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Ph.D thesis

Cosmological constraints on time-varying fundamental
constant and extra component
(物理定数の時間変化と余剰成分に関する宇宙観測
制限)

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Abstract

Lambda Cold Dark Matter (Λ CDM) model, which predicts the cosmological constant Λ as dark energy and cold dark matter (CDM), has successfully explained various properties and observations of the universes. It not only explains the current accelerated expansion of the universe revealed by local observations but also explains early universe phenomena such as the Cosmic Microwave Background (CMB) and Big Bang Nucleosynthesis (BBN), effectively describing 13.7 billion years of the universe history. However, as observational precision has improved, a discrepancy exceeding 4σ has been reported between the direct measurement of the Hubble constant from local observations and the value derived from distant observations assuming the Λ CDM model.

The shortened sound horizon scale at the recombination epoch by introducing the time-varying fundamental constant, extra radiation, or early dark energy (EDE) is a simple approach to this discrepancy (so-called the Hubble tension). We examine them with paying attention to the fit to BBN. We find that a smaller electron mass is preferred in BBN era because the greater electron mass leads to a larger helium mass fraction. We also find that a fit to BBN in EDE models also is somewhat poorer than that in the Λ CDM model, because the increased inferred baryon asymmetry leads to smaller deuterium abundance.

Another important cosmological measurement is baryon acoustic oscillation (BAO). The results of the latest BAO measurements by Dark Energy Spectroscopic Instrument (DESI) survey have been reported. We present constraints on the varying electron mass model and early dark energy models in new DESI data as well as cosmic microwave background by Planck and the conventional BAO data from 6dF, MGS, and DR12 and supernovae light curve data into analysis. Since new DESI BAO data indicate a slightly longer sound horizon $r_d h$ than the other BAO observations, for the two models, we find a higher Hubble constant is indicated.

Chapter 1

Introduction

Cosmological standard model, the Λ CDM model has been successful to explain the cosmological observations at various redshifts. This model assumes the existence of a cosmological constant Λ for dark energy and cold dark matter (CDM). Using this model, we can explain not only the current accelerating expansion of the universe, as proven by local observations, but also phenomena in the early universe, such as the cosmic microwave background radiation (CMB) and Big Bang nucleosynthesis (BBN).

However, as observations have become more precise, a discrepancy has emerged between the Hubble constants from distant (high redshifts) observations and local (low redshifts) observations. In distant observations, assuming Λ CDM model, Planck team, who observed the cosmic microwave background (CMB), reported $H_0 = 67.36 \pm 0.54$ km/s/Mpc [1], and the combination of baryon acoustic oscillations (BAO) and Big Bang nucleosynthesis (BBN) also reported $H_0 = 67.6 \pm 1.0$ km/s/Mpc [2]. Other distant observations [3, 4, 5] also report a lower Hubble constant than local observations. Meanwhile, in local observations, Cepheid variables and Type Ia supernovae observations by SH0ES reported $H_0 = 73.04 \pm 1.04$ km/s/Mpc [6], and lensed quasars observations by H0LiCOW also reported $H_0 = 73.30^{+1.7}_{-1.8}$ km/s/Mpc [7], though Tip of the Red Giant Branch (TRGB) observations reported a slightly lower Hubble constant, $H_0 = 69.6 \pm 2.5$ km/s/Mpc [8]. This discrepancy is about 5σ and is known as a serious problem called Hubble tension.

To solve the Hubble tension, several ideas have been proposed (For a review, see the following paper [9, 10, 11, 12, 13, 14]). Among the various ideas to solve the Hubble tension [9, 10, 11, 12, 13], in this paper, we primarily focus on the varying electron mass (varying m_e) model which is regarded as

the most promising solution [13]. If we assume that the electron mass was larger by a few percent than the present value $m_{e0} = 511$ keV before the recombination epoch $z \gtrsim 1000$, then the varying electron mass model seems a potential solution to the Hubble tension [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25], because a shorter sound horizon scale at the recombination epoch indicates a higher H_0 .

In this paper, we also examine other possible consequences of the time-varying electron mass and fine structure constant model. An important event in the early Universe is the BBN where light elements were synthesized. The helium mass fraction has been reported as $Y_p = 0.2449 \pm 0.0040$ from the recent observation data [26]. The deuterium abundance D/H has been measured as $(D/H) = (2.527 \pm 0.030) \times 10^{-5}$ [27]. However, if the electron mass is larger than the present value, then the abundance of synthesized light elements would be altered [28, 29, 30, 31, 20]. One reason is that the decay rate of the neutron n to the proton p^+ , electron e^- and neutrino ν also depends on the electron mass as [32]

$$\Gamma(n \rightarrow p^+ e^- \nu) = \frac{G_F^2}{2\pi^3} (1 + 3g_A^2) m_e^5 \lambda_0(q), \quad (1.1)$$

$$\lambda_0(q) = \int_1^q dx x(x-q)^2(x^2-1)^{1/2}, \quad (1.2)$$

with $q = Q/m_e$ where G_F is the Fermi constant, g_A is the axial-vector coupling of the nucleon and $Q = m_n - m_p$ is the difference between the proton mass m_p and the neutron mass m_n . The rate of weak interactions such as $\nu + n \leftrightarrow p^+ + e^-$ also depends on the electron mass [33]. Thus, it causes earlier (later) decoupling of $n \leftrightarrow p$ interactions for a larger (smaller) electron mass and results the change of the neutron to proton ratio. The prediction of the BBN is sensitive to the variation of the neutron decay rate and the weak interaction rate [34, 35].

The early dark energy (EDE) model [36, 37, 38, 39, 40, 41, 42, 43, 44] is another popular scenario as a possible solution to the H_0 tension and has been intensively studied. The sound horizon in the EDE models is shorter than that in the standard Λ CDM model due to a temporal extra energy of EDE before the recombination epoch. As a result, larger H_0 is inferred than the Λ CDM. We note that the EDE models should be constrained not to increase the amplitude of matter fluctuation σ_8 [36] or the baryon density $\Omega_b h^2$ [45, 46]. Many models of EDE are proposed with different potentials, but in this paper, we focus on the axionlike EDE [36], which could explain the

cosmic birefringence [47, 48] suggested by the recent analyses of the Planck data [49, 50].

In EDE models, although the negligible energy density of the EDE component at the BBN epoch appears not to change the BBN prediction, the baryon abundance inferred from the CMB would be different from that in the Λ CDM model to adjust the CMB spectrum and hence the resultant light element abundance could be affected. In this paper, by taking the fit with BBN into account, we evaluate and compare those scenarios of additional relativistic degrees of freedoms and the EDEs.

Dark Energy Spectroscopic Instrument (DESI) survey has reported the results of the latest BAO measurements [51, 52] and it also might be needed to modify the Λ CDM model to explain the results. In their paper, the cosmological constant dark energy $(w, w_a) = (-1, 0)$ is 2.1σ discrepant with the combined analysis of DESI+SDSS+CMB+PantheonPlus [53], where w_a is the derivative of the equation of the parameter w with respect to the scale factor a . To explain this result, many extended models of dark energy have been discussed [54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65]. In this paper, we discuss the implication of the DESI BAO results to two promising cosmological scenarios motivated by the Hubble tension; the varying m_e model and the axionlike EDE model. Since those two scenarios of the varying electron mass model and EDE models take different approaches to realize the shorter sound horizon, i.e. the former alters the energy level of recombination, while the latter introduces an extra energy density, the comparison of those two seems to be insightful for further consideration.

This paper is organized as follows. We present the explanation of the standard cosmology in Part I, the method and the datasets of our analysis in Part 6, the results in Sec. II, III, and the summary in Part IV.

Part I

standard cosmology

In this part, we like to give an overview of the cosmological standard model with referencing to the textbooks [66, 67]. In section 2, we derive the Friedmann equations, and in section 3 and 4, we give an overview of BBN and CMB, respectively.

Chapter 2

Friedmann Equation

As a first step to discuss the evolution of our universe, we give the action of general relativity:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} (R - 2\Lambda) + \mathcal{L}_m \right], \quad (2.1)$$

where $\kappa^2 = 8\pi G$ and G is the gravitational constant, and Λ is a cosmological constant. The Euler-Lagrange equation for this action is the Einstein equation¹, which is given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \quad (2.2)$$

where R and $R_{\nu\sigma}$ are Ricci scalar and Ricci tensor given by

$$R_{\nu\rho\sigma}^\lambda \equiv \Gamma_{\nu\sigma,\rho}^\lambda - \Gamma_{\nu\rho,\sigma}^\lambda + \Gamma_{\nu\sigma}^\alpha \Gamma_{\alpha\rho}^\lambda - \Gamma_{\nu\rho}^\alpha \Gamma_{\alpha\sigma}^\lambda, \quad (2.3)$$

$$R_{\nu\sigma} \equiv R_{\nu\rho\sigma}^\rho = \Gamma_{\nu\sigma,\rho}^\rho - \Gamma_{\nu\rho,\sigma}^\rho + \Gamma_{\nu\sigma}^\alpha \Gamma_{\alpha\rho}^\rho - \Gamma_{\nu\rho}^\alpha \Gamma_{\alpha\sigma}^\rho, \quad (2.4)$$

$$R \equiv g^{\nu\sigma} R_{\nu\sigma} = g^{\nu\sigma} R_{\nu\rho\sigma}^\rho = g^{\nu\sigma} (\Gamma_{\nu\sigma,\rho}^\rho - \Gamma_{\nu\rho,\sigma}^\rho + \Gamma_{\nu\sigma}^\alpha \Gamma_{\alpha\rho}^\rho - \Gamma_{\nu\rho}^\alpha \Gamma_{\alpha\sigma}^\rho), \quad (2.5)$$

and $T_{\mu\nu}$ is a energy momentum tensor given by

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \left\{ -\frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial g^{\mu\nu}} + \frac{\partial}{\partial x^\alpha} \left(\frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial g_{,\alpha}^{\mu\nu}} \right) \right\}. \quad (2.6)$$

¹See Appendix A for the detail calculation.

In order to solve the Eq. (2.2), we should give a form of the metric. Here, we assume a homogeneous and isotropic flat universe, which is described by Friedmann-Lemaître-Robertson-Walker metric (FLRW metric) as

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \quad (2.7)$$

The function of time $a(t)$ represents the relative size of the universe, and it is conventional to set its current value to 1. Considering the existence of structures such as galaxy clusters, it doesn't seem that our universe is perfectly homogeneous and isotropic. However, when we extend our view to scales of about tens of mega parsecs ($1\text{pc} = 3 \times 10^{16}\text{m}$), the universe can be considered homogeneous and isotropic. For example, the temperature of the CMB photon coming from different directions are almost same. We assume that the universe is homogeneous and isotropic in this sense. This assumption implies that our location in the universe is not special, and it is known as the cosmological principle. The deviations from homogeneity and isotropy (fluctuations) will be introduced as perturbations in the later sections.

The energy momentum tensor of a perfect fluid is given by

$$T^\mu_\nu(t) = \sum_i \begin{pmatrix} -\rho_i(t) & 0 & 0 & 0 \\ 0 & p_i(t) & 0 & 0 \\ 0 & 0 & p_i(t) & 0 \\ 0 & 0 & 0 & p_i(t) \end{pmatrix}, \quad (2.8)$$

where p_i is a pressure of the fluid, ρ_i is a energy density of the fluid, and index $i = \text{DM, b, r}$ denotes dark matter (DM), baryon, and radiation, respectively.

Using this energy momentum tensor Eq. (2.8), FLRW metric Eq. (2.7), and Einstein equation (2.2) and doing a little algebra we obtain the Friedmann Equation

$$H^2(t) = \sum_i \frac{\kappa^2}{3} \rho_i(t), \quad (2.9)$$

$$\dot{H}(t) = - \sum_i \frac{\kappa^2}{2} (\rho_i(t) + p_i(t)), \quad (2.10)$$

where $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$ is a Hubble parameter. Here, dot denotes the cosmic time derivative.

The Hubble parameter can convert to

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \lim_{\Delta t \rightarrow 0} \frac{a(t + \Delta t) - a(t)}{a(t)\Delta t},$$

and expresses the expansion rate of the universe per unit of time.

Meanwhile, the energy-momentum conservation law (continuity equation) is given by,

$$T_{i;\mu}^{\mu\nu} = \dot{\rho}_i + 3H(\rho_i + p_i) = 0, \quad (2.11)$$

and the equation of state w_i is given by

$$w_i = \frac{p_i}{\rho_i} \quad (2.12)$$

For the radiation, $w_r = 1/3$ and for the dark matter and baryon $w_{DM} = w_b = 0$.

Therefore, when we assume that our universe is dominated by the component that has constant equation of state w_i , we obtain

$$\rho_i \propto a(t)^{-3(1+w_i)}. \quad (2.13)$$

Using this equation and Eq. (2.9), we can solve the scale factor of our universe $a(t)$ as a function of the time t ,

$$a(t) = (t - t_i)^{2/(3(1+w_i))}, \quad (2.14)$$

where t_i is the initial time. From this equation, we find that the early universe is very small and high density like a “fireball” and our universe starts at the “big bang”.

Chapter 3

big bang nucleosynthesis

When the universe temperature exceeds 1 MeV but lower than 1 GeV, neutron-proton conversion occurs as follows:

$$n + \nu_e \leftrightarrow p + e^-, \quad (3.1)$$

$$n + e^+ \leftrightarrow p + \bar{\nu}_e, \quad (3.2)$$

$$n \leftrightarrow p + e^- + \bar{\nu}_e. \quad (3.3)$$

These reactions decouple at the temperature $T_d \simeq 0.8\text{MeV}$ and then, the β -decay progresses as,

$$n \rightarrow p + e^- + \bar{\nu}_e. \quad (3.4)$$

The life-time of the β decay is $\tau_n \simeq 880.2\text{s}$. After the production of protons, so-called big bang nucleosynthesis (BBN) occurs. In this section, I will mainly focus on the synthesis process of ^4He and provide an overview of this nucleosynthesis.

When the reaction 3.3 is in thermal equilibrium at temperature T , the number densities of neutrons and protons $n_n(T), n_p(T)$ should follow the Maxwell-Boltzmann distribution as follows,

$$\begin{cases} n_n(T) = g_n \left(\frac{m_n T}{2\pi}\right)^{3/2} e^{-(m_n - \mu_n)/T} \\ n_p(T) = g_p \left(\frac{m_p T}{2\pi}\right)^{3/2} e^{-(m_p - \mu_p)/T} \end{cases} \quad (3.5)$$

Here, $g_n = g_p = 2$ are spin degrees and m_n, m_p, μ_n, μ_p are the mass and chemical potential of a neutron and proton, respectively. Thus, at the decoupling temperature (T_d), the number density fraction is given by

$$\frac{n_n(T_d)}{n_p(T_d)} \simeq \exp[-(m_n - m_p)/T_d] \simeq 0.2. \quad (3.6)$$

On the other hand, in the lower temperature ($T < T_d$) β -decay (3.4) progresses and the number densities should be

$$\begin{cases} n_n(t) = n_n(T_d)e^{-t/\tau_n} \\ n_p(t) = n_p(T_d) + n_n(T_d)(1 - e^{-t/\tau_n}) \end{cases}, \quad (3.7)$$

and the number density fraction is given by

$$\frac{n_n}{n_p}(t) \simeq \frac{e^{-t/\tau_n}}{6 - e^{-t/\tau_n}}. \quad (3.8)$$

When there are sufficient protons, neutrons couple to protons and deuterium are produced as,

$$n + p \leftrightarrow D + \gamma.$$

Since the deuterium atoms are unstable, deuterium couple to each other and Helium atoms ${}^4\text{He}$ are produced as,

$$D + D \rightarrow \gamma + {}^4\text{He},$$

$$D + D \rightarrow p + {}^3\text{He}, \quad {}^3\text{He} + D \rightarrow n + {}^4\text{He},$$

$$D + D \rightarrow n + {}^3\text{He}, \quad {}^3\text{He} + D \rightarrow n + {}^4\text{He}.$$

We can find the time evolution of the mass fraction of the synthesized atoms by following up the nucleon reactions and we obtain Fig. 3.1 using the numerical code `ParthENoPE` [68, 69, 70].

In the numerical code, for example, before the deuterium production ($T > 0.1\text{MeV}$), the neutron number fraction X_n and the proton number fraction X_p are related as,

$$\frac{dX_n}{dt} = -\Gamma(n \rightarrow p)X_n + \Gamma(p \rightarrow n)X_p \quad (3.9)$$

Here, $\Gamma(n \rightarrow p)$ and $\Gamma(p \rightarrow n)$ are the decay width given by

$$\Gamma(n \rightarrow p) = A \int \sqrt{1 - \frac{m_e^2}{(Q+q)^2}} \frac{(Q+q)^2 q^2 dq}{(1 + e^{q/kT_\nu})(1 + e^{-(Q+q)/kT})}, \quad (3.10)$$

$$\Gamma(p \rightarrow n) = A \int \sqrt{1 - \frac{m_e^2}{(Q+q)^2}} \frac{(Q+q)^2 q^2 dq}{(1 + e^{-q/kT_\nu})(1 + e^{(Q+q)/kT})}, \quad (3.11)$$

where k is the Boltzman constant, $Q = m_n - m_p = 1.293\text{MeV}$, m_p , m_n , and m_e are the mass of protons, neutrons, and electrons, respectively. The integrals over q run from $-\infty$ to $+\infty$, except for the range from $-Q - m_e$ to $-q + m_e$. A is defined as,

$$A \equiv \frac{G_{\text{wk}}^2 (1 + 3g_A^2) \cos^2 \theta_C}{2\pi^3 \hbar}, \quad (3.12)$$

where G_{wk} is the weak coupling constant, g_A is the axial vector coupling of beta decay, and θ_C is the Cabibbo angle.

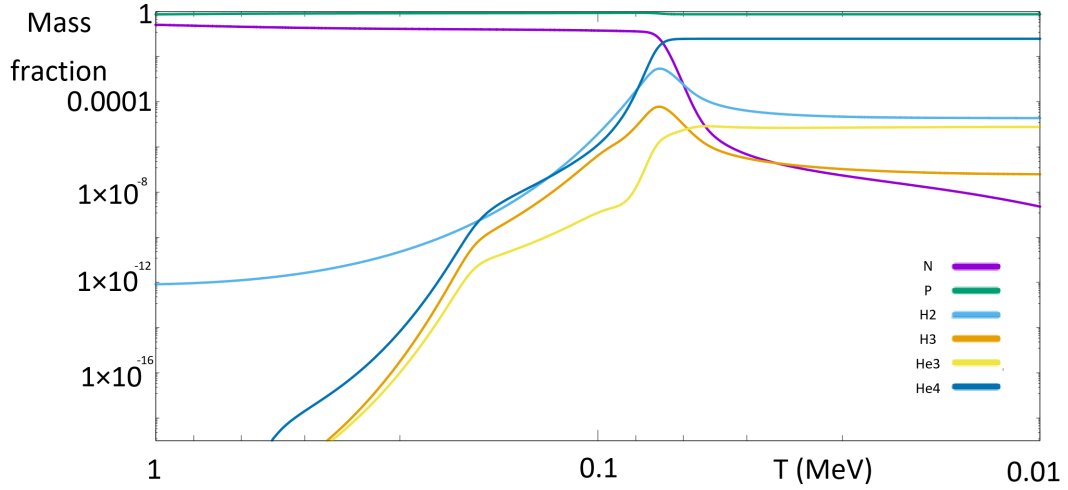


Figure 3.1: The mass fraction of the each elements synthesized in the BBN era as a function of a temerature. This figure is illustrated using the numerically code: `PARthENoPE` [70].

At time $t \simeq 250\text{s} (\simeq 4\text{min}) \equiv t_{\text{He}}$, we can find that nearly all neutrons are incorporated into ${}^4\text{He}$, and the synthesis of helium ${}^4\text{He}$, freezes. The proton-neutron mass ratio at t_{He} can be derived from Eq. (3.8) as,

$$\frac{n_n}{n_p}(t_{\text{He}}) \simeq 0.144 \simeq \frac{1}{7}. \quad (3.13)$$

Therefore, the mass fraction of the synthesized helium is given by

$$Y_P \equiv \frac{\text{mass energy density of } {}^4\text{He}}{\text{mass energy density of baryons}} \simeq \frac{4(n_n/2)}{n_n + n_p} \simeq \frac{2}{1+7} = 0.25. \quad (3.14)$$

In the previous work [71], the helium mass fraction is measured as $Y_P = 0.2453 \pm 0.0034 (1\sigma)$. This value is extremely close to the theoretical approximation 3.14, serving as one of the strong evidence of the BBN scenario. For other light elements as well (except lithium), the theoretical and observational values are in good agreement. Therefore, when attempting to modify the standard model, we must take care not to disrupt this BBN scenario.

Chapter 4

cosmic microwave background

After the Big Bang nucleosynthesis, the universe is too hot, and the electrons are too energetic to be bound into atoms. As a result, photons and electrons repeatedly undergo Thomson scattering, preventing photons from traveling straight. As the universe expanded and cooled, photons were eventually able to travel without scattering. The light that traveled straight from the last scattering surface is known as the Cosmic Microwave Background Radiation (CMB).

4.1 Saha equation

In this subsection, we discuss when CMB photons were last scattered. The scattering of photons and electrons ceases when free electrons become bound into Hydrogen and Helium atoms, which process is called recombination.¹

We start our calculation from the time when protons, electrons, hydrogen atoms, and helium atoms are in thermal equilibrium at the same temperature as the radiation. The number density of non-relativistic particles i is given by the Maxwell-Boltzmann distribution as follows:

$$n_i = g_i (2\pi\hbar)^{-3} \exp\left[\frac{\mu_i}{k_B T}\right] \int d^3\mathbf{q} \exp\left[\frac{-(m_i + q^2/2m_i)}{k_B T}\right], \quad (4.1)$$

where m_i is the mass of particle i , g_i is the internal degrees of the freedom (number of the spin states), and μ_i is the chemical potential. Here, we

¹Although electrons and protons had not previously combined before this time, in astrophysics, the process of electrons being captured by protons is conventionally referred to as "re"combination.

consider protons (p), electrons (e^-), and hydrogen atoms in the 1s, 2s, and 2p states as the particle i . About one-quarter of the baryon mass (about 24%) exists in the form of neutral helium atoms during the relatively low-temperature period of interest ($T \leq 4400\text{K}$). Therefore, helium does not significantly affect the ionization and recombination reactions we are considering. Since electrons and protons have a half ($1/2$) spin, the number of the spin state is $g_p = g_{e^-} = 2$. The hydrogen atom in the 1s ground state has two hyperfine states with spin 0 or 1, and $g_{1s} = 1 + 3 = 4$.

Since the radiative transition $p + e^- \rightleftharpoons \text{H}_{1s} + \gamma$ is a rapid reaction, the chemical potential satisfies the following conservation laws:

$$\mu_p + \mu_{e^-} = \mu_{1s}. \quad (4.2)$$

Using the formula of a Gaussian integration

$$(2\pi\hbar)^{-3} \int d^3\mathbf{q} \exp\left[\frac{-q^2}{2mk_B T}\right] = \left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2},$$

Eq. (4.1) gives

$$n_i = g_i \left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2} \exp\left[\frac{(\mu_i - m_i)}{k_B T}\right].$$

Noting the approximation $m_H/m_p = 1$,

$$\frac{n_{1s}}{n_p n_e} = \left(\frac{mk_B T}{2\pi\hbar^2}\right)^{-3/2} \exp\left(\frac{B_1}{k_B T}\right) \quad (4.3)$$

Here, $B_1 \equiv m_p + m_e - m_H = 13.6\text{eV}$ is the binding energy of the 1s ground state of the hydrogen atom.

From the electric-neutral condition of our universe, the number density of electrons and protons are same:

$$n_e = n_p. \quad (4.4)$$

The number density of the excited states are suppressed by the factor $\exp(-\Delta E/k_B T)$ comparing with the ground state. The energy difference ΔE is greater than 10.2eV, which is the energy difference between the energy state of $n = 1$ and $n = 2$. When a temperature of the universe is lower than 4200K, this factor is smaller than 6×10^{-13} and we can ignore it.

From the discussion on Big Bang nucleosynthesis, 76% mass of the baryon exists as the form of neutral hydrogen or ionized hydrogen, while the remaining 24% exists as the form of neutral helium,

$$n_p + n_{1s} = 0.76n_B. \quad (4.5)$$

Here, n_B is a number density of baryon (neutrons and protons). Therefore, free electron fraction defined by

$$X \equiv \frac{n_p}{n_p + n_{1s}} \quad (4.6)$$

should be satisfy the following Saha equation:

$$X(1 + SX) = 1. \quad (4.7)$$

S should be defined as

$$S \equiv \frac{(n_p + n_{1s})n_{1s}}{n_p^2} = 0.76n_B \left(\frac{mk_BT}{2\pi\hbar^2} \right)^{-3/2} \exp(B_1/k_BT). \quad (4.8)$$

Here, we use Eqs. (4.3), (4.4), (4.5) and we can check Eq. (4.7) using the following equation

$$1 + SX = 1 + \frac{(n_p + n_{1s})n_{1s}}{n_p^2} \frac{n_p}{n_p + n_{1s}} = \frac{n_p + n_{1s}}{n_p}.$$

Here, physical constant are numerically given by

$$\frac{m_e k_B}{2\pi\hbar^2} = \frac{m_e c^2 k_B}{c^2 \hbar 2\pi\hbar} = \frac{8.2 \times 10^{-14}(\text{J}) \times 2 \times 10^{10}(\text{Hz/K})}{(3 \times 10^{10}(\text{cm/s}))^2 \times 1 \times 10^{-34}(\text{J} \cdot \text{s})} \simeq 2 \times 10^{10}(/K \cdot \text{cm}^2),$$

and the present baryon number density is given by

$$n_{B0} = \frac{3\Omega_B H_0^2}{8\pi G m_N} = 1.123 \times 10^{-5} \Omega_B h^2 (/ \text{cm}^3).$$

Thus, the baryon number density of the given temperature is

$$n_B = n_{B0}(a_0/a)^3 = n_{B0}(T/T_{\gamma 0})^3$$

Noting that

$$\frac{B_1}{k_B} = 157894\text{K}, \quad T_{\gamma 0} = 2.725\text{K},$$

$T(\text{K})$	4500	4200	4000	3800	3600	3400	3200	3000	2.7
$\Omega_B h^2 = 0.02$	0.998	0.981	0.900	0.634	0.290	0.094	0.0240	0.00491	2.0×10^{-12571}
$\Omega_B h^2 = 0.03$	0.997	0.971	0.863	0.565	0.244	0.078	0.0196	0.00401	1.6×10^{-12571}

Table 4.1: The value of X

we obtain

$$\begin{aligned}
S &= 0.76 \times 1.23 \times 10^{-5} \times \left(\frac{m_e k_B}{2\pi\hbar^2} \right)^{-3/2} \times \frac{T^{3/2}}{T_{\gamma 0}^3} \Omega_B h^2 \\
&= 1.747 \times 10^{-22} e^{157894/T} T^{3/2} \Omega_B h^2,
\end{aligned} \tag{4.9}$$

and using Eq. (4.7), we obtain the table 4.1

As we know the fraction $X(z)$, we can calculate when the Thomson scattering rate $\Gamma = n_e \sigma_{TC}$ becomes equal to the Hubble parameter H . This is the epoch of recombination (or "decoupling") when the universe becomes transparent, and its redshift is around $z_* \simeq 1120$.

Table 4.1 approximately shows the rapid decrease in the ionization degree around 3800K, but it does not show the details accurately. This is because, as the ionization degree (the fraction of free protons) decreases, the system departs from equilibrium.

Specifically, photons emitted during the recombination of H_{1s} can re-excite nearby H_{1s} atoms (the likelihood of which increases as the ionization degree decreases), preventing a net increase in the number of H_{1s} atoms. To increase the number of H_{1s} atoms via the transition $H_{2p} \rightarrow H_{1s}$, we must wait for the emitted photons to lose their energy through the redshift by the expansion of the universe. The H_{2s} state can only transition to the ground state by emitting two photons, which have lower individual energies and thus cannot re-excite ground state atoms. Therefore, H_{2s} transitions do not need to be considered in this context. When these corrections are taken into account, recombination and the decoupling of the universe are delayed, resulting in the last-scattering redshift $z_* \simeq 1090$.

4.2 CMB Temperature

In this subsection, we will discuss the current temperature of the CMB radiation and the energy density ratio in the present universe. From the discussion

of the statistical mechanics, in a thermal equilibrium at temperature T , the number density of photons which frequency is between ν and $\nu + d\nu$ is given by the black-body radiation spectrum

$$n_T(\nu)d\nu = \frac{g_* 4\pi\nu^2 d\nu}{c^3 [\exp(h\nu/k_B T) - 1]}. \quad (4.10)$$

Here h is the Planck constant, k_B is the Boltzmann constant, c is the speed of light, g_* is the inner freedom ($g_* = 2$ for photon).

