Doctoral Dissertation

Flavor structures from string compactification and moduli stabilization

（超弦理論のコンパクト化およびモジュライ安定化から導かれるフレーバー構造）

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March 2025

Acknowledgement

I would like to express my greatest appreciation to my supervisor, Tatsuo Kobayashi, for the physics he has taught me, for his generosity of time and spirit, and for his continual encouragement. I am also very grateful to many of my collaborators, Hikaru Uchida, Shota Kikuchi, Shohei Takada, Riku Sakuma, Tim Jeric, Hironobu Mita, Ryusei Nishida, Hajime Otsuka, Tetsutaro Higaki, Yusuke Yamada, Shohei Uemura, Junichiro Kawamura, Yoshihiko Abe, Komei Goto, Shuta Funakoshi, Kouki Hoshiya, and Yuya Ogawa for useful discussions and insightful comments. I also thank all members of the theoretical particle and cosmological physics group at Hokkaido University and the people involved with me in the Ph.D. course. The study in the Ph.D. course would not have been possible without the help of them. I have been financially supported by Grant-in-Aid for JSPS Research Fellows No. JP24KJ0249 and JST SPRING Grant No. JPMJSP2119.

Abstract

We study the realization of flavor structures in the Standard Model of particle physics from the string-inspired effective field theories. Firstly, we study magnetized toroidal compactifications including T^4 and $T^4/\mathbb{Z}_N$ orbifold models within ten-dimensional $\mathcal{N}=1$ super Yang-Mills theory. Chiral fermion zero-mode wavefunctions and their degeneracies are explicitly analyzed. Conditions needed to realize three-generation models are clarified. We use modular transformation to count the number of zero-modes systematically. Secondly, we study the stabilization of the complex structure modulus τ of a toroidal extra-dimensional space since its vacuum expectation value is related to flavor structures in four-dimensional modular flavor symmetric models. We analyze the one-loop Coleman-Weinberg potential which results from the radiative corrections of matter fields whose mass terms are written by modular forms. We have discussed the possibility of slightly shifting the modulus vacuum expectation value from the $\mathbb{Z}_3$ fixed-point $\tau = e^{2\pi i/3}$ where the tree-level potential is minimized. Indeed, such a small deviation from the fixed point can naturally explain the hierarchical masses and mixing angles of quarks and leptons within modular flavor symmetric models. Depending on the choice of modular forms, the number of species contributing to the radiative corrections, etc., modulus can slightly deviate from the fixed point. Lastly, we study spontaneous CP violation through moduli stabilization within four-dimensional $\mathcal{N}=1$ supergravity model. In a CP invariant modular flavor model, the only source of CP violation is the vacuum expectation values of complex structure moduli. We have succeeded in realizing CP-breaking vacua with or without supersymmetry by the combined effects of Type IIB three-form flux and modular symmetric couplings between matter fields and the moduli. We have also revealed some useful properties of CP and modular invariant scalar potential which are model-independent.

Contents

Chapter 1

Introduction

For a long time since Democritus (460-370 B.C.), the most fundamental constituents of matter have been hypothesized as point particles. Indeed, based on the "particle hypothesis", a successful theory, the Standard Model of particle physics was established. The Standard Model quite accurately explains various experimental data. For instance, its perturbative computation reproduces the experimentally measured electron g-factor to the accuracy $\mathcal{O}(10^{-12})$. Nevertheless, the Standard Model is imperfect and some fundamental problems remain. One of the most important puzzles lies in the origin of the flavor structures of quarks and leptons. The reason behind the appearance of three generations, hierarchies in mass ratios, patterns in mixing angles, and CP violation are left unanswered. Furthermore, the origin of the Standard Model's gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ is also unclear. Another serious issue is that the Standard Model is incapable of describing quantum gravity.

To answer these problems, we need to study theories beyond the Standard Model. Superstring theory is the leading candidate with the potential for a consistent description of quantum gravity [1–4]. It is defined at the Planck scale $M_{\rm pl}$ and assumes that strings are the fundamental constituents of matter and forces. If string theory were the ultimate unified theory, we expect that the Standard Model can be derived as its low-energy effective field theory. Thus, the origins of the aforementioned flavor and gauge structures should be unraveled. However, it is not straightforward. This is because we need to consider string compactifications. Since superstring theory requires 10-dimensional (10D) space-time for its theoretical consistency, six-dimensional (6D) extra space dimensions should be compact. So far, we have no principle to choose a unique compact space (vacuum). This is because the vacua of superstring theory are infinitely degenerate at the perturbative level. Non-perturbative effects of superstring theory would provide a hint but unfortunately are not well-understood. For each string vacuum, we have a corresponding four-dimensional (4D) effective field theory, hence we are left with enormous collections of models. Nevertheless, we could still ask the following question: "What kinds of string compactifications can realize the Standard Model as a low-energy 4D effective field theory? (or is there any?)" To tackle the question, we have two approaches, i.e., top-down and bottom-up approaches.

In the top-down approach, we start with the superstring theory or its 10D effective field theories. By choosing certain compact space $\mathcal{M}_6$ and through dimensional reduction, 4D low-energy effective field theories are obtained. Then, we analyze the 4D models to evaluate how well they can realize the phenomenological data obtained in the electro-weak scale $M_{\rm EW}$.

One of the most important requirements for a realistic 4D model is to be a chiral theory. The Standard Model is chiral since the left-handed quarks and leptons transform as doublets under $SU(2)_L$ whereas the right-handed ones are singlets. Also, $U(1)_Y$ charges are different between the left and right. This means that the supersymmetry (SUSY) needs to be reduced down to either 4D $\mathcal{N}=1$ or $\mathcal{N}=0$. Particularly, compactifications that preserve 4D $\mathcal{N}=1$ SUSY are attractive since the absence of tachyonic states is guaranteed. Note that superstring theory has 10D $\mathcal{N}=1,(\mathcal{N}=2)$ SUSY which is equivalent to 4D $\mathcal{N}=4,(\mathcal{N}=8)$ SUSY. Various model buildings have been studied based on superstring theories such as Heterotic, Type IIA, Type IIB, etc [5]. For instance, Calabi-Yau compactifications of heterotic string theories have captured attention because the $SU(3)$ holonomy naturally leads to 4D $\mathcal{N}=1$ SUSY [6]. In the Type IIB or Type I side, it is possible to control the amount of SUSY preserved in the 4D theories by introducing magnetic fluxes on D-branes [7–12]. This kind of model is known as the magnetized D-brane model and is quite interesting. Remarkably, 4D chiral theories can be realized even when we have chosen simpler compact spaces such as T^6 in magnetized D-brane models.

In this paper, we discuss magnetized D-brane models within 10D $\mathcal{N}=1$ super Yang-Mills theory, an effective field theory of superstring theory [13]. (See also [14–17].) One of the simplest models is the factorized torus $T^2\times T^2\times T^2$ with background 2-form Yang-Mills magnetic flux. By appropriately choosing the size of magnetic flux on the torus, semi-realistic gauge groups and three generations of chiral fermions can be obtained. Note that the size of generations corresponds to the number of degenerated zero-modes in the compact space. Furthermore, thanks to the simple geometry of two-dimensional (2D) torus T^2, we can analytically compute Yukawa coupling constants by the overlap integral of fermion zero-mode wavefunctions and boson wavefunction in the extra dimension [13][1]. In the magnetized T^2 model, wavefunctions are written by the Jacobi-theta function with characteristics [13]. Owing to the quasi-localized profiles of those wavefunctions, non-trivial overlaps lead to hierarchical values of Yukawa coupling constants. Nevertheless, magnetized $T^2\times T^2\times T^2$ models are simple toy models. We generally find exotic modes such as the open string moduli which are phenomenologically unwanted [19]. We also find only a limited number of realistic three-generation models. More possibilities for model buildings need to be studied.

Hence, magnetized orbifold $T^2/\mathbb{Z}_N,\ (N=2,3,4,6)$ models have gathered attention [19–27]. $T^2/\mathbb{Z}_N$ orbifolds are defined by imposing $\mathbb{Z}_N$ twist identification on T^2. This allows greater possibilities for realistic model buildings since there are more varieties of three-generation models and unwanted exotic modes can be projected out. The size of generations depends on the magnitude of magnetic flux, $\mathbb{Z}_N$ eigenvalue associated with the $\mathbb{Z}_N$ twist, and the constant

[1] Higher order coupling constants can also be analytically computed [18].

gauge field, i.e., Scherk-Schwarz phases. The zero-mode number have been analyzed using various methods. One of the most efficient ways for zero-mode counting is to apply the modular transformation as in Ref. [28], where conditions for three-generation models are explicitly identified.[2] Phenomenological aspects of the orbifold models have been studied intensively [19–24] and flavor structures of quarks and leptons were realized quite successfully.

Another interesting aspect of magnetized toroidal compactification is the modular symmetry $SL(2,\mathbb{Z})$ associated with the arbitrariness in choosing the basis vectors defining the shape of the torus. Under the modular transformation, matter wavefunctions and Yukawa coupling constants transform non-trivially like modular forms [30–38]. Then, we can identify modular symmetry as a flavor symmetry. Their corresponding modular weights and representations depend on the size of magnetic fluxes, $\mathbb{Z}_N$ eigenvalues, and Scherk-Schwarz phases. Indeed, phenomenologically attractive non-Abelian discrete flavor symmetries can emerge from the modular group.

It is intriguing to extend the analyses to T^{2n} and $T^{2n}/\mathbb{Z}_N, (n = 2,3)$ with background magnetic fluxes including "oblique components" [13, 14, 42]. This is because there is no reason to assume that the 6D extra dimension is factorized into 2D spaces, e.g. $T^2 \times T^2 \times T^2$, other than it will make the analyses simple. Another motivation for the generalization is that it will open up new possibilities for obtaining realistic three-generation models. Moreover, modular symmetries such as $Sp(2n,\mathbb{Z})$ larger than $SL(2,\mathbb{Z})$ can appear under such compactifications and they may become important to understand the origin of flavor structures of the Standard Model. In the case of magnetized T^{2n}, zero-mode wavefunctions are written by Riemann-theta functions [13, 14].

In this paper, we will explicitly analyze the zero-mode wavefunctions and degeneracy on magnetized T^4 and $T^4/\mathbb{Z}_N$ models along Refs. [43–45] since this is the first step towards realistic model buildings. We assume that the 6D compact space is of the form $T^4 \times \mathcal{M}_2$ or $T^4/\mathbb{Z}_N \times \mathcal{M}_2$. Here, $\mathcal{M}_2$ denotes an unspecified 2D compact space on which there is only a single zero-mode. Hence, the zero-mode number is solely determined by the T^4 or $T^4/\mathbb{Z}_N$ parts. We show the explicit form of zero-mode wavefunctions with non-vanishing Scherk-Schwarz phases. We discuss some details about the negative chirality wavefunctions since we succeed in generalizing the result presented in Ref. [14]. For systematic zero-mode countings in $T^4/\mathbb{Z}_N$, $(N > 2)$, we apply the $Sp(4,\mathbb{Z})$ modular transformation of zero-mode wavefunctions. Conditions needed to realize three-generation models will be clarified.[3] By the overlap integral of these wavefunctions, Yukawa couplings can be computed [14, 17], although, comprehensive model building is still under development.

In the bottom-up approach, extensions of the Standard Model are studied by imposing non-Abelian discrete flavor symmetries which could have resulted from the modular symmetry, e.g. $SL(2,\mathbb{Z})$ of the extra dimension such as the torus. The bottom-up model buildings have revealed an interesting relationship between the shapes of compact spaces and flavor structures

[2]Zero-mode counting by index theorem is also studied [29, 39, 40].

[3]Zero-mode counting by index theorem is also studied [41].

providing hints for phenomenologically preferred string compactifications. Recently, modular flavor symmetric models have been attracting great attention. Finite modular symmetries Γ_N including $\Gamma_2 \simeq S_3, \Gamma_3 \simeq A_4, \Gamma_4 \simeq S_4$, and $\Gamma_5 \simeq A_5$ are identified as the flavor symmetry [46]. Under the modular transformation, Yukawa coupling constants and matter fields transform non-trivially, while the Lagrangian is invariant. Yukawa coupling constants are modular forms which is a holomorphic function of the complex structure modulus $\tau \in \mathbb{C}$, $(\mathrm{Im}\tau > 0)$ parametrizing the shape of the compact space. Three generations of quarks and leptons are (reducible or irreducible) three-dimensional representations of the finite modular group and may carry non-vanishing modular weights. Model buildings with S_3 [47], A_4 [48], S_4 [49], and A_5 [50, 51] symmetries have been performed. Furthermore, generalizations to higher levels and covering groups have been studied [36, 52–56]. See Refs. [57, 58] for reviews.

When the modulus τ acquires its vacuum expectation value (VEV), the Yukawa coupling constants are fixed. In general, $SL(2, \mathbb{Z})$ modular symmetry is completely broken by the VEV. However, at the fixed points $\tau = i, \omega := e^{2\pi i/3}$ and $i\infty$, modular S, ST and T symmetries are preserved and we have $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_N$ residual symmetries, respectively [59]. Here, N denotes the level of the finite modular group, Γ_N. It has been shown that the values of modular forms can become hierarchical near the fixed points depending on their charges under the residual symmetries. Modular forms behave like $Y(\tau) \sim \varepsilon^n$ where $|\varepsilon| \ll 1$ denotes the deviation from the fixed points and $n \in \{0, 1, \cdots\}$ is related to the residual charge of $Y(\tau)$. This behavior is useful to realize the hierarchical masses and mixing angles of quarks and leptons without fine-tuning. In Refs. [60, 61], lepton flavor structures have been studied without severe fine-tuning. Similarly, quark flavors have been studied with modular forms of A_4 [62, 63], S'_4 [64], $S'_4 \times S_3$ [65], $\Gamma_6 \simeq S_3 \times A_4$ [66], and $A_4 \times A_4 \times A_4$ [67]. (See also [68].) Furthermore, simultaneous realizations of both quark and lepton flavors have been studied in Refs. [69–71].

CP violation is another important flavor structure of quarks and leptons. In CP invariant modular flavor models where CP symmetry is implemented as an outer automorphism of the modular group, the only source of CP violation is the VEV of the complex structure modulus τ [72]. In Ref. [72], it was also shown that CP is broken by the modulus VEV, if τ and $\tau^{(\mathrm{CP})} := -\bar{\tau}$ cannot be related by any of the elements of the modular group, i.e., $-\bar{\tau} \neq \gamma\tau$, $(\forall\gamma \in SL(2, \mathbb{Z}))$.[4] Recently, it has been pointed out that the simultaneous realization of both quark hierarchical structures and sizable CP violation is quite difficult with single modulus τ [62, 63, 67]. Nevertheless, such difficulty can be lifted by considering multi-moduli VEVs [62, 63, 67, 74].

In short, bottom-up approaches have revealed phenomenologically preferred shapes of (toroidal) compact spaces. That is, VEVs of the complex structure moduli lie in the vicinity of the fixed points, and CP is violated by at least one of the moduli. We emphasize that in most modular flavor symmetric models, those moduli VEVs are chosen by hand as free parameters to realize the flavor structures. Hence, we are led to ask, "What are the dynamic or actual string compactifications that lead to the phenomenologically desirable moduli VEVs?"

[4] Generalizations to CP invariant $Sp(2n, \mathbb{Z})$ Siegel modular flavor models have been studied in [73].

There have been numerous studies on this topic, i.e., moduli stabilization (see for review [5]). In general, moduli refer to massless scalar fields which parametrize string vacua. Note that moduli stabilization is not only important to fix the values of coupling constants but also for providing masses to the scalars so that any new contributions to gravitational interaction, i.e., "fifth force" are avoided. In the type IIB orientifold compactification, complex structure moduli, and axio-dilaton can be stabilized by the three-form flux-induced potential known as the Gukov-Vafa-Witten superpotential [75]. In Ref. [76], type IIB three-form flux compactifications on isotropic $T^6/\mathbb{Z}_2$ and $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2')$ are studied. The results show that stabilization to the $\mathbb{Z}_3$ fixed point $\tau = \omega$ is most statistically favored under the tadpole cancellation conditions. There are also other mechanisms leading to the stabilization of τ at an exact fixed point $\tau = \omega$ [77, 78]. Now, we recall that a slight deviation of the modulus VEV is needed to explain mass hierarchies of quarks and leptons in modular flavor models without fine-tunings. Hence, it is phenomenologically important to study mechanisms that would generate a slight deviation of the modulus VEV from the exact fixed point.

In this paper, we study the possibilities of obtaining such deviations by the radiative corrections due to couplings between matter fields and complex structure modulus τ [79]. We evaluate the possible shift of the VEV from the fixed points by analyzing the Coleman-Weinberg potential at the 1-loop level [80]. Illustrating examples with A_4 modular symmetry are presented, based on which we study the conditions needed to generate the slight shifts. Indeed, various scenarios can successfully stabilize the modulus VEV near the fixed points $\tau = i, \omega, i\infty$ are known [81–85].

In contrast, realization of CP violation through moduli stabilization has not been so straightforward. We stress here that CP should be broken spontaneously. This is because 4D CP can be embedded into the 10D proper Lorentz symmetry and a gauge symmetry of superstring theory [86, 87]. In Ref. [88], the possibility of spontaneous CP violation was studied based on three-form flux compactification with $T^6/\mathbb{Z}_2$ and $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2')$ orientifolds. The background three-form fluxes were constrained by CP symmetry. Then, it was found that vacuum solutions are CP-conserving or include flat directions where CP-conserving and violating vacua are degenerate. This implies that spontaneous CP violation is hardly realized solely by the flux on the toroidal compact space.[5] Other approaches to moduli stabilization were also discussed. For instance, gaugino condensation in the hidden sector was studied within modular invariant supergravity models [90, 91]. However, the obtained VEVs of τ do not break CP. Another interesting approach starts with a general form of modular and CP invariant supergravity Lagrangian Refs. [78,82,83]. (See Refs. [92–94] for early works on moduli stabilization by modular invariant superpotential and Refs. [95–98] for CP violations.) In most cases, spontaneous CP violation hardly occurs, although an exception was discovered in Ref. [82] where CP-breaking VEV of τ is found near the fixed point $\tau = \omega$. However, the obtained vacuum is anti-de Sitter, hence the effect of uplifting may or may not change the occurrence of CP violation.

[5]Such analysis was extended to Calabi-Yau manifolds [89]. The result is essentially the same, although one exceptional example of spontaneous CP violation was found.

In this paper, we study the realization of spontaneous CP violation through the combined effects of CP symmetric Type IIB flux compactifications [88] and modular symmetric matter-moduli couplings [84, 99]. For illustration, we consider factorized $T^6/\mathbb{Z}_2$ orbifold with three complex structure moduli τ_i, $(i = 1, 2, 3)$. Our discussion is based on Ref. [100]. We also clarify some general properties of CP and modular invariant scalar potential V. This allows us to understand why the local minimum of V is typically CP-conserving. We will see that partial breakings or modifications of modular symmetries can favor spontaneous CP violation.

This paper is organized as follows. In chapter 2, we study magnetized torus and orbifold models from the top-down. After briefly reviewing compactification with background magnetic fluxes of 10D $\mathcal{N} = 1$ super Yang-Mills theory in section 2.1, we review magnetized T^2 and $T^2/\mathbb{Z}_N$ orbifold models in sections 2.2 and 2.3. We develop the discussions to magnetized T^4 and $T^4/\mathbb{Z}_N$ orbifold models in sections 2.4, 2.5, and 2.6. In chapter 3, we discuss the stabilization of the complex structure modulus τ. To emphasize the significance of the modulus VEV in modular flavor symmetric models, we review $A_4 \times A_4 \times A_4$ symmetric quark flavor model in section 3.1. In section 3.2, we study radiative corrections on the modulus stabilization of τ to realize phenomenologically preferred modulus VEV. In section 3.3, we illustrate our A_4 model and analyze the 1-loop effective potential numerically and analytically. In chapter 4, we study spontaneous CP violation through moduli stabilization. In sections 4.1 and 4.2, we review CP transformation in higher dimensional theories and 4D modular flavor models, respectively. In section 4.3, we study the general properties of CP and modular symmetric scalar potential. In sections 4.4, 4.5, and 4.6, we study CP invariant moduli stabilization and discuss how to realize spontaneous CP violation. Chapter 5 is devoted to our summary. In Appendix A, we explain more details about magnetized T^4 and $T^4/\mathbb{Z}_N$ models, including negative chirality zero-mode wavefunctions, $Sp(4, \mathbb{Z})$ transformation of zero-modes, and the quantization of Scherk-Schwarz phases on magnetized $T^4/\mathbb{Z}_3$. In Appendix B, we mainly review the A_4 group and its modular forms. In Appendix C, we study another model that can generate spontaneous CP violation through moduli stabilization.

Chapter 2

Magenetized torus and orbifold models

In this chapter, we discuss magnetized torus and orbifold models motivated by superstring theory. We introduce background 2-form Yang-Mills magnetic field on the toroidal compact space. This opens up the possibilities for building realistic 4D effective field theories from the top down in the sense that both flavor and gauge structures of the Standard Model may be realized. One of the most important criteria for realistic models is the three-generation of quarks and leptons. Hence, we explicitly analyze the conditions to produce three-generation models under the magnetized torus and orbifold compactifications. Moreover, we present analytical forms of zero-mode wavefunctions on the magnetized compact spaces which are crucial for computations of Yukawa coupling constants.

2.1 Compactification with magnetic fluxes

In section 2.1, we review the dimensional reduction of 10D $\mathcal{N}=1$ super Yang-Mills theory [13].[1]

2.1.1 10D $\mathcal{N}=1$ super Yang-Mills with background magnetic flux

10D $\mathcal{N}=1$ super Yang-Mills theory is an effective field theory of superstring theory. The action is given by

$$S_{10} = \int d^{10}w(\mathcal{L}_B + \mathcal{L}_F), \tag{2.1}$$

where w denotes 10D spacetime real coordinates. $\mathcal{L}_B$ and $\mathcal{L}_F$ stand for the bosonic and fermionic parts of the Lagrangian, respectively. They are given by

$$\mathcal{L}_B = -\frac{1}{4g^2}\mathrm{Tr}\left\{F^{MN}F_{MN}\right\}, \quad \mathcal{L}_F = \frac{i}{2g^2}\mathrm{Tr}\left\{\bar{\lambda}\Gamma^M D_M\lambda\right\}. \tag{2.2}$$

The trace is taken over the adjoint representation of gauge group G. For the concreteness of our discussion, we consider $G = U(N)$.[2] 10D spacetime indices are labelled by $M, N \in \{0, \cdots 9\}$

[1] See also [15, 16].

[2] When a stack of N D-branes coincide, $U(N)$ gauge symmetry can be realized in string theory.

and Γ^M are the gamma matrices. Note that the gauge coupling constant g has mass dimension $(\text{mass})^{-3}$ in 10D. The gauge field strength is defined as

$$F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N], \tag{2.3}$$

where A_M is the gauge field of $U(N)$ in the adjoint representation. The covariant derivative acting on gaugino λ is defined as

$$D_M \lambda = \partial_M \lambda - i[A_M, \lambda], \tag{2.4}$$

where we take vanishing spin connection because we will focus on flat space such as toroidal compact space. Note that the gaugino λ is a 10D Majorana-Weyl spinor. The action is invariant under the $U(N)$ gauge transformation

$$A_M \to U A_M U^{-1} - i(\partial_M U)U^{-1}, \tag{2.5}$$

$$\lambda \to U \lambda U^{-1}, \tag{2.6}$$

where U denotes a coordinate-dependent transformation matrix in the adjoint representation.

To proceed, we take the Lie algebra basis of $U(N)$ as $(U_a)_{ij} = \delta_{ai}\delta_{aj}$, $(e_{ab})_{ij} = \delta_{ai}\delta_{bj}$ where $a, b \in \{1, \cdots, N\}$. These basis satisfy the following commutation relations:

$$[U_a, U_b] = 0, \quad [U_a, e_{bc}] = (\delta_{ab} - \delta_{ac})e_{bc}, \quad [e_{ab}, e_{cd}] = \delta_{bc}e_{ad} - \delta_{ad}e_{cb}. \tag{2.7}$$

One can see that U_a's construct the Cartan subalgebra. Both the gauge field and gaugino are expanded as

$$A_M = B_M + W_M = B_M^a U_a + W_M^{ab} e_{ab}, \tag{2.8}$$

$$\lambda = \xi + \Psi = \xi^a U_a + \Psi^{ab} e_{ab}. \tag{2.9}$$

Since the Lie algebra of $U(N)$ is hermitian, we have $A_M^\dagger = A_M$ and $\lambda^\dagger = \lambda$. This imposes both B_M and ξ to be real and $W_M^{ab} = (W_M^{ba})^*$, $\Psi^{ab} = (\Psi^{ba})^*$. By substituting the expansions into the fermionic part of the Lagrangian eq. (2.2) we obtain

$$\mathcal{L}_F = \mathcal{L}_F^{(2)'} + \mathcal{L}_Y' + \mathcal{L}_F', \tag{2.10}$$

where

$$\mathcal{L}_F^{(2)'} = \frac{i}{2g^2}\mathrm{Tr}\{\bar{\Psi}\Gamma^M \partial_M \Psi - i\bar{\Psi}\Gamma^M [B_M, \Psi]\}, \tag{2.11}$$

$$\mathcal{L}_Y' = \frac{1}{2g^2}\mathrm{Tr}\{\bar{\Psi}\Gamma^M [W_M, \Psi]\}, \tag{2.12}$$

$$\mathcal{L}_F' = \frac{i}{2g^2}\mathrm{Tr}\{\bar{\xi}\Gamma^M \partial_M \xi - i\bar{\xi}\Gamma^M [W_M, \Psi] - i\bar{\Psi}\Gamma^M [W_M, \xi]\}\}. \tag{2.13}$$

Similarly, the bosonic part of the Lagrangian can be expanded as

$$\mathcal{L}_B = -\frac{1}{2g^2}\mathrm{Tr}\{D_M W_N D^M W^N - D_M W_N D^N W^M - iG_{MN}[W^M, W^N]\} + \cdots, \tag{2.14}$$

where the ellipsis corresponds to irrelevant terms. G_{MN} and $D_M W_N$ are defined as

$$G_{MN} = \partial_M B_N - \partial_N B_M, \tag{2.15}$$

$$D_M W_N = \partial_M W_N - i[B_M, W_N]. \tag{2.16}$$

Our purpose is to realize realistic 4D low-energy effective theories that can reproduce the Standard Model. As we will see later, introducing background 2-form Yang-Mills magnetic flux is one of the effective ways to achieve this. Let us decompose the spacetime indices into $\mu \in \{0, \cdots, 3\}$ of $\mathbb{R}^{1,3}$ and $i \in \{4, \cdots, 9\}$ of the compact space $\mathcal{M}_6$. Gauge fields with polarizations along the extra-dimensional space can have non-vanishing vacuum expectation values without breaking the 4D Poincáre invariance. Hence, we consider

$$B_i^a(w) = \langle B_i^a\rangle(y) + C_i^a(w), \tag{2.17}$$

$$W_i^{ab}(w) = \langle W_i^{ab}\rangle(y) + \Phi_i^{ab}(w), \tag{2.18}$$

where $y = (y^4, \cdots y^9)$ denotes the coordinate of compact space $\mathcal{M}_6$. In what follows, we assume $\langle W_i^{ab}\rangle = 0$, namely we restrict to Abelian magnetic flux. Then, the gauge group is broken to $U(N) \to \prod_i U(n_\alpha)$ while preserving the rank, i.e., $\sum_\alpha n_\alpha = N$. Eq. (2.11) can be rewritten as

$$\mathcal{L}_F^{(2)'} = \frac{i}{2g^2}\bar{\Psi}^{ba}\Gamma^\mu D_\mu \Psi^{ab} + \frac{i}{2g^2}\bar{\Psi}^{ba}\Gamma^i \tilde{D}_i \Psi^{ab}, \tag{2.19}$$

where

$$\tilde{D}_i \Psi^{ab} = \partial_i \Psi^{ab} - i(B_i^a - B_i^b)\Psi^{ab}. \tag{2.20}$$

Similarly, eq. (2.12) can be rewritten as

$$\begin{aligned}\mathcal{L}'_Y = &\frac{1}{2g^2}\left(\bar{\Psi}^{ab}\Gamma^i \Phi_i^{bd}\Psi^{da} - \bar{\Psi}^{ab}\Gamma^i \Phi_i^{ca}\Psi^{bc}\right) \\ &+ \frac{1}{2g^2}\left(\bar{\Psi}^{ab}\Gamma^\mu \Phi_\mu^{bd}\Psi^{da} - \bar{\Psi}^{ab}\Gamma^\mu \Phi_\mu^{ca}\Psi^{bc}\right).\end{aligned} \tag{2.21}$$

The first line is the three-point couplings between two fermions and a 4D scalar, hence relevant for Yukawa couplings. The second line corresponds to the couplings between fermions and a 4D vector. One can also check that (2.14) can be expanded as

$$\begin{aligned}\mathcal{L}_B = &\frac{i}{4g^2}\left(G_{ij}^a - G_{ij}^b\right)\left(\Phi^{i,ab}\Phi^{j,ba} - \Phi^{j,ab}\Phi^{i,ba}\right) - \frac{1}{2g^2}\Big[\left(D_\mu \Phi_i^{ba} D^\mu \Phi^{i,ab}\right) \\ &+ \left(\tilde{D}_i \Phi_j^{ba}\tilde{D}^i \Phi^{j,ab}\right) - 2\left(\tilde{D}_i W_\mu^{ba} D^\mu \Phi^{i,ab}\right) - \left(\tilde{D}_i \Phi_j^{ba}\tilde{D}^j \Phi^{i,ab}\right)\Big] + \cdots.\end{aligned} \tag{2.22}$$

To derive the action of 4D effective field theory, we expand the 10D fields by the Kaluza-Klein decomposition:

$$\Psi^{ab}(w) = \sum_n \chi_n^{ab}(x) \otimes \psi_n^{ab}(y), \tag{2.23}$$

$$\Phi_i^{ab}(w) = \sum_n \varphi_{i,n}^{ab}(x) \otimes \phi_{i,n}^{ab}(y), \tag{2.24}$$

where $x = (x^0, x^1, x^2, x^3)$ denotes the real coordinates of $\mathbb{R}^{3,1}$ and $n = \{0, 1, 2, \cdots\}$ denotes the Kaluza-Klein excitation number. The wavefunctions $\psi_n^{ab}(y)$ and $\phi_n^{ab}(y)$ are eigenstates of the Dirac operator and Klein-Gordon operator defined on the magnetized compact space, i.e.,

$$i\tilde{\not{D}}_6 \psi_n^{ab}(y) = i\Gamma^{(4)}\Gamma^j \tilde{D}_j \psi_n^{ab}(y) = m_n \psi_n^{ab}(y), \tag{2.25}$$

$$\Delta_6 \phi_{i,n}^{ab}(y) = -\tilde{D}_j \tilde{D}^j \phi_{i,n}^{ab}(y) = M_{i,n}^2 \phi_{i,n}^{ab}(y), \tag{2.26}$$

where $\Gamma^{(4)} := i\Gamma^0\Gamma^1\Gamma^2\Gamma^3$. Noting that the 10D Dirac equation of motion for $\Psi^{ab}(w)$ is given by

$$i\Gamma^M D_M \Psi^{ab}(w) = 0, \tag{2.27}$$

the 4D fermionic fields χ_n^{ab} satisfy

$$i\Gamma^{(4)}\Gamma^\mu D_\mu \chi_n^{ab}(x) = -m_n \chi_n^{ab}(x). \tag{2.28}$$

For the definiteness of our notation, we set $m_0 = 0 < |m_1| < |m_2| < \cdots$. Hence, χ_0^{ab} is identified as 4D massless fermion. Through a similar discussion, one finds that the 4D bosonic fields $\varphi_{i,n}^{ab}$ satisfy

$$\Delta_4 \varphi_{i,n}^{ab}(x) = M_{i,n}^2 \varphi_{i,n}^{ab}(x) - 2i\langle {G^{aj}}_i - {G^{bj}}_i \rangle \varphi_{j,n}^{ab}(x), \tag{2.29}$$

under the gauge fixing condition $\tilde{D}^i \Phi_i^{ab} = 0$.

In the energy scale sufficiently lower than the Kaluza-Klein scale, only light modes are relevant. The massless 4D fields include the gauge fields A_μ^α, gaugino λ_α, scalar C_i^α in the adjoint representation of the unbroken gauge group $U(n_\alpha)$, and bifundamental fermions χ_0^{ab}. Note that χ_0^{ab} are candidates to be identified as quarks and leptons in the Standard Model. It is also important to notice that three generations of quarks and leptons can be realized if there are three degeneracies in the zero-modes of the internal Dirac operator $i\tilde{\not{D}}_6$. On the other hand, the lightest bifundamental scalar modes $\varphi_{i,0}^{ab}$ can be massive, massless, or tachyonic depending on the choices of string vacua, e.g. size of the background magnetic flux and the geometry of the extra dimensions. $\varphi_{i,0}^{ab}$ is a candidate to be identified as Higgs boson in the Standard Model. If the compactification preserves 4D $\mathcal{N} \geq 1$ supersymmetry, the bifundamental scalar modes are massless and are the superpartners of χ_0^{ab}. We will discuss more details in the next section.

The 3-point couplings among the lightest modes of bifundamental fermions χ_0^{ab} and the bifundamental scalars φ_0^{ab} are identified as the Yukawa couplings in the Standard Model. From eqs. (2.21), (2.23) and (2.24), we obtain the Yukawa coupling terms:

$$
\begin{aligned}
S_Y =& \frac{1}{2g^2} \sum_{I,J,K} \left\{ \int d^4x \; \bar{\chi}_I^{ab} \varphi_{J,i}^{bd} \chi_K^{da} \cdot \int_{\mathcal{M}_6} d^6y \; \psi_I^{ab\dagger} \phi_{J,i}^{bd} \Gamma^i \psi_K^{da} \right. \\
& \left. - \int d^4x \; \bar{\chi}_I^{ab} \varphi_{J,i}^{ca} \chi_K^{bc} \cdot \int_{\mathcal{M}_6} d^6y \; \psi_I^{ab\dagger} \phi_{J,i}^{ca} \Gamma^i \psi_K^{bc} \right\},
\end{aligned} \tag{2.30}
$$

where I, J, and K label the degeneracy of the lightest modes. We have omitted the sub-index $n = 0$. Eq. (2.30) shows that overlap integrals of lightest wavefunctions over the compact space $\mathcal{M}_6$ give Yukawa coupling constants in 4D effective field theory,

$$
Y_{IJK}^{(1)} = \frac{1}{2g^2} \int_{\mathcal{M}_6} d^6y \; \psi_I^{ab\dagger} \phi_{J,i}^{bd} \Gamma^i \psi_K^{da}, \quad Y_{IJK}^{(2)} = \frac{1}{2g^2} \int_{\mathcal{M}_6} d^6y \; \psi_I^{ab\dagger} \phi_{J,i}^{ca} \Gamma^i \psi_K^{bc}. \tag{2.31}
$$

In this paper, we consider $\mathcal{M}_6$ to be a torus or its orbifolds. This simplification allows one to analytically compute the wavefunctions and Yukawa coupling constants [13].

2.2 Magnetized T^2 model

In this section, we assume that the 6D compact space is a direct product of three magnetized 2D tori, i.e., $\mathcal{M}_6 = T^2 \times T^2 \times T^2$. Under this setup, we can focus on only one T^2 for the following reasons. Since three magnetized tori are "factorized", we can solve the zero-mode Dirac equation on each T^2 independently. As a result, wavefunctions including bifundamental zero-modes ψ_0^{ab} are simply written by the products of zero-mode wavefunctions on three tori as

$$
{\psi_0}^{ab} = {\psi_0^{(1)}}^{ab} \otimes {\psi_0^{(2)}}^{ab} \otimes {\psi_0^{(3)}}^{ab}, \tag{2.32}
$$

where ${\psi_0^{(r)}}^{ab}$, $(r = 1, 2, 3)$ denotes a zero-mode on the r-th torus. We require three degeneracies of zero-mode fermions in the extra dimension to realize three generations of quarks and leptons. There must be three degeneracies only in one of the 2D torus because 3 is a prime number. This implies that the flavor structure primarily originates from only one T^2 and the remaining two tori would only affect the overall factor of the Yukawa coupling constant independent of the flavor (generation) indices, $\{I, J, K\}$. Therefore, we will next review magnetized T^2 compactifications following Ref. [13].

2.2.1 Definition of T^2

The 2D flat torus T^2 is defined as a quotient of complex plane $\mathbb{C}$ by a lattice Λ defined by two independent basis $e_1, e_2 \in \mathbb{C}$,

$$
\Lambda = \{n_1 e_1 + n_2 e_2 | n_1, n_2 \in \mathbb{Z}\}. \tag{2.33}
$$

This means that two independent translations on the complex plane are identified $u \sim u + e_i, (i = 1, 2)$ where u is a complex coordinate of $\mathbb{C}$. Hence, it is often expressed as $T^2 \simeq \mathbb{C}/\Lambda$. We introduce a complex coordinate of T^2 by $z = u/e_1$. Then, we have identifications on z coordinate as

$$z \sim z + 1, \quad z \sim z + \tau, \tag{2.34}$$

where we defined $\tau := e_2/e_1$. This complex parameter τ is known as the complex structure modulus of T^2. We can set $\mathrm{Im}\tau > 0$ without loss of generality.[3] Since the metric of $\mathbb{C}$ is given by $ds^2 = du d\bar{u}$, that of T^2 is written as

$$ds^2 = 2h_{\mu\bar{\nu}} dz^\mu d\bar{z}^{\bar{\nu}}, \tag{2.35}$$

where

$$h = |e_1|^2 \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}. \tag{2.36}$$

Background magnetic flux

We introduce the abelian background 2-form Yang-Mills magnetic flux:

$$F = F_{z\bar{z}}\, dz \wedge d\bar{z} = \frac{\pi i}{\mathrm{Im}\tau} \begin{pmatrix} M_a \mathbf{1}_{N_a \times N_a} & & \\ & M_b \mathbf{1}_{N_b \times N_b} & \\ & & M_c \mathbf{1}_{N_c \times N_c} \end{pmatrix} dz \wedge d\bar{z}, \tag{2.37}$$

where $\mathbf{1}_{N_a \times N_a}$ denotes $(N_a \times N_a)$ identity matrix. $M_i, (i = a, b, c)$ is a real constant which will be quantized as we will see. The magnetic flux background in eq. (2.37) corresponds to the vector potential, $A = A_z dz + A_{\bar{z}} d\bar{z}$ where

$$\begin{aligned} A_z &= -\frac{\pi i}{2\mathrm{Im}\tau} \begin{pmatrix} M_a(\bar{z} + \bar{\zeta}_a)\mathbf{1}_{N_a \times N_a} & & \\ & M_b(\bar{z} + \bar{\zeta}_b)\mathbf{1}_{N_b \times N_b} & \\ & & M_c(\bar{z} + \bar{\zeta}_c)\mathbf{1}_{N_c \times N_c} \end{pmatrix}, \\ A_{\bar{z}} &= \frac{\pi i}{2\mathrm{Im}\tau} \begin{pmatrix} M_a(z + \zeta_a)\mathbf{1}_{N_a \times N_a} & & \\ & M_b(z + \zeta_b)\mathbf{1}_{N_b \times N_b} & \\ & & M_c(z + \zeta_c)\mathbf{1}_{N_c \times N_c} \end{pmatrix}. \end{aligned} \tag{2.38}$$

$\zeta_{a,b,c}$ are complex constants known as Wilson lines. In what follows, we consider vanishing Wilson lines because one can always absorb these constant degrees of freedom into Scherk-Schwarz phases which will be defined shortly [26]. Let us denote the gauge field in the adjoint of $U(N_a)$ by A^{aa}. One can easily check the following boundary conditions:

$$\begin{aligned} A^{aa}(z + 1) &= A^{aa}(z) + d\chi_1^a(z), \\ A^{aa}(z + \tau) &= A^{aa}(z) + d\chi_\tau^a(z), \end{aligned} \tag{2.39}$$

[3] If $\mathrm{Im}\tau < 0$, one can just exchange e_1 and e_2 to get positive imaginary part. If $\mathrm{Im}\tau = 0$, T^2 is reduced to a one-dimensional sphere S^1, hence we do not consider it.

where χ_1^a and χ_τ^a are defined as

$$\chi_1^a(z) := \frac{\pi M_a}{\mathrm{Im}\tau}\mathrm{Im}z, \quad \chi_\tau^a(z) := \frac{\pi M_a}{\mathrm{Im}\tau}\mathrm{Im}(\bar{\tau}z). \tag{2.40}$$

Similarly, we can write the boundary conditions of A^{bb} and A^{cc}.

Spinors on magnetized T^2

Here, we consider the spinor fields on T^2 with background magnetic flux. To simplify our discussion, we concentrate on the $U(N_a) \times U(N_b)$ block in the gauge space. We have the two-component spinor on T^2,

$$\Psi(z,\bar{z}) = \begin{pmatrix} \psi_+(z,\bar{z}) \\ \psi_-(z,\bar{z}) \end{pmatrix}, \quad \psi_\pm = \begin{pmatrix} \psi_\pm^{aa} & \psi_\pm^{ab} \\ \psi_\pm^{ba} & \psi_\pm^{bb} \end{pmatrix}, \tag{2.41}$$

where ψ_+ and ψ_- denote the positive and negative chirality components. $\psi_\pm^{aa}$ transforms as the adjoint representation of $U(N_a)$, whereas $\psi_\pm^{ab}$ transforms as the bifundamental representation of $U(N_a) \times U(N_b)$. Representations of $\psi_\pm^{bb}$ and $\psi_\pm^{ba}$ are similarly understood. Eq. (2.39) implies that boundary conditions of $\Psi(z,\bar{z})$ under the lattice shifts are given by

$$\begin{aligned} \Psi(z+1) &= U_1 \Psi(z) U_1^\dagger, \\ \Psi(z+\tau) &= U_\tau \Psi(z) U_\tau^\dagger, \end{aligned} \tag{2.42}$$

where U_1, U_τ are unitary matrices defined as

$$\begin{aligned} U_1 &= \begin{pmatrix} e^{i\chi_1^a(z)+2\pi i\alpha^a} & 0 \\ 0 & e^{i\chi_1^b(z)+2\pi i\alpha^b} \end{pmatrix}, \\ U_\tau &= \begin{pmatrix} e^{i\chi_\tau^a(z)+2\pi i\beta^a} & 0 \\ 0 & e^{i\chi_\tau^b(z)+2\pi i\beta^b} \end{pmatrix}, \end{aligned} \tag{2.43}$$

with real constants $\alpha^k, \beta^k \in \mathbb{R}$, $(k = a, b)$ known as the Scherk-Schwarz phases. Hence, spinors in each gauge sector will satisfy the following boundary conditions:

$$\psi_\pm^{aa}(z+1) = \psi_\pm^{aa}(z), \quad \psi_\pm^{aa}(z+\tau) = \psi_\pm^{aa}(z), \tag{2.44}$$

$$\psi_\pm^{ab}(z+1) = e^{i\chi_1^{ab}(z)+2\pi i\alpha^{ab}}\psi_\pm^{ab}(z), \quad \psi_\pm^{ab}(z+\tau) = e^{i\chi_\tau^{ab}(z)+2\pi i\beta^{ab}}\psi_\pm^{ab}(z), \tag{2.45}$$

$$\psi_\pm^{ba}(z+1) = e^{i\chi_1^{ba}(z)+2\pi i\alpha^{ba}}\psi_\pm^{ba}(z), \quad \psi_\pm^{ba}(z+\tau) = e^{i\chi_\tau^{ba}(z)+2\pi i\beta^{ba}}\psi_\pm^{ba}(z), \tag{2.46}$$

$$\psi_\pm^{bb}(z+1) = \psi_\pm^{bb}(z), \quad \psi_\pm^{bb}(z+\tau) = \psi_\pm^{bb}(z), \tag{2.47}$$

where we have defined

$$\chi_1^{ab}(z) := \frac{\pi M_{ab}}{\mathrm{Im}\tau}\mathrm{Im}z, \quad \chi_\tau^{ab}(z) := \frac{\pi M_{ab}}{\mathrm{Im}\tau}\mathrm{Im}(\bar{\tau}z), \tag{2.48}$$

with

$$\begin{aligned}M_{ab} &:= M_a - M_b,\\ \alpha^{ab} &:= \alpha^a - \alpha^b,\\ \beta^{ab} &:= \beta^a - \beta^b.\end{aligned} \tag{2.49}$$

We observe that the bifundamental fermions $\psi_{\pm}^{ab}, \psi_{\pm}^{ba}$ experience the difference in the flux quanta between $U(N_a)$ and $U(N_b)$ sectors. By requiring the uniqueness of the wavefunctions under a contractible loop in T^2, the magnetic flux gets quantized as

$$M_{ab} \in \mathbb{Z}. \tag{2.50}$$

This is referred to as Dirac's charge quantization. On the other hand, the gauginos $\psi_{\pm}^{aa}, \psi_{\pm}^{bb}$ of the unbroken gauge groups do not feel any flux.

