Title	On the topology of the complements of plane algebraic curves
Author(s)	菅原, 朔見
Degree Grantor	北海道大学
Degree Name	博士(理学)
Dissertation Number	甲第16082号
Issue Date	2024-09-25
DOI	https://doi.org/10.14943/doctoral.k16082
Doc URL	https://hdl.handle.net/2115/93604
Type	doctoral thesis
File Information	Sakumi_Sugawara.pdf

On the topology of the complements of plane algebraic curves
（平面代数曲線補集合のトポロジーについて）

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Doctoral Thesis
September, 2024

Abstract

The complement of a plane algebraic curve has been studied from topological and algebro-geometric viewpoints for a long time. One important technique for studying the topology of the complements is called braid monodromy. It determines the specific description of the homotopy type of the complement and embedded topology. In this thesis, we describe the handle decomposition and the Kirby diagram of the complement by refining the braid monodromy technique.

Contents

1 Introduction **2**

2 Handle decompositions and Kirby diagrams **4**
2.1 Handle decompositions 4
2.2 Kirby diagrams 5
2.2.1 1-handles 5
2.2.2 2-handles 6
2.2.3 3- and 4-handles 7
2.2.4 Dotted circle description of 1-handles 7

3 Topology of plane algebraic curves **9**

4 Handle decompositions of plane curve complements **12**
4.1 Stable maps on 3-manifolds 12
4.2 Handle decomposition 12

5 Kirby diagrams and examples **22**

6 Future issues **29**
6.1 Stein and contact structures 29
6.2 Projective plane curves 30
6.3 Zariski pairs 30
6.4 Milnor fibers 31

Acknowledgements **32**

References **33**

1 Introduction

The topology of algebraic varieties, especially, of algebraic curves and surfaces has been studied by many authors from various viewpoints, and there are many excellent works. Among them, one of the most remarkable results may be that of Lefschetz in 1924. In his book [Lef24], Lefschetz studied the topology of projective algebraic varieties using the notion of pencils and proved the Lefschetz hyperplane section theorem.

A few years later, Zariski studied the fundamental group of the complement of a plane algebraic curve [Za29]. His original motivation was the existence problem of algebraic functions with a given plane curve as a branch. For the application to this problem, he studied the fundamental group and pointed out that it could be computed by using the projection to a generic line and the monodoromy around its critical values. His idea was completed by van Kampen's calculation [vK33] and the method is now called the Zariski–van Kampen method (see also [Shim07]).

The Zariski–van Kampen method evolved into the notion of the braid monodromy, which was introduced by Moishezon [Mo81]. In the braid monodromy, the monodromy around each critical value is considered as an element of the braid group. Using this notion, he computed the fundamental group of the complements of plane curves appearing as branch curves of holomorphic mappings between certain algebraic surfaces, and succeeded in giving the normal form of the surfaces. Since then, the braid monodromy has been a primary tool in the study of the topology of algebraic curves and has contributed to various results. In fact, the homotopy type of the complement of an affine plane algebraic curve can be explicitly described by using the braid monodromy [Lib86]. Moreover, it also determines the embedded topology of the corresponding projective plane curve [ACC03].

On the other hand, the complement of an affine plane algebraic curve admits a Stein structure. For an affine plane curve C, the complement $M = \mathbb{C}^2 \setminus C$ is holomorphically embedded in $\mathbb{C}^3$ as a closed subset by the map $(x, y) \mapsto (x, y, 1/f(x, y))$, where $f(x, y)$ is the defining polynomial of C. Thus M is indeed a Stein surface. It is known that for any complex n-dimensional Stein manifold, the handle decomposition (as a real $2n$-dimensional manifold) that is compatible with the Stein structure has no handle of index greater than n [AF59]. In addition, when $n = 2$, there is a detailed study on the topology of Stein surfaces by Gompf [Go98]. He described by using Kirby diagrams all the handle decompositions that arise as compatible handle decompositions of Stein surfaces. In this sense, the topology of Stein surfaces has already been characterised (when $n > 2$, the topological characterisation was given by Eliashberg [E90]). However, determining the diffeomorphism type of a given Stein surface is another important problem. Indeed, after Gompf's work, Stein surfaces have been one of the central objects in 4-dimensional topology, and various examples have been explicitly described by Lefschetz fibrations and Kirby diagrams (see for example [GS99, LP01, OS04]). Therefore, it is natural to ask about the diffeomorphism type of the complement M as a 4-manifold.

In this doctoral thesis, we discuss the diffeomorphism type of the complement of an affine plane algebraic curve. The main purpose is to give an explicit handle decomposition and to describe the Kirby diagram of it. The idea of our method is inspired by Moishezon's braid monodromy and Libgober's theorem [Lib86]. Let us briefly recall Libgober's work. Combining the idea of braid monodromy and presentations of knot groups, he gave the homotopy type of the complement of an affine plane algebraic curve, which is more refined information than the fundamental group. More specifically, he obtained the presentation of the fundamental group of the complement whose associated 2-dimensional CW complex is homotopy equivalent to the complement itself. Then it is natural to expect that this CW complex is the core of a handle decomposition consisting only of 0, 1, and 2-handles. However, it seems difficult to describe more information than the homotopy type, because its construction uses deformation retractions to the link complement in S^3 around the singularity of the curve.

So we take a different strategy in order to obtain the handle decomposition. Instead of constructing a deformation retraction, we decompose the neighborhood of each singularity into several parts. We then study how these parts are attached. The 1-handlebody is relatively easy to find. We extend it by an isotopy as far as possible. Next, we need to find the 2-handles. The key point at this stage is to focus on the stable map obtained as the restriction of the projection $\mathbb{C}^2 \ni (x, y) \mapsto x \in \mathbb{C}$ to the boundary of a regular neighborhood of the curve. We can show that each connected component of the critical point set corresponds to a 2-handle and its attachment to the 1-handlebody is naturally described in terms of braid monodromy.

Now we mention the recent work of the author and Yoshinaga [SY22]. In their paper, handle decompositions and Kirby diagrams of the complement of complexified real line arrangements were obtained. This result corresponds to a special case of our main result. However, their method is based on the Lefschetz hyperplane section theorem and depends heavily on the combinatorial structure of a real line arrangement, so it cannot be extended to general plane algebraic curves. On the other hand, the handle decomposition in [SY22] has a good property. It is compatible with the Stein structure, since it is obtained by a strictly plurisubharmonic Morse function (see [Yo07]). We do not know whether this is the case for the handle decomposition obtained in this thesis, so it would be a future issue to determine it.

This thesis is organized as follows. Sections 2 and 3 are devoted to the preparation of the main construction. We review the basics of handle decompositions and Kirby diagrams in Section 2, and the notion of braid monodromy for plane algebraic curves in Section 3. In Section 4, after recalling the basics of stable maps, we give an explicit handle decomposition for the complement of an affine plane algebraic curve. This is the main construction. In Section 5, we give an algorithm for drawing Kirby diagrams which describe the handle decompositions obtained in the main construction. We also give some examples of Kirby diagrams using the algorithm. In Section 6, we will discuss further topics and open problems.

2 Handle decompositions and Kirby diagrams

In this section, we review handle decompositions of smooth manifolds and Kirby diagrams for 4-dimensional manifolds (see [Ki89, GS99, Ak16] for details). Throughout this section, we assume that a manifold is smooth and if there are corners, they are appropriately smoothed.

2.1 Handle decompositions

Definition 2.1. Let X be a compact n-dimensional manifold with non-empty boundary and k an integer with $0 \leq k \leq n$. By attaching $D^k \times D^{n-k}$ along an embedding $\varphi : \partial D^k \times D^{n-k} \to \partial X$, we obtain a manifold $X \cup_\varphi (D^k \times D^{n-k})$. The copy of attached $D^k \times D^{n-k}$ is called a *n-dimensional k-handle*. We call the integer k the *index*, the map φ the *attaching map*, $D^k \times \{0\}$ the *core*, $\partial D^k \times D^{n-k}$ the *attaching region*, $\partial D^k \times \{0\}$ the *attaching sphere*, $\{0\} \times D^{n-k}$ the *cocore*, and $\{0\} \times \partial D^{n-k}$ the *belt sphere*. When $k = 2$, the attaching sphere is homeomorphic to S^1, thus we call this the *attaching circle*.

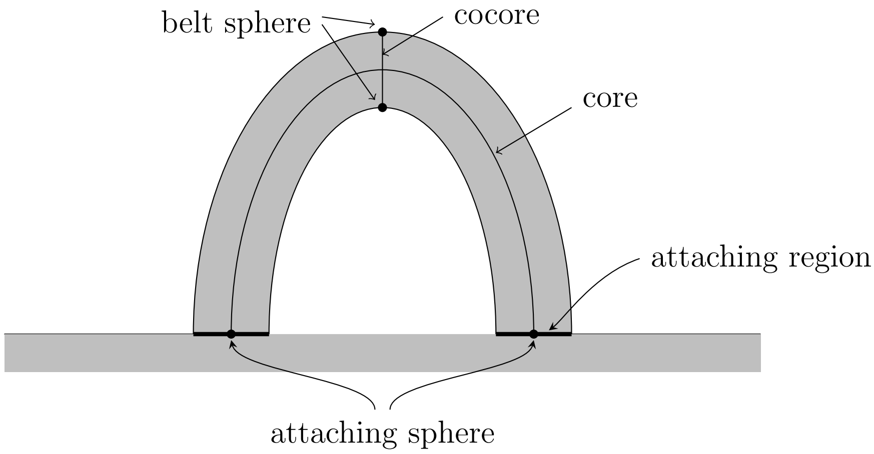


Figure 1: Attaching a handle (2-dimensional 1-handle)

Remark 2.2. Since the attaching region of a 0-handle is an empty set, attaching a 0-handle is equivalent to taking the disjoint union with D^n.

Definition 2.3. A *handlebody* is a manifold obtained from an empty set by attaching handles. A handlebody with handles whose indices are all less than or equal to k is called a *k-handlebody*. A *handle decomposition* of a manifold X is an identification of X with a handlebody.

Remark 2.4. Due to the Morse theory, every compact manifold has a handle decomposition.

Furthermore, we have the following propositions on the order of handle attachments (see [Mi65]).

Proposition 2.5. By performing an isotopy for the attaching maps, we can rearrange the handle decomposition so that handles are attached such that the indices are decreasing. In addition, handles with the same index may be attached in any order.

Proposition 2.6. Every compact manifold admits a handle decomposition with only one 0-handle. Moreover, a closed n-dimensional manifold admits a handle decomposition that has a unique 0-handle and a unique n-handle.

In this thesis, we consider only handle decompositions with the following conditions:

- There is only one 0-handle.
- If the manifold is closed and n-dimensional, then an n-handle is also unique.
- Handles are attached such that the indices are decreasing.

Proposition 2.7. Let X be an n-dimensional manifold with boundary. Suppose that $(k-1)$-handle h^{k-1} and k-handle h^k are attached to X so that the belt sphere of h^{k-1} and the attaching sphere of h^k intersect transversally at one point. Then, the manifold with these handles attached is diffeomorphic to X.

Definition 2.8. The pair of handles h^{k-1} and h^k satisfying the assumption of Proposition 2.7 is called a *canceling pair*.

2.2 Kirby diagrams

A handle decomposition of an orientable 4-dimensional manifold is described as a framed link diagram in $\mathbb{R}^3$, called a *Kirby diagram*. In this section, we will introduce how to draw the Kirby diagram from a handle decomposition of a 4-dimensional manifold.

A handle decomposition of a 4-dimensional manifold consists of handles whose indices are at most 4. We now assume that there is only one 0-handle D^4 with the boundary $S^3 = \mathbb{R}^3 \cup \{\infty\}$. The Kirby diagram is constructed by drawing the attachments of the remaining handles to the part $\mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\}$ of the boundary.

2.2.1 1-handles

The attaching region of a 1-handle is $S^0 \times D^3 = \{-1, 1\} \times D^3$. Thus, we draw a pair of two 3-balls into $\mathbb{R}^3$. Since we assume that a manifold is orientable, the manifold with a 1-handle attached is diffeomorphic to $S^1 \times D^3$, which can be obtained also by identifying two 3-balls on the boundary of D^4. These two 3-balls are lying in the interior of the resulting manifold. Thus, the boundary of the manifold with a 1-handle attached can be described by identifying two 2-spheres in the diagram. The identification of these two spheres is given by the reflection of a plane Π perpendicularly bisecting the segment joining their centers (see Figure 2).

