_{p,n+1}(k) = p\hat{D}(k)\hat{C}^\tau_{p,n}(k) + \sum_{m=2}^{n+1} \hat{\pi}^\tau_{p,m}(k)\hat{C}^\tau_{p,n+1-m}(k) \tag{3.1.2}$$

is in the form of (3.1.1) with

$$\begin{aligned} f_m(k;p) &= \hat{C}^\tau_{p,m}(k) \ \ (m \geq 0), \quad g_1(k;p) = p\hat{D}(k), \\ g_m(k;p) &= \hat{\pi}^\tau_{p,m}(k) \ \ (m \geq 2), \quad e_m(k;p) = 0 \ \ (m \geq 1). \end{aligned} \tag{3.1.3}$$

There are four conditions which they introduce for a Gaussian behavior of the solution to (3.1.1): Assumptions S,D,G and E. Assumption S is about the symmetries of the functions in (3.1.1) with respect to the underlying lattice symmetries, as well as the uniform boundedness in $k \in [-\pi,\pi]^d$ of f_n for each n. We can easily check that Assumption S holds for the self-avoiding walk and the memory-τ walk. Since $e_n \equiv 0$ $(n \geq 1)$ for the self-avoiding walk and the memory-τ walk, we do not have to check Assumption E. They proved in [29, Appendix] that D defined by (1.1.1) satisfies Assumption D. In [28, Proposition 4.1], van der Hofstad and Slade checked that Assumption G holds for the self-avoiding walk. Assumptions D and G are the following:

Assumption D [29]. For some $\epsilon > 0$, D has $2+2\epsilon$ moments, i.e.,

$$\sum_{x\in\mathbb{Z}^d} |x|^{2+2\epsilon} D(x) < \infty, \tag{3.1.4}$$

and there exist constants C_1, C_2, c_1, c_2 and η such that

$$\sigma^2 := \sum_{x\in\mathbb{Z}^d} |x|^2 D(x) \le C_1 L^2, \qquad \|D\|_\infty \le C_2\beta, \tag{3.1.5}$$

$$c_1 L^2|k|^2 \le 1-\hat{D}(k) \le c_2 L^2|k|^2 \quad (\|k\|_\infty \le L^{-1}), \tag{3.1.6}$$

$$1-\hat{D}(k) > \eta \quad (\|k\|_\infty \ge L^{-1}), \tag{3.1.7}$$

$$1-\hat{D}(k) < 2-\eta \quad (k\in[-\pi,\pi]^d), \tag{3.1.8}$$

where the right inequality of (3.1.6) holds for $k\in[-\pi,\pi]^d$.

Assumption G [29]. There is an L_0, an interval $I\subset[1-\delta,1+\delta]$ with $\delta\in(0,1)$ and a function $K_f\mapsto C_g(K_f)$, such that if the bounds

$$\left\|\hat{D}^2 f_m(\cdot;p)\right\|_1 \le K_f\beta m^{-\frac{d}{2}}, \quad |f_m(0;p)|\le K_f, \quad |\nabla^2 f_m(0;p)|\le K_f\sigma^2 m \tag{3.1.9}$$

hold for some $K_f>1$, $L\ge L_0$, $p\in I$ and for all $1\le m\le n$, then for that L and p, and for all $k\in[-\pi,\pi]^d$ and $2\le m\le n+1$, the following bounds hold:

$$|g_m(k;p)|\le C_g(K_f)\beta m^{-\frac{d}{2}}, \qquad |\nabla^2 g_m(0;p)|\le C_g(K_f)\sigma^2\beta m^{-\frac{d-2}{2}}, \tag{3.1.10}$$

$$|\partial_p g_m(0;p)| = mp^{-1}g_m(0;p)\le C_g(K_f)\beta m^{-\frac{d-2}{2}}, \tag{3.1.11}$$

$$\begin{aligned}&|g_m(0;p)-g_m(k;p)-[1-\hat{D}(k)]\sigma^{-2}\nabla^2 g_m(0;p)|\\ &\qquad \le C_g(K_f)\beta[1-\hat{D}(k)]^{1+\epsilon'} m^{-\frac{d-2-2\epsilon'}{2}}\end{aligned} \tag{3.1.12}$$

where $\epsilon'\in[0,\epsilon]$ with ϵ given in (3.1.4).

Since (3.1.9) are shown to be true for all $m\ge 1$ by the inductive method in [29], all bounds in Assumption G, i.e., (3.1.10)–(3.1.12), also hold for all $m\ge 2$.

From now on, we consider only the self-avoiding walk and drop ∞ from the notation. The following is the summary of the results in [29] which we need for our analysis for the self-avoiding walk:

Theorem 3.1.1 ([29]). *Let $d>4$ and $L\gg 1$. For $n\ge 1$, the followings hold:*

- *Theorem 1.1(a)(d): Fix $\gamma\in(0,1\wedge(d-4)/4\wedge\epsilon)$ and $\delta\in(0,(1\wedge(d-4)/4\wedge\epsilon)-\gamma)$, where ϵ is defined by (3.1.4). Then,*

$$\hat{C}_{p_c,n}\left(k/\sqrt{v\sigma^2 n}\right) = e^{-\frac{k^2}{2d}}\left[1+O(|k|^2 n^{-\delta})+O(\beta n^{-\frac{d-4}{2}})\right] \tag{3.1.13}$$

where

$$v=\frac{\sigma^2+\frac{1}{p_c}\sum_{x\in\mathbb{Z}^d}|x|^2\Pi_{p_c}(x)}{\sigma^2(1+\partial_p\hat{\Pi}_{p_c})}. \tag{3.1.14}$$

The error term is estimated uniformly in $\{k\in\mathbb{R}^d : 1-\hat{D}(k/\sqrt{v\sigma^2 n})\le\gamma n^{-1}\log n\}$.

- *(H4): For $k \in \{k \in [-\pi,\pi]^d : 1-\hat{D}(k) > \gamma n^{-1}\log n\}$,*

$$\left|\hat{C}_{p_c,n}(k)\right| \le C_1'(1-\hat{D}(k))^{-2-\rho}n^{-\frac{d}{2}}, \tag{3.1.15}$$

$$\left|\hat{C}_{p_c,n}(k) - \hat{C}_{p_c,n-1}(k)\right| \le C_2'(1-\hat{D}(k))^{-1-\rho}n^{-\frac{d}{2}}. \tag{3.1.16}$$

  where C_1', C_2' are independent of β and τ and satisfy that $C_1' \gg C_2'$.

The results in [29] also hold for the memory-τ walk. To prove them, it suffices to verify Assumptions G for the memory-τ walk. To do so, we can easily follow the same proof in [28, Proposition 4.1], in which van der Hofstad and Slade verified Assumption G for the self-avoiding walk.

By (3.1.10) and (3.1.12) with $g_m(k;p) = \hat{\pi}_{p,m}(k)$ $(m \ge 2)$, we can conclude the following lemma.

Lemma 3.1.2 ([29]). *There are constants $C_g(K_f), C_g' < \infty$ for $d > 4$ and $L \gg 1$ such that for all $k \in [-\pi,\pi]^d$,*

$$|\hat{\pi}_{p_c,m}(k)| \le C_g(K_f)\beta m^{-\frac{d}{2}} \quad (m \ge 2), \tag{3.1.17}$$

$$\left|\hat{\Pi}_{p_c}\right| \le C_g'\beta, \qquad \left|\sum_{x\in\mathbb{Z}^d}|x|^2\Pi_{p_c}(x)\right| \le C_g'\beta\sigma^2, \tag{3.1.18}$$

$$\left|\hat{\Pi}_{p_c} - \hat{\Pi}_{p_c}(k) - (1-\hat{D}(k))\sigma^{-2}\sum_{x\in\mathbb{Z}^d}|x|^2\Pi_{p_c}(x)\right| \le C_g'\beta(1-\hat{D}(k)). \tag{3.1.19}$$

3.1.2 Quantities at the critical point

Here, we define the Fourier representation of the two-point function at $p = p_c$. Since $G_{p_c}(x)$ is not summable, we can not define $\hat{G}_{p_c}(k)$ in definition of (0.0.8). To understand $\hat{G}_{p_c}(k)$, we take $p \uparrow p_c$ limit of $G_p(x)$. Let

$$J_p(x) = p\left(D(x) + p^{-1}\Pi_p(x)\right). \tag{3.1.20}$$

Notice that $\hat{J}_{p_c}(0) = 1$. It is known as the infrared bound

$$1 - \hat{J}_p(k) \ge K_0|k|^2 \quad (p \le p_c). \tag{3.1.21}$$

In [14, Appendix A], Hara proved the left-continuity of $\hat{\Pi}_p(k)$ for $p \le p_c$. By (3.1.21), the dominated convergence theorem and the left-continuity of $\hat{J}_p(k)$,

$$\begin{aligned} G_{p_c}(x) = \lim_{p \uparrow p_c} G_p(x) &= \lim_{p \uparrow p_c} \int_{[-\pi,\pi]^d} \frac{1}{1-\hat{J}_p(k)} e^{-ik\cdot x} \frac{d^dk}{(2\pi)^d} \\ &= \int_{[-\pi,\pi]^d} \frac{1}{1-\hat{J}_{p_c}(k)} e^{-ik\cdot x} \frac{d^dk}{(2\pi)^d} \\ &= \int_{[-\pi,\pi]^d} e^{-ik\cdot x} \int_0^\infty e^{-t\left[1-\hat{J}_{p_c}(k)\right]} dt \frac{d^dk}{(2\pi)^d}. \end{aligned} \tag{3.1.22}$$

We define $\hat{G}_{p_c}(k)$ by the integrand of (3.1.22). Next lemma is important for our analysis through the rest of Chapter 3.

Lemma 3.1.3. *Let $d > 4$ and $L \gg 1$. Then, for all $k \in [-\pi,\pi]^d$,*

$$\left|\hat{G}_{p_c}(k)\right| \le \frac{1+O(\beta)}{1-\hat{D}(k)}. \tag{3.1.23}$$

Consequently, there is a positive constant $K_1 < \infty$ independent of β such that

$$\left\|(\hat{G}_{p_c})^2\right\|_1 = \int_{[-\pi,\pi]^d} \left|\hat{G}_{p_c}(k)\right|^2 \frac{d^dk}{(2\pi)^d} \le 1 + K_1\beta. \tag{3.1.24}$$

Furthermore,

$$\|D * G_{p_c}\|_\infty = O(\beta), \qquad \left\|D * G_{p_c}^{*2}\right\|_\infty = O(\beta). \tag{3.1.25}$$

Proof of Lemma 3.1.3 A rewrite of $1-\hat{J}_{p_c}(k)$:

$$1-\hat{J}_{p_c}(k) = (1-\hat{D}(k))\left\{p_c + \frac{\hat{\Pi}_{p_c} - \hat{\Pi}_{p_c}(k)}{(1-\hat{D}(k))}\right\} \tag{3.1.26}$$

yields

$$\begin{aligned} \hat{G}_{p_c}(k) - \frac{1}{1-\hat{D}(k)} &= \frac{1}{1-\hat{D}(k)}\left\{\frac{1}{p_c + \frac{\hat{\Pi}_{p_c}-\hat{\Pi}_{p_c}(k)}{(1-\hat{D}(k))}} - 1\right\} \\ &\overset{(3.0.6)}{=} \frac{1}{1-\hat{D}(k)}\left\{\frac{\hat{\Pi}_{p_c} - \frac{\hat{\Pi}_{p_c}-\hat{\Pi}_{p_c}(k)}{(1-\hat{D}(k))}}{p_c + \frac{\hat{\Pi}_{p_c}-\hat{\Pi}_{p_c}(k)}{(1-\hat{D}(k))}}\right\}. \end{aligned} \tag{3.1.27}$$

Therefore, by (3.1.18)–(3.1.19) and (3.1.27), we can conclude (3.1.23). We obtain (3.1.24) as consequences of (1.1.5) and (3.1.23). By using

$$G_p(x) = \delta_{o,x} + \sum_{n=1}^{\infty} C_{p,n}(x) \le \delta_{o,x} + (D * G_p)(x) \quad (p \le p_c) \tag{3.1.28}$$

with (1.1.5) and (3.1.23), we can show (3.1.25). ■

Let

$$\sigma_J^2 := \sum_{x \in \mathbb{Z}^d} |x|^2 J_{p_c}(x) = p_c \left(\sigma^2 + p_c^{-1} \sum_{x \in \mathbb{Z}^d} |x|^2 \Pi_{p_c}(x) \right). \tag{3.1.29}$$

Then, the following lemma holds.

Lemma 3.1.4.

$$\sigma_J^2 L^{-2} = \Sigma_U^2 + O(L^{-1}) \qquad (L \uparrow \infty), \tag{3.1.30}$$

where Σ_U^2 is defined in (1.1.7).

Proof of Lemma 3.1.4 By combining (3.1.29) and (3.1.18) with (1.2.1) for the self-avoiding walk, we obtain

$$\sigma_J^2 = \sigma^2 (1 + O(\beta)). \tag{3.1.31}$$

To approximate σ_J^2, we use the following lemma, which we prove in Appendix, the end of Chapter 3.

Lemma 3.1.5.

$$\frac{\sigma^2}{(L+1/2)^2} = \Sigma_U^2 + O(L^{-2}) \qquad (L \uparrow \infty). \tag{3.1.32}$$

By (3.1.31) and Lemma 3.1.5, we complete the proof of Lemma 3.1.4. ■

3.2 Estimate of the proportionality constant

We investigate $\hat{\Pi}^{(1)}_{p_c,>\tau} \equiv \hat{\Pi}^{\infty,(1)}_{p_c^\infty,>\tau}$, from which we obtain the dominant term in terms of both β and τ of (3.0.2).

Since $J^{(1)}[0,n]$ can be regarded as the indicator function, which is 1 if there are no self-intersections except for $\omega(0) = \omega(n) = o$, and is 0 otherwise, we can rewrite $\hat{\Pi}^{(1)}_{p_c,>\tau}$ as

$$\hat{\Pi}^{(1)}_{p_c,>\tau} = \sum_{n=\tau+1}^{\infty} \sum_{\omega \in \mathcal{W}_n(o,o)} W_{p_c}(\omega) J^{(1)}[0,n] = p_c \sum_{n=\tau+1}^{\infty} (D * C_{p_c,n-1})(o)$$
$$= p_c \int_{[-\pi,\pi]^d} \hat{D}(k) \hat{C}_{p_c,\geq\tau}(k) \frac{d^d k}{(2\pi)^d} \tag{3.2.1}$$

where

$$\hat{C}_{p_c,\geq\tau}(k) = \sum_{n=\tau}^{\infty} \hat{C}_{p_c,n}(k). \tag{3.2.2}$$

First we show the following lemma, which gives a rewrite of $\hat{C}_{p_c,\geq\tau}(k)$.

Lemma 3.2.1. *For $\tau \geq 2$,*

$$\hat{C}_{p_c,\geq\tau}(k) = \hat{G}_{p_c}(k) \left[\hat{C}_{p_c,\tau}(k) - \hat{\pi}_{p_c,\tau}(k) + \hat{E}_{p_c:\tau}(k) \right] \tag{3.2.3}$$

where

$$\hat{E}_{p_c:\tau}(k) = \sum_{m=2}^{\tau-1} \sum_{n=\tau+1-m}^{\tau-1} \hat{\pi}_{p_c,m}(k) \hat{C}_{p_c,n}(k) + \sum_{m=\tau}^{\infty} \sum_{n=0}^{\tau-1} \hat{\pi}_{p_c,m}(k) \hat{C}_{p_c,n}(k). \tag{3.2.4}$$

Proof of Lemma 3.2.1. By summing the both sides of (3.1.2) over $n \geq \tau$, we obtain

$$\left[1 - p_c \hat{D}(k) \right] \hat{C}_{p_c,\geq\tau}(k) = \hat{C}_{p_c,\tau}(k) + \sum_{n=\tau+1}^{\infty} \hat{\pi}_{p_c,n}(k) + \sum_{n=\tau}^{\infty} \sum_{m=2}^{n} \hat{\pi}_{p_c,m}(k) \hat{C}_{p_c,n+1-m}(k). \tag{3.2.5}$$

The last term of RHS of (3.2.5) can be rewritten as

$$\sum_{n=\tau}^{\infty} \sum_{m=2}^{n} \hat{\pi}_{p_c,m}(k) \hat{C}_{p_c,n+1-m}(k) = \sum_{m=2}^{\infty} \sum_{n=\tau \vee m}^{\infty} \hat{\pi}_{p_c,m}(k) \hat{C}_{p_c,n+1-m}(k)$$
$$= \left(\sum_{m=2}^{\tau-1} \sum_{n=\tau}^{\infty} + \sum_{m=\tau}^{\infty} \sum_{n=m}^{\infty} \right) \hat{\pi}_{p_c,m}(k) \hat{C}_{p_c,n+1-m}(k). \tag{3.2.6}$$

Since

$$\begin{aligned}\sum_{m=2}^{\tau-1}\sum_{n=\tau}^{\infty}\hat{\pi}_{p_c,m}(k)\hat{C}_{p_c,n+1-m}(k) &= \sum_{m=2}^{\tau-1}\sum_{n=\tau+1-m}^{\infty}\hat{\pi}_{p_c,m}(k)\hat{C}_{p_c,n}(k)\\ &= \sum_{m=2}^{\tau-1}\left(\sum_{n=\tau}^{\infty}+\sum_{n=\tau+1-m}^{\tau-1}\right)\hat{\pi}_{p_c,m}(k)\hat{C}_{p_c,n}(k), \qquad (3.2.7)\\ \sum_{m=\tau}^{\infty}\sum_{n=m}^{\infty}\hat{\pi}_{p_c,m}(k)\hat{C}_{p_c,n+1-m}(k) &= \sum_{m=\tau}^{\infty}\sum_{n=1}^{\infty}\hat{\pi}_{p_c,m}(k)\hat{C}_{p_c,n}(k)\\ &= \sum_{m=\tau}^{\infty}\left(\sum_{n=0}^{\tau-1}+\sum_{n=\tau}^{\infty}\right)\hat{\pi}_{p_c,m}(k)\hat{C}_{p_c,n}(k)-\sum_{m=\tau}^{\infty}\hat{\pi}_{p_c,m}(k), \qquad (3.2.8)\end{aligned}$$

the RHS of (3.2.6) is equal to

$$\sum_{m=2}^{\infty}\hat{\pi}_{p_c,m}(k)\hat{C}_{p_c,\geq\tau}(k)+\hat{E}_{p_c:\tau}(k)-\sum_{m=\tau}^{\infty}\hat{\pi}_{p_c,m}(k). \qquad (3.2.9)$$

Therefore, by substituting (3.2.9) into (3.2.5), we obtain

$$\underbrace{\left[1-p_c\hat{D}(k)-\sum_{m=2}^{\infty}\hat{\pi}_{p_c,m}(k)\right]}_{=\hat{G}_{p_c}(k)^{-1}}\hat{C}_{p_c,\geq\tau}(k)=\hat{C}_{p_c,\tau}(k)-\hat{\pi}_{p_c,\tau}(k)+\hat{E}_{p_c:\tau}(k), \qquad (3.2.10)$$

which concludes (3.2.3). ■

From (3.2.1) and Lemma 3.2.1, we can decompose $\hat{\Pi}^{(1)}_{p_c,>\tau}$ as

$$\hat{\Pi}^{(1)}_{p_c,>\tau}=p_c\left(M_1-M_2+M_3\right), \qquad (3.2.11)$$

where

$$M_1=\int_{[-\pi,\pi]^d}\hat{D}(k)\hat{C}_{p_c,\tau}(k)\hat{G}_{p_c}(k)\frac{d^dk}{(2\pi)^d}, \qquad (3.2.12)$$

$$M_2=\int_{[-\pi,\pi]^d}\hat{D}(k)\hat{\pi}_{p_c,\tau}(k)\hat{G}_{p_c}(k)\frac{d^dk}{(2\pi)^d}, \qquad (3.2.13)$$

$$M_3=\int_{[-\pi,\pi]^d}\hat{D}(k)\hat{E}_{p_c:\tau}(k)\hat{G}_{p_c}(k)\frac{d^dk}{(2\pi)^d}. \qquad (3.2.14)$$

Let

$$S_n=\left\{k\in[-\pi,\pi]^d:1-\hat{D}(k)\leq\gamma n^{-1}\log n\right\}, \qquad (3.2.15)$$

$$L_n=\left\{k\in[-\pi,\pi]^d:1-\hat{D}(k)>\gamma n^{-1}\log n\right\}, \qquad (3.2.16)$$

where γ is defined by Theorem 3.1.1. M_1 can be decomposed as

$$M_1 = \int_{S_\tau} \hat{D}(k)\hat{C}_{p_c,\tau}(k)\hat{G}_{p_c}(k)\frac{d^dk}{(2\pi)^d} + \int_{L_\tau} \hat{D}(k)\hat{C}_{p_c,\tau}(k)\hat{G}_{p_c}(k)\frac{d^dk}{(2\pi)^d}. \tag{3.2.17}$$

We will show the following two lemmas in Section 3.2.1 and in Section 3.2.2 respectively, by which we can conclude Theorem 3.0.1.

Lemma 3.2.2. *Let $d > 4$ and $L \gg 1$. Fix γ and δ defined in Theorem 3.1.1. Then, for $\varepsilon > 0$, as $\tau \uparrow \infty$,*

$$\begin{aligned}&\int_{S_\tau} \hat{D}(k)\hat{C}_{p_c,\tau}(k)\hat{G}_{p_c}(k)\frac{d^dk}{(2\pi)^d}\\ &\qquad = \frac{2}{d-2}\left(\frac{d}{2\pi\Sigma_U^2}\right)^{\frac{d}{2}}\beta\tau^{-\frac{d-2}{2}}\left[1 + O(L^{-1}) + O(\tau^{-\left(\frac{\gamma C_0}{4dc_2}\wedge\delta\right)}) + o(\beta^{\varepsilon}\tau^{-\frac{d\varepsilon}{2}})\right]\end{aligned} \tag{3.2.18}$$

where the last error term is valid for any $\varepsilon > 0$, and c_2, C_0 appearing in the 3rd term are the constants that satisfy (3.1.6) and

$$\sigma_J^2 L^{-2} \geq C_0 \tag{3.2.19}$$

with σ_J^2 defined by (3.1.29), respectively.

Lemma 3.2.3. *Let $d > 4$ and $L \gg 1$. As $\tau \uparrow \infty$,*

$$\int_{L_\tau} \hat{D}(k)\hat{C}_{p_c,\tau}(k)\hat{G}_{p_c}(k)\frac{d^dk}{(2\pi)^d} = O\left(\beta\right)\tau^{-\frac{d-2}{2}}\left[O((\log\tau)^{-1}) + O(\beta^{\frac{1}{2}})\right], \tag{3.2.20}$$

$$M_2 = O(\beta^2)\tau^{-\frac{d}{2}}, \qquad M_3 = O(\beta^{\frac{3}{2}})\tau^{-\frac{d-2}{2}}. \tag{3.2.21}$$

Proof of Theorem 3.0.1. Summarizing (1.2.1), (3.0.19), (3.2.11), (3.2.17), Lemma 3.2.2 and Lemma 3.2.3, we complete the proof of Theorem 3.0.1. ■

3.2.1 Detailed analysis of the main term

In this section, we are going to show Lemma 3.2.2.

