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Doctoral Dissertation

# Non-very generic intersections and their combinatorics

(ノンベリージジェネリック交叉とその組合せ論)

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# *Abstract*

Faculty of Science

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Doctor of Philosophy

**Non-very generic intersections and their combinatorics**

by Pragnya Das

The discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A})$ defined by Manin and Schechtman is an arrangement of hyperplanes constructed from a generic arrangement $\mathcal{A}$ of $n$ hyperplanes in a $k$-dimensional space generalizing the classical braid arrangement. In particular, $\mathcal{B}(n, 1, \mathcal{A})$ is the Braid arrangement.

The combinatorics of $\mathcal{B}(n, k, \mathcal{A})$ is known to be constant when $\mathcal{A}$ varies in an open Zariski set $\mathcal{Z}$ in the space of all generic arrangements. In particular $\mathcal{A}$ is called very generic if $\mathcal{A} \in \mathcal{Z}$ and non-very generic otherwise. In the later case, when $\mathcal{A}$ is a non-very generic arrangement, the combinatorics of the discriminantal arrangement can be studied by exploiting its relation with the special configurations of points in the space.

In this thesis, we begin by reviewing the definition of a discriminantal arrangement and some basic properties of its combinatorics.

We rewrite the Sylvester's and the orchard problem in terms of the combinatorics of $\mathcal{B}(n, 2, \mathcal{A})$ and then we build two arrangements of 12 lines in the projective real plane having, respectively, the minimum number of ordinary points and the maximum number of multiplicity 3 intersections.

Finally we distinguish between two types of non-very generic intersections and study the *Arithmetic non-very generic intersections* which can be classified using the combinatorics of the discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A})$.

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# Chapter 1

# Introduction

The discriminantal arrangement $\mathcal{B}(n,k,\mathcal{A})$ defined by Manin and Schechtman ( in 1987, see[15]) is an arrangement of hyperplanes constructed from a generic arrangement $\mathcal{A}$ of $n$ hyperplanes in a $k$-dimensional space generalizing the classical braid arrangement. In particular, $\mathcal{B}(n,1,\mathcal{A})$ is the Braid arrangement.

The combinatorics of $\mathcal{B}(n,k,\mathcal{A})$ is known to be constant when $\mathcal{A}$ varies in an open Zariski set $\mathcal{Z}$ in the space of all generic arrangements. In particular $\mathcal{A}$ is called very generic if $\mathcal{A} \in \mathcal{Z}$ and non-very generic otherwise. In the later case, when $\mathcal{A}$ is a non-very generic arrangement, the combinatorics of the dicriminantal arrangement can be studied by exploiting its relation with the special configurations of points in the space.

The study of special configurations of lines and points in the plane is a classic subject even if several problems have been solved only recently or are still open. In this thesis two of these problems are correlated to the combinatorics of the dicriminantal problem and those are the generalized Sylvester's problem and orchard problem.

In 1893 Sylvester [20] posed the problem "*prove that it is not possible to arrange any finite number of real points so that a right line through every two of them shall pass through a third, unless they all lie in the same right line*". This problem was solved, among others, in 1944 by Gallai (see [11]) and his solution is known nowadays as the Sylvester-Gallai Theorem. If we call a line passing trough exactly two points *ordinary*, a generalization of this problem is to count the minimum number of ordinary lines given $n$ points in $\mathbb{R}^2$. Green and Tao in 2013, [12], described the most advanced result about ordinary lines proving that if $n$ is sufficiently large, then there are at least $\left[\frac{n}{2}\right]$ ordinary lines. The Sylvester's problem can also be posed in its dual form: "given $n$ lines in the real projective plane, there must be a point of intersection of just two lines", and the point is called *ordinary point*. The dual of Green and Tao's Theorem states that $n$ lines in projective plane intersect in at least $\left[\frac{n}{2}\right]$ ordinary points if $n$ is large enough.

The orchard problem was posed around 200 years ago and asks which is the maximum number $t_3(n)$ of 3-point lines attainable with a configuration of $n$ points. Its dual form is equivalent to ask which is the maximum number of multiplicity 3 intersections attainable with a configuration of $n$ lines. So far the most advanced known result is due to Burr, Grünbaum and Sloane (see [13]) in 1974, where the authors provided a lower bound for $t_3(n)$.

In this thesis we connect those two problems to the study of the combinatorics of the discriminantal arrangement associated to non-very generic arrangements of $n$ lines in the affine and projective real plane. This part of my thesis is based on a joint work with E.Palezzato and S.Settepanella (see [6])

In [1], Athanasiadis provided a description of the combinatorics of dicriminantal arrangement $\mathcal{B}(n,k,\mathcal{A})$ when $\mathcal{A} \in \mathcal{Z}$ proving a conjecture by Bayer and Brandt, [2], who called $\mathcal{A}$ *very generic* if $\mathcal{A} \in \mathcal{Z}$ and *non-very generic* otherwise. While proving the conjecture by Bayer and Brandt, Athanasiadis also provided a condition needed to be satisfied by the combinatorics of the discriminantal arrangement in order for the arrangement to be very generic. We call this condition as the BBA-condition. We will keep those names throughout the thesis.

Few years before the work of Manin and Schechtman, Crapo [4] introduced a notion equivalent to the discriminantal arrangement, one starting from realizable matroids and called them *geometry of circuits*. In his paper Crapo provided an example of an arrangement $\mathcal{A}$ of 6 lines in the real plane such that $\mathcal{A} \notin \mathcal{Z}$, pointing out how the directions at infinity of the lines in such arrangement give rise to a special configuration of 6 points in the projective line.

In 2016, Libgober and Settepanella [14] gave a sufficient geometric condition for an arrangement $\mathcal{A}$ to be non-very generic. In particular, in the case $k = 3$, their result shows that multiplicity 3 codimension 2 intersections of hyperplanes in $\mathcal{B}(n,3,\mathcal{A})$ appear (i.e. $\mathcal{A} \notin \mathcal{Z}$) if and only if collinearity conditions for points at infinity of lines, intersections of certain planes in $\mathcal{A}$, are satisfied. Thus confirming what already pointed out by Crapo, i.e. the combinatorics of the discriminantal arrangement $\mathcal{B}(n,k,\mathcal{A})$ for $\mathcal{A} \notin \mathcal{Z}$ is strongly related to the problem of special configurations of points in the space. Two subsequent papers by Sawada, Yamagata and Settepanella (see [16], [17]) went in the same direction connecting the combinatorics of $\mathcal{B}(6,3,\mathcal{A})$, for $\mathcal{A} \notin \mathcal{Z}$, to the Pappus's hexagon Theorem and the Hesse configuration in the complex case.

The study of the combinatorics of the discriminantal arrangement $\mathcal{B}(n,k,\mathcal{A})$ when $\mathcal{A}$ is non-very generic, is connected to the study of special configurations of points in the space (see [4], [14], [16], [17], [18], [19]). Exploiting this connection, in this work we firstly rewrite the orchard problem in terms of the combinatorics of $\mathcal{B}(n,2,\mathcal{A})$ then, starting from the non-very generic arrangement described in [16] and [17], we build two arrangements of 12 lines in the projective real plane having, respectively, the minimum number of ordinary points and the maximum number of multiplicity 3 intersections. In particular the latter is combinatorially isomorphic to the well known Böröczky arrangement (see [3], [10]). Finally we provide a purely combinatorial way to build such examples and a conjecture which, if proved, would allow to build arrangements of $n$ lines in the real projective plane with the minimum number of ordinary points or the maximum number of multiplicity 3 intersections starting from a non-very generic arrangement of $m(<< n)$ lines.

The content of the thesis is as follows. In Chapter 2, we recall the definition of the discriminantal arrangement and few basic known results. We also discuss about the various results known till now about the combinatorics of the discriminantal arrangement when $\mathcal{A} \in \mathcal{Z}$ and otherwise.

In Chapter 3, which is based on a joint work with E.Palezzato and S.Settepanella (see [6]), we describe how the orchard problem can be stated in terms of the combinatorics of $\mathcal{B}(n,2,\mathcal{A})$, with $\mathcal{A}$ non-very generic arrangement in the real plane, we provide a geometric way to construct, starting from a non-very generic arrangement of 6 lines in the projective plane, the two examples of configurations of 12 lines having, respectively, minimum number of ordinary points and maximum number of multiplicity 3 intersections, and finally we provide a purely combinatorial way to build the same configurations and we state the main theorem of the thesis and a conjecture.

We describe a purely combinatorial way to build, from a non-very generic arrangement of 6 planes in the real space, two special configuration of 12 lines in the projective real plane one with minimum number of ordinary points and one with maximum number of multiplicity 3 intersections.

In Chapter 4, which is based on a joint work with T.Saito and S.Settepanella (see [7]), we discuss the *Bayer-Brandth-Athanasiadis*-condition provided by Athanasiadis in [1] and provide conditions required for the existence of *Arithmetic non-very generic* intersection in the discriminantal arrangement $\mathcal{B}(n,k,\mathcal{A})$.

# Chapter 2

# Preliminaries

## 2.1 Prerequisites

Throughout this dissertation we will be using the following notation:

| | |
|---|---|
| $\mathbb{N}$ | the set of Natural number>0 |
| $\mathbb{Z}$ | the set of integers |
| $\mathbb{R}$ | the set of real numbers |
| $\mathbb{C}$ | the set of complex numbers |
| $V = \mathbb{K}^n$ | $V$ is an affine space, $\mathbb{K}$ is a field and in this dissertation $\mathbb{K} = \mathbb{R}, \mathbb{C}$ |
| $\mathbb{P}(V)$ | the projective space of the vector space $V$ |
| $[m]$ | the set $\{1, \ldots, m\}$ where m is a natural number. |

A hyperplane $H$ is an $n-1$ dimensional affine subspace of an affine space $V$. That is $H = \{v \in V \colon \alpha \cdot v = 0\}$, where $\alpha$ is a fixed nonzero vector in $V$ and $\alpha \cdot v$ is the usual dot product:

$$(\alpha_1, \ldots, \alpha_n) \cdot (v_1, \ldots, v_n) = \sum_{i=1}^{n} \alpha_i v_i.$$

An *arrangement of hyperplanes* $\mathcal{A} = \{H_1, \ldots, H_m\}$ is a set of $m$ hyperplanes in an affine space $V$, where $H_i \subset V = \mathbb{K}^n$. The dimension of $\mathcal{A} = \dim(V) = n$. The *rank* of a hyperplane arrangement $\mathcal{A}$ is defined as $\operatorname{rank}(\mathcal{A}) = \dim\langle \alpha_1, \ldots, \alpha_m \rangle$, where $\alpha_i$ is a vector orthogonal to $H_i$ and $\langle \alpha_1, \ldots, \alpha_m \rangle$ is the space spanned by $\alpha_1, \ldots, \alpha_m$. An arrangement $\mathcal{A}$ is called *essential* if $\operatorname{rank}(\mathcal{A}) = \dim(\mathcal{A})$.

