PRESENTATIONS OF SCHUR COVERS OF BRAID GROUPS

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Abstract. In this paper, we consider several basic facts of Schur covers of the symmetric groups and braid groups. In particular, we give explicit presentations of Schur covers of braid groups.

1. Introduction

For a group G, consider a central group extension

$$0 \to H_2(G) \to E \xrightarrow{\varpi} G \to 1$$

of G by the integral second homology $H_2(G)$ of G. If the cohomology class $u \in H^2(G, H_2(G))$ corresponding to this central extension is mapped to $\mathrm{id}_{H_2(G)}$ by the homomorphism $\kappa : H^2(G, H_2(G)) \to \mathrm{Hom}_{\mathbb{Z}}(H_2(G), H_2(G))$ coming from the universal coefficient theorem, the group E is called a Schur cover of G. A concept of Schur covers was originally introduced by Schur in a series of his works [9, 10] in the study of projective representations of finite groups. In particular, he determined the Schur cover of the symmetric groups, and gave a finite presentation for it. (See Theorem 2.2.)

For $n \geq 1$, let B_n be the braid group of n-strands. The group B_n was studied in his original works by Artin [1] who gave its first finite presentation. Let $\mathfrak{S}_n$ be the symmetric group of degree n. There is a natural surjective homomorphism from B_n to $\mathfrak{S}_n$. In other words, B_n is an extension of $\mathfrak{S}_n$. From a viewpoint of Artin groups, $\mathfrak{S}_n$ is the Coxeter group of B_n. Now the study of the braid groups has achieved a good progress, and has a broad range of remarkable results by quite many authors. For the study of the Schur covers of B_n, it turns out that there is a unique non-trivial central extension of B_n by $H_2(B_n)$ up to isomorphism from the universal coefficient theorem. (For details, see Subsection 2.2.) Namely, the Schur cover of B_n is essentially unique, and denote it by C_n. On the other hand, Huebschmann [5] studied the homomorphism $\varpi : C_n \to B_n$ from a viewpoint of a crossed module, and showed that it is a free B_n-crossed module generated by a single generator of B_n. Thus the group C_n has two interesting features. One is the non-trivial central extension, and the other is a free crossed module.

So far, any explicit presentation of the Schur cover of B_n has not been obtained. In the main theorem of this paper, we give it as follows.

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Theorem 1. *For any $n \geq 4$, a presentation for the Schur cover C_n of the braid group B_n is given by generators*

$$\sigma_1^*, \cdots, \sigma_{n-1}^*, z^*$$

subject to relations

$$\begin{aligned}
&(z^*)^2 = 1, \\
&\sigma_i^* \sigma_{i+1}^* \sigma_i^* = \sigma_{i+1}^* \sigma_i^* \sigma_{i+1}^* \quad (1 \leq i \leq n-2), \\
&\sigma_i^* \sigma_j^* = z^* \sigma_j^* \sigma_i^* \quad (i < j-1), \\
&\sigma_i^* z^* = z^* \sigma_i^* \quad (1 \leq i \leq n-1).
\end{aligned}$$

Here $\varpi\colon C_n \to B_n$ satisfies $\varpi(\sigma_i^*) = \sigma_i$ $(1 \leq i \leq n-1)$, $\varpi(z^*) = 1$.

In order to show this, we use 2-cocycles of B_n and combinatorial group theory. In Section 5, as an appenfix, we give explicit constructions of generators of $H^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ for $n \geq 4$. These cocycles are used to construct those of B_n. This paper is based on the master's thesis ([8]) of the second author at Hokkaido University in 2022.

2. Schur covers

First of all, we review and fix notations for Artin's braid groups and its related groups. For $n \geq 1$, let B_n be the braid group of n strands. The groups B_n were introduced by Artin in 1925, and Artin gave its finite presentations for them in 1947. More precisely, Artin [1] obtained the first finite presentation of B_n as follows:

$$\begin{aligned}
B_n = \langle\ \sigma_1, \ldots, \sigma_{n-1} \,|\, &\sigma_{i+1}\sigma_i\sigma_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_i^{-1} \ \ (1 \leq i \leq n-2), \\
&[\sigma_i, \sigma_j] \ \ (1 \leq i < j \leq n-1,\ |i-j| \geq 2)\ \rangle.
\end{aligned} \tag{1}$$

By the universal mapping poperty there is unique homomorphism onto $\mathfrak{S}_n$. It is also well-known that $\mathfrak{S}_n$ has the following finite presentation.

$$\begin{aligned}
\mathfrak{S}_n = \langle\ s_1, \ldots, s_{n-1} \,|\, &a_i := s_{i+1}s_is_{i+1}s_i^{-1}s_{i+1}^{-1}s_i^{-1} = 1 \ \ (1 \leq i \leq n-2), \\
&b_{ij} := [s_i, s_j] = 1 \ \ (1 \leq i < j \leq n-1,\ |i-j| \geq 2), \\
&c_i := s_i^2 = 1 \ \ (1 \leq i \leq n-1)\ \rangle.
\end{aligned} \tag{2}$$

Thus we see that there is the standard surjective homomorphism $\pi : B_n \to \mathfrak{S}_n$ such that $\pi(\sigma_i) = s_i$ for $1 \leq i \leq n-1$.

Here for a group G, we review the correspondence between central extensions and 2-cocycles of G. Let A be an abelian group and

$$0 \to A \xrightarrow{i} E \xrightarrow{\varpi} G \to 1 \tag{3}$$

a central extension of G by A. Namely, $i(A)$ is contained in the center of E. Choose a set-theoretic section $s\colon G \to E$ of ϖ such that $s(1) = 1$. Let $f_E : G \times G \to A$ be the map defined by $(g, h) \mapsto s(g)s(h)(s(gh))^{-1}$. Then f_E is a normalized 2-cocycle of G to A.