Next, let us write down the Dirac equation. We adopt gamma matrices of the following form,

$$\Gamma^z = \frac{1}{e_1}\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix},\ \Gamma^{\bar{z}} = \frac{1}{\bar{e}_1}\begin{pmatrix}0 & 0\\ 2 & 0\end{pmatrix}, \tag{2.51}$$

satisfying $\{\Gamma^z, \Gamma^{\bar{z}}\} = 2h^{z\bar{z}}\mathbf{1}_2 = (4/|e_1|^2)\mathbf{1}_2$. We define the covariant derivative operators as

$$\begin{aligned}\nabla_z &= \partial_z - iA_z,\\ \nabla_{\bar{z}} &= \bar{\partial}_{\bar{z}} - iA_{\bar{z}}.\end{aligned} \tag{2.52}$$

Then, the Dirac operator is given by

$$i\not{D} = i\Gamma^z\nabla_z + i\Gamma^{\bar{z}}\nabla_{\bar{z}} = i\begin{pmatrix}0 & \frac{2}{e_1}(\partial_z - iA_z)\\ \frac{2}{\bar{e}_1}(\bar{\partial}_{\bar{z}} - iA_{\bar{z}}) & 0\end{pmatrix} =: i\begin{pmatrix}0 & -\mathcal{D}^\dagger\\ \mathcal{D} & 0\end{pmatrix}. \tag{2.53}$$

Since fermion zero-modes are most relevant in low-energy effective theories, we focus on the massless Dirac equation, i.e., $i\not{D}\Psi(z,\bar{z}) = 0$. This leads to

$$\mathcal{D}\psi_+(z,\tau) = 0, \tag{2.54}$$

$$\mathcal{D}^\dagger\psi_-(z,\tau) = 0. \tag{2.55}$$

Writing (2.54) more explicitly, we find

$$\begin{aligned}\mathcal{D}\psi_+(z,\tau) &= \frac{2}{\bar{e}_1}(\bar{\partial}_{\bar{z}}\psi_+ - i[A_{\bar{z}}, \psi_+])\\ &= \frac{2}{\bar{e}_1}\begin{pmatrix}\bar{\partial}_{\bar{z}}\psi_+^{aa} & \left(\bar{\partial}_{\bar{z}} + \frac{\pi M_{ab}}{2\mathrm{Im}\tau}z\right)\psi_+^{ab}\\ \left(\bar{\partial}_{\bar{z}} - \frac{\pi M_{ab}}{2\mathrm{Im}\tau}z\right)\psi_+^{ba} & \bar{\partial}_{\bar{z}}\psi_+^{bb}\end{pmatrix} = 0,\end{aligned} \tag{2.56}$$

$$\mathcal{D}^\dagger\psi_-(z,\tau) = \frac{2}{e_1}\begin{pmatrix}\partial_z\psi_-^{aa} & \left(\partial_z - \frac{\pi M_{ab}}{2\mathrm{Im}\tau}\bar{z}\right)\psi_-^{ab}\\ \left(\partial_z + \frac{\pi M_{ab}}{2\mathrm{Im}\tau}\bar{z}\right)\psi_-^{ba} & \partial_z\psi_-^{bb}\end{pmatrix} = 0. \tag{2.57}$$

Since any (anti-)holomorphic functions which satisfy the periodic boundary conditions (2.44) and (2.47), diagonal components are constants:

$$\psi_\pm^{aa} = \text{const.}, \quad \psi_\pm^{bb} = \text{const.} \tag{2.58}$$

On the other hand, bifundamental zero-mode wavefunctions satisfying the boundary conditions (2.45) or (2.46) are non-trivial. If $M_{ab} > 0$, ψ_+^{ab} and ψ_-^{ba} have M_{ab} genenerated zero-modes. They are given by

$$\begin{aligned} \psi_+^{ab}: \quad & \psi_+^{(j+\alpha^{ab},\beta^{ab}),M_{ab}}(z,\tau) := \frac{M_{ab}^{1/4}}{\mathcal{A}^{1/2}} e^{\pi i M_{ab} z \frac{\mathrm{Im}z}{\mathrm{Im}\tau}}\, \vartheta \begin{bmatrix} \frac{j+\alpha^{ab}}{M_{ab}} \\ -\beta^{ab} \end{bmatrix} (M_{ab}z, M_{ab}\tau), \\ \psi_-^{ba}: \quad & \psi_-^{(j+\alpha^{ba},\beta^{ba}),M_{ba}}(z,\tau) := \left(\psi_+^{(j+\alpha^{ab},\beta^{ab}),M_{ab}}(z,\tau)\right)^*, \end{aligned} \tag{2.59}$$

where the $j = 0, 1, \cdots, |M_{ab}| - 1$ labels the degeneracy of the zero-modes. ϑ denotes the Jacobi-theta function with characteristic:

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\nu,\tau) := \sum_{l\in\mathbb{Z}} e^{\pi i (l+a)^2 \tau} e^{2\pi i (l+a)(\nu+b)}, \quad a, b \in \mathbb{R},\ \nu \in \mathbb{C},\ \tau \in \mathbb{H}, \tag{2.60}$$

where $\mathbb{H}$ represents the upper-half plane, i.e., $\mathrm{Im}\tau > 0$. The area of T^2 is denoted by $\mathcal{A} := |e_1|^2(\mathrm{Im}\tau)$. Note that ψ_+^{ab} and ψ_-^{ba} correspond to particles and anti-particles with each other in the 4D field theory. Remarkably, one can check that the other components of bi-fundamental zero-modes must be vanishing, i.e., $\psi_+^{ba} = \psi_-^{ab} = 0$. This property is crucial for the realization of chiral 4D effective field theories.

If $M_{ab} < 0$, ψ_+^{ba} and ψ_-^{ab} have $|M_{ab}|$ degenerated zero-modes. Corresponding wavefunctions are given by flipping the gauge indices $a \leftrightarrow b$ in eq. (2.59). The remaining components are vanishing, i.e., $\psi_+^{ab} = \psi_-^{ba} = 0$.

Lastly, the zero-mode wavefunctions (2.59) satisfy the following normalization condition:[4]

$$\int_{T^2} dz d\bar{z} \sqrt{|\det h|}\, \psi_\pm^{(j+\alpha_1^{ab},\alpha_\tau^{ab}),M_{ab}}(z,\tau) \left(\psi_\pm^{(k+\alpha_1^{ab},\alpha_\tau^{ab}),M_{ab}}(z,\tau)\right)^* = (2\mathrm{Im}\tau)^{-1/2} \delta_{j,k}. \tag{2.61}$$

One can check that this normalization condition is modular invariant.

2.2.2 Bosonic modes on magnetized T^2

Here, we briefly review the bosonic fields. For the sake of concreteness, let us consider the case $M_{ab} > 0$. Then, wavefunctions in (2.59) are eigenfunctions of the Laplacian on magnetized T^2

[4] We have adopted a normalization condition following [32] which slightly differs from the convention along [13]. Note also that our Kronecker delta is defined as

$$\delta_{j,k} = \begin{cases} 1 & (j \equiv k \mod M_{ab}), \\ 0 & (j \not\equiv k \mod M_{ab}). \end{cases}$$

as follows. First, Laplacian on the magnetized T^2 is given by

$$\Delta_{T^2} := -\nabla_z \nabla^z + \nabla_{\bar{z}} \nabla^{\bar{z}} = -\frac{2}{|e_1|^2}\{\nabla_z, \nabla_{\bar{z}}\} = \frac{1}{2}\{\mathcal{D}^\dagger, \mathcal{D}\}. \tag{2.62}$$

Second, we notice that the square of the Dirac operator on magnetized T^2 can be written as

$$(i\not{D})^2 = \frac{1}{2}\{\mathcal{D}^\dagger, \mathcal{D}\} + \frac{1}{2}\begin{pmatrix} [\mathcal{D}^\dagger, \mathcal{D}] & 0 \\ 0 & -[\mathcal{D}^\dagger, \mathcal{D}] \end{pmatrix} = \Delta_{T^2} + \frac{1}{2|e_1|^2}\begin{pmatrix} F_{z\bar{z}} & 0 \\ 0 & F_{\bar{z}z} \end{pmatrix}. \tag{2.63}$$

Then, one obtains

$$\Delta_{T^2}\psi_+^{(j+\alpha^{ab},\beta^{ab}),M_{ab}}(z,\tau) = \frac{2\pi M_{ab}}{\mathcal{A}}\psi_+^{(j+\alpha^{ab},\beta^{ab}),M_{ab}}(z,\tau). \tag{2.64}$$

Hence, wavefunctions of the Laplacian Δ_{T^2} are given by the eigenstates of the Dirac operator on magnetized T^2. Note that appropriate boundary conditions are also satisfied. It can also be checked that $\frac{2\pi M_{ab}}{\mathcal{A}}$ is in fact the lowest eigenvalue of Δ_{T^2}.

To compute the mass spectrum of the bosonic modes in 4D, we need to consider three magnetized tori, $T_1^2 \times T_2^2 \times T_3^2$. From eq. (2.29), one can show that scalar modes polarized along the first torus T_1^2, i.e., $\varphi_z^{ab}(x), \varphi_{\bar{z}}^{ab}(x)$ acquire squared masses given by

$$\mp\frac{2\pi M_{ab}^{(1)}}{\mathcal{A}_1} + \frac{2\pi|M_{ab}^{(2)}|}{\mathcal{A}_2} + \frac{2\pi|M_{ab}^{(3)}|}{\mathcal{A}_3}, \tag{2.65}$$

respectively. Here, $M_{ab}^{(r)}$ $(r = 1,2,3)$ denotes the background magnetic flux on the r-th 2D torus. Similarly, $\mathcal{A}_r$ stands for the area of the r-th torus. This shows that the bosonic modes can be tachyonic, massless, or massive depending on the relative sizes of magnetic fluxes and areas of three tori. One can generalize the mass spectrum for vector fields polarized along the r-th torus. Particularly, the lowest squared mass is given by

$$M_{\text{lowest}}^2 = 2\pi\left(\sum_{s\neq r}\frac{|M_{ab}^{(s)}|}{\mathcal{A}_s} - \frac{|M_{ab}^{(r)}|}{\mathcal{A}_r}\right), \tag{2.66}$$

for general integer-valued magnetic fluxes $M_{ab}^{(s)} \in \mathbb{Z}, (s = 1,2,3)$. The 4D $\mathcal{N} = 1$ supersymmetry (SUSY) in the ab-sector is preserved under the magnetized torus compactification only if there exists a bosonic mode with $M_{\text{lowest}}^2 = 0$.[5]

2.2.3 Phenomenological remarks

Here, we comment on the phenomenological aspects of toroidal compactifications with background magnetic fluxes. First, one of the most important effects of the magnetic flux is that

[5]If we allow certain field charged under a $U(1)$ gauge symmetry to develop its VEV, it contributes to the D-term as a Fayet-Iliopoulos term. Then, the SUSY-preserving condition can be relaxed. See Ref. [101] where sneutrino develops its VEV in a semi-realistic model building based on magnetized $T^2/\mathbb{Z}_2$ compactification.

it allows us to obtain chiral 4D theories. This is because depending on the sign of the magnetic flux M_{ab}, only positive or negative chirality zero-modes on T^2 can exist. Let us briefly explain. For simplicity, we consider a 6D spacetime $\mathbb{R}^{1,3} \times T^2$. Since the gaugino in the 6D super-Yang Mills field satisfies the Weyl condition, its total chirality is fixed to positive (or negative). Hence, we have the following expansion,

$$\Psi^{ab}(x,z) = \sum_j \psi_L^{(ab)j}(x) \otimes \psi_-^{(ab)j}(z) + \sum_j \psi_R^{(ab)j}(x) \otimes \psi_+^{(ab)j}(z), \tag{2.67}$$

where Ψ^{ab} denotes the gaugino in 6D which transforms as a bi-fundamental representation under $U(N_a) \times U(N_b)$. Fields $\psi_{L(R)}^{(ab)j}(x)$ are corresponding 4D fermionic modes with left (right) chirality. The index j denotes the degeneracy of the zero-modes $\psi_\pm^{(ab)j}(z)$ on magnetized T^2. We have omitted Kaluza-Klein modes since they are irrelevant to our discussion. As an example, let us choose $M_{ab} < 0$. Then, we have non-trivial solution for $\psi_-^{(ab)j}$ whereas $\psi_+^{(ab)j} = 0$. Hence, in 4D, only the left-handed fermion $\psi_L^{(ab)j}$ can exist, hence chiral. We can generalize the discussion to 10D with magnetized $T^2 \times T^2 \times T^2$ compactifications. To realize 4D chiral theories, 4D $\mathcal{N} = 4$ SUSY needs to be reduced to $\mathcal{N} = 1$ or 0. This can be achieved by turning on non-vanishing background magnetic fluxes to each torus. Second, it is remarkable that the generation of fermions is realized, cf. comment under (2.29). Particularly, if $|M_{ab}^{(1)}| = 3$, $|M_{ab}^{(2)}| = |M_{ab}^{(3)}| = 1$, three-generation models which are chiral can be obtained. 4D chiral fermions corresponding to the degenerated zero-modes in the extra dimension carry identical charges under gauge transformations, while their masses can be different. This is because Yukawa couplings depend on the profiles of degenerated zero-mode wavefunctions. Lastly, the introduction of magnetic fluxes allows us to break the $U(N)$ gauge symmetries into semi-realistic ones.

On the other hand, there remain some challenges in magnetized $T^2 \times T^2 \times T^2$ compactification. For example, there have been attempts to realize the flavor structure of quarks and leptons along Pati-Salam models [19]. If we require three-generation models possessing at least 4D $\mathcal{N} = 1$ SUSY and full-rank Yukawa coupling constants, unwanted massless exotic modes generally appear. These modes include vector-like matters and diagonal adjoint fields known as open-string moduli, which are severely constrained by various experiments in low-energies.

2.2.4 $SL(2,\mathbb{Z})$ modular symmetry

Toroidal spaces are endowed with discrete geometrical symmetries, called modular symmetries. It has great applications in toroidal compactifications.

Modular transformation

Here, we briefly review the modular symmetry of $T^2 \simeq \mathbb{C}/\Lambda$. (See e.g. [102, 103].) Modular symmetry originates from the arbitrariness in choosing the basis vectors defining the lattice Λ in eq. (2.33). Without changing Λ, we can transform the basis vectors from e_i to e'_i, $(i = 1, 2)$

as

$$\begin{pmatrix} e_2 \\ e_1 \end{pmatrix} \to \begin{pmatrix} e'_2 \\ e'_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_2 \\ e_1 \end{pmatrix}, \tag{2.68}$$

if $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$ are satisfied. The set of these transformations forms a group known as the modular group:

$$\Gamma = SL(2, \mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}. \tag{2.69}$$

There are two generators of $SL(2, \mathbb{Z})$ which are usually chosen as

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.70}$$

The generators S and T satisfy the following relations:

$$S^2 = -\mathbb{I}, \quad S^4 = (ST)^3 = \mathbb{I}, \tag{2.71}$$

where $\mathbb{I}$ denotes the identity. Actions of $\gamma \in \Gamma$ on the complex coordinate z and the complex structure modulus τ of T^2 are given by

$$\begin{aligned} \gamma : z = \frac{u}{e_1} \to z' = \frac{u}{e'_1} = \frac{z}{c\tau + d}, \\ \gamma : \tau = \frac{e_2}{e_1} \to \tau' = \frac{e'_2}{e'_1} = \frac{a\tau + b}{c\tau + d}, \end{aligned} \tag{2.72}$$

respectively. In particular, the actions of S and T are as follows:

$$S : (z, \tau) \to \left(-\frac{z}{\tau}, -\frac{1}{\tau} \right), \quad T : (z, \tau) \to (z, \tau + 1). \tag{2.73}$$

Modular transformation of zero-mode wavefunctions

We review modular transformations of zero-mode wavefunctions in the bi-fundamental representation. In what follows, we will omit the gauge indices and express $M_{ab}, \alpha_{ab}, \beta^{ab}$ as M, α, β, respectively to simplify our notations. The modular transformation of zero-mode wavefuctions have been studied in Refs. [28, 31, 32] when the Scherk-Schwarz phases are vanishing. Generalizations to include the Scherk-Schwarz phases or Wilson lines are studied in Refs. [33, 34].

Under the T transformation, positive chirality zero-modes transform as

$$\psi_+^{(j+\alpha,\beta),M}(z, \tau + 1) = e^{\frac{\pi i}{M}(j+\alpha)(j-\alpha+x)} \psi_+^{(j+\alpha,\, \beta-\alpha+x/2),M}(z, \tau), \tag{2.74}$$

where

$$x = \begin{cases} 0 & \text{if } M \equiv 0 \pmod 2, \\ 1 & \text{if } M \equiv 1 \pmod 2. \end{cases} \tag{2.75}$$

Under the S transformation, they transform as

$$\psi_+^{(j+\alpha,\beta),M}\left(-\frac{z}{\tau},-\frac{1}{\tau}\right)=(-\tau)^{1/2}\sum_{k=0}^{M-1}\frac{e^{\frac{\pi i}{4}}}{\sqrt{M}}e^{\frac{2\pi i}{M}((j+1)k+(1-\alpha)\beta)}\psi_+^{(k+\beta,1-\alpha),M}(z,\tau), \tag{2.76}$$

where we choose the branch such that the real part of $(-\tau)^{1/2}$ becomes positive. We can see that the zero-mode wavefunctions on T^2 behave like modular forms of weight 1/2.[6]

2.3 Magnetized $T^2/\mathbb{Z}_N$

In this section, we review magnetized $T^2/\mathbb{Z}_N$ orbifold models [25, 26, 28, 29]. Motivations to study orbifolds are as follows. Firstly, if we consider orbifolds, there are more varieties in three-generation models, offering a wider range of semi-realistic model buildings. We have seen that three-generation can be realized only when the magnetic flux M satisfies $|M|=3$, whereas on orbifolds, $|M|$ can take various integers. In this section, we will review the conditions to realize three-generation models in the orbifold compactifications. Another motivation is that most of the exotic modes that appear in magnetized T^2 compactification can be projected out owing to the boundary conditions of fields on orbifolds.

2.3.1 Setup of $T^2/\mathbb{Z}_N$ models

Introducing orbifolds

$T^2/\mathbb{Z}_N$ orbifolds are defined as quotient of complex plane $\mathbb{C}$ by imposing the following identifications:

$$z\sim z+1,\quad z\sim z+\tau,\quad z\sim\rho z, \tag{2.77}$$

where ρ is the $\mathbb{Z}_N$ twist:

$$\rho=e^{2\pi i/N}. \tag{2.78}$$

It is known that there exist only four kinds of orbifolds of T^2, i.e., $T^2/\mathbb{Z}_N, (N=2,3,4,6)$.[7] For the shift and twist identifications to be consistent, the complex structure modulus τ of $T^2/\mathbb{Z}_N$ orbifold is fixed, except for $T^2/\mathbb{Z}_2$.

$$\begin{aligned}&T^2/\mathbb{Z}_2:\tau\in\mathbb{C},\ \mathrm{Im}(\tau)>0,\\&T^2/\mathbb{Z}_3:\tau=e^{2\pi i/3}=:\omega,\\&T^2/\mathbb{Z}_4:\tau=i,\\&T^2/\mathbb{Z}_6:\tau=\omega.\end{aligned} \tag{2.79}$$

[6]Modular symmetries of holomorphic Yukawa couplings are also studied in [36,37] and emerging non-Abelian finite discrete flavor groups are investigated.

[7]This can be understood by the analysis of crystallography [104]. Using a more elementary method, chapter 3 of Ref. [105] gives an alternative proof.

Note that the fixed values of the modulus $\tau = i$ and $\tau = \omega$ are S and ST invariant points, respectively.

Let us comment on the metric of $T^2/\mathbb{Z}_N$. In the bulk of the orbifolds, we have the flat metric same as T^2, given by $ds^2 = |e_1|^2 dz d\bar{z}$. We can see that the orbifold twist $z \to \rho z$ does not change the form of the metric, namely it is an isometry. There are singular points on orbifolds where the curvature is localized and the metric is not well-defined. In this paper, we do not discuss such localized effects. [8]

Boundary conditions

We consider the boundary conditions of fields on magnetized $T^2/\mathbb{Z}_N$ orbifolds. First, fermionic zero-modes $\psi_{\pm,T^2/\mathbb{Z}_N}$ on $T^2/\mathbb{Z}_N$ satisfy the following boundary conditions:

$$\psi_{\pm,\mathbb{Z}_N}(z+1) = U_1(z)\psi_{\pm,\mathbb{Z}_N}(z), \tag{2.80}$$

$$\psi_{\pm,\mathbb{Z}_N}(z+\tau) = U_\tau(z)\psi_{\pm,\mathbb{Z}_N}(z), \tag{2.81}$$

$$\psi_{+,\mathbb{Z}_N}(\rho z) = \eta\, \psi_{+,\mathbb{Z}_N}(z), \tag{2.82}$$

$$\psi_{-,\mathbb{Z}_N}(\rho z) = \rho\, \eta\, \psi_{-,\mathbb{Z}_N}(z), \tag{2.83}$$

where $\eta = \rho^l = e^{2\pi i l/N}$, $(l = 0, \cdots, N-1)$ stands for the $\mathbb{Z}_N$ eigenvalues. The 1-form vector potential A on $T^2/\mathbb{Z}_N$ is still given by (2.38). Hence, it satisfies

$$A_z(\rho z) = \bar{\rho} A_z(z), \quad A_{\bar{z}}(\rho z) = \rho A_{\bar{z}}(z), \tag{2.84}$$

under the $\mathbb{Z}_N$ twist, i.e., $z \to \rho z$. In fact, we can write down the $\mathbb{Z}_N$ eigenstate $\psi_{+,\mathbb{Z}_N}$ explicitly,

$$\psi^{(j+\alpha,\beta),M}_{+,\mathbb{Z}_N,\eta}(z) \propto \sum_{k=0}^{N-1} \bar{\eta}^k\, \psi^{(j+\alpha,\beta),M}_{+}(\rho^k z), \tag{2.85}$$

where we have assumed $M > 0$. Although we can construct the $\mathbb{Z}_N$ eigenstates as in eq. (2.85), evaluating the number of independent modes with $\mathbb{Z}_N$ charge η is not always straightforward. This is because not all of $\psi^{(j+\alpha,\beta),M}_{+,\mathbb{Z}_N,\eta}(z)$, $(j = 0, \cdots, M-1)$ are linearly independent, in general.

Scherk-Schwarz phases on magnetized $T^2/\mathbb{Z}_N$

On magnetized $T^2/\mathbb{Z}_N$ orbifolds, Scherk-Schwarz phases become quantized [26] as shown in Table 2.1. Note that in $T^2/\mathbb{Z}_3$ and $T^2/\mathbb{Z}_6$ orbifolds, allowed values of Scherk-Schwarz phases depend on the even-odd of the background magnetic flux M. On the other hand, the allowed values of Scherk-Schwarz phases do not depend on the magnetic flux in $T^2/\mathbb{Z}_2$ and $T^2/\mathbb{Z}_4$.

[8] Studies on localized effects can be found at [106].

Orbifold	M	Scherk-Schwarz phases (α, β) (mod 1)
$T^2/\mathbb{Z}_2$	-	$(0,0),\ (\frac{1}{2},0),\ (0,\frac{1}{2}),\ (\frac{1}{2},\frac{1}{2})$
$T^2/\mathbb{Z}_3$	even	$(0,0),\ (\frac{1}{3},\frac{1}{3}),\ (\frac{2}{3},\frac{2}{3})$
	odd	$(\frac{1}{6},\frac{1}{6}),\ (\frac{1}{2},\frac{1}{2}),\ (\frac{5}{6},\frac{5}{6})$
$T^2/\mathbb{Z}_4$	-	$(0,0),\ (\frac{1}{2},\frac{1}{2})$
$T^2/\mathbb{Z}_6$	even	$(0,0)$
	odd	$(\frac{1}{2},\frac{1}{2})$

Table 2.1: The list of allowed Scherk-Schwarz phases on $T^2/\mathbb{Z}_N$ with magnetic flux [26].

2.3.2 $T^2/\mathbb{Z}_2$

Here, we analyze the zero-mode number on magnetized $T^2/\mathbb{Z}_2$. We focus on the case with positive magnetic flux $M > 0$. Hence, only the positive chirality modes are non-zero. We have $\mathbb{Z}_2$ eigenstates with eigenvalues either $+1$ or -1. We call those states as $\mathbb{Z}_2$ even- or odd-modes, respectively. These eigenstates are constructed by the linear combinations of $\psi^{(j+\alpha,\beta),M}_{T^2+}$ $(j = 0, \cdots M-1)$ as

$$\psi^{(j+\alpha,\beta),M}_{T^2/\mathbb{Z}_2+,\pm1}(z) \propto \psi^{(j+\alpha,\beta),M}_{+}(z) \pm \psi^{(j+\alpha,\beta),M}_{+}(-z), \tag{2.86}$$

where $\psi^{(j+\alpha,\beta),M}_{T^2/\mathbb{Z}_2+,\pm1}(z)$ satisfy

$$\psi^{(j+\alpha,\beta),M}_{T^2/\mathbb{Z}_2+,\pm1}(-z) = \pm\psi^{(j+\alpha,\beta),M}_{T^2/\mathbb{Z}_2+,\pm1}(z). \tag{2.87}$$

To count the number of $\mathbb{Z}_2$ eigenstates, following relation is useful,

$$\psi^{(j+\alpha,\beta),M}_{+}(-z) = \psi^{(M-(j+\alpha),-\beta),M}_{+}(z) = e^{-4\pi i(j+\alpha)\beta/M}\psi^{(M-(j+\alpha),\beta),M}_{+}(z), \tag{2.88}$$

which holds under $(\alpha, \beta) = (0,0), (1/2,0), (0,1/2), (1/2,1/2)$ (mod 1) [26]. For instance, let us consider the case with vanishing Scherk-Schwarz phases, i.e., $(\alpha, \beta) = (0,0)$ (mod 1). Then, (2.88) is reduced to

$$\psi^{(j,0),M}_{+}(-z) = \psi^{(M-j,0),M}_{+}(z). \tag{2.89}$$

We find that when the magnetic flux is even $M \equiv 0$ (mod 2), zero-modes $\psi^{(j,0),M}_{T^2+}$ with labels $j = 0, M/2$ (mod M) are invariant under the $\mathbb{Z}_2$ twist, hence they are $\mathbb{Z}_2$ even modes by themselves. Remaining $(M-2)$ modes $\psi^{(j,0),M}_{T^2+}$ with labels $j \neq 0, M/2$ (mod M) are not $\mathbb{Z}_2$ eigenstates by themselves. Hence, they must be paired up as shown in (2.86). This produces additional $(M-2)/2$ independent $\mathbb{Z}_2$ even-modes and $(M-2)/2$ independent $\mathbb{Z}_2$ odd-modes. Thus, the total zero-mode number of $\mathbb{Z}_2$ even sector is given by

$$n_+ = 2 + \frac{M-2}{2} = \frac{M}{2} + 1, \tag{2.90}$$

and that of $\mathbb{Z}_2$-odd sector is given by

$$n_- = \frac{M}{2} - 1. \tag{2.91}$$

On the other hand, when the magnetic flux is odd $M \equiv 1 \pmod 2$, only $j \equiv 0 \pmod M$ is $\mathbb{Z}_2$ invariant mode among $\psi^{(j,0),M}_{T^2+}$ $(j = 0, \cdots M-1)$. Hence, the numbers of zero-modes in $\mathbb{Z}_2$-even and odd sector are given by

$$n_+ = 1 + \frac{M-1}{2} = \frac{M+1}{2}, \quad n_- = \frac{M-1}{2}, \tag{2.92}$$

respectively. One can extend the same method to other choices of Scherk-Schwarz phases. That is, we first identify $\mathbb{Z}_2$ eigenstate(s) among $\psi^{(j+\alpha,\beta),M}_+$ and then pair up the remaining modes as in (2.86). The result is summarized in Table 2.2.

(α,β)	M	n_+	n_-
$(0,0)$	even	$M/2+1$	$M/2-1$
	odd	$(M+1)/2$	$(M-1)/2$
$(\frac{1}{2},0)$	even	$M/2$	$M/2$
	odd	$(M+1)/2$	$(M-1)/2$
$(0,\frac{1}{2})$	even	$M/2$	$M/2$
	odd	$(M+1)/2$	$(M-1)/2$
$(\frac{1}{2},\frac{1}{2})$	even	$M/2$	$M/2$
	odd	$(M-1)/2$	$(M+1)/2$

Table 2.2: Number of positive chirality zero-modes in magnetized $T^2/\mathbb{Z}_2$ model [26]. We have assumed $M > 0$ for concreteness.

2.3.3 $T^2/\mathbb{Z}_4$

Here, we analyze the zero-mode number on magnetized $T^2/\mathbb{Z}_4$ by using modular transformations following the method in Refs. [28, 29]. For concreteness, we focus on the case with positive magnetic flux $M > 0$.

First, it is important to notice that the $\mathbb{Z}_4$ twist can be realized by the modular S transformation. Since the modulus τ is fixed to the S-invariant point $\tau = i$, the S transformation acts as

$$S: \ (z, \tau = i) \to (iz, \tau = i), \tag{2.93}$$

which is indeed the $\mathbb{Z}_4$ twist. We may see this as an implication of the algebraic relation, $S^4 = \mathbb{I}$.

Second, we note that any $\mathbb{Z}_4$ eigenstates can be written by linear combinations of zero-modes on magnetized T^2 as,

$$\psi^{(j+\alpha,\beta),M}_{+,\mathbb{Z}_4,\eta}(z,i) = \sum_{k=0}^{M-1} a_k\, \psi^{(k+\alpha,\beta),M}_{+}(z,i), \quad a_k \in \mathbb{C}, \tag{2.94}$$

because $\psi^{(j+\alpha,\beta),M}_{+,\mathbb{Z}_4,\eta}(z,i)$ must also satisfy both the Dirac equation (2.54) and the boundary conditions on T^2 (2.43). Let us denote the representation matrix of S transformation on $\psi^{(j+\alpha,\beta)}_{+}$ by $\rho(S)$, namely

$$\psi^{(j+\alpha,\beta),M}_{+}(iz,i) = \sum_{k=0}^{M-1} \rho(S)_{jk} \psi^{(k+\alpha,\beta),M}_{+}(z,i). \tag{2.95}$$

Then, the constant coefficients a_k need to form an eigenvector of $\rho(S)^{\mathrm{T}}$ with an eigenvalue η for (2.94) to behave as

$$\begin{aligned}
\psi^{(j+\alpha,\beta),M}_{+,\mathbb{Z}_4,\eta}(iz,i) &= \sum_{j=0}^{M-1} a_j\, \psi^{(j+\alpha,\beta),M}_{+}(iz,i) \\
&= \sum_{j=0}^{M-1}\sum_{k=0}^{M-1} [\rho(S)^{\mathrm{T}}]_{kj}\, a_j\, \psi^{(k+\alpha,\beta),M}_{+}(z,i) \\
&= \eta \sum_{k=0}^{M-1} a_k\, \psi^{(k+\alpha,\beta),M}_{+}(z,i) \\
&= \eta\, \psi^{(j+\alpha,\beta),M}_{+,\mathbb{Z}_4,\eta}(z,i).
\end{aligned} \tag{2.96}$$

This shows that the zero-mode counting of $\mathbb{Z}_4$ eigenstates is equivalent to counting the multiplicity of eigenvalues η in $\rho(S)$.

Now, let us denote the number of zero-modes with $\mathbb{Z}_4$ eigenvalues $\eta = \pm 1, \pm i$ by $n_{(\pm 1)}, n_{(\pm i)}$, respectively. They satisfy the following conditions:

$$n_{(+1)} + n_{(+i)} + n_{(-1)} + n_{(-i)} = M, \tag{2.97}$$

$$n_{(+1)} + i\, n_{(+i)} - n_{(-1)} - i\, n_{(-i)} = \mathrm{tr}\rho(S). \tag{2.98}$$

If both quantities M and $\mathrm{tr}\rho(S)$ are given, eqs. (2.97) and (2.98) constrain three real degrees of freedom. To determine $n_{(\pm 1)}, n_{(\pm i)}$, one more independent condition is needed. That is provided by the number of zero-modes in magnetized $T^2/\mathbb{Z}_2$ shown in Table 2.2. Since two consecutive

$\mathbb{Z}_4$ twists are equivalent to the $\mathbb{Z}_2$ twist, the sum of the number of $\mathbb{Z}_4$ zero-modes corresponding to $\mathbb{Z}_4$ charge ± 1 is equal to the number of $\mathbb{Z}_2$ even zero-modes,

$$n_{(+1)} + n_{(-1)} = n_+. \tag{2.99}$$

In what follows, we demonstrate explicit zero-mode counting on magnetized $T^2/\mathbb{Z}_4$.

$T^2/\mathbb{Z}_4$ with vanishing Scherk-Schwarz phases $(\alpha, \beta) = (0, 0)$

Here, we evaluate the number of zero-modes on magnetized $T^2/\mathbb{Z}_4$ with vanishing Scherk-Schwarz phases. From eq. (2.76), positive chirality zero-modes on magnetized T^2 transform as

$$\psi_+^{(j,0),M}(iz, i) = \sum_{k=0}^{M-1} \frac{1}{\sqrt{M}} e^{\frac{2\pi i}{M} jk} \psi_+^{(k,0),M}(z, i), \tag{2.100}$$

under S transformation where we have fixed to $\tau = i$. We compute the trace of $\rho(S)$ given by

$$\mathrm{tr}\rho(S) = \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} e^{\frac{2\pi i}{M} k^2}. \tag{2.101}$$

For explicit computation, the generalized Landsberg-Schaar relation is useful.[9] It is given by

$$S(a, b, c) = \sqrt{\left|\frac{c}{a}\right|} \exp\left[\frac{\pi i}{4}\left(\mathrm{sgn}(ac) - \frac{b^2}{ac}\right)\right] S(-c, -b, a), \tag{2.102}$$

which holds under the conditions $a, b, c \in \mathbb{Z}$ with a and c positive, and $ac + b \equiv 0 \pmod 2$. Here, $S(a, b, c)$ is defined as

$$S(a, b, c) := \sum_{n=0}^{|c|-1} \exp\left(\frac{\pi i(an^2 + bn)}{c}\right), \quad a, b, c \in \mathbb{Z}. \tag{2.103}$$

We also note that sgn denotes the sign function, i.e., $\mathrm{sgn}(x) = \pm 1$ when $x \gtrless 0$, respectively. Then, $\mathrm{tr}\rho(S)$ can be evaluated for arbitrary M as follows,

$$\begin{aligned} \mathrm{tr}\rho(S) &= \frac{1}{\sqrt{M}} S(2, 0, M) \\ &= \frac{1}{\sqrt{2}} e^{\frac{\pi i}{4}} S(-M, 0, 2) \\ &= \frac{e^{\frac{\pi i}{4}}}{\sqrt{2}} (1 + e^{-\frac{\pi i M}{2}}). \end{aligned} \tag{2.104}$$

[9]See, for example, [107] and references therein.

The generalized Landsberg-Schaar relation is applied in the second equality of (2.104). We find that the value of $\mathrm{tr}\rho(S)$ depends on M modulo 4,

$$\mathrm{tr}\rho(S) = \begin{cases} 1+i & M \equiv 0 \pmod 4, \\ 1 & M \equiv 1 \pmod 4, \\ 0 & M \equiv 2 \pmod 4, \\ i & M \equiv 3 \pmod 4. \end{cases} \tag{2.105}$$

From eqs. (2.97)-(2.99), (2.105) and Table 2.2, one obtains the zero-mode number in each $\mathbb{Z}_4$ sector. Results are summarized in Table 2.3.

M	$n_{(+1)}$	$n_{(-1)}$	$n_{(+i)}$	$n_{(-i)}$
$M \equiv 0 \pmod 4$	$M/4+1$	$M/4$	$M/4$	$M/4-1$
$M \equiv 1 \pmod 4$	$(M+3)/4$	$(M-1)/4$	$(M-1)/4$	$(M-1)/4$
$M \equiv 2 \pmod 4$	$(M+2)/4$	$(M+2)/4$	$(M-2)/4$	$(M-2)/4$
$M \equiv 3 \pmod 4$	$(M+1)/4$	$(M+1)/4$	$(M+1)/4$	$(M-3)/4$

Table 2.3: Number of positive chirality zero-modes on magnetized $T^2/\mathbb{Z}_4$ with vanishing Scherk-Schwarz phases $(\alpha,\beta)=(0,0)$.

$T^2/\mathbb{Z}_4$ with Scherk-Schwarz phases $(\alpha,\beta)=(1/2,1/2)$

Here, we evaluate the number of zero-modes on magnetized $T^2/\mathbb{Z}_4$ with non-vanishing Scherk-Schwarz phases, i.e., $(\alpha,\beta)=(1/2,1/2)$. From eq. (2.76), positive chirality zero-modes on magnetized T^2 transform as

$$\psi_+^{(j+1/2,1/2),M}(iz,i) = \sum_{k=0}^{M-1} \frac{1}{\sqrt{M}} e^{\frac{2\pi i}{M}((j+1)k+1/4)} \psi_+^{(k,0),M}(z,i), \tag{2.106}$$

under S transformation where we have fixed $\tau=i$. In this case, $\mathrm{tr}\rho(S)$ is evaluated as

$$\begin{aligned} \mathrm{tr}\rho(S) &= \frac{1}{\sqrt{M}} e^{\frac{\pi i}{2M}} \sum_{k=0}^{M-1} e^{\frac{2\pi i}{M}(k^2+k)} \\ &= \frac{1}{\sqrt{M}} e^{\frac{\pi i}{2M}} S(2,2,M) \\ &= \frac{1}{\sqrt{2}} e^{\frac{\pi i}{4}} S(-M,-2,2) \\ &= \frac{e^{\frac{\pi i}{4}}}{\sqrt{2}} (1-e^{-\frac{\pi i M}{2}}), \end{aligned} \tag{2.107}$$

where we have used the generalized Landsberg-Schaar relation in the third equality. We find that the value of $\mathrm{tr}\rho(S)$ depends on M modulo 4,

$$\mathrm{tr}\rho(S) = \begin{cases} 0 & M \equiv 0 \pmod 4, \\ i & M \equiv 1 \pmod 4, \\ 1+i & M \equiv 2 \pmod 4, \\ 1 & M \equiv 3 \pmod 4. \end{cases} \tag{2.108}$$

From eqs. (2.97)-(2.99), (2.108) and Table 2.2, one obtains the zero-mode number in each $\mathbb{Z}_4$ sector. Results are summarized in Table 2.4.

M	$n_{(+1)}$	$n_{(-1)}$	$n_{(+i)}$	$n_{(-i)}$
$M \equiv 0 \pmod 4$	$M/4$	$M/4$	$M/4$	$M/4$
$M \equiv 1 \pmod 4$	$(M-1)/4$	$(M+3)/4$	$(M-1)/4$	$(M-1)/4$
$M \equiv 2 \pmod 4$	$(M+2)/4$	$(M-2)/4$	$(M+2)/4$	$(M-2)/4$
$M \equiv 3 \pmod 4$	$(M+1)/4$	$(M-3)/4$	$(M+1)/4$	$(M+1)/4$

Table 2.4: Number of positive chirality zero-modes on magnetized $T^2/\mathbb{Z}_4$ with non-vanishing Scherk-Schwarz phases $(\alpha, \beta) = (1/2, 1/2)$.