An arrangement $\mathcal{A} = \{H_1, \ldots, H_m\}$ in $V = \mathbb{K}^n$ is called a *generic arrangement* if

- $\dim(H_{i_1} \cap \ldots \cap H_{i_p}) = n - p$ if $p \leq n$,
- $H_{i_1} \cap \ldots \cap H_{i_p} = \phi$ if $p > n$.

An arrangement $\mathcal{A}$ is called a central arrangement if $0 \in \bigcap_{H \in \mathcal{A}} H$.

The *intersection set* $\mathcal{L}(\mathcal{A})$ of an arrangement $\mathcal{A}$ is the set of all intersections of hyperplanes in $\mathcal{A}$ that is, $\mathcal{L}(\mathcal{A}) = \left\{X \subset V : X = \bigcap_{i \in I} H_i, I \subset [m]\right\} \cup \left\{V\right\}$. The intersection set is a poset with order $X, Y \in \mathcal{L}(\mathcal{A}), X \leq Y \iff X \supseteq Y$.

For a central arrangement $\mathcal{A}$ if $\mathcal{L}(\mathcal{A})$ is a lattice with $V$ as the minimum and $\bigcap_{i=1}^{m} H_i$ as the maximum elements, then $\mathcal{L}(\mathcal{A})$ is called the intersection lattice of $\mathcal{A} = \{H_1, \ldots, H_m\}$. The rank of an element $X \in \mathcal{L}(\mathcal{A})$ is $\mathrm{rank}(X) = \mathrm{codim}(X) = \dim(V) - \dim(X)$.

## 2.2 Preliminaries

### 2.2.1 Discriminantal arrangement

Let $\mathcal{A} = \{H_i^0, i = 1, ..., n\}$ be a generic arrangement in $\mathbb{C}^k, k < n$ i.e. a collection of hyperplanes such that $\mathrm{codim}(\bigcap_{i \in K, |K|=p} H_i^0) = p$. The space of parallel translates $\mathbb{S}(H_1^0, ..., H_n^0)$ (or simply $\mathbb{S}$ when the dependence on $H_i^0$ is clear or not essential) is the space of $n$-tuples $H_1, ..., H_n$ such that either $H_i \cap H_i^0 = \emptyset$ or $H_i = H_i^0$ for any $i = 1, ..., n$. One can identify $\mathbb{S}$ with the $n$-dimensional affine space $\mathbb{C}^n$ in such a way that $(H_1^0, ..., H_n^0)$ corresponds to the origin. In particular, an ordering of hyperplanes in $\mathcal{A}$ determines the coordinate system in $\mathbb{S}$ (see [14]).

We will use the compactification of $\mathbb{C}^k$ viewing it as $\mathbb{P}^k(\mathbb{C}) \setminus H_\infty$ endowed with collection of hyperplanes $\bar{H}_i^0$ which are projecive closures of affine hyperplanes $H_i^0$.

Given a generic arrangement $\mathcal{A}$ in $\mathbb{C}^k$ formed by hyperplanes $H_i, i = 1, ..., n$ *the trace at infinity*, denoted by $\mathcal{A}_\infty$, is the arrangement formed by hyperplanes $H_{\infty,i} = \bar{H}_i^0 \cap H_\infty$ in $H_\infty$.

The trace $\mathcal{A}_\infty$ of an arrangement $\mathcal{A}$ determines the space of parallel translates $\mathbb{S}$ (as a subspace in the space of $n$-tuples of hyperplanes in $\mathbb{P}^k$) (see [15]).

Fixed a generic arrangement $\mathcal{A}$, consider the closed subset of $\mathbb{S}$ formed by those collections which fail to form a generic arrangement. This subset of $\mathbb{S}$ is a union of hyperplanes $D_L \subset \mathbb{S}$ (see [15]). Each hyperplane $D_L$ corresponds to a subset $L = \{i_1, \ldots, i_{k+1}\} \subset [n] := \{1, \ldots, n\}$ and it consists of $n$-tuples of translates of hyperplanes $H_1^0, \ldots, H_n^0$ in which translates of $H_{i_1}^0, \ldots, H_{i_{k+1}}^0$ fail to form a general position arrangement. The arrangement $\mathcal{B}(n, k, \mathcal{A})$ of hyperplanes $D_L$ is called *discriminantal arrangement* and has been introduced by Manin and Schechtman in [15]. Notice that $\mathcal{B}(n, k, \mathcal{A})$ depends on the trace at infinity $\mathcal{A}_\infty$ and hence it is sometimes more properly denoted by $\mathcal{B}(n, k, \mathcal{A}_\infty)$.

We can also explain the discriminantal arrangement in the following way; let $\alpha_i = (a_{i1}, \dots, a_{ik})$ be the normal vectors of hyperplanes $H_i$, $1 \leq i \leq n$, in the generic arrangement $\mathcal{A}$ in $\mathbb{C}^k$. Normal here is intended with respect to the usual dot product

$$(a_1, \dots, a_k) \cdot (v_1, \dots, v_k) = \sum_i a_i v_i.$$

Then the normal vectors to the hyperplanes $D_L$, $L = \{s_1 < \cdots < s_{k+1}\} \subset [n]$ in $\mathbb{S} \simeq \mathbb{C}^n$ are nonzero vectors of the form

$$\alpha_L = \sum_{i=1}^{k+1} (-1)^i \det(\alpha_{s_1}, \dots, \hat{\alpha_{s_i}}, \dots, \alpha_{s_{k+1}}) e_{s_i}, \tag{2.1}$$

where $\{e_j\}_{1 \leq j \leq n}$ is the standard basis of $\mathbb{C}^n$ (cf. [2]).

### 2.2.2 Combinatorics of the discriminantal arrangement

It is well known (see, among others [4],[15]) that there exists an open Zariski set $\mathcal{Z}$ in the space of generic arrangements of $n$ hyperplanes in $\mathbb{R}^k(\mathbb{C}^k)$, such that the intersection lattice of the discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A})$ is independent from the choice of the arrangement $\mathcal{A} \in \mathcal{Z}$. Bayer and Brandt in [2] call the arrangements $\mathcal{A} \in \mathcal{Z}$ *very generic* and the ones which are not in $\mathcal{Z}$, *non-very generic*. We will use their terminology in the rest of this paper. The name very generic is motivated by the fact that, in this case, the number of intersection of rank $r$ in the intersection lattice $\mathcal{L}(\mathcal{B}(n, k, \mathcal{A}))$ is the largest possible for any $r$ and any arrangement $\mathcal{A}$ of $n$ hyperplanes in $\mathbb{R}^k(\mathbb{C}^k)$.

In [4] Crapo proved that the intersection lattice of $\mathcal{B}(n, k, \mathcal{A}), \mathcal{A} \in \mathcal{Z}$ is isomorphic to the Dilworth completion of the $k$-times lower-truncated Boolean algebra $B_n$ ( see Theorem 2. page 149 ). A more precise description of this lattice in the real case is due to Athanasiadis who proved in [1] a conjecture by Bayer and Brandt which stated that the intersection lattice of the discriminantal arrangement is isomorphic to the collection of all sets $\{S_1, \dots, S_m\}$, where $S_i$ are subsets of $[n] = \{1, \dots, n\}$, each of cardinality at least $k + 1$, such that

$$\mid \bigcup_{i \in I} S_i \mid > k + \sum_{i \in I} (\mid S_i \mid - k) \text{ for all } I \subset [m] = \{1, \dots, m\}, \mid I \mid \geq 2. \tag{2.2}$$

We call the condition provided in Equation 2.2 as the Bayer-Brandth-Athanasiadis condition or the *BBA*-condition.

The isomorphism is the natural one which associate to the set $S_i$ the space $D_{S_i} = \bigcap_{L\subset S_i,|L|=k+1} D_L$ intersection of all hyperplanes in the discriminantal arrangement indexed in $S_i$. In particular if the condition in Equation (2.2) is satisfied, this implies that the subspaces $D_{S_i}, i = 1, \dots, m$ intersect transversally or, equivalently, that rank $\bigcap_{i=1}^{m} D_{S_i} = \sum_{i=1}^{m}(| S_i | -k)$ being $\operatorname{rank}(D_{S_i}) =| S_i | -k$ (as proved in Corollary 3.6 in [1]).

Notice that while there are two different descriptions of the intersection lattice in the very generic case, very few is known about it in the non-very generic case.

An intersection in $\mathcal{B}(n, k, \mathcal{A})$ when $\mathcal{A} \in \mathcal{Z}$ is called a very generic intersection and an intersection which doesn't exist in $\mathcal{B}(n, k, \mathcal{A})$ when $\mathcal{A} \in \mathcal{Z}$ is called a non-very generic intersection. In [1], by proving a conjecture by Bayer and Brandt (see [2]), Athanasiadis provided a complete description of the combinatorics of the discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A})$ when $\mathcal{A}$ is very generic. That is, they provided a condition which is satisfied by the combinatorics of all the intersections in $\mathcal{L}(\mathcal{B}(n, k, \mathcal{A}))$ when $\mathcal{A}$ is a very generic arrangement which we call the Bayer-Brandt-Athanasiadis condition or the BBA-condition. The BBA-condition is a global condition for very genericity, and using BBA-condition it is not possible to determine locally whether an intersection in a discriminantal arrangement is very generic. Untill recently, for all the known examples of non-very generic arrangements $\mathcal{A}$, the BBA-condition was not satisfied by the combinatorics of any non-very generic intersections in $\mathcal{L}(\mathcal{B}(n, k, \mathcal{A}))$.