The equilibrium of photons continued until the photons no longer scatter with the electrons. After the equilibrium era, considering that photons could move freely, we need to discuss how this number density changes. We assume that the scattering stopped at a certain time t_L (L : Last scattering). From the Table 4.1, we already find that the last scattering time t_L can be set when the temperature is around $T_L \simeq 3300K$. The physical frequency ν at a given time t is related to the physical frequency ν_L at the time of last scattering t_L by the following relationship:

$$a(t_L)\nu_L = a(t)\nu.$$

thus, we obtain the description of ν_L as

$$\nu_L = \nu \frac{a(t)}{a(t_L)}. \quad (4.11)$$

As the number density is proportional to a^{-3} , a comoving number density n_T is given by

$$a^3(t) \times n(\nu, t)d\nu = a^3(t_L) \times n_{T_L}(\nu_L, t_L)d\nu_L.$$

Therefore,

$$\begin{aligned} n(\nu, t)d\nu &= \left(\frac{a(t_L)}{a(t)} \right)^3 n_{T_L}(\nu_L, t_L)d\nu_L \\ &= \left(\frac{a(t_L)}{a(t)} \right)^3 \frac{4\pi g_* (\nu a(t_L)/a(t))^2 d(\nu a(t_L)/a(t))}{c^3 \left[\exp \left(\frac{h\nu a(t_L)/a(t)}{k_B T} \right) - 1 \right]}, \end{aligned}$$

then, the factor $(a(t_L)/a(t))$ vanish and we obtain

$$n(\nu, t)d\nu = \frac{4\pi g_* \nu^2 d\nu}{c^3 [\exp(h\nu/k_B T(t)) - 1]}, \quad (4.12)$$

where the temperature of CMB $T(t)$ is defined as

$$T(t) \equiv T_L \frac{a(t_L)}{a(t)}. \quad (4.13)$$

In practice, what we observe is the energy density spectrum which is multiplied the number density Eq. (4.12) by $h\nu$. From the Cosmic Background Explorer (COBE) observation, the present temperature of CMB is measured as,

$$T_{\text{CMB today}} \simeq 2.73\text{K}. \quad (4.14)$$

Since we know the temperature of CMB, we can calculate the energy density of the radiation not only photons but neutrinos. Multiplying Eq. (4.12) by $h\nu$, the energy density would be

$$\rho(t) = \int_0^\infty \frac{h\nu \times 4\pi g_* \nu^2 d\nu}{c^3 [\exp(h\nu/k_B T(t)) - 1]}$$

Noting the natural unit,

$$\begin{aligned} \frac{2\pi\nu}{c} &= \frac{2\pi}{\lambda} = \frac{p}{\hbar} = p, \\ h\nu &= \hbar \frac{2\pi\nu}{c} = p, \end{aligned}$$

we obtain the result

$$\begin{aligned} \rho(t) &= \frac{g_*}{2\pi^2} \int_0^\infty \frac{p^3 dp}{\exp(p/k_B T(t)) - 1} \\ &= \frac{g_*}{2\pi^2} (k_B T(t))^4 \int_0^\infty \frac{x^3 dx}{e^x - 1}. \end{aligned}$$

When we consider a fermion like a neutrino, we should replace the factor $e^x - 1$ to $e^x + 1$ and the result would be changed by a factor of 7/8. The energy densities for the massless particles are given by

$$\rho(t) = \frac{\pi^2 g_* k_B^4 T^4(t)}{30(\hbar c)^3} \times \begin{cases} 1 & \text{boson} \\ 7/8 & \text{fermion} \end{cases}. \quad (4.15)$$

For performing the integration, we use the following Riemann zeta function formula,

$$\int_0^\infty \frac{x^3}{e^x \pm 1} dx = \begin{cases} \frac{\pi^2}{15} & - \\ \frac{7\pi^2}{120} & + \end{cases} \quad (4.16)$$

From the results of statistical physics, the relation between the free energy $\mathfrak{f}$ and the nergy density ρ is given by

$$\mathfrak{f} = -\frac{\rho}{3}. \quad (4.17)$$

On the other hand, from a formula of thermodynamics, we know that the relation between the free energy and the pressure is $\mathfrak{f} = -p$. Therefore, we can derive the equation of state of radiation $w_r = p/\rho = \frac{1}{3}$. Noting that $\rho = \mathfrak{f} + Ts$, entropy density s is also derived as

$$s = \frac{4}{3} \frac{\rho}{T}.$$

Therefore, the entropy of the universe, whose volume is $a^3(t)$, $S(t)$ is given by

$$S(t) = \frac{2\pi^2 g_* k_B^4 T^3(t)}{45(\hbar c)^3} a^3(t) \times \begin{cases} 1 & \text{boson} \\ 7/8 & \text{fermion} \end{cases}. \quad (4.18)$$

Radiation compose of a photon (boson, $g_* = 2$), neutrino, and anti-neutrino (fermion, $g_* = 1$). In the very early universe, an electron and positron (fermion, $g_* = 2$) are also relativistic. These all particles keep a thermal equilibrium, weak interacting with each other. This condition keep until the decoupling temperature $T_{\text{de}} \simeq 1.7 \times 10^{10} K$. When the temperature T_1 (or scale factor a_1) is higher than T_{de} , $S(a_1)$ is given by

$$S(a_1) = \frac{2\pi^2 g_* k_B^4 T_1^3}{45(\hbar c)^3} a_1^3 \left[2 + \frac{7}{8}(2 + 2 + 3 \times 2) \right] = \frac{43}{90} \pi^2 k_B^4 T_1^3 a_1^3. \quad (4.19)$$

Then, after the universe get cooler than $T_{NR} \equiv \frac{m_e c^2}{k_B} = 5.9 \times 10^9 K$, positrons and electrons become non-relativistic and the pair annihilate as

$$e^+ + e^- \rightarrow \gamma + \gamma. \quad (4.20)$$

In this era, photon and neutrino doesn't interact with each other. Then, we should note that the photon temperature T_γ is different from the neutrino

temperature T_ν , after T_{NR} . Assuming that the scale of the universe is a_2 , when the temperature of photon and neutrino are T_γ , and T_ν , we obtain

$$S(a_2) = \frac{2\pi^2 g_* k_B^4}{45(\hbar c)^3} a_2^3 \left(2T_\gamma^3 + \frac{7}{8} 6T_\nu^3 \right) \quad (4.21)$$

On the other hand, the conservation law of the entropy is given by

$$S(a_1) = S(a_2).$$

We also note that $\rho_\nu \propto a^{-4} \propto T_\nu^4$, which leads to ²

$$a_1 T_1 = a_2 T_\nu.$$

Then, we obtain the relation

$$43(a_2 T_\nu)^3 = 4a_2^3 \left(2T_\gamma^3 + \frac{21}{4} T_\nu^3 \right)$$

and finally we achieve at the temperature ratio between neutrino and photon.

$$\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11} \right)^{1/3}.$$

Thus, from Eq. (4.15), in the case that $T < T_{NR} = 5.9 \times 10^9 K$, the energy density ratio is given by

$$\frac{\rho_\nu}{\rho_\gamma} = \frac{7}{8} \left(\frac{4}{11} \right)^{1/3} N_{\text{eff}} \simeq 0.2771 N_{\text{eff}}. \quad (4.22)$$

Here, N_{eff} is a relativistic degrees of freedom.

Then, from Eq. (4.14), (4.15), we find the present energy density of photon,

$$\rho_{\gamma 0} = \frac{\pi^2}{15} \frac{k_B^4}{\hbar^3 c^3} T_0^4 = \frac{4.17 \times 10^{-20} (\text{J} \cdot \text{cm}^{-3})}{c^2 (\text{m}^2/\text{s}^2)} \times 1000 (g/kg) \simeq 4.64 \times 10^{-34} (\text{g} \cdot \text{cm}^{-3}). \quad (4.23)$$

²The temperature of photon doesn't satisfy this relation, because there is a energy flow due to the reaction 4.20.

Translating this result into the Ω parameter, which is the ratio to the critical energy density $\rho_{crit}^{(0)}$:

$$\rho_{crit}^{(0)} = \frac{3H_0^2}{\kappa^2} = 1.88h^2 \times 10^{-29} (\text{g} \cdot \text{cm}^{-3}), \quad (4.24)$$

we obtain

$$\Omega_\gamma h^2 = \frac{\rho_{\gamma 0}}{\rho_{crit}^{(0)}} h^2 \simeq 2.47 \times 10^{-5}. \quad (4.25)$$

Assuming $h = 0.68$, $N_{\text{eff}} = 3$, Ω parameter of the radiation, Ω_r is given by

$$\Omega_r = \Omega_\gamma (1 + 0.2771 N_{\text{eff}}) = 8.98 \times 10^{-5}. \quad (4.26)$$

Therefore, we find that the proportion of radiation energy (including neutrinos) in the present Universe is about 10^{-4} .

4.3 The dipole anisotropy

In the previous section, we confirmed that the temperature of the CMB is about 2.725 K. However, this value is an average over the whole-sky CMB, and the temperature slightly varies depending on the angle where we observe. In the following sections, we will discuss such temperature fluctuations. First, we will discuss the temperature fluctuations caused by the motion of the Earth relative to the CMB.

In the previous section, we discuss using the number density per unit frequency $n(\nu, t)d\nu$, but for the following discussion, it is more convenient to use the number density per momentum unit for each polarization $N_\gamma(\mathbf{p})d^3p$.

The relation of momentum and frequency is given by

$$p = \frac{h}{\lambda} = h \frac{\nu}{c}$$

Since the number of the particles in the equivalent infinitesimal intervals is the same, we find:

$$N_\gamma(\mathbf{p})d^3p = N_\gamma(\mathbf{p})4\pi p^2 dp = \frac{1}{2}n(\nu, t)d\nu$$

The factor $1/2$ takes account of the number of the polarizations. Thus, Eq. (4.12) gives,

$$\begin{aligned} N_\gamma(\mathbf{p}) &= \frac{1}{2} \frac{n(\nu, t)}{4\pi p^2} \frac{d\nu}{dp} = \frac{\frac{4\pi(\frac{c}{h}p)^2}{c^3[\exp(h(\frac{c}{h}p)/k_B T(t)) - 1]}}{4\pi p^2} \frac{c}{h} \\ &= \frac{1}{h^3[\exp(pc/k_B T) - 1]}. \end{aligned} \quad (4.27)$$

Hereafter, we omit (t) of $T(t)$. We should note that this description of the number density is for the fixed observer to the CMB. On the other hand, we move toward to the CMB. We assume its speed β (the speed of light is the unit), and we distinguish quantities for the Earth observer by a prime symbol. When we denote the angle between the trajectory of the CMB photons and the velocity of the Earth by θ , and assume that the Earth is moving in the x -direction, then by applying the Lorentz transformation, we obtain:

$$\begin{pmatrix} p \\ p_x \\ p_{y,z} \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 \\ -\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p' \\ p' \cos \theta \\ p' \sin \theta \end{pmatrix}, \quad (4.28)$$

$$p = \gamma p' (1 + \beta \cos \theta). \quad (4.29)$$

Noticing that $p_0 = |\mathbf{p}| = p$, since photons. Thus, from Eq. (4.27), (4.28), considering that a number density is scalar, we find

$$N'_\gamma(\mathbf{p}') = N_\gamma(\mathbf{p}) = \frac{1}{h^3[\exp(p'c/kT') - 1]}. \quad (4.30)$$

Here, we define the CMB observed temperature on the Earth by

$$T' = \frac{T}{\gamma(1 + \beta \cos \theta)}. \quad (4.31)$$

Therefore, the observed temperature of the CMB differs depending on whether CMB photons are coming from the same direction as the Earth's motion or in the opposite direction. In practice, COBE satellite find $\pm 3mK$ difference between a temperature of the pole and the average value.

4.4 Evolution of the scalar perturbation

In this section, we fix the synchronous gauge ³. Considering the symmetry of the space, any perturbation can be expanded in the Fourier series as follows:

$$A(\mathbf{x}, t) = \sum_n \int d^3q \alpha_n(\mathbf{q}) A_{nq}(t) e^{i\mathbf{q} \cdot \mathbf{x}} \quad (4.32)$$

The subscript n represents various independent solutions (modes). $\alpha_n(\mathbf{q})$ is the probabilistic weight of each mode and is common to each perturbation. This can be interpreted as the initial condition value at $t = 0$, depending on q , which includes angular dependence. When it did not depend on q , the space would be completely isotropic, and galaxies would not be formed. On the other hand, $A_{nq}(t)$, which represents time evolution, does not depend on q , including angular dependence. While the initial condition values exhibit anisotropy within the range of first-order perturbations, the established physics is said to maintain isotropy even within the range of first-order perturbations.

Let's interpret this more precisely. Performing a Fourier expansion corresponds to interpreting fluctuations (perturbation quantities) as a superposition of sine functions. For example, let's compare the proportional coefficients of $\sin x \sin y \sin 2z$ and $\sin x \sin 2y \sin z$. The fluctuation specific to the z -axis ($\sin x \sin y \sin 2z$) and the fluctuation specific to the y -axis ($\sin x \sin y \sin 2z$) will have different values for their initial conditions (corresponding to the part of $\alpha_n(\mathbf{q})$). However, the way these evolve over time (corresponding to the part of $A_{nq}(t)$) is said to be isotropic.

For example, the metric [B.73](#) is given by

$$\delta g_{00} = 0, \delta g_{0i} = 0 \quad (4.33)$$

$$\delta g_{ij}(\mathbf{x}, t) = a^2(t) \int d^3q \alpha_n(\mathbf{q}) [A_{nq}(t) \delta_{ij} - q_i q_j B_{nq}(t)] e^{i\mathbf{q} \cdot \mathbf{x}}, \quad (4.34)$$

³Please see the detailed discussion of the gauge and perturbations.

because δg_{ij} can be calculated as,

$$\begin{aligned}
\delta g_{ij}(\mathbf{x}, t) &= a^2(t) [A(\mathbf{x}, t)\delta_{ij} + B(\mathbf{x}, t)_{,ij}] \\
&= a^2(t) \left\{ \sum_n \int d^3q \alpha_n(\mathbf{q}) A_{nq}(t) e^{i\mathbf{q}\cdot\mathbf{x}} \delta_{ij} + \left[\sum_n \int d^3q \alpha_n(\mathbf{q}) B_{nq}(t) e^{i\mathbf{q}\cdot\mathbf{x}} \right]_{,ij} \right\} \\
&= a^2(t) \sum_n \int d^3q \alpha_n(\mathbf{q}) [A_{nq}(t)\delta_{ij} - q_i q_j B_{nq}(t)] e^{i\mathbf{q}\cdot\mathbf{x}}.
\end{aligned}$$

Cold Dark Matter

To remove the extra degree of the gauge freedom, we should note that $\delta u_D(\mathbf{x}, t) = 0$. Assuming the "cold" dark matter, the values p , δp , and π vanish. Then, from Eq. (B.79), we obtain the energy-momentum conservation:

$$\dot{\delta\rho}_D(\mathbf{x}, t) + 3H(t)\delta\rho_D(\mathbf{x}, t) + \bar{\rho}_D(t)\partial_0 \left(\frac{3A(\mathbf{x}, t) + \nabla^2 B(\mathbf{x}, t)}{2} \right) = 0.$$

Considering the eq. (4.34), this equation is rewritten as

$$\begin{aligned}
&\int d^3q \alpha(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{x}} \dot{\delta\rho}_{Dq}(t) + 3H(t) \int d^3q \alpha(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{x}} \delta\rho_{Dq}(t) \\
&+ \bar{\rho}_D(t) \partial_0 \left\{ \frac{3 \int d^3q \alpha(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{x}} A_q(t) + \nabla^2 [\int d^3q \alpha(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{x}} B_q(t)]}{2} \right\} = 0,
\end{aligned}$$

and we obtain the

$$\dot{\delta\rho}_{Dq}(t) + 3H(t)\delta\rho_{Dq}(t) = -\bar{\rho}_D(t)\psi_q(t). \quad (4.35)$$

Here, we define ψ as

$$\psi_q(t) \equiv \frac{3\dot{A}_q(t) - q^2 \dot{B}_q(t)}{2}. \quad (4.36)$$

baryon plasma

Due to the electromagnetic interaction between electrons and atomic nuclei (baryons), the system behaves as a perfect fluid plasma. Here, we will discuss

the case where electrons and baryons behave non-relativistically. At this point, similar to cold dark matter, pressure p , pressure perturbations δp , and the anisotropic stress π can be neglected. However, δu_B is not zero. Thus, the time evolution equation of the baryon fraction B.79 can be rewritten as,

$$\dot{\delta\rho_{Bq}}(t) + 3H(t)\delta\rho_{Bq}(t) - \frac{q^2}{a^2(t)}\bar{\rho}_B(t)\delta u_{Bq}(t) = -\bar{\rho}_{Bq}(t)\psi_q(t). \quad (4.37)$$

In the baryon plasma case, we have the two variables: $\delta\rho$ and δu . To obtain the solutions of two values, we need one more equation. For example, we can rewrite the momentum-conservation B.78 as,

$$\delta p_\gamma(t) + q^2\pi_{\gamma q}^S(t) + (\partial_0 + 3H(t)) \left[\bar{\rho}_B(t)\delta u_{Bq}(t) + \frac{4}{3}\bar{\rho}_\gamma(t)\delta u_{\gamma q}(t) \right] = 0. \quad (4.38)$$

The baryon plasma and photons interact through Thomson scattering. In other words, they are not independent, so the momentum conservation law accounts for the combined contributions of the baryon plasma and photons. On the other hand, it has already been confirmed that, approximately, energy conservation holds independently for both photons and the baryon plasma.

photon

The number density of a photon is given by

$$n_\gamma^{ij}(\mathbf{x}, \mathbf{p}, t) = \frac{1}{2}\bar{n}_\gamma(a(t)p^0(\mathbf{x}, \mathbf{p}, t)) \left[g^{ij}(\mathbf{x}, t) - \frac{g^{ik}(\mathbf{x}, t)g^{jl}(\mathbf{x}, t)p_k p_l}{[p^0(\mathbf{x}, \mathbf{p}, t)]^2} \right] + \delta n_\gamma^{ij}(\mathbf{x}, \mathbf{p}, t), \quad (4.39)$$

$$p^0(\mathbf{x}, \mathbf{p}, t) = \sqrt{g^{ij}(\mathbf{x}, t)p_i p_j} = a(t)p,$$

$$\bar{n}_\gamma(p) = \frac{1}{(2\pi)^3} [\exp(p/k_B a(t)\bar{T}(t)) - 1]^{-1}.$$

The Boltzmann equation of a photon is given by

$$\begin{aligned} & \frac{\partial \delta n_\gamma^{ij}(\mathbf{x}, \mathbf{p}, t)}{\partial t} + \frac{\hat{p}_k}{a(t)} \frac{\partial \delta n_\gamma^{ij}(\mathbf{x}, \mathbf{p}, t)}{\partial x^k} + 2H(t)\delta n_\gamma^{ij}(\mathbf{x}, \mathbf{p}, t) \\ & - \frac{1}{4a^2(t)} p \bar{n}'_\gamma(p) \hat{p}_k \hat{p}_l \frac{\partial}{\partial t} (a^{-2}(t)\delta g_{kl}(\mathbf{x}, t)) (\delta_{ij} - \hat{p}_i \hat{p}_j) \end{aligned}$$

$$\begin{aligned}
&= -\omega_c(t)\delta n_\gamma^{ij}(\mathbf{x}, \mathbf{p}, t) + \frac{3\omega_c(t)}{8\pi} \int d^2\hat{p}_1 \\
&\times [\delta n_\gamma^{ij}(\mathbf{x}, p\hat{p}_1, t) - \hat{p}_i\hat{p}_k\delta n_\gamma^{ij}(\mathbf{x}, p\hat{p}_1, t) - \hat{p}_j\hat{p}_k\delta n_\gamma^{ij}(\mathbf{x}, p\hat{p}_1, t) + \hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l\delta n_\gamma^{kl}(\mathbf{x}, p\hat{p}_1, t)] \\
&- \frac{\omega_c(t)}{2a^3(t)} p_k \delta u_{Bk}(\mathbf{x}, t) \bar{n}'_\gamma(p) (\delta_{ij} - \hat{p}_i\hat{p}_j)
\end{aligned} \tag{4.40}$$

The number density of photon is given by $g_{ij}(\mathbf{x}, \mathbf{p}, t)n_\gamma^{ij}(\mathbf{x}, \mathbf{p}, t)$ and the energy-momentum tensor of photon can be written as

$$T_{\gamma\nu}^\mu = \frac{1}{\sqrt{\det g}} \iiint dp_1 dp_2 dp_3 g_{ij}(\mathbf{x}, \mathbf{p}, t) n_\gamma^{ij}(\mathbf{x}, \mathbf{p}, t) \frac{p^\mu p_\nu}{p_0}. \tag{4.41}$$

Noting that $p^0 = -p_0 = \sqrt{g^{ij}(\mathbf{x}, t)p_i p_j}$, $p^i = g^{ij}p_j$, the first perturbation of the energy-momentum tensors are given by

$$\delta T_{\gamma j}^i = \frac{1}{a^4(t)} \iiint (\Pi_{k=1}^3 dp_k) a^2(t) \delta n_\gamma^{ll}(\mathbf{x}, \mathbf{p}, t) \frac{p_i p_j}{\sqrt{p_k p_k}}, \tag{4.42}$$

$$\delta T_{\gamma j}^0 = \frac{1}{a^3(t)} \iiint (\Pi_{k=1}^3 dp_k) a^2(t) \delta n_\gamma^{ll}(\mathbf{x}, \mathbf{p}, t) p_j, \tag{4.43}$$

$$\delta T_{\gamma 0}^0 = -\frac{1}{a^4(t)} \iiint (\Pi_{k=1}^3 dp_k) a^2(t) \delta n_\gamma^{ll}(\mathbf{x}, \mathbf{p}, t) \sqrt{p_m p_m}. \tag{4.44}$$

For the simplicity, we define the matrix J_{ij} as,

$$a^4(t) \bar{\rho}_\gamma(t) J^{ij}(\mathbf{x}, \hat{p}, t) \equiv a^2(t) \int_0^\infty \delta n_\gamma^{ij}(\mathbf{x}, \mathbf{p}, t) 4\pi p^3 dp. \tag{4.45}$$

$$\bar{\rho}_\gamma(t) \equiv \frac{1}{a^4(t)} \int 4\pi p^3 \bar{n}_\gamma(p) dp \tag{4.46}$$

For example, ij components of Eq. (4.42) can be rewritten as

$$\begin{aligned}
\delta T_{\gamma j}^i &= \frac{1}{a^4(t)} \iiint (\Pi_{k=1}^3 dp_k) a^2(t) \delta n_\gamma^{ll}(\mathbf{x}, \mathbf{p}, t) \frac{p_i p_j}{\sqrt{p_k p_k}} \\
&= \frac{1}{a^4(t)} \int \int p^2 dp d^2\hat{p} a^2(t) \delta n_\gamma^{ll}(\mathbf{x}, \mathbf{p}, t) \frac{p_i}{p} \frac{p_j}{p} p \\
&= \frac{1}{a^4(t)} \int d^2\hat{p} \frac{\hat{p}_i \hat{p}_j}{4\pi} a^2(t) \int 4\pi p^3 dp \delta n_\gamma^{ll}(\mathbf{x}, \mathbf{p}, t) \\
&= \bar{\rho}_\gamma(t) \int \frac{d^2\hat{p}}{4\pi} J_{ll} \hat{p}_i \hat{p}_j.
\end{aligned} \tag{4.47}$$

$$\delta T_{\gamma 0}^0 = -\bar{\rho}_\gamma(t) \int \frac{d^2 \hat{p}}{4\pi} J_u. \quad (4.48)$$

$$\delta T_{\gamma j}^0 = a(t) \bar{\rho}_\gamma(t) \int \frac{d^2 \hat{p}}{4\pi} J_u \hat{p}_j. \quad (4.49)$$

For the preparation to solve the Eq. (4.40), let the physical quantities in Eq. (4.40) be a Fourier expansion. SVT decomposing $J_{ij}(\mathbf{x}, \mathbf{p}, t)$ and choosing scalar mode, we can decompose as,

$$J_{ij}(\mathbf{x}, \hat{p}, t) = A_1(\mathbf{x}, \hat{p}, t) \delta_{ij} + \partial_i \partial_j A_2(\mathbf{x}, \hat{p}, t).$$

From Eq. (4.32), since it's proportional to $\alpha(\mathbf{q})$,

$$J_{ij}(\mathbf{q}, \hat{p}, t) = \alpha(\mathbf{q}) [A_1(q, \hat{p}, t) \delta_{ij} - q_i q_j A_2(q, \hat{p}, t)]$$

$$\ast J_{ij}(\mathbf{x}, \hat{p}, t) = \int d^3 q J_{ij}(\mathbf{q}, \hat{p}, t) e^{i\mathbf{q} \cdot \mathbf{x}} \quad (4.50)$$

Since $J_{ij}(\mathbf{q}, \hat{p}, t) \sim \delta n_{ij}(\mathbf{x}, \mathbf{p}, t)$, To be orthogonal to $\mathbf{p}$, it should be proportional to the factor $\delta_{ij} - \hat{p}_i \hat{p}_j$. Multiplying $\delta_{ij} - \hat{p}_i \hat{p}_j$ by δ_{ik} or $q_i q_k = q^2 \hat{q}_i \hat{q}_k$, we obtain $\delta_{kj} - \hat{p}_k \hat{p}_j$ or $q^2 \hat{q}_k (\hat{q}_j - (\hat{q} \cdot \hat{p}) \hat{p}_j)$. Since $J_{jk}(\mathbf{q}, \hat{p}, t)$ is a symmetric tensor ($J_{jk}(\mathbf{q}, \hat{p}, t) = J_{kj}(\mathbf{q}, \hat{p}, t)$), we rewrite $q^2 \hat{q}_k (\hat{q}_j - (\hat{q} \cdot \hat{p}) \hat{p}_j)$ as $(\hat{q}_k - (\hat{q} \cdot \hat{p}) \hat{p}_k) (\hat{q}_j - (\hat{q} \cdot \hat{p}) \hat{p}_j)$. To sum up the discussion $J_{jk}(\mathbf{q}, \hat{p}, t)$ consist of the part proportional to $\delta_{jk} - \hat{p}_j \hat{p}_k$ and the part proportional to $(\hat{q}_j - (\hat{q} \cdot \hat{p}) \hat{p}_j) (\hat{q}_k - (\hat{q} \cdot \hat{p}) \hat{p}_k)$.

Thus, the terms that affect the trace J_{ii} appearing in the energy-momentum tensor are divided into two components: $\Delta_T^{(S)}(q, \hat{q} \cdot \hat{p}, t)$, which influences the trace, and $\Delta_P^{(S)}(q, \hat{q} \cdot \hat{p}, t)$, which does not.

$$J_{ij}(\mathbf{q}, \hat{p}, t) = \alpha(\mathbf{q}) \left\{ \frac{1}{2} [\Delta_T^{(S)}(q, \hat{q} \cdot \hat{p}, t) - \Delta_P^{(S)}(q, \hat{q} \cdot \hat{p}, t)] (\delta_{ij} - \hat{p}_i \hat{p}_j) \right.$$

$$\left. + \Delta_P^{(S)}(q, \hat{q} \cdot \hat{p}, t) \left[\frac{(\hat{q}_i - (\hat{q} \cdot \hat{p}) \hat{p}_i) (\hat{q}_j - (\hat{q} \cdot \hat{p}) \hat{p}_j)}{1 - (\hat{p} \cdot \hat{q})^2} \right] \right\}. \quad (4.51)$$

Here, we write $1 - (\hat{p} \cdot \hat{q})^2$ for the normalization factor. $\Delta_T^{(S)}$ is a function of $(q, \hat{p}, t)$, because only $\alpha(\mathbf{q})$ has the angular dependence of $\mathbf{q}$ and the time evolution part has only absolute value dependence of $\mathbf{q}$. Considering the specialty of the direction of motion of the photon $\hat{p}$ from $\mathbf{q}$, $\Delta_T^{(S)}$ is also the function of $\hat{q} \cdot \hat{p}$. Therefore, we set $\Delta_T^{(S)}$ as the function of $(q, \hat{q} \cdot \hat{p}, t)$.

Reconsidering the SVT decomposition, we dived the integration of $J_{ij}(\mathbf{q}, \hat{p}, t)$ as,

$$\int \frac{d^2 \hat{p}}{4\pi} J_{ij}(\mathbf{q}, \hat{p}, t) = \alpha(\mathbf{q}) [\delta_{ij} \Phi(q, t) + \frac{1}{2} \hat{q}_i \hat{q}_j \Pi(q, t)]. \quad (4.52)$$

We can connect $\Phi(q, t), \Pi(q, t)$ with $\Delta_T^{(S)}(q, \hat{q} \cdot \hat{p}, t), \Delta_P^{(S)}(q, \hat{q} \cdot \hat{p}, t)$ in the relation, which will be Eq. (4.81).