2.3.4 $T^2/\mathbb{Z}_3$

Here, we analyze the zero-mode number on magnetized $T^2/\mathbb{Z}_3$. It is important to notice that the $\mathbb{Z}_3$ twist can be realized by the modular ST transformation. Since the modulus τ is fixed to the ST-invariant point $\tau = \omega$, ST transformation acts as

$$ST: \ (z, \tau = \omega) \to (\omega z, \tau = \omega), \tag{2.109}$$

which is indeed the $\mathbb{Z}_3$ twist. We can see this as an implication of the algebraic relation, $(ST)^3 = \mathbb{I}$. We note that $\mathbb{Z}_3$ eigenstates can be written by linear combinations of zero-modes $\psi_+^{(k+\alpha,\beta),M}(z,\omega)$ on magnetized T^2 where τ is fixed to ω. Hence, the zero-mode counting of $\mathbb{Z}_3$ eigenstates is equivalent to counting the multiplicity of eigenvalues $\eta = 1, \omega, \omega^2$ in the representation matrix of ST transformation which we denote by $\rho(ST)$, i.e.,

$$\psi_+^{(j+\alpha,\beta),M}(\omega z, \omega) = \sum_{k=0}^{M-1} \rho(ST)_{jk} \psi_+^{(k+\alpha,\beta),M}(z,\omega). \tag{2.110}$$

Let us denote the number of zero-modes with $\mathbb{Z}_3$ eigenvalues $1, \omega, \omega^2$ by $n_{(1)}, n_{(\omega)}, n_{(\omega^2)}$, respectively. They satisfy the following conditions:

$$n_{(1)} + n_{(\omega)} + n_{(\omega^2)} = M, \tag{2.111}$$

$$n_{(1)} + \omega\, n_{(\omega)} + \omega^2\, n_{(\omega^2)} = \mathrm{tr}\rho(ST). \tag{2.112}$$

If both quantities M and $\mathrm{tr}\rho(ST)$ are given, eqs. (2.111) and (2.112) constrain three real degrees of freedom, hence we can determine $n_{(1)}, n_{(\omega)}$ and $n_{(\omega^2)}$. For simplicity, we assume vanishing Scherk-Schwarz phases, i.e., $(\alpha, \beta) = (0, 0)$.[10] Then, the magnetic flux M is constrained to even integers, c.f., Table 2.1,

$$M = 2M', \quad M' \in \mathbb{Z}. \tag{2.113}$$

From eqs. (2.74) and (2.76), positive chirality zero-modes on magnetized T^2 transform as

$$\psi_+^{(j,0),M}(\omega z, \omega) = \frac{e^{-\frac{\pi i}{12}}}{\sqrt{M}} \sum_{k=0}^{M-1} e^{\frac{2\pi i}{M}jk} e^{\frac{\pi i}{M}k^2} \psi_+^{(k,0),M}(z, \omega), \tag{2.114}$$

under ST transformation where we have fixed to $\tau = \omega$. We can evaluate the trace $\mathrm{tr}\rho(ST)$ as follows

$$\mathrm{tr}\rho(ST) = \frac{e^{-\frac{\pi i}{12}}}{\sqrt{M}} \sum_{k=0}^{M-1} e^{\frac{3\pi i k^2}{M}} = \frac{e^{-\frac{\pi i}{12}}}{\sqrt{M}} S(3, 0, M) = \frac{e^{\frac{\pi i}{6}}}{\sqrt{3}} S(-M, 0, 3). \tag{2.115}$$

In the third equality, we have used the generalized Landsberg-Schaar relation. We find that the value of $\mathrm{tr}\rho(ST)$ depends on M modulo 6,

$$\mathrm{tr}\rho(ST) = \begin{cases} 2+\omega & M \equiv 0 \pmod{6}, \\ -\omega & M \equiv 2 \pmod{6}, \\ \omega & M \equiv 4 \pmod{6}. \end{cases} \tag{2.116}$$

From eqs. (2.111), (2.112) and (2.116), one obtains the zero-mode number in each $\mathbb{Z}_3$ sector. Results are summarized in Table 2.5.

$M = 2M'$	$n_{(1)}$	$n_{(\omega)}$	$n_{(\omega^2)}$
$M' \equiv 0 \pmod{3}$	$M/3+1$	$M/3$	$M/3-1$
$M' \equiv 1 \pmod{3}$	$(M+1)/3$	$(M-2)/3$	$(M+1)/3$
$M' \equiv 2 \pmod{3}$	$(M-1)/3$	$(M+2)/3$	$(M-1)/3$

Table 2.5: Number of positive chirality zero-modes on magnetized $T^2/\mathbb{Z}_3$ with vanishing Scherk-Schwarz phases.

2.3.5 $T^2/\mathbb{Z}_6$

Here, we present the zero-mode number on magnetized $T^2/\mathbb{Z}_6$. It is important to notice that the $\mathbb{Z}_6$ twist can be realized by the modular $T^{-1}S$ transformation. Since the modulus τ is fixed to the $T^{-1}S$-invariant point $\tau = \omega$, $T^{-1}S$ transformation acts as

$$T^{-1}S: \ (z, \tau = \omega) \to (\kappa z, \tau = \omega), \tag{2.117}$$

[10] For the analyses with non-vanishing Scherk-Schwarz phases, see for example [29].

which is indeed a $\mathbb{Z}_6$ twist because $\kappa := e^{\pi i/3}$. We can see this as an implication of the algebraic relation, $(T^{-1}S)^6 = \mathbb{I}$. We note that any $\mathbb{Z}_6$ eigenstates can be written by linear combinations of zero-modes $\psi_+^{(k+\alpha,\beta),M}(z,\omega)$ on magnetized T^2 where τ is fixed to ω. Hence, the zero-mode counting of $\mathbb{Z}_6$ eigenstates is equivalent to counting the multiplicity of eigenvalues $\eta = \kappa^r$ $(r = 0, 1, \cdots, 5)$ in the representation matrix of $T^{-1}S$ transformation which we denote by $\rho(T^{-1}S)$,

$$\psi_+^{(j+\alpha,\beta),M}(\kappa z,\omega) = \sum_{k=0}^{M-1} \rho(T^{-1}S)_{jk} \psi_+^{(k+\alpha,\beta),M}(z,\omega). \tag{2.118}$$

Let us denote the number of zero-modes with $\mathbb{Z}_6$ eigenvalues $\eta = \kappa^r$ $(r = 0, 1, \cdots, 5)$ by $n_{(\kappa^r)}$. They satisfy the following conditions:

$$n_{(1)} + n_{(\kappa)} + n_{(\kappa^2)} + n_{(\kappa^3)} + n_{(\kappa^4)} + n_{(\kappa^5)} = M, \tag{2.119}$$

$$n_{(1)} + \kappa n_{(\omega)} + \kappa^2 n_{(\kappa^2)} + \kappa^3 n_{(\kappa^3)} + \kappa^4 n_{(\kappa^4)} + \kappa^5 n_{(\kappa^5)} = \mathrm{tr}\rho(T^{-1}S). \tag{2.120}$$

For simplicity, we assume vanishing Scherk-Schwarz phases, i.e., $(\alpha,\beta) = (0,0)$. Then, the magnetic flux M is constrained to even integers, c.f., Table 2.1,

$$M = 2M', \quad M' \in \mathbb{Z}. \tag{2.121}$$

From eq. eqs. (2.74) and (2.76), we obtain $\mathrm{tr}\rho(T^{-1}S)$. This can be evaluated as

$$\mathrm{tr}\rho(T^{-1}S) = \frac{e^{\frac{\pi i}{12}}}{\sqrt{M}} \sum_{k=0}^{M-1} e^{\frac{\pi i}{M}k^2} = \frac{e^{\frac{\pi i}{12}}}{\sqrt{M}} S(1,0,M) = e^{\pi i/3} S(1,0,M) = \kappa, \tag{2.122}$$

where we have used the generalized Landsberg-Schaar relation in the third equality. Even when M and $\mathrm{tr}\rho(T^{-1}S)$ are given, eqs. (2.119) and (2.120) can constrain only three real degrees of freedom. Hence we need more constraints to fully determine $n_{(\kappa^r)}$ $(r = 0, 1, \cdots, 5)$. They are provided by the number of zero-modes in magnetized $T^2/\mathbb{Z}_2$ and $T^2/\mathbb{Z}_3$ which we have already analyzed in previous subsections. Since three consecutive $\mathbb{Z}_6$ twists are equivalent to the $\mathbb{Z}_2$ twist, we find

$$n_{(1)} + n_{(\kappa^2)} + n_{(\kappa^4)} = \frac{M}{2} + 1. \tag{2.123}$$

Since two consecutive $\mathbb{Z}_6$ twists are equivalent to the $\mathbb{Z}_3$ twist, we find

$$\begin{aligned}
n_{(1)} + n_{(\kappa^3)} &= \begin{cases} \frac{M}{3} + 1, & \text{if } M \equiv 0 \pmod 6, \\ \frac{M+1}{3}, & \text{if } M \equiv 2 \pmod 6, \\ \frac{M-1}{3}, & \text{if } M \equiv 4 \pmod 6, \end{cases} \\
n_{(\kappa)} + n_{(\kappa^4)} &= \begin{cases} \frac{M}{3}, & \text{if } M \equiv 0 \pmod 6, \\ \frac{M-2}{3}, & \text{if } M \equiv 2 \pmod 6, \\ \frac{M+2}{3}, & \text{if } M \equiv 4 \pmod 6. \end{cases}
\end{aligned} \tag{2.124}$$

From eqs. (2.119)-(2.124), one obtains the zero-mode number in each $\mathbb{Z}_6$ sector. Results are summarized in Table 2.6.

$M = 2M'$	$n_{(1)}$	$n_{(\kappa)}$	$n_{(\kappa^2)}$	$n_{(\kappa^3)}$	$n_{(\kappa^4)}$	$n_{(\kappa^5)}$
$M' \equiv 0 \pmod 3$	$M/6+1$	$M/6$	$M/6$	$M/6$	$M/6$	$M/6-1$
$M' \equiv 1 \pmod 3$	$(M+4)/6$	$(M-2)/6$	$(M+4)/6$	$(M-2)/6$	$(M-2)/6$	$(M-2)/6$
$M' \equiv 2 \pmod 3$	$(M+2)/6$	$(M+2)/6$	$(M+2)/6$	$(M-4)/6$	$(M+2)/6$	$(M-4)/6$

Table 2.6: Number of positive chirality zero-modes on magnetized $T^2/\mathbb{Z}_6$ with vanishing Scherk-Schwarz phases.

2.4 Magnetized T^4 model

In this section, we discuss magnetized T^4 models [13, 14, 17, 43, 44]. One of our motivations is that there will be greater possibilities of obtaining realistic models. Since we have no guarantee that the 6D compact space is factorized into three two-dimensional spaces such as $T^2 \times T^2 \times T^2$, magnetized T^4 is a good starting point for generalizations. Moreover, background magnetic fluxes generally contain 'oblique' flux components which were not taken into account in the previous sections. This also motivates us to abandon the factorized structure.

2.4.1 Geometry of T^4

We extend the analyses to a 4-dimensional flat torus T^4 which is defined as the quotient of two complex planes $\mathbb{C}^2$ by a 4-dimensional lattice Λ, i.e., $T^4 \simeq \mathbb{C}^2/\Lambda$. Let us denote the complex coordinates of the covering space $\mathbb{C}^2$ by $\vec{u} = (u^1, u^2)^{\mathrm{T}}$. We consider a lattice Λ which is spanned by the following four basis vectors whose components in the $\vec{u}$ coordinate system are given by

$$\vec{v}_1 = 2\pi R_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = 2\pi \begin{pmatrix} R_1\Omega_{11} \\ R_2\Omega_{21} \end{pmatrix}, \quad \vec{v}_3 = 2\pi R_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \vec{v}_4 = 2\pi \begin{pmatrix} R_1\Omega_{12} \\ R_2\Omega_{22} \end{pmatrix}. \tag{2.125}$$

Here, Ω_{ij} denotes the (ij)-component of a 2×2 complex matrix Ω such that $\mathrm{Im}\Omega \neq 0$. $R_1, R_2 > 0$ correspond to scale factors. This means that four independent translations in $\mathbb{C}^2$ are identified as $\vec{u} \sim \vec{u} + \vec{v}_i$, $(i = 1, 2, 3, 4)$, i.e., $T^4 \sim \mathbb{C}^2/\Lambda$. We introduce complex coordinates of T^4 by $\vec{z} = (\vec{v}_1, \vec{v}_2)^{-1}\vec{u}$. Then, we have the identifications:

$$\vec{z} \sim \vec{z} + \vec{e}_j, \quad \vec{z} \sim \vec{z} + \Omega\vec{e}_j, \quad (j = 1, 2), \tag{2.126}$$

where $\vec{e}_j$ denotes the following orthogonal unit vectors:

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.127}$$

Since the metric of $\mathbb{C}^2$ is given by $ds^2 = \sum_{j=1,2} du^j d\bar{u}^j$, induced metric of T^4 in the $\vec{z}$ coordinate is

$$ds^2 = 2h_{i\bar{j}} dz^i d\bar{z}^j, \tag{2.128}$$

where

$$h_{i\bar{j}} = \frac{1}{2}(2\pi R_i)^2 \delta_{i\bar{j}}. \tag{2.129}$$

2.4.2 Background magnetic flux on T^4

We introduce the background magnetic flux on T^4:

$$F = \frac{1}{2} p_{x^i x^j} dx^i \wedge dx^j + \frac{1}{2} p_{y^i y^j} dy^i \wedge dy^j + p_{x^i y^j} dx^i \wedge dy^j, \tag{2.130}$$

where x^i, y^i $(i = 1, 2)$ are real coordinates defined by $z^i = x^i + \Omega_{ij} y^j$. The coefficients $p_{x^i x^j}, p_{y^i y^j}$, and $p_{x^i y^j}$ take real values. If we transform the coordinate to the $\vec{z}$ system, we find

$$F = \frac{1}{2} F_{z^i z^j} dz^i \wedge dz^j + \frac{1}{2} F_{\bar{z}^i \bar{z}^j} d\bar{z}^i \wedge d\bar{z}^j + F_{z^i \bar{z}^j} (i dz^i \wedge d\bar{z}^j), \tag{2.131}$$

where[11]

$$\begin{aligned}
F_{z^i z^j} &= \left[(\bar{\Omega} - \Omega)^{-1 \mathrm{T}} (\bar{\Omega}^{\mathrm{T}} p_{xx} \bar{\Omega} + p_{yy} + p_{xy}^{\mathrm{T}} \bar{\Omega} - \bar{\Omega}^{\mathrm{T}} p_{xy}) (\bar{\Omega} - \Omega)^{-1} \right]_{ij}, \\
F_{\bar{z}^i \bar{z}^j} &= \left[(\bar{\Omega} - \Omega)^{-1 \mathrm{T}} (\Omega^{\mathrm{T}} p_{xx} \Omega + p_{yy} + p_{xy}^{\mathrm{T}} \Omega - \Omega^{\mathrm{T}} p_{xy}) (\bar{\Omega} - \Omega)^{-1} \right]_{ij}, \\
F_{z^i \bar{z}^j} &= \left[i (\bar{\Omega} - \Omega)^{-1 \mathrm{T}} \left(\bar{\Omega}^{\mathrm{T}} p_{xx} \Omega + p_{yy} + p_{xy}^{\mathrm{T}} \Omega - \bar{\Omega}^{\mathrm{T}} p_{xy} \right) (\bar{\Omega} - \Omega)^{-1} \right]_{ij}.
\end{aligned} \tag{2.132}$$

One of the necessary conditions to conserve $\mathcal{N} = 1$ supersymmetry in the 4D space-time field theory is that F is constrained to a $(1,1)$-form, i.e., $F_{z^i z^j} = F_{\bar{z}^i \bar{z}^j} = 0$.[12] This is satisfied if

$$\Omega^{\mathrm{T}} p_{xx} \Omega + p_{yy} + p_{xy}^{\mathrm{T}} \Omega - \Omega^{\mathrm{T}} p_{xy} = 0. \tag{2.133}$$

We demand (2.133) which leads to

$$F = -\frac{1}{2} \left[(p_{xx} \Omega - p_{xy}) (\mathrm{Im}\Omega)^{-1} \right]_{ij} (i dz^i \wedge d\bar{z}^j). \tag{2.134}$$

Following [14, 42–44], we will simplify our discussions by assuming

$$p_{xx} = p_{yy} = 0. \tag{2.135}$$

As a result, the background magnetic flux (2.134) is reduced to

$$F = \pi \left[\mathbf{N}^{\mathrm{T}} (\mathrm{Im}\Omega)^{-1} \right]_{ij} (i dz^i \wedge d\bar{z}^j), \tag{2.136}$$

[11] Overall sign of $F_{z^i \bar{z}^j}$ in (2.132) differs from [13], while consistent with [17].

[12] This can be understood as vanishing F-term in the superfield description of 10D Super Yang-Mills theory with magnetized extra dimensions [108].

where we have defined a 2×2 real matrix $\mathbf{N}$ by

$$\mathbf{N} = \frac{1}{2\pi} p_{xy}^{\mathrm{T}}. \tag{2.137}$$

Similar to the Dirac quantization in the magnetized T^2 model, c.f. (2.50), $\mathbf{N}$ is quantized to integer matrices. The $F_{(2,0)} = F_{(0,2)} = 0$ condition (2.133) is rewritten as

$$(\mathbf{N}\Omega)^{\mathrm{T}} = \mathbf{N}\Omega. \tag{2.138}$$

Vector potential which corresponds to the magnetic flux (2.136) is given by

$$\begin{aligned} A(\vec{z}, \bar{\vec{z}}) &= \pi \mathrm{Im} \left\{ [\mathbf{N}(\bar{\vec{z}} + \bar{\vec{\zeta}})]^{\mathrm{T}} (\mathrm{Im}\Omega)^{-1} d\vec{z} \right\} \\ &= -\frac{\pi i}{2} \left\{ [\mathbf{N}(\bar{\vec{z}} + \bar{\vec{\zeta}})]^{\mathrm{T}} (\mathrm{Im}\Omega)^{-1} \right\}_i dz^i + \frac{\pi i}{2} \left\{ [\mathbf{N}(\vec{z} + \vec{\zeta})]^{\mathrm{T}} (\mathrm{Im}\Omega)^{-1} \right\}_i d\bar{z}^i \\ &=: A_{z^i} dz^i + A_{\bar{z}^i} d\bar{z}^i, \end{aligned} \tag{2.139}$$

where $\vec{\zeta}$ is a 2-dimensional complex constant vector. These constants in vector potentials are called Wilson lines. The boundary conditions of $A(\vec{z}, \bar{\vec{z}})$ are given by

$$\begin{aligned} A(\vec{z} + \vec{e}_k) &= A(\vec{z}) + d\chi_{\vec{e}_k}(\vec{z}), \\ A(\vec{z} + \Omega\vec{e}_k) &= A(\vec{z}) + d\chi_{\Omega\vec{e}_k}(\vec{z}), \end{aligned} \tag{2.140}$$

where

$$\chi_{\vec{e}_k}(\vec{z}) := \pi[\mathbf{N}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}\mathrm{Im}(\vec{z} + \vec{\zeta})]_k, \quad \chi_{\Omega\vec{e}_k}(\vec{z}) := \pi \mathrm{Im}[\mathbf{N}\bar{\Omega}(\mathrm{Im}\Omega)^{-1}(\vec{z} + \vec{\zeta})]_k. \tag{2.141}$$

Hereafter, we set $\vec{\zeta} = 0$ because one can always absorb it into Scherk-Schwarz phases following the method in Ref. [26].

2.4.3 Dirac equation on magnetized T^4

Here, we write down the zero-mode Dirac equation on magnetized T^4. We adopt the following representation of the gamma matrices,

$$\begin{aligned} (2\pi R_1)\Gamma^{z^1} &= \sigma_+ \otimes \sigma^3 = \begin{pmatrix} 0 & 2 & & \\ 0 & 0 & & \\ & & 0 & -2 \\ & & 0 & 0 \end{pmatrix}, \quad (2\pi R_2)\Gamma^{z^2} = I_2 \otimes \sigma_+ = \begin{pmatrix} & & 2 & 0 \\ & & 0 & 2 \\ 0 & 0 & & \\ 0 & 0 & & \end{pmatrix}, \\ (2\pi R_1)\Gamma^{\bar{z}^1} &= \sigma_- \otimes \sigma^3 = \begin{pmatrix} 0 & 0 & & \\ 2 & 0 & & \\ & & 0 & 0 \\ & & -2 & 0 \end{pmatrix}, \quad (2\pi R_2)\Gamma^{\bar{z}^2} = I_2 \otimes \sigma_- = \begin{pmatrix} & & 0 & 0 \\ & & 0 & 0 \\ 2 & 0 & & \\ 0 & 2 & & \end{pmatrix}, \end{aligned} \tag{2.142}$$

where σ^a, $(a=1,2,3)$ denote the Pauli matrices and $\sigma_\pm := \sigma^1 \pm i\sigma^2$ and I_2 represents a 2×2 unit matrix. We can check $\{\Gamma^{z^i}, \Gamma^{\bar{z}^j}\} = 2h^{i\bar{j}}$. The chirality matrix Γ^5 is given by

$$\Gamma^5 = \sigma^3 \otimes \sigma^3 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}. \tag{2.143}$$

The Dirac operator on T^4 is given by

$$i\not{D} = i\Gamma^{z^j} D_{z^j} + i\Gamma^{\bar{z}^j} \bar{D}_{\bar{z}^j} = 2i \begin{pmatrix} 0 & \rho_1 D_{z^1} & \rho_2 D_{z^2} & 0 \\ \rho_1 \bar{D}_{\bar{z}^1} & 0 & 0 & \rho_2 D_{z^2} \\ \rho_2 \bar{D}_{\bar{z}^2} & 0 & 0 & -\rho_1 D_{z^1} \\ 0 & \rho_2 \bar{D}_{\bar{z}^2} & -\rho_1 \bar{D}_{\bar{z}^1} & 0 \end{pmatrix}, \tag{2.144}$$

where we defined $\rho_j := 1/(2\pi R_j)$, $(j=1,2)$. The covariant derivatives are written as

$$\begin{aligned} D_{z^j} &= \partial_{z^j} - iA_{z^j} = \partial_{z^j} - \frac{\pi}{2}([\mathbf{N}\vec{\bar{z}}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1})_j, \\ \bar{D}_{\bar{z}^j} &= \bar{\partial}_{\bar{z}^j} - iA_{\bar{z}^j} = \bar{\partial}_{\bar{z}^j} + \frac{\pi}{2}([\mathbf{N}\vec{z}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1})_j. \end{aligned} \tag{2.145}$$

Fermion zero-modes satisfy the Dirac equation,

$$i\not{D}\Psi(\vec{z}, \vec{\bar{z}}) = 0, \tag{2.146}$$

where Ψ denotes the 4-component spinor on T^4,

$$\Psi(\vec{z}, \vec{\bar{z}}) = \begin{pmatrix} \psi_+^1 \\ \psi_-^2 \\ \psi_-^1 \\ \psi_+^2 \end{pmatrix}. \tag{2.147}$$

We have two positive and negative chirality components denoted by ψ_+^r and ψ_-^r, $(r=1,2)$, respectively. Then, one can rewrite (2.146) as

$$\rho_1 \bar{D}_{\bar{z}^1}\psi_+^1 + \rho_2 D_{z^2}\psi_+^2 = 0, \tag{2.148}$$
$$\rho_2 \bar{D}_{\bar{z}^2}\psi_+^1 - \rho_1 D_{z^1}\psi_+^2 = 0, \tag{2.149}$$
$$\rho_1 D_{z^1}\psi_-^2 + \rho_2 D_{z^2}\psi_-^1 = 0, \tag{2.150}$$
$$\rho_2 \bar{D}_{\bar{z}^2}\psi_-^2 - \rho_1 \bar{D}_{\bar{z}^1}\psi_-^1 = 0. \tag{2.151}$$

Recalling the boundary conditions of the gauge background A in eq. (2.140), bi-fundamental matter field Ψ should transform as

$$\begin{aligned} \Psi(\vec{z} + \vec{e}_k) &= e^{2\pi i\alpha_k} e^{i\chi_{\vec{e}_k}(\vec{z})}\Psi(\vec{z}), \\ \Psi(\vec{z} + \Omega\vec{e}_k) &= e^{2\pi i\beta_k} e^{i\chi_{\Omega\vec{e}_k}(\vec{z})}\Psi(\vec{z}), \end{aligned} \tag{2.152}$$

under the lattice shifts. Here, $\alpha_k, \beta_k \in \mathbb{R}$ denote the Scherk-Schwarz phases.

2.4.4 Solving zero-mode Dirac equation

Let us solve the zero-mode Dirac equation. For this purpose, the following commutation relations are useful,

$$\begin{aligned} [D_{z^i}, D_{z^j}] &= F_{z^i z^j} = 0, \\ [\bar{D}_{\bar{z}^i}, \bar{D}_{\bar{z}^j}] &= F_{\bar{z}^i \bar{z}^j} = 0, \\ [D_{z^i}, \bar{D}_{\bar{z}^j}] &= F_{z^i \bar{z}^j}. \end{aligned} \tag{2.153}$$

We define Laplace operator on the magnetized T^4 as[13]

$$\Delta_{T^4} = -2 \sum_{i=1,2} \rho_i^2 \{D_{z^i}, \bar{D}_{\bar{z}^i}\}. \tag{2.154}$$

From eqs. (2.148) and (2.149), one obtains

$$\Delta_{T^4} \psi_+^1 = 2(\rho_1^2 F_{z^1\bar{z}^1} + \rho_2^2 F_{z^2\bar{z}^2})\psi_+^1, \tag{2.155}$$

$$\Delta_{T^4} \psi_+^2 = -2(\rho_1^2 F_{z^1\bar{z}^1} + \rho_2^2 F_{z^2\bar{z}^2})\psi_+^2, \tag{2.156}$$

by use of the commutation relations (2.153). In what follows, we restrict to the case

$$\rho_1 = \rho_2 =: \rho, \quad (R_1 = R_2 =: R), \tag{2.157}$$

for simplicity. Note that eigenvalues of the Laplace operator on a compact space are non-negative.[14] Hence, if

$$F_{z^1\bar{z}^1} + F_{z^2\bar{z}^2} = \mathrm{tr}(\mathbf{N}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}) > 0, \tag{2.158}$$

the second positive chirality mode is vanishing, $\psi_+^2 = 0$. Then, (2.148) and (2.149) are reduced to

$$\bar{D}_{\bar{z}^j} \psi_+^1 = 0, \quad (j = 1, 2). \tag{2.159}$$

On the other hand, if $F_{z^1\bar{z}^1} + F_{z^2\bar{z}^2} = \mathrm{tr}(\mathbf{N}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}) < 0$, we find $\psi_+^1 = 0$ and

$$D_{z^j} \psi_+^2 = 0, \quad (j = 1, 2). \tag{2.160}$$

Let us focus on solving (2.159) for ψ_+^1 under the boundary conditions (2.152). In fact,

$$\psi(\vec{z}) := e^{\frac{\pi i}{2}\mathrm{Im}[(\mathbf{N}\vec{z})^{\mathrm{T}} \cdot (\mathrm{Im}\mathbf{N}\Omega)^{-1} \cdot \mathbf{N}\vec{z}]} \theta(\vec{z}), \tag{2.161}$$

satisfies the boundary conditions (2.152) if θ behaves as

$$\begin{aligned} \theta(\vec{z} + \vec{e}_k) &= \theta(\vec{z}), \\ \theta(\vec{z} + \Omega\vec{e}_k) &= e^{-\pi i \vec{e}_k^{\mathrm{T}} \cdot \mathrm{Re}(\mathbf{N}\Omega) \cdot \vec{e}_k} e^{-2\pi i \vec{e}_k^{\mathrm{T}} \cdot \mathrm{Re}(\mathbf{N}\vec{z})} \theta(\vec{z}). \end{aligned} \tag{2.162}$$

[13]This definition is consistent with $\Delta_6 = -\tilde{D}^j \tilde{D}_j$ in eq. (2.26).
[14]See for example [109].

Since θ is periodic along $\vec{e}_k$, we can Fourier expand as

$$\theta(\vec{x},\vec{y}) = \sum_{\vec{n}\in\mathbb{Z}^2} c_{\vec{n}}(\vec{y})\cdot e^{2\pi i\vec{n}^{\mathrm{T}}\cdot\vec{x}}, \tag{2.163}$$

where $c_{\vec{n}}(\vec{y})$ denote $\vec{y}$-dependent coefficients. Due to the pseudo-periodicity of θ along $\Omega\vec{e}_k$, we find

$$c_{\vec{n}}(\vec{y}+\vec{e}_k) = c_{\vec{n}+\mathbf{N}^{\mathrm{T}}\vec{e}_k}(\vec{y})e^{-\pi i\vec{e}_k^{\mathrm{T}}\cdot(\mathrm{Re}\mathbf{N}\Omega)\cdot(\vec{e}_k+2\vec{y})}. \tag{2.164}$$

To determine $c_{\vec{n}}(\vec{y})$, we need to solve the equation of motion. By substituting eq. (2.161) to the Dirac equation (2.159), we obtain

$$\frac{\partial c_{\vec{n}}(\vec{y})}{\partial y_k} + (2\pi\mathrm{Im}[\mathbf{N}\vec{z}]_k - 2\pi i n_j\Omega_{jk})\, c_{\vec{n}}(\vec{y}) = 0, \quad (k=1,2). \tag{2.165}$$

The solution is given by

$$c_{\vec{n}}(\vec{y}) = K_{\vec{n}}\exp\left[-\pi\vec{y}^{\mathrm{T}}\cdot\mathrm{Im}(\mathbf{N}\Omega)\cdot\vec{y} + 2\pi i\vec{n}^{\mathrm{T}}\cdot\Omega\vec{y}\right], \tag{2.166}$$

where $K_{\vec{n}}$'s are unspecified constants. From eq. (2.166), we obtain

$$\frac{c_{\vec{n}+\mathbf{N}^{\mathrm{T}}\vec{e}_k}(\vec{y})}{c_{\vec{n}}(\vec{y}+\vec{e}_k)} = \frac{K_{\vec{n}+\mathbf{N}^{\mathrm{T}}\vec{e}_k}}{K_{\vec{n}}}\cdot\exp\left[2\pi i\vec{e}_k^{\mathrm{T}}\cdot(\mathrm{Re}\mathbf{N}\Omega)\cdot\vec{y} + \pi\vec{e}_k^{\mathrm{T}}\cdot(\mathrm{Im}\mathbf{N}\Omega)\cdot\vec{y} - 2\pi i\vec{n}^{\mathrm{T}}\cdot\Omega\vec{e}_k\right]. \tag{2.167}$$

From eqs. (2.164) and (2.167), we find the following relation,

$$K_{\vec{n}}\cdot e^{-\pi i(\mathbf{N}^{-1\mathrm{T}}\cdot\vec{n})^{\mathrm{T}}\cdot\mathbf{N}\Omega\cdot(\mathbf{N}^{-1\mathrm{T}}\vec{n})} = K_{\vec{n}+\mathbf{N}^{\mathrm{T}}\vec{e}_k}\cdot e^{-\pi i(\mathbf{N}^{-1\mathrm{T}}\cdot\vec{n}+\vec{e}_k)^{\mathrm{T}}\cdot\mathbf{N}\Omega\cdot(\mathbf{N}^{-1\mathrm{T}}\cdot\vec{n}+\vec{e}_k)}. \tag{2.168}$$

This shows that $K_{\vec{n}}$ and $K_{\vec{n}+\mathbf{N}^{\mathrm{T}}\vec{e}_k}$ are dependent, hence we have only $|\mathrm{det}\mathbf{N}|$ independent zero-modes. Let us introduce $f(\vec{n})$ defined by

$$f(\vec{n}) := K_{\vec{n}}\cdot e^{-\pi i(\mathbf{N}^{-1\mathrm{T}}\cdot\vec{n})^{\mathrm{T}}\cdot\mathbf{N}\Omega\cdot(\mathbf{N}^{-1\mathrm{T}}\vec{n})}. \tag{2.169}$$

One can check that $f(\vec{n})$ is periodic,

$$f(\vec{n}+\mathbf{N}^{\mathrm{T}}\vec{e}_k) = f(\vec{n}), \quad (k=1,2). \tag{2.170}$$

As a result, we obtain explicit forms of fermion zero-mode wavefunctions:

$$\begin{aligned}\psi &= e^{\pi i[\mathbf{N}\vec{z}]^{\mathrm{T}}\cdot(\mathrm{Im}\Omega)^{-1}\cdot\mathrm{Im}\vec{z}}\cdot\sum_{\vec{n}\in\mathbb{Z}^2} f(\vec{n})e^{\pi i(\mathbf{N}^{-1\mathrm{T}}\vec{n})^{\mathrm{T}}\cdot\mathbf{N}\Omega\cdot(\mathbf{N}^{-1\mathrm{T}}\vec{n})}e^{2\pi i\vec{n}^{\mathrm{T}}\vec{z}} \\ &= e^{\pi i[\mathbf{N}\vec{z}]^{\mathrm{T}}\cdot(\mathrm{Im}\Omega)^{-1}\cdot\mathrm{Im}\vec{z}}\cdot\sum_{\vec{r}\in\mathbb{Z}^2}\sum_{\vec{J}\in\Lambda_{\mathbf{N}}} f(\mathbf{N}^{\mathrm{T}}\vec{r}+\vec{J})e^{\pi i[\mathbf{N}^{-1\mathrm{T}}(\mathbf{N}^{\mathrm{T}}\vec{r}+\vec{J})]^{\mathrm{T}}\cdot\mathbf{N}\Omega\cdot[\mathbf{N}^{-1\mathrm{T}}(\mathbf{N}^{\mathrm{T}}\vec{r}+\vec{J})]}e^{2\pi i(\mathbf{N}^{\mathrm{T}}\vec{r}+\vec{J})^{\mathrm{T}}\vec{z}} \\ &= e^{\pi i[\mathbf{N}\vec{z}]^{\mathrm{T}}\cdot(\mathrm{Im}\Omega)^{-1}\cdot\mathrm{Im}\vec{z}}\cdot\sum_{\vec{J}\in\Lambda_{\mathbf{N}}} f(\vec{J})\cdot\vartheta\begin{bmatrix}\vec{J}^{\mathrm{T}}\mathbf{N}^{-1}\\ \vec{0}\end{bmatrix}(\mathbf{N}\vec{z},\mathbf{N}\Omega),\end{aligned} \tag{2.171}$$

where $\vec{J} \in \mathbb{Z}^2$ denote integer points which lie inside the cell $\Lambda_{\mathbf{N}}$ spanned by

$$\mathbf{N}^{\mathrm{T}}\vec{e}_k, \qquad (k=1,2). \tag{2.172}$$

ϑ is the Riemann theta-function defined as

$$\vartheta \begin{bmatrix} \vec{a}^{\mathrm{T}} \\ \vec{b}^{\mathrm{T}} \end{bmatrix} (\vec{\nu}, \tilde{\Omega}) = \sum_{\vec{l}\in\mathbb{Z}^2} e^{\pi i(\vec{l}+\vec{a})^{\mathrm{T}}\cdot\tilde{\Omega}\cdot(\vec{l}+\vec{a})} e^{2\pi i(\vec{l}+\vec{a})^{\mathrm{T}}\cdot(\vec{\nu}+\vec{b})}, \quad \vec{a},\vec{b}\in\mathbb{R}^2, \vec{\nu}\in\mathbb{C}^2,\ \tilde{\Omega}^{\mathrm{T}}=\tilde{\Omega},\ \mathrm{Im}\tilde{\Omega}>0. \tag{2.173}$$

Let us summarize the result we obtained. The first positive chirality zero-modes ψ_+^1 are written as

$$\psi_+^1: \quad \psi^{\vec{J},\mathbf{N}}(\vec{z},\Omega) = \mathcal{N}_{\vec{J}} \cdot e^{\pi i[\mathbf{N}\vec{z}]^{\mathrm{T}}\cdot(\mathrm{Im}\Omega)^{-1}\cdot\mathrm{Im}\vec{z}} \cdot \vartheta \begin{bmatrix} \vec{J}^{\mathrm{T}}\mathbf{N}^{-1} \\ \vec{0} \end{bmatrix} (\mathbf{N}\vec{z}, \mathbf{N}\Omega), \tag{2.174}$$

where $\mathcal{N}_{\vec{J}}$ denotes the normalization constant which will be determined soon. It should be noted that (2.177) is normalizable only if $\mathrm{Im}(\mathbf{N}\Omega)$ is positive definite,[15]

$$\mathrm{Im}(\mathbf{N}\Omega) > 0. \tag{2.175}$$

If $\mathrm{Im}(\mathbf{N}\Omega)$ is negative definite, i.e., $\mathrm{Im}(\mathbf{N}\Omega) < 0$, the second positive chirality component ψ_+^2 becomes non-vanishing, while $\psi_+^1 = 0$. Its explicit form is given by

$$\psi_+^2: \quad \chi^{\vec{J},\mathbf{N}}(\bar{\vec{z}},\bar{\Omega}) = \mathcal{N}_{\vec{J}} \cdot e^{\pi i[\mathbf{N}\bar{\vec{z}}]^{\mathrm{T}}\cdot(\mathrm{Im}\bar{\Omega})^{-1}\cdot\mathrm{Im}\bar{\vec{z}}} \cdot \vartheta \begin{bmatrix} \vec{J}^{\mathrm{T}}\mathbf{N}^{-1} \\ \vec{0} \end{bmatrix} (\mathbf{N}\bar{\vec{z}}, \mathbf{N}\bar{\Omega}). \tag{2.176}$$

When the matrix $\mathrm{Im}(\mathbf{N}\Omega)$ is indefinite, negative chirality components ψ_-^1, ψ_-^2 become non-trivial. We explain details about the negative chirality zero-modes in Appendix A.1, where we have generalized the results in Ref. [14] to the case when $\mathrm{Re}\Omega \neq 0$.

Scherk-Schwarz phases

We can extend the analyses under the presence of Scherk-Schwarz phases, $\vec{\alpha} = (\alpha_1, \alpha_2)^{\mathrm{T}}, \vec{\beta} = (\beta_1, \beta_2)^{\mathrm{T}}$. For later convenience, we will use the following basis for fermion zero-mode wave-functions:

$$\psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(\vec{z},\Omega) = \mathcal{N}_{(\vec{J}+\vec{\alpha},\vec{\beta})} \cdot e^{\pi i[\mathbf{N}\vec{z}]^{\mathrm{T}}\cdot(\mathrm{Im}\Omega)^{-1}\cdot\mathrm{Im}\vec{z}} \cdot \vartheta \begin{bmatrix} (\vec{J}+\vec{\alpha})^{\mathrm{T}}\mathbf{N}^{-1} \\ -\vec{\beta}^{\mathrm{T}} \end{bmatrix} (\mathbf{N}\vec{z}, \mathbf{N}\Omega), \tag{2.177}$$

where $\vec{J} \in \mathbb{Z}^2$ denote integer points inside the cell defined by eq. (2.172). The degeneracy of the zero-modes is $|\mathrm{det}\mathbf{N}|$.

[15] Positive definiteness of $\mathrm{Im}(\mathbf{N}\Omega) > 0$ leads to (2.158). If we multiply ${(\mathrm{Im}\Omega)^{-1}}^{\mathrm{T}}$ from the left and $(\mathrm{Im}\Omega)^{-1}$ from the right of $\mathrm{Im}(\mathbf{N}\Omega)$, we obtain ${(\mathrm{Im}\Omega)^{-1}}^{\mathrm{T}}\mathbf{N} > 0$. Since the transpose of a positive definite matrix is still positive definite, we obtain $\mathbf{N}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1} > 0$. This immediately gives us (2.158).

Three-generation models in magnetized T^4

In magnetized T^4 model, three-generation is realized when $|\mathrm{det}\mathbf{N}| = 3$. Although we have infinite number of integer matrices $\mathbf{N}$ with $|\mathrm{det}\mathbf{N}| = 3$, the zero-mode label $\vec{J}^{\,\mathrm{T}}\mathbf{N}^{-1}$ can be classified into 4 types [42, 43].