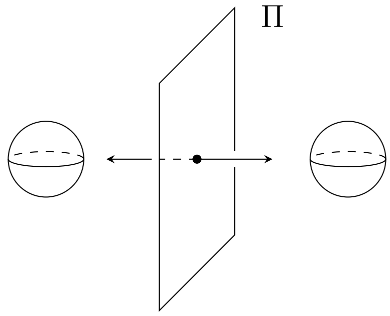


Figure 2: Attaching sphere of a 1-handle

2.2.2 2-handles

The attaching circle and the attaching region of a 2-handle are $\partial D^2 \times \{0\} = S^1$ and $\partial D^2 \times D^2 = S^1 \times D^2$, respectively. Thus, the attaching circle is a knot K in the boundary 3-manifold of a 1-handlebody. The attaching region is a tubular neighborhood of the knot. Let $x \in D^2 \setminus \{0\}$. Then, $\partial D^2 \times \{x\}$ is attached to a knot K' which runs parallel to K. The *framing coefficient* of the 2-handle is defined as the linking number $lk(K, K')$. We write the framing coefficient near the attaching circle in the diagram (see Figure 3). Figure 4 shows an example of the Kirby diagram of a 2-handlebody.

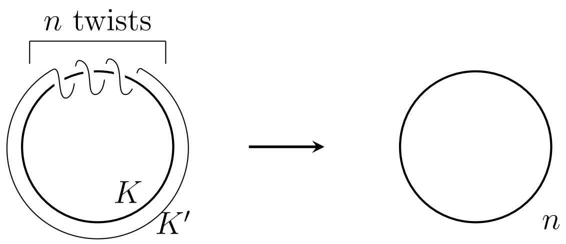


Figure 3: Framing of a 2-handle

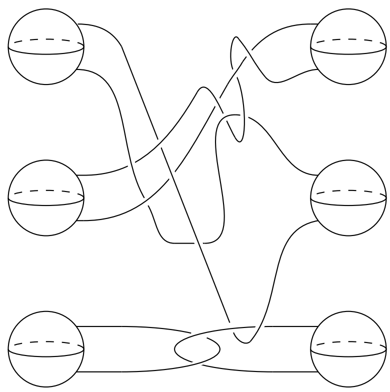


Figure 4: Handle attaching up to 2-handles

2.2.3 3- and 4-handles

Suppose that the 4-manifold is closed. The boundary of a 3-handle is divided into $(\partial D^3 \times D^1) \cup (D^3 \times \partial D^1)$. Since there is only one 4-handle by our assumption, $D^3 \times \partial D^1$ is attached to the 4-handle. Thus, the union of 3-handles and the 4-handle is diffeomorphic to $\natural_k S^1 \times D^3$, where k is the number of 3-handles. Since its boundary is $\sharp_k S^1 \times S^2$, the boundary of 2-handlebody must be diffeomorphic to $\sharp_k S^1 \times S^2$. It is known that every self diffeomorphism of $\sharp_k S^1 \times S^2$ is extended to $\natural_k S^1 \times D^3$ [LP72]. Therefore, the diffeomorphism type of a closed 4-manifold is determined by its 2-handlebody. We just write the number of 3- and 4-handles in the diagram. Even when the case of a manifold is not closed, we also write the numbers of 3 and 4-handles in the diagram.

Remark 2.9. A 4-dimensional 2-handlebody is not necessarily the 2-handlebody of a closed 4-manifold, since if it is, its boundary must be diffeomorphic to $\sharp_k S^1 \times S^2$.

2.2.4 Dotted circle description of 1-handles

The attachment of a 1-handle is described as a pair of two spheres identified by a reflection of the plane Π. We locate these two spheres closer to the plane. Then we forget the two spheres, draw a dotted circle on the plane Π, and bind the attaching circles of 2-handles passing through this 1-handle. We consider this dotted circle as the 1-handle attachment (see Figure 5).

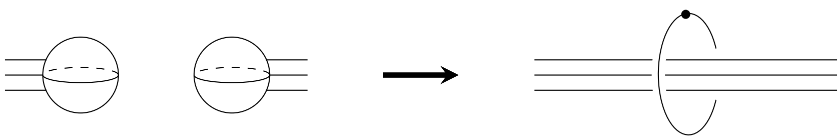


Figure 5: Dotted circle description of a 1-handle

There is another viewpoint on the dotted circle description of 1-handle attachments. Let us consider a 4-manifold X with boundary and attach a 1-handle to X. Attaching a canceling 2-handle to this manifold, the resulting manifold is diffeomorphic to X. Therefore, attaching a 1-handle is equivalent to removing this canceling 2-handle. This 2-handle can be considered as a tubular neighborhood of its cocore which is trivially and properly embedded 2-disk. Therefore, removing the tubular neighborhood of a trivially and properly embedded 2-disk from X is equivalent to attaching 1-handle (Here, a properly embedded 2-disk in X is an embedded disk whose boundary circle is embedded in the boundary ∂X). From this viewpoint of 1-handle attaching, the dotted circle of the 1-handle corresponds to the boundary circle of the embedded 2-disk (or equivalently, the belt sphere of 2-handle).

Remark 2.10. We can read off the fundamental group of a 4-manifold from the Kirby diagram. As is well known, in a CW complex, 1-cells correspond to the generators of the fundamental group and the attachments of 2-cells correspond to the relations. Similarly, 1-handles correspond to the generators and the attachments of 2-handles correspond to the relations. Let us consider a Kirby diagram with a dotted circle description of 1-handles. For each dotted circle, consider its bounding disk. Each disk corresponds to a generator of the fundamental group. We also give an orientation for each attaching circle of 2-handles. For each attaching circle, read the generators appearing as the intersection with these disks along the oriented attaching circle. This gives the corresponding relation in the fundamental group.

Example 2.11. The Kirby diagrams of $S^2 \times S^2$ and $\mathbb{C}P^2$ are described in Figure 6.

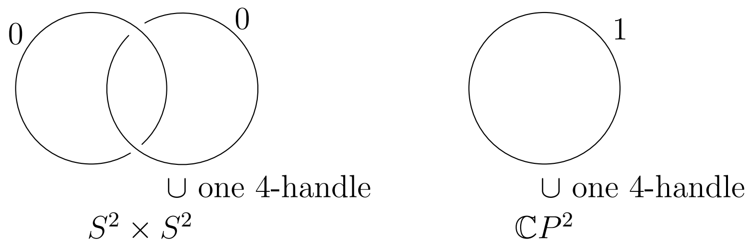


Figure 6: Examples of Kirby diagrams

3 Topology of plane algebraic curves

In this section, we recall the braid monodromy for plane algebraic curves (for details, see [ACC03, CS97, Mo81, Lib86]).

Let (x, y) be the complex coordinates on $\mathbb{C}^2$ and C be a plane algebraic curve defined by a reduced polynomial $f(x, y)$ of degree n such that the coefficient of y^n is non-zero. We denote by $M = \mathbb{C}^2 \setminus C$ the complement. Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be the projection to the first factor and we set $L_x = \pi^{-1}(x)$ for $x \in \mathbb{C}$. Let $X = \{p_1, \dots, p_N\} \subset \mathbb{C}$ be the set of points whose fiber L_{p_i} contains a singularity of C or tangents to C (see Figure 7). We may assume that the projection π is generic so that L_{p_i} contains at most one singularity for each $p_i \in X$.

In this setting, we have the following proposition.

Proposition 3.1. (Lemma 3.3.5 in [Di92]). The restriction $\pi : \mathbb{C}^2 \setminus (C \cup \pi^{-1}(X)) \to \mathbb{C} \setminus X$ is a locally trivial fibration.

Proof. We identify $\mathbb{C}^2$ with the affine chart $\{[x : y : z] \in \mathbb{C}P^2 \mid z \neq 0\}$. Then, $\mathbb{C}P^2 = \mathbb{C}^2 \cup L_\infty$, where $L_\infty = \{z = 0\}$ is the infinity line. The projection π is extended to $\overline{\pi} : \mathbb{C}P^2 \setminus \{[0 : 1 : 0]\} \to \mathbb{C}P^1$ as $\overline{\pi}([x : y : z]) = [x : z]$. It is enough to show that the restriction $\overline{\pi} : \mathbb{C}P^2 \setminus (C \cup \overline{\pi}^{-1}(\overline{X})) \to \mathbb{C}P^1 \setminus \overline{X}$ is a locally trivial fibration, where $\overline{X} = X \cup \{[1 : 0]\}$. Let Z be the blow-up of $\mathbb{C}P^2$ at the point $[0 : 1 : 0]$ and $\varphi : Z \to \mathbb{C}P^2$ be the canonical projection. We denote the exceptional divisor by E. The composition $\overline{\pi} \circ \varphi : Z \setminus E \to \mathbb{C}P^1$ can be extended to the map $\widetilde{\pi} : Z \to \mathbb{C}P^1$. Note that this is a proper map between smooth manifolds. Let $\widetilde{C}$ and $\widetilde{X}$ be the strict transformations of the projective closure of C and $\overline{\pi}^{-1}(\overline{X})$, respectively. We set

$$Z_0 = Z \setminus \widetilde{X},\ Z_\infty = Z_0 \cap E,\ \widetilde{C_0} = \widetilde{C} \cap Z_0.$$

Then, the restriction $\widetilde{\pi} : Z_0 \to \mathbb{C}P^1 \setminus \overline{X}$ is a proper submersion. Also, $\widetilde{\pi} : \widetilde{C_0} \cup Z_\infty \to \mathbb{C}P^1 \setminus \overline{X}$ is a proper submersion. By Ehresmann's fibration theorem, $\widetilde{\pi}$ induce a locally trivial fibration of the pair $(Z_0, \widetilde{C_0} \cup Z_\infty)$ over $\mathbb{C}P^1 \setminus \overline{X}$. Thus, we have the locally trivial fibration $\widetilde{\pi} : Z_0 \setminus (\widetilde{C_0} \cup Z_\infty) \to \mathbb{C}P^1 \setminus \overline{X}$. Since $Z_0 \setminus (\widetilde{C_0} \cup Z_\infty) \cong \mathbb{C}P^2 \setminus (C \cup \overline{\pi}^{-1}(\overline{X}))$, this completes the proof. $\square$

Definition 3.2. A system of arcs $s_1, \dots, s_N$ in $\mathbb{C}$ is called a *non-intersecting arc system* with respect to the base point $p_0 \in \mathbb{C}$ and a finite set $\{p_1, \dots, p_N\} \subset \mathbb{C}$ if the following two conditions are satisfied.

- Each arc s_i is smooth and connects p_0 and p_i.
- There are no self-intersections in each s_i, and $s_i \cap s_j = \{p_0\}$ if $i \neq j$.

For $1 \leq i \leq N$, let U_i be a small open disk centered at p_i. Fix a point $p_i' \in \partial U_i$ for each i. Take a point $p_0 \in \mathbb{C} \setminus X$ and a non-intersecting arc system $s_1, \dots, s_N$ in $\mathbb{C}$ with respect to the base point p_0 and $\{p_1', \dots, p_N'\}$. Let $\gamma_i \in \pi_1(\mathbb{C} \setminus X, p_0)$ be the element which represents the loop based at p_0 obtained by concatenating s_i, ∂U_i (going counter-clockwise) and s_i^{-1}.

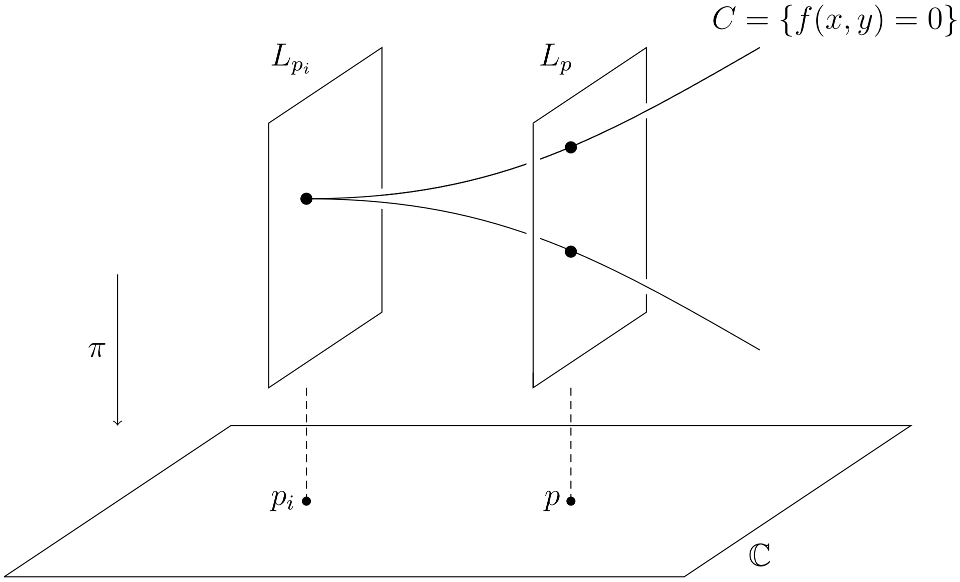


Figure 7: The projection $\pi : \mathbb{C}^2 \to \mathbb{C}$

By trivializing the fibration $\pi : \mathbb{C}^2 \setminus (C \cup \pi^{-1}(X)) \to \mathbb{C} \setminus X$ along each loop γ_i, we obtain an automorphism of L_{p_0} which preserves $L_{p_0} \cap C$. The trivialization along the loops defines a homomorphism

$$\Phi : \pi_1(\mathbb{C} \setminus X, p_0) \to B(L_{p_0}, L_{p_0} \cap C),$$

where $B(L_{p_0}, L_{p_0} \cap C)$ denotes the mapping class group of L_{p_0} preserving the set $L_{p_0} \cap C$, which is isomorphic to the Artin's braid group B_n with n strands. This homomorphism Φ is called the braid monodromy of the algebraic curve. We define $\beta_i = \Phi(\gamma_i)$ as the monodromy along the loop γ_i.