The scalar Fourier mode of $\delta u_{Bi}(\mathbf{x}, t)$ is

$$\delta u_{Bi}(\mathbf{x}, t) = \int d^3 q \delta u_{Bi}(\mathbf{q}, t) e^{i\mathbf{q} \cdot \mathbf{x}}, \quad (4.53)$$

$$\delta u_{Bi}(\mathbf{q}, t) = i\alpha(\mathbf{q}) q_i \delta u_{Bq}(t). \quad (4.54)$$

Since our preparation is done, let's calculate the Eq. (4.40). Multimplying the equation by $4\pi p^3$ and integrating with p , we get

$$\begin{aligned} & \frac{\partial [\int dp 4\pi p^3 \delta n_\gamma^{ij}(\mathbf{x}, \mathbf{p}, t)]}{\partial t} + \frac{\hat{p}_k}{a(t)} \frac{\partial [\int dp 4\pi p^3 \delta n_\gamma^{ij}(\mathbf{x}, \mathbf{p}, t)]}{\partial x^k} + 2H(t) \left[\int dp 4\pi p^3 \delta n_\gamma^{ij}(\mathbf{x}, \mathbf{p}, t) \right] \\ & - \frac{1}{4a^2(t)} \left[\int dp 4\pi p^3 p \bar{n}'_\gamma(p) \right] \hat{p}_k \hat{p}_l \frac{\partial}{\partial t} (a^{-2}(t) \delta g_{kl}(\mathbf{x}, t)) (\delta_{ij} - \hat{p}_i \hat{p}_j) \\ & = -\omega_c(t) \left[\int dp 4\pi p^3 \delta n_\gamma^{ij}(\mathbf{x}, \mathbf{p}, t) \right] + \frac{3\omega_c(t)}{8\pi} \int d^2 \hat{p}_1 \\ & \times \left\{ \left[\int dp 4\pi p^3 \delta n_\gamma^{ij}(\mathbf{x}, p \hat{p}_1, t) \right] - \hat{p}_i \hat{p}_k \left[\int dp 4\pi p^3 \delta n_\gamma^{kj}(\mathbf{x}, p \hat{p}_1, t) \right] \right. \\ & \left. - \hat{p}_j \hat{p}_k \left[\int dp 4\pi p^3 \delta n_\gamma^{ik}(\mathbf{x}, p \hat{p}_1, t) \right] + \hat{p}_i \hat{p}_j \hat{p}_k \hat{p}_l \left[\int dp 4\pi p^3 \delta n_\gamma^{kl}(\mathbf{x}, p \hat{p}_1, t) \right] \right\} \\ & - \frac{\omega_c(t)}{2a^3(t)} \hat{p}_k \delta u_{Bk}(\mathbf{x}, t) \left[\int dp 4\pi p^3 p \bar{n}'_\gamma(p) \right] (\delta_{ij} - \hat{p}_i \hat{p}_j). \quad (4.55) \end{aligned}$$

Using the definition 4.45, 4.46 and integrating by parts, we obtain the formula

$$\int dp 4\pi p^4 \bar{n}'_\gamma(p) = -16\pi \int_0^\infty dp p^3 \bar{n}_\gamma(p) = -4a^4(t) \bar{\rho}_\gamma(t), \quad (4.56)$$

and we can calculate Eq. 4.55 as,

$$\begin{aligned} & \frac{\partial[a^2(t) \bar{\rho}_\gamma(t) J^{ij}(\mathbf{x}, \hat{p}, t)]}{\partial t} + \frac{\hat{p}_k}{a(t)} \frac{\partial[a^2(t) \bar{\rho}_\gamma(t) J^{ij}(\mathbf{x}, \hat{p}, t)]}{\partial x^k} + 2H(t)[a^2(t) \bar{\rho}_\gamma(t) J^{ij}(\mathbf{x}, \hat{p}, t)] \\ & + a^2(t) \bar{\rho}_\gamma(t) \hat{p}_k \hat{p}_l \frac{\partial}{\partial t} (a^{-2}(t) \delta g_{kl}(\mathbf{x}, t)) (\delta_{ij} - \hat{p}_i \hat{p}_j) \\ & = -\omega_c(t)[a^2(t) \bar{\rho}_\gamma(t) J^{ij}(\mathbf{x}, \hat{p}, t)] + \frac{3\omega_c(t)}{8\pi} \int d^2 \hat{p}_1 \\ & \times \{[a^2(t) \bar{\rho}_\gamma(t) J^{ij}(\mathbf{x}, \hat{p}_1, t)] - \hat{p}_i \hat{p}_k [a^2(t) \bar{\rho}_\gamma(t) J^{kj}(\mathbf{x}, \hat{p}_1, t)] \\ & - \hat{p}_j \hat{p}_k [a^2(t) \bar{\rho}_\gamma(t) J^{ik}(\mathbf{x}, \hat{p}_1, t)] + \hat{p}_i \hat{p}_j \hat{p}_k \hat{p}_l [a^2(t) \bar{\rho}_\gamma(t) J^{kl}(\mathbf{x}, \hat{p}_1, t)]\} \\ & - \frac{\omega_c(t)}{2a^3(t)} \hat{p}_k \delta u_{Bk}(\mathbf{x}, t) [-4a^4(t) \bar{\rho}_\gamma(t)] (\delta_{ij} - \hat{p}_i \hat{p}_j). \end{aligned} \quad (4.57)$$

Considering the time derivative of the energy conservation $\dot{\bar{\rho}}_r + 4H\bar{\rho}_r = 0$, we find

$$\frac{\partial[a^2(t) \bar{\rho}_\gamma(t)]}{\partial t} = [2H(t) - 4H(t)]a^2(t) \bar{\rho}_\gamma(t) = -2H(t)a^2(t) \bar{\rho}_\gamma(t).$$

Noting this equation and dividing Eq. 4.57 by $a^2(t) \bar{\rho}_\gamma(t)$, we find

$$\begin{aligned} & \frac{\partial J^{ij}(\mathbf{x}, \hat{p}, t)}{\partial t} + \frac{\hat{p}_k}{a(t)} \frac{\partial J^{ij}(\mathbf{x}, \hat{p}, t)}{\partial x^k} + \hat{p}_k \hat{p}_l \frac{\partial}{\partial t} (a^{-2}(t) \delta g_{kl}(\mathbf{x}, t)) (\delta_{ij} - \hat{p}_i \hat{p}_j) \\ & = -\omega_c(t) J^{ij}(\mathbf{x}, \hat{p}, t) + \frac{3\omega_c(t)}{8\pi} \int d^2 \hat{p}_1 \\ & \times \{J^{ij}(\mathbf{x}, \hat{p}_1, t) - \hat{p}_i \hat{p}_k J^{kj}(\mathbf{x}, \hat{p}_1, t) - \hat{p}_j \hat{p}_k J^{ik}(\mathbf{x}, \hat{p}_1, t) + \hat{p}_i \hat{p}_j \hat{p}_k \hat{p}_l J^{kl}(\mathbf{x}, \hat{p}_1, t)\} \\ & + \frac{2\omega_c(t)}{a(t)} \hat{p}_k \delta u_{Bk}(\mathbf{x}, t) (\delta_{ij} - \hat{p}_i \hat{p}_j). \end{aligned} \quad (4.58)$$

Substituting the Fourier decomposition Eqs. (4.34), (4.50), (4.53), we find

$$\begin{aligned}
& \frac{\partial}{\partial t} \left[\int d^3 q J_{ij}(\mathbf{q}, \hat{p}, t) e^{i\mathbf{q} \cdot \mathbf{x}} \right] + \frac{\hat{p}_k}{a(t)} \frac{\partial}{\partial x^k} \left[\int d^3 q J_{ij}(\mathbf{q}, \hat{p}, t) e^{i\mathbf{q} \cdot \mathbf{x}} \right] \\
& + \hat{p}_k \hat{p}_l \frac{\partial}{\partial t} \left(\int d^3 q \alpha(\mathbf{q}) [A_q(t) \delta_{kl} - q_k q_l B_q(t)] e^{i\mathbf{q} \cdot \mathbf{x}} \right) (\delta_{ij} - \hat{p}_i \hat{p}_j) \\
& = -\omega_c(t) \left[\int d^3 q J_{ij}(\mathbf{q}, \hat{p}, t) e^{i\mathbf{q} \cdot \mathbf{x}} \right] + \frac{3\omega_c(t)}{8\pi} \int d^2 \hat{p}_1 \\
& \times \left\{ \left[\int d^3 q J_{ij}(\mathbf{q}, \hat{p}_1, t) e^{i\mathbf{q} \cdot \mathbf{x}} \right] - \hat{p}_i \hat{p}_k \left[\int d^3 q J_{kj}(\mathbf{q}, \hat{p}_1, t) e^{i\mathbf{q} \cdot \mathbf{x}} \right] \right. \\
& \left. - \hat{p}_j \hat{p}_k \left[\int d^3 q J_{ik}(\mathbf{q}, \hat{p}_1, t) e^{i\mathbf{q} \cdot \mathbf{x}} \right] + \hat{p}_i \hat{p}_j \hat{p}_k \hat{p}_l \left[\int d^3 q J_{kl}(\mathbf{q}, \hat{p}_1, t) e^{i\mathbf{q} \cdot \mathbf{x}} \right] \right\} \\
& + \frac{2\omega_c(t)}{a(t)} \hat{p}_k \left[\int d^3 q \delta u_{Bk}(\mathbf{q}, t) e^{i\mathbf{q} \cdot \mathbf{x}} \right] (\delta_{ij} - \hat{p}_i \hat{p}_j). \tag{4.59}
\end{aligned}$$

For eqch $\mathbf{q}$, we find

$$\begin{aligned}
& \frac{\partial}{\partial t} J_{ij}(\mathbf{q}, \hat{p}, t) + i \frac{\hat{p}_k}{a(t)} q_k J_{ij}(\mathbf{q}, \hat{p}, t) + \alpha(\mathbf{q}) [\dot{A}_q(t) - (\mathbf{q} \cdot \hat{p})^2 \dot{B}_q(t)] (\delta_{ij} - \hat{p}_i \hat{p}_j) \\
& = -\omega_c(t) J_{ij}(\mathbf{q}, \hat{p}, t) + \frac{3\omega_c(t)}{8\pi} \int d^2 \hat{p}_1 \\
& \times \{ J_{ij}(\mathbf{q}, \hat{p}, t) - \hat{p}_i \hat{p}_k J_{kj}(\mathbf{q}, \hat{p}, t) - \hat{p}_j \hat{p}_k J_{ik}(\mathbf{q}, \hat{p}, t) + \hat{p}_i \hat{p}_j \hat{p}_k \hat{p}_l J_{kl}(\mathbf{q}, \hat{p}, t) \} \\
& + \frac{2\omega_c(t)}{a(t)} \hat{p}_k \delta u_{Bk}(\mathbf{q}, t) [\delta_{ij} - \hat{p}_i \hat{p}_j] \tag{4.60}
\end{aligned}$$

Substituting the SVT decomposition, Eqs. (4.51), (4.52), (4.54), we find

$$\begin{aligned}
& \frac{\partial}{\partial t} \alpha(\mathbf{q}) \left\{ \frac{1}{2} [\Delta_T^{(S)}(q, \mu, t) - \Delta_P^{(S)}(q, \mu, t)] (\delta_{ij} - \hat{p}_i \hat{p}_j) + \Delta_P^{(S)}(q, \mu, t) \left[\frac{(\hat{q}_i - \mu \hat{p}_i)(\hat{q}_j - \mu \hat{p}_j)}{1 - \mu^2} \right] \right\} \\
& + i \frac{\hat{p}_k}{a(t)} q_k \alpha(\mathbf{q}) \left\{ \frac{1}{2} [\Delta_T^{(S)}(q, \mu, t) - \Delta_P^{(S)}(q, \mu, t)] (\delta_{ij} - \hat{p}_i \hat{p}_j) + \Delta_P^{(S)}(q, \mu, t) \left[\frac{(\hat{q}_i - \mu \hat{p}_i)(\hat{q}_j - \mu \hat{p}_j)}{1 - \mu^2} \right] \right\} \\
& + \alpha(\mathbf{q}) [\dot{A}_q(t) - (\mathbf{q} \cdot \hat{p})^2 \dot{B}_q(t)] (\delta_{ij} - \hat{p}_i \hat{p}_j) \\
& = -\omega_c(t) \alpha(\mathbf{q}) \left\{ \frac{1}{2} [\Delta_T^{(S)}(q, \mu, t) - \Delta_P^{(S)}(q, \mu, t)] (\delta_{ij} - \hat{p}_i \hat{p}_j) + \Delta_P^{(S)}(q, \mu, t) \left[\frac{(\hat{q}_i - \mu \hat{p}_i)(\hat{q}_j - \mu \hat{p}_j)}{1 - \mu^2} \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{3\omega_c(t)}{2} \times \{ \alpha(\mathbf{q}) [\delta_{ij}\Phi(q, t) + \frac{1}{2}\hat{q}_i\hat{q}_j\Pi(q, t)] - \hat{p}_i\hat{p}_k\alpha(\mathbf{q}) [\delta_{kj}\Phi(q, t) + \frac{1}{2}\hat{q}_k\hat{q}_j\Pi(q, t)] \\
& - \hat{p}_j\hat{p}_k\alpha(\mathbf{q}) [\delta_{ik}\Phi(q, t) + \frac{1}{2}\hat{q}_i\hat{q}_k\Pi(q, t)] + \hat{p}_i\hat{p}_j\hat{p}_k\hat{p}_l\alpha(\mathbf{q}) [\delta_{kl}\Phi(q, t) + \frac{1}{2}\hat{q}_k\hat{q}_l\Pi(q, t)] \} \\
& + \frac{2\omega_c(t)}{a(t)} \hat{p}_k i\alpha(\mathbf{q}) q_k \delta u_{Bq}(t) [\delta_{ij} - \hat{p}_i\hat{p}_j]. \tag{4.61}
\end{aligned}$$

For the simplicity, we define

$$\hat{q} \cdot \hat{p} = \mu$$

Considering the case of $i = j$, we should note that

$$\frac{(\hat{q}_i - \mu\hat{p}_i)(\hat{q}_i - \mu\hat{p}_i)}{1 - \mu^2} = 1 \tag{4.62}$$

$$\delta_{ii} - \hat{p}_i\hat{p}_i = 2 \tag{4.63}$$

When we set $i = j$, in the Eq. (4.61), and using Eqs. (4.62), (4.63), we find

$$\begin{aligned}
& \dot{\Delta}_T^{(S)}(q, \mu, t) + i \frac{q\mu}{a(t)} \Delta_T^{(S)}(q, \mu, t) \\
& = -2[\dot{A}_q(t) - \mu^2 q^2 \dot{B}_q(t)] - \omega_c(t) \Delta_T^{(S)}(q, \mu, t) + 3\omega_c(t) \Phi(q, t) \\
& + \frac{3}{4} \omega_c(t) (1 - \mu^2) \Pi(q, t) + 4i \frac{q\mu}{a(t)} \omega_c(t) \delta u_{Bq}(t) \tag{4.64}
\end{aligned}$$

In the Eq. (4.61), subtracting (4.64) $\times (\delta_{ij} - \hat{p}_i\hat{p}_j)/2$,

$$\begin{aligned}
& \frac{\partial}{\partial t} \left\{ \frac{1}{2} [-\Delta_P^{(S)}(q, \mu, t)] (\delta_{ij} - \hat{p}_i\hat{p}_j) + \Delta_P^{(S)}(q, \mu, t) \left[\frac{(\hat{q}_i - \mu\hat{p}_i)(\hat{q}_j - \mu\hat{p}_j)}{1 - \mu^2} \right] \right\} \\
& + i \frac{\hat{p}_k}{a(t)} q_k \left\{ \frac{1}{2} [-\Delta_P^{(S)}(q, \mu, t)] (\delta_{ij} - \hat{p}_i\hat{p}_j) + \Delta_P^{(S)}(q, \mu, t) \left[\frac{(\hat{q}_i - \mu\hat{p}_i)(\hat{q}_j - \mu\hat{p}_j)}{1 - \mu^2} \right] \right\} \\
& = -\omega_c(t) \left\{ \frac{1}{2} [-\Delta_P^{(S)}(q, \mu, t)] (\delta_{ij} - \hat{p}_i\hat{p}_j) + \Delta_P^{(S)}(q, \mu, t) \left[\frac{(\hat{q}_i - \mu\hat{p}_i)(\hat{q}_j - \mu\hat{p}_j)}{1 - \mu^2} \right] \right\} \\
& - \frac{3\omega_c(t)}{4} \Pi(q, t) \left\{ -\frac{1}{2} (1 - \mu^2) (\delta_{ij} - \hat{p}_i\hat{p}_j) + [(\hat{q}_i - \mu\hat{p}_i)(\hat{q}_j - \mu\hat{p}_j)] \right\} \tag{4.65}
\end{aligned}$$

Divided by $\{-\frac{1}{2}(\delta_{ij} - \hat{p}_i\hat{p}_j) + [\frac{(\hat{q}_i - \mu\hat{p}_i)(\hat{q}_j - \mu\hat{p}_j)}{1 - \mu^2}]\}$, we obtain

$$\dot{\Delta}_P^{(S)}(q, \mu, t) + i \frac{q\mu}{a(t)} \Delta_P^{(S)}(q, \mu, t) = -\omega_c(t) \Delta_P^{(S)}(q, \mu, t) + \frac{3\omega_c(t)}{4} (1 - \mu^2) \Pi(q, t). \quad (4.66)$$

Eq. (4.64) and Eq. (4.66) get simpler, it's still difficult to solve. Especially, we haven't clarify the relation between $\Phi(q, t)$, $\Pi(q, t)$ and $\Delta_T^{(S)}(q, \mu, t)$, $\Delta_P^{(S)}(q, \mu, t)$. Then, we will decompose $\Delta_T^{(S)}(q, \mu, t)$, $\Delta_P^{(S)}(q, \mu, t)$ using Legendre functions as,

$$\Delta_T^{(S)}(q, \mu, t) = \sum_{l=0}^{\infty} i^{-l} (2l+1) P_l(\mu) \Delta_{T,l}^{(S)}(q, t), \quad (4.67)$$

$$\Delta_P^{(S)}(q, \mu, t) = \sum_{l=0}^{\infty} i^{-l} (2l+1) P_l(\mu) \Delta_{P,l}^{(S)}(q, t). \quad (4.68)$$

Legendre function is defined as,

$$P_n(x) \equiv \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n]. \quad (4.69)$$

Multiplying Eq. (4.66) by $P_m(\mu)$ and integrating over μ , we find

$$\begin{aligned} & \int_{-1}^1 d\mu P_m(\mu) \sum_{l=0}^{\infty} i^{-l} (2l+1) P_l(\mu) \dot{\Delta}_{P,l}^{(S)}(q, t) + i \int_{-1}^1 d\mu \frac{q\mu}{a(t)} P_m(\mu) \sum_{l=0}^{\infty} i^{-l} (2l+1) P_l(\mu) \Delta_{P,l}^{(S)}(q, t) \\ &= - \int_{-1}^1 d\mu \omega_c(t) P_m(\mu) \sum_{l=0}^{\infty} i^{-l} (2l+1) P_l(\mu) \Delta_{P,l}^{(S)}(q, t) + \frac{3\omega_c(t)}{4} \int_{-1}^1 d\mu P_m(\mu) (1 - \mu^2) \Pi(q, t). \end{aligned} \quad (4.70)$$

Considering the formula about Legendre function:

$$(2l+1)\mu P_l(\mu) = (l+1)P_{l+1}(\mu) + lP_{l-1}(\mu), \quad (4.71)$$

$$\int_{-1}^1 P_l(\mu) P_m(\mu) d\mu = \frac{2}{2m+1} \delta_{lm}, \quad (4.72)$$

$$\begin{cases} P_0(\mu) = 1 \\ P_1(\mu) = \mu \\ P_2(\mu) = \frac{1}{2}(3\mu^2 - 1) \end{cases}, \quad (4.73)$$

Therefore,

$$(1 - \mu^2) = -\frac{2}{3}P_2(\mu) + \frac{2}{3}P_0(\mu) \quad (4.74)$$

The each terms of Eq. (4.70) are rewritten as,

$$\begin{aligned} & \int_{-1}^1 d\mu P_m(\mu) \sum_{l=0}^{\infty} i^{-l} (2l+1) P_l(\mu) \dot{\Delta}_{P,l}^{(S)}(q, t) = 2i^{-m} \dot{\Delta}_{P,m}^{(S)}(q, t) \\ & \quad i \int_{-1}^1 d\mu \frac{q\mu}{a(t)} P_m(\mu) \sum_{l=0}^{\infty} i^{-l} (2l+1) P_l(\mu) \Delta_{P,l}^{(S)}(q, t) \\ & = \frac{q}{a(t)} \Delta_{P,l}^{(S)}(q, t) \int_{-1}^1 d\mu P_m(\mu) \sum_{l=0}^{\infty} i^{-l+1} [(l+1)P_{l+1}(\mu) + lP_{l-1}(\mu)] \\ & = \frac{q}{a(t)} \Delta_{P,l}^{(S)}(q, t) \sum_{l=0}^{\infty} i^{-l+1} [(l+1) \frac{2}{2m+1} \delta_{m,l+1} + l \frac{2}{2m+1} \delta_{m,l-1}] \\ & = \frac{q}{a(t)} [mi^{-m+2} \frac{2}{2m+1} \Delta_{P,m-1}^{(S)}(q, t) + (m+1)i^{-m} \frac{2}{2m+1} \Delta_{P,m+1}^{(S)}(q, t)] \\ & \int_{-1}^1 d\mu \omega_c(t) P_m(\mu) \sum_{l=0}^{\infty} i^{-l} (2l+1) P_l(\mu) \Delta_{P,l}^{(S)}(q, t) = 2\omega_c(t) i^{-m} \Delta_{P,m}^{(S)}(q, t). \\ & \int_{-1}^1 d\mu P_m(\mu) (1-\mu^2) = \int_{-1}^1 d\mu P_m(\mu) [-\frac{2}{3}P_2(\mu) + \frac{2}{3}P_0(\mu)] = \frac{2}{3} [i^{-m} \frac{2}{5} \delta_{m2} + 2i^{-m} \delta_{m0}] \end{aligned}$$

We used $i^{-m} = 1$ ($m = 0$) or $i^{-m} = -1$ ($m = 2$) in the last line. Returning to Eq. (4.70) and divided by $2i^{-m}$, we obtain

$$\begin{aligned} & \dot{\Delta}_{P,l}^{(S)}(q, t) + \frac{q}{a(t)} \frac{1}{2l+1} [-l \Delta_{P,l-1}^{(S)}(q, t) + (l+1) \Delta_{P,l+1}^{(S)}(q, t)] \\ & = -\omega_c(t) \Delta_{P,l}^{(S)}(q, t) + \frac{1}{2} \left(\delta_{l0} + \frac{\delta_{l2}}{5} \right) \omega_c(t) \Pi(q, t). \end{aligned} \quad (4.75)$$

For the Eq. (4.64), we also obtain

$$\begin{aligned} & \dot{\Delta}_{T,l}^{(S)}(q, t) + \frac{q}{a(t)} \frac{1}{2l+1} [-l \Delta_{T,l-1}^{(S)}(q, t) + (l+1) \Delta_{T,l+1}^{(S)}(q, t)] \\ & = -\omega_c(t) \Delta_{T,l}^{(S)}(q, t) - 2\dot{A}_q(t) \delta_{l0} + 2q^2 \dot{B}_q(t) \left(\frac{\delta_{l0}}{3} - \frac{2\delta_{l2}}{15} \right) \end{aligned}$$

$$+ \omega_c(t)[3\Phi(q, t) + \frac{1}{2}\Pi(q, t)]\delta_{l0} + \frac{1}{10}\omega_c(t)\Pi(q, t)\delta_{l0} - \frac{4}{3}\frac{q}{a(t)}\omega_c(t)\delta u_{Bq}(t)\delta_{l1}. \quad (4.76)$$

Before the first term of the second line and the term of Π are similar as Eq. (4.75). In other terms, we should have uses Eq. (4.72), (4.73) or

$$\begin{cases} \mu^2 = \frac{P_0(\mu)}{3} + 2\frac{P_2(\mu)}{3} \\ \mu = P_1(\mu) \end{cases}.$$

Let's derive the relation between $\Phi(q, t)$, $\Pi(q, t)$ and $\Delta_T^{(S)}(q, \mu, t)$, $\Delta_P^{(S)}(q, \mu, t)$. Using the formula:

$$\frac{1}{4\pi} \int d^2\hat{p} f(\hat{q} \cdot \hat{p}) \hat{p}_i \hat{p}_j = \mathcal{A} \delta_{ij} + \mathcal{B} \hat{q}_i \hat{q}_j, \quad (4.77)$$

$$\ast \begin{cases} \mathcal{A} = \frac{1}{4} \int_{-1}^1 d\mu f(\mu)(1 - \mu^2) = \frac{1}{6} \int_{-1}^1 d\mu f(\mu)[P_0(\mu) - P_2(\mu)] \\ \mathcal{B} = \frac{1}{4} \int_{-1}^1 d\mu f(\mu)(3\mu^2 - 1) = \frac{1}{2} \int_{-1}^1 d\mu f(\mu)P_2(\mu) \end{cases}, \quad (4.78)$$

To prove the Eq. (4.77), (4.78), we should decompose the equation, multiplying by $\hat{q}_i \hat{q}_j$ we find

$$\frac{1}{4\pi} \int d^2\hat{p} f(\mu) \mu^2 = \frac{1}{2} \int_{-1}^1 d\mu f(\mu) \mu^2 = \mathcal{A} + \mathcal{B}, \quad (4.79)$$

or multiplying by δ_{ij} we find

$$\frac{1}{4\pi} \int d^2\hat{p} f(\mu) = \frac{1}{2} \int_{-1}^1 d\mu f(\mu) = 3\mathcal{A} + \mathcal{B}. \quad (4.80)$$

Substituting Eq. (4.51) to Eq. (4.52), we find

$$\begin{aligned} & \int \frac{d^2\hat{p}}{4\pi} \alpha(\mathbf{q}) \left\{ \frac{1}{2} [\Delta_T^{(S)}(q, \hat{q} \cdot \hat{p}, t) - \Delta_P^{(S)}(q, \hat{q} \cdot \hat{p}, t)] (\delta_{ij} - \hat{p}_i \hat{p}_j) \right. \\ & \quad \left. + \Delta_P^{(S)}(q, \hat{q} \cdot \hat{p}, t) \left[\frac{(\hat{q}_i - (\hat{q} \cdot \hat{p}) \hat{p}_i)(\hat{q}_j - (\hat{q} \cdot \hat{p}) \hat{p}_j)}{1 - (\hat{p} \cdot \hat{q})^2} \right] \right\} \\ & = \alpha(\mathbf{q}) \int_{-1}^1 d\mu (\Delta_T^{(S)}(q, \mu, t) - \Delta_P^{(S)}(q, \mu, t)) \left[\left(\frac{1}{6} P_0(\mu) + \frac{1}{12} P_2(\mu) \right) \delta_{ij} - \frac{1}{4} P_2(\mu) \hat{q}_i \hat{q}_j \right] \end{aligned}$$

$$\begin{aligned}
& +\alpha(\mathbf{q}) \int_{-1}^1 d\mu \Delta_P^{(S)}(q, \mu, t) \left[\left(\frac{1}{12} P_0(\mu) + \frac{1}{6} P_2(\mu) \right) \delta_{ij} + \left(\frac{1}{4} P_0(\mu) - \frac{1}{2} P_2(\mu) \right) \hat{q}_i \hat{q}_j \right] \\
& = \alpha(\mathbf{q}) \left[\delta_{ij} \Phi(q, t) + \frac{1}{2} \hat{q}_i \hat{q}_j \Pi(q, t) \right]
\end{aligned}$$

Substituting Eq. (4.67), (4.68) and using the formula (4.72), we find

$$\begin{aligned}
& \alpha(\mathbf{q}) \int_{-1}^1 d\mu \left\{ \sum_{l=0}^{\infty} i^{-l} (2l+1) P_l(\mu) \Delta_{T,l}^{(S)}(q, t) - \sum_{l=0}^{\infty} i^{-l} (2l+1) P_l(\mu) \Delta_{P,l}^{(S)}(q, t) \right\} \\
& \quad \times \left[\left(\frac{1}{6} P_0(\mu) + \frac{1}{12} P_2(\mu) \right) \delta_{ij} - \frac{1}{4} P_2(\mu) \hat{q}_i \hat{q}_j \right] \\
& + \alpha(\mathbf{q}) \int_{-1}^1 d\mu \sum_{l=0}^{\infty} i^{-l} (2l+1) P_l(\mu) \Delta_{P,l}^{(S)}(q, t) \left[\left(\frac{1}{12} P_0(\mu) + \frac{1}{6} P_2(\mu) \right) \delta_{ij} + \left(\frac{1}{4} P_0(\mu) - \frac{1}{2} P_2(\mu) \right) \hat{q}_i \hat{q}_j \right] \\
& = \alpha(\mathbf{q}) \left[\delta_{ij} \frac{1}{6} [2\Delta_{T,0}^{(S)}(q, t) - \Delta_{P,0}^{(S)}(q, t) - \Delta_{T,2}^{(S)}(q, t) - \Delta_{P,2}^{(S)}(q, t)] \right. \\
& \quad \left. + \frac{1}{2} \hat{q}_i \hat{q}_j (\Delta_{P,0}^{(S)}(q, t) + \Delta_{T,2}^{(S)}(q, t) + \Delta_{P,2}^{(S)}(q, t)) \right] \\
& = \alpha(\mathbf{q}) \left[\delta_{ij} \Phi(q, t) + \frac{1}{2} \hat{q}_i \hat{q}_j \Pi(q, t) \right].
\end{aligned}$$

$$\begin{cases} \Phi(q, t) = \frac{1}{6} [2\Delta_{T,0}^{(S)}(q, t) - \Delta_{P,0}^{(S)}(q, t) - \Delta_{T,2}^{(S)}(q, t) - \Delta_{P,2}^{(S)}(q, t)] \\ \Pi(q, t) = \Delta_{P,0}^{(S)}(q, t) + \Delta_{T,2}^{(S)}(q, t) + \Delta_{P,2}^{(S)}(q, t) \end{cases} \quad (4.81)$$

Here, we also write down the useful formula,

$$\frac{1}{4\pi} \int d^2 \hat{p} f(\hat{q} \cdot \hat{p}) \hat{p}_i = \frac{\hat{q}_i}{2} \int_{-1}^1 d\mu f(\mu) P_1(\mu). \quad (4.82)$$

We only find the term which is proportional to q_i , when we Fourier expand, SVT decompose, and take the scalar mode of the equation. Then multiplying $\hat{q}_i$ and comparing the coefficients, we can find the above formula.

Substituting Eq. (4.51) to Eq. (4.47)-(4.48), using $J_l(\mathbf{x}, \mathbf{p}, t) = \Delta_T^{(S)}(\mathbf{x}, \mathbf{p}, t)$ and Eq. (4.76), (4.82), we find

$$\begin{aligned}
\delta T_{\gamma j}^i(\mathbf{x}, t) &= \bar{\rho}_\gamma(t) \int \frac{d^2 \hat{p}}{4\pi} J_{il}(\mathbf{x}, \mathbf{p}, t) \hat{p}_i \hat{p}_j \\
&= \bar{\rho}_\gamma(t) \int d^3 q e^{i\mathbf{q} \cdot \mathbf{x}} \alpha(\mathbf{q}) \int_{-1}^1 d\mu \Delta_T^{(S)}(q, \mu, t) \times \left[\frac{1}{6} (P_0(\mu) - P_2(\mu)) \delta_{ij} + \frac{1}{2} P_2(\mu) \hat{q}_i \hat{q}_j \right]
\end{aligned} \tag{4.83}$$

$$\begin{aligned}
\delta T_{\gamma j}^0 &= a(t) \bar{\rho}_\gamma(t) \int \frac{d^2 \hat{p}}{4\pi} J_{il}(\mathbf{x}, \mathbf{p}, t) \hat{p}_j \\
&= a(t) \bar{\rho}_\gamma(t) \int d^3 q e^{i\mathbf{q} \cdot \mathbf{x}} \alpha(\mathbf{q}) \frac{\hat{q}_j}{2} \int_{-1}^1 d\mu \Delta_T^{(S)}(q, \mu, t) P_1(\mu)
\end{aligned} \tag{4.84}$$

$$\begin{aligned}
\delta T_{\gamma 0}^0 &= -\bar{\rho}_\gamma(t) \int \frac{d^2 \hat{p}}{4\pi} J_{il}(\mathbf{x}, \mathbf{p}, t) \\
&= -\frac{1}{2} \bar{\rho}_\gamma(t) \int d^3 q e^{i\mathbf{q} \cdot \mathbf{x}} \alpha(\mathbf{q}) \int_{-1}^1 d\mu \Delta_T^{(S)}(q, \mu, t) P_0(\mu)
\end{aligned} \tag{4.85}$$

Substituting Eq. (4.67), (4.68) and comparing Eq. (B.35), we find

$$\begin{aligned}
\delta T_{\gamma j}^i &= \bar{\rho}_\gamma(t) \int d^3 q e^{i\mathbf{q} \cdot \mathbf{x}} \alpha(\mathbf{q}) \int_{-1}^1 \sum_{l=0}^{\infty} i^{-l} (2l+1) P_l(\mu) \Delta_{T,l}^{(S)}(q, t) \times \left[\frac{1}{6} (P_0(\mu) - P_2(\mu)) \delta_{ij} + \frac{1}{2} P_2(\mu) \hat{q}_i \hat{q}_j \right] \\
&= \int d^3 q e^{i\mathbf{q} \cdot \mathbf{x}} \alpha(\mathbf{q}) \left[\delta_{ij} \frac{\bar{\rho}_\gamma}{3} (\Delta_{T,0}^{(S)} + \Delta_{T,2}^{(S)}) - \hat{q}_i \hat{q}_j \bar{\rho}_\gamma \Delta_{T,2}^{(S)} \right] \\
&= \int d^3 q e^{i\mathbf{q} \cdot \mathbf{x}} \alpha(\mathbf{q}) \left[\delta_{ij} \delta p_q - \hat{q}_i \hat{q}_j q^2 \pi_{\gamma q}^S \right]. \\
\delta T_{\gamma j}^0 &= a(t) \bar{\rho}_\gamma(t) \int d^3 q e^{i\mathbf{q} \cdot \mathbf{x}} \alpha(\mathbf{q}) \frac{\hat{q}_j}{2} \int_{-1}^1 \sum_{l=0}^{\infty} i^{-l} (2l+1) P_l(\mu) \Delta_{T,l}^{(S)}(q, t) P_1(\mu) \\
&= \int d^3 q e^{i\mathbf{q} \cdot \mathbf{x}} \alpha(\mathbf{q}) i q_j \frac{-3a(t)}{4q} \Delta_{T,1}^{(S)} \frac{4}{3} \bar{\rho}_\gamma \\
&= \int d^3 q e^{i\mathbf{q} \cdot \mathbf{x}} \alpha(\mathbf{q}) i q_j \delta u_q \frac{4}{3} \bar{\rho}_\gamma. \\
\delta T_{\gamma 0}^0 &= -\frac{1}{2} \bar{\rho}_\gamma(t) \int d^3 q e^{i\mathbf{q} \cdot \mathbf{x}} \alpha(\mathbf{q}) \int_{-1}^1 \sum_{l=0}^{\infty} i^{-l} (2l+1) P_l(\mu) \Delta_{T,l}^{(S)}(q, t) P_0(\mu)
\end{aligned}$$

$$\begin{aligned}
&= - \int d^3q e^{i\mathbf{q}\cdot\mathbf{x}} \alpha(\mathbf{q}) \bar{\rho}_\gamma \Delta_{T,0}^{(S)} \\
&= \int d^3q e^{i\mathbf{q}\cdot\mathbf{x}} \alpha(\mathbf{q}) (-\delta\rho_{\gamma q}). \\
&\begin{cases} \delta p_{\gamma q} = \frac{\bar{\rho}_\gamma}{3} (\Delta_{T,0}^{(S)} + \Delta_{T,2}^{(S)}) \\ q^2 \pi_{\gamma q}^S = \bar{\rho}_\gamma \Delta_{T,2}^{(S)} \\ \delta\rho_{\gamma q} = \bar{\rho}_\gamma \Delta_{T,0}^{(S)} \\ \frac{q}{a(t)} \delta u_{\gamma q} = -\frac{3}{4} \Delta_{T,1}^{(S)} \end{cases}. \tag{4.86}
\end{aligned}$$

These results lead to $\delta T_{\gamma\lambda}^\lambda = 0$, considering $\delta T_\lambda^\lambda = 3\delta p - \delta\rho + \nabla^2 \pi^S$, which is the last term of Eq. (B.35).

