Doctoral Thesis

On representatives of hyperfunctions with several growth orders via relative Čech-Dolbeault cohomology
(相対チェックドルボーコホモロジーによる佐藤超関数の種々の増大度を持つ代表元について)

Tatsuki Nishida

Department of Mathematics, Graduate School of Science,
Hokkaido University

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Abstract

Honda, Izawa and Suwa define the relative Čech-Dolbeault representation of hyperfunctions in HIS[4]. This article aims to characterize relative Čech-Dolbeault representatives of C^∞ functions and distributions as hyperfunctions.

In Part I, we explain the relationship between relative Čech cohomology and relative Čech-Dolbeault cohomology. Then, we consider well-behaved relative Čech-Dolbeault representatives called "quasi-Whitney" forms. The correspondence between these representatives and C^∞ functions is demonstrated via relative Čech cohomology.

In Part II, we extended relative Čech-Dolbeault cohomology to sheaves on subanalytic sites. Representatives of distributions are characterized by those of tempered distributions, which can be treated within the framework of subanalytic sheaf theory.

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* Department of Mathematics Hokkaido University, Sapporo 060-0810, Japan

Introduction

Hyperfunctions play an essential role in the theory of linear partial differential equations. Mikio Sato invented the hyperfunction theory in 1958 (Sato[20],[21]), and this theory founded the field of mathematics known as algebraic analysis (see SKK[22], KKK[8], Kashiwara-Schapira[9],[12]). The sheaf of hyperfunctions is defined by local cohomology with coefficients in the holomorphic functions. A section of the sheaf cohomology is usually calculated by employing the theory of relative Čech cohomology, and the relative Čech representation of hyperfunctions is well-researched subject (see KKK[8], Kaneko[7], AKY[1], Komori-Umeta[15]). However, to manipulate the hyperfunction theory, we often have to use sophisticated techniques of complex analysis in several variables. Recently, N. Honda, T. Izawa and T. Suwa introduce a new representation of hyperfunctions in HIS[4] based on the theory of relative Čech-Dolbeault cohomology.

This article consists of two parts. The following is an overview of each part.

In Part I, We define "quasi-Whitney" forms as well-behaved relative Čech-Dolbeault representatives of C^∞ functions as hyperfunctions. To verify the correspondence between these representatives and C^∞ functions, we must focus on how C^∞ functions are represented in relative Čech cohomology. Such correspondences in the relative Čech representation have been thoroughly studied since the early days of hyperfunction theory. For example, Kaneko[7] and Komatsu[14] are useful references. Note that Part I is a modified version of the author's paper [18] "*Embedding of infinitely differentiable functions into the space of hyperfunctions via Čech-Dolbeault cohomology and its inverse map*".

Part I is organized as follows: Section 1 is the introduction for Part I. In Section 2, we recall notations and definitions of hyperfunctions and hyperforms.

In Section 3, we recall hyperfunctions and their representations. Firstly, in Subsection 3.1, we prepare the relative Čech-Dolbeault representation of hyperfunctions. Next, Subsection 3.2 explains the relationship between the relative Čech representation and the relative Čech-Dolbeault representation. Then, in Subsection 3.3, we introduce infinitesimal wedges and a twisted Radon kernel, which plays an important role in the definition of an embedding of distributions in the relative Čech representation. Lastly, in Subsection 3.4, we define an embedding of distributions in the relative Čech-Dolbeault representation.

Section 4 treats the main subject of Part I. Subsections 4.1 and 4.2 are preparations for Subsection 4.3. Subsection 4.1 defines limits of $(n-1)$-forms. Subsection 4.2 gives an inverse map of C^∞ functions with compact support. Subsection 4.3 shows our main results, which are Theorems 4.10 and 4.11. These theorems give a simple expression of the inverse of the embedding map, where the limits of forms are essential for this expression.

In Part II, we deal with the conditions that relative Čech-Dolbeault representatives of distributions. To establish a more general method than the C^∞ function case, we develop the relative Čech-Dolbeault cohomology theory on subanalytic sites. This approach leads to the relative Čech-Dolbeault representation of tempered distributions and allows us to determine the conditions for representatives of distributions as hyperfunctions. For the subanalytic sheaf theory, Kashiwara-Schapira[11], Prelli[19] and Guillermou-Schapira[3] are useful references.

Part II is organized as follows: Section 1 is the introduction for Part II. Section 2 gives notations and examples of subanalytic sheaves.

Section 3 explains relative Čech-Dolbeault cohomology on subanalytic sites. In Subection 3.1, we define U-soft sheaves on subanalytic sites and give properties and examples of soft sheaves. Note that soft sheaves on subanalytic sites is introduced by Guillermou[2]. In Subection 3.2, we define relative Čech derived functors. Relative Čech-Dolbeault cohomology is defined by these functors. In Subection 3.3, we introduce localized version of the relative Čech derived functors.

In Section 4, we give a representation of tempered distributions $\mathscr{D}b^t_{M_{sa}}$ by using relative Čech-Dolbeault cohomology. The representation of tempered distributions can be obtained from relative Čech-Dolbeault cohomology of tempered Dolbeault complexes (Definition 4.7), and its definition is very similar to that of hyperfunctions. This representation of $\mathscr{D}b^t_{M_{sa}}$ provides conditions for classifying representatives of distributions $\mathscr{D}b_M$ (Theorem 4.13).

Part I

C^∞ functions as hyperfunctions via relative Čech-Dolbeault cohomology

1 Introduction for Part I

Let M be a real analytic, n-dimensional, oriented manifold and X its complexification. In the relative Čech representation, a hyperfunction is represented by a formal sum of holomorphic functions such as $\bigoplus_i (-1)^i F_{\hat{i}}$. On the other hand, the relative Čech-Dolbeault representative of a hyperfunction is a pair of C^∞ differential forms (μ_1, μ_{01}), where μ_1 is a $(0,n)$-form on X, and μ_{01} is a $(0, n-1)$-form on $X \setminus M$. Remarkably, only two differential forms suffice to represent a hyperfunction. Additionally, by the softness of the sheaf of C^∞ forms, we can employ helpful tools such as cutoff functions or partitions of unity.

Since HIS[4] defines an embedding of distributions, we can regard C^∞ functions as hyperfunctions in the relative Čech-Dolbeault representation. This part aims to characterize a relative Čech-Dolbeault representative which corresponds to the image of a C^∞ function by the embedding map and also to construct the inverse map of the embedding of C^∞ functions.

In Čech representation, an inverse map of the embedding of C^∞ functions is defined by a sum of limits $\bigoplus_i (-1)^i F_{\hat{i}}(x + \sqrt{-1}\, y) \to \sum_i (-1)^i F_{\hat{i}}(x)$ $(y \to 0)$. Morimoto and Kaneko established this inverse map in Čech representation (Morimoto[17], Kaneko[7]). However, since non-trivial Čech-Dolbeault representatives always diverge on M, we cannot consider the limit $\lim\limits_{r \to 0} \mu_{01}(x + \sqrt{-1}\, r\omega)$ as the one for a Čech representative. To overcome this difficulty, we introduce a new class of forms called "quasi-Whitney", for which we can consider the limit of the form in an appropriate sense. In Lemma 4.14, we show that there always exists a representative belonging to the quasi-Whitney class for the hyperfunction image of a C^∞ function. Following this, for such a representative, we provide a simple expression of the inverse map in Theorems 4.10 and 4.11, which is our main result.

Part I is organized as follows: In Section 2, we recall notations and definitions of hyperfunctions and hyperforms.

In Section 3, we recall hyperfunctions and their representations. Firstly, in Subsection 3.1, we prepare the relative Čech-Dolbeault representation of hyperfunctions. Next, Subsection 3.2 explains the relationship between the relative Čech representation and the relative Čech-Dolbeault representation. Then, in Subsection 3.3, we introduce infinitesimal wedges and a twisted Radon kernel, which plays an important role in the definition of an embedding of distributions in the relative Čech representation. Lastly, in Subsection 3.4, we define an embedding of distributions in the relative Čech-Dolbeault representation.

Section 4 treats the main subject of Part I. Subsections 4.1 and 4.2 are preparations for Subsection 4.3. Subsection 4.1 defines limits of $(n-1)$-forms. Subsection 4.2 gives an inverse map of C^{∞} functions with compact support. Subsection 4.3 shows our main results, which are Theorems 4.10 and 4.11. These theorems give a simple expression of the inverse of the embedding map, where the limits of forms are essential for this expression.

2 Preliminary

2.1 Notations

Notation 2.1 Let X be a topological space. $\mathrm{Op}(X)$ denotes the set of open subsets in X. For any subset $K \subset X$, we write an interior of K as $\mathrm{Int}(K)$. For any $U, U' \in \mathrm{Op}(X)$, $U' \subset\subset U$ means that U' is a relatively compact subset of U.

Notation 2.2 For any subset K in a topological space X and for any sheaf $\mathscr{F}$ on X, we define $\mathscr{F}[K]$ by

$$\mathscr{F}[K] = \varinjlim_{K \subset U} \mathscr{F}(U),$$

where $U \subset X$ runs through open neighborhoods of K. This means that $\mathscr{F}[K]$ is a set of sections defined in open neighborhoods of K.

Let M be a real analytic manifold of dimension n and X its complexification.

Remark 2.3 In Part I, we assume that manifolds are always countable at infinity, and that sheaves are always those of abelian groups.

Notation 2.4 For any $\Omega \in \mathrm{Op}(M)$ and for any $U \in \mathrm{Op}(X)$, we say that U is a complex neighborhood of Ω if $\Omega \subset U$ and Ω is a closed set in U. Moreover, if a complex neighborhood U of Ω is a Stein open set, we call U a Stein neighborhood of Ω.

Notation 2.5 $\mathbb{Z}_M$ (resp. $\mathbb{Z}_X$) denotes the sheaf of $\mathbb{Z}$ valued locally constant functions on M (resp. X).

Notation 2.6 We define $\mathscr{A}_M$ to be the sheaf of real analytic functions on M and $\mathscr{O}_X$ to be the sheaf of holomorphic functions on X.

Remark 2.7 For any $\Omega \in \mathrm{Op}(M)$, we define the sections of real analytic functions on M by

$$\mathscr{A}_M(\Omega) = \varinjlim_{U} \mathscr{O}_X(U),$$

where U runs through complex neighborhoods of Ω.

Definition 2.8 Let K be a closed set in M. We define $\mathscr{D}b_M$ as the sheaf of distributions on M and also define $\mathscr{D}b_K$ to be the sheaf of distributions supported by K.

Notation 2.9 For any $p, q \in \{0, 1, \cdots, n\}$, we define that $\mathscr{A}_M^{(p)}$ is the sheaf of real analytic p-forms on M, $\mathscr{O}_X^{(p)}$ is the sheaf of holomorphic p-forms on X, and $C_X^{\infty,(p,q)}$ is the sheaf of C^∞ (p,q)-forms with C^∞ coefficients on X. For other $p, q \in \mathbb{Z}$, we set $\mathscr{A}_M^{(p)} = 0$, $\mathscr{O}_X^{(p)} = 0$ and $C_X^{\infty,(p,q)} = 0$.

Remark 2.10 $\mathscr{A}_M^{(0)}$ (resp. $\mathscr{O}_X^{(0)}$) is nothing but $\mathscr{A}_M$ (resp. $\mathscr{O}_X$).

Notation 2.11 The symbol $\widehat{\bullet}$ means to omit the corresponding letter in a sequence or a family of sets, etc. For example, we employ the following notations:

- $(i_0, i_1, \cdots, \widehat{i_\ell}, \cdots, i_k) = (i_0, i_1, \cdots, i_{\ell-1}, i_{\ell+1}, \cdots, i_k)$,
- $U_0 \cap U_1 \cap \cdots \cap \widehat{U_\ell} \cap \cdots \cap U_k = U_0 \cap U_1 \cap \cdots \cap U_{\ell-1} \cap U_{\ell+1} \cap \cdots \cap U_k$,

where $k, \ell \in \mathbb{Z}_{\geq 0} = \{i \in \mathbb{Z} \mid i \geq 0\}$, $\ell \leq k$, $(i_0, i_1, \cdots, i_k) \in \{0, 1, \cdots, n\}^{k+1}$ and $U_0, U_1, \cdots, U_k \in \mathrm{Op}(X)$.

2.2 Hyperfunctions and hyperforms

Let M be a real analytic, n-dimensional, oriented manifold and X its complexification, and let $\Omega \in \mathrm{Op}(M)$ and $U \in \mathrm{Op}(X)$ such that U is a complex neighborhood of Ω.

Notation 2.12 (Relative cohomology) For any sheaf $\mathscr{F}$ on X and for any $k \in \mathbb{Z}$, $H_\Omega^k(U; \mathscr{F})$ denotes the k-th relative cohomology group of $\mathscr{F}$ with supports in Ω.

See KKK[8] for the relative cohomology groups of a sheaf.

Definition 2.13 (Hyperfunctions) We define the space of hyperfunctions on Ω by

$$\mathscr{B}_M(\Omega) = H_\Omega^n(U; \mathscr{O}_X) \underset{\mathbb{Z}_M(\Omega)}{\otimes} or_{M/X}(\Omega),$$

where $or_{M/X}(\Omega) = H_\Omega^n(U; \mathbb{Z}_X)$ is the sections of the relative orientation sheaf on Ω.

Definition 2.14 (Hyperforms) The space of p-hyperforms on Ω is given by

$$\mathscr{B}_M^{(p)}(\Omega) = H_\Omega^n(U; \mathscr{O}_X^{(p)}) \underset{\mathbb{Z}_M(\Omega)}{\otimes} or_{M/X}(\Omega).$$

Note that $\mathscr{B}_M(\Omega) = \mathscr{B}_M^{(0)}(\Omega)$. Similarly, we define hyperforms and hyperfunctions with closed support as follows.

Definition 2.15 For any closed set K in Ω, the space of p-hyperforms supported by K is defined by

$$\mathscr{B}^{(p)}_K(\Omega) = H^n_K(U; \mathscr{O}^{(p)}_X) \underset{\mathbb{Z}_M(\Omega)}{\otimes} or_{M/X}(\Omega).$$

The space $\mathscr{B}_K(\Omega)$ of hyperfunctions supported by K is defined in the same way.

Remark 2.16 $\mathscr{B}^{(p)}_M = \{\mathscr{B}^{(p)}_M(\Omega)\}_{\Omega\in \mathrm{Op}(M)}$ and $\mathscr{B}^{(p)}_K = \{\mathscr{B}^{(p)}_{K\cap\Omega}(\Omega)\}_{\Omega\in \mathrm{Op}(M)}$ form sheaves. For more details, refer to KKK[8].

3 Relative Čech cohomology and relative Čech-Dolbeault cohomology

A concrete expression of a hyperfunction is usually realized by using either the theory of relative Čech cohomology or that of relative Čech-Dolbeault cohomology. Both of these cohomology theories have their own advantages. In this section, we briefly review their definitions and establish the canonical isomorphism between them. Additionally, we consider the embedding of distributions into hyperfunctions from the viewpoint of both cohomology theories.

3.1 Relative Čech-Dolbeault cohomology and hyperfunctions

Let M be a real analytic, n-dimensional, oriented manifold and X its complexification, and let $\Omega \in \mathrm{Op}(M)$ and $U \in \mathrm{Op}(X)$ such that U is a complex neighborhood of Ω.

Definition 3.1 For any $p, q \in \mathbb{Z}$, we define

$$C^{\infty,(p,q)}_X(U, U \setminus \Omega) = C^{\infty,(p,q)}_X(U) \oplus C^{\infty,(p,q-1)}_X(U \setminus \Omega),$$

and

$$\begin{array}{cccc} \bar{\vartheta} \ : & C^{\infty,(p,q)}_X(U, U \setminus \Omega) & \longrightarrow & C^{\infty,(p,q+1)}_X(U, U \setminus \Omega) \\ & \cup\!\!\!\cup & & \cup\!\!\!\cup \\ & (\mu_1, \mu_{01}) & \longmapsto & (\bar{\partial}\mu_1,\, \mu_1|_{U\setminus\Omega} - \bar{\partial}\mu_{01}). \end{array}$$

Then, $(C^{\infty,(p,\bullet)}_X(U, U \setminus \Omega), \bar{\vartheta})$ is a complex of vector spaces. $H^{p,q}_{\bar{\vartheta}}(U, U \setminus \Omega)$ denotes the q-th cohomology group of this complex.

If S is a closed set in U, we can define a complex $(C^{\infty,(p,\bullet)}_X(U, U \setminus S), \bar{\vartheta})$ and a cohomology $H^{p,q}_{\bar{\vartheta}}(U, U \setminus S)$ in the same way.

Theorem 3.2 (Relative Dolbeault theorem) For any closed set S in U, we have the canonical isomorphism

$$H^{p,q}_{\bar{\vartheta}}(U, U \setminus S) \simeq H^q_S(U; \mathscr{O}^{(p)}_X).$$

In Theorem A.3, we provide a proof of this theorem using the framework of the derived category. Additionally, Theorem 3.23 establishes an isomorphism between Čech cohomology and Čech-Dolbeault cohomology, and its proof may offer further insights. For more details, refer to HIS[4] and Suwa[25].

Corollary 3.3 Let K be a closed set in Ω. There are isomorphisms

$$\mathscr{B}_M^{(p)}(\Omega) \simeq H_{\bar{\vartheta}}^{p,n}(U, U \setminus \Omega) \underset{\mathbb{Z}_M(\Omega)}{\otimes} or_{M/X}(\Omega)$$

and

$$\mathscr{B}_K^{(p)}(\Omega) \simeq H_{\bar{\vartheta}}^{p,n}(U, U \setminus K) \underset{\mathbb{Z}_M(\Omega)}{\otimes} or_{M/X}(\Omega).$$

Definition 3.4 (Integration of $\boldsymbol{n}$-forms) Let K be a compact set in Ω, and let or_M denote the orientation sheaf of M. We define the integration

$$\int_\Omega : \mathscr{B}_K^{(n)}(\Omega) \underset{\mathbb{Z}_M(\Omega)}{\otimes} or_M(\Omega) \to \mathbb{C}$$

as follows: For any $u \in \mathscr{B}_K^{(n)}(\Omega)$ represented by $(\mu_1, \mu_{01}) \in C_X^{\infty,(n,n)}(U) \oplus C_X^{\infty,(n,n-1)}(U \setminus K)$, we define

$$\int_\Omega u = \int_D \mu_1 - \int_{\partial D} \mu_{01},$$

where D is an open set with C^∞ boundary satisfying $K \subset D \subset \overline{D} \subset U$, and ∂D is the boundary of D.

Remark 3.5 We choose $(y_1, \cdots, y_n, x_1, \cdots, x_n)$ as a positive coordinate system of $\mathbb{C}^n$ with $z_i = x_i + \sqrt{-1}y_i$ $(i = 1, \cdots, n)$ through this article. Additionally, ∂D is oriented as follows: $\overline{D}$ is a manifold with C^∞ boundary ∂D. Let $p \in \partial D$. There exist a neighborhood V of p and a coordinate system $(p_1, \cdots, p_{2n})$ on V such that $\overline{D} \cap V = \{p \in V \mid p_1 \leq 0\}$. If $(p_1, p_2, p_3, \cdots, p_{2n})$ is a positive coordinate system on $\overline{D} \cap V$, we define that $(p_2, p_3, \cdots, p_{2n})$ is a positive coordinate system on $\partial D \cap V$.

Definition 3.6 (Integration of hyperfunction) Assume $M = \mathbb{R}^n$ and $X = \mathbb{C}^n$ with the coordinates $(z_1, \cdots, z_n)$, and let K be a compact set in Ω. Then, for any $u \in \mathscr{B}_K(\Omega)$ represented by $(\mu_1, \mu_{01}) \in C_X^{\infty,(0,n)}(U) \oplus C_X^{\infty,(0,n-1)}(U \setminus K)$, we define

$$\int_\Omega u\, dx = \int_D \mu_1 \wedge dz - \int_{\partial D} \mu_{01} \wedge dz,$$

where $dz = dz_1 \wedge \cdots \wedge dz_n$, D is an open set with C^∞ boundary satisfying $K \subset D \subset \overline{D} \subset U$, and ∂D is the boundary of D.

Theorem 3.7 ([4, Corollary 6.12]) Let $i : M \hookrightarrow X$ be the canonical embedding, and let K be a compact set in Ω. Then, $\mathscr{B}_K^{(p)}(\Omega)$ and $\mathscr{A}_M^{(n-p)}[K]$ become FS and DFS spaces, respectively, and

$$\mathscr{B}_K^{(p)}(\Omega) \times (\mathscr{A}_M^{(n-p)}[K] \otimes or_M(\Omega)) \longrightarrow H_{\bar{\vartheta}}^{n,n}(U, U \setminus K) \otimes i^{-1}or_X(\Omega) \xrightarrow{\int} \mathbb{C}$$

is topologically non-degenerate. Here, an FS space means a Fréchet-Schwartz topological vector space, and a DFS space is a dual Fréchet-Schwartz topological vector space (see Komatsu[14]). Hence, we have

$$\mathscr{B}_K^{(p)}(\Omega) \simeq (\mathscr{A}_M^{(n-p)}[K] \otimes or_M(\Omega))'.$$

Note that $(\bullet)'$ denotes the strong dual of a topological vector space.

Proposition 3.8 (Excision, [4, Proposition 4.8]) For any complex neighborhood $\widetilde{U}$ of Ω, there is an isomorphism

$$H_{\bar{\vartheta}}^{p,q}(U, U \setminus \Omega) \simeq H_{\bar{\vartheta}}^{p,q}(\widetilde{U}, \widetilde{U} \setminus \Omega).$$

Let us define an embedding of real analytic functions in Čech-Dolbeault representation. Since M is orientable, we can choose a generator $e_{M/X} \in or_{M/X}(\Omega)$ such that $e_{M/X,x}$ generates a stalk $or_{M/X,x}$ over $\mathbb{Z}$ for any $x \in \Omega$. By considering the canonical morphism

$$or_{M/X}(\Omega) \;\; = \;\; H_\Omega^n(U; \mathbb{Z}_X) \longrightarrow H_\Omega^n(U; \mathscr{O}_X),$$

we get a representative $\nu \in C_X^{\infty,(0,n)}(U, U \setminus \Omega)$ of the image of $e_{M/X} \in or_{M/X}(\Omega)$ through the above morphism. Then, we set the constant function $1 \in \mathscr{B}_M(\Omega)$ as $[\nu] \otimes e_{M/X}$. Let ν be such a representative.

Definition 3.9 (Embedding of $\mathscr{A}_M$ in Čech-Dolbeault representation) We define an embedding $\iota_{\mathscr{A}_M(\Omega)}^{CD} : \mathscr{A}_M(\Omega) \to H_{\bar{\vartheta}}^{0,n}(U, U \setminus \Omega) \otimes or_{M/X}(\Omega) \simeq \mathscr{B}_M(\Omega)$ by

$$\begin{array}{cccccc} \iota_{\mathscr{A}_M(\Omega)}^{CD} : & \mathscr{A}_M(\Omega) & \longrightarrow & H_{\bar{\vartheta}}^{0,n}(\widetilde{U}, \widetilde{U} \setminus \Omega) \otimes or_{M/X}(\Omega) & \simeq & H_{\bar{\vartheta}}^{0,n}(U, U \setminus \Omega) \otimes or_{M/X}(\Omega). \\ & \cup\!\!\!\mid & & \cup\!\!\!\mid & & \\ & f & \longmapsto & \left[\widetilde{F}\nu|_{\widetilde{U}}\right] \otimes e_{M/X} & & \end{array}$$

Here $\widetilde{U} \subset U$ is a complex neighborhood of Ω on which f can be extended to a holomorphic function $\widetilde{F} \in \mathscr{O}_X(\widetilde{U})$.

Proposition 3.10 $\iota_{\mathscr{A}_M}^{CD} = \{\iota_{\mathscr{A}_M(\Omega)}^{CD}\}_{U \in \mathrm{Op}(M)}$ is a sheaf morphism from $\mathscr{A}_M$ to $\mathscr{B}_M$.

Let us define boundary value morphisms. If $W \in \mathrm{Op}(X)$ satisfies good geometrical conditions, we can regard holomorphic functions on W as hyperfunctions.

Recall that U is a complex neighborhood of $\Omega \in \mathrm{Op}(M)$. We take $W \in \mathrm{Op}(X)$ satisfying the following two conditions:

(B_1) $\overline{W} \supset \Omega$.

(B_2) The inclusion $(U \setminus W) \setminus M \hookrightarrow U \setminus W$ gives a homotopy equivalence.

Proposition 3.11 ([4, Lemma 7.10]) We can take a representative $\nu = (\nu_1, \nu_{01}) \in C_X^{\infty,(0,n)}(U, U \setminus \Omega)$ introduced before Definition 3.9 such that $\mathrm{Supp}_U(\nu_1) \subset W$ and $\mathrm{Supp}_{U\setminus\Omega}(\nu_{01}) \subset W$.

Definition 3.12 (Boundary value morphism, [4, Subsection 7.2]) Let $\nu \in C_X^{\infty,(0,n)}(U, U \setminus \Omega)$ be a representative given in Proposition 3.11. We define

$$b_W : \mathscr{O}_X(W) \to \mathscr{B}_M(\Omega) = H_{\bar\vartheta}^{0,n}(U, U \setminus \Omega) \otimes_{\mathbb{Z}_M(\Omega)} or_{M/X}(\Omega)$$

by

$$b_W(F) = [F\nu] \otimes e_{M/X} \quad (F \in \mathscr{O}_X(W)).$$

3.2 Relative Čech cohomology and isomorphism b_Ω

This subsection aims to define a map from the relative Čech-Dolbeault representation to the relative Čech representation of hyperfunctions, which is given in Definition 3.22. We refer the reader to Kaneko[7] and KKK[8] for the theory of Čech cohomology.

Assume $M = \mathbb{R}^n$ and $X = \mathbb{C}^n$. Let $\Omega \in \mathrm{Op}(M)$ and let U be a Stein neighborhood of Ω. Here, by Grauert's theorem (see Theorem 1.2.3 of KKK[8]), we can take a Stein neighborhood U of Ω. In what follows, we fix a section $e_{M/X} \in or_{M/X}(\Omega)$ such that it generates $or_{M/X}$ over $\mathbb{Z}_X$ on Ω and determines the same relative orientation along M on each connected component of Ω.

Take $v_0, v_1, \cdots, v_{n-1}$ as basis vectors of $\mathbb{R}^n$, and set $v_n = -(v_0 + v_1 + \cdots + v_{n-1})$. Moreover, we assume that the orientation of the frame $v_0, v_1, \cdots, v_{n-1}$ is the same as the one determined by $e_{M/X}$. Let $H_i = \{y \in \mathbb{R}^n \mid \langle y, v_i \rangle > 0\}$ and $U_i = (\mathbb{R}^n + \sqrt{-1}\,H_i) \cap U$ $(0 \le i \le n)$. Furthermore, we define $U_{n+1} = U$, $\Lambda' = \{0, 1, \cdots, n\}$ and $\Lambda = \{0, 1, \cdots, n+1\}$. Then, $\mathcal{U}' = \{U_i\}_{i\in\Lambda'}$ is a covering of $U \setminus \Omega$ and $\mathcal{U} = \{U_i\}_{i\in\Lambda} = \mathcal{U}' \cup \{U\}$ is a covering of U. Since $\mathbb{R}^n + \sqrt{-1}\,H_0, \cdots, \mathbb{R}^n + \sqrt{-1}\,H_n$ and U are Stein open sets, $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ is a Stein covering of U, that is, each U_i is a Stein open set.

Definition 3.13 (Čech cohomology) Let $\mathscr{F}$ be a sheaf on X. We define

$$C^k(\mathcal{U}, \mathcal{U}'; \mathscr{F}) = \left\{ \{F_{i_0,\cdots,i_k}\}_{(i_0,\cdots,i_k)\in\Lambda^{k+1}} \;\middle|\; \begin{array}{l} F_{i_0,\cdots,i_k} \in \mathscr{F}(U_{i_0} \cap \cdots \cap U_{i_k}), \\ F_{i_0,\cdots,i_\ell,i_{\ell+1},\cdots,i_k} = -F_{i_0,\cdots,i_{\ell+1},i_\ell,\cdots,i_k}, \\ (i_0, \cdots, i_k) \in \Lambda'^{k+1} \Rightarrow F_{i_0,\cdots,i_k} = 0 \end{array} \right\},$$

and a map $\delta : C^k(\mathcal{U},\mathcal{U}';\mathscr{F}) \to C^{k+1}(\mathcal{U},\mathcal{U}';\mathscr{F})$ is defined by

$$\delta\left(\{F_{i_0,\cdots,i_k}\}_{(i_0,\cdots,i_k)\in\Lambda^{k+1}}\right) = \left\{\sum_{\ell=0}^{k+1}(-1)^\ell F_{j_0,\cdots,\widehat{j_\ell},\cdots,j_{k+1}}\Big|_{U_{j_0}\cap\cdots\cap U_{j_{k+1}}}\right\}_{(j_0,\cdots,j_{k+1})\in\Lambda^{k+2}},$$

where $\widehat{\bullet}$ means to omit the corresponding letter in a sequence. $(C^k(\mathcal{U},\mathcal{U}';\mathscr{F}),\delta)$ is a complex, and we write its k-th cohomology group as $H^k(\mathcal{U},\mathcal{U}';\mathscr{F})$.

Remark 3.14 We have

$$C^k(\mathcal{U},\mathcal{U}';\mathscr{O}) \subset \bigoplus_{(i_0,\cdots,i_k)\in\Lambda^{k+1}} \mathscr{F}(U_{i_0}\cap\cdots\cap U_{i_k}).$$

In Čech cohomology, the sorting order of subscripts is important for the sign of representatives because of the condition $F_{i_0,\cdots,i_\ell,i_{\ell+1},\cdots,i_k} = -F_{i_0,\cdots,i_{\ell+1},i_\ell,\cdots,i_k}$. To emphasize this point, Kaneko[7] uses the symbol $\mathscr{F}(U_{i_0}\wedge\cdots\wedge U_{i_k})$ instead of $\mathscr{F}(U_{i_0}\cap\cdots\cap U_{i_k})$. In this article, we do not use this notation. However, we pay attention to the order of subscripts.