Let $L_{p_0} \cap C = \{q_{0,1}, \ldots, q_{0,n}\}$. Take a base point $q_0 \in L_{p_0} \setminus C$. Then we can define a non-intersecting arc system with respect to the base point q_0 and $\{q_{0,1}, \ldots, q_{0,n}\}$. Let $e_1, \ldots, e_n \in \pi_1(L_{p_0} \setminus C, q_0)$ be the elements which represent the loops defined from the non-intersecting arc system (Figure 8). The fundamental group $\pi_1(L_{p_0} \setminus C, q_0)$ is free of rank n and generated by $e_1, \ldots, e_n$.

The mapping class group $B(L_{p_0}, L_{p_0} \cap C)$ acts on the fundamental group $\pi_1(L_{p_0} \setminus C, q_0)$, which can be seen as the Artin action of the braid group B_n to the free group F_n.

For a point $p \in \partial U_i$, the fiber $L_p \cap C$ is a set of n points. We suppose that m_i of them collapse into one when the point p reaches p_i. Then the integer m_i coincides with the usual multiplicity of the singularity of C on L_{p_i} if L_{p_i} intersects C transversely. Let $L_{p_i'} \cap C = \{q_{i,1}, \ldots, q_{i,m_i}, q_{i,m_i+1}, \ldots q_{i,n}\}$ and we assume that first m_i points collapse. By trivializing along s_i, each collapsing point $q_{i,j}$ reaches a point in $L_{p_0} \cap C$, and write it $q_{0,i_j} \in L_{p_0} \cap C$. Let $I_i = \{i_1, \ldots, i_{m_i - 1}\}$ be the corresponding index set.

In this setting, Libgober proved the following theorem.

Theorem 3.3. [Lib86] *The complement $M = \mathbb{C}^2 \setminus C$ is homotopy equivalent to a 2-dimensional CW complex associated with the following presentation of the funda-*

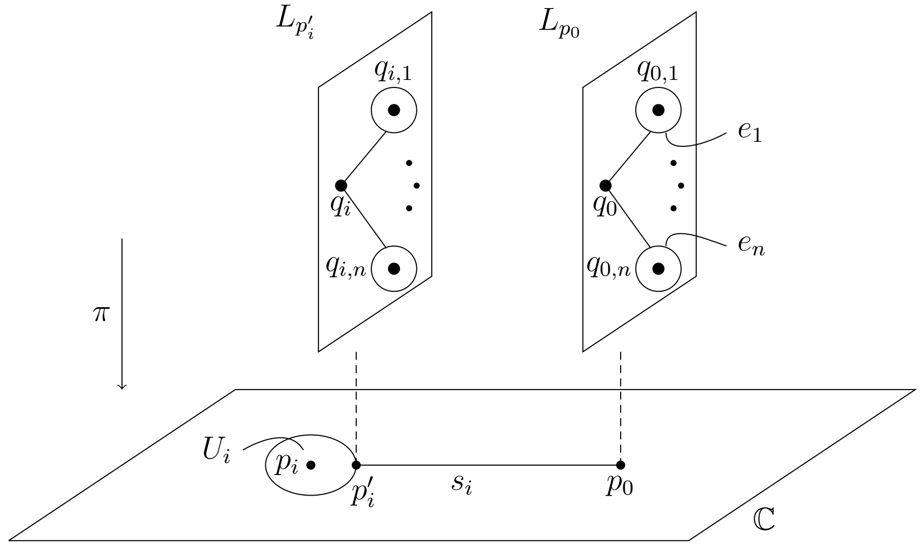


Figure 8: Non-intersection arc systems in the fibers

mental group:

$$\pi_1(\mathbb{C}^2 \setminus C) \cong \langle e_1, \ldots, e_n \mid \beta_i \cdot e_j = e_j, i = 1, \ldots, N, j \in I_i \rangle,$$

where $\cdot$ denotes the Artin action. Recall that β_i is the monodromy around the loop γ_i.

It is convenient to consider the monodromy around each point $p_i \in X$ and arc s_i separately, when we calculate and describe Kirby diagrams in the forthcoming sections. Let $\overline{\beta_i} \in B(L_{p'_i}, L_{p'_i} \cap C)$ the monodromy around ∂U_i and $\phi_i : B(L_{p_0}, L_{p_0} \cap C) \to B(L_{p'_i}, L_{p'_i} \cap C)$ be the isomorphism induced by the trivialization along s_i. The monodromy $\overline{\beta_i}$ is called the local monodromy around $p_i \in X$.

Remark 3.4. Let us consider the local monodromy around $p_i \in X$. Let $\gamma_i : [0,1] \to \partial U_i$ be a smooth parametrization with $\gamma_i(0) = \gamma_i(1) = p'_i$ that orients ∂U_i counter-clockwise. For fixed $0 \leq t_0 \leq 1$, $L_{\gamma_i(t_0)} \cap C$ is expressed as $\{\widetilde{\gamma}_{i,1}(t_0), \ldots, \widetilde{\gamma}_{i,n}(t_0)\}$, where $\widetilde{\gamma}_{i,j} : [0,1] \to \pi^{-1}(\partial U_i)$ is the lift of γ_i such that $\widetilde{\gamma}_{i,j}(0) = q_{i,j}$. Therefore, $\pi^{-1}(\partial U_i) \cap C = \bigcup_{j=1}^{n} \bigcup_{t \in [0,1]} \widetilde{\gamma}_{i,j}(t)$ forms a braid with n strands in $\pi^{-1}(\partial U_i)$. This braid corresponds to the local monodromy $\overline{\beta_i} \in B(L_{p'_i}, L_{p'_i} \cap C)$.

4 Handle decompositions of plane curve complements

4.1 Stable maps on 3-manifolds

Before going to the main result, let us review stable maps on 3-manifolds (see [Lev65] for details). We will need this later to describe the 2-handle attachment.

Let N be a closed orientable 3-manifold and P be a 2-dimensional manifold without boundary. We denoted by $C^\infty(N,P)$ the space of smooth maps from N to P with Whitney C^∞-topology. A map $f \in C^\infty(N,P)$ is called a *stable map* if there exist a neighborhood U of f in $C^\infty(N,P)$ such that every $g \in U$ is right-left equivalent to f, that is, there exist diffeomorphisms $\varphi : N \to N$ and $\Phi : P \to P$ such that $g = \Phi \circ f \circ \varphi^{-1}$. It is known that the set of stable maps is open and dense in $C^\infty(N,P)$. For $f \in C^\infty(N,P)$, let us denote the critical point set by $C(f)$. It is known that a map f is stable if and only if it satisfies the following local and global conditions: For $p \in C(f)$, there exist local coordinates (u,x,y) centered at p such that f is expressed as one of the following forms:

(i) $f(u,x,y) = (u, x^2 + y^2)$ (definite fold point)

(ii) $f(u,x,y) = (u, x^2 - y^2)$ (indefinite fold point)

(iii) $f(u,x,y) = (u, y^2 + ux - x^3)$ (cusp point)

and

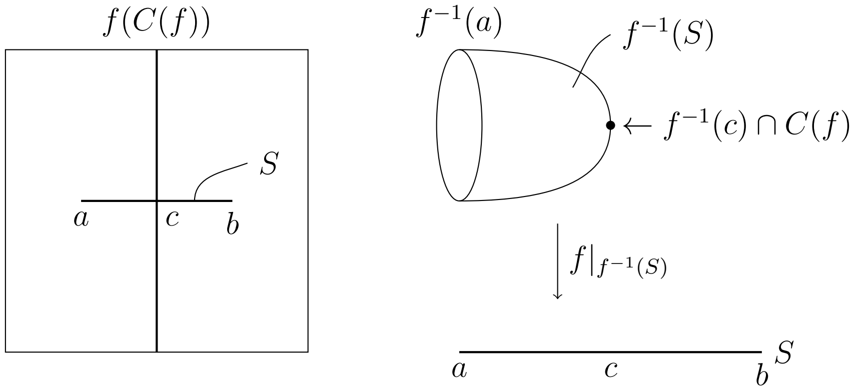
(a) the restriction of f to $C(f) \setminus \{\text{cusp points}\}$ is an immersion with normal crossings,

(b) for a cusp point p, $f^{-1}(f(p)) \cap C(f) = \{p\}$.

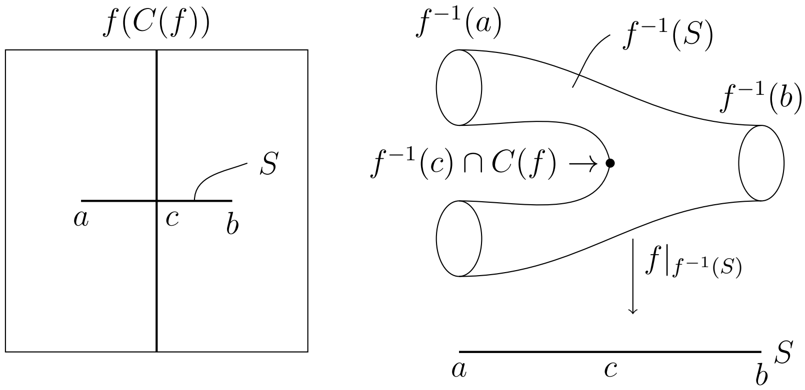
The preimage of a regular value is called a regular fiber. Since a regular fiber is a 1-manifold, it is a link in N. Around the definite fold and indefinite fold points, the fiber changes as in Figure 9. Note that around a definite fold point, one of the connected components of the regular fiber is collapsed, whereas, around an indefinite fold point, two connected components become one.

4.2 Handle decomposition

In this section, we give a handle decomposition of the complement $M = \mathbb{C}^2 \setminus C$. Let $D \subset \mathbb{C}$ be a sufficiently large closed disk such that a non-intersecting arc system $s_1, s_2, \ldots, s_N$ is contained in D. Take a small regular neighborhood $\nu(C)$ of C such that $L_q \cap \nu(C)$ consists of n connected components for all $q \in \partial U_i$ (see Figure 10). We also assume that $\nu(C)$ is generic so that the restriction of π to the boundary $\partial\nu(C)$ is a stable map. We define the disk D_y in the fiber complex line such that $L_p \cap \nu(C)$ is contained in $\{p\} \times D_y$ for all $p \in D$. Clearly, $D \times D_y$ is diffeomorphic to the standard 4-ball.



(i) definite fold point



(ii) indefinite fold point

Figure 9: Fibers around critical points of stable maps

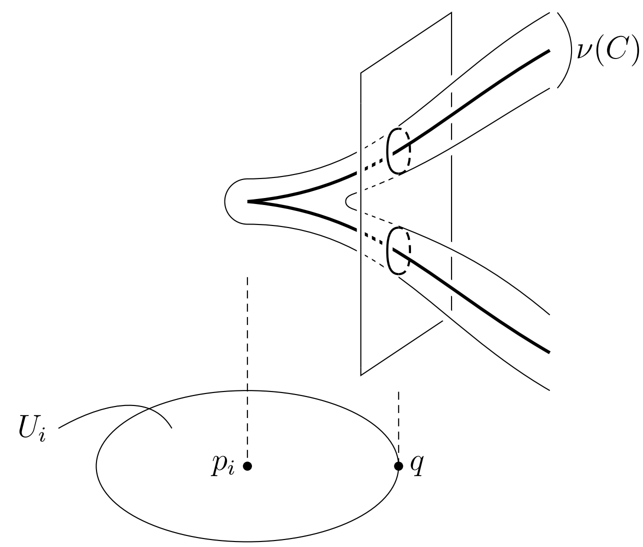


Figure 10: The neighborhoods U_i and $\nu(C)$

We set $M_0 = (D \times D_y) \setminus \nu(C)$. The manifold M_0 is a compact 4-manifold with boundary and the interior of M_0 is diffeomorphic to M. We shall give a handle decomposition of M_0.