4.5 Neutrino

The number density of neutrinos is given by,

$$n_\nu(\mathbf{x}, \mathbf{p}, t) = \bar{n}_\nu(a(t)p^0(\mathbf{x}, \mathbf{p}, t)) + \delta n_\nu(\mathbf{x}, \mathbf{p}, t), \tag{4.87}$$

$$\bar{n}_\nu(p) = \frac{1}{(2\pi)^3} \frac{1}{\exp(\frac{p}{k_B a(t) T(t)}) + 1}. \tag{4.88}$$

Here, p^0 is given by $p^0(\mathbf{x}, \mathbf{p}, t) = \sqrt{g^{ij}(\mathbf{x}, t)p_i p_j}$. The perturbation Boltzmann equation is given by

$$\frac{\partial \delta n_\nu(\mathbf{x}, \mathbf{p}, t)}{\partial t} + \frac{\hat{p}_i}{a(t)} \frac{\partial \delta n_\nu(\mathbf{x}, \mathbf{p}, t)}{\partial x^i} = \frac{p \bar{n}'_\nu(p)}{2} \hat{p}_i \hat{p}_j \frac{\partial}{\partial t} \left(\frac{\delta g_{ij}(\mathbf{x}, t)}{a^2(t)} \right), \tag{4.89}$$

where $\bar{n}'_\nu(p) = \frac{\partial \bar{n}_\nu(p)}{\partial p}$.

As with the case of photons, we define the intensity function. However, for neutrinos, we consider the number of neutrino species as N_ν , and neutrinos and antineutrinos are counted separately,

$$a^4(t) \bar{\rho}_\nu(t) J(\mathbf{x}, \hat{p}, t) \equiv N_\nu \int_0^\infty \delta n_\nu(\mathbf{x}, \mathbf{p}, t) 4\pi p^3 dp. \tag{4.90}$$

$$\bar{\rho}_\nu(t) \equiv \frac{N_\nu}{a^4(t)} \int 4\pi p^3 \bar{n}_\nu(p) dp \tag{4.91}$$

Then, we rewrite the energyf momentum tensor as,

$$\delta T_{\nu j}^i = \bar{\rho}_\nu(t) \int \frac{d^2 \hat{p}}{4\pi} J(\mathbf{x}, \hat{p}, t) \hat{p}_i \hat{p}_j, \quad (4.92)$$

$$\delta T_{\nu j}^0 = a(t) \bar{\rho}_\nu(t) \int \frac{d^2 \hat{p}}{4\pi} J(\mathbf{x}, \hat{p}, t) \hat{p}_j, \quad (4.93)$$

$$\delta T_{\nu 0}^0 = -\bar{\rho}_\nu(t) \int \frac{d^2 \hat{p}}{4\pi} J(\mathbf{x}, \hat{p}, t). \quad (4.94)$$

When expressing the intensity function as a Fourier integral, unlike in the case of photons, it is a scalar rather than a tensor, we obtain

$$J(\mathbf{x}, \hat{p}, t) = \int d^3 q \alpha(\mathbf{q}) \Delta_\nu^{(S)}(q, \hat{q} \cdot \hat{p}, t) e^{i\mathbf{q} \cdot \mathbf{x}}. \quad (4.95)$$

Multiplying it by $4\pi p^3$ and integrating over p , we obtain the equation that the value of $\Delta_\nu^{(S)}(q, \hat{q} \cdot \hat{p}, t)$ should satisfy as,

$$\frac{\partial \Delta_\nu^{(S)}(q, \hat{q} \cdot \hat{p}, t)}{\partial t} + i \frac{q\mu}{a(t)} \Delta_\nu^{(S)}(q, \hat{q} \cdot \hat{p}, t) = -2\dot{A}_q(t) + 2q^2 \mu^2 \dot{B}_q(t). \quad (4.96)$$

For $\Delta_\nu^{(S)}(q, \hat{q} \cdot \hat{p}, t)$, we find,

$$\Delta_\nu^{(S)}(q, \mu, t) = \sum_{l=0}^{\infty} i^{-l} (2l+1) P_l(\mu) \Delta_{\nu, l}^{(S)}(q, t), \quad (4.97)$$

where $\mu = \hat{q} \cdot \hat{p}$. Then we achieve the following equations:

$$\begin{aligned} & \dot{\Delta}_{\nu, l}^{(S)}(q, t) + \frac{q}{a(t)} \frac{1}{2l+1} [-l \Delta_{\nu, l-1}^{(S)}(q, t) + (l+1) \Delta_{\nu, l+1}^{(S)}(q, t)] \\ &= -2\dot{A}_q(t) \delta_{l0} + 2q^2 \dot{B}_q(t) \left(\frac{\delta_{l0}}{3} - \frac{2\delta_{l2}}{15} \right), \end{aligned} \quad (4.98)$$

$$\begin{cases} \delta p_{\nu q} = \frac{\bar{\rho}_\nu}{3} (\Delta_{\nu, 0}^{(S)} + \Delta_{\nu, 2}^{(S)}) \\ q^2 \pi_{\nu q}^S = \bar{\rho}_\nu \Delta_{\nu, 2}^{(S)} \\ \delta \rho_{\nu q} = \bar{\rho}_\nu \Delta_{\nu, 0}^{(S)} \\ \frac{q}{a(t)} \delta u_{\nu q} = -\frac{3}{4} \Delta_{\nu, 1}^{(S)} \end{cases}. \quad (4.99)$$

Gravity

Finally we should write down the equation that describes the time evolution of $\dot{A}$ and $\dot{B}$. From the Eq. (B.77), using Eq. (4.36):

$$\psi_q \equiv \frac{3\dot{A}_q - q^2\dot{B}_q}{2},$$

we obtain

$$-4\pi G a^2 (\delta\rho_{Dq} + \delta\rho_{Bq} + \delta\rho_{\gamma q} + \delta\rho_{\nu q} + 3\delta p_{\gamma q} + 3\delta p_{\nu q} - q^2\pi_{\gamma q}^S - q^2\pi_{\nu q}^S) = \frac{\partial}{\partial t} [a^2\psi_q]. \quad (4.100)$$

To solve only $\dot{A}$, rewriting Eq. (B.76), we should use

$$8\pi G \left(\frac{4}{3}\bar{\rho}_\gamma \delta u_{\gamma q} + \frac{4}{3}\bar{\rho}_\nu \delta u_{\nu q} + \bar{\rho}_B \delta u_{Bq} \right) = \dot{A}_q \quad (4.101)$$

and solve $\dot{A}$. Then we can also find $\dot{B}$ from Eq. (4.36).

Initial condition

In order to solve differential equations, initial values should be determined.

For the neutrino we set the initial time t_1 when the temperature is $\bar{T}_1 = 10^9(K)$ and the initial condition is given as,

$$\delta n_\nu(\mathbf{x}, \mathbf{p}, t_1) = -p \frac{d\bar{n}_\nu(p)}{dp} \left[\frac{\delta T(\mathbf{x}, t_1)}{\bar{T}} + \frac{\hat{p}_k \cdot \delta u_{Bk}(\mathbf{x}, t_1)}{a(t_1)} \right]. \quad (4.102)$$

Neutrinos maintained equilibrium through weak interactions, but they decoupled from equilibrium at extremely high temperatures ($T_{de} \sim 10^{10}K$). It was necessary to choose the temperature shortly after decoupling, not too far from T_{de} , as the initial value, so $\bar{T}_1 = 10^9(K)$ was selected.

We can do a similar discussion for photons, and it is sufficient to set the initial time t_1 before the time of recombination. For instance, setting $\bar{T}_1 > 10^5(K)$ will work (cf. Saha's equation). Following the same approach as with neutrinos, we apply a correction for the term where multiplying g_{ij} perpendicular to p_i results in 1.

$$\delta n_\gamma^{ij}(\mathbf{x}, \mathbf{p}, t_1) = -\frac{1}{2} \frac{1}{a^2(t)} (\delta_{ij} - \hat{p}_i \hat{p}_j) p \frac{d\bar{n}_\gamma(p)}{dp} \left[\frac{\delta T(\mathbf{x}, t_1)}{\bar{T}} + \frac{\hat{p}_k \cdot \delta u_{Bk}(\mathbf{x}, t_1)}{a(t_1)} \right]. \quad (4.103)$$

Multiplying this by $4\pi p^3$ and integrating over p , from Eq. (4.45), we find the initial condition of J_{ij} ,

$$\begin{aligned}
J_{ij}(\mathbf{x}, \hat{p}, t_1) &= \frac{a^2(t_1)}{a^4(t_1)\bar{\rho}_\gamma(t_1)} \int_0^\infty \delta n_\gamma^{ij}(\mathbf{x}, \mathbf{p}, t_1) 4\pi p^3 dp \\
&= -\frac{1}{2a^4(t_1)} \frac{1}{\bar{\rho}_\gamma(t_1)} \int_0^\infty (\delta_{ij} - \hat{p}_i \hat{p}_j) p \frac{d\bar{n}_\nu(p)}{dp} \left[\frac{\delta T(\mathbf{x}, t_1)}{\bar{T}} + \frac{\hat{p}_k \cdot \delta u_{Bk}(\mathbf{x}, t_1)}{a(t_1)} \right] 4\pi p^3 dp \\
&= 2(\delta_{ij} - \hat{p}_i \hat{p}_j) \left[\frac{\delta T(\mathbf{x}, t_1)}{\bar{T}} + \frac{\hat{p}_k \cdot \delta u_{Bk}(\mathbf{x}, t_1)}{a(t_1)} \right]. \tag{4.104}
\end{aligned}$$

In the last equuall, we have used Eq. (4.56).

4.6 multipole expansion of the correlation function of temperature fluctuations

In this section, we will specifically calculate the multipole expansion of the correlation function of temperature fluctuations due to scalar-type perturbations. We give the temperature of the background radiation in the direction of the unit vector $\hat{\mathbf{n}}$ as $T(\hat{\mathbf{n}})$, the angular mean temperature as,

$$T_0 = \frac{1}{4\pi} \int d^2\hat{\mathbf{n}} T(\hat{\mathbf{n}}),$$

and its fluctuation as

$$\Delta T(\hat{\mathbf{n}}) = T(\hat{\mathbf{n}}) - T_0. \tag{4.105}$$

Particularly, we will calculate multiple expansion coefficients for temperature fluctuation angle correlation functions,

$$C_{TT,l} = \frac{1}{4\pi} \int d^2\hat{\mathbf{n}} \int d^2\hat{\mathbf{n}}' P_l(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}') < \Delta T(\mathbf{n}) \Delta T(\hat{\mathbf{n}}') >. \tag{4.106}$$

The $<>$ stands for the mean value.

Considering the Eq. (4.45), which is the definition of J_{ij} ,

$$\begin{aligned}
\int 4\pi p^3 \bar{n}_\gamma(p) dp &= \bar{T}_{\gamma 0}^0 \propto T_0^4 \\
a^2(t) \int 4\pi p^3 \delta n_\gamma^{ii}(\mathbf{x}, \mathbf{p}, t) dp &= \delta T_{\gamma 0}^0 \propto 4T_0^3 \Delta T(\mathbf{n})
\end{aligned}$$

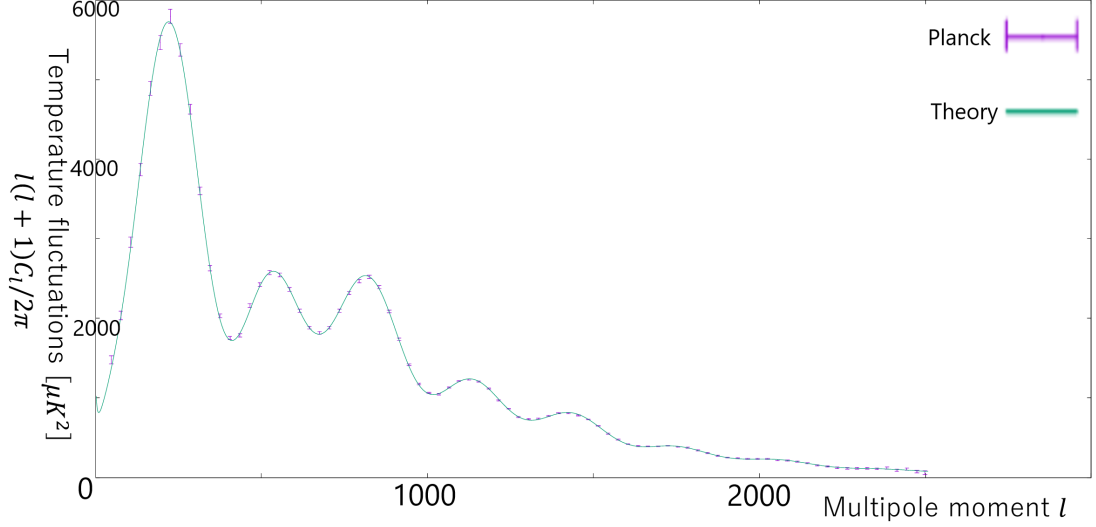


Figure 4.1: The temperature fluctuations $l(l+1)C_l/2\pi(\mu K^2)$ as a function of a multipole moment l . The green curve shows the theoretical prediction and the purple error bars show the binned data from Planck [1]. This figure is illustrated by CAMB [72].

At time t_0 and position $\mathbf{x} = 0$, we write the difference between the temperature of the CMB coming from the direction of $\mathbf{n}$ and the mean temperature T_0 as $\Delta T(\mathbf{n})$. Since the direction of motion of the photon is $-\mathbf{n}$,

$$T_0^4 J_{ii}(\mathbf{x} = 0, -\mathbf{n}, t_0) = 4T_0^3 \Delta T(\mathbf{n}).$$

Then, from Eqs. (4.50), (4.51), we obtain

$$\frac{\Delta T(\mathbf{n})}{T_0} = \frac{1}{4} J_{ii}(0, -\mathbf{n}, t_0) \quad (4.107)$$

$$= \frac{1}{4} \int d^3 q \alpha(\mathbf{q}) \Delta_T^{(S)}(q, -\hat{\mathbf{q}} \cdot \hat{\mathbf{n}}, t_0) \quad (4.108)$$

The result of the numerical solution to the power spectrum is given in the Fig. 4.1. This figure shows the peaks and the dampings in the CMB power spectrum.

sound horizon

The positions of the CMB power spectrum is depend on the length of the sound horizon at the last scattering

$$r_s(z_*) \equiv \int_0^{t(z_*)} \frac{c_s d\tilde{t}}{a(\tilde{t})} = \int_{z_*}^{\infty} \frac{c_s dz}{H(z)}. \quad (4.109)$$

Here, z_* is the redshifts of the last scattering, and c_s is the speed of sound of the fluctuations of the photon baryon plasma fluid

$$c_s = \sqrt{\frac{1}{3(1 + R_S)}}, \quad R_S = \frac{3\rho_b}{4\rho_\gamma}. \quad (4.110)$$

The angular diameter distance to the lastscattering surface is given by

$$D_M(z) \equiv \int_{t(z)}^0 \frac{d\tilde{t}}{a(\tilde{t})}. \quad (4.111)$$

Therefore, using the phase corrections ϕ_i [73], the peak locations and corresponding angle are given by

$$l_n = \frac{\pi}{\theta_A} (n - \phi_n), \quad (4.112)$$

$$\theta_A = \frac{r_s(z_*)}{D_M(z_*)}. \quad (4.113)$$

damping

Fig. 4.1 shows that as the angular wavenumber l increases, the amplitude decreases. This is due to two effects: Silk damping, which suppresses shorter-wavelength fluctuations due to the free streaming of photons, and Landau damping, which dissipates photon momentum into baryons. These effects are approximately evaluated as,

$$\frac{l(l+1)C_{TT,l}}{2\pi} \sim \exp(-l^2 d_D^2 / D_M^2(z_*)), \quad (4.114)$$

$$d_D^2 = d_{\text{sil}}^2 + d_{\text{Landau}}^2, \quad (4.115)$$

and the the silk damping length and the Landau damping length are given by

$$d_{\text{silk}}^2 = a_L^2 \int_0^{t_L} \frac{t_\gamma}{6a^2(1+R)} \left\{ \frac{16}{15} + \frac{R^2}{(1+R)} \right\} dt \quad (4.116)$$

$$d_{\text{Landau}}^2 = \frac{\sigma_t^2}{6(1+R_S(t_*))}, \quad (4.117)$$

where t_γ is the photon mean free time.

Part II

early dark energy and extra radiation

Chapter 5

model

Before we discuss the varying fundamental constant, we treat early dark energy and extra radiation, which are more simple models to alleviate the Hubble tension.

As we explain in the Sec. 2, the expansion rate of our Universe, which is called the Hubble parameter, is defined as

$$H(t) = \frac{\dot{a}}{a}, \quad (5.1)$$

where $a(t)$ is the scale factor and dot denotes the derivative with respect to the cosmic time t . In this section, we use the scale factor a instead of time t as a “time” variable. Then, we regard the Hubble parameter as a function of a as $H(a)$ and normalize the scale factor as $a(t_0) = a_0 = 1$ with t_0 being the age of the Universe.

5.1 Extra radiation

Increasing the effective number of neutrinos N_{eff} is one of the simple model to alleviate the Hubble tension. This model use the one additional parameter N_{eff} , which is expressed as

$$\Omega_r = \left(1 + \frac{7}{8} \left(\frac{4}{11} \right)^{4/3} N_{\text{eff}} \right) \Omega_\gamma. \quad (5.2)$$

Here,

$$\Omega_i = \frac{\rho_i}{3M_P^2 H^2} \Big|_{t=t_0}, \quad (5.3)$$

with M_P being the reduced Planck mass, are the present values of density parameters for i species. γ and ν stand for CMB photon and neutrino, respectively. The factor $\frac{7}{8} \left(\frac{4}{11}\right)^{4/3}$ is coming from the energy fraction between photon and neutrino, as explained in Eq. (4.22). In the following, we call this model N_{eff} model or extra radiation model.

We note that a large ΔN_{eff} to alleviate the Hubble tension can be easily compatible with Big Bang Nucleosynthesis if there is large lepton asymmetry [74, 75, 76, 77] or assuming the ΔN_{eff} production after BBN [78].

5.2 Early dark energy

In literature, the EDE scenarios are modeled by variety of functional forms of scalar field such as axion potential-like [79, 36, 37], polynomial [39], α -attractor-like [38] and others [42, 41, 42, 43, 44, 40, 80, 81]. These properties and differences due to a variety of potential forms have been studied in these literature. In this work, we propose a more generalized early dark energy model that has two parameters. In order to consider a time dependent dark energy and examine the early dark energy, we give the dark energy equation of state $w(a) = p_{DE}(a)/\rho_{DE}(a)$ as a function of the cosmic scale. By integrating the continuity equation

$$\dot{\rho}_{DE} + 3H\rho_{DE}(1 + w(a)) = 0, \quad (5.4)$$

we obtain [82]

$$\rho_{DE}(a) = \rho_{DE}(a_0) \exp \left(- \int_{a_0}^a 3(1 + w(\hat{a})) \frac{d\hat{a}}{\hat{a}} \right). \quad (5.5)$$

We find the cosmic expansion by solving the Friedman equation for the EDE models

$$\frac{H^2(a)}{H_0^2} = (\Omega_c + \Omega_b)a^{-3} + \Omega_r a^{-4} + \Omega_{DE}(a), \quad (5.6)$$

$$\Omega_{DE}(a) = \Omega_{DE}(a_0) \exp \left(- \int_{a_0}^a 3(1 + w(\hat{a})) \frac{d\hat{a}}{\hat{a}} \right). \quad (5.7)$$

Indices of each Ω parameter, c, b, r, DE , stand for cold dark matter, baryons, radiation and dark energy (DE), respectively. Here, DE is the sum of the

EDE component and the cosmological constant Λ which is responsible for the today's accelerating expansion of our universe: $\rho_{DE}(a) = \rho_{EDE}(a) + \rho_\Lambda$.

In the EDE scenarios, the energy component of dark energy becomes subdominant around the moment of the matter-radiation equality, which is surely earlier than the cosmological constant dark energy becomes subdominant. Hence, we call this model “early” dark energy. Particularly, it's interesting to consider the case that EDE contributes to about several percents of the total energy density. After the phase transition time, the EDE component decreases as $\rho_{DE} \propto a^{-n}$ and faster than the background energy densities do. Here, we introduce two parameters a_c and a_m . One parameter a_c is the scale factor at the moment when the EDE start to decrease like radiation ($n = 4$) or kination ($n = 6$). The other parameter a_m is the scale factor at the moment when the EDE component gets as small as the cosmological constant. Phenomenologically, we parameterize the equation of state $w(a)$ as

$$w_n(a) = -1 + \frac{n}{3} \frac{(a_c^n + a_m^n)a^n}{(a_c^n + a^n)(2a_c^n + a_m^n + a^n)}, \quad (5.8)$$

and the resultant Ω parameter is given by

$$\Omega_{DEn}(a) = \Omega_\Lambda \frac{a^n + 2a_c^n + a_m^n}{a^n + a_c^n}. \quad (5.9)$$

For $a < a_c$ and $a_m < a$, the dark energy behaves like a cosmological constant. Typical evolution of $w_n(a)$ and $\rho_{DEn}(a)$ normalized by ρ_Λ are shown in Fig. 5.1. We note the above $\Omega_{DEn}(a)$ is proportional to the f_{EDE} parameter often used in literature as

$$f_{EDE}(a) = \Omega_{DEn}(a) \frac{H_0^2}{H^2(a)}, \quad (5.10)$$

and we show the typical evolutions of $f_{EDE}(a)$ in Fig. 5.2.

Since Eq. (5.9) can be approximated as

$$\Omega_{DEn}(a)|_{a=1} \simeq \Omega_\Lambda(1 + a_m^n) = \Omega_\Lambda + \Omega_{EDE}, \quad (5.11)$$

we will use $\Omega_{EDE}/\Omega_\Lambda$ as a variable parameter instead of a_m .

In the case we assume $n = 4$, which case is expressed as EDE4 hereafter, we fix $a_c = 10^{-4}$ and we sample $\Omega_{EDE}/\Omega_\Lambda \in [1 \times 10^{-7}, 1 \times 10^{-5}]$. In the case we assume $n = 6$, which case is expressed as EDE6 hereafter, we fix $a_c = 3 \times 10^{-4}$ and we sample $\Omega_{EDE}/\Omega_\Lambda \in [1 \times 10^{-14}, 5 \times 10^{-12}]$. Those values of a_c are motivated by the previous work [37].

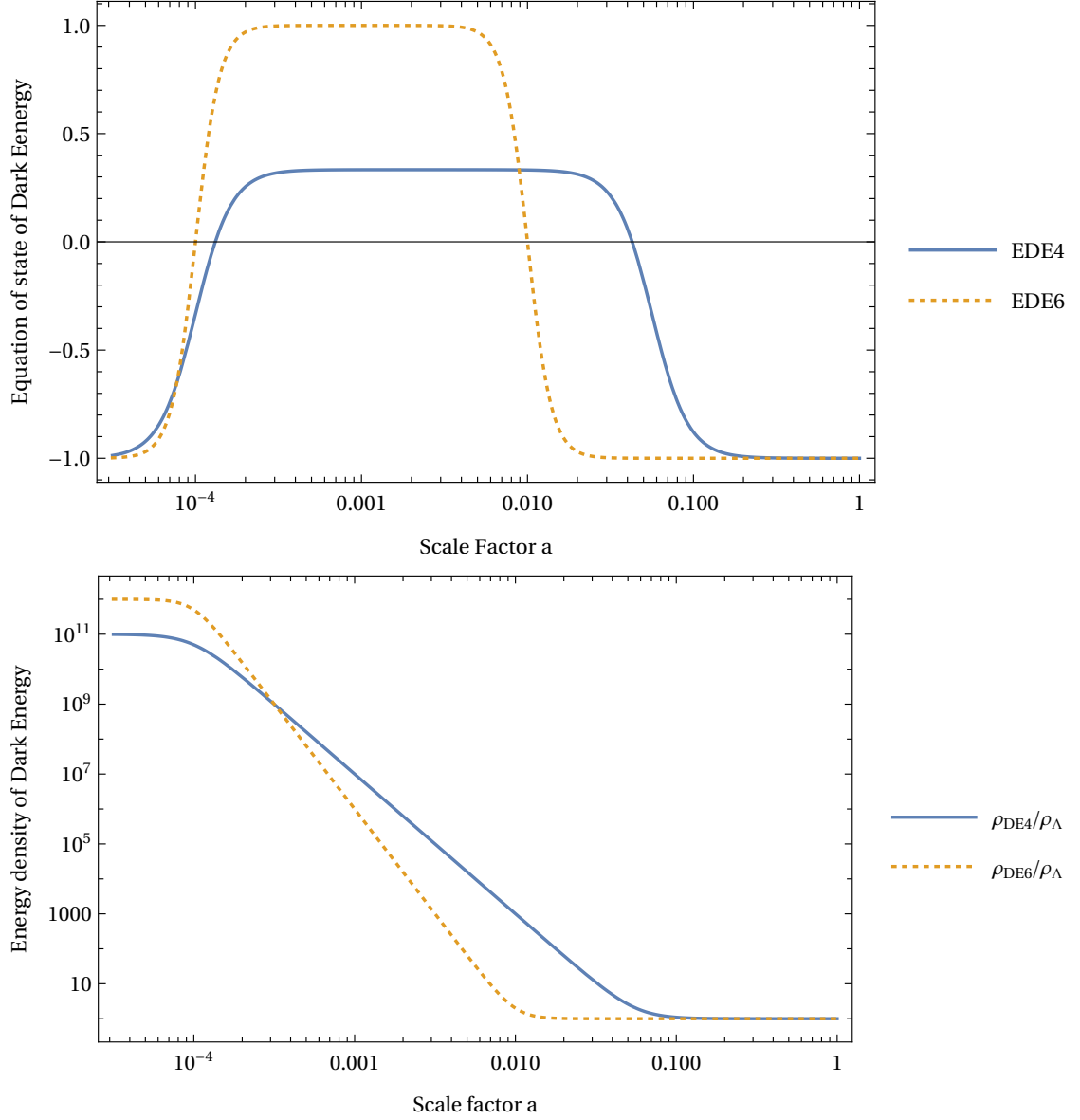


Figure 5.1: We plot the evolution of the equation of state parameter w_n and $\rho_{DEn}(a)/\rho_\Lambda$ for $n = 4$ and 6 . For definiteness, in these plots, we take $a_c = 10^{-4}$, $a_m = 10^{-5/4}$ for $n = 4$ and $a_c = 10^{-4}$, $a_m = 10^{-2}$ for $n = 6$. We can confirm that ρ and w behave like radiation or kination for $a_c < a < a_m$ and like a cosmological constant for $a < a_c$ and $a_m < a$.

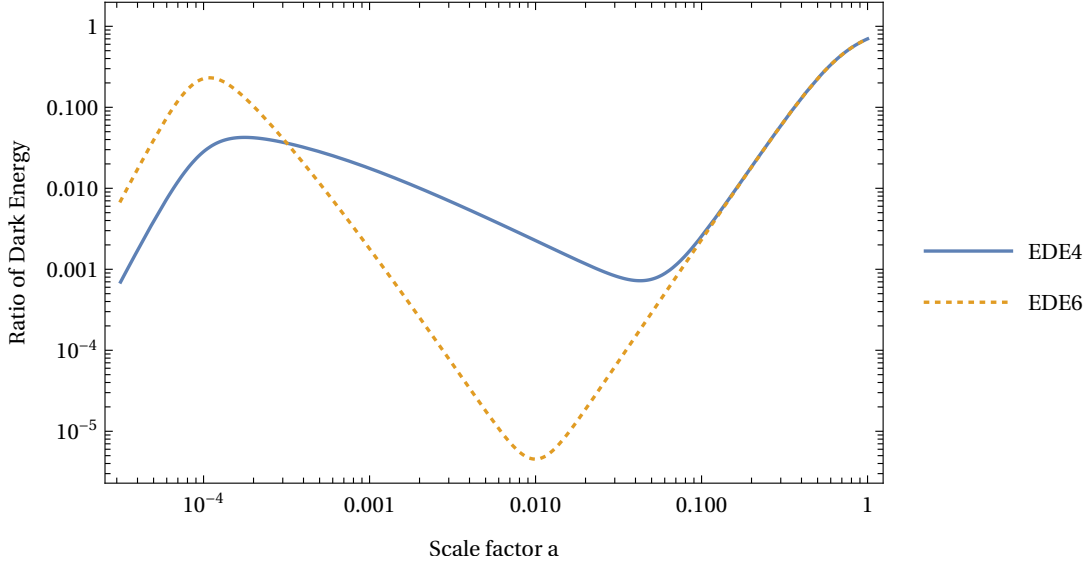


Figure 5.2: We plot $f_{\text{EDE}}(a)$. The choice of a_c and a_m are same as in Fig. 5.1.

5.3 axionlike EDE

Beside the generalized EDE, which we explained in the previous subsection, we also examine the axionlike EDE [36], which could explain the cosmic birefringence [47, 48] suggested by the recent analyses of the Planck data [49, 50].

The potential of the axionlike EDE takes the form [36]

$$V(\phi) = \Lambda^4 \left(1 - \cos \left(\frac{\phi}{f} \right) \right)^n, \quad (5.12)$$

where Λ is the energy scale of the potential, f is the breaking scale of the shift symmetry, and n is the power index of the cosine function. As in the previous paper by Poulin et al [37], we use the three phenomenological parameters: z_c , Θ_i , and $f_{\text{de}}(z_c) \equiv \frac{\rho_{\text{de}}(z_c)}{\rho_{\text{tot}}(z_c)}$, which stand for the redshift when ϕ starts to oscillate, the initial value of the scalar field ϕ/f , and the energy fraction of EDE to the total energy density ρ_{tot} at z_c . After the transition $z < z_c$, the energy density of EDE decreases as $\rho_{\text{de}} \propto a^{-4}$ for $n = 2$, which time dependence is similar to the EDE4 model.¹

¹Dark radiation, which also decreases as $\rho_{\text{de}} \propto a^{-4}$ is discussed in Ref. [78], and they

We should note that EDE can worsen the S_8 tension, as is discussed in the previous work [37]. Since we should keep the value of $\theta_d \propto r_d h$, there is a degeneracy between r_d and H_0 . The only way to increase the Hubble constant H_0 keeping both the degeneracy and the CMB fit is increasing the matter density $\omega_m \equiv \Omega_m h^2$ which leads to a larger S_8 , as is illustrated in Ref. [14].