Proof of Lemma 3.2.2. By (3.1.13), the LHS of (3.2.18) can be decomposed as

$$\int_{S_\tau} \hat{D}(k)\hat{C}_{p_c,\tau}(k)\hat{G}_{p_c}(k)\frac{d^dk}{(2\pi)^d} = \left[1 + O\left(\beta\tau^{-\frac{(d-4)}{2}}\right)\right] M_{1.1} + M_{1.2} \tag{3.2.22}$$

where

$$M_{1.1} = \int_0^\infty \int_{S_\tau} \hat{D}(k)e^{-\tau\frac{v\sigma^2|k|^2}{2d}}e^{-t\left[1-\hat{J}_{p_c}(k)\right]}\frac{d^dk}{(2\pi)^d}dt, \tag{3.2.23}$$

$$M_{1.2} = \int_{S_\tau} \hat{D}(k)e^{-\tau\frac{v\sigma^2|k|^2}{2d}}\hat{G}_{p_c}(k)O(L^2|k|^2\tau^{1-\delta})\frac{d^dk}{(2\pi)^d}. \tag{3.2.24}$$

We begin with (3.2.23). Since $1-\hat{D}(k) \le \gamma\tau^{-1}\log\tau$ for $k \in S_\tau$, we obtain

$$M_{1.1} = \left[1 + O(\tau^{-1}\log\tau)\right] \underbrace{\int_0^\infty \int_{S_\tau} e^{-\tau\frac{v\sigma^2|k|^2}{2d}}e^{-t\left[1-\hat{J}_{p_c}(k)\right]}\frac{d^dk}{(2\pi)^d}dt}_{:=M'_{1.1}}. \tag{3.2.25}$$

We shortly show that for any $\varepsilon > 0$

$$M'_{1.1} - \int_0^\infty \int_{S_\tau} e^{-(\tau+t)\frac{\sigma_J^2|k|^2}{2d}}\frac{d^dk}{(2\pi)^d}dt = \int_\tau^\infty I_{t,4}dt = o(\beta^{1+\varepsilon}t^{-\frac{d(1+\varepsilon)}{2}}), \tag{3.2.26}$$

where

$$I_{t,4} = \int_{S_\tau}\left\{e^{-\tau\frac{v\sigma^2|k|^2}{2d}}e^{-(t-\tau)\left[1-\hat{J}_{p_c}(k)\right]} - e^{-t\frac{\sigma_J^2|k|^2}{2d}}\right\}\frac{d^dk}{(2\pi)^d}. \tag{3.2.27}$$

Hence, we first consider

$$\int_0^\infty \int_{S_\tau} e^{-(\tau+t)\frac{\sigma_J^2|k|^2}{2d}}\frac{d^dk}{(2\pi)^d}dt = \int_\tau^\infty \underbrace{\int_{S_\tau} e^{-t\frac{\sigma_J^2|k|^2}{2d}}\frac{d^dk}{(2\pi)^d}}_{:=I_{t,1}-I_{t,2}-I_{t,3}}dt \tag{3.2.28}$$

where

$$I_{t,1} = \int_{\mathbb{R}^d} e^{-t\frac{\sigma_J^2|k|^2}{2d}}\frac{d^dk}{(2\pi)^d}, \tag{3.2.29}$$

$$I_{t,2} = \int_{\mathbb{R}^d\setminus[-\pi,\pi]^d} e^{-t\frac{\sigma_J^2|k|^2}{2d}}\frac{d^dk}{(2\pi)^d}, \tag{3.2.30}$$

$$I_{t,3} = \int_{L_\tau} e^{-t\frac{\sigma_J^2|k|^2}{2d}}\frac{d^dk}{(2\pi)^d}. \tag{3.2.31}$$

The first integral $I_{t,1}$ is calculated exactly as

$$I_{t,1} = \left(\frac{d}{2\pi\sigma_J^2 t}\right)^{\frac{d}{2}} \overset{(3.1.30)}{=} \left(\frac{d}{2\pi\Sigma_U^2}\right)^{\frac{d}{2}}\beta\left[1+O(L^{-1})\right]t^{-\frac{d}{2}}. \tag{3.2.32}$$

For $I_{t,2}$ and $I_{t,3}$, by using $\int_{|k|\geq b} e^{-a|k|^2} \frac{d^dk}{(2\pi)^d} \leq \exp(-ab^2/2)(2\pi a)^{-d/2}$ $(\forall a, b > 0)$ and $t \geq \tau$, we can bound each of them as

$$I_{t,2} \leq \int_{|k|>\pi} e^{-t\frac{\sigma_J^2|k|^2}{2d}} \frac{d^dk}{(2\pi)^d} \leq \left(\frac{d}{\pi\sigma_J^2 t}\right)^{\frac{d}{2}} e^{-\tau\frac{\pi^2\sigma_J^2}{4d}} \leq O(\beta)e^{-\tau L^2\frac{\pi^2 C_0}{4d}} t^{-\frac{d}{2}}, \tag{3.2.33}$$

$$I_{t,3} \leq \int_{|k|^2\geq\gamma L^2\tau(c_2\log\tau)^{-1}} e^{-t\frac{\sigma_J^2|k|^2}{2d}} \frac{d^dk}{(2\pi)^d} \leq \left(\frac{d}{\pi\sigma_J^2 t}\right)^{\frac{d}{2}} \tau^{-\frac{\gamma\sigma_J^2}{4dc_2L^2}} \leq O(\beta)\tau^{-\frac{\gamma C_0}{4dc_2}} t^{-\frac{d}{2}}, \tag{3.2.34}$$

where the first inequality of (3.2.34) is due to (3.1.6). Therefore, we obtain

$$\int_\tau^\infty (I_{t,1} - I_{t,2} - I_{t,3})\, dt$$
$$= \frac{2}{d-2}\left(\frac{d}{2\pi\Sigma_U^2}\right)^{\frac{d}{2}} \beta\tau^{-\frac{d-2}{2}}\left[1 + O(L^{-1}) + O(e^{-\tau L^2\frac{\pi^2 C_0}{4d}}) + O(\tau^{-\frac{\gamma C_0}{4dc_2}})\right]. \tag{3.2.35}$$

Next we show (3.2.26). By the change of variables from k to $l := L^{1+\varepsilon}\sqrt{t^{1+\varepsilon}}k$ for any $\varepsilon > 0$, we obtain

$$I_{t,4} = \beta^{1+\varepsilon} t^{-\frac{d(1+\varepsilon)}{2}} \tilde{I}_{t,4} \tag{3.2.36}$$

where

$$\tilde{I}_{t,4} = \int_{\tilde{S}_\tau} \left\{ e^{-\tau\frac{v\sigma^2|l|^2}{2dL^{2+2\varepsilon}t^{1+\varepsilon}}} e^{-(t-\tau)\left[1-\hat{J}_{p_c}\left(\frac{l}{L^{1+\varepsilon}\sqrt{t^{1+\varepsilon}}}\right)\right]} - e^{-\frac{\sigma_J^2|l|^2}{2dL^{2+2\varepsilon}t^{\varepsilon}}} \right\} \frac{d^dl}{(2\pi)^d} \tag{3.2.37}$$

with $\tilde{S}_\tau$ being the domain of l corresponding to S_τ. To apply the dominated convergence theorem, by using (3.1.21), we get the uniform upper bound on $\tilde{I}_{t,4}$ as

$$|\tilde{I}_{t,4}| \leq \int_{\tilde{S}_\tau} \left| e^{-\tau\frac{v\sigma^2|l|^2}{2dL^{2+2\varepsilon}t^{1+\varepsilon}}} e^{-(t-\tau)\frac{K_0|l|^2}{L^{2+2\varepsilon}t^{1+\varepsilon}}} + e^{-\frac{\sigma_J^2|l|^2}{2dL^{2+2\varepsilon}t^{\varepsilon}}} \right| \frac{d^dl}{(2\pi)^d} < \infty. \tag{3.2.38}$$

Notice that

$$\exp\left(-\frac{\tau v\sigma^2|l|^2}{2dL^{2+2\varepsilon}t^{1+\varepsilon}}\right) = \exp\left(-\frac{\tau\sigma_J^2|l|^2}{2dL^{2+2\varepsilon}t^{1+\varepsilon}}\right)\{1 - E_1\}, \tag{3.2.39}$$

$$\exp\left(-(t-\tau)\left[1 - \hat{J}_{p_c}\left(\frac{l}{L^{1+\varepsilon}\sqrt{t^{1+\varepsilon}}}\right)\right]\right) = \exp\left(-\frac{(t-\tau)\sigma_J^2|l|^2}{2dL^{2+2\varepsilon}t^{1+\varepsilon}}\right)\{1 - E_2\}. \tag{3.2.40}$$

where

$$E_1 = 1 - \exp\left(-\tau\frac{(v\sigma^2 - \sigma_J^2)l^2}{2dL^{2+2\varepsilon}t^{1+\varepsilon}}\right), \tag{3.2.41}$$

$$E_2 = 1 - \exp\left(-(t-\tau)\left[1 - \hat{J}_{p_c}\left(\frac{l}{L^{1+\varepsilon}\sqrt{t^{1+\varepsilon}}}\right) - \frac{\sigma_J^2|l|^2}{2dL^{2+2\varepsilon}t^{1+\varepsilon}}\right]\right). \tag{3.2.42}$$

Therefore the integrand of $\tilde{I}_{t,4}$ can be decomposed into three terms which have the common term $e^{-(\sigma_J^2 l^2)/2dL^{2+2\varepsilon}t^{\varepsilon}}$ and at least one of the two extra factors: E_1 or E_2. Since there is a positive constant $c < \infty$ such that

$$\begin{aligned}\hat{\Pi}_{p_c} - p_c \partial_p \hat{\Pi}_{p_c} = \sum_{m=2}^{\infty} (1-m)\hat{\pi}_{p_c,m}(0) &= \sum_{m=2}^{\infty} (m-1)\hat{\pi}^{(1)}_{p_c,m}(0) + O(\beta^2) \\ &\overset{(3.1.17)}{\geq} c\beta, \end{aligned} \tag{3.2.43}$$

we obtain

$$\begin{aligned} v\sigma^2 - \sigma_J^2 \overset{(3.1.14)}{=} \frac{1}{p_c\left(1+\partial_p \hat{\Pi}_{p_c}\right)}\sigma_J^2 - \sigma_J^2 &\overset{(3.0.6)}{=} \frac{\hat{\Pi}_{p_c} - p_c \partial_p \hat{\Pi}_{p_c}}{p_c\left(1+\partial_p \hat{\Pi}_{p_c}\right)}\sigma_J^2 \\ &\geq O(\beta)\sigma_J^2, \end{aligned} \tag{3.2.44}$$

where the constant in $O(\beta)$ is positive. Therefore, E_1 goes to zero as $t \uparrow \infty$ or $L \uparrow \infty$ with fixed $l \in \mathbb{R}^d$. For E_2, the Taylor expansion

$$e^{ix} = 1 + ix - \frac{1}{2}x^2 - x^2 \int_0^1 (1-s)(e^{isx} - 1)ds \quad (x \in \mathbb{R}) \tag{3.2.45}$$

yields

$$\begin{aligned} &1 - \hat{J}_{p_c}\left(\frac{l}{L^{1+\varepsilon}\sqrt{t^{1+\varepsilon}}}\right) - \frac{\sigma_J^2 |l|^2}{2dL^{2+2\varepsilon}t^{1+\varepsilon}} \\ &\quad = \sum_{x\in\mathbb{Z}^d} J_{p_c}(x)\left(1 - e^{\frac{il\cdot x}{L^{1+\varepsilon}\sqrt{t^{1+\varepsilon}}}}\right) - \frac{\sigma_J^2 |l|^2}{2dL^{2+2\varepsilon}t^{1+\varepsilon}} \\ &\quad = \sum_{x\in\mathbb{Z}^d} J_{p_c}(x) \frac{(l\cdot x)^2}{L^{2+2\varepsilon}t^{1+\varepsilon}} \int_0^1 (1-s)(e^{\frac{isl\cdot x}{L^{1+\varepsilon}\sqrt{t^{1+\varepsilon}}}} - 1)ds, \end{aligned} \tag{3.2.46}$$

which goes to zero as $t \uparrow \infty$ or $L \uparrow \infty$ with fixed $l \in \mathbb{R}^d$. Thus, the dominated convergence theorem implies $\tilde{I}_{t,4} = o(1)$ as $t \uparrow \infty$ or $L \uparrow \infty$. Thus, we can conclude (3.2.26) as required.

Finally we consider $M_{1.2}$. By using (3.1.23) and the change of variables from k to $l := L\sqrt{\tau}k$, we can bound $M_{1.2}$ as

$$\begin{aligned} |M_{1.2}| &\leq \int_{S_\tau} e^{-\tau\frac{v\sigma^2|k|^2}{2d}} |\hat{G}_{p_c}(k)| O(L^2|k|^2\tau^{1-\delta}) \frac{d^dk}{(2\pi)^d} \\ &\leq (1+O(\beta)) \int_{S_\tau} e^{-\tau\frac{v\sigma^2|k|^2}{2d}} \frac{1}{1-\hat{D}(k)} O(L^2|k|^2\tau^{1-\delta}) \frac{d^dk}{(2\pi)^d} \\ &\leq O(\beta)\tau^{-\frac{d}{2}-\delta} \int_{\tilde{S}_\tau} e^{-\frac{v\sigma^2|l|^2}{2dL^2}} \frac{1}{1-\hat{D}(l/L\sqrt{\tau})} O(|l|^2) \frac{d^dl}{(2\pi)^d} \end{aligned} \tag{3.2.47}$$

with $\tilde{S}_\tau$ being the domain of l corresponding to S_τ. By (3.1.6) and (3.1.7), we obtain

$$\frac{1}{1-\hat{D}(l/(L\sqrt{\tau}))} \le \begin{cases} \frac{1}{c_1}\tau|l|^{-2} & (\|l\|_\infty \le \sqrt{\tau}), \\ \eta^{-1} & (\|l\|_\infty \ge \sqrt{\tau}). \end{cases} \tag{3.2.48}$$

Therefore, the integral of (3.2.47) can be bounded uniformly, which conclude

$$M_{1,2} = O(\beta)\tau^{-\frac{d-2+2\delta}{2}}. \tag{3.2.49}$$

By summarizing (3.2.26), (3.2.35) and (3.2.49), we finish the proof of Lemma 3.2.2. ∎

3.2.2 Estimates of the error terms

In this section, we prove Lemma 3.2.3. The following is the key lemma for Lemma 3.2.3.

Lemma 3.2.4. *Let $d > 4$ and $L \gg 1$. For $m \ge 2$, $n \ge 0$,*

$$\int_{[-\pi,\pi]^d} \hat{D}(k)\hat{\pi}_{p_c,m}(k)\hat{C}_{p_c,n}(k)\hat{G}_{p_c}(k)\frac{d^dk}{(2\pi)^d} = \begin{cases} O(\beta^2)m^{-\frac{d}{2}} & (n=0), \\ O(\beta^{\frac{3}{2}})m^{-\frac{d}{2}}n^{-\frac{d-2}{2}} & (n \ge 1). \end{cases} \tag{3.2.50}$$

Especially, for $n \ge 1$,

$$\int_{S_n} \left|\hat{D}(k)\hat{\pi}_{p_c,m}(k)\hat{C}_{p_c,n}(k)\hat{G}_{p_c}(k)\right| \frac{d^dk}{(2\pi)^d} = O(\beta^2)m^{-\frac{d}{2}}n^{-\frac{d-2}{2}}, \tag{3.2.51}$$

$$\int_{L_n} \left|\hat{D}(k)\hat{\pi}_{p_c,m}(k)\hat{C}_{p_c,n}(k)\hat{G}_{p_c}(k)\right| \frac{d^dk}{(2\pi)^d} = O(\beta^{\frac{3}{2}})m^{-\frac{d}{2}}n^{-\frac{d-2}{2}}(\log n)^{-1}. \tag{3.2.52}$$

Assuming Lemma 3.2.4, we first prove Lemma 3.2.3 and then we prove Lemma 3.2.4.

Proof of Lemma (3.2.20), assuming Lemma 3.2.4. First we consider (3.2.20). By using (3.1.2), the LHS of (3.2.20) can be decomposed as

$$\int_{L_\tau} \hat{D}(k)\hat{C}_{p_c,\tau}(k)\hat{G}_{p_c}(k)\frac{d^dk}{(2\pi)^d} = p_c M_{1,3} + M_{1,4}, \tag{3.2.53}$$

where

$$M_{1,3} = \int_{L_\tau} \hat{D}(k)^2\hat{C}_{p_c,\tau-1}(k)\hat{G}_{p_c}(k)\frac{d^dk}{(2\pi)^d}, \tag{3.2.54}$$

$$M_{1,4} = \sum_{m=2}^{\tau}\int_{L_\tau} \hat{D}(k)\hat{\pi}_{p_c,m}(k)\hat{C}_{p_c,\tau-m}(k)\hat{G}_{p_c}(k)\frac{d^dk}{(2\pi)^d}. \tag{3.2.55}$$

First we consider $M_{1.3}$. To use (3.1.15)–(3.1.16), we decompose $M_{1.3}$ as

$$M_{1.3} = \int_{L_\tau} \hat{D}(k)^2 \hat{C}_{p_c,\tau}(k) \hat{G}_{p_c}(k) \frac{d^d k}{(2\pi)^d} + \int_{L_\tau} \hat{D}(k)^2 \left[\hat{C}_{p_c,\tau-1}(k) - \hat{C}_{p_c,\tau}(k)\right] \hat{G}_{p_c}(k) \frac{d^d k}{(2\pi)^d}, \tag{3.2.56}$$

which is bounded above by

$$C_1' \tau^{-\frac{d}{2}} \int_{L_\tau} \hat{D}(k)^2 \left(\frac{1}{1-\hat{D}(k)}\right)^{1+\rho} \left[1 + \left(\frac{1}{1-\hat{D}(k)}\right)\right] \underbrace{\left|\hat{G}_{p_c}(k)\right|}_{\overset{(3.1.23)}{\leq} \frac{1+O(\beta)}{1-\hat{D}(k)}} \frac{d^d k}{(2\pi)^d}$$

$$\leq O\left(\tau^{-\frac{d-2}{2}} (\log \tau)^{-1}\right) \int_{L_\tau} \hat{D}(k)^2 \left(\frac{1}{1-\hat{D}(k)}\right)^{2+\rho} \frac{d^d k}{(2\pi)^d}, \tag{3.2.57}$$

where for the last inequality we use $1-\hat{D}(k) \geq \gamma\tau^{-1}\log\tau$. When $\|k\|_\infty \geq L^{-1}$, by (3.1.7), the integrand of (3.2.57) can be bounded above by $\hat{D}(k)^2$ with proportionality constant. Thus,

$$\int_{L_\tau, \|k\|_\infty \geq L^{-1}} \hat{D}(k)^2 \left(\frac{1}{1-\hat{D}(k)}\right)^{2+\rho} \frac{d^d k}{(2\pi)^d} \leq \frac{1}{\eta^{2+\rho}} \int_{[-\pi,\pi]^d} \hat{D}(k)^2 \frac{d^d k}{(2\pi)^d} \overset{(1.1.3)}{\leq} O(\beta). \tag{3.2.58}$$

For $\|k\|_\infty \leq L^{-1}$, the change of variables from k to $l := kL$ and (3.1.6) yields

$$\int_{L_\tau, \|k\|_\infty \leq L^{-1}} \underbrace{\hat{D}(k)^2}_{\leq 1} \left(\frac{1}{1-\hat{D}(k)}\right)^{2+\rho} \frac{d^d k}{(2\pi)^d} \leq \frac{1}{c_1^{2+\rho}} \beta \int_{\|l\|_\infty \leq 1} |l|^{-4-2\rho} \frac{d^d l}{(2\pi)^d} = O(\beta). \tag{3.2.59}$$

The integral is bounded because $\rho \leq (d-4)/2$. Therefore, we conclude

$$M_{1.3} = O(\beta)\, \tau^{-\frac{d-2}{2}} (\log \tau)^{-1}. \tag{3.2.60}$$

Next we consider $M_{1.4}$. When $m = \tau$, by (3.1.17) and the Schwarz inequality, we obtain

$$\int_{L_\tau} \hat{D}(k) \hat{\pi}_{p_c,\tau}(k) \hat{G}_{p_c}(k) \frac{d^d k}{(2\pi)^d} \leq O(\beta) \tau^{-\frac{d}{2}} \int_{[-\pi,\pi]^d} \left|\hat{D}(k)\hat{G}_{p_c}(k)\right| \frac{d^d k}{(2\pi)^d} \leq O(\beta)\tau^{-\frac{d}{2}} \underbrace{\left\|\hat{D}^2\right\|_1^{\frac{1}{2}}}_{\overset{(1.1.3)}{=} O(\beta^{1/2})} \underbrace{\left\|\hat{G}_{p_c}^2\right\|_1^{\frac{1}{2}}}_{\overset{(3.1.24)}{<} (1+K_1\beta)^{1/2}} = O(\beta^{\frac{3}{2}})\tau^{-\frac{d}{2}}. \tag{3.2.61}$$

For $m < \tau$, by using (3.2.51)–(3.2.52), we obtain

$$\int_{L_\tau} \hat{D}(k)\hat{\pi}_{p_c,m}(k)\hat{C}_{p_c,\tau-m}(k)\hat{G}_{p_c}(k)\frac{d^dk}{(2\pi)^d}$$
$$\leq \int_{L_{\tau-m}\cup S_{\tau-m}} \left|\hat{D}(k)\hat{\pi}_{p_c,m}(k)\hat{C}_{p_c,\tau-m}(k)\hat{G}_{p_c}(k)\right| \frac{d^dk}{(2\pi)^d} = O(\beta^{\frac{3}{2}})m^{-\frac{d}{2}}(\tau-m)^{-\frac{d-2}{2}}. \tag{3.2.62}$$

By (3.2.61)–(3.2.62), (3.2.55) can be estimated as

$$\begin{aligned} M_{1.4} &= O(\beta^{\frac{3}{2}})\sum_{m=2}^{\tau} m^{-\frac{d}{2}}(\tau-m+1)^{-\frac{d-2}{2}} \\ &\leq O(\beta^{\frac{3}{2}})\left\{O(\tau^{-\frac{d-2}{2}})\sum_{m=2}^{\lfloor(\tau+1)/2\rfloor} m^{-\frac{d}{2}} + O(\tau^{-\frac{d}{2}})\sum_{m=\lfloor(\tau+1)/2\rfloor+1}^{\tau}(\tau-m+1)^{-\frac{d-2}{2}}\right\} \\ &= O(\beta^{\frac{3}{2}})\tau^{-\frac{d-2}{2}}, \end{aligned} \tag{3.2.63}$$

where we bound either m or $\tau-m+1$ longer bound below by $\tau/2$ for the second inequality.