Let $\pi_0 := \pi|_{M_0} : M_0 \to D$ be the restriction of the projection $\pi : \mathbb{C}^2 \to \mathbb{C}$. Note that the restriction of π_0 to $M_0 \setminus \bigcup_{i=1}^N \pi_0^{-1}(U_i)$ is again a locally trivial fibration. Now we prepare some notations (see Figure 11).

- s_0 : an arc connecting p_0 and a point at ∂D such that $\{s_0, s_1, \ldots, s_N\}$ is a non-intersecting arc system.
- $\Gamma = \bigcup_{i=0}^N s_i$
- $\nu(s_i)$: a small tubular neighborhood of s_i for $i = 0, 1, \ldots, N$.
- $\nu(\Gamma) = \bigcup_{i=0}^N \nu(s_i) \cup \bigcup_{i=1}^N U_i$
- $M_1 = \pi_0^{-1}(D \setminus \nu(\Gamma))$
- $D_n = D^2 \setminus \{\text{disjoint } n \text{ small disks}\}$

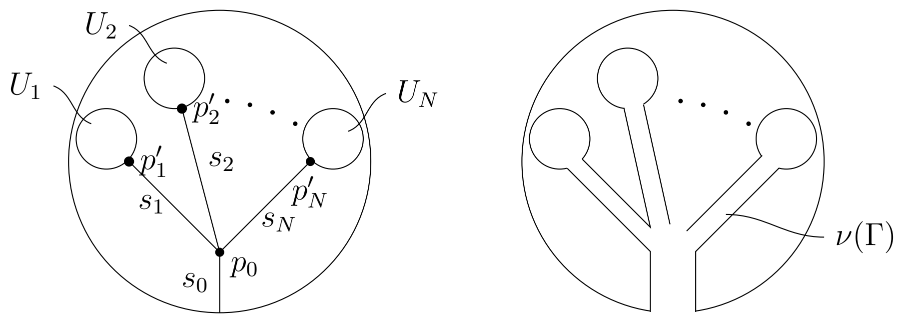


Figure 11: The non-intersecting system, the graph Γ and its neighborhood $\nu(\Gamma)$

In this setting, $D^2 \setminus \nu(\Gamma)$ is diffeomorphic to D^2, and thus contractible. Therefore, the restriction $\pi_0 : M_1 \to D \setminus \nu(\Gamma)$ is a trivial fibration. Each fiber is diffeomorphic to D_n, and hence, $M_1 \cong D_n \times D^2$. Thus, we have the following proposition.

Proposition 4.1. The manifold M_1 is diffeomorphic to a 4-manifold obtained from a 4-ball B^4 by attaching n 1-handles.

Since $M_0 = M_1 \cup \pi_0^{-1}(\nu(\Gamma))$, we need to see how $\pi_0^{-1}(\nu(\Gamma))$ is attached to M_1. We will see the attachment of $\pi_0^{-1}(\nu(\Gamma))$ by decomposing it into some parts. Here, we provide the following lemma, which is needed to describe the change of topology when each part is attached. The proof is easy and we can check it by hand.

Lemma 4.2. *Let X and Y be compact manifolds with boundary, such that $Y \subset \partial X$ and $\dim Y = \dim \partial X$. Then, we have the following (see Figure 12).*

(1) The maniofld X is diffeomorphic to the manifold $X \cup_Y (Y \times [0,1])$ which is obtained from X by attaching $Y \times [0,1]$ along $Y \times \{0\} \cong Y$. In other words, attaching $Y \times [0,1]$ to X does not change the diffeomorphism type of X.

(2) Suppose that $[0,1] \times Y$ is embedded in X with $\{1\} \times Y \subset \partial X$. Then, X is diffeomorphic to the manifold $X \setminus ((0,1) \times \mathrm{Int}(Y))$. In other words, removing $Y \times [0,1]$ from X does not change the diffeomorphism type of X.

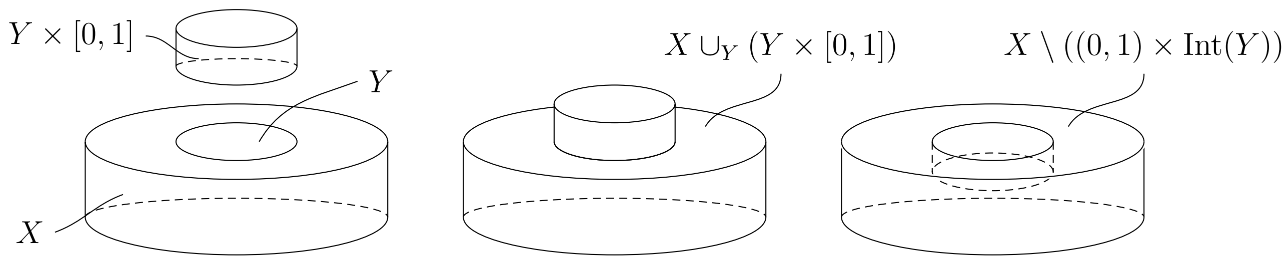


Figure 12: Attachment and removing of $Y \times [0,1]$. In this picture, $X \cong D^3$ and $Y \cong D^2$.

For each p_i, define $V_i \subset U_i$ as a small disk centered at p_i such that $\pi_0^{-1}(a) \cong D_{n-(m_i-1)}$ for all $a \in V_i$ (see Figure 13). Recall that m_i is the multiplicity of the singularity of C contained in L_{p_i}.

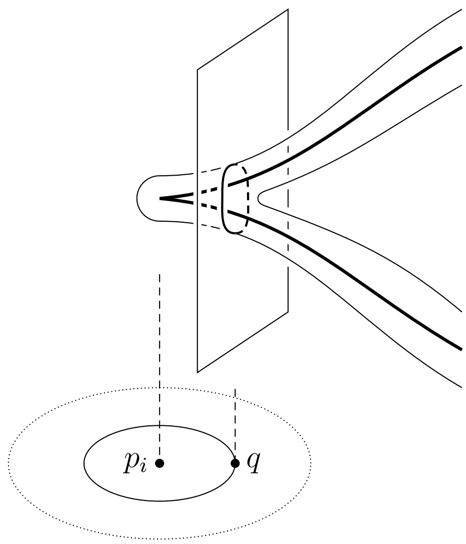


Figure 13: The neighborhood V_i

We set $c_i = \partial U_i \setminus \nu(s_i)$. For $a \in \partial U_i$, we denote by ℓ_a the intersection of $\overline{U_i \setminus V_i}$ and the segment connecting p_i with a. We denote the union of segments ℓ_a by $A = \bigcup_{a \in c_i} \ell_a$ and set $B = \overline{U_i \setminus (A \cup V_i)}$ (Figure 14).

Each fiber $\pi_0^{-1}(\ell_a)$ is diffeomorphic to $\natural_n S^1 \times D^2$, so we have

$$\pi_0^{-1}(A) \cong \pi_0^{-1}(\ell_a) \times [0,1] \cong (\natural_n S^1 \times D^2) \times [0,1].$$

The part $\pi_0^{-1}(A)$ is attached to M_1 along $\pi_0^{-1}(c_i) \cong \natural_n S^1 \times D^2$. Therefore, by Lemma 4.2 (1), attaching $\pi_0^{-1}(A)$ to M_1 does not change the diffeomorphism type.

Since $\pi_0^{-1}(p) \cong D_{n-(m_i-1)}$ for all $p \in V_i$, we have

$$\pi_0^{-1}(V_i) \cong D_{n-(m_i-1)} \times D^2 \cong (D_{n-(m_i-1)} \times [0,1]) \times [0,1].$$

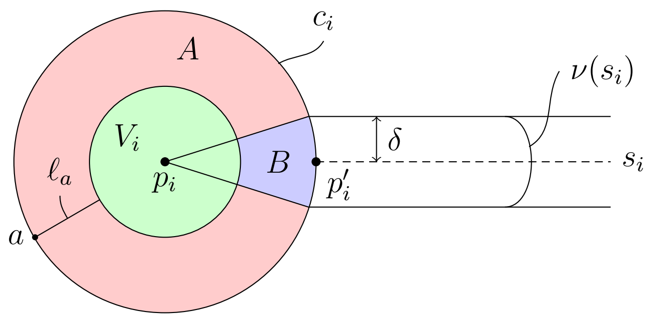


Figure 14: The regions A and B

The part $\pi_0^{-1}(V_i)$ is attached to $\pi_0^{-1}(A)$ along $\pi_0^{-1}(V_i \cap A) \cong D_{n-(m_i-1)} \times [0,1]$. Again, by Lemma 4.2 (1), attaching $\pi_0^{-1}(V_i)$ to $\pi_0^{-1}(A)$ does not change the diffeomorphism type. From these observations, we have the following proposition.

Proposition 4.3. The manifold $M_1 \cup \pi_0^{-1}(A \cup V_i)$ is diffeomorphic to M_1.

Let us denote $M_2 = M_1 \cup \pi_0^{-1}(A \cup V_i)$. Next, we see the attachment of $\pi_0^{-1}(B)$. We have the following theorem for this attachment.

Theorem 4.4. *Attaching $\pi_0^{-1}(B)$ to M_2 is equivalent to attaching (m_i-1) 2-handles.*

From now on, we give the proof of this theorem. For simplicity, we suppose $m_i = 2$. First, we consider the restriction of π_0 to the boundary ∂M_0. We have the following proposition about the critical point of $\pi_0|_{\partial M_0}$ (see Figure 15).

Proposition 4.5. The map $\pi_0|_{\partial M_0}$ has only indefinite folds as the critical points in $\pi_0^{-1}(B)$. For each segment ℓ_a $(a \in B \cap \partial U_i)$, there is just one critical point in the fiber $\pi_0^{-1}(\ell_a)$.

Proof. First, since we take the neighborhood $\nu(C)$ to be small, the number of the connected components of each regular fiber of $\pi_0|_{\partial M_0}$ is at most two. (If there are three or more, then $D_y \cap \nu(C)$ consists of three or more disks for some $y \in B$. Among these disks, there is a disk that does not intersect C, since C and D_y intersect at no more than two points. In this case, we can retake a smaller neighborhood by removing this disk, but this contradicts that $\nu(C)$ is sufficiently small).

Therefore, the number of connected components of the fiber decreases from two to one as a point on the segment ℓ_a approaches p_i. The two connected components are joined together and become one. At this time, since we assume that $\pi_0|_{\partial M_0}$ is a stable map, there is one fiber that contains a critical point and the critical point is an indefinite fold. □

Therefore, by using appropriate local coordinates (X, Y, v) of ∂M_0 and (u, v) of B, the map $\pi_0|_{\partial M_0}$ is described as follows around the critical points:

$$\pi_0|_{\partial M_0} : (X, Y, v) \mapsto (X^2 - Y^2, v), \tag{1}$$

and the intersection S of the critical value set and B is described as $\{(u, v) \mid u = 0\}$.

Let α_r be the intersection of a curve $\{(u, v) \mid u = v^2 + r\}$ and B. Then, there exist $\xi_1 < 0$ and $\xi_2 > 0$ such that $B = \bigcup_{\xi_1 \leq r \leq \xi_2} \alpha_r$ and $\pi_0^{-1}(B) = \bigcup_{\xi_1 \leq r \leq \xi_2} \pi_0^{-1}(\alpha_r)$. To

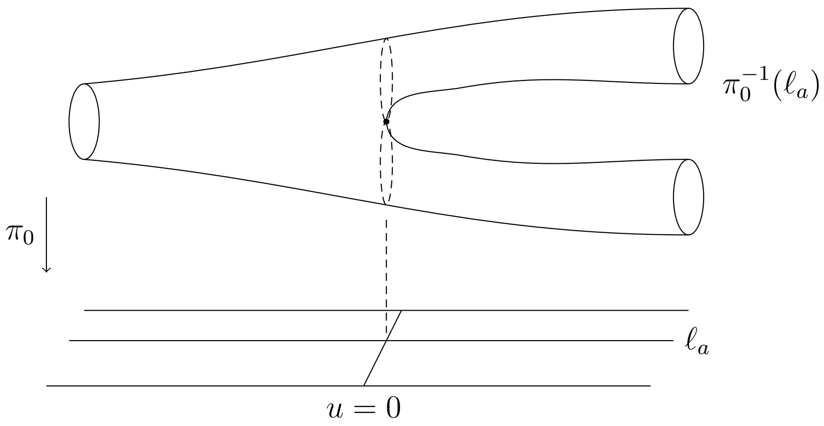


Figure 15: The critical point of $\pi_0|_{\partial M_0}$

see the attachment of $\pi_0^{-1}(B)$ to M_2, we observe that of $\pi_0^{-1}(\alpha_r)$ for each $\xi_1 \le r \le \xi_2$ (see Figure 16).