Conversely, if a normalized 2-cocycle $f : G \times G \to A$ of G is given, then f defines a central extension of G by A as follows. Let E_f be the set $A \times G$ equipped with the product defined by $(a, g)(b, h) = (a+b+f(g, h), gh)$. From the 2-cocycle condition of f, E_f become a group. Let $\mathcal{E}(G, A)$ be the set of equivalence classes of central extensions of G by A. It is well-known that the correspondence $[E] \mapsto [f_E]$ defines a bijection $\mathcal{E}(G, A) \to H^2(G, A)$, whose inverse is given by $[f] \mapsto [E_f]$.

We review the Schur covers of the symmetric groups and the braid groups. Here, for the simplicity, for a group G we will abbreviate the integral homology group $H_p(G,\mathbb{Z})$ of G as $H_p(G)$. For any $p \geq 1$ and any trivial G-module M, let us consider the universal coefficients theorem

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(H_{p-1}(G), M) \to H^p(G, M) \xrightarrow{\kappa} \mathrm{Hom}_{\mathbb{Z}}(H_p(G), M) \to 0.$$

Definition 2.1. *For groups G and E, let*

$$0 \to H_2(G) \to E \xrightarrow{\varpi} G \to 1 \tag{4}$$

be a central extension, and $u \in H^2(G, H_2(G))$ the cohomology class corresponding to this central extension. The central extension E is called a Schur cover of G if $\kappa(u) = \mathrm{id}_{H_2(G)}$.

We remark that a Schur cover is not unique in general. If G is a perfect group, there exists a unique Schur cover up to isomorphism. Then (4) is called the universal central extension of G.

2.1. **Schur covers of the symmetric groups.** Here, we review Schur covers of the symmetric group $\mathfrak{S}_n$. For any $n \geq 4$, it is known that

$$H_1(\mathfrak{S}_n) \cong \mathbb{Z}/2\mathbb{Z}, \quad H_2(\mathfrak{S}_n) \cong \mathbb{Z}/2\mathbb{Z}.$$

Hence, a Schur cover E of $\mathfrak{S}_n$ satisfies

$$0 \to \mathbb{Z}/2\mathbb{Z} \to E \to \mathfrak{S}_n \to 1. \tag{5}$$

By using the universal coefficient theorem and

$$\begin{aligned} &\mathrm{Hom}_{\mathbb{Z}}(H_2(\mathfrak{S}_n), \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}, \\ &\mathrm{Ext}^1_{\mathbb{Z}}(H_1(\mathfrak{S}_n), \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}, \end{aligned}$$

we have $H^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Hence, there exist four non-equivalent central extensions which satisfy (5). Since $\mathrm{Hom}_{\mathbb{Z}}(H_2(\mathfrak{S}_n), H_2(\mathfrak{S}_n)) \cong \mathbb{Z}/2\mathbb{Z}$ consists of two elements, which are the identity map $\mathrm{id}_{H_2(\mathfrak{S}_n)}$ and the zero map, there exists two elements of $H^2(\mathfrak{S}_n, H_2(\mathfrak{S}_n))$ which are mapped to $\mathrm{id}_{H_2(\mathfrak{S}_n)}$. Each of them induces the extension (5) and the following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, equal] & H^2(\mathfrak{S}_n, H_2(\mathfrak{S}_n)) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(H_2(\mathfrak{S}_n), H_2(\mathfrak{S}_n)) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

Therefore, there exist two Schur covers of $\mathfrak{S}_n$. The 2-cocycles which corresponds to the two Schur covers are given by the following theorem.

Theorem 2.2 (Schur [9, 10] (See also [2].))**.** *Consider $\mathbb{Z}/2\mathbb{Z}$ as a multiplicative group, and let z be its generator.*
(i) For any $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z} = \{1, z\}$, there exists the 2-cocycle $\tau_{[\alpha,\beta]} : \mathfrak{S}_n \times \mathfrak{S}_n \to \mathbb{Z}/2\mathbb{Z}$ such that

$$\tau_{[\alpha,\beta]}(s_i, s_i) = \alpha, \quad \tau_{[\alpha,\beta]}(s_k, s_l) = \begin{cases} 1 & (k < l) \\ \beta & (l < k). \end{cases}$$

Then we have $H^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z}) = \{[\tau_{[\alpha,\beta]}] \mid \alpha, \beta \in \{1, z\}\}$.
(ii) Let

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} E_{[\alpha,\beta]} \xrightarrow{\varpi} \mathfrak{S}_n \to 1$$

be the central extension corresponding to $[\tau_{[\alpha,\beta]}]$*. Then* $E_{\tau_{[\alpha,\beta]}}$ *has a presentation whose generators are* $t_1, \ldots, t_{n-1}, z$ *subject to relations*

$$\begin{aligned} &t_i^2 = \alpha,\ z^2 = 1,\ zt_i = t_iz,\ t_jt_{j+1}t_j = t_{j+1}t_jt_{j+1},\ t_kt_l = \beta t_lt_k \\ &(1 \le i \le n-1,\ 1 \le j \le n-2,\ 1 \le k < l-1 \le n-2). \end{aligned} \tag{6}$$

Here $\iota(z) = z$*,* $\varpi(t_i) = s_i$ *and* $\varpi(z) = 1$*.*
(iii) If $\beta = z$*, then for any* $\alpha \in \mathbb{Z}/2\mathbb{Z}$*, the central extension* $E_{[\alpha,\beta]}$ *is a Schur cover of* $\mathfrak{S}_n$*.*

In Section 5, we give explicit calculations to obtain the 2-cocycles $\tau_{[\alpha,\beta]}$ by using combinatorial group theory since there is no explicit calculations in [9].