To examine the axionlike EDE, we use the CAMB [72] where axionlike EDE is already implemented and perform the MCMC analysis, sampling $f_{\text{de}}(z_c) \in [0.00001, 0.15]$, $z_c \in [1000, 50000]$, and $\Theta_i \equiv \phi_{\text{ini}}/f \in [0.01, 3.14]$ in addition to the 6 standard parameters.

report that a larger amount of dark radiation ΔN_{eff} and a higher Hubble constant H_0 is favored, when considering DESI BAO data.

Chapter 6

datasets and methodology

In our study, we perform a MCMC analysis of several models, using the public MCMC code `CosmoMC-planck2018` [83], requiring the convergence $R-1 < 0.03$. We analyze the models by referring to the following cosmological observations:

- CMB from Planck [1]. We use the temperature and polarization likelihoods for high l `plik` ($l = 30$ to 2508 in TT and $l = 30$ to 1997 in EE and TE) and low l `Commander` and lowE `SimAll` ($l = 2$ to 29). We also include CMB lensing [84].
- BAO from 6dF [85], MGS [86], and DR12 [87] (see the left Tab. 6.1 for the detail).
- Light curve of SNeIa from *Pantheon* [88].
- DESI BAO [53] (see the right Tab. 6.1 for the detail).
- SH0ES(R19) [89] of the local measurement of the Hubble constant
- helium mass fraction Y_P measurement [26] and deuterium abundance D/H measurement [27]

z_{eff}		z_{eff}	
6DF [85] 0.106	$r_s/D_V=0.336\pm0.015$	DESI [53] 0.30	$D_V/r_d=7.93\pm0.15$
MGS [86] 0.105	$D_V/r_s=4.466\pm0.168$	0.51	$D_M/r_d=13.62\pm0.25$
DR12 [87] 0.38	$D_M/r_s=1512.39\pm24.99$	0.51	$D_H/r_d=20.98\pm0.61$
0.38	$H(z)r_s=81.2087\pm2.3683$	0.71	$D_M/r_d=16.85\pm0.32$
0.51	$D_M/r_s=1975.22\pm30.10$	0.71	$D_H/r_d=20.08\pm0.60$
0.51	$H(z)r_s=90.9029\pm2.3288$	0.93	$D_M/r_d=21.71\pm0.28$
0.61	$D_M/r_s=2306.68\pm37.08$	0.93	$D_H/r_d=17.88\pm0.35$
0.61	$H(z)r_s=98.9647\pm 2.5019$	1.32	$D_M/r_d=27.79\pm0.69$
		1.32	$D_H/r_d=13.82\pm0.42$
		1.49	$D_V/r_d=26.07\pm0.67$
		2.33	$D_M/r_d=39.71\pm0.94$
		2.33	$D_H/r_d=8.52\pm0.17$

Table 6.1: The distance and expansion rete data from 6DF, MGS, and DR12 (left). The distant data from DESI (right).

Chapter 7

results

7.1 Big Bang Nucleosynthesis

Although we have examined both the EDE4 and the EDE6, we have confirmed that the EDE6 model shows slightly better fit than the EDE4 model, as has been pointed out in previous works. Thus, we show a posterior distribution for only the EDE6 model in Fig. 7.1 and the posterior distribution for an N_{eff} model in Fig. 7.2. In addition to above two models those have been studied in literature, we also consider the model where both the extra radiation and the EDE component exist, hereafter we call a co-existing model or $\text{EDE6} + N_{\text{eff}}$, motivated by the fact that those are in principle independent sectors.

We show, for comparison between models, combined plots of the EDE4, the EDE6, a co-existing model with the EDE6 plus N_{eff} , N_{eff} together with the ΛCDM for reference in Fig. 7.5. This results are also summarized in Tab. 7.1. It is clear that both the EDE6 and a co-existing model similarly prefer a larger value of H_0 than other models do. We could confirm that the EDE4 model shows the most poor improvement for the H_0 tension, as shown in Fig. 7.5.

For reference, we also show the same plot of the analysis without including Pantheon and R19 data [in other words (CMB + BAO + BBN)] in Fig. 7.4. One can confirm that the results for N_{eff} in Fig. 6 almost reproduce Fig. 35 in Ref. [1]. The constraints on EDE in Fig. 7.5 and Fig. 7.4 do not differ significantly. On the other hand, we can find sizable shifts of the posterior and the central values for N_{eff} and the coexistence model, depending on whether

Model	EDE6	EDE6+ N_{eff}	Λ CDM	N_{eff}
$\Omega_{\text{EDE}}/\Omega_{\Lambda}$	$1.00^{+0.45}_{-0.65} \times 10^{-12}$ (8.00×10^{-13})	$0.79^{+0.27}_{-0.71} \times 10^{-12}$ (5.80×10^{-13})	Λ CDM	Λ CDM
N_{eff}	3.046	3.23 ± 0.13 (3.18946)	3.046	3.28 ± 0.12 (3.28)
H_0 [km/s/Mpc]	68.80 ± 0.52 (68.50)	69.73 ± 0.82 (69.42)	68.19 ± 0.39 (68.22)	69.54 ± 0.80 (69.38)
CMB:lensing χ^2	8.83	8.81	8.52	9.17
CMB:TTTEEE χ^2	2350.05	2352	2348.86	2354.72
CMB:low l χ^2	22.04	21.48	22.83	21.93
CMB:lowE χ^2	395.88	397.82	399.52	396.51
Cooke χ^2	0.65	0.18	0.30	0.0037
Aver χ^2	0.23	0.92	0.22	1.56
SH0ES χ^2	15.18	10.56	16.75	10.71
JLA Pantheon18 χ^2	1034.85	1034.74	1034.77	1034.77
BAO χ^2	5.35	5.41	5.24	5.32
prior χ^2	5.89	3.71	4.51	3.30
total χ^2	3838.95	3835.63	3841.52	3838.00

Table 7.1: Constraints (68%) and the best fit values in parentheses on main parameters based on CMB+BAO+Pantheon+R19+BBN. The values of χ^2 in lower rows are for the best value points in each model.

we include R19 data. If one pays attention to only the central values, it looks like the EDE indicates the largest value of H_0 and models with N_{eff} indicate even smaller values than Λ CDM does. However, the constraints of a certain confidence level on the N_{eff} and coexistence models are much weaker than those for the EDE model. In addition, one should be aware of the presence of the prior dependence in Fig. 7.4, where the energy density of EDE is positive definite, while we allow $N_{\text{eff}} < 3.046$ for the models with N_{eff} .

As is well known, increase of N_{eff} affects the fit with the observation of light elements, because it contributes the cosmic expansion at BBN epoch and alters the proton to neutron p/n ratio. This leads to increasing both the helium mass fraction Y_P and the deuterium abundance D/H. By increasing N_{eff} , the CMB fit simultaneously indicates a larger $\Omega_b h^2$ which reduces the D/H. In total, the enhancement of D/H is suppressed. As the result, the χ^2 of Y_P observation, χ^2_{Aver} , increases, while the χ^2 of D/H observation, χ^2_{Cooke} ,

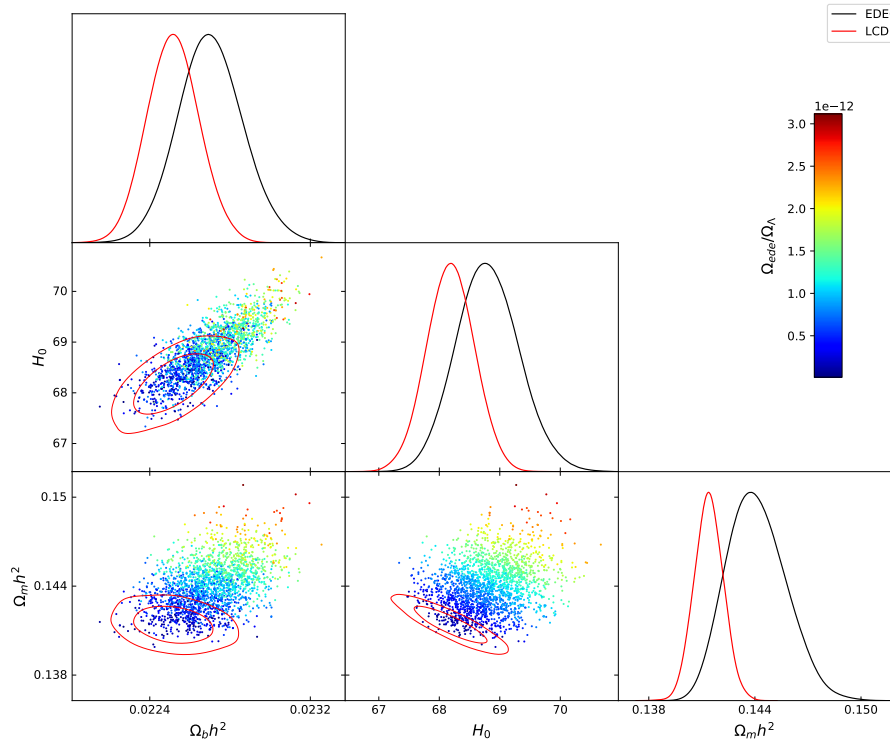
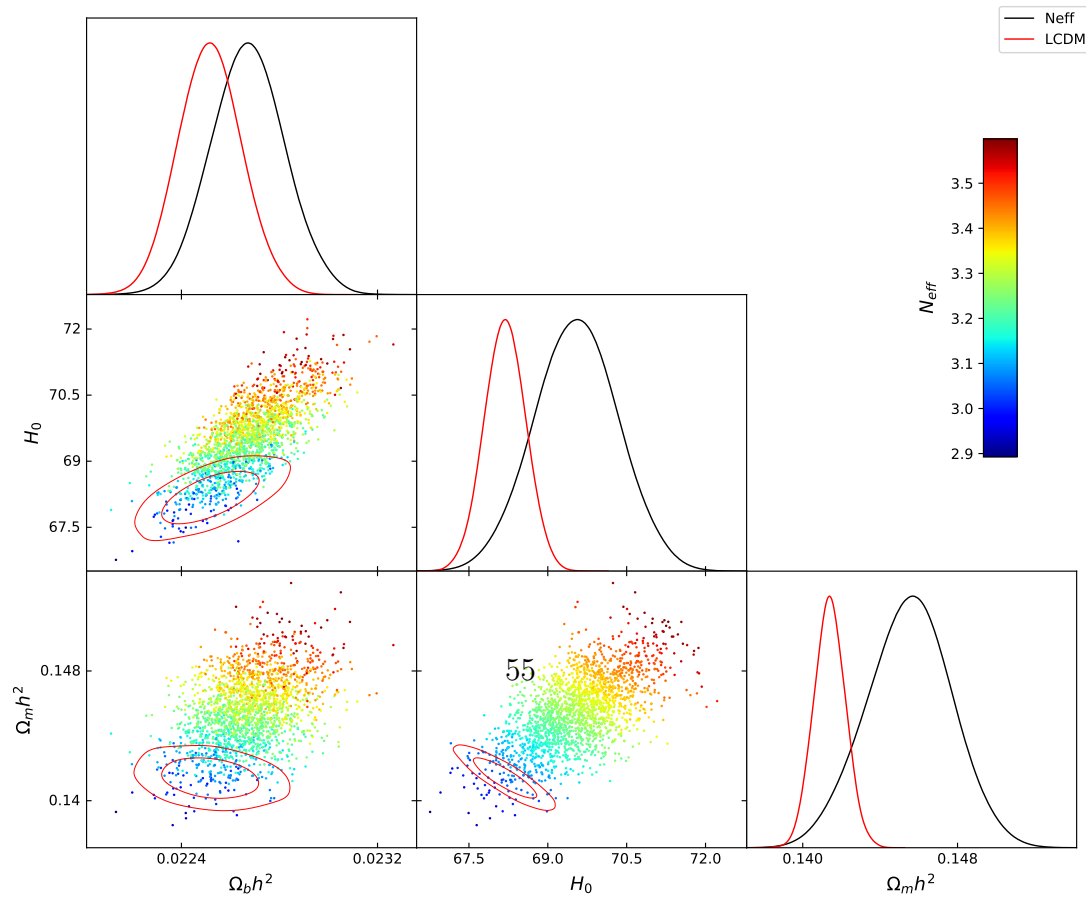


Figure 7.1: The posterior and the constraints on a EDE6 model. Red curves are for the Λ CDM model, for reference.



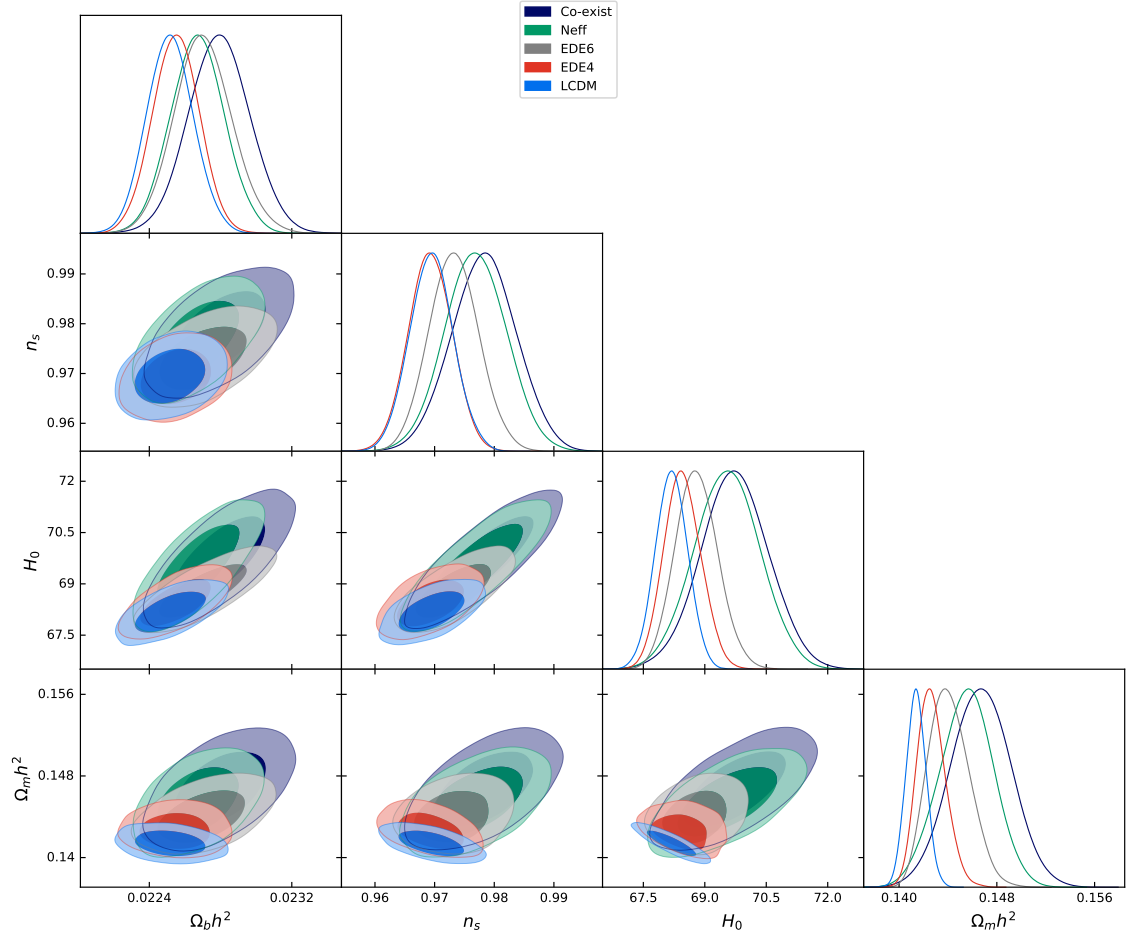


Figure 7.3: The posterior and the constraints of several models.

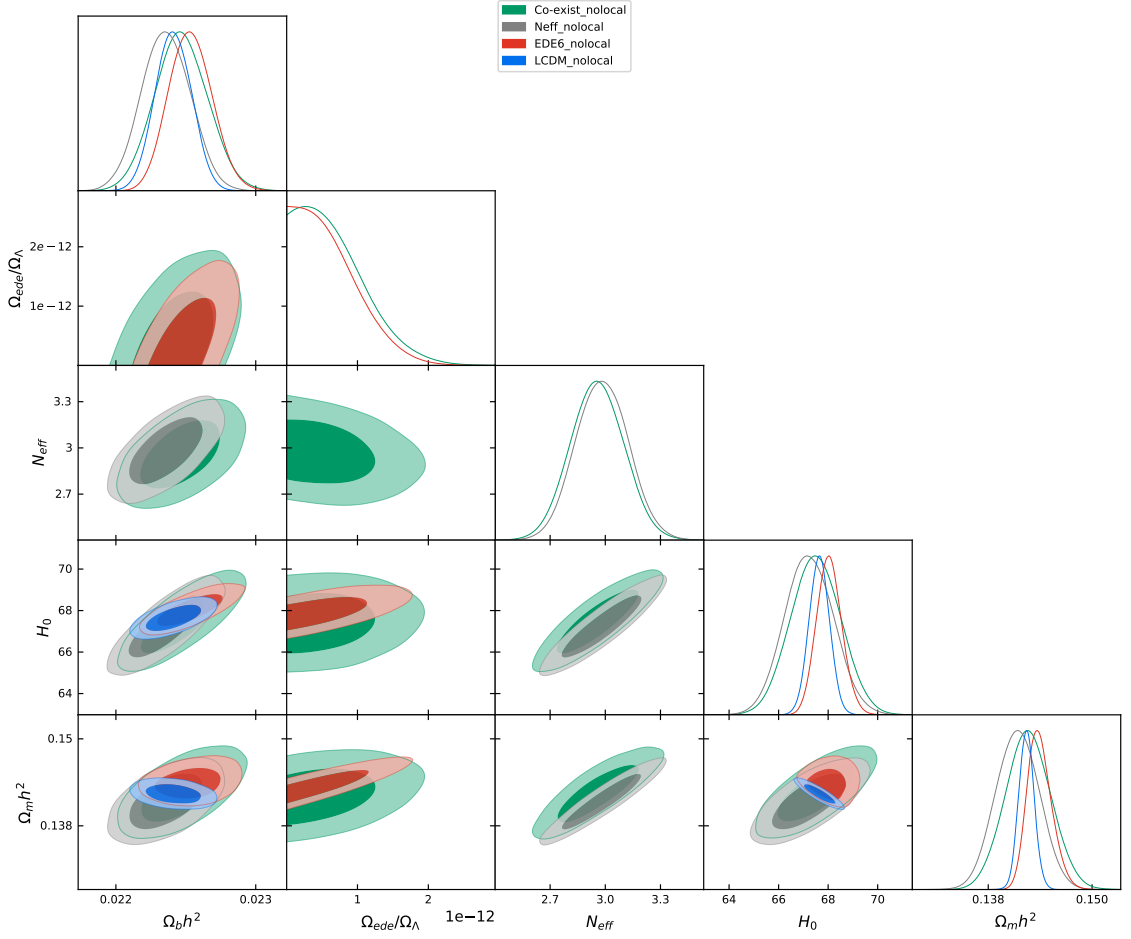


Figure 7.4: The posterior and the constraints of several models without including local data

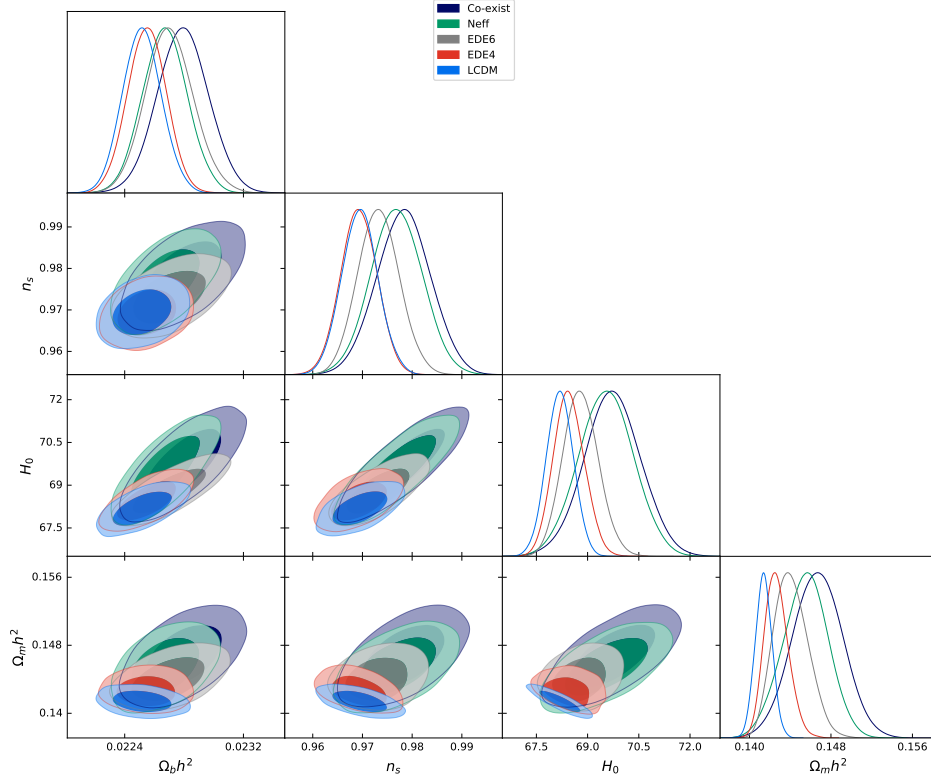


Figure 7.5: The posterior and the constraints of several models.

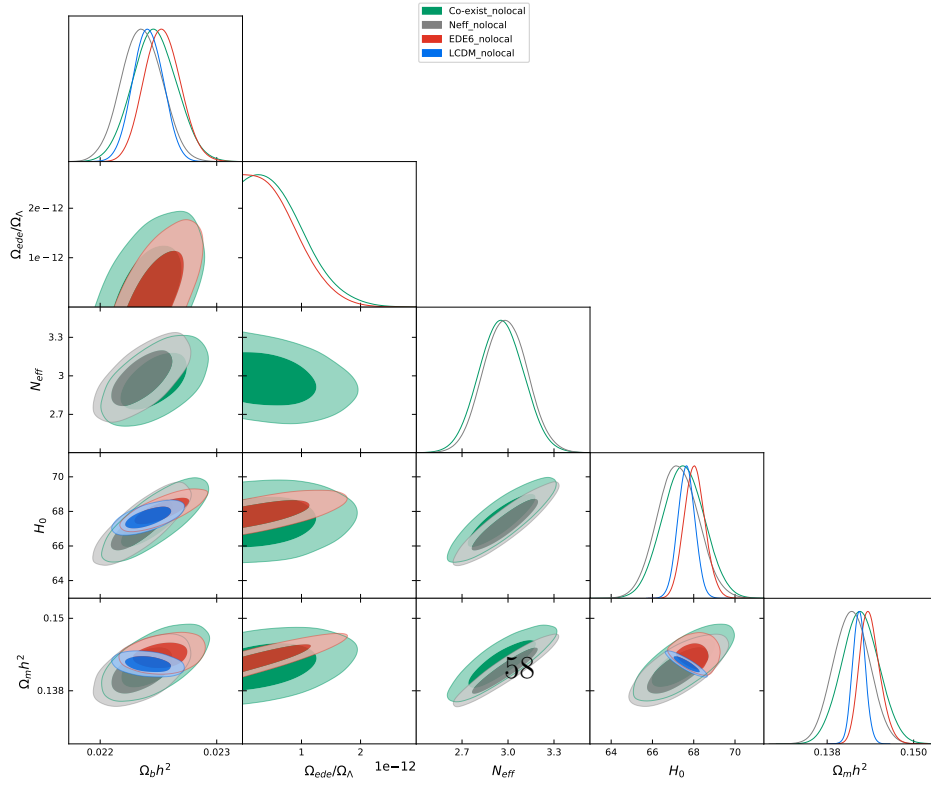


Figure 7.6: The posterior and the constraints of several models without including R19 data.

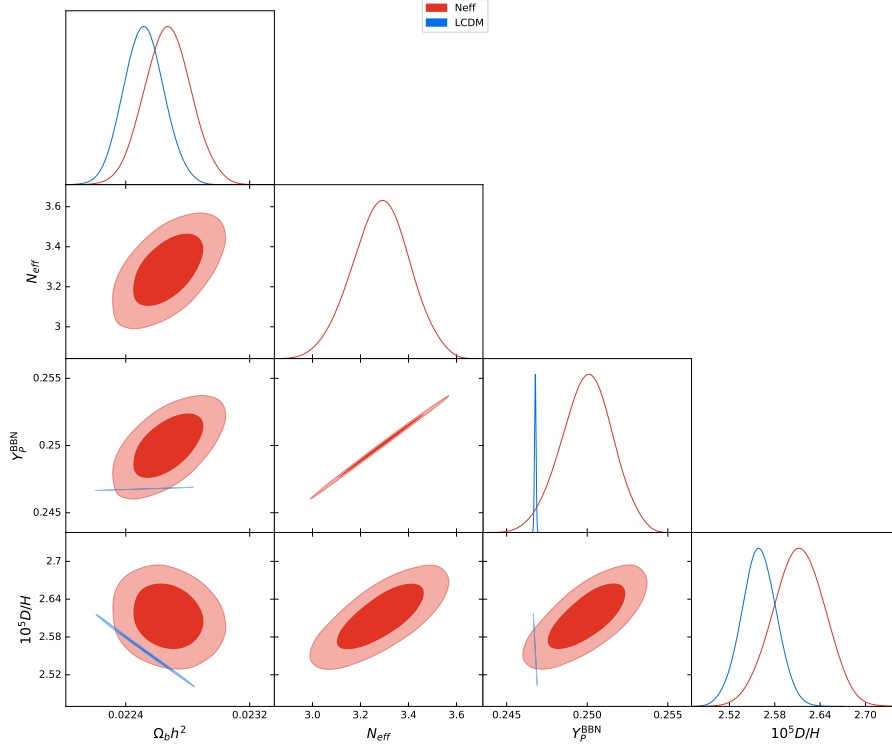


Figure 7.7: The posterior and its BBN dependence in the N_{eff} model.

does a little. In fact, the value of χ_{Cooke}^2 of a N_{eff} model is smaller than that in the ΛCDM model. This can be seen in Fig. 7.7 and Tab. 7.1. On the other hand, in the EDE scenarios, increasing $\Omega_b h^2$ to adjust the CMB fit reduces the D/H abundance significantly. Thus, the χ_{Aver}^2 increases little, while the χ_{Cooke}^2 does. This can be seen in Fig. 7.8 and Tab. 7.1. The trade-off relation between the fit to the helium mass fraction Y_P and the deuterium abundance D/H exists. This relation can be seen in Tab 7.1 more clearly, when we compare with the co-existing model.

The constraints on N_{eff} from BBN and BAO data only without including CMB nor SH0SE has been derived as $N_{\text{eff}} = 2.88 \pm 0.16$ which is less than 3 [35]. As is well known, a larger N_{eff} is disfavored from BBN. This is consistent with results in Tab. 7.1. What we additionally found is that the EDE models without ΔN_{eff} is also limited from BBN, because too large $\Omega_b h^2$ inferred by CMB gives rises to too little D/H abundance.

In literature on the new physics interpretation of the Hubble tension, data

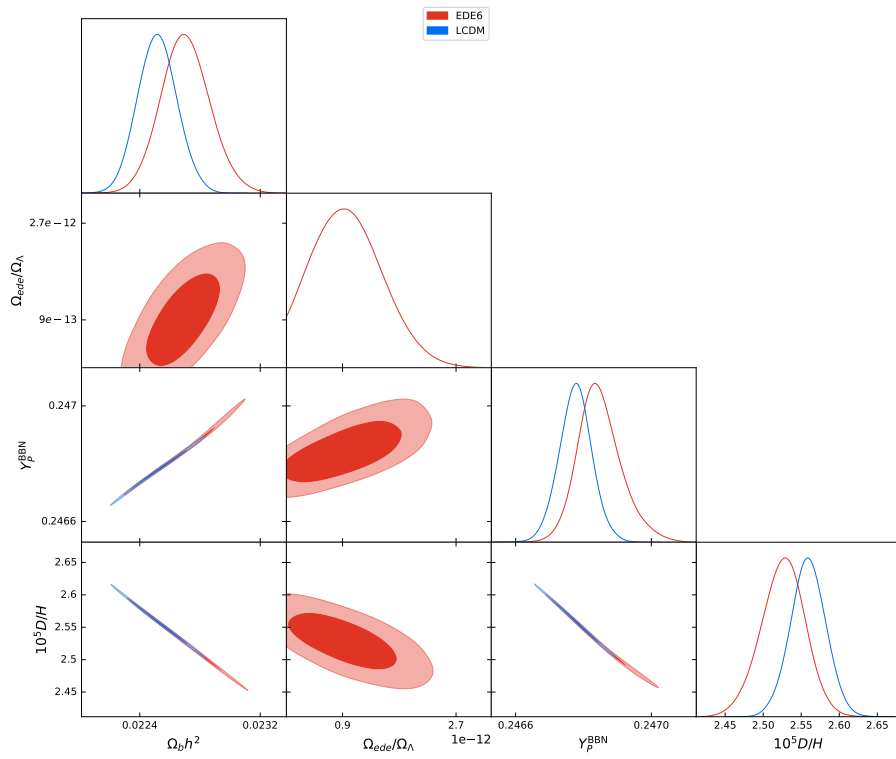


Figure 7.8: The posterior and its BBN dependence in the EDE6 model.

sets have not included BBN data. We have derived constraints from the data set with and without BBN data for each model. As can be expected, data sets without BBN indicates larger values of H_0 and N_{eff} . The magnitude of the differences is shown in Fig. 7.9.

7.2 DESI BAO

We present the results of axionlike early dark energy, which is given in Fig. 7.10 and Table. 7.2. We find a higher Hubble constant and a lower Ω_m when DESI BAO data are included. This result is in agreement with the previous paper [62], while authors of Ref. [62] have considered the $n = 3$ model and excluded other BAO data.