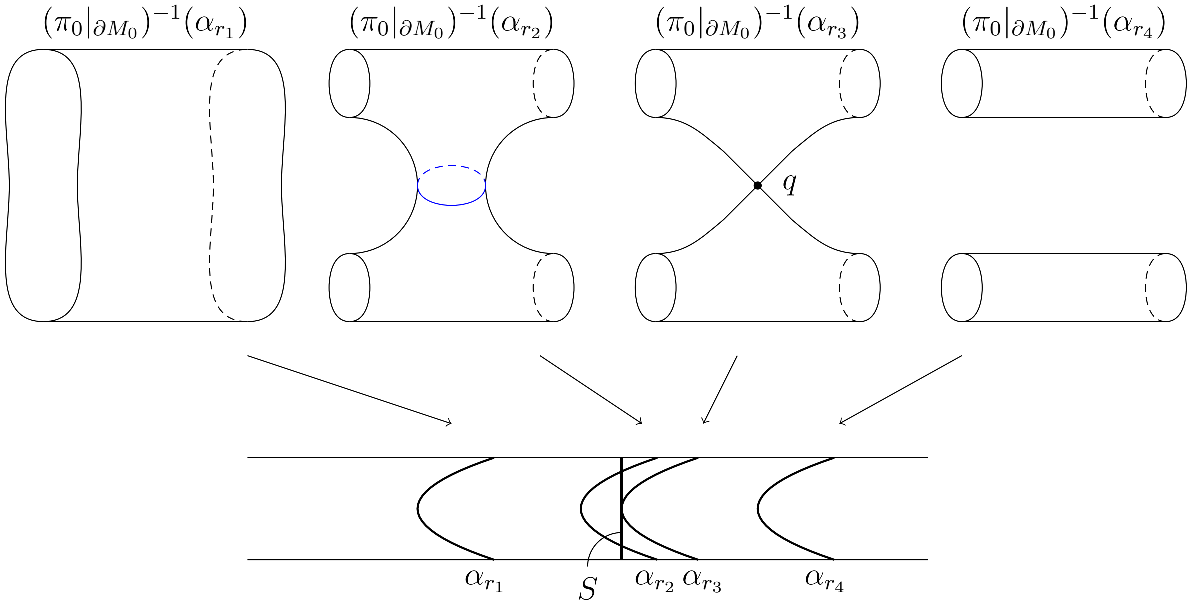


Figure 16: The fibers of $\pi_0|_{\partial M}$ over B

Let η be a sufficiently small positive number. At first, attaching $\bigcup_{\xi_1 \le r \le -\eta} \pi_0^{-1}(\alpha_r)$ to M_2 does not change the diffeomorphism type, by Lemma 4.2 (2).

Next, we consider the part $-\eta \le r \le \eta$. Let q be the intersection point of the critical point set of $\pi_0|_{\partial M_0}$ and $\pi_0^{-1}(\alpha_0)$. Let (s, t) be the real coordinates of the fiber complex line for each $(u, v) \in B$. That is, the projection is expressed as $\pi(s, t, u, v) = (u, v)$. On the other hand, $\pi_0|_{\partial M_0}$ is expressed as $(X, Y, v) \mapsto (X^2 - Y^2, v)$ by using the local coordinates of ∂M_0. Therefore, by using these coordinates, the embedding $F : \partial M_0 \to \mathbb{C}^2 \cong \mathbb{R}^4$ is expressed as

$$F : (X, Y, v) \mapsto (f(X, Y, v), g(X, Y, v), X^2 - Y^2, v),$$

where f and g are smooth functions. Let us fix $v = 0$ and define a smooth map $F_0 : \mathbb{R}^2 \to \mathbb{R}^2$ as $F_0 : (X, Y) \mapsto (s = f(X, Y, 0), t = g(X, Y, 0))$. Since F induces the embedding $(X, Y) \mapsto (f(X, Y, 0), g(X, Y, 0), X^2 - Y^2)$ and $(X, Y) = (0, 0)$ is a critical point of $X^2 - Y^2$, dF_0 is a rank two matrix at (0.0). Hence, by the inverse function theorem, F_0 is locally a diffeomorphism around a neighborhood U_0 of $(X, Y) = (0, 0)$. Similarly, it follows that $(X, Y) \mapsto (f(X, Y, v_0), g(X, Y, v_0))$ is a diffeomorphism on U_0 for fixed small v_0. Therefore, there exists a small neighborhood $U_1 \subset \mathbb{C}^2$ of q such that the map

$$\varphi : U_1 \to U_0 \times U_2;\ (s, t, u, v) \mapsto (X, Y, u, v)$$

is a diffeomorphism, where $s = f(X, Y, v)$ and $t = g(X, Y, v)$ and $U_2 \subset B$ is a neighborhood of $(u, v) = (0, 0)$. Thus, after composing φ, the projection π is expressed as $\pi(X, Y, u, v) = (u, v)$ and ∂M_0 is locally described as

$$\{(X, Y, u, v) \mid u = X^2 - Y^2\}.$$

From now on, we will use the coordinates (X, Y, u, v) around the critical point, and instead of u, we use the coordinate $r = u - v^2$ by a coordinate transformation $(u, v) \mapsto (u - v^2, v)$. Around the critical point in $\bigcup_{-\eta \le r \le \eta} \pi_0^{-1}(\alpha_r)$, ∂M_0 is described as follows (note that $u = v^2 + r$):

$$\{(X, Y, r, v) \mid X^2 - Y^2 = v^2 + r,\ -\eta \le r \le \eta\}.$$

Therefore, around the critical point, the part N which is obtained from M_2 by attaching $\bigcup_{-\eta \le r \le \eta} \pi_0^{-1}(\alpha_r)$ is described as follows:

$$N = \{(X, Y, r, v) \mid X^2 - Y^2 \le v^2 + r, -\eta \le r \le \eta\}.$$

We decompose N into the following two parts N_1 and N_2:

$$\begin{aligned} N_1 &= \{(X, Y, r, v) \mid X^2 - Y^2 - v^2 \le -\eta,\ -\eta \le r \le \eta\}, \\ N_2 &= \{(X, Y, r, v) \mid -\eta \le X^2 - Y^2 - v^2 \le r \le \eta\}. \end{aligned}$$

Clearly, attaching N_1 does not change the topology. On attaching N_2, we have the following proposition.

Proposition 4.6. N_2 is diffeomorphic to a 4-ball.

Proof. Define the region N_2' in $\mathbb{R}^3$ as follows

$$N_2' = \{(X, Y, v) \mid -\eta \le X^2 - Y^2 - v^2 \le \eta\}.$$

Then, N_2' is diffeomorphic to a 3-ball D^3 after smoothing corners. Let $p : N_2 \to N_2'$ be the projection $p(X, Y, r, v) = (X, Y, v)$. Then, it is a surjective map and each fiber $\{X^2 - Y^2 - v^2 \le r \le \eta\}$ is diffeomorphic to D^1, over the interior of N_2'. Therefore, by smoothing corners, we obtain that $N_2 \cong N_2' \times D^1 \cong D^3 \times D^1 \cong D^4$. □

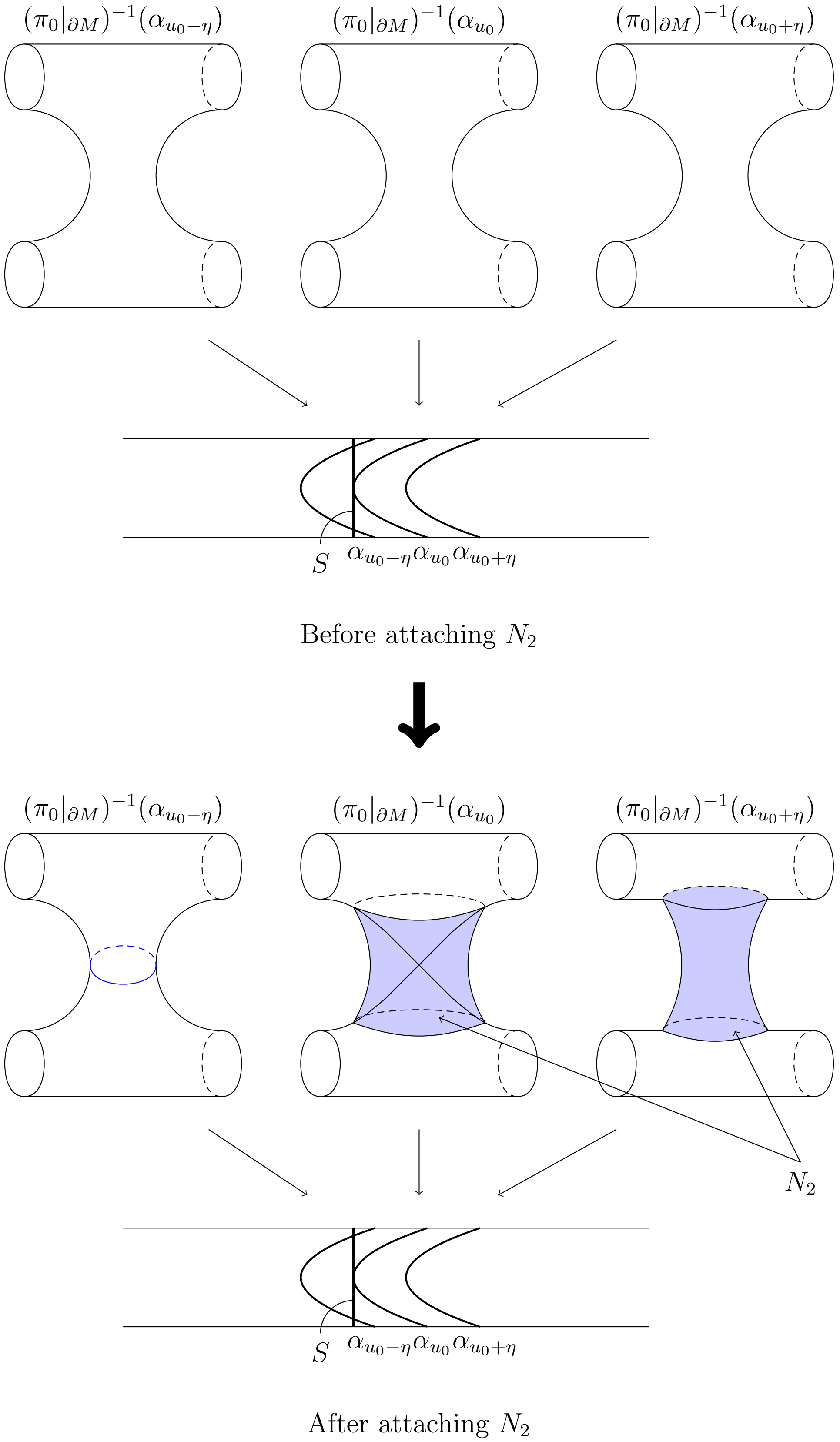


Figure 17: Attachment of N_2

The parts N_1 and N_2 are attached along N_3 which is defined as

$$N_3 = \{(X, Y, r, v) \mid X^2 - Y^2 - v^2 = -\eta,\ -\eta \leq r \leq \eta\}$$

This attaching area N_3 is diffeomorphic to $S^1 \times D^2$. Therefore, attaching N_2 is equivalent to attaching a 2-handle.

The remaining part, attaching $\bigcup_{\eta \leq r \leq \xi_2} \pi_0^{-1}(\alpha_r)$ does not change the diffeomorphism type for the same reason as in the part $\xi_1 \leq r \leq -\eta$. Therefore, attaching $\pi_0^{-1}(B)$ is equivalent to attaching a 2-handle (see Figure 17 for the attaching of N_2). When $m_i \geq 3$, the critical value set is $(m_i - 1)$ disjoint union of segments. Similar to the above discussion, a 2-handle attaching appears each time α_r crosses the critical value segment. Therefore, attaching $\pi_0^{-1}(B)$ is equivalent to attaching $(m_i - 1)$ 2-handles. This completes the proof of Theorem 4.4.