We next show (3.2.21). Recall the definition of M_2 and M_3:

$$M_2 = \int_{[-\pi,\pi]^d} \hat{D}(k)\hat{\pi}_{p_c,\tau}(k)\hat{G}_{p_c}(k)\frac{d^dk}{(2\pi)^d}, \tag{3.2.64}$$

$$M_3 = M_{3.1} + M_{3.2}, \tag{3.2.65}$$

where

$$M_{3.1} = \sum_{m=2}^{\tau-1}\sum_{n=\tau+1-m}^{\tau-1}\int_{[-\pi,\pi]^d} \hat{D}(k)\hat{\pi}_{p_c,m}(k)\hat{C}_{p_c,n}(k)\hat{G}_{p_c}(k)\frac{d^dk}{(2\pi)^d}, \tag{3.2.66}$$

$$M_{3.2} = \sum_{m=\tau}^{\infty}\sum_{n=0}^{\tau-1}\int_{[-\pi,\pi]^d} \hat{D}(k)\hat{\pi}_{p_c,m}(k)\hat{C}_{p_c,n}(k)\hat{G}_{p_c}(k)\frac{d^dk}{(2\pi)^d}. \tag{3.2.67}$$

Since $\hat{C}_{p_c,0}(k) = 1$, the case of $m=\tau, n=0$ of (3.2.50) immediately implies

$$M_2 = O(\beta^2)\tau^{-\frac{d}{2}}. \tag{3.2.68}$$

For $M_{3.1}$, by (3.2.50), it is estimated in a similar way to (3.2.63) as

$$\begin{aligned} M_{3.1} &= O(\beta^{\frac{3}{2}})\sum_{m=2}^{\tau-1}\sum_{n=\tau+1-m}^{\tau-1} m^{-\frac{d}{2}}n^{-\frac{d-2}{2}} \\ &\leq O(\beta^{\frac{3}{2}})\left\{O(\tau^{-\frac{d-2}{2}})\sum_{m=2}^{\lfloor(\tau+1)/2\rfloor} m^{-\frac{d-2}{2}} + O(\tau^{-\frac{d}{2}})\sum_{m=\lfloor(\tau+1)/2\rfloor+1}^{\tau-1}\sum_{n=\tau+1-m}^{\tau-1} n^{-\frac{d-2}{2}}\right\} \\ &= O(\beta^{\frac{3}{2}})\tau^{-\frac{d-2}{2}}. \end{aligned} \tag{3.2.69}$$

where we bound either m or n longer bound below by $(\tau+1)/2$ for the second inequality.

We finally consider $M_{3.2}$. The integral of (3.2.67) can be rewritten in terms of convolutions on $\mathbb{Z}^d$ and then $M_{3.2}$ is estimated as

$$\begin{aligned} M_{3.2} &= \sum_{m=\tau}^{\infty}\sum_{y\in\mathbb{Z}^d}\pi_{p_c,m}(y)\left(D*\sum_{n=0}^{\tau-1}C_{p_c,n}*G_{p_c}\right)(y) \\ &\le \underbrace{\left\|D*G_{p_c}^{*2}\right\|_{\infty}}_{\stackrel{(3.1.25)}{=}O(\beta)}\sum_{m=\tau}^{\infty}\underbrace{|\hat{\pi}_{p_c,m}(0)|}_{\stackrel{(3.1.17)}{\le}O(\beta)m^{-d/2}} = O(\beta^2)\tau^{-\frac{d-2}{2}}. \end{aligned} \tag{3.2.70}$$

Summarzing (3.2.69)–(3.2.70), we obtain

$$M_3 = O(\beta^{\frac{3}{2}})\tau^{-\frac{d-2}{2}}. \tag{3.2.71}$$

Then we finish the proof of Lemma 3.2.3. ■

Finally we show Lemma 3.2.4.

Proof of Lemma 3.2.4 First of all, we consider $n=0$ case, i.e., $\hat{C}_{p_c,0}(k)=1$. By (3.1.17), it is estimated as

$$\begin{aligned} \int_{[-\pi,\pi]^d}\hat{D}(k)\hat{\pi}_{p_c,m}(k)\hat{G}_{p_c}(k)\frac{d^dk}{(2\pi)^d} &= \sum_{x\in\mathbb{Z}^d}\pi_{p_c,m}(x)(D*G_{p_c})(x) \\ &\le \underbrace{\|D*G_{p_c}\|_{\infty}}_{\stackrel{(3.1.25)}{=}O(\beta)}|\hat{\pi}_{p_c,m}(0)| \le O(\beta^2)m^{-d/2}. \end{aligned} \tag{3.2.72}$$

For $n\ge 1$, we split the domain of integration into S_n and L_n. First we show (3.2.51). By using (3.1.13) and (3.1.22), we can bound the LHS of (3.2.51) as

$$\begin{aligned} &\int_{S_n}\left|\hat{D}(k)\hat{\pi}_{p_c,m}(k)\hat{C}_{p_c,n}(k)\hat{G}_{p_c}(k)\right|\frac{d^dk}{(2\pi)^d} \\ &\stackrel{(3.1.17)}{\le} O(\beta)m^{-\frac{d}{2}}\int_{S_n}\left|\hat{D}(k)\hat{C}_{p_c,n}(k)\hat{G}_{p_c}(k)\right|\frac{d^dk}{(2\pi)^d} \\ &\stackrel{(3.1.13)(3.1.22)}{\le} O(\beta)m^{-\frac{d}{2}}\Bigg\{\left[1+O\left(\beta n^{-\frac{(d-4)}{2}}\right)\right]\underbrace{M'_{1.1}(n)}_{\stackrel{(3.2.26)(3.2.35)}{=}O(\beta)n^{-\frac{d-2}{2}}} \\ &\qquad + \underbrace{\int_{S_n}e^{-n\frac{v\sigma^2|k|^2}{2d}}|\hat{G}_{p_c}(k)|O(L^2|k|^2n^{1-\delta})\frac{d^dk}{(2\pi)^d}}_{\stackrel{(3.2.47)}{=}O(\beta)o(n^{-\frac{d-2}{2}})}\Bigg\} \\ &= O(\beta^2)m^{-\frac{d}{2}}n^{-\frac{d-2}{2}} \end{aligned} \tag{3.2.73}$$

where $M'_{1.1}(n)$ is defined by (3.2.25) in which τ replaced n.

By (3.1.15) and (3.1.23), the LHS of (3.2.52) can be bounded as

$$\begin{aligned}
&\int_{L_n} \left|\hat{D}(k)\hat{\pi}_{p_c,m}(k)\hat{C}_{p_c,n}(k)\hat{G}_{p_c}(k)\right| \frac{d^dk}{(2\pi)^d} \\
&\le O(\beta)m^{-\frac{d}{2}}n^{-\frac{d}{2}} \int_{L_n} \left|\hat{D}(k)\right| \left(\frac{1}{1-\hat{D}(k)}\right)^{3+\rho} \frac{d^dk}{(2\pi)^d} \\
&\le O(\beta)m^{-\frac{d}{2}}n^{-\frac{d-2}{2}}(\log n)^{-1} \int_{L_n} \left|\hat{D}(k)\right| \left(\frac{1}{1-\hat{D}(k)}\right)^{2+\rho} \frac{d^dk}{(2\pi)^d},
\end{aligned} \tag{3.2.74}$$

where for the last inequality we use $1-\hat{D}(k) \ge \gamma n^{-1}\log n$. By (3.1.6) and (3.1.7), the integral of (3.2.74) can be bounded above by

$$\begin{aligned}
&\int_{L_n,\|k\|_\infty \le L^{-1}} \left(\frac{1}{c_1 L^2|k|^2}\right)^{2+\rho} \frac{d^dk}{(2\pi)^d} + \frac{1}{\eta^{2+\rho}} \int_{L_n,\|k\|_\infty \ge L^{-1}} \left|\hat{D}(k)\right| \frac{d^dk}{(2\pi)^d} \\
&\le O(\beta) \underbrace{\int_{\|l\|_\infty \le 1} |l|^{-4-2\rho} \frac{d^dl}{(2\pi)^d}}_{<\infty} + \frac{1}{\eta^{2+\rho}} \underbrace{\left\|\hat{D}^2\right\|_1^{\frac{1}{2}}}_{\overset{(1.1.3)}{=} O(\beta^{1/2})},
\end{aligned} \tag{3.2.75}$$

where we use the change of variables from k to $l := kL$ for the first term and the Schwarz inequality for the second term. By combining (3.2.74) and (3.2.75), we obtain (3.2.52). This completes the proof of Lemma 3.2.4. ■

Appendix: Proof of Lemma 3.1.5.

This Appendix give the proof of Lemma 3.1.5, which is used for approximation of σ^2.

Proof of Lemma 3.1.5. For $x \in \mathbb{Z}^d$, we difine

$$D_o(x) = \frac{\mathbb{1}_{\{\|x\|_\infty \le L\}}}{(2L+1)^d}. \tag{3.2.76}$$

It is known that the Fourier transform of D_o and U have the simple product expressions [26, Section 4] as follows:

$$\hat{U}(k) = \int_{\mathbb{R}^d} U(x)e^{ik\cdot x}d^dx = \prod_{j=1}^{d} \frac{\sin k_j}{k_j}, \tag{3.2.77}$$

$$\hat{D}_o(k) = \sum_{x\in\mathbb{Z}^d} D_o(x)e^{ik\cdot x} = \prod_{j=1}^{d} \frac{\sin[(2L+1)\frac{k_j}{2}]}{(2L+1)\sin\frac{k_j}{2}} = \frac{\hat{U}((L+\frac{1}{2})k)}{\hat{U}(\frac{k}{2})}. \tag{3.2.78}$$

Notice that we can rewrite $\sigma^2/(L+1/2)^d$ and Σ_U^2 as

$$\frac{\sigma^2}{(L+1/2)^2} = -d\,\partial_{k_1}^2 \hat{D}\left(\frac{k}{L+1/2}\right)\Big|_{k=0}, \quad \Sigma_U^2 = -d\,\partial_{k_1}^2 \hat{U}\,(k)\Big|_{k=0}. \tag{3.2.79}$$

Therefore, to prove Lemma 3.1.5, it suffices to show

$$\begin{aligned}\partial_{k_1}^2 \hat{D}\left(\frac{k}{L+1/2}\right)\Big|_{k=0} - \partial_{k_1}^2 \hat{D}_o\left(\frac{k}{L+1/2}\right)\Big|_{k=0} &= -\frac{\sum_{x\in\mathbb{Z}^d} x_1^2\,(D(x)-D_o(x))}{\left(L+\frac{1}{2}\right)^2}\\ &= O(\beta),\end{aligned} \tag{3.2.80}$$

and

$$\partial_{k_1}^2 \hat{D}_o\left(\frac{k}{L+1/2}\right)\Big|_{k=0} - \partial_{k_1}^2 \hat{U}\,(k)\Big|_{k=0} = O(L^{-2}). \tag{3.2.81}$$

Since

$$D(x) - D_o(x) = \frac{\mathbb{1}_{||x||_\infty=0}}{(2L+1)^d - 1} + \frac{\mathbb{1}_{||x||_\infty\le L}}{(2L+1)^d[(2L+1)^d - 1]}, \tag{3.2.82}$$

by using (3.1.5) we obtain

$$\begin{aligned}\frac{\sum_{x\in\mathbb{Z}^d} x_1^2\,(D(x)-D_o(x))}{\left(L+\frac{1}{2}\right)^2} &= (2L+1)^{-d}\underbrace{\left(L+\frac{1}{2}\right)^{-2}\sum_{x\in\mathbb{Z}^d} x_1^2 D(x)}_{<\infty}\\ &= O(\beta),\end{aligned} \tag{3.2.83}$$

which shows (3.2.80).

Next we consider (3.2.81). Using (3.2.78), we get

$$\partial_{k_1}^2 \hat{D}_o\left(\frac{k}{L+1/2}\right)\Big|_{k=0} = \partial_{k_1}^2 \frac{\hat{U}(k)}{\hat{U}(\frac{k}{2L+1})}\Bigg|_{k=0} = \partial_{k_1}^2 \hat{U}(k)\Big|_{k=0} - \partial_{k_1}^2 \hat{U}\left(\frac{k}{2L+1}\right)\Big|_{k=0} \tag{3.2.84}$$

where for the last equation we use the facts that $\partial_{k_1}\hat{U}(k)\Big|_{k=0} = \partial_{k_1}\hat{U}(\frac{k}{2L+1})\Big|_{k=0} = 0$ because of the reflection, and that $\hat{U}(0) = 1$. The second term of (3.2.84) can be calculated directly as

$$\partial_{k_1}^2 \hat{U}\left(\frac{k}{2L+1}\right)\Big|_{k=0} = -(2L+1)^{-2}\underbrace{\int_{\mathbb{R}^d} U(x)x_1^2 d^d x}_{=\Sigma_U^2/d\ <\infty} = O(L^{-2}). \tag{3.2.85}$$

Therefore we have (3.2.81) and this completes the proof of Lemma 3.1.5. ■

3.3 Notes

The author identifies the leading-order asymptotic of the difference of p_c^τ and p_c^∞ in terms of τ for the uniform spread-out model in dimension $d > 4$. The main reasons why the author restricts the models to the uniform spread-out model are as follows. First, due to the results by van der Hofstad and Slade [29] (Lemma 3.1.1 in this thesis), which are proved only for the spread-out model, the analysis can be simplified to the computation problem. Second, for the uniform spread-out models, the estimate of Riemann-sum approximation is simple because U and D_o have the simple product expression (see Proof of Lemma 3.1.5). The results of [29] can be applied to the general spread-out model (not only the uniform spread out model), where 1-step transition probability of underlying randomwalk is defined by

$$D(x) = \frac{h(x/L)}{\sum_{x\in\mathbb{Z}^d} h(x/L)} \tag{3.3.1}$$

with h being a probability distribution over $\mathbb{R}^d \backslash o$, which is invariant under rotations by $\pi/2$ and reflections in the coordination hyperplanes, and almost everywhere continuous. Therefore, at least, the results of Theorem 3.0.1 should be extended to the general spread out model. Below, possibilities about the extension of Theorem 3.0.1 to other models are summarized.

Weakly self-avoiding walk The weakly self-avoiding walk is a model for walks in which self-intersection is allowed but assigned probability penalty. Let

$$c_n(x) = \sum_{\substack{\omega\in\mathcal{W}(o,x)\\|\omega|=n}} \prod_{0\le s<t\le n} (1-\lambda_{st}\delta_{\omega_s,\omega_t}), \quad c_n := \sum_{x\in\mathbb{Z}^d} c_n(x) \tag{3.3.2}$$

where λ_{st} is defined in [24] and [5] respectively as follows: for $0 < \beta \ll 1$ and $p \ge 0$,

$$\lambda_{st} = \begin{cases} 1 - e^{-\frac{\beta}{|s-t|^p}} & \text{in [24]}, \\ 1 - e^{-\beta} & \text{in [5]}. \end{cases} \tag{3.3.3}$$

For the weakly memory-τ walk, we need the condition $|s-t| \le \tau$ in the product of (3.3.2). In [24], van der Hofstad, Hollander and Slade consider the weakly self-avoiding walk where loops of length m are penalis zed by a factor $e^{-\beta/m^p}$ for $d > 4,\ p \ge 0$, and for $d \le 4,\ p > (4-d)/2$, which is so-called elastic self-avoiding walk, while Bolthausen and Ritzmann [5] consider general weakly self-avoiding walk, where any loops are penalized with the same probability weight uniformly. In [24], van der Hofstad, Hollander and Slade proved that for $d > 4,\ p \ge 0$, or, for $d \le 4,\ p > (4-d)/2$, there is a $\beta_0 = \beta_0(d,p)$ such that for $\beta < \beta_0$,

$$\frac{1}{c_n}\hat{c}_n\left(\frac{k}{\sqrt{Dn}}\right) = e^{-\frac{k^2}{2d}\left[1+O(n^{-\delta'})\right]}, \quad \frac{c_n(x)}{c_n} = 2\left(\frac{d}{2\pi Dn}\right)^{\frac{d}{2}} e^{-\frac{dx^2}{2Dn}}[1+o(1)] \tag{3.3.4}$$

where $\delta' \in (0, 1 \wedge \frac{2p+d-4}{4})$ and D is identified with the lace-expansion coefficient. Notice that the second result in (3.3.4) for $d > 4$, $p = 0$, which is a local central limit theorem, was not proved by them. For $d > 4$, $p = 0$, where this case is equivalent to the general weakly self-avoiding walk for $d > 4$, they showed weaker results: $\sup_x \frac{c_n(x)}{c_n} \leq O(n^{-d/2})$. On the other hand, in [5], Bolthausen and Ritzmann proved that for $d > 4$,

$$\left| \frac{c_n(x)}{c_n} - 2\varphi_{\delta n} \right| \leq \left[n^{-\frac{1}{2}} \varphi_{\nu n}(x) + n^{-\frac{d}{2}} \sum_{j=1}^{n/2} j \varphi_{\nu j}(x) \right], \tag{3.3.5}$$

where

$$\varphi_\eta(x) := (2\pi\eta)^{-\frac{d}{2}} e^{-\frac{x^2}{2\eta}} \tag{3.3.6}$$

and δ is identified with the lace-expansion coefficient and ν is a dimension-dependent constant. Their estimates in [5] are in x-space not in Fourier space, so they did not show the gaussian behavior in Fourier space such as the first results in (3.3.4).

In light of the above, using the results (3.3.4) or (3.3.5), we should be able to obtain the leading-order asymptotics of the difference between the critcal points for the weakly self-avoiding walk and the weakly memory-τ walk, and identify the expression of the coefficient of the leading term. For the elastic self-avoiding walk, due to the Fourier estimate of (3.3.4), we can follow the same proof in [33] to obatain such results. Furthermore, for the elastic self-avoiding walk, we can obtain the results in $d \leq 4$. On the other hand, since we do not have the results in Fourier space for the general weakly self-avoiding walk, the proof would not be the same and we will have to estimate carefully the error which comes from the right hand side of (3.3.5).

Nearest-neighbor model For the nearest-neighbor model, the underlying random walk takes steps only in $\Lambda = \{x \in \mathbb{Z}^d : \sum_{i=1}^d |x_i| = 1\}$. In [37], Madras and Slade proved that for the nearest-neighbor self avoiding walk, there is a constant B such that for sufficiently high dimension bigger than 5,

$$\sup_{x \in \mathbb{Z}^d} C^\infty_{p_c^\infty, n}(x) \leq B n^{-d/2}. \tag{3.3.7}$$

This is a weaker version of the statements such as (3.3.4)–(3.3.6), but the best estimate related to the behavior of $C^\infty_{p_c^\infty, n}(x)$ so far. If we consider using (3.3.7) for the estimate of the difference of the critical points between memory-τ walk and self-avoiding walk, we shoud show that the upper bound of it is of order $\tau^{-\frac{d-2}{2}}$ for sufficiently high dimension bigger than 5. For such results, we would not need conditions about the relation between τ and d while for Kesten's result (1.2.4) we have to suppose $d \geq \max\{13\tau - 13, 5\}$. However, it seems difficult to obtain the exact expresson of the coefficient of the leading term such as C_d in Theorem 1.2.1 because (3.3.7) is just an upper bound, which is insufficient for our goal.

To extend the results of Theorem 1.2.1 to the nearest-neighbor model, we seem to need a kind of the local central limit theorem for the nearest-neighbor self-avoiding walk and memory-τ walk. The proof for such results obtained for the spread-out or weakly self-avoiding walk so far (e.g.,[29, 1, 5, 24]) relies on the fact that we can take a sufficiently small parameter while we can not for the nearest-neighbor model. Also, we have the same difficulty for the long-range model. For the long-range model, we require h in (3.3.1) to satisfy $|x|^{d+\alpha}h(x)$ is bounded away from zero and infinity where $\alpha > 0$ is the characteristic index. The author has not known such results on the nearest-neighbor or long-range self-avoiding walk, even though these are more natural models for self-avoiding walk.

第4章 Estimates of the critical points for the lattice trees and the lattice animals

This chapter gives the proof of the following main theorem stated in Section 1.2.2. All items in this chapter that do not specify a source are from [34].

Theorem 4.0.1. *For both models with $d > 8$ and $L \uparrow \infty$,*

$$p_c = \frac{1}{e} + C\beta + O(\beta L^{-1}), \tag{4.0.1}$$

where the model-dependent constant C has the following random-walk representation:

$$C_{\mathrm{LT}} = \sum_{n=2}^{\infty} \frac{n+1}{2e} U^{*n}(o), \qquad C_{\mathrm{LA}} = C_{\mathrm{LT}} - \frac{1}{2e^2} \sum_{n=3}^{\infty} U^{*n}(o). \tag{4.0.2}$$

As we mentioned in Section 1.2.2, the proof of Theorem 4.0.1 is based on the lace expansion for the 2-point function and detailed analysis of the 1-point function, similarly to the previous work by Miranda and Slade [40]. The different point of our method from that of them is to introduce a new base point p_1 defined in (4.1.1) below, as $p_1 g_{p_1} = 1$. It is to estimate various generating functions in terms of massless random walks. For the spread-out models of self-avoiding walk, percolation, oriented percolation and the contact process, van der Hofstad and Sakai [26] simply used the base point $p_1 = 1$, because of the unity of the 1-point function for those models. Since the analysis in terms of the underlying random walks is very simple, we do not have to know in detail the lace expansion; the exception is in Lemma 4.1.1 below, where we investigate the first lace-expansion coefficient $\hat{\pi}_p^{(1)}$ to prove $p_c - p_1 = O(\beta^2)$. However, the basic facts (summarized in Proposition 4.1.2 below) and a *minimum* definition about the lace expansion coefficients should be enough to read the proof, which we hope makes this paper more accessible to wider audience.

Our method can be applied to the nearest-neighbor model as well to identify the coefficient of $(2d)^{-1}$, as we can use the same method (i.e., Lemma 4.1.1 below) to conclude $p_c - p_1 = O(d^{-2})$, but this limit the accuracy our method can achieve. Therefore, to identify the higher-order coefficients, we may need investigate the lace expansion coefficients at p_c more carefully as Miranda and Slade do in [40].

4.1 Results due to the lace expansion

In this section, we approximate p_{c} by p_1 that is defined for both models by the identity

$$p_1 g_{p_1} = 1. \tag{4.1.1}$$

Lemma 4.1.1. *For both models with $d > 8$ and $L \uparrow \infty$,*

$$0 < p_{\mathrm{c}} - p_1 = O(\beta^2). \tag{4.1.2}$$

The key to the proof is the following collection of the lace-expansion results [16, 35].