2.2. **Schur covers of braid groups.** Here we consider a Schur cover of the braid group B_n. For any $n \geq 4$, it is well-known that

$$H_1(B_n) \cong \mathbb{Z}, \quad H_2(B_n) \cong \mathbb{Z}/2\mathbb{Z}.$$

A Schur cover of B_n satisfies

$$0 \to \mathbb{Z}/2\mathbb{Z} \to E \xrightarrow{\varpi} B_n \to 1. \tag{7}$$

By using the universal coefficient theorem and

$$\begin{aligned} &\mathrm{Hom}(H_2(B_n), \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}, \\ &\mathrm{Ext}^1_{\mathbb{Z}}(H_1(B_n), \mathbb{Z}/2\mathbb{Z}) = 0, \end{aligned}$$

we have $H^2(B_n, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. In this case, there are two non-equivalent central extensions of (7). Furthermore, since $\kappa : H^2(B_n, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(H_2(B_n), \mathbb{Z}/2\mathbb{Z})$ is an isomorphism, the non-trivial central extension of B_n by $\mathbb{Z}/2\mathbb{Z}$ is the unique Schur cover. For the standard surjection $\pi : B_n \to \mathfrak{S}_n$, consider the four 2-cocycles $\mu_{[\alpha,\beta]} := \tau_{[\alpha,\beta]} \circ (\pi \times \pi)\colon B_n \times B_n \to \mathbb{Z}/2\mathbb{Z}$ of B_n, and the following commutative diagram induced from the universal coefficient theorem:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(H_1(\mathfrak{S}_n), \mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d] & H^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d, two heads, "\varpi^*"] & \mathrm{Hom}(H_2(\mathfrak{S}_n), \mathbb{Z}/2\mathbb{Z}) \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & H^2(B_n, \mathbb{Z}/2\mathbb{Z}) \arrow[r, "\cong"] & \mathrm{Hom}(H_2(B_n), \mathbb{Z}/2\mathbb{Z}) \arrow[r] & 0 \end{tikzcd}$$

For any $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z} = \{1, z\}$, let

$$0 \to \mathbb{Z}/2\mathbb{Z} \to C_{[\alpha,\beta]} \to B_n \to 1$$

be a central extension corresponding to $\mu_{[\alpha,\beta]}$. We obtain the following Proposition.

Proposition 2.3. *As notation above,*
(i) If $\beta = 1$*, then the central extension is trivial.*
(ii) $C_{[1,z]}$ *and* $C_{[z,z]}$ *are equivalent central extensions of* B_n*, and each of them is a Schur cover of* B_n*.*

3. Presentations of Extensions of Groups

In this section, according to [6], we review a method to obtain a presentation for an extension of presented groups. Let

$$1 \to K \xrightarrow{\iota} G \xrightarrow{q} N \to 1$$

be a group extension, and assume that presentations for K and N are given by

$$K = \langle Y \mid S \rangle, \quad N = \langle X \mid R \rangle$$

respectively. Set $X^* := \{x^* \mid x \in X\}$ and $Y^* := \{y^* \mid y \in Y\}$. Let F be the free group on $X^* \cup Y^*$, and set $S^* := \{(y_1^*)^{e_1} \cdots (y_k^*)^{e_k} \in F \mid {y_1}^{e_1} \cdots {y_k}^{e_k} \in S\}$. For any $x \in X$, take $x' \in q^{-1}(x)$, and set $X' := \{x' \in G \mid x \in X\}$. Let $\varphi\colon F \to G$ be the surjective homomorphism defined by $x^* \mapsto x'$ and $y^* \mapsto \iota(y)$ for any $x^* \in X^*$ and $y^* \in Y^*$. For any $r = {x_1}^{f_1} \cdots {x_l}^{f_l} \in R$, set $r^* := (x_1^*)^{f_1} \cdots (x_l^*)^{f_l} \in F$. Then $\varphi(r^*) = (x_1')^{f_1} \cdots (x_l')^{f_l} \in \mathrm{Ker}(q) = \mathrm{Im}(\iota)$. Hence we can write $\varphi(r^*) = y_{r,1}^{e_1} \cdots y_{r,k}^{e_k}$ for some $y_{r,i} \in Y$ and $e_i = \pm 1$. Set $v_r := (y_{r,1}^*)^{e_1} \cdots (y_{r,k}^*)^{e_k} \in F$, and $R^* := \{r^* {v_r}^{-1} \in F \mid r \in R\}$. Since $\mathrm{Im}(\iota)$ is the normal subgroup of G, we see $\varphi((x^*)^{-1} y^* x^*) \in \mathrm{Im}(\iota)$ for any $x^* \in X^*$ and $y^* \in Y^*$. Thus we can write $\varphi((x^*)^{-1} y^* x^*) = \iota(y_{x,y,1}^{a_1} \cdots y_{x,y,k}^{a_k})$ for some $y_{x,y,i} \in Y$ and $a_i = \pm 1$. Set $w_{x,y} := (y_{x,y,1}^*)^{a_1} \cdots (y_{x,y,k}^*)^{a_k} \in F$, and $T^* := \{(x^*)^{-1} y^* x^* w_{x,y}^{-1} \in F \mid x^* \in X^*$ and $y^* \in Y^*\}$.

Theorem 3.1 (Presentations of Extensions of Groups)**.** *As notation above, we have*

$$G = \langle X^* \cup Y^* \mid R^* \cup S^* \cup T^* \rangle.$$

For a proof, see [6].