Finally, only the part $\nu(\Gamma) \setminus U_i$ is left. However, attaching $\pi^{-1}(\nu(\Gamma) \setminus U_i)$ does not change the topology again by Lemma 4.2 (2). This completes the construction of a handle decomposition of M_0.

Remark 4.7. The attaching circle of the 2-handle is written as $\{(X_0, Y, r_0, v) \mid Y^2 + v^2 = X_0^2 + \eta\}$ by using fixed X_0 and r_0. This can be moved into $\pi_0^{-1}(\alpha_3) \subset \partial M_1$ by the following procedure (see Figure 18). The resulting attaching circle is denoted by c_i'.

(i) Move the attaching circle c_i into the fiber $(\pi_0|_{\partial M_2})^{-1}(\alpha_1)$.

(ii) Corresponding to moving the arc from α_1 to α_2 in the base space, move the attaching circle into $(\pi_0|_{\partial(M_1 \cup \pi_0^{-1}(A))})^{-1}(\alpha_2)$.

(iii) Move the attaching circle in $\pi_0^{-1}(\alpha_2)$ so that the fiber of the endpoints of α_2 link to the attaching circle and except for the endpoints of α_2, the attaching circle runs parallel to the outside cylinder of $\pi_0^{-1}(\alpha_2)$. Remark that $\pi_0^{-1}(\alpha_2)$ is contained in $\partial(M_1 \cup \pi_0^{-1}(A))$.

(iv) Corresponding to moving the arc from α_2 to α_3 in the base space, move the attaching circle into $(\pi_0|_{\partial M_1})^{-1}(\alpha_3)$. Here, B denotes the braid representing the local monodromy.

For example, if the local equations are $\{y^2 = x^2\}$ and $\{y^3 = x^2\}$, the attaching circles of 2-handles are described as in Figures 19 and 20, respectively. Later, we will consider M_1 as the 1-handlebody of M_0 and use these perturbed attaching circles when we describe the Kirby diagrams.

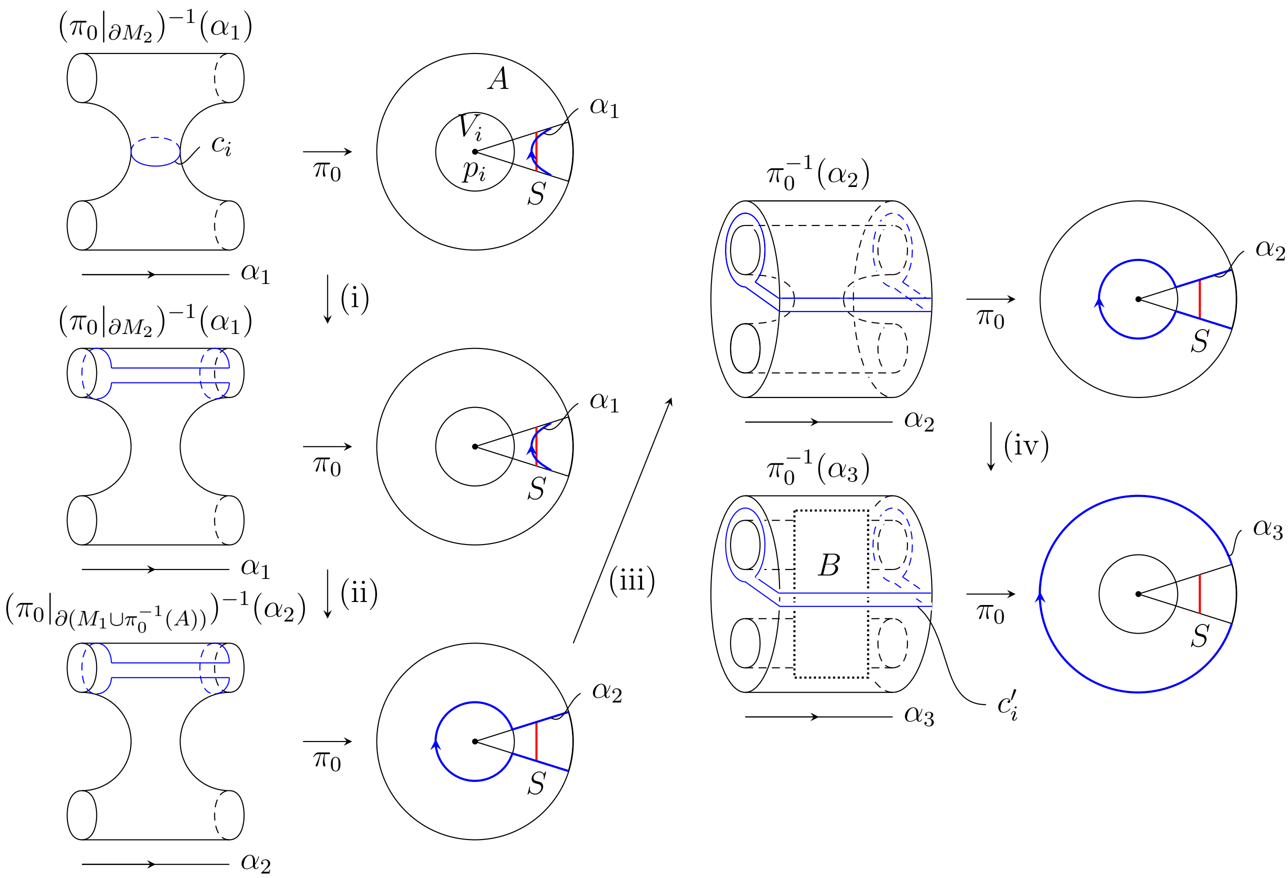


Figure 18: Moving of the attaching circle

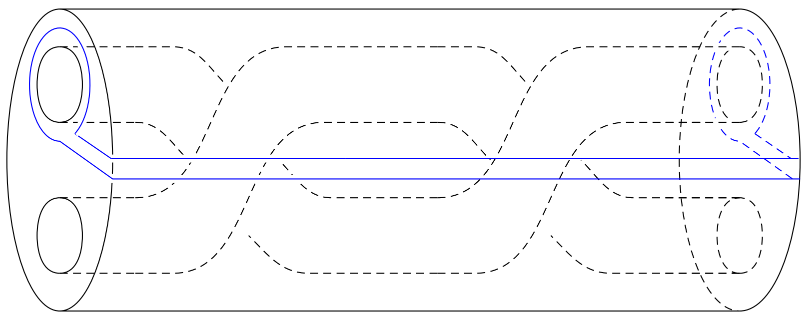


Figure 19: Attaching 2-handles around $\{y^2 = x^2\}$

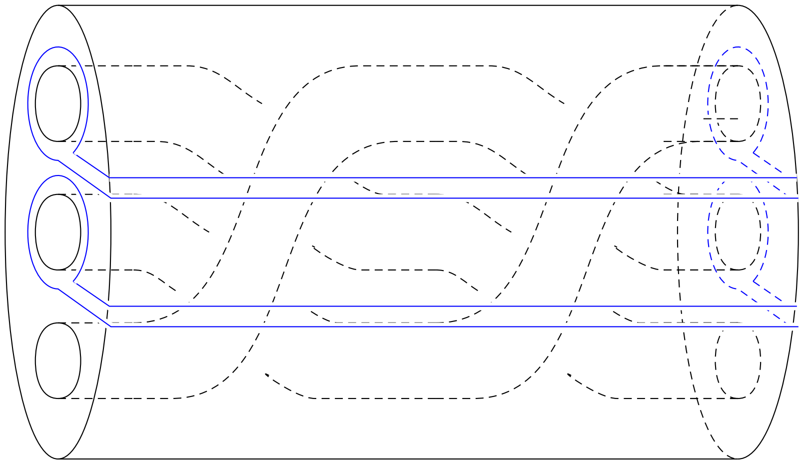


Figure 20: Attaching 2-handles around $\{y^3 = x^2\}$

5 Kirby diagrams and examples

In this section, we see how to obtain Kirby diagrams and give some examples. First, a 1-handle is presented as a dotted circle in a Kirby diagram. Removing the tubular neighborhood of a trivially and properly embedded D^2 in a 4-ball B^4 is equivalent to attaching a 1-handle. The dotted circle of the 1-handle obtained in this way is the boundary of the disk embedded in S^3. The 1-handlebody M_1 of M_0 is obtained from $\pi^{-1}(D \setminus \nu(\Gamma)) \cong B^4$ by removing $\pi^{-1}(D \setminus \nu(\Gamma)) \cap \nu(C)$. The Kirby diagram is drawn in $\partial(\pi^{-1}(D \setminus \nu(\Gamma))) \cong S^3$. Since the removed part $\pi^{-1}(D \setminus \nu(\Gamma)) \cap \nu(C)$ is a disjoint union of tubular neighborhood of n properly embedded D^2, dotted circles of these 1-handles are $\pi^{-1}(\partial(D \setminus \nu(\Gamma))) \cap C$ (trivial link of n-components). As noted in Remark 3.4, around the fiber of tangent points and singularities of C, $\pi^{-1}(\partial U_i) \cap C$ is a braid which represents the local monodromy. Therefore, this braid also appears as the dotted circle of the 1-handles. In Remark 4.7, we moved the attaching circle of the 2-handle so that it is on ∂M_1. Thus, the Kirby diagram is locally described as in Figure 21. To describe the global Kirby diagram, we need the relationship among local monodromies. These are determined by the trivialization along each path s_i. The braid defined by the trivialization appears as the dotted circles of 1-handles and this connects each local monodromy.

Next, we have to consider the framing coefficient of each 2-handle. The attaching circle of the 2-handle is written as $C_0 = \{(X_0, Y, r_0, v) \mid Y^2 + v^2 = X_0^2 + \eta\} \subset N_3 \subset N_2$ for appropriately fixed X_0 and r_0. A slightly perturbed circle is expressed as $C_\varepsilon = \{(X_0 + \varepsilon, Y, r_0, v) \mid Y^2 + v^2 = (X_0 + \varepsilon)^2 + \eta\}$ for sufficiently small $\varepsilon > 0$. In Remark 4.7, we move the attaching circle into ∂M_1 and let C'_0 be the perturbed attaching circle here. Similarly, we can move the perturbed circle C_ε into ∂M_1 and denote it by C'_ε (see Figure 22). Perturbed circles C'_0 and C'_ε are both located in $\partial\left(\pi_0^{-1}(\alpha_3) \cup (\pi^{-1}(\alpha_3) \cap \nu(C))\right) \cong \partial B^3 \cong S^2$. This 2-sphere is contained in $\pi^{-1}(\partial(D \setminus \nu(\Gamma))) \cong S^3$, in which the Kirby diagram is drawn. Therefore, C_0 and C_ε are located in the same 2-sphere embedded in 3-sphere. Thus, we can separate C_0 and C_ε and the linking number is equal to 0. Therefore, the framing coefficient is equal to 0 for every 2-handle.

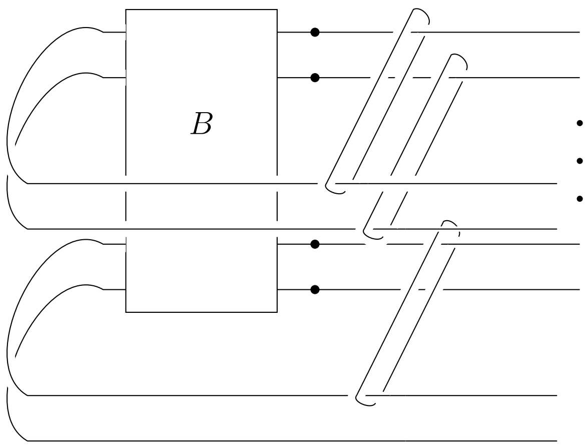


Figure 21: Local description of the Kirby diagram. There are n 1-handles and m_i 2-handles and B is the braid representing the local monodromy.