2.4.5 Normalization

Here, we show that the wavefunctions in (2.177) satisfy the following orthonormalization condition:

$$\int_{T^4} d^2z d^2\bar{z}\sqrt{|\mathrm{det}h|}\ \psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}\left(\psi^{(\vec{K}+\vec{\alpha},\vec{\beta}),\mathbf{N}}\right)^* = |\mathrm{det}(2\mathrm{Im}\Omega)|^{-1/2}\delta_{\vec{J},\vec{K}}, \tag{2.178}$$

if we choose

$$|\mathcal{N}_{(\vec{J}+\vec{\alpha},\vec{\beta})}| = [\mathrm{Vol}(T^4)]^{-1/2}|\mathrm{det}\mathbf{N}|^{1/4}, \tag{2.179}$$

where $\mathrm{Vol}(T^4) := (2\pi R)^4|\mathrm{det}(\mathrm{Im}\Omega)|$ denotes the volume of T^4. For the proof, we begin with re-expressing (2.177) in terms of $\mathrm{Re}\vec{z}$ and $\mathrm{Im}\vec{z}$ as,

$$\begin{aligned}\psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(\vec{z},\Omega) &= \mathcal{N}_{(\vec{J}+\vec{\alpha},\vec{\beta})}\, e^{\pi i[\mathbf{N}\cdot(\mathrm{Re}\vec{z}+i\mathrm{Im}\vec{z})]^{\mathrm{T}}\cdot(\mathrm{Im}\Omega)^{-1}\cdot\mathrm{Im}\vec{z}} \\ &\cdot\sum_{\vec{l}\in\mathbb{Z}^2} e^{\pi i[\vec{l}^{\mathrm{T}}+(\vec{J}+\vec{\alpha})^{\mathrm{T}}\mathbf{N}^{-1}]\cdot(\mathbf{N}\Omega)\cdot[\vec{l}+\mathbf{N}^{-1\,\mathrm{T}}(\vec{J}+\vec{\alpha})]}\cdot e^{2\pi i[\vec{l}^{\mathrm{T}}+(\vec{J}+\vec{\alpha})^{\mathrm{T}}\mathbf{N}^{-1}]\cdot(\mathbf{N}\mathrm{Re}\vec{z}+i\mathbf{N}\mathrm{Im}\vec{z}-\vec{\beta})}.\end{aligned} \tag{2.180}$$

Hence, we obtain

$$\begin{aligned}&\psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(\psi^{(\vec{K}+\vec{\alpha},\vec{\beta}),\mathbf{N}})^* = \mathcal{N}_{(\vec{J}+\vec{\alpha},\vec{\beta})}\left(\mathcal{N}_{(\vec{K}+\vec{\alpha},\vec{\beta})}\right)^* e^{-2\pi\mathrm{Im}\vec{z}^{\mathrm{T}}\cdot\mathbf{N}(\mathrm{Im}\Omega)^{-1}\cdot\mathrm{Im}\vec{z}} \\ &\cdot\sum_{\vec{l},\vec{m}\in\mathbb{Z}^2} e^{\pi i[\vec{l}^{\mathrm{T}}+(\vec{J}+\vec{\alpha})^{\mathrm{T}}\mathbf{N}^{-1}]\cdot(\mathbf{N}\Omega)\cdot[\vec{l}+\mathbf{N}^{-1\,\mathrm{T}}(\vec{J}+\vec{\alpha})]}\cdot e^{-\pi i[\vec{m}^{\mathrm{T}}+(\vec{K}+\vec{\alpha})^{\mathrm{T}}\mathbf{N}^{-1}]\cdot(\mathbf{N}\bar{\Omega})\cdot[\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}(\vec{K}+\vec{\alpha})]} \\ &\cdot e^{2\pi i[\{\vec{l}^{\mathrm{T}}+(\vec{J}+\vec{\alpha})^{\mathrm{T}}\mathbf{N}^{-1}\}-\{\vec{m}^{\mathrm{T}}+(\vec{K}+\vec{\alpha})^{\mathrm{T}}\mathbf{N}^{-1}\}]\cdot(\mathbf{N}\mathrm{Re}\vec{z}-\vec{\beta})} e^{-2\pi[\{\vec{l}^{\mathrm{T}}+(\vec{J}+\vec{\alpha})^{\mathrm{T}}\mathbf{N}^{-1}\}+\{\vec{m}^{\mathrm{T}}+(\vec{K}+\vec{\alpha})^{\mathrm{T}}\mathbf{N}^{-1}\}]\cdot\mathbf{N}\mathrm{Im}\vec{z}}.\end{aligned} \tag{2.181}$$

We denote the integrating region, i.e., the fundamental region of T^4 generated by four lattice vectors as $\mathcal{D}_{T^4}$. This can be written as

$$\mathcal{D}_{T^4}: \qquad t_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + t_2\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + t_3\begin{pmatrix}\mathrm{Re}\Omega_{11}\\ \mathrm{Im}\Omega_{11}\\ \mathrm{Re}\Omega_{21}\\ \mathrm{Im}\Omega_{21}\end{pmatrix} + t_4\begin{pmatrix}\mathrm{Re}\Omega_{12}\\ \mathrm{Im}\Omega_{12}\\ \mathrm{Re}\Omega_{22}\\ \mathrm{Im}\Omega_{22}\end{pmatrix}, \quad (0\le t_i\le 1), \tag{2.182}$$

where we express components in the $(\mathrm{Re}z^1, \mathrm{Im}z^1, \mathrm{Re}z^2, \mathrm{Im}z^2)^{\mathrm{T}}$ real 4 dimensional coordinate system. By noticing that the integrand in (2.178) is periodic under the translations $\vec{z} \to \vec{z}+\vec{e}_i, (i = 1, 2)$, we can shift $\mathcal{D}_{T^4}$ by $f_1\vec{e}_1$ and $f_2\vec{e}_2$ where $f_1, f_2 \in \mathbb{R}$ without changing the result of integration. Since the periodicity $\vec{z} \to \vec{z} + \vec{e}_i, (i = 1, 2)$ holds for any values of t_3, t_4, both f_1 and f_2 can be arbitrary smooth functions of t_3, t_4. Hence, we may evaluate the integral over the following region:

$$\mathcal{D}'_{T^4}: \qquad u_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + u_3 \begin{pmatrix} 0 \\ \mathrm{Im}\Omega_{11} \\ 0 \\ \mathrm{Im}\Omega_{21} \end{pmatrix} + u_4 \begin{pmatrix} 0 \\ \mathrm{Im}\Omega_{12} \\ 0 \\ \mathrm{Im}\Omega_{22} \end{pmatrix}, \quad (0 \le u_i \le 1), \tag{2.183}$$

which is simpler than $\mathcal{D}_{T^4}$. By integrating Eq.(2.181) with respect to $\mathrm{Re}z^n, (n = 1, 2)$, we get

$$\begin{aligned} &\int_{\mathcal{D}'_{T^4}} d^2z d^2\bar{z} \sqrt{|\mathrm{det}h|}\ \psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}} (\psi^{(\vec{K}+\vec{\alpha},\vec{\beta}),\mathbf{N}})^* \\ &= \mathcal{N}_{(\vec{J}+\vec{\alpha},\vec{\beta})} \left(\mathcal{N}_{(\vec{K}+\vec{\alpha},\vec{\beta})}\right)^* (2\pi R)^4 \int d^2\mathrm{Im}z\ e^{-2\pi \mathrm{Im}\vec{z}^{\mathrm{T}} \cdot \mathbf{N} (\mathrm{Im}\Omega)^{-1} \cdot \mathrm{Im}\vec{z}} \\ &\cdot \sum_{\vec{l},\vec{m} \in \mathbb{Z}^2} \delta_{\vec{l}+\mathbf{N}^{-1}\vec{J}, \vec{m}+\mathbf{N}^{-1}\vec{K}} \cdot e^{\pi i [\vec{l}^{\mathrm{T}} + (\vec{J}^{\mathrm{T}} + \vec{\alpha}^{\mathrm{T}})\mathbf{N}^{-1}] \cdot (\mathbf{N}\Omega) \cdot [\vec{l} + \mathbf{N}^{-1\mathrm{T}}(\vec{J}+\vec{\alpha})]} e^{-\pi i [\vec{m}^{\mathrm{T}} + (\vec{K}+\vec{\alpha})^{\mathrm{T}}\mathbf{N}^{-1}] \cdot (\mathbf{N}\bar{\Omega}) \cdot [\vec{m} + \mathbf{N}^{-1\mathrm{T}}(\vec{K}+\vec{\alpha})]} \\ &\cdot e^{-2\pi i [(\vec{l}^{\mathrm{T}} + \vec{J}^{\mathrm{T}}\mathbf{N}^{-1}) - (\vec{m}^{\mathrm{T}} + \vec{K}^{\mathrm{T}}\mathbf{N}^{-1})]\vec{\beta}} e^{-2\pi [\vec{l}^{\mathrm{T}} + (\vec{J}+\vec{\alpha})^{\mathrm{T}}\mathbf{N}^{-1} + \vec{m}^{\mathrm{T}} + (\vec{K}+\vec{\alpha})^{\mathrm{T}}\mathbf{N}^{-1}]\mathbf{N}\mathrm{Im}\vec{z}}. \end{aligned} \tag{2.184}$$

Notice that $\delta_{\vec{l}+\mathbf{N}^{-1\mathrm{T}}\vec{J}, \vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{K}} = \delta_{\vec{l},\vec{m}} \cdot \delta_{\vec{J},\vec{K}}$.[16] Then, we obtain

$$\begin{aligned} &\int_{T^4} d^2z d^2\bar{z} \sqrt{|\mathrm{det}h|}\ \psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}} (\psi^{(\vec{K}+\vec{\alpha},\vec{\beta}),\mathbf{N}})^* \\ &= \delta_{\vec{J},\vec{K}} \cdot |\mathcal{N}_{(\vec{J}+\vec{\alpha},\vec{\beta})}|^2 (2\pi R)^4 \int d^2\mathrm{Im}z\ e^{-2\pi \mathrm{Im}\vec{z}^{\mathrm{T}} \cdot \mathbf{N} (\mathrm{Im}\Omega)^{-1} \cdot \mathrm{Im}\vec{z}} \\ &\cdot \sum_{\vec{l} \in \mathbb{Z}^2} e^{-2\pi [\vec{l}^{\mathrm{T}} + (\vec{J}+\vec{\alpha})^{\mathrm{T}}\mathbf{N}^{-1}] \cdot (\mathbf{N}\mathrm{Im}\Omega) \cdot [\vec{l} + \mathbf{N}^{-1\mathrm{T}}(\vec{J}+\vec{\alpha})]} \cdot e^{-4\pi [\vec{l}^{\mathrm{T}} + (\vec{J}+\vec{\alpha})^{\mathrm{T}}\mathbf{N}^{-1}]\mathbf{N}\mathrm{Im}\vec{z}}. \end{aligned} \tag{2.185}$$

This shows the orthogonality of the zero-mode wavefunctions. Our remaining task is determining the normalization constant satisfying Eq. (2.178). By setting $\vec{J} = \vec{K}$ in (2.185), we obtain

$$|\mathcal{N}_{(\vec{J}+\vec{\alpha},\vec{\beta})}|^2 (2\pi R)^4 \int d^2\mathrm{Im}z \sum_{\vec{l} \in \mathbb{Z}^2} e^{-2\pi [\vec{y}^{\mathrm{T}} + \vec{l}^{\mathrm{T}} + (\vec{J}+\vec{\alpha})^{\mathrm{T}}\mathbf{N}^{-1}] \cdot \mathbf{N} (\mathrm{Im}\Omega) \cdot [\vec{y} + \vec{l} + \mathbf{N}^{-1\mathrm{T}}(\vec{J}+\vec{\alpha})]} = |\mathrm{det}(2\mathrm{Im}\Omega)|^{-1/2}, \tag{2.186}$$

where we defined $\vec{y} := (\mathrm{Im}\Omega)^{-1}\mathrm{Im}\vec{z}$. Note that

$$d\mathrm{Im}z^1 d\mathrm{Im}z^2 = |\mathrm{det}(\mathrm{Im}\Omega)| dy^1 dy^2, \tag{2.187}$$

[16] Here, we have used the fact that $\vec{J}$ and $\vec{K}$ are integer vectors inside the cell $\Lambda_{\mathbf{N}}$ spanned by (2.172).

and the range of the integral over $y^n, (n = 1, 2)$ is $0 - 1$. Hence, we find

$$\mathrm{Vol}(T^4)|\mathcal{N}_{(\vec{J}+\vec{\alpha},\vec{\beta})}|^2 \sum_{\vec{l}\in\mathbb{Z}^2} \int_0^1 d^2y\; e^{-2\pi[\vec{y}^{\,\mathrm{T}}+\vec{l}^{\,\mathrm{T}}+(\vec{J}+\vec{\alpha})^{\mathrm{T}}\mathbf{N}^{-1}]\cdot\mathbf{N}(\mathrm{Im}\Omega)\cdot[\vec{y}+\vec{l}+\mathbf{N}^{-1\,\mathrm{T}}(\vec{J}+\vec{\alpha})]} = |\det(2\mathrm{Im}\Omega)|^{-1/2}, \tag{2.188}$$

where $\mathrm{Vol}(T^4) := (2\pi R)^4|\det(\mathrm{Im}\Omega)|$. We can rewrite this as

$$\mathrm{Vol}(T^4)|\mathcal{N}_{(\vec{J}+\vec{\alpha},\vec{\beta})}|^2 \int_{-\infty}^{\infty} du^1 du^2 e^{-2\pi[\vec{u}^{\,\mathrm{T}}+(\vec{J}+\vec{\alpha})^{\mathrm{T}}\mathbf{N}^{-1}]\cdot\mathbf{N}(\mathrm{Im}\Omega)\cdot[\vec{u}+\mathbf{N}^{-1\,\mathrm{T}}(\vec{J}+\vec{\alpha})]} = |\det(2\mathrm{Im}\Omega)|^{-1/2}, \tag{2.189}$$

where we defined $\vec{u} := \vec{y} + \vec{l}$. The remaining computation is straightforward as it is a Gaussian integral. We have verified eq. (2.179).

2.5 Magnetized $T^4/\mathbb{Z}_2$

In this section, we study magnetized $T^4/\mathbb{Z}_2$ models [42, 43]. We define $T^4/\mathbb{Z}_2$ orbifold from T^4 by imposing the twist identification given by

$$\vec{z} \sim -\vec{z}. \tag{2.190}$$

As in the case of $T^2/\mathbb{Z}_2$ model, the zero-mode wave functions are either $\mathbb{Z}_2$ even $\psi_{\mathbb{Z}_2,+}$ or odd $\psi_{\mathbb{Z}_2,-}$ satisfying

$$\psi_{\mathbb{Z}_2,\pm}(-\vec{z},\Omega) = \pm\psi_{\mathbb{Z}_2,\pm}(\vec{z},\Omega). \tag{2.191}$$

For concreteness, we focus on the case when only the positive chirality mode ψ_+^1 is non-zero, namely, we consider the case (2.175). The $\mathbb{Z}_2$ eigenstates $\psi_{\mathbb{Z}_2,\pm}$ are written by linear combinations of the zero-modes wavefunctions on T^4 with the same magnetic flux. This is because $\mathbb{Z}_2$ eigenstates also need to satisfy the Dirac equation (2.146) and the shift boundary conditions (2.152).

$\mathbb{Z}_2$ eigenstates

The $\mathbb{Z}_2$ twist boundary condition (2.191) can be satisfied if we take the following linear combinations,

$$\psi_{\mathbb{Z}_2,+}^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(\vec{z},\Omega) \propto \psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(\vec{z},\Omega) + \psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(-\vec{z},\Omega), \tag{2.192}$$

$$\psi_{\mathbb{Z}_2,-}^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(\vec{z},\Omega) \propto \psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(\vec{z},\Omega) - \psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(-\vec{z},\Omega). \tag{2.193}$$

We require $\psi_{\mathbb{Z}_2,+}^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(\vec{z},\Omega)$ to satisfy the boundary conditions under the shifts (2.152). Hence, $\psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(-\vec{z},\Omega)$ must satisfy

$$\begin{aligned} \psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(-(\vec{z}+\vec{e}_k),\Omega) &= e^{-2\pi i\alpha_k}\cdot e^{i\chi_{\vec{e}_k}(\vec{z})}\cdot\psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(-\vec{z},\Omega), \\ \psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(-(\vec{z}+\Omega\vec{e}_k),\Omega) &= e^{-2\pi i\beta_k}\cdot e^{i\chi_{\Omega\vec{e}_k}(\vec{z})}\cdot\psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(-\vec{z},\Omega). \end{aligned} \tag{2.194}$$

This requires $e^{-2\pi i\alpha_k} = e^{2\pi i\alpha_k}$ and $e^{-2\pi i\beta_k} = e^{2\pi i\beta_k}$. Thus, the Scherk-Schwarz phases have the following 16 patterns [43]:

$$\begin{aligned}\vec{\alpha} &= (0 \text{ or } 1/2,\ 0 \text{ or } 1/2)\,, \quad \text{mod } 1,\\ \vec{\beta} &= (0 \text{ or } 1/2,\ 0 \text{ or } 1/2)\,, \quad \text{mod } 1.\end{aligned} \tag{2.195}$$

2.5.1 Zero-mode counting on magnetized $T^4/\mathbb{Z}_2$

Here, we study the number of degenerated $\mathbb{Z}_2$ even and odd zero-modes on magnetized $T^4/\mathbb{Z}_2$. As in (2.192), $\mathbb{Z}_2$ eigenstates can be written by the linear combinations of two wavefunctions on T^4. It should be noted however that for some particular zero-modes on T^4 with label $\vec{J}$, we have

$$\psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(\vec{z},\Omega) = \psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(-\vec{z},\Omega). \tag{2.196}$$

In such cases, $\mathbb{Z}_2$ even mode $\psi_{\mathbb{Z}_2,+}$ is given by a single wavefunction $\psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}$. We denote the independent number of such invariant mode $\psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}$ by C_+. Similarly, for some particular zero-modes on T^4 with label $\vec{J}$, we have

$$\psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(\vec{z},\Omega) = -\psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(-\vec{z},\Omega). \tag{2.197}$$

This shows that the corresponding $\mathbb{Z}_2$ odd mode $\psi_{\mathbb{Z}_2,-}$ is written by a single wavefunction $\psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}$. We denote the number of such odd modes by C_-. Remaining $\{|\det\mathbf{N}| - (C_+ + C_-)\}$ zero-modes on T^4 are paired up to form $\mathbb{Z}_2$ eigenstates. Hence, the number of the zero-modes of $\mathbb{Z}_2$ even and odd sectors are given by the following formulae respectively,

$$\begin{aligned}N_{\text{even}} &= \frac{|\det\mathbf{N}| - (C_+ + C_-)}{2} + C_+ = \frac{|\det\mathbf{N}|}{2} + \frac{1}{2}(C_+ - C_-),\\ N_{\text{odd}} &= \frac{|\det\mathbf{N}| - (C_+ + C_-)}{2} + C_- = \frac{|\det\mathbf{N}|}{2} - \frac{1}{2}(C_+ - C_-).\end{aligned} \tag{2.198}$$

This motivates us to determine C_+ and C_-. The following relation is useful for the analysis,

$$\psi^{(\vec{J}+\vec{\alpha},\vec{\beta}),\mathbf{N}}(-\vec{z},\Omega) = \psi^{(\mathbf{N}^{\mathrm{T}}\vec{e}-\vec{J}-\vec{\alpha},-\vec{\beta}),\mathbf{N}}(\vec{z},\Omega) = e^{-4\pi i(\vec{J}+\vec{\alpha})^{\mathrm{T}}\cdot\mathbf{N}^{-1}\cdot\vec{\beta}}\psi^{(\mathbf{N}^{\mathrm{T}}\vec{e}-\vec{J}-\vec{\alpha},\vec{\beta}),\mathbf{N}}(\vec{z},\Omega), \tag{2.199}$$

where $\vec{e} := k_1\vec{e}_1 + k_2\vec{e}_2,\ k_1, k_2 \in \mathbb{Z}$.[17] In the second equality of eq. (2.199), we have used the fact that Scherk-Schwarz phases are given by (2.195).

Firstly, let us study the simplest setup where the Scherk-Schwarz phases are vanishing ($\vec{\alpha} = \vec{\beta} = \vec{0}$). Then, eq. (2.199) is reduced to

$$\psi^{(\vec{J},\vec{0}),\mathbf{N}}(-\vec{z},\Omega) = \psi^{(\mathbf{N}^{\mathrm{T}}\vec{e}-\vec{J},\vec{0}),\mathbf{N}}(\vec{z},\Omega). \tag{2.200}$$

From this, we find $C_- = 0$. It also shows that zero-modes with labels $\vec{J} \in \mathbb{Z}^2$ satisfying

$$2\vec{J} = \mathbf{N}^{\mathrm{T}}(k_1\vec{e}_1 + k_2\vec{e}_2), \quad k_1, k_2 \in \mathbb{Z}, \tag{2.201}$$

[17] $\vec{e}_1$ and $\vec{e}_2$ are 2-dimensional standard unit vectors, i.e., $\vec{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$.

are invariant under the $\mathbb{Z}_2$ twist and contribute to C_+. To determine C_+, we first note that there are at most 4 independent solutions to (2.201) which are

$$\begin{aligned} &\vec{J}=\vec{0}, \qquad \vec{J}=\frac{1}{2}\mathbf{N}^{\mathrm{T}}\vec{e}_1=\begin{pmatrix} n_{11}/2 \\ n_{12}/2 \end{pmatrix}, \\ &\vec{J}=\frac{1}{2}\mathbf{N}^{\mathrm{T}}\vec{e}_2=\begin{pmatrix} n_{21}/2 \\ n_{22}/2 \end{pmatrix}, \quad \vec{J}=\frac{1}{2}\mathbf{N}^{\mathrm{T}}(\vec{e}_1+\vec{e}_2)=\begin{pmatrix} (n_{11}+n_{21})/2 \\ (n_{12}+n_{22})/2 \end{pmatrix}. \end{aligned} \tag{2.202}$$

We recall that only when these labels are integer-valued, $\vec{J}\in\mathbb{Z}^2$, they correspond to the zero-mode states and contribute to C_+. Obviously, $\vec{J}=\vec{0}$ is always a zero-mode state. On the other hand, the remaining 3 labels in (2.202) are integer-valued only when $\mathbf{N}$ has a certain structure. $\vec{J}=\frac{1}{2}\mathbf{N}\vec{e}_1$ is integer-valued only if $n_{11}\equiv 0, n_{12}\equiv 0$ (mod2). $\vec{J}=\frac{1}{2}\mathbf{N}\vec{e}_2$ is integer-valued only if $n_{22}\equiv 0, n_{21}\equiv 0$ (mod2). $\vec{J}=\frac{1}{2}\mathbf{N}^{\mathrm{T}}(\vec{e}_1+\vec{e}_2)$ is integer-valued only if $n_{11}+n_{21}\equiv 0, n_{12}+n_{22}\equiv 0$ (mod2). In this way, C_+ depends on the mod 2 structure of $\mathbf{N}$. We have classified 16 different patterns of the mod 2 structure. We find $C_+=4$, only if all matrix elements of $\mathbf{N}$ are even numbers,

$$C_+=4: \quad \mathbf{N}=\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}\equiv\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ (mod 2)}. \tag{2.203}$$

We find $C_+=2$ only if the mod 2 structure of the flux matrix $\mathbf{N}$ is as follows:

$C_+=2:$

$$\mathbf{N}\equiv\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ (mod 2)}; \quad \vec{J}=\vec{0},\ \frac{1}{2}\mathbf{N}^{\mathrm{T}}\vec{e}_2, \tag{2.204}$$

$$\mathbf{N}\equiv\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \text{ (mod 2)}; \quad \vec{J}=\vec{0},\ \frac{1}{2}\mathbf{N}^{\mathrm{T}}\vec{e}_1, \tag{2.205}$$

$$\mathbf{N}\equiv\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ (mod 2)}; \quad \vec{J}=\vec{0},\ \frac{1}{2}\mathbf{N}^{\mathrm{T}}(\vec{e}_1+\vec{e}_2), \tag{2.206}$$

where we have also shown the corresponding $\mathbb{Z}_2$ invariant modes among eq. (2.202). The remaining 6 patterns in the mod 2 structure of $\mathbf{N}$ correspond to the case $C_+=1$ where the $\mathbb{Z}_2$-invariant mode is $\vec{J}=\vec{0}$:

$C_+=1:$

$$\mathbf{N}\equiv\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ (mod 2)}. \tag{2.207}$$

Secondly, let us consider non-vanishing $\vec{\beta}$, while we still set $\vec{\alpha}=\vec{0}$. For instance, under the choice $\vec{\alpha}=\vec{0},\ \vec{\beta}=(0,\frac{1}{2})$, eq. (2.199) is reduced to

$$\psi^{(\vec{J},\vec{\beta}=(0,\frac{1}{2})),\mathbf{N}}(-\vec{z},\Omega)=e^{-2\pi i\vec{J}^{\mathrm{T}}\mathbf{N}^{-1}\vec{e}_2}\psi^{(\vec{e}-\vec{J},\vec{\beta}=(0,\frac{1}{2})),\mathbf{N}}(\vec{z},\Omega), \tag{2.208}$$

where $\vec{e} = k_1\vec{e}_1 + k_2\vec{e}_2,\ (k_1, k_2 \in \mathbb{Z})$. This shows that modes with label $\vec{J} = \frac{1}{2}\mathbf{N}^{\mathrm{T}}\vec{e}_1$ and $\vec{J} = \frac{1}{2}\mathbf{N}^{\mathrm{T}}(\vec{e}_1 + \vec{e}_2)$ are now $\mathbb{Z}_2$-odd modes if $\vec{J} \in \mathbb{Z}^2$, because $2\vec{J}^{\mathrm{T}}\mathbf{N}^{-1}\vec{e}_2 \equiv 1 \pmod 2$. On the other hand, $\vec{J} = \vec{0}$ and $\vec{J} = \frac{1}{2}\mathbf{N}^{\mathrm{T}}\vec{e}_1$ are still $\mathbb{Z}_2$-even modes if $\vec{J} \in \mathbb{Z}^2$, because $2\vec{J}^{\mathrm{T}}\mathbf{N}^{-1}\vec{e}_2 \equiv 0 \pmod 2$. Hence, there are only simple modifications from the previous results eqs. (2.203) – (2.207). For example, the mod 2 structure of $\mathbf{N}$ in eq. (2.203) now corresponds to $C_+ = C_- = 2$. We also find that 3 patterns of the mod 2 structure in (2.204) now correspond to $C_+ = C_- = 1$. Similarly, we can analyze other choices of $\vec{\beta}$, i.e., $\vec{\beta} = (\frac{1}{2}, 0)$ and $\vec{\beta} = (\frac{1}{2}, \frac{1}{2})$. Results are summarized in Table2.7.

<table>
<tr><th>mod 2 structure
$(n_{11}, n_{12}, n_{21}, n_{22})$</th><th>$\vec{\beta} = (0,0)$</th><th>$\vec{\beta} = (0,\frac{1}{2})$</th><th>$\vec{\beta} = (\frac{1}{2},0)$</th><th>$\vec{\beta} = (\frac{1}{2},\frac{1}{2})$</th></tr>
<tr><td>$(0,0,0,0)$</td><td>$C_+ = 4$
$C_- = 0$</td><td colspan="3">$C_+ = 2$
$C_- = 2$</td></tr>
<tr><td>$(0,0,0,1)$
$(0,0,1,0)$
$(0,0,1,1)$</td><td rowspan="3">$C_+ = 2$
$C_- = 0$</td><td>$C_+ = 2$
$C_- = 0$</td><td>$C_+ = 1$
$C_- = 1$</td><td>$C_+ = 1$
$C_- = 1$</td></tr>
<tr><td>$(0,1,0,0)$
$(1,0,0,0)$
$(1,1,0,0)$</td><td>$C_+ = 1$
$C_- = 1$</td><td>$C_+ = 2$
$C_- = 0$</td><td>$C_+ = 1$
$C_- = 1$</td></tr>
<tr><td>$(0,1,0,1)$
$(1,0,1,0)$
$(1,1,1,1)$</td><td>$C_+ = 1$
$C_- = 1$</td><td>$C_+ = 1$
$C_- = 1$</td><td>$C_+ = 2$
$C_- = 0$</td></tr>
<tr><td>$(0,1,1,0)$
$(1,0,0,1)$
$(0,1,1,1)$
$(1,0,1,1)$
$(1,1,0,1)$
$(1,1,1,0)$</td><td colspan="4">$C_+ = 1$
$C_- = 0$</td></tr>
</table>

Table 2.7: Values of $C_\pm$ classified by the mod 2 structure of $\mathbf{N}$ when $\vec{\alpha} = (0,0)$.

Lastly, we determine C_+ and C_- when the Scherk-Schwarz phase $\vec{\alpha}$ is non-vanishing. For instance, under the choice $\vec{\alpha} = (0, \frac{1}{2}), \vec{\beta} = (0,0)$, eq. (2.199) is reduced to

$$\psi^{(\vec{J}+\vec{e}_2/2,\vec{0}),\mathbf{N}}(-\vec{z},\Omega) = \psi^{(\vec{e}-\vec{J}-\vec{e}_2/2,\vec{0}),\mathbf{N}}(\vec{z},\Omega), \tag{2.209}$$

where $\vec{e} = k_1\vec{e}_1 + k_2\vec{e}_2,\ (k_1, k_2 \in \mathbb{Z})$. We immediately find $C_- = 0$. Eq. (2.209) also shows that zero-mode with labels $\vec{J} \in \mathbb{Z}^2$ satisfying

$$2\vec{J} + \vec{e}_2 = \mathbf{N}^{\mathrm{T}}(k_1\vec{e}_1 + k_2\vec{e}_2), \quad k_1, k_2 \in \mathbb{Z}. \tag{2.210}$$

are invariant under the $\mathbb{Z}_2$ twist and contribute to C_+. When $k_1 \equiv k_2 \equiv 0 \pmod 2$, we obtain $\vec{J} = -\frac{1}{2}\vec{e}_1$, however it is not integer-valued. When $k_1 \equiv 1, k_2 \equiv 0 \pmod 2$, we obtain $\vec{J} =$

$\frac{1}{2}(n_{11}, n_{12}-1)^{\mathrm{T}}$ which is integer-valued only if $n_{11} \equiv 0, n_{12} \equiv 1 \pmod 2$. When $k_1 \equiv 0, k_2 \equiv 1 \pmod 2$, we obtain $\vec{J} = \frac{1}{2}(n_{21}, n_{22}-1)^{\mathrm{T}}$ which is integer-valued only if $n_{21} \equiv 0, n_{22} \equiv 1 \pmod 2$. When $k_1 \equiv k_2 \equiv 1 \pmod 2$, we obtain $\vec{J} = \frac{1}{2}(n_{11}+n_{21}, n_{12}+n_{22}-1)^{\mathrm{T}}$ which is integer-valued only if $n_{11}+n_{21} \equiv 0, n_{12}+n_{22} \equiv 1 \pmod 2$. Let us summarize the results. We have $C_+ = 2$ only if the mod 2 structure of the flux matrix $\mathbf{N}$ is as follows:

$$
\begin{aligned}
&C_+ = 2: \\
&\mathbf{N} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2; \quad \vec{J} = \frac{1}{2}(n_{11}, n_{12}-1)^{\mathrm{T}},\ \frac{1}{2}(n_{11}+n_{21}, n_{12}+n_{22}-1)^{\mathrm{T}}, \\
&\mathbf{N} \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \pmod 2; \quad \vec{J} = \frac{1}{2}(n_{21}, n_{22}-1)^{\mathrm{T}},\ \frac{1}{2}(n_{11}+n_{21}, n_{12}+n_{22}-1)^{\mathrm{T}}, \\
&\mathbf{N} \equiv \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \pmod 2; \quad \vec{J} = \frac{1}{2}(n_{11}, n_{12}-1)^{\mathrm{T}},\ \frac{1}{2}(n_{21}, n_{22}-1)^{\mathrm{T}},
\end{aligned}
\tag{2.211}
$$

where we have also shown the corresponding $\mathbb{Z}_2$ invariant modes. We have $C_+ = 1$ only if the mod 2 structure of the flux matrix is as follows:

$$
\begin{aligned}
&C_+ = 1: \\
&\mathbf{N} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \pmod 2, \quad \vec{J} = \frac{1}{2}(n_{11}, n_{12}-1)^{\mathrm{T}} \\
&\mathbf{N} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \pmod 2; \quad \vec{J} = \frac{1}{2}(n_{21}, n_{22}-1)^{\mathrm{T}}, \\
&\mathbf{N} \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \pmod 2, \quad \vec{J} = \frac{1}{2}(n_{11}+n_{21}, n_{12}+n_{22}-1)^{\mathrm{T}},
\end{aligned}
\tag{2.212}
$$

where we have also shown the corresponding $\mathbb{Z}_2$ invariant modes. The remaining 7 patterns of the mod 2 structure correspond to the $C_+ = 0$ case.

In the same way, we can determine C_+ and C_- under other choices of the Scherk-Schwarz phases. Results are summarized in Table 2.8.

2.5.2 Three generation models of $T^4/\mathbb{Z}_2$

Here, we study the conditions to realize three generations. To obtain 3 degenerated zero-modes in the $\mathbb{Z}_2$ even sector, $|\mathrm{det}\mathbf{N}|$ needs to satisfy,

$$|\mathrm{det}\mathbf{N}| = 6 - (C_+ - C_-) =: D_+. \tag{2.213}$$

To obtain 3 degenerated zero-modes in the $\mathbb{Z}_2$ odd sector, $|\mathrm{det}\mathbf{N}|$ needs to satisfy

$$|\mathrm{det}\mathbf{N}| = 6 + (C_+ - C_-) =: D_-. \tag{2.214}$$

<table>
<tr><th colspan="3">mod 2 structure $(n_{11}, n_{12}, n_{21}, n_{22})$</th><th rowspan="2">$\vec{\beta} = (0,0)$</th><th rowspan="2">$\vec{\beta} = (0,\frac{1}{2})$</th><th rowspan="2">$\vec{\beta} = (\frac{1}{2},0)$</th><th rowspan="2">$\vec{\beta} = (\frac{1}{2},\frac{1}{2})$</th></tr>
<tr><th>$\vec{\alpha} = (0,\frac{1}{2})$</th><th>$\vec{\alpha} = (\frac{1}{2},0)$</th><th>$\vec{\alpha} = (\frac{1}{2},\frac{1}{2})$</th></tr>
<tr><td>$(0,0,0,1)$</td><td>$(0,0,1,0)$</td><td>$(0,0,1,1)$</td><td rowspan="3">$C_+ = 2$
$C_- = 0$</td><td>$C_+ = 0$
$C_- = 2$</td><td>$C_+ = 1$
$C_- = 1$</td><td>$C_+ = 1$
$C_- = 1$</td></tr>
<tr><td>$(0,1,0,0)$</td><td>$(1,0,0,0)$</td><td>$(1,1,0,0)$</td><td>$C_+ = 1$
$C_- = 1$</td><td>$C_+ = 0$
$C_- = 2$</td><td>$C_+ = 1$
$C_- = 1$</td></tr>
<tr><td>$(0,1,0,1)$</td><td>$(1,0,1,0)$</td><td>$(1,1,1,1)$</td><td>$C_+ = 1$
$C_- = 1$</td><td>$C_+ = 1$
$C_- = 1$</td><td>$C_+ = 0$
$C_- = 2$</td></tr>
<tr><td>$(1,0,1,1)$
$(1,1,1,0)$</td><td>$(0,1,1,1)$
$(1,1,0,1)$</td><td>$(0,1,1,0)$
$(1,0,0,1)$</td><td rowspan="3">$C_+ = 1$
$C_- = 0$</td><td>$C_+ = 0$
$C_- = 1$</td><td>$C_+ = 0$
$C_- = 1$</td><td>$C_+ = 1$
$C_- = 0$</td></tr>
<tr><td>$(1,0,0,1)$
$(1,1,0,1)$</td><td>$(0,1,1,0)$
$(1,1,1,0)$</td><td>$(1,0,1,1)$
$(0,1,1,1)$</td><td>$C_+ = 0$
$C_- = 1$</td><td>$C_+ = 1$
$C_- = 0$</td><td>$C_+ = 0$
$C_- = 1$</td></tr>
<tr><td>$(0,1,1,0)$
$(0,1,1,1)$</td><td>$(1,0,0,1)$
$(1,0,1,1)$</td><td>$(1,1,1,0)$
$(1,1,0,1)$</td><td>$C_+ = 1$
$C_- = 0$</td><td>$C_+ = 0$
$C_- = 1$</td><td>$C_+ = 0$
$C_- = 1$</td></tr>
<tr><td>$(0,0,0,0)$
$(0,0,1,0)$
$(1,0,0,0)$
$(1,0,1,0)$
$(0,0,1,1)$
$(1,1,0,0)$
$(1,1,1,1)$</td><td>$(0,0,0,0)$
$(0,0,0,1)$
$(0,1,0,0)$
$(0,1,0,1)$
$(0,0,1,1)$
$(1,1,0,0)$
$(1,1,1,1)$</td><td>$(0,0,0,0)$
$(0,0,1,0)$
$(1,0,0,0)$
$(1,0,1,0)$
$(0,0,0,1)$
$(0,1,0,0)$
$(0,1,0,1)$</td><td colspan="4">$C_+ = 0$
$C_- = 0$</td></tr>
</table>

Table 2.8: Values of $C_\pm$ classified by the mod 2 structure of $\mathbf{N}$ when $\vec{\alpha} = (0,\frac{1}{2}), (\frac{1}{2},0), (\frac{1}{2},\frac{1}{2})$.

We have denoted the required values of $|\det\mathbf{N}|$ for the realization of three-generation in $\mathbb{Z}_2$ even and odd sectors by D_+, D_-, respectively. Table 2.9 shows the required assignments of mod 2 structure in $\mathbf{N}$, determinant $|\det\mathbf{N}|$, $\mathbb{Z}_2$ parity, and $\vec{\beta}$ under the choice $\vec{\alpha} = (0,0)^{\mathrm{T}}$.

For non-vanishing choices of $\vec{\alpha}$, Table 2.10 shows the conditions. From these tables, we find that three degenerated zero-modes can be produced when $|\det\mathbf{N}| = 4, 5, 6, 7, 8$ in the first positive chirality component ψ_+^1 of magnetized $T^4/\mathbb{Z}_2$ model. We also note that when the mod 2 structure of the flux $\mathbf{N}$ is $(0,0,0,0)$, we cannot obtain three degenerated zero-modes. This comes from the fact that $|\det\mathbf{N}|$ is always 0 modulo 4.

<table>
<tr><th>mod 2 structure
$(n_{11}, n_{12}, n_{21}, n_{22})$</th><th>$\vec{\beta} = (0,0)$</th><th>$\vec{\beta} = (0,\frac{1}{2})$</th><th>$\vec{\beta} = (\frac{1}{2},0)$</th><th>$\vec{\beta} = (\frac{1}{2},\frac{1}{2})$</th></tr>
<tr><td>$(0,0,0,0)$</td><td colspan="4">no solution</td></tr>
<tr><td>$(0,0,0,1)$
$(0,0,1,0)$
$(0,0,1,1)$</td><td rowspan="3">$D_+ = 4$
$D_- = 8$</td><td>$D_+ = 4$
$D_- = 8$</td><td>$D_+ = 6$
$D_- = 6$</td><td>$D_+ = 6$
$D_- = 6$</td></tr>
<tr><td>$(0,1,0,0)$
$(1,0,0,0)$
$(1,1,0,0)$</td><td>$D_+ = 6$
$D_- = 6$</td><td>$D_+ = 4$
$D_- = 8$</td><td>$D_+ = 6$
$D_- = 6$</td></tr>
<tr><td>$(0,1,0,1)$
$(1,0,1,0)$
$(1,1,1,1)$</td><td>$D_+ = 6$
$D_- = 6$</td><td>$D_+ = 6$
$D_- = 6$</td><td>$D_+ = 4$
$D_- = 8$</td></tr>
<tr><td>$(0,1,1,0)$
$(1,0,0,1)$
$(0,1,1,1)$
$(1,0,1,1)$
$(1,1,0,1)$
$(1,1,1,0)$</td><td colspan="4">$D_+ = 5$
$D_- = 7$</td></tr>
</table>

Table 2.9: Required values of $|\det\mathbf{N}|$ to produce three degenerated zero-modes in magnetized $T^4/\mathbb{Z}_2$ under $\vec{\alpha} = (0,0)$.

2.6 Magnetized $T^4/\mathbb{Z}_N$

In this section, we further extend the analysis of zero-mode counting to magnetized $T^4/\mathbb{Z}_N$, $(N = 3, 4, 5, 6, \text{etc.})$ orbifolds [44, 45]. We assume that the Scherk-Schwarz phases are vanishing, i.e., $\vec{\alpha} = \vec{\beta} = \vec{0}$. As it is explicitly presented in the following sections, complex structure moduli Ω of $T^4/\mathbb{Z}_N$, $(N = 3, 4, 5, 6, \text{etc.})$ are frozen to specific values. This means that the lattice shape is fixed. In particular, we focus on orbifolds whose lattice shapes correspond to the simply-laced root lattice of Lie groups. We will see that Ω corresponding to such orbifolds can be chosen

<table>
<tr><th colspan="3">mod 2 structure $(n_{11}, n_{12}, n_{21}, n_{22})$</th><th rowspan="2">$\vec{\beta} = (0,0)$</th><th rowspan="2">$\vec{\beta} = (0, \frac{1}{2})$</th><th rowspan="2">$\vec{\beta} = (\frac{1}{2}, 0)$</th><th rowspan="2">$\vec{\beta} = (\frac{1}{2}, \frac{1}{2})$</th></tr>
<tr><th>$\vec{\alpha} = (0, \frac{1}{2})$</th><th>$\vec{\alpha} = (\frac{1}{2}, 0)$</th><th>$\vec{\alpha} = (\frac{1}{2}, \frac{1}{2})$</th></tr>
<tr><td>$(0,0,0,1)$</td><td>$(0,0,1,0)$</td><td>$(0,0,1,1)$</td><td rowspan="3">$D_+ = 4$
$D_- = 8$</td><td>$D_+ = 8$
$D_- = 4$</td><td>$D_+ = 6$
$D_- = 6$</td><td>$D_+ = 6$
$D_- = 6$</td></tr>
<tr><td>$(0,1,0,0)$</td><td>$(1,0,0,0)$</td><td>$(1,1,0,0)$</td><td>$D_+ = 6$
$D_- = 6$</td><td>$D_+ = 8$
$D_- = 4$</td><td>$D_+ = 6$
$D_- = 6$</td></tr>
<tr><td>$(0,1,0,1)$</td><td>$(1,0,1,0)$</td><td>$(1,1,1,1)$</td><td>$D_+ = 6$
$D_- = 6$</td><td>$D_+ = 6$
$D_- = 6$</td><td>$D_+ = 8$
$D_- = 4$</td></tr>
<tr><td>$(1,0,1,1)$
$(1,1,1,0)$</td><td>$(0,1,1,1)$
$(1,1,0,1)$</td><td>$(0,1,1,0)$
$(1,0,0,1)$</td><td rowspan="3">$D_+ = 5$
$D_- = 7$</td><td>$D_+ = 7$
$D_- = 5$</td><td>$D_+ = 7$
$D_- = 5$</td><td>$D_+ = 5$
$D_- = 7$</td></tr>
<tr><td>$(1,0,0,1)$
$(1,1,0,1)$</td><td>$(0,1,1,0)$
$(1,1,1,0)$</td><td>$(1,0,1,1)$
$(0,1,1,1)$</td><td>$D_+ = 7$
$D_- = 5$</td><td>$D_+ = 5$
$D_- = 7$</td><td>$D_+ = 7$
$D_- = 5$</td></tr>
<tr><td>$(0,1,1,0)$
$(0,1,1,1)$</td><td>$(1,0,0,1)$
$(1,0,1,1)$</td><td>$(1,1,1,0)$
$(1,1,0,1)$</td><td>$D_+ = 5$
$D_- = 7$</td><td>$D_+ = 7$
$D_- = 5$</td><td>$D_+ = 7$
$D_- = 5$</td></tr>
<tr><td>$(0,0,0,0)$</td><td>$(0,0,0,0)$</td><td>$(0,0,0,0)$</td><td colspan="4">no solution</td></tr>
<tr><td>$(0,0,1,0)$
$(1,0,0,0)$
$(1,0,1,0)$
$(0,0,1,1)$
$(1,1,0,0)$
$(1,1,1,1)$</td><td>$(0,0,0,1)$
$(0,1,0,0)$
$(0,1,0,1)$
$(0,0,1,1)$
$(1,1,0,0)$
$(1,1,1,1)$</td><td>$(0,0,1,0)$
$(1,0,0,0)$
$(1,0,1,0)$
$(0,0,0,1)$
$(0,1,0,0)$
$(0,1,0,1)$</td><td colspan="4">$D_+ = 6$
$D_- = 6$</td></tr>
</table>

Table 2.10: Required values of $|\det \mathbf{N}|$ to produce three degenerated zero-modes in magnetized $T^4/\mathbb{Z}_2$ under $\vec{\alpha} = (0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2})$.

to be symmetric $\Omega^{\mathrm{T}} = \Omega$ with positive definite imaginary part, i.e., $\mathrm{Im}\Omega > 0$. Hence, we parametrize Ω as

$$\Omega = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix}, \tag{2.215}$$

where $\tau_i \in \mathbb{C}$, $(i = 1, 2, 3)$ satisfying $\mathrm{Im}\Omega > 0$. The set of such symmetric complex 2×2 matrices Ω with positive definite imaginary part is called Siegel upper-half-space of genus 2 denoted by $\mathcal{H}_2$ [103].

As we saw in the magnetized $T^2/\mathbb{Z}_N$ models, modular transformation is a powerful tool for systematic analyses of zero-mode number. Hence, we will first review the Siegel modular group $Sp(4,\mathbb{Z})$ which is an extension of $SL(2,\mathbb{Z})$ as we consider higher dimensional torus T^4.

2.6.1 $Sp(4,\mathbb{Z})$ modular symmetry

The Siegel modular group $Sp(4,\mathbb{Z})$ is composed of 4×4 integer matrices

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.216}$$

satisfying the symplectic condition $\gamma^{\mathrm{T}} J \gamma = J$ (see for instance [103]). Here, A, B, C, D are 2×2 integer matrices and J is defined as

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}. \tag{2.217}$$

There are 4 generators of $Sp(4,\mathbb{Z})$,

$$S = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \quad T_i = \begin{pmatrix} I_2 & B_i \\ 0 & I_2 \end{pmatrix}, \ (i = 1, 2, 3), \tag{2.218}$$

where I_2 denotes the 2×2 identity matrix and B_i, $(i = 1, 2, 3)$ are symmetric integer matrices given by

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.219}$$

Under $\gamma \in Sp(4,\mathbb{Z})$, the complex coordinates and the complex structure $\Omega \in \mathcal{H}_2$ are transformed as

$$(\vec{z}, \Omega) \to ((C\Omega + D)^{-1^{\mathrm{T}}} \vec{z}, (A\Omega + B)(C\Omega + D)^{-1}). \tag{2.220}$$

This can be understood by checking the transformation of lattice vectors defining T^4. This can be done in the same way as we have derived eq. (2.72) in T^2 models.