4. Presentations of Schur covers of Braid groups

In this section, we prove our main theorem. By Theorem 2.2, for any $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z} = \{1, z\}$, the 2-cocyle $\mu_{[\alpha,\beta]}\colon B_n \times B_n \to \mathbb{Z}/2\mathbb{Z}$ defined in Subsection 2.2 satisfies the following equations.

$$\begin{cases} \mu_{[\alpha,\beta]}(\sigma_i, \sigma_i) = \alpha, & \\ \mu_{[\alpha,\beta]}(\sigma_k, \sigma_l) = 1 & (k < l), \\ \mu_{[\alpha,\beta]}(\sigma_l, \sigma_k) = \beta & (k < l). \end{cases}$$

Let

$$1 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} C_{[\alpha,\beta]} \xrightarrow{\varpi} B_n \to 1$$

be the central extension corresponding to $\mu_{[\alpha,\beta]}$. The group $C_{[\alpha,\beta]}$ can be identified with $\mathbb{Z}/2\mathbb{Z} \times B_n$ as a set. Under this identification, the group operation on $C_{[\alpha,\beta]}$ is represented as

$$(x, \sigma) \cdot (y, \tau) = (x \cdot y \cdot \mu_{[\alpha,\beta]}(\sigma, \tau), \sigma\tau),$$

and the homomorphisms $\iota : \mathbb{Z}/2\mathbb{Z} \to C_{[\alpha,\beta]}$, $\varpi : C_{[\alpha,\beta]} \to B_n$ are represented as

$$\iota(s) = (s, 1), \quad \varpi(x, \sigma) = \sigma.$$

Furthermore, there exists a standard set-theoretic section

$$B_n \to C_{[\alpha,\beta]}, \quad \sigma \mapsto (1, \sigma).$$

Consider a presentation $\langle z \mid z^2 \rangle$ of $\mathbb{Z}/2\mathbb{Z}$ and Artin's presentation of B_n as mentioned in (1). For simplicity, we write the above presentations as $\mathbb{Z}/2 = \langle Y \mid S \rangle, B_n = \langle X \mid R \rangle$.

Let $F(X)$ be the free group on X, and $\varphi : F(X) \to B_n$ the standard surjection. Since $\varphi|_X : X \to B_n$ is injective, we regard X as a subset of B_n through φ, and we write $\varphi(x)$ as x for any $x \in X$ by abuse of language. In a similar way, we consider $Y \subset \mathbb{Z}/2\mathbb{Z}$.

We use Theorem 3.1 to obtain a presentation of $C_{[\alpha,\beta]}$. First, set $X^* \coloneqq \{{\sigma_1}^*, \cdots, \sigma_{n-1}^*\}$ and $Y^* \coloneqq \{z^*\}$. Let F be the free group $F(X^* \cup Y^*)$ on $X^* \cup Y^*$. Here $S^* = \{(z^*)^2\} \subset F$. Set $X' \coloneqq \{(1, \sigma_i) \mid 1 \leq i \leq n-1\}$. The standard homomorphism $\psi \colon F \to C_{[\alpha,\beta]}$ is defined by $z^* \mapsto z$ and $\sigma_i^* \mapsto (1, \sigma_i)$. For each i and j such that $1 \leq i < j \leq n-1$, define $r_{ij} \in R$ and $r_{ij}^* \in F$ by

$$r_{ij} \coloneqq \begin{cases} \sigma_i\sigma_{i+1}\sigma_i(\sigma_{i+1}\sigma_i\sigma_{i+1})^{-1} & (i = j-1), \\ \sigma_i\sigma_j(\sigma_j\sigma_i)^{-1} & (i < j-1), \end{cases}$$

$$r_{ij}^* \coloneqq \begin{cases} \sigma_i^*\sigma_{i+1}^*\sigma_i^*(\sigma_{i+1}^*\sigma_i^*\sigma_{i+1}^*)^{-1} & (i = j-1), \\ \sigma_i^*\sigma_j^*(\sigma_j^*\sigma_i^*)^{-1} & (i < j-1). \end{cases}$$

Then we have

$$\varphi(r_{ij}^*) = \begin{cases} (1,\sigma_i)(1,\sigma_{i+1})(1,\sigma_i)(1,\sigma_{i+1})^{-1}(1,\sigma_i)^{-1}(1,\sigma_{i+1})^{-1} & (1 \leq i \leq n-2), \\ (1,\sigma_i)(1,\sigma_j)(1,\sigma_i)^{-1}(1,\sigma_j)^{-1} & (i < j-1). \end{cases}$$

Consider the element $(1,\sigma_i)(1,\sigma_j)(1,\sigma_i)^{-1}(1,\sigma_j)^{-1}$ for $i < j-1$. By using equations

$$(1,\sigma_i)(1,\sigma_j) = (1 \cdot 1 \cdot \mu_{[\alpha,\beta]}(\sigma_i,\sigma_j), \sigma_i\sigma_j) = (1, \sigma_i\sigma_j)$$

and

$$(1,\sigma_j)(1,\sigma_i) = (1 \cdot 1 \cdot \mu_{[\alpha,\beta]}(\sigma_j,\sigma_i), \sigma_j\sigma_i) = (\beta, \sigma_i\sigma_j) = (\beta, 1)(1, \sigma_i\sigma_j),$$

we obtain

$$\begin{aligned} (1,\sigma_i)(1,\sigma_j)(1,\sigma_i)^{-1}(1,\sigma_j)^{-1} &= (1,\sigma_i\sigma_j)(1,\sigma_i\sigma_j)^{-1}(\beta,1)^{-1} \\ &= (\beta,1)^{-1} = (\beta,1). \end{aligned}$$

Next, we consider the element $(1,\sigma_i)(1,\sigma_{i+1})(1,\sigma_i)(1,\sigma_{i+1})^{-1}(1,\sigma_i)^{-1}(1,\sigma_{i+1})^{-1}$ for $1 \leq i \leq n-2$. Similarly, we have

$$\begin{aligned} (1,\sigma_i)(1,\sigma_{i+1})(1,\sigma_i) &= (1 \cdot 1 \cdot \mu_{[\alpha,\beta]}(\sigma_i,\sigma_{i+1})(1,\sigma_i) \\ &= (1,\sigma_i\sigma_{i+1})(1,\sigma_i) \\ &= (1 \cdot 1 \cdot \mu_{[\alpha,\beta]}(\sigma_i\sigma_{i+1},\sigma_i), \sigma_i\sigma_{i+1}\sigma_i) \end{aligned}$$

and

$$\begin{aligned} (1,\sigma_{i+1})(1,\sigma_i)(1,\sigma_{i+1}) &= (1,\sigma_{i+1})(1 \cdot 1 \cdot \mu(\sigma_i\sigma_{i+1}), \sigma_i\sigma_{i+1}) \\ &= (1,\sigma_{i+1})(1,\sigma_i\sigma_{i+1}) \\ &= (1 \cdot 1 \cdot \mu_{[\alpha,\beta]}(\sigma_{i+1},\sigma_i\sigma_{i+1}), \sigma_{i+1}\sigma_i\sigma_{i+1}) \\ &= (\mu_{[\alpha,\beta]}(\sigma_{i+1},\sigma_i\sigma_{i+1}), \sigma_{i+1}\sigma_i\sigma_{i+1}). \end{aligned}$$