Theorem 3.15 (Leray) There is the canonical isomorphism

$$H^k(\mathcal{U},\mathcal{U}';\mathscr{O}_X) \simeq H^k_\Omega(U;\mathscr{O}_X).$$

To prove Leray's theorem, $\mathcal{U}$ must be a Stein covering. We omit the proof of Theorem 3.15. See Theorem 5.5.6 of Kaneko[7] or Theorem 1.2.1 of KKK[8]. Theorem 3.15 means that a formal direct sum of holomorphic functions represents a hyperfunction, i.e.,

$$\bigoplus_{0\le i\le n} F_{\widehat{i}} \in \bigoplus_{0\le i\le n} \mathscr{O}_X(U_{\widehat{i}})$$

gives a hyperfunction on Ω, where $U_{\widehat{i}} = U_0\cap\cdots\cap\widehat{U_i}\cap\cdots\cap U_{n+1}$ and $F_{\widehat{i}} = F_{0,\cdots,\widehat{i},\cdots,n+1}$.

Remark 3.16 We must clarify the sequence order of the $\widehat{i}$. In what follows, we write $\widehat{i}$ for the increasing sequence $0,1,2,\cdots,i-1,i+1,\cdots,n+1 \in \Lambda^{n+1}$.

Proposition 3.17 (Excision) For any Stein neighborhood $\widetilde{U}$ of Ω, we define $\widetilde{\mathcal{U}} = \{\widetilde{U}_i\}_{i\in\Lambda}$ and $\widetilde{\mathcal{U}}' = \{\widetilde{U}_i\}_{i\in\Lambda'}$ by

$$\widetilde{U}_0 = (\mathbb{R}^n + \sqrt{-1}\,H_0)\cap\widetilde{U},\ \cdots,\ \widetilde{U}_n = (\mathbb{R}^n + \sqrt{-1}\,H_0)\cap\widetilde{U},\ \text{and}\ \widetilde{U}_{n+1} = \widetilde{U}.$$

Then, there exists the canonical isomorphism

$$H^n(\mathcal{U},\mathcal{U}';\mathscr{O}_X) \simeq H^n(\widetilde{\mathcal{U}},\widetilde{\mathcal{U}}';\mathscr{O}_X). \tag{3.1}$$

Furthermore, we have the canonical isomorphism

$$H^n(\mathcal{U},\mathcal{U}';\mathscr{O}_X) \simeq \varinjlim_{\Omega\subset\widetilde{U}} H^n(\widetilde{\mathcal{U}},\widetilde{\mathcal{U}}';\mathscr{O}_X), \tag{3.2}$$

where $\widetilde{U}$ runs through complex neighborhoods of Ω.

Proof. We may assume $\widetilde{U} \subset U$. Now, we have a commutative diagram

$$\begin{array}{ccc} H^n_\Omega(U;\mathscr{O}_X) & \longrightarrow & H^n_\Omega(\widetilde{U};\mathscr{O}_X) \\ \downarrow & & \downarrow \\ H^n(\mathcal{U},\mathcal{U}';\mathscr{O}_X) & \longrightarrow & H^n(\widetilde{\mathcal{U}},\widetilde{\mathcal{U}}';\mathscr{O}_X). \end{array}$$

By Theorem 3.15 (Leray's theorem), the vertical morphisms are isomorphic. Additionally, by the excision theorem of local cohomology, the first horizontal morphism is an isomorphism. Then, the equation (3.1) holds. By Grauert's theorem, the equation (3.2) is also valid. □

Note that the excision theorem of local cohomology is explained after Definition 1.1.9 of KKK[8], and that we refer the reader to Lemma 5.5.12 of Kaneko[7] for more details of Proposition 3.17.

Definition 3.18 (Embedding of $\mathscr{A}_M$ in Čech representation) For any $f \in \mathscr{A}_M(\Omega)$, we define $\iota^C_{\mathscr{A}_M(\Omega)} : \mathscr{A}_M(\Omega) \to H^n(\mathcal{U},\mathcal{U}';\mathscr{O}_X) \otimes or_{M/X}(\Omega) \simeq \mathscr{B}_M(\Omega)$ by

$$\begin{array}{cccccc} \iota^C_{\mathscr{A}_M(\Omega)} : \mathscr{A}_M(\Omega) & \longrightarrow & H^n(\widetilde{\mathcal{U}},\widetilde{\mathcal{U}}';\mathscr{O}_X) \otimes or_{M/X}(\Omega) & \simeq & H^n(\mathcal{U},\mathcal{U}';\mathscr{O}_X) \otimes or_{M/X}(\Omega), \\ \Psi & & \Psi & & \\ f & \longmapsto & \left[\, \widetilde{F}|_{U_{\widehat{k}}} \right] \otimes [e_{\widehat{k}}] & & \end{array}$$

for some $k \in \{0, 1, \cdots, n\}$. Here $\widetilde{U} \subset U$ is a Stein neighborhood of Ω on which f can be extended to a holomorphic function $\widetilde{F} \in \mathscr{O}_X(\widetilde{U})$, we set

$$\begin{aligned} &\widetilde{\mathcal{U}}' = \left\{ \widetilde{U}_0 = (\mathbb{R}^n + \sqrt{-1}H_0) \cap \widetilde{U},\ \cdots,\ \widetilde{U}_n = (\mathbb{R}^n + \sqrt{-1}H_n) \cap \widetilde{U} \right\} \\ &\text{and } \widetilde{\mathcal{U}} = \widetilde{\mathcal{U}}' \cup \left\{ \widetilde{U}_{n+1} = \widetilde{U} \right\} \end{aligned}$$

as in Proposition 3.17, and $[e_{\widehat{k}}] \in or_{M/X}(\Omega) = H^n(\mathcal{U},\mathcal{U}';\mathbb{Z}_X) \simeq H^n(\widetilde{\mathcal{U}},\widetilde{\mathcal{U}}';\mathbb{Z}_X)$ is represented by

$$e_{\widehat{k}} = \{G_{i_0,\cdots,i_n}\}_{(i_0,\cdots,i_n)\in\Lambda_k{}^{n+1}} \in \bigoplus_{(i_0,\cdots,i_n)\in\Lambda_k{}^{n+1}} \mathbb{Z}_X(U_{i_0} \cap \cdots \cap U_{i_n}),$$

where $\Lambda_k = \{0, 1, \cdots, k-1, k+1, \cdots, n+1\}$, $G_{\widehat{k}} = G_{0,\cdots,k-1,k+1,\cdots,n+1} = 1$ and $G_{i_0,\cdots,i_\ell,i_{\ell+1},\cdots,i_n} = -G_{i_0,\cdots,i_{\ell+1},i_\ell,\cdots,i_n}$ for any ℓ.

Note that $[(-1)^k e_{\widehat{k}}] = e_{M/X}$ holds for any $k = 0, 1, \cdots, n$.

Proposition 3.19 $\iota^C_{\mathscr{A}_M(\Omega)}$ does not depend on the choices of $k \in \{0, 1, \cdots, n\}$ and $\iota^C_{\mathscr{A}_M} = \{\iota^C_{\mathscr{A}_M(\Omega)}\}_{\Omega\in\mathrm{Op}(M)}$ becomes a sheaf morphism from $\mathscr{A}_M$ to $\mathscr{B}_M$.

Remark 3.20 When we fix a generator $e_{M/X} \in or_{M/X}(\Omega)$, for any hyperfunction $u = \left[\bigoplus_i (-1)^i F_{\widehat{i}}\right] \otimes e_{M/X} \in H^n(\mathcal{U},\mathcal{U}';\mathscr{O}_X) \otimes or_{M/X}(\Omega) = \mathscr{B}_M(\Omega)$, we call

$\bigoplus_i (-1)^i F_{\widehat{i}} \in \bigoplus_i \mathscr{O}_X(U_{\widehat{i}})$ a Čech representative of u. This expression is different from the one given in Kaneko[7], which is explained in Remark 3.29.

Now, by the map b_W in Definition 3.12, we can define an isomorphism from Čech representation to Čech-Dolbeault representation. Let us construct the special representative ν whose support is contained in $U_{\widehat{i}}$. Recall that $\Omega \in \mathrm{Op}(M)$ and that U is a Stein neighborhood of Ω.

Example 3.21 ([4, Example 7.14]) Let $\varphi_0, \varphi_1, \cdots, \varphi_n$ be C^∞ functions on $U \backslash \Omega$ which satisfy

(1) $\mathrm{Supp}_{U \backslash \Omega}(\varphi_i) \subset U_i \qquad (i = 0, 1, \cdots, n)$,

(2) $\displaystyle\sum_{i=0}^{n} \varphi_i = 1$ on $U \setminus \Omega$.

For any $i = 0, 1, \cdots, n$, we define

$$\chi_{U_{\widehat{i}}}(z) = \begin{cases} 1 & \text{if } z \in U_{\widehat{i}}\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

and

$$\nu_{i,01} = \begin{cases} (-1)^i (n-1)!\, \chi_{U_{\widehat{i}}}\, \bar{\partial}\varphi_0 \wedge \cdots \wedge \widehat{\bar{\partial}\varphi_i} \wedge \cdots \wedge \bar{\partial}\varphi_{n-1} & (i = 0, 1, \cdots, n-1), \\ (-1)^n (n-1)!\, \chi_{U_{\widehat{n}}}\, \bar{\partial}\varphi_0 \wedge \cdots \wedge \bar{\partial}\varphi_{n-2} & (i = n). \end{cases}$$

Then, $\nu_i = (0, \nu_{i,01})$ satisfies $[\nu_i] \otimes e_{M/X} = 1 \in \mathscr{B}_M(U)$ and $\mathrm{Supp}_{U \backslash \Omega}(\nu_{i,01}) \subset U_{\widehat{i}}$.

Definition 3.22 For any $i = 0, 1, \cdots, n$, we take ν_i as $(0, \nu_{i,01})$ given in Example 3.21, and we set the map $b_{U_{\widehat{i}}}$ by $b_{U_{\widehat{i}}}(F_{\widehat{i}}) = [F_{\widehat{i}} \nu_i] \otimes e_{M/X}$ as in Definition 3.12. Then, we define a map $b_\Omega : H^n(\mathcal{U}, \mathcal{U}'; \mathscr{O}_X) \otimes or_{M/X}(\Omega) \to H^{0,n}_{\bar{\vartheta}}(U, U \setminus \Omega) \otimes or_{M/X}(\Omega)$ by

$$b_\Omega \left(\left[\bigoplus_{i=0}^{n} (-1)^i F_{\widehat{i}} \right] \otimes e_{M/X} \right) = \sum_{i=0}^{n} b_{U_{\widehat{i}}}(F_{\widehat{i}}),$$

where $e_{M/X}$ is the section of $or_{M/X}(\Omega)$ specified at the beginning of Subsection 3.2.

Example 7.17 of HIS[4] is helpful for the reader in understanding the map b_Ω.

Theorem 3.23 The map b_Ω is an isomorphism.

Proof. We have a double complex

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^0(\mathcal{U}, \mathcal{U}'; \mathscr{O}_X) & \hookrightarrow & C^0(\mathcal{U}, \mathcal{U}'; C_X^{\infty,(0,0)}) & \xrightarrow{\bar{\partial}^{0,0}} & C^0(\mathcal{U}, \mathcal{U}'; C_X^{\infty,(0,1)}) & \xrightarrow{\bar{\partial}^{0,1}} & \cdots \\
 & & \Big\downarrow{\scriptstyle \delta^0} & & \Big\downarrow{\scriptstyle \delta^{0,0}} & & \Big\downarrow{\scriptstyle \delta^{0,1}} & & \\
0 & \longrightarrow & C^1(\mathcal{U}, \mathcal{U}'; \mathscr{O}_X) & \hookrightarrow & C^1(\mathcal{U}, \mathcal{U}'; C_X^{\infty,(0,0)}) & \xrightarrow{\bar{\partial}^{1,0}} & C^1(\mathcal{U}, \mathcal{U}'; C_X^{\infty,(0,1)}) & \xrightarrow{\bar{\partial}^{1,1}} & \cdots\,. \\
 & & \Big\downarrow{\scriptstyle \delta^1} & & \Big\downarrow{\scriptstyle \delta^{1,0}} & & \Big\downarrow{\scriptstyle \delta^{1,1}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Note that $\bar\partial^{p,q}$ is the Dolbeault operator $\bar\partial$. Since $\mathcal{U}$ is a Stein covering, each row is exact (Corollary 2.5.12 of Hörmander[6]). By the Weil procedure, the first column is quasi-isomorphic to the complex $\operatorname{tot}(C^\bullet(\mathcal{U},\mathcal{U}';C_X^{\infty,(0,\bullet)}))$ (Theorem 12.5.4 of Kashiwara-Schapira[12]). The complex $\operatorname{tot}(C^\bullet(\mathcal{U},\mathcal{U}';C_X^{\infty,(0,\bullet)}))$ is defined as follows: For any $p,q\in\mathbb{Z}$, we write $A^{p,q}$ for $C^p(\mathcal{U},\mathcal{U}';C_X^{\infty,(0,q)})$ and let A denote a double complex $\{A^{p,q},\delta^{p,q},\bar\partial^{p,q}\}_{p,q\in\mathbb{Z}}$. For any $k\in\mathbb{Z}$, let us define

$$\operatorname{tot}(A)^k=\bigoplus_{p+q=k}A^{p,q},$$

and

$$d^k_{\operatorname{tot}(A)}\Big|_{A^{p,q}}=\delta^{p,q}+(-1)^p\bar\partial^{p,q}.$$

Here the simple complex $\big(\operatorname{tot}(A),d_{\operatorname{tot}(A)}\big)$ is called the total complex of a double complex A.

On the other hand, for any $q\in\mathbb{Z}$, we have isomorphisms

$$\begin{aligned}C_X^{\infty,(0,q)}(U)&\simeq C^0(\mathcal{U},\mathcal{U}';C_X^{\infty,(0,q)}),\\ C_X^{\infty,(0,q-1)}(U\setminus\Omega)&\simeq\operatorname{Ker}\delta^{1,q-1}\subset C^1(\mathcal{U},\mathcal{U}';C_X^{\infty,(0,q-1)}),\end{aligned}$$

since $C_X^{\infty,(0,q-1)}$ is a sheaf. Then, a morphism

$$\begin{array}{rcccl}\bar\vartheta^q &:& C^0(\mathcal{U},\mathcal{U}';C_X^{\infty,(0,q)})\oplus\operatorname{Ker}\delta^{1,q-1} & \longrightarrow & C^0(\mathcal{U},\mathcal{U}';C_X^{\infty,(0,q+1)})\oplus\operatorname{Ker}\delta^{1,q}\\ && \Cup && \Cup\\ && (\mu_1,\mu_{01}) & \longmapsto & (\bar\partial^{0,q}\mu_1,\delta^{0,q}\mu_1-\bar\partial^{1,q-1}\mu_{01}),\end{array}$$

defines the differential of Čech-Dolbeault complex in Definition 3.1.

Now, let us prove that the diagram chasing on the double complex $C^\bullet(\mathcal{U},\mathcal{U}';C_X^{\infty,(0,\bullet)})$ sends a cocycle of $C^n(\mathcal{U},\mathcal{U}';\mathscr{O}_X)$ to a cocycle of $C^0(\mathcal{U},\mathcal{U}';C_X^{\infty,(0,n)})\oplus\operatorname{Ker}\delta^{1,n-1}$ which belongs to the same cohomology class as the one given by b_Ω in Definition 3.22. For any $i\in\Lambda'=\{0,1,\cdots,n\}$, let φ_i be a cutoff function defined in Example 3.21. Firstly, we regard $\bigoplus_{i=0}^n F_{\hat\imath}\in\operatorname{Ker}\delta^n\subset C^n(\mathcal{U},\mathcal{U}';\mathscr{O}_X)$ as an element $\omega^{n,0}$ in $C^n(\mathcal{U},\mathcal{U}';C_X^{\infty,(0,0)})$. Secondly, for any $\alpha=(\alpha_0,\alpha_1,\cdots,\alpha_{n-1})\in\Lambda^n=\{0,1,\cdots,n+1\}^n$, we define

$$\omega_\alpha^{n-1,0}=\sum_{i\notin\alpha}\varphi_iF_{i\alpha_0\cdots\alpha_{n-1}}\in C_X^{\infty,(0,0)}(U_{\alpha_0}\cap U_{\alpha_1}\cap\cdots\cap U_{\alpha_{n-2}}).$$

Then, $\omega^{n-1,0}=\{\omega_\alpha^{n-1,0}\}_{\alpha\in\Lambda^n}\in C^{n-1}(\mathcal{U},\mathcal{U}';C_X^{\infty,(0,0)})$ satisfies $\delta^{n-1,0}(\omega^{n-1,0})=\omega^{n,0}$. Since $\bar\partial F_{\hat\imath}=0$ for any $i\in\Lambda'$, we have

$$\bar\partial\omega_\alpha^{n-1,0}=\sum_{i\notin\alpha}\bar\partial\varphi_iF_{i\alpha_0\cdots\alpha_{n-2}}.$$

Finally, we set $\omega^{n-1,1}=\bar\partial\omega^{n-1,0}\in C^{n-1}(\mathcal{U},\mathcal{U}';C_X^{\infty,(0,1)})$. If we repeat this procedure, we get $\omega^{1,n-1}=\sum_{i=0}^n F_{\hat\imath}\,\nu_{i,01}$, where the representative $\nu_i=(0,\nu_{i,01})\in C_X^{\infty,(0,n-1)}(U,U\setminus\Omega)$ is defined in Example 3.21. Summing up, $b_\Omega\left(\left[\bigoplus_{i=0}^{n}(-1)^i F_{\widehat{i}}\right]\otimes e_{M/X}\right)=[(0,\omega^{1,n-1})]\otimes e_{M/X}\in H_{\bar\vartheta}^{0,n}(U,U\setminus\Omega)\otimes or_{M/X}(\Omega)$ holds, and the above diagram chasing induces the same morphism from $H^n(\mathcal{U},\mathcal{U}';\mathscr{O}_X)\otimes or_{M/X}(\Omega)\to H_{\bar\vartheta}^{0,n}(U,U\setminus\Omega)\otimes or_{M/X}(\Omega)$ as the one given by the quasi-isomorphisms

$$C^\bullet(\mathcal{U},\mathcal{U}';\mathscr{O}_X)\xrightarrow{\sim}\operatorname{tot}(C^\bullet(\mathcal{U},\mathcal{U}';C_X^{\infty,(0,\bullet)}))\xleftarrow{\sim}C_X^{\infty,(0,\bullet)}(U,U\setminus\Omega).$$

The Weil procedure shows that the left morphism is quasi-isomorphic, as at the beginning of this proof. The right quasi-isomorphism is explained in HIS[4] and Suwa[25]. This completes the proof. □

3.3 Infinitesimal wedges and embedding of distributions via relative Čech cohomology

In this subsection, we define an embedding of distributions via Čech cohomology. Kaneko[7] and AKY[1] are helpful for readers. We use the same notations as Subsection 3.2, i.e., we keep the notations below:

- $M=\mathbb{R}^n$, $X=\mathbb{C}^n$,
- $\Omega\in\operatorname{Op}(M)$, $U\in\operatorname{Op}(X)$ is a Stein neighborhood of Ω,
- $v_0,v_1,\cdots,v_{n-1}$ are basis vectors of $\mathbb{R}^n$, which has the same orientation as that of $e_{M/X}$.
- $v_n=-(v_0+v_1+\cdots+v_{n-1})$,
- $H_i=\{y\in\mathbb{R}^n\mid\langle y,v_i\rangle>0\}$ $(0\le i\le n)$,
- $U_i=(\mathbb{R}^n+\sqrt{-1}H_i)\cap U$ $(0\le i\le n)$, $U_{n+1}=U$,
- $\Lambda'=\{0,1,\cdots,n\}$, $\Lambda=\{0,1,\cdots,n+1\}$,
- $\mathcal{U}'=\{U_i\}_{i\in\Lambda'}$, $\mathcal{U}=\{U_i\}_{i\in\Lambda}$,
- $U_{\widehat{i}}=U_0\cap\cdots\cap\widehat{U_i}\cap\cdots\cap U_{n+1}$ $(0\le i\le n)$.

We set $\Gamma_i=H_0\cap\cdots\cap\widehat{H_i}\cap\cdots\cap H_n$ for any $i=0,1,\cdots,n$, and we have $U_{\widehat{i}}=(\Omega+\sqrt{-1}\Gamma_i)\cap U$.

Definition 3.24 (Proper cone and Dual cone) A subset $\Gamma\subset\mathbb{R}^n$ is called a cone if Γ satisfies

$$\{ry\in\mathbb{R}^n\mid\ y\in\Gamma,r>0\}\subset\Gamma.$$

We define a proper cone $\Gamma\subset\mathbb{R}^n$ as a cone $\Gamma\subset\mathbb{R}^n$ satisfying

$$\overline{\Gamma}\setminus\{0\}\subset\{y\in\mathbb{R}^n\mid\ \langle\xi,y\rangle>0\}$$

for some $\xi\in\mathbb{R}^n$. For any cone $\Gamma\subset\mathbb{R}^n$, its dual cone Γ° is defined by

$$\Gamma^\circ=\{\xi\in\mathbb{R}^n\mid\ \langle\xi,y\rangle\ge0\ (\forall y\in\Gamma)\}.$$

Definition 3.25 (Infinitesimal wedge, [7, Definition 2.2.9]) Let $\Gamma \subset \mathbb{R}^n$ be an open cone and $\Omega \in \mathrm{Op}(\mathbb{R}^n)$. An open set $W \subset \mathbb{C}^n$ is called an infinitesimal wedge of type $\Omega + \sqrt{-1}\,\Gamma\, 0$ if it satisfies the following conditions:

(a) $W \subset \Omega + \sqrt{-1}\,\Gamma$ holds.
(b) For any open subcone $\Gamma' \subset \Gamma$ with $\overline{\Gamma'} \setminus \{0\} \subset \Gamma$ and for any compact set $K \subset \Omega$, there exists $\delta > 0$ such that

$$K + \sqrt{-1}\,\left(\Gamma' \cap \left\{y \in \mathbb{R}^n \;\middle|\; |y| < \delta\right\}\right) \subset W\,.$$

Ω is called the edge of the infinitesimal wedge W of type $\Omega + \sqrt{-1}\,\Gamma\, 0$, and Γ is called the opening of the infinitesimal wedge W of type $\Omega + \sqrt{-1}\,\Gamma\, 0$.

$F(z) \in \mathscr{O}_X(\Omega + \sqrt{-1}\,\Gamma\, 0)$ means that $F(z)$ is holomorphic in an infinitesimal wedge of type $\Omega + \sqrt{-1}\,\Gamma\, 0$. We set, for $0 \le i < j \le n$,

$$\Gamma_{ij} = H_0 \cap \cdots \cap \widehat{H_i} \cap \cdots \cap \widehat{H_j} \cap \cdots \cap H_n,$$

and define the morphism $\rho : \bigoplus\limits_{0\le i<j\le n} \mathscr{O}_X(\Omega + \sqrt{-1}\Gamma_{ij}0) \to \bigoplus\limits_{0\le i\le n} \mathscr{O}_X(\Omega + \sqrt{-1}\Gamma_i 0)$ by

$$\sum_{0\le i<j\le n} F_{\widehat{ij}} \longmapsto \sum_{0\le i\le n} \left(\sum_{k=0}^{i-1} (-1)^k F_{\widehat{ki}}\big|_{\Omega+\sqrt{-1}\Gamma_i 0} - \sum_{k=i+1}^{n} (-1)^k F_{\widehat{ik}}\big|_{\Omega+\sqrt{-1}\Gamma_i 0} \right).$$

Here, for any $0 \le k < i$ (resp. $i < k \le n$), $F_{\widehat{ki}} \in \mathscr{O}_X(\Omega + \sqrt{-1}\Gamma_{ki}0)$ (resp. $F_{\widehat{ik}} \in \mathscr{O}_X(\Omega + \sqrt{-1}\Gamma_{ki}0)$), and $\widehat{ki}$ (resp. $\widehat{ik}$) means $0, 1, \cdots, k-1, k+1, \cdots, i-1, i+1, \cdots, n+1 \in \Lambda^n$ (resp. $0, 1, \cdots, i-1, i+1, \cdots, k-1, k+1, \cdots, n+1 \in \Lambda^n$) as in Remark 3.16.

Definition 3.26 We define

$$\hat{H}(\{\Omega + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X) = \frac{\bigoplus\limits_{0\le i\le n} \mathscr{O}_X(\Omega + \sqrt{-1}\Gamma_i 0)}{\rho\left(\bigoplus\limits_{0\le i<j\le n} \mathscr{O}_X(\Omega + \sqrt{-1}\Gamma_{ij} 0)\right)}.$$

Remark 3.27 Since $\hat{H}(\{\Omega + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X)$ is a variant of Čech cohomology group, correctly speaking, we regard an element $F_{\hat{i}} \in \mathscr{O}_X(\Omega + \sqrt{-1}\Gamma_i 0)$ as an alternative section like the definition of Čech cohomology. In this article, we identify $F_{\hat{i}}$ with the alternative section $\{\widetilde{F}_\alpha\}_{\alpha \in \{0,1,\cdots,i-1,i+1,\cdots,n+1\}^{n+1}}$ satisfying $\widetilde{F}_{\hat{i}} = \widetilde{F}_{0,1,\cdots,i-1,i+1,\cdots,n+1} = F_{\hat{i}}$.

Theorem 3.28 There exists the canonical isomorphism

$$H^n(\mathcal{U}, \mathcal{U}'; \mathscr{O}_X) \simeq \hat{H}(\{\Omega + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X).$$

Proof. Let $V \in \mathrm{Op}(\mathbb{C}^n)$ be a Stein neighborhood of Ω. For a sufficiently small $\delta > 0$, we define an open covering $\mathcal{V}' = \{V_0, V_1, \cdots, V_n\}$ of $V \setminus \Omega$ by

$$V_i = V \cap \left\{ z \in \mathbb{C}^n \;\middle|\; x \in \Omega, \delta|y| < \langle y, v_i \rangle \right\} \quad (i = 0, 1, \cdots, n),$$

and an open covering $\mathcal{V}$ of V by

$$\mathcal{V} = \{V_0, V_1, \cdots, V_n, V_{n+1} = V\}.$$

Note that each V_i is also a Stein open set. By the definition of infinitesimal wedges, the restriction morphism induces the canonical morphisms

$$H^n(\mathcal{U}, \mathcal{U}', \mathscr{O}_X) \xrightarrow{\rho_1} \hat{H}(\{\Omega + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X) \xrightarrow{\rho_2} \varinjlim_{\Omega \subset V} H^n(\mathcal{V}, \mathcal{V}', \mathscr{O}_X),$$

where V runs through Stein neighborhoods of Ω. Using the same arguments as in the proof of Proposition 3.17, all the morphisms in the following commutative diagram are isomorphic:

$$\begin{CD}
H^n_\Omega(U; \mathscr{O}_X) @>>> \varinjlim\limits_{\Omega \subset V} H^n_\Omega(V; \mathscr{O}_X) \\
@VVV @VVV \\
H^n(\mathcal{U}, \mathcal{U}'; \mathscr{O}_X) @>>> \varinjlim\limits_{\Omega \subset V} H^n(\mathcal{V}, \mathcal{V}'; \mathscr{O}_X).
\end{CD}$$

This means that $\rho_2 \circ \rho_1$ is isomorphic. It follows from the theorem of the edge of the wedge by A. Martineau (see Corollary 4.6.13 [7]) that ρ_2 is injective. Thus we conclude that ρ_1 is an isomorphism. □

Note that Komori-Umeta[15] explains the general cases of Theorem 3.28.

Remark 3.29 As in the case of Remark 3.20, we write a representative of a hyperfunction $u = \left[\bigoplus_i (-1)^i F_{\widehat{i}}\right] \otimes e_{M/X} \in \hat{H}(\{\Omega + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X) \otimes or_{M/X}(\Omega)$ as $\bigoplus_i (-1)^i F_{\widehat{i}} \in \bigoplus_i \mathscr{O}_X(\Omega + \sqrt{-1}\Gamma_i 0)$ for a fixed generator $e_{M/X} \in or_{M/X}(\Omega)$. Since the equation

$$\left[\bigoplus_{i=0}^n (-1)^i F_{\widehat{i}} \right] \otimes e_{M/X} = \sum_{i=0}^n \left[F_{\widehat{i}} \right] \otimes [e_i] \in \hat{H}(\{\Omega + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X) \otimes or_{M/X}(\Omega)$$

holds, Kaneko[7] writes a Čech representative of a hyperfunction $\sum_i \left[F_{\widehat{i}}\right] \otimes [e_i]$ as $\sum_i F_{\widehat{i}}(x + \sqrt{-1}\Gamma_i 0)$. Here, $[e_i] \in or_{M/X}(\Omega)$ is defined in Definition 3.18.