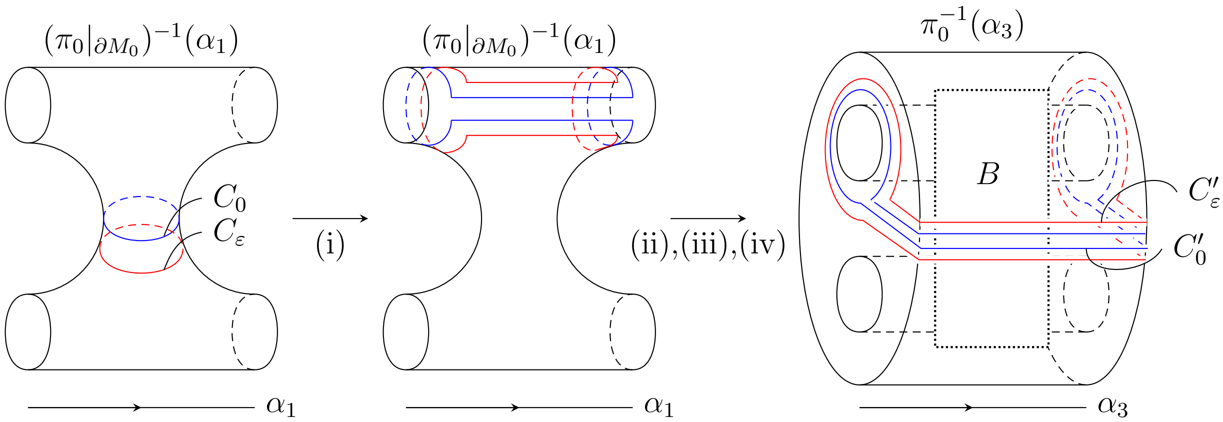


Figure 22: Perturbation of the attaching circle. The procedures (i), (ii), (iii) and (iv) are the same as Figure 18.

Example 5.1. (Brieskorn polynomials) Let $f(x, y) = x^p - y^q$ $(2 \leq p \leq q)$. Then $X = \{0\}$. Take a graph Γ as in Figure 23. It suffices to observe the monodromy around $x = 0$. When x goes counterclockwise around 0, the y-coordinate in the fiber rotates by $2p\pi/q$. This monodromy can be presented by the following braid $B_{p,q} = (\sigma_1\sigma_2 \cdots \sigma_{q-1})^p$ as in Figure 24, where σ_j is a canonical generator, presenting the half twist of the j-th point and the $j+1$ point (see also [O78] for the presentation of the fundamental group of the complement of the curve $x^p = y^q$). Therefore, the Kirby diagram of the complement is described as in Figure 25. There are q 1-handles and $(q-1)$ 2-handles. For example, let us suppose $p = q = 2$. Then, the Kirby diagram is as in Figure 26, so we can see that it is diffeomorphic to $T^2 \times D^2$. This proves the well-known fact that the complement of $\{x^2 - y^2 = 0\}$ is diffeomorphic to the interior of $T^2 \times D^2$.

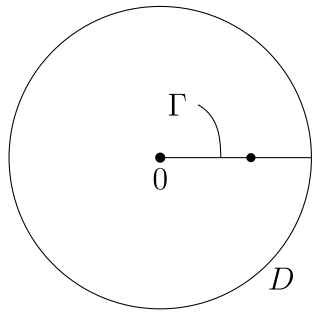


Figure 23: A graph Γ for $f(x, y) = x^p - y^q$

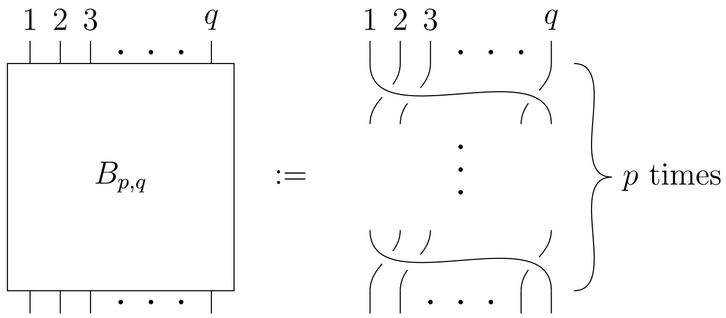


Figure 24: A braid $B_{p,q}$

Example 5.2. (Line arrangement of generic 3 lines) Let $f(x, y) = (x + y)(x - y)(y - 1)$. The curve $C = \{f(x, y) = 0\}$ is the union of three generic lines. Then, $X = \{-1, 0, 1\}$. Each point in X corresponds to the intersection of two lines. Take a graph Γ as in Figure 27. Each local monodromy around the point in X is isomorphic to the one at the origin of $\{x^2 - y^2 = 0\}$. Since Γ runs parallel to the real axis close enough except around the points in X, non-trivial braidings do not appear over this part. When Γ makes a half turn around a point in X, a half twist appears in the Kirby diagram as the half of the monodromy. Therefore, the Kirby diagram is as in Figure 28. Here, B is the braid which becomes trivial after composing with the product of all other braids presenting 1-handles in the Kirby diagram.

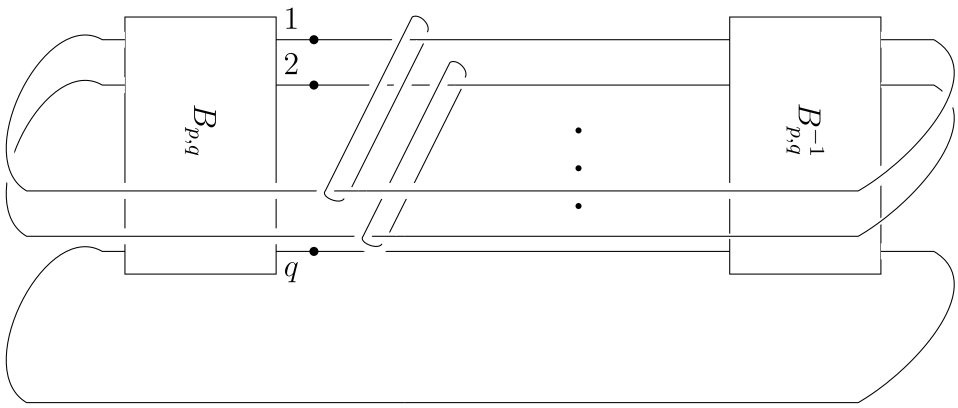


Figure 25: The Kirby diagram obtained from the graph Γ

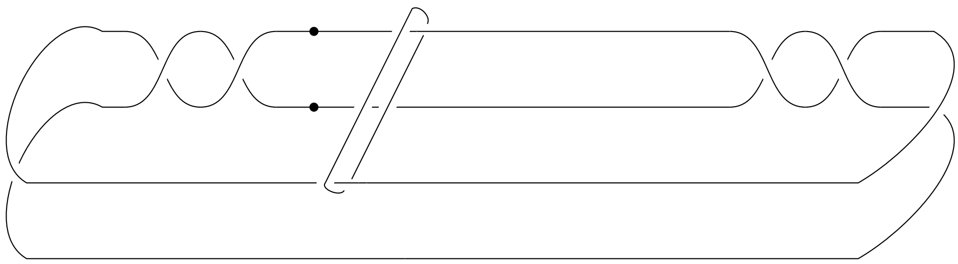


Figure 26: The case $p = q = 2$

Also, we can take a graph Γ' as in Figure 29. By this graph, we can obtain the handle decomposition as follows, although in a little different way from the one described in the main result. First, attach a 2-handle corresponding to the fiber around -1. Next, after trivializing along the path connecting -1 and 0, attach a 2-handle corresponding to $0 \in X$. At this time, dotted circles look different from the ones obtained from the graph Γ, that is, the local monodromy around 0 splits into two half rotations. Similarly, we can attach the remaining 2-handle corresponding to $1 \in X$. Finally, the resulting Kirby diagram is described as in Figure 30.

Remark 5.3. In [SY22], Kirby diagrams for complements of complexified real line arrangements are obtained. We can see that the Kirby diagram described in the above example is obtained from the one in [SY22] by handle sliding.

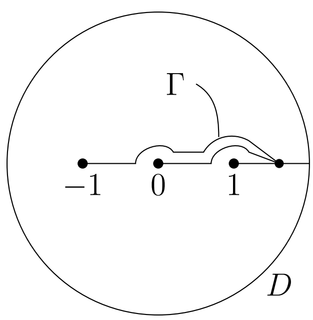


Figure 27: A graph Γ for $f(x, y) = (x + y)(x - y)(y - 1)$

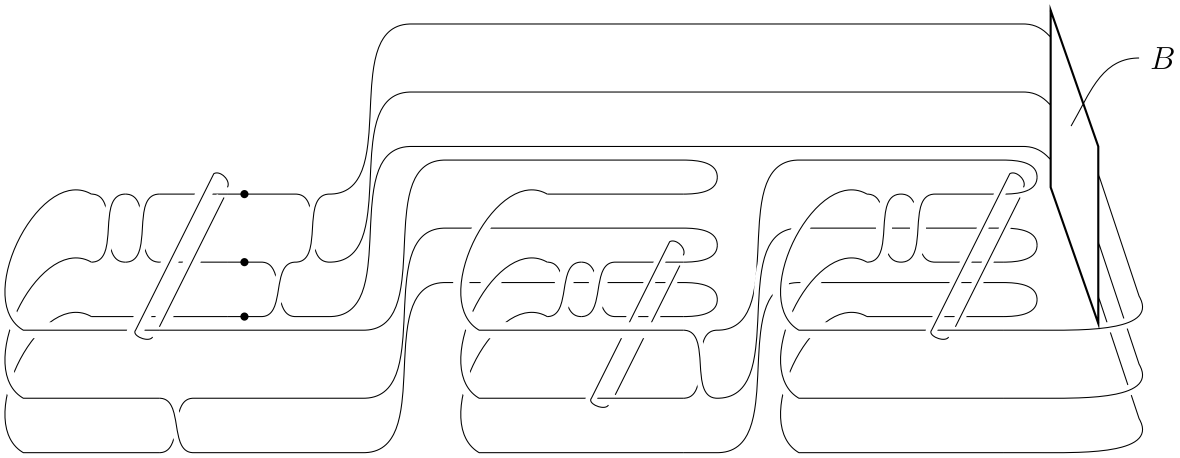


Figure 28: The Kirby diagram obtained from the graph Γ

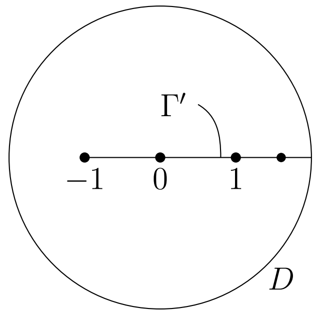


Figure 29: Another graph Γ' for $f(x, y) = (x + y)(x - y)(y - 1)$

Example 5.4. (Fermat curves) Let $f(x, y) = x^n+y^n-1$. Then, $X = \{1, \zeta, \zeta^2, \ldots, \zeta^{n-1}\}$, where $\zeta^k = e^{2k\pi\sqrt{-1}/n}$. Take the graph Γ as in Figure 31. The complement $D \setminus \nu(\Gamma)$ of the neighborhood of the graph can be deformed into D' defined as the following procedure. First, define subsets $U_0, s_0^{\pm}, \gamma_0$ in $\mathbb{C}$ as follows:

$$\begin{aligned} U_0 &= \{x \in \mathbb{C} \mid |x - 1| \leq \varepsilon, \arg(x) \notin (-\delta, \delta)\}, \\ s_0^{\pm} &= \{x = u + \sqrt{-1}v \mid 1 - \varepsilon \cos\delta \leq u \leq 2, v = \pm \sin\delta\}, \\ \gamma_0 &= \{x \in \mathbb{C} \mid |x| = 2, \arg(x) \in [\delta, (2\pi/n) - \delta]\}. \end{aligned}$$

Let $U_k, s_k^{\pm}, \gamma_k$ $(k = 1, \ldots, n-1)$ be the sets obtained from $U_0, s_0^{\pm}, \gamma_0$ by rotating $2k\pi/n$ degrees. At this time, define D' as the domain containing the origin rounded by the curve $\bigcup_{k=0}^{n-1}(\partial U_k \cup s_k^+ \cup s_k^- \cup \gamma_k)$ (see Figure 32). Each 2-handle is attached at the fiber of the area $2k\pi/n - \delta \leq \theta \leq 2k\pi/n + \delta$ around $\zeta^k \in X$. By symmetry, we will describe the handle attachment corresponding to just $1 \in X$. For each other $\zeta^k \in X$, we can similarly obtain the handle attachment.