Lemma 4.1. *For any $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z} = \{1, z\}$ and any $1 < i < n-1$, $\mu_{[\alpha,\beta]}(\sigma_i\sigma_{i+1},\sigma_i) = \mu_{[\alpha,\beta]}(\sigma_{i+1},\sigma_i\sigma_{i+1})$.*

Proof. We use the same notation as in Theorem 2.2. Consider the central extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} E_{[\alpha,\beta]} \xrightarrow{\varpi} \mathfrak{S}_n \to 1$$

corresponding to the 2-cocycle $\tau_{[\alpha,\beta]}\colon \mathfrak{S}_n \times \mathfrak{S}_n \to \mathbb{Z}/2\mathbb{Z}$. The group $E_{[\alpha,\beta]}$ can be identified with $\mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_n$ as a set. Thus, the operation on $E_{[\alpha,\beta]}$ is given by

$$(x, s_i) \cdot (y, s_j) = (x \cdot y \cdot \tau_{[\alpha,\beta]}(s_i, s_j), s_i s_j).$$

In this case, $(1, s_i)(1, s_{i+1})(1, s_i) = (1, s_{i+1})(1, s_i)(1, s_{i+1})$ follows from Schur [9]. Furthermore we see $\tau_{[\alpha,\beta]}(s_i s_{i+1}, s_i) = \tau_{[\alpha,\beta]}(s_{i+1}, s_i s_{i+1})$ from

$$\begin{aligned}
&(\text{LHS}) = (\tau_{[\alpha,\beta]}(s_i, s_{i+1}), s_i s_{i+1})(1, s_i) = (1, s_i s_{i+1})(1, s_i) = (\tau_{[\alpha,\beta]}(s_i s_{i+1}, s_i), s_i s_{i+1} s_i),\\
&(\text{RHS}) = (1, s_{i+1})(\tau_{[\alpha,\beta]}(s_i, s_{i+1}), s_i s_{i+1}) = (\tau_{[\alpha,\beta]}(s_{i+1}, s_i s_{i+1}), s_{i+1} s_i s_{i+1}).
\end{aligned}$$

Since $\mu_{[\alpha,\beta]} = \tau_{[\alpha,\beta]} \circ (\pi \times \pi)$, we obtain the required result. $\square$

From Lemma 4.1,

$$\begin{aligned}
&(1, \sigma_i)(1, \sigma_{i+1})(1, \sigma_i)(1, \sigma_{i+1})^{-1}(1, \sigma_i)^{-1}(1, \sigma_{i+1})^{-1}\\
&= (\mu_{[\alpha,\beta]}(\sigma_i\sigma_{i+1}, \sigma_i), \sigma_i\sigma_{i+1}\sigma_i)(\mu_{[\alpha,\beta]}(\sigma_{i+1}, \sigma_i\sigma_{i+1}), \sigma_{i+1}\sigma_i\sigma_{i+1})^{-1} = (1, 1).
\end{aligned}$$

Set

$$R^* \coloneqq \{\sigma_i^*\sigma_{i+1}^*\sigma_i^*(\sigma_{i+1}^*\sigma_i^*\sigma_{i+1}^*)^{-1} \mid 1 \le i \le n-2\} \cup \{\sigma_i^*\sigma_j^*(\sigma_j^*\sigma_i^*)^{-1}(z^*)^{-1} \mid i < j-1\}.$$

We have $\varphi((\sigma_i^*)^{-1}z^*\sigma_i^*) \in \mathrm{Im}(\iota)$ for any $\sigma_i^* \in X^*$ and $z^* \in Y^*$, and

$$\begin{aligned}
\varphi((\sigma_i^*)^{-1}z^*\sigma_i^*) &= (1, \sigma_i)^{-1}(z, 1)(1, \sigma_i)\\
&= (z, \sigma_i^{-1})(z, 1)(1, \sigma_i)\\
&= (1 \cdot 1, \sigma_i^{-1})(1, \sigma_i)\\
&= (z, 1) = \iota(z).
\end{aligned}$$

Hence, the set T^* of Theorem 3.1 is

$$T^* = \{(\sigma_i^*)^{-1}z^*\sigma_i^*(z^*)^{-1} \in F \mid \sigma_i^* \in X^*, z^* \in Y^*\}.$$

From the above calculations and Theorem 3.1, we obtain the following theorem.

Theorem 4.2. *For any $n \ge 4$ and $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z} = \{1, z\}$, a presentation for $C_{[\alpha,\beta]}$ is given by generators*

$$\sigma_1^*, \cdots, \sigma_{n-1}^*, z^*$$

subject to relations

$$\begin{aligned}
&(z^*)^2 = 1,\\
&\sigma_i^*\sigma_{i+1}^*\sigma_i^* = \sigma_{i+1}^*\sigma_i^*\sigma_{i+1}^* \quad (1 \le i \le n-2),\\
&\sigma_i^*\sigma_j^* = \begin{cases} \sigma_j^*\sigma_i^* & (\beta = 1,\ i < j-1),\\ z^*\sigma_j^*\sigma_i^* & (\beta = z,\ i < j-1), \end{cases}\\
&\sigma_i^* z^* = z^*\sigma_i^* \quad (1 \le i \le n-1).
\end{aligned}$$

Here $\varpi(\sigma_i^*) = \sigma_i$ $(1 \le i \le n-1)$, $\varpi(z^*) = 1$, $\iota(z) = z^*$.