Fix a sufficiently small $\delta > 0$ and set

$$H_{\delta,i} = \left\{ y \in \mathbb{R}^n \;\middle|\; \delta|y| < \langle y, v_i \rangle \right\} \qquad (i = 0, 1, \cdots, n).$$

We also define $H_{\delta,\widehat{i}}$ in the same way as $U_{\widehat{i}}$, that is,

$$H_{\delta,\widehat{i}} = H_{\delta,0} \cap \cdots \cap \widehat{H_{\delta,i}} \cap \cdots \cap H_{\delta,n}.$$

Let $\nu_i = (0, \nu_{i,01})$ $(i = 0, 1, \cdots, n)$ be the one constructed in Example 3.21, where we replace the condition (1) in Example 3.21 with the following $(1)^*$:

$(1)^*$ $\mathrm{Supp}_{U\setminus\Omega}(\varphi_i) \subset \Omega + \sqrt{-1}H_{\delta,i} \qquad (i = 0, 1, \cdots, n)$,

Then, in this case, the resulting ν_i satisfies

$$\mathrm{Supp}_{U\setminus\Omega}(\nu_{i,01}) \subset \Omega + \sqrt{-1}H_{\delta,\widehat{i}} \qquad (i = 0, 1, \cdots, n).$$

Note that $H_{\delta,\widehat{i}}$ is a convex cone properly contained in Γ_i. By the definition of the infinitesimal wedge, for any infinitesimal wedge W of type $\Omega+\sqrt{-1}\Gamma_i$, there exists an complex neighborhood V of Ω such that

$$\mathrm{Supp}_{U\setminus\Omega}(\nu_{i,01}) \cap V \subset (\Omega + \sqrt{-1}H_{\delta,\widehat{i}}) \cap V \subset W,$$

from which we have the well-defined morphism

$$\bigoplus_{0\le i\le n} \mathscr{O}_X(\Omega + \sqrt{-1}\Gamma_i 0) \otimes or_{M/X}(\Omega) \longrightarrow \varinjlim_{V} H^{0,n}_{\bar\vartheta}(V, V\setminus\Omega) \otimes or_{M/X}(\Omega)$$

by

$$\left(\bigoplus_{0\le i\le n} (-1)^i F_{\widehat{i}}\right) \otimes e_{M/X} \longmapsto \left[\sum_{0\le i\le n} F_{\widehat{i}}\big|_V \, \nu_i\big|_V\right] \otimes e_{M/X},$$

where V runs through Stein neighborhoods of Ω. The above morphism sends $\rho\left(\bigoplus_{0\le i<j\le n} \mathscr{O}_X(\Omega + \sqrt{-1}\Gamma_{ij}0)\right) \otimes or_{M/X}(\Omega)$ to 0. Hence we have the morphism

$$\hat{H}(\{\Omega + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X) \otimes or_{M/X}(\Omega) \longrightarrow \varinjlim_{\Omega\subset V} H^{0,n}_{\bar\vartheta}(V, V\setminus\Omega) \otimes or_{M/X}(\Omega).$$

Thanks to Theorems 3.23 and 3.28, we obtain the theorem below.

Theorem 3.30 We have the following commutative diagram whose morphisms are all isomorphic:

$$\begin{array}{ccc}
H^n(\mathcal{U}, \mathcal{U}'; \mathscr{O}_X) \otimes or_{M/X}(\Omega) & \xrightarrow{\quad b_\Omega \quad} & H^{0,n}_{\bar\vartheta}(U, U\setminus\Omega) \otimes or_{M/X}(\Omega) \\
\downarrow & & \downarrow \\
\hat{H}(\{\Omega + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X) \otimes or_{M/X}(\Omega) & \longrightarrow & \varinjlim_{\Omega\subset V} H^{0,n}_{\bar\vartheta}(V, V\setminus\Omega) \otimes or_{M/X}(\Omega).
\end{array}$$

Proof. It follows from Theorem 3.28 that the first vertical morphism is isomorphic. By Proposition 3.8, the second vertical morphism is also isomorphic. Since the map b_Ω is an isomorphism by Theorem 3.23, the second horizontal morphism is also isomorphic, which completes the proof. $\square$

Definition 3.31 The second horizontal isomorphism in the above theorem

$$\hat{H}(\{\Omega + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X) \otimes or_{M/X}(\Omega) \longrightarrow \varinjlim_{\Omega \subset V} H_{\bar{\vartheta}}^{0,n}(V, V \setminus \Omega) \otimes or_{M/X}(\Omega)$$

is also denoted by b_Ω.

Now then, let us consider an embedding of distributions via Čech cohomology. For any z, $\zeta \in \mathbb{C}^n$, we use the notations $z\zeta = z_1\zeta_1 + \cdots + z_n\zeta_n$ and $\zeta^2 = \zeta_1^2 + \cdots + \zeta_n^2$.

Definition 3.32 For any (z, ζ), the twisted Radon kernel $W(z, \zeta)$ is defined by

$$W(z,\zeta) = \frac{(n-1)!}{(-2\pi\sqrt{-1})^n} \frac{(1-\sqrt{-1}z\zeta/\sqrt{\zeta^2})^{n-1} - (1-\sqrt{-1}z\zeta/\sqrt{\zeta^2})^{n-2}(z^2-(z\zeta)^2/\zeta^2)}{\{z\zeta + \sqrt{-1}(z^2\sqrt{\zeta^2} - (z\zeta)^2/\sqrt{\zeta^2})\}^n}.$$

Here, we define $\sqrt{\bullet} : \mathbb{C} \to \mathbb{C}$ by

$$\sqrt{z} = \sqrt{r}\, e^{\sqrt{-1}\theta/2} \quad (r > 0,\ -\pi < \theta < \pi,\ z = re^{\sqrt{-1}\theta}).$$

Lemma 3.33 ([7, Lemma 2.3.4]) We write $z = x + \sqrt{-1}y$ and $\zeta = \xi + \sqrt{-1}\eta$. Let $\Gamma \subset \mathbb{R}^n$ be an arbitrary proper convex open cone and Γ° its dual cone. There exists a $2n$ dimensional open convex cone $\Delta \subset \mathbb{R}_y^n \times \mathbb{R}_\eta^n$ such that $\Delta \cap \{\eta = 0\} \supset \Gamma$ and $W(z,\zeta) \in \mathscr{O}_{X\times X}\left(\mathbb{R}^n \times \mathrm{Int}(\Gamma^\circ) + \sqrt{-1}\Delta 0\right)$.

Moreover, $W(z, \zeta)$ can be analytically continued to a neighborhood of the real open set $(\mathbb{R}^n \setminus \{0\}) \times \mathrm{Int}(\Gamma^\circ)$. In particular, if we fix $\zeta = \xi \in \mathbb{R}^n \setminus \{0\}$, $W(z,\zeta)$ is holomorphic as a function of z in the following infinitesimal wedge with the half-space $\{y \in \mathbb{R}^n \mid y\xi > 0\}$ as the opening:

$$\mathbb{R}^n + \sqrt{-1}\left\{y \in \mathbb{R}^n \;\middle|\; y\xi > y^2|\xi| - (y\xi)^2/|\xi|\right\}.$$

Furthermore, it is analytically continued to the following complex neighborhood of $\mathbb{R}^n \setminus \{0\}$:

$$\left\{x + \sqrt{-1}\,y \in \mathbb{C}^n \;\middle|\; |y\xi| + (y^2|\xi| - (y\xi)^2/|\xi|) < \frac{1}{4}\left(|x\xi| + 2(x^2|\xi| - (x\xi)/|\xi|)\right)\right\}.$$

Lemma 3.34 ([7, Lemma 2.3.5]) Let $f(x)$ be a C^n function on Ω supported by a compact set $K \subset \Omega$. We define

$$F(z,\zeta) = \int_{\mathbb{R}^n} f(w)W(z-w,\zeta)\,dw.$$

Then, $F(z,\zeta)$ satisfies the following conditions:

(1) For any proper convex open cone $\Gamma \subset \mathbb{R}^n$, F is a holomorphic function of (z,ζ) in the infinitesimal wedge of type $\mathbb{R}^n \times \mathrm{Int}(\Gamma^\circ) + \sqrt{-1}\Delta 0$ where Δ is defined in Lemma 3.33.

(2) F can be analytically continued to a neighborhood of $(\Omega \setminus K) \times (\mathbb{R}^n \setminus \{0\})$.

(3) F can be continuously extended to $\Omega \times (\mathbb{R}^n \setminus \{0\})$, and the continuously extended function $F(x,\xi)$ satisfies

$$f(x) = \int_{S^{n-1}} F(x,\xi)\,d\xi.$$

Lemma 3.35 ([7, Corollary 2.3.6]) We have the equation

$$\int_{S^{n-1}} W(z,\xi)\, d\xi = 0$$

in a neighborhood of $\mathbb{R}^n \setminus \{0\}$. This integral converges locally uniformly in z.

Definition 3.36 For any non-empty proper cone $\Gamma \subset \mathbb{R}^n$, we define

$$W(z,\Gamma^\circ) = \int_{\Gamma^\circ \cap S^{n-1}} W(z,\xi)\, d\xi \in \mathscr{O}(\mathbb{R}^n + \sqrt{-1}\Gamma 0).$$

Definition 3.37 (Embedding of $\mathscr{D}b_K$ in Čech representation) Let K be a compact set in Ω. For any $u \in \mathscr{D}b_K(\Omega)$ and for any $i = 0, 1, \cdots, n$, we set

$$F_{\hat{i}}(z) = \langle u(w), W(z-w, \Gamma_{\hat{i}}^\circ)\, dw \rangle \in \mathscr{O}(\Omega + \sqrt{-1}\Gamma_i 0),$$

and let us define $\iota^C_{\mathscr{D}b_K(\Omega)} : \mathscr{D}b_K(\Omega) \to \mathscr{B}_K(\Omega)$ by $u \mapsto \left[\bigoplus_i (-1)^i F_{\hat{i}}\right] \otimes e_{M/X}$. Here $e_{M/X}$ is the section of $or_{M/X}(\Omega)$ specified at the beginning of Subsection 3.2.

Proposition 3.38 $\mathrm{Supp}(\iota^C_{\mathscr{D}b_K(\Omega)}(u)) \subset \mathrm{Supp}(u)$ holds.

This proposition easily follows from Lemma 3.35.

Definition 3.39 (Embedding of $\mathscr{D}b$ in Čech representation) Let $u \in \mathscr{D}b_M(\Omega)$. Decompose u into a locally finite sum as $u = \sum_\lambda u_\lambda$, where $u_\lambda \in \mathscr{D}b_{K_\lambda}(\Omega)$ and K_λ is a compact set in Ω. We define $\iota^C_{\mathscr{D}b_M(\Omega)}(u) = \sum_\lambda \iota^C_{\mathscr{D}b_{K_\lambda}(\Omega)}(u_\lambda) \in \mathscr{B}_M(\Omega)$.

Remark 3.40 Since $\langle u(w), W(z-w, \Gamma_{\hat{i}}^\circ)\, dw \rangle \in \mathscr{O}(\Omega + \sqrt{-1}\Gamma_i 0)$ for any $i = 0, 1, \cdots, n$, $\iota^C_{\mathscr{D}b_M(\Omega)}$ is a map from $\mathscr{D}b_M(\Omega)$ to $\hat{H}(\{\Omega + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X) \otimes or_{M/X}(\Omega) = \mathscr{B}_M(\Omega)$.

Proposition 3.41 $\iota^C_{\mathscr{D}b_M(\Omega)}$ is well-defined, and $\iota^C_{\mathscr{D}b_M} = \{\iota^C_{\mathscr{D}b_M(\Omega)}\}_{\Omega \in \mathrm{Op}(M)}$ is a sheaf morphism from $\mathscr{D}b_M$ to $\mathscr{B}_M$.

Proposition 3.42 ([7, Theorem 3.5.5]) The diagram below commutes.

$$\begin{tikzcd} \mathscr{A}_M(\Omega) \arrow[r, hook] \arrow[d, "\iota^C_{\mathscr{A}_M(\Omega)}"] & \mathscr{D}b_M(\Omega), \arrow[dl, "\iota^C_{\mathscr{D}b_M(\Omega)}"] \\ H^n(\mathcal{U},\mathcal{U}';\mathscr{O}_X) \otimes or_{M/X}(\Omega) & \end{tikzcd}$$

where $\mathscr{A}_M(\Omega) \hookrightarrow \mathscr{D}b_M(\Omega)$ is the canonical embedding, and $\iota^C_{\mathscr{A}_M(\Omega)}$ is defined in Definition 3.18. Note that we identify $H^n(\mathcal{U},\mathcal{U}';\mathscr{O}_X)$ with $\hat{H}(\{\Omega + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X)$ by Theorem 3.28.

3.4 Embedding of distributions via relative Čech-Dolbeault cohomology

HIS[4] defines an embedding $\mathscr{D}b_M \hookrightarrow \mathscr{B}_M$ in the framework of Čech-Dolbeault cohomology. Let $M = \mathbb{R}^n$, $X = \mathbb{C}^n$ and $\Omega \in \mathrm{Op}(M)$. Since $M = \mathbb{R}^n$ is oriented, there is an isomorphism $or_M \simeq \mathbb{Z}_M$ which also gives the isomorphism

$$\mathscr{B}_M(\Omega) \simeq H^n_\Omega(U; \mathscr{O}_X),$$

where U is a complex neighborhood of Ω. By this isomorphism, we omit the section of the relative orientation sheaf in the definition of $\mathscr{B}_M(\Omega)$. Now, let $\theta(z,t)$ be a C^∞ function on $X \times \Omega$ which satisfies the following conditions:

(1) $\theta(z,t)$ is identically 1 in a neighborhood of $\{0\} \times \Omega$.
(2) $\mathrm{Supp}(\theta) \subset T$, where T is an open set in $X \times \Omega$ such that

$$\{0\} \times \Omega \subset T \subset \left\{(z,t) \in X \times \Omega \;\middle|\; |z| < 3^{-1} \min\{1\,,\, \mathrm{dist}(t, M \setminus \Omega)\}\right\}.$$

Here, $\mathrm{dist}(t, M \setminus \Omega)$ means the distance from t to $M \setminus \Omega$ in M, and we set $\mathrm{dist}(t, \emptyset) = +\infty$.

Definition 3.43 ([4, Theorem 8.1]) Let $\tau = (\tau_1, \tau_{01}) \in C_X^{\infty,(0,n)}(X, X \setminus M)$ be a Čech-Dolbeault representative of Dirac's delta function. For any $u \in \mathscr{D}b_M(\Omega)$, we define

$$\iota^{CD}_{\mathscr{D}b_M(\Omega)}(u) = \left[\left(\int_\Omega (\theta\tau_1 + \bar{\partial}_z\theta \wedge \tau_{01})(z-t,t)u(t)dt, \int_\Omega (\theta\tau_{01})(z-t,t)u(t)dt\right)\right].$$

Proposition 3.44 ([4, Lemma 8.19]) $\iota^{CD}_{\mathscr{D}b_M} = \{\iota^{CD}_{\mathscr{D}b_M(\Omega)}\}_{\Omega \in \mathrm{Op}(M)}$ is a sheaf morphism from $\mathscr{D}b_M$ to $\mathscr{B}_M$.

We omit the proof of this proposition. For more details, see Section 8 of HIS[4]. Thanks to Proposition 3.44, by gluing the embedding morphisms on each local chart, we can obtain the embedding morphism $\iota^{CD}_{\mathscr{D}b_M} : \mathscr{D}b_M \to \mathscr{B}_M$ for a real analytic manifold M.

4 Embedding of C^∞ functions

4.1 Limits of $(n-1)$-forms

Let M be a real analytic, n-dimensional, oriented manifold and X its complexification. We define a normal bundle $T_M X$ by $\mathrm{Coker}\,(TM \to M \underset{X}{\times} TX)$ as a vector bundle, and we write the coordinate of $T_M X$ as (x, η). Moreover, we equip $T_M X$ with a bundle metric and fix it in what follows. By the tubular neighborhood theorem, we can find an isomorphism $T_M X \simeq X$

with the following commutative diagram:

$$\begin{tikzcd} T_M X \arrow[r, "\sim"] & X. \\ M \arrow[u] \arrow[ur] & \end{tikzcd}$$

Here $M \to T_M X$ is the embedding which regards M as the zero section of $T_M X$ and $M \to X$ is the closed embedding. Note that we replace X with a small open neighborhood of M in X if necessary, and that we identify X with $T_M X$ in this subsection.

We define $B = \{(x, \eta) \in T_M X = X \mid |\eta| \leq 1\}$ and $S = \partial B$. Let $\tau \colon S \to M$ be the restriction of the canonical projection $X \simeq T_M X \to M$, and let i be an embedding $S \hookrightarrow X \setminus M$. Set $S_x = S \cap \tau^{-1}(x)$ for $x \in M$. The projection $\tau : S \to M$ induces maps

$$\tau_* : TS \to S \underset{M}{\times} TM \quad \text{and} \quad \tau^* : S \underset{M}{\times} T^*M \to T^*S.$$

For any $\lambda > 0$ and for any $(x, \eta) \in T_M X \simeq X$, let us define a map $\phi_\lambda : X \setminus M \to X \setminus M$ by $\phi_\lambda(x, \eta) = (x, \lambda\eta)$. We note that the $\operatorname{Ker} \tau_*$ consists of a vector field on S which is tangent to S_x for any $x \in M$.

Definition 4.1 For any $\mu \in \bigwedge^{n-1} T^*(X \setminus M)$ and for any $v \in \bigwedge^{n-1} \operatorname{Ker} \tau_*$, if the limit

$$\lim_{\lambda \to +0} \langle i^* \phi_\lambda^* \mu, v \rangle$$

exists and converges locally uniformly on S, we say that μ converges locally uniformly on S. Then, there exists a form $\tilde{\mu} \in \bigwedge^{n-1} \operatorname{Coker} \tau^*$ such that

$$\langle \tilde{\mu}, v \rangle = \lim_{\lambda \to +0} \langle i^* \phi_\lambda^* \mu, v \rangle$$

for any $v \in \bigwedge^{n-1} \operatorname{Ker} \tau_*$. We define $L(\mu) = -2^{n-1} \tilde{\mu}$.

Since a form in $\operatorname{Im} \tau^* \subset T^*S$ becomes zero on $\operatorname{Ker} \tau_* \subset TS$, we can regard $L(\mu)|_{S_x}$ as an $(n-1)$-form on S_x. Hence we may consider the integration of $L(\mu)$ on S_x and $I(\mu)(x) = \displaystyle\int_{S_x} L(\mu)$ is a continuous function on M. By a system of local coordinates, we can describe Definition 4.1 concretely as follows:

Remark 4.2 We identify S with $M \times \sqrt{-1} S^{n-1}$ and take $\omega = (\omega_1, \cdots, \omega_n)$ as a system of homogeneous coordinates of S^{n-1}. We define $i_x : S_x \to X \setminus M$ by

$$\begin{array}{cccc} i_x : & S_x & \longrightarrow & X \setminus M \\ & \cup & & \cup \\ & \omega & \longmapsto & x + \sqrt{-1}\omega \end{array}$$

for any $x \in M$. Let us take

$$\mu(z) = \sum_{|I|+|J|=n-1} f_{I,J}(z)\, dz_I \wedge d\bar{z}_J \in \bigwedge^{n-1} T^*(X \setminus M)\,,$$

and we define $\varepsilon_{I,J}$ by

$$\varepsilon_{I,J} = \begin{cases} \operatorname{sgn}\begin{pmatrix} i_1 & \cdots & i_p & j_1 & \cdots & j_q \\ k_1 & \cdots & k_p & k_{p+1} & \cdots & k_{n-1} \end{pmatrix} & \begin{array}{l}\text{if } i_1, \cdots, i_p, j_1, \cdots, j_q \text{ are} \\ \text{mutually distinct,}\end{array} \\ 0 & \text{otherwise,} \end{cases}$$

where $p, q \in \{0, 1, 2, \cdots, n\}$, $p+q = n-1$, $I = (i_1, \cdots, i_p) \in \{1, 2, \cdots, n\}^p$, $J = (j_1, \cdots, j_q) \in \{1, 2, \cdots, n\}^q$, $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, $d\bar{z}_J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$, $\{k_1, \cdots, k_{n-1}\} = I \cup J$ and $k_1 < k_2 < \cdots < k_{n-1}$.

If the limit

$$\begin{aligned} &\lim_{\lambda \to +0} (i_x^* \circ \phi_\lambda^*(\mu))(\omega) \\ &= \lim_{\lambda \to +0} (\sqrt{-1}\,\lambda)^{n-1} \sum_{\substack{|I|+|J|=n-1, \\ k \in \{1, \cdots, n\} \setminus (I \cup J)}} (-1)^{|J|+k+1}\, \varepsilon_{I,J}\, \omega_k\, f_{I,J}(x + \sqrt{-1}\lambda\omega) \end{aligned}$$

exists and converges to a continuous function locally uniformly on S for any $(x, \sqrt{-1}\omega) \in M \times \sqrt{-1}S^{n-1}$, we say that μ converges locally uniformly on S. Then $L(\mu)$ exists and it is given by

$$\begin{aligned} L(\mu)(x, \omega) &= -2^{n-1} \lim_{\lambda \to +0} (i_x^* \circ \phi_\lambda^*(\mu))(\omega) \\ &= \lim_{\lambda \to +0} (2\sqrt{-1}\,\lambda)^{n-1} \sum_{\substack{|I|+|J|=n-1, \\ k \in \{1, \cdots, n\} \setminus (I \cup J)}} (-1)^{|J|+k}\, \varepsilon_{I,J}\, \omega_k\, f_{I,J}(x + \sqrt{-1}\lambda\omega)\, ds. \end{aligned}$$

Here $ds = \sum_{i=1}^{n} (-1)^{i+1} \omega_i\, d\omega_1 \wedge \cdots \wedge \widehat{d\omega_i} \wedge \cdots \wedge d\omega_n$.

4.2 Reconstruction of C^∞ functions with compact support

Let $M = \mathbb{R}^n$ and $X = \mathbb{C}^n$, and let $\Omega \in \mathrm{Op}(M)$ and $U \in \mathrm{Op}(X)$ such that U is a complex neighborhood of Ω. Since $M = \mathbb{R}^n$ is oriented, we omit the section of the relative orientation sheaf in the definition of $\mathscr{B}_M(\Omega)$.

Theorem 4.3 Let K be a compact set in Ω, $f(x) \in C^n(\Omega)$ whose support is contained in K, and let $\mu = (\mu_1, \mu_{01}) \in C_X^{\infty,(0,n)}(U, U \setminus K)$ be a representative of $\iota^{CD}_{\mathscr{D}b_M(\Omega)}(f)$. We define

$$\widetilde{\mathcal{R}}(z, \zeta) = \int_D W(z - w, \zeta)\, \mu_1(w) \wedge dw - \int_{\partial D} W(z - w, \zeta)\, \mu_{01}(w) \wedge dw,$$

where D is an open set with C^∞ boundary satisfying $K \subset D \subset \overline{D} \subset U$ and $z \notin \overline{D}$. Then, $\widetilde{\mathcal{R}}$ does not depend on the choices of a representative μ and a domain D, and satisfies the following conditions:

(1) For any proper convex open cone $\Gamma \subset \mathbb{R}^n$, $\widetilde{\mathcal{R}}$ is a holomorphic function of (z, ζ) in the infinitesimal wedge of type $\mathbb{R}^n \times \mathrm{Int}(\Gamma^\circ) + \sqrt{-1}\Delta 0$ where Δ is defined in Lemma 3.33.
(2) $\widetilde{\mathcal{R}}$ can be analytically continued to a neighborhood of $(\Omega \setminus K) \times (\mathbb{R}^n \setminus \{0\})$.
(3) $\widetilde{\mathcal{R}}$ can be continuously extended to $\Omega \times (\mathbb{R}^n \setminus \{0\})$, and the continuously extended function $\widetilde{\mathcal{R}}(x, \xi)$ satisfies

$$f(x) = \int_{S^{n-1}} \widetilde{\mathcal{R}}(x, \xi)\, d\xi.$$

Proof. Firstly, we show that $\mu \in \mathrm{Im}\,\bar{\vartheta} \Rightarrow \widetilde{\mathcal{R}} = 0$. By the assumption, there exists (τ_1, τ_{01}) such that $\mu_1 = \bar{\partial}\tau_1$ on U, $\mu_{01} = \tau_1 - \bar{\partial}\tau_{01}$ on $U \setminus K$. Then, we have the following calculation:

$$\begin{aligned}
\widetilde{\mathcal{R}}(z, \zeta) &= \int_D W(z-w, \zeta)\, \bar{\partial}\tau_1(w) \wedge dw - \int_{\partial D} W(z-w, \zeta)\, (\tau_1(w) - \bar{\partial}\tau_{01}(w)) \wedge dw \\
&= \int_{\partial D} W(z-w, \zeta)\, \bar{\partial}\tau_{01}(w) \wedge dw \qquad \text{(the Stokes formula and } d = \partial + \bar{\partial}) \\
&= 0 \qquad (\partial\partial D = \emptyset).
\end{aligned}$$

Hence $\widetilde{\mathcal{R}}$ does not depend on the choice of representatives.

Secondly, we prove that $\widetilde{\mathcal{R}}$ does not depend on a domain D. Let $K \subset D' \subset D \subset U$ and $z \notin \overline{D}$. Since

$$\begin{aligned}
\widetilde{\mathcal{R}}(z, \zeta) &= \int_D W(z-w, \zeta)\, \mu_1(w) \wedge dw - \int_{\partial D} W(z-w, \zeta)\, \mu_{01}(w) \wedge dw \\
&= \int_{D'} W(z-w, \zeta)\, \mu_1(w) \wedge dw - \int_{\partial D'} W(z-w, \zeta)\, \mu_{01}(w) \wedge dw \\
&\quad + \int_{D \setminus D'} W(z-w, \zeta)\, \mu_1(w) \wedge dw - \int_{\partial(D \setminus D')} W(z-w, \zeta)\, \mu_{01}(w) \wedge dw,
\end{aligned}$$

it is sufficient to prove that

$$\int_{D \setminus D'} W(z-w, \zeta)\, \mu_1(w) \wedge dw = \int_{\partial(D \setminus D')} W(z-w, \zeta)\, \mu_{01}(w) \wedge dw.$$

This equation follows from $\bar{\partial}\mu_{01} = \mu_1$ on $D \setminus D' \subset U \setminus K$ and the Stokes formula.

Finally, let us prove

$$\widetilde{\mathcal{R}}(z, \zeta) = \int_\Omega W(z-t, \zeta) f(t) dt\,.$$

We choose Bochner-Martinelli type $(0, -(-1)^{\frac{n(n+1)}{2}}\beta)$ as a representative of Dirac's δ, where β is defined by

$$\beta(z) = (-1)^{\frac{n(n-1)}{2}} \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \frac{1}{\|z\|^{2n}} \sum_{i=1}^n (-1)^{i-1} \bar{z}_i\, d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_i} \wedge \cdots \wedge d\bar{z}_n.$$

We take $\theta(z,t) \in C^\infty(U \times \Omega)$ satisfying the conditions of Definition 3.43. By the definition of $\iota^{CD}_{\mathscr{D}b_M(\Omega)}(f)$, we can choose a representative μ as follows:

$$\mu = -(-1)^{\frac{n(n+1)}{2}} \left(\int_\Omega f(t)\bar{\partial}_z \theta(z-t,t) \wedge \beta(z-t)\, dt\, ,\, \int_\Omega f(t)\theta(z-t,t)\beta(z-t)\, dt \right).$$

Using the Bochner-Martinelli formula (Lemma A.5), we get

$$\begin{aligned}
\widetilde{\mathcal{R}}(z,\zeta) &= -(-1)^{\frac{n(n+1)}{2}} \int_D W(z-w,\zeta) \left(\int_\Omega f(t)\bar{\partial}_w \theta(w-t,t) \wedge \beta(w-t)\, dt \right) \wedge dw \\
&\qquad + (-1)^{\frac{n(n+1)}{2}} \int_{\partial D} W(z-w,\zeta) \left(\int_\Omega f(t)\theta(w-t,t)\beta(w-t)\, dt \right) \wedge dw \\
&= \int_\Omega f(t)\, dt \left((-1)^{\frac{n(n+1)}{2}} \int_{\partial D} W(z-w,\zeta)\theta(w-t,t)\beta(w-t) \wedge dw \right. \\
&\qquad \left. -(-1)^{\frac{n(n+1)}{2}} \int_D W(z-w,\zeta)\bar{\partial}_w \theta(w-t,t) \wedge \beta(w-t) \wedge dw \right) \\
&= \int_\Omega f(t) W(z-t,\zeta)\theta(0,t)\, dt \quad = \int_\Omega f(t) W(z-t,\zeta)\, dt.
\end{aligned}$$

By Lemma 3.34, $f(x) = \displaystyle\int_{S^{n-1}} \widetilde{\mathcal{R}}(x,\xi)\, d\xi$ holds. □

Definition 4.4 Let K be a compact set in Ω. For any $u = [\mu] = [(\mu_1, \mu_{01})] \in \mathscr{B}_K(\Omega)$, we define

$$\widetilde{\mathcal{R}}[\mu](z,\zeta) = \int_D W(z-w,\zeta)\, \mu_1(w) \wedge dw - \int_{\partial D} W(z-w,\zeta)\, \mu_{01}(w) \wedge dw,$$

where D is an open set with C^∞ boundary satisfying $K \subset D \subset \overline{D} \subset U$ and $z \notin \overline{D}$.

Since $\widetilde{\mathcal{R}}[\mu]$ does not depend on the choice of μ by the same arguments as in the proof of Theorem 4.3, for any $u = [\mu] \in \mathscr{B}_K(\Omega)$, we write $\widetilde{\mathcal{R}}[u] = \widetilde{\mathcal{R}}[\mu]$. Recall that $M = \mathbb{R}^n$, $X = \mathbb{C}^n$ and $\Omega \in \mathrm{Op}(M)$.