Since the defining equation is $y^n = x^n - 1$, $L_1 \cap C$ is the set consisting of one point to which n points of the intersection between the fiber of a point in the neighborhood of $1 \in X$ and C degenerates. Around the tangent point $(1, 0)$ in $L_1 \cap C$, C is locally isomorphic to $\{x = y^n\}$. From easy computation, the fiber coordinate y rotates

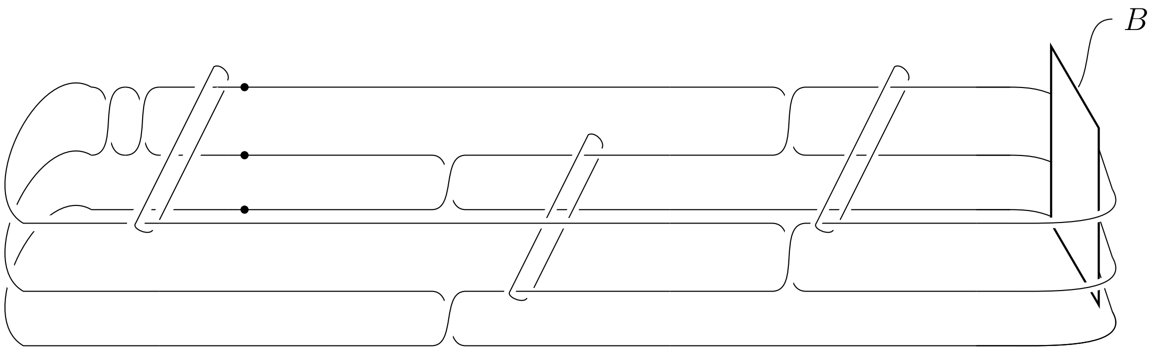


Figure 30: The Kirby diagram obtained from the graph Γ'

$2\pi/n$ when x turn around ∂U_0. Also, when x moves along the path γ_0, the fiber coordinate y also rotates $2\pi/n$. Therefore, we can obtain the picture of $\pi^{-1}(\partial D')\cap C$ and the Kirby diagram of the complement. (See Figure 33 for the case $n = 3$). For general n, we can describe a similar diagram, and moving by an isotopy, we obtain the Kirby diagram as in Figure 34.

Remark 5.5. For a fixed degree, all non-singular curves that intersect the line at infinity transversely have diffeomorphic complements. Therefore, the Kirby diagram obtained in the above example gives that of the complement of a non-singular curve intersecting the line at infinity transversely.

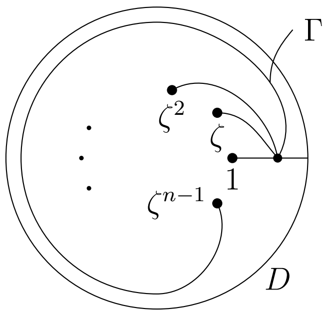


Figure 31: A graph Γ for $f(x, y) = y^n - x^n - 1$

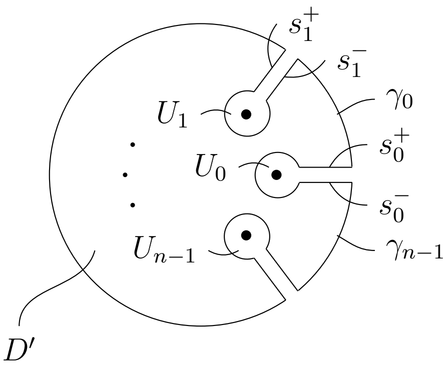


Figure 32: The disk D'

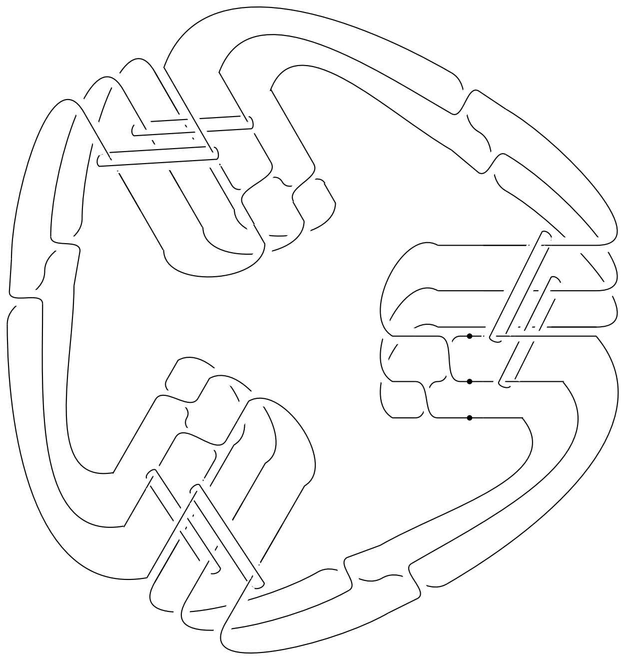


Figure 33: The Kirby diagram obtained from the graph Γ' ($n = 3$)

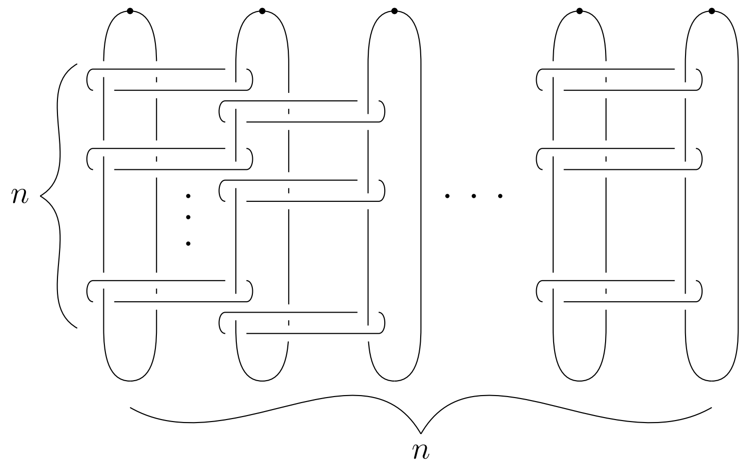


Figure 34: The Kirby diagram for general n

6 Future issues

In this section, we will discuss some further problems related to our results.

6.1 Stein and contact structures

As we noted in Introduction, the complement of an affine plane algebraic curve admits a Stein structure. Recall that a complex manifold X is Stein if there exists a closed holomorphic embedding into a complex Euclidean space. It is known that a complex manifold is Stein if and only if it admits a strictly plurisubharmonic function (shortly, s.p.s.h. function). A smooth function $f : X \to \mathbb{R}$ on a complex manifold is s.p.s.h. if the complex Hessian

$$\left(\frac{\partial^2 f}{\partial z_j \partial \overline{z}_k} \right)$$

is positive definite for every point $p \in X$, where $(z_1, \ldots, z_n)$ is a local complex coordinate around $p \in X$. A compact complex manifold X with boundary is called Stein domain if X is expressed as the sublevel set $f^{-1}\left((-\infty, c]\right)$ for a regular value $c \in \mathbb{R}$ and $\partial X = f^{-1}(c)$ for a s.p.s.h. function f. The boundary of a Stein domain carries a canonical contact structure defined as the hyperplane field $\xi = (TX \cap J(TX))|_{\partial X}$, where J is the complex structure on X. We can generically assume that a s.p.s.h. function is a Morse function. Thus, we can obtain a handle decomposition of a Stein manifold by the s.p.s.h. function. Such a handle decomposition is called a compatible handle decomposition with the Stein structure.

Since we have obtained the handle decomposition by not using a s.p.s.h. function, we do not know whether the handle decomposition in the main result is compatible with the Stein structure. Then it is natural to ask the following problem.

Problem 6.1. *Determine the handle decomposition of the complement that is compatible with the Stein structure.*

At this time, we can expect that M_0 defined in Section 4 is a Stein domain. Then, the boundary carries a canonical contact structure mentioned above. Thus, we can ask about the contact structure that arises on the boundary.

Problem 6.2. *Determine the contact structure of the boundary 3-manifold of M_0.*

6.2 Projective plane curves

In this thesis, we discussed the complements of affine plane algebraic curves. We can also consider diffeomorphism types of the complements of projective plane curves. For any affine plane curve C, we can define the projective closure $\overline{C}$ in $\mathbb{C}P^2$. We set $\overline{M} = \mathbb{C}P^2 \setminus \overline{C}$.

Problem 6.3. *Describe the handle decomposition and the Kirby diagram of $\overline{M}$.*

Note that the "projective version" of Libgober's theorem on the homotopy type is still unknown. However, the fundamental group of the projective complement can be computed by just adding the relation "the product of all generators = 1" to the presentation of the fundamental group of the affine plane curve complement. Hence one may expect that the CW complex corresponding to this presentation is homotopy equivalent to the projective complement, however, it is not true in general. If it were true, the Euler characteristic of the projective complement would increase by one. However, the following proposition implies that this does not happen in general.

Proposition 6.4. Let C be a plane algebraic curve in $\mathbb{C}^2$ and $\overline{C}$ be its projective closure in $\mathbb{C}P^2$. Denote their complements by M and $\overline{M}$, respectively. Then we have

$$\chi(\overline{M}) = \chi(M) - n + 2,$$

where $\chi(M)$ and $\chi(\overline{M})$ are the Euler characteristics and $n = \#(L_\infty \cap \overline{C})$.

Proof. Let $U = \overline{M} \setminus L_\infty$, and V be the tubular neighborhood of $L_\infty \setminus (L_\infty \cap \overline{C})$. Then, U and V are open sets in $\mathbb{C}P^2$ and $U \cong M$. The intersection $U \cap V$ is the total space of S^1-bundle over $L_\infty \setminus (L_\infty \cap \overline{C})$. Since $L_\infty \setminus (L_\infty \cap \overline{C}) = \mathbb{C}P^1 \setminus \{n \text{ points}\}$, this bundle is trivial and thus $U \cap V = (L_\infty \setminus (L_\infty \cap \overline{C})) \times S^1$. The union $U \cup V \cong \overline{M}$. Thus, from the Mayer-Vietoris exact sequence, we have the following:

$$\begin{aligned} \chi(U \cup V) &= \chi(U) + \chi(V) - \chi(U \cap V), \\ \chi(\overline{M}) &= \chi(M) - n + 2. \end{aligned}$$

□

6.3 Zariski pairs

A pair of projective plane algebraic curves (C_1, C_2) is called a Zariski pair if their tubular neighborhoods $T(C_1)$ and $T(C_2)$ are homeomorphic but there is no homeomorphism $(\mathbb{C}P^2, C_1) \to (\mathbb{C}P^2, C_2)$. The first example of a Zariski pair was given in [Za29]. In that example, two sextic curves with six cusps were considered. One has all six cusps on the same conic, while the other does not. Although they have diffeomorphic tubular neighborhoods, they have different embedded topology because the fundamental groups of the complements are not isomorphic. Their fundamental groups were explicitly computed by Oka [O92]. Since then, a lot of examples have been found and Zariski pairs have become important objects in studying the topology of algebraic curves [ACT08]. It is also known that there exists a Zariski pair such that the fundamental groups of the complements are isomorphic [Shim03, Shir17]. Such a pair is called a π_1-equivalent Zariski pair. The π_1-equivalent Zariski pairs are primarily studied by using algebro-geometric invariants. On the other hand, the main result of this thesis describes the diffeomorphism type of the complement. We expect that our result can be applied to studying Zariski pairs.

Problem 6.5. *Find a Zariski pair by using handle decompositions or Kirby diagrams of the complement.*

6.4 Milnor fibers

Some Milnor fibers are related to the complements of plane curves. Let $Q(x, y, z)$ be a homogeneous complex polynomial of degree n and $F = \{(x, y, z) \in \mathbb{C}^3 \mid Q(x, y, z) = 1\}$ be the Milnor fiber. Since Q is homogeneous, it also defines a projective plane curve $\overline{C} = \{[x : y : z] \in \mathbb{C}P^2 \mid Q(x, y, z) = 0\}$. It is known that the Hopf fibration $\pi : \mathbb{C}^3 \setminus \{(0, 0, 0)\} \to \mathbb{C}P^2$ induces a n-fold covering map $\pi : F \to \mathbb{C}P^2 \setminus \overline{C}$. Thus, the Milnor fiber is a covering space of a projective curve complement. When Q contains a linear factor, the projective curve decomposes into an affine plane curve C and line at infinity L_∞. Therefore the Milnor fiber is a covering space of the complement of an affine plane curve in that case.

Problem 6.6. *Using covering space theory, describe handle decompositions and Kirby diagrams for such Milnor fibers.*

Acknowledgements

First of all, I would like to express my sincere gratitude to Professor Naohiko Kasuya for his valuable advice and persistent support as my supervisor. I appreciate his careful review and comments on this thesis.

I would like to express my sincere gratitude to Professor Masahiko Yoshinaga. This research was preceded by a joint work with Professor Yoshinaga [SY22]. His advice and discussions during the joint work motivated this research.

I would like to express my gratitude to my colleagues and friends. The daily mathematical discussions with them have stimulated my research life.

I would like to express my sincere gratitude to my family, especially, to my wife, who has persistently supported and encouraged me.

My research is supported by the Japan Society for the Promotion of Science (JSPS) KAKENHI Grant Number 22KJ0114. Thanks to the financial support, I have been able to lead a fulfilling research life.

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