Proposition 4.1.2 ([16, 35])**.** *For both models in $d > 8$, there is a model-dependent $L_0 < \infty$ such that, for all $L \geq L_0$, the following holds for all $p \leq p_{\mathrm{c}}$:*

1. *The 1-point function is bounded away from zero and infinity. In fact,*

$$1 \leq g_p \leq 4. \tag{4.1.3}$$

2. *There is a constant $K < \infty$ such that the lace-expansion coefficients $\pi_p^{(n)}(x)$ $(n \in \mathbb{N})$ defined in (2.2.10) are bounded as*

$$\forall x \in \mathbb{Z}^d, \quad \pi_p^{(n)}(x) \leq \frac{KL^{-6}(K\beta)^{n-1}}{(\|x\| \vee L)^{2d-6}}. \tag{4.1.4}$$

As a consequence of (2.2.9) and (4.1.4), it is concluded that there is a $K' < \infty$ such that

$$\forall x \neq o, \quad \tau_{p_{\mathrm{c}}}(x) \leq \frac{K'L^{-2}}{(\|x\| \vee L)^{d-2}}, \qquad \chi_p \underset{p \uparrow p_{\mathrm{c}}}{\asymp} (p_{\mathrm{c}} - p)^{-1/2}, \tag{4.1.5}$$

where the latter means $\chi_p/(p_{\mathrm{c}} - p)^{-1/2}$ is bounded away from 0 and ∞ as $p \uparrow p_{\mathrm{c}}$.

The above results for lattice trees are proven in [35] by following the same line of proof as in [16] and using the convolution bounds in [9, Lemma 3.2] instead of the weaker ones in [16, Proposition 1.7]. The same strategy applies to lattice animals, and we refrain from showing details.

Consequently, for any $p \le p_c$,

$$\hat{\pi}_p^{(n)} = \sum_{x\in\mathbb{Z}^d} \pi_p^{(n)}(x) \overset{(4.1.4)}{\le} K(K\beta)^{n-1}\bigg(\sum_{x:\|x\|\le L} L^{-2d} + \sum_{x:\|x\|>L} \frac{L^{-6}}{\|x\|^{2d-6}}\bigg) = O(\beta)^n. \qquad (4.1.6)$$

Moreover, by subadditivity (i.e., forgetting edge-disjointness among paths from o to x),

$$\hat{h}_p \le \sum_{x\ne o} \tau_p(x)^2 \overset{(4.1.5)}{\le} (K'L^{-2})^2\bigg(\sum_{x:\|x\|\le L} L^{2(2-d)} + \sum_{x:\|x\|>L} \|x\|^{2(2-d)}\bigg) = O(\beta). \qquad (4.1.7)$$

By summing (2.2.9) over $x \in \mathbb{Z}^d$, solving the resulting equation for χ_p and then using the fact that χ_p diverges as $p \uparrow p_c$, we obtain the identity

$$p_c = \frac{1}{g_{p_c} + \hat{h}_{p_c} + \hat{\pi}_{p_c}} = \bigg(g_{p_c} + \sum_{x\ne o} h_{p_c}(x) + \sum_x \pi_{p_c}(x)\bigg)^{-1}, \qquad (4.1.8)$$

where $\hat{h}_p = \sum_{x\in\mathbb{Z}^d} h_p(x)$. Substituting (4.1.6)-(4.1.7) to (4.1.8) yields[1]

$$p_c = \frac{1}{g_{p_c}}\bigg(1 + \frac{\hat{h}_{p_c} - \hat{\pi}_{p_c}^{(1)}}{g_{p_c}} + O(\beta^2)\bigg)^{-1} = \frac{1}{g_{p_c}}\bigg(1 + \frac{\hat{\pi}_{p_c}^{(1)} - \hat{h}_{p_c}}{g_{p_c}}\bigg) + O(\beta^2), \qquad (4.1.10)$$

which is the starting point of the analysis.

Proof of Lemma 4.1.1. First we show $p_1 < p_c$. Since pg_p is increasing in p with $p_1 g_{p_1} = 1$, it suffices to show $p_c g_{p_c} > 1$. By (4.1.3) and (4.1.10), it then suffices to show that $\hat{\pi}_{p_c}^{(1)} - \hat{h}_{p_c}$ is bounded from below by β times a positive constant for large L. Here, and only here, we use the actual definition of the lace-expansion coefficient $\hat{\pi}_p^{(1)}$ (see, e.g., [18]). We can easily check that $\hat{\pi}_p^{(1)}$ for both models is larger than the sum of triangles consisting only of three distinct edges: $\hat{\pi}_p^{(1)} \ge |\Lambda|(|\Lambda|-1)(p/|\Lambda|)^3$, which is enough for lattice trees because $\hat{h}_p \equiv 0$. For lattice animals, we show below $\hat{h}_p \le \frac{1}{4}\hat{\pi}_p^{(1)} + O(\beta^2)$ for $p \le p_c$ in high dimensions $d > 8$. The aforementioned sufficient condition for $p_c g_{p_c} > 1$ is now verified.

Next we show $p_c - p_1 = O(\beta^2)$ for lattice animals by induction. The same induction also works for lattice trees with $A = T$ and $\hat{h}_{p_c} \equiv 0$. Let $\{\ell_n\}_{n\in\mathbb{N}}$ be the following increasing sequence bounded above by 2:

$$\ell_1 = 1, \qquad\qquad \ell_{j+1} = 1 + \frac{\ell_j}{2} \qquad [j \in \mathbb{N}]. \qquad (4.1.11)$$

[1] In [35], Liang investigated $\hat{\pi}_{p_c}^{(1)}$ in (4.1.10) for lattice trees and showed that, for all $d > 8$, $p_c g_{p_c}$ rather than p_c exhibits

$$p_c g_{p_c} = 1 + \frac{\beta}{e}\sum_{n=2}^{\infty}\binom{n+1}{2} U^{*n}(o) + O(\beta/L) \qquad \text{as } L \uparrow \infty. \qquad (4.1.9)$$

This may be a bit of surprise, as the coefficient of β is much larger than that in (4.0.1)–(4.0.2).

Since $p_c = O(1)$ (see,e.g., (1.2.10) or [16, Proposition 2.2]) and $p_1 g_{p_1} = p_c(g_{p_c} + \hat{h}_{p_c} + \hat{\pi}_{p_c}) = 1$, we have

$$0 < 1 - \frac{p_1}{p_c} = 1 - \frac{g_{p_c} + \hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1}} = -\underbrace{\frac{g_{p_c} - g_{p_1}}{g_{p_1}}}_{\geq 0} - \frac{\hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1}}, \tag{4.1.12}$$

which is bounded above by $-\hat{\pi}_{p_c}/g_{p_1} = O(\beta)$ (due to (4.1.6)), confirming $p_c - p_1 = O(\beta^{\ell_1})$.

Now we suppose $p_c - p_1 = O(\beta^{\ell_j})$. Notice that $g_{p_c} - g_{p_1}$ can be rewritten as

$$\begin{aligned} g_{p_c} - g_{p_1} &= \sum_{A \in \mathcal{A}_o} \left(1 - \left(\frac{p_1}{p_c}\right)^{|E_A|}\right) W_{p_c}(A) = \left(1 - \frac{p_1}{p_c}\right) \underbrace{\sum_{A \in \mathcal{A}_o} \sum_{n=0}^{|E_A|-1} \left(\frac{p_1}{p_c}\right)^n W_{p_c}(A)}_{=:F} \\ &\overset{(4.1.12)}{=} \left(-\frac{g_{p_c} - g_{p_1}}{g_{p_1}} - \frac{\hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1}}\right) F. \end{aligned} \tag{4.1.13}$$

Solving this for $g_{p_c} - g_{p_1}$ yields

$$g_{p_c} - g_{p_1} = -\frac{\hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1} + F} F, \tag{4.1.14}$$

which is bounded above by $-\hat{\pi}_{p_c} = O(\beta)$ (due to (4.1.6)) for both models. By substituting (4.1.14) to (4.1.12), we obtain

$$\begin{aligned} p_c - p_1 = p_c \left(\frac{1}{g_{p_1}} \frac{\hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1} + F} F - \frac{\hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1}} \right) &= -p_c \frac{\hat{h}_{p_c} + \hat{\pi}_{p_c}}{g_{p_1} + F} \\ &\overset{(4.1.6)}{=} p_c \frac{\hat{\pi}^{(1)}_{p_c} - \hat{h}_{p_c}}{g_{p_1} + F} + O(\beta^2). \end{aligned} \tag{4.1.15}$$

Recall the definition of F in (4.1.13). Since $(p_1/p_c)^n W_{p_c}(A) = (p_c/p_1)^{|E_A|-n} W_{p_1}(A)$, which is also true for lattice trees, we have

$$F = \sum_{A \in \mathcal{A}_o} \sum_{n=1}^{|E_A|} \left(\frac{p_c}{p_1}\right)^n W_{p_1}(A) \overset{p_1 < p_c}{\geq} \sum_{A \in \mathcal{A}_o} |E_A| W_{p_1}(A) \overset{|V_A| \leq 2|E_A|}{\geq} \frac{\chi_{p_1}}{2}. \tag{4.1.16}$$

By (4.1.15) and (4.1.16), we can estimate $p_c - p_1$ as

$$\begin{aligned} p_c - p_1 = p_c \frac{\hat{\pi}^{(1)}_{p_c} - \hat{h}_{p_c}}{g_{p_1} F^{-1} + 1} F^{-1} + O(\beta^2) &= O(\beta) \chi_p^{-1} + O(\beta^2) \\ &= O(\beta)(p_c - p_1)^{\frac{1}{2}} + O(\beta^2), \end{aligned} \tag{4.1.17}$$

where, for the last inequality, we use $\chi_{p_1} \asymp (p_c - p_1)^{-1/2}$ for both models in dimensions $d > 8$. Applying the inductive hypothesis $p_c - p_1 = O(\beta^{\ell_j})$ to (4.1.17), we obtain $p_c - p_1 = O(\beta^{\ell_{j+1}})$. Therefore the induction completed. Since $\lim_{j \uparrow \infty} \ell_j = 2$, this proves $p_c - p_1 = O(\beta^2)$, as required. ■

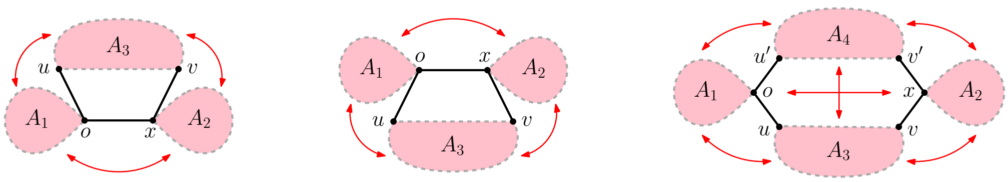
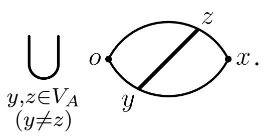


図 4.1: Schematic representations of the three terms in (4.1.21). The black line segments are pivotal for $o \Longleftrightarrow x$ in A. Removal of those edges results in the animals $\{A_j\}_{j=1}^3$ or $\{A_j\}_{j=1}^4$ that are mutually avoiding, as indicated by the red arrows. The vertices in Λ are ordered in an arbitrary way (counter-clockwise in the above 2-dimensional figures).

Proof of $\hat{h}_p \le \frac{1}{4}\hat{\pi}_p^{(1)} + O(\beta^2)$ for lattice animals. First we recall that

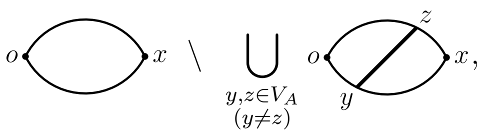
$$\hat{h}_p = \sum_{x \neq o} \sum_{A \in \mathcal{D}_{o,x}} \left(\frac{p}{|\Lambda|}\right)^{|E_A|} = \sum_{x \neq o} \sum_{A \in \mathcal{A}_o} \left(\frac{p}{|\Lambda|}\right)^{|E_A|} \mathbb{1}_{\{o \Longleftrightarrow x\}}. \tag{4.1.18}$$

We split the sum into two depending on whether or not there are distinct vertices $y, z \in V_A$ such that $o \longleftrightarrow y$, $y \longleftrightarrow x$, $o \longleftrightarrow z$, $z \longleftrightarrow x$ and $y \longleftrightarrow z$ occur in A edge-disjointly, i.e., those connections occur in distinct sets of E_A. (We note that, if $y = o$, for example, then we should interpret this as $o \longleftrightarrow x$, $o \Longleftrightarrow z$ and $z \longleftrightarrow x$ occuring edge-disjointly.) Intuitively,



(4.1.19)

Using submultiplicativity and the x-space bound in (4.1.5), we can show that the contribution from this case is $O(\beta^2)$. On the other hand, if there are no such vertices $y, z \in V_A$, i.e.,



(4.1.20)

then there are exactly two edge-disjoint connections between o and x, with two pivotal edges from o, say $\{o, u\}, \{o, u'\}$, and two from x, say $\{v, x\}, \{v', x\}$, one of which may coincide with either $\{o, u\}$ or $\{o, u'\}$, for the double connection $o \Longleftrightarrow x$ in A. Suppose that there is order among vertices in Λ. If $u \in \Lambda$ is earlier than $u' \in \Lambda$ in this order, we write $u \prec u'$. Let $\Lambda(x) = \{v \in V : v - x \in \Lambda\}$. Then, the contribution to $\hat{h}_p$ from (4.1.20) is bounded above by (see Figure 4.1)

$$\sum_{x\neq o}\sum_{\substack{A_1\in\mathcal{A}_o\\ A_2\in\mathcal{A}_x}} W_p(A_1)\,W_p(A_2)\Bigg(\mathbb{1}_{\{x\in\Lambda\}}\frac{p}{|\Lambda|}\sum_{\substack{u\in\Lambda\\(x\prec u)}}\sum_{\substack{v\in\Lambda(x)\\(v\neq o)}}\bigg(\frac{p}{|\Lambda|}\bigg)^2\sum_{A_3\in\mathcal{A}_{u,v}} W_p(A_3)$$

$$+\,\mathbb{1}_{\{x\in\Lambda\}}\frac{p}{|\Lambda|}\sum_{\substack{u\in\Lambda\\(u\prec x)}}\sum_{\substack{v\in\Lambda(x)\\(v\neq o)}}\bigg(\frac{p}{|\Lambda|}\bigg)^2\sum_{A_3\in\mathcal{A}_{u,v}} W_p(A_3)$$

$$+\sum_{\substack{u,u'\in\Lambda\\(u\prec u')}}\sum_{\substack{v,v'\in\Lambda(x)\\(v\neq v')}}\bigg(\frac{p}{|\Lambda|}\bigg)^4\sum_{\substack{A_3\in\mathcal{A}_{u,v}\\A_4\in\mathcal{A}_{u',v'}}} W_p(A_3)\,W_p(A_4)\Bigg)\prod_{i\neq j}\mathbb{1}_{\{V_{A_i}\cap V_{A_j}=\varnothing\}}. \tag{4.1.21}$$

Since Λ is symmetric with respect to the underlying lattice symmetry, the first and second terms are the same. Due to the same reason, the third term remains unchanged when the restriction $u\prec u'$ is replaced by $u'\prec u$. Therefore, (4.1.21) equals

$$\sum_{x\neq o}\sum_{\substack{A_1\in\mathcal{A}_o\\ A_2\in\mathcal{A}_x}} W_p(A_1)\,W_p(A_2)\Bigg(\mathbb{1}_{\{x\in\Lambda\}}\frac{p}{|\Lambda|}\sum_{\substack{u\in\Lambda\\(u\neq x)}}\sum_{\substack{v\in\Lambda(x)\\(v\neq o)}}\bigg(\frac{p}{|\Lambda|}\bigg)^2\sum_{A_3\in\mathcal{A}_{u,v}} W_p(A_3)$$

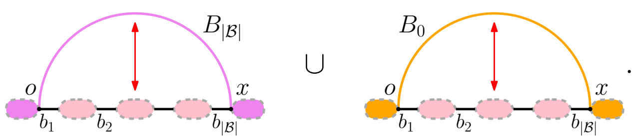
$$+\,\frac{1}{2}\sum_{\substack{u,u'\in\Lambda\\(u\neq u')}}\sum_{\substack{v,v'\in\Lambda(x)\\(v\neq v')}}\bigg(\frac{p}{|\Lambda|}\bigg)^4\sum_{\substack{A_3\in\mathcal{A}_{u,v}\\A_4\in\mathcal{A}_{u',v'}}} W_p(A_3)\,W_p(A_4)\Bigg)\prod_{i\neq j}\mathbb{1}_{\{V_{A_i}\cap V_{A_j}=\varnothing\}}. \tag{4.1.22}$$

Now we compare (4.1.22) with $\hat{\pi}_p^{(1)}$ for lattice animals, which is defined as (see Figure 4.2)

$$\hat{\pi}_p^{(1)}=\sum_x\sum_{\substack{\mathcal{B}=\{(\underline{b}_i,\overline{b}_i)\}_{i=1}^{|\mathcal{B}|}\\(|\mathcal{B}|\geq 1)}}\bigg(\frac{p}{|\Lambda|}\bigg)^{|\mathcal{B}|}\prod_{j=0}^{|\mathcal{B}|}\sum_{B_j\in\mathcal{A}_{\overline{b}_j,\underline{b}_{j+1}}} W_p(B_j)\,\mathbb{1}_{\{\overline{b}_j\Longleftrightarrow\underline{b}_{j+1}\text{ in }B_j\}}$$

$$\times\mathbb{1}_{\{B_0\cap B_{|\mathcal{B}|}\neq\varnothing\}}\prod_{\substack{0\leq k<l\leq|\omega|\\((k,l)\neq(0,|\mathcal{B}|))}}\mathbb{1}_{\{B_k\cap B_l=\varnothing\}}, \tag{4.1.23}$$

where we have abused the notation $\overline{b}_0=o$ and $\underline{b}_{|\mathcal{B}|+1}=x$. This can be bounded below by restricting the sum over $\mathcal{B}$ to those satisfying $\underline{b}_1=o$ and $\overline{b}_{|\mathcal{B}|}=x$ (so that $\mathcal{A}_{\overline{b}_0,\underline{b}_1}=\mathcal{A}_o$ and $\mathcal{A}_{\overline{b}_{|\mathcal{B}|},\underline{b}_{|\mathcal{B}|+1}}=\mathcal{A}_x$) and then by restricting the sum over $B_0\in\mathcal{A}_o$ to $B_0=\{o\}$ (so that $\mathbb{1}_{\{B_0\cap B_{|\mathcal{B}|}\neq\varnothing\}}=\mathbb{1}_{\{o\in B_{|\mathcal{B}|}\}}$) or restricting the sum over $B_{|\mathcal{B}|}\in\mathcal{A}_x$ to $B_{|\mathcal{B}|}=\{x\}$ (so that $\mathbb{1}_{\{B_0\cap B_{|\mathcal{B}|}\neq\varnothing\}}=\mathbb{1}_{\{x\in B_0\}}$):



. (4.1.24)

Those two terms are basically the same. Splitting the sum over ω into two depending on whether $|\mathcal{B}|=1$ (so that $\mathcal{B}=\{b_1\}$, where $b_1=(o,x)$) or $|\mathcal{B}|\geq 2$ and then, for the latter,

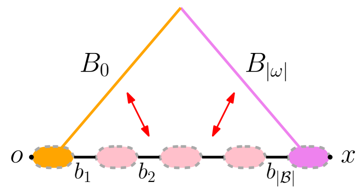


図 4.2: Schematic representation of $\pi_p^{(1)}(x)$. The sequence of edges $\mathcal{B} = \{b_1, \dots, b_{|\mathcal{B}|}\}$ joined by the animals $B_0, \dots, B_{|\mathcal{B}|}$ form the backbone from o to x in A. The terminal animals B_0 and $B_{|\mathcal{B}|}$ share a vertex (due to $\mathbb{1}_{\{B_0 \cap B_{|\mathcal{B}|} \neq \varnothing\}}$ in (4.1.23)), otherwise those animals are mutually avoiding (due to the product of indicators in (4.1.23)). Each animal B_j contains a double connection between $\overline{b}_j$ and $\underline{b}_{j+1}$, i.e., $B_j \in \mathcal{D}_{\overline{b}_j, \underline{b}_{j+1}}$.

by summing over the animals $B_1, \dots, B_{|\mathcal{B}|-1}$ (to form an animal $A_3 \in \mathcal{A}_{\overline{b}_1, \underline{b}_{|\mathcal{B}|}}$), we obtain

$$\hat{\pi}_p^{(1)} \geq 2 \sum_{x \neq o} \sum_{B \in \mathcal{A}_{o,x}} W_p(B) \Bigg(\mathbb{1}_{\{(o,x) \notin E_B\}} \mathbb{1}_{\{x \in \Lambda\}} \frac{p}{|\Lambda|} + \sum_{\substack{u \in \Lambda \\ (u \neq x)}} \sum_{\substack{v \in \Lambda(x) \\ (v \neq o)}} \left(\frac{p}{|\Lambda|} \right)^2 \sum_{A_3 \in \mathcal{A}_{u,v}} W_p(A_3) \, \mathbb{1}_{\{V_B \cap V_{A_3} = \varnothing\}} \Bigg). \tag{4.1.25}$$

We further bound this below by restricting the sum over $B \in \mathcal{A}_{o,x}$ to smaller animals $B = (V_B, E_B)$ with either

(i) $V_B = V_{A_1} \cup V_{A_2}$, $E_B = E_{A_1} \cup \{(o,x)\} \cup E_{A_2}$ for some $A_1 \in \mathcal{A}_o$, $A_2 \in \mathcal{A}_x$ (as in the left and middle figures of Figure 4.1), or

(ii) $V_B = V_{A_1} \cup V_{A_2} \cup V_{A_4}$, $E_B = E_{A_1} \cup \{(o,u')\} \cup E_{A_4} \cup \{(v',x)\} \cup E_{A_2}$ for some $A_1 \in \mathcal{A}_o$, $A_2 \in \mathcal{A}_x$, $u' \in \Lambda$, $v' \in \Lambda(x)$, $A_4 \in \mathcal{A}_{u',v'}$ (as in the right figure of Figure 4.1).