Lemma 4.5 We assume that Γ_i is defined as in Subsection 3.3 for any $i = 0, 1, \cdots, n$. Let K be a compact set in Ω, U a Stein neighborhood of Ω and $u = [\mu] = [(\mu_1, \mu_{01})] \in \mathscr{B}_K(\Omega)$ ($\mu \in C_X^{\infty,(0,n)}(U, U \setminus K)$). We define $\widetilde{\mathcal{R}}[\mu](z, \Gamma_i^\circ)$ by

$$\widetilde{\mathcal{R}}[\mu](z,\Gamma_i^\circ) = \int_{\Gamma_i^\circ \cap S^{n-1}} \widetilde{\mathcal{R}}[\mu](z,\xi)\, d\xi \,\in \mathscr{O}_X(\Omega + \sqrt{-1}\Gamma_i 0).$$

Then, $\displaystyle\bigoplus_{i=0}^{n} (-1)^i \widetilde{\mathcal{R}}[\mu](z,\Gamma_i^\circ)$ is a Čech representative of u, and the morphism

$$\begin{aligned}
\mathcal{R} : H^{0,n}_{\bar{\vartheta}}(U, U \setminus K) \otimes or_{M/X}(\Omega) &\to \hat{H}(\{\Omega + \sqrt{-1}\Gamma_i 0\}_{i=0}^n; \mathscr{O}_X) \otimes or_{M/X}(\Omega) \\
&\simeq H^n(\mathcal{U}, \mathcal{U}'; \mathscr{O}_X) \otimes or_{M/X}(\Omega)
\end{aligned}$$

defined by

$$\mathcal{R}(u) = \mathcal{R}(\mu) = \left[\bigoplus_{i=0}^{n} (-1)^i \widetilde{\mathcal{R}}[\mu](z, \Gamma_i^\circ)\right]$$

satisfies $b_\Omega \circ \mathcal{R} = \mathrm{id}$ on $\mathscr{B}_K(\Omega)$, where b_Ω is given in Definition 3.31.

Proof. Since $\widetilde{\mathcal{R}}[\mu](z, \Gamma_i^\circ)$ can be analytically continued to a complex neighborhood U' of $\Omega \setminus K$ and $\sum_{i=0}^{n} (-1)^i \widetilde{\mathcal{R}}[\mu](z, \Gamma_i^\circ) = 0$ holds in U', for any $\psi \in \mathscr{A}_M[K]$, we can perform the integration of the hyperfunction $\psi \mathcal{R}(u)$ as the one defined by Kaneko (see Section 3.4 of [7]). Let us show

$$\langle \mathcal{R}(\mu), \psi dx \rangle = \langle \mu, \psi dx \rangle \quad (\psi \in \mathscr{A}_M[K]).$$

Here, the above symbols $\langle \bullet, \bullet \rangle$ on the left-hand side and the right-hand side are cohomological pairings realized by the integrations of Čech cohomology and Čech-Dolbeault cohomology, respectively.

Let $U'' \in \mathrm{Op}(X)$ be the intersection of U and the domain of $\psi \in \mathscr{A}_M[K]$, and let W_i $(i = 0, \cdots, n)$ be the infinitesimal wedge of type $\Omega + \sqrt{-1}\Gamma_i 0$ on which $\widetilde{\mathcal{R}}[\mu](z, \Gamma_i^\circ)$ is defined. By the definition of the integration of Čech representation, we have

$$\begin{aligned}
&\langle \mathcal{R}(\mu), \psi dx \rangle \\
&= \sum_i \int_{(-1)^i(\Omega' + \sqrt{-1}\,\varepsilon_i)} \psi(z)\,(-1)^i\, \widetilde{\mathcal{R}}[\mu](z, \Gamma_i^\circ)\, dz \\
&= \sum_i \int_{\Omega' + \sqrt{-1}\,\varepsilon_i} \psi(z)\, \widetilde{\mathcal{R}}[\mu](z, \Gamma_i^\circ)\, dz \\
&= \sum_i \int_{\Omega' + \sqrt{-1}\,\varepsilon_i} \psi(z) \int_{\Gamma_i^\circ \cap S^{n-1}} \Big(\int_D W(z - w, \xi)\, \mu_1(w) \wedge dw \\
&\qquad\qquad - \int_{\partial D} W(z - w, \xi)\, \mu_{01}(w) \wedge dw \Big)\, d\xi\, dz \\
&= \int_D \Big(\sum_i \int_{\Gamma_i^\circ \cap S^{n-1}} \int_{\Omega' + \sqrt{-1}\,\varepsilon_i} \psi(z)\, W(z - w, \xi)\, dz\, d\xi \Big)\, \mu_1(w) \wedge dw \\
&\quad - \int_{\partial D} \Big(\sum_i \int_{\Gamma_i^\circ \cap S^{n-1}} \int_{\Omega' + \sqrt{-1}\,\varepsilon_i} \psi(z)\, W(z - w, \xi)\, dz\, d\xi \Big)\, \mu_{01}(w) \wedge dw \\
&= \int_D \psi(w)\, \mu_1(w) \wedge dw - \int_{\partial D} \psi(w)\, \mu_{01}(w) \wedge dw \\
&= \langle \mu, \psi dx \rangle\,,
\end{aligned}$$

where $\Omega'\,(\supset K\,)$ is an open set in Ω such that $\Omega' \subset\subset \Omega$, $\varepsilon_i : \Omega \to \mathbb{R}^n$ is a smooth mapping such that

$$\begin{cases} \varepsilon_i(t) = 0 & (t \in \Omega \setminus \Omega'), \\ t + \sqrt{-1}\, r\, \varepsilon_i(t) \in (\Omega' + \sqrt{-1}\Gamma_i) \cap W_i \cap U'' & (t \in \Omega',\ 0 < r \leq 1), \end{cases}$$

$\Omega' + \sqrt{-1}\,\varepsilon_i = \{t + \sqrt{-1}\,\varepsilon_i(t) \mid t \in \Omega'\}$ and $-(\Omega' + \sqrt{-1}\,\varepsilon_i)$ is just $\Omega' + \sqrt{-1}\,\varepsilon_i$ with the opposite orientation, and $D \in \mathrm{Op}(X)$ satisfies $K \subset D \subset \overline{D} \subset U''$ and $(\Omega' + \sqrt{-1}\,\varepsilon_i) \cap D = \emptyset$.

Since $\mathscr{B}_K(\Omega)$ is the dual topological vector space of $\mathscr{A}_M[K]$ by Theorem 3.7, $\langle \mathcal{R}(\mu), \psi dx\rangle = \langle \mu, \psi dx\rangle$ means that $\mathcal{R}(\mu)$ and μ represent the same hyperfunction. Because b_Ω is isomorphic, we get $b_\Omega \circ \mathcal{R} = \mathrm{id}$ on $\mathscr{B}_K(\Omega)$. □

Corollary 4.6 Let $\iota_{\mathscr{A}_M(\Omega)} : \mathscr{A}_M(\Omega) \to \mathscr{D}b_M(\Omega)$ be the canonical embedding. The diagram below commutes.

$$\begin{tikzcd}
\mathscr{A}_M(\Omega) \arrow[rr, bend left, "\iota_{\mathscr{A}_M(\Omega)}"] \arrow[dr, "\iota^C_{\mathscr{A}_M(\Omega)}"] \arrow[ddr, bend right, "\iota^{CD}_{\mathscr{A}_M(\Omega)}"'] & \text{(T)} & \mathscr{D}b_M(\Omega). \arrow[dl, "\iota^C_{\mathscr{D}b_M(\Omega)}"'] \arrow[ddl, bend left, "\iota^{CD}_{\mathscr{D}b_M(\Omega)}"] \\
\text{(L)} & H^n(\mathcal{U},\mathcal{U}';\mathscr{O}_X)\otimes or_{M/X}(\Omega) \arrow[d, "b_\Omega"] & \text{(R)} \\
 & H^{0,n}_{\bar\vartheta}(U, U\setminus\Omega)\otimes or_{M/X}(\Omega) &
\end{tikzcd}$$

Note that we identify $H^n(\mathcal{U},\mathcal{U}';\mathscr{O}_X)$ with $\hat{H}(\{\Omega+\sqrt{-1}\Gamma_i 0\}_{i=0}^n;\mathscr{O}_X)$ by Theorem 3.28.

Proof. By Proposition 3.42, (T) commutes. The commutativity of (L) is obvious from the definition. Let us prove that (R) commutes. Since all morphisms are induced from the sheaf morphisms and $\mathscr{D}b_M$ is a soft sheaf, it is enough to prove the claim locally. Hence, we may consider only sections with compact support.

Let Ω be an open set in M, K a compact set in Ω and u a section of $\mathscr{D}b_K(\Omega)$. Using the same arguments as in the proof of Theorem 4.3, we have

$$\widetilde{\mathcal{R}}[\mu](z,\Gamma_i^\circ) = \langle u(t), W(z-t,\Gamma_i^\circ)dt\rangle \quad (i = 0, 1, \cdots, n),$$

where μ is a Čech-Dolbeault representative of $\iota^{CD}_{\mathscr{D}b_M(\Omega)}(u)$. By Definition 3.37, the diagram below commutes.

$$\begin{tikzcd}
H^n(\mathcal{U},\mathcal{U}';\mathscr{O}_X)\otimes or_{M/X}(\Omega) & \mathscr{D}b_K(\Omega). \arrow[l, "\iota^C_{\mathscr{D}b_M(\Omega)}"'] \arrow[dl, "\iota^{CD}_{\mathscr{D}b_M(\Omega)}"] \\
H^{0,n}_{\bar\vartheta}(U, U\setminus K)\otimes or_{M/X}(\Omega) \arrow[u, "\mathcal{R}"] &
\end{tikzcd}$$

Then, we obtain the commutativity of (R) by Lemma 4.5. □

By Corollary 4.6, $\iota^{CD}_{\mathscr{D}b_M}\big|_{\mathscr{A}_M}$ coincides with $\iota^{CD}_{\mathscr{A}_M}$.

4.3 Inverse map of embedding of C^∞ functions

Let us consider the characterization of $\iota^{CD}_{\mathscr{D}b_M}(C^\infty)$ and the inverse map from $\iota^{CD}_{\mathscr{D}b_M}(C^\infty)$ to C^∞.

Let M be a real analytic, n-dimensional, oriented manifold and X its complexification, and let $\Omega \in \mathrm{Op}(M)$ and $U \in \mathrm{Op}(X)$ such that U is a complex neighborhood of Ω. Since M is oriented, we omit the section of the relative orientation sheaf in the definition of $\mathscr{B}_M(\Omega)$.

Throughout this subsection, we use the same notations as in Subsection 4.1: We write $B = \{(x,\eta) \in X \mid |\eta| \le 1\}$, $S = \partial B$ and $S_x = \partial B \cap \tau^{-1}(x)$ for any $x \in \Omega$. We may assume $B \subset U$. Remember that the projection $\tau : S \to \Omega$ is associated with the projection $U \simeq T_\Omega U \to \Omega$ which is given by the tubular neighborhood theorem. The projection $\tau : S \to \Omega$ induces maps

$$\tau_* : TS \to S \underset{\Omega}{\times} T\Omega \quad \text{and} \quad \tau^* : S \underset{\Omega}{\times} T^*\Omega \to T^*S.$$

In what follows, since the problem is local, we regard U (resp. Ω) as X (resp. M). Note that we can regard S_x and S as S^{n-1} and $\Omega \times \sqrt{-1}S^{n-1}$, respectively.

Definition 4.7 A form $\mu_{01} \in C_X^{\infty,(0,n-1)}(U \setminus \Omega)$ is said to belong to $C_X^{\infty,\mathrm{qw},(0,n-1)}(U \setminus \Omega)$ if the following conditions are satisfied.

(1) $\mu_{01} \in \bigwedge^{n-1} T^*(U \setminus \Omega)$ converges to a continuous form $L(\mu_{01}) \in \bigwedge^{n-1} \mathrm{Coker}\,\tau^*$ locally uniformly on S.

(2) $L(\mu_{01})$ is a C^∞ form on S.

Here, $L(\mu_{01})$ is defined in Subsection 4.1.

Note that we call $C_X^{\infty,\mathrm{qw},(0,n-1)}(U \setminus \Omega)$ a space of quasi-Whitney $(0, n-1)$-forms. By Remark 4.2, we can rephrase Definition 4.7 as follows:

Remark 4.8 A form $\mu_{01} \in C_X^{\infty,(0,n-1)}(U \setminus \Omega)$ is said to belong to $C_X^{\infty,\mathrm{qw},(0,n-1)}(U \setminus \Omega)$ if the following conditions are satisfied.

(1) For any $x \in \Omega$ and $\omega \in S^{n-1}$, the limit

$$\lim_{\lambda \to +0} (-2\sqrt{-1}\,\lambda)^{n-1} \sum_{i=1}^{n} (-1)^i \omega_i f_i(x + \sqrt{-1}\lambda\omega)$$

exists and converges to a continuous function locally uniformly with respect to $(x, \sqrt{-1}\omega) \in \Omega \times \sqrt{-1}S^{n-1}$, where $f_1, \cdots, f_n$ are coefficients of μ_{01}, that is,

$$\mu_{01} = \sum_{i=1}^{n} f_i \, d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_i} \wedge \cdots \wedge d\bar{z}_n.$$

(2) $L(\mu_{01})$ is a C^∞ form on $\Omega \times \sqrt{-1}S^{n-1}$.

Recall that we can write

$$L(\mu_{01})(x,\omega) = \lim_{\lambda\to+0}(-2\sqrt{-1}\,\lambda)^{n-1}\sum_{i=1}^{n}(-1)^i\,\omega_i\,f_i(x+\sqrt{-1}\lambda\omega)\,ds, \tag{4.1}$$

and $ds = \sum_{i=1}^{n}(-1)^{i+1}\omega_i\,d\omega_1\wedge\cdots\wedge\widehat{d\omega_i}\wedge\cdots\wedge d\omega_n$.

Definition 4.9 Let us define

$$C_X^{\infty,\mathrm{qw},(0,n)}(U,U\setminus\Omega) = \left\{\mu=(\mu_1,\mu_{01})\in C_X^{\infty,(0,n)}(U,U\setminus\Omega)\;\middle|\;\mu_{01}\in C_X^{\infty,\mathrm{qw},(0,n-1)}(U\setminus\Omega)\right\}.$$

The following theorem is our main result.

Theorem 4.10 Let $x_0\in M$. There exist an open neighborhood Ω of x_0 and a complex neighborhood U of Ω for which the following 2 conditions are equivalent. For any $u\in\mathscr{B}_M(\Omega)$,

(1) There is a C^∞ function f on Ω such that $u=\iota^{CD}_{\mathscr{D}b_M(\Omega)}(f)$.

(2) There is a Čech-Dolbeault representative $\mu=(\mu_1,\mu_{01})$ of u such that $\mu\in C_X^{\infty,\mathrm{qw},(0,n)}(U,U\setminus\Omega)$.

Additionally,

$$I(\mu)(x)=\int_{S_x}L(\mu_{01})(x,\omega)\in C^\infty(\Omega)$$

is well-defined and we get $I(\mu)=f$.

We give a global version of Theorem 4.10 for $M=\mathbb{R}^n$.

Theorem 4.11 Assume $M=\mathbb{R}^n$ and $X=\mathbb{C}^n$. For any $\Omega\in\mathrm{Op}(M)$, for any $U\in\mathrm{Op}(X)$ which is a complex neighborhood of Ω and for any $u\in\mathscr{B}_M(\Omega)$, the conditions (1) and (2) as in Theorem 4.10 are equivalent. Additionally, we define $I(\mu)$ as the same integration in Theorem 4.10. Then, $I(\mu)$ is well-defined and $I(\mu)=f$ holds.

Considering a local coordinate system, it is enough to prove Theorem 4.11 for Theorem 4.10. To show Theorem 4.11, we prove the following 3 lemmas (Lemma 4.12, Lemma 4.13 and Lemma 4.14). Hereafter, let us assume $M=\mathbb{R}^n$, $X=\mathbb{C}^n$, and let $\Omega\in\mathrm{Op}(M)$ and $U\in\mathrm{Op}(X)$ such that U is a complex neighborhood of Ω.

Lemma 4.12 Let u be a hyperfunction on Ω, and let μ and μ' be representatives of u. If $\mu,\,\mu'\in C_X^{\infty,\mathrm{qw},(0,n)}(U,U\setminus\Omega)$, then $I(\mu)=I(\mu')$.

Proof. Fix $x_0\in\Omega$. Let us prove that there exists an open neighborhood Ω_0 of x_0 such that $I(\mu)=I(\mu')$ on Ω_0. Let $x_0\in U''\subset\subset U'\subset\subset U$, $\Omega''=U''\cap\Omega$, and $\Omega'=U'\cap\Omega$. We can choose representatives $\tilde{\mu}$ and $\tilde{\mu}'$ satisfying the following conditions (see Lemma A.4):

(1) $\tilde{\mu}$ and $\tilde{\mu}'$ determine the same hyperfunction on Ω.

(2) $\tilde{\mu} = \mu$ on U'' and $\tilde{\mu}' = \mu'$ on U''.

(3) $\operatorname{Supp}\tilde{\mu} \subset U'$ and $\operatorname{Supp}\tilde{\mu}' \subset U'$.

It is clear that $\tilde{\mu}|_{U''}, \tilde{\mu}'|_{U''} \in C_X^{\infty,\mathrm{qw},(0,n)}(U'', U'' \setminus \Omega'')$. Then, we set

$$\widetilde{\mathcal{R}}[\tilde{\mu}](z,\zeta) = \int_D W(z-w,\zeta)\,\tilde{\mu}_1(w) \wedge dw - \int_{\partial D} W(z-w,\zeta)\,\tilde{\mu}_{01}(w) \wedge dw,$$
$$\widetilde{\mathcal{R}}[\tilde{\mu}'](z,\zeta) = \int_D W(z-w,\zeta)\,\tilde{\mu}'_1(w) \wedge dw - \int_{\partial D} W(z-w,\zeta)\,\tilde{\mu}'_{01}(w) \wedge dw,$$

with the notations of Theorem 4.3. We may assume $U'' \subset\subset D$. Since $\tilde{\mu}$ and $\tilde{\mu}'$ satisfy the condition (1), we can prove $\widetilde{\mathcal{R}}[\tilde{\mu}](z,\zeta) = \widetilde{\mathcal{R}}[\tilde{\mu}'](z,\zeta)$ in the same way as the proof of Theorem 4.3. Hence, it is enough to prove that there exists an open neighborhood $\Omega_0 \subset \Omega''$ of x_0 such that $\widetilde{\mathcal{R}}[\tilde{\mu}](z,\zeta)$ can be continuously extended to $\Omega_0 \times (\mathbb{R}^n \setminus \{0\})$ and we have

$$\int_{S^{n-1}} \widetilde{\mathcal{R}}[\tilde{\mu}](x,\xi)d\xi = I(\mu)(x) \qquad (x \in \Omega_0).$$

Let D' be an open subset of U such that $D \cap (\Omega \setminus \Omega'') \subset D'$ and $x_0 \in \operatorname{Int}(\Omega'' \setminus D')$. We define D'' as an open subset of Ω'' such that $D'' = \operatorname{Int}(\Omega'' \setminus D')$. By deforming D, we get

$$\begin{aligned}\widetilde{\mathcal{R}}[\tilde{\mu}](z,\zeta) = &\int_{D'} W(z-w,\zeta)\,\tilde{\mu}_1(w) \wedge dw - \int_{\partial D'} W(z-w,\zeta)\,\tilde{\mu}_{01}(w) \wedge dw \\ &+ \int_{D''} W(z-t,\zeta)\left(\int_{S^{n-1}} L(\mu_{01})(t,\omega)\right) dt,\end{aligned}$$

for any $z \notin \overline{D' \cup D''}$. Note that, by $\tilde{\mu} = \mu$ on U'', we have $L(\tilde{\mu}_{01})(t,\omega) = L(\mu_{01})(t,\omega)$ $(t \in D'', \omega \in S^{n-1})$. The deformation of D is illustrated in Figure 1.

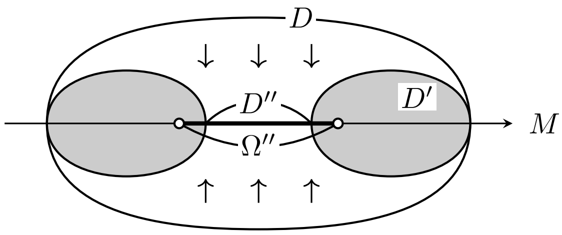

Figure1 Deformation of the integration domain D

From Lemma 3.33, the first and the second integrals can be analytically continued to a complex neighborhood of D'', and the third integral can be continuously extended to $D'' \times$

$(\mathbb{R}^n \setminus \{0\})$ by Theorem 4.3 (3). Hence, for any $x \in D''$, the equation

$$\begin{aligned}\int_{S^{n-1}} \widetilde{\mathcal{R}}[\tilde{\mu}](x,\xi)\, d\xi = &\int_{S^{n-1}} \left(\int_{D'} W(x-w,\xi)\, \tilde{\mu}_1(w) \wedge dw \right) d\xi \\ &- \int_{S^{n-1}} \left(\int_{\partial D'} W(x-w,\xi)\, \tilde{\mu}_{01}(w) \wedge dw \right) d\xi \\ &+ \int_{S^{n-1}} \int_{D''} W(x-t,\xi) \left(\int_{S^{n-1}} L(\mu_{01})(t,\omega) \right) dt\, d\xi\end{aligned}$$

holds. By Lemma 3.35, the first and the second integrals become 0. Then, we get the equation

$$\int_{S^{n-1}} \widetilde{\mathcal{R}}[\tilde{\mu}](x,\xi)\, d\xi = \int_{S^{n-1}} \int_{D''} W(x-t,\xi)\, I(\mu)(t)\, dt\, d\xi \qquad (x \in D'').$$

By applying Lemmas 3.34 and 3.35 to the case $f = I(\mu)$, we obtain

$$\int_{S^{n-1}} \int_{D''} W(x-t,\xi)\, I(\mu)(t)\, dt\, d\xi = I(\mu)(x) \qquad (x \in D'').$$

This completes the proof. □

By Lemma 4.12, $I(\mu)$ does not depend on the choice of $\mu \in C_X^{\infty,\mathrm{qw},(0,n)}(U, U \setminus \Omega)$. If a hyperfunction u is represented by an element μ of $C_X^{\infty,\mathrm{qw},(0,n)}(U, U \setminus \Omega)$, we write $I(u)$ for $I(\mu)$ hereafter. We can regard I as a map

$$I : \frac{C_X^{\infty,\mathrm{qw},(0,n)}(U, U \setminus \Omega)}{C_X^{\infty,\mathrm{qw},(0,n)}(U, U \setminus \Omega) \cap \bar{\vartheta}(C_X^{\infty,(0,n-1)}(U, U \setminus \Omega))} \longrightarrow C^\infty(\Omega) \ .$$

Lemma 4.13 Let $f \in C^\infty(\Omega)$. If $\iota^{CD}_{\mathscr{D}b_M(\Omega)}(f)$ has a representative $\mu \in C_X^{\infty,\mathrm{qw},(0,n)}(U, U \setminus \Omega)$, then $I \circ \iota^{CD}_{\mathscr{D}b_M(\Omega)}(f) = f$.

Proof. Let $x \in \Omega'' \subset\subset \Omega' \subset\subset \Omega$ and $\psi \in C_0^\infty(\Omega)$ such that

$$\psi(t) = \begin{cases} 1 & \text{on } \Omega'', \\ 0 & \text{on } \Omega \setminus \Omega'. \end{cases}$$

We write complex neighborhoods of Ω'', Ω', and Ω as $U'' \subset\subset U' \subset\subset U$, respectively. By Lemma A.4, there exists a representative $\tilde{\mu}$ of $\iota^{CD}_{\mathscr{D}b_M(\Omega)}(\psi f)$ such that

$$\tilde{\mu} = \mu \text{ on } U'' \quad \text{and} \quad \tilde{\mu} = 0 \text{ on } U \setminus U' \,.$$

Since $\mathrm{Supp}(\iota^{CD}_{\mathscr{D}b_M(\Omega)}(\psi f)) \subset \overline{\Omega'}$, it follows from Theorem 4.3 and Lemma 4.12 that

$$I(\mu)(x) = \int_{S^{n-1}} \widetilde{\mathcal{R}}[\tilde{\mu}](x,\xi)\, d\xi = \psi(x) f(x) = f(x)$$

hold for any $x \in \Omega''$. This completes the proof. □

Remember that $\Omega \in \mathrm{Op}(\mathbb{R}^n)$ and that $U \in \mathrm{Op}(\mathbb{C}^n)$ is a complex neighborhood of Ω.

Lemma 4.14 Let u be a hyperfunction on Ω. The following conditions are equivalent.

(i) There exists $f \in C^\infty(\Omega)$ such that $u = \iota^{CD}_{\mathscr{D}b_M(\Omega)}(f)$.

(ii) There exists a representative μ of u such that $\mu \in C^{\infty,\mathrm{qw},(0,n)}_X(U, U \setminus \Omega)$.

Proof. First, we prove (i) $\Rightarrow$ (ii). We define $(\tau_1, \tau_{01}) = \left(0, \dfrac{(n-1)!\,\bar\partial\varphi_1 \wedge \cdots \wedge \bar\partial\varphi_{n-1}}{(2\pi\sqrt{-1})^n\, z_1 \cdots z_n}\right)$ as a representative of Dirac's delta function. Here $\varphi_i \in C^\infty(X \setminus M)$ $(i = 1, \cdots, n)$ is defined as follows:

> Let $\{\widetilde\varphi_i\}_{i=1}^n$ be the partition of unity of S^{n-1} such that $\widetilde\varphi_i = 0$ on a neighborhood of $S^{n-1} \cap \{y_i = 0\}$ $(i = 1, \cdots, n)$. Then, we define
>
> $$\varphi_i(x + \sqrt{-1}r\omega) = \widetilde\varphi_i(\omega)$$
>
> for any $x \in M$, $r > 0$, $\omega \in S^{n-1}$ such that $x + \sqrt{-1}r\omega \in X$.

This representative is explained in Example 7.26 of HIS[4]. By the definition of $\iota^{CD}_{\mathscr{D}b_M(\Omega)}(f) = [\mu] = [(\mu_1, \mu_{01})]$, we obtain

$$\mu = \left(\int_\Omega (\theta\tau_1 + \bar\partial_z\theta \wedge \tau_{01})(z - t, t) f(t) dt, \int_\Omega (\theta\tau_{01})(z - t, t) f(t) dt\right).$$

Then, it is sufficient to prove that $\mu = (\mu_1, \mu_{01})$ satisfies $\mu_{01} \in C^{\infty,\mathrm{qw},(0,n-1)}_X(U \setminus \Omega)$. Note that we have

$$\mu_{01} = \frac{(n-1)!\,\bar\partial\varphi_1 \wedge \cdots \wedge \bar\partial\varphi_{n-1}}{(2\pi\sqrt{-1})^n} \int_\Omega \frac{\theta(z-t,t)f(t)}{(z_1 - t_1)\cdots(z_n - t_n)} dt.$$

Let $z = x + \sqrt{-1}y$, $r = \sqrt{y_1^2 + \cdots + y_n^2}$ and

$$(\omega_1, \cdots, \omega_n) = \left(\frac{y_1}{r}, \cdots, \frac{y_n}{r}\right) \in S^{n-1} \subset \mathbb{R}^n.$$

The embedding $i : S^{n-1} \to \mathbb{R}^n$ induces the map

$$i_* : TS^{n-1} \to S^{n-1} \underset{\mathbb{R}^n}{\times} T\mathbb{R}^n,$$

that is, by considering the $(n-1)$-sphere S^{n-1} as a subset of $\mathbb{R}^n$, a vector field on S^{n-1} can be described by a vector field on $\mathbb{R}^n$. The space $\mathrm{Im}\, i_*$ is generated by

$$\frac{\partial}{\partial\omega_i} = \frac{\partial}{\partial y_i} - \frac{y_i}{r^2}\sum_{j=1}^n y_j \frac{\partial}{\partial y_j} \quad (i = 1, 2, \cdots, n)$$

with the relation

$$\omega_1 \frac{\partial}{\partial\omega_1} + \cdots + \omega_n \frac{\partial}{\partial\omega_n} = 0,$$

and we have $\rho_*\left(\dfrac{\partial}{\partial y_i}\right) = r^{-1}\dfrac{\partial}{\partial \omega_i}$, where $\rho : Y' = \mathbb{R}^n \setminus \{0\} \ni y \mapsto y/r \in S^{n-1}$ and $\rho_* : TY' \to Y' \underset{S^{n-1}}{\times} TS^{n-1}$ is induced by ρ. Under the above consideration, we define

$$J_n = \begin{pmatrix} \dfrac{\partial \varphi_1}{\partial \bar{z}_1} & \cdots & \dfrac{\partial \varphi_1}{\partial \bar{z}_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \varphi_{n-1}}{\partial \bar{z}_1} & \cdots & \dfrac{\partial \varphi_{n-1}}{\partial \bar{z}_n} \end{pmatrix} \text{ and } \widetilde{J}_n = \begin{pmatrix} \dfrac{\partial \widetilde{\varphi}_1}{\partial \omega_1} & \cdots & \dfrac{\partial \widetilde{\varphi}_1}{\partial \omega_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \widetilde{\varphi}_{n-1}}{\partial \omega_1} & \cdots & \dfrac{\partial \widetilde{\varphi}_{n-1}}{\partial \omega_n} \end{pmatrix}.$$

Note that $\varphi_i(z) = \widetilde{\varphi}_i(\omega)$. Additionally, an $(n-1)\times(n-1)$ matrix J_{ni} (resp. $\widetilde{J}_{ni}$) denotes the submatrix of J_n (resp. $\widetilde{J}_n$) from which the i-th column is omitted. The $(0, n-1)$-form $\bar{\partial}\varphi_1 \wedge \cdots \wedge \bar{\partial}\varphi_{n-1}$ is written as

$$\left(\sum_{i=1}^n \frac{\partial \varphi_1}{\partial \bar{z}_i} d\bar{z}_i\right) \wedge \cdots \wedge \left(\sum_{i=1}^n \frac{\partial \varphi_{n-1}}{\partial \bar{z}_i} d\bar{z}_i\right) = \sum_{i=1}^n \det J_{ni}\ d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_i} \wedge \cdots \wedge d\bar{z}_n.$$

Since $\det J_{ni} = (-2\sqrt{-1}r)^{-n+1} \det \widetilde{J}_{ni}$, the coefficient of $L(\mu_{01})$ is represented by

$$\lim_{r\to+0} \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \sum_{i=1}^n (-1)^i\,\omega_i \det \widetilde{J}_{ni} \int_\Omega \frac{\theta(x+\sqrt{-1}r\omega - t, t) f(t)}{(x_1+\sqrt{-1}r\omega_1 - t_1)\cdots(x_n+\sqrt{-1}r\omega_n - t_n)} dt.$$

Here, $L(\mu_{01})$ is defined by the equation (4.1) in Remark 4.8.

