博士学位論文

Study on a Generalization of Boltzmann's Entropy Formula for Non-compact Classical Spin Systems
（非コンパクトな古典スピン系におけるボルツマンのエントロピー公式の拡張についての研究）

加治佐　貴大

北海道大学大学院理学院数学専攻

2024 年 9 月

Study on a Generalization of Boltzmann's Entropy Formula for Non-compact Classical Spin Systems

Takahiro Kajisa [1]

2024/8/25

[1]Department of Mathematics, Hokkaido University, Sapporo, Japan. E-mail: kajisa.takahiro.n7@elms.hokudai.ac.jp

Preface

We prove Boltzmann's entropy formula $S = k \log W$ in classical non-compact spin systems. The formula not only qualifies a thermodynamic limit point of microcanonical ensembles to be a Gibbs state, but we also find that it defines an elusive topology on the states. The proofs are heavily based on a shape of potentials, the negative exchanging interactions, and convexity of pressure and entropy. The former structure is an uncommon condition, but it brings regularities in such thermodynamic functions. Moreover, this assumption can be removed after taking the suitable thermodynamic limit. Our prominent example is lattice quantum scalar fields, especially, φ^4_ν model.

Contents

Chapter 1

Introduction

1.1 Introduction

In statistical mechanics, Boltzmann's entropy formula $S = k \log W$ plays a fundamental role in connecting thermodynamic entropy S and the number of microscopic states W in a given energy band. Ruelle [28] and Lanford [18] gave well-posed statements of Boltzmann's formula in classical compact spin systems, see also [27, 32]. Their conclusions enable us to establish the equivalence as probability measures between microcanonical states and Gibbs states in terms of thermodynamic functions. Such a problem is called the equivalence of ensembles, and that have been a central theme in statistical mechanics, see a physical review [34].

Boltzmann's formula can be analyzed from a view point of the large deviation theory, see [6, 21, 22] for classical systems and [24] for quantum systems. One can also prove the above equivalence of ensembles through large deviation principle [9, 10, 13, 21], but it merely claims that the concentration of two ensembles happens on the set of unique Gibbs phases. Hence, it is hard to prove the equivalence of ensembles in this manner when a phase transition, that means the non-uniqueness of the Gibbs phases at same inverse temperature, occurs. However, the original proof [28] is independent of the structure of the Gibbs phases, an extension in that direction can be considered as clues to cut into the hard situation.

The objective of this dissertation is to extend Boltzmann's formula to classical non-compact spin systems. While classical non-compact spin systems have been widely investigated [2, 3, 7, 13, 14, 16, 19, 29], an endeavor to generalize Boltzmann's formula, to the best of our knowledge, may be unprecedented. In tackling this issue, we must be careful of the following two points: First, we have to restrict a class of unbounded potentials appropriately. For example, superstability and suitable regularity are often imposed on unbounded potentials to ensure the existence of their pressures, see [19, 29] for classical cases, and [25] for quantum cases, but this is not very effective in controlling microcanonical entropy. Hence, we must find a more reasonable potential class that ensures the existence and convexity of the pressure and the entropy. Second, since our reference measure on a single spin configuration space is just a σ-finite measure on a non-compact Polish space, then good properties disappear that existed in the compact spin systems. For instance, the tightness of states indexed by increasing regions would be difficult to show, and the semicontinuity of entropy is lost [13].

In this study, we take the following procedure: we first introduce a class of potentials with super-stable, negative and finite range interactions. The negativity ensures the existence of pressure and microcanonical entropy, regardless of its finiteness [32, Theorem II.2.4], and the concavity of the microcanonical entropy. Then, we prove Boltzmann's formula for non-compact spin systems by the convexity (concavity) of the pressure (entropy). Our proof is based on a cut-off technique for a single spin configuration space. By that method, we obtain cut-off thermodynamic functions that can be regarded as those in certain bounded spin systems. The cut-off dependence after taking the thermodynamic limit can be removed by a very simple argument: the cut-off pressures and its sub-derivatives converge as we remove the cut-off. It is noteworthy to emphasize that our findings can be applied to several models in Euclidean quantum scalar field theory [11]. Although it may have a positive interaction, the lattice Laplacian, and seems not to be handled in the current framework, we can transform it into a negatively interacting potential without changing the value of pressure.

Chapter 2

Preliminaries

2.1 Convex analysis

At first, we cite several results of convex analysis. A real and finite function $F : I \to \mathbb{R}$ defined on an interval I of $\mathbb{R}$ is *convex* if and only if

$$F(\theta x + (1-\theta)y) \leq \theta F(x) + (1-\theta)F(y)$$

holds for $x, y \in I$ and $\theta \in [0, 1]$. This condition is equivalent to the case $\theta = 1/2$, if F is Lebesgue measurable [1]. Also a function F is concave if $-F$ is convex.

Proposition 2.1.1. [26, Theorem 24.1.] A convex function f has at most a countable number of points of the non-differentiability. Moreover, for any non-differentiable point $x \in \mathbb{R}$ of f, we can approximate the right and left differential $D^{\pm} f$ on x: Let $\{x_n^+\}_n$, $\{x_n^-\}_n$ be sets of differentiable points of f such as $x_n^+ \downarrow x$ and $x_n^- \uparrow x$ monotonically. Then, it follows that

$$D^{\pm} f(x) = \lim_{n\to\infty} Df(x_n^{\pm})$$

Proposition 2.1.2. [32, Proposition I.3.2.] Let $(f_n)_n$ be a sequence of convex functions on $\mathbb{R}$ with $f_n \to f$ as $n \to \infty$ pointwisely. Then, f is convex and for any fixed x_0,

$$(D^- f)(x_0) \leq \liminf_{n\to\infty} D^- f_n(x_0) \leq \limsup_{n\to\infty} D^+ f_n(x_0) \leq (D^+ f)(x_0).$$

Especially if f is differentiable at x_0, we have $(Df)(x_0) = \lim_{n\to\infty} D^{\pm} f_n(x_0)$.

Proposition 2.1.3. [26, Corollary 24.2.1] Let f be a convex function on $\mathbb{R}$ and α be an extended real number such as $\lim_{x\to\infty} D^- f(x) = \alpha$. Then, $\lim_{x\to\infty} f(x)/x = \alpha$.

2.2 Subadditive functions

A function $f : \mathbb{N} \to \mathbb{R}$ is called subadditive if and only if $f(n+m) \leq f(n) + f(m)$.

Proposition 2.2.1. [32, Theorem II.2.3.] Let $f : \mathbb{N}^\nu \to \mathbb{R}$ be a subadditive function on each argument. Then, the following limit exists in the extended real numbers

$$\lim_{n_1 \to \infty} \frac{f(n_1, \cdots, n_\nu)}{(n_1 \cdots n_\nu)} = \inf_{n_1 \in \mathbb{N}} \frac{f(n_1, \cdots, n_\nu)}{(n_1 \cdots n_\nu)}.$$

Chapter 3

The Spin Systems: General Settings

3.1 Configurations

Let S be a Polish space, $\mathscr{S}$ be a Borel field on S, and μ_0 be a Borel σ-finite measure on $\mathscr{S}$. The element $s \in S$ is called a single spin *configuration* and the measure space $(S, \mathscr{S}, \mu_0)$ is called a single spin configuration space. For each $x \in \mathbb{Z}^\nu$ we correspond $(S_x, \mathscr{S}_x, \mu_{0,x})$ as a copy of $(S, \mathscr{S}, \mu_0)$. Then, we define the entire configuration space Ω as a product space $\Omega = S^{\mathbb{Z}^\nu}$, $\nu \in \mathbb{N}$, and $\mathscr{F}$ be its Borel field. It is well known that $\mathscr{F}$ equals to the Kolmogorv σ-field, i.e., $\mathscr{F}$ includes all the cylinder events $\mathscr{S}_\Delta := \prod_{x\in\Delta} \mathscr{S}_x$ for $\Delta \Subset \mathbb{Z}^\nu$, where $A \Subset B$ means that A is a finite subset of B. Especially if S is compact and μ_0 is a finite measure, we call $(S, \mathscr{S}, \mu_0)$ a compact spin. In this study, we treat a non-compact spin system given by a non-compact polish space S and σ-finite measure μ_0 such as $\mu_0(S) = \infty$. For the system, we approximate it by a compact system $(S^{(k)}, \mu_0)$ such as the limit set of an increasing family $\{S^{(k)}\}_k$ is S. An example is $(S, \mu_0) := (\mathbb{R}, \lambda)$ and $S^{(k)} = (-k, k)$, where λ is Lebesgue measure.