From Proposition 2.3, we have the following corollary.

Corollary 4.3. *For any $n \ge 4$, a presentation for the Schur cover C_n of the braid group B_n is given by generators*

$$\sigma_1^*, \cdots, \sigma_{n-1}^*, z^*$$

subject to relations

$$
\begin{aligned}
&(z^*)^2 = 1,\\
&\sigma_i^*\sigma_{i+1}^*\sigma_i^* = \sigma_{i+1}^*\sigma_i^*\sigma_{i+1}^* \quad (1 \le i \le n-2),\\
&\sigma_i^*\sigma_j^* = z^*\sigma_j^*\sigma_i^* \quad (i < j-1),\\
&\sigma_i^* z^* = z^*\sigma_i^* \quad (1 \le i \le n-1).
\end{aligned}
$$

Here $\varpi\colon C_n \to B_n$ *satisfies* $\varpi(\sigma_i^*) = \sigma_i$ $(1 \le i \le n-1)$, $\varpi(z^*) = 1$.

5. Appendix: Constructions of generators of $H^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z})$

In this section, by using the presentation of $\mathfrak{S}_n$ and combinatorial group theory, we give explicit constructions of generators of $H^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ for $n \ge 4$. Here we do not show $H_2(\mathfrak{S}_n) \cong \mathbb{Z}/2\mathbb{Z}$, but the order of $H_2(\mathfrak{S}_n)$ is at most 2. Hence by combining the fact that $H_2(\mathfrak{S}_n) \cong \mathbb{Z}/2\mathbb{Z}$, we obtain an explicit generator of $H_2(\mathfrak{S}_n)$. By using this, we construct generators of $H^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z})$.

5.1. **A generator of $H_2(\mathfrak{S}_n, \mathbb{Z})$.** Recall the presentation (2) of the symmetric group $\mathfrak{S}_n$. Let F be the free group on $\{s_1, \dots, s_{n-1}\}$, and $\varphi : F \to \mathfrak{S}_n$ the standard surjection defined by the correspondence $s_i \mapsto (i\ i{+}1)$. The kernel $R := \mathrm{Ker}(\varphi)$ of φ is the normal closure of all of a_is, b_{ij}s and c_is in F. The homological five-term exact sequence of the group extension

$$1 \to R \to F \to \mathfrak{S}_n \to 1 \tag{8}$$

is

$$0 = H_2(F, \mathbb{Z}) \to H_2(\mathfrak{S}_n, \mathbb{Z}) \to H_1(R, \mathbb{Z})_F \xrightarrow{\iota_*} H_1(F, \mathbb{Z}) \xrightarrow{\varphi_*} H_1(\mathfrak{S}_n, \mathbb{Z}) \to 0.$$

Here $\iota : R \to F$ is the inclusion map, and the maps ι_* and φ_* are the induced homomorphisms by ι and φ respectively. Since F is a free group, we have $H_2(F, \mathbb{Z}) = 0$, and hence $H_2(\mathfrak{S}_n, \mathbb{Z}) \cong \mathrm{Ker}(\iota_*)$. Since the signature map $\mathrm{sgn} : \mathfrak{S}_n \to \{\pm 1\}$ induces the abelianization of $\mathfrak{S}_n$, we identify $H_1(\mathfrak{S}_n, \mathbb{Z})$ with $\{\pm 1\}$ via the map sgn. Consider the natural identifications $H_1(R, \mathbb{Z})_F = R/[F, R]$ and $H_1(F, \mathbb{Z}) = F/[F, F]$. For any $\sigma \in F$, set $\sigma' := \sigma \pmod{[F, F]}$. Then $F/[F, F]$ is the free abelian group of rank $n-1$ with basis $s_1', \dots, s_{n-1}'$. The map φ_* maps each s_i' to -1, and it it easily seen that $\mathrm{Ker}(\varphi_*)$ is the free abelian group of rank $n-1$ with basis $s_2'(s_1')^{-1}, s_3'(s_2')^{-1}, \dots, s_{n-1}'(s_{n-2}')^{-1}, c_{n-1}'$.

Next, we find a minimal generating set of $R/[F, R]$. For any $r \in R$, denote by r'' the coset class of r in $R/[F, R]$.

Lemma 5.1. *For $n \ge 4$, $R/[F, R]$ is generated by $a_1'', \dots, a_{n-2}'', c_{n-1}'', b_{13}''$ as an abelian group.*

Proof. Notice that for any $r \in R$ and $x \in F$, $xrx^{-1} \equiv r \pmod{[F, R]}$. Thus $R/[F, R]$ is generated by the coset classes of all of relators

$$a_i \ (1 \le i \le n-2), \quad b_{ij} \ (1 \le i < j \le n-1,\ |i-j| \ge 2), \quad c_i \ (1 \le i \le n-1)$$

as an abelian group. We reduce these generators. Observe that

$$s_i a_i s_i^{-1} = a_i^{-1} \cdot (s_{i+1} s_i s_{i+1}^2 s_i^{-1} s_{i+1}^{-1}) \cdot s_i^{-2},$$

and hence

$$c_i \equiv a_i^{-2} c_{i+1} \pmod{[F, R]}$$

for any $1 \leq i \leq n-2$. Therefore we can remove the generators $c''_1, \dots, c''_{n-2}$ from the generating set.