Let us prove that the coefficient of $L(\mu_{01})$ converges to a continuous function locally uniformly on $\Omega \times \sqrt{-1}S^{n-1}$ (the condition (1) of Definition 4.7). The $\det \widetilde{J}_{ni}$ vanishes in a neighborhood of the set $\{y_1 y_2 \cdots y_n = 0\}$, since $\bar{\partial}\varphi_1 \wedge \cdots \wedge \bar{\partial}\varphi_{n-1}$ is 0 in a conic open neighborhood of the set $\{y_1 y_2 \cdots y_n = 0\}$. Therefore, we consider the above limit on $\{\omega_1 \cdots \omega_n \neq 0\}$. By Lebesgue's dominated convergence theorem, the integral

$$\int_\Omega \frac{\partial^n(\theta(x-t,t)f(t))}{\partial t_1 \cdots \partial t_n} \log(x_1 - t_1)\cdots\log(x_n - t_n) dt$$

is continuous on $x \in \Omega$. Then, as was in the proof of Corollary 2.3.2 of Kaneko[7], by employing the partial integration with respect to the variables $t_1, t_2, \cdots, t_n$ respectively, the coefficient of $L(\mu_{01})$ converges to a continuous function

$$\frac{(n-1)!}{(2\pi\sqrt{-1})^n} \sum_{i=1}^n (-1)^i\,\omega_i \det \widetilde{J}_{ni} \int_\Omega \frac{\partial^n(\theta(x-t,t)f(t))}{\partial t_1 \cdots \partial t_n} \log(x_1 - t_1)\cdots\log(x_n - t_n) dt$$

with respect to $(x, \sqrt{-1}\omega) \in \Omega \times \sqrt{-1}S^{n-1}$, which implies the condition (1) of Definition 4.7. By the repetition of partial integrations several times in the same way as above, we can also show that $L(\mu_{01})$ is a C^∞ form on $\Omega \times \sqrt{-1}S^{n-1}$ (the condition (2) of Definition 4.7).

Next, let us prove (ii) $\Rightarrow$ (i). It suffices to show $\iota^{CD}_{\mathscr{D}b_M(\Omega)} \circ I(\mu)$ and μ give the same equivalent class. Let Ω' be a sufficiently small open neighborhood of x, U' a complex neighborhood of Ω'

such that $U' \subset U$, and $\chi_{\Omega'}$ the characteristic function of Ω'. By considering $\chi_{\Omega'} I(\mu)$ instead of $I(\mu)$ and noticing that $\iota^{CD}_{\mathscr{D}b_M(\Omega)}$ is induced from a sheaf morphism, we have

$$\begin{aligned}
&\iota^{CD}_{\mathscr{D}b_M(\Omega)}(I(\mu))\Big|_{\Omega'} = \iota^{CD}_{\mathscr{D}b_M(\Omega)}(\chi_{\Omega'} I(\mu))\Big|_{\Omega'} \\
&= \left[\left(\int_{\Omega'} (\theta\tau_1 + \bar{\partial}_z\theta \wedge \tau_{01})(z-t,t) I(\mu)(t) dt, \int_{\Omega'} (\theta\tau_{01})(z-t,t) I(\mu)(t) dt\right)\right]\Bigg|_{\Omega'}.
\end{aligned}$$

Now we take (τ_1, τ_{01}) to be a representative of $b_M\left(\bigoplus_i (-1)^i \int_{\Gamma_i^\circ \cap S^{n-1}} W(z,\xi) d\xi\right)$. Note that b_M is defined in Definition 3.31 and that $\bigoplus_i (-1)^i \int_{\Gamma_i^\circ \cap S^{n-1}} W(z,\xi) d\xi$ is a Čech representative of Dirac's delta. Since U' is small enough, we can take the θ so that

$$(\tau_1(z), \tau_{01}(z)) = ((\theta\tau_1 + \bar{\partial}_z\theta \wedge \tau_{01})(z,t), \theta\tau_{01}(z,t)) \quad ((z,t) \in U' \times \Omega')$$

holds. Then, in the definition of b_Ω (Definitions 3.22 and 3.31), we take the specific φ_i's which are independent of the variables x such as the ones given in the first part of this proof. For such a b_Ω, we can interchange the order of b_Ω and the integrations with respect to the variable t, and hence, we have

$$\iota^{CD}_{\mathscr{D}b_M(\Omega)} \circ I(\mu)\Big|_{\Omega'} = \left[b_{\Omega'}\left(\bigoplus_i (-1)^i \int_{\Gamma_i^\circ \cap S^{n-1}} \int_{\Omega'} W(z-t,\xi) I(\mu)(t)\, dt\, d\xi\right)\right].$$

By Lemma 4.5, it is enough to prove that

$$\left[\bigoplus_i (-1)^i \widetilde{\mathcal{R}}[\tilde{\mu}](z, \Gamma_i^\circ)\right] = \left[\bigoplus_i (-1)^i \int_{\Gamma_i^\circ \cap S^{n-1}} \int_{\Omega'} W(z-t,\xi) I(\mu)(t)\, dt\, d\xi\right] \text{ on } \Omega' \tag{4.2}$$

holds as Čech representation, where $\tilde{\mu}$ is a Čech-Dolbeault representative such that $\tilde{\mu}|_{U'} = \mu|_{U'}$ and $\tilde{\mu} \in C_X^{\infty,(0,n)}(U, U \setminus K)$ for a compact set K with $\overline{\Omega'} \subset \mathrm{Int}(K) \subset K \subset \Omega$. $\widetilde{\mathcal{R}}[\tilde{\mu}](z, \Gamma_i^\circ)$ is defined by

$$\widetilde{\mathcal{R}}[\tilde{\mu}](z, \Gamma_i^\circ) = \int_{\Gamma_i^\circ \cap S^{n-1}} \left(\int_D W(z-w,\xi) \tilde{\mu}_1(w) \wedge dw - \int_{\partial D} W(z-w,\xi) \tilde{\mu}_{01}(w) \wedge dw\right) d\xi\,.$$

By deforming D suitably and using the same arguments as in the proof of Lemma 4.12, we can prove the equation (4.2). This completes the proof. □

A Appendix

Let X be a complex manifold, U an open set in X and S a closed set in U.

Definition A.1 For any complex $(\mathscr{F}^\bullet, d^\bullet)$ of sheaves on X, we define

$$\mathscr{F}^q(U, U \setminus S) = \mathscr{F}^q(U) \oplus \mathscr{F}^{q-1}(U \setminus S),$$

and

$$\begin{array}{rcccl} \vartheta^q & : & \mathscr{F}^q(U, U \setminus S) & \longrightarrow & \mathscr{F}^{q+1}(U, U \setminus S) \\ & & \cup & & \cup \\ & & (s_1, s_{01}) & \longmapsto & (d^q(s_1),\, s_1|_{U\setminus S} - d^{q-1}(s_{01})). \end{array}$$

Then, $(\mathscr{F}^\bullet(U, U \setminus S), \vartheta^\bullet)$ is a complex of abelian groups. $H^q(U, U \setminus S; \mathscr{F}^\bullet)$ denotes the q-th cohomology group of this complex.

By the definition, we can also write $\mathscr{F}^q(U, U \setminus S) = \Gamma(U; \mathscr{F}^q) \oplus \Gamma(U \setminus S; \mathscr{F}^{q-1})$.

Lemma A.2 Let $\mathscr{F}^\bullet$ and $\mathscr{G}^\bullet$ be complexes bounded below consisting of soft sheaves on X. If $\mathscr{F}^\bullet$ is quasi-isomorphic to $\mathscr{G}^\bullet$, then $\mathscr{F}^\bullet(U, U \setminus S)$ is quasi-isomorphic to $\mathscr{G}^\bullet(U, U \setminus S)$.

Proof. By Lemma 1 and Lemma 2 of KKK[8], we may assume that there is a morphism from $\mathscr{F}^\bullet$ to $\mathscr{G}^\bullet$. Here, we note that flabby sheaves are soft. Let us consider the following commutative diagram of short exact sequences of complexes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathscr{F}^\bullet[-1])^q(U \setminus S) & \xrightarrow{i_2} & \mathscr{F}^q(U) \oplus \mathscr{F}^{q-1}(U \setminus S) & \xrightarrow{p_1} & \mathscr{F}^q(U) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\mathscr{G}^\bullet[-1])^q(U \setminus S) & \xrightarrow{i_2} & \mathscr{G}^q(U) \oplus \mathscr{G}^{q-1}(U \setminus S) & \xrightarrow{p_1} & \mathscr{G}^q(U) & \longrightarrow & 0, \end{array}$$

where $(\bullet)[-1]$ is the shift functor, $q \in \mathbb{Z}$, i_2 is the inclusion map into the second component and p_1 is the projection onto the first component. Then, we get the following commutative diagram of long exact sequences.

$$\begin{array}{ccccccccc} \cdots \to & H^{q-1}(U \setminus S; \mathscr{F}^\bullet) & \to & H^q(U, U \setminus S; \mathscr{F}^\bullet) & \to & H^q(U; \mathscr{F}^\bullet) & \xrightarrow{\iota} & H^q(U \setminus S; \mathscr{F}^\bullet) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & H^{q-1}(U \setminus S; \mathscr{G}^\bullet) & \to & H^q(U, U \setminus S; \mathscr{G}^\bullet) & \to & H^q(U; \mathscr{G}^\bullet) & \xrightarrow{\iota} & H^q(U \setminus S; \mathscr{G}^\bullet) & \to \cdots, \end{array}$$

where $q \in \mathbb{Z}$ and ι is induced by restriction. These long exact sequences are discussed in Suwa[24]. Since soft sheaves are $\Gamma(V; \bullet)$-injective for any $V \in \mathrm{Op}(X)$ (see Definition 1.8.2, Proposition 2.5.10 and Exercise II.6 of Kashiwara-Schapira[9]), we have $H^q(V; \mathscr{F}^\bullet) \simeq H^q(V; \mathscr{G}^\bullet)$ for any $V \in \mathrm{Op}(X)$ and $q \in \mathbb{Z}$. Applying the 5-lemma to the above diagram, we obtain $H^q(U, U \setminus S; \mathscr{F}^\bullet) \simeq H^q(U, U \setminus S; \mathscr{G}^\bullet)$ for any $q \in \mathbb{Z}$. $\square$

Theorem A.3 For any $p \in \mathbb{Z}$, we have the quasi-isomorphisms

$$C_X^{\infty,(p,\bullet)}(U, U \setminus S) \;\simeq\; \mathscr{I}^\bullet(U, U \setminus S) \;\simeq\; \mathbf{R}\Gamma_S(U; \mathscr{O}_X^{(p)}),$$

where $(\mathscr{I}^\bullet, d^\bullet)$ is a flabby resolution of $\mathscr{O}_X^{(p)}$.

Proof. Since $\mathscr{I}^\bullet$ is a flabby resolution of $\mathscr{O}_X^{(p)}$, $\mathbf{R}\Gamma_S(U;\mathscr{O}_X^{(p)})$ is quasi-isomorphic to $\Gamma_S(U;\mathscr{I}^\bullet)$. Moreover, by Lemma A.2, $C_X^{\infty,(p,\bullet)}(U,U\setminus S)$ is quasi-isomorphic to $\mathscr{I}^\bullet(U,U\setminus S)$. Then, it suffices to show that $\Gamma_S(U;\mathscr{I}^\bullet)$ is quasi-isomorphic to $\mathscr{I}^\bullet(U,U\setminus S)$.

The canonical morphism $\phi:\Gamma_S(U;\mathscr{I}^\bullet)\to\mathscr{I}^\bullet(U,U\setminus S)$ is defined by

$$\begin{array}{ccc} \Gamma_S(U;\mathscr{I}^q) & \longrightarrow & \mathscr{I}^q(U,U\setminus S) \\ \Downarrow & & \Downarrow \\ s & \longmapsto & (s,0) \end{array} \qquad (q\in\mathbb{Z}).$$

By $\mathrm{supp}(s)\subset S$, ϕ forms a complex morphism and it induces morphisms between the cohomology groups. It is easy to see that each $\phi^q:H_S^q(U;\mathscr{I}^\bullet)\to H^q(U,U\setminus S;\mathscr{I}^\bullet)$ is injective. Hence, let us show that for any $q\in\mathbb{Z}$, ϕ^q is surjective.

Since each $\mathscr{I}^q$ is flabby, for any $[(s_1,s_{01})]\in H^q(U,U\setminus S;\mathscr{I}^\bullet)$, there exists $s'\in\Gamma(U,\mathscr{I}^{q-1})$ such that $s'|_{U\setminus S}=s_{01}$. Then, we get

$$[(s_1,s_{01})]=[(s_1,s_{01})-\vartheta(s',0)]=[(s_1-d^{q-1}(s'),0)]$$

and we define $\tilde{s}=s_1-d^{q-1}(s')$. By $(s_1,s_{01})\in\operatorname{Ker}\vartheta^q$, we have $\tilde{s}|_{U\setminus S}=s_1|_{U\setminus S}-d^{q-1}(s_{01})=0$ and $[\tilde{s}]\in H_S^q(U;\mathscr{I}^\bullet)$. This means that ϕ^q is surjective, and it has been shown that $H_S^q(U;\mathscr{I}^\bullet)\simeq H^q(U,U\setminus S;\mathscr{I}^\bullet)$ for any $q\in\mathbb{Z}$. This completes the proof. $\square$

For the proof above, Honda-Komori[5] is a useful reference. If the readers are interested in the specific flabby resolution $\mathscr{B}^{(p,\bullet)}$ of $\mathscr{O}_X^{(p)}$, Komatsu[13] will be a useful reference.

Let M be a real analytic, n-dimensional, oriented manifold and X its complexification, and let $\Omega\in\mathrm{Op}(M)$ and U a complex neighborhood of Ω. Since M is oriented, we omit the section of the relative orientation sheaf in the definition of $\mathscr{B}_M(\Omega)$.

Lemma A.4 (a representative with compact support) Assume $\Omega\subset\subset M$. Let $\mu\in C_X^{\infty,(0,n)}(U,U\setminus\Omega)$ be a representative of a hyperfunction and $U',U''\in\mathrm{Op}(X)$ such that $U'\subset\subset U\subset\subset U''$. Then, there exists a representative $\tilde{\mu}\in C_X^{\infty,(0,n)}(X,X\setminus M)$ such that

$$\tilde{\mu}=\begin{cases}\mu \text{ on } U',\\ 0 \text{ on } X\setminus\overline{U''}.\end{cases}$$

Proof. Since $\mathscr{B}_M$ is flabby, we can find a hyperfunction $\tilde{u}\in\mathscr{B}_M(M)$ such that $\tilde{u}|_\Omega=[\mu]$, $\tilde{u}|_{M\setminus\overline{\Omega}}=0$. Hence, there exists $\mu_0\in C_X^{\infty,(0,n)}(X,X\setminus M)$ such that

$$\mu_0|_U+\bar{\vartheta}\tau'=\mu\,,$$
$$\mu_0|_{X\setminus\overline{U}}+\bar{\vartheta}\tau''=0\,.$$

Here $\tau'\in C_X^{\infty,(0,n-1)}(U,U\setminus\Omega)$ and $\tau''\in C_X^{\infty,(0,n-1)}(X\setminus\overline{U},(X\setminus\overline{U})\setminus(M\setminus\overline{\Omega}))$.

Let U_0' and U_0'' be satisfying $U' \subset\subset U_0' \subset\subset U \subset\subset U_0'' \subset\subset U''$. We define $\varphi', \varphi'' \in C_0^\infty(X)$ by

$$\varphi' = \begin{cases} 1 \text{ on } U', \\ 0 \text{ on } X \setminus \overline{U_0'}, \end{cases} \qquad \varphi'' = \begin{cases} 1 \text{ on } X \setminus \overline{U''}, \\ 0 \text{ on } U_0''. \end{cases}$$

Then, if we set $\tilde{\mu} = \mu_0 + \bar{\vartheta}(\varphi'\tau') + \bar{\vartheta}(\varphi''\tau'')$, we get

$$\begin{aligned} \tilde{\mu}|_{U'} &= \mu_0|_{U'} + \bar{\vartheta}\tau'|_{U'} &&= \mu|_{U'}\,, \\ \tilde{\mu}|_{X\setminus\overline{U''}} &= \mu_0|_{X\setminus\overline{U''}} + \bar{\vartheta}\tau''|_{X\setminus\overline{U''}} &&= 0\,. \end{aligned}$$

This completes the proof. $\square$

Lemma A.5 (Bochner-Martinelli formula) Let $D \subset \mathbb{C}^n$ be a bounded domain with C^1 boundary. For any $z \in D$ and for any $g \in C^1(\overline{D})$, we have

$$g(z) = (-1)^{\frac{n(n+1)}{2}} \int_{\partial D} g(\zeta)\beta(\zeta - z) \wedge d\zeta - (-1)^{\frac{n(n+1)}{2}} \int_D \bar{\partial} g(\zeta) \wedge \beta(\zeta - z) \wedge d\zeta,$$

where

$$\beta(z) = (-1)^{\frac{n(n-1)}{2}} \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \frac{1}{\|z\|^{2n}} \sum_{i=1}^{n} (-1)^{i-1} \bar{z}_i \, d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_i} \wedge \cdots \wedge d\bar{z}_n.$$

We omit the proof of Lemma A.5 (see Krantz[16]). Note that we choose $(y_1, \cdots, y_n, x_1, \cdots, x_n)$ as a positive coordinate system on $\mathbb{C}^n$. Since the conventional positive coordinate system is $(x_1, y_1, \cdots, x_n, y_n)$, we have to multiply the formula by $(-1)^{\frac{n(n+1)}{2}}$.

Part II

Relative Čech-Dolbeault cohomology on subanalytic sites

1 Introduction for Part II

The concept of sheaves on sites offers a generalization of classical sheaves. Kashiwara-Schapira[11] established this powerful theory and Prelli[19] significantly contributes to the understanding of sheaves on subanalytic sites. Subsequently, Guillermou-Schapira[3] study sheaves on (linear) subanalytic sites more. The subanalytic sheaf theory allows us to handle objects that are inaccessible to the classical sheaf theory. For example, tempered distribution is not a classical sheaf but a subanalytic sheaf.

On the other hand, Suwa[25] establishes relative Čech-Dolbeault cohomology for classical sheaves and Honda-Izawa-Suwa[4] gives a new representation of hyperfunctions. Relative Čech-Dolbeault cohomology is a representation of local cohomology using soft sheaves. By the softness of relative Čech-Dolbeault representatives, we can employ helpful tools such as cutoff functions or partitions of unity.

In this part, we extend the theory of relative Čech-Dolbeault cohomology to sheaves on subanalytic sites. For that purpose, we construct relative Čech derived functors, where we also need the notion of softness for sheaves on subanalytic sites, which was first introduced by Guillermou[2]. Here, to make the computation of relative Čech derived functors applicable to much wider classes of sheaves, we introduce the concept of U-soft, which is a much more localized notion than softness. By U-soft sheaves, we can develop the relative Čech-Dolbeault cohomology theory more precisely. As an application of the theory of relative Čech-Dolbeault cohomology on subanalytic shaeves, we can provide explicit conditions under which relative Čech-Dolbeault representatives of hyperfunctions belong to the class of distributions.

Part II is organized as follows: Section 2 gives notations and examples of subanalytic sheaves.

Section 3 explains relative Čech-Dolbeault cohomology on subanalytic sites. In Subection 3.1, we define U-soft sheaves on subanalytic sites and give properties and examples of soft sheaves. Note that soft sheaves on subanalytic sites is introduced by Guillermou[2]. In Subection 3.2, we define relative Čech derived functors. Relative Čech-Dolbeault cohomology is defined by these functors. In Subection 3.3, we introduce localized version of the relative Čech derived functors.

In Section 4, we give a representation of tempered distributions $\mathscr{D}b^t_{M_{sa}}$ by using relative Čech-Dolbeault cohomology. The representation of tempered distributions can be obtained from relative Čech-Dolbeault cohomology of tempered Dolbeault complexes (Definition 4.7), and its definition is very similar to that of hyperfunctions. This representation of $\mathscr{D}b^t_{M_{sa}}$ provides conditions for classifying representatives of distributions $\mathscr{D}b_M$ (Theorem 4.13).

2 Preliminary

2.1 Notations

Let X be a real analytic manifold.

Remark 2.1 In Part II, we assume that manifolds are always countable at infinity, and that sheaves are always those of abelian groups.

Notation 2.2 For any $V, V' \in \mathrm{Op}(X)$, $V' \subset\subset V$ means that V' is a relatively compact subset of V.

Notation 2.3 $\mathrm{Mod}(\mathbb{Z})$ denotes the set of abelian groups.

Notation 2.4 $\mathrm{Op}(X)$ denotes the set of open subsets in X and $\mathrm{Op}(X_{sa})$ denotes the set of subanalytic open subsets in X.

Notation 2.5 (Subanalytic sites, [19]) $\mathrm{Op}(X_{sa})$ is regarded as a category with restriction morphisms. $\mathrm{Op}(X_{sa})$ is endowed with the following topology: $S \subset \mathrm{Op}(X_{sa})$ is a covering of $U \in \mathrm{Op}(X_{sa})$ if for any compact set $K \subset X$, there exists a finite subset S_0 of S such that $K \cap \bigcup_{V \in S_0} V = K \cap U$. We will call X_{sa} a subanalytic site, and $U_{X_{sa}}$ denotes the category $\mathrm{Op}(X_{sa}) \cap U$ endowed with the topology induced from X_{sa}.

$U_{X_{sa}}$ and U_{sa} are different. Remark 1.1.1 in Prelli[19] is a helpful example.

Notation 2.6 $\mathrm{Cov}(U)$ denotes the set of open coverings of $U \subset X$ and $\mathrm{Cov}(U_{X_{sa}})$ denotes the set of subanalytic open coverings of $U_{X_{sa}}$.

Notation 2.7 ([11, Notation 6.1.2]) $\rho_{sa} : X \to X_{sa}$ denotes the natural functor of sites.

Notation 2.8 $\mathrm{Mod}(\mathbb{Z}_X)$ denotes the set of sheaves on X and $\mathrm{Mod}(\mathbb{Z}_{X_{sa}})$ denotes the set of sheaves on X_{sa}.

Sheaves on X_{sa} are also called subanalytic sheaves. For more details, see Prelli[19].

Notation 2.9 Let $\mathscr{F}$ be a sheaf on X_{sa}. For any locally closed subanalytic subset $Z \subset X$, we define $\mathscr{F}(Z) = \varinjlim_{Z \subset U} \mathscr{F}(U)$, where $U \subset X$ runs through open subanalytic neighborhoods of Z.

This means that $\mathscr{F}(Z)$ is a set of sections defined in open subanalytic neighborhoods of Z. Note that, for any closed subanalytic subset $Z \subset X$, $\mathscr{F}(Z) = \Gamma(Z, \mathscr{F}|_Z)$ holds. Proposition 2.5.1 in Kashiwara-Schapira[9] is a useful reference.

Notation 2.10 Let $\mathscr{F}$ be a sheaf on X_{sa}, U an open subanalytic subset in X and Z a locally closed subanalytic subset in U. We define $\rho_{Z,U} : \mathscr{F}(U) \to \mathscr{F}(Z)$ by $\varinjlim_{V} \rho_{V,U}$, where $V \subset U$ runs through open subanalytic neighborhoods of Z and $\rho_{V,U}$ is the restriction $\mathscr{F}(U) \to \mathscr{F}(V)$.

We also write $(\bullet)|_Z = \rho_{Z,U}(\bullet)$. For any sheaf $\mathscr{F}$ on X, $\mathscr{F}(Z)$ and $\rho_{Z,U}(\bullet) = (\bullet)|_Z$ are defined in the same way as the previous definition.

Definition 2.11 (Soft sheaves on X) $\mathscr{F} \in \mathrm{Mod}(\mathbb{Z}_X)$ is said to be soft if and only if, for any closed subset $Z \subset X$, the restriction morphism

$$\mathscr{F}(X) \to \mathscr{F}(Z)$$

is surjective.

Definition 2.12 (c-soft sheaves on X) $\mathscr{F} \in \mathrm{Mod}(\mathbb{Z}_X)$ is said to be c-soft if and only if, for any compact subset $K \subset X$, the restriction morphism

$$\mathscr{F}(X) \to \mathscr{F}(K)$$

is surjective.

Since X is locally compact and countable at infinity, any c-soft sheaf is soft. See Exercise II.6 in Kashiwara-Scharipra[9].

Definition 2.13 (Fine sheaves on X) $\mathscr{F} \in \mathrm{Mod}(\mathbb{Z}_X)$ is said to be fine if and only if, $\mathscr{F}$ is a sheaf of $\mathscr{R}$-module, where $\mathscr{R}$ is a sheaf of rings with unity such that, for any $\{U_i\} \in \mathrm{Cov}(X)$, there exists a partition of unity subordinate to $\{U_i\}$.

2.2 Examples of subanalytic sheaves

Let X be a real analytic manifold.

Definition 2.14 (Tempered C^∞ functions, [10, Section 10.1], [3, Definition 4.1]) Let $V \in \mathrm{Op}(X_{sa})$. $f \in C_X(V)$ is said to have polynomial growth at $p \in X \setminus V$ if it satisfies the following condition. For a local coordinate system $x = (x_1, \cdots, x_n)$ around p, there exist a sufficiently small compact neighborhood K of p, a positive number C and a positive integer N such that

$$|f(x)| \le C \,\mathrm{dist}(x, K \setminus V)^{-N} \quad (x \in K \cap V).$$

Here, $\mathrm{dist}(x, K \setminus V)$ is the distance from x to $K \setminus V$. We say that $f \in C_X^\infty(V)$ belongs to $C_{X_{sa}}^{\infty,t}(V)$ if all its derivatives have polynomial growth at any $p \in X \setminus V$.

Definition 2.15 For any open subanalytic subset $V \subset X$, $f \in C_X^\infty(V)$ belongs to $C_{X_{sa}}^{\infty,\mathrm{w}}(V)$ if it satisfies one of the following equivalent conditions:

1. There exists an open subanalytic covering $\{V_i\}$ of $V_{X_{sa}}$ such that each $f|_{V_i}$ can extend to X as a C^∞ function.
2. For any open subanalytic covering $\{V_i\}$ of $V_{X_{sa}}$ with every V_i being cohomologically trivial, each $f|_{V_i}$ can extend to X as a C^∞ function.
3. For any open subanalytic subset $V' \subset\subset X$, any higher derivative of f is uniformly bounded on $V \cap V'$.

Then $\{C_{X_{sa}}^{\infty,t}(V)\}$ and $\{C_{X_{sa}}^{\infty,\mathrm{w}}(V)\}$ form sheaves $C_{X_{sa}}^{\infty,t}$ and $C_{X_{sa}}^{\infty,\mathrm{w}}$ on X_{sa}.

Definition 2.16 Let $U \subset X$ be an open subanalytic subset. For any open subanalytic subset $V \subset X$, we say that $f \in C_X^\infty(U \cap V)$ belongs to $C_{X_{sa},U}^\infty(V)$ if there exists an open subanalytic covering $\{V_i\}$ of $V_{X_{sa}}$ such that each $f|_{U\cap V_i}$ can be extended to a C^∞ function $\tilde{f}$ on X, where any higher derivative of $\tilde{f}$ is 0 on $X \setminus U$.

Then $\{C_{X_{sa},U}^\infty(V)\}$ forms a sheave $C_{X_{sa},U}^\infty$ on X_{sa}. Note that $C_{X_{sa},U}^\infty$ is a subsheaf of $\mathscr{H}om(\mathbb{C}_U, C_{X_{sa}}^{\infty,\mathrm{w}})$.

3 Soft sheaves and relative Čech-Dolbeault cohomology

3.1 Soft sheaves on subanalytic sites

Let X be a real analytic manifold. We fix $U \in \mathrm{Op}(X_{sa})$.

Definition 3.1 (U-soft sheaves) $\mathscr{F} \in \mathrm{Mod}(\mathbb{Z}_{X_{sa}})$ is said to be U-soft if and only if, for any closed subanalytic subset $Z \subset X$, the restriction morphism

$$\mathscr{F}(U) \to \mathscr{F}(U \cap Z)$$

is surjective.

Proposition 3.2 For any soft sheaf $\mathscr{F}$ on X, $\rho_{sa\,*}(\mathscr{F})$ is X-soft on X_{sa}.

Proof. Let $Z \subset X$ be a closed subanalytic subset. Since $\rho_{sa\,*}(\mathscr{F})(Z) = \mathscr{F}(\rho_{sa}^{-1}(Z)) = \mathscr{F}(Z)$, the restriction $\rho_{sa\,*}(\mathscr{F})(X) \to \rho_{sa\,*}(\mathscr{F})(Z)$ is surjective. □

Lemma 3.3 Let $V, W \subset X$ be subanalytic open subsets and $\mathscr{F}$ a $(V \cap W)$-soft sheaf on X_{sa}. Then, $\Gamma_V(\mathscr{F})$ is a W-soft sheaf on X_{sa}.