In the axionlike EDE ($n = 2$) model, we also find that when the DESI BAO data are included, a larger amount of the EDE resulting in a higher Hubble constant and a larger $r_d h$ is preferred as,

$$\begin{cases} f_{de}(z_c) = 0.0117^{+0.0039}_{-0.010} \\ H_0 = 68.74^{+0.62}_{-0.90} \text{ km/s/Mpc} \\ \Omega_m = 0.3078 \pm 0.0057 \\ r_d h = 100.02 \pm 0.73 \text{ km/s} \end{cases} \quad (\text{CMB} + \text{BAO} + \text{Pantheon}), \quad (7.1)$$

$$\begin{cases} f_{de}(z_c) = 0.0123^{+0.0049}_{-0.0092} \\ H_0 = 69.19^{+0.60}_{-0.84} \text{ km/s/Mpc} \\ \Omega_m = 0.3026 \pm 0.0046 \\ r_d h = 100.71 \pm 0.61 \text{ km/s} \end{cases} \quad (\text{CMB} + \text{BAO} + \text{Pantheon} + \text{DESI BAO}). \quad (7.2)$$

This indicates that all of the shorter sound horizon solutions to the Hubble tension can be preferred when considering DESI BAO measurements, caused by a larger total energy density of the early Universe.

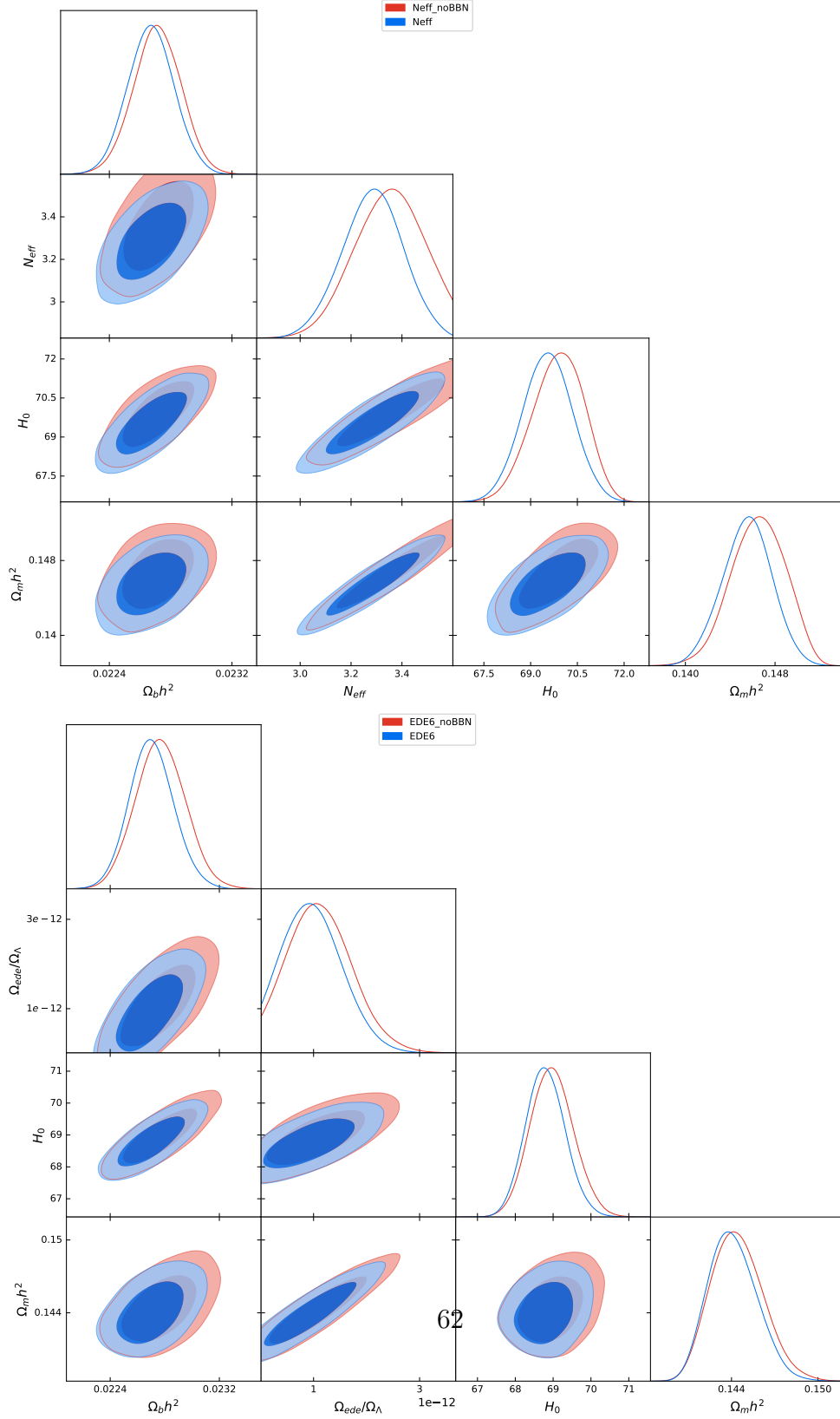


Figure 7.9: Comparison of constraints based on data sets with and without BBN for the N_{eff} model (upper) and the EDE6 (lower).

Parameter	m_e	$m_e + \Omega_K$	EDE
$\Omega_b h^2$	0.02251 ± 0.00016	$0.02288^{+0.00045}_{-0.00050}$	0.02253 ± 0.00016
$\Omega_c h^2$	0.1205 ± 0.0019	0.1216 ± 0.0022	$0.1222^{+0.0016}_{-0.0026}$
m_e/m_{e0}	1.0054 ± 0.0065	1.019 ± 0.018	-
Ω_K	-	-0.0041 ± 0.0049	-
$f_{de}(z_c)$	-	-	$0.0117^{+0.0039}_{-0.010}$
H_0	68.6 ± 1.1	69.7 ± 1.7	$68.74^{+0.62}_{-0.90}$
Ω_m	0.3054 ± 0.0078	0.299 ± 0.011	0.3078 ± 0.0057
S_8	0.824 ± 0.010	0.824 ± 0.010	0.824 ± 0.011
T_{H_0}	3.10σ	1.81σ	3.51σ
T_{s8}	2.43σ	2.43σ	2.37σ

Parameter	m_e	$m_e + \Omega_K$	EDE
$\Omega_b h^2$	0.02260 ± 0.00014	0.02269 ± 0.00036	0.02259 ± 0.00015
$\Omega_c h^2$	0.1210 ± 0.0018	0.1213 ± 0.0021	$0.1216^{+0.0016}_{-0.0026}$
m_e/m_{e0}	1.0092 ± 0.0055	1.013 ± 0.014	-
Ω_K	-	$-0.0009^{+0.0032}_{-0.0036}$	-
$f_{de}(z_c)$	-	-	$0.0123^{+0.0049}_{-0.0092}$
H_0	69.44 ± 0.84	69.7 ± 1.4	$69.19^{+0.60}_{-0.84}$
Ω_m	0.2993 ± 0.0053	0.2978 ± 0.0085	0.3026 ± 0.0046
S_8	0.8205 ± 0.0097	0.8201 ± 0.0096	0.8178 ± 0.0096
T_{H_0}	2.89σ	2.06σ	3.23σ
T_{s8}	2.27σ	2.26σ	2.14σ

Table 7.2: 68% constraints and Gaussian tension to other measurements from the dataset CMB + BAO + Pabtheon (upper) and CMB + BAO + Pantheon + DESI (lower)

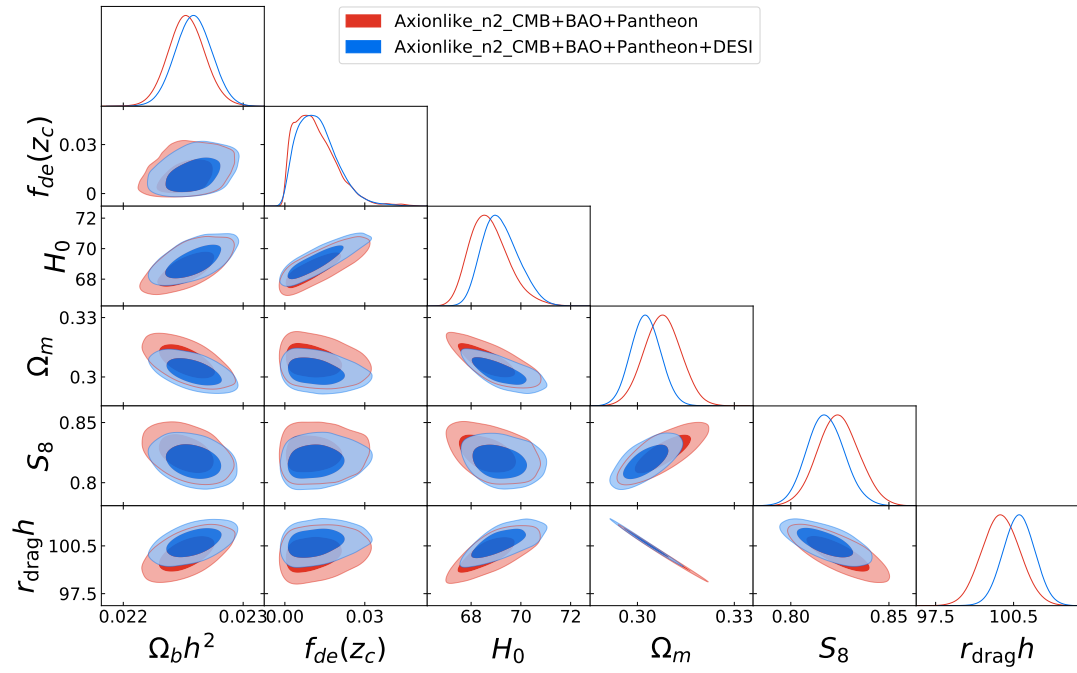


Figure 7.10: Posterior distributions of some parameters for axionlike EDE ($n = 2$) model. Different colors of reddish and bluish contours stand for different datasets.

Part III

Varying fundamental constant

Chapter 8

model

In this part, we will discuss varying fundamental constant, especially focusing on the electron mass and the fine structure constant.

8.1 Varying fine structure constant effect on BBN

We apply the following modifications to implement the effects of varying fine structure constant in the calculation of BBN referring to the previous papers [90, 91, 92, 93, 94].

Corrections to the weak reaction rate due to the mass difference variation

By increasing the value of α , the mass difference between neutrons and protons $Q \equiv m_n - m_p$ decreases. The response of Q to the the variation of α is given by [95]

$$Q = \left(-0.76 \frac{\alpha}{\alpha_0} + 2.05 \right) \text{ MeV}, \quad (8.1)$$

where $\alpha_0 \simeq 1/137.036$ is the present value of the fine structure constant.

The change of Q affects the weak reaction rate during the BBN era as

$$\Gamma(n \rightarrow p^+ e^- \nu_e)_{\text{BBN}} = \Gamma_{n0} \frac{\lambda_{n \rightarrow p^+}(Q)}{\lambda_{n \rightarrow p^+}(Q_0)}, \quad (8.2)$$

where $\Gamma_{n0} = 1/\tau_{n0} \simeq 1/(879.4 \text{ s})$ and $Q_0 \simeq 1.29 \text{ MeV}$ are the present values of the neutron decay rate and the mass difference between the neutron and proton, respectively. λ is the phase space integral [33], which is given by

$$\lambda_{n \rightarrow p^+}(Q) \propto \left(\int_{-\infty}^{-m_e - Q} \frac{x^2(Q+x)^2 \sqrt{1 - \frac{m_e^2}{(Q+x)^2}}}{\left(e^{\frac{x}{T_\nu}} + 1\right) \left(e^{-\frac{Q+x}{T}} + 1\right)} dx + \int_{m_e - Q}^{\infty} \frac{x^2(Q+x)^2 \sqrt{1 - \frac{m_e^2}{(Q+x)^2}}}{\left(e^{\frac{x}{T_\nu}} + 1\right) \left(e^{-\frac{Q+x}{T}} + 1\right)} dx \right), \quad (8.3)$$

with T_ν being the temperature of neutrinos, and a similar expression holds for the $p^+ \rightarrow n$ weak interaction rate.

Although we assume that α does not change during the BBN epoch, it should eventually evolve to the present value at some point in the course of the evolution of the Universe. We have implemented the above change in `ParthENoPE` [68, 69, 70], which we use to calculate the abundance of light elements such as D and ^4He .

Reaction rates corrections

Effects of the time variation of α also appears in the Coulomb barrier penetrability. For the reactions involving charged particles, the expression of the cross section $\sigma(E)$ is given by

$$\sigma(E) = \frac{S(E)}{E} \exp(-2\pi\alpha Z_i Z_j \sqrt{\mu/2E}), \quad (8.4)$$

where $S(E)$ is the astrophysical S -factor, μ is the reduced mass, and $Z_{i,j}$ are the atomic number of the nuclei. Since a smaller α increases the cross section, the nuclear reactions proceed more quickly. We have modified `ParthENoPE3.0` to implement the variation of α in the above cross section, following the treatment of Ref. [90].

Other corrections

We have also taken account of other minor effects [91]: corrections in final-state Coulomb interaction, radiative capture rate and the contributions of electromagnetic mass ΔM_{EM} .

Neumerical Evaluation

In Fig. 8.1, we show the density plots of D_p (left) and Y_p (right) in the $\eta - \Delta\alpha/\alpha_0$ plane where η is the baryon-to-photon ratio and $\Delta\alpha = \alpha - \alpha_0$. Re-

gions between red lines in the left and right panels correspond to 1σ allowed region from measurements of D_p [27] and the EMPRESS Y_p [96], respectively. For comparison, in the right panel, another recent helium abundance measurement from Aver et al. [26] is also shown as region between green dashed lines. As shown in the figure, the abundance of ^4He becomes smaller as the fine structure constant α decreases. Therefore, the value of Y_p reported by the EMPRESS can be explained by a smaller value of α . The abundance of D is also affected by the change of α , however, its effect is partially offset by the change of η as we explain below.

The response of Y_p against the change of α can be understood by noticing that the mass difference Q decreases by increasing α and the neutron decay rate becomes larger (the neutron lifetime becomes shorter), which makes the abundance of ^4He larger. The changes of Y_p and D_p in response to the variation of α and η , denoted as $\Delta\alpha$ and $\Delta\eta$, can be understood by the following scaling relation [97]:

$$\frac{\Delta D_p}{D_p} = 3.6 \frac{\Delta\alpha}{\alpha} - 1.6 \frac{\Delta\eta}{\eta}, \quad (8.5)$$

$$\frac{\Delta Y_p}{Y_p} = 1.9 \frac{\Delta\alpha}{\alpha} + 0.04 \frac{\Delta\eta}{\eta}. \quad (8.6)$$

The α dependence in D_p can be somewhat canceled by the change of η . This behavior indeed can be seen in the left panel of Fig. 8.1. On the other hand, the ^4He abundance is not so sensitive with respect to the variation of η , and hence α almost controls the value of Y_p , which can be observed in the right panel of Fig. 8.1.

8.2 Varying electron mass effect on CMB and BAO

In the varying electron mass model, we assume that the value of the electron mass at early Universe differs from its present value and it dramatically changes to the current value after the recombination is complete. As discussed in the previous work [15], the primary contribution of an increased electron mass is the increasing energy level of hydrogen and Lyman alpha photon, both of which are directly proportional to the electron mass $E \propto m_e$. Since the energy level of hydrogen is higher, a photon energy at the standard

recombination temperature is too small to excite the hydrogen. In a theory with large electron mass, the recombination takes place earlier than the standard and the resultant sound horizon becomes shorter. Thus, to reproduce the same measured multiple moment of acoustic peaks of the CMB power spectrum, the larger electron mass results in a shorter diameter distance and a higher H_0 .

Another consequence of the increased electron mass is the Thomson scattering rate σ_T , which is proportional to m_e^{-2} and suppresses the peak of high- l CMB power spectrum. The other minor contributions include the photoionization cross sections, the recombination coefficient, K factors, Einstein A coefficients, and the two-photon decay rates (see Ref. [15] for details).

The varying electron mass model can significantly relieve the Hubble tension without spoiling the CMB fitting because there are the degeneracy between cold dark matter density $\omega_c = \Omega_c h^2$, baryon density $\omega_b = \Omega_b h^2$, and the sound horizon scale in the CMB spectrum, which transformed into the degeneracy between m_e and H_0 .

On the other hand, the BAO bound can break this degeneracy, since a greater electron mass leads to a larger ω_c , ω_b and H_0 resulting in a larger $\theta_d(z) \equiv r_d/D_M$, as has been shown in Ref. [18]¹. Here, we define the sound horizon and diameter distances

$$r_d = \int_{z_d}^{\infty} \frac{c_s(z)}{H(z)} dz, \quad (8.7)$$

$$D_M(z) = \frac{c}{H_0 \sqrt{\Omega_K}} \sinh \left[\sqrt{\Omega_K} \int_0^z \frac{dz'}{H(z')/H_0} \right], \quad (8.8)$$

$$D_H(z) = \frac{c}{H(z)}, \quad (8.9)$$

and $D_V = (z D_M(z)^2 D_H(z))^{1/3}$, where z_d is the redshift of the drag epoch. However, a negative Ω_K model with positive spatial curvature $K > 0$ weakens the BAO constraints on variation of m_e , since we increase one degree of the freedom to fit BAO. We should note that, in the Λ CDM model, the Planck paper [1] reports a negative Ω_K which is nearly 2σ away from $\Omega_K = 0$,² while the DESI paper [53] reports zero-consistent results. Therefore, it is also important to carefully consider the combination of varying $m_e + \Omega_K$.

¹The analysis including the latest SDSS BAO data [98] on varying m_e and EDE is recently discussed [22].

²Using the last Planck data release (PR4), they obtain somewhat closer results to the flat Universe [99, 100].

To examine the varying electron mass model, we make all the above mentioned modifications to the recombination code **recfast** [101] and perform the Markov-chain Monte Carlo (MCMC) analysis, sampling $m_e \in [0.9 \text{ } 1.2]$ in addition to the six standard parameters.

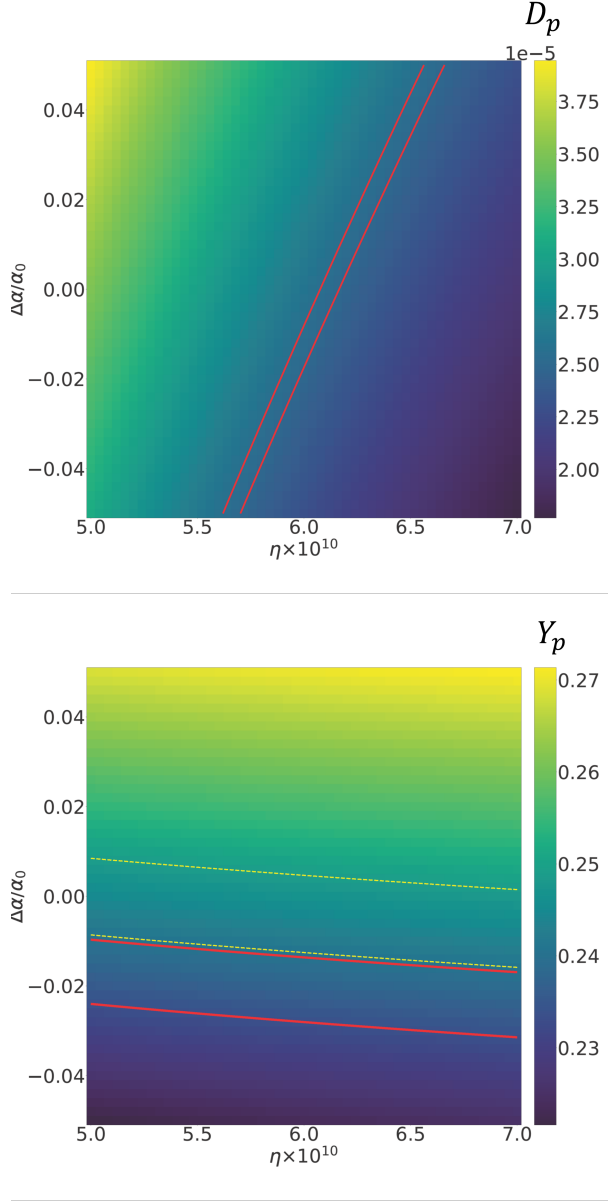


Figure 8.1: Density plots of D_p (left) and Y_p (right) in the $\eta - \Delta\alpha/\alpha_0$ plane. In this figure, other parameters are fixed as $\Delta N_{\text{eff}} = 0$ (the deviation from the standard value of N_{eff}) and $\xi_e = 0$. Regions between red lines in the left and right panels correspond to 1σ allowed region from measurements of D_p [27] and the EMPRESS Y_p [96], respectively. Region between green dashed lines in the right panel correspond to 1σ allowed region from another Y_p measurement by Aver et al. [26].

Chapter 9

results

9.1 Big Bang Neucleosynthesis

We investigate the impact of the EMPRESS result of Y_p on models beyond the standard assumption, with particular emphasis on the time-varying α . Since it has been discussed that the EMPRESS result may suggest non-standard values of the effective number of neutrino N_{eff} and non-zero lepton asymmetry, characterized by the chemical potential (the degeneracy parameter) of the electron neutrino ξ_e [96], we also vary N_{eff} and ξ_e in models with varying α in some cases.

To study the preferred ranges for α and η , we calculate the value of χ^2 fitted to the observed values of Y_p by the EMPRESS [96] and D_p [27]. We also adopt the prior for the baryon density derived from the Planck CMB measurement [1]. The total value of χ^2 is evaluated as

$$\chi^2(\alpha, \eta) = \chi_{Y_p}^2(\alpha, \eta) + \chi_{D_p}^2(\alpha, \eta), \quad (9.1)$$

where $\chi_{Y_p}^2$ and $\chi_{D_p}^2$ are the ones for Y_p by the EMPRESS [96] and D_p [27], which are respectively given as

$$\chi_{Y_p}^2(\alpha, \eta) = \frac{(Y_{p,\text{obs}} - Y_p^{(\text{th})}(\alpha, \eta))^2}{\sigma_{Y_p,\text{obs}}^2 + \sigma_{Y_p,\text{th}}^2}, \quad (9.2)$$

$$\chi_{D_p}^2(\alpha, \eta) = \frac{(D_{p,\text{obs}} - D_p^{(\text{th})}(\alpha, \eta))^2}{\sigma_{D_p,\text{obs}}^2 + \sigma_{D_p,\text{th}}^2}. \quad (9.3)$$

Here $Y_{p,\text{obs}} = 0.2370$ and $\sigma_{Y_p,\text{obs}} = 0.00335$ are the mean value and 1σ error from the observation of Y_p [96]. For comparison, we also evaluate the fit

with the Y_p measurement by Aver et al. [26], $Y_p = 0.2449 \pm 0.040$, in some cases. For deuterium abundance, we take $D_{p,\text{obs}} = 2.527 \times 10^{-5}$ and $\sigma_{D_{p,\text{obs}}} = 0.030 \times 10^{-5}$ from Cooke et al. [27]. The theoretical uncertainty for Y_p is given by $\sigma_{Y_p,\text{th}} = \sqrt{0.00003^2 + 0.00012^2}$, which originates from the nuclear rates and the neutron lifetime, corresponding to the first and the second term in the square root respectively [70]. For the theoretical uncertainty of D_p , those for the nuclear reaction rates give $\sigma_{D_p,\text{th}} = \sqrt{0.06^2 \times 10^{-10}}$ [70]. $Y_p^{(\text{th})}(\alpha, \eta)$ and $D_p^{(\text{th})}(\alpha, \eta)$ are theoretically predicted values of Y_p and D_p calculated by using PARthENoPE [68, 69, 70] which we have modified to implement the variation of α .

We calculate χ^2 given by Eq. (9.1) to obtain constraints on $\alpha, N_{\text{eff}}, \xi_e$ and η . In the following, we present constraints on a two dimensional plane by marginalizing over the other parameter(s), which is in practice done by determining (fixing) the redundant parameter to the one minimizing the value of χ^2 . When we use η as one of variables in a two dimensional plane.

Now, we present constraints on the parameters obtained by fitting to the data of Y_p from the EMPRESS [96] and D_p from Cooke et al. [27].

We show the constraint for models with fixed $(\Delta N_{\text{eff}}, \xi_e) = (0, 0)$ in the η - $\Delta\alpha/\alpha_0$ plane of Fig. 9.1 where 1σ and 2σ allowed regions are shown with dark and light red shading, respectively. For comparison, we also display the same constraints with unshaded black contours in the case that we adopt the Y_p measurement from Aver et al. [26] instead of the EMPRESS data, and the same D_p data. The figure indicates that the best-fit value for the EMPRESS+Cooke et al. data is $\alpha/\alpha_0 \simeq 0.98$, and $\alpha/\alpha_0 \simeq 1$ is excluded more than 2σ . This indicates that the fine structure constant should be decreased by 2 % during BBN epoch compared to the present value when the EMPRESS Y_p is adopted. On the other hand, no such deviation is necessary if we adopt the Y_p data by Aver et al. After marginalizing over η , the bound for α is obtained as $-2.6\% < \Delta\alpha/\alpha_0 < -1.4\% (1\sigma)$ (EMPRESS+Cooke et al.) and $-1.2\% < \Delta\alpha/\alpha_0 < 0.4\% (1\sigma)$ (Aver et al.+Cooke et al.).

We also obtain the bound for m_e as $-5.1\% < \Delta m_e/m_{e0} < 1.4\% (1\sigma)$ (Aver et al.+Cooke et al.), since a larger electron mass gives rise to the smaller weak interaction rate for the proton and neutron conversion, which affects the light element abundance.

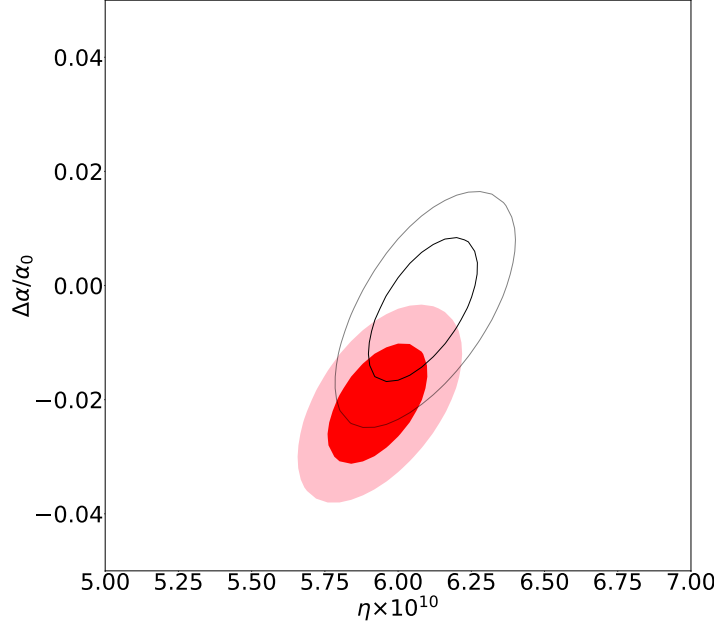


Figure 9.1: 1σ and 2σ constraints in the η - $\Delta\alpha/\alpha_0$ plane from the EMPRESS measurements of Y_p and D_p (red shaded). We also show the case where we adopt the Y_p measurement by Aver et al. and D_p (unshaded black contours).

9.2 DESI BAO

We present the results of models we have examined. The 1D and 2D marginalized posteriors of different cosmological parameters are shown in Fig. 9.3 and the 68% confidence level constraints are listed in Table 7.2.

We find that the DESI BAO results disfavor lower m_e and indicate slightly larger electron mass $m_e/m_{e0} = 1.0092 \pm 0.0055$ (CMB+BAO+DESI BAO). This indicates a shorter sound horizon at the recombination and larger H_0 than the analysis without DESI BAO. This is due to the measured sound horizon r_d relative to the diameter distance r_d/D by DESI BAO. Diameter distance D is inversely proportional to the Hubble constant H_0 . Therefore, in the BAO measurements, a product of the sound horizon and the Hubble constant $r_d h$ is constrained. For the greater electron mass, the Hubble

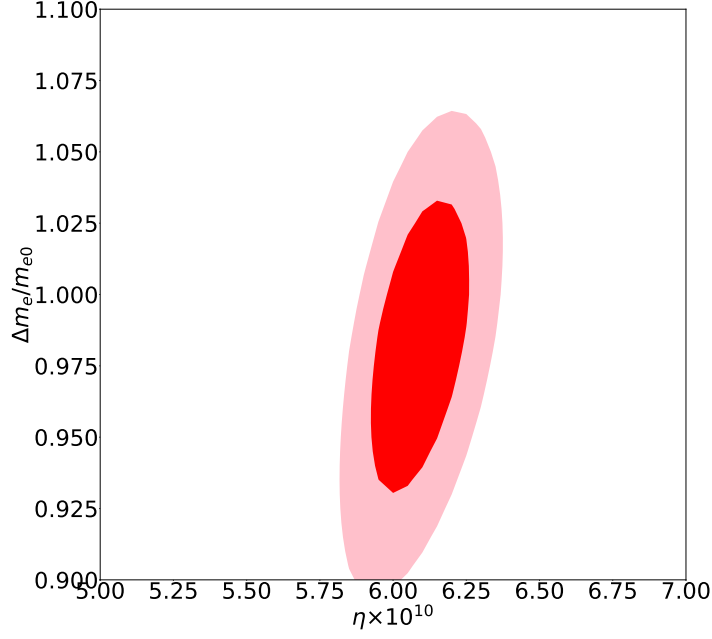


Figure 9.2: 1σ and 2σ constraints in the η - $\Delta m_e/m_{e0}$ plane from the Y_p measurement by Aver et al. and D_p (red shaded).

constant H_0 increases while the sound horizon r_d slightly decreases, and the product $r_d h$ increases, which is preferred from DESI BAO measurements as in Fig. 9.3.

We show the best-fit point of $(\Omega_m, r_d h) = (0.3094, 99.773 \text{ km/s})$ for $m_e/m_{e0} = 0.99$ and $(\Omega_m, r_d h) = (0.2979, 101.364 \text{ km/s})$ for $m_e/m_{e0} = 1.01$ for the data combination of CMB+BAO+Pantheon+DESI BAO, in the Table 7.2. As a result, using only DESI BAO analysis, we quote $(\Omega_m, r_d h) = (0.295, 101.8 \text{ km/s})$ for the best-fit point of Λ CDM model from the DESI paper [53]. By comparing those, for the smaller electron mass, the Hubble constant H_0 becomes too small, and the sound horizon relative to the diameter distance $r_d/D \propto r_d h$ becomes too short, which leads to poor DESI fitting. The difference in the χ^2 values accounts for $\Delta\chi^2_{\text{DESI, bestfit}(m_e/m_{e0}=0.99)} - \Delta\chi^2_{\text{DESI, bestfit}(m_e/m_{e0}=1.01)} \simeq 4.7$ as in Table ?? . Figs. 9.4, 9.5, and 9.6 show the comoving diameter distance over the sound horizon at the drag epoch

$D_M/r_d(z)$, the Hubble distance over the sound horizon at the drag epoch $D_H/r_d(z)$ and the angle-average distance over the sound horizon at the drag epoch $D_V/r_d(z)$ of $m_e/m_{e0} = 1.01$ and 0.99 models, respectively, as the function of redshifts with the error bar of the DESI measurements.