Next, for any $1 \leq i < j \leq n-2$ such that $|i-j| \geq 2$, observe

$$\begin{aligned} s_j s_{j+1} b_{ij} s_{j+1}^{-1} s_j^{-1} &= [s_j s_{j+1} s_i s_{j+1}^{-1} s_j^{-1}, s_j s_{j+1} s_j s_{j+1}^{-1} s_j^{-1}] \\ &= [[s_j s_{j+1}, s_i] s_i, a_j^{-1} s_{j+1}]. \end{aligned}$$

By using commutator formulae

$$[x, yz] = [x,y][x,z][[z,x],y], \quad [xy, z] = [x,[y,z]][y,z][x,z], \tag{9}$$

since $[s_j s_{j+1}, s_i]$, $a_j^{-1} \in R$, we have

$$[[s_j s_{j+1}, s_i] s_i, a_j^{-1} s_{j+1}] \equiv [s_i, s_{j+1}] \pmod{[F,R]},$$

and hence $b''_{ij} = b''_{i,j+1} \in R/[F,R]$. Similarly, we can see $b''_{i-1,j} = b''_{i,j} \in R/[F,R]$ for any $2 \leq i < j \leq n-1$ such that $|i-j| \geq 2$. Thus we obtain $b''_{ij} = b''_{13} \in R/[F,R]$ for any $|i-j| \geq 2$, and the required result. □

Since ι_* satisfies

$$\iota_*(a''_1) = s'_2(s'_1)^{-1}, \dots, \iota_*(a''_{n-2}) = s'_{n-1}(s'_{n-2})^{-1}, \iota_*(c''_{n-1}) = c'_{n-1},$$

it turns out that the subgroup of $R/[F,R]$ generated by $a''_1, \dots, a''_{n-2}, c''_{n-1}$ is a free abelian group of rank $n-1$, and that

$$R/[F,R] \cong \langle a''_1, \dots, a''_{n-2}, c''_{n-1} \rangle \oplus \langle b''_{13} \rangle.$$

We see that $(b''_{ij})^2 = 1$ is induced from the observation

$$s_i b_{ij} s_i^{-1} = s_i^2 \cdot (s_j s_i^{-2} s_j^{-1}) \cdot b_{ij}^{-1} \in R.$$

By considering Hopf's formula, we see

$$H_2(\mathfrak{S}_n, \mathbb{Z}) \cong (R \cap [F,F])/[F,R] = \langle b''_{13} \rangle.$$

Since $H_2(\mathfrak{S}_n) \cong \mathbb{Z}/2\mathbb{Z}$, we see that b''_{13} is a non-trivial 2-torsion, and it generates $H_2(\mathfrak{S}_n)$.

5.2. **Generators of $H^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z})$.** Here we give explicit constructions of generators of $H^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ for $n \geq 4$. The $\mathbb{Z}/2\mathbb{Z}$-coefficient cohomological five-term exact sequence of the group extension (8) is

$$\begin{aligned} 0 \to H^1(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z}) &\xrightarrow{\varphi^*} H^1(F, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\iota^*} H^1(R, \mathbb{Z}/2\mathbb{Z})^F \\ &\xrightarrow{\mathrm{tg}} H^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z}) \to H^2(F, \mathbb{Z}/2\mathbb{Z}) = 0. \end{aligned}$$

Here the maps ι^* and φ^* are the induced homomorphisms by ι and φ respectively. Since F is a free group, we have $H^2(F, \mathbb{Z}/2\mathbb{Z}) = 0$, and hence $H^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Coker}(\iota^*)$. Notice that

$$H^1(F, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(F/[F,F], \mathbb{Z}/2\mathbb{Z}), \quad H^1(R, \mathbb{Z}/2\mathbb{Z})^F \cong \mathrm{Hom}_{\mathbb{Z}}(R/[F,R], \mathbb{Z}/2\mathbb{Z}).$$

Consider $F/[F,F]$ and $R/[F,R]$ as additive groups. For any $1 \leq i \leq n-1$, let $s_i^* : F/[F,F] \to \mathbb{Z}/2\mathbb{Z}$ be the homomorphism defined by

$$k_1 s'_1 + \dots + k_{n-1} s'_{n-1} \mapsto \overline{k_i},$$

where $\overline{k} \in \mathbb{Z}/2\mathbb{Z}$ means the coset class of $k \in \mathbb{Z}$. Then $s_1^*, \dots, s_{n-1}^*$ form a basis of $H^1(F, \mathbb{Z}/2\mathbb{Z})$ as a $\mathbb{Z}/2\mathbb{Z}$-vector space. Recall that $R/[F,R] \cong \langle a_1'', \dots, a_{n-2}'', c_{n-1}'' \rangle \oplus \langle b_{13}'' \rangle$. Any element $x \in R/[F,R]$ is uniquely written as

$$x = l_1 a_1'' + \cdots + l_{n-2} a_{n-2}'' + l_{n-1} c_{n-1}'' + \bar{l} b_{13}''$$

for some $l_i \in \mathbb{Z}$ and $\bar{l} \in \mathbb{Z}/2\mathbb{Z}$. Define a_i^*, c_{n-1}^* and $b_{13}^* \in \mathrm{Hom}_{\mathbb{Z}}(R/[F,R], \mathbb{Z}/2\mathbb{Z})$ by

$$a_i^*(x) := \overline{l_i} \quad (1 \leq i \leq n-2), \quad c_{n-1}^*(x) := \overline{l_{n-1}}, \quad b_{13}^*(x) := \bar{l}.$$

Then $a_1^*, \dots, a_{n-2}^*, c_{n-1}^*, b_{13}^*$ is a basis of $\mathrm{Hom}_{\mathbb{Z}}(R/[F,R], \mathbb{Z}/2\mathbb{Z})$ as a $\mathbb{Z}/2\mathbb{Z}$-vector space. We have

$$\iota^*(s_i^*) = \begin{cases} -a_1^* & (i = 1), \\ a_{i+1}^* - a_i^* & (2 \leq i \leq n-2), \\ a_{n-2}^* & (i = n-1), \end{cases}$$

and see $\mathrm{Im}(\iota^*) = \langle a_1^*, \dots, a_{n-2}^* \rangle$. Hence we obtain that $H^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z})$ is generated by $\mathrm{tg}(c_{n-1}^*)$ and $\mathrm{tg}(b_{13}^*)$. Furthermore, since $H^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2$, the elements $\mathrm{tg}(c_{n-1}^*)$, $\mathrm{tg}(b_{13}^*)$ form a basis of $H^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z})$ as a $\mathbb{Z}/2\mathbb{Z}$-vector space.