Proof. Let $Z \subset X$ be a closed subanalytic subsets. We have a commutative diagram

$$\begin{array}{ccc} \Gamma_V(\mathscr{F})(W) & \xrightarrow{\quad\sim\quad} & \mathscr{F}(V \cap W) \\ \downarrow & & \downarrow \\ \varinjlim\limits_{W \cap Z \subset T} \Gamma_V(\mathscr{F})(W \cap \overline{T}) & \xrightarrow{\varphi} & \varinjlim\limits_{W \cap Z \subset T} \mathscr{F}(V \cap W \cap \overline{T}), \end{array}$$

where T runs through open subanalytic neighborhoods of $W \cap Z$ and all morphism is induced by restriction. By the $(V \cap W)$-softness of $\mathscr{F}$, the second vertical morphism is surjective.

Let us prove that the second horizontal morphism φ is injective. Let $s \in \varinjlim\limits_{W \cap Z \subset T} \Gamma_V(\mathscr{F})(W \cap \overline{T})$ with $\varphi(s) = 0$. We use a notation $\varphi = \varinjlim\limits_{W \cap Z \subset T} \varphi_T$, where $\varphi_T : \Gamma_V(\mathscr{F})(W \cap \overline{T}) \to \mathscr{F}(V \cap W \cap \overline{T})$ is also induced by restriction. From the definition of the inductive limit, there exists T_0 and s_0 such that $\varphi_{T_0}(s_0) \in \mathscr{F}(V \cap W \cap \overline{T_0})$. By the hereditarily paracompactness of X, we can take $T_1 \in \mathrm{Op}(X_{sa})$ such that $W \cap Z \subset T_1 \subset\subset T_0$. Since $\varphi_{T_0}(s_0)|_{T_0} \in \mathscr{F}(V \cap W \cap T_0) = \Gamma_V(\mathscr{F})(W \cap T_0)$, $s_1 = \varphi_{T_0}(s_0)|_{\overline{T_1}}$ can be regarded as an element of $\Gamma_V(\mathscr{F})(W \cap \overline{T_1})$. Since φ is induced by restriction, s_1 satisfies $s = [s_1]$ and $s_1 = 0$. This implies that $s = 0$.

From the above consideration, we obtain the surjectivity of the first vertical morphism. By $\Gamma_V(\mathscr{F})(W \cap Z) \simeq \varinjlim\limits_{W \cap Z \subset T} \Gamma_V(\mathscr{F})(W \cap \overline{T})$, it can be concluded that $\Gamma_V(\mathscr{F})$ is W-soft. □

The following proposition is important for the localized relative Čech functor $\rho_{sa}^{-1} \circ C_{\mathcal{U},\mathcal{U}'}$, which is explained in Subsection 3.3.

Proposition 3.4 Let $\mathscr{F}$ be a U-soft sheaf on X_{sa}. Then, $\rho_{sa}^{-1} \circ \Gamma_U(\mathscr{F})$ is a soft sheaf on X.

Proof. By applying Lemma 3.3 with $V = U$ and $W = X$, it becomes clear that $\Gamma_U(\mathscr{F})$ is X-soft. Hence, it is sufficient to prove that $\rho_{sa}^{-1}(\mathscr{F})$ is a c-soft sheaf on X for any X-soft sheaf $\mathscr{F}$.

Let $K \subset X$ be a compact subset. For any positive integer i, K can be covered by a finite number of open balls of radius $1/i$ around each point. Let $\{B_{i,j}\}_{j:\text{finite}}$ be this covering. We define $K_i = \bigcup_j \overline{B_{i,j}}$. The set of closed subanalytic subsets $\{K_i\}_{i=0}^{\infty}$ satisfies $K = \bigcap_i K_i$ and $K \subset K_{i+1} \subset K_i$. Then, we get $\rho_{sa}^{-1}(\mathscr{F})(K) = \varinjlim\limits_i \rho_{sa}^{-1}(\mathscr{F})(K_i)$.

Let $s \in \rho_{sa}^{-1}(\mathscr{F})(K)$. By $\rho_{sa}^{-1}(\mathscr{F})(K) = \varinjlim\limits_i \rho_{sa}^{-1}(\mathscr{F})(K_i)$, there exists $s_i \in \rho_{sa}^{-1}(\mathscr{F})(K_i)$ such that $s_i|_K = s$. Since $\mathscr{F}$ is X-soft, $\mathscr{F}(X) \to \mathscr{F}(K_i)$ is surjective. Hence, $\rho_{sa}^{-1}(\mathscr{F})(X) \to \rho_{sa}^{-1}(\mathscr{F})(K_i)$ is surjective and there exists $\tilde{s} \in \rho_{sa}^{-1}(\mathscr{F})(X)$ such that $\tilde{s}|_K = s$. Note that, for any closed subanalytic subset $Z \subset X$, $\rho_{sa}^{-1}(\mathscr{F})(Z) = \mathscr{F}(Z)$ holds. This completes the proof. □

Definition 3.5 ($\mathcal{U}$-soft sheaves) For any open subanalytic covering $\mathcal{U} = \{U_i\}_{i \in \Lambda}$ of $U_{X_{sa}}$,

$\mathscr{F} \in \mathrm{Mod}(\mathbb{Z}_{X_{sa}})$ is said to be $\mathcal{U}$-soft if and only if, for any $k \geq 0$, $(\alpha_0, \cdots, \alpha_k) \in \Lambda^{k+1}$, $\mathscr{F}$ is $(U_{\alpha_0} \cap \cdots \cap U_{\alpha_k})$-soft.

Definition 3.6 (Soft sheaves on X_{sa}, [2, Definition 4.1]) $\mathscr{F} \in \mathrm{Mod}(\mathbb{Z}_{X_{sa}})$ is said to be soft if and only if, for any open subanalytic subset $V \subset X$, $\mathscr{F}$ is V-soft.

The following proposition is important for the localized relative Čech functor $C_{\mathcal{U},\mathcal{U}'}$, which is defined in Definition 3.47.

Proposition 3.7 Let $\mathscr{F}$ be a soft sheaf on X_{sa}. For any open subanalytic subset $V \subset X$, $\Gamma_V(\mathscr{F})$ is a soft sheaf on X_{sa}.

Proof. Let $V \subset X$ be an open subanalytic subset. By Lemma 3.3, for any $W \in \mathrm{Op}(X_{sa})$, $\Gamma_V(\mathscr{F})$ is W-soft. In other words, $\Gamma_V(\mathscr{F})$ is a soft sheaf on X_{sa}. $\square$

Definition 3.8 (Flabby sheaves on X_{sa}, [3, Definition 2.16]) $\mathscr{F} \in \mathrm{Mod}(\mathbb{Z}_{X_{sa}})$ is said to be flabby if and only if, for any open subanalytic subset $V \subset X$ and for any open subanalytic subset $W \subset V$, the restriction morphism

$$\mathscr{F}(V) \to \mathscr{F}(W)$$

is surjective.

Note that $\mathscr{F}$ is flabby if and only if, for any open subanalytic subset $V \subset X$, the restriction $\mathscr{F}(X) \to \mathscr{F}(V)$ is surjective.

Proposition 3.9 ([3, Lemma 2.17 (i)]) Injective sheaves are flabby on X_{sa}.

Proposition 3.10 For any subanalytic sheaf $\mathscr{F}$, a flabby resolution $\mathscr{I}^\bullet$ exists.

Proof. $\mathscr{F}$ has an injective resolution $\mathscr{I}^\bullet$. By Proposition 3.9, for any $k \in \mathbb{Z}$, each $\mathscr{I}^k$ are flabby. $\square$

Proposition 3.11 Flabby sheaves are soft on X_{sa}. In particular, flabby sheaves are U-soft on X_{sa}.

Proof. Let $\mathscr{F}$ be a flabby sheaf, $V \subset X$ an open subanalytic subset and $Z \subset X$ a closed subanalytic subset. By the definition of $\mathscr{F}(V \cap Z)$, for any $s \in \mathscr{F}(V \cap Z)$, there exists $V \cap Z \subset W \in \mathrm{Op}(X_{sa})$ and $s' \in \mathscr{F}(W)$ such that $s'|_{V \cap Z} = s$. Since $\mathscr{F}$ is a flabby sheaf, we get $\tilde{s} \in \mathscr{F}(V)$ such that $\tilde{s}|_W = s'$. Then $\tilde{s}|_{V \cap Z} = s$ holds and the restriction $\mathscr{F}(V) \to \mathscr{F}(V \cap Z)$ is surjective. $\square$

Example 3.12 $C^\infty_{X_{sa},U}$ is U-soft.

Proof. Let $Z \subset X$ be a closed subanalytic subset and let $f \in C^\infty_{X_{sa},U}(U \cap Z)$. By the definition, we can find an open subanalytic covering $\{V_i\}$ of Z and each $f|_{U\cap V_i}$ can be extended to $g_i \in C^\infty_X(X)$ with $g_i|_{X\setminus U} = 0$. The collection $\{g_i|_{(U\cap V_i)\cup(X\setminus U)}\}$ can be glued together to form a function $g \in C^\infty(X \setminus (U \setminus Z))$. This construction ensures that $g|_{U\cap Z} = f$ and $g|_{X\setminus U} = 0$.

Next, let $(Z \subset)V \in \mathrm{Op}(X_{sa})$. We can find $\psi \in C^\infty_X(X)$ such that $\psi = 1$ on Z and $\psi = 0$ on $X \setminus V$. Then, we define $\tilde{g} = \psi g \in C^\infty_X(X)$ and $\tilde{f} = \tilde{g}|_U \in C^\infty_X(U)$. Since $\tilde{f}$ can be extended to a C^∞_X function $\tilde{g}$ on X with $\tilde{g}|_{X\setminus U} = 0$ for the covering $\{U\}$ of $U_{X_{sa}}$, $\tilde{f}$ is a $C^\infty_{X_{sa},U}$ function on U with $\tilde{f}|_{U\cap Z} = f$. □

Example 3.13 Let $\mathscr{F}$ be a sheaf of $C^{\infty,t}_{X_{sa}}$-module on X_{sa}. Then, $\mathscr{F}$ is soft. In particular, $C^{\infty,t}_{X_{sa}}$ is soft.

Proof. Let $V \subset X$ be an open subanalytic subset, $Z \subset X$ a closed subanalytic subset and let $f \in \mathscr{F}(V \cap Z)$. There exist $W \in \mathrm{Op}(X_{sa})$ and $\tilde{f} \in \mathscr{F}(W)$ such that $Z \subset W \subset V$ and $\tilde{f}|_Z = f$. By Lemma 10.3 in [10], we get $\psi \in C^{\infty,t}_X(V)$ such that $\psi = 1$ on Z and $\psi = 0$ on $V \setminus W$. Then, $\psi\tilde{f}$ belongs to $\mathscr{F}(V)$ with $(\psi\tilde{f})|_Z = f$. □

Proposition 3.14 Let $\mathscr{F}$ be a fine sheaf on X. Then, $\rho_{sa\,*}(\mathscr{F})$ is soft on X_{sa}.

To prove this proposition, use the same arguments as in the proof of Example 3.13.

Example 3.15 $C^{\infty,\mathrm{w}}_{X_{sa}}$ is not soft.

Proof. We may assume that the open set U is contained in a coordinate neighborhood. Let $X = \mathbb{R}^n$. We define $U = \mathbb{R}^n \setminus \{0\}$ and $Z = \mathbb{R} \times \{0\}^{n-1}$, and we choose $f \in C^{\infty,\mathrm{w}}_{X_{sa}}(U \cap Z)$ as

$$f(x) = \begin{cases} 0 & x \in (-\infty, 0) \times \{0\}^{n-1}, \\ 1 & x \in (0, \infty) \times \{0\}^{n-1}, \end{cases}$$

Since the limit of $f(x)$ at $0 \in \mathbb{R}^n$ is not uniquely determined, f does not extend to U. Hence, $C^{\infty,\mathrm{w}}_{X_{sa}}$ is not soft. □

Lemma 3.16 Let $\mathscr{F}, \mathscr{G}$ and $\mathscr{H}$ be sheaves on X_{sa}. If $0 \longrightarrow \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H}$ is exact, then

$$0 \longrightarrow \mathscr{F}(V) \xrightarrow{f_V} \mathscr{G}(V) \xrightarrow{g_V} \mathscr{H}(V)$$

is exact for any $V \in \mathrm{Op}(X_{sa})$.

Proof. Let $V \in \mathrm{Op}(X_{sa})$. We can find an open subanalytic covering $\{V_i\}$ of $V_{X_{sa}}$ such that

$$0 \longrightarrow \mathscr{F}(V_i) \xrightarrow{f_{V_i}} \mathscr{G}(V_i) \xrightarrow{g_{V_i}} \mathscr{H}(V_i) \qquad (\forall i)$$

is exact. Then, we have $\mathrm{Ker}(f_{V_i}) = 0$ and $\mathrm{Im}(f_{V_i}) \subset \mathrm{Ker}(g_{V_i})$. Since $f : \mathscr{F} \to \mathscr{G}$ and $g : \mathscr{G} \to \mathscr{H}$ are sheaf morphisms, $\mathrm{Ker}(f_V) = 0$ and $\mathrm{Im}(f_V) \subset \mathrm{Ker}(g_V)$ hold.

Let us show $\mathrm{Im}(f_V) \supset \mathrm{Ker}(g_V)$. Let $t \in \mathrm{Ker}(g_V)$. By $g_{V_i}(t|_{V_i}) = 0$, we get $s_i \in \mathscr{F}(V_i)$ such that $f_{V_i}(s_i) = t|_{V_i}$. Since we have $\mathrm{Ker}(f_{V_i}) = 0$ and $f_{V_i \cap V_j}(s_i|_{V_i \cap V_j} - s_j|_{V_i \cap V_j}) = 0$, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ holds. Hence, there exists $s \in \mathscr{F}(V)$ such that $s|_V = t$. □

Lemma 3.17 Let $\mathscr{F}, \mathscr{G}$ and $\mathscr{H}$ be sheaves on X_{sa}, and let Z be a closed subanalytic set. If $0 \longrightarrow \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \longrightarrow 0$ is exact and $\mathscr{F}, \mathscr{G}$ is U-soft, then

$$0 \longrightarrow \mathscr{F}(U \cap Z) \xrightarrow{f_{U\cap Z}} \mathscr{G}(U \cap Z) \xrightarrow{g_{U\cap Z}} \mathscr{H}(U \cap Z) \longrightarrow 0$$

is exact.

Proof. We first show the result assuming that Z is compact. By Lemma 3.16 and the fact that $\mathscr{F}(U \cap Z) = \varinjlim_{Z \subset V} \mathscr{F}(U \cap V)$,

$$0 \longrightarrow \mathscr{F}(U \cap Z) \xrightarrow{f_{U\cap Z}} \mathscr{G}(U \cap Z) \xrightarrow{g_{U\cap Z}} \mathscr{H}(U \cap Z)$$

is exact. Let us show that $\mathscr{G}(U \cap Z) \to \mathscr{H}(U \cap Z)$ is surjective. Let $u \in \mathscr{H}(U \cap Z)$, and let $\{Z_i\}_{i=1}^n$ be a finite closed subanalytic covering of Z such that there exists $t_i \in \mathscr{G}(U \cap Z_i)$ with $g_{U \cap Z_i}(t_i) = u|_{U \cap Z_i}$. Since $g_{U\cap Z_1 \cap Z_2}\left(t_1|_{U \cap Z_1 \cap Z_2} - t_2|_{U \cap Z_1 \cap Z_2}\right) = 0$, there exists $s_{12} \in \mathscr{F}(U \cap Z_1 \cap Z_2)$ such that $f_{U \cap Z_1 \cap Z_2}(s_{12}) = t_1|_{U \cap Z_1 \cap Z_2} - t_2|_{U \cap Z_1 \cap Z_2}$. Since $\mathscr{F}$ is soft, there exists $s_2 \in \mathscr{F}(U \cap Z_2)$ such that $s_2|_{U \cap Z_1 \cap Z_2} = s_{12}$. Then, $t_2' = t_2 + f_{U \cap Z_2}(s_2) \in \mathscr{G}(U \cap Z_2)$ satisfies $t_2'|_{U \cap Z_1 \cap Z_2} = t_1|_{U \cap Z_1 \cap Z_2}$. Hence, by the gluing axiom, we get $\tilde{t} \in \mathscr{G}(U \cap (Z_1 \cup Z_2))$ such that $\tilde{t}\big|_{U \cap Z_1} = t_1$ and $\tilde{t}\big|_{U \cap Z_2} = t_2'$. Thus the induction proceeds and we can take $t \in \mathscr{G}(U \cap Z)$ with $g_{U \cap Z}(t) = u|_{U \cap Z}$.

Next, we show the result in the general case of Z. Let $\{K_n\}_{n=1}^{\infty}$ be an exhaustion by compact sets of X and we define $Z_n = Z \cap K_n$. By the compactness of Z_n,

$$0 \longrightarrow \mathscr{F}(U \cap Z_n) \xrightarrow{f_{U\cap Z_n}} \mathscr{G}(U \cap Z_n) \xrightarrow{g_{U\cap Z_n}} \mathscr{H}(U \cap Z_n) \longrightarrow 0$$

is exact. Since $\mathscr{F}$ is U-soft, for any $n \leq q$, we have $\rho_{U \cap Z_n, U \cap Z_p}(\mathscr{F}(U \cap Z_p)) = \mathscr{F}(U \cap Z_n)$. Thus, $\{\mathscr{F}(U \cap Z_n), \rho_{U \cap Z_n, U \cap Z_p}\}$ satisfies the Mittag-Leffler condition. By Proposition 1.12.3 in Kashiwara-Schapira[9],

$$0 \longrightarrow \varprojlim_n \mathscr{F}(U \cap Z_n) \xrightarrow{\varprojlim_n f_{U\cap Z_n}} \varprojlim_n \mathscr{G}(U \cap Z_n) \xrightarrow{\varprojlim_n g_{U\cap Z_n}} \varprojlim_n \mathscr{H}(U \cap Z_n) \longrightarrow 0$$

is exact. This completes the proof. □

The above proof of Lemma 3.17 is based on Proposition 1.5.2 in Prelli[19].

Lemma 3.18 Let $\mathscr{F}, \mathscr{G}$ and $\mathscr{H}$ be sheaves on X_{sa}, and let $0 \longrightarrow \mathscr{F} \xrightarrow{f} \mathscr{G} \xrightarrow{g} \mathscr{H} \longrightarrow 0$ be an exact sequence. If $\mathscr{F}$ and $\mathscr{G}$ are U-soft, $\mathscr{H}$ is U-soft.

Proof. By Lemma 3.17, g_U and $g_{U\cap Z}$ are surjective in the following diagram.

$$\begin{tikzcd}
\mathscr{G}(U) \arrow[r, "g_U"] \arrow[d, "\rho_{U\cap Z,U}"] & \mathscr{H}(U) \arrow[d, "\rho'_{U\cap Z,U}"] \\
\mathscr{G}(U\cap Z) \arrow[r, "g_{U\cap Z}"] & \mathscr{H}(U\cap Z).
\end{tikzcd}$$

Since $\mathscr{G}$ is U-soft, $\rho_{U\cap Z,U}$ is surjective. Then, we get the surjectivity of $\rho'_{U\cap Z,U}$. This means that $\mathscr{H}$ is U-soft. $\square$

Lemma 3.19 Let $\mathscr{F}$ be a sheaf on X_{sa}. Assume $\mathscr{F}$ is U-soft. Then we have

$$H^k(U;\mathscr{F}) = 0 \quad (k \neq 0).$$

In particular, if $\mathscr{F}$ is soft, $H^k(V;\mathscr{F}) = 0 \quad (k \neq 0)$ holds for any $V \in \mathrm{Op}(X_{sa})$.

Proof. We can find a injective sheaf $\mathscr{G}$ and a monomorphism $f : \mathscr{F} \to \mathscr{G}$. Let us consider an exact sequence

$$0 \longrightarrow \mathscr{F} \xrightarrow{\ f\ } \mathscr{G} \longrightarrow \operatorname{Coker} f \longrightarrow 0.$$

Since injective sheaves are U-soft, $\mathscr{G}$ is U-soft. By Lemma 3.18, $\mathscr{H}$ is U-soft. Hence, for any injective resolution $\mathscr{I}^\bullet$ of $\mathscr{F}$, we have the decomposition of the long exact sequence into the short exact sequences of U-soft sheaves:

$$\begin{tikzcd}
 & & & 0 \arrow[dr] & & 0 & & \\
 & & & & \operatorname{Coker}\iota \arrow[ur] \arrow[dr, "d'^0"] & & & \\
0 \arrow[r] & \mathscr{F} \arrow[rr, "\iota"] \arrow[dr, equal] & & \mathscr{I}^0 \arrow[ur] \arrow[rr, "d^0"] & & \mathscr{I}^1 \arrow[rr, "d^1"] \arrow[dr] & & \cdots. \\
 & & \mathscr{F} \arrow[ur] \arrow[dr] & & & & \operatorname{Coker} d'^0 \arrow[ur] \arrow[dr] & \\
 & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

Thus, we can conclude $H^k(U;\mathscr{F}) = 0$ for any $k \neq 0$. $\square$

Corollary 3.20 U-soft sheaves are $\Gamma(U;\bullet)$-acyclic.

By Lemma 3.19, the following example also means that $C^{\infty,\mathrm{w}}_{X_{sa}}$ is not soft.

Example 3.21 Assume $X = \mathbb{C}$ and $U = \left\{z \in \mathbb{C} \;\middle|\; 0 < |z| < 1\right\}$. Then, we have

$$H^k(U; C^{\infty,\mathrm{w}}_{X_{sa}}) \neq 0 \quad (k = 0, 1).$$

Proof. Let

$$U_1 = \left\{ z \in U \;\middle|\; -\frac{3}{4}\pi < \arg(z) < \frac{3}{4}\pi \right\} \quad \text{and} \quad U_2 = \left\{ z \in U \;\middle|\; \frac{1}{4}\pi < \arg(z) < \frac{7}{4}\pi \right\}.$$

Assume $H^1(U; C^{\infty,\mathrm{w}}_{X_{sa}}) = 0$. By the Mayer-Vietoris sequence, we get the exact sequence

$$0 \longrightarrow C^{\infty,\mathrm{w}}_{X_{sa}}(U) \xrightarrow{\phi} C^{\infty,\mathrm{w}}_{X_{sa}}(U_1) \oplus C^{\infty,\mathrm{w}}_{X_{sa}}(U_2) \xrightarrow{\psi} C^{\infty,\mathrm{w}}_{X_{sa}}(U_1 \cap U_2) \longrightarrow 0,$$

where $\phi : f \mapsto (f|_{U_1}, f|_{U_2})$ and $\psi : (f, g) \mapsto f|_{U_1 \cap U_2} - g|_{U_1 \cap U_2}$. Since ψ is surjective, we can find $f_1 \in C^{\infty,\mathrm{w}}_{X_{sa}}(U_1)$ and $f_2 \in C^{\infty,\mathrm{w}}_{X_{sa}}(U_2)$ such that

$$\left(f_1|_{U_1 \cap U_2} - f_2|_{U_1 \cap U_2} \right)(x) = \begin{cases} 0 & z \in \left\{ z \in U \;\middle|\; \frac{1}{4}\pi < \arg(z) < \frac{3}{4}\pi \right\}, \\ 1 & z \in \left\{ z \in U \;\middle|\; \frac{5}{4}\pi < \arg(z) < \frac{7}{4}\pi \right\}. \end{cases}$$

The function f_1 (resp. f_2) converges to a constant c_1 (resp. c_2) at $z = 0$ in $\overline{U_1}$ (resp. $\overline{U_2}$). Hence, $f_1|_{U_1 \cap U_2} - f_2|_{U_1 \cap U_2}$ converges to a constant $c_1 - c_2$ at $z = 0$. This gives a contradiction and we get $H^1(U; C^{\infty,\mathrm{w}}_{X_{sa}}) \neq 0$. $\square$

Lemma 3.22 For any soft sheaf $\mathscr{F}$ on X_{sa} and for any open subanalytic subset $V \subset X$, we have

$$H^k(\Gamma_V(\mathscr{F})) = 0 \quad (k \neq 0).$$

Proof. By Proposition 3.7 and Lemma 3.19, we have

$$H^k(W; \Gamma_V(\mathscr{F})) = 0 \quad (k \neq 0)$$

for any $W \in \mathrm{Op}(X_{sa})$. Therefore, for any $k \neq 0$, the sheafification of $W \mapsto H^k(W; \Gamma_V(\mathscr{F}))$ is the 0 sheaf on X_{sa}. $\square$

Corollary 3.23 Soft sheaves are Γ_V-acyclic for any open subanalytic subset $V \subset X$.

Lemma 3.24 For any U-soft sheaf $\mathscr{F}$ on X_{sa}, we have

$$H^k(\rho_{sa}^{-1} \circ \Gamma_U(\mathscr{F})) = 0 \quad (k \neq 0).$$

Proof. By Proposition 3.4, $\rho_{sa}^{-1} \circ \Gamma_U(\mathscr{F})$ is a soft sheaf on X. Thus, we obtain the desired result. $\square$

Corollary 3.25 U-soft sheaves are $\rho_{sa}^{-1} \circ \Gamma_U$-acyclic.

3.2 Relative Čech derived functors for subanalytic sheaves

Let $U \subset X$ be an open subanalytic subset and $Z \subset X$ a closed subanalytic subset. We take a finite open subanalytic covering $\mathcal{U} = \{U_i\}_{i \in \Lambda}$ of U and a finite open subanalytic covering $\mathcal{U}' = \{U_i\}_{i \in \Lambda'} \subset \mathcal{U}$ of $U \setminus Z$. Note that $\Lambda' \subset \Lambda$.

Definition 3.26 (Relative Čech cohomology for subanalytic sheaves) Let $\mathscr{F} \in \mathrm{Mod}(\mathbb{Z}_{X_{sa}})$. For any $k = 0, 1, 2, \cdots$, we define

$$C^k(\mathcal{U},\mathcal{U}';\mathscr{F}) = \left\{ \bigoplus_{(\alpha_0,\cdots,\alpha_k)\in\Lambda^{k+1}} s_{\alpha_0,\cdots,\alpha_k} \;\middle|\; \begin{array}{l} s_{\alpha_0,\cdots,\alpha_k} \in \mathscr{F}(U_{\alpha_0}\cap\cdots\cap U_{\alpha_k}), \\ s_{\alpha_0,\cdots,\alpha_\ell,\alpha_{\ell+1},\cdots,\alpha_k} = -s_{\alpha_0,\cdots,\alpha_{\ell+1},\alpha_\ell,\cdots,\alpha_k}, \\ (\alpha_0,\ \cdots,\alpha_k) \in {\Lambda'}^{k+1} \Rightarrow s_{\alpha_0,\cdots,\alpha_k} = 0 \end{array} \right\},$$

and a map $\delta^k_{\mathscr{F}} : C^k(\mathcal{U},\mathcal{U}';\mathscr{F}) \to C^{k+1}(\mathcal{U},\mathcal{U}';\mathscr{F})$ is defined by

$$\delta^k_{\mathscr{F}}\left(\bigoplus_{(\alpha_0,\cdots,\alpha_k)\in\Lambda^{k+1}} s_{\alpha_0,\cdots,\alpha_k}\right) = \bigoplus_{(\beta_0,\cdots,\beta_{k+1})\in\Lambda^{k+2}} \left(\sum_{\ell=0}^{k+1}(-1)^\ell \left(s_{\beta_0,\cdots,\widehat{\beta_\ell},\cdots,\beta_{k+1}}\right)\Big|_{U_{\beta_0}\cap\cdots\cap U_{\beta_{k+1}}}\right),$$

where $\widehat{\bullet}$ means to omit the corresponding letter in a sequence. $(C^\bullet(\mathcal{U},\mathcal{U}';\mathscr{F}), \delta^\bullet_{\mathscr{F}})$ is a complex, and we write its k-th cohomology group as $H^k(\mathcal{U},\mathcal{U}';\mathscr{F})$. If $\mathcal{U}' = \emptyset$, we express it simply as $C^k(\mathcal{U};\mathscr{F}) = C^k(\mathcal{U},\emptyset;\mathscr{F})$ and $H^k(\mathcal{U};\mathscr{F}) = H^k(\mathcal{U},\emptyset;\mathscr{F})$.

Definition 3.27 Let $\mathscr{F},\mathscr{G} \in \mathrm{Mod}(\mathbb{Z}_{X_{sa}})$ and $f = \{f_U : \mathscr{F}(U) \to \mathscr{G}(U)\}_{U\in\mathrm{Op}(X_{sa})}$ a sheaf morphism from $\mathscr{F}$ to $\mathscr{G}$. We define $C^k(\mathcal{U},\mathcal{U}';f)$ by

$$\begin{array}{cccc} C^k(\mathcal{U},\mathcal{U}';f) & : & C^k(\mathcal{U},\mathcal{U}';\mathscr{F}) \longrightarrow & C^k(\mathcal{U},\mathcal{U}';\mathscr{G}) \\ & & \cup & \cup \\ & & \displaystyle\bigoplus_{(\alpha_0,\cdots,\alpha_k)\in\Lambda^{k+1}} s_{\alpha_0,\cdots,\alpha_k} \longmapsto & \displaystyle\bigoplus_{(\alpha_0,\cdots,\alpha_k)\in\Lambda^{k+1}} f_{U_{\alpha_0}\cap\cdots\cap U_{\alpha_k}}(s_{\alpha_0,\cdots,\alpha_k}). \end{array}$$

$C^\bullet(\mathcal{U},\mathcal{U}';f) = \{C^k(\mathcal{U},\mathcal{U}';f)\}_{k\in\mathbb{Z}}$ forms a morphism of complexes.

From now on, we will consider $\mathrm{Mod}(\mathbb{Z})$, $\mathrm{Mod}(\mathbb{Z}_X)$ and $\mathrm{Mod}(\mathbb{Z}_{X_{sa}})$ as categories.