3.2 Observables

The local observables are given by continuous functions $C(S^X)$ and bounded continuous functions $C_b(S^X)$ for bounded regions $X \Subset \mathbb{Z}^\nu$. Furthermore, the quasi-local observables are provided by the inductive limit of $C_b(S^X)$ completed with the uniform convergence topology: $C_b(\Omega) := \overline{\cup_{X\Subset\mathbb{Z}^\nu} C_b(S^X)}$. Also, the *translation map* $\theta_a : C_b(\Omega) \to C_b(\Omega)$, $a \in \mathbb{Z}^\nu$ is defined by $\theta_a f(s_{X+a}) := f(s_X)$.

3.3 Potentials

A *potential* is a set of local measurable functions $\Phi := \{\Phi_X \in C(S^X)\}_{X \Subset \mathbb{Z}^\nu}$ with translational invariance $\theta_a \Phi_X = \Phi_{X+a}$. Let $A_\Phi = \sum_{X \ni 0} |X|^{-1} \Phi_X$. A Hamiltonian on Λ with open boundary, free boundary condition is defined by $H^{\Phi}_{\Lambda}(s_\Lambda \times s_{\Lambda^c}) := \sum_{X \cap \Lambda \neq \varnothing} \Phi_X$ and $H^{F,\Phi}_{\Lambda}(s_\Lambda) := \sum_{X \subset \Lambda} \Phi_X$ respectively. Also, a periodic Hamiltonian $H^{P,\Phi}_{\Lambda}$ on Λ is defined by excluding the terms of $|X| > |\Lambda|$ from $H^{\Phi}_{\Lambda}(s_\Lambda \times \prod_{a \in \mathbb{Z}^\nu} \theta_{a\Lambda}(s_\Lambda))$.

Definition 3.3.1. The set of the potentials satisfy $(\mathrm{R}-1,2,3)$ is called *Ruelle class* and denoted by $\mathcal{R}$. Also, the set of the potentials with $(\mathrm{R}-1,2,3,4)$ is called *Negativity class* and written by $\mathcal{N}$.

(R-1) (*Superstability*). For all $\Lambda \Subset \mathbb{Z}^\nu$, there exist $\gamma \in 2\mathbb{N},\ A > 0,\ C \in \mathbb{R}$ such that

$$H^{\Phi}_{\Lambda}(s_\Lambda) \geq \sum_{x \in \Lambda} (A s_x^\gamma - C) \,.$$

(R-2) (*Strong regularity*). There exists a monotonically decreasing function $J : \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that

$$\sum_{x \in \mathbb{Z}^\nu} J(|x|) = A_\gamma, \qquad A_\gamma < \begin{cases} \infty & (\gamma > 2), \\ A & (\gamma = 2), \end{cases}$$

and for disjoint $\Lambda_1,\ \Lambda_2 \Subset \mathbb{Z}^\nu$,

$$|\Phi_{\Lambda_1 \cup \Lambda_2}(s_{\Lambda_1} \times s_{\Lambda_2})| \leq \frac{1}{2} \sum_{x \in \Lambda_1} \sum_{y \in \Lambda_2} J(|x-y|)(s_x^2 + s_y^2).$$

(R-3) (*Of finite range*). There exists $R \in \mathbb{N}$ such that $J(n) = 0$ for all $n \geq R$.

(R-4) (*Negativity*). $\Phi_X \leq 0$ for $|X| \geq 2$.

Remark 3.3.2. Since Assumption (R-3) is just for the simplicity, this restriction should be removed if necessary. The negativity (R-4) is not often used, but as we will see later it yields many regularities of thermodynamic functions. This condition can be removed by a proper treatment of Hamiltonian boundary conditions, see Example 5.1.3 and Remark 5.1.4.

Remark 3.3.3. Our assumptions except for the negativity are used only to prove the consistency of periodic boundary pressure and normal pressure function. Hence, if we do not wish to restore the potential class from $\mathcal{N}$ to $\mathcal{R}$, more broad potentials can be handled.

3.4 States

The probability measures on Ω is called *states* and denoted by $\mathcal{P}(\Omega)$. A *phase* is an element of the translationally invariant subset of $\mathcal{P}(\Omega)$ and denoted by $\mathcal{P}^I(\Omega)$. An important subset of states is *Gibbs states* for a potential Φ. First, a finite volume Gibbs states for open boundary Hamiltonian γ_Λ^Φ is defined by

$$(\gamma_\Lambda^\Phi f)(s_{\Lambda^c}) = \frac{1}{Z_\Lambda^\Phi(s_{\Lambda^c})} \int_{S^\Lambda} f(s_\Lambda \times s_{\Lambda^c}) \exp\left[-H_\Lambda^\Phi(s_\Lambda \times s_{\Lambda^c})\right] d\mu_{0,\Lambda},$$

where $\Lambda \Subset \mathbb{Z}$, $f \in C_b(\Omega)$, $\mu_{0,\Lambda} = \prod_{x\in\Lambda} \mu_{0,x}$ and Z_Λ^Φ is the normalization constant such as $(\gamma_\Lambda^\Phi(S^\Lambda))(s_{\Lambda^c}) = 1$. This is defined in the same way for free and periodic Hamiltonian and written by $\gamma_\Lambda^{F,\Phi}, \gamma_\Lambda^{P,\Phi}$. In the infinite volume, Gibbs states are characterized by the DLR equation: for all $\Lambda \Subset \mathbb{Z}$, $g \in C_b(\Omega)$,

$$\mu(\gamma_\Lambda^\Phi g) = \mu(g).$$

The set of $\mu \in \mathcal{P}(\Omega)$ satisfying DLR equation for Φ is denoted by $\mathcal{G}(\Phi)$. The corresponding Gibbs phases is $\mathcal{G}^I(\Phi) := \mathcal{G}(\Phi) \cap \mathcal{P}^I(\Omega)$. In our setting, any tight sequence of finite volume Gibbs states has limit points in the set of Gibbs states even if Ω is non-compact. The following facts are basic:

Proposition 3.4.1. [19, Section 4.] Let $\Phi \in \mathcal{R}$ and $\{\gamma_\Lambda\}_\Lambda$ be a sequence of Gibbs states of Φ with one of the free, periodic, or constant boundary conditions, indexed by increasing parallelpiped regions Λ. Then, $\{\gamma_\Lambda\}_\Lambda$ is tight. Also, let γ be an accumulation point of $\{\gamma_\Lambda\}_\Lambda$, then $\gamma \in \mathcal{G}(\Phi)$. In particular, if $\{\gamma_\Lambda\}_\Lambda$ has periodic boundary, its accumulation points belong to $\mathcal{G}^I(\Phi)$.

Remark 3.4.2. Boundary conditions that can be selected are even more general than those described here.

Let $\mathscr{T}_\Lambda := \mathscr{S}_{\mathbb{Z}^\nu \setminus \Lambda}$ and $\mathscr{T} := \cap_{\Lambda \Subset \mathbb{Z}^\nu} \mathscr{T}_\Lambda$. This is called the tail σ-field.

Proposition 3.4.3. [12, Theorem 7.7.] $\mathcal{G}(\Phi) \subset \mathcal{P}(\Omega)$ is a convex set. Also, $\mu \in \mathcal{P}(\Omega)$ is an extreme element of $\mathcal{G}(\Phi)$ if and only if μ is trivial on $\mathscr{T}$, i.e., $\mu(A) = \{0, 1\}$ for any $A \in \mathscr{T}$. Moreover, if $\mu_1, \mu_2 \in \mathcal{G}(\Phi)$ coincide on $\mathscr{T}$, then $\mu_1 = \mu_2$.