The contribution from (i) to the right-hand side of (4.1.25) is

$$2 \sum_{x \neq o} \sum_{\substack{A_1 \in \mathcal{A}_o \\ A_2 \in \mathcal{A}_x}} W_p(A_1) \, W_p(A_2) \mathbb{1}_{\{x \in \Lambda\}} \frac{p}{|\Lambda|} \sum_{\substack{u \in \Lambda \\ (u \neq x)}} \sum_{\substack{v \in \Lambda(x) \\ (v \neq o)}} \left(\frac{p}{|\Lambda|} \right)^2 \sum_{A_3 \in \mathcal{A}_{u,v}} W_p(A_3) \prod_{i \neq j} \mathbb{1}_{\{V_{A_i} \cap V_{A_j} = \varnothing\}}, \tag{4.1.26}$$

while the contribution from (ii) is

$$2 \sum_{x \neq o} \sum_{\substack{A_1 \in \mathcal{A}_o \\ A_2 \in \mathcal{A}_x}} W_p(A_1) \, W_p(A_2) \Bigg(\mathbb{1}_{\{x \in \Lambda\}} \frac{p}{|\Lambda|} \sum_{\substack{u \in \Lambda \\ (u \neq x)}} \sum_{\substack{v \in \Lambda(x) \\ (v \neq o)}} \left(\frac{p}{|\Lambda|} \right)^2 \sum_{A_3 \in \mathcal{A}_{u,v}} W_p(A_3) + \sum_{\substack{u,u' \in \Lambda \\ (u \neq u')}} \sum_{\substack{v,v' \in \Lambda(x) \\ (v \neq v')}} \left(\frac{p}{|\Lambda|} \right)^4 \sum_{\substack{A_3 \in \mathcal{A}_{u,v} \\ A_4 \in \mathcal{A}_{u',v'}}} W_p(A_3) \, W_p(A_4) \Bigg) \prod_{i \neq j} \mathbb{1}_{\{V_{A_i} \cap V_{A_j} = \varnothing\}}. \tag{4.1.27}$$

Notice that the sum of (4.1.26) and (4.1.27) is four times as large as (4.1.22). This completes the proof of $\hat{h}_p \leq \frac{1}{4}\hat{\pi}_p^{(1)} + O(\beta^2)$. ■

4.2 Detailed analysis of the 1-point function for lattice trees

To complete the proof of Theorem 4.0.1, it remains to investigate $p_1 = 1/g_{p_1}$ (due to (4.1.1) and (4.1.2)). In this section, we concentrate our attention to lattice trees and show the following:

Lemma 4.2.1. *For lattice trees with $d > 8$ and $L \uparrow \infty$,*

$$g_{p_1} = e\bigg(1 - \sum_{n=2}^{\infty} \frac{n+1}{2} D^{*n}(o)\bigg) + O(\beta^2). \tag{4.2.1}$$

Consequently,

$$p_1 = \frac{1}{e} + \sum_{n=2}^{\infty} \frac{n+1}{2e} D^{*n}(o) + O(\beta^2). \tag{4.2.2}$$

To prove Lemma 4.2.1, we first rewrite g_{p_1} by identifying the connected neighbors Y of the origin as

$$\begin{aligned} g_{p_1} &= \sum_{T \in \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_T|} \\ &= 1 + \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 1)}} \sum_{T \in \mathcal{T}_{Y \cup \{o\}}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_T|} \\ &= 1 + \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 1)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in \mathcal{T}_y \setminus \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|} \prod_{\substack{u,v \in Y \\ (u \neq v)}} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} = \varnothing\}}, \end{aligned} \tag{4.2.3}$$

where, and from now on, $\sum_{Y \subset \Lambda}$ is the sum over sets Y of distinct vertices of Λ (we recall that o is not included in Λ) and

$$\prod_{y \in Y} \sum_{R_y \in \mathcal{T}_y \setminus \mathcal{T}_o} (p_1/|\Lambda|)^{|E_{R_y}|} = \sum_{R_{y_1} \in \mathcal{T}_{y_1} \setminus \mathcal{T}_o} (p_1/|\Lambda|)^{|E_{R_{y_1}}|} \cdots \sum_{R_{y_n} \in \mathcal{T}_{y_n} \setminus \mathcal{T}_o} (p_1/|\Lambda|)^{|E_{R_{y_n}}|} \tag{4.2.4}$$

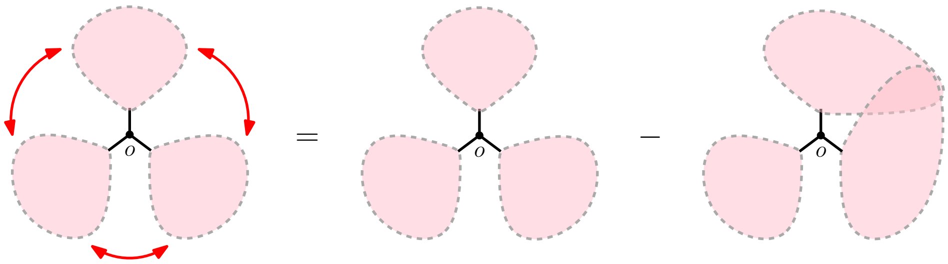


図 4.3: Intuitive explanation of (4.2.5). The double-headed arrows on the left ($= g_{p_1}$) represents mutual avoidance among subtrees. In the first term on the right ($= G$), those subtrees are independently summed over $\mathcal{T}_y \setminus \mathcal{T}_o$, $y \in Y$, where Y is the set of connected neighbors of the origin. In the second term on the right ($= H$), there is at least one pair of subtrees that share vertices.

for $Y = \{y_1, \ldots, y_n\}$. By convention, $\prod_{u \neq v} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} = \varnothing\}}$ is regarded as 1 when $|Y| = 1$. Let (see Figure 4.3)

$$g_{p_1} = G - H, \tag{4.2.5}$$

where

$$G = 1 + \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 1)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in \mathcal{T}_y \setminus \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|}, \tag{4.2.6}$$

$$H = \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 2)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in \mathcal{T}_y \setminus \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|} \left(1 - \prod_{\substack{u,v \in Y \\ (u \neq v)}} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} = \varnothing\}}\right). \tag{4.2.7}$$

We investigate those G and H in Sections 4.2.1 and 4.2.2, respectively (cf., Lemmas 4.2.2 and 4.2.4 below).

4.2.1 Detailed analysis of G

From now on, we frequently use

$$S_{\geq t}(x) = \sum_{n=t}^{\infty} D^{*n}(x), \tag{4.2.8}$$

where $D^{*0}(x) = \delta_{o,x}$ by convention. The following is what we are going to show in this section:

Lemma 4.2.2. *For lattice trees with $d > 8$ and $L \uparrow \infty$,*

$$G = e\left(1 - \frac{1}{2}D^{*2}(o) - S_{\geq 2}(o)\right) + O(\beta^2). \tag{4.2.9}$$

Proof. Since $p_1 g_{p_1} = 1$, we can rewrite G as

$$G = 1 + \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 1)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \prod_{y \in Y} \Bigg(\underbrace{\sum_{R_y \in \mathcal{T}_y} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|}}_{g_{p_1}} - \underbrace{\sum_{R_y \in \mathcal{T}_{o,y}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|}}_{\tau_{p_1}(y)} \Bigg)$$

$$= 1 + \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 1)}} \left(\frac{1}{|\Lambda|}\right)^{|Y|} \prod_{y \in Y} \left(1 - \frac{\tau_{p_1}(y)}{g_{p_1}}\right). \tag{4.2.10}$$

If we replace $\prod_{y \in Y}(1 - \tau_{p_1}(y)/g_{p_1})$ by 1, then we obtain

$$G_0 := 1 + \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 1)}} \left(\frac{1}{|\Lambda|}\right)^{|Y|} = \left(1 + \frac{1}{|\Lambda|}\right)^{|\Lambda|}. \tag{4.2.11}$$

Since $k \log(1 + 1/k) = 1 - 1/(2k) + O(k^{-2})$ as $k \uparrow \infty$, and since $|\Lambda|^{-1} = D^{*2}(o)$, we obtain

$$G_0 = e\left(1 - \frac{1}{2|\Lambda|}\right) + O(|\Lambda|^{-2}) = e\left(1 - \frac{1}{2} D^{*2}(o)\right) + O(\beta^2). \tag{4.2.12}$$

Next we consider the remainder, which is

$$G - G_0 = \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 1)}} \left(\frac{1}{|\Lambda|}\right)^{|Y|} \left(\prod_{y \in Y} \left(1 - \frac{\tau_{p_1}(y)}{g_{p_1}}\right) - 1\right)$$

$$= \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 1)}} \left(\frac{1}{|\Lambda|}\right)^{|Y|} \sum_{\substack{Z \subset Y \\ (|Z| \geq 1)}} \prod_{y \in Z} \frac{-\tau_{p_1}(y)}{g_{p_1}}. \tag{4.2.13}$$

Changing the order of sums yields

$$G - G_0 = \sum_{\substack{Z \subset \Lambda \\ (|Z| \geq 1)}} \prod_{y \in Z} \frac{-\tau_{p_1}(y)}{g_{p_1}} \left(\frac{1}{|\Lambda|}\right)^{|Z|} \sum_{Z \subset Y \subset \Lambda} \left(\frac{1}{|\Lambda|}\right)^{|Y \setminus Z|}$$

$$= \sum_{\substack{Z \subset \Lambda \\ (|Z| \geq 1)}} \prod_{y \in Z} \frac{-\tau_{p_1}(y)\, D(y)}{g_{p_1}} \left(1 + \frac{1}{|\Lambda|}\right)^{|\Lambda \setminus Z|}$$

$$= G_1 + G_2, \tag{4.2.14}$$

where

$$G_1 = \sum_{\substack{Z \subset \Lambda \\ (|Z| = 1)}} \prod_{y \in Z} \frac{-\tau_{p_1}(y)\, D(y)}{g_{p_1}} \left(1 + \frac{1}{|\Lambda|}\right)^{|\Lambda \setminus Z|} = \frac{-(\tau_{p_1} * D)(o)}{g_{p_1}} \frac{G_0}{(1 + 1/|\Lambda|)}, \tag{4.2.15}$$

$$G_2 = \sum_{\substack{Z \subset \Lambda \\ (|Z| \geq 2)}} \prod_{y \in Z} \frac{-\tau_{p_1}(y)\, D(y)}{g_{p_1}} \left(1 + \frac{1}{|\Lambda|}\right)^{|\Lambda \setminus Z|}. \tag{4.2.16}$$

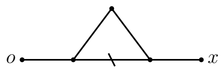
We make use of reflection symmetry for (4.2.14)–(4.2.16), and will use it below frequently. To estimate G_1 and G_2, we use the following lemma, which will be proven after the proof of Lemma 4.2.2 is completed.

Lemma 4.2.3. *For any $d > 2$ and $x \neq o$, the lattice-tree 2-point function satisfies*

$$0 \leq S_{\geq 1}(x) - \frac{\tau_{p_1}(x)}{g_{p_1}} \leq \sum_{\substack{y,z \in \mathbb{Z}^d \\ (y \neq z)}} S_{\geq 0}^{*2}(z-y)\, S_{\geq 0}(y)\, S_{\geq 1}(z-y)\, S_{\geq 0}(x-z). \qquad (4.2.17)$$

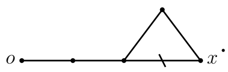
Remark. The right-hand side of (4.2.17) is diagrammatically represented by

$$\sum_{\substack{y,z \in \mathbb{Z}^d \\ (y \neq z)}} S_{\geq 0}^{*2}(z-y)\, S_{\geq 0}(y)\, S_{\geq 1}(z-y)\, S_{\geq 0}(x-z) = \qquad (4.2.18)$$



where an unslashed (resp., slashed) line represents $S_{\geq 0}$ (resp., $S_{\geq 1}$) and an unlabelled vertex is summed over $\mathbb{Z}^d$. Due to translation invariance, we can change the order of terms in (4.2.17) for a given $x \neq o$. Then (4.2.18) is also equal to

$$\sum_{\substack{y \in \mathbb{Z}^d \\ (y \neq x)}} S_{\geq 0}^{*2}(y)\, S_{\geq 0}^{*2}(x-y)\, S_{\geq 1}(x-y) = \qquad (4.2.19)$$



These diagramatic representations will be used in the proof of Lemma 4.2.4 below.

First we estimate G_2. By the first inequality in (4.2.17) and (1.1.3), we can show

$$\frac{(\tau_{p_1} * D)(o)}{g_{p_1}} \leq \sup_{x \in \Lambda} \frac{\tau_{p_1}(x)}{g_{p_1}} \leq \sup_{x \in \Lambda} S_{\geq 1}(x) \leq \sum_{n=1}^{\infty} \|D^{*n}\|_\infty \stackrel{d \geq 2}{=} O(\beta). \qquad (4.2.20)$$

Therefore,

$$|G_2| \leq G_0 \sum_{n=2}^{|\Lambda|} \sum_{\substack{Z \subset \Lambda \\ (|Z|=n)}} \prod_{y \in Z} \frac{\tau_{p_1}(y)\, D(y)}{g_{p_1}} \leq G_0 \sum_{n=2}^{\infty} \left(\frac{(\tau_{p_1} * D)(o)}{g_{p_1}} \right)^n = O(\beta^2). \qquad (4.2.21)$$

Next we estimate G_1 in (4.2.15). By using (4.2.17), we have

$$
\begin{aligned}
0 \le S_{\ge 2}(o) - \frac{(\tau_{p_1} * D)(o)}{g_{p_1}} &= \sum_{x\in\mathbb{Z}^d} D(x) \left(S_{\ge 1}(x) - \frac{\tau_{p_1}(x)}{g_{p_1}} \right) \\
&\overset{(4.2.17)}{\le} \sum_{x\in\mathbb{Z}^d} D(x) \sum_{\substack{y,z\in\mathbb{Z}^d \\ (y\neq z)}} S_{\ge 0}^{*2}(z-y)\, S_{\ge 0}(y)\, S_{\ge 1}(z-y)\, S_{\ge 0}(x-z) \\
&\le \sum_{\substack{y,z\in\mathbb{Z}^d \\ (y\neq z)}} S_{\ge 0}^{*2}(z-y)\, S_{\ge 0}(y)\, S_{\ge 1}(z-y)\, S_{\ge 1}(z) \\
&\le \sup_{w\neq o} S_{\ge 0}^{*2}(w)\ \left(S_{\ge 0} * S_{\ge 1}^{*2}\right)(o). \qquad (4.2.22)
\end{aligned}
$$

By (1.1.4), we can estimate each term as

$$
S_{\ge 0}^{*2}(w) = \sum_{s,t=0}^{\infty} D^{*(s+t)}(w) \overset{w\neq o}{\le} \sum_{n=1}^{\infty} (n+1) D^{*n}(w) \overset{d\ge 4}{=} O(\beta), \qquad (4.2.23)
$$

$$
\left(S_{\ge 0} * S_{\ge 1}^{*2}\right)(o) = \sum_{s=0}^{\infty} \sum_{t,u=1}^{\infty} D^{*(s+t+u)}(o) = \sum_{n=2}^{\infty} \binom{n}{2} D^{*n}(o) \overset{d\ge 6}{=} O(\beta), \qquad (4.2.24)
$$

so that

$$
0 \le S_{\ge 2}(o) - \frac{(\tau_{p_1} * D)(o)}{g_{p_1}} = O(\beta^2). \qquad (4.2.25)
$$

Therefore, by (4.2.12),

$$
\begin{aligned}
G_1 &= (-\underbrace{S_{\ge 2}(o)}_{\overset{(4.2.20)}{=} O(\beta)} + O(\beta^2)) \Big(e - \underbrace{\frac{e}{2} D^{*2}(o)}_{\overset{(??)}{=} O(\beta)} + O(\beta^2) \Big) \\
&= -e S_{\ge 2}(o) + O(\beta^2). \qquad (4.2.26)
\end{aligned}
$$

Summarizing (4.2.12), (4.2.14), (4.2.21) and (4.2.26), we complete the proof of Lemma 4.2.2. ■

Proof of Lemma 4.2.3. First we recall

$$
\frac{\tau_{p_1}(x)}{g_{p_1}} = \frac{1}{g_{p_1}} \sum_{T\in\mathcal{T}_{o,x}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_T|}. \qquad (4.2.27)
$$

Since a tree $T \in \mathcal{T}_{o,x}$ can be divided into a unique path $\omega = (\omega_0, \omega_1, \dots, \omega_{|\omega|})$ from $\omega_0 = o$ to $\omega_{|\omega|} = x$, called a backbone, and disjoint subtrees $R_j \in \mathcal{T}_{\omega_j}$, called ribs (see Figure 1.2), we can rewrite the above expression as

$$
\frac{\tau_{p_1}(x)}{g_{p_1}} = \frac{1}{g_{p_1}} \sum_{\omega:o\to x} \left(\frac{p_1}{|\Lambda|}\right)^{|\omega|} \prod_{j=0}^{|\omega|} \sum_{R_j\in\mathcal{T}_{\omega_j}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_j}|} \prod_{s<t} \mathbb{1}_{\{V_{R_s}\cap V_{R_t}=\varnothing\}}. \qquad (4.2.28)
$$

If we replace the indicator $\prod_{s<t} \mathbb{1}_{\{V_{R_s} \cap V_{R_t} = \varnothing\}}$ by 1, then we obtain

$$\frac{1}{g_{p_1}} \sum_{\omega : o \to x} \left(\frac{p_1}{|\Lambda|}\right)^{|\omega|} \prod_{j=0}^{|\omega|} \underbrace{\sum_{R_j \in \mathcal{T}_{\omega_j}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_j}|}}_{g_{p_1}} \overset{(4.1.1)}{=} \sum_{\omega : o \to x} \left(\frac{1}{|\Lambda|}\right)^{|\omega|} \overset{x \neq o}{=} S_{\geq 1}(x). \tag{4.2.29}$$

Next we consider the remainder. Since $1 - \prod_{j=1}^n a_j \leq \sum_{j=1}^n (1 - a_j)$ as long as $0 \leq a_j \leq 1$ for all j, we can bound the remainder as

$$\begin{aligned} S_{\geq 1}(x) - \frac{\tau_{p_1}(x)}{g_{p_1}} &= \frac{1}{g_{p_1}} \sum_{\omega : o \to x} \left(\frac{p_1}{|\Lambda|}\right)^{|\omega|} \prod_{j=0}^{|\omega|} \sum_{R_j \in \mathcal{T}_{\omega_j}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_j}|} \left(1 - \prod_{s<t} \mathbb{1}_{\{V_{R_s} \cap V_{R_t} = \varnothing\}}\right) \\ &\leq \frac{1}{g_{p_1}} \sum_{\omega : o \to x} \left(\frac{p_1}{|\Lambda|}\right)^{|\omega|} \prod_{j=0}^{|\omega|} \sum_{R_j \in \mathcal{T}_{\omega_j}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_j}|} \sum_{s<t} \mathbb{1}_{\{V_{R_s} \cap V_{R_t} \neq \varnothing\}}. \end{aligned} \tag{4.2.30}$$

If $V_{R_s} \cap V_{R_t} \neq \varnothing$, then there must be a $w \in \mathbb{Z}^d$ that is shared by those two ribs. Therefore, the remainder is further bounded above as

$$\begin{aligned} S_{\geq 1}(x) - \frac{\tau_{p_1}(x)}{g_{p_1}} &\leq \frac{1}{g_{p_1}} \sum_{\omega : o \to x} \left(\frac{p_1}{|\Lambda|}\right)^{|\omega|} \sum_{s<t} \underbrace{\prod_{j \neq s,t} \sum_{R_j \in \mathcal{T}_{\omega_j}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_j}|}}_{g_{p_1}^{|\omega|-1}} \\ &\qquad \times \sum_{w \in \mathbb{Z}^d} \underbrace{\sum_{R_s \in \mathcal{T}_{\omega_s, w}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_s}|}}_{\tau_{p_1}(w - \omega_s)} \underbrace{\sum_{R_t \in \mathcal{T}_{\omega_t, w}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_t}|}}_{\tau_{p_1}(\omega_t - w)} \\ &= \frac{1}{g_{p_1}^2} \sum_{\omega : o \to x} \left(\frac{1}{|\Lambda|}\right)^{|\omega|} \sum_{s<t} \sum_{w \in \mathbb{Z}^d} \tau_{p_1}(w - \omega_s)\, \tau_{p_1}(\omega_t - w) \\ &= \sum_{\substack{y, z \in \mathbb{Z}^d \\ (y \neq z)}} \frac{\tau_{p_1}^{*2}(z - y)}{g_{p_1}^2} \sum_{\omega : o \to y \to z \to x} \left(\frac{1}{|\Lambda|}\right)^{|\omega|} \\ &= \sum_{\substack{y, z \in \mathbb{Z}^d \\ (y \neq z)}} \frac{\tau_{p_1}^{*2}(z - y)}{g_{p_1}^2} S_{\geq 0}(y)\, S_{\geq 1}(z - y)\, S_{\geq 0}(x - z). \end{aligned} \tag{4.2.31}$$

The proof of (4.2.17) is completed by applying (4.2.28)–(4.2.29) to $(\tau_{p_1}/g_{p_1})^{*2}$ in the above bound. ∎

4.2.2 Detailed analysis of H

To complete the proof of Lemma 4.2.1, it suffices to show the following:

Lemma 4.2.4. *For lattice trees with $d > 8$ and $L \uparrow \infty$,*

$$H = e \sum_{n=3}^{\infty} \frac{n-1}{2} D^{*n}(o) + O(\beta^2). \tag{4.2.32}$$

Proof. Recall the definition (4.2.7) of H:

$$H = \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 2)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in \mathcal{T}_y \setminus \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|} \left(1 - \prod_{\substack{u,v \in Y \\ (u \neq v)}} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} = \varnothing\}}\right). \tag{4.2.33}$$

First we split the indicator $1 - \prod_{u \neq v} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} = \varnothing\}}$ in (4.2.7) by introducing order among pairs of distinct vertices in Λ, called bonds. If a bond b is earlier than another bond b' in that order, we denote it by $b < b'$. Then we have

$$\begin{aligned}
&1 - \prod_{\{u,v\} \subset Y} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} = \varnothing\}} \\
&= \sum_{\{u,v\} \subset Y} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \varnothing\}} \prod_{\substack{\{u',v'\} \subset Y \\ (\{u',v'\} < \{u,v\})}} \mathbb{1}_{\{V_{R_{u'}} \cap V_{R_{v'}} = \varnothing\}} \\
&= \sum_{\{u,v\} \subset Y} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \varnothing\}} - \sum_{\{u,v\} \subset Y} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \varnothing\}} \Bigg(1 - \prod_{\substack{\{u',v'\} \subset Y \\ (\{u',v'\} < \{u,v\})}} \mathbb{1}_{\{V_{R_{u'}} \cap V_{R_{v'}} = \varnothing\}}\Bigg),
\end{aligned} \tag{4.2.34}$$

where the second sum on the right is zero when $|Y| = 2$. Let H_1 be the contribution from the first sum on the right:

$$H_1 = \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 2)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \prod_{y \in Y} \sum_{R_y \in \mathcal{T}_y \setminus \mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|} \sum_{\{u,v\} \subset Y} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \varnothing\}}. \tag{4.2.35}$$

We will later show (after the derivation of (4.2.32); see (4.2.53)) that

$$H_2 := H_1 - H = O(\beta^2). \tag{4.2.36}$$

Next we investigate H_1. Let H_1' be the contribution from the case of $|Y| = 2$:

$$H_1' = \sum_{\{u,v\} \subset \Lambda} \left(\frac{p_1}{|\Lambda|}\right)^2 \sum_{\substack{R_u \in \mathcal{T}_u \setminus \mathcal{T}_o \\ R_v \in \mathcal{T}_v \setminus \mathcal{T}_o}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_u}| + |E_{R_v}|} \mathbb{1}_{\{V_{R_u} \cap V_{R_v} \neq \varnothing\}}. \tag{4.2.37}$$

By subadditivity, we already know that $H_1' = O(\beta)$ for $d > 4$. By changing the order of sums, we can rewrite $H_1 - H_1'$ as

$$\begin{aligned}
H_1 - H_1' &= \sum_{\substack{Y\subset\Lambda \\ (|Y|\geq 3)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \prod_{y\in Y} \sum_{R_y\in\mathcal{T}_y\setminus\mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|} \sum_{\{u,v\}\subset Y} \mathbb{1}_{\{V_{R_u}\cap V_{R_v}\neq\varnothing\}} \\
&= \underbrace{\sum_{\{u,v\}\subset\Lambda} \left(\frac{p_1}{|\Lambda|}\right)^{2} \sum_{\substack{R_u\in\mathcal{T}_u\setminus\mathcal{T}_o \\ R_v\in\mathcal{T}_v\setminus\mathcal{T}_o}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_u}|+|E_{R_v}|} \mathbb{1}_{\{V_{R_u}\cap V_{R_v}\neq\varnothing\}}}_{H_1'} \\
&\qquad \times \sum_{\substack{Y'\subset\Lambda\setminus\{u,v\} \\ (|Y'|\geq 1)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y'|} \prod_{y'\in Y'} \underbrace{\sum_{R_{y'}\in\mathcal{T}_{y'}\setminus\mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_{y'}}|}}_{g_{p_1}-\tau_{p_1}(y')}. \qquad (4.2.38)
\end{aligned}$$