Next we review the transgression map $\mathrm{tg} : H^1(R, \mathbb{Z}/2\mathbb{Z})^F \to H^2(\mathfrak{S}_n, \mathbb{Z}/2\mathbb{Z})$. For any $[f] \in H^1(R, \mathbb{Z}/2\mathbb{Z})^F$, we examine $\mathrm{tg}([f])$. Let $T \subset F$ be a set of left coset representatives of R in F. We assume that T contains 1, $s_1, \dots, s_{n-1}$ and $s_i s_j$ for any $1 \leq i < j \leq n-1$, and fix T. In the following, we define elements $m_\sigma \in \mathbb{Z}/2\mathbb{Z}$ for any $\sigma \in F$. First, for any $\tau \in T$, since $[\tau \cdot f] = [f] \in H^1(R, \mathbb{Z}/2\mathbb{Z})^F$, there exists some $m_\tau \in \mathbb{Z}/2\mathbb{Z}$ such that

$$(\tau \cdot f - f)(x) = x \cdot m_\tau - m_\tau$$

for any $x \in R$. Since F acts on $\mathbb{Z}/2\mathbb{Z}$ trivially, we may choose $m_\tau = 0$ for any $\tau \in T$. For any $\sigma \in F$, there exist unique elements $\tau \in T$ and $r \in R$ such that $\sigma = \tau r$. Set $m_\sigma := m_\tau + \tau \cdot (f(r'')) = f(r'')$. By using m_σs for all $\sigma \in F$, define the map $\alpha_f : F \times F \to \mathbb{Z}/2\mathbb{Z}$ by

$$\alpha_f(\sigma, \rho) := m_{\sigma\rho} - \sigma \cdot m_\rho - m_\sigma = m_{\sigma\rho} - m_\rho - m_\sigma.$$

The map α_f satisfies the following.

- For any $\sigma, \rho, \mu \in G$,

$$\alpha_f(\rho, \mu) - \alpha_f(\sigma\rho, \mu) + \alpha_f(\sigma, \rho\mu) - \alpha_f(\sigma, \rho) = 0.$$

- For any $\sigma, \rho \in G$ and any $r, l \in R$,

$$\alpha_f(\sigma r, \rho l) = \alpha_f(\sigma, \rho).$$

Thus, we see that α_f induces the 2-cocycle $\overline{\alpha_f} : \mathfrak{S}_n \times \mathfrak{S}_n \to \mathbb{Z}/2\mathbb{Z}$ by $\overline{\alpha_f}(\varphi(\sigma), \varphi(\rho)) := \alpha_f(\sigma, \rho)$, and have $\mathrm{tg}([f]) = [\overline{\alpha_f}]$.

We give a few examples of calculations of images of $\overline{\alpha_f}$ for $f = c_{n-1}^*$ and b_{13}^*. First, from the 2-cocycle condition of $\overline{\alpha_f}$, we have

$$\overline{\alpha_f}(\varphi(\sigma), 1) = \overline{\alpha_f}(1, \varphi(\sigma)) = \overline{\alpha_f}(1, 1) = m_1 = 0$$

for any $\sigma \in F$. Notice that $m_{s_i} = 0$ for any $1 \leq i \leq n-1$ since $s_i \in T$. Hence we have

$$\begin{aligned} \overline{\alpha_f}(\varphi(s_i s_{i+1}), \varphi(s_i)) &= m_{s a_i s_{i+1} s_i} - m_{s_i s_{i+1}} = m_{s_{i+1} s_i s_{i+1}} - m_{s_i s_{i+1}} \\ &= \overline{\alpha_f}(\varphi(s_{i+1}), \varphi(s_i s_{i+1})). \end{aligned}$$

Next we consider $\overline{\alpha_f}(\varphi(s_i), \varphi(s_j))$. For any $1 \leq i, j \leq n-1$, we can write $s_i s_j \in F$ as τr for $\tau \in T$ and $r \in R$ as follows:

$$s_i s_j = \begin{cases} s_j s_i \cdot (s_i^{-1} s_j^{-1} [s_i, s_j] s_j s_i) & (j < i), \\ 1 \cdot s_i^2 & (i = j), \\ s_i s_j \cdot 1 & (i < j). \end{cases}$$

Thus, for the case where $f = c^*_{n-1}$, since $c''_i = (a''_i)^{-2}(a''_{i+1})^{-2} \cdots (a''_{n-2})^{-2} c''_{n-1}$, we have

$$m_{s_i s_j} = \begin{cases} 1 & (i = j), \\ 0 & (i \neq j), \end{cases}$$

and hence

$$\overline{\alpha_{c^*_{n-1}}}(\varphi(s_i), \varphi(s_j)) = \begin{cases} 1 & (i = j), \\ 0 & (i \neq j). \end{cases}$$

On the other hand, for the case where $f = b^*_{13}$, we have

$$m_{s_i s_j} = \begin{cases} 1 & (i < j), \\ 0 & (i \geq j), \end{cases}$$

and hence

$$\overline{\alpha_{b^*_{13}}}(\varphi(s_i), \varphi(s_j)) = \begin{cases} 1 & (j < i), \\ 0 & (i \leq j). \end{cases}$$

From these facts, it is easily seen that there exists the 2-cocycle of $\mathfrak{S}_n$ appearing Theorem 2.2.

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