Algebraic relations

Let us present algebraic relations satisfied by the generators of $Sp(4,\mathbb{Z})$. One can check the following:

$$\begin{aligned}
&S^4 = I_4, \\
&(ST_1T_2)^3 = I_4, \\
&(ST_1T_3)^5 = I_4,\ (ST_2T_3)^5 = I_4, \\
&(ST_3)^6 = I_4,\ (ST_1T_2^{-1})^6 = I_4,\ ((T_1T_2)^{-1}S)^6 = I_4, \\
&(ST_1T_2^{-1}T_3^{-1})^8 = I_4, \\
&(ST_1)^{12} = I_4,\ (ST_2)^{12} = I_4.
\end{aligned} \tag{2.221}$$

These algebraic relations will be useful when analyzing the zero-mode number on $T^4/\mathbb{Z}_N$ orbifolds.

2.6.2 $Sp(4,\mathbb{Z})$ transformation of background magnetic flux

Here, we study the transformation properties of the magnetic flux components under $Sp(4,\mathbb{Z})$. In general, the flux components p_{xy}, p_{xx}, p_{yy} behave non-trivially under the modular transformation.[18] To determine the behavior, it suffices to consider the generators of $Sp(4,\mathbb{Z})$, namely S and T_i transformations.

S transformation

Under the S transformation, the complex coordinates and the complex structure moduli are transformed as

$$\begin{aligned}
\vec{z} &\xrightarrow{S} \vec{z}_{(S)} = -\Omega^{-1}\vec{z}, \\
\Omega &\xrightarrow{S} \Omega_{(S)} = -\Omega^{-1}.
\end{aligned} \tag{2.222}$$

The complex coordinates and the complex structure moduli after the S transformation are denoted by $\vec{z}_{(S)}$ and $\Omega_{(S)}$, respectively. Eq. (2.222) can be translated to the real coordinates $\vec{x}, \vec{y}$ along the lattice vectors, i.e., $\vec{z} = \vec{x} + \Omega\vec{y}$. The results are

$$\begin{aligned}
\vec{x}_{(S)} &= -\vec{y}, \\
\vec{y}_{(S)} &= \vec{x},
\end{aligned} \tag{2.223}$$

where we have defined $\vec{x}_{(S)}$ and $\vec{y}_{(S)}$ by $\vec{z}_{(S)} =: \vec{x}_{(S)} + \Omega_{(S)}\vec{y}_{(S)}$. Then, components of the background 2-form magnetic flux are given by

$$\begin{aligned}
F &= \frac{1}{2}(p_{xx}^{(S)})_{ij} dx_{(S)}^i \wedge dx_{(S)}^j + \frac{1}{2}(p_{yy}^{(S)})_{ij} dy_{(S)}^i \wedge dy_{(S)}^j + (p_{xy}^{(S)})_{ij} dx_{(S)}^i \wedge dy_{(S)}^j \\
&= \frac{1}{2}(p_{xx}^{(S)})_{ij} dy^i \wedge dy^j + \frac{1}{2}(p_{yy}^{(S)})_{ij} dx^i \wedge dx^j + (p_{xy}^{(S)})_{ij}^{\mathrm{T}} dx^i \wedge dy^j,
\end{aligned} \tag{2.224}$$

[18] This is in contrast to T^2 models where the flux component M is always invariant under $SL(2,\mathbb{Z})$.

where $p_{xy}^{(S)}, p_{xx}^{(S)}, p_{yy}^{(S)}$ denote flux quanta in the S transformed coordinate system. We find the transformation properties of the flux components as

$$\begin{aligned} p_{xx}^{(S)} &= p_{yy}, \\ p_{yy}^{(S)} &= p_{xx}, \\ p_{xy}^{(S)} &= (p_{xy})^{\mathrm{T}}. \end{aligned} \tag{2.225}$$

This shows that the flux matrix $\mathbf{N}$ defined in (2.137) gets transposed,

$$\mathbf{N} \xrightarrow{S} \mathbf{N}_{(S)} = \mathbf{N}^{\mathrm{T}}. \tag{2.226}$$

We also notice that the condition $p_{xx} = p_{yy} = 0$ in eq. (2.135) is preserved under the S-transformation. Moreover, the SUSY condition is also preserved,

$$(\mathbf{N}_{(S)}\Omega_{(S)})^{\mathrm{T}} = \mathbf{N}_{(S)}\Omega_{(S)}. \tag{2.227}$$

T_i transformations

Under the T_i transformation, the complex coordinates and the complex structure moduli are transformed as

$$\begin{aligned} \vec{z} &\xrightarrow{T_i} \vec{z}_{(T_i)} = \vec{z}, \\ \Omega &\xrightarrow{T_i} \Omega_{(T_i)} = \Omega + B_i. \end{aligned} \tag{2.228}$$

The complex coordinates and the complex structure moduli after the T_i transformation are denoted by $\vec{z}_{(T_i)}$ and $\Omega_{(T_i)}$, respectively. Notice that we can easily generalize to compositions of T_i transformations. We will call such transformations as T transformation. If we consider T transformation $T := T_1^{p_1}T_2^{p_2}T_3^{p_3}$, $(p_1, p_2, p_3 \in \mathbb{Z})$, transformation properties are obtained by replacing B_i with $B := \sum_{i=1}^{3} p_i B_i$ in (2.228).[19]

Eq. (2.228) can be translated to the real coordinates $\vec{x}, \vec{y}$ along the lattice vectors, i.e., $\vec{z} = \vec{x} + \Omega\vec{y}$. The results are

$$\begin{aligned} \vec{x}_{(T)} &= \vec{x} - B\vec{y}, \\ \vec{y}_{(T)} &= \vec{y}. \end{aligned} \tag{2.229}$$

We have defined $\vec{x}_{(T)}$ and $\vec{y}_{(T)}$ by $\vec{z}_{(T)} = \vec{x}_{(T)} + \Omega_{(T)}\vec{y}_{(T)}$ where $\Omega_{(T)} = \Omega + B$. Then, real components of the background 2-form magnetic flux are given by

$$\begin{aligned} F &= \frac{1}{2}(p_{xx}^{(T)})_{ij} dx_{(T)}^i \wedge dx_{(T)}^j + \frac{1}{2}(p_{yy}^{(T)})_{ij} dy_{(T)}^i \wedge dy_{(T)}^j + (p_{xy}^{(T)})_{ij} dx_{(T)}^i \wedge dy_{(T)}^j \\ &= \frac{1}{2}(p_{xx}^{(T)})_{ij} dx^i \wedge dx^j + \frac{1}{2}[(p_{yy}^{(T)}) + (Bp_{xx}^{(T)}B) - (Bp_{xy}^{(T)}) + (Bp_{xy}^{(T)})^{\mathrm{T}}]_{ij} dy^i \wedge dy^j \\ &+ [(p_{xy}^{(T)}) - (p_{xx}^{(T)}B)]_{ij} dx^i \wedge dy^j, \end{aligned} \tag{2.230}$$

[19]Since T_i, $(i = 1, 2, 3)$ are commutative, ordering of T_i is irrelevant.

where $p_{xy}^{(T)}, p_{xx}^{(T)}, p_{yy}^{(T)}$ denote flux quanta in the T transformed coordinate system. We find the transformation properties of the flux components as

$$\begin{aligned}
p_{xx}^{(T)} &= p_{xx}, \\
p_{yy}^{(T)} &= p_{yy} + (Bp_{xx}B) + (Bp_{xy}) - (Bp_{xy})^{\mathrm{T}}, \\
p_{xy}^{(T)} &= p_{xy} + (p_{xx}B).
\end{aligned} \tag{2.231}$$

We can see that the condition $p_{xx} = p_{yy} = 0$ is no longer maintained unless Bp_{xy} is symmetric,

$$(Bp_{xy})^{\mathrm{T}} = Bp_{xy}. \tag{2.232}$$

Under the condition (2.232), the magnetic flux $\mathbf{N}$ is invariant,

$$\mathbf{N} \xrightarrow{T} \mathbf{N}_{(T)} = \mathbf{N}. \tag{2.233}$$

At last, we check the SUSY condition eq. (2.138) under T. One obtains

$$(\mathbf{N}_{(T)}\Omega_{(T)})^{\mathrm{T}} = \mathbf{N}_{(T)}\Omega_{(T)}, \tag{2.234}$$

using $(\mathbf{N}\Omega)^{\mathrm{T}} = \mathbf{N}\Omega$. This shows that the SUSY condition is consistent with the T transformation under eq. (2.232).

2.6.3 $Sp(4,\mathbb{Z})$ transformation of zero-mode wavefunctions

Here, we study the $Sp(4,\mathbb{Z})$ transformation of the positive chirality zero-mode wavefunctions of magnetized T^4 models.

S transformation of zero-modes

The positive chirality zero-mode wavefunctions in (2.174) transform as

$$\psi^{\vec{J},\mathbf{N}}(-\Omega^{-1}\vec{z}, -\Omega^{-1}) = \sqrt{\det(\mathbf{N}^{-1}\Omega/i)} \sum_{\vec{K}\in\Lambda_{\mathbf{N}}} e^{2\pi i \vec{J}^{\mathrm{T}}\cdot\mathbf{N}^{-1}\cdot\vec{K}} \psi^{\vec{K},\mathbf{N}}(\vec{z},\Omega), \tag{2.235}$$

under the S transformation, if $\mathbf{N}^{\mathrm{T}} = \mathbf{N}$. We have assumed that the Scherk-Schwarz phases are vanishing. We give a proof of (2.235) in appendix A.2.1.

Let us comment on the metric. In general, metric components (2.129) generally transform non-trivially under S transformation. We recall that the flat metric was assumed when solving the Dirac equation. Hence, one might wonder whether the zero-mode wavefunctions in the $\vec{z}_{(S)}$ coordinates system are still given by $\psi^{\vec{J},\mathbf{N}}(\vec{z}_{(S)}, \Omega_{(S)})$. Nevertheless, we do not need to worry about it for our purpose, i.e., zero-mode counting on magnetized $T^4/\mathbb{Z}_N$, $(N \geq 3)$ orbifolds. This is because the complex structure moduli corresponding to those orbifolds are fixed to specific values which make the metric components invariant under S.

T_i transformations of zero-modes

Here, we study $T = T_1^{p_1} T_2^{p_2} T_3^{p_3}$, $(p_1, p_2, p_3 \in \mathbb{Z})$ transformation of the zero-mode wavefunctions on magnetized T^4 under the condition (2.232), which can be rewritten as

$$(\mathbf{N}B)^{\mathrm{T}} = \mathbf{N}B. \tag{2.236}$$

Then, the zero-mode wavefunctions in eq. (2.177) transform as,

$$\psi^{\vec{J},\mathbf{N}}(\vec{z}, \Omega + B) = e^{2\pi i \vec{J}^{\mathrm{T}} \cdot (\mathbf{N}^{-1}B) \cdot \vec{J}} \psi^{\vec{J},\mathbf{N}}(\vec{z}, \Omega), \tag{2.237}$$

if all of the diagonal matrix elements of $\mathbf{N}B$ are even. We give a proof of eq.(2.237) in Appendix A.2.2.

2.6.4 $T^4/\mathbb{Z}_4$

We explicitly analyze the zero-mode number on magnetized $T^4/\mathbb{Z}_4$. We still assume that the Scherk-Schwarz phases are vanishing. We begin with the algebraic relation,

$$S^4 = I_4. \tag{2.238}$$

This implies that S transformation can be regarded as the $\mathbb{Z}_4$ twist defining a $T^4/\mathbb{Z}_4$ orbifold. Since the orientations of the lattice vectors defining T^4 should be invariant under the $\mathbb{Z}_4$ twist, let us determine the S-invariant complex structure moduli $\Omega \in \mathcal{H}_2$. Such Ω must satisfy

$$\Omega^2 = \begin{pmatrix} \tau_1^2 + \tau_3^2 & \tau_3(\tau_1 + \tau_2) \\ \tau_3(\tau_1 + \tau_2) & \tau_2^2 + \tau_3^2 \end{pmatrix} = -I_2. \tag{2.239}$$

By comparing the off-diagonal elements we obtain $\tau_3 = 0$ or $\tau_1 = -\tau_2$. Let us take $\tau_3 = 0$ because we require $\Omega \in \mathcal{H}_2$. Then, from the diagonal elements, the moduli parameters are fixed as, $\tau_1^2 = -1$, $\tau_2^2 = -1$. Hence, the relevant solution is given by

$$\Omega_{(S)} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \tag{2.240}$$

where $\Omega_{(S)}$ stands for the S-invariant moduli. When Ω is fixed to $\Omega_{(S)}$, the complex coordinate of T^4 is transformed under S as

$$\vec{z} \xrightarrow{S} i\vec{z}. \tag{2.241}$$

This is nothing but a $\mathbb{Z}_4$ twist. Hence, we can define $T^4/\mathbb{Z}_4$ orbifold by imposing the identification:

$$\vec{z} \sim i\vec{z}, \tag{2.242}$$

on T^4 whose moduli are given by $\Omega = \Omega_{(S)}$. Note that the twist $\vec{z} \to i\vec{z}$ does not change the form of the flat metric (2.129), namely an isometry. The fixed moduli correspond to the following orientations of lattice vectors:

$$e_1 = 2\pi R \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ e_2 = 2\pi R \begin{pmatrix} i \\ 0 \end{pmatrix}, \ e_3 = 2\pi R \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ e_4 = 2\pi R \begin{pmatrix} 0 \\ i \end{pmatrix}, \tag{2.243}$$

where the components are of the coordinate system $\vec{z}$. One can check that the $\mathbb{Z}_4$ twist (S transformation) rotates the lattice vectors as follows:

$$\begin{pmatrix} e_2^{(S)} \\ e_4^{(S)} \\ e_1^{(S)} \\ e_3^{(S)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_2 \\ e_4 \\ e_1 \\ e_3 \end{pmatrix}. \tag{2.244}$$

This is identified as the generalized Coxeter element of $SO(4)^2$ with $\mathbb{Z}_2$ outer automorphism exchanging $e_1 \leftrightarrow e_2$ and $e_3 \leftrightarrow e_4$ [110, 111]. Here, $e_a^{(S)}$, $(a = 1, 2, 3, 4)$ denote the lattice vectors after the S transformation.

Magnetic flux N on $T^4/\mathbb{Z}_4$

On $T^4/\mathbb{Z}_4$ where the moduli values are fixed to $\Omega_{(S)}$, the SUSY condition (2.138) is satisfied if and only if $\mathbf{N}$ is symmetric,

$$\mathbf{N} = \begin{pmatrix} n_1 & m \\ m & n_2 \end{pmatrix}, \quad n_1, n_2, m \in \mathbb{Z}. \tag{2.245}$$

Then, $\mathbf{N}$ is S-invariant, cf. (2.226). This ensures that $\mathbf{N}$ (2.245) is consistent with the $\mathbb{Z}_4$ symmetry that defines the orbifold. For concreteness, we focus on the case when the positive definite condition (2.175) is satisfied. This corresponds to

$$\mathrm{det}\mathbf{N} > 0, \quad \mathrm{tr}\mathbf{N} > 0. \tag{2.246}$$

Then, only the first positive chirality component ψ_+^1 is non-zero.

Zero-mode counting on magnetized $T^4/\mathbb{Z}_4$

We can compute the degeneracy of zero-modes in magnetized $T^4/\mathbb{Z}_4$ model by using S transformation. The method is essentially the same as $T^2/\mathbb{Z}_4$ model reviewed in subsection 2.3.3.

Zero-mode counting of $\mathbb{Z}_4$ eigenstates is equivalent to counting the multiplicity of eigenvalues $\eta = \pm 1, \pm i$ in the representation matrix $\rho(S)$. Here, $\rho(S)$ is the $(\mathrm{det}\mathbf{N} \times \mathrm{det}\mathbf{N})$ matrix which is given by setting $\Omega = \Omega_{(S)}$ in eq. (2.235), i.e.,

$$\rho(S)_{\vec{J}\vec{K}} = \frac{1}{\sqrt{\mathrm{det}\mathbf{N}}} e^{2\pi i \vec{J}^{\mathrm{T}}.\mathbf{N}^{-1}.\vec{K}}. \tag{2.247}$$

Now, let us denote the number of zero-modes with $\mathbb{Z}_4$ eigenvalues $\eta = \pm 1, \pm i$ by $D_{(\pm 1)}, D_{(\pm i)}$, respectively. They satisfy the following conditions:

$$D_{(+1)} + D_{(+i)} + D_{(-1)} + D_{(-i)} = \mathrm{det}\mathbf{N}, \tag{2.248}$$

$$D_{(+1)} + iD_{(+i)} - D_{(-1)} - iD_{(-i)} = \mathrm{tr}\rho(S). \tag{2.249}$$

Eqs. (2.248) and (2.249) constrain three real degrees of freedom. One more independent constraint is needed to determine $D_{(\pm 1)}, D_{(\pm i)}$. That is provided by the number of zero-modes in magnetized $T^4/\mathbb{Z}_2$. Since two consecutive $\mathbb{Z}_4$ twists are equivalent to $\mathbb{Z}_2$ twist, the sum of the number of $\mathbb{Z}_4$ zero-modes with $\mathbb{Z}_4$ charge $\pm 1, (\pm i)$ is equal to the number of $\mathbb{Z}_2$ even (odd) zero-modes. Thus, we obtain

$$N_{\text{even}} - N_{\text{odd}} = C_+ - C_- = D_{(+1)} - D_{(+i)} + D_{(-1)} - D_{(-i)}$$
$$= \begin{cases} 1: & \text{if } \det\mathbf{N} \equiv 1 (\text{mod} 2), \\ 2: & \text{if } \det\mathbf{N} \equiv 0\, (\text{mod} 2) \quad \text{and} \quad \gcd(n_i, m) \equiv 1 (\text{mod} 2) \text{ for } i = 1 \text{ or } 2, \\ 4: & \text{if } \gcd(n_i, m) \equiv 0 \ (\text{mod} 2) \text{ for both } i = 1 \text{ and } 2, \end{cases} \tag{2.250}$$

where N_{even} and N_{odd} denote the number of $\mathbb{Z}_2$ even and odd zero-modes, respectively, cf. (2.198). The values of C_+ and C_- are summarized in Table 2.7.

Evaluating $\mathrm{tr}\rho(S)$

For the zero-mode counting, we need to compute $\mathrm{tr}\rho(S)$:

$$\begin{aligned} \mathrm{tr}[\rho(S)] &= \frac{1}{\sqrt{\det\mathbf{N}}} \sum_{\vec{K} \in \Lambda_{\mathbf{N}}} e^{2\pi i \vec{K}^{\mathrm{T}}.\mathbf{N}^{-1}.\vec{K}} \\ &= \frac{1}{\sqrt{\det\mathbf{N}}} \sum_{\vec{K} \in \Lambda_{\mathbf{N}}} e^{\frac{2\pi i}{\det\mathbf{N}}(n_2 K_1^2 + n_1 K_2^2 - 2m K_1 K_2)}. \end{aligned} \tag{2.251}$$

Even when we fix the vallue of $\det\mathbf{N}$, there exist infinitely many $\mathbf{N}$ of the form (2.245) satisfying (2.246).[20] In addition, the lattice $\Lambda_{\mathbf{N}}$ characterizing the region of summation over $\vec{K}$ in eq.(2.251) is dependent on individual $\mathbf{N}$. Hence, it is a non-trivial task to systematically evaluate the trace $\mathrm{tr}\rho(S)$ for all the possibilities of $\mathbf{N}$.

One of the effective computational methods is to use the Krazer formula [21]. Let us denote a non-singular symmetric integer $(r \times r)$ matrix by A and let a, b be positive integers. Then, the Krazer formula tells us that the following relation

$$b^{-r/2} \sum_{\vec{x} \in (\mathbb{Z}/b\mathbb{Z})^r} e^{\frac{\pi i a}{b} \vec{x}^{\mathrm{T}} A \vec{x}} = e^{\frac{\pi i \sigma(A)}{4}} a^{-r/2} s^{1/2 - r} \sum_{\vec{y} \in (\mathbb{Z}/sa\mathbb{Z})^r} e^{-\frac{\pi i b}{a} \vec{y}^{\mathrm{T}} A^{-1} \vec{y}}, \tag{2.252}$$

holds under the conditions $ab \equiv 0 \pmod 2$ or $A_{ii} \equiv 0 \pmod 2$ for $\forall i = 1, \cdots, r$. Here, $\sigma(A)$ is the signature of A (i.e., the difference between the numbers of positive and negative eigenvalues

[20] Consider a 2×2 symmetric integer matrix $\mathbf{N}$ with $\mathrm{sgn}(\mathrm{tr}\mathbf{N}) = +1$. Let us define $\mathbf{N}' := P^{\mathrm{T}}\mathbf{N}P$ where $P \in GL(2, \mathbb{Z})$. Clearly, $\mathbf{N}'$ is an integer matrix of the form (2.245). Moreover, we find that both the determinant and the sign of the trace are conserved, i.e., $\det\mathbf{N}' = \det\mathbf{N}$, $\mathrm{sgn}(\mathrm{tr}\mathbf{N}') = +1$. Since $GL(2, \mathbb{Z})$ is an infinite group, we obtain infinite variations of $\mathbf{N}'$. This means one has an infinite number of $\mathbf{N}$ satisfying (2.246) even when we fix the value of the determinant.

[21] For instance, see Refs. [112, 113] and references therein.

of A) and $s := |\mathrm{det}A|$. We can check

$$\sum_{\vec{y}\in(\mathbb{Z}/sa\mathbb{Z})^r} e^{-\frac{\pi ib}{a}\vec{y}^{\mathrm{T}}A^{-1}\vec{y}} = s^{r-1}\sum_{\vec{y}\in\Lambda_{aA}} e^{-\frac{\pi ib}{a}\vec{y}^{\mathrm{T}}A^{-1}\vec{y}}. \tag{2.253}$$

Here, Λ_{aA} denotes integer points which lie inside the cell spanned by $aA\vec{e}_k$, $(k = 1, \cdots, r)$ where $\vec{e}_k$ denotes the standard unit vectors, i.e., $(\vec{e}_k)_l = \delta_{kl}$. Hence, by setting $A' := aA$, we find

$$b^{-r/2}\sum_{x\in(\mathbb{Z}/b\mathbb{Z})^r} e^{\frac{\pi i}{b}x^{\mathrm{T}}A'x} = \frac{e^{\frac{\pi i\sigma(A')}{4}}}{\sqrt{|\mathrm{det}A'|}}\sum_{y\in\Lambda_{A'}} e^{-\pi iby^{\mathrm{T}}(A')^{-1}y}, \tag{2.254}$$

which holds under the conditions $b \equiv 0 \pmod 2$ or $A'_{ii} \equiv 0 \pmod 2$.

Now, let us apply the formula (2.254) to compute $\mathrm{tr}\rho(S)$. By setting $b = 2$, $A' = \mathbf{N}$, $r = 2$ and by taking complex conjugation of (2.254), we obtain

$$\frac{1}{\sqrt{|\mathrm{det}\mathbf{N}|}}\sum_{\vec{K}\in\Lambda_{\mathbf{N}}} e^{2\pi i\vec{K}^{\mathrm{T}}\mathbf{N}^{-1}\vec{K}} = \frac{e^{-\frac{\pi i\sigma(\mathbf{N})}{4}}}{2}\sum_{\vec{L}\in(\mathbb{Z}/2\mathbb{Z})^2} e^{-\frac{\pi i}{2}\vec{L}^{\mathrm{T}}\mathbf{N}\vec{L}}. \tag{2.255}$$

Thus, we can evaluate $\mathrm{tr}\rho(S)$ as

$$\mathrm{tr}\rho(S) = \frac{i}{2}\left(1 + e^{-\frac{\pi i}{2}n_1} + e^{-\frac{\pi i}{2}n_2} + e^{-\frac{\pi i}{2}(n_1+n_2+2m)}\right), \tag{2.256}$$

where we have taken $\sigma(\mathbf{N}) = 2$ because of (2.246). Eq. (2.256) shows that $\mathrm{tr}\rho(S)$ depends on $n_i\,(i = 1, 2)$ modulo 4 and m modulo 2. Using this formula, we have analyzed the number of zero-modes on magnetized $T^4/\mathbb{Z}_4$. The results are summarized in Table 2.11.

det$\mathbf{N}$	conditions	$D_{(+1)}$	$D_{(-1)}$	$D_{(+i)}$	$D_{(-i)}$
1		1	0	0	0
2		1	1	0	0
3	$n_1 \equiv n_2 \equiv 2 \pmod 4$	1	1	0	1
	otherwise	1	1	1	0
4	$n_i \equiv 1 \pmod 4$, $(i = 1 \text{ or } 2)$	2	1	1	0
	$n_i \equiv 2 \pmod 4$, $(i = 1 \text{ or } 2)$	2	2	0	0
5	$n_i \equiv 1 \pmod 4$, $(i = 1 \text{ or } 2)$	2	1	1	1
	$n_i \equiv 3 \pmod 4$, $(i = 1 \text{ or } 2)$	1	2	1	1
6		2	2	1	1
7		2	2	2	1
8	$n_i \equiv 1 \pmod 4$, $(i = 1 \text{ or } 2)$	3	2	2	1
	$n_i \equiv 3 \pmod 4$, $(i = 1 \text{ or } 2)$	2	3	2	1
	$n_i \equiv 2 \pmod 4$, $(i = 1 \text{ or } 2)$	3	3	1	1
9	$n_i \equiv 1 \pmod 4$, $(i = 1 \text{ or } 2)$	3	2	2	2
	$n_i \equiv 3 \pmod 4$, $(i = 1 \text{ or } 2)$	2	3	2	2
10		3	3	2	2
11	$n_1 \equiv n_2 \equiv 2 \pmod 4$	3	3	2	3
	otherwise	3	3	3	2
12	$n_i \equiv 1 \pmod 4$, $(i = 1 \text{ or } 2)$	4	3	3	2
	$n_i \equiv 3 \pmod 4$, $(i = 1 \text{ or } 2)$	3	4	3	2
	$n_i \equiv 2 \pmod 4$, $(i = 1 \text{ or } 2)$	4	4	2	2
	$n_1 \equiv n_2 \equiv 0 \pmod 4$	4	4	3	1
13	$n_i \equiv 1 \pmod 4$, $(i = 1 \text{ or } 2)$	4	3	3	3
	$n_i \equiv 3 \pmod 4$, $(i = 1 \text{ or } 2)$	3	4	3	3
14		4	4	3	3
15		4	4	4	3
16	$n_i \equiv 1 \pmod 4$, $(i = 1 \text{ or } 2)$	5	4	4	3
	$n_i \equiv 2 \pmod 4$, $(i = 1 \text{ or } 2)$	5	5	3	3
	$n_1 \equiv n_2 \equiv 0 \pmod 4$	5	5	4	2
17	$n_i \equiv 1 \pmod 4$, $(i = 1 \text{ or } 2)$	5	4	4	4
	$n_i \equiv 3 \pmod 4$, $(i = 1 \text{ or } 2)$	4	5	4	4
18		5	5	4	4
19	$n_1 \equiv n_2 \equiv 2 \pmod 4$	5	5	4	5
	otherwise	5	5	5	4

Table 2.11: Number of positive chirality zero-modes in magnetized $T^4/\mathbb{Z}_4$ models. We have only shown the range ($1 \leq \det\mathbf{N} \leq 19$) because three-generation models do not appear for larger values of det$\mathbf{N}$.

Derivation

In what follows, we derive our results in Table 2.11. We will separately consider 4 cases classified by the value of det$\mathbf{N}$ modulo 4.

det$\mathbf{N} \equiv 1 \pmod 4$:

Let us evaluate tr$\rho(S)$ when det$\mathbf{N} \equiv 1 \pmod 4$. As is clear from (2.256), we need to consider values of n_i, $(i = 1, 2)$ modulo 4 and m modulo 2. We find that the value of $n_1 n_2$ modulo 4 can only take

$$n_1 n_2 \equiv 1 + m^2 \equiv \begin{cases} 1 \pmod 4, & \text{if } m \equiv 0 \pmod 2, \\ 2 \pmod 4, & \text{if } m \equiv 1 \pmod 2. \end{cases} \tag{2.257}$$

For example, when $n_1 n_2 \equiv 1 \pmod 4$ with $m \equiv 0 \pmod 2$, possible values are $(n_1, n_2) \equiv (1, 1)$ or $(3, 3) \pmod 4$. Then, (2.256) gives us tr$\rho(S) = +1$ and -1, respectively. Similarly, we can evaluate tr$\rho(S)$ when $n_1 n_2 \equiv 2 \pmod 4$. Table 2.12 shows the values of tr$\rho(S)$ under the possible choices of n_1 and n_2 modulo 4. Note that $n_1 n_2 \equiv 0, 3 \pmod 4$ is impossible, hence we have added cross signs to the corresponding places. Since the value of tr$\rho(S)$ is invariant under the permutation $n_1 \leftrightarrow n_2$, we leave a blank in the lower triangle of Table 2.12.

		n_2			
		0	1	2	3
n_1	0	$\times$	$\times$	$\times$	$\times$
	1		$+1$	$+1$	$\times$
	2			$\times$	-1
	3				-1

Table 2.12: tr$\rho(S)$ under the possible choices of n_1, n_2 modulo 4 when det$\mathbf{N} \equiv 1 \pmod 4$. Cross signs denote the inconsistent choices of n_1, n_2. Since tr$\rho(S)$ is symmetric under the permutation $n_1 \leftrightarrow n_2$, we leave a blank in the lower triangle.

As in Table 2.12 if $n_i \equiv 1 \pmod 4$, $(i = 1 \,\text{or}\, 2)$, we obtain tr$\rho(S) = +1$. As a result, we find

$$D_{(+1)} = \frac{\det\mathbf{N} + 3}{4}, \quad D_{(-1)} = D_{(+i)} = D_{(-i)} = \frac{\det\mathbf{N} - 1}{4}, \tag{2.258}$$

from eqs. (2.248), (2.249) and (2.250). On the other hand, if $n_i \equiv 3 \pmod 4$, $(i = 1 \,\text{or}\, 2)$, we obtain tr$\rho(S) = -1$. As a result, we find

$$D_{(-1)} = \frac{\det\mathbf{N} + 3}{4}, \quad D_{(+1)} = D_{(+i)} = D_{(-i)} = \frac{\det\mathbf{N} - 1}{4}. \tag{2.259}$$

$\det\mathbf{N} \equiv 3 \pmod 4$:

Let us evaluate $\mathrm{tr}\rho(S)$ when $\det\mathbf{N} \equiv 3 \pmod 4$. We find that the value of $n_1 n_2$ modulo 4 can only take

$$n_1 n_2 \equiv 3 + m^2 \equiv \begin{cases} 3 \pmod 4, & \text{if } m \equiv 0 \pmod 2, \\ 0 \pmod 4, & \text{if } m \equiv 1 \pmod 2. \end{cases} \tag{2.260}$$

It is impossible to have $n_1 n_2 \equiv 1, 2 \pmod 4$. Hence, we have added cross signs to the corresponding places in Table 2.13.

		n_2			
		0	1	2	3
n_1	0	$+i$	$+i$	$+i$	$+i$
	1		$\times$	$\times$	$+i$
	2			$-i$	$\times$
	3				$\times$

Table 2.13: $\mathrm{tr}\rho(S)$ under the possible choices of n_1, n_2 modulo 4 when $\det\mathbf{N} \equiv 3 \pmod 4$. Cross signs denote the inconsistent choices of n_1, n_2. Since $\mathrm{tr}\rho(S)$ is symmetric under the permutation $n_1 \leftrightarrow n_2$, we leave a blank in the lower triangle.

As summarized in the Table 2.13, if $n_1 \equiv n_2 \equiv 2 \pmod 4$, we obtain $\mathrm{tr}\rho(S) = -i$. This gives us

$$D_{(+i)} = \frac{\det\mathbf{N} - 3}{4}, \quad D_{(+1)} = D_{(-1)} = D_{(-i)} = \frac{\det\mathbf{N} + 1}{4}, \tag{2.261}$$

Otherwise, we obtain $\mathrm{tr}\rho(S) = +i$, yielding

$$D_{(-i)} = \frac{\det\mathbf{N} - 3}{4}, \quad D_{(+1)} = D_{(-1)} = D_{(+i)} = \frac{\det\mathbf{N} + 1}{4}. \tag{2.262}$$

One last comment. When $\det\mathbf{N} \equiv 7 \pmod 8$, we cannot obtain $\mathrm{tr}\rho(S) = -i$. This is simply because there is no 2×2 symmetric integer matrix $\mathbf{N}$ satisfying $n_1 \equiv n_2 \equiv 2 \pmod 4$ if $\det\mathbf{N} \equiv 7 \pmod 8$. It can be proven using proof by contradiction. Suppose that we have an integer matrix $\mathbf{N}$, where $n_1 \equiv n_2 \equiv 2 \pmod 4$ and $\det\mathbf{N} \equiv 7 \pmod 8$. Then, the relation $\det\mathbf{N} \equiv 7 \pmod 8$ can be rewritten as $m^2 \equiv 5 \pmod 8$. This is a contradiction because 5 is not a quadratic residue modulo 8. Hence, we have presented only the case $\mathrm{tr}\rho(S) = +i$ for $\det\mathbf{N} = 7, 15$ in Table 2.11.

$\det\mathbf{N} \equiv 2 \pmod 4$:

Let us evaluate $\mathrm{tr}\rho(S)$ when $\det\mathbf{N} \equiv 2 \pmod 4$. We find that the value of $n_1 n_2$ modulo 4 can only take

$$n_1 n_2 \equiv 2 + m^2 \equiv \begin{cases} 2 \pmod 4, & \text{if } m \equiv 0 \pmod 2, \\ 3 \pmod 4, & \text{if } m \equiv 1 \pmod 2. \end{cases} \tag{2.263}$$

Since having $n_1 n_2 \equiv 1, 2 \pmod 4$ is impossible, we have added cross signs to the corresponding places in Table 2.14. We have found that $\mathrm{tr}\rho(S)$ is always zero when $\det\mathbf{N} \equiv 2 \pmod 4$. Hence, we obtain

$$D_{(+1)} = D_{(-1)} = \frac{\det\mathbf{N} + 2}{4}, \quad D_{(+i)} = D_{(-i)} = \frac{\det\mathbf{N} - 2}{4}, \tag{2.264}$$

from eqs. (2.248), (2.249) and (2.250).

		n_2 0	1	2	3
n_1	0	$\times$	$\times$	$\times$	$\times$
	1		$\times$	0	0
	2			$\times$	0
	3				$\times$

Table 2.14: $\mathrm{tr}\rho(S)$ under the possible choices of n_1, n_2 modulo 4 when $\det\mathbf{N} \equiv 2 \pmod 4$. Cross signs denote the inconsistent choices of n_1, n_2. Since $\mathrm{tr}\rho(S)$ is symmetric under the permutation $n_1 \leftrightarrow n_2$, we leave a blank in the lower triangle.

$\det\mathbf{N} \equiv 0 \pmod 4$**:**

Let us evaluate $\mathrm{tr}\rho(S)$ when $\det\mathbf{N} \equiv 0 \pmod 4$. We find that the value of $n_1 n_2$ modulo 4 can only take

$$n_1 n_2 \equiv m^2 \equiv \begin{cases} 0 \pmod 4, & \text{if } m \equiv 0 \pmod 2, \\ 1 \pmod 4, & \text{if } m \equiv 1 \pmod 2. \end{cases} \tag{2.265}$$

Values of $\mathrm{tr}\rho(S)$ under the possible choices of n_1, n_2 modulo 4 are summarized in Table 2.15. Since having $n_1 n_2 \equiv 2, 3 \pmod 4$ is impossible, we put cross signs to the corresponding places.

		n_2 0	1	2	3
n_1	0	$+2i$	$1+i$	0	$-1+i$
	1		$1+i$	$\times$	$\times$
	2			0	$\times$
	3				$-1+i$

Table 2.15: $\mathrm{tr}\rho(S)$ under the possible choices of n_1, n_2 modulo 4 when $\det\mathbf{N} \equiv 0 \pmod 4$. Cross signs denote the inconsistent choices of n_1, n_2. Since $\mathrm{tr}\rho(S)$ is symmetric under the permutation $n_1 \leftrightarrow n_2$, we leave a blank in the lower triangle.

If n_1 or $n_2 \equiv 1 \pmod 4$, we obtain $\mathrm{tr}\rho(S) = 1 + i$, leading to

$$D_{(+1)} = \frac{\det\mathbf{N} + 4}{4}, \quad D_{(-1)} = D_{(+i)} = \frac{\det\mathbf{N}}{4}, \quad D_{(-i)} = \frac{\det\mathbf{N} - 4}{4}. \tag{2.266}$$

If n_1 or $n_2 \equiv 3 \pmod 4$, we obtain $\mathrm{tr}\rho(S) = -1+i$, leading to

$$D_{(-1)} = \frac{\det\mathbf{N}+4}{4}, \quad D_{(+1)} = D_{(+i)} = \frac{\det\mathbf{N}}{4}, \quad D_{(-i)} = \frac{\det\mathbf{N}-4}{4}. \tag{2.267}$$

In case of $n_1 \equiv n_2 \equiv 0 \pmod 2$, the flux matrix $\mathbf{N}$ can be always written as

$$\mathbf{N} = \begin{pmatrix} n_1 & m \\ m & n_2 \end{pmatrix} = 2\begin{pmatrix} n_1' & m' \\ m' & n_2' \end{pmatrix} = 2\mathbf{N}', \tag{2.268}$$

where $n_1', n_2', m' \in \mathbb{Z}$. When n_1' or $n_2' \equiv 1 \pmod 2$, we obtain $\mathrm{tr}\rho(S) = 0$, which leads to

$$D_{(+1)} = D_{(-1)} = \frac{\det\mathbf{N}+4}{4}, \quad D_{(+i)} = D_{(-i)} = \frac{\det\mathbf{N}-4}{4}. \tag{2.269}$$

When $n_1' \equiv n_2' \equiv 0 \pmod 2$, we obtain $\mathrm{tr}\rho(S) = 2i$, which leads to

$$D_{(+1)} = D_{(-1)} = \frac{\det\mathbf{N}+4}{4}, \quad D_{(+i)} = \frac{\det\mathbf{N}-4}{4}, \quad D_{(-i)} = \frac{\det\mathbf{N}-8}{4}. \tag{2.270}$$

Note that $\mathrm{tr}\rho(S) = 2i$ is possible only if $\det\mathbf{N} \equiv 0$ or $12 \pmod{16}$. This comes from

$$\det\mathbf{N}' = n_1' n_2' - (m')^2 \equiv -(m')^2 \equiv 0 \text{ or } 3 \pmod 4, \tag{2.271}$$

where we have used the fact that quadratic residues of 4 are 0 and 1 (mod 4).

One last comment. In Table 2.11, one might wonder why we do not list the case $\mathrm{tr}\rho(S) = -1+i$ for $\det\mathbf{N} = 4, 16$. This is simply because there is no symmetric positive definite integer matrix $\mathbf{N}$ such that $n_1 \equiv 3 \pmod 4$ or $n_2 \equiv 3 \pmod 4$ if $\det\mathbf{N} = 4, 16$. We give a detailed proof in Appendix A.4.

Three generation in $T^4/\mathbb{Z}_4$

From table 2.11, we find that three-generation can be realized when $8 \leq \det\mathbf{N} \leq 16$. One would question whether or not three-generation models can appear for larger $\det\mathbf{N}$. The following analysis shows that there are no such cases. For a given value of $\det\mathbf{N} > 0$, our previous analyses show that the minimum number of $\mathbb{Z}_4$ eigenstates is given by

$$D_{(-i)} = \frac{\det\mathbf{N}-8}{4}, \tag{2.272}$$

in eq. (2.270). Then, the necessary condition to obtain three-generation is

$$\frac{\det\mathbf{N}-8}{4} \leq 3, \tag{2.273}$$

which gives us $\det\mathbf{N} \leq 20$. Recalling that eq. (2.270) is possible only if $\det\mathbf{N} \equiv 0, 12 \pmod{16}$, we cannot obtain three-generation when $\det\mathbf{N} = 20$. Hence, table 2.11 captures all three-generation models.

2.6.5 $T^4/\mathbb{Z}_3$

We analyze the zero-mode number on magnetized $T^4/\mathbb{Z}_3$. We begin with the algebraic relation,

$$(ST_1T_2)^3 = I_4. \tag{2.274}$$

This implies that ST_1T_2 transformation can be related to the $\mathbb{Z}_3$ twist defining a $T^4/\mathbb{Z}_3$ orbifold. The complex structure moduli behave under ST_1T_2 as,

$$\Omega \xrightarrow{T_2} \Omega + B_2 \xrightarrow{T_1} \Omega + B_1 + B_2 \xrightarrow{S} -(\Omega + I_2)^{-1}. \tag{2.275}$$

We can straightforwardly find ST_1T_2 invariant $\Omega \in \mathcal{H}_2$ as,

$$\Omega_{(ST_1T_2)} = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}, \quad \omega = e^{\frac{2\pi i}{3}}. \tag{2.276}$$

When Ω is fixed to $\Omega_{(ST_1T_2)}$, the complex coordinate is transformed under ST_1T_2 as

$$\vec{z} \xrightarrow{ST_1T_2} \omega \vec{z}. \tag{2.277}$$

This is nothing but a $\mathbb{Z}_3$ twist. Hence, we can define $T^4/\mathbb{Z}_3$ orbifold by imposing the identification:

$$\vec{z} \sim \omega \vec{z}, \tag{2.278}$$

on T^4 whose moduli are fixed to $\Omega = \Omega_{(ST_1T_2)}$. Note that the twist $\vec{z} \to \omega\vec{z}$ does not change the form of the flat metric (2.129), namely an isometry. The fixed moduli correspond to the following orientations of lattice vectors:

$$e_1 = 2\pi R \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; e_2 = 2\pi R \begin{pmatrix} \omega \\ 0 \end{pmatrix}, \; e_3 = 2\pi R \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; e_4 = 2\pi R \begin{pmatrix} 0 \\ \omega \end{pmatrix}, \tag{2.279}$$

which can be identified as the root lattice of $SU(3) \times SU(3)$ Lie group. Fig. 2.1 shows the corresponding Dynkin diagram.

Figure 2.1: Lattice vectors of $T^4/\mathbb{Z}_3$ expressed as a Dynkin diagram.

Under the $\mathbb{Z}_3$ twist realized by the ST_1T_2 transformation, the lattice vectors transform as

$$\begin{pmatrix} e_2^{(ST_1T_2)} \\ e_4^{(ST_1T_2)} \\ e_1^{(ST_1T_2)} \\ e_3^{(ST_1T_2)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} e_2 \\ e_4 \\ e_1 \\ e_3 \end{pmatrix}. \tag{2.280}$$

This $\mathbb{Z}_3$ twist is the Coxeter element of $SU(3)$ [110, 111]. Here, $e_a^{(ST_1T_2)}$, $(a = 1, 2, 3, 4)$ denote the lattice vectors after the ST_1T_2 transformation.