Similarly to the proof of Lemma 4.2.2, we can show that the last line is estimated as

$$\begin{aligned}
&\sum_{\substack{Y'\subset\Lambda\setminus\{u,v\} \\ (|Y'|\geq 1)}} \left(\frac{1}{|\Lambda|}\right)^{|Y'|} \prod_{y'\in Y'} \left(1 - \frac{\tau_{p_1}(y')}{g_{p_1}}\right) \\
&= \left(1 + \frac{1}{|\Lambda|}\right)^{|\Lambda|-2} - 1 + \sum_{\substack{Y'\subset\Lambda\setminus\{u,v\} \\ (|Y'|\geq 1)}} \left(\frac{1}{|\Lambda|}\right)^{|Y'|} \left(\prod_{y'\in Y'} \left(1 - \frac{\tau_{p_1}(y')}{g_{p_1}}\right) - 1\right) \\
&= e - 1 + O(\beta). \qquad (4.2.39)
\end{aligned}$$

Therefore,

$$H_1 - H_1' = H_1'\big(e - 1 + O(\beta)\big), \qquad (4.2.40)$$

or equivalently

$$H_1 = eH_1' + O(\beta^2). \qquad (4.2.41)$$

Next we investigate H_1'. To do so, we first rewrite $\mathbb{1}_{\{V_{R_u}\cap V_{R_v}\neq\varnothing\}}$ in (4.2.37) by introducing order among vertices in $\mathbb{Z}^d$. For a vertex set V and an element $x \in V$, we denote by $V^{<x}$ the set of vertices in V that are earlier than x in that order. By identifying the earliest element x among V_{R_u} that is also in V_{R_v} (so that $V_{R_u}^{<x} \cap V_{R_v} = \varnothing$), we can rewrite $\mathbb{1}_{\{V_{R_u}\cap V_{R_v}\neq\varnothing\}}$ as

$$\begin{aligned}
\mathbb{1}_{\{V_{R_u}\cap V_{R_v}\neq\varnothing\}} &= \sum_{x\in V_{R_u}} \mathbb{1}_{\{x\in V_{R_v}\}} \mathbb{1}_{\{V_{R_u}^{<x}\cap V_{R_v}=\varnothing\}} \\
&= \sum_{x\in\mathbb{Z}^d} \mathbb{1}_{\{x\in V_{R_u}\cap V_{R_v}\}} - \sum_{x\in V_{R_u}\cap V_{R_v}} \Big(1 - \mathbb{1}_{\{V_{R_u}^{<x}\cap V_{R_v}=\varnothing\}}\Big). \qquad (4.2.42)
\end{aligned}$$

Let H_1'' be the contribution from the first sum in the last line:

$$\begin{aligned}
H_1'' &= \sum_{\{u,v\}\subset\Lambda}\left(\frac{p_1}{|\Lambda|}\right)^2\sum_{x\in\mathbb{Z}^d}\sum_{\substack{R_u\in\mathcal{T}_{u,x}\setminus\mathcal{T}_o\\ R_v\in\mathcal{T}_{v,x}\setminus\mathcal{T}_o}}\left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_u}|+|E_{R_v}|}\\
&= \sum_{\{u,v\}\subset\Lambda}\left(\frac{p_1}{|\Lambda|}\right)^2\sum_{x\in\mathbb{Z}^d}\Big(\tau_{p_1}(u-x)-\tau_{p_1}^{(3)}(o,u,x)\Big)\Big(\tau_{p_1}(v-x)-\tau_{p_1}^{(3)}(o,v,x)\Big)\\
&\overset{(4.1.1)}{=} \sum_{\{u,v\}\subset\Lambda}\left(\frac{1}{|\Lambda|}\right)^2\sum_{x\in\mathbb{Z}^d}\left(\frac{\tau_{p_1}(u-x)}{g_{p_1}}-\frac{\tau_{p_1}^{(3)}(o,u,x)}{g_{p_1}}\right)\left(\frac{\tau_{p_1}(v-x)}{g_{p_1}}-\frac{\tau_{p_1}^{(3)}(o,v,x)}{g_{p_1}}\right),
\end{aligned}\tag{4.2.43}$$

where $\tau_p^{(3)}(o,u,x)$ is a 3-point function, defined as

$$\tau_p^{(3)}(o,u,x)=\sum_{T\in\mathcal{T}_{o,u,x}}W_p(T).\tag{4.2.44}$$

We will later show that

$$H_2'':=H_1''-H_1'=O(\beta^2).\tag{4.2.45}$$

Finally we investigate H_1''. The dominant contribution to H_1'' comes from the product of 2-point functions:

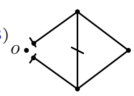
$$\begin{aligned}
H_1''' &:= \sum_{\{u,v\}\subset\Lambda}\left(\frac{1}{|\Lambda|}\right)^2\sum_{x\in\mathbb{Z}^d}\frac{\tau_{p_1}(u-x)}{g_{p_1}}\frac{\tau_{p_1}(v-x)}{g_{p_1}}\\
&= 2\sum_{\{u,v\}\subset\Lambda}\left(\frac{1}{|\Lambda|}\right)^2\frac{\tau_{p_1}(u-v)}{g_{p_1}}+\sum_{\{u,v\}\subset\Lambda}\left(\frac{1}{|\Lambda|}\right)^2\sum_{x\neq u,v}\frac{\tau_{p_1}(u-x)}{g_{p_1}}\frac{\tau_{p_1}(v-x)}{g_{p_1}},
\end{aligned}\tag{4.2.46}$$

where we have used the identity $\tau_p(o)=g_p$. We will later show that the other contribution to H_1'' which involves 3-point functions is estimated as

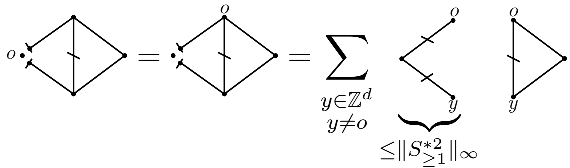
$$H_2''':=H_1''-H_1'''=O(\beta^2).\tag{4.2.47}$$

By Lemma 4.2.3, the first term in (4.2.46) is estimated as

$$\begin{aligned}
&2\sum_{\{u,v\}\subset\Lambda}\left(\frac{1}{|\Lambda|}\right)^2\frac{\tau_{p_1}(u-v)}{g_{p_1}}\\
&=\sum_{\substack{u,v\in\Lambda\\(u\neq v)}}\left(\frac{1}{|\Lambda|}\right)^2S_{\ge1}(u-v)+\sum_{\substack{u,v\in\Lambda\\(u\neq v)}}\left(\frac{1}{|\Lambda|}\right)^2\left(\frac{\tau_{p_1}(u-v)}{g_{p_1}}-S_{\ge1}(u-v)\right)\\
&=S_{\ge3}(o)-\underbrace{\frac{1}{|\Lambda|}S_{\ge1}(o)}_{O(\beta^2)\text{ for }d>2}+\underbrace{\sum_{\substack{u,v\in\Lambda\\(u\neq v)}}\left(\frac{1}{|\Lambda|}\right)^2\left(\frac{\tau_{p_1}(u-v)}{g_{p_1}}-S_{\ge1}(u-v)\right)}_{\overset{(4.2.17)-(4.2.18)}{\le}}
\end{aligned}\tag{4.2.48}$$



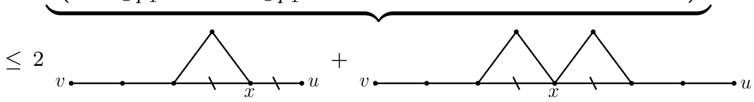
where a gap next to the origin in the last diagram represents $1/|\Lambda|$. By translation invariance and (4.2.23)-(4.2.24), the last term is bounded above by



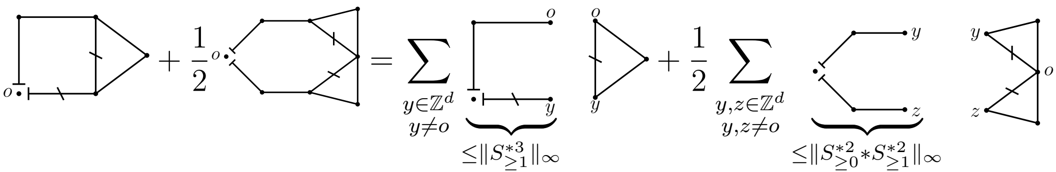
$$\le \|S^{*2}_{\ge 1}\|_\infty (S^{*2}_{\ge 0} * S_{\ge 1})(o) = O(\beta^2). \tag{4.2.49}$$

Similarly, the second term in (4.2.46) is estimated as

$$\begin{aligned}
&\sum_{\{u,v\}\subset\Lambda} \left(\frac{1}{|\Lambda|}\right)^2 \sum_{x\neq u,v} \frac{\tau_{p_1}(u-x)}{g_{p_1}} \frac{\tau_{p_1}(v-x)}{g_{p_1}} \\
&= \frac{1}{2} S^{*2}_{\ge 2}(o) - \underbrace{\frac{1}{2|\Lambda|} S^{*2}_{\ge 1}(o)}_{O(\beta^2) \text{ for } d>4} \\
&+ \frac{1}{2} \sum_{\substack{u,v\in\Lambda \\ u\neq v}} \left(\frac{1}{|\Lambda|}\right)^2 \sum_{x\neq u,v} \underbrace{\left(\frac{\tau_{p_1}(u-x)}{g_{p_1}} \frac{\tau_{p_1}(v-x)}{g_{p_1}} - S_{\ge 1}(u-x) S_{\ge 1}(v-x)\right)} \quad .
\end{aligned} \tag{4.2.50}$$



By Lemma 4.2.3, (4.2.19) and the translation invariance, the last term of (4.2.50) is bounded above by



$$\begin{aligned}
&\le \underbrace{\|S^{*3}_{\ge 1}\|_\infty (S^{*2}_{\ge 0} * S_{\ge 1})(o)}_{O(\beta^2) \text{ for } d>6} + \frac{1}{2} \underbrace{\|S^{*2}_{\ge 0} * S^{*2}_{\ge 1}\|_\infty (S^{*2}_{\ge 0} * S_{\ge 1})(o)^2}_{O(\beta^3) \text{ for } d>8} \\
&= O(\beta^2).
\end{aligned} \tag{4.2.51}$$

Therefore,

$$\begin{aligned}
H_1''' = S_{\ge 3}(o) + \frac{1}{2} S^{*2}_{\ge 2}(o) + O(\beta^2) &= \sum_{n=3}^{\infty} D^{*n}(o) + \frac{1}{2} \sum_{n,m=2}^{\infty} D^{*(n+m)}(o) + O(\beta^2) \\
&= \sum_{n=3}^{\infty} \frac{n-1}{2} D^{*n}(o) + O(\beta^2).
\end{aligned} \tag{4.2.52}$$

Summarizing all the above estimates, we arrive at

$$\begin{aligned}
H \overset{(4.2.36)}{=} H_1 + O(\beta^2) \overset{(4.2.41)}{=} eH_1' + O(\beta^2) \overset{(4.2.45)}{=} eH_1'' + O(\beta^2) \overset{(4.2.47)}{=} eH_1''' + O(\beta^2) \\
\overset{(4.2.52)}{=} e \sum_{n=3}^{\infty} \frac{n-1}{2} D^{*n}(o) + O(\beta^2),
\end{aligned} \tag{4.2.53}$$

as required. It remains to show (4.2.36), (4.2.45) and (4.2.47). ■

Proof of (4.2.36): bounding H_2. First we recall that H_2 is the contribution from the second sum on the right of (4.2.34):

$$H_2 = \sum_{\substack{Y\subset\Lambda \\ (|Y|\geq 3)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \prod_{y\in Y} \sum_{R_y\in\mathcal{T}_y\setminus\mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|} \sum_{\{u,v\}\subset Y} \mathbb{1}_{\{V_{R_u}\cap V_{R_v}\neq\varnothing\}} \times\Bigg(1-\prod_{\substack{\{u',v'\}\subset Y \\ (\{u',v'\}<\{u,v\})}} \mathbb{1}_{\{V_{R_{u'}}\cap V_{R_{v'}}=\varnothing\}}\Bigg), \tag{4.2.54}$$

which is nonnegative. Since we get an upper bound

$$1-\prod_{\substack{\{u',v'\}\subset Y \\ (\{u',v'\}<\{u,v\})}} \mathbb{1}_{\{V_{R_{u'}}\cap V_{R_{v'}}=\varnothing\}} \leq \sum_{\substack{\{u',v'\}\subset Y \\ (\{u',v'\}<\{u,v\})}} \mathbb{1}_{\{V_{R_{u'}}\cap V_{R_{v'}}\neq\varnothing\}} \tag{4.2.55}$$

in a same manner as (4.2.34), we can bound H_2 as

$$\begin{aligned} H_2 &\leq \sum_{\substack{Y\subset\Lambda \\ (|Y|\geq 3)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \prod_{y\in Y} \sum_{R_y\in\mathcal{T}_y\setminus\mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|} \sum_{\substack{\{u,v\}\subset Y \\ \{u',v'\}\subset Y \\ (\{u',v'\}<\{u,v\})}} \mathbb{1}_{\{V_{R_u}\cap V_{R_v}\neq\varnothing\}}\mathbb{1}_{\{V_{R_{u'}}\cap V_{R_{v'}}\neq\varnothing\}} \\ &= \frac{1}{2}\sum_{\substack{Y\subset\Lambda \\ (|Y|\geq 3)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \prod_{y\in Y} \sum_{R_y\in\mathcal{T}_y\setminus\mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|} \sum_{\substack{\{u,v\}\subset Y \\ \{u',v'\}\subset Y \\ (\{u',v'\}\neq\{u,v\})}} \mathbb{1}_{\{V_{R_u}\cap V_{R_v}\neq\varnothing\}}\mathbb{1}_{\{V_{R_{u'}}\cap V_{R_{v'}}\neq\varnothing\}}. \end{aligned} \tag{4.2.56}$$

Since $\{u,v\}\neq\{u',v'\}$, the union $\{u,v\}\cup\{u',v'\}$ consists of either three or four distinct vertices. We denote the contribution from the former by $H_{2,3}$, and that from the latter by $H_{2,4}$ and then we obtain

$$H_2 \leq \frac{1}{2}(H_{2,3}+H_{2,4}). \tag{4.2.57}$$

First we investigate $H_{2,4}$, which is bounded as (see Figure 4.4)

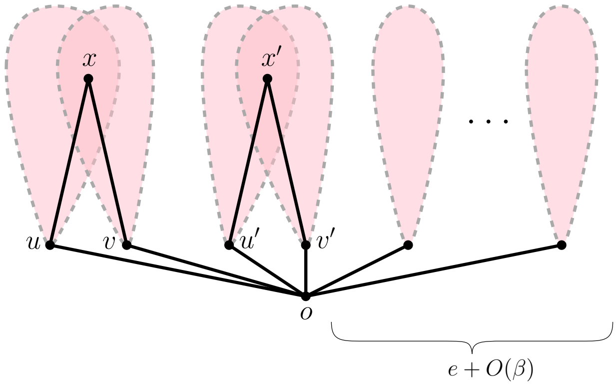


図 4.4: Schematic representation of $H_{2,4}$.

$$
\begin{aligned}
H_{2,4} &= \sum_{\substack{Y\subset\Lambda \\ (|Y|\geq 4)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \prod_{y\in Y} \sum_{R_y\in\mathcal{T}_y\setminus\mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|} \sum_{\substack{u,v,u',v'\in Y \\ \text{(distinct)}}} \mathbb{1}_{\{V_{R_u}\cap V_{R_v}\neq\varnothing\}} \mathbb{1}_{\{V_{R_{u'}}\cap V_{R_{v'}}\neq\varnothing\}} \\
&= \sum_{\substack{u,v,u',v'\in\Lambda \\ \text{(distinct)}}} \left(\frac{p_1}{|\Lambda|}\right)^{4} \sum_{\substack{R_u\in\mathcal{T}_u\setminus\mathcal{T}_o \\ R_v\in\mathcal{T}_v\setminus\mathcal{T}_o \\ R_{u'}\in\mathcal{T}_{u'}\setminus\mathcal{T}_o \\ R_{v'}\in\mathcal{T}_{v'}\setminus\mathcal{T}_o}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_u}|+|E_{R_v}|+|E_{R_{u'}}|+|E_{R_{v'}}|} \mathbb{1}_{\{V_{R_u}\cap V_{R_v}\neq\varnothing\}} \mathbb{1}_{\{V_{R_{u'}}\cap V_{R_{v'}}\neq\varnothing\}} \\
&\quad\times \left(1 + \sum_{\substack{Y'\subset\Lambda\setminus\{u,v,u',v'\} \\ (|Y'|\geq 1)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y'|} \prod_{y\in Y'} \underbrace{\sum_{R_y\in\mathcal{T}_y\setminus\mathcal{T}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|}}_{g_{p_1}-\tau_{p_1}(y)}\right).
\end{aligned}
\tag{4.2.58}
$$

By (4.1.1) (i.e., $p_1 g_{p_1} = 1$), the last line is equal to

$$
1 + \sum_{\substack{Y'\subset\Lambda\setminus\{u,v,u',v'\} \\ (|Y'|\geq 1)}} \left(\frac{1}{|\Lambda|}\right)^{|Y'|} = \left(1+\frac{1}{|\Lambda|}\right)^{|\Lambda|-4} = e + O(\beta). \tag{4.2.59}
$$

Then, by ignoring the constraint that $\{u,v\}$ and $\{u',v'\}$ are disjoint pairs and using the trivial inequality $\mathbb{1}_{\{V_{R_u}\cap V_{R_v}\neq\varnothing\}} \leq \sum_x \mathbb{1}_{\{x\in V_{R_u}\cap V_{R_v}\}}$ as well as the relation $\mathcal{T}_{u,x}\setminus\mathcal{T}_o \subset \mathcal{T}_{u,x}$, $H_{2,4}$ is further bounded above as

$$
\begin{aligned}
H_{2,4} &\leq \big(e+O(\beta)\big) \Bigg(\sum_{\{u,v\}\subset\Lambda} \left(\frac{p_1}{|\Lambda|}\right)^2 \sum_{\substack{R_u\in\mathcal{T}_u\setminus\mathcal{T}_o \\ R_v\in\mathcal{T}_v\setminus\mathcal{T}_o}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_u}|+|E_{R_v}|} \mathbb{1}_{\{V_{R_u}\cap V_{R_v}\neq\varnothing\}} \Bigg)^2 \\
&\leq \big(e+O(\beta)\big) \Bigg(\sum_{\{u,v\}\subset\Lambda} \left(\frac{p_1}{|\Lambda|}\right)^2 \sum_{x\in\mathbb{Z}^d} \underbrace{\sum_{\substack{R_u\in\mathcal{T}_{u,x} \\ R_v\in\mathcal{T}_{v,x}}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_u}|+|E_{R_v}|}}_{\tau_{p_1}(u-x)\,\tau_{p_1}(v-x)} \Bigg)^2
\end{aligned}
$$

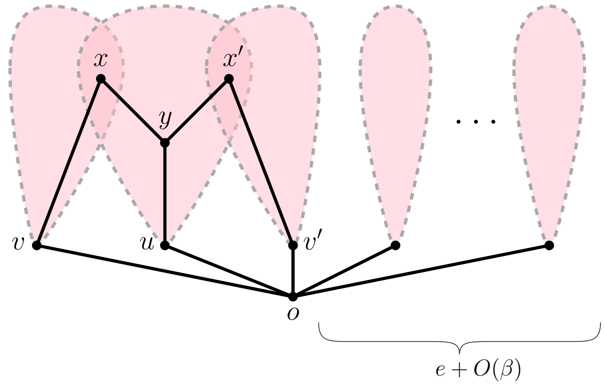


図 4.5: Schematic representation of $H_{2,3}$.

$$\overset{(4.1.1)}{=} \big(e+O(\beta)\big)\left(\sum_{\{u,v\}\subset\Lambda}\left(\frac{1}{|\Lambda|}\right)^2\sum_{x\in\mathbb{Z}^d}\frac{\tau_{p_1}(u-x)}{g_{p_1}}\frac{\tau_{p_1}(v-x)}{g_{p_1}}\right)^2. \tag{4.2.60}$$

Finally, by using $\tau_{p_1}(u-x)/g_{p_1}\le S_{\ge 1}(u-x)$ for $x\neq u$ (cf., the first inequality in (4.2.17)) and $\tau_p(o)=g_p$ for $x=u$, we arrive at

$$H_{2,4}\le\big(e+O(\beta)\big)\left(\sum_{u,v\in\Lambda}\left(\frac{1}{|\Lambda|}\right)^2 S_{\ge 0}^{*2}(u-v)\right)^2\le\big(e+O(\beta)\big)\,S_{\ge 1}^{*2}(o)^2\overset{d\ge 4}{=}O(\beta^2). \tag{4.2.61}$$

Next we investigate $H_{2,3}$, which is bounded in a similar way to (4.2.60) as (see Figure 4.5)

$$\begin{aligned}
H_{2,3}&=\sum_{\substack{Y\subset\Lambda\\(|Y|\ge 3)}}\left(\frac{p_1}{|\Lambda|}\right)^{|Y|}\prod_{y\in Y}\sum_{R_y\in\mathcal{T}_y\setminus\mathcal{T}_o}\left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|}\sum_{\substack{u,v,v'\in Y\\(\text{distinct})}}\mathbb{1}_{\{V_{R_u}\cap V_{R_v}\neq\varnothing\}}\mathbb{1}_{\{V_{R_u}\cap V_{R_{v'}}\neq\varnothing\}}\\
&\le\big(e+O(\beta)\big)\sum_{\substack{u,v,v'\in\Lambda\\(\text{distinct})}}\left(\frac{p_1}{|\Lambda|}\right)^3\sum_{x,x'\in\mathbb{Z}^d}\sum_{\substack{R_u\in\mathcal{T}_{u,x,x'}\\R_v\in\mathcal{T}_{v,x}\\R_{v'}\in\mathcal{T}_{v',x'}}}\left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_u}|+|E_{R_v}|+|E_{R_{v'}}|}\\
&=\big(e+O(\beta)\big)g_{p_1}^2\sum_{\substack{u,v,v'\in\Lambda\\(\text{distinct})}}\left(\frac{1}{|\Lambda|}\right)^3\sum_{x,x'\in\mathbb{Z}^d}\frac{\tau_{p_1}^{(3)}(u,x,x')}{g_{p_1}^3}\frac{\tau_{p_1}(v-x)}{g_{p_1}}\frac{\tau_{p_1}(v'-x')}{g_{p_1}}.
\end{aligned}\tag{4.2.62}$$