Remark 3.4.4. The densities of any Gibbs measure $\mu \in \mathcal{G}(\Phi)$ on the cylinder events on $\mathscr{S}_\Lambda$ can be written down $\int \mu(d\bar{\omega})(d\gamma_\Lambda/d\mu_{0,\Lambda})(\omega \times \bar{\omega})$ by the DLR equation. Hence, the difference among the elements of $\mathcal{G}(\Phi)$ is reduced to its behavior on $\mathscr{T}$.

Remark 3.4.5. A Gibbs measure μ being trivial on the tail field is analogous to the Kolmogorov 0-1 law holding for the identical distributions.

3.5 Thermodynamic functions

The local pressure function is

$$p_\Lambda(\beta) := \frac{1}{|\Lambda|} \log \int_{S^\Lambda} e^{-\beta H_\Lambda^\Phi(s_\Lambda)} \mu_{0,\Lambda}(ds_\Lambda),$$

and local microcanonical entropy is

$$m_\Lambda(u) := \frac{1}{|\Lambda|} \log \mu_{0,\Lambda} \left\{ s_\Lambda \in S^\Lambda \mid H_\Lambda^\Phi(s_\Lambda) \le |\Lambda| u \right\}.$$

Usually, it needs enormous effort to prove the existence of the each limit and its inherited properties. By a negativity assumption on a potential, that procedure can be surprisingly simplified:

Lemma 3.5.1. Let $\Phi \in \mathcal{N}$. Then, $p_\Lambda(\beta)$ and $m_\Lambda(u)$ are superadditive functions on Λ, and the limits $p(\beta) = \lim_{\Lambda \to \mathbb{Z}^\nu} p_\Lambda(\beta)$ and $m(u) = \lim_{\Lambda \to \mathbb{Z}^\nu} m_\Lambda(u)$ exist. Moreover, $p(\beta)$ is convex on β and $m(u)$ is concave on u.

Proof. Fix $\Phi \in \mathcal{N}$ and abbreviate symbols from a Hamiltonian. At first, from the negativity we have $H_\Lambda \le H_{\Lambda_1} + H_{\Lambda_2}$ and

$$e^{-\beta H_\Lambda} \ge e^{-\beta H_{\Lambda_1}} e^{-\beta H_{\Lambda_2}}$$

for a disjoint decomposition $\Lambda = \Lambda_1 \cup \Lambda_2$.

Then, we have the superadditivity:

$$\log \int_{S^\Lambda} e^{-\beta H_\Lambda} d\mu_{0,\Lambda} \ge \log \int_{S^{\Lambda_1}} e^{-\beta H_{\Lambda_1}} d\mu_{0,\Lambda_1} + \log \int_{S^{\Lambda_2}} e^{-\beta H_{\Lambda_2}} d\mu_{0,\Lambda_2}.$$

From Proposition 2.2.1 we have the limit $p(\beta) = \lim_{\Lambda \to \mathbb{Z}^\nu} p_\Lambda(\beta)$. Also by the Schwarz inequality, we have the convexity $p_\Lambda(\frac{\beta_1+\beta_2}{2}) \le \frac{1}{2} p_\Lambda(\beta_1) + \frac{1}{2} p_\Lambda(\beta_2)$, and by Proposition 2.1.2, $p(\beta)$ is convex.

On a microcanonical entropy, since

$$\log \mu_{0,\Lambda} \{H_\Lambda \le |\Lambda| u\} \ge \log \mu_{0,\Lambda_1} \{H_{\Lambda_1} \le |\Lambda_1| u\} + \log \mu_{0,\Lambda_2} \{H_{\Lambda_2} \le |\Lambda_2| u\}$$

holds, then Proposition 2.2.1 yields the existence of $m(u) = \lim_{\Lambda \to \mathbb{Z}^\nu} m_\Lambda(u)$. Also, let $u = (u_1 + u_2)/2$ and $|\Lambda_1| = |\Lambda_2| = |\Lambda|/2$, we obtain

$$\begin{aligned} &\log \mu_{0,\Lambda} \{H_\Lambda \le |\Lambda| u\} \\ &\ge \log \mu_{0,\Lambda} \{H_{\Lambda_1} + H_{\Lambda_2} \le |\Lambda_1| u_1 + |\Lambda_2| u_2\} \\ &\ge \log \mu_{0,\Lambda_1} \{H_{\Lambda_1} \le |\Lambda_1| u_1\} + \log \mu_{0,\Lambda_2} \{H_{\Lambda_2} \le |\Lambda_2| u_2\}. \end{aligned}$$

Dividing both sides by $|\Lambda|$ and taking the limit $\Lambda \to \mathbb{Z}^\nu$, we obtain the midpoint concavity $m(u) \ge \frac{m(u_1)+m(u_2)}{2}$. That completes the proof since $m(u)$ is Lebesgue measurable and bounded by $p(\beta) + \beta u$, see the beginning of the proof of Theorem 5.1.1. □

Various boundary conditioned pressures can be considered. As an important example, we treat the periodic version:

$$p_\Lambda^P(\beta) = \frac{1}{|\Lambda|} \log \int_{S^\Lambda} e^{-\beta H_\Lambda^P(\Phi)} d\mu_{0,\Lambda}.$$

The next fact is basic:

Proposition 3.5.2. [19, Theorem 3.1.] Let $\Phi \in \mathcal{R}$, then the infinite volume limit exists and $\lim_{\Lambda \to \mathbb{Z}^\nu} p_\Lambda^P(\beta) = p(\beta)$.

Remark 3.5.3. This claim is the only one in which the superstability condition (R-1) is used. We will see about this later.

The local relative entropy $\mathscr{H}_\Lambda$ is defined by

$$\mathscr{H}_\Lambda(\mu \mid \nu) = \begin{cases} \nu(f_\Lambda \log f_\Lambda) & \text{if } \mu \ll \nu \text{ on } \mathscr{S}_\Lambda \\ \infty & \text{otherwise,} \end{cases}$$

where f_Λ is Radon-Nikodym derivative of the restrictions $d(\mu|_{\mathscr{S}_\Lambda})/d(\nu|_{\mathscr{S}_\Lambda})$. The limit

$$h(\mu \mid \nu) := \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \mathscr{H}_\Lambda(\mu \mid \nu)$$

exists for all $\mu, \nu \in \mathcal{P}^I(\Omega)$ because in this case $\mathscr{H}_\Lambda(\mu \mid \nu)$ is superadditive, see [13, Section 15.2]. Especially, let $\nu|_{\mathscr{S}_\Lambda} := \mu_{0,\Lambda}$, we define entropy functional as

$$\mathscr{H}_\Lambda(\mu) := -\mathscr{H}_\Lambda(\mu \mid \mu_{0,\Lambda})$$

and its density limit

$$h(\mu) := \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \mathscr{H}_\Lambda(\mu).$$

The relation between the Gibbs states and the thermodynamic functions is as follows:

Proposition 3.5.4. [17, Section 4.] Let $\Phi \in \mathcal{R}$. Then, $\mu \in \mathcal{G}^I(\beta\Phi)$ if and only if μ is a solution of the variational principle:

$$p(\beta) = \sup_{\rho \in \mathcal{P}^I} \left\{ h(\rho) - \beta \rho(A_\Phi) \right\}.$$

In particular, let β be a point of differentiability of p, there exists $u \in \mathbb{R}$ uniquely and it follows that

$$(Dp)(\beta) = -\rho(A_\Phi) = -u.$$

3.5.1 Notes on the pressure

Now, we will make graphical explanations of the variational principle. The ground state energy is defined by

$$e_{\inf}(\Phi) := \lim_{\Lambda\to\mathbb{Z}^\nu} \inf_{s_\Lambda \in S^\Lambda} \frac{H_\Lambda(\Phi)(s_\Lambda)}{|\Lambda|}.$$

At first, the convex function $p(\beta)$ for $\Phi \in \mathcal{R}$ is defined on $[0, \infty)$, and the negative slope value u of p takes a value in $(e_{\inf}, \infty)$ due to Lemma 5.2.2. The variational principle states that the tangents to p at β is given by $l(\rho) := h(\rho) - \beta\rho(A_\Phi)$ where $\rho \in \mathcal{G}^I(\beta\Phi)$, see Figure 1.