Notation 3.28 Let $\mathcal{C}$ be $\mathrm{Mod}(\mathbb{Z})$, $\mathrm{Mod}(\mathbb{Z}_X)$ or $\mathrm{Mod}(\mathbb{Z}_{X_{sa}})$. $\mathbf{C}^+(\mathcal{C})$ denotes the category of bounded below complexes consisting of objects in $\mathcal{C}$. In addition, $\mathbf{K}^+(\mathcal{C})$ (resp. $\mathbf{D}^+(\mathcal{C})$) denotes the triangulated (resp. derived) category of $\mathbf{C}^+(\mathcal{C})$.

Notation 3.29 Let $\mathcal{C}$ be $\mathrm{Mod}(\mathbb{Z})$, $\mathrm{Mod}(\mathbb{Z}_X)$ or $\mathrm{Mod}(\mathbb{Z}_{X_{sa}})$. $\mathbf{C}^2(\mathcal{C})$ denotes the set of double complexes consisting of objects in $\mathrm{Mod}(\mathcal{C})$. $\mathbf{C}^2_f(\mathrm{Mod}(\mathbb{Z}))$ denotes the full subcategory of $\mathbf{C}^2(\mathcal{C})$ consisting of objects A such that $\left\{(i,j)\in\mathbb{Z}^2 \;\middle|\; i+j=k \text{ and } A^{i,j}\neq 0\right\}$ is finite for all $k\in\mathbb{Z}$.

For the theory of derived categories, it is useful to refer to [9] or [12].

Definition 3.30 (Relative Čech complexes for subanalytic sheaf complexes) Let $\mathscr{F}^\bullet \in$

$\mathbf{C}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))$. We define a double complex $C^\bullet(\mathcal{U},\mathcal{U}';\mathscr{F}^\bullet)$ by

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^0(\mathcal{U},\mathcal{U}';\mathscr{F}^0) & \xrightarrow{d_{\mathscr{F}}^{0,0}} & C^0(\mathcal{U},\mathcal{U}';\mathscr{F}^1) & \xrightarrow{d_{\mathscr{F}}^{0,1}} & \cdots \\ & & \downarrow{\scriptstyle \delta_{\mathscr{F}}^{0,0}} & & \downarrow{\scriptstyle \delta_{\mathscr{F}}^{0,1}} & & \\ 0 & \longrightarrow & C^1(\mathcal{U},\mathcal{U}';\mathscr{F}^0) & \xrightarrow{d_{\mathscr{F}}^{1,0}} & C^1(\mathcal{U},\mathcal{U}';\mathscr{F}^1) & \xrightarrow{d_{\mathscr{F}}^{1,1}} & \cdots, \\ & & \downarrow{\scriptstyle \delta_{\mathscr{F}}^{1,0}} & & \downarrow{\scriptstyle \delta_{\mathscr{F}}^{1,1}} & & \\ & & \vdots & & \vdots & & \ddots \end{array}$$

where $d_{\mathscr{F}}^{p,q} = C^p(\mathcal{U},\mathcal{U}';d_{\mathscr{F}^\bullet}^q)$ and $\delta_{\mathscr{F}}^{p,q} = \delta_{\mathscr{F}^q}^p$.

Definition 3.31 (Total complexes of double complexes) Let A denote a double complex $\{A^{p,q},\delta^{p,q},d^{p,q}\}_{p,q\in\mathbb{Z}} \in \mathbf{C}_f^2(\mathrm{Mod}(\mathbb{Z}))$. For any $k \in \mathbb{Z}$, we define

$$\mathrm{tot}(A)^k = \bigoplus_{p+q=k} A^{p,q},$$

and

$$d_{\mathrm{tot}(A)}^k\Big|_{A^{p,q}} = \delta^{p,q} \oplus (-1)^p d^{p,q}.$$

Here the simple complex $\big(\mathrm{tot}(A), d_{\mathrm{tot}(A)}\big)$ is called the total complex of a double complex A.

For more details, see Section 11.5 and Section 12.5 in Kashiwara-Schapira[12].

Definition 3.32 (Relative Čech functors) We define a functor $C(\mathcal{U},\mathcal{U}') : \mathbf{C}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}})) \to \mathbf{C}^+(\mathrm{Mod}(\mathbb{Z}))$ by

$$\begin{aligned} C(\mathcal{U},\mathcal{U}')(\mathscr{F}^\bullet) &= \mathrm{tot}\left(C^\bullet(\mathcal{U},\mathcal{U}';\mathscr{F}^\bullet)\right), \\ C(\mathcal{U},\mathcal{U}')(f^\bullet) &= \bigoplus_{p+q=k} C^p(\mathcal{U},\mathcal{U}';f^q), \end{aligned}$$

where $\mathscr{F}^\bullet,\mathscr{G}^\bullet \in \mathrm{Ob}\left(\mathbf{C}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))\right)$ and $f^\bullet \in \mathrm{Hom}_{\mathbf{C}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))}(\mathscr{F}^\bullet,\mathscr{G}^\bullet)$.

Proposition 3.33 The functor $C(\mathcal{U},\mathcal{U}')$ sends morphisms of homotopic to zero into morphisms of homotopic to zero.

Proof. Let $f \in \mathrm{Hom}_{\mathbf{C}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))}(\mathscr{F}^\bullet,\mathscr{G}^\bullet)$ be homotopic to zero. There exists a morphism $s^k : \mathscr{F}^k \to \mathscr{G}^{k-1}$ in $\mathrm{Mod}(\mathbb{Z}_{X_{sa}})$ such that

$$f^k = s^{k+1} \circ d_{\mathscr{F}}^k + d_{\mathscr{G}}^{k-1} \circ s^k$$

for any $k \in \mathbb{Z}$. Then, the morphism $\widetilde{s}^k \colon C(\mathcal{U},\mathcal{U}')(\mathscr{F}^\bullet)^k \to C(\mathcal{U},\mathcal{U}')(\mathscr{G}^\bullet)^{k-1}$ in $\mathrm{Mod}(\mathbb{Z})$ is defined by

$$\widetilde{s}^k = \bigoplus_{p+q=k} (-1)^p C^p(\mathcal{U},\mathcal{U}';s^q)$$

and this morphism satisfies

$$C(\mathcal{U},\mathcal{U}')(f)^k = \widetilde{s}^{k+1} \circ d^k_{C(\mathcal{U},\mathcal{U}')(\mathscr{F})} + d^{k-1}_{C(\mathcal{U},\mathcal{U}')(\mathscr{G})} \circ \widetilde{s}^k.$$

Note that, since s^k is a sheaf morphism for each $k \in \mathbb{Z}$, we have the following commutative diagram:

$$\begin{tikzcd}
C^{p-1}(\mathcal{U},\mathcal{U}';\mathscr{F}^{q+1}) \arrow[r, "\delta^{p-1}_{\mathscr{F}^{q+1}}"] \arrow[d, "C^{p-1}(\mathcal{U},\mathcal{U}';s^{q+1})"'] & C^p(\mathcal{U},\mathcal{U}';\mathscr{F}^{q+1}) \arrow[d, "C^p(\mathcal{U},\mathcal{U}';s^{q+1})"] \\
C^{p-1}(\mathcal{U},\mathcal{U}';\mathscr{G}^q) \arrow[r, "\delta^{p-1}_{\mathscr{G}^q}"] & C^p(\mathcal{U},\mathcal{U}';\mathscr{G}^q).
\end{tikzcd}$$

Hence, $C(\mathcal{U},\mathcal{U}')(f)$ is homotopic to zero. □

Proposition 3.34 The functor $C(\mathcal{U},\mathcal{U}')$ is a triangulated functor from $\mathbf{K}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))$ to $\mathbf{K}^+(\mathrm{Mod}(\mathbb{Z}))$.

Proof. $C(\mathcal{U},\mathcal{U}')$ is additive, and we have $C(\mathcal{U},\mathcal{U}')(\mathscr{F})[1] = C(\mathcal{U},\mathcal{U}')(\mathscr{F}[1])$ for any $\mathscr{F} \in \mathbf{C}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))$. Moreover, by Proposition 3.33, $C(\mathcal{U},\mathcal{U}')$ sends morphisms of homotopic to zero into morphisms of homotopic to zero. Let us prove that $C(\mathcal{U},\mathcal{U}')$ sends distinguished triangles of $\mathrm{Mod}(\mathbb{Z}_{X_{sa}})$ into distinguished triangles of $\mathrm{Mod}(\mathbb{Z}_{X_{sa}})$.

Let $f \in \mathrm{Hom}_{\mathbf{C}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))}(\mathscr{F},\mathscr{G})$. The mapping cone $M(f)$ is defined by

$$\begin{cases} M(f)^k = \mathscr{F}[1]^k \oplus \mathscr{G}^k, \\ d^k_{M(f)} = \begin{pmatrix} d^k_{\mathscr{F}[1]} & 0 \\ f^{k+1} & d^k_{\mathscr{G}} \end{pmatrix}, \end{cases}$$

where $\mathscr{F}[1]^k = \mathscr{F}^{k+1}$ and $d^k_{\mathscr{F}[1]} = -d^{k+1}_{\mathscr{F}}$. We define the morphisms $\alpha(f) : \mathscr{G} \to M(f)$ and $\beta(f) : M(f) \to \mathscr{F}[1]$ by

$$\alpha(f)^k = \begin{pmatrix} 0 \\ \mathrm{id}_{\mathscr{G}^k} \end{pmatrix},$$
$$\beta(f)^k = \begin{pmatrix} \mathrm{id}_{\mathscr{F}^{k+1}} & 0 \end{pmatrix}.$$

It is enough to prove that

$$C(\mathcal{U},\mathcal{U}')(M(f)) = M(C(\mathcal{U},\mathcal{U}')(f)),$$
$$d^k_{C(\mathcal{U},\mathcal{U}')(M(f))} = d^k_{M(C(\mathcal{U},\mathcal{U}')(f))}.$$

We can easily show the above equations by the following calculations.

$$\begin{aligned}
C(\mathcal{U},\mathcal{U}')(M(f))^k &= \bigoplus_{p+q=k} C^p(\mathcal{U},\mathcal{U}';\mathscr{F}^{q+1}\oplus\mathscr{G}^q)\\
&= \bigoplus_{p+q=k} \left(C^p(\mathcal{U},\mathcal{U}';\mathscr{F}^{q+1})\oplus C^p(\mathcal{U},\mathcal{U}';\mathscr{G}^q)\right)\\
&= \bigoplus_{p+q=k+1} C^p(\mathcal{U},\mathcal{U}';\mathscr{F}^{q})\oplus \bigoplus_{p+q=k} C^p(\mathcal{U},\mathcal{U}';\mathscr{G}^q)\\
&= M(C(\mathcal{U},\mathcal{U}')(f))^k
\end{aligned}$$

Since $C^p(\mathcal{U},\mathcal{U}';\mathscr{F}^{q+1})\oplus C^p(\mathcal{U},\mathcal{U}';\mathscr{G}^q) = C^p(\mathcal{U},\mathcal{U}';M(f)^q)$, we get

$$\begin{aligned}
&d^k_{M(C(\mathcal{U},\mathcal{U}')(f))}\Big|_{C^p(\mathcal{U},\mathcal{U}';\mathscr{F}^{q+1})\oplus C^p(\mathcal{U},\mathcal{U}';\mathscr{G}^q)}\\
&= \left(-d^{k+1}_{C(\mathcal{U},\mathcal{U}')(\mathscr{F})}\oplus\left(C(\mathcal{U},\mathcal{U}')(f)^{k+1}+d^k_{C(\mathcal{U},\mathcal{U}')(\mathscr{G})}\right)\right)\Big|_{C^p(\mathcal{U},\mathcal{U}';\mathscr{F}^{q+1})\oplus C^p(\mathcal{U},\mathcal{U}';\mathscr{G}^q)}\\
&= -\left(\delta^{p,q+1}_{\mathscr{F}}+(-1)^p d^{p,q+1}_{\mathscr{F}}\right)\oplus\left(C^p(\mathcal{U},\mathcal{U}';f^{q+1})+\delta^{p,q}_{\mathscr{G}}+(-1)^p d^{p,q}_{\mathscr{G}}\right)\\
&= \left(-\delta^{p,q+1}_{\mathscr{F}}\oplus\delta^{p,q}_{\mathscr{G}}\right)\oplus(-1)^p\left(-d^{p,q+1}_{\mathscr{F}}\oplus\left(C^p(\mathcal{U},\mathcal{U}';f^{q+1})+d^{p,q}_{\mathscr{G}}\right)\right)\\
&= \delta^{p,q}_{M(f)}\oplus(-1)^p d^{p,q}_{M(f)}\\
&= d^k_{C(\mathcal{U},\mathcal{U}')(M(f))}\Big|_{C^p(\mathcal{U},\mathcal{U}';M(f)^q)}.
\end{aligned}$$

This completes the proof. □

Now, let us give a useful criterium that ensures the existence of derived functors.

Lemma 3.35 ([9, Remark 1.8.10]) Let F be a functor of triangulated categories from $\mathbf{K}^+(\mathcal{C})$ to $\mathbf{K}^+(\mathcal{C}')$. Assume to be given a full triangulated subcategory $\mathcal{I}$ of $\mathbf{K}^+(\mathcal{C})$ such that

(1) For any $\mathscr{F}^\bullet\in\mathrm{Ob}(\mathbf{K}^+(\mathcal{C}))$, there is a quasi-isomorphism $\mathscr{F}^\bullet\to\mathscr{G}^\bullet$ with a complex $\mathscr{G}^\bullet\in\mathcal{I}$,

(2) If $\mathscr{F}^\bullet\in\mathrm{Ob}(\mathcal{I})$ is quasi-isomorphic to 0, then $F(\mathscr{F}^\bullet)$ is quasi-isomorphic to 0.

Then, we can define $\mathbf{R}F:\mathbf{D}^+(\mathcal{C})\to\mathbf{D}^+(\mathcal{C}')$.

Proposition 3.36 (Relative Čech derived functors) There exists the right derived functor $\mathbf{R}C(\mathcal{U},\mathcal{U}'):\mathbf{D}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))\to\mathbf{D}^+(\mathrm{Mod}(\mathbb{Z}))$.

We define $F=C(\mathcal{U},\mathcal{U}')$, $\mathcal{C}=\mathrm{Mod}(\mathbb{Z}_{X_{sa}})$, $\mathcal{C}'=\mathrm{Mod}(\mathbb{Z})$ and let $\mathcal{I}$ be the full triangulated subcategory of $\mathbf{K}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))$ whose complexes consisting of $\mathcal{U}$-soft sheaves on X_{sa}. Then, we can use Lemma 3.35, and it is enough to prove the following lemma. Note that the condition (1) in Lemma 3.35 is obvious since the existence of flabby resolutions and the fact that flabby sheaves are $\mathcal{U}$-soft sheaves.

Lemma 3.37 If $\mathscr{F}^\bullet\in\mathbf{K}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))$ consisting of $\mathcal{U}$-soft sheaves is quasi-isomorphic to 0, $C(\mathcal{U},\mathcal{U}')(\mathscr{F}^\bullet)$ is quasi-isomorphic to 0.

Proof. Let $\mathscr{F}^\bullet \in \mathbf{K}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))$ satisfy the assumptions of the lemma. $C(\mathcal{U})(\mathscr{F}^\bullet)$ is quasi-isomorphic to 0 by Lemma 3.19. Note that $C(\mathcal{U})(\bullet) = \mathrm{tot}\,(C^\bullet(\mathcal{U};\bullet))$ is the direct sum of $\Gamma(U_\alpha;\bullet)$ for any $j \geq 0$, $\alpha \in \Lambda^{j+1}$. From the short exact sequence

$$0 \longrightarrow C(\mathcal{U},\mathcal{U}')(\mathscr{F}^\bullet) \longrightarrow C(\mathcal{U})(\mathscr{F}^\bullet) \longrightarrow C(\mathcal{U}')(\mathscr{F}^\bullet) \longrightarrow 0,$$

there is a long exact sequence

$$0 \to H^0(\mathcal{U},\mathcal{U}')(\mathscr{F}^\bullet) \to H^0(\mathcal{U})(\mathscr{F}^\bullet) \to H^0(\mathcal{U}')(\mathscr{F}^\bullet) \to H^1(\mathcal{U},\mathcal{U}')(\mathscr{F}^\bullet) \to \cdots.$$

Hence, $H^k(\mathcal{U},\mathcal{U}')(\mathscr{F}^\bullet)$ holds for any $k \in \mathbb{Z}$. □

Lemma 3.38 Let $V \in \mathrm{Op}(X_{sa})$, and let $\mathcal{V} = \{V_i\}_{i\in\Lambda}$ be a finite open subanalytic covering of V. If $\mathscr{F} \in \mathrm{Mod}(\mathbb{Z}_{X_{sa}})$ satisfies $H^k(V_{\alpha_0} \cap \cdots \cap V_{\alpha_j};\mathscr{F}) = 0$ $(k \neq 0)$ for any $j \geq 0$, $\alpha = (\alpha_0,\cdots,\alpha_j) \in \Lambda^{j+1}$, we have

$$H^k(\mathcal{V};\mathscr{F}) = \begin{cases} \mathscr{F}(V) & (k=0),\\ 0 & (k\neq 0),\end{cases}$$

that is, $\iota_V : \Gamma(V;\mathscr{F}) \to C^\bullet(\mathcal{V};\mathscr{F})$ is a quasi-isomorphism.

Proof. For any $\ell \geq 0$, $\alpha = (\alpha_0,\cdots,\alpha_j) \in \Lambda^{j+1}$ and for any $V' \in \mathrm{Op}(X_{sa})$, we set

$$\begin{aligned} V_\alpha &= V_{\alpha_0} \cap V_{\alpha_1} \cap \cdots \cap V_{\alpha_j},\\ V^{(\ell)} &= V_0 \cup V_1 \cap \cdots \cup V_\ell,\\ \mathcal{V}_\ell &= \{V_0, V_1, \cdots, V_\ell\},\\ \mathcal{V}_\ell \cap V' &= \{V_0 \cap V', V_1 \cap V', \cdots, V_\ell \cap V'\}. \end{aligned}$$

Let us prove by induction on $\ell \geq 0$. It is obvious that $\Gamma(V^{(0)};\mathscr{F}) \to C^\bullet(\mathcal{V}_0;\mathscr{F})$ and $\Gamma(V^{(0)} \cap V_\alpha;\mathscr{F}) \to C^\bullet(\mathcal{V}_0 \cap V_\alpha;\mathscr{F})$ are quasi-isomorphisms for any $j \geq 0$, $\alpha \in \Lambda^{j+1}$.

Assume that $\Gamma(V^{(\ell)};\mathscr{F}) \to C^\bullet(\mathcal{V}_\ell;\mathscr{F})$ and $\Gamma(V^{(\ell)} \cap V_\alpha;\mathscr{F}) \to C^\bullet(\mathcal{V}_\ell \cap V_\alpha;\mathscr{F})$ are quasi-isomorphisms for any $j \geq 0$, $\alpha \in \Lambda^{j+1}$. We show that $\Gamma(V^{(\ell+1)};\mathscr{F}) \to C^\bullet(\mathcal{V}_{\ell+1};\mathscr{F})$ and $\Gamma(V^{(\ell+1)} \cap V_\alpha;\mathscr{F}) \to C^\bullet(\mathcal{V}_{\ell+1} \cap V_\alpha;\mathscr{F})$ are quasi-isomorphisms for any $j \geq 0$, $\alpha \in \Lambda^{j+1}$. By the Mayer-Vietoris sequence, there is the following diagram.

$$\begin{array}{ccccc} 0 \longrightarrow & H^0(V^{(\ell+1)};\mathscr{F}) & \longrightarrow & H^0(V^{(\ell)};\mathscr{F}) \oplus H^0(V_{\ell+1};\mathscr{F}) \\ & \downarrow & & \downarrow \\ 0 \longrightarrow & H^0(\mathcal{V}_{\ell+1};\mathscr{F}) & \longrightarrow & H^0(\mathcal{V}_\ell;\mathscr{F}) \oplus H^0(V_{\ell+1};\mathscr{F}) \end{array}$$

$$\begin{array}{ccccc} \longrightarrow & H^0(V^{(\ell)} \cap V_{\ell+1};\mathscr{F}) & \longrightarrow & H^1(V^{(\ell+1)};\mathscr{F}) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & \\ \longrightarrow & H^0(\mathcal{V}_\ell \cap V_{\ell+1};\mathscr{F}) & \longrightarrow & H^1(\mathcal{V}_{\ell+1};\mathscr{F}) & \longrightarrow \cdots. \end{array}$$

By the inductive hypothesis and the five lemma, we get $H^k(V^{(\ell+1)};\mathscr{F}) \simeq H^k(\mathcal{V}_{\ell+1};\mathscr{F})$ for any $k \in \mathbb{Z}$. Similarly, for any $j \geq 0$, $\alpha \in \Lambda^{j+1}$, there is the following diagram.

$$\begin{array}{ccccc} 0 \longrightarrow & H^0(V^{(\ell+1)} \cap V_\alpha;\mathscr{F}) & \longrightarrow & H^0(V^{(\ell)} \cap V_\alpha;\mathscr{F}) \oplus H^0(V_{\ell+1} \cap V_\alpha;\mathscr{F}) & \\ & \downarrow & & \downarrow & \\ 0 \longrightarrow & H^0(\mathcal{V}_{\ell+1} \cap V_\alpha;\mathscr{F}) & \longrightarrow & H^0(\mathcal{V}_\ell \cap V_\alpha;\mathscr{F}) \oplus H^0(V_{\ell+1} \cap V_\alpha;\mathscr{F}) & \end{array}$$

$$\begin{array}{ccccc} \longrightarrow & H^0(V^{(\ell)} \cap (V_{\ell+1} \cap V_\alpha);\mathscr{F}) & \longrightarrow & H^1(V^{(\ell+1)} \cap V_\alpha;\mathscr{F}) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & \\ \longrightarrow & H^0(\mathcal{V}_\ell \cap (V_{\ell+1} \cap V_\alpha);\mathscr{F}) & \longrightarrow & H^1(\mathcal{V}_{\ell+1} \cap V_\alpha;\mathscr{F}) & \longrightarrow \cdots . \end{array}$$

Then, we also get $H^k(V^{(\ell+1)} \cap V_\alpha;\mathscr{F}) \simeq H^k(\mathcal{V}_{\ell+1} \cap V_\alpha;\mathscr{F})$ for any $k \in \mathbb{Z}$. □

Corollary 3.39 Let $\mathscr{F}$ be a subanalytic sheaf on X, and let $V \in \mathrm{Op}(X_{sa})$. For any finite open subanalytic covering $\mathcal{V}$, $\mathcal{W}$ of V, if $\mathscr{F}$ is $\mathcal{V}$-soft and $\mathcal{W}$-soft, there is the canonical isomorphism

$$H^k(\mathcal{V};\mathscr{F}) \simeq H^k(\mathcal{W};\mathscr{F}) \quad (k \in \mathbb{Z}).$$

Corollary 3.40 Complexes consisting of $\mathcal{U}$-soft sheaves on X_{sa} are $C(\mathcal{U})$-acyclic.

Proposition 3.41 We have the canonical quasi-isomorphism

$$\mathbf{R}\Gamma_Z(U;\bullet) \xrightarrow[QIS]{\sim} \mathbf{R}C(\mathcal{U},\mathcal{U}')(\bullet).$$

Proof. For any complex $\mathscr{I}^\bullet$ of flabby sheaves, we prove

$$\Gamma_Z(U;\mathscr{I}^\bullet) \xrightarrow{\sim} C(\mathcal{U},\mathcal{U}')(\mathscr{I}^\bullet).$$

By Corollary 3.39, it is enough to prove the above statement with $\mathcal{U} = \{U, U \setminus Z\}$ and $\mathcal{U}' = \{U \setminus Z\}$. Note that $\mathscr{I}^\bullet$ is a complex of $\{U, U \setminus Z\}$-soft sheaves since flabby sheaves are soft sheaves.

We define $C(U, U \setminus Z) = C(\{U, U \setminus Z\}, \{U \setminus Z\})$ and it is easy to show

$$\begin{aligned} C(U, U \setminus Z)(\mathscr{I}^\bullet)^k &= \Gamma(U;\mathscr{I}^k) \oplus \Gamma(U \setminus Z;\mathscr{I}^{k-1}), \\ d^k_{C(U,U\setminus Z)(\mathscr{I}^\bullet)}\left((s_1, s_{01})\right) &= \left(d^k_{\mathscr{I}^\bullet(U)}(s_1),\, s_1|_{U\setminus Z} - d^{k-1}_{\mathscr{I}^\bullet(U\setminus Z)}(s_{01})\right), \end{aligned}$$

where $k \in \mathbb{Z}$ and $(s_1, s_{01}) \in \Gamma(U;\mathscr{I}^k) \oplus \Gamma(U \setminus Z;\mathscr{I}^{k-1})$.

The canonical morphism $\iota : \Gamma_Z(U;\mathscr{I}^\bullet) \to C(U, U \setminus Z)(\mathscr{I}^\bullet)$ is defined by

$$\begin{array}{ccc} \Gamma_Z(U;\mathscr{I}^k) & \longrightarrow & C(U, U \setminus Z)(\mathscr{I}^\bullet)^k \\ \cup\!\!\!\!\!\! \ \ \uplus & & \uplus \\ s & \longmapsto & (s, 0) \end{array} \qquad (k \in \mathbb{Z}).$$

It follows from $\mathrm{supp}(s) \subset Z$ that ι forms a morphism of complexes.

Since each $\mathscr{I}^k$ is flabby, for any $(s_1, s_{01}) \in C(U, U \setminus Z)(\mathscr{I}^\bullet)^k$, there exists $\tilde{s}_{01} \in \Gamma(U; \mathscr{I}^{k-1})$ such that $\tilde{s}_{01}|_{U\setminus Z} = s_{01}$. We have

$$[(s_1, s_{01})] = \Big[(s_1, s_{01}) - d^{k-1}_{C(U,U\setminus Z)(\mathscr{I}^\bullet)}(\tilde{s}_{01}, 0)\Big] = [(s_1 - d^{k-1}_{\mathscr{I}^\bullet(U)}(\tilde{s}_{01}), 0)] \in H^k(U, U \setminus Z; \mathscr{I}^\bullet).$$

Here, $H^k(U, U \setminus Z; \mathscr{I}^\bullet)$ is the k-th cohomology group of $C(U, U \setminus Z)(\mathscr{I}^\bullet)$. If there is another $\tilde{s}'_{01}$ such that $\tilde{s}_{01}|_{U\setminus Z} = s_{01}$, then $(s_1 - d^{k-1}_{\mathscr{I}^\bullet(U)}(\tilde{s}_{01}), 0)$ and $(s_1 - d^{k-1}_{\mathscr{I}^\bullet(U)}(\tilde{s}'_{01}), 0)$ give the same equivalent class. Hence, the morphism

$$\begin{array}{ccc} H^k(U, U \setminus Z)(\mathscr{I}^\bullet) & \longrightarrow & H^k_Z(U; \mathscr{I}^\bullet) \\ \Psi & & \Psi \\ [(s_1, s_{01})] & \longmapsto & [s_1 - d^{k-1}_{\mathscr{I}^\bullet(U)}(\tilde{s}_{01})] \end{array} \qquad (k \in \mathbb{Z})$$

is well-defined and this is nothing but the inverse of

$$\begin{array}{ccc} H^k_Z(U; \mathscr{I}^\bullet) & \longrightarrow & H^k(U, U \setminus Z)(\mathscr{I}^\bullet) \\ \Psi & & \Psi \\ [s] & \longmapsto & [(s, 0)] \end{array} \qquad (k \in \mathbb{Z}).$$

This completes the proof. □

Remark 3.42 We denote $C(\{U, U \setminus Z\}, \{U \setminus Z\})$ by $C(U, U \setminus Z)$.

Corollary 3.43 Let $\mathscr{F}^\bullet$ be a complex of sheaves on X_{sa}. By Proposition 3.36 and Proposition 3.41, if $\mathscr{I}^\bullet$ is a $\mathcal{U}$-soft resolution of $\mathscr{F}^\bullet$, we have

$$H^k(\mathbf{R}\Gamma_Z(U; \mathscr{F}^\bullet)) \simeq H^k(\mathbf{R}C(\mathcal{U}, \mathcal{U}')(\mathscr{F}^\bullet)) \simeq H^k(\mathcal{U}, \mathcal{U}'; \mathscr{I}^\bullet)$$

for any $k \in \mathbb{Z}$.

3.3 Localized relative Čech derived functors

Let $U \subset X$ be an open subanalytic subset and $Z \subset X$ a closed subanalytic subset. We take a finite open subanalytic covering $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ of U and a finite open subanalytic covering $\mathcal{U}' = \{U_i\}_{i\in\Lambda'} \subset \mathcal{U}$ of $U \setminus Z$. Note that $\Lambda' \subset \Lambda$.