One of the very first examples of such a non-very generic arrangement is the Crapo's construction of 6 lines in $\mathbb{R}^2$ (see [5]) where the combinatorics of all the intersections in $\mathcal{L}(B(6, 2, \mathcal{A}))$ fails the BBA-condition. For example, the construction of this arrangement (see Figure 2.1) provides a non-very generic intersection $X = \bigcap_{i=1}^{4} D_{L_i}$ of multiplicity 4 in rank 3 in $\mathcal{L}(\mathcal{B}(6, 2, \mathcal{A}))$, where $D_{L_i}$ are hyperplanes in $\mathcal{B}(6, 2, \mathcal{A})$ and $L_1 = \{1, 2, 3\}, L_2 = \{1, 4, 6\}, L_3 = \{3, 4, 5\}, L_4 = \{2, 5, 6\}$ are the set of index subsets. Here the set $\{L_1, L_2, L_3, L_4\}$ doesn't satisfy the BBA-condition.

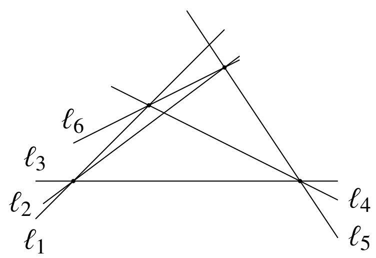


Figure 2.1

Contrary to the Crapo's construction, Yamagata and Settepanella provided two examples of non-very generic arrangements (see Examples 4.2 and 4.3 in [19]) containing intersection in $\mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ whose combinatorics satisfy the BBA-condition. That is to say, Example 4.2 in [19] is an arrangement of 16 hyperplanes in $\mathbb{R}^{11}$. The construction of this example provided an intersection $X = \bigcap_{i=1}^{4} D_{L_i}$ of multiplicity 4 in rank 3 in $\mathcal{L}(\mathcal{B}(16,11,\mathcal{A}))$, where $D_{L_i}$ are hyperplanes in $\mathcal{B}(16,11,\mathcal{A})$ and $L_1 = [16]\setminus\{13,14,15,16\}, L_2 = [16]\setminus\{9,10,11,12\}, L_3 = [16]\setminus\{5,6,7,8\}, L_4 = [16]\setminus\{1,2,3,4\}$ are the set of index subsets. The set $\{L_1, L_2, L_3, L_4\}$ satisfies the BBA-condition. The authors also constructed another example, Example 4.3, an arrangement with 10 hyperplanes in $\mathbb{R}^3$. The construction of this example provided an intersection $X = \bigcap_{i=1}^{5} D_{L_i}$ of multiplicity 5 in rank 4 in $\mathcal{L}(\mathcal{B}(10,3,\mathcal{A}))$, where $D_{L_i}$ are hyperplanes in $\mathcal{B}(10,3,\mathcal{A})$ and $L_1 = \{1,2,3,4\}, L_2 = \{1,5,6,7\}, L_3 = \{2,5,8,9\} L_4 = \{3,6,8,9\}, L_5 = \{4,7,9,10\}$ are the set of index subsets. The set $\{L_1, L_2, L_3, L_4, L_5\}$ satisfies the BBA-condition. The authors showed that, for both these examples, there exists a non-very generic intersection whose combinatorics satisfy the BBA-condition. That is, the failing of BBA-condition can only be sufficient but not necessary for the existence of non-very generic intersection and hence for an arrangement to be non-very generic.

Taking motivation from the above examples, we distinguish between the two types of non-very generic intersections. We call a non-very generic intersection an *arithmetic non-very generic* or *ANVG* intersection if its combinatorics doesn't satisfies the BBA-condition. On the other hand, we call a non-very generic intersection a *geometric non-very generic* or *GNVG* intersection if it satifies the BBA-condition. For example, the Crapo's construction (see [4] and refer to 2.1) contains only ANVG intersections while the two non-very generic intersections (in Examples 4.2 and 4.3) provided in [19] are GNVG intersection.

**Remark 2.2.1.** *We call an arrangement arithmetic non-very generic or ANVG arrangement if all its non-very generic intersections in $\mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ are ANVG intersections. On the other hand, we call an arrangement geometric non-very generic or GNVG arrangement if there exists a non-very generic intersection in $\mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ which is a GNVG intersection.*

Furthermore, in Lemma 4.1.5, we specify the conditions on an intersection $X \in \mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ such that the failing of the BBA-condition by the combinatorics of $X$ is necessary and sufficient for $X$ to be a non-very generic intersection. In particular, we study the conditions required for a non-very generic intersection in $\mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ to be an ANVG intersection.

# Chapter 3

# Generalised orchard and Sylvester's problem via discriminantal arrangements

## 3.1 The orchard problem and the combinatorics of $\mathcal{B}(n,2,\mathcal{A})$.

The study of maximum number of intersections of multiplicity 3 can be rewritten in terms of combinatorics of the discriminantal arrangement $\mathcal{B}(n,2,\mathcal{A})$. In this section we introduce the notation and the definitions needed to formulate the orchard problem in terms of combinatorics of the discriminantal arrangement.

### 3.1.1 Non-very generic intersections

Let $\mathcal{A}$ be a generic arrangement of $n$ lines in $\mathbb{R}^2$ (equivalently $\mathbb{C}^2$) and denote by $\mathcal{L}_r(\mathcal{B}(n,2,\mathcal{A}))$ the rank $r$ elements in the intersection lattice of $\mathcal{B}(n,2,\mathcal{A})$. In this case hyperplanes $D_L \in \mathcal{B}(n,2,\mathcal{A})$ are indexed by subsets $L \subset [n]$ ($\mid L \mid = 3$). By combinatorics of very generic discriminantal arrangement, we get that $r$ hyperplanes $D_{L_i}(i = 1,\dots,r)$ either intersect transversally or there exists a partition $\{S_1,...,S_m\}$ of $\{1,\dots,r\}, |S_i| \geq 3$ which satisfies Athanasiadis condition (in Equation (2.2)) such that $\cap_{i=1}^{r} D_{L_i} = \cap_{i=1}^{m} D_{S_i}$ . Notice that in this case $D_{S_i}$ intersects transversally, and $\mathrm{rank}(\cap_{i=1}^{r} D_{L_i}) = \sum_{i=1}^{m}(\mid S_i \mid -2)$ (see Corollary 3.6 in [1]). On the other hand, in the non-very generic case we can have $r$ hyperplanes which do not intersect transversally even if for any subset $I \subset [r], \mid I \mid \geq 2$ we have the intersection of hyperplanes $\cap_{i\in I} D_{L_i} \neq D_K \in \mathcal{L}(\mathcal{B}(n,2,\mathcal{A}))$ where $K \subset [n], \mid K \mid > 3$. We recall the definition of simple intersection given in [18].

**Definition 3.1.1.** *An element $X \in \mathcal{L}_r(\mathcal{B}(n,2,\mathcal{A}))$ is called a simple intersection if $X = \cap_{i=1}^m D_{L_i}$, $D_{L_i} \in \mathcal{B}(n,2,\mathcal{A})$, $m \geq r$ and for every subset $I \subset [r], \mid I \mid \geq 2$ we have the intersection of hyperplanes $\cap_{i\in I} D_{L_i} \neq D_K \in \mathcal{L}(\mathcal{B}(n,2,\mathcal{A}))$ where $K \subset [n], \mid K \mid > 3$. In particular if $m > r$ we call $X$ a non-very generic simple intersection. We call $m$ the multiplicity of $X$.*

**Remark 3.1.2.** *Notice that an element $\mathcal{A}'$ in a simple intersection $X \in \mathcal{L}_r(\mathcal{B}(n,2,\mathcal{A}))$ corresponds to a translate of the generic arrangement $\mathcal{A}$ containing only double and triple intersection points. Indeed an arrangement $\mathcal{A}'$ of $n$ lines in $\mathbb{R}^2(\mathbb{C}^2)$ in which lines $l_i$'s intersect in points $P = \cap_{i\in K} l_i$, $\mid K \mid > 3$ belongs to $D_K = \underset{\substack{L\subset K \\ |L|=3}}{\cap} D_L$ by definition.*
*Vice versa if a translate $\mathcal{A}'$ of a generic arrangement $\mathcal{A}$ of $n$ lines in $\mathbb{R}^2(\mathbb{C}^2)$ contains lines intersecting only in double points and triple points $P_i(i = 1,\ldots,m)$ of the form $P_i = \cap_{j\in L_i} l_j (L_i \subset [n], \mid L_i \mid = 3)$ then $\mathcal{A}'$ is an element in the simple intersection $X = \cap_{i=1}^m D_{L_i} \in \mathcal{L}_r(\mathcal{B}(n,2,\mathcal{A}))$.*

Based on the above Remark 3.1.2 and the description of the combinatorics of very generic arrangement provided by Athanasiadis, the following proposition holds.

**Proposition 3.1.3.** *If a generic arrangement $\mathcal{A}$ of $n$ lines in $\mathbb{R}^2(\mathbb{C}^2)$ is very generic, then all simple intersection $X = \cap_{i=1}^m D_{L_i} \in \mathcal{L}_r(\mathcal{B}(n,2,\mathcal{A}))$ have multiplicity $m$ equals to their rank $r$.*

*Proof.* If $\mathcal{A}$ is a very generic arrangement of $n$ lines in $\mathbb{R}^2(\mathbb{C}^2)$ then any simple intersection $X = \cap_{i=1}^m D_{L_i} \in \mathcal{L}_r(\mathcal{B}(n,2,\mathcal{A}))$ satisfies the condition $r = \operatorname{rank}(\cap_{i=1}^m D_{L_i}) = \sum_{i=1}^m (\mid L_i \mid -k) = \sum_{i=1}^m (3-2) = m$ (see corollary 3.6 in [1]), that is $r = m$.

□

By the definition of simple intersection $X \in \mathcal{L}_r(\mathcal{B}(n,2,\mathcal{A}))$, it follows that if the multiplicity $m$ is $m = r$ then $X$ is transversal intersection of exactly $m = r$ hyperplanes, that is $m = r$ is the minimum value of $m$ when the arrangement $\mathcal{A}$ varies among all generic arrangements of $n$ lines in $\mathbb{R}^2$ ( equivalently $\mathbb{C}^2$). One could ask which is the maximum value for $m$. Proposition 3.1.3 states that when the arrangement $\mathcal{A}$ is very generic then all simple intersections $X \in \mathcal{L}_r(\mathcal{B}(n,2,\mathcal{A}))$ have multiplicity $m = r$. We can also deduce from proposition 3.1.3 that if $\mathcal{A}$ is an arrangement such that $\mathcal{L}_r(\mathcal{B}(n,2,\mathcal{A}))$ admits simple intersections of multiplicity $m > r$, then $\mathcal{A}$ has to be non-very generic. While the fact that $m = r$ is the minimum value follows trivially from the definition of simple intersection, to study the maximum value for $m$ is quite difficult but interesting problem. Indeed the following proposition holds.