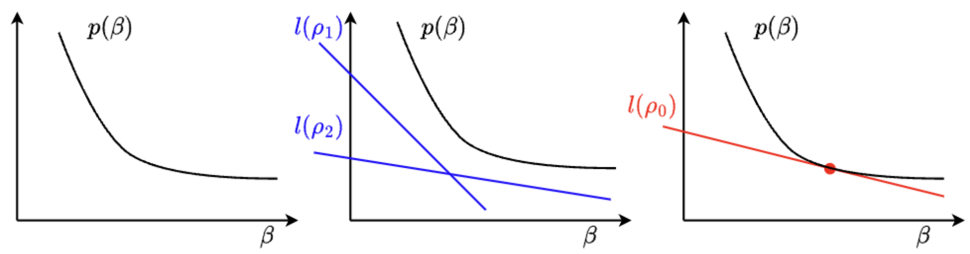

Figure 3.1: A look at the variational principle.

Since there exist a correspondence $u \mapsto \beta_u$ by the convexity, we can write

$$p(\beta_u) = h(\rho) - \beta_u\rho(A_\Phi), \quad \text{where} \quad \rho(A_\Phi) = u, \ \rho \in \mathcal{G}^I(\beta_u\Phi).$$

Hence if we find a solution of the variational problem $\rho \in \mathcal{P}^I(\Omega)$ with $\rho(A_\Phi) = u$, it is automatically a Gibbs phase of the potential $\beta_u\Phi$.

One of the prominent examples of such a ρ, except for the limit of finite volume Gibbs states, is the limit microcanonical states: let

$$D_\Lambda(u,\delta) = \left\{ s_\Lambda \in S^\Lambda \;\middle|\; \frac{H_\Lambda^\Phi}{|\Lambda|} \in (u-\delta, u+\delta) \right\},$$

$$D_\Lambda^P(u,\delta) = \left\{ s_\Lambda \in S^\Lambda \;\middle|\; \frac{H_\Lambda^{P,\Phi}}{|\Lambda|} \in (u-\delta, u+\delta) \right\},$$

then a finite volume microcanonical state on Λ is defined by the conditional expectation with respect to $D_\Lambda(u,\delta)$,

$$\mu_\Lambda(\cdot) := \mu_{0,\Lambda}(\cdot \mid D_\Lambda(u,\delta)).$$

It is believed that the thermodynamic limits μ of μ_Λ solve the variational problem, and indeed they do in the compact spin systems. In this case, any μ belongs to $\beta_u\Phi$ and thus, μ has a family of the Gibbsian local densities. This is so called the equivalence of ensembles. Our main theorem in this study is a very tool to compute the entropy density $h(\mu)$ in non-compact systems.

We will see the case of compact systems in the next section.

Chapter 4

Results on the compact spin systems

4.1 Review from the viewpoint of the large deviation theory

Let $\mathcal{B}$ be a class of absolutely summable potentials Φ such as

$$|||\Phi||| = \sum_{X \ni 0} \|\Phi_X\|_\infty < \infty,$$

for the sup norm $\|\cdot\|_\infty$ on $C_b(\Omega)$. Only in this section, we assume that the configuration space Ω is compact. We define the periodic empirical distribution

$$s_\Lambda \in S^\Lambda \mapsto R_\Lambda(s_\Lambda) := \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \theta_{-x} * \delta_{s_\Lambda},$$

where the boundary condition is periodic, δ_x is Dirac delta function, and the translation $\theta_{-x}* : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ is defined by $(\theta_{-x} * \mu)(f) = \mu(\theta_x f)$. The next proposition is called the large deviation principle (LDP):

Proposition 4.1.1. [12, Theorem 15.45.] Let $\gamma \in \mathcal{G}^I(\beta\Phi)$ and $U \subset \mathcal{P}^I(\Omega)$, then

$$\begin{aligned} -\inf_{\mu \in U^\circ} h(\mu \mid \gamma) &\leq \liminf_{\Lambda \to \mathbb{Z}} \frac{1}{|\Lambda|} \log \left[\inf_{y \in \Omega} \gamma_\Lambda \left(R_\Lambda \in U \mid y \right) \right] \\ &\leq \limsup_{\Lambda \to \mathbb{Z}} \frac{1}{|\Lambda|} \log \left[\sup_{y \in \Omega} \gamma_\Lambda \left(R_\Lambda \in U \mid y \right) \right] \leq -\inf_{\mu \in \bar{U}} h(\mu \mid \gamma), \end{aligned}$$

where $U^\circ, \bar{U}$ denote the closure and the interior in the weak topology.

These inequality hold independently of the way γ is taken. Actually, $h(\mu \mid \gamma)$ is represented by

$$h(\mu \mid \gamma) = p(\beta\Phi) + \mu(A_{\beta\Phi}) - h(\mu).$$

By the definition $h(\mu \mid \gamma) \geq 0$, and by the variational principle the equality $h(\mu \mid \gamma) = 0$ holds only for $\mu \in \mathcal{G}^I(\beta\Phi)$. Therefore, $h(\mu \mid \gamma)$ defines a regular like topology on $\mathcal{P}^I(\Omega)$ that separate $\mathcal{G}^I(\beta\Phi)$ and any other point $\mu \notin \mathcal{G}^I(\beta\Phi)$. This is why LDP is not effective when a phase transition occurs.

Corollary 4.1.2. [12, Corollary 15.48.] Let γ_Λ be the finite volume Gibbs state of $\beta(u)\Phi$, and $\delta > 0$, then

$$\lim_{\Lambda \to \mathbb{Z}} \gamma_\Lambda \left(D_\Lambda^P(u, \delta) \right) = 1.$$

Proof. Note that for any $s_\Lambda \in S^\Lambda$, we can write $R_\Lambda(A_\Phi) = H_\Lambda^{P,\Phi}(s_\Lambda)/|\Lambda|$. Let U be a subset of $\mathcal{P}$ with $\mu(A_\Phi) \in (-\infty, u - \delta] \cup [u + \delta, \infty)$. Clearly $U \cap \mathcal{G}(\beta(u)\Phi) = \varnothing$ and $h := \inf_{\mu \in \bar{U}} h(\mu \mid \gamma) > 0$. Thus by the large deviation, we obtain $\gamma_\Lambda(R_\Lambda \in U) \sim e^{-|\Lambda| h} \to 0$, which completes the proof. □

Remark 4.1.3. A boundary condition of γ_Λ can be freely selected even if it is in the form of an integral like $\int \mu(dt)\gamma_\Lambda(\cdot \mid t)$.

Remark 4.1.4. Such a $\delta > 0$ is essential because an element of $\bar{U}$ may touch $\mathcal{G}(\beta(u)\Phi)$ if $\delta = 0$. Later, we will find that δ can be removed through continuity of thermodynamic functions, after taking the infinite volume limit.

Remark 4.1.5. The statement of the large deviation principle can be interpreted as the convergence of the probability measure of the set of probability measure, i.e., the convergence on $\mathcal{P}(\mathcal{P}(\Omega))$. In this sense, [13] does not study the weak convergence of probability measures directly.

With the help of LDP, we derive the Boltzmann's entropy formula for the compact spin systems, which means that $p(\beta)$ is the Legendre transformation of a convex function $p^*(u) := -m(-u)$ at β. Only for the compact spin systems, we define $e_{\mathrm{av}} := D^+p(0)$ since the pressures do not diverge at $\beta = 0$.