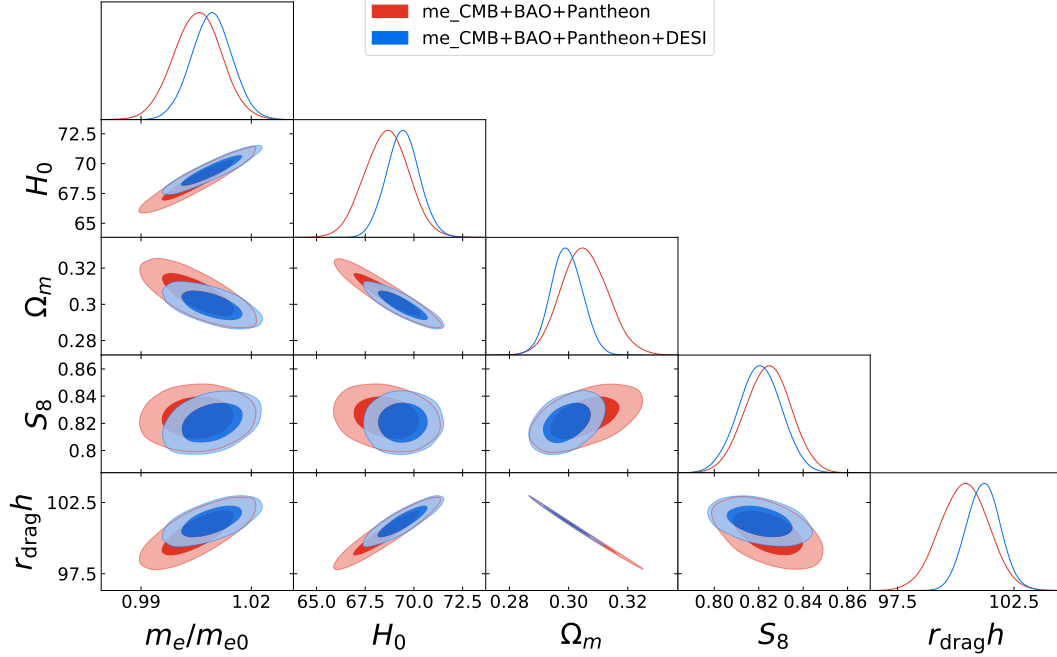


Figure 9.3: Posterior distributions of some parameters for the varying m_e model. Different colors of reddish and bluish contours stand for different datasets.

Next, we discuss the results of varying m_e in a curved Universe (the $m_e + \Omega_K$ model), which results are summarized in Fig. 9.7 and Table. 7.2. We find that a positive curvature ($\Omega_k \lesssim 0.01$) is only allowed in the red contour (not including DESI BAO data) of Fig. 9.7, and we obtain the 68% constraint $|\Omega_k| < 0.0045$ (68% $\mathcal{D}$ +DESI) from Tab. 7.2. This is greatly consistent with the results for the Λ CDM model from DESI paper [53], and we quote $\Omega_k = -0.0102 \pm 0.0054$ [68% CMB(Planck+ACT)] while $\Omega_k = 0.0024 \pm 0.0016$ (68% CMB+DESI). Therefore, we conclude that a spatially flat Universe is preferred in the Λ CDM model and the varying electron mass model from DESI BAO data. For the Hubble constant, the central value shifts little, but

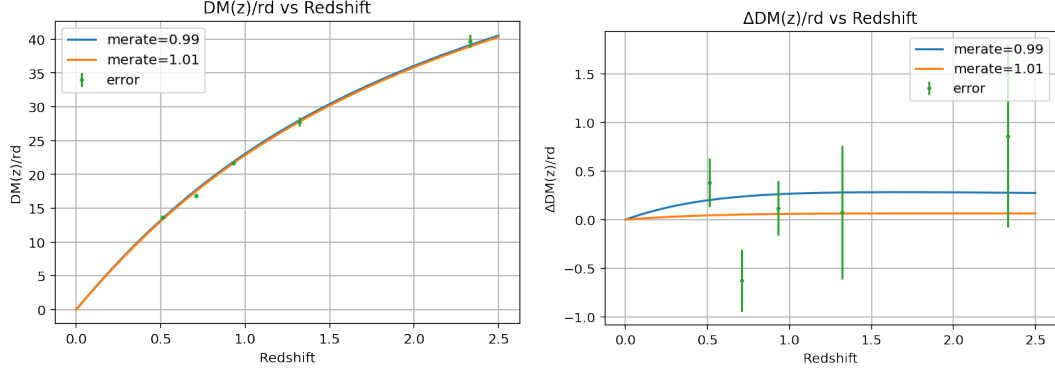


Figure 9.4: The comoving diameter distance over the sound horizon at the drag epoch $D_M/r_d(z)$ as a function of redshift. We use the best-fit value at $m_e = 1.01$ for the orange curve and $m_e = 0.99$ for the blue curve. The green error bars represent the data from the DESI BAO measurements. In the right panel, we illustrate the difference from the $D_M(z)/r_d$ assuming $(\Omega_M, r_d h) = (0.295, 101.8)$, which is the mean value of the only DESI BAO constraint [53].

the size of the error decreases when DESI BAO data are incorporated, since both the positive curvature and the smaller electron mass is disfavored from DESI BAO data.

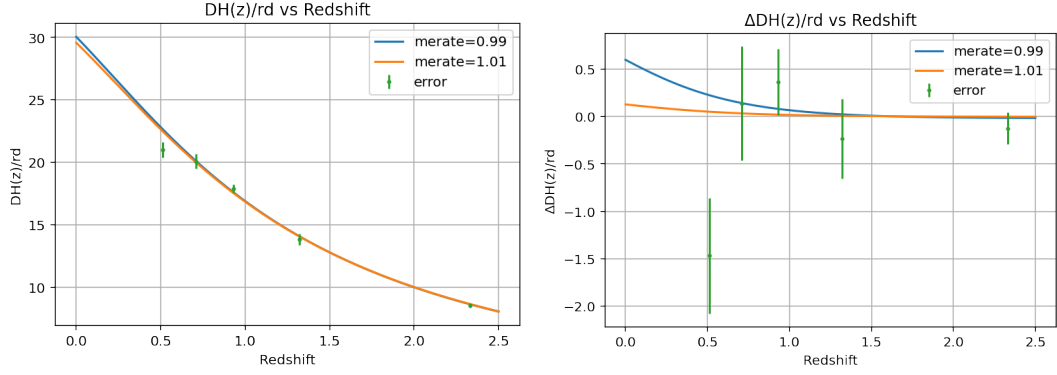


Figure 9.5: Same as Fig. 9.4, for the Hubble distance over the sound horizon at the drag epoch $D_H/r_d(z)$ as a function of redshift.

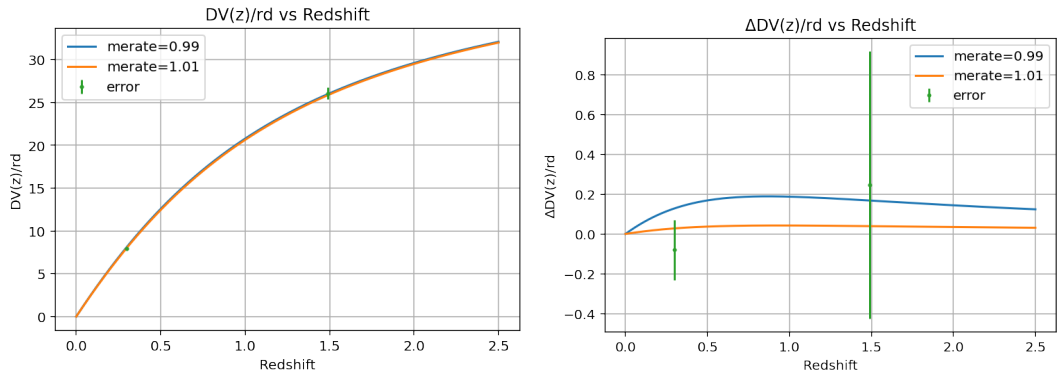


Figure 9.6: Same as Fig. 9.4, for the angle-average distance over the sound horizon at the drag epoch $D_V/r_d(z)$ as a function of redshift.

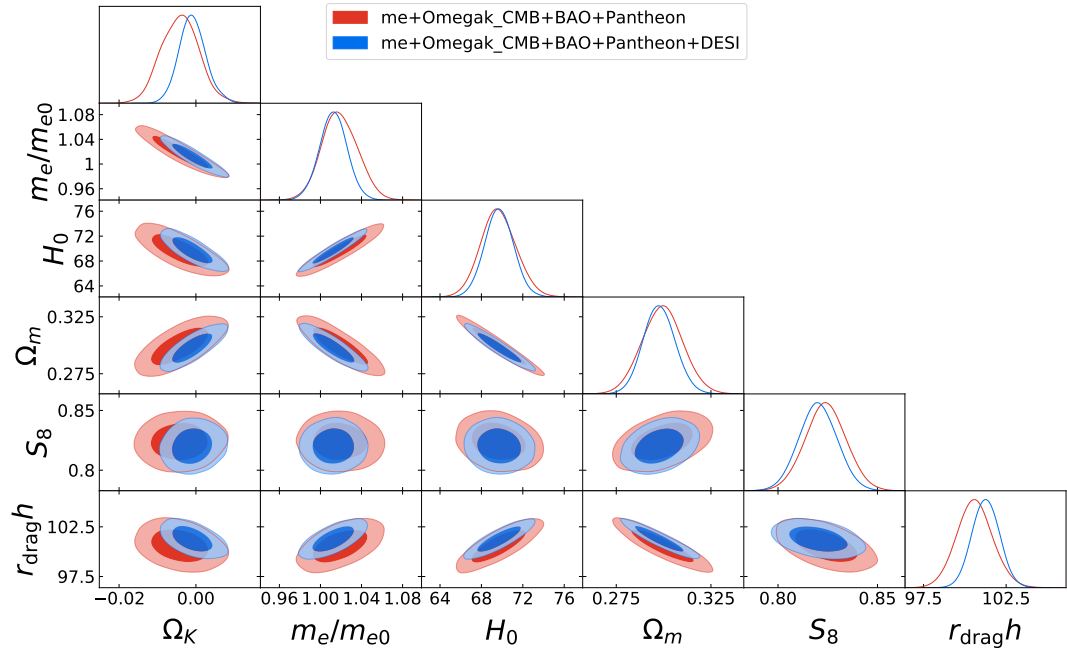


Figure 9.7: Posterior distributions of some parameters for the $m_e + \Omega_K$ model. Different colors of reddish and bluish contours stand for different datasets.

Part IV

Conclusion and discussion

We have investigated the impact of the time variation of α to the recent measurement of the primordial helium abundance by the EMPRESS [96].

Our analysis suggests that the recent measurement of Y_p by the EMPRESS may indicate a time variation of the fine structure constant, which would give a significant implication for the fundamental physics, since it means that some fundamental constant could change in time. Although we have assumed that only the fine structure constant changes with time in this paper, in some theories, other constants can simultaneously be altered. It would be interesting to pursue such possibilities in the light of the EMPRESS Y_p result, which is left for the future work.

We also examine the varying electron mass model and the axionlike early dark energy model, considering the recent DESI BAO measurements. In the analysis using a combination of CMB, conventional BAO and light curves (Pantheon), as in the previous studies, we find that the electron mass before the CMB era is in agreement with the current value in a 1σ confidence level. However, when the DESI BAO data are included, we find that the larger electron mass is preferred as,

$$\begin{aligned} &\begin{cases} m_e/m_{e0} = 1.0054 \pm 0.0065 \\ H_0 = 68.6 \pm 1.1 \text{ km/s/Mpc} \\ \Omega_m = 0.3054 \pm 0.0078 \end{cases} \quad (\text{CMB} + \text{BAO} + \text{Pantheon}), \quad (9.4) \\ &\begin{cases} m_e/m_{e0} = 1.0092 \pm 0.0055 \\ H_0 = 69.44 \pm 0.84 \text{ km/s/Mpc} \\ \Omega_m = 0.2993 \pm 0.0053 \end{cases} \quad (\text{CMB} + \text{BAO} + \text{Pantheon} + \text{DESI BAO}). \quad (9.5) \end{aligned}$$

Although the electron mass in the BAO era is assumed to be the same as today, the larger electron mass at CMB era leads to a higher Hubble constant H_0 and a product of a sound horizon and Hubble constant $r_d h$ to fit the CMB power spectrum, bringing it closer to the DESI best-fit value. This is the reason why DESI BAO data prefer the larger electron mass in the CMB era. Though larger values of m_e/m_{e0} and H_0 are indicated in a closed Universe in the previous studies, we find that such a case is not preferred once we take the DESI BAO measurements into account.

In the axionlike EDE ($n = 2$) model, we also find that when the DESI BAO data are included, a larger amount of the EDE resulting in a higher

Hubble constant and a larger $r_d h$ is preferred as,

$$\begin{cases} f_{de}(z_c) = 0.0117^{+0.0039}_{-0.010} \\ H_0 = 68.74^{+0.62}_{-0.90} \text{ km/s/Mpc} \\ \Omega_m = 0.3078 \pm 0.0057 \\ r_d h = 100.02 \pm 0.73 \text{ km/s} \end{cases} \quad (\text{CMB} + \text{BAO} + \text{Pantheon}), \quad (9.6)$$

$$\begin{cases} f_{de}(z_c) = 0.0123^{+0.0049}_{-0.0092} \\ H_0 = 69.19^{+0.60}_{-0.84} \text{ km/s/Mpc} \\ \Omega_m = 0.3026 \pm 0.0046 \\ r_d h = 100.71 \pm 0.61 \text{ km/s} \end{cases} \quad (\text{CMB} + \text{BAO} + \text{Pantheon} + \text{DESI BAO}). \quad (9.7)$$

This indicates that all of the shorter sound horizon solutions to the Hubble tension can be preferred when considering DESI BAO measurements, whether caused by an earlier recombination or larger total energy density of the early Universe.

Part V

Appendix

Appendix A

Review of general relativity

metric and christoffel symbol

Our 4-dimensional spacetime is described using x^μ ($\mu = 0, 1, 2, 3$) and the distance between two neighboring points are given by

$$ds^2 = < dx^\mu \vec{e}_\mu(x), dx^\nu \vec{e}_\nu(x) > \equiv g_{\mu\nu}(x) dx^\mu dx^\nu, \quad (\text{A.1})$$

and we call $g_{\mu\nu}(x)$ metric. For example, the metric of 4-dimensional Minkowski space $\eta_{\mu\nu}$ is given by

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We define the determinant of the metric $g_{\mu\nu}$ as $g \equiv \det(g_{\mu\nu})$, which always takes a negative value. We also define the inverse matrix of the metric $g_{\mu\nu}$ as $g^{\mu\nu}$, which leads to

$$g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma, \quad (\text{A.2})$$

or derivative with respect to x^σ

$$0 = \delta^\alpha_{\nu,\sigma} = (g^{\alpha\mu} g_{\mu\nu})_{,\sigma} = g^{\alpha\mu}_{,\sigma} g_{\mu\nu} + g^{\alpha\mu} g_{\mu\nu,\sigma}. \quad (\text{A.3})$$

The squared value of the contravariant vector A^μ is defined as

$$g_{\mu\nu} A^\mu A^\nu, \quad (\text{A.4})$$

and this value is invariant under the coordinate transformation. The covariance vector A_μ is defined as

$$A_\mu \equiv g_{\mu\nu} A^\nu \quad (\text{A.5})$$

Multiplying Eq. (A.3), by $g^{\beta\nu}$ and using Eq. (A.2), we obtain

$$g_{,\sigma}^{\alpha\beta} = -g^{\alpha\mu} g^{\beta\nu} g_{\mu\nu,\sigma},$$

and multiplying this by δx^σ , we obtain

$$\delta g^{\alpha\beta} = -g^{\alpha\mu} g^{\beta\nu} \delta g_{\mu\nu} \quad (\text{A.6})$$

or multiplying this by $-g_{\alpha\mu} g_{\beta\nu}$, we obtain

$$\delta g_{\mu\nu} = -g_{\alpha\mu} g_{\beta\nu} \delta g^{\alpha\beta} \quad (\text{A.7})$$

These formula A.6, A.7 relate a variation of $g^{\alpha\beta}$ and a variation of $g_{\mu\nu}$.

Another important quantity, Christoffel symbol is defined as

$$\Gamma_{\mu\nu\sigma} \equiv \frac{1}{2}(g_{\mu\nu,\sigma} + g_{\mu\sigma,\nu} - g_{\sigma\nu,\mu}), \quad (\text{A.8})$$

and by definition, we obtain

$$\Gamma_{\mu\nu\sigma} + \Gamma_{\nu\mu\sigma} = g_{\mu\nu,\sigma}. \quad (\text{A.9})$$

Christoffel symbol with upper index and a covariance derivative of a matrix $Y_{\mu\nu}^\alpha$ are defined by

$$\Gamma_{\nu\sigma}^\mu = g^{\mu\lambda} \Gamma_{\lambda\nu\sigma} = \frac{1}{2} g^{\mu\lambda} (g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\sigma\nu,\lambda}), \quad (\text{A.10})$$

$$Y_{\mu\nu;\sigma}^\alpha = Y_{\mu\nu,\sigma}^\alpha - \Gamma_{\mu\sigma}^\gamma Y_{\gamma\nu}^\alpha - \Gamma_{\nu\sigma}^\gamma Y_{\mu\gamma}^\alpha + \Gamma_{\gamma\sigma}^\alpha Y_{\mu\nu}^\gamma. \quad (\text{A.11})$$

Substituting $g_{\mu\nu}$ into Y and noting the Eq. (A.12), we obtain

$$\begin{aligned} g_{\mu\nu;\sigma} &= g_{\mu\nu,\sigma} - \Gamma_{\mu\sigma}^\alpha g_{\alpha\nu} - \Gamma_{\nu\sigma}^\alpha g_{\mu\alpha} \\ &= g_{\mu\nu,\sigma} - \Gamma_{\nu\mu\sigma} - \Gamma_{\mu\nu\sigma} \\ &= 0 \end{aligned} \quad (\text{A.12})$$

$$\therefore 0 = (\delta_\nu^\rho)_{;\sigma} = (g^{\rho\mu} g_{\mu\nu})_{;\sigma} = g_{;\sigma}^{\rho\mu} g_{\mu\nu} + g^{\rho\mu} g_{\mu\nu;\sigma} = g_{;\sigma}^{\rho\mu} g_{\mu\nu},$$

We use the Leibniz product rule of covariance derivative at the third equal and we achieve at

$$g_{;\sigma}^{\rho\mu} = 0, \quad (\text{A.13})$$

which indicate that a covariance derivative of a matrix is always zero.

Supplemental calculation $\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$

Theorem

When $\tilde{A}$ is the cofactor matrix of square matrix A ,

$$A\tilde{A} = \tilde{A}A = (\det A)I \quad (I \text{ is the identity matrix})$$

Let $\tilde{g}_{\alpha\beta}$ be cofactor matrix of $g_{\alpha\beta}$ and using the above theorem, we obtain

$$\begin{aligned}\tilde{g}^{\mu\nu}g_{\nu\alpha} &= g\delta_{\alpha}^{\mu} = gg^{\mu\nu}g_{\nu\alpha}, \\ \therefore \tilde{g}^{\mu\nu} &= gg^{\mu\nu},\end{aligned}$$

considering the cofactor expansion of the determinant g , we obtain

$$\frac{\partial g}{\partial g_{\alpha\beta}} = \tilde{g}^{\alpha\beta} = gg^{\alpha\beta}. \quad (\text{A.14})$$

$$\therefore \delta g = \tilde{g}^{\alpha\beta}\delta g_{\alpha\beta} = gg^{\alpha\beta}\delta g_{\alpha\beta}. \quad (\text{A.15})$$

Noting that $0 = \delta(g^{\alpha\beta}g_{\alpha\beta}) = g^{\alpha\beta}\delta g_{\alpha\beta} + \delta g^{\alpha\beta}g_{\alpha\beta}$, we obtain

$$\delta g = -gg_{\alpha\beta}\delta g^{\alpha\beta},$$

and finally we arrive at

$$\delta(\sqrt{-g}) = \frac{\delta g}{-2\sqrt{-g}} = \frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}. \quad (\text{A.16})$$

Supplemental calculation $g^{\mu\nu}\delta R_{\mu\nu} = (g^{\alpha\nu}\delta\Gamma_{\alpha\nu}^{\mu} - g^{\mu\nu}\delta\Gamma_{\nu\alpha}^{\alpha})_{;\mu}$

From the definition of a Ricci tensor Eq. (2.2), we obtain

$$\delta R_{\mu\nu} = \delta\Gamma_{\mu\nu,\alpha}^{\alpha} - \delta\Gamma_{\mu\alpha,\nu}^{\alpha} + \delta\Gamma_{\mu\nu}^{\gamma}\Gamma_{\gamma\alpha}^{\alpha} + \Gamma_{\mu\nu}^{\gamma}\delta\Gamma_{\gamma\alpha}^{\alpha} - \delta\Gamma_{\mu\alpha}^{\gamma}\Gamma_{\nu\gamma}^{\alpha} - \Gamma_{\mu\alpha}^{\gamma}\delta\Gamma_{\nu\gamma}^{\alpha} \quad (\text{A.17})$$

Noting that Christoffel symbol is a tensor¹ and we obtain the covariance derivative of christoffel symbol,

$$(\delta\Gamma_{\mu\nu}^{\alpha})_{;\alpha} = (\delta\Gamma_{\mu\nu}^{\alpha})_{,\alpha} + \Gamma_{\gamma\alpha}^{\alpha}\delta\Gamma_{\mu\nu}^{\gamma} - \Gamma_{\mu\alpha}^{\gamma}\delta\Gamma_{\gamma\nu}^{\alpha} - \Gamma_{\nu\alpha}^{\gamma}\delta\Gamma_{\mu\gamma}^{\alpha}, \quad (\text{A.18})$$

$$(\delta\Gamma_{\mu\alpha}^{\alpha})_{;\nu} = (\delta\Gamma_{\mu\alpha}^{\alpha})_{,\nu} + \Gamma_{\gamma\nu}^{\alpha}\delta\Gamma_{\mu\alpha}^{\gamma} - \Gamma_{\mu\nu}^{\gamma}\delta\Gamma_{\gamma\alpha}^{\alpha} - \Gamma_{\alpha\nu}^{\gamma}\delta\Gamma_{\mu\gamma}^{\alpha}, \quad (\text{A.19})$$

¹we can confirm it using quotient theorem

noting that $\Gamma_{\nu\sigma}^\mu = \Gamma_{\sigma\nu}^\mu$, we obtain Eq. (A.17) = Eq. (A.18) – Eq. (A.19):

$$\delta R_{\mu\nu} = (\delta\Gamma_{\mu\nu}^\alpha)_{;\alpha} - (\delta\Gamma_{\mu\alpha}^\alpha)_{;\nu}. \quad (\text{A.20})$$

Noting that covariance derivative of a metric should be 0,

$$\begin{aligned} g^{\mu\nu} \delta R_{\mu\nu} &= g^{\mu\nu} (\delta\Gamma_{\mu\nu}^\alpha)_{;\alpha} - g^{\mu\nu} (\delta\Gamma_{\mu\alpha}^\alpha)_{;\nu} \\ &= (g^{\mu\nu} \delta\Gamma_{\mu\nu}^\alpha)_{;\alpha} - (g^{\mu\nu} \delta\Gamma_{\mu\alpha}^\alpha)_{;\nu} \\ &= (g^{\alpha\nu} \delta\Gamma_{\alpha\nu}^\mu - g^{\mu\nu} \delta\Gamma_{\nu\alpha}^\alpha)_{;\mu}. \end{aligned}$$

Then, we finally achieve at

$$g^{\mu\nu} \delta R_{\mu\nu} = X_{;\mu}^\mu, \quad (\text{A.21})$$

where $X^\mu \equiv g^{\alpha\nu} \delta\Gamma_{\alpha\nu}^\mu - g^{\mu\nu} \delta\Gamma_{\nu\alpha}^\alpha$.

Einstein equation

In this subsection, we like to derive the Einstein equation from the action 2.1 using the above supplemental calculations.

For the variation of a metric $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$, we obtain the variation of the gravity action 2.1,

$$\begin{aligned} \delta S_{\text{Gravity}} &= \frac{1}{2\kappa^2} \int d^4x \left((\delta\sqrt{-g}) (R - 2\Lambda) + \sqrt{-g} (\delta g^{\mu\nu}) R_{\mu\nu} + \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \right) \\ &= \frac{1}{2\kappa^2} \int d^4x \left(\left(-\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \right) (R - 2\Lambda) + \sqrt{-g} (\delta g^{\mu\nu}) R_{\mu\nu} + \sqrt{-g} X_{;\mu}^\mu \right) \\ &= \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(\left(-\frac{1}{2} g_{\mu\nu} \delta g^{\mu\nu} \right) (R - 2\Lambda) + (\delta g^{\mu\nu}) R_{\mu\nu} + X_{;\mu}^\mu \right) \\ &= \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(\delta g^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} \right) + X_{;\mu}^\mu \right) \end{aligned} \quad (\text{A.22})$$

The last term of this equation should vanish, due to the divergence theorem. Otherwise, matter term should be

$$\begin{aligned}
\delta S_{\text{matter}} &= \int d^4x \delta (\sqrt{-g} \mathcal{L}_m) \\
&= \int d^4x \left(\frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial g^{\mu\nu}} \delta g^{\mu\nu} + \left(\frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial g_{,\alpha}^{\mu\nu}} \right) \delta g_{,\alpha}^{\mu\nu} \right) \\
&= \int d^4x \left(\frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial g^{\mu\nu}} \delta g^{\mu\nu} - \frac{\partial}{\partial x^\alpha} \left(\frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial g_{,\alpha}^{\mu\nu}} \right) \delta g^{\mu\nu} \right) \\
&= -\frac{1}{2} \int d^4x \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu},
\end{aligned}$$

where the energy momentum tensor is defined as,

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \left\{ -\frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial g^{\mu\nu}} + \frac{\partial}{\partial x^\alpha} \left(\frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial g_{,\alpha}^{\mu\nu}} \right) \right\}. \quad (\text{A.23})$$

Therefore, we finally achieve at the Einstein equation $\delta S_{\text{Gravity}} + \delta S_{\text{matter}} = 0$:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}. \quad (\text{A.24})$$

Appendix B

Formulation of the perturbations

B.1 Perturbation

We obtained the FLRW metric assuming homogeneous and isotropic universe. Hereafter, we write this metric $\bar{g}_{\mu\nu}$ and similarly write other background quantities with $\bar{}$ as well. On the other hand, there is somewhere galaxy cluster exists and somewhere it doesn't. Therefore, as far as smaller scale than galaxy clustering, we cannot claim that our universe is homogeneous and isotropic. Comparing to the universe scale, this scale is severely small and we can ignore it, before this section. However, there definitely exist the deviation fluctuation and we write the metric with it as follows:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \quad (\text{B.1})$$

We treat perturbations $h_{\mu\nu}$ as infinitesimal values.

In general the inverse matrix of $M + \delta M$ is given by $M^{-1} - M^{-1}(\delta M)M^{-1}$. Therefore, the perturbation of inverse matrix of the metric $g_{\mu\nu}$ is given by

$$h^{\mu\nu} \equiv g^{\mu\nu} - \bar{g}^{\mu\nu} = -\bar{g}^{\mu\rho}\bar{g}^{\sigma\nu}h_{\rho\sigma}. \quad (\text{B.2})$$

We note that $h^{\mu\nu} \neq \bar{g}^{\mu\rho}\bar{g}^{\sigma\nu}h_{\rho\sigma}$.

Since the Christoffel symbol and Ricci tensor are defined as

$$\Gamma_{\nu\sigma}^{\mu} = g^{\mu\lambda}\Gamma_{\lambda\nu\sigma} = \frac{1}{2}g^{\mu\lambda}(g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\sigma\nu,\lambda}), \quad (\text{B.3})$$

$$R_{\nu\sigma} \equiv R_{\nu\rho\sigma}^{\rho} = \Gamma_{\nu\sigma,\rho}^{\rho} - \Gamma_{\nu\rho,\sigma}^{\rho} + \Gamma_{\nu\sigma}^{\alpha}\Gamma_{\alpha\rho}^{\rho} - \Gamma_{\nu\rho}^{\alpha}\Gamma_{\sigma\alpha}^{\rho}, \quad (\text{B.4})$$

its perturbations are given by

$$\delta\Gamma_{\nu\sigma}^{\mu} = \frac{1}{2}\bar{g}^{\mu\lambda}(-2h_{\lambda\alpha}\bar{\Gamma}_{\nu\sigma}^{\alpha} + h_{\lambda\nu,\sigma} + h_{\lambda\sigma,\nu} - h_{\sigma\nu,\lambda}) \quad (\text{B.5})$$

$$\delta R_{\nu\sigma} = \delta\Gamma_{\nu\sigma,\rho}^{\rho} - \delta\Gamma_{\nu\rho,\sigma}^{\rho} + \delta\Gamma_{\nu\sigma}^{\alpha}\bar{\Gamma}_{\alpha\rho}^{\rho} + \bar{\Gamma}_{\nu\sigma}^{\alpha}\delta\Gamma_{\alpha\rho}^{\rho} - \delta\Gamma_{\nu\rho}^{\alpha}\bar{\Gamma}_{\sigma\alpha}^{\rho} - \bar{\Gamma}_{\nu\rho}^{\alpha}\delta\Gamma_{\sigma\alpha}^{\rho} \quad (\text{B.6})$$

$$\delta R_{00} = \frac{1}{2a^2}h_{00,ii} + \frac{3}{2}Hh_{00,0} - \frac{1}{a^2}h_{i0,i0} + \frac{1}{2a^2}\left[h_{ii,00} - 2Hh_{ii,0} + 2\left(H^2 - \frac{\ddot{a}}{a}\right)h_{ii}\right] \quad (\text{B.7})$$

$$\delta R_{0j} = \delta R_{j0} = Hh_{00,j} + \frac{1}{2a^2}(h_{j0,ii} - h_{i0,ij}) - \left(\frac{\ddot{a}}{a} + 2H^2\right)h_{j0} + \frac{1}{2}\left[\frac{1}{a^2}(h_{kk,j} - h_{kj,k})\right]_{,0} \quad (\text{B.8})$$

$$\begin{aligned} \delta R_{ij} = & -\frac{1}{2}h_{00,kj} - (2\dot{a}^2 + a\ddot{a})\delta_{jk}h_{00} - \frac{1}{2}a\ddot{a}\delta_{jk}h_{00,0} + \frac{1}{2a^2}(h_{jk,ii} - h_{ik,ij} - h_{ij,ik} + h_{ii,jk}) \\ & - \frac{1}{2}h_{jk,00} + \frac{1}{2}H(h_{jk,0} - \delta_{jk}h_{ii,0}) + H^2(-2h_{jk} + \delta_{jk}h_{ii}) + H\delta_{jk}h_{i0,i} \\ & + \frac{1}{2}(h_{k0,0j} + h_{j0,0k}) + \frac{1}{2}H(h_{k0,j} + h_{j0,k}) \end{aligned}$$

On the other hand, Einstein equation is

$$R_{\mu\nu} = -8\pi GS_{\mu\nu} \quad (\text{B.9})$$

Here, $S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}T_{\rho\sigma}$ and then,

$$\delta S_{\mu\nu} = \delta T_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\delta T_{\lambda}^{\lambda} - \frac{1}{2}h_{\mu\nu}\bar{g}^{\rho\sigma}\bar{T}_{\rho\sigma}, \quad (\text{B.10})$$

where T_{ρ}^{ρ} is given by $T_{\rho}^{\rho} \equiv g^{\rho\sigma}T_{\rho\sigma}$.