**Proposition 3.1.4.** *A generic arrangement $\mathcal{A}$ of $n$ lines in $\mathbb{R}^2(\mathbb{C}^2)$ admits a translate $\mathcal{A}'$ which is an arrangement of $n$ lines in $\mathbb{R}^2(\mathbb{C}^2)$ with maximum number of multiplicity 3 intersection points if and only if it exists a simple intersection $X \in \mathcal{L}_{n-3}(\mathcal{B}(n,2,\mathcal{A}))$ with maximum multiplicity $m$, i.e., for any generic arrangement $\overline{\mathcal{A}}$ of $n$ lines in $\mathbb{R}^2(\mathbb{C}^2)$ and simple intersections $\overline{X} \in \mathcal{L}_{n-3}(\mathcal{B}(n,2,\overline{\mathcal{A}}))$, multiplicity of $\overline{X}$ is equal to or smaller than $m$.*

*Proof.* By Remark 3.1.2 we know that an arrangement of $n$ lines in $\mathbb{R}^2(\mathbb{C}^2)$ with only double and triple intersection points is an element in a simple intersection $X = \cap_{i=1}^{m} D_{L_i} \in \mathcal{L}_r(\mathcal{B}(n,2,\mathcal{A}))$ with multiplicity $m$ exactly equals to the number of triple intersection points. Hence the maximum number of triple points that an arrangement of $n$ lines in $\mathbb{R}^2(\mathbb{C}^2)$ with only double and triple intersection can have is exactly equal to the maximum multiplicity that a simple intersection can have when both rank $r$ and $\mathcal{A}$ vary. Since $n-3$ is the maximal rank in which non central arrangements appear, then the proof follows. □

Proposition 3.1.4 states the equivalence between the problem of studying simple non-very generic intersections of $\mathcal{B}(n,2,\mathcal{A})$ and the orchard problem.

**Example 3.1.5.** *Consider the arrangement $\mathcal{A}^t$ shown in Figure 3.1, translate of a generic arrangement $\mathcal{A}$. Lines $\ell_1,\dots,\ell_7$ in $\mathcal{A}^t$ have normal vectors*

$$\alpha_1 = (-2,2), \alpha_2 = (-3,4), \alpha_3 = (0,6), \alpha_4 = (2,4), \alpha_5 = (3,2), \alpha_6 = (-1,2) \text{ and } \alpha_7 = (-1.2,1.8).$$

*Then $\mathcal{A}^t$ is an element in the non-empty simple intersection $X = \cap_{L\in I} D_L$, $I = \{\{1,2,3\},\{1,4,6\},\{1,5,7\},\{2,4,7\}\{2,5,6\},\{3,4,5\}\}$. $X$ is a simple non-very generic intersection in rank $7-3=4$ having multiplicity $6$, the maximum multiplicity attainable for such intersections when $\mathcal{A} \notin \mathcal{Z}$.*

## 3.2 Construction of 12 lines with 19 triple points from Pappus's configuration.

In this section we will consider a generic arrangement $\mathcal{A}$ of 6 planes in $\mathbb{R}^3$ and its trace at infinity $\mathcal{A}_\infty$, a generic arrangement of lines in projective plane.

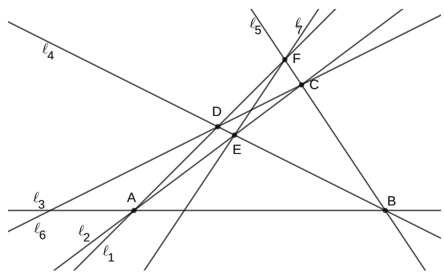


FIGURE 3.1: Seven lines arrangement

We begin by considering a generic arrangement $\mathcal{A}$ of 6 planes in $\mathbb{R}^3$ whose trace at infinity $\mathcal{A}_\infty = \{l_{i_1}, l_{i_2}, ..., l_{i_6}\}$ of 6 lines in $\mathbb{P}^2\mathbb{R}$ is represented in Figure 3.2(A). The lines in $\mathcal{A}_\infty$

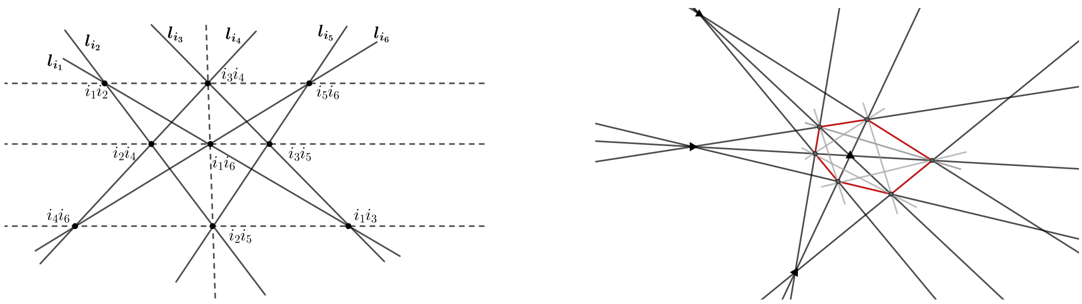


FIGURE 3.2: (A)-left: Generic arrangement $\mathcal{A}_\infty$ and (B)-right: the rank 2 section of its discriminantal arrangement $\mathcal{B}(6, 3, \mathcal{A}_\infty)$

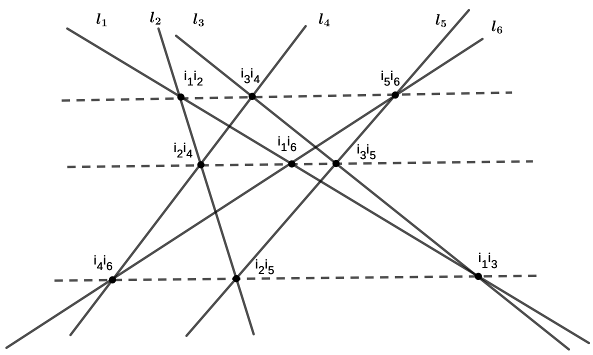


FIGURE 3.3: Arrangement $\mathcal{P}_\infty$ corresponding to Pappus's configuration with 3 collinearity conditions.

intersect in 15 double points satisfying 4 collinearity conditions. For obvious reasons in the rest of this section we will refer to this configuration as Pappus's configuration with 4 collinearities and denote it by $\mathcal{P}^c_\infty$, while $\mathcal{P}_\infty$ will denote the arrangement in Figure 3.3, i.e., the Pappus's configuration with 3 collinearities.

In [14] the authors proved that collinearity conditions in $\mathcal{A}_\infty$ correspond to multiplicity 3 intersections in $\mathcal{L}_2(\mathcal{B}(n, 3, \mathcal{A}_\infty))$. In particular a generic planar section of $\mathcal{B}(6, 3, \mathcal{P}^c_\infty)$ is represented in Figure 3.2(B).

Our goal in this section is to show how the study of arrangements of projective lines with collinearities is connected to the study of arrangements of lines with maximum number of triple points and that of arrangements of lines with minimum number of double points. We will do this studying the case of 12 lines in the plane builded along the following steps:

1. Consider the Pappus's configuration $\mathcal{P}^c_\infty$ (resp. $\mathcal{P}_\infty$ ) with 4 (resp. 3) collinearity conditions with the additional condition that the three lines corresponding to the three classical collinearities of Pappus's hexagon configuration are concurrent ( as depicted in Figure 3.4 where the three collinear lines are the parallel lines $l'_1, l'_3$ and the line at infinity $l'_2$).

2. Add to this arrangement six more lines $\mathcal{P}'_\infty = \{l'_1, ....., l'_6\}$ as follows:

   (a) lines $l'_1, l'_2, l'_3$ are the three concurrent lines added in correspondence of the three Pappus's collinearities;

   (b) lines $l'_4, l'_5, l'_6$ are added so that each one of them contains exactly two different double intersections of $\mathcal{P}^c_\infty$ (resp. $\mathcal{P}_\infty$ ) and that each double point is contained in only one line $l'_i, i = 1, \dots, 6$.

   We call the so built arrangement $\mathcal{P}'_\infty$ a *completion* of $\mathcal{P}^c_\infty$ (resp. $\mathcal{P}_\infty$ ).

There are 3 different completions $\mathcal{P}'_\infty$ of $\mathcal{P}^c_\infty$ and $\mathcal{P}_\infty$ . Indeed since lines in $\mathcal{P}^c_\infty$ ( resp. $\mathcal{P}_\infty$ ) intersect in 15 double points, 9 of which already belong to $l'_1, l'_2, l'_3$, the remaining 6 double points[1] can be partitioned in 3 different ways as sets of couples. Two different arrangements $\mathcal{P}_\infty \cup \mathcal{P}'_\infty$ are depicted in Figure 3.4 and Figure 3.6 while Figure 3.7 contains the arrangement $\mathcal{P}^c_\infty \cup \mathcal{P}'_\infty$ with the last possible alternative[2]. With above notations, the following proposition holds.

**Proposition 3.2.1.** *The arrangements of lines $\mathcal{P}_\infty \cup \mathcal{P}'_\infty$ are arrangements of 12 lines in $\mathbb{P}^2\mathbb{R}$ with maximum number of triple points and $\mathcal{P}^c_\infty \cup \mathcal{P}'_\infty$ is an arrangement of lines with minimum number of double points.*

[1] Notice that, by construction, any three of those 6 points cannot be collinear, i.e. the construction in (2)(b) gives rise to three distinct lines.

[2] In order to to simplify the figures we have chosen the line $l'_2$ to be the line at infinity.

*Proof.* It is well known that an arrangement of 12 lines in $\mathbb{P}^2\mathbb{R}$ has at most 19 triple points (see [13] for details) or at minimum 6 double points (see [12] for more details).