Theorem 4.1.6. Let $\Phi \in \mathcal{B} \cap \mathcal{N}$ and $u \in (e_{\inf}, e_{\mathrm{av}})$. Then, we obtain

$$p(\beta) = -\beta u + m(u). \tag{4.1}$$

Remark 4.1.7. From Proposition 3.5.4, it follows that $m(u) = h(\gamma)$, where $\gamma \in \mathcal{G}^I(\beta(u)\Phi)$. Hence, we may conclude that in the thermodynamic limit the logarithmic density of states $m(u)$ equals the thermodynamic entropy $h(\gamma)$.

Proof. First, by a direct calculation,

$$\begin{aligned}
&\log \int_{S^\Lambda} e^{-\beta H_\Lambda} d\mu_{0,\Lambda} \\
&\geq \log \int_{\{s \in S^\Lambda : H_\Lambda(s) \leq |\Lambda| u\}} e^{-\beta H_\Lambda} d\mu_{0,\Lambda} \\
&\geq -\beta |\Lambda| u + \log \mu_{0,\Lambda}\{s \in S^\Lambda : H_\Lambda(s) \leq |\Lambda| u\}.
\end{aligned}$$

Dividing $|\Lambda|$ and sending $\Lambda \to \mathbb{Z}^\nu$ we have

$$p(\beta) \geq -\beta u + m(u).$$

Because of the boundedness of $\Phi \in \mathcal{B}$, for all $\varepsilon > 0$ there exists Λ_0 such that for $\Lambda \Supset \Lambda_0$,

$$D^P_\Lambda(u, \delta) \leq D_\Lambda(u, \delta + \varepsilon).$$

From Corollary 4.1.2, we have

$$\left| \frac{1}{|\Lambda|} \log \int_{D^P_\Lambda(u,\delta)} e^{-\beta H_\Lambda} d\mu_{0,\Lambda} - p_\Lambda \right| = \left| \frac{1}{|\Lambda|} \log \gamma^\beta_\Lambda(D^P_\Lambda(u,\delta)) \right| \to 0,$$

as $\Lambda \to \mathbb{Z}^\nu$. Thus, we obtain

$$\begin{aligned}
p(\beta) &= \lim_{\Lambda \to \mathbb{Z}^\nu} \frac{1}{|\Lambda|} \log \int_{D^P_\Lambda(u,\delta)} e^{-\beta H_\Lambda} d\mu_{0,\Lambda} \\
&\leq \lim_{\Lambda \to \mathbb{Z}^\nu} \frac{1}{|\Lambda|} \log \int_{D_\Lambda(u,\delta+\varepsilon)} e^{-\beta H_\Lambda} d\mu_{0,\Lambda} \\
&\leq -\beta(u - \delta - \varepsilon) + m(u + \delta + \varepsilon).
\end{aligned}$$

By the continuity of m, we obtain

$$p(\beta) \leq -\beta u + m(u).$$

Combining the two inequalities, we complete the proof. $\square$

Chapter 5

Main results

Our goal it to extend the Boltzmann's formula to non-compact spin systems. The results are cited from [15].

At the beginning, we have three strategies:

(A) To introduce the total variation distance $\|\cdot\|_{\mathrm{TV}}$ on $\mathcal{P}(\Omega)$ and prove the compactness of a net $\{\gamma_\Lambda\}_\Lambda$. In this case, the claim of Corollary 4.1.2 becomes obvious. However, to make a nice bound of total variance seems to be difficult even if Ω is compact; for instance, the Pinsker's inequality is well-known:

$$\|\mu_1 - \mu_2\|_{\mathrm{TV}} \leq \sqrt{\mathscr{H}(\mu_1 \mid \mu_2)}.$$

Since we only know the convergence of the specific relative entropy $h(\mu_1 \mid \mu_2)$, the right-hand side is hard to analyze in the thermodynamic limit. Worse yet, because this topology reflects the values on the tail field, we can not make an accurate bound unless we know the structure of the Gibbs states completely.

(B) In contrast to (A), the large deviation principle holds by making use of $h(\mu_1 \mid \mu_2)$. In that sense, the convergence by the large deviation may be weaker than the total variation. Then, our another idea is to generalize LDP to non-compact spin systems. However, as far as I know, this would require the hyper mixing condition proposed in the context of Markov processes [8]; that is, for all $n \in \mathbb{N}$,

$$\|f_1 \cdots f_n\|_{L^1(\mu)} \leq \prod_{m=1}^{n} \|f_m\|_{L^{\delta(R)}(\mu)}$$

where $\delta : \mathbb{N} \to [1, \infty)$ with $\lim \delta(R) = 1$, and the supports of $\{f_m\}$ are separated from each other by at least R. This condition is equivalent to the hypercontractivity of Gibbsian density, see [8, Section 6], and some examples are find in [4, 5, 31].

(C) As we see in the previous paragraph, an extension of non-compact LDP imposes more difficult condition on the correlation functions. Our final idea is to control generating functions of Gibbs states, such as pressures and entropies. From the probabilistic viewpoint, the pressure of the potential Φ can be regarded as a generalization of the cumulant generating function of the random variable A_Φ, and the entropy of a phase is as the generalized total variation distance from the Gibbs phase $\gamma^\Phi \in \mathcal{G}^I(\Phi)$. Hence, we try to lift up the introduced results of the compact spin systems to certain non-compact cases. The tools that we will use are just the convexity of each function and the compact space approximation of Ω. Finally, I am of the opinion that such a methodology could be useful when tackling different problems of non-compact spin systems.

5.1 Main results

The following is our main theorem:

Theorem 5.1.1. For $\Phi \in \mathcal{N}$ and $u \in (e_{\inf}, \infty)$, we obtain

$$p(\beta) = -\beta u + m(u). \tag{5.1}$$

Next, we find a prescription to deal with a certain class of positively interacting and superstable potentials.

Corollary 5.1.2. Let H^P_Λ be the periodic boundary conditioned Hamiltonian of Φ and u as above. Then,

$$m(u) = \lim_{\Lambda \to \mathbb{Z}^\nu} \frac{1}{|\Lambda|} \log \mu_{0,\Lambda}\{s \in S^\Lambda : H^P_\Lambda(s) \leq |\Lambda| u\}. \tag{5.2}$$

Example 5.1.3. (*the lattice φ^4_ν model*). Let us consider a single spin space $(\mathbb{R}, \lambda)$ equipped with a $\mathbb{R}$-valued spin variable φ. We define a φ^4_ν-type potential $\Phi' \in \mathcal{R} \backslash \mathcal{N}$ as

$$\begin{cases} \Phi'_{\{x\}} = A_4 \varphi_x^4 + A_2 \varphi_x^2, \\ \Phi'_{\{x,y;|x-y|=1\}} = \lambda(\varphi_x - \varphi_y)^2, \end{cases}$$

for $A_4 > 0, A_2 \in \mathbb{R}, \lambda \geq 0$, and otherwise zero. By a simple calculation

$$(\varphi_x - \varphi_y)^2 = 2\varphi_x^2 + 2\varphi_y^2 - (\varphi_x + \varphi_y)^2$$

we get another potential $\Phi \in \mathcal{N}$, that is physically equivalent to Φ' in the sense of [12, Definition 2.33], i.e.,

$$H_\Lambda^{P,\Phi}(\phi_\Lambda) = H_\Lambda^{P,\Phi'}(\phi_\Lambda) \tag{5.3}$$

holds for the periodic boundary conditioned Hamiltonians $H_\Lambda^{P,\Phi}$ and $H_\Lambda^{P,\Phi'}$. Also, Φ is explicitly written as

$$\begin{cases} \Phi_{\{x\}} = A_4\varphi_x^4 + (A_2 + 4\nu\lambda)\varphi_x^2, \\ \Phi_{\{x,y;|x-y|=1\}} = -\lambda(\varphi_x + \varphi_y)^2, \end{cases}$$

and otherwise zero.