Magnetic flux N on $T^4/\mathbb{Z}_3$

On $T^4/\mathbb{Z}_3$ where the moduli values are fixed to $\Omega_{(ST_1T_2)}$, the SUSY condition (2.138) is satisfied if and only if $\mathbf{N}$ is symmetric. Then, we note T_1T_2 transformation is consistent with the condition $p_{xx} = p_{yy} = 0$, cf. eq. (2.236). The T_1T_2 transformation of the wavefunctions in eq. (2.237) is valid only when all diagonal elements of $\mathbf{N}$ are even.[22] Hence, we restrict ourselves to

$$\mathbf{N} = \begin{pmatrix} n_1 & m \\ m & n_2 \end{pmatrix} = \begin{pmatrix} 2n_1' & m \\ m & 2n_2' \end{pmatrix}, \quad n_1', n_2', m \in \mathbb{Z}. \tag{2.281}$$

Note that $\mathbf{N}$ in (2.281) is ST_1T_2-invariant. This ensures that $\mathbf{N}$ is consistent with the $\mathbb{Z}_3$ symmetry that defines the orbifold. We also note that $\det\mathbf{N} \equiv -m^2 \equiv 0, 1 \pmod 4$. For concreteness, we focus on the case when the positive definite condition (2.175) is satisfied. Then, $\mathbf{N}$ is positive definite.

Zero-mode counting method

We can compute the degeneracy of zero-modes in magnetized $T^4/\mathbb{Z}_3$ model by using ST_1T_2 transformation. Zero-mode counting of $\mathbb{Z}_3$ eigenstates is equivalent to counting the multiplicity of eigenvalues $\eta = 1, \omega, \omega^2$ in the representation matrix $\rho(ST_1T_2)$. One can check that $\rho(ST_1T_2)$ is given by

$$\rho(ST_1T_2)_{\vec{J}\vec{K}} = \frac{e^{-\frac{\pi i}{6}}}{\sqrt{\det\mathbf{N}}} e^{3\pi i \vec{J}^{\mathrm{T}}.\mathbf{N}^{-1}.\vec{K}}, \tag{2.282}$$

by using eqs. (2.235) and (2.237) under the choice $\Omega = \Omega_{(ST_1T_2)}$.

Now, let us denote the number of zero-modes with $\mathbb{Z}_3$ eigenvalues $\eta = 1, \omega, \omega^2$ by $D_{(+1)}, D_{(\omega)}, D_{(\omega^2)}$, respectively. They satisfy the following conditions:

$$D_{(+1)} + D_{(\omega)} + D_{(\omega^2)} = \det\mathbf{N}, \tag{2.283}$$

$$D_{(+1)} + \omega D_{(\omega)} + \omega^2 D_{(\omega^2)} = \mathrm{tr}\rho(ST_1T_2). \tag{2.284}$$

Evaluating $\mathrm{tr}\rho(ST_1T_2)$

For the zero-mode counting, we need to compute $\mathrm{tr}\rho(ST_1T_2)$:

$$\mathrm{tr}[\rho(ST_1T_2)] = \frac{e^{-\frac{\pi i}{6}}}{\sqrt{\det\mathbf{N}}} \sum_{\vec{K}\in\Lambda_{\mathbf{N}}} e^{3\pi i \vec{K}^{\mathrm{T}}.\mathbf{N}^{-1}.\vec{K}}. \tag{2.285}$$

[22] More rigoros understanding is as follows. From the consistency of shifts and twist boundary conditions in our $T^4/\mathbb{Z}_3$ model, magnetic flux is constrained to $n_1 \equiv n_2 \equiv 0 \pmod 2$ if we require vanishing Scherk-Schwarz phases, i.e., $\vec{\alpha} = \vec{\beta} = \vec{0}$. We present the proof in Appendix A.3.

By use of the Krazer formula, one obtains

$$\mathrm{tr}[\rho(ST_1T_2)] = \frac{e^{\frac{\pi i}{3}}}{3} \sum_{L_1=0}^{2} \sum_{L_2=0}^{2} e^{-\frac{2\pi i}{3}(n'_1 L_1^2 + n'_2 L_2^2 + m L_1 L_2)}. \tag{2.286}$$

This shows that $\mathrm{tr}\rho(ST_1T_2)$ depends on n'_1, n'_2 and m modulo 3. Using eq. (2.286), we have analyzed the number of zero-modes on magnetized $T^4/\mathbb{Z}_3$. The results are summarized in table 2.16.

det**N**	conditions	$D_{(+1)}$	$D_{(\omega)}$	$D_{(\omega^2)}$
3	-	2	0	1
4	-	1	1	2
7	-	2	2	3
8	-	3	3	2
11	-	4	4	3
12	$n'_i \equiv 1 \pmod 3, \quad (i = 1 \text{ or } 2)$	5	3	4
	$n'_i \equiv 2 \pmod 3, \quad (i = 1 \text{ or } 2)$	3	5	4
15	$n'_i \equiv 1 \pmod 3, \quad (i = 1 \text{ or } 2)$	6	4	5
	$n'_i \equiv 2 \pmod 3, \quad (i = 1 \text{ or } 2)$	4	6	5
16	-	5	5	6
19	-	6	6	7
20	-	7	7	6
23	-	8	8	7
24	$n'_i \equiv 1 \pmod 3, \quad (i = 1 \text{ or } 2)$	9	7	8
	$n'_i \equiv 2 \pmod 3, \quad (i = 1 \text{ or } 2)$	7	9	8
27	$n'_i \equiv 1 \pmod 3, \quad (i = 1 \text{ or } 2)$	10	8	9
	$n'_1 \equiv n'_2 \equiv 0 \pmod 3$	10	10	7
28	-	9	9	10
31	-	10	10	11
32	-	11	11	10
35	-	12	12	11

Table 2.16: Number of positive chirality zero-modes in magnetized $T^4/\mathbb{Z}_3$ models. We have only shown the range (det**N** ≤ 35) because three-generation models do not appear for larger values of det**N**.

Derivation

In what follows, we derive our results in Table 2.16. We will separately consider 3 cases classified by the value of det**N** modulo 3.

$\det\mathbf{N} \equiv 1 \pmod 3$:

Let us evaluate $\mathrm{tr}\rho(ST_1T_2)$ when $\det\mathbf{N} \equiv 1 \pmod 3$. We find that the value of $n_1'n_2'$ modulo 3 can only take

$$n_1'n_2' \equiv 1 + m^2 \equiv \begin{cases} 1 \pmod 3, & \text{if } m \equiv 0 \pmod 3, \\ 2 \pmod 3, & \text{if } m \equiv \pm 1 \pmod 3. \end{cases} \tag{2.287}$$

Table 2.17 shows the values of $\mathrm{tr}\rho(ST_1T_2)$ under the possible choices of n_1' and n_2' modulo 3.[23] Note that $n_1'n_2' \equiv 0 \pmod 3$ is impossible, hence we have added cross signs to the corresponding places. Since $\mathrm{tr}\rho(ST_1T_2)$ is invariant under the permutation $n_1' \leftrightarrow n_2'$, we leave a blank in the lower triangle of Table 2.12.

		n_2' 0	1	2
	0	$\times$	$\times$	$\times$
n_1'	1		ω^2	ω^2
	2			ω^2

Table 2.17: $\mathrm{tr}\rho(ST_1T_2)$ under the possible choices of n_1', n_2' modulo 3 when $\det\mathbf{N} \equiv 1 \pmod 3$. Cross signs denote the inconsistent choices of n_1', n_2'. Since $\mathrm{tr}\rho(ST_1T_2)$ is symmetric under the permutation $n_1' \leftrightarrow n_2'$, we leave a blank in the lower triangle.

As in Table 2.17, we obtain $\mathrm{tr}\rho(ST_1T_2) = \omega^2$. Then, we find

$$D_{(+1)} = D_{(\omega)} = \frac{\det\mathbf{N} - 1}{3}, \quad D_{(\omega^2)} = \frac{\det\mathbf{N} + 2}{3}. \tag{2.288}$$

$\det\mathbf{N} \equiv 2 \pmod 3$:

Let us evaluate $\mathrm{tr}\rho(ST_1T_2)$ when $\det\mathbf{N} \equiv 2 \pmod 3$. We find that the value of $n_1'n_2'$ modulo 3 can only take

$$n_1'n_2' \equiv 2 + m^2 \equiv \begin{cases} 2 \pmod 3, & \text{if } m \equiv 0 \pmod 3, \\ 0 \pmod 3, & \text{if } m \equiv \pm 1 \pmod 3. \end{cases} \tag{2.289}$$

Table 2.18 shows the values of $\mathrm{tr}\rho(ST_1T_2)$ under the possible choices of n_1' and n_2' modulo 3.

[23] Although $\mathrm{tr}\rho(ST_1T_2)$ is dependent on $m \pmod 3$, we have found that both $m \equiv 1$ and $m \equiv -1$ lead to the same value of the trace when $n_1'n_2' \equiv 2 \pmod 3$.

		$n'_2 = 0$	$n'_2 = 1$	$n'_2 = 2$
n'_1	0	$1+\omega$	$1+\omega$	$1+\omega$
	1		$\times$	$1+\omega$
	2			$\times$

Table 2.18: $\mathrm{tr}\rho(ST_1T_2)$ under the possible choices of n'_1, n'_2 modulo 3 when $\det\mathbf{N} \equiv 2 \pmod 3$. Cross signs denote the inconsistent choices of n'_1, n'_2. Since $\mathrm{tr}\rho(ST_1T_2)$ is symmetric under the permutation $n'_1 \leftrightarrow n'_2$, we leave a blank in the lower triangle.

As in Table 2.18, we obtain $\mathrm{tr}\rho(ST_1T_2) = 1+\omega$. Then, we find

$$D_{(+1)} = D_{(\omega)} = \frac{\det\mathbf{N}+1}{3}, \quad D_{(\omega^2)} = \frac{\det\mathbf{N}-2}{3}. \tag{2.290}$$

$\det\mathbf{N} \equiv 0 \pmod 3$**:**

Let us evaluate $\mathrm{tr}\rho(ST_1T_2)$ when $\det\mathbf{N} \equiv 0 \pmod 3$. We find that the value of $n'_1n'_2$ modulo 3 can only take

$$n'_1n'_2 \equiv m^2 \equiv \begin{cases} 0 \pmod 3, & \text{if } m \equiv 0 \pmod 3, \\ 1 \pmod 3, & \text{if } m \equiv \pm 1 \pmod 3. \end{cases} \tag{2.291}$$

Table 2.19 shows the values of $\mathrm{tr}\rho(ST_1T_2)$ under the possible choices of n'_1 and n'_2 modulo 3.

		$n'_2 = 0$	$n'_2 = 1$	$n'_2 = 2$
n'_1	0	$3(1+\omega)$	$2+\omega^2$	$2\omega+\omega^2$
	1		$2+\omega^2$	$\times$
	2			$2\omega+\omega^2$

Table 2.19: $\mathrm{tr}\rho(ST_1T_2)$ under the possible choices of n'_1, n'_2 modulo 3 when $\det\mathbf{N} \equiv 0 \pmod 3$. Cross signs denote the inconsistent choices of n'_1, n'_2. Since $\mathrm{tr}\rho(ST_1T_2)$ is symmetric under the permutation $n'_1 \leftrightarrow n'_2$, we leave a blank in the lower triangle.

If n'_1 or $n'_2 \equiv 1 \pmod 3$, we obtain $\mathrm{tr}\rho(ST_1T_2) = 2+\omega^2$, which leads to

$$D_{(+1)} = \frac{\det\mathbf{N}+3}{3}, \quad D_{(\omega)} = \frac{\det\mathbf{N}-3}{3}, \quad D_{(\omega^2)} = \frac{\det\mathbf{N}}{3}. \tag{2.292}$$

If n'_1 or $n'_2 \equiv 2 \pmod 3$, we obtain $\mathrm{tr}\rho(ST_1T_2) = 2\omega+\omega^2$, which leads to

$$D_{(+1)} = \frac{\det\mathbf{N}-3}{3}, \quad D_{(\omega)} = \frac{\det\mathbf{N}+3}{3}, \quad D_{(\omega^2)} = \frac{\det\mathbf{N}}{3}. \tag{2.293}$$

In Table 2.16, one might wonder why we do not list the case $\mathrm{tr}\rho(ST_1T_2) = 2\omega+\omega^2$ when $\det\mathbf{N} = 3,\, 27$. This is simply because there is no symmetric positive definite integer matrix

$\mathbf{N}$ such that $n'_1 \equiv 2 \pmod 3$ or $n'_2 \equiv 2 \pmod 3$ if det$\mathbf{N}$ is 3 or 27. We give detailed proof in Appendix A.4.

If $n'_1 \equiv n'_2 \equiv 0 \pmod 3$, we obtain $\mathrm{tr}\rho(ST_1T_2) = 3(1+\omega)$, which leads to

$$D_{(+1)} = \frac{\det\mathbf{N} - 3}{3}, \quad D_{(\omega)} = \frac{\det\mathbf{N} + 3}{3}, \quad D_{(\omega^2)} = \frac{\det\mathbf{N}}{3}. \tag{2.294}$$

Note that det$\mathbf{N}$ must be a multiple of 9 in this case.

2.6.6 $T^4/\mathbb{Z}_5$

We analyze the zero-mode number on magnetized $T^4/\mathbb{Z}_5$. We begin with the algebraic relation,

$$(ST_1T_3)^5 = I_4. \tag{2.295}$$

This implies that ST_1T_3 transformation can be related to the $\mathbb{Z}_5$ twist defining a $T^4/\mathbb{Z}_5$ orbifold. The complex structure moduli behave under ST_1T_3 as,

$$\Omega \xrightarrow{T_3} \Omega + B_3 \xrightarrow{T_1} \Omega + B_1 + B_3 \xrightarrow{S} -(\Omega + B_1 + B_3)^{-1}. \tag{2.296}$$

We have found ST_1T_3 invariant $\Omega \in \mathcal{H}_2$ as,

$$\Omega_{(ST_1T_3)} = \begin{pmatrix} -\frac{1}{2} + \frac{i}{2}\sqrt{\frac{5+2\sqrt{5}}{5}} & -\frac{1}{2} + \frac{i}{2}\sqrt{\frac{5+2\sqrt{5}}{5}} - i\sqrt{\frac{5+\sqrt{5}}{10}} \\ -\frac{1}{2} + \frac{i}{2}\sqrt{\frac{5+2\sqrt{5}}{5}} - i\sqrt{\frac{5+\sqrt{5}}{10}} & i\sqrt{\frac{5+\sqrt{5}}{10}} \end{pmatrix}. \tag{2.297}$$

Since $\Omega^5_{(ST_1T_3)} = I_2$, this is nothing but a $\mathbb{Z}_5$ twist. Hence, we can define $T^4/\mathbb{Z}_5$ orbifold by imposing the identification:

$$\vec{z} \sim \Omega_{(ST_1T_3)}\vec{z}, \tag{2.298}$$

on T^4 whose moduli are fixed to $\Omega = \Omega_{(ST_1T_3)}$. Note that the twist $\vec{z} \to \Omega_{(ST_1T_3)}\vec{z}$ does not change the form of the flat metric (2.129), namely an isometry. The fixed moduli correspond to the following orientations of lattice vectors:

$$\begin{aligned} e_1 &= 2\pi R \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ e_2 = 2\pi R \begin{pmatrix} -\frac{1}{2} + \frac{i}{2}\sqrt{\frac{5+2\sqrt{5}}{5}} \\ -\frac{1}{2} + \frac{i}{2}\sqrt{\frac{5+2\sqrt{5}}{5}} - i\sqrt{\frac{5+\sqrt{5}}{10}} \end{pmatrix}, \\ e_3 &= 2\pi R \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ e_4 = 2\pi R \begin{pmatrix} -\frac{1}{2} + \frac{i}{2}\sqrt{\frac{5+2\sqrt{5}}{5}} - i\sqrt{\frac{5+\sqrt{5}}{10}} \\ i\sqrt{\frac{5+\sqrt{5}}{10}} \end{pmatrix}. \end{aligned} \tag{2.299}$$

which can be identified as the root lattice of $SU(5)$ Lie group. Fig. 2.2 shows the corresponding Dynkin diagram.

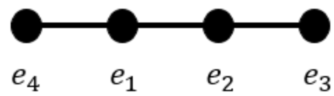

Figure 2.2: Lattice vectors of $T^4/\mathbb{Z}_5$ expressed as a Dynkin diagram.

Under the $\mathbb{Z}_5$ twist realized by the ST_1T_3 transformation, the lattice vectors transform as

$$\begin{pmatrix} e_2^{(ST_1T_3)} \\ e_4^{(ST_1T_3)} \\ e_1^{(ST_1T_3)} \\ e_3^{(ST_1T_3)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} e_2 \\ e_4 \\ e_1 \\ e_3 \end{pmatrix}. \tag{2.300}$$

This $\mathbb{Z}_5$ twist is the Coxeter element of $SU(5)$ root lattice. Here, $e_a^{(ST_1T_3)}$, $(a = 1, 2, 3, 4)$ denote the lattice vectors after the ST_1T_3 transformation.

Magnetic flux N on $T^4/\mathbb{Z}_5$

When the complex structure moduli are fixed to $\Omega_{(ST_1T_3)}$, the SUSY condition eq. (2.138) restricts $\mathbf{N}$ to

$$\mathbf{N} = \begin{pmatrix} n_1 & n_1 - n_2 \\ n_1 - n_2 & n_2 \end{pmatrix}, \ n_1, n_2 \in \mathbb{Z}. \tag{2.301}$$

Then, we note that T_1T_3 transformation is consistent with the the condition $p_{xx} = p_{yy} = 0$, cf. eq.(2.236). The T_1T_3 transformation of the wavefunctions in eq. (2.237) is valid only when all diagonal elements of $\mathbf{N}$ are even. Hence, we restrict ourselves to

$$\mathbf{N} = 2 \begin{pmatrix} n_1' & n_1' - n_2' \\ n_1' - n_2' & n_2' \end{pmatrix}, \ n_1', n_2' \in \mathbb{Z}. \tag{2.302}$$

Note that $\mathbf{N}$ in (2.302) is ST_1T_3-invariant. This ensures that $\mathbf{N}$ is consistent with the $\mathbb{Z}_5$ symmetry that defines the orbifold. For concreteness, we focus on the case when the positive definite condition (2.175) is satisfied. Then, $\mathbf{N}$ is positive definite.

Zero-mode counting method

We can compute the degeneracy of zero-modes in magnetized $T^4/\mathbb{Z}_5$ model by using ST_1T_3 transformation. Zero-mode counting of $\mathbb{Z}_5$ eigenstates is equivalent to counting the multiplicity of eigenvalues $\eta = \zeta^k, (k = 0, 1, \cdots, 4)$ in the representation matrix $\rho(ST_1T_3)$ where $\zeta := e^{\frac{2\pi i}{5}}$. Let us denote the number of degeneracy in zero-modes by $D_{(\zeta^k)}$, $(k = 0, 1, 2, 3, 4)$. They satisfy the following conditions:

$$\sum_{k=0}^{4} D_{(\zeta^k)} = \det\mathbf{N}, \tag{2.303}$$

$$\sum_{k=0}^{4} \zeta^k D_{(\zeta^k)} = \mathrm{tr}\rho(ST_1T_3). \tag{2.304}$$

By using eqs. (2.235) and (2.237) under the choice $\Omega = \Omega_{(ST_1T_3)}$, $\mathrm{tr}\rho(ST_1T_3)$ is written as

$$\mathrm{tr}[\rho(ST_1T_3)] = \frac{e^{-\frac{\pi i}{10}}}{\sqrt{\det\mathbf{N}}} \sum_{\vec{K}\in\Lambda_{\mathbf{N}}} e^{2\pi i\vec{K}^{\mathrm{T}}\mathbf{N}^{-1}\vec{K}} e^{\pi i\vec{K}^{\mathrm{T}}[\mathbf{N}^{-1}(B_1+B_3)]\vec{K}}. \tag{2.305}$$

If the value of $\mathrm{tr}\rho(ST_1T_3)$ is given, eq. (2.304) determines $D_{(\zeta^k)}$ up to $D_{(1)} = D_{(\zeta)} = \cdots = D_{(\zeta^4)} = 1$.[24] This arbitrariness can be eliminated once we fixed $\det\mathbf{N}$. Therefore, eqs.(2.303) and (2.304) provide sufficient information to determine $D_{(\zeta^k)}$.

Evaluating $\mathrm{tr}\rho(ST_1T_3)$

For the zero-mode counting, we need to compute $\mathrm{tr}\rho(ST_1T_3)$ in eq. (2.305) which can be rewritten as

$$\mathrm{tr}[\rho(ST_1T_3)] = \frac{e^{-\frac{\pi i}{10}}}{\sqrt{\det\mathbf{N}}} \sum_{\vec{K}\in\Lambda_{\mathbf{N}}} e^{5\pi i\vec{K}^{\mathrm{T}}\mathbf{N}'^{-1}\vec{K}}. \tag{2.306}$$

$\mathbf{N}'$ is defined as

$$\mathbf{N}' := \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \mathbf{N} = 2\begin{pmatrix} n_1' + n_2' & 2n_1' - 3n_2' \\ 2n_1' - 3n_2' & 4n_2' - n_1' \end{pmatrix}. \tag{2.307}$$

At first sight, we cannot apply the Krazer formula (2.254) to evaluate (2.306). The problem is that $\mathbf{N}'$ in the exponential is different from $\mathbf{N}$ which defines $\Lambda_{\mathbf{N}}$ where $\vec{K}$ is summed over. To resolve the difficulty, we consider changing the summation region from $\Lambda_{\mathbf{N}}$ to $\Lambda_{\mathbf{N}'}$. To make things consistent, one needs to multiply $1/5$ as a correction factor. This is because $\Lambda_{\mathbf{N}'}$ is 5 times larger than $\Lambda_{\mathbf{N}}$ and $e^{5\pi i\vec{K}^{\mathrm{T}}\mathbf{N}'^{-1}\vec{K}}$ is invariant under the shift $\vec{K} \to \vec{K} + \mathbf{N}\vec{e}_k$. Then, we obtain

$$\begin{aligned} \mathrm{tr}[\rho(ST_1T_3)] &= \frac{e^{-\frac{\pi i}{10}}}{5\sqrt{\det\mathbf{N}}} \sum_{\vec{K}\in\Lambda_{\mathbf{N}'}} e^{5\pi i\vec{K}^{\mathrm{T}}\mathbf{N}'^{-1}\vec{K}} \\ &= \frac{e^{-\frac{\pi i}{10}} i}{5\sqrt{5}} \sum_{K_1=0}^{4}\sum_{K_2=0}^{4} \exp\left[-\frac{2\pi i}{5}\left\{(n_1'+n_2')K_1^2 - (n_1'+n_2')K_2^2 + 4(n_1'+n_2')K_1K_2\right\}\right], \end{aligned} \tag{2.308}$$

[24]Suppose $a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4 = 0$ where a_n's are non-negative integers. Then we find that $a_0 = a_1 = \cdots = a_4$ needs to be satisfied. The arbitrariness is due to $\sum_{k=0}^{4}\zeta^k = 0$.

where the Krazer formula is applied in the second equality. This shows that $\mathrm{tr}\rho(ST_1T_3)$ depends on n'_1, n'_2 modulo 5. Using (2.308), we have analyzed the number of zero-modes on magnetized $T^4/\mathbb{Z}_5$. The results are summarized in Table 2.20. We find that three generation models appear when $\det\mathbf{N} = 16, 20$. It is obvious that when we adopt the algebraic relation $(ST_2T_3)^5 = I_4$ instead of eq.(2.295), we obtain essentially the same result. This is because those are simply related by the interchange of two complex coordinates z_1 and z_2.

$\det\mathbf{N}$	$D_{(+1)}$	$D_{(\zeta)}$	$D_{(\zeta^2)}$	$D_{(\zeta^3)}$	$D_{(\zeta^4)}$
4	1	0	1	1	1
16	3	4	3	3	3
20	5	4	5	3	3
36	7	8	7	7	7

Table 2.20: Number of positive chirality zero-modes in magnetized $T^4/\mathbb{Z}_5$ model. We have only shown the range ($\det\mathbf{N} \leq 36$) because three-generation models do not appear for larger values of $\det\mathbf{N}$.

Derivation

In what follows, we derive our results in Table 2.20. We will separately consider 5 cases classified by the value of $\det\mathbf{N}$ modulo 5.

$\det\mathbf{N} \equiv 1 \pmod 5$**:**

Let us evaluate $\mathrm{tr}\rho(ST_1T_3)$ when $\det\mathbf{N} \equiv 1 \pmod 5$. We find that the value of $n'_1 n'_2$ modulo 5 can only take

$$n'_1 n'_2 \equiv -1 + (n'_1 - n'_2)^2 \equiv \begin{cases} 4 \pmod 5, & \text{if } n'_1 - n'_2 \equiv 0 \pmod 5, \\ 0 \pmod 5, & \text{if } n'_1 - n'_2 \equiv \pm 1 \pmod 5, \\ 3 \pmod 5, & \text{if } n'_1 - n'_2 \equiv \pm 2 \pmod 5. \end{cases} \tag{2.309}$$

Table 2.21 shows the values of $\mathrm{tr}\rho(ST_1T_3)$ under the possible choices of n'_1 and n'_2 modulo 5. Cross signs denote inconsistent choices of n'_1 and n'_2 under the condition $\det\mathbf{N} \equiv 1 \pmod 5$. Since $\mathrm{tr}\rho(ST_1T_3)$ is invariant under the permutation $n'_1 \leftrightarrow n'_2$, we leave a blank in the lower triangle of Table 2.21.

		n'_2: 0	1	2	3	4
n'_1	0	$\times$	ζ	$\times$	$\times$	ζ
	1		$\times$	$\times$	ζ	$\times$
	2			ζ	$\times$	ζ
	3				ζ	$\times$
	4					$\times$

Table 2.21: $\mathrm{tr}\rho(ST_1T_3)$ under the possible choices of n'_1, n'_2 modulo 5 when $\det\mathbf{N} \equiv 1 \pmod 5$. Cross signs denote the inconsistent choices of n'_1, n'_2. Since $\mathrm{tr}\rho(ST_1T_3)$ is symmetric under the permutation $n'_1 \leftrightarrow n'_2$, we leave a blank in the lower triangle.

As in Table 2.21, we obtain $\mathrm{tr}\rho(ST_1T_3) = \zeta$, which leads to

$$D_{(\zeta)} = \frac{\det\mathbf{N} + 4}{5}, \quad D_{(+1)} = D_{(\zeta^2)} = D_{(\zeta^3)} = D_{(\zeta^4)} = \frac{\det\mathbf{N} - 1}{5}. \tag{2.310}$$

$\det\mathbf{N} \equiv 2 \pmod 5$**:**

There is no flux matrix $\mathbf{N}$ of the form eq. (2.302) when $\det\mathbf{N} \equiv 2 \pmod 5$. Let us briefly explain. We notice that the possible value of $n'_1 n'_2$ modulo 5 is narrowed down to the following three cases:

$$n'_1 n'_2 \equiv 3 + (n'_1 - n'_2)^2 \equiv \begin{cases} 3 \pmod 5, & \text{if } n'_1 - n'_2 \equiv 0 \pmod 5, \\ 4 \pmod 5, & \text{if } n'_1 - n'_2 \equiv \pm 1 \pmod 5, \\ 2 \pmod 5, & \text{if } n'_1 - n'_2 \equiv \pm 2 \pmod 5. \end{cases} \tag{2.311}$$

One notices the inconsistency between $n'_1 n'_2 \equiv 3 \pmod 5$ and $n'_1 - n'_2 \equiv 0 \pmod 5$. Hence, $n'_1 n'_2$ can never be 3 modulo 5. Similarly, one can check $n'_1 n'_2$ cannot take 2 or 4 modulo 5. In short, there is no $\mathbf{N}$ satisfying $\det\mathbf{N} \equiv 2 \pmod 5$.

$\det\mathbf{N} \equiv 3 \pmod 5$**:**

There is no flux matrix $\mathbf{N}$ of the form eq. (2.302) when $\det\mathbf{N} \equiv 3 \pmod 5$.

$\det\mathbf{N} \equiv 4 \pmod 5$**:**

Let us evaluate $\mathrm{tr}\rho(ST_1T_3)$ when $\det\mathbf{N} \equiv 4 \pmod 5$. We find that the value of $n'_1 n'_2$ modulo 5 can only take

$$n'_1 n'_2 \equiv 1 + (n'_1 - n'_2)^2 \equiv \begin{cases} 1 \pmod 5, & \text{if } n'_1 - n'_2 \equiv 0 \pmod 5, \\ 2 \pmod 5, & \text{if } n'_1 - n'_2 \equiv \pm 1 \pmod 5, \\ 0 \pmod 5, & \text{if } n'_1 - n'_2 \equiv \pm 2 \pmod 5. \end{cases} \tag{2.312}$$

Table 2.22 shows the values of $\mathrm{tr}\rho(ST_1T_3)$ under the possible choices of n'_1 and n'_2 modulo 5. Cross signs denote inconsistent choices of n'_1 and n'_2 under the condition $\det\mathbf{N} \equiv 4 \pmod 5$. Since $\mathrm{tr}\rho(ST_1T_3)$ is invariant under the permutation $n'_1 \leftrightarrow n'_2$, we leave a blank in the lower triangle of Table 2.22.

		n'_2				
		0	1	2	3	4
n'_1	0	$\times$	$\times$	$-\zeta$	$-\zeta$	$\times$
	1		$-\zeta$	$-\zeta$	$\times$	$\times$
	2			$\times$	$\times$	$\times$
	3				$\times$	$-\zeta$
	4					$-\zeta$

Table 2.22: $\mathrm{tr}\rho(ST_1T_3)$ under the possible choices of n'_1, n'_2 modulo 5 when $\det\mathbf{N} \equiv 4 \pmod 5$. Cross signs denote the inconsistent choices of n'_1, n'_2. Since $\mathrm{tr}\rho(ST_1T_3)$ is symmetric under the permutation $n'_1 \leftrightarrow n'_2$, we leave a blank in the lower triangle.

As in Table 2.21, we obtain $\mathrm{tr}\rho(ST_1T_3) = -\zeta$, which leads to

$$D_{(\zeta)} = \frac{\det\mathbf{N} - 4}{5}, \quad D_{(+1)} = D_{(\zeta^2)} = D_{(\zeta^3)} = D_{(\zeta^4)} = \frac{\det\mathbf{N} + 1}{5}. \tag{2.313}$$

$\det\mathbf{N} \equiv 0 \pmod 5$**:**

Let us evaluate $\mathrm{tr}\rho(ST_1T_3)$ when $\det\mathbf{N} \equiv 0 \pmod 5$. We find that the value of $n'_1n'_2$ modulo 5 can only take

$$n'_1n'_2 \equiv (n'_1 - n'_2)^2 \equiv \begin{cases} 0 \pmod 5, & \text{if } n'_1 - n'_2 \equiv 0 \pmod 5, \\ 1 \pmod 5, & \text{if } n'_1 - n'_2 \equiv \pm 1 \pmod 5, \\ 4 \pmod 5, & \text{if } n'_1 - n'_2 \equiv \pm 2 \pmod 5. \end{cases} \tag{2.314}$$

Table 2.23 shows the values of $\mathrm{tr}\rho(ST_1T_3)$ under the possible choices of n'_1 and n'_2 modulo 5. Cross signs denote inconsistent choices of n'_1 and n'_2 under the condition $\det\mathbf{N} \equiv 0 \pmod 5$. Since $\mathrm{tr}\rho(ST_1T_3)$ is invariant under the permutation $n'_1 \leftrightarrow n'_2$, we leave a blank in the lower triangle of Table 2.23.

		n'_2				
		0	1	2	3	4
n'_1	0	$2+\zeta+2\zeta^2$	$\times$	$\times$	$\times$	$\times$
	1		$\times$	$\times$	$\times$	$2+\zeta+2\zeta^2$
	2			$\times$	$\times$	$\times$
	3				$\times$	$\times$
	4					$\times$

Table 2.23: $\mathrm{tr}\rho(ST_1T_3)$ under the possible choices of n'_1, n'_2 modulo 5 when $\det\mathbf{N} \equiv 0 \pmod 5$. Cross signs denote the inconsistent choices of n'_1, n'_2. Since $\mathrm{tr}\rho(ST_1T_3)$ is symmetric under the permutation $n'_1 \leftrightarrow n'_2$, we leave a blank in the lower triangle.

As in Table 2.23, we obtain $\mathrm{tr}\rho(ST_1T_3) = 2+\zeta+2\zeta^2$, which leads to

$$D_{(+1)} = D_{(\zeta^2)} = \frac{\det\mathbf{N}+5}{5}, \quad D_{(\zeta)} = \frac{\det\mathbf{N}}{5}, \quad D_{(\zeta^3)} = D_{(\zeta^4)} = \frac{\det\mathbf{N}-5}{5}. \tag{2.315}$$

2.6.7 $T^4/\mathbb{Z}_6$

We analyze the zero-mode number on magnetized $T^4/\mathbb{Z}_6$. We begin with the algebraic relation,

$$((T_1T_2)^{-1}S)^6 = I_4. \tag{2.316}$$

This implies that $(T_1T_2)^{-1}S$ transformation can be related to the $\mathbb{Z}_6$ twist defining a $T^4/\mathbb{Z}_6$ orbifold. We have found $(T_1T_2)^{-1}S$ invariant $\Omega \in \mathcal{H}_2$ as

$$\Omega_{((T_1T_2)^{-1}S)} = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}, \quad \omega = e^{\frac{2\pi i}{3}}, \tag{2.317}$$

which is identical to $\Omega_{(ST_1T_2)}$ in eq. (2.276). When Ω is fixed to $\Omega_{((T_1T_2)^{-1}S)}$, the complex coordinate is transformed under $((T_1T_2)^{-1}S)$ as

$$\vec{z} \xrightarrow{(T_1T_2)^{-1}S} \kappa\vec{z}, \quad \kappa := e^{\frac{\pi i}{3}}. \tag{2.318}$$

This is nothing but a $\mathbb{Z}_6$ twist. Hence, we can define $T^4/\mathbb{Z}_6$ orbifold by imposing the identification:

$$\vec{z} \sim \kappa\vec{z}, \tag{2.319}$$

on T^4 whose moduli are fixed to $\Omega = \Omega_{((T_1T_2)^{-1}S)}$. Note that the twist $\vec{z} \to \kappa\vec{z}$ does not change the form of the flat metric (2.129), namely an isometry. The lattice vectors corresponding to $\Omega_{((T_1T_2)^{-1}S)}$ are given by (2.279). Under the $\mathbb{Z}_6$ twist realized by $(T_1T_2)^{-1}S$ transformation, the lattice vectors transform as

$$\begin{pmatrix} e_2^{((T_1T_2)^{-1}S)} \\ e_4^{((T_1T_2)^{-1}S)} \\ e_1^{((T_1T_2)^{-1}S)} \\ e_3^{((T_1T_2)^{-1}S)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_2 \\ e_4 \\ e_1 \\ e_3 \end{pmatrix}, \tag{2.320}$$

where $e_a^{((T_1T_2)^{-1}S)}$, $(a = 1, 2, 3, 4)$ denote the lattice vectors after the $(T_1T_2)^{-1}S$ transformation. One can check that two consecutive $\mathbb{Z}_6$ twists are identical to the $\mathbb{Z}_3$ twist (2.279). Moreover, three consecutive $\mathbb{Z}_6$ twists are equivalent to the $\mathbb{Z}_2$ twist, $\vec{z} \to -\vec{z}$.

Magnetic flux N on $T^4/\mathbb{Z}_6$

Magnetic flux matrix $\mathbf{N}$ consistent with the $\mathbb{Z}_6$ twist is given by eq. (2.281) in the $T^4/\mathbb{Z}_3$ models. This is because the complex structure moduli $\Omega_{((T_1T_2)^{-1}S)}$ of our $T^4/\mathbb{Z}_6$ orbifold are identical to that of $T^4/\mathbb{Z}_3$.

Zero-mode counting on magnetized $T^4/\mathbb{Z}_6$

We can compute the degeneracy of zero-modes in magnetized $T^4/\mathbb{Z}_6$ model by using $(T_1T_2)^{-1}S$ transformation. Zero-mode counting of $\mathbb{Z}_6$ eigenstates is equivalent to counting the multiplicity of eigenvalues $\eta = \kappa^k, (k = 0, 1, \cdots, 5)$ in the representation matrix $\rho((T_1T_2)^{-1}S)$ where $\kappa = e^{\frac{\pi i}{3}}$. Let us denote the number of degeneracy in zero-modes by $D_{(\kappa^k)}$, $(k = 0, 1, \cdots, 5)$. They satisfy the following conditions:

$$\sum_{k=0}^{5} D_{(\kappa^k)} = \det\mathbf{N}, \tag{2.321}$$

$$\sum_{k=0}^{5} \kappa^k D_{(\kappa^k)} = \mathrm{tr}\rho((T_1T_2)^{-1}S). \tag{2.322}$$

Eqs. (2.321) and (2.321) are not sufficient to determine $D_{(\kappa^k)}$, hence we need additional constraints. We recall that three $\mathbb{Z}_6$ twists are equivalent to the $\mathbb{Z}_2$ twist. From this fact, we obtain

$$D_{(+1)} + D_{(\kappa^2)} + D_{(\kappa^4)} = N_{\text{even}}, \tag{2.323}$$

where N_{even} denotes the number of $\mathbb{Z}_2$ even zero-modes, cf. (2.198). Similarly, by recalling that two consecutive $\mathbb{Z}_6$ twists are equivalent to the $\mathbb{Z}_3$ twist, one obtains

$$\begin{aligned} D_{(+1)} + D_{(\kappa^3)} &= D_{\mathbb{Z}_3,(+1)}, \\ D_{(\kappa)} + D_{(\kappa^4)} &= D_{\mathbb{Z}_3,(\omega)}. \end{aligned} \tag{2.324}$$

Here, $D_{Z_3,(\omega^k)}$ denotes the number of zero-modes with $\mathbb{Z}_3$ charges ω^k on the $T^4/\mathbb{Z}_3$ orbifold, cf. table 2.16.

Evaluating $\mathrm{tr}\rho((T_1T_2)^{-1}S)$

To analyze the number of zero-modes, we evaluate the trace of $\rho((T_1T_2)^{-1}S)$:

$$\mathrm{tr}[\rho((T_1T_2)^{-1}S)] = \frac{e^{\frac{\pi i}{6}}}{\sqrt{\det\mathbf{N}}} \sum_{\vec{K}\in\Lambda_{\mathbf{N}}} e^{\pi i \vec{K}^{\mathrm{T}}.\mathbf{N}^{-1}.\vec{K}}. \tag{2.325}$$

By use of the Krazer formula, one can confirm

$$\mathrm{tr}[\rho((T_1T_2)^{-1}S)] = e^{2\pi i/3}, \tag{2.326}$$

for all $\mathbf{N}$ in (2.281). Using this result, we have analyzed the number of zero-modes on magnetized $T^4/\mathbb{Z}_6$ as summarized in Table 2.24. We find that three generation models appear when $12 \leq \det N \leq 27$.

det$\mathbf{N}$	conditions	$+1$	κ	κ^2	κ^3	κ^4	κ^5
3	-	1	0	1	1	1	0
4	-	1	0	2	0	1	0
7	-	1	1	2	1	1	1
8	-	2	1	2	1	2	0
11	-	2	2	2	2	2	1
12	$n'_i\equiv 1 \pmod 3$, $(i=1 \text{ or } 2)$	3	1	3	2	2	1
	$n'_i\equiv 2 \pmod 3$, $(i=1 \text{ or } 2)$	2	2	3	1	3	1
15	$n'_i\equiv 1 \pmod 3$, $(i=1 \text{ or } 2)$	3	2	3	3	2	2
	$n'_i\equiv 2 \pmod 3$, $(i=1 \text{ or } 2)$	2	3	3	2	3	2
16	-	3	2	4	2	3	2
19	-	3	3	4	3	3	3
20	-	4	3	4	3	4	2
23	-	4	4	4	4	4	3
24	$n'_i\equiv 1 \pmod 3$, $(i=1 \text{ or } 2)$	5	3	5	4	4	3
	$n'_i\equiv 2 \pmod 3$, $(i=1 \text{ or } 2)$	4	4	5	3	5	3
27	$n'_i\equiv 1 \pmod 3$, $(i=1 \text{ or } 2)$	5	4	5	5	4	4
27	$n'_1\equiv n'_2 \equiv 0 \pmod 3$	5	5	4	5	5	3
28	-	5	4	6	4	5	4
31	-	5	5	6	5	5	5
32	-	6	5	6	5	6	4
35	-	6	6	6	6	6	5

Table 2.24: Number of positive chirality zero-modes in magnetized $T^4/\mathbb{Z}_6$ models. We have only shown the range (det$\mathbf{N} \leq 35$) because three-generation models do not appear for larger values of det$\mathbf{N}$.

2.6.8 $T^4/\mathbb{Z}_8$

We study the magnetized $T^4/\mathbb{Z}_8$ model. We begin with the algebraic relation,

$$(ST_1T_2^{-1}T_3^{-1})^8 = I_4. \tag{2.327}$$

This implies that $ST_1T_2^{-1}T_3^{-1}$ transformation can be related to the $\mathbb{Z}_8$ twist defining a $T^4/\mathbb{Z}_8$ orbifold. We have found $ST_1T_2^{-1}T_3^{-1}$ invariant $\Omega \in \mathcal{H}_2$ as

$$\Omega_{(ST_1T_2^{-1}T_3^{-1})} = \begin{pmatrix} -\frac{1}{2} + \frac{i}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} + \frac{i}{\sqrt{2}} \end{pmatrix}. \tag{2.328}$$

When Ω is fixed to $\Omega_{(ST_1T_2^{-1}T_3^{-1})}$, the complex coordinate is transformed under $ST_1T_2^{-1}T_3^{-1}$ as

$$\vec{z} \xrightarrow{ST_1T_2^{-1}T_3^{-1}} \Omega_{(ST_1T_2^{-1}T_3^{-1})}\vec{z}. \tag{2.329}$$

This is nothing but a $\mathbb{Z}_8$ twist. Note that the twist $\mathbb{Z}_8$ twist does not change the form of the flat metric (2.129), namely an isometry. Hence, we can define $T^4/\mathbb{Z}_8$ orbifold by imposing the identification:

$$\vec{z} \sim \Omega_{(ST_1T_2^{-1}T_3^{-1})}\vec{z}, \tag{2.330}$$

on T^4 whose moduli are fixed to $\Omega = \Omega_{(ST_1T_2^{-1}T_3^{-1})}$. Let us understand the orientations of the lattice vectors defining the geometry of the orbifold. For later convenience, we consider $T_2^{-1}T_3^{-1}$ transformation to $\Omega_{(ST_1T_2^{-1}T_3^{-1})}$. Then, the complex structure moduli are transformed under $T_2^{-1}T_3^{-1}$ as

$$\Omega' \equiv \Omega_{(ST_1T_2^{-1}T_3^{-1})} - B_2 - B_3 = \begin{pmatrix} -\frac{1}{2} + \frac{i}{\sqrt{2}} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} + \frac{i}{\sqrt{2}} \end{pmatrix}. \tag{2.331}$$

It is important to note that the lattice points are kept invariant under $T_2^{-1}T_3^{-1}$, hence we are still with the same lattice points defined by $\Omega_{(ST_1T_2^{-1}T_3^{-1})}$. Lattice vectors which correspond to Ω' are given by

$$e_1 = 2\pi R \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ e_2 = 2\pi R \begin{pmatrix} -\frac{1}{2} + \frac{i}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix},\ e_3 = 2\pi R \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ e_4 = 2\pi R \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} + \frac{i}{\sqrt{2}} \end{pmatrix}, \tag{2.332}$$

which can be identified as the root lattice of $SO(8)$ Lie group as shown in Figure 2.3.