Due to submultiplicativity, we can bound $\tau_p^{(3)}(u,x,x')$ as

$$\tau_p^{(3)}(u,x,x')\le\sum_{y\in\mathbb{Z}^d}\tau_p(u-y)\,\tau_p(x-y)\,\tau_p(x'-y). \tag{4.2.63}$$

Then, by using $\tau_{p_1}(u-x)/g_{p_1} \le S_{\ge 1}(u-x)$ for $x \ne u$ and $\tau_p(o) = g_p$ for $x = u$, we can bound the sum in (4.2.62) as

$$\begin{aligned}
&\sum_{\substack{u,v,v'\in\Lambda\\ \text{(distinct)}}} \left(\frac{1}{|\Lambda|}\right)^3 \sum_{y,x,x'\in\mathbb{Z}^d} \frac{\tau_{p_1}(u-y)}{g_{p_1}} \frac{\tau_{p_1}(x-y)}{g_{p_1}} \frac{\tau_{p_1}(x'-y)}{g_{p_1}} \frac{\tau_{p_1}(v-x)}{g_{p_1}} \frac{\tau_{p_1}(v'-x')}{g_{p_1}} \\
&\le \sum_{y,x,x'\in\mathbb{Z}^d} S_{\ge 0}(x-y)\, S_{\ge 0}(x'-y) \underbrace{\sum_{u,v,v'\in\Lambda} \left(\frac{1}{|\Lambda|}\right)^3 S_{\ge 0}(u-y)\, S_{\ge 0}(v-x)\, S_{\ge 0}(v'-x')}_{S_{\ge 1}(y)\, S_{\ge 1}(x)\, S_{\ge 1}(x')} \\
&= \sum_{y\in\mathbb{Z}^d} (S_{\ge 0} * S_{\ge 1})(y)^2 \; S_{\ge 1}(y) \\
&\le \|S_{\ge 0} * S_{\ge 1}\|_\infty \, (S_{\ge 0} * S_{\ge 1}^{*2})(o) \overset{d\ge 6}{=} O(\beta^2).
\end{aligned} \tag{4.2.64}$$

This together with (4.2.57) and (4.2.61) implies

$$H_2 \le \frac{1}{2}(H_{2,3} + H_{2,4}) = O(\beta^2), \tag{4.2.65}$$

as required. ∎

Proof of (4.2.45): bounding H_2''. First we recall that H_2'' is the contribution to H_1' from the second sum on the right of (4.2.42):

$$H_2'' = \sum_{\{u,v\}\subset\Lambda} \left(\frac{p_1}{|\Lambda|}\right)^2 \sum_{x\in\mathbb{Z}^d} \sum_{\substack{R_u\in\mathcal{T}_{u,x}\setminus\mathcal{T}_o\\ R_v\in\mathcal{T}_{v,x}\setminus\mathcal{T}_o}} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_u}|+|E_{R_v}|} \Big(1 - \mathbb{1}_{\{V_{R_u}^{<x}\cap V_{R_v}=\varnothing\}}\Big). \tag{4.2.66}$$

Notice that

$$1 - \mathbb{1}_{\{V_{R_u}^{<x}\cap V_{R_v}=\varnothing\}} \le \sum_{x'\in\mathbb{Z}^d\setminus\{x\}} \mathbb{1}_{\{x'\in V_{R_u}\cap V_{R_v}\}}. \tag{4.2.67}$$

By the inclusion relation $\mathcal{T}_{u,x,x'} \setminus \mathcal{T}_o \subset \mathcal{T}_{u,x,x'}$ and using (4.2.63), (4.1.1) and (4.2.17), we can bound H_2'' as (see Figure 4.6)

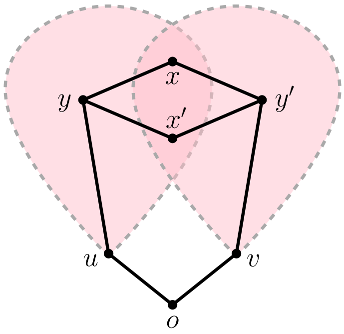


図 4.6: Schematic representation of the bound on H_2'' due to (4.2.67).

$$\begin{aligned}
H_2'' &\le \sum_{\{u,v\}\subset\Lambda} \bigg(\frac{p_1}{|\Lambda|}\bigg)^2 \sum_{\substack{x,x'\in\mathbb{Z}^d\\(x\neq x')}} \underbrace{\sum_{\substack{R_u\in\mathcal{T}_{u,x,x'}\\R_v\in\mathcal{T}_{v,x,x'}}} \bigg(\frac{p_1}{|\Lambda|}\bigg)^{|E_{R_u}|+|E_{R_v}|}}_{\tau_{p_1}^{(3)}(u,x,x')\,\tau_{p_1}^{(3)}(v,x,x')} \\
&\le g_{p_1}^4 \sum_{\substack{x,x',y,y'\in\mathbb{Z}^d\\(x\neq x')}} \frac{\tau_{p_1}(y-x)}{g_{p_1}} \frac{\tau_{p_1}(y-x')}{g_{p_1}} \frac{\tau_{p_1}(y'-x)}{g_{p_1}} \frac{\tau_{p_1}(y'-x')}{g_{p_1}} \\
&\qquad \times \frac{1}{2} \sum_{u,v\in\Lambda} \bigg(\frac{1}{|\Lambda|}\bigg)^2 \frac{\tau_{p_1}(u-y)}{g_{p_1}} \frac{\tau_{p_1}(v-y')}{g_{p_1}} \\
&\le \frac{g_{p_1}^4}{2} \sum_{\substack{x,x',y,y'\in\mathbb{Z}^d\\(x\neq x')}} S_{\ge0}(y-x)\,S_{\ge0}(y-x')\,S_{\ge0}(y'-x)\,S_{\ge0}(y'-x')\,S_{\ge1}(y)\,S_{\ge1}(y').
\end{aligned} \tag{4.2.68}$$

Shifting the variables by $-x'$ and changing the variables $x-x', y-x', y'-x'$ to the new ones w, z, z', respectively, we can rewrite the above sum as

$$\sum_{\substack{w,z,z'\in\mathbb{Z}^d\\(w\neq o)}} S_{\ge0}(z-w)\,S_{\ge0}(z)\,S_{\ge0}(z'-w)\,S_{\ge0}(z') \underbrace{\sum_{x'\in\mathbb{Z}^d} S_{\ge1}(z+x')\,S_{\ge1}(z'+x')}_{S_{\ge1}^{*2}(z-z')}, \tag{4.2.69}$$

which is bounded above by

$$\begin{aligned}
\|S_{\ge1}^{*2}\|_\infty \sum_{w\neq o} S_{\ge0}^{*2}(w)^2 &= \|S_{\ge1}^{*2}\|_\infty \sum_{w\neq o} \bigg(\sum_{n=1}^\infty (n+1)D^{*n}(w)\bigg)^2 \\
&\le \|S_{\ge1}^{*2}\|_\infty \sum_{t=2}^\infty D^{*t}(o) \underbrace{\sum_{n=1}^{t-1}(n+1)(t-n+1)}_{O(t^3)} \overset{d\ge8}{=} O(\beta^2),
\end{aligned} \tag{4.2.70}$$

as required. ■

Proof of (4.2.47): bounding H_2'''. First we recall that H_2''' is the contribution to H_1'' which involves 3-point functions (cf., (4.2.43)):

$$H_2''' = \sum_{\{u,v\}\subset\Lambda} \left(\frac{1}{|\Lambda|}\right)^2 \sum_{x\in\mathbb{Z}^d} \frac{\tau_{p_1}^{(3)}(o,u,x)}{g_{p_1}} \left(\frac{\tau_{p_1}^{(3)}(o,v,x)}{g_{p_1}} - 2\frac{\tau_{p_1}(v-x)}{g_{p_1}}\right). \tag{4.2.71}$$

By (4.1.1), (4.2.63) and (4.2.17), we can readily conclude that

$$\begin{aligned}
|H_2'''| &\le \frac{g_{p_1}^2}{2} \sum_{u,v\in\Lambda} \left(\frac{1}{|\Lambda|}\right)^2 \sum_{x,y\in\mathbb{Z}^d} \frac{\tau_{p_1}(y)}{g_{p_1}} \frac{\tau_{p_1}(y-u)}{g_{p_1}} \frac{\tau_{p_1}(y-x)}{g_{p_1}} \\
&\qquad \times \left(g_{p_1}^2 \sum_{z\in\mathbb{Z}^d} \frac{\tau_{p_1}(z)}{g_{p_1}} \frac{\tau_{p_1}(z-v)}{g_{p_1}} \frac{\tau_{p_1}(z-x)}{g_{p_1}} + 2\frac{\tau_{p_1}(v-x)}{g_{p_1}}\right) \\
&\le g_{p_1}^2 \sum_{x,y\in\mathbb{Z}^d} S_{\ge 0}(y)\, S_{\ge 1}(y)\, S_{\ge 0}(y-x) \\
&\qquad \times \left(\frac{g_{p_1}^2}{2} \sum_{z\in\mathbb{Z}^d} S_{\ge 0}(z)\, S_{\ge 1}(z)\, S_{\ge 0}(z-x) + S_{\ge 1}(x)\right) \\
&= g_{p_1}^2 \sum_{y\in\mathbb{Z}^d} S_{\ge 0}(y)\, S_{\ge 1}(y) \left(\frac{g_{p_1}^2}{2} \sum_{z\in\mathbb{Z}^d} S_{\ge 0}(z)\, S_{\ge 1}(z)\, S_{\ge 0}^{*2}(y-z) + (S_{\ge 0} * S_{\ge 1})(y)\right) \\
&\le g_{p_1}^2 \underbrace{(S_{\ge 0} * S_{\ge 1})(o)}_{O(\beta)\text{ for } d>4} \left(\frac{g_{p_1}^2}{2} (S_{\ge 0} * S_{\ge 1})(o) \underbrace{\|S_{\ge 0}^{*2}\|_\infty}_{O(1)\text{ for } d>4} + \underbrace{\|S_{\ge 0} * S_{\ge 1}\|_\infty}_{O(\beta)\text{ for } d>4}\right) \\
&= O(\beta^2),
\end{aligned} \tag{4.2.72}$$

as required. ■

4.3 Difference between lattice trees and lattice animals

Finally we prove Theorem 4.0.1 for lattice animals. Recall that, by Lemma 4.1.1, it suffices to investigate $p_1 = 1/g_{p_1}$ (cf., (4.1.1)). The following is the key lemma:

Lemma 4.3.1. *For lattice animals with $d > 8$ and $L \uparrow \infty$,*

$$g_{p_1} = e\left(1 - \sum_{n=2}^{\infty} \frac{n+1}{2} D^{*n}(o)\right) + \frac{1}{2} S_{\ge 3}(o) + O(\beta^2). \tag{4.3.1}$$

Consequently,

$$p_1 = \frac{1}{e} + \sum_{n=2}^{\infty} \frac{n+1}{2e} D^{*n}(o) - \frac{1}{2e^2} S_{\ge 3}(o) + O(\beta^2). \tag{4.3.2}$$

Proof. As a first step, we want a similar decomposition to (4.2.3) for lattice animals. To do so, we identify the connected neighbors Y of the origin, just as done in (4.2.3). Then, we introduce $\Gamma(Y)$, which is the set of all partitions of Y. For example, if $Y = \{1, 2, 3\}$, then

$$\Gamma(Y) = \Big\{\{Y\},\ \{\{1,2\},\{3\}\},\ \{\{1,3\},\{2\}\},\ \{\{1\},\{2,3\}\},\ \{\{1\},\{2\},\{3\}\}\Big\}. \qquad (4.3.3)$$

For a partition $\gamma \in \Gamma(Y)$, we denote by $|\gamma|$ the number of sets in γ, so that $\gamma = \{\gamma_j\}_{j=1}^{|\gamma|}$. We can rewrite g_{p_1} as

$$g_{p_1} = 1 + \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 1)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \sum_{\gamma \in \Gamma(Y)} \prod_{j=1}^{|\gamma|} \sum_{R_j \in \mathcal{A}_{\gamma_j} \setminus \mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_j}|} \prod_{i<j} \mathbb{1}_{\{V_{R_i} \cap V_{R_j} = \varnothing\}}. \qquad (4.3.4)$$

The contribution from the maximum partition $\bar{\gamma} = \{\{y\}\}_{y \in Y}$ (i.e., $|\bar{\gamma}| = |Y|$) is equal to (4.2.3) (with $\mathcal{T}$ replaced by $\mathcal{A}$) and can be decomposed into G and H as in (4.2.5) (with R_y regarded as animals instead of trees). Let I be the contribution from the remaining partitions $\gamma \in \Gamma(Y)$ with $|\gamma| < |Y|$, which is zero for lattice trees:

$$\begin{aligned} I &= g_{p_1} - (G - H) \\ &= \sum_{\substack{Y \subset \Lambda \\ (|Y| \geq 2)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \sum_{\substack{\gamma \in \Gamma(Y) \\ (|\gamma| < |Y|)}} \prod_{j=1}^{|\gamma|} \sum_{R_j \in \mathcal{A}_{\gamma_j} \setminus \mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_j}|} \prod_{i<j} \mathbb{1}_{\{V_{R_i} \cap V_{R_j} = \varnothing\}}. \end{aligned} \qquad (4.3.5)$$

To evaluate G, H and I for lattice animals, we cannot apply Lemma 4.2.3, which is a powerful tool for lattice trees to identify the coefficients of β as well as to estimate the error terms of $O(\beta^2)$. For the latter purpose for lattice animals, we will use the infrared bound (4.1.5) (and monotonicity in p, i.e., $\tau_{p_1} \leq \tau_{p_c}$); for the former purpose, we will use the following bounds that correspond to (4.2.25), (4.2.48) and (4.2.50), respectively:

Lemma 4.3.2. *For lattice animals with $d > 8$ and $L \uparrow \infty$,*

$$\left| \sum_{u \in \Lambda} \frac{1}{|\Lambda|} \frac{\tau_{p_1}(u)}{g_{p_1}} - S_{\geq 2}(o) \right| = O(\beta^2), \qquad (4.3.6)$$

$$\left| \sum_{\{u,v\} \subset \Lambda} \left(\frac{1}{|\Lambda|}\right)^2 \frac{\tau_{p_1}(u-v)}{g_{p_1}} - \frac{1}{2} S_{\geq 3}(o) \right| = O(\beta^2), \qquad (4.3.7)$$

$$\left| \sum_{\{u,v\} \subset \Lambda} \left(\frac{1}{|\Lambda|}\right)^2 \sum_{x \neq u,v} \frac{\tau_{p_1}(u-x)}{g_{p_1}} \frac{\tau_{p_1}(x-v)}{g_{p_1}} - \frac{1}{2} S^{*2}_{\geq 2}(o) \right| = O(\beta^2). \qquad (4.3.8)$$

We will prove Lemma 4.3.2 after the proof of Lemma 4.3.1 is completed.

Now we resume the proof of Lemma 4.3.1 assuming the bounds in Lemma 4.3.2. First we recall $G = G_0 + G_1 + G_2$ (cf., (4.2.14)), where G_0 is independent of the models and estimated as (4.2.12); G_1 is defined as (4.2.15) and here we use (4.3.6) to show (4.2.26); G_2 is defined as (4.2.16) and obeys the same bound as (4.2.21). As a result, Lemma 4.2.2 also holds for lattice animals. Similarly, we can show $H = e(H_1''' + H_2''' - H_2'' - H_2) + O(\beta^2)$ (cf., (4.2.36), (4.2.41), (4.2.45) and (4.2.47)), where H_1''' is defined in (4.2.46) and here we use (4.3.7)–(4.3.8) to show (4.2.52); H_2 is bounded by $H_{2,3} + H_{2,4}$, and $H_{2,3}$ and $H_{2,4}$ are further bounded as (4.2.58)–(4.2.60) and (4.2.62)–(4.2.64) (with $\mathcal{T}$ replaced by $\mathcal{A}$), and here we use the infrared bound (4.1.5) and the convolution bound on power functions [9, Lemma 3.2(i)] to show $H_2 = O(\beta^2)$, such as

$$\begin{aligned}
&\sum_{\{u,v\}\subset\Lambda}\left(\frac{1}{|\Lambda|}\right)^2\sum_{x\in\mathbb{Z}^d}\tau_{p_1}(u-x)\,\tau_{p_1}(v-x)\\
&\leq \sum_{\{u,v\}\subset\Lambda}\left(\frac{1}{|\Lambda|}\right)^2\sum_{x\in\mathbb{Z}^d}\frac{O(L^{-2})}{(\|u-x\|\vee L)^{d-2}}\frac{O(L^{-2})}{(\|v-x\|\vee L)^{d-2}}\\
&\leq \sum_{\{u,v\}\subset\Lambda}\left(\frac{1}{|\Lambda|}\right)^2\frac{O(L^{-4})}{(\|u-v\|\vee L)^{d-4}} = O(\beta^2).
\end{aligned}\tag{4.3.9}$$

Similarly we can show that H_2'' and H_2''' are both $O(\beta^2)$ by using the infrared bound and the convolution bound, instead of bounding τ_{p_1}/g_{p_1} by $S_{\geq 0}$ or $S_{\geq 1}$, just as done for lattice trees. As a result, Lemma 4.2.4 also holds for lattice animals.

Next we investigate I, which is unique for lattice animals. Let I_1 be the contribution from $\gamma \in \Gamma(Y)$ with $|\gamma| = |Y| - 1$, i.e., consisting of a pair $\{u, v\}$ and $|Y| - 2$ singletons $\{y\}_{y\in Y\setminus\{u,v\}}$:

$$\begin{aligned}
I_1 = &\sum_{\substack{Y\subset\Lambda\\(|Y|\geq 2)}}\left(\frac{p_1}{|\Lambda|}\right)^{|Y|}\sum_{\{u,v\}\subset Y}\sum_{R\in\mathcal{A}_{u,v}\setminus\mathcal{A}_o}\left(\frac{p_1}{|\Lambda|}\right)^{|E_R|}\prod_{y\in Y\setminus\{u,v\}}\sum_{R_y\in\mathcal{A}_y\setminus\mathcal{A}_o}\left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|}\\
&\times\prod_{y\in Y\setminus\{u,v\}}\mathbb{1}_{\{V_R\cap V_{R_y}=\varnothing\}}\prod_{\substack{y,z\in Y\setminus\{u,v\}\\(y\neq z)}}\mathbb{1}_{\{V_{R_y}\cap V_{R_z}=\varnothing\}},
\end{aligned}\tag{4.3.10}$$

where an empty product is regarded as 1. The dominant contribution to I_1, denoted I_1', comes from when the last line is replaced by 1. By the tree-graph inequality (4.2.63), which is also true for lattice animals due to subadditivity, and then using the infrared bound (4.1.5), it is estimated as (see Figure 4.7)

$$I_1' = \sum_{\{u,v\}\subset\Lambda}\left(\frac{p_1}{|\Lambda|}\right)^2\Big(\tau_{p_1}(u-v) - \tau_{p_1}^{(3)}(o,u,v)\Big)\underbrace{\sum_{Y'\subset\Lambda\setminus\{u,v\}}\left(\frac{1}{|\Lambda|}\right)^{|Y'|}\prod_{y\in Y'}\left(1-\frac{\tau_{p_1}(y)}{g_{p_1}}\right)}_{e+O(\beta)}$$

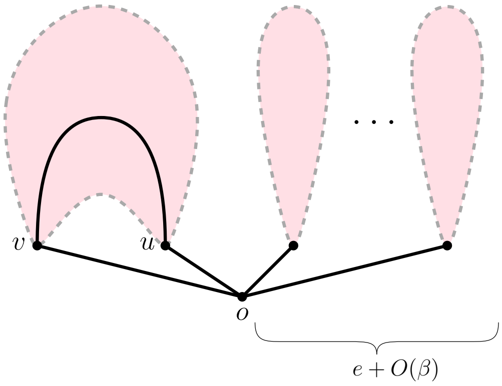


図 4.7: Schematic representation of the dominant contribution to I_1'.

$$
\begin{aligned}
&= \underbrace{p_1\big(e+O(\beta)\big)}_{1+O(\beta)} \sum_{\{u,v\}\subset\Lambda} \left(\frac{1}{|\Lambda|}\right)^2 \left(\frac{\tau_{p_1}(u-v)}{g_{p_1}} - \frac{\tau_{p_1}^{(3)}(o,u,v)}{g_{p_1}}\right) \\
&= \sum_{\{u,v\}\subset\Lambda} \left(\frac{1}{|\Lambda|}\right)^2 \frac{\tau_{p_1}(u-v)}{g_{p_1}} + O(\beta^2) \\
&\overset{(4.3.7)}{=} \frac{1}{2} S_{\geq 3}(o) + O(\beta^2). \qquad (4.3.11)
\end{aligned}
$$

On the other hand, by using $1-ab \leq (1-a)+(1-b)$ for any $a,b \in \{0,1\}$, we can bound the difference $I_1' - I_1$ (≥ 0) as

$$
\begin{aligned}
I_1' - I_1 &\leq \sum_{\{u,v\}\subset\Lambda} \left(\frac{p_1}{|\Lambda|}\right)^2 \sum_{R\in\mathcal{A}_{u,v}\setminus\mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_R|} \sum_{Y'\subset\Lambda\setminus\{u,v\}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y'|} \prod_{y\in Y'} \sum_{R_y\in\mathcal{A}_y\setminus\mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|} \\
&\qquad \times \left(1 - \prod_{y\in Y'} \mathbb{1}_{\{V_R\cap V_{R_y}=\varnothing\}} + 1 - \prod_{\substack{y,z\in Y' \\ (y\neq z)}} \mathbb{1}_{\{V_{R_y}\cap V_{R_z}=\varnothing\}}\right) \\
&\leq \sum_{\{u,v\}\subset\Lambda} \left(\frac{p_1}{|\Lambda|}\right)^2 \sum_{R\in\mathcal{A}_{u,v}\setminus\mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_R|} \sum_{Y'\subset\Lambda\setminus\{u,v\}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y'|} \prod_{y\in Y'} \sum_{R_y\in\mathcal{A}_y\setminus\mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_y}|} \\
&\qquad \times \left(\sum_{y\in Y'} \mathbb{1}_{\{V_R\cap V_{R_y}\neq\varnothing\}} + \sum_{\substack{y,z\in Y' \\ (y\neq z)}} \mathbb{1}_{\{V_{R_y}\cap V_{R_z}\neq\varnothing\}}\right). \qquad (4.3.12)
\end{aligned}
$$

This is $O(\beta^2)$, as the contribution from the former (resp., latter) sum in the last line can be estimated in a similar way to showing $H_{2,3} = O(\beta^2)$ (resp., $H_{2,4} = O(\beta^2)$); see Figure 4.8. As a result,

$$
I_1 = \frac{1}{2} S_{\geq 3}(o) + O(\beta^2). \qquad (4.3.13)
$$

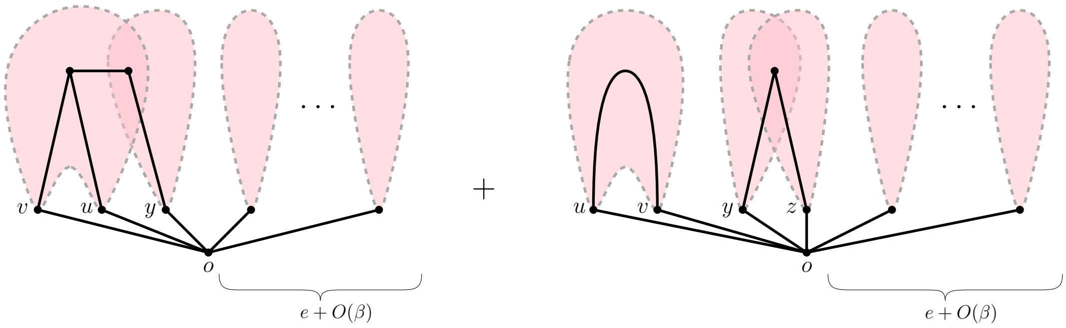


図 4.8: Schematic representation of the bound on (4.3.12).