By Proposition 3.5.2 and Equation (5.3), we obtain

$$\begin{aligned} p(\beta\Phi) &= \lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \log \int_{S^\Lambda} e^{-\beta H_\Lambda^P(\Phi)} d\mu_{0,\Lambda} \\ &= \lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \log \int_{S^\Lambda} e^{-\beta H_\Lambda^P(\Phi')} d\mu_{0,\Lambda} = p(\beta\Phi'). \end{aligned}$$

Then, by Corollary 5.1.2 and Theorem 5.1.1 yield

$$m(u,\Phi') = m(u,\Phi) = p(\beta\Phi) + \beta u = p(\beta\Phi') + \beta u.$$

This is exactly the Boltzmann's formula for a positively interacting φ_ν^4-type potential $\Phi' \in \mathcal{R}\backslash\mathcal{N}$.

Remark 5.1.4. The physically equivalent transformation $\Phi' \notin \mathcal{N} \mapsto \Phi \in \mathcal{N}$ is not so common: For example, a potential Ψ provided with $\Psi(\{x,y\}) := |\varphi_x - \varphi_y|$ has no such correspondence.

5.2 Proofs of main results

Let $\{S_k\}_{k\in\mathbb{N}}$ be a sequence of increasing compact sets with $\mu_0(S_k) < \infty$ and $\cup_{k\in\mathbb{N}} S_k = S$. Also, let $\Omega_k := S_k^{\mathbb{Z}^\nu}$ and Φ_k be the restriction of Φ to Ω_k which is defined by restricting the function Φ_X to S_k^X for each $X \Subset \mathbb{Z}^\nu$, and we denote the cut-off Hamiltonian by $H_{\Lambda,k} := H_\Lambda(\Phi_k)$. Clearly, Ω_k constitute a compact configuration space and Φ_k is a bounded potential on Ω_k. For $\Phi \in \mathcal{N}$, we define the cut-off local pressure $p_{\Lambda,k}$ by

$$p_{\Lambda,k}(\beta) := \frac{1}{|\Lambda|} \log \int_{S_k^\Lambda} e^{-\beta H_{\Lambda,k}} d\mu_{0,\Lambda}$$

and the cut-off pressure p_k by

$$p_k(\beta) := \lim_{\Lambda \to \mathbb{Z}^\nu} \frac{1}{|\Lambda|} \log \int_{S_k^\Lambda} e^{-\beta H_{\Lambda,k}} d\mu_{0,\Lambda}.$$

Also, we define

$$e_{\mathrm{inf},k} := \lim_{\Lambda \to \mathbb{Z}^\nu} \inf_{s \in \Omega_k} |\Lambda|^{-1} H_{\Lambda,k}(s),$$

$$e_{\mathrm{av},k} := -\lim_{\beta \to 0} D^+ p_k(\beta).$$

The relation between the pressure and the ground state energy is described below.

Proposition 5.2.1. [32, Theorem II.2.6] Assume the support of μ_0 is all of Ω_k. Then,

$$\lim_{\beta \to \infty} \frac{p_k(\beta)}{\beta} = -e_{\mathrm{inf},k}.$$

The next is our main technical lemma that shows how the cut-off dependence removed:

Lemma 5.2.2.

(i) For the double sequence $\{p_{\Lambda,k}\}_{\Lambda \Subset \mathbb{Z}^\nu, k \in \mathbb{N}}$, there exists a monotonically increasing sequence of regions Λ_j such that the doubly indexed sequence $\{p_{\Lambda_j,k}\}_{j,k}$ forms a doubly monotonically increasing sequence. As a consequence, we have the monotone convergence of the cut-off pressures:

$$p(\beta) = \lim_{k \to \infty} p_k(\beta).$$

(ii) Choose $c_k \in \mathbb{R}$ such that $D^- p_k(\beta) \le c_k \le D^+ p_k(\beta)$. Assume that p is differentiable at β. Then we have $Dp(\beta) = \lim_{k \to \infty} c_k$.

(iii) We get

$$\lim_{\beta \to \infty} D^- p(\beta) = -\lim_{k \to \infty} e_{\mathrm{inf},k} = -e_{\mathrm{inf}}, \tag{5.4}$$

$$\lim_{\beta \to 0} D^+ p(\beta) = -\lim_{k \to \infty} e_{\mathrm{av},k} = -\infty. \tag{5.5}$$

Proof. (i) As β is fixed, the β-dependence of pressure is omitted in this proof. First, since Φ_k is in $\mathcal{N}$, we have $p_{\Lambda,k} \le p_k$ for all Λ, k by Proposition 2.2.1. For $k = 1$, an increasing subsequence $\{p_{\Lambda_1(i),1}\}_i$ can be taken such as $p_{\Lambda_1(i),1} \uparrow p_1$ as $i \to \infty$. Next in the case of $k = 2$, we find $p_{\Lambda_1(i),2} \le p_2$ for all i and can also take an increasing subsequence $\{p_{\Lambda_2(i),2}\}_i$ with $p_{\Lambda_2(i),2} \uparrow p_2$. By repeating these procedures and setting $\Lambda_j := \Lambda_j(j)$, we obtain the sequence $\{p_{\Lambda_j,k}\}_j$ which is monotonically increasing for all k. From the construction method, it is evident that $p_{\Lambda_j,k}$ is monotonically increasing with respect to k for each j. Thus, its iterated limits are interchangeable and in particular we have $p(\beta) = \lim_{k\to\infty} p_k(\beta)$.

(ii) Since $p_k(\beta)$ converges to $p(\beta)$ as $k \to \infty$ pointwisely, we can apply Proposition 2.1.2.

(iii) First, by Lemma 5.2.2-(i), Proposition 2.1.2, and the monotonicity of the left derivative, we have

$$D^- p(\beta) \le \liminf_{k\to\infty} D^- p_k(\beta) \le \liminf_{k\to\infty} \lim_{\beta\to\infty} D^- p_k(\beta) = \liminf_{k\to\infty}(-e_{\inf,k}).$$

On the other hand, Proposition 2.1.3, Proposition 5.2.1, and the monotonicity of the cut-offs pressure yield

$$\lim_{\beta\to\infty} D^- p(\beta) = \lim_{\beta\to\infty} \frac{p(\beta)}{\beta} \ge \lim_{\beta\to\infty} \frac{p_k(\beta)}{\beta} = \lim_{\beta\to\infty} D^- p_k(\beta) = -e_{\inf,k}.$$

Hence, we obtain

$$\lim_{\beta\to\infty} D^- p(\beta) = \lim_{k\to\infty}(-e_{\inf,k}).$$

We remark that, $e_{\inf,k} = \lim_{\Lambda\to\mathbb{Z}^\nu} \inf_{s\in\Omega_k} |\Lambda|^{-1} H_{\Lambda,k}$ exists and equals $\inf_{\Lambda\subset\mathbb{Z}^\nu} \inf_{s\in\Omega_k} |\Lambda|^{-1} H_{\Lambda,k}$ since $\inf_{s\in\Omega_k} H_{\Lambda,k}$ is subadditive: $\inf_{s\in\Omega_k} H_{\Lambda_1\cup\Lambda_2,k} \le \inf_{s\in\Omega_k} H_{\Lambda_1,k} + \inf_{s\in\Omega_k} H_{\Lambda_2,k}$ [32, Theorem II.2.3]. Thus by interchanging the infimums, we get

$$\begin{aligned}
\lim_{k\to\infty} e_{\inf,k} &= \lim_{k\to\infty} \inf_{\Lambda\subset\mathbb{Z}^\nu} \inf_{s\in\Omega_k} |\Lambda|^{-1} H_\Lambda(s) \\
&= \lim_{k\to\infty} \inf_{s\in\Omega_k} \inf_{\Lambda\subset\mathbb{Z}^\nu} |\Lambda|^{-1} H_\Lambda(s) \\
&= \inf_{s\in\Omega} \inf_{\Lambda\subset\mathbb{Z}^\nu} |\Lambda|^{-1} H_\Lambda(s) \\
&= e_{\inf},
\end{aligned}$$

where the third equality is due to the lower boundedness of H_Λ. Thus, the claim for $D^- p$ is proved. Next, by using the inequality $p \ge p_{\Lambda,k}$ and applying Fatou's lemma for $p_{\Lambda,k}$, we obtain

$$\liminf_{\beta\to 0} p(\beta) \ge \liminf_{\beta\to 0} p_{\Lambda,k}(\beta) = \log \mu_0(S_k).$$