The arrangement $\mathcal{P}^c_\infty \cup \mathcal{P}'_\infty$ represented in Figure 3.7 has exactly 6 double intersections. Indeed an intersection of multiplicity 6 added to the 15 intersections of multiplicity 3 leaves exactly 6 intersections of multiplicity 2 by the classical formula

$$\binom{s}{2} = \sum_{k\geq 2} t_k \binom{k}{2},$$

where $s$ is the number of mutually distinct lines and $t_k$ is the number of intersection points of multiplicity $k$ (see Equation (1) in [8]).

In the case depicted in Figure 3.4, i.e., $\mathcal{P}_\infty \cup \mathcal{P}'_\infty$, if we denote by $P_{ij}$ the double point given by the intersection $l_i \cap l_j$, the line $l'_5$ is chosen to be the line passing through double points $P_{23}$ and $P_{15}$, $l'_6$ the one passing through $P_{36}$ and $P_{45}$ and, finally, $l'_4$ is the one containing $P_{14}$ and $P_{26}$. This construction adds 6 triple points to the 10 ones obtained adding the lines $l'_1, l'_2$ and $l'_3$, for a total of 16 triple points. What is left to prove is that whenever we join lines in this way, if we denote by $P'_{ij} = l'_i \cap l'_j$, the intersections $P'_{56}, P'_{45}$ and $P'_{46}$ are triple points, that is, more precisely, $P'_{56} \in l'_2, P'_{45} \in l'_1$ and $P'_{46} \in l'_3$. In order to prove this we remark that the line $l'_5$ is parallel to $l'_6$ since, by simple geometric consideration, we have that the trapezium(or trapezoid) formed by $l'_5, l'_6, l_3, l_5$ has the parallel sides $\overline{P_{23}P_{36}}$ and $\overline{P_{15}P_{45}}$ of equal length making it a parallelogram. As a consequence $l'_2 \cap l'_5 \cap l'_6$ is not empty, that is $P'_{56} \in l'_2$. The same argument applies to the proof that $P'_{45} \in l'_1$ ( resp. $P'_{46} \in l'_3$ ) as soon as we consider the same configuration choosing the line $l'_1$ ( resp. $l'_3$ ) as the line at infinity as depicted in the left configuration of Figure 3.5 (resp. right configuration of Figure 3.5). The same argument applies to the arrangement $\mathcal{P}_\infty \cup \mathcal{P}'_\infty$ depicted in Figure 3.6. □

The arrangement $\mathcal{P}_\infty \cup \mathcal{P}'_\infty$ is combinatorially isomorphic to the well known Böröczky arrangement (see Example 2 of [10] for reference). In [10] the construction is purely geometric based on choosing $n$ diagonals of regular $2n$-gons, while in our case we start with any six lines in Pappus's configurations and using a purely combinatorial method we achieve the arrangement $\mathcal{P}_\infty \cup \mathcal{P}'_\infty$ as illustrated in the next section.

In the next subsection we will provide a way to build the completion $\mathcal{P}'_\infty$ that depends on the combinatorics of $\mathcal{B}(6,3,\mathcal{P}^c_\infty)$ and $\mathcal{B}(6,3,\mathcal{P}_\infty)$. This gives a purely combinatorial construction that can be generalized to a conjecture which we state in Subsection 5.3.

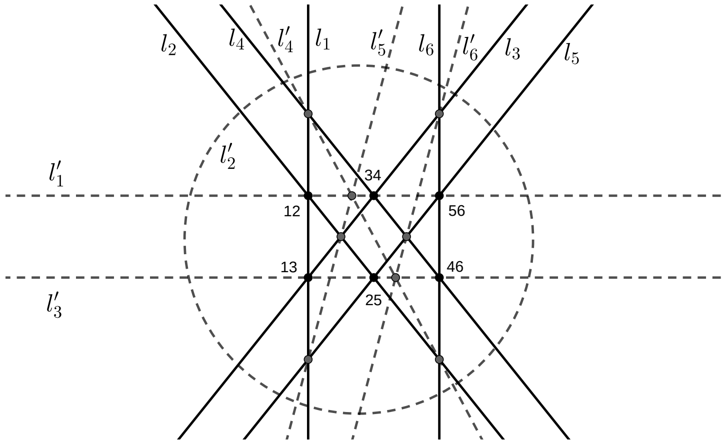


FIGURE 3.4: Arrangement $\mathcal{P}_\infty \cup \mathcal{P}'_\infty$ with 19 3-points ($P_{ij}$ is written as $ij$).

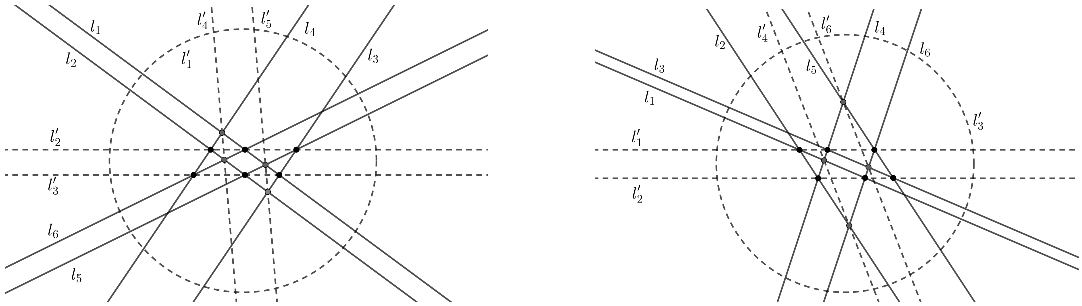


FIGURE 3.5: on the left $\mathcal{P}_\infty \cup \mathcal{P}'_\infty$ with $l'_1$ as line at infinity .On the right $l'_3$ is the line at infinity.

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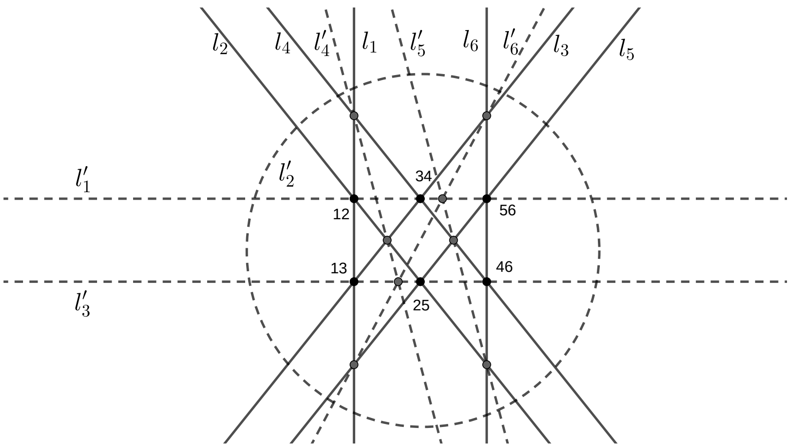


FIGURE 3.6: An arrangement $\mathcal{P}_\infty \cup \mathcal{P}'_\infty$ with 19 3-points. $P_{ij}$ is written as $ij$.

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### 3.2.1 Strong action on points $P \in \mathcal{L}_2(\mathcal{A}_\infty)$ and the $\sigma$-completion

Let's consider a generic arrangement $\mathcal{A}$ of $n$ planes in $\mathbb{R}^3$ with trace at infinity $\mathcal{A}_\infty = \{l_1, l_2, ..., l_n\}$ of $n$ lines in $\mathbb{P}^2\mathbb{R}$. Let $S \subseteq S_n$ be the subset of the symmetric group defined by

$$S = \{\sigma \in S_n \mid \sigma = \tau_1\tau_2....\tau_k, \quad \tau_i : \text{ disjoint transpositions}\} \quad .$$

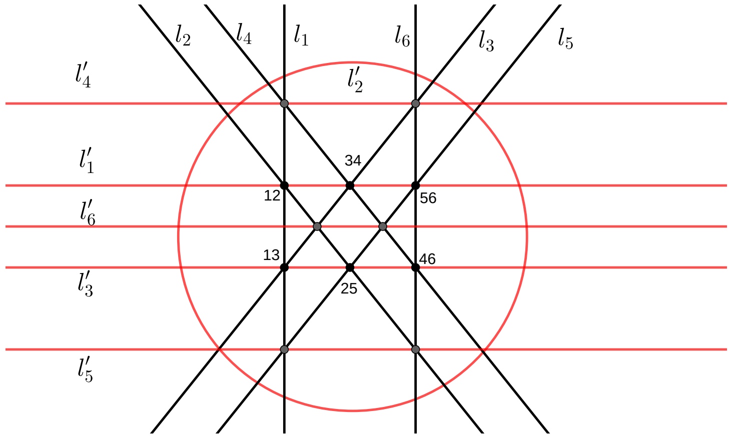


FIGURE 3.7: The arrangement $\mathcal{P}^c_\infty \cup \mathcal{P}'_\infty$ with 6 2-points. $P_{ij}$ is written as $ij$.

The elements $\sigma \in S$ acts naturally on elements $P = l_m \cap l_n \in \mathcal{L}_2(\mathcal{A}_\infty)$ as $\sigma.P = l_{\sigma.m} \cap l_{\sigma.n}$. We say that an element $\sigma \in S$ *acts strongly* or that the action of $\sigma$ is *strong* on $\mathcal{A}_\infty$ if it fixes non-trivial collinearities in $\mathcal{L}_2(\mathcal{A}_\infty)$. That is if the points $\{P_1, P_2, \ldots, P_m\}, m \geq 3$ in $\mathcal{L}_2(\mathcal{A}_\infty)$ are collinear and belonging to different lines, then $\sigma.P_i \in \{P_1, \ldots, P_m\}$ for any $i \in \{1, \ldots, m\}$. With the above notations, the following lemma holds.

**Lemma 3.2.2.** *An element $\sigma \in S \subseteq S_6$ acts strongly on $\mathcal{L}_2(\mathcal{P}_\infty)$ if and only if $\sigma$ is a product of exactly three transpositions each one fixing a point in one of the three distinct collinearities.*

*Proof.* Let the three classical collinearity conditions in the Pappus's configuration be $C_1 = \{P_{i_1i_2}, P_{i_3i_4}, P_{i_5i_6}\}$, $C_2 = \{P_{i_2i_4}, P_{i_1i_6}, P_{i_3i_5}\}$ and $C_3 = \{P_{i_1i_3}, P_{i_2i_5}, P_{i_4i_6}\}$ as depicted in Figure 3.3.