Since the background energy momentum tensor of the perfect fluid doesn't change from Eq. (2.8) and Friedmann equation is still

$$\bar{\rho} = \frac{3}{8\pi G}H^2, \quad (\text{B.11})$$

$$\bar{p} = -\frac{1}{8\pi G}\left(2\frac{\ddot{a}}{a} + H^2\right). \quad (\text{B.12})$$

Then, the perturbative equation is given by

$$\bar{g}^{\rho\sigma}\bar{T}_{\rho\sigma} = 3\bar{p} - \bar{\rho} = \frac{-3}{4\pi G} \left(\frac{\ddot{a}}{a} + H^2 \right), \quad (\text{B.13})$$

or writing down each component separately,

$$\begin{cases} \delta S_{00} = \delta T_{00} + \frac{1}{2}\delta T_{\lambda}^{\lambda} + \frac{3}{8\pi G} \left(\frac{\ddot{a}}{a} + H^2 \right) h_{00} \\ \delta S_{0i} = \delta T_{i0} + \frac{3}{8\pi G} \left(\frac{\ddot{a}}{a} + H^2 \right) h_{i0} \\ \delta S_{jk} = \delta T_{jk} - \frac{a^2}{2}\delta_{jk}\delta T_{\lambda}^{\lambda} + \frac{3}{8\pi G} \left(\frac{\ddot{a}}{a} + H^2 \right) h_{jk} \end{cases}. \quad (\text{B.14})$$

The perturbative parts of Eq. (B.9) are given by

$$\begin{aligned} \frac{1}{2a^2}h_{00,ii} + \frac{3}{2}Hh_{00,0} - \frac{1}{a^2}h_{i0,i0} + \frac{1}{2a^2}[h_{ii,00} - 2Hh_{ii,0} + 2\left(H^2 - \frac{\ddot{a}}{a}\right)h_{ii}] \\ = -8\pi G \left(\delta T_{00} + \frac{1}{2}\delta T_{\lambda}^{\lambda} \right) - 3 \left(\frac{\ddot{a}}{a} + H^2 \right) h_{00} \end{aligned} \quad (\text{B.15})$$

$$-8\pi G\delta T_{j0} = Hh_{00,j} + \frac{1}{2a^2}(h_{j0,ii} - h_{i0,ij}) + \left(2\frac{\ddot{a}}{a} + H^2 \right) h_{j0} + \frac{1}{2} \left[\frac{1}{a^2}(h_{kk,j} - h_{kj,k}) \right]_{,0} \quad (\text{B.16})$$

$$\begin{aligned} -8\pi G \left(\delta T_{jk} - \frac{a^2}{2}\delta_{jk}\delta T_{\lambda}^{\lambda} \right) = & -\frac{1}{2}h_{00,kj} - (2\dot{a}^2 + a\ddot{a})\delta_{jk}h_{00} - \frac{1}{2}a\dot{a}\delta_{jk}h_{00,0} \\ & + \frac{1}{2a^2}(h_{jk,ii} - h_{ik,ij} - h_{ij,ik} + h_{ii,jk}) - \frac{1}{2}h_{jk,00} \\ & + \frac{1}{2}H(h_{jk,0} - \delta_{jk}h_{ii,0}) + H^2(h_{jk} + \delta_{jk}h_{ii}) + H\delta_{jk}h_{i0,i} \\ & + \frac{1}{2}(h_{k0,0j} + h_{j0,0k}) + \frac{1}{2}H(h_{k0,j} + h_{j0,k}) + 3\frac{\ddot{a}}{a}h_{jk} \end{aligned} \quad (\text{B.17})$$

In the last of this section, we calculate the energy momentum conservation,

$$T^{\mu\nu}_{;\mu} = 0. \quad (\text{B.18})$$

The perturbative part of this equation is given by

$$(\delta T_{\nu}^{\mu})_{,\mu} + \bar{\Gamma}_{\mu\lambda}^{\mu}\delta T_{\nu}^{\lambda} - \bar{\Gamma}_{\mu\nu}^{\lambda}\delta T_{\lambda}^{\mu} + \delta\Gamma_{\mu\lambda}^{\mu}\bar{T}_{\nu}^{\lambda} - \delta\Gamma_{\mu\nu}^{\lambda}\bar{T}_{\lambda}^{\mu} = 0,$$

and we obtain the energy conservation and momentum conservation from $\nu = 0, j$, respectively:

$$\delta T_{0,0}^0 + \delta T_{0,i}^i + 3H\delta T_0^0 - H\delta T_i^i - \left(\frac{\bar{\rho} + \bar{p}}{2a^2}\right)(-2Hh_{ii} + \dot{h}_{ii}) = 0, \quad (\text{B.19})$$

$$\delta T_{j,0}^0 + \delta T_{j,i}^i + 2H\delta T_j^0 - a\dot{a}\delta T_0^j - (\bar{\rho} + \bar{p})\left(\frac{1}{2}h_{00,j} - Hh_{j0}\right) = 0. \quad (\text{B.20})$$

B.2 Scalar-Vector-Tensor decomposition

The derived equation in the previous section like Eqs. (B.15), (B.16), (B.17), (B.19), (B.20) are very complicated and these equations are severely difficult to solve. Therefore, in this subsection, we decompose the perturbation to scalar term vector term and tensor term and simplify the equations. We call this procedure Scalar-Vector-Tensor (SVT) decomposition.

First, we decompose the perturbed metric $h_{\mu\nu}$. Since h_{00} is a scalar and we cannot decompose it. So, we just rename as follows,

$$h_{00} = -E. \quad (\text{B.21})$$

Next, we consider the 3-dimensional vector, h_{0i} . According to Helmholtz's theorem, any vector can be decomposed into a divergence term and a curl (vortex) term. Using this guiding principle, we decompose it into a vector component and a scalar component,

$$h_{i0} = a(F_{,i} + G_i). \quad (\text{B.22})$$

Here the divergence of G_i should be zero,

$$\sum_i G_{i,i} = 0. \quad (\text{B.23})$$

We consider the physics on curved space time, and so we need to replace the differential term $G_{,i}^i$ to the covariance differential term $G_{|i}^i \equiv G_{,i}^i + \Gamma_{ji}^i G^j$. However, $\bar{\Gamma}_{jk}^i$ is 0, and we don't need this procedure. We can apply the similar principle into the 3-dimensional tensor h_{ij} ,

$$h_{ij} = a^2(A\delta_{ij} + B_{,ij} + C_{i,j} + C_{j,i} + D_{ij}). \quad (\text{B.24})$$

Here, there is another constraints

$$\begin{aligned}\sum_i C_{i,i} &= 0, \\ \sum_i D_{ij,i} &= 0, \\ \sum_i D_{ii} &= 0.\end{aligned}\tag{B.25}$$

Next we decompose the momentum energy tensor into the scalar, vector, and tensor mode. Our energy momentum tensor is given by

$$T_{\mu\nu} = pg_{\mu\nu} + (p + \rho)u_\mu u_\nu + t_{\mu\nu}.\tag{B.26}$$

Each physical quantity consist of background value (which is denoted with bar) and perturbed value like,

$$u_\mu = \bar{u}_\mu + \delta u_\mu.$$

On the other hand, since $g_{\mu\nu}u^\mu u^\nu = -1$ ¹,

$$h_{00} + \bar{g}_{00}(\bar{u}^0 + \delta u^0)(\bar{u}^0 + \delta u^0) + \bar{g}_{00} = -1,$$

and noting that $\bar{u}_i = 0$ $\bar{u}_0 = -1$, we obtain

$$\delta u^0 = \frac{h_{00}}{2},\tag{B.27}$$

$$\delta u_0 \equiv \delta(g_{0\mu}u^\mu) = h_{00}\bar{u}^0 + \bar{g}_{00}\delta u^0 = \frac{h_{00}}{2}(\neq \bar{g}_{0\mu}\delta u^\mu).\tag{B.28}$$

Let's calculate the energy momentum tensor of (0, 0) component,

$$\begin{aligned}\delta T_{00} &= \bar{p}h_{00} - \delta p + (\delta p + \delta \rho) - 2(\bar{p} + \bar{\rho})\frac{h_{00}}{2} \\ &= -\bar{\rho}h_{00} + \delta \rho.\end{aligned}$$

Similarly, we obtain

$$\delta T_{0i} = \bar{p}h_{i0} - (\bar{p} + \bar{\rho})\delta u_i,\tag{B.29}$$

¹It is evident that this equation is correct in a fixed coordinate system. Transforming it to a general coordinate system, we can verify this.

$$\delta T_{ij} = \bar{p}h_{ij} + a^2\delta_{ij}\delta p + t_{ij}. \quad (\text{B.30})$$

Decomposing δu_i of Eq. (B.29) into scalar, vector mode and t_{ij} of Eq. (B.30) into scalar, vector, tensor mode, we obtain

$$\delta T_{0i} = \bar{p}h_{i0} - (\bar{p} + \bar{\rho})(\delta u_i^V + \delta u_{,i}) \quad (\text{B.31})$$

$$\delta T_{ij} = \bar{p}h_{ij} + a^2(\delta_{ij}\delta p + \pi_{,ij}^S + \pi_{j,i}^V + \pi_{i,j}^V + \pi_{ij}^T) \quad (\text{B.32})$$

Here, we have another constraints

$$\pi_{i,i}^V = \delta u_{i,i}^V = 0 \quad \pi_{ij,i}^T = 0 \quad \pi_{ii}^T = 0 \quad (\text{B.33})$$

Given the definition of δu_i , we note that

$$\delta u^i \equiv \delta(g^{i\mu}u_\mu) \neq g^{i\mu}\delta u_\mu. \quad (\text{B.34})$$

Writing down each momentum tensor, we obtain

$$\begin{cases} \delta T_j^i = \delta_{ij}\delta p + \pi_{,ij}^S + \pi_{j,i}^V + \pi_{i,j}^V + \pi_{ij}^T \\ \delta T_0^i = \frac{1}{a^2}(\bar{p} + \bar{\rho})(aF_{,i} + aG_i - \delta u_i^V - \delta u_{,i}) \\ \delta T_i^0 = (\bar{p} + \bar{\rho})(\delta u_i^V + \delta u_{,i}) \\ \delta T_0^0 = -\delta\rho \\ \delta T_\lambda^\lambda = 3\delta p - \delta\rho + \nabla^2\pi^S \end{cases}. \quad (\text{B.35})$$

Next we consider the perturbative equation of motion, like Eq. (B.16). Using Eqs. (B.12), (B.25), we obtain

$$8\pi G(\bar{\rho} + \bar{p})\delta u_j^V + 8\pi G(\bar{\rho} + \bar{p})\partial_j\delta u = \frac{1}{2a}\nabla^2 G_j - \frac{1}{2}\nabla^2 \dot{C}_j - H\partial_j E + \partial_j \dot{A}.$$

The equation from the curl component (divergence-free vector) and the scalar component (divergence or compression) satisfy the equality separately. Therefore, writing them separately, we obtain:

$$8\pi G(\bar{\rho} + \bar{p})a\delta u_j^V = \frac{1}{2}\nabla^2 G_j - \frac{a}{2}\nabla^2 \dot{C}_j,$$

$$8\pi Ga(\bar{\rho} + \bar{p})\partial_j\delta u = -\dot{a}\partial_j E + a\partial_j \dot{A}.$$

By combining the results from the other perturbed Einstein equations and the equations derived from the conservation of energy and momentum, we obtain the following set of equations:

For the scalar mode, from i, j components of the Einstein equation (B.17), the terms proportional to δ_{ij} are

$$\begin{aligned} & -4\pi G a^2 [\delta\rho - \delta p - \nabla^2 \pi^S] \\ & = \frac{1}{2} a \dot{a} \dot{E} + (2\dot{a}^2 + a\ddot{a})E + \frac{1}{2} \nabla^2 A - \frac{1}{2} a^2 \ddot{A} - 3a\dot{a}\dot{A} - \frac{1}{2} a\dot{a} \nabla^2 \dot{B} + \dot{a} \nabla^2 F, \end{aligned} \quad (\text{B.36})$$

from the i, j components of the Einstein equation (B.17), the terms proportional to $\partial_i \partial_j$ are

$$\partial_i \partial_j [16\pi G a^2 \pi^S + E + A - a^2 \ddot{B} - 3a\dot{a}\dot{B} + 2a\dot{F} + 4\dot{a}F] = 0, \quad (\text{B.37})$$

from the $0, j$ components of the Einstein equation (B.16), the terms proportional to ∂_j are

$$8\pi G a (\bar{\rho} + \bar{p}) \partial_j \delta u = -\dot{a} \partial_j E + a \partial_j \dot{A}, \quad (\text{B.38})$$

from the $0, 0$ components of the Einstein equation (B.15), the scalar mode terms are

$$\begin{aligned} & -4\pi G (\delta\rho + 3\delta p + \nabla^2 \pi^S) \\ & = -\frac{1}{2a^2} \nabla^2 E - \frac{3}{2} H \dot{E} - \frac{1}{a} \nabla^2 \dot{F} - \frac{\dot{a}}{a^2} \nabla^2 F + \frac{3}{2} \ddot{A} + 3H\dot{A} - \frac{3\ddot{a}}{a} E + \frac{1}{2} \nabla^2 \ddot{B} + H \nabla^2 \dot{B}, \end{aligned} \quad (\text{B.39})$$

from the momentum conservation law (B.20), the scalar mode terms are

$$\partial_j \left[\delta p + \nabla^2 \pi^S + \partial_0 [(\bar{\rho} + \bar{p}) \delta u] + 3H(\bar{\rho} + \bar{p}) \delta u + \frac{1}{2} (\bar{\rho} + \bar{p}) E \right] = 0, \quad (\text{B.40})$$

from the energy conservation law (B.19), the scalar mode terms are

$$\begin{aligned} & \dot{\delta\rho} + 3H(\delta\rho + \delta p) + \nabla^2 [-a^{-1}(\bar{\rho} + \bar{p})F + a^{-2}(\bar{\rho} + \bar{p})\delta u + H\pi^S] \\ & + \frac{1}{2} (\bar{\rho} + \bar{p}) \partial_0 [3A + \nabla^2 B] = 0. \end{aligned} \quad (\text{B.41})$$

For the vector mode, from the i, j components of the Einstein equation (B.17), the terms proportional to ∂_k are

$$\partial_k [16\pi G a^2 \pi_j^V - a^2 \ddot{C}_j - 3a\dot{a}\dot{C}_j + a\dot{G}_j + 2\dot{a}G_j] = 0 \quad (\text{B.42})$$

from the $0, j$ components of the Einstein equation (B.16), the vector mode terms are

$$8\pi G (\bar{\rho} + \bar{p}) a \delta u_j^V = \frac{1}{2} \nabla^2 G_j - \frac{a}{2} \nabla^2 \dot{C}_j \quad (\text{B.43})$$

from the momentum conservation law (B.20), the vector mode terms are

$$\nabla^2 \pi_j^V + \partial_0[(\bar{\rho} + \bar{p})\delta u_j^V] + 3H(\bar{\rho} + \bar{p})\delta u_j^V = 0 \quad (\text{B.44})$$

In the case of a perfect fluid, $\pi_j^V = 0$ and from Eq. (B.44), we obtain

$$(\bar{\rho} + \bar{p})\delta u_j^V \sim a^{-3} \quad (\text{B.45})$$

and using Eq. (B.43), we also obtain

$$G_j - a\dot{C}_j \sim a^{-2}. \quad (\text{B.46})$$

For the tensor mode, from the i, j components of the Einstein equation (B.17), the tensor mode terms are

$$-16\pi G a^2 \pi_{ij}^T = \nabla^2 D_{ij} - a^2 \ddot{D}_{ij} - 3a\dot{a}\dot{D}_{ij}. \quad (\text{B.47})$$

Using the SVT decomposition, useful formulas can also be derived from the number density conservation law,

$$(nu^\mu)_{;\mu} = (nu^\mu)_{,\mu} + n\Gamma_{\nu\mu}^\mu u^\nu = 0. \quad (\text{B.48})$$

In the background level,

$$\dot{\bar{n}} = -3H\bar{n}. \quad (\text{B.49})$$

This indicate that $\bar{n}$ is proportional to a^{-3} .

In the perturbative level,

$$(\bar{n}\delta u^\mu)_{,\mu} + (\delta n \bar{u}^\mu)_{,\mu} + \delta n \bar{\Gamma}_{\nu\mu}^\mu \bar{u}^\nu + \bar{n}\delta \Gamma_{\nu\mu}^\mu \bar{u}^\nu + \bar{n}\bar{\Gamma}_{\nu\mu}^\mu \delta u^\nu = 0. \quad (\text{B.50})$$

The first term of Eq. (B.50) is

$$\begin{aligned} (\bar{n}\delta u^\mu)_{,\mu} &= (\bar{n}\delta u^0)_{,0} + (\bar{n}\delta u^i)_{,i} = \bar{n}_{,0} \left(-\frac{E}{2}\right) + \bar{n} \left(-\frac{E_{,0}}{2}\right) + \bar{n}[\delta(g^{i\mu}u_\mu)]_{,i} \\ &= \bar{n}_{,0} \left(-\frac{E}{2}\right) + \bar{n} \left(-\frac{E_{,0}}{2}\right) + \bar{n} \left[-h_{,i}^{i0} + \frac{1}{a^2}(\delta u^{iV} + \partial^i \delta u)_{,i}\right] \\ &= \bar{n}_{,0} \left(-\frac{E}{2}\right) + \bar{n} \left(-\frac{E_{,0}}{2}\right) + \bar{n} \left(\frac{1}{a^2}\nabla^2 \delta u - \frac{1}{a}\nabla^2 F\right) \end{aligned} \quad (\text{B.51})$$

We use the Eqs. (B.21), (B.27), (B.34) for the second equal sign and Eqs. (B.23), (B.33) for the fourth equal sign.

Noting Eq.(B.49), the second to fourth terms of the eq B.50 are calculated as,

$$\begin{aligned}
(\delta n \bar{u}^\mu)_{,\mu} + \delta n \bar{\Gamma}_{\nu\mu}^\mu \bar{u}^\nu &= \delta n_{,0} + \delta n \bar{\Gamma}_{0\mu}^\mu \\
&= \delta n_{,0} + 3H\delta n \\
&= \delta n_{,0} - \delta n \frac{\bar{n}_{,0}}{\bar{n}} \\
\bar{n} \delta \Gamma_{\nu\mu}^\mu \bar{u}^\nu &= \bar{n} \delta \Gamma_{0\mu}^\mu \\
&= \bar{n} \left(\frac{E_{,0}}{2} + \frac{3A_{,0} + \nabla^2 B_{,0}}{2} \right) \\
\bar{n} \bar{\Gamma}_{\nu\mu}^\mu \delta u^\nu &= 3H\bar{n}\delta u^0 \\
&= \frac{1}{2}\bar{n}_{,0}E
\end{aligned}$$

Summing up the above results, we obtain the continuity equation for the first-order perturbation of the number density:

$$\partial_t \left(\frac{\delta n}{\bar{n}} \right) + \frac{1}{a^2} \nabla^2 \delta u + \frac{1}{2} (3\dot{A} + \nabla^2 \dot{B}) - \frac{1}{a} \nabla^2 F = 0. \quad (\text{B.52})$$

B.3 gauge transformation

The definition of the perturbations through SVT decomposition in the previous section is extremely general and contains components beyond the degrees of freedom. In other words, as a result of choosing an unnecessarily complex coordinate system to understand the prospects of perturbations, we have allowed the presence of gravitational potentials that lack much physical meaning (so-called "redundant" potentials). For instance, considering an unnecessary accelerated frame can give rise to an apparent gravitational potential. Therefore, we would like to eliminate these redundant potentials by choosing an appropriate coordinate system. Naturally, this leads us to consider coordinate transformations. Since we are specifically discussing first-order perturbations, it would be useful to consider the infinitesimal coordinate transformations (gauge transformations) for first-order perturbations.

$$x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu(x) \quad (\text{B.53})$$

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = g_{\lambda\kappa}(x) \frac{\partial x^\lambda}{\partial x'^\mu} \frac{\partial x^\kappa}{\partial x'^\nu} \quad (\text{B.54})$$

How does the first-order perturbation of the metric tensor $h_{\mu\nu}$ transform under a general coordinate transformation (as in B.53) At this point, we need to be cautious about the nature of our coordinate transformation. What we are interested in comparing are quantities like $\rho'(x' = x)$ and $\rho(x = x)$. Under an ordinary coordinate transformation, background quantities would also undergo a transformation. However, we are stating that only the first-order perturbation changes, and the background spacetime quantities are not allowed to contain perturbative components. Therefore, the background spacetime quantities remain unchanged. It may appear as though we are observing physical quantities at different spacetime points, but the difference is understood as arising from the change in the perturbative components under the coordinate transformation.

$$\begin{aligned}\Delta h_{\mu\nu}(x) &\equiv h'_{\mu\nu}(x' = x) - h_{\mu\nu}(x = x) \\ &= g'_{\mu\nu}(x' = x) - g_{\mu\nu}(x = x) - (\bar{g}'_{\mu\nu}(x' = x) - \bar{g}_{\mu\nu}(x = x))\end{aligned}$$

considering $\bar{g}'_{\mu\nu}(x' = x) - \bar{g}_{\mu\nu}(x = x) = 0$ and then

$$\begin{aligned}\Delta h_{\mu\nu}(x) &= g'_{\mu\nu}(x' = x) - g_{\mu\nu}(x = x) \\ &= g'_{\mu\nu}(x' - \epsilon) - g_{\mu\nu}(x) \\ &= g'_{\mu\nu}(x') - \bar{g}_{\mu\nu,\lambda}\epsilon^\lambda - g_{\mu\nu}(x) \\ &= g_{\lambda\kappa}(x) \frac{\partial x^\lambda}{\partial x'^\mu} \frac{\partial x^\kappa}{\partial x'^\nu} - \bar{g}_{\mu\nu,\lambda}\epsilon^\lambda - g_{\mu\nu}(x) \\ &= -\bar{g}_{\lambda\mu}(x)\epsilon^\lambda_{,\nu} - \bar{g}_{\nu\lambda}(x)\epsilon^\lambda_{,\mu} - \bar{g}_{\mu\nu,\lambda}\epsilon^\lambda.\end{aligned}\tag{B.55}$$

We use Eq. (B.54) in the second last equall.

The subscription of ϵ^μ is raised and lowered using the metric $g_{\mu\nu}$. Writing the each $\Delta h_{\mu\nu}(x)$,

$$\Delta h_{ij} = -\epsilon_{i,j} - \epsilon_{j,i} + 2a\dot{a}\delta_{ij}\epsilon_0,\tag{B.56}$$

$$\Delta h_{i0} = -\epsilon_{i,0} - \epsilon_{0,i} + 2H\epsilon_i,\tag{B.57}$$

$$\Delta h_{00} = -2\epsilon_{0,0}.\tag{B.58}$$

We should note that $\epsilon_\mu = g_{\mu\nu}\epsilon^\nu$, $\epsilon_0 = -\epsilon^0$, $\epsilon_i = a^2\epsilon^i$.

The previous discussion did not specifically rely on the fact that the subject was the metric. Therefore, a similar argument can also be applied to the energy-momentum tensor.

$$\Delta\delta T_{\mu\nu} = -\bar{T}_{\lambda\mu}\epsilon_{,\nu}^\lambda - \bar{T}_{\nu\lambda}\epsilon_{,\mu}^\lambda - \bar{T}_{\mu\nu,\lambda}\epsilon^\lambda, \quad (\text{B.59})$$

$$\Delta\delta T_{ij} = -\bar{p}\epsilon_{i,j} - \bar{p}\epsilon_{j,i} + (a^2\bar{p})_{,0}\delta_{ij}\epsilon_0, \quad (\text{B.60})$$

$$\Delta\delta T_{i0} = -\bar{p}\epsilon_{i,0} + \bar{\rho}\epsilon_{0,i} + 2\bar{p}H\epsilon_i, \quad (\text{B.61})$$

$$\Delta\delta T_{00} = 2\bar{\rho}\epsilon_{0,0} + \dot{\bar{\rho}}\epsilon_0. \quad (\text{B.62})$$

Furthermore, decomposing ϵ_i into vorticity and divergence modes,

$$\epsilon_i = \epsilon_{,i}^S + \epsilon_i^V. \quad (\text{B.63})$$

Here, we should note that the divergence of the vector mode is zero,

$$\epsilon_{i,i}^V = 0. \quad (\text{B.64})$$

Next, let's examine the specific component transformations. For example, from the Eqs.(B.22),(B.57),

$$\begin{aligned} \Delta h_{i0} &= a\Delta(F_{,i} + G_i) \\ &= -(\epsilon_{,i}^S + \epsilon_i^V)_{,0} - \epsilon_{0,i} + 2H(\epsilon_{,i}^S + \epsilon_i^V) \\ &\begin{cases} \Delta F = \frac{1}{a}(-\dot{\epsilon}^S - \epsilon_0 + 2H\epsilon^S) \\ \Delta G_i = \frac{1}{a}(-\dot{\epsilon}_i^V + 2H\epsilon_i^V) \end{cases} \end{aligned}$$

Of course, F undergoes a transformation corresponding to scalar modes, while G undergoes a transformation corresponding to vector modes. Similarly,

$$\begin{cases} \Delta A = 2H\epsilon_0 \\ \Delta B = -\frac{2}{a^2}\epsilon^S \\ \Delta C_i = -\frac{1}{a^2}\epsilon_i^V \\ \Delta D_{ij} = 0 \\ \Delta E = 2\dot{\epsilon}_0 \\ \Delta F = \frac{1}{a}(-\dot{\epsilon}^S - \epsilon_0 + 2H\epsilon^S) \\ \Delta G_i = \frac{1}{a}(-\dot{\epsilon}_i^V + 2H\epsilon_i^V) \end{cases} \quad (\text{B.65})$$

$$\begin{cases} \Delta\delta p = \dot{\bar{p}}\epsilon_0 \\ \Delta\delta\rho = \dot{\bar{\rho}}\epsilon_0 \\ \Delta\delta u = -\epsilon_0 \end{cases} \quad (\text{B.66})$$

$$\Delta\pi^S = \Delta\pi_i^V = \Delta\pi_{ij}^T = \Delta\delta u_i^V = 0 \quad (\text{B.67})$$

The conditions for a perfect fluid, such as $\pi^S = \pi_i^V = \pi_{ij}^T = 0$ and the condition that the velocity field is a potential flow $\delta u_i^V = 0$, are gauge-invariant. Under any coordinate transformation, a scalar quantity $s(x)$ satisfies $s(x) = s'(x')$, but perturbations transform under gauge transformations as follows:

$$\begin{aligned} \Delta\delta s(x) &= s'(x) - s(x) = s'(x) - s'(x') \\ &= s'(x) - s'(x + \epsilon) = -\bar{s}_{,\mu}\epsilon^\mu(x) = -\dot{\bar{s}}(t)\epsilon_0 \end{aligned} \quad (\text{B.68})$$

The final step in the derivation uses the fact that background quantities do not change with coordinates (i.e., they have no fluctuations).

B.4 Selecting Gauge

Vector perturbations decay with cosmic expansion. Tensor perturbations, however, are gauge-invariant because gauge transformations only affect vector and scalar modes. Therefore, the perturbations of interest and those where gauge considerations are meaningful are the scalar perturbations A, B, E, F . Although there are four perturbations, their number can be reduced by appropriately choosing ϵ .

Newton Gauge

We choose ϵ^S to be $B = 0$ and ϵ_0 to be $F = 0$ and rewrite A, E in the following convention,

$$\begin{aligned} A &\equiv -2\Psi, \\ E &\equiv 2\Phi. \end{aligned} \quad (\text{B.69})$$

Therefore, we can write the scalar perturbation as

$$g_{00} = -1 - 2\Phi, \quad (\text{B.70})$$

$$g_{0i} = 0, \quad (\text{B.71})$$

$$g_{ij} = a^2\delta_{ij}(1 - 2\Psi). \quad (\text{B.72})$$

Synchronous gauge

We choose ϵ_0 to be $E = 0$ and ϵ^S to be $F = 0$. Then we obtain the scalar metric,

$$g_{00} = -1, g_{0i} = 0, g_{ij} = a^2[(1 + A)\delta_{ij} + B_{,ij}]. \quad (\text{B.73})$$

When we choose the synchronous gauge, the Einstein equations and energy-momentum conservation Eq. (B.36)-(B.41) should be

$$-4\pi G a^2[\delta\rho - \delta p - \nabla^2\pi^S] = \frac{1}{2}\nabla^2 A - \frac{1}{2}a^2\ddot{A} - 3a\dot{a}\dot{A} - \frac{1}{2}a\dot{a}\nabla^2\dot{B} \quad (\text{B.74})$$

$$16\pi G a^2\pi^S + A - a^2\ddot{B} - 3a\dot{a}\dot{B} = 0 \quad (\text{B.75})$$

$$8\pi G a(\bar{\rho} + \bar{p})\delta u = a\dot{A} \quad (\text{B.76})$$

$$-4\pi G(\delta\rho + 3\delta p + \nabla^2\pi^S) = \frac{3}{2}\ddot{A} + 3H\dot{A} + \frac{1}{2}\nabla^2\ddot{B} + H\nabla^2\dot{B} \quad (\text{B.77})$$

$$\delta p + \nabla^2\pi^S + \partial_0[(\bar{\rho} + \bar{p})\delta u] + 3H(\bar{\rho} + \bar{p})\delta u = 0 \quad (\text{B.78})$$

$$\dot{\delta\rho} + 3H(\delta\rho + \delta p) + \nabla^2[a^{-2}(\bar{\rho} + \bar{p})\delta u + H\pi^S] + \frac{1}{2}(\bar{\rho} + \bar{p})\partial_0[3A + \nabla^2 B] = 0 \quad (\text{B.79})$$

There still exists the gauge freedom. Considering the gauge transformation

$$\epsilon_0 = -\tau(\mathbf{x}) \epsilon^S = a^2(t)\tau(\mathbf{x}) \int a^{-2}(t)dt, \quad (\text{B.80})$$

and the Eqs. (B.65), (B.66), we obtain

$$\Delta E = 0 \text{ and } \Delta F = 0.$$

Therefore, this transformation keep the satisfaction of the conditions of the synchronous gauge, but

$$\Delta A = -\frac{2\dot{a}\tau}{a} \Delta B = -2\tau \int a^{-2}(t)dt$$

$$\Delta\delta p = -\dot{\bar{p}}\tau \Delta\delta\rho = -\dot{\bar{p}}\tau \Delta\delta u = \tau$$

We would have to individually verify whether the values of quantities such as A that we obtained are physical or not. This degree of freedom can be removed when there exists the matter D , which moves slower than the speed of light,

$$\bar{p}_D = 0 \bar{\rho}_D \propto a^{-3} \delta p_D = \pi_D^S = 0$$

From the Eq. (B.78), δu shouldn't be depend of the time and we can remove the degree as we set $\tau = -\delta u$.

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