Hence, we have

$$\limsup_{\beta\to 0} D^+ p(\beta) \le \limsup_{\beta\to 0} \frac{p(1) - p(\beta)}{1-\beta} \le p(1) - \log \mu_0(S_k).$$

Because $\mu_0(S_k) \to \infty$ as $k \to \infty$, we have $\lim_{\beta\to 0} D^+p(\beta) = -\infty$. In addition, Proposition 2.1.2 yields

$$\limsup_{k\to\infty}(-e_{\mathrm{av},k}) = \limsup_{k\to\infty} D^+p_k(0) \leq \limsup_{\beta\to 0} D^+p(\beta) \leq -\infty.$$

Therefore we conclude that $\lim_{\beta\to 0} D^+p(\beta) = -\lim_{k\to\infty} e_{\mathrm{av},k} = -\infty$. □

The Boltzmann's formula for the cut-off systems has been obtained:

Lemma 5.2.3. Let $u_k \in (e_{\inf,k}, e_{\mathrm{av},k})$ with $\beta := \beta_{u_k}$, and define

$$m_k(u_k) := \lim_{\Lambda\to\mathbb{Z}^\nu} |\Lambda|^{-1} \log \mu_{0,\Lambda}\{s \in S_k^\Lambda \mid H_{\Lambda,k}(s) \leq |\Lambda|u_k\}.$$

Then,

$$m_k(u_k) = p_k(\beta) + \beta u_k.$$

Proof. The proof is same as Theorem 4.1.6 □

Now, we get to the proof of Theorem 5.1.1:

Proof of Theorem 5.1.1. Since

$$\begin{aligned}
&\log \int_{S^\Lambda} e^{-\beta H_\Lambda} d\mu_{0,\Lambda} \\
&\geq \log \int_{\{s\in S^\Lambda : H_\Lambda(s)\leq|\Lambda|u\}} e^{-\beta H_\Lambda} d\mu_{0,\Lambda} \\
&\geq -\beta|\Lambda|u + \log \mu_{0,\Lambda}\{s \in S^\Lambda : H_\Lambda(s) \leq |\Lambda|u\}
\end{aligned}$$

holds, we have the following upper bound:

$$m(u) \leq p(\beta) + \beta u. \tag{5.6}$$

Next, let $\alpha > 0$ be a differentiable point of p with its derivative $Dp(\alpha) = -v$. Then, we can take $v_k \in (e_{\min,k}, e_{\mathrm{av},k})$ such that $-v_k \in [(D^-p_k)(\alpha), (D^+p_k)(\alpha)]$ for each $k \in \mathbb{N}$. Readers can infer from Figure 5.1 that p is approximated by p_k at α.

By applying Lemma 5.2.2-(ii), we have

$$v = \lim_{k\to\infty} v_k. \tag{5.7}$$

Thus, for all $\varepsilon > 0$ there exists a k_0 such that for any $k \geq k_0$, we have $v_k - v < \varepsilon$ and

$$\mu_{0,\Lambda}\{s \in S^\Lambda : H_\Lambda(\Phi)(s) \leq |\Lambda|(v+\varepsilon)\} \geq \mu_{0,\Lambda}\{s \in S_k^\Lambda : H_\Lambda(\Phi_k)(s) \leq |\Lambda|v_k\}. \tag{5.8}$$

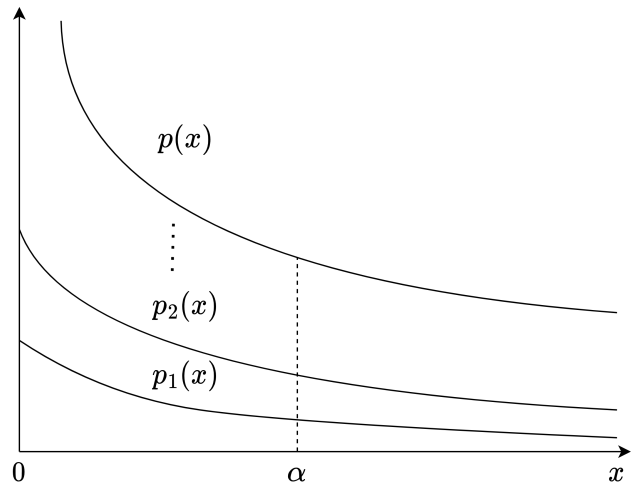

Figure 5.1: graphical description of the approximation to p.

Therefore,

$$m(v+\varepsilon) \geq m_k(v_k) = p_k(\alpha) + \alpha v_k.$$

By sending $k \to \infty$, Lemma 5.2.2-(i) and Equation (5.7) yield

$$m(v+\varepsilon) \geq p(\alpha) + \alpha v.$$

Finally, by the concavity of m, we get

$$m(v) \geq p(\alpha) + \alpha v. \tag{5.9}$$

On the non-differentiable point β of p, we set the right and left derivatives as $-v^{\pm} = D^{\pm}p(\beta)$. By Proposition 2.1.1, we can take sequences of the differentiable points $\{\beta_n^{\pm}\}$ of p with $\beta_n^{\pm} \to \beta$. Here, let $Dp(\beta_n^{\pm}) = -v_n^{\pm}$, then $v_n^{\pm} \to v^{\pm}$ holds as $n \to \infty$, respectively. By the continuity of m and Equation (5.9), we have

$$m(v^{\pm}) = \lim_{n\to\infty} m(v_n^{\pm}) \geq \lim_{n\to\infty} \left(p(\beta_n^{\pm}) + \beta_n^{\pm}(v_n^{\pm})\right) = p(\beta) + \beta v^{\pm}. \tag{5.10}$$

In this case, for $v \in (v^-, v^+)$ there exists a $t \in (0,1)$ such that $v = tv^- + (1-t)v^+$. Also, $\beta_v = \beta$ holds for all $t \in (0,1)$. Then, by the concavity of $m(v)$ and Equation (5.10),we have

$$\begin{aligned} m(v) &\geq tm(v^-) + (1-t)m(v^+) \\ &\geq p(\beta) + \beta\left(tv^- + (1-t)v^+\right) \\ &= p(\beta) + \beta v. \end{aligned}$$

Thus, we have $m(u) \geq p(\beta) + \beta u$ for all $u \in (e_{\inf}, \infty)$, which completes the proof. $\square$

Proof of Corollary 5.1.2. Let

$$\overline{m}^P(u) := \limsup_{\Lambda \to \mathbb{Z}^\nu} \frac{1}{|\Lambda|} \log \mu_{0,\Lambda}\{s \in S^\Lambda : H_\Lambda^P(s) \leq |\Lambda| u\},$$

$$\underline{m}^P(u) := \liminf_{\Lambda \to \mathbb{Z}^\nu} \frac{1}{|\Lambda|} \log \mu_{0,\Lambda}\{s \in S^\Lambda : H_\Lambda^P(s) \leq |\Lambda| u\}.$$

The upper bound $\overline{m}^P(u) \leq m(u)$ can be derived along the same line as Equation (5.6) by using Proposition 3.5.2. On the other hand, since $H_\Lambda^P(\Phi) \leq H_\Lambda(\Phi)$ for $\Phi \in \mathcal{N}$, we have $\{H_\Lambda^P(\Phi) \leq |\Lambda| u\} \supset \{H_\Lambda(\Phi) \leq |\Lambda| u\}$ and thus $\underline{m}^P(u) \geq m(u)$. These prove the claim.

□

Chapter 6

Conclusion

In this study, we derived the Boltzmann's entropy formula for the negatively interacting and superstable non-compact spin systems. The analysis was based on the compact approximation and the convexity of thermodynamic functions.