Definition 3.44 Let $\mathscr{F} \in \mathrm{Mod}(\mathbb{Z}_{X_{sa}})$. For any $k = 0, 1, 2, \cdots$ and for any $V \in \mathrm{Op}(X_{sa})$, we define

$$\begin{aligned} C^k_{\mathcal{U},\mathcal{U}'}(\mathscr{F})(V) &= \left\{ \bigoplus_{(\alpha_0,\cdots,\alpha_k)\in\Lambda^{k+1}} s_{\alpha_0,\cdots,\alpha_k} \;\middle|\; \begin{array}{l} s_{\alpha_0,\cdots,\alpha_k} \in \Gamma_{U_{\alpha_0}\cap\cdots\cap U_{\alpha_k}}(\mathscr{F})(V), \\ s_{\alpha_0,\cdots,\alpha_\ell,\alpha_{\ell+1},\cdots,\alpha_k} = -s_{\alpha_0,\cdots,\alpha_{\ell+1},\alpha_\ell,\cdots,\alpha_k}, \\ (\alpha_0, \cdots, \alpha_k) \in \Lambda'^{k+1} \Rightarrow s_{\alpha_0,\cdots,\alpha_k} = 0 \end{array} \right\} \\ &= C^k(\mathcal{U} \cap V, \mathcal{U}' \cap V; \mathscr{F}), \end{aligned}$$

and a map $\delta^k_{\mathscr{F},V} : C^k_{\mathcal{U},\mathcal{U}'}(\mathscr{F})(V) \to C^{k+1}_{\mathcal{U},\mathcal{U}'}(\mathscr{F})(V)$ is defined by

$$\delta^k_{\mathscr{F},V}\left(\bigoplus_{(\alpha_0,\cdots,\alpha_k)\in\Lambda^{k+1}} s_{\alpha_0,\cdots,\alpha_k}\right) = \bigoplus_{(\beta_0,\cdots,\beta_{k+1})\in\Lambda^{k+2}}\left(\sum_{\ell=0}^{k+1}(-1)^\ell\left(s_{\beta_0,\cdots,\widehat{\beta_\ell},\cdots,\beta_{k+1}}\right)\Big|_{U_{\beta_0}\cap\cdots\cap U_{\beta_{k+1}}\cap V}\right),$$

where $\widehat{\bullet}$ means to omit the corresponding letter in a sequence. $(C^\bullet_{\mathcal{U},\mathcal{U}'}(\mathscr{F})(V), \delta^\bullet_{\mathscr{F},V})$ is a complex of $\mathrm{Mod}(\mathbb{Z})$. Here, it is easy to show the following statement.

- $C^k_{\mathcal{U},\mathcal{U}'}(\mathscr{F}) = \{C^k_{\mathcal{U},\mathcal{U}'}(\mathscr{F})(V)\}_{V\in\mathrm{Op}(X_{sa})}$ forms a sheaf on X_{sa}.
- $\delta^k_{\mathscr{F}} = \{\delta^k_{\mathscr{F},V}\}_{V\in\mathrm{Op}(X_{sa})}$ forms a sheaf morphism from $C^k_{\mathcal{U},\mathcal{U}'}(\mathscr{F})$ to $C^{k+1}_{\mathcal{U},\mathcal{U}'}(\mathscr{F})$.

$(C^\bullet_{\mathcal{U},\mathcal{U}'}(\mathscr{F}), \delta^\bullet_{\mathscr{F}})$ is a sheaf complex, and we write its k-th cohomology sheaf as $H^k(C^\bullet_{\mathcal{U},\mathcal{U}'}(\mathscr{F}))$. If $\mathcal{U}' = \emptyset$, we express it simply as $C^k_{\mathcal{U}}(\mathscr{F})(V) = C^k_{\mathcal{U},\emptyset}(\mathscr{F})(V)$ and $H^k(C^\bullet_{\mathcal{U}}(\mathscr{F})) = H^k(C^\bullet_{\mathcal{U},}(\mathscr{F}))$.

Definition 3.45 Let $\mathscr{F},\mathscr{G} \in \mathrm{Mod}(\mathbb{Z}_{X_{sa}})$ and f a sheaf morphism from $\mathscr{F}$ to $\mathscr{G}$. For any $V \in \mathrm{Op}(X_{sa})$, we define

$$C^k_{\mathcal{U},\mathcal{U}'}(f_V) = C^k(\mathcal{U},\mathcal{U}'; f_V).$$

$C^k_{\mathcal{U},\mathcal{U}'}(f) = \{C^k_{\mathcal{U},\mathcal{U}'}(f_V)\}_{V\in\mathrm{Op}(X_{sa})}$ forms a sheaf morphism, and $C^\bullet_{\mathcal{U},\mathcal{U}'}(f) = \{C^k_{\mathcal{U},\mathcal{U}'}(f)\}_{k\in\mathbb{Z}}$ forms a morphism of sheaf complexes.

Definition 3.46 Let $\mathscr{F}^\bullet \in \mathbf{C}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))$. We define a double complex $C^\bullet_{\mathcal{U},\mathcal{U}'}(\mathscr{F}^\bullet)$ by

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^0_{\mathcal{U},\mathcal{U}'}(\mathscr{F}^0) & \xrightarrow{d^{0,0}_{\mathscr{F}}} & C^0_{\mathcal{U},\mathcal{U}'}(\mathscr{F}^1) & \xrightarrow{d^{0,1}_{\mathscr{F}}} & \cdots \\
 & & \downarrow{\scriptstyle\delta^{0,0}_{\mathscr{F}}} & & \downarrow{\scriptstyle\delta^{0,1}_{\mathscr{F}}} & & \\
0 & \longrightarrow & C^1_{\mathcal{U},\mathcal{U}'}(\mathscr{F}^0) & \xrightarrow{d^{1,0}_{\mathscr{F}}} & C^1_{\mathcal{U},\mathcal{U}'}(\mathscr{F}^1) & \xrightarrow{d^{1,1}_{\mathscr{F}}} & \cdots, \\
 & & \downarrow{\scriptstyle\delta^{1,0}_{\mathscr{F}}} & & \downarrow{\scriptstyle\delta^{1,1}_{\mathscr{F}}} & & \\
 & & \vdots & & \vdots & & \ddots
\end{array}$$

where $d^{p,q}_{\mathscr{F}} = C^p_{\mathcal{U},\mathcal{U}'}(d^q_{\mathscr{F}^\bullet})$ and $\delta^{p,q}_{\mathscr{F}} = \delta^p_{\mathscr{F}^q}$.

Definition 3.47 (Localized relative Čech functors) We define a functor $C_{\mathcal{U},\mathcal{U}'} : \mathbf{C}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}})) \to \mathbf{C}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))$ by

$$C_{\mathcal{U},\mathcal{U}'}(\mathscr{F}^\bullet) = \mathrm{tot}\left(C^\bullet_{\mathcal{U},\mathcal{U}'}(\mathscr{F}^\bullet)\right),$$
$$C_{\mathcal{U},\mathcal{U}'}(f^\bullet) = \bigoplus_{p+q=k} C^p_{\mathcal{U},\mathcal{U}'}(f^q),$$

where $\mathscr{F}^\bullet,\mathscr{G}^\bullet \in \mathrm{Ob}\left(\mathbf{C}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))\right)$ and $f^\bullet \in \mathrm{Hom}_{\mathbf{C}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))}(\mathscr{F}^\bullet,\mathscr{G}^\bullet)$.

In what follows, we will prove the existence of the right derived functor $\mathbf{R}C_{\mathcal{U},\mathcal{U}'}$ (resp. $\rho_{sa}^{-1} \circ \mathbf{R}C_{\mathcal{U},\mathcal{U}'}$). Many of the proofs can be obtained by replacing $\Gamma(U;\bullet)$ with Γ_U (resp. $\rho_{sa}^{-1} \circ \Gamma_U$) and $C(\mathcal{U},\mathcal{U}';\bullet)$ with $C_{\mathcal{U},\mathcal{U}'}$ (resp. $\rho_{sa}^{-1} \circ C_{\mathcal{U},\mathcal{U}'}$) in the proofs in Subsection 3.2. Therefore, we will only refer to the corresponding propositions.

Proposition 3.48 The functor $C_{\mathcal{U},\mathcal{U}'}$ (resp. $\rho_{sa}^{-1} \circ C_{\mathcal{U},\mathcal{U}'}$) sends morphisms of homotopic to zero into morphisms of homotopic to zero.

See the proof of Proposition 3.33.

Proposition 3.49 The functor $C_{\mathcal{U},\mathcal{U}'}$ (resp. $\rho_{sa}^{-1} \circ C_{\mathcal{U},\mathcal{U}'}$) is a triangulated functor from $\mathbf{K}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))$ to $\mathbf{K}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))$ (resp. $\mathbf{K}^+(\mathrm{Mod}(\mathbb{Z}_X))$).

See the proof of Proposition 3.34.

Proposition 3.50 (Localized relative Čech derived functors) There exists the right derived functor $\mathbf{R}C_{\mathcal{U},\mathcal{U}'} : \mathbf{D}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}})) \to \mathbf{D}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))$ (resp. $\rho_{sa}^{-1} \circ \mathbf{R}C_{\mathcal{U},\mathcal{U}'} : \mathbf{D}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}})) \to \mathbf{D}^+(\mathrm{Mod}(\mathbb{Z}_X))$).

The proof of this proposition is similar to that of Proposition 3.36. We define $F = C_{\mathcal{U},\mathcal{U}'}$ (resp. $F = \rho_{sa}^{-1} \circ C_{\mathcal{U},\mathcal{U}'}$), $\mathcal{C} = \mathrm{Mod}(\mathbb{Z}_{X_{sa}})$, $\mathcal{C}' = \mathrm{Mod}(\mathbb{Z}_{X_{sa}})$ (resp. $\mathcal{C}' = \mathrm{Mod}(\mathbb{Z}_X)$) and let $\mathcal{I}$ be the full triangulated subcategory of $\mathbf{K}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))$ whose complexes consisting of soft (resp. $\mathcal{U}$-soft) sheaves on X_{sa}. Then, we can use Lemma 3.35, and it is enough to prove the following lemma.

Lemma 3.51 If $\mathscr{F}^\bullet \in \mathbf{K}^+(\mathrm{Mod}(\mathbb{Z}_{X_{sa}}))$ consisting of soft (resp. $\mathcal{U}$-soft) sheaves is quasi-isomorphic to 0, $C_{\mathcal{U},\mathcal{U}'}(\mathscr{F}^\bullet)$ (resp. $\rho_{sa}^{-1} \circ C_{\mathcal{U},\mathcal{U}'}(\mathscr{F}^\bullet)$) is quasi-isomorphic to 0.

See the proof of Lemma 3.37.

Lemma 3.52 Let $V \in \mathrm{Op}(X_{sa})$ and let $\mathcal{V} = \{V_i\}_{i\in\Lambda}$ be a finite open subanalytic covering of V. If $\mathscr{F} \in \mathrm{Mod}(\mathbb{Z}_{X_{sa}})$ satisfies $H^k(\Gamma_{V_{\alpha_0}\cap\cdots\cap V_{\alpha_j}}(\mathscr{F})) = 0$ (resp. $H^k(\rho_{sa}^{-1} \circ \Gamma_{V_{\alpha_0}\cap\cdots\cap V_{\alpha_j}}(\mathscr{F})) = 0$) $(k \neq 0)$ for any $j \geq 0$, $\alpha = (\alpha_0, \cdots, \alpha_j) \in \Lambda^{j+1}$, we have

$$H^k(C_{\mathcal{V}}(\mathscr{F})) = \begin{cases} \Gamma_V(\mathscr{F}) & (k = 0), \\ 0 & (k \neq 0), \end{cases}$$

$$\left(\quad \text{resp. } H^k(\rho_{sa}^{-1} \circ C_{\mathcal{V}}(\mathscr{F})) = \begin{cases} \rho_{sa}^{-1} \circ \Gamma_V(\mathscr{F}) & (k = 0), \\ 0 & (k \neq 0) \end{cases} \right).$$

See the proof of Lemma 3.38.

Corollary 3.53 Let $\mathscr{F}$ be a subanalytic sheaf on X, and let $V \in \mathrm{Op}(X_{sa})$. For any finite open subanalytic covering $\mathcal{V}$, $\mathcal{W}$ of V, if $\mathscr{F}$ is soft (resp. $\mathcal{V}$-soft and $\mathcal{W}$-soft), there is the canonical isomorphism

$$H^k(C_{\mathcal{V}}(\mathscr{F})) \simeq H^k(C_{\mathcal{W}}(\mathscr{F})) \quad (k \in \mathbb{Z}),$$

$$\Big(\quad \text{resp. } H^k(\rho_{sa}^{-1} \circ C_{\mathcal{V}}(\mathscr{F})) \simeq H^k(\rho_{sa}^{-1} \circ C_{\mathcal{W}}(\mathscr{F})) \quad (k \in \mathbb{Z}) \quad \Big).$$

Corollary 3.54 Complexes consisting of soft (resp. $\mathcal{U}$-soft) sheaves on X_{sa} are $C_{\mathcal{U}}$-acyclic (resp. $\rho_{sa}^{-1} \circ C_{\mathcal{U}}$-acyclic).

Proposition 3.55 We have the canonical quasi-isomorphism

$$\mathbf{R}\Gamma_{U \cap Z}(\bullet) \xrightarrow[QIS]{\sim} \mathbf{R}C_{\mathcal{U},\mathcal{U}'}(\bullet),$$

$$\left(\quad \text{resp. } \quad \rho_{sa}^{-1} \circ \mathbf{R}\Gamma_{U \cap Z}(\bullet) \xrightarrow[QIS]{\sim} \rho_{sa}^{-1} \circ \mathbf{R}C_{\mathcal{U},\mathcal{U}'}(\bullet) \quad \right).$$

Proof. We can prove $\mathbf{R}\Gamma_{U \cap Z}(\bullet) \xrightarrow[QIS]{\sim} \mathbf{R}C_{\mathcal{U},\mathcal{U}'}(\bullet)$ by the same argument as in the proof of Proposition 3.41. Then, since ρ_{sa}^{-1} is an exact functor, $\rho_{sa}^{-1} \circ \mathbf{R}\Gamma_{U \cap Z}(\bullet) \xrightarrow[QIS]{\sim} \rho_{sa}^{-1} \circ \mathbf{R}C_{\mathcal{U},\mathcal{U}'}(\bullet)$ also holds. □

Remark 3.56 We denote $C_{\{U,U \setminus Z\},\{U \setminus Z\}}$ by $C_{U,U \setminus Z}$.

Corollary 3.57 Let $\mathscr{F}^\bullet$ be a complex of sheaves on X_{sa}. By Proposition 3.50 and Proposition 3.55, if $\mathscr{I}^\bullet$ is a soft (resp. $\mathcal{U}$-soft) resolution of $\mathscr{F}^\bullet$, we have

$$H^k(\mathbf{R}\Gamma_{U \cap Z}(\mathscr{F}^\bullet)) \simeq H^k(\mathbf{R}C_{\mathcal{U},\mathcal{U}'}(\mathscr{F}^\bullet)) \simeq H^k(C_{\mathcal{U},\mathcal{U}'}(\mathscr{I}^\bullet))$$

$$\Big(\text{ resp. } H^k(\rho_{sa}^{-1} \circ \mathbf{R}\Gamma_{U \cap Z}(\mathscr{F}^\bullet)) \simeq H^k(\rho_{sa}^{-1} \circ \mathbf{R}C_{\mathcal{U},\mathcal{U}'}(\mathscr{F}^\bullet)) \simeq H^k(\rho_{sa}^{-1} \circ C_{\mathcal{U},\mathcal{U}'}(\mathscr{I}^\bullet)) \Big)$$

for any $k \in \mathbb{Z}$.

Theorem 3.58 Let $\mathscr{F}^\bullet$ be a complex of sheaves on X_{sa}. For any $V \in \mathrm{Op}(X_{sa})$, we have

$$\mathbf{R}\Gamma(V; \mathbf{R}C_{\mathcal{U},\mathcal{U}'}(\mathscr{F}^\bullet)) = \mathbf{R}C(\mathcal{U} \cap V, \mathcal{U}' \cap V; \mathscr{F}^\bullet).$$

This theorem follows from Proposition 3.7.

4 Application

4.1 Tempered distributions

Let M be a real analytic, n-dimensional, oriented manifold and X its complexification. We fix an isomorphism between $or_{M/X}$ and $\mathbb{Z}_M$, and will often omit the orientation sheaf hereafter.

Notation 4.1 We write the sheaf of distributions on M as $\mathscr{D}b_M$.

Definition 4.2 ([19, Definition 3.3.1]) For any open subanalytic subset $\Omega \subset M$, we define

$$\mathscr{D}b^t_{M_{sa}}(\Omega) = \mathscr{D}b_M(M)/\Gamma_{M\setminus\Omega}(M;\mathscr{D}b_M).$$

Then, $\mathscr{D}b^t_{M_{sa}} = \{\mathscr{D}b^t_{M_{sa}}(\Omega)\}_{\Omega\in\mathrm{Op}(M_{sa})}$ forms a sheaf on M_{sa}.

For more details of $\mathscr{D}b^t_{M_{sa}}$, we also refer to Schapira[23].

Notation 4.3 For any $\Omega \in \mathrm{Op}(M)$ and for any $U \in \mathrm{Op}(X)$, we say that U is a complex neighborhood of Ω if $\Omega \subset U$ and Ω is a closed set in U.

Notation 4.4 For any $q \in \{0, 1, \cdots, n\}$, we define that $C^{\infty,t,(0,q)}_{X_{sa}}$ is the sheaf of $(0, q)$-forms with $C^{\infty,t}_{X_{sa}}$ coefficients on X_{sa}. For other $q \in \mathbb{Z}$, $C^{\infty,t,(0,q)}_{X_{sa}} = 0$. We write the Dolbeault complex of the sheaf $C^{\infty,t}_{X_{sa}}$ as $(C^{\infty,t,(0,\bullet)}_{X_{sa}}, \bar\partial)$.

Notation 4.5 The derived sheaf $\mathscr{O}^t_{X_{sa}}$ is defined by the complex $(C^{\infty,t,(0,\bullet)}_{X_{sa}}, \bar\partial)$. $\mathscr{O}^t_{X_{sa}}$ is called the derived sheaf of tempered homlomorphic functions.

Proposition 4.6 Let $\Omega \in \mathrm{Op}(X_{sa})$. We have

$$\mathscr{D}b^t_{M_{sa}} \simeq H^n(\mathbf{R}\Gamma_M(\mathscr{O}^t_{X_{sa}}))|_M \underset{\mathbb{Z}_M}{\otimes} or_{M/X} \simeq H^n(\mathbf{R}\Gamma_M(\mathscr{O}^t_{X_{sa}}))|_M,$$

where $or_{M/X}$ is the relative orientation sheaf.

By Example 3.13, $C^{\infty,t,(0,q)}_{X_{sa}}$ is soft for any $q \in \mathbb{Z}$. Then, by the definition of $\mathscr{O}^t_{X_{sa}}$ and Corollary 3.57, we have

$$H^n(\mathbf{R}\Gamma_M(\mathscr{O}^t_{X_{sa}}))|_M \simeq H^n(\mathbf{R}\Gamma_M(C^{\infty,t,(0,\bullet)}_{X_{sa}}))|_M \simeq H^n(C_{X,X\setminus M}(C^{\infty,t,(0,\bullet)}_{X_{sa}}))|_M.$$

Considering Theorem 3.58, $\mathscr{D}b^t_{M_{sa}}(\Omega)$ can be computed as follows.

$$\begin{aligned}\mathscr{D}b^t_{M_{sa}}(\Omega) &\simeq H^n(C_{X,X\setminus M}(C^{\infty,t,(0,\bullet)}_{X_{sa}}))|_M(\Omega)\\ &\simeq \varinjlim_U H^n(U; C_{X,X\setminus M}(C^{\infty,t,(0,\bullet)}_{X_{sa}}))\\ &\simeq \varinjlim_U H^n(U, U\setminus\Omega; C^{\infty,t,(0,\bullet)}_{X_{sa}}),\end{aligned}$$

where $U \in \mathrm{Op}(X_{sa})$ runs through complex neighborhoods of Ω. By Corollary 3.43 and $\Gamma_\Omega(U;\bullet) \simeq \Gamma_\Omega(V;\bullet)$ for any complex neighborhoods U and V of Ω, we have $\mathscr{D}b^t(\Omega) \simeq H^n(U, U\setminus\Omega; C^{\infty,t,(0,\bullet)}_{X_{sa}})$ for any complex neighborhood U of Ω. Now, let us recall the definition of the cohomology group $H^n(U, U\setminus\Omega; C^{\infty,t,(0,\bullet)}_{X_{sa}})$.

Definition 4.7 For any $q \in \mathbb{Z}$ and for any complex neighborhood U of Ω, we define

$$C^{\infty,t,(0,q)}_{X_{sa}}(U, U\setminus\Omega) = C^{\infty,t,(0,q)}_{X_{sa}}(U) \oplus C^{\infty,t,(0,q-1)}_{X_{sa}}(U\setminus\Omega),$$

and

$$\begin{array}{rccc} \bar{\vartheta} \;:\; & C^{\infty,t,(0,q)}_{X_{sa}}(U, U \setminus \Omega) & \longrightarrow & C^{\infty,t,(0,q+1)}_{X_{sa}}(U, U \setminus \Omega) \\ & \Downarrow & & \Downarrow \\ & (\mu_1, \mu_{01}) & \longmapsto & (\bar{\partial}\mu_1,\, \mu_1|_{U\setminus\Omega} - \bar{\partial}\mu_{01}). \end{array}$$

Then, $(C^{\infty,t,(0,\bullet)}_{X_{sa}}(U, U \setminus \Omega), \bar{\vartheta})$ is a complex of vector spaces. $H^q(U, U \setminus \Omega; C^{\infty,t,(0,\bullet)}_{X_{sa}})$ denotes the q-th cohomology group of this complex.

Note that, for any $u \in \mathscr{D}b^t_{M_{sa}}(\Omega)$ and for any complex neighborhood U of Ω, u is represented by a pair $(\mu_1, \mu_{01}) \in C^{\infty,t,(0,n)}_{X_{sa}}(U) \oplus C^{\infty,t,(0,n-1)}_{X_{sa}}(U \setminus \Omega)$ such that $\mu_1 = \bar{\partial}\mu_{01}$. This representation is called the relative Čech-Dolbeault representation of tempered distributions.

Additionally, hyperfunctions can also be represented in the following way as the relative Čech-Dolbeault representation.

Notation 4.8 We define $\mathscr{O}_X$ to be the sheaf of holomorphic functions on X.

Notation 4.9 For any $q \in \{0, 1, \cdots, n\}$, we define that $C^{\infty,(0,q)}_X$ is the sheaf of $(0,q)$-forms with C^∞ coefficients on X. For other $q \in \mathbb{Z}$, $C^{\infty,(0,q)}_X = 0$. We write the Dolbeault complex as $(C^{\infty,(0,\bullet)}_X, \bar{\partial})$.

Definition 4.10 For any open subset $\Omega \subset M$, we define

$$\mathscr{B}_M = H^n(\mathbf{R}\Gamma_M(\mathscr{O}_X))|_M \underset{\mathbb{Z}_M}{\otimes} or_{M/X} \simeq H^n(\mathbf{R}\Gamma_M(\mathscr{O}_X))|_M,$$

where $or_{M/X}$ is the relative orientation sheaf.

Since $C^{\infty,(0,\bullet)}_X$ is a soft resolution of $\mathscr{O}_X$, we have

$$H^n(\mathbf{R}\Gamma_M(\mathscr{O}_X))|_M \simeq H^n(C_{X,X\setminus M}(C^{\infty,(0,\bullet)}_X))|_M.$$

Using the same argument in Definition 4.7, for any $u \in \mathscr{B}_M(\Omega)$ and for any complex neighborhood U of Ω, u is represented by a pair $(\mu_1, \mu_{01}) \in C^{\infty,(0,n)}_X(U) \oplus C^{\infty,(0,n-1)}_X(U \setminus \Omega)$ such that $\mu_1 = \bar{\partial}\mu_{01}$. This representation of hyperfunctions is thoroughly explained in Honda-Izawa-Suwa[4]. In that paper, $H^n(U, U \setminus \Omega; C^{\infty,(0,\bullet)}_X)$ is written as $H^{0,n}_{\bar{\vartheta}}(U, U \setminus \Omega)$.

Moreover, using the fact that $\mathscr{D}b_M = \rho_{sa}^{-1}\mathscr{D}b^t_{M_{sa}}$, we can also give a representation of distributions in the framework of the relative Čech-Dolbeault cohomology.

Proposition 4.11 We have

$$\mathscr{D}b_M(\Omega) = \varprojlim_{\Omega' \subset\subset \Omega,\, \Omega' \in \mathrm{Op}(X_{sa})} \mathscr{D}b^t_{M_{sa}}(\Omega').$$

This proposition follows from the following lemma.

Lemma 4.12 ([19, Proposition 1.1.12]) Let $\mathscr{F} \in \mathrm{Mod}(\mathbb{Z}_{M_{sa}})$ and let $\Omega \in \mathrm{Op}(M)$. Then

$$\rho_{sa}^{-1}\mathscr{F}(\Omega) = \varprojlim_{\Omega' \subset\subset \Omega,\, \Omega' \in \mathrm{Op}(X_{sa})} \mathscr{F}(\Omega').$$

By Proposition 4.11, we get the following theorem.

Theorem 4.13 Let $\Omega \in \mathrm{Op}(M)$ and $u \in \mathscr{B}_M(\Omega)$. The following 2 conditions are equivalent.

(1) $u \in \mathscr{D}b_M(\Omega)$.

(2) There exists a complex neighborhood U of Ω and a representative $\mu = (\mu_1, \mu_{01}) \in C_X^{\infty,(0,n)}(U, U \setminus \Omega)$ of $u \in \mathscr{B}_M(\Omega)$ such that, for any $\Omega' \subset\subset \Omega$, we can find a complex neighborhood $U' \subset\subset U$ of Ω' such that $\mu_1|_{U'} \in C_{X_{sa}}^{\infty,t,(0,n)}(U' \setminus \Omega')$ and $\mu_{01}|_{U' \setminus \Omega'} \in C_{X_{sa}}^{\infty,t,(0,n-1)}(U' \setminus \Omega')$.

Proof. (2) $\Longrightarrow$ (1) is obvious. Let us prove (1) $\Longrightarrow$ (2). We take U' and U'' such that $U' \subset\subset U'' \subset\subset U$. Set $\Omega' = U' \cap M$ and $\Omega'' = U'' \cap M$. We can find representatives $\mu' \in C_{X_{sa}}^{\infty,t,(0,n)}(U', U' \setminus \Omega')$ and $\mu'' \in C_{X_{sa}}^{\infty,t,(0,n)}(U'', U'' \setminus \Omega'')$ such that $u|_{\Omega'} = [\mu']$ and $u|_{\Omega''} = [\mu'']$. By Lemma 4.14 that will be explained after this theorem, for any V' and V'' such that $V' \subset\subset U' \subset\subset V'' \subset\subset U''$, there exists $\widetilde{\mu} \in C_{X_{sa}}^{\infty,t,(0,n)}(U'', U'' \setminus \Omega'')$ satisfies the following conditions:

- $\widetilde{\mu} = \mu'$ on V'.
- $\widetilde{\mu} = \mu''$ on $U'' \setminus \overline{V''}$.
- $[\widetilde{\mu}] = u|_{\Omega''}$.

Since V' can be taken as an open set sufficiently close to U', we can glue the representatives together. This completes the proof. $\square$

Lemma 4.14 Let $\Omega \in \mathrm{Op}(M_{sa})$ be a relatively compact subset, $U \in \mathrm{Op}(X_{sa})$ a complex neighborhood of Ω and $\mu \in C_{X_{sa}}^{\infty,t,(0,n)}(U, U \setminus \Omega)$ representatives of a tempered distribution $u \in \mathscr{D}b_{M_{sa}}^t(\Omega)$. We take $U', U'' \in \mathrm{Op}(X)$ such that $U' \subset\subset U \subset\subset U''$. Then, there exists a representative $\tilde{\mu} \in C_{X_{sa}}^{\infty,t,(0,n)}(X, X \setminus M)$ such that

$$\tilde{\mu} = \begin{cases} \mu \text{ on } U', \\ 0 \text{ on } X \setminus \overline{U''}. \end{cases}$$

Proof. By the definition of $\mathscr{D}b_{M_{sa}}^t$ and the softness of $\mathscr{D}b_{M_{sa}}^t$, we can find $\tilde{u} \in \mathscr{D}b_{M_{sa}}^t(M)$ and $\Omega', \Omega'' \in \mathrm{Op}(M_{sa})$ such that

- $\Omega' \subset \Omega \subset \Omega''$,
- $\tilde{u}|_{\Omega'} = u|_{\Omega'}$ and $\tilde{u}|_{M \setminus \overline{\Omega''}} = 0$.

Let U' and U'' be a complex neighborhood of Ω' and Ω'', respectively. We may assume $U' \subset\subset U \subset\subset U''$. Then, there exist $\nu \in C^{\infty,t,(0,n)}_{X_{sa}}(X, X \setminus M)$, $\tau' \in C^{\infty,t,(0,n-1)}_{X_{sa}}(U', U' \setminus \Omega)$ and $\tau'' \in C^{\infty,t,(0,n-1)}_{X_{sa}}(X \setminus \overline{U''}, (X \setminus \overline{U''}) \setminus (M \setminus \overline{\Omega''}))$ such that

$$\begin{aligned} \nu|_{U'} + \bar{\vartheta}\tau' &= \mu|_{U'}\,, \\ \nu|_{X\setminus\overline{U''}} + \bar{\vartheta}\tau'' &= 0\,. \end{aligned}$$

Let V' and V'' be satisfying $U' \subset\subset V' \subset\subset U \subset\subset V'' \subset\subset U''$. We define $\varphi', \varphi'' \in C^\infty_0(X)$ by

$$\varphi' = \begin{cases} 1 \text{ on } U', \\ 0 \text{ on } X \setminus \overline{V'}, \end{cases} \qquad \varphi'' = \begin{cases} 1 \text{ on } X \setminus \overline{U''}, \\ 0 \text{ on } V''. \end{cases}$$

Then, if we set $\tilde{\mu} = \nu + \bar{\vartheta}(\varphi'\tau') + \bar{\vartheta}(\varphi''\tau'')$, we get

$$\begin{aligned} \tilde{\mu}|_{U'} &= \nu|_{U'} + \bar{\vartheta}\tau'|_{U'} &&= \mu|_{U'}\,, \\ \tilde{\mu}|_{X\setminus\overline{U''}} &= \nu|_{X\setminus\overline{U''}} + \bar{\vartheta}\tau''|_{X\setminus\overline{U''}} &&= 0\,. \end{aligned}$$

This completes the proof. $\square$

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