Finally we estimate the difference $I - I_1$:

$$I - I_1 = \sum_{\substack{Y\subset\Lambda \\ (|Y|\geq 3)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \sum_{\substack{\gamma\in\Gamma(Y) \\ (|\gamma|\leq|Y|-2)}} \prod_{j=1}^{|\gamma|} \sum_{R_j\in\mathcal{A}_{\gamma_j}\setminus\mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_j}|} \prod_{i<j} \mathbb{1}_{\{V_{R_i}\cap V_{R_j}=\varnothing\}}. \tag{4.3.14}$$

Since $|\gamma| \leq |Y| - 2$, there are two possibilities: (i) there is a set in γ which includes at least 3 distinct neighbors of the origin, or (ii) there are at least two disjoint sets in γ both of which include exactly two distinct neighbors of the origin. Therefore,

$$\begin{aligned} I - I_1 &\leq \sum_{\substack{Y\subset\Lambda \\ (|Y|\geq 3)}} \left(\frac{p_1}{|\Lambda|}\right)^{|Y|} \sum_{\gamma\in\Gamma(Y)} \prod_{j=1}^{|\gamma|} \sum_{R_j\in\mathcal{A}_{\gamma_j}\setminus\mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_j}|} \prod_{i<j} \mathbb{1}_{\{V_{R_i}\cap V_{R_j}=\varnothing\}} \\ &\qquad\qquad \times \Big(\mathbb{1}_{\{\exists j,\, |\gamma_j|\geq 3\}} + \mathbb{1}_{\{\exists i\neq j,\, |\gamma_i|=|\gamma_j|=2\}}\Big) \\ &= I_3 + I_2, \end{aligned} \tag{4.3.15}$$

where I_3 and I_2 are the contributions from $\mathbb{1}_{\{\exists j,\, |\gamma_j|\geq 3\}}$ and $\mathbb{1}_{\{\exists i\neq j,\, |\gamma_i|=|\gamma_j|=2\}}$, respectively.

For I_2, we split the set Y of neighbors of the origin into U, V and $Y' = Y \setminus (U \cup V)$, where $U \cap V = \varnothing$ and $|U| = |V| = 2$. Partially ignoring the avoidance constraint among animals, we can bound I_2 as

$$\begin{aligned} I_2 \leq &\sum_{\substack{U\subset\Lambda \\ (|U|=2)}} \left(\frac{p_1}{|\Lambda|}\right)^2 \sum_{R\in\mathcal{A}_U\setminus\mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_R|} \sum_{\substack{V\subset\Lambda\setminus U \\ (|V|=2)}} \left(\frac{p_1}{|\Lambda|}\right)^2 \sum_{R'\in\mathcal{A}_V\setminus\mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R'}|} \\ &\times \sum_{Y'\subset\Lambda\setminus(U\cup V)} \left(\frac{p_1}{|\Lambda|}\right)^{|Y'|} \sum_{\gamma\in\Gamma(Y')} \prod_{j=1}^{|\gamma|} \sum_{R_j\in\mathcal{A}_{\gamma_j}\setminus\mathcal{A}_o} \left(\frac{p_1}{|\Lambda|}\right)^{|E_{R_j}|} \prod_{i<j} \mathbb{1}_{\{V_{R_i}\cap V_{R_j}=\varnothing\}}. \end{aligned} \tag{4.3.16}$$

Notice that the second line is almost identical to g_{p_1}; the only difference is the domain of summation over Y', and therefore it is bounded above by g_{p_1}. Since $p_1 g_{p_1} = 1$, we obtain

$$I_2 \le g_{p_1}\Bigg(\sum_{\{u,v\}\subset\Lambda}\Big(\frac{p_1}{|\Lambda|}\Big)^2 \underbrace{\sum_{R\in\mathcal{A}_{u,v}}\Big(\frac{p_1}{|\Lambda|}\Big)^{|E_R|}}_{\tau_{p_1}(u-v)}\Bigg)^2 \overset{(4.3.7)}{\le} p_1\Big(\frac{1}{2}S_{\ge 3}(o)+O(\beta^2)\Big)^2 = O(\beta^2). \tag{4.3.17}$$

For I_3, we split the set Y into X and $Y' = Y \setminus X$, where X includes at least 3 distinct vertices $x, y, z \in \Lambda$. Again, by partially ignoring the avoidance constraint among animals, we can bound I_3 as

$$\begin{aligned} I_3 \le &\sum_{\{x,y,z\}\subset\Lambda}\ \sum_{\substack{X\subset\Lambda \\ (X\ni x,y,z)}}\Big(\frac{p_1}{|\Lambda|}\Big)^{|X|}\sum_{R\in\mathcal{A}_X\setminus\mathcal{A}_o}\Big(\frac{p_1}{|\Lambda|}\Big)^{|E_R|} \\ &\times \sum_{Y'\subset\Lambda\setminus X}\Big(\frac{p_1}{|\Lambda|}\Big)^{|Y'|}\sum_{\gamma\in\Gamma(Y')}\prod_{j=1}^{|\gamma|}\sum_{R_j\in\mathcal{A}_{\gamma_j}\setminus\mathcal{A}_o}\Big(\frac{p_1}{|\Lambda|}\Big)^{|E_{R_j}|}\prod_{i<j}\mathbb{1}_{\{V_{R_i}\cap V_{R_j}=\varnothing\}}. \end{aligned} \tag{4.3.18}$$

Notice again that the second line is bounded above by g_{p_1}. Using the relation $\mathcal{A}_X \setminus \mathcal{A}_o \subset \mathcal{A}_{x,y,z}$ and splitting X into $\{x, y, z\}$ and $X' = X \setminus \{x, y, z\}$, we obtain

$$I_3 \le g_{p_1}\sum_{\{x,y,z\}\subset\Lambda}\Big(\frac{p_1}{|\Lambda|}\Big)^3 \underbrace{\sum_{R\in\mathcal{A}_{x,y,z}}\Big(\frac{p_1}{|\Lambda|}\Big)^{|E_R|}}_{\tau^{(3)}_{p_1}(x,y,z)} \underbrace{\sum_{X'\subset\Lambda\setminus\{x,y,z\}}\Big(\frac{p_1}{|\Lambda|}\Big)^{|X'|}}_{\le(1+p_1/|\Lambda|)^{|\Lambda|}}. \tag{4.3.19}$$

By the tree-graph inequality (4.2.63), we can show that

$$\begin{aligned} &\sum_{\{x,y,z\}\subset\Lambda}\Big(\frac{p_1}{|\Lambda|}\Big)^3\tau^{(3)}_{p_1}(x,y,z) \\ &\le \sum_{\{x,y,z\}\subset\Lambda}\Big(\frac{1}{|\Lambda|}\Big)^3\sum_{w\in\mathbb{Z}^d}\frac{\tau_{p_1}(x-w)}{g_{p_1}}\frac{\tau_{p_1}(y-w)}{g_{p_1}}\frac{\tau_{p_1}(z-w)}{g_{p_1}} \\ &= \sum_{\{x,y,z\}\subset\Lambda}\Big(\frac{1}{|\Lambda|}\Big)^3\Bigg(\sum_{w\ne x,y,z}\frac{\tau_{p_1}(x-w)}{g_{p_1}}\frac{\tau_{p_1}(y-w)}{g_{p_1}}\frac{\tau_{p_1}(z-w)}{g_{p_1}}+\frac{3}{g_{p_1}}\frac{\tau_{p_1}(x-z)}{g_{p_1}}\frac{\tau_{p_1}(y-z)}{g_{p_1}}\Bigg) \\ &\le \frac{3}{g_{p_1}}\Big(\underbrace{\|D*\tau_{p_1}\|_\infty+\|D\|_\infty}_{O(\beta)\ (\because(4.1.5))}\Big)\underbrace{\sum_{\{x,y\}\subset\Lambda}\Big(\frac{1}{|\Lambda|}\Big)^2\sum_{w\ne x,y}\frac{\tau_{p_1}(x-w)}{g_{p_1}}\frac{\tau_{p_1}(y-w)}{g_{p_1}}}_{O(\beta)\ (\because(4.3.8))}, \end{aligned} \tag{4.3.20}$$

hence $I_3 = O(\beta^2)$. This completes the proof of $I = \frac{1}{2}S_{\ge 3}(o) + O(\beta^2)$, hence the proof of Lemma 4.3.1. ∎

Proof of Lemma 4.3.2. First we prove (4.3.6). By the inverse Fourier transfom, we have the rewrite

$$\sum_{u\in\Lambda}\frac{1}{|\Lambda|}\frac{\tau_{p_1}(u)}{g_{p_1}}=\Big(D*\frac{\tau_{p_1}(u)}{g_{p_1}}\Big)(o)=\int_{[-\pi,\pi]^d}\hat{D}(k)\,\frac{\hat{\tau}_{p_1}(k)}{g_{p_1}}\frac{\mathrm{d}^dk}{(2\pi)^d}. \tag{4.3.21}$$

Notice that the Fourier transform of the recursion equation (2.2.9) yields

$$\hat{\tau}_p(k)=\frac{g_p+\hat{h}_p(k)+\hat{\pi}_p(k)}{1-(g_p+\hat{h}_p(k)+\hat{\pi}_p(k))p\hat{D}(k)}. \tag{4.3.22}$$

We use this identity at $p_1=1/g_{p_1}$. Let

$$H(x)=\frac{h_{p_1}(x)+\pi_{p_1}(x)}{g_{p_1}}. \tag{4.3.23}$$

Thanks to the symmetry, the Fourier transform $\hat{H}(k)$ is real. Moreover, by (4.1.6)–(4.1.7), we can show that, for $d>8$ and $L\gg 1$, $|\hat{H}(k)|=O(\beta)$ uniformly in k. Then, we can rewrite $\hat{\tau}_{p_1}(k)/g_{p_1}$ as

$$\begin{aligned}\frac{\hat{\tau}_{p_1}(k)}{g_{p_1}}&=\frac{1+\hat{H}(k)}{1-(1+\hat{H}(k))\hat{D}(k)}\\&=\frac{1}{1-\hat{D}(k)}+\frac{\hat{H}(k)}{1-\hat{D}(k)}\underbrace{\frac{1}{1-(1+\hat{H}(k))\hat{D}(k)}}_{=:\hat{F}(k)}.\end{aligned} \tag{4.3.24}$$

Applying this to (4.3.21) yields the main term $S_{\geq 2}(o)$ as

$$\sum_{u\in\Lambda}\frac{1}{|\Lambda|}\frac{\tau_{p_1}(u)}{g_{p_1}}=\underbrace{\int_{[-\pi,\pi]^d}\frac{\hat{D}(k)}{1-\hat{D}(k)}\frac{\mathrm{d}^dk}{(2\pi)^d}}_{S_{\geq 1}(o)\ \ (=S_{\geq 2}(o))}+\int_{[-\pi,\pi]^d}\frac{\hat{D}(k)\hat{H}(k)}{1-\hat{D}(k)}\hat{F}(k)\frac{\mathrm{d}^dk}{(2\pi)^d}. \tag{4.3.25}$$

It remains to show that the second term on the right is $O(\beta^2)$. To do so, we want an effective bound on $\hat{F}(k)$. We will show at the end of the proof that, for $d>8$ and $L\gg 1$, there is an L-independent constant $C<\infty$ such that

$$0<\hat{F}(k)\leq\frac{C}{1-\hat{D}(k)}, \tag{4.3.26}$$

uniformly in k. However, to use (4.3.26) for the second term of (4.3.25), we have to bound $\hat{D}(k)$ above by the absolute value of it, which makes it difficult to show the error being $O(\beta^2)$. Instead, we first rewrite $\hat{F}(k)$ as

$$\hat{F}(k)=\frac{1}{1-\hat{D}(k)}+\frac{\hat{D}(k)\hat{H}(k)}{1-\hat{D}(k)}\hat{F}(k). \tag{4.3.27}$$

Then, the second term on the right of (4.3.25) equals

$$\underbrace{\int_{[-\pi,\pi]^d}\frac{\hat{D}(k)\hat{H}(k)}{(1-\hat{D}(k))^2}\frac{\mathrm{d}^dk}{(2\pi)^d}}_{(D*S^{*2}_{\geq 0}*H)(o)}+\int_{[-\pi,\pi]^d}\bigg(\frac{\hat{D}(k)\hat{H}(k)}{1-\hat{D}(k)}\bigg)^2\hat{F}(k)\frac{\mathrm{d}^dk}{(2\pi)^d}. \tag{4.3.28}$$

Notice that, due to the identity (4.3.27), we can make $\hat{D}(k)^2$, which is always positive, in the second term of (4.3.28). The first term is readily bounded by $\|D*S^{*2}_{\geq 0}\|_\infty\hat{H}(0)=O(\beta^2)$. For the second term, we use $|\hat{H}(k)|=O(\beta)$ and (4.3.26) to obtain that

$$\int_{[-\pi,\pi]^d}\bigg(\frac{\hat{D}(k)\hat{H}(k)}{1-\hat{D}(k)}\bigg)^2\hat{F}(k)\frac{\mathrm{d}^dk}{(2\pi)^d}\leq O(\beta^2)\underbrace{\int_{[-\pi,\pi]^d}\frac{\hat{D}(k)^2}{(1-\hat{D}(k))^3}\frac{\mathrm{d}^dk}{(2\pi)^d}}_{(D^{*2}*S^{*3}_{\geq 0})(o)}=O(\beta^3). \tag{4.3.29}$$

This completes the proof of (4.3.6).

We can also prove (4.3.7)–(4.3.8) in a similar manner by assuming (4.3.26). Hence we here prove only (4.3.7). By the inverse Fourier transform, we can rewrite the sum in (4.3.7) as

$$\begin{aligned}\sum_{\{u,v\}\subset\Lambda}\bigg(\frac{1}{|\Lambda|}\bigg)^2\frac{\tau_{p_1}(u-v)}{g_{p_1}}&=\frac{1}{2}\bigg(D^{*2}*\frac{\tau_{p_1}}{g_{p_1}}\bigg)(o)-\frac{1}{2}D^{*2}(o)\\&=\frac{1}{2}\int_{[-\pi,\pi]^d}\hat{D}(k)^2\frac{\hat{\tau}_{p_1}(k)}{g_{p_1}}\frac{\mathrm{d}^dk}{(2\pi)^d}-\frac{1}{2}D^{*2}(o).\end{aligned} \tag{4.3.30}$$

Then, by the identity (4.3.24), we can extract the main term $\frac{1}{2}S_{\geq 3}(o)$ as

$$\underbrace{\frac{1}{2}\int_{[-\pi,\pi]^d}\frac{\hat{D}(k)^2}{1-\hat{D}(k)}\frac{\mathrm{d}^dk}{(2\pi)^d}-\frac{1}{2}D^{*2}(o)}_{\frac{1}{2}S_{\geq 3}(o)}+\frac{1}{2}\int_{[-\pi,\pi]^d}\frac{\hat{D}(k)^2\hat{H}(k)}{1-\hat{D}(k)}\hat{F}(k)\frac{\mathrm{d}^dk}{(2\pi)^d}. \tag{4.3.31}$$

Similarly to (4.3.29), the second term is bounded as

$$\bigg|\frac{1}{2}\int_{[-\pi,\pi]^d}\frac{\hat{D}(k)^2\hat{H}(k)}{1-\hat{D}(k)}\hat{F}(k)\frac{\mathrm{d}^dk}{(2\pi)^d}\bigg|\overset{(4.3.26)}{\leq}O(\beta)\underbrace{\int_{[-\pi,\pi]^d}\frac{\hat{D}(k)^2}{(1-\hat{D}(k))^2}\frac{\mathrm{d}^dk}{(2\pi)^d}}_{(D^{*2}*S^{*2}_{\geq 0})(o)}=O(\beta^2), \tag{4.3.32}$$

hence the completion of the proof of (4.3.7).

Finally we prove the inequality (4.3.26), for $\|k\|\geq\frac{1}{L}$ and $\|k\|\leq\frac{1}{L}$ separately. We begin with the former case. It is known (cf., e.g., [26]) that our D satisfies [29, Assumption D]; in particular, there is an L-independent constant $\eta\in(0,1)$ such that

$$-1+\eta\overset{\forall k}{\leq}\hat{D}(k)\overset{\|k\|\geq\frac{1}{L}}{\leq}1-\eta. \tag{4.3.33}$$

Since $|\hat{H}(k)| = O(\beta)$, we obtain that, for $L \gg 1$,

$$-1 + \frac{\eta}{2} \overset{\forall k}{\leq} \big(1 + \hat{H}(k)\big)\hat{D}(k) \overset{\|k\| \geq \frac{1}{L}}{\leq} 1 - \frac{\eta}{2}, \tag{4.3.34}$$

hence

$$0 < \frac{1}{2 - \eta/2} \overset{\forall k}{\leq} \hat{F}(k) \overset{\|k\| \geq \frac{1}{L}}{\leq} \frac{2}{\eta} \overset{\forall k}{\leq} \frac{2}{\eta} \frac{2 - \eta}{1 - \hat{D}(k)}. \tag{4.3.35}$$

It remains to show that $\hat{F}(k)$ is bounded above by a multiple of $(1 - \hat{D}(k))^{-1}$ uniformly in $\|k\| \leq \frac{1}{L}$. We note that

$$\begin{aligned} \hat{F}(k)^{-1} &= 1 - \big(1 + \hat{H}(0)\big) + \big(1 + \hat{H}(0)\big)\big(1 - \hat{D}(k)\big) + \big(\hat{H}(0) - \hat{H}(k)\big)\hat{D}(k) \\ &= -\hat{H}(0) + \left(1 + \hat{H}(0) + \frac{\hat{H}(0) - \hat{H}(k)}{1 - \hat{D}(k)}\hat{D}(k)\right)\big(1 - \hat{D}(k)\big). \end{aligned} \tag{4.3.36}$$

Since $-\hat{H}(0)$ is bounded below by a positive multiple of β (as explained in the beginning of the proof of Lemma 4.1.1), ignoring this term yields a lower bound on $\hat{F}(k)^{-1}$. Moreover, since $|k \cdot x| \leq \|k\|\|x\| \leq 1$ for $x \in \Lambda$ and $\|k\| \leq \frac{1}{L}$, and since $1 - \cos t \geq \frac{2}{\pi^2}t^2$ for $|t| \leq 1$, there is a $c > 0$ such that

$$1 - \hat{D}(k) = \sum_{x \in \Lambda} \frac{1 - \cos(k \cdot x)}{|\Lambda|} \geq \frac{2}{\pi^2} \sum_{x \in \Lambda} \frac{(k \cdot x)^2}{|\Lambda|} = \frac{2\|k\|^2}{d\pi^2} \underbrace{\sum_{x \in \Lambda} \frac{\|x\|^2}{|\Lambda|}}_{\geq cL^2}. \tag{4.3.37}$$

On the other hand, by $1 - \cos t \leq \frac{1}{2}t^2$ for any t and using the x-space bounds (4.1.4) and (4.1.7), we have

$$|\hat{H}(0) - \hat{H}(k)| \leq \sum_x \frac{(k \cdot x)^2}{2} |H(x)| \leq \frac{\|k\|^2}{2d} \underbrace{\sum_x \|x\|^2 |H(x)|}_{O(L^2\beta)}. \tag{4.3.38}$$

Therefore, by taking L sufficiently large, $\hat{F}(k)^{-1}$ is bounded below by a positive multiple of $1 - \hat{D}(k)$, uniformly in $\|k\| \leq \frac{1}{L}$. Combined with (4.3.35), this completes the proof of the inequality (4.3.26), hence the completion of the proof of Lemma 4.3.2. ■

4.4 Notes

Having accomplished the estimate of the critical point for the spread-out lattice tree and lattice animals, the evaluation of p_c for several major statistical mechanical models

Model	**Spread-out model**	**Nearest-neighbor model**	d_c
Lattice trees	**[I]** Kawamoto, Sakai [34]	Miranda, Slade [40]	8
Lattice animals			
Oriented percolation	van der Hofstad, Sakai [26]	Cox, Durrett [10]*	4
Contact process		Liggett [36]*	
Self-avoiding walk		Hara, Slade [22]	
Percolation			6

表 4.1: The authors who achived the best estimate of p_c in $d > d_c$ so far

for the spread-out model has been concluded. For the nearest-neighbor model, the best estimate for oriented percolation (OP) so far was achieved by Cox and Durrett [10]:

$$1 + \frac{1}{2}\left(\frac{1}{2d}\right)^2 + o(d^{-2}) \leq p_c \leq 1 + \left(\frac{1}{2d}\right)^2 + O(d^{-3}), \tag{4.4.1}$$

and for contact process (CP) so far, it was obtained by Ligget [36]:

$$p_c = 1 + O\left(d^{-1}\right). \tag{4.4.2}$$

Both best estimates (4.4.1) and (4.4.2) appear to have room for improvement because they did not identify the exact expression of the error term. For the lace expansion for oriented percolation, three derivations are known[17, 41, 46], but Sakai's derivation [46] is particularly simple. It is known that the contact process can be approximated by an oriented percolation model (see e.g., [2, 3]), and thus the lace expansion derived by Sakai can be applied to contact process. As we do in Chapter 3 and Chapter 4, the critical points for both OP and CP can be described with the lace-expansion coefficient. However, while for the spread-out model, the second error term from the mean-field value arises due to the Riemann-sum approximation (see (1.1.8)), in the case of the nearest-neighbor model, the second error term comes from the lace-expansion coefficient. Therefore, we need to investigate the lace-expansion coefficinet more carefully. Hara and Slade [22], investigated the critical point for the nearest-neighbor self-avoiding walk and percolation, and Miranda and Slade [40] investigated the critical point for the nearest-neighbor lattice tree and lattice animals. The main idea used in [22, 40] has the recursive structure. They first get coarse estimate of p_c, and then by the coarse estimate, they obtain a more accurate estimate. Then, using the estimate obtained by the first procedure, they obtain a more accurate estimate. Iterating this procedure will results in increasingly accurate estimates of p_c. The same strategy used in [22, 40] should be applied to the estimate of p_c for the nearest-neighbor oriented percolation and contact process.

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