Assume that $\sigma$ fixes a line of $\mathcal{P}_\infty$. Without loss of generalities, we can assume that $\sigma$ fixes $l_{1_1}$. Then for any point $P_{i_1j} \in l_{i_1}$, we have $\sigma.P_{i_1j} = P_{i_1k} \in l_{i_1}$, for some $k \neq j$. As a consequence, $\sigma$ cannot act strongly on all non-trivial collinearities as no collinearity contains two points in the same line in $\mathcal{P}_\infty$. It follows that if $\sigma$ is strong then $\sigma$ cannot fix any line, i.e., it is a product of exactly three disjoint transpositions.

Moreover, if $\sigma$ fixes all the three points in a non-trivial collinearity then it is easy to check that $\sigma$ sends at least one point of one collinearity to a point in another, that is, $\sigma$ does not act strongly on $\mathcal{L}_2(\mathcal{P}_\infty)$.

Analogously, it is not difficult to check that if $\sigma$ permutes all the three points in a non-trivial collinearity then $\sigma$ will be a product of non-disjoint transpositions. So $\sigma$ has to be the product of three disjoint transpositions that fixes exactly one point in each collinearity. □

Let $\mathcal{A}$ be a generic arrangement of $n$ planes in $\mathbb{R}^3(\mathbb{C}^3)$ with trace at infinity $\mathcal{A}_\infty$, and $\sigma \in S_n$ an element that acts strongly on $\mathcal{A}_\infty$. We call *$\sigma$-completion* of $\mathcal{A}_\infty$, and denote it by $\mathcal{A}_\infty^\sigma$, the arrangement $\mathcal{A}_\infty^\sigma = \{l'_1, \ldots, l'_m\}$ satisfying the following conditions:

- $l'_i$ is the line $P\sigma.P$, where $P \in \mathcal{L}_2(\mathcal{A}_\infty)$.
- For any point $P \in \mathcal{L}_2(\mathcal{A}_\infty)$, there exists exactly one line $l'_i \in \mathcal{A}_\infty^\sigma$ such that $P \in l'_i$.

The Lemma 3.2.2 implies that we have 3 different $\sigma$-completions $\mathcal{P}_\infty^\sigma$ of $\mathcal{P}_\infty$.

As a consequence, for any given permutation $(i_1, i_2, \ldots, i_6)$, of the indices $\{1, \ldots, 6\}$, the set of $\sigma$ acting strongly on $\mathcal{P}_\infty$ is

$$S_{\mathcal{P}_\infty} = \{\sigma_1 = \tau_{i_1 i_2}\tau_{i_3 i_5}\tau_{i_4 i_6}, \sigma_2 = \tau_{i_1 i_3}\tau_{i_2 i_4}\tau_{i_5 i_6}, \sigma_3 = \tau_{i_1 i_6}\tau_{i_2 i_5}\tau_{i_3 i_4}\}.$$

The collinearities are $C_1 = \{P_{i_1 i_2}, P_{i_3 i_4}, P_{i_5 i_6}\}, C_2 = \{P_{i_2 i_4}, P_{i_1 i_6}, P_{i_3 i_5}\}$ and $C_3 = \{P_{i_1 i_3}, P_{i_2 i_5}, P_{i_4 i_6}\}$ as depicted in Figure 3.3. Remark that $\sigma_3$ is the only element which acts strongly on $\mathcal{P}_\infty^c$ too and the following proposition, equivalent to Proposition 3.2.1 holds.

**Proposition 3.2.3.** *If $\mathcal{P}_\infty$ and $\mathcal{P}_\infty^c$ satisfy the additional condition that three collinearities of the classical Pappus's configuration are concurrent, then for $\sigma \in S_{\mathcal{P}_\infty}$ we have that*

1. *$\mathcal{P}_\infty^c \cup \mathcal{P}_\infty^\sigma$ is an arrangement with minimum number of 2-points if $\sigma$ is the element that acts strongly on $\mathcal{P}_\infty^c$,*
2. *$\mathcal{P}_\infty \cup \mathcal{P}_\infty^\sigma$ is an arrangement with maximum number of 3-points if $\sigma$ doesn't act strongly on $\mathcal{P}_\infty^c$.*

### 3.2.2 Strong action on $\mathcal{B}(n, 3, \mathcal{A}_\infty)$ and the line arrangement $\mathcal{A}^\sigma$

The results in Subsection 3.2.1 can be easily expressed in terms of the combinatorics of the discriminantal arrangement $\mathcal{B}(n, k, \mathcal{A}_\infty)$. The symmetric group $S_n$ acts naturally on the hyperplanes $D_L \in \mathcal{B}(n, k, \mathcal{A}_\infty)(L = \{s_1, \ldots, s_{k+1}\} \subset [n])$ as

$$\sigma.D_L = D_{\sigma.L} \in \mathcal{B}(n, k, \mathcal{A}_\infty),\ \sigma.L = \{\sigma.s_1, \ldots, \sigma.s_{k+1}\}.$$

Moreover, since each collinearity condition in $\mathcal{A}_\infty$ corresponds to a point $P$ of multiplicity 3 in $\mathcal{L}_2(\mathcal{B}(n,3,\mathcal{A}_\infty))$ (see [14]), $\sigma$ acts strongly on $\mathcal{A}_\infty$ if and only if it fixes the corresponding intersection $P \in \mathcal{L}_2(\mathcal{B}(n,3,\mathcal{A}_\infty))$[3]. Accordingly, we will say that $\sigma$ acts strongly on $\mathcal{B}(n,3,\mathcal{A}_\infty)$ if it fixes all triple intersections in $\mathcal{L}_2(\mathcal{B}(n,3,\mathcal{A}_\infty))$. Notice that this definition too has meaning only when the arrangement $\mathcal{A}$ is non-very generic.

In order to rewrite Proposition 3.2.3 only in terms of the combinatorics of the discriminantal arrangement we need two more steps. First we give the following definition.

**Definition 3.2.4.** *Two intersections $P, P' \in \mathcal{L}_2(\mathcal{B}(n,3,\mathcal{A}_\infty))$ are independent if and only if any hyperplane $D_L \in \mathcal{B}(n,3,\mathcal{A}_\infty)$ which contains $P$ does not contain the point $P'$.*
*Given three pairwise independent intersections $P_i \in \mathcal{L}_2(\mathcal{B}(n,3,\mathcal{A}_\infty))$ the intersection $P \in \mathcal{L}_2(\mathcal{B}(n,3,\mathcal{A}_\infty))$ is purely dependent from $P_i$'s if it is the intersection of three hyperplanes each one containing exactly one $P_i, i = 1,2,3$ (as in Figure 3.8).*

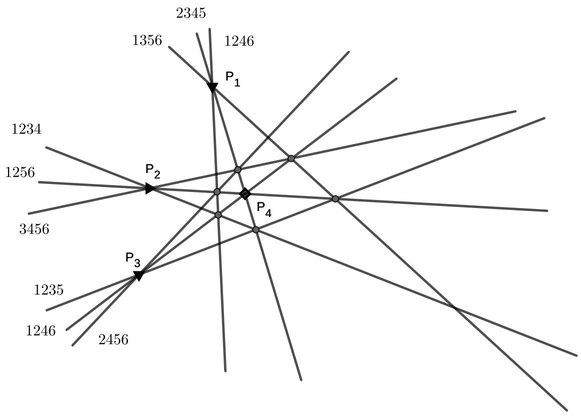


FIGURE 3.8: Discriminantal arrangement corresponding to the Pappus's configuration with 4 multiplicity 3 points

Remark that purely dependent intersections only exist when $\mathcal{A}$ is non-very generic arrangement. Furthermore it is a simple remark that $\mathcal{B}(6,3,\mathcal{A}_\infty)$ contains at most 3 independent intersections, realized when $\mathcal{A}_\infty = \mathcal{P}_\infty$, and a purely dependent one as in $\mathcal{P}^c_\infty$.

Secondly let's denote by $\mathcal{A}^\sigma$ the arrangement in $\mathbb{R}^2(\mathbb{C}^2)$ whose projective closure is the $\sigma$-completion $\mathcal{A}^\sigma_\infty$ of $\mathcal{A}_\infty$. It is a simple remark that the $\sigma$ completion $\mathcal{P}^\sigma_\infty$ at point $(2)$ of Proposition 3.2.3 is an arrangement of 6 lines with 4 triple points, that is the maximum possible ( see Figure 3.4). Hence, for a suitable choice of the line at infinity, $\mathcal{P}^\sigma$ is an arrangement in a simple non-very generic intersection $X \in \mathcal{L}_3(\mathcal{B}(6,2,\mathcal{A}))$ of maximum multiplicity and

[3] In details the collinearity condition $C = \{P_{i_1i_2}, P_{i_3i_4}, P_{i_5i_6}\}$ in $\mathcal{A}_\infty$ corresponds to the 3-point $P = \cap_{i=1}^3 D_{L_i} \in \mathcal{L}_2(\mathcal{B}(6,3,\mathcal{A}_\infty))$ with $L_1 = \{i_3,i_4,i_5,i_6\}, L_2 = \{i_1,i_2,i_5,i_6\}, L_3 = \{i_1,i_2,i_3,i_4\}$.

the following theorem equivalent to Proposition 3.2.3 holds.

**Theorem 3.2.5.** *Let $\mathcal{B}(6,3,\mathcal{A}_\infty)$ be a discriminantal arrangement with maximum number of independent intersections in $\mathcal{L}_2(\mathcal{B}(6,3,\mathcal{A}_\infty))$, and $\sigma \in S$ an element that acts strongly on $\mathcal{B}(6,3,\mathcal{A}_\infty)$. Then*

1. *the arrangement $\mathcal{A}_\infty \cup \mathcal{A}_\infty^\sigma$ is an arrangement with minimum number of intersections of multiplicity 2 if and only if $\mathcal{L}_2(\mathcal{B}(6,3,\mathcal{A}_\infty))$ contains a purely dependent intersection fixed by $\sigma$ and $\mathcal{A}_\infty^\sigma$ is central.*

2. *$\mathcal{A}_\infty \cup \mathcal{A}_\infty^\sigma$ is an arrangement with maximum number of intersections of multiplicity 3 if and only if $\mathcal{A}^\sigma \in X \in \mathcal{L}_3(B(6,2,\mathcal{A}^\sigma))$ is simple non-very generic intersection of multiplicity* 4 .