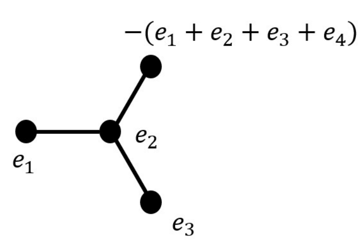

Figure 2.3: Lattice vectors of $T^4/\mathbb{Z}_8$ expressed as a Dynkin diagram.

We note that the $\mathbb{Z}_8$ twist in eq. (2.329) is identified as the generalized Coxeter element of $SO(8)$ including the $\mathbb{Z}_2$ outer automorphism [110, 111].

Magnetic flux N on $T^4/\mathbb{Z}_8$

When the complex structure moduli are fixed to $\Omega_{(ST_1T_2^{-1}T_3^{-1})}$, the SUSY condition eq. (2.138) restricts $\mathbf{N}$ to

$$\mathbf{N} = \begin{pmatrix} n_1 & \frac{n_2-n_1}{2} \\ \frac{n_2-n_1}{2} & n_2 \end{pmatrix}, \quad n_1, n_2 \in \mathbb{Z}, \tag{2.333}$$

where $n_1 \equiv n_2 \pmod 2$. The $T_1T_2^{-1}T_3^{-1}$ transformation of the wavefunctions in eq. (2.237) is valid only when all diagonal matrix elements of $\mathbf{N}(B_1 - B_2 - B_3)$ are even. Hence, we require

$$n_1 + n_2 \equiv 0 \pmod 4. \tag{2.334}$$

Note that our flux $\mathbf{N}$ is $ST_1T_2^{-1}T_3^{-1}$ invariant. This ensures that $\mathbf{N}$ is consistent with the $\mathbb{Z}_8$ symmetry that defines the orbifold.

Zero-mode counting

Number of zero-mode eigenstates $D_{(\xi^0)}, .., D_{(\xi^7)}$ $(\xi = e^{\pi i/4})$ can be analyzed as before if we focus on the case when the positive definite condition (2.175) is satisfied. That is, $\mathbf{N}$ is positive definite. Note that two and four consecutive $\mathbb{Z}_8$ twists are $\mathbb{Z}_4$ and $\mathbb{Z}_2$ twists, respectively. Such zero-mode counting has been performed in Ref. [45] and three-generation models are obtained.

2.6.9 $T^4/\mathbb{Z}_{12}$

We study the zero-mode number on magnetized $T^4/\mathbb{Z}_{12}$. We begin with the algebraic relation,

$$(ST_1)^{12} = I_4. \tag{2.335}$$

This implies that ST_1 transformation can be regarded as the $\mathbb{Z}_{12}$ twist defining a $T^4/\mathbb{Z}_{12}$ orbifold. We can straightforwardly find ST_1 invariant $\Omega \in \mathcal{H}_2$ as,

$$\Omega_{(ST_1)} = \begin{pmatrix} \omega & 0 \\ 0 & i \end{pmatrix}, \quad \omega = e^{\frac{2\pi i}{3}}. \tag{2.336}$$

When Ω is fixed to $\Omega_{(ST_1)}$, the complex coordinate is transformed under ST_1 as

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \xrightarrow{ST_1} \begin{pmatrix} \omega z_1 \\ i z_2 \end{pmatrix}, \tag{2.337}$$

which is clearly a $\mathbb{Z}_{12}(\cong \mathbb{Z}_3 \times \mathbb{Z}_4)$ twist. Hence, we can define $T^4/\mathbb{Z}_{12}$ orbifold by imposing the identification:

$$z_1 \sim \omega z_1, \quad z_2 \sim i z_2. \tag{2.338}$$

on T^4 whose moduli are fixed to $\Omega = \Omega_{(ST_1)}$. The fixed moduli correspond to the following orientations of lattice vectors:

$$e_1 = 2\pi R \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ e_2 = 2\pi R \begin{pmatrix} \omega \\ 0 \end{pmatrix}, \ e_3 = 2\pi R \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ e_4 = 2\pi R \begin{pmatrix} 0 \\ i \end{pmatrix}, \tag{2.339}$$

which can be identified as the root lattice of $SU(3) \times SU(2) \times SU(2)$.

Magnetic flux N on $T^4/\mathbb{Z}_{12}$

On the $T^4/\mathbb{Z}_{12}$ orbifold , background magnetic flux can be consistently introduced if $\mathbf{N}$ is of the form,

$$\mathbf{N} = \begin{pmatrix} 2n_1' & 0 \\ 0 & n_2 \end{pmatrix}, \quad n_1', n_2 \in \mathbb{Z}. \tag{2.340}$$

The SUSY condition (2.138) is satisfied only if $\mathbf{N}$ is diagonal. Then, the flux is ST_1 invariant. The $(1,1)$-component of $\mathbf{N}$ needs to be even for the T_1 transformation of zero-modes as eq. (2.237) to be valid.[25]

Discussion

We have seen that both the complex structure moduli Ω and flux $\mathbf{N}$ are diagonal, cf. eqs. (2.336) and (2.340). This means that the $T^4/\mathbb{Z}_{12}$ is factorized into magnetized $T^2/\mathbb{Z}_3 \times T^2/\mathbb{Z}_4$ whose size of flux quanta are $M_1 = 2n_1'$ and $M_2 = n_2$, respectively. Zero-modes on these geometries are studied in Ref. [28] and we have reviewed them in section 2.3. Thus, zero-mode numbers can be determined straightforwardly by noting that zero-mode wavefunctions on magnetized $T^2/\mathbb{Z}_3 \times T^2/\mathbb{Z}_4$ are given by the product of those on each orbifold.

It is obvious that when we adopt the algebraic relation $(ST_2)^2 = I_4$ instead of eq.(2.335), we obtain essentially the same result. This is because those are simply related by the interchange of two complex coordinates z_1 and z_2.

[25] The flux needs to be even integer owing to the boundary conditions of $T^2/\mathbb{Z}_3$ orbifold Ref. [26].

Chapter 3

Radiative modulus stabilization in modular flavor models

In this chapter, we study the radiative correction to the stabilization of complex structure modulus τ. We briefly review 4D modular flavor symmetric models to show there is a phenomenologically preferred value of τ for the realization of hierarchical flavor structures in the Standard Model. After that, we analyze the 1-loop corrected effective potential which may realize the desired vacuum expectation value of τ along Ref. [79].

3.1 Modular flavor models

Here, we review the general setup of 4D $\mathcal{N}=1$ supersymmetric modular flavor models. To begin with, we introduce the following infinite group, known as the principal congruence subgroup:

$$\Gamma(N)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL(2,\mathbb{Z}),\ \begin{pmatrix} a & b \\ c & d\end{pmatrix}\equiv\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\pmod{N}\right\}, \tag{3.1}$$

where N denotes a positive integer[1]. Since the actions of γ and $-\gamma$ on τ are identical, we define the quotient, $\overline{\Gamma}(N)\equiv\Gamma(N)/\{\mathbb{I},-\mathbb{I}\}$. Note that $\overline{\Gamma}(N>2)=\Gamma(N)$ because $-\mathbb{I}\notin\Gamma(N>2)$. Conventionally, $\overline{\Gamma}(1)$ is often denoted as $\overline{\Gamma}=PSL(2,\mathbb{Z})$. Next, we define the finite modular subgroups Γ_N by the quotient $\Gamma_N\equiv\overline{\Gamma}/\overline{\Gamma}(N)$. When $N\leq 5$, we have the following presentations,

$$\Gamma_N=\langle S,T|S^2=\mathbb{I},(ST)^3=\mathbb{I},T^N=\mathbb{I}\rangle. \tag{3.2}$$

It is worth noting that Γ_N is isomorphic to the non-Abelian discrete symmetry groups S_3,A_4,S_4,A_5 for $N=2,3,4,5$ respectively [46]. We identify these finite non-Abelian groups as flavor symmetries. [2]

[1]For more details, see, for example Ref. [102].

[2]For model building with these non-Abelian discrete flavor symmetries, see Refs. [114–122].

In Γ_N modular flavor models, 4D matter chiral superfields Φ_I transform as "weighted" multiplets with modular weight $-k_I$ as [48, 123, 124],

$$(\Phi_I)_i \to (c\tau + d)^{-k_I} \rho_I(\gamma)_{ij} (\Phi_I)_j, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}, \tag{3.3}$$

where $\rho_I(\gamma)$ denotes a unitary representation of Γ_N. The modulus superfield τ transforms as

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad \gamma \in \overline{\Gamma}. \tag{3.4}$$

Having reviewed modular symmetry, let us consider modular symmetric 4D $\mathcal{N} = 1$ global SUSY Lagrangian. In superfield formalism, it can be expressed as

$$\mathcal{L} = \int d^2\theta d^2\overline{\theta}\ K(\tau, \overline{\tau}, \Phi_I, \overline{\Phi}_I) + \left[\int d^2\theta\ W(\tau, \Phi_I) + \text{h.c.} \right], \tag{3.5}$$

where K and W denote the Kähler potential and superpotential, respectively. θ and $\overline{\theta}$ are Grassmann variables. Since we are interested in the flavor structure of matter fields, we omit gauge sectors. By requiring the invariance of our Lagrangian under the transformations eqs. (3.3) and (3.4), Kähler potential must be invariant up to Kähler transformation,

$$K(\tau, \overline{\tau}, \Phi_I, \overline{\Phi}_I) \xrightarrow{\gamma} K(\tau, \overline{\tau}, \Phi_I, \overline{\Phi}_I) + f_K(\tau, \Phi_I) + \overline{f_K}(\overline{\tau}, \overline{\Phi_I}), \quad \gamma \in \overline{\Gamma}, \tag{3.6}$$

where $f_K(\tau, \Phi)$ and $\overline{f_K}(\overline{\tau}, \overline{\Phi_I})$ are chiral superfield and anti-chiral superfield, respectively. We consider the following simple Kähler potential:

$$K(\tau, \overline{\tau}, \Phi_I, \overline{\Phi}_I) = -\Lambda_0^2 \ln\left(-i(\tau - \overline{\tau})\right) + \sum_I \frac{|\Phi_I|^2}{(i(\overline{\tau} - \tau))^{k_I}}, \tag{3.7}$$

where Λ_0 is a mass parameter. The superpotential must be invariant under the modular transformation,

$$W(\tau, \Phi_I) \xrightarrow{\gamma} W(\tau, \Phi_I), \quad \gamma \in \overline{\Gamma}. \tag{3.8}$$

This implies that superpotential W is constructed by extracting Γ_N trivial singlet $\mathbf{1}$ terms out of the products of holomorphic functions $Y_{I_1 \ldots I_m}(\tau)$ and matter chiral superfields, i.e.,

$$W(\tau, \Phi_I) = \sum_m \sum_{\{I_1, \ldots, I_m\}} \left(Y_{I_1 \ldots I_m}(\tau) \Phi_{I_1} \ldots \Phi_{I_m} \right)_{\mathbf{1}}. \tag{3.9}$$

To obtain non-vanishing terms in the superpotential eq. (3.9), $Y_{I_1, \ldots, I_m}(\tau)$ must be a modular form of level N with appropriate modular weight k_Y. The modular forms of weight k_Y and level N are holomorphic functions of τ which transform under $\overline{\Gamma}(N)$ as follows:

$$Y_{I_1 \ldots I_m}(\gamma\tau) = (c\tau + d)^{k_Y} Y_{I_1 \ldots I_m}(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}(N), \tag{3.10}$$

where k_Y is a non-negative even integer. Then, the modular forms $Y_{I_1...I_m}$ transform under $\gamma \in \bar{\Gamma}$ as

$$[Y_{I_1...I_m}(\gamma\tau)]_i = (c\tau + d)^{k_Y} \rho(\gamma)_{ij}[Y_{I_1...I_m}(\tau)]_j, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}, \tag{3.11}$$

where $\rho(\gamma)$ denotes a unitary representation matrix of $\gamma \in \Gamma_N$. Eq. (3.8) requires that the modular weight of superpotential must be vanishing, hence $Y_{I_1 \cdots I_m}$ needs to carry modular weight $k_Y = k_{I_1} + ... + k_{I_m}$. Furthermore, the tensor products of the representations of superfields and modular forms must contain a trivial singlet $\mathbf{1}$ to obtain a non-vanishing superpotential.

3.1.1 Hierarchies in quark masses and mixing angles without fine-tuning

Here, we show the effectiveness of the modular flavor models by presenting $A_4 \times A_4 \times A_4$ model which can realize hierarchies in quark masses and mixing angles without fine-tuning [67]. For simplicity, we assume that superfields carry the same modular weights under all A_4's. We denote the matter chiral superfields and their modular weights as follows.

- quark doublets $Q = (Q^1, Q^2, Q^3)$ are three-dimensional representation (reducible or irreducible) of A_4 with modular weight $-k_Q$.
- up sector quark singlets $u_R = (u_R^1, u_R^2, u_R^3)$ are three-dimensional representation (reducible or irreducible) of A_4 with modular weight $-k_u$.
- down sector quark singlets $d_R = (d_R^1, d_R^2, d_R^3)$ are three-dimensional representation (reducible or irreducible) of A_4 with weight $-k_d$.
- both up and down sector Higgs fields $H_{u,d}$ are one-dimensional representations of A_4 with modular weight $-k_{H_{u,d}}$.

Note that these matter fields can belong to different representations under each A_4. The super potential for the up-type quark sector is given by

$$W_u = \sum_{\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3} \left[Y_{\mathbf{r}_1}^{(k_{Y_u})} Y_{\mathbf{r}_2}^{(k_{Y_u})} Y_{\mathbf{r}_3}^{(k_{Y_u})} (Q^1 \ Q^2 \ Q^3) \begin{pmatrix} \alpha^{11}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \alpha^{12}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \alpha^{13}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} \\ \alpha^{21}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \alpha^{22}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \alpha^{23}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} \\ \alpha^{31}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \alpha^{32}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \alpha^{33}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} \end{pmatrix} \begin{pmatrix} u_R^1 \\ u_R^2 \\ u_R^3 \end{pmatrix} H_u \right]_{\mathbf{1}}, \tag{3.12}$$

where $Y_{\mathbf{r}_n}^{(k_{Y_u})}(\tau)$, $(n = 1, 2, 3)$ represent a modular form of level 3 with modular weight k_{Y_u} which transform non-trivially, cf. (3.11), as an irreducible representation $\mathbf{r}_n$ of n-th A_4. Note that the modular invariance requires $k_{Y_u} = k_Q + k_u + k_{H_u}$. Coefficients $\alpha^{ij}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3}$ are constants. Similarly, one can write down the superpotential for the down-type quark sector,

$$W_d = \sum_{\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3} \left[Y_{\mathbf{r}_1}^{(k_{Y_d})} Y_{\mathbf{r}_2}^{(k_{Y_d})} Y_{\mathbf{r}_3}^{(k_{Y_d})} (Q^1 \ Q^2 \ Q^3) \begin{pmatrix} \beta^{11}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \beta^{12}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \beta^{13}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} \\ \beta^{21}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \beta^{22}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \beta^{23}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} \\ \beta^{31}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \beta^{32}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \beta^{33}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} \end{pmatrix} \begin{pmatrix} d_R^1 \\ d_R^2 \\ d_R^3 \end{pmatrix} H_d \right]_{\mathbf{1}}, \tag{3.13}$$

where $\beta^{ij}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3}$ are constant coefficients and we require $k_{Y_d} = k_Q + k_d + k_{H_d}$. After the Higgs fields as well as the scalar component of τ acquire their vacuum expectation values $\langle H_u \rangle$, $\langle H_d \rangle$, and $\langle \tau \rangle$, we obtain mass matrices of up-type quarks M_u,

$$\begin{aligned}(Q^1\ Q^2\ Q^3) M_u \begin{pmatrix} u_R^1 \\ u_R^2 \\ u_R^3 \end{pmatrix} \\ = \sum_{\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3} \left[\prod_{n=1}^{3} Y_{\mathbf{r}_n}^{(k_{Y_u})}(\langle\tau\rangle)(Q^1\ Q^2\ Q^3) \begin{pmatrix} \alpha^{11}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \alpha^{12}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \alpha^{13}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} \\ \alpha^{21}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \alpha^{22}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \alpha^{23}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} \\ \alpha^{31}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \alpha^{32}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \alpha^{33}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} \end{pmatrix} \begin{pmatrix} u_R^1 \\ u_R^2 \\ u_R^3 \end{pmatrix} \langle H_u \rangle \right]_{\mathbf{1}},\end{aligned} \tag{3.14}$$

and down-type quarks M_d,

$$\begin{aligned}(Q^1\ Q^2\ Q^3) M_d \begin{pmatrix} d_R^1 \\ d_R^2 \\ d_R^3 \end{pmatrix} \\ = \sum_{\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3} \left[\prod_{n=1}^{3} Y_{\mathbf{r}_n}^{(k_{Y_d})}(\langle\tau\rangle)(Q^1\ Q^2\ Q^3) \begin{pmatrix} \beta^{11}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \beta^{12}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \beta^{13}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} \\ \beta^{21}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \beta^{22}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \beta^{23}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} \\ \beta^{31}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \beta^{32}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} & \beta^{33}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} \end{pmatrix} \begin{pmatrix} d_R^1 \\ d_R^2 \\ d_R^3 \end{pmatrix} \langle H_d \rangle \right]_{\mathbf{1}}.\end{aligned} \tag{3.15}$$

For the realization of quark masses and mixing angles, we do not allow fine-tuning of coupling constants $\alpha^{ij}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3}$ and $\beta^{ij}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3}$. To emphasize this point, we restrict the constant coefficients to $+1$ or -1,

$$\alpha^{ij}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} = \pm 1, \quad \beta^{ij}_{\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3} = \pm 1. \tag{3.16}$$

Then, the source of hierarchies should emerge from the values of the modular forms. In fact, when the complex structure modulus τ takes its VEV close to modular fixed points, $\tau = i, \omega$, and $i\infty$, the values of modular forms can become hierarchical [61,69]. This is due to the residual symmetries at the fixed points. When τ acquires its VEV, A_4 symmetry is generally broken, whereas there remain $\mathbb{Z}_3$ symmetry at $\tau = \omega, i\infty$ and $\mathbb{Z}_2$ symmetry at $\tau = i$.

Here, we focus on the case when τ is at nearby ω and explain how hierarchies in the values of modular forms emerge. First, we note that $\tau = \omega$ is invariant under the ST transformation,

$$\tau = \omega \xrightarrow{T} \omega + 1 \xrightarrow{S} -\frac{1}{\omega + 1} = \omega. \tag{3.17}$$

Hence, ST generates the $\mathbb{Z}_3$ residual symmetry. Let us consider singlet modular forms of A_4. There are three singlets, $\mathbf{1}, \mathbf{1}'$ and $\mathbf{1}''$ in A_4. Representation matrices of the generators S and T are given by

$$\rho_{\mathbf{1}}(S) = \rho_{\mathbf{1}}(T) = 1, \quad \rho_{\mathbf{1}'}(S) = 1,\ \rho_{\mathbf{1}'}(T) = \omega, \quad \rho_{\mathbf{1}''}(S) = 1,\ \rho_{\mathbf{1}''}(T) = \omega^2. \tag{3.18}$$

Under the ST transformation, the singlet modular forms with modular weight k_Y behave as

$$Y_{\mathbf{r}}^{(k_Y)}(\tau) \xrightarrow{ST} Y_{\mathbf{r}}^{(k_Y)}(-1/(\tau+1)) = (-1-\tau)^{k_Y}\rho_{\mathbf{r}}(ST)Y_{\mathbf{r}}^{(k_Y)}(\tau), \tag{3.19}$$

where $\mathbf{r} \in \{\mathbf{1}, \mathbf{1}', \mathbf{1}''\}$. One obtains $\rho_{\mathbf{1}}(ST) = 1$, $\rho_{\mathbf{1}'}(ST) = \omega$ and $\rho_{\mathbf{1}''}(ST) = \omega^2$ from eq. (3.18). For convenience, let us define $\tilde{\rho}_{\mathbf{r}} := \omega^{-k_Y}\rho_{\mathbf{r}}$ and a complex variable

$$u := \frac{\tau-\omega}{\tau-\omega^2}, \tag{3.20}$$

which parametrizes the deviation of τ from ω [61]. Then, (3.19) can be rewritten as

$$Y_{\mathbf{r}}^{(k_Y)}(\omega^2 u) = \left(\frac{1-\omega^2 u}{1-u}\right)^{k_Y} \tilde{\rho}_{\mathbf{r}}(ST)Y_{\mathbf{r}}^{(k_Y)}(u), \tag{3.21}$$

by noting $u \xrightarrow{ST} \omega^2 u$. If we define $\tilde{Y}_{\mathbf{r}}^{(k_Y)}$ by

$$\tilde{Y}_{\mathbf{r}}^{(k_Y)}(u) := (1-u)^{-k_Y}Y_{\mathbf{r}}^{(k_Y)}(\tau), \tag{3.22}$$

one immediately obtains

$$\tilde{Y}_{\mathbf{r}}^{(k_Y)}(\omega^2 u) = \tilde{\rho}_{\mathbf{r}}(ST)Y_{\mathbf{r}}^{(k_Y)}(u). \tag{3.23}$$

Taylor expansion with respect to u, $(|u| \ll 1)$ leads to

$$(\omega^{2l} - \tilde{\rho}_{\mathbf{r}}(ST))\frac{d^l\tilde{Y}_{\mathbf{r}}^{(k_Y)}(u)}{du^l}\bigg|_{u=0} = 0, \quad (l = 0, 1, 2, \cdots). \tag{3.24}$$

This shows that when $\omega^{2l} - \tilde{\rho}_{\mathbf{r}}(ST) \neq 0$, the l-th derivative of $\tilde{Y}_{\mathbf{r}}^{(k_Y)}(u)$ with respect to u vanishes at $u = 0, (\tau = \omega)$. In other words, the relation implies that the modular forms behave as $\tilde{Y}_{\mathbf{r}}^{(k_Y)} \sim Y_{\mathbf{r}}^{(k_Y)} \sim \mathcal{O}(|u|^l)$ in the vicinity of $\tau = \omega$ if $l (= 0, 1, 2)$ satisfies $\omega^{2l} - \tilde{\rho}_{\mathbf{r}}(ST) = 0$. It is known that the $k_Y = 8$ is the lowest weight at which all three singlet A_4 modular forms exist. Hence, let us closely look at weight 8 singlet A_4 modular forms which we denote by

$$Y_{\mathbf{1}}^{(8)}(\tau), \quad Y_{\mathbf{1}'}^{(8)}(\tau), \quad Y_{\mathbf{1}''}^{(8)}(\tau), \tag{3.25}$$

where they are explicitly defined in Appendix B.2. When the modular weight is lower than 8, some of the singlet representations are lacking. By noting the following relationship between the derivatives of $Y_{\mathbf{r}}^{(k_Y)}(\tau)$ and those of $\tilde{Y}_{\mathbf{r}}^{(k_Y)}(u)$,

$$\frac{dY_{\mathbf{r}}^{(k_Y)}(\tau)}{d\tau} = \frac{(1-u)^{k_Y+2}}{\sqrt{3}i}\left[\frac{d\tilde{Y}_{\mathbf{r}}^{(k_Y)}}{du} - \frac{k_Y}{1-u}\tilde{Y}_{\mathbf{r}}^{(k_Y)}\right], \tag{3.26}$$

$$\frac{d^2Y_{\mathbf{r}}^{(k_Y)}(\tau)}{d\tau^2} = -\frac{(1-u)^{k_Y+4}}{3}\left[\frac{d^2\tilde{Y}_{\mathbf{r}}^{(k_Y)}}{du^2} - 2\frac{k_Y+1}{1-u}\frac{d\tilde{Y}_{\mathbf{r}}^{(k_Y)}}{du} + \frac{k_Y^2+k_Y}{(1-u)^2}\tilde{Y}_{\mathbf{r}}^{(k_Y)}\right], \tag{3.27}$$

we obtain the Taylor expansions of the singlet modular forms (3.25):

$$Y_{\mathbf{1}}^{(8)}(\tau) = -\frac{1}{6}\frac{d^2\tilde{Y}_{\mathbf{1}}^{(8)}(u)}{du^2}\bigg|_{u=0}(\tau-\omega)^2 + \mathcal{O}((\tau-\omega)^3), \tag{3.28}$$

$$Y_{\mathbf{1}'}^{(8)}(\tau) = \frac{1}{\sqrt{3}i}\frac{d\tilde{Y}_{\mathbf{1}'}^{(8)}(u)}{du}\bigg|_{u=0}(\tau-\omega) + \mathcal{O}((\tau-\omega)^2), \tag{3.29}$$

$$Y_{\mathbf{1}''}^{(8)}(\tau) = \tilde{Y}_{\mathbf{1}''}^{(8)}(0) - \frac{8\tilde{Y}_{\mathbf{1}''}^{(8)}(0)}{\sqrt{3}i}(\tau-\omega) + \mathcal{O}((\tau-\omega)^2). \tag{3.30}$$

We have summarized the behaviors of A_4 singlet modular forms weight $k_Y = 8$ near $\tau = \omega$ in Table 3.1. We can see that their values are hierarchical depending on $\rho_{\mathbf{r}}(ST)$, i.e., "ST-charge".

singlet modular form	$Y_{\mathbf{1}}^{(8)}$	$Y_{\mathbf{1}'}^{(8)}$	$Y_{\mathbf{1}''}^{(8)}$
$\rho_{\mathbf{r}}(ST)$	1	ω	ω^2
order	ε^2	ε	1

Table 3.1: Behaviors of A_4 singlet modular forms with weight $k_Y = 8$ in the vicinity of $\tau = \omega$. We have defined the small parameter ε by $\varepsilon := \tau - \omega$.

Through the analogous discussion, the values of modular forms can become hierarchical when $\tau \sim i\infty$ and $\tau \sim i$ [61].

Concrete model

Having understood that modular forms become hierarchical in the vicinity of the fixed point, let us present one of the best-fit models of $A_4 \times A_4 \times A_4$ modular flavor models studied in Ref. [67] which successfully reproduces quark mass ratios and mixing angles without fine-tuning. The setup of the best-fit model is as follows. Modular weights of modular forms and chiral superfields are assigned as,

$$k_{Y_u} = k_{Y_d} = 8, \quad k_Q = k_u = k_d = 4, \quad k_{H_u} = k_{H_d} = 0. \tag{3.31}$$

Higgs fields H_u and H_d have vanishing modular weights and are assumed to be trivial singlet $\mathbf{1}$ under $A_4 \times A_4 \times A_4$. Hence, the VEV of τ is the only source of modular symmetry breaking. We choose the VEV of τ in the vicinity of ω as

$$\tau = \omega + 0.051i. \tag{3.32}$$

The $A_4 \times A_4 \times A_4$ representations of the chiral superfields are assigned as

$$(Q^1, Q^2, Q^3) = (\mathbf{1'}_1 \otimes \mathbf{1'}_2 \otimes \mathbf{1'}_3, \mathbf{1}_1 \otimes \mathbf{1''}_2 \otimes \mathbf{1}_3, \mathbf{1''}_1 \otimes \mathbf{1''}_2 \otimes \mathbf{1''}_3), \tag{3.33}$$

$$(u_R^1, u_R^2, u_R^3) = (\mathbf{1''}_1 \otimes \mathbf{1''}_2 \otimes \mathbf{1''}_3, \mathbf{1''}_1 \otimes \mathbf{1}_2 \otimes \mathbf{1''}_3, \mathbf{1''}_1 \otimes \mathbf{1''}_2 \otimes \mathbf{1''}_3), \tag{3.34}$$

$$(d_R^1, d_R^2, d_R^3) = (\mathbf{1''}_1 \otimes \mathbf{1''}_2 \otimes \mathbf{1}_3, \mathbf{1''}_1 \otimes \mathbf{1}_2 \otimes \mathbf{1'}_3, \mathbf{1''}_1 \otimes \mathbf{1''}_2 \otimes \mathbf{1''}_3), \tag{3.35}$$

where $\mathbf{r}_n$, $(n = 1, 2, 3)$ denotes the representation $\mathbf{r} \in \{\mathbf{1}, \mathbf{1}', \mathbf{1}''\}$ under the n-th A_4. The corresponding mass matrices of the best-fit model are as follows

$$
\begin{aligned}
M_u &= \begin{pmatrix} Y^{(8)}_{\mathbf{1}} Y^{(8)}_{\mathbf{1}} Y^{(8)}_{\mathbf{1}} & Y^{(8)}_{\mathbf{1}} Y^{(8)}_{\mathbf{1}''} Y^{(8)}_{\mathbf{1}} & Y^{(8)}_{\mathbf{1}} Y^{(8)}_{\mathbf{1}} Y^{(8)}_{\mathbf{1}} \\ Y^{(8)}_{\mathbf{1}'} Y^{(8)}_{\mathbf{1}''} Y^{(8)}_{\mathbf{1}'} & Y^{(8)}_{\mathbf{1}'} Y^{(8)}_{\mathbf{1}'} Y^{(8)}_{\mathbf{1}'} & -Y^{(8)}_{\mathbf{1}'} Y^{(8)}_{\mathbf{1}''} Y^{(8)}_{\mathbf{1}'} \\ Y^{(8)}_{\mathbf{1}''} Y^{(8)}_{\mathbf{1}''} Y^{(8)}_{\mathbf{1}''} & -Y^{(8)}_{\mathbf{1}''} Y^{(8)}_{\mathbf{1}'} Y^{(8)}_{\mathbf{1}''} & -Y^{(8)}_{\mathbf{1}''} Y^{(8)}_{\mathbf{1}''} Y^{(8)}_{\mathbf{1}''} \end{pmatrix}, \\
M_d &= \begin{pmatrix} Y^{(8)}_{\mathbf{1}} Y^{(8)}_{\mathbf{1}} Y^{(8)}_{\mathbf{1}''} & Y^{(8)}_{\mathbf{1}} Y^{(8)}_{\mathbf{1}''} Y^{(8)}_{\mathbf{1}'} & Y^{(8)}_{\mathbf{1}} Y^{(8)}_{\mathbf{1}} Y^{(8)}_{\mathbf{1}} \\ Y^{(8)}_{\mathbf{1}'} Y^{(8)}_{\mathbf{1}''} Y^{(8)}_{\mathbf{1}} & -Y^{(8)}_{\mathbf{1}'} Y^{(8)}_{\mathbf{1}'} Y^{(8)}_{\mathbf{1}''} & Y^{(8)}_{\mathbf{1}'} Y^{(8)}_{\mathbf{1}''} Y^{(8)}_{\mathbf{1}'} \\ Y^{(8)}_{\mathbf{1}''} Y^{(8)}_{\mathbf{1}''} Y^{(8)}_{\mathbf{1}'} & -Y^{(8)}_{\mathbf{1}''} Y^{(8)}_{\mathbf{1}'} Y^{(8)}_{\mathbf{1}} & -Y^{(8)}_{\mathbf{1}''} Y^{(8)}_{\mathbf{1}''} Y^{(8)}_{\mathbf{1}''} \end{pmatrix},
\end{aligned}
\tag{3.36}
$$

where we have chosen the constant coefficients α^{ij} and β^{ij} as

$$
\begin{pmatrix} \alpha^{11} & \alpha^{12} & \alpha^{13} \\ \alpha^{21} & \alpha^{22} & \alpha^{23} \\ \alpha^{31} & \alpha^{32} & \alpha^{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} \beta^{11} & \beta^{12} & \beta^{13} \\ \beta^{21} & \beta^{22} & \beta^{23} \\ \beta^{31} & \beta^{32} & \beta^{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}. \tag{3.37}
$$

At $\tau = \omega + 0.051i$, the hierarchical structures of the mass matrices in (3.36) are numerically evaluated as

$$
|M_u/M_u^{33}| = \begin{pmatrix} 1.04 \times 10^{-5} & 4.76 \times 10^{-4} & 1.04 \times 10^{-5} \\ 2.18 \times 10^{-2} & 3.22 \times 10^{-3} & 2.18 \times 10^{-2} \\ 1.00 & 1.48 \times 10^{-1} & 1.00 \end{pmatrix} \tag{3.38}
$$

$$
\sim \begin{pmatrix} \mathcal{O}(\varepsilon^6) & \mathcal{O}(\varepsilon^4) & \mathcal{O}(\varepsilon^6) \\ \mathcal{O}(\varepsilon^2) & \mathcal{O}(\varepsilon^3) & \mathcal{O}(\varepsilon^2) \\ \mathcal{O}(1) & \mathcal{O}(\varepsilon) & \mathcal{O}(1) \end{pmatrix}, \tag{3.39}
$$

$$
|M_d/M_d^{33}| = \begin{pmatrix} 4.76 \times 10^{-4} & 3.22 \times 10^{-3} & 1.04 \times 10^{-5} \\ 3.22 \times 10^{-3} & 2.18 \times 10^{-2} & 2.18 \times 10^{-2} \\ 1.48 \times 10^{-1} & 3.22 \times 10^{-3} & 1.00 \end{pmatrix} \tag{3.40}
$$

$$
\sim \begin{pmatrix} \mathcal{O}(\varepsilon^4) & \mathcal{O}(\varepsilon^3) & \mathcal{O}(\varepsilon^6) \\ \mathcal{O}(\varepsilon^3) & \mathcal{O}(\varepsilon^2) & \mathcal{O}(\varepsilon^2) \\ \mathcal{O}(\varepsilon) & \mathcal{O}(\varepsilon^3) & \mathcal{O}(1) \end{pmatrix}, \tag{3.41}
$$

where M_u^{33}, M_d^{33} denote the $(3,3)$-element of M_u and M_d, respectively. Here, we show the orders in ε. One can compute quark mass ratios and Cabbibo-Kobayashi-Maskawa (CKM) mixing matrix elements from these mass matrices. Results are summarized in Table 3.2. We can see that $A_4 \times A_4 \times A_4$ model successfully reproduces the hierarchical structures in the quark sector without fine-tuning α^{ij} and β^{ij}.

	$\frac{m_u}{m_t}\times 10^6$	$\frac{m_c}{m_t}\times 10^3$	$\frac{m_d}{m_b}\times 10^4$	$\frac{m_s}{m_b}\times 10^2$	$\lvert V_{\mathrm{CKM}}^{us}\rvert$	$\lvert V_{\mathrm{CKM}}^{cb}\rvert$	$\lvert V_{\mathrm{CKM}}^{ub}\rvert$
obtained values	10.3	4.52	13.29	2.27	0.202	0.0420	0.00319
GUT scale values	5.39	2.80	9.21	1.82	0.225	0.0400	0.00353
1σ errors	± 1.68	± 0.12	± 1.02	± 0.10	± 0.0007	± 0.0008	± 0.00013

Table 3.2: The mass ratios of the quarks and the absolute values of the CKM matrix elements at the benchmark point $\tau = \omega + 0.051i$. GUT scale values at 2×10^{16} GeV with $\tan\beta = 5$ [125,126] and 1σ errors are shown.

3.1.2 The vacuum expectation value of τ

We have seen that the vacuum expectation value of complex structure modulus τ is important in modular flavor models. Particularly, slight deviation of τ from the fixed points such as $\tau = \omega$ plays a crucial role in reproducing the hierarchical flavor structures. In most modular flavor models including our $A_4 \times A_4 \times A_4$ model, the vacuum expectation value of τ is chosen by hand. However, in UV theory, we expect that some dynamics govern the determination of the modulus VEV, i.e., modulus stabilization. In the next section, we discuss modulus stabilization to answer the question, "How can the modulus τ stabilize in the vicinity of the fixed point $\tau = \omega$?"

3.2 Radiative modulus stabilization

In this section, we study modulus stabilization of τ due to 1-loop radiative corrections resulting from couplings between τ and matter fields following Ref. [79]. We consider the following superpotential within 4D $\mathcal{N} = 1$ global SUSY,

$$W(\tau, \Phi_I) = \frac{1}{2}\langle\phi\rangle Y_{\mathbf{r}}^{(k_Y)}(\tau)\sum_{I=1}^{n}\Phi_I^2, \tag{3.42}$$

where $\langle\phi\rangle$ is a constant with mass dimension 1.[3] Φ_I denotes matter chiral superfields with an integral modular weight $-k_I$. For simplicity, we assume that Φ_I belongs to a singlet representation of Γ_N. Note that if we require $\bar{\Gamma}$ modular invariance of W, the modular weight of the matter field is constrained to $2k_I = k_Y$. In general, $\langle\phi\rangle$ may carry non-trivial modular weight as a spurion. In such cases, modular symmetry is broken and we have $2k_I \neq k_Y$. We consider

[3]Superpotential of the form (3.42) can be generated by D-brane instanton effect. In such a case, $\langle\phi\rangle$ corresponds to $\sim e^{-S_{\mathrm{cl}}}M_{\mathrm{com}}$ where S_{cl} and M_{com} denote the classical action of the D-brane instanton and compactification scale, respectively [5]. We may also consider $\langle\phi\rangle$ as the VEV of a GUT Higgs superfield.

the following Kähler potential:

$$K(\tau,\overline{\tau},\Phi_I,\overline{\Phi}_I) = -\Lambda_0^2 \ln\left(-i(\tau-\overline{\tau})\right) + \sum_{I=1}^{n} \frac{|\Phi_I|^2}{(i(\overline{\tau}-\tau))^{k_I}}. \tag{3.43}$$

One can check that the above Lagrangian does not yield a tree-level scalar potential of τ because it is identically vanishing under $\langle\Phi_I\rangle = 0$. However, we obtain non-vanishing scalar potential by taking into account radiative corrections. Of course, unless SUSY is broken, radiative corrections are exactly canceled between the superpartners. Whereas in a realistic setup, SUSY should be broken at some energy scale, hence we expect non-trivial radiative corrections to the effective potential. Here, we simply introduce soft SUSY-breaking terms and do not consider the details of the breaking mechanism. Let us denote the one-loop correction to the scalar potential by V_1. Since there are n-species of matter flavors Φ_I the effective potential is written as

$$V_{\text{eff}} = V_0 + nV_1, \tag{3.44}$$

where V_0 denotes a tree-level potential that could have been induced through scenarios such as the three-form flux compactification in superstring theory. In fact, three-form flux compactification of toroidal orientifold statistically favors the point $\tau = \omega$ [76]. In addition, some other mechanisms can also lead to the stabilization $\tau = \omega$ [77]. Hence, let us approximate the tree-level potential as

$$V_0 = m_\tau^4 |\tau-\omega|^2, \tag{3.45}$$

where m_τ is a mass parameter. As long as the deviation of τ from ω is sufficiently small, our approximation of the tree-level potential would not change our conclusion.

3.2.1 Coleman-Weinberg potential

By computing the one-loop radiative correction, one obtains the following Coleman-Weinberg potential:

$$V_1 = \frac{1}{32\pi^2}\left[(M^2+m_0^2)^2 \ln\left(\frac{M^2+m_0^2}{\sqrt{e}\Lambda^2}\right) - M^4 \ln\left(\frac{M^2}{\sqrt{e}\Lambda^2}\right)\right], \tag{3.46}$$

where

$$M^2 := (2\mathrm{Im}\tau)^{2k_I} \langle\phi\rangle^2 \left|Y_{\mathbf{r}}^{(k_Y)}\right|^2. \tag{3.47}$$

We have adopted the cut-off regularization.[4] The cut-off scale is denoted by Λ. For concreteness, we identify the cut-off scale to be the compactification scale. Note that we have introduced the soft SUSY-breaking mass m_0 to the scalar component of Φ_I by hand. In this paper, we assume

[4]Terms that vanish in the limit $\Lambda^2 \to \infty$ are neglected [80]. Hence, we require $M^2 + m_0^2 \ll \Lambda^2$ to trust the expression in eq. (3.46) as a good approximation. In particular, we demand $\max(M^2, m_0^2)/\Lambda^2 \lessapprox 0.01$ to ensure the validity of our expression.

m_0 is a constant independent of τ.[5] Note also that our Coleman-Weinberg potential (3.46) is invariant under the modular transformation eq. (3.4) if $2k_I = k_Y$ is satisfied.

For later convenience, let us analytically perform differentiation of the Coleman-Weinberg potential with respect to the modulus. The first derivatives of V_1 by $x := \mathrm{Re}\tau$ and $y := \mathrm{Im}\tau$ are given by

$$\begin{aligned} \frac{\partial V_1}{\partial x} &= \frac{1}{32\pi^2}\frac{\partial M^2}{\partial x}\, C(M^2, m_0^2), \\ \frac{\partial V_1}{\partial y} &= \frac{1}{32\pi^2}\frac{\partial M^2}{\partial y}\, C(M^2, m_0^2), \end{aligned} \tag{3.48}$$

where

$$C(M^2, m_0^2) := 2(M^2 + {m_0}^2)\ln\left(\frac{M^2+m_0^2}{\sqrt{e}\Lambda^2}\right) - 2M^2\ln\left(\frac{M^2}{\sqrt{e}\Lambda^2}\right) + {m_0}^2. \tag{3.49}$$

The second derivatives are given by

$$\begin{aligned} \frac{\partial^2 V_1}{\partial x^2} &= \frac{1}{32\pi^2}\left[\frac{\partial^2 M^2}{\partial x^2}\, C(M^2, m_0^2) + 2\left(\frac{\partial M^2}{\partial x}\right)^2 \ln\left(\frac{M^2+m_0^2}{M^2}\right)\right], \\ \frac{\partial^2 V_1}{\partial y^2} &= \frac{1}{32\pi^2}\left[\frac{\partial^2 M^2}{\partial y^2}\, C(M^2, m_0^2) + 2\left(\frac{\partial M^2}{\partial y}\right)^2 \ln\left(\frac{M^2+m_0^2}{M^2}\right)\right], \\ \frac{\partial^2 V_1}{\partial x\partial y} &= \frac{1}{32\pi^2}\left[\frac{\partial^2 M^2}{\partial x\partial y}\, C(M^2, m_0^2) + 2\left(\frac{\partial M^2}{\partial x}\right)\left(\frac{\partial M^2}{\partial y}\right) \ln\left(\frac{M^2+m_0^2}{M^2}\right)\right], \end{aligned} \tag{3.50}$$

where

$$\begin{aligned} \frac{\partial M^2}{\partial x} &= 2\langle\phi\rangle^2 (2y)^{2k_I}\ \mathrm{Re}\{(\partial_\tau Y)Y^*\}, \\ \frac{\partial M^2}{\partial y} &= 2^{2k_I}\langle\phi\rangle^2 y^{2k_I-1}\left[-2y\ \mathrm{Im}\{(\partial_\tau Y)Y^*\} + 2k_I|Y|^2\right], \end{aligned} \tag{3.51}$$

and

$$\begin{aligned} \frac{\partial^2 M^2}{\partial x^2} &= 2\langle\phi\rangle^2 (2y)^{2k_I}\left[\mathrm{Re}\{(\partial_\tau^2 Y)Y^*\} + |\partial_\tau Y|^2\right], \\ \frac{\partial^2 M^2}{\partial y^2} &= 2^{2k_I}\langle\phi\rangle^2 y^{2k_I-2}\left[-2y^2(\mathrm{Re}\{(\partial_\tau^2 Y)Y^*\} - |\partial_\tau Y|^2) - 8k_I y\mathrm{Im}\{(\partial_\tau Y)Y^*\} + 2k_I(2k_I-1)|Y|^2\right], \\ \frac{\partial^2 M^2}{\partial x\partial y} &= 2^{2k_I+1}\langle\phi\rangle^2 y^{2k_I-1}\left[2k_I\mathrm{Re}\{(\partial_\tau Y)Y^*\} - y\mathrm{Im}\{(\partial_\tau^2 Y)Y^*\}\right]. \end{aligned} \tag{3.52}$$

Note that the singlet modular form $Y_{\mathbf{r}}^{(k_Y)}$ is simply written as Y in eqs. (3.51) and (3.52).