Our method is more advantageous in two respects than applications of LDP:

(A) First, our approach is a range of naive probability theory: Just as the convergence of a characteristic function is essential to show the weak convergence of pedagogical probability distributions, the convergence of the pressure is important to prove the weak convergence of each Gibbs state since the pressure is one of the generalization of the cumulant generating function. For instance, let us consider a single spin space $(\mathbb{R}, \lambda)$ and a potential $A_{\beta\Phi} := \beta s^2$ with a perturbation ts, $t \in \mathbb{R}$. Then, by the independence of spin variables, the pressure is calculated as follows:
$$p(\beta s^2 + ts) = \log \int_{\mathbb{R}} e^{-\beta s^2 + ts} \ \lambda(ds).$$
Clearly, this is the cumulant generating function of the gaussian distribution up to a constant. Hence, a Gibbs state can be regarded as a distribution of a sequence of dependent random variables $\{\theta_x(A_\Phi)\}_{x \in \mathbb{Z}^\nu}$. In fact, the first and the second derivatives of pressures are corresponding to the mean and the series of the correlation function between $\theta_x(A_\Phi)$ and $\theta_y(A_\Phi)$, see [13, Section 16.2]. In summary, the analysis of the pressure has much in common with the cumulants, and it enables us to study the structure inside the set of Gibbs phases.

(B) Second, our analysis can reflect the shape of potentials, including the sign of the exchange interactions, whereas the large deviation principle values correlation functions rather than information of potentials. As we see, the negativity assumption simplifies the complex proofs that frequently appear in the rigorous statistical mechanics. Another benefit of the assumption is that similar proofs hold for quantum systems with unbounded spin variables as unbounded operators on Hilbert spaces. Recently, the last obstacle of the proof of Proposition 3.5.2 in quantum cases has been cleared away, which the superstability condition imposed only for in the classical spin systems. And now, such a condition is rewritten by a more simple one; e^{-H_Λ} is a trace class and the interaction term can be transformed in the same way as Example 5.1.3. Consequently, our theorem extends to major models in condensed matter physics, but whether this result can be linked to the non-commutative equilibrium states, the KMS states, is still open.

Bibliography

[1] Blumberg, H. On Convex Functions. *Trans. Amer. Math. Soc.* **20**, 40-44 (1919)

[2] Campanino, M., Capocaccia, D. & Olivieri, E. Analyticity for one-dimensional systems with long-range superstable interactions. *Journal Of Statistical Physics.* **33**, 437-476 (1983)

[3] Cassandro, M., Olivieri, E., Pellegrinotti, A. & Presutti, E. Existence and uniqueness of DLR measures for unbounded spin systems. *Zeitschrift Für Wahrscheinlichkeitstheorie Und Verwandte Gebiete.* **41**, 313-334 (1978)

[4] Caputo, P., and Deuschel, J. D., Large Deviations and Variational Principle for Harmonic Crystals. *Communications in Mathematical Physics*, **209**, 595-632 (2000)

[5] Caputo, P., and Deuschel, Jean-Dominique. Critical large deviations in harmonic crystals with long-range interactions. *The Annals of Probability*, **29.1**, 242-287 (2001)

[6] Cuneo, N., Jakšić, V., Pillet, C.A. and Shirikyan, A. Large deviations and fluctuation theorem for selectively decoupled measures on shift spaces. *Reviews in Mathematical Physics*, **31(10)**, 1950036 (2019)

[7] Dobrushin, R. Prescribing a system of random variables by conditional distributions. *Theory Of Probability & Its Applications.* **15**, 458-486 (1970)

[8] Deuschel, J. D., Stroock, D. W., *Large deviations.* American Mathematical Soc. (2001)

[9] Deuschel, Jean-Dominique, Daniel W. Stroock, and Hans Zessin. Microcanonical distributions for lattice gases. *Communications in Mathematical Physics.* **139.1**, 83-101 (1991)

[10] Ellis, R. S., Haven, K., and Turkington, B. Large deviation principles and complete equivalence and nonequivalence results for pure and mixed ensembles. *Journal of Statistical Physics*, **101**, 999-1064 (2000)

[11] Fernández, R., Fröhlich, J. & Sokal, A. *Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory.* Springer Science & Business Media (1992)

[12] Georgii, H. O. *Gibbs Measures and Phase Transitions.* de Gruyter (2011)

[13] Georgii, H. O., and Zessin, H. Large deviations and the maximum entropy principle for marked point random fields. *Probability theory and related fields*, **96(2)**, 177-204 (1993)

[14] Béllissard, J. & Høegh-Krohn, R. Compactness and the maximal Gibbs state for random Gibbs fields on a lattice. *Communications In Mathematical Physics.* **84**, 297-327 (1982)

[15] Kajisa, T., Boltzmann' s mean entropy formula for unbounded spin systems. *Reviews in Mathematical Physics*, **2450021**, (2024).

[16] Holley, R. & Stroock, D. The DLR conditions for translation invariant Gaussian measures on L($\mathbb{R}^d$). *Zeitschrift Für Wahrscheinlichkeitstheorie Und Verwandte Gebiete.* **53**, 293-304 (1980)

[17] Künsch, H. Almost sure entropy and the variational principle for random fields with unbounded state space., *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* **58.1**, 69-85 (1981)

[18] Lanford, O. E. Entropy and equilibrium states in classical statistical mechanics. *In Statistical Mechanics and Mathematical Problems* (pp. 1-113). Berlin, Heidelberg: Springer Berlin Heidelberg (2007)

[19] Lebowitz, J. & Presutti, E. Statistical mechanics of systems of unbounded spins. *Communications in Mathematical Physics.* **50**, 195-218 (1976)

[20] Lebowitz, J. & Presutti, E. Erratum: "Statistical mechanics of systems of unbounded spins" . *Communications in Mathematical Physics.* **78**, 151-151 (1980)

[21] Lewis, J. T., Pfister, C. E., & Sullivan, W. Entropy, concentration of probability and conditional limit theorems. *Markov Processes and Related Fields.* **1**, 319-386 (1995)

[22] Lewis, J. T., Pfister, C. E., & Sullivan, W. The equivalence of ensembles for lattice systems: some examples and a counterexample. *Journal of Statistical Physics*, **77**, 397-419 (1994)

[23] Müller, M. P., et al. Thermalization and canonical typicality in translation-invariant quantum lattice systems. *Communications in Mathematical Physics.* **340**, 499-561 (2015)

[24] Ogata, Y. and Rey-Bellet, L. Ruelle-Lanford functions and large deviations for asymptotically decoupled quantum systems. *Reviews in Mathematical Physics.* **23(2)** 211-232 (2011)

[25] Park, Y. M., and Hyun, J. Y. Uniqueness and clustering properties of Gibbs states for classical and quantum unbounded spin systems. *Journal of statistical physics.* **80**, 223-271 (1995)

[26] Rockafellar, R. Tyrrell. *Convex Analysis.* Princeton university press (1970)

[27] Ruelle, D. *Statistical Mechanics: Rigorous Results.* World Scientific (1999)

[28] Ruelle, D. Correlation functionals. *Journal of Mathematical Physics.* **6(2)**, 201-220 (1965)

[29] Ruelle, D. Superstable interactions in classical statistical mechanics. *Communications in Mathematical Physics.* **18**, 127-159 (1970)

[30] Ruelle, D. Probability estimates for continuous spin systems. *Communications in Mathematical Physics.* **50**, 189-194 (1976)

[31] Simon, B. $P(\phi)_2$ *Euclidean (Quantum) Field Theory.* Princeton University Press. (1974)

[32] Simon, B. *The Statistical Mechanics of Lattice Gases, Volume I.* Princeton University Press (1993)

[33] Slawny, J. Ergodic properties of equilibrium states. *Communications in Mathematical Physics.* **80**, 477-483 (1981)

[34] Touchette, H. Equivalence and nonequivalence of ensembles: thermodynamic, macrostate, and measure levels. *Journal of Statistical Physics.* **159(5)**, 987-1016 (2015)