The above statement only depends on the combinatorics of discriminantal arrangement allowing us to generalize it into the following conjecture.

**Conjecture 3.2.6.** *Let $\mathcal{B}(n,3,\mathcal{A}_\infty)$ be a discrimanantal arrangement with maximum number of independent intersections in $\mathcal{L}_2(\mathcal{B}(n,3,\mathcal{A}_\infty))$, and $\sigma \in S$ an element which acts strongly on $\mathcal{B}(n,3,\mathcal{A}_\infty)$. Then*

1. *the arrangement $\mathcal{A}_\infty \cup \mathcal{A}_\infty^\sigma$ is an arrangement with minimum number of intersections of multiplicity 2 if and only if $\mathcal{L}_2(\mathcal{B}(n,3,\mathcal{A}_\infty))$ also contains the maximum number of purely dependent intersections fixed by $\sigma$ and $\mathcal{A}_\infty^\sigma$ is central;*

2. *$\mathcal{A}_\infty \cup \mathcal{A}_\infty^\sigma$ is an arrangement with maximum number of intersections of multiplicity 3 if and only if $\mathcal{A}^\sigma \in X \in \mathcal{L}_{n-3}(B(n,2,\mathcal{A}^\sigma))$ is simple non-very generic intersection of maximum multiplicity.*

# Chapter 4

# Arithmetic non-very generic intersection

## 4.1 Athanasiadis rank for non-very generic intersections

An arrangement $\mathcal{A}$ is either very generic or non-very generic depending on the intersection lattice of $\mathcal{B}(n,k,\mathcal{A})$. Hence, from this section onwards we will focus on the study of intersections $X \in \mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$.

### 4.1.1 Canonical presentation of intersections in $\mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$

The following definition will play a central role in the rest of the paper.

**Definition 4.1.1.** *Given an intersection $X \in \mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ we call $D_S$ the **components** of $X$ if $X \subset D_S$ and $X \not\subset D_{S\cup\{j\}}$ for all $j \in [n] \setminus S$. In particular for an intersection $X = \bigcap_{i=1}^m D_{S_i} \in \mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$, the set of indices $\{S_1,\ldots,S_m\}$ corresponding to all the components $\{D_{S_i},\ldots,D_{S_m}\}$ of $X$ is called the **canonical presentation** of $X$.*

Let's remark that while there are several ways to represent an element $X \in \mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$, its components are unique and so is its canonical presentation. This motivates the following definition.

**Definition 4.1.2** (Athanasiadis rank)**.** *Given an intersection $X = \bigcap_{i=1}^m D_{S_i} \in \mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ with canonical presentation $\{S_1,\ldots,S_m\}$, we call the natural number $a_X = \sum_{i=1}^m(|S_i| - k)$ the **Athanasiadis rank** of $X$.*

By Equation (2.2) if an arrangement $\mathcal{A}$ is very generic then all the intersections $X \in \mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ satisfy $\operatorname{rank}(X) = a_X$.

**Definition 4.1.3** (Non-very generic intersection)**.** *Given an intersection $X = \bigcap_{i=1}^{m} D_{S_i} \in \mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ with canonical presentation $\{S_1, \dots, S_m\}$, we call $X$ very generic if* $\mathrm{rank}(X) = a_X$*, and non-very generic otherwise.*

While the first known examples of non-very generic intersections all failed the BBA-condition defined in Equation 2.2, as shown in [19], there are non-very generic intersections which satisfy it. Hence we further distinguish between the non-very generic intersections as follows.

**Definition 4.1.4.** *We call a non-very generic intersection $X = \bigcap_{i=1}^{m} D_{S_i} \in \mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ geometric non-very generic or GNVG if its canonical presentation $\{S_1, \dots, S_m\}$ satisfies the BBA-condition and arithmetic non-very generic or ANVG otherwise.*

It is important to remark here that in [21], Lemma 3.8, the authors show that if $\mathcal{A}$ is a non-very generic arrangement then for any intersection $X \in \mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ which is GNVG, it exists an intersection $Y \subset X$ which is ANVG. It follows that we can have non-very generic arrangements $\mathcal{A}$ which are *arithmetic non-very generic*, i.e., all non-very generic intersections in $\mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ are ANVG, but there are not *geometric non-very generic* arrangements.
While the study of GNVG intersections seem to be very difficult, the one of the ANVG is simply related to the failing of the BBA-condition, i.e., to be an arithmetic non-very generic arrangement is completely classified by the BBA-condition. Hence our goal is to study under which conditions a non-very generic arrangement $\mathcal{A}$ is arithmetic. In the rest of this section we will focus on the study of conditions for which a non-very generic intersection is ANVG. In order to simplify the exposition, from now on whenever we write $X = \bigcap_{i=1}^{m} D_{S_i} \in \mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ we intend that $\{S_1, \dots, S_m\}$ is the canonical presentation of $X$, unless differently specified.

**Lemma 4.1.5.** *Let $X = \bigcap_{i=1}^{m} D_{S_i}$ be an intersection in $\mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ and $\mathbb{T} \in P(n,k)$ be a set which satisfies the condition:*

$$\forall\, S_i \in \{S_1, \dots, S_m\}\, \exists\, T \in \mathbb{T} \text{ such that } S_i \subset T, \quad \text{and } \mathrm{rank}(X) \geq \sum_{T \in \mathbb{T}} (|T| - k) - |\mathbb{T}_0|, \tag{4.1}$$

*where $\mathbb{T}_0 := \{T \in \mathbb{T} \mid |\{i \mid S_i \subset T\}| \geq 2\}$.*
*Then $X$ is a non-very generic intersection if and only if it is an ANVG intersection.*

*Proof.* If canonical presentation $\{S_1, \dots, S_m\}$ of the intersection $X = \bigcap_{i=1}^{m} D_{S_i} \in \mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ doesn't satisfy the BBA-condition, then $X$ is a non-very generic intersection.

Let $X = \bigcap_{i=1}^{m} D_{S_i} \in \mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ be a non-very generic intersection. We define:

$$X|T := \bigcap_{S_i \subset T} D_{S_i} \text{ and } \epsilon_T := \begin{cases} 0 & \text{if } T \notin \mathbb{T}_0 \\ 1 & \text{if } T \in \mathbb{T}_0 \end{cases}$$

for $T \in \mathbb{T}$. Let us assume that *BBA*- condition is satisfied then

$$a_{X|T} \le \left| \bigcup_{S_i \subset T} S_i \right| - k - \epsilon_T \le |T| - k - \epsilon_T.$$

Thus $a_X = \sum_{T\in\mathbb{T}} a_{X|T} \le \sum_{T\in\mathbb{T}} (\left|\bigcup_{S_i\subset T} S_i\right| - k - \epsilon_T) \le \sum_{T\in\mathbb{T}} (|T| - k - \epsilon_T) = \sum_{T\in\mathbb{T}} (|T| - k) - |\mathbb{T}_0| \le \operatorname{rank}(X)$ which is a contradiction. Hence the *BBA*-condition fails. □

A trivial example of this lemma are intersections $X = \bigcap_{i=1}^{m} D_{S_i}$ with $\operatorname{rank}(X) = 3$ in an arrangement of 6 lines in the real plane. In this case, if we fix $\mathbb{T} = \mathbb{T}_0 = \{123456\}$, then all intersections $X = \bigcap_{i=1}^{m} D_{S_i}$ trivially satisfy the condition that $S_i \in \mathbb{T}$, and if $\operatorname{rank}(X) = 3$, then the condition $\operatorname{rank}(X) \ge \sum_{T\in\mathbb{T}} (|T| - 2) - |\mathbb{T}_0| = (6 - 2) - 1 = 3$ is satisfied. That is all the non-very generic intersections in the Crapo's arrangement of 6 lines in $\mathbb{R}^2$ fail the BBA-condition as in the example described in the Introduction (see Figure 2.1).

**Lemma 4.1.6.** *Let $X = \bigcap_{i=1}^{m} D_{S_i}$ be an intersection of rank $n-k-1$ in $\mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$. If $X$ is a non-very generic intersection, then $\left|\bigcup_{i=1}^{m} S_i\right| = |\bigcup_{X\subset D_L} L| = n$.*

*Proof.* Let $X = \bigcap_{i=1}^{m} D_{S_i}$ be a non-very generic intersection of rank $n-k-1$ in $\mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ and assume that $\left|\bigcup_{i=1}^{m} S_i\right| = |\bigcup_{X\subset D_L} L| < n$. Then it exists a subset $L \in \binom{[n]}{k+1}$ such that $L \not\subset \bigcup_{i=1}^{m} S_i$ and hence $X \not\subset D_L$. It follows that the $\operatorname{rank}(X \cap D_L) = \operatorname{rank}(X) + \operatorname{rank}(D_L) = n-k$, that is $X \cap D_L$ is central. On the other hand $X$ is a non-very generic intersection so the $\operatorname{rank}(X \cap D_L) < a_X + 1 = a_{X\cap D_L}$. Hence $X \cap D_L$ is a central and non-very generic intersection which is absurd. □

The above lemma and Lemma 4.1.5 when $|\mathbb{T}| = 1$ give us the following theorem:

**Theorem 4.1.7.** *Let $X = \bigcap_{i=1}^{m} D_{S_i}$ be an intersection of rank $n-k-1$ in $\mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$. Then $X$ is a non-very generic intersection if and only if it is ANVG.*

**Corollary 4.1.8.** *An arrangement $\mathcal{A}$ with $n+3$ hyperplanes in $\mathbb{C}^n$ is non-very generic if and only if it is an ANVG arrangement.*

## 4.2 Description of rank 2 intersections in the discrminantal arrangements

In this section, we correct the result on rank 2 intersections in discriminantal arrangements provided by Libgober and Settepanella in [14]. Libgober and Settepanella said that the multiplicity of the rank 2 intersections in discriminantal arrangements $\mathcal{B}(n,k,\mathcal{A})$ is $2,3$ or $k+2$ (see [14]). We add on to that result by showing that we can construct intersections of rank 2 with multiplicity upto $\frac{n}{n-k-1}$ by considering arrangements having possible restrictions used by Libgober and Settepanella in [14].

**Theorem 4.2.1.** *Let* $X=\bigcap_{i=1}^{m} D_{L_i}$ *be an intersection in* $\mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$. *Suppose* $\bigcup_{i=1}^{m} L_i=[n]$. *If* $\operatorname{rank}(X)=2$*, then* $X$ *satisfies* $a_X \leq \frac{n}{n-k-1}$. *Conversely, if* $\frac{n}{n-k-1}$ *is an integer, then there exists an arrangement consisting of* $n$ *hyperplanes in* $\mathbb{R}^k$ *with non-very generic intersection* $X$ *satisfying* $a_X=\frac{n}{n-k-1}$.

We will give the proof of this later. Note that every rank 2 non-very generic intersections in $\mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$ are simple, that is, the canonical presentation of $X=\bigcap_{i=1}^{m} D_{L_i}$ is $\{L_1,\ldots,L_m\}$ with $|L_i|=k+1$ for $i=1,\ldots,m$ and $a_X=m$. Also, the multiplicity of simple non-very generic intersection equals to its Athanasiadis rank.
Before we give general construction and proof of inequality, we construct an arrangement of 8 hyperplanes in $\mathbb{R}^5$ which is the first example containing rank 2 intersection whose multiplicity is different from the ones mentioned in the result by Libgober and Settepanella.

**Example 4.2.2.** *Let*

$$(\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8)=\begin{pmatrix}1&0&0&0&0&1&2&3\\0&1&0&0&0&1&-2&-1\\0&0&1&0&0&1&3&2\\0&0&0&1&0&1&-1&-2\\0&0&0&0&1&1&1&1\end{pmatrix}$$

*and let* $\mathcal{A}=\{H_1,\ldots,H_8\}$ *be the central arrangement where* $H_i$ *is orthogonal to* $\alpha_i$. *The arrangement* $\mathcal{A}$ *is generic. Consider the intersection* $X$ *of hyperplanes*

$$D_{123456}, D_{123478}, D_{125678}, D_{345678}$$

*in a discriminantal arrangement $\mathcal{B}(8,5,\mathcal{A})$. The canonical presentation of $X = D_{123456} \cap D_{123478} \cap D_{125678} \cap D_{345678}$ is {{1,2,3,4,5,6},{1,2,3,4,7,8},{1,2,5,6,7,8},{3,4,5,6,7,8}}. Hence $a_X = 4$.*

*We obtain*

$$\begin{pmatrix} \alpha_{123456} \\ \alpha_{123478} \\ \alpha_{125678} \\ \alpha_{345678} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 1 & -1 \\ -8 & -8 & 0 & 0 & -4 & 4 & -4 & 4 \\ 0 & 0 & 8 & 8 & 4 & -4 & -4 & 4 \end{pmatrix}$$

*and its rank equals 2. Thus* $\operatorname{rank}(X) = 2 \neq a_X - 1 = 3$.

Recall Falk's adjoint construction to consider a general construction.

**Definition 4.2.3.** *Let $\{\alpha_1, \dots, \alpha_n\}$ be a set of vectors in $\mathbb{R}^k$. The **Gale diagram** of $\{\alpha_1, \dots, \alpha_n\}$ is a set $\{\beta_1, \dots, \beta_n\}$ of vectors in $\mathbb{R}^{n-k}$ such that the row space of $(\beta_1 \dots \beta_n)$ is the orthogonal complement of the row space of $(\alpha_1 \dots \alpha_n)$.*

Remark that, if $\{\alpha_1, \dots \alpha_n\}$ is in general position, then its Gale diagram is also in general position. Also, it is known that taking Gale diagram is a dual operation, i.e., taking Gale diagram twice for a set of vectors returns to the original one. For more properties of Gale diagrams, refer to [22].

**Theorem 4.2.4** ([9])**.** *Let $\mathcal{A}$ be a generic arrangement with normals $\{\alpha_1, \dots \alpha_n\}$ in $\mathbb{R}^k$. The essential part of $\mathcal{B}(n,k,\mathcal{A})$ is linearly equivalent to the arrangement consisting of all hyperplanes spanned by $n-k-1$ elements in a Gale diagram of $\{\alpha_1, \dots \alpha_n\}$.*

Thus, we can see that the inequality in Theorem 4.2.1 is optimal if we can construct an arrangement generated by $n$ points $\beta_1, \dots, \beta_n$ in $n-k$-space that has intersection $X$ of rank 2 with the multiplicity $\frac{n}{n-k-1}$ when $\frac{n}{n-k-1}$ is an integer. Let us consider the case when $n-k-1 = 2$.

**Example 4.2.5.** *Let $n \geq 6$ be an even number and let $k = n-3$. Take*

$$(\beta_1 \beta_2 \dots \beta_n) = \begin{pmatrix} \cos\frac{2\pi}{n} & \cos\frac{2\cdot 2\pi}{n} & \dots & \cos\frac{j\cdot 2\pi}{n} & \dots & \cos\frac{n\cdot 2\pi}{n} \\ \sin\frac{2\pi}{n} & \sin\frac{2\cdot 2\pi}{n} & \dots & \sin\frac{j\cdot 2\pi}{n} & \dots & \sin\frac{n\cdot 2\pi}{n} \\ 1 & 1 & \dots & 1 & \dots & 1 \end{pmatrix}.$$

*It is easy to see that* $\det(\beta_{i_1}\beta_{i_2}\beta_{i_3}) \neq 0$ *for all* $1 \leq i_1 < i_2 < i_3 \leq n$. *Hence* $\beta_1, \dots, \beta_n$ *are in general position. Now, the* $1$*-dimensional subspace* $X = \langle (0,0,1)^{\mathsf{T}} \rangle$ *is included by all* $\langle \beta_i, \beta_{i+\frac{n}{2}} \rangle$ *for* $i = 1, \dots, \frac{n}{2}$. *Hence the arrangement generated by* $\beta_1, \dots, \beta_n$ *has the rank* $2$ *intersection* $X$ *of multiplicity* $\frac{n}{2} = \frac{n}{n-k-1}$. *Taking the Gale diagram* $\{\alpha_1, \dots, \alpha_n\}$ *of* $\{\beta_1, \dots, \beta_n\}$, *we obtain the arrangement* $\{H_1, \dots, H_n\}$ *as we needed.*

More generally, we provide a construction as follows.

**Example 4.2.6.** *Suppose* $n-k-1$ *devides* $n$. *Let* $X$ *be a linear subspace of dimension* $n-k-2$ *in* $\mathbb{R}^{n-k}$. *Take* $\frac{n}{n-k-1}$ *hyperplanes* $D_1, \dots, D_{\frac{n}{n-k-1}}$ *in* $\mathbb{R}^{n-k}$ *with the condition that* $D_i \supset X$ *for each* $i = 1, \dots, \frac{n}{n-k-1}$. *We want to take* $n-k-1$ *vectors* $\beta_{i,1}, \dots, \beta_{i,n-k-1}$ *that are not contained by* $X$ *in each hyperplane* $D_i$. *Here, we require* $\{\beta_{i,j} \mid 1 \leq i \leq \frac{n}{n-k-1}, 1 \leq j \leq n-k-1\}$ *to be in the general position, but since the set of generic arrangements in the space of arrangements is a Zariski open set, they can always be in the general position. Hence the arrangement consisting of* $D_i = \langle \beta_{i,1}, \dots, \beta_{i,n-k-1} \rangle$ *has the rank* $2$ *intersection* $X$ *of multiplicity* $\frac{n}{n-k-1}$.

Thus, we obtain that Theorem 4.2.1 is an optimal evaluation. Let us show the inequality of Athanasiadis rank for the rank 2 intersections.

*proof of Theorem 4.2.1.* If $X$ is very generic, then the statement holds. Suppose $X = \bigcap_{i=1}^m D_{L_i}$ be a rank 2 non-very generic intersection with the canonical presentation $\{L_1, \dots, L_m\}$ in $\mathcal{L}(\mathcal{B}(n,k,\mathcal{A}))$.

Firstly, we show that

$$L_{i_1} \cup L_{i_2} = \bigcup_{i=1}^m L_i \tag{4.2}$$

for $1 \leq i_1 < i_2 \leq m$. If there exists an index $j \in \bigcup_{i=1}^m L_i \setminus (L_{i_1} \cup L_{i_2})$ and $i_o \in [m]$ such that $L_{i_0} \ni j$, then $D_{L_{i_0}} \supset X = D_{L_{i_1}} \cap D_{L_{i_2}} \supset D_{L_{i_1} \cup L_{i_2}}$. It is a contradiction.

By Equation (4.2), for each $i, j \in [m]$, we get $L_i \cup L_j = [n]$, or equivalently, $[n] \setminus L_i \subset L_j$. Thus, we obtain

$$n = \big|[n]\big| = \sum_{i=1}^m \big|[n] \setminus L_i\big| + \left|\bigcap_{i=1}^m L_i\right| \geq \sum_{i=1}^m \big|[n] \setminus L_i\big| = m(n-k-1).$$

That is $a_X \leq \frac{n}{n-k-1}$.

□

**Remark 4.2.7.** *Only the part 3 in Theorem 3.9 in [14] requires modification. The parts 1,2,4 in Theorems 3.9 hold even for the examples in the present paper. The part 2 in Theorems 3.9 is non-trivial, however it follows from Equation* (4.2). *Equation* (4.2) *states that when the three hyperplanes* $D_{L_1}, D_{L_2}, D_{L_3}$ *intersect with rank* 2*, then* $L_1 \subset L_2 \cup L_3$*,* $L_2 \subset L_1 \cup L_3$*, and* $L_3 \subset L_1 \cup L_2$ *hold. Therefore, by setting* $L'_i = L_i \setminus \bigcap_{j=1}^{3} L_j$*, we obtain the good-*3*-partition* $\{L'_1 \cap L'_2, L'_1 \cap L'_3, L'_2 \cap L'_3\}$*, which corresponds to the multiplicity* 3 *intersection of rank* 2 *in the discriminantal arrangement determined by the dependent restriction* $\mathcal{A}^X$*, where* $X = \bigcap_{i \in L_1 \cap L_2 \cap L_3} H_i$*. Moreover, from the perspective of the Gale dual, considering the restriction* $\mathcal{A}^X$*, where* $X = \bigcap_{i \in L_1 \cap L_2 \cap L_3} H_i$*, is equivalent to considering the deletion of* $\{\beta_i \mid i \in \bigcap_{j=1}^{3} L_j\}$ *from* $\{\beta_i \mid i = 1, \ldots, n\}$.

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