[5] We have assumed a scenario where the dominant source of SUSY breaking is mediated by a field that does not couple to τ, and its modular weight should be zero. The modular symmetry of soft SUSY breaking terms is discussed in Ref. [127].

Canonical basis of the modulus

Here, we comment on using a non-canonical field τ when analyzing the effective potential. From the Kähler potential in eq. (3.7), we obtain the following kinetic term of the modulus τ,

$$-\frac{\Lambda_0^2}{(2\mathrm{Im}\tau)^2}|\partial_\mu\tau|^2 = -\frac{\Lambda_0^2}{(2y)^2}\left[(\partial_\mu x)^2 + (\partial_\mu y)^2\right], \tag{3.53}$$

where $x = \mathrm{Re}\tau$ and $y = \mathrm{Im}\tau$ which are clearly non-canonical.

To evaluate the stability of the vacua, one would expect a conversion to canonically normalized fields and potential should be differentiated with respect to them. However, we will see that the signs of the second derivatives of the potential evaluated at its stationary points are unaffected under the conversion. Hence, a local minimum (maximum) of a potential V plotted as a function of x and y corresponds to a local minimum (maximum) of V in the canonical basis.

This can be explicitly checked as follows. Suppose a stationary point of a potential $V(x, y)$ is at $\tau_* = x_* + iy_*$. Then, we can expand the kinetic term (3.53) around τ_* as

$$-\Lambda_0^2\left[\frac{1}{(2y_*)^2} + \mathcal{O}(\Delta y)\right]\cdot\left[(\partial_\mu\Delta x)^2 + (\partial_\mu\Delta y)^2\right], \tag{3.54}$$

where $\Delta x := x - x_*$ and $\Delta y := y - y_*$ denote small deviations. On the other hand, if we introduce χ, ψ as canonical fields, we have

$$\mathcal{L}_{\mathrm{kin}} = -\frac{1}{2}\left[(\partial_\mu\chi)^2 + (\partial_\mu\psi)^2\right]. \tag{3.55}$$

Comparison between eq. (3.54) and eq. (3.55) gives us

$$\begin{aligned} \chi(\Delta x, \Delta y) &= \frac{\Lambda_0}{\sqrt{2}y_*}\Delta x + \mathcal{O}(\Delta^2), \\ \psi(\Delta x, \Delta y) &= \frac{\Lambda_0}{\sqrt{2}y_*}\Delta y + \mathcal{O}(\Delta^2), \end{aligned} \tag{3.56}$$

where $\mathcal{O}(\Delta^2)$ denotes second or higher order terms of small deviations Δx and Δy. Then, the second derivatives of V about canonical and the non-canonical basis are related as

$$\begin{aligned} \left.\frac{\partial^2 V}{\partial\chi^2}\right|_{\tau=\tau_*} &= \frac{2y_*^2}{\Lambda_0^2}\left.\frac{\partial^2 V}{\partial x^2}\right|_{\tau=\tau_*}, \\ \left.\frac{\partial^2 V}{\partial\psi^2}\right|_{\tau=\tau_*} &= \frac{2y_*^2}{\Lambda_0^2}\left.\frac{\partial^2 V}{\partial y^2}\right|_{\tau=\tau_*}, \\ \left.\frac{\partial^2 V}{\partial\chi\partial\psi}\right|_{\tau=\tau_*} &= \frac{2y_*^2}{\Lambda_0^2}\left.\frac{\partial^2 V}{\partial x\partial y}\right|_{\tau=\tau_*}, \end{aligned} \tag{3.57}$$

at the stationary point $\tau = \tau_*$. This shows that the second derivatives of potential around the minimum/maximum are the same up to a positive overall constant. Thus, one can discuss the stability using non-canonical fields.

3.3 Vacuum structure with A_4 modular forms

In this section, we discuss concrete models by specifying the singlet modular forms in the superpotential eq. (3.42). Following Ref. [79], we assume A_4 singlet modular forms with weight $k_Y = 8$. Since there are three possible singlet representations $\mathbf{r} = \mathbf{1}, \mathbf{1}', \mathbf{1}''$ cf. (3.25), we analyze the 1-loop Coleman-Weinberg potential V_1 for each singlet representation, particularly around the fixed point $\tau = \omega$. We also discuss the possibility of obtaining stable vacua close to ω under the effective potential $V_{\text{eff}} = V_0 + nV_1$.

3.3.1 A_4 trivial singlet $\mathbf{1}$

We study the radiative corrections under the choice of A_4 modular form $Y_{\mathbf{1}}^{(8)}(\tau)$ in the trivial singlet $\mathbf{r} = \mathbf{1}$ with weight $k_Y = 8$. By substituting the Taylor expansion of the modular form around $\tau = \omega$ (3.28) into the squared mass (3.47), one obtains

$$M^2 = \frac{3^{k_I}\langle\phi\rangle^2}{36}\left|\left.\frac{d^2\tilde{Y}_{\mathbf{1}}^{(8)}(u)}{du^2}\right|_{u=0}\right|^2 |\tau-\omega|^4 + \cdots\,, \tag{3.58}$$

where the ellipsis denotes higher-order correction terms which are negligible for our analyses. Then, one notices from eqs. (3.48), (3.50) that both first and second derivatives of V_1 with respect to x, y vanish at $\tau = \omega$,

$$\begin{aligned} \left.\frac{\partial V_1}{\partial x}\right|_{\tau=\omega} &= \left.\frac{\partial V_1}{\partial y}\right|_{\tau=\omega} = 0, \\ \left.\frac{\partial^2 V_1}{\partial x^2}\right|_{\tau=\omega} &= \left.\frac{\partial^2 V_1}{\partial y^2}\right|_{\tau=\omega} = \left.\frac{\partial^2 V_1}{\partial x \partial y}\right|_{\tau=\omega} = 0, \end{aligned} \tag{3.59}$$

while the value of 1-loop potential is $V_1(\omega) = \frac{1}{32\pi^2} m_0^4 \ln\left(\frac{m_0^2}{\sqrt{e}\Lambda^2}\right)$. In Fig. 3.1, we present a numerical illustration of V_1 in the vicinity of $\tau = \omega$ under the choices of parameter values explicitly shown in the caption. The result clearly shows that $\tau = \omega$ is an unstable point of V_1. In fact, the fixed point $\tau = \omega$ is a local maximum of V_1 independent of the values of k_I, $\langle\phi\rangle$ and m_0. This can be understood by expanding V_1 in terms of $\frac{M^2}{m_0^2}$ which is sufficiently small since $M^2 \to 0$ as $\tau \to \omega$. Then, the bosonic contribution dominates the potential V_1 and leads to

$$V_1 \simeq \frac{m_0^4}{32\pi^2}\left[\ln\left(\frac{m_0^2}{\sqrt{e}\Lambda^2}\right) + \frac{2M^2}{m_0^2}\ln\left(\frac{m_0^2}{\Lambda^2}\right)\right], \tag{3.60}$$

which is a good approximation if τ is sufficiently close to ω. The second term of eq. (3.60) is important because the first term is just a constant. Then, one notices that V_1 generally behaves as an inverted quartic potential maximized at $\tau = \omega$ by noting $\ln(m_0^2/\Lambda^2) < 0$ and the Taylor expansion of M^2 in eq. (3.58).

Thus, taking into account the tree-level potential $V_0 = m_\tau^4|\tau-\omega|^2$, the effective potential $V_{\text{eff}} = V_0 + nV_1$ has a local minimum at $\tau = \omega$. The one-loop potential nV_1 would hardly generate the desired small deviation from the fixed point $\tau = \omega$.

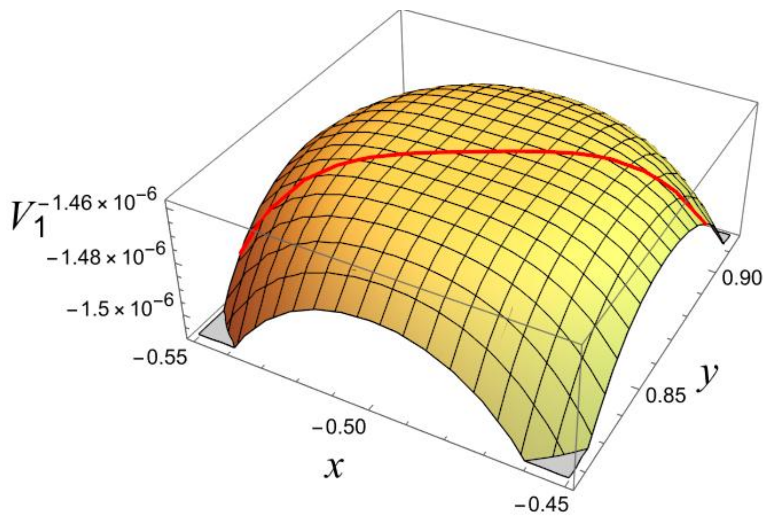

Figure 3.1: The one-loop potential V_1 with $\mathbf{r} = \mathbf{1}$ and $k_Y = 8$ in the vicinity of $\tau = \omega$. The parameters are chosen as $k_I = 4, \langle\phi\rangle = 10^{-2}, m_0 = 10^{-1}$ under $\sqrt{e}\Lambda^2 = 1$. The red line shows the arc $|\tau| = 1$.

Let us briefly comment on the one-loop potential far away from the fixed point $\tau = \omega$. According to eq. (3.48), there would also be potential minima at the values of τ satisfying $C(M^2, m_0^2) = 0$. However, such points are generally unphysical since they conflict with our effective field theory description which is valid under $M^2 \ll \Lambda^2$.

3.3.2 A_4 non-trivial singlet $\mathbf{1}'$

We study the radiative corrections under the choice of A_4 modular form $Y_{\mathbf{1}'}^{(8)}(\tau)$ in the singlet $\mathbf{r} = \mathbf{1}'$ with weight $k_Y = 8$. By substituting the Taylor expansion of the modular form around $\tau = \omega$ (3.29) into the squared mass (3.47), one obtains

$$M^2 = \frac{3^{k_I}|D|^2\langle\phi\rangle^2}{3}|\tau-\omega|^2\left[1+\delta y\left(\frac{4k_I}{\sqrt{3}}-6\sqrt{3}\right)+\mathcal{O}(|\tau-\omega|^2)\right], \tag{3.61}$$

where $D := \frac{d\tilde{Y}_{\mathbf{1}'}^{(8)}(u)}{du}\Big|_{u=0} \simeq -10.6 + 18.3i$ and $\delta y := y - \frac{\sqrt{3}}{2}$. From eq. (3.48), we find that the first derivative of V_1 with respect to x, y vanish at $\tau = \omega$,

$$\left.\frac{\partial V_1}{\partial x}\right|_{\tau=\omega} = \left.\frac{\partial V_1}{\partial y}\right|_{\tau=\omega} = 0, \tag{3.62}$$

whereas the second derivatives are non-vanishing,

$$\left.\frac{\partial^2 V_1}{\partial x^2}\right|_{\tau=\omega} = \left.\frac{\partial^2 V_1}{\partial y^2}\right|_{\tau=\omega} = \frac{1}{24\pi^2}3^{k_I}\,|D|^2\,\langle\phi\rangle^2 m_0^2\ln\left(\frac{m_0^2}{\Lambda^2}\right), \tag{3.63}$$

$$\left.\frac{\partial^2 V_1}{\partial x\partial y}\right|_{\tau=\omega} = 0. \tag{3.64}$$

We find that the fixed point $\tau = \omega$ is a local maximum of V_1 since we have $\frac{\partial^2 V_1}{\partial x^2}\big|_{\tau=\omega} < 0$ and $\frac{\partial^2 V_1}{\partial y^2}\big|_{\tau=\omega} < 0$ independent of the values of $k_I, \langle\phi\rangle$ and m_0^2.[6] In Fig. 3.2, we present a numerical illustration of V_1 in the vicinity of $\tau = \omega$ under the choices of parameters $k_I = 4, \langle\phi\rangle = 10^{-2}, m_0 = 10^{-1}$.

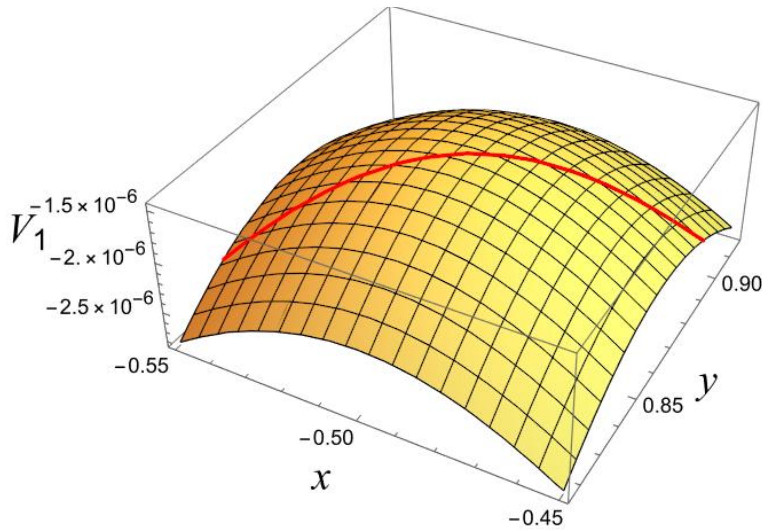

Figure 3.2: The one-loop effective potential V_1 in the vicinity of $\tau = \omega$ for $\mathbf{r} = \mathbf{1}'$ with the weight $k_Y = 8$. The parameters are chosen as $k_I = 4, \langle\phi\rangle = 10^{-2}, m_0 = 10^{-1}$ under $\sqrt{e}\Lambda^2 = 1$. The red line shows the arc $|\tau| = 1$.

Since the second derivatives of V_1 at $\tau = \omega$ are non-zero and negative, there is a possibility to obtain a local minimum of the effective potential $V_{\text{eff}} = V_0 + nV_1$ slightly deviated from the fixed point. Nevertheless, the 1-loop correction is small in general, and one of the ways to generate the deviation is to consider a sufficiently large number of species contributing to the 1-loop correction. One can estimate the number of species n required for the 1-loop correction nV_1 to generate slight deviations as

$$n \geq n_c \equiv \frac{-48\pi^2 m_\tau^4}{3^{k_I}|D|^2\langle\phi\rangle^2 m_0^2 \ln(m_0^2/\Lambda^2)}, \tag{3.65}$$

where n_c is defined by

$$\frac{\partial^2}{\partial x^2}(V_0 + n_c V_1) = 0,\ \frac{\partial^2}{\partial y^2}(V_0 + n_c V_1) = 0, \quad (\text{at } \tau = \omega). \tag{3.66}$$

We can see from eq. (3.65) that n_c becomes larger as the soft mass m_0 and the mass parameter $\langle\phi\rangle$ become smaller. This is because, when m_0 is reduced, SUSY-breaking effects are suppressed, and the radiative correction per unit species gets smaller. Hence, more species are needed for

[6] Note that we have $m_0^2 \ll \Lambda^2$.

the deviations. Similarly, when $\langle\phi\rangle$ is reduced, the couplings between the modulus τ and matter species become smaller leading to a smaller radiative correction per unit species.

We emphasize that our perturbative approach is still valid even though the "tree-level potential" and the one-loop potential become comparable. This is because the origin of tree-level potential V_0 we assume has nothing to do with that of the one-loop contribution. Possible origins of the "tree-level potential" include three-form flux or non-perturbative effects such as gaugino condensation. On the other hand, the origin of the Coleman-Weinberg potential is the loops of matter fields coupled with τ through their "Yukawa couplings". We further note that the coupling is small $|Y(\tau)| \leq \mathcal{O}(0.1)$[7] if the modulus is in the vicinity of $\tau = \omega$. Thus, the one-loop corrections can be comparable with the "tree-level potential" without problems of strong couplings.

Nevertheless, perturbativity of gravitational interactions sets a bound on the number of species, known as the "species bound" [128]. Although we expect that the one-loop correction to the Newton constant is relaxed thanks to SUSY, a naive application of the species bound requires

$$\Lambda < \frac{M_{\rm pl}}{\sqrt{n}}, \tag{3.67}$$

where $M_{\rm pl}$ denotes the Planck scale. For instance, if we assume $M_{\rm pl} \simeq 10^{19}$GeV and the cut-off Λ close to the compactification scale $M_{\rm com} \simeq 10^{17}$GeV, "species bound" is satisfied for $n < 10^4$. We will see in numerical examples that our model is consistent with the "species bound" if m_τ is reasonably small.

Note that our effective potential $V_{\rm eff}$ is invariant under $\delta x \to -\delta x$. This can be checked by noticing that our one loop correction V_1 is symmetric under both CP transformation $\tau \to -\bar{\tau}$ and modular T transformation. Spontaneous CP violation is a phenomenologically important aspect of the flavor structure of quarks and leptons.[8] However, as will be shown in eq. (3.68), such a violation would not occur in our model near $\tau = \omega$. Physically relevant potential minima are found on the line $x = -\frac{1}{2}$. Thus, we primarily focus on the behavior of the effective potential along the line $x = -\frac{1}{2}$.

Let us present some numerical illustrations of $V_{\rm eff}$. First, we choose $(k_I, \langle\phi\rangle, m_0) = (4, 10^{-2}, 10^{-1})$. Under this choice of parameters, the critical species number n_c is estimated as $n_c/m_\tau^4 \simeq 3180$ in the unit satisfying $\sqrt{e}\Lambda^2 = 1$. Hence, we have shown $V_{\rm eff}$ along $x = -\frac{1}{2}$ for $n/m_\tau^4 = 3300$ (Fig. 3.3) and $n/m_\tau^4 = 3400$ (Fig. 3.4). The deviation $\delta y_* := y_* - \frac{\sqrt{3}}{2}$ is numerically obtained as $|\delta y_*| \simeq 0.0273$ and $|\delta y_*| \simeq 0.0362$, respectively. For clarity, we show explicit values of the critical species number with $e^{1/4}\Lambda = 10^{17}$GeV. When m_τ is 10^{17}GeV, 0.5×10^{17}GeV, 0.2×10^{17}GeV, the corresponding critical species number is given by $n_c \simeq 3180, 200, 5$, respectively. The species bound (3.67) is satisfied in these examples.

[7]The convergence of the perturbative expansion is further improved by the loop factor.

[8]CP-violation is caused by the VEV of τ if it lies neither on the lines $2x \equiv 0 \pmod 1$ nor the arc $|\tau| = 1$.

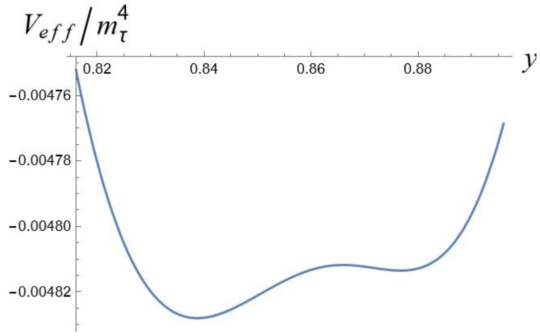

Figure 3.3: $n/m_\tau^4 = 3300$

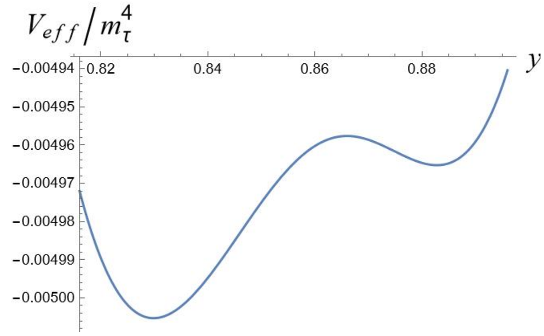

Figure 3.4: $n/m_\tau^4 = 3400$

To confirm the stability of the vacuum, let us check the (x, y) dependence of $V_{\rm eff}$. In Fig. 3.5, we have shown the numerical illustration of $V_{\rm eff}$ around the fixed point $\tau = \omega$ where we have used the same parameters as in Fig. 3.4. We can see that the local minimum is on the "CP invariant" line $\delta x = 0$. The shape of 1-loop corrected effective potential generally takes the "tilted wine-bottle" shape as the one presented in Fig. 3.5.

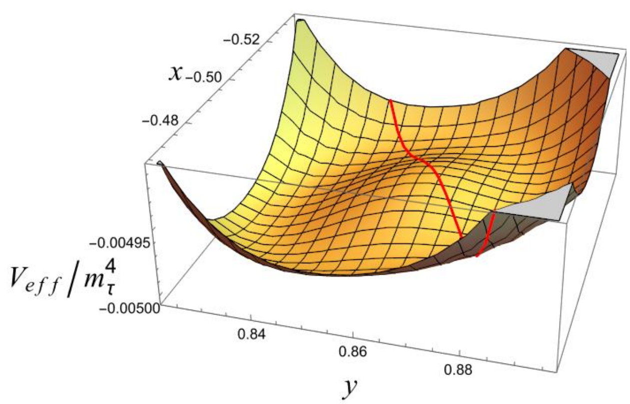

Figure 3.5: Three-dimensional plot of $V_{\rm eff}/m_\tau^4$ in the vicinity of $\tau = \omega$. The chosen parameters are identical to those in Fig. 3.4, namely $k_I = 4, \langle\phi\rangle = 10^{-2}, m_0 = 10^{-1}, n/m_\tau^4 = 3400$ in the unit $\sqrt{e}\Lambda^2 = 1$. The red line shows the arc $|\tau| = 1$.

Let us also consider a different parameter set, $(k_I, \langle\phi\rangle, m_0) = (4, 10^{-2}, 10^{-3})$. Under this choice of parameters, the critical species number n_c is estimated as $n_c/m_\tau^4 \simeq 9.8 \times 10^6$ in the unit satisfying $\sqrt{e}\Lambda^2 = 1$. Hence, we have shown $V_{\rm eff}$ for $n/m_\tau^4 = 2.2 \times 10^7$ (Fig. 3.6) or $n/m_\tau^4 = 2.5 \times 10^7$ (Fig. 3.7). The deviation is numerically obtained as $|\delta y_*| \simeq 0.0244$ and $|\delta y_*| \simeq 0.0356$, respectively.

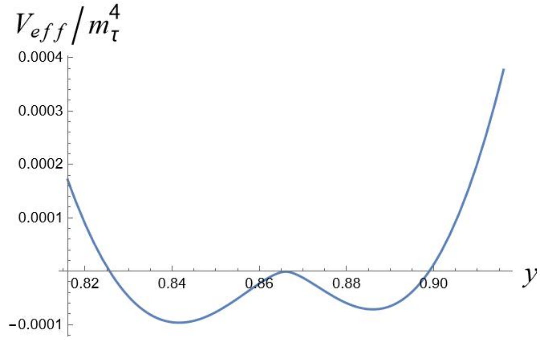

Figure 3.6: $n/m_\tau^4 = 2.2 \times 10^7$

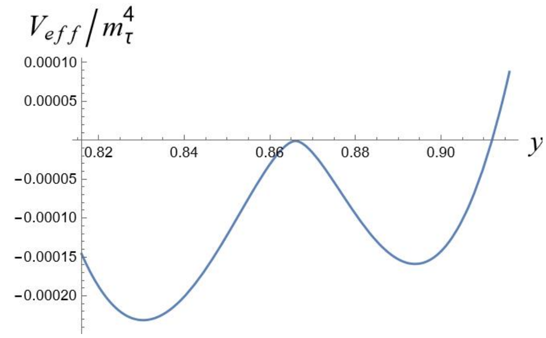

Figure 3.7: $n/m_\tau^4 = 2.5 \times 10^7$

For clarity, we note that the critical species numbers are $n_c \simeq 9.8 \times 10^6, 980, 61$ for $m_\tau = 10^{17}$GeV, 10^{16}GeV, 0.5×10^{16}GeV, respectively, where we have taken $e^{1/4}\Lambda = 10^{17}$GeV. Concerning the consistency with (3.67), we have shown the relation between m_τ and n_c with the "species bound" in Fig. 3.8. The blue line plots the relation $n_c/m_\tau^4 = 9.8 \times 10^6$ while the orange line denotes the upper bound of the species number $n \leq (\Lambda^2/M_{\rm pl})^2 \simeq 10^4$. When m_τ is reasonably small (e.g. $m_\tau \lessapprox 1.5 \times 10^{16}$GeV), our models are compatible with the bound.

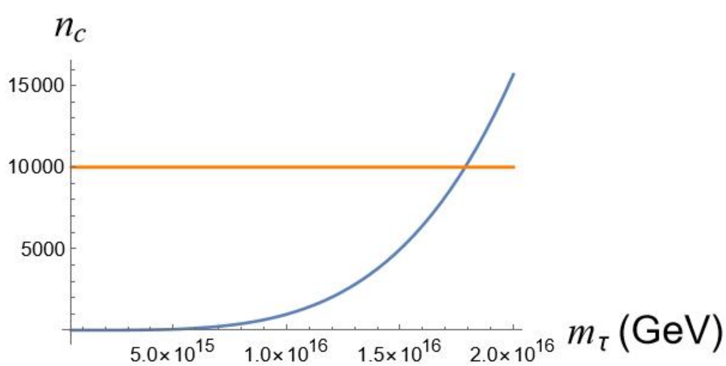

Figure 3.8: The relation between n_c and m_τ. We have shown $n_c/(m_\tau)^4 = 9.8 \times 10^6$ (blue curve) and also the upper bound on the species number $(\Lambda^2/M_{\rm pl})^2 \simeq 10^4$ (orange line). We have used $e^{1/4}\Lambda = 10^{17}$GeV.

As another example, we have considered the parameter set $(k_I, \langle\phi\rangle, m_0) = (4, 10^{-3}, 10^{-1})$. In this case, we have $n_c/m_\tau^4 \simeq 3.2 \times 10^5$ in the unit satisfying $\sqrt{e}\Lambda^2 = 1$. For clarity, we note that when m_τ is 10^{17}GeV, 0.5×10^{17}GeV, 10^{16}GeV, the corresponding critical species number is given by $n_c \simeq 3.2 \times 10^5, 2.0 \times 10^4, 32$, respectively where we have taken $e^{1/4}\Lambda = 10^{17}$GeV. We have shown numerical illustration of $V_{\rm eff}$ for $n/m_\tau^4 = 3.3 \times 10^5$ in Fig. 3.9. We find a new vacuum $\tau = \tau_*$ apart from the original vacuum $\tau = \omega$. Numerically, we have found the shift to be $|\delta\tau_*| = |\tau_* - \omega| \sim 0.2$ which is phenomenologically unattractive for realizing hierarchical flavor structures. Although we have changed the species number n, we could hardly control the position of the vacuum. Hence, we could not realize a small deviation from the fixed point $\tau = \omega$

for this parameter region. Later, we will analytically clarify such a behavior. For completeness, we have numerically found one of the local minima at $\tau_* \simeq -0.44 + 0.67i$ in Fig. 3.9. This is a CP-violating point, however, it is out of the validity of our approximation. Hence, we cannot regard this as a physically relevant vacuum.

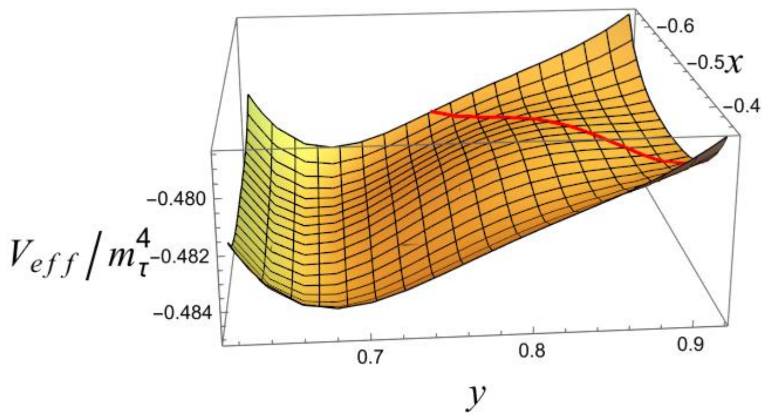

Figure 3.9: Three-dimensional plot of $V_{\rm eff}/m_\tau^4$. The parameters are chosen as $k_I = 4, \langle\phi\rangle = 10^{-3}, m_0 = 10^{-1}, n/m_\tau^4 = 3.3 \times 10^5$ under $\sqrt{e}\Lambda^2 = 1$. The red line shows the arc $|\tau| = 1$.

Approximation

Here, we analytically discuss $V_{\rm eff}$ near the fixed point $\tau = \omega$ to confirm the behavior numerically observed in Figs. 3.4-3.9. For this purpose, we expand the one-loop potential V_1 in terms of $\delta\tau = \tau - \omega$. By substituting the expansion (3.61) into (3.46), one finds

$$\begin{aligned}
&32\pi^2 V_1 \\
\simeq&(\xi^2|\delta\tau|^2 + m_0^2)\left[(\xi^2|\delta\tau|^2 + m_0^2)\ln\left(\frac{\xi^2|\delta\tau|^2 + m_0^2}{\sqrt{e}\Lambda^2}\right) + 2\xi^2|\delta\tau|^2\delta y\left(\frac{4k_I - 18}{\sqrt{3}}\right)\ln\left(\frac{\xi^2|\delta\tau|^2 + m_0^2}{\Lambda^2}\right)\right] \\
&- (\xi^2|\delta\tau|^2)^2\Big[\ln\left(\frac{\xi^2|\delta\tau|^2}{\sqrt{e}\Lambda^2}\right) + 2\delta y\left(\frac{4k_I - 18}{\sqrt{3}}\right)\ln\left(\frac{\xi^2|\delta\tau|^2}{\Lambda^2}\right)\Big],
\end{aligned} \tag{3.68}$$

where $\xi^2 := \frac{3^{k_I}|D|^2\langle\phi\rangle^2}{3}$ (see Appendix B.3 for its derivation). For $k_I < 9/2$, we find that the value of V_1 increases as we increase δy while keeping $|\delta\tau|$ fixed. This confirms the "tilted wine-bottle"" shape of $V_{\rm eff} = V_0 + nV_1$ in Fig. 3.5.

Next, let us analytically confirm the deviations $\delta\tau$ from the fixed point $\tau = \omega$ which we have numerically evaluated before. In particular, we consider two different parametric regimes $M^2 \gg m_0^2$ and $M^2 \ll m_0^2$.

For $M^2 \gg m_0^2$ corresponding to ones in Figs. 3.6 and 3.7, assuming $\langle\phi\rangle^2 \gg m_0^2$, V_1 can be further simplified as

$$V_1\Big|_{x=-1/2} \sim \frac{1}{16\pi^2} m_0^2 \xi^2 (\delta y)^2 \ln\left(\frac{\xi^2 (\delta y)^2}{\Lambda^2}\right), \tag{3.69}$$

where we have fixed $x = -1/2$. By substituting (3.69) into the extremum condition $\partial_y(V_0 + nV_1) = 0$, the local minimum is obtained at

$$|\delta y_*| \sim \left(\frac{3\Lambda^2}{3^{k_I} e|D|^2\langle\phi\rangle^2}\right)^{\frac{1}{2}} \exp\left(-\frac{24\pi^2 m_\tau^4}{3^{k_I}|D|^2 n m_0^2\langle\phi\rangle^2}\right), \tag{3.70}$$

where $\delta y_* = y_* - \frac{\sqrt{3}}{2}$. Let us estimate the deviation δy_* from this analytical expression and compare it with the previous numerical examples in Figs. 3.6 and 3.7. Substituting $(k_I, \langle\phi\rangle, m_0) = (4, 10^{-2}, 10^{-3})$ with $n/m_\tau^4 = 2.2 \times 10^7$ (Figs. 3.6) or $n/m_\tau^4 = 2.5 \times 10^7$ (Figs. 3.7) into (3.70), we find

$$\begin{aligned} |\delta y_*| &\sim 0.022, \quad (n/m_\tau^4 = 2.2 \times 10^7), \\ |\delta y_*| &\sim 0.031, \quad (n/m_\tau^4 = 2.5 \times 10^7). \end{aligned} \tag{3.71}$$

in the unit satisfying $\sqrt{e}\Lambda^2 = 1$. On the other hand, numerically we have found $|\delta y_*| \simeq 0.0273$ and $|\delta y_*| \simeq 0.0362$, respectively. Hence, our analytical estimates confirm the numerical results. To further confirm the consistencies, we check the value of $\Delta V_{\text{eff}} = |V_{\text{eff}}(\omega) - V_{\text{eff}}(\omega + i\delta y_*)|$. The linear order approximation gives us

$$-\Delta V_{\text{eff}} \simeq \delta y_* \cdot \frac{\partial}{\partial y} V_{\text{eff}}\Big|_{(\delta x, \delta y)=(0, \frac{\delta y_*}{2})}. \tag{3.72}$$

Using the approximation eq. (3.70), we obtain

$$\Delta V_{\text{eff}} \sim \frac{\ln 4}{16\pi^2 e} \Lambda^2 n m_0^2 \, \exp\left(-\frac{48\pi^2 m_\tau^4}{3^{k_I}|D|^2 n m_0^2\langle\phi\rangle^2}\right). \tag{3.73}$$

In our numerical illustrations of Figs. 3.6 and 3.7, we numerically obtain $\Delta V_{\text{eff}} \simeq 9.6 \times 10^{-5}$ and $\simeq 2.3 \times 10^{-4}$, respectively. On the other hand, our analytical approximation yields $\Delta V_{\text{eff}} \simeq 1.1 \times 10^{-4}$ and $\simeq 2.6 \times 10^{-4}$, respectively. Hence, our analytical approximation again confirms the numerical results.

Next, let us focus on the regime $M^2 \ll m_0^2$ corresponding to the case shown in Fig. 3.9. Then, we can approximate the effective potential V_{eff} as

$$V_{\text{eff}} \simeq \frac{n}{32\pi^2} m_0^4 \ln\left(\frac{m_0^2}{\sqrt{e}\Lambda^2}\right) + m_\tau^4\left(1 - \frac{n}{n_c}\right)|\delta\tau|^2 - m_\tau^4 \frac{n}{n_c}\left(\frac{4k_I}{\sqrt{3}} - 6\sqrt{3}\right)|\delta\tau|^2\delta y. \tag{3.74}$$

This shows the approximate behavior of Fig. 3.9 at $|\delta\tau| \ll \mathcal{O}(1)$. By solving the extremum condition $\partial_\tau(V_0 + nV_1) = 0$, one finds

$$\begin{aligned} \delta y_* &\simeq \frac{2(n_c - n)}{3n\left(\frac{4k_I}{\sqrt{3}} - 6\sqrt{3}\right)}, \quad (\text{when } \delta x_* = 0), \\ \delta y_* &\simeq \frac{(n_c - n)}{n\left(\frac{4k_I}{\sqrt{3}} - 6\sqrt{3}\right)}, \quad (\text{when } \delta x_* \neq 0). \end{aligned} \tag{3.75}$$

Then, we have $\delta y_* > 0$ under $n > n_c$ and $k_I < 9/2$. Hence, the solutions (3.75) correspond to unstable vacua as shown in Fig. 3.9. Since we do not find a solution with $\delta y_* < 0$, one can hardly produce a local minimum near $\tau = \omega$ in the parameter regime $M^2 \ll m_0^2$.

3.3.3 A_4 non-trivial singlet $\mathbf{1}''$

We study the radiative corrections under the choice of A_4 modular form $Y^{(8)}_{\mathbf{1}''}(\tau)$ in the singlet $\mathbf{r} = \mathbf{1}''$ with weight $k_Y = 8$. By substituting the Taylor expansion of the modular form around $\tau = \omega$ (3.30) into the squared mass (3.47), one obtains

$$M^2 = 3^{k_I}|E|^2\langle\phi\rangle^2\left[1 + \frac{4k_I - 16}{\sqrt{3}}\delta y - \frac{8}{3}(\delta x)^2 + \frac{2}{3}(4k_I^2 - 34k_I + 68)(\delta y)^2 + \cdots\right], \tag{3.76}$$

where $E := Y^{(8)}_{\mathbf{1}''}(\omega) \simeq -2.05 - 3.55i$. From eq. (3.48), we find that the first derivative of V_1 by $x = \mathrm{Re}\tau$ vanishes at $\tau = \omega$, while the derivative by $y = \mathrm{Im}\tau$ is non-vanishing unless $k_I = 4$,

$$\begin{aligned} \left.\frac{\partial V_1}{\partial x}\right|_{\tau=\omega} &= 0, \\ \left.\frac{\partial V_1}{\partial y}\right|_{\tau=\omega} &= \frac{k_I - 4}{8\sqrt{3}\pi^2}[M^2\ C(M^2, m_0^2)]_{\tau=\omega}. \end{aligned} \tag{3.77}$$

The second derivatives of V_1 at $\tau = \omega$ are given by

$$\begin{aligned} \left.\frac{\partial^2 V_1}{\partial x^2}\right|_{\tau=\omega} &= -\frac{1}{6\pi^2}[M^2 C(M^2, m_0^2)]_{\tau=\omega}, \\ \left.\frac{\partial^2 V_1}{\partial y^2}\right|_{\tau=\omega} &= \frac{1}{24\pi^2}\left[(4k_I^2 - 34k_I + 68)M^2 C(M^2, m_0^2) + 8(k_I - 4)^2 M^4 \ln\left(1 + \frac{m_0^2}{M^2}\right)\right]_{\tau=\omega}, \\ \left.\frac{\partial^2 V_1}{\partial x \partial y}\right|_{\tau=\omega} &= 0. \end{aligned} \tag{3.78}$$

Note that if $k_I = 4$, the 1-loop potential V_1 is symmetric under the modular transformation eq. (3.4).[9] In this case, one finds $\left.\frac{\partial^2 V_1}{\partial x^2}\right|_{\tau=\omega} = \left.\frac{\partial^2 V_1}{\partial y^2}\right|_{\tau=\omega} = -\frac{1}{6\pi^2}[M^2 C(M^2, m_0^2)]_{\tau=\omega} > 0$, hence

[9] To be more precise, $k_I \neq 4$ means that $\langle\phi\rangle$ carries a non-vanishing weight under the modular transformation. Then, modular symmetry is spontaneously broken by the VEV $\langle\phi\rangle$. On the other hand, if $k_I = 4$, modular symmetry is preserved until τ acquires its VEV.

$\tau = \omega$ is a local minimum of V_1. The positivity of the second derivatives can be checked easily. If we differentiate $C(M^2, m_0^2)$ with respect to m_0^2 we find

$$\frac{\partial C(M^2, m_0^2)}{\partial m_0^2} = 2 \ln \left(\frac{e(M^2 + m_0^2)}{\Lambda^2} \right), \tag{3.79}$$

which is negative, because $M^2 + m_0^2 \ll \Lambda^2$. Reminding ourselves $C(M^2, 0) = 0$ from eq. (3.49), we find $C(M^2, m_0^2) < 0$ which confirms the positivity.

In Figs. 3.10 and 3.11, we show the numerical illustrations of V_1 where we have chosen $k_I = 2$ and $k_I = 4$, respectively. We confirm $\tau = \omega$ is a stable point of V_1 when $k_I = 4$, whereas unstable for $k_I = 2$. We will further discuss the case $k_I \neq 4$ in the next sub-subsection.

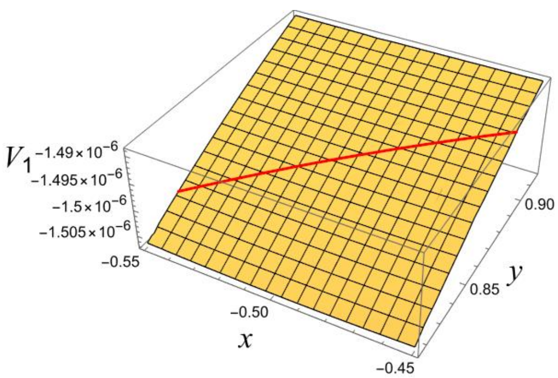

Figure 3.10: The one-loop effective potential V_1 in the vicinity of $\tau = \omega$ for $\mathbf{r} = \mathbf{1}''$ with the weight $k_Y = 8$. The parameters are chosen as $(k_I, \langle\phi\rangle, m_0) = (2, 10^{-3}, 10^{-1})$. The red line shows the arc $|\tau| = 1$.

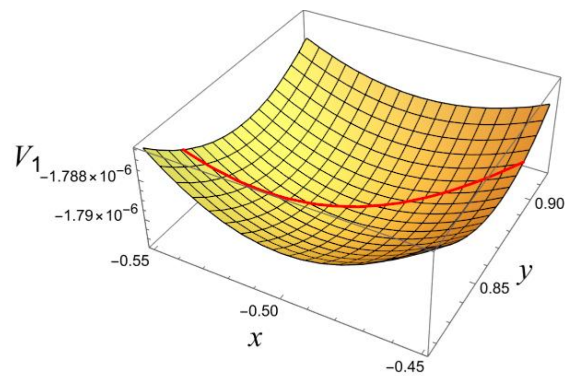

Figure 3.11: The one-loop effective potential V_1 in the vicinity of $\tau = \omega$ for $\mathbf{r} = \mathbf{1}''$ with the weight $k_Y = 8$. The parameters are chosen as $(k_I, \langle\phi\rangle, m_0) = (4, 10^{-3}, 10^{-1})$. The red line shows the arc $|\tau| = 1$.

Deviation from $\tau = \omega$

The case $k_I \neq 4$ is phenomenologically interesting because the fixed point $\tau = \omega$ is not a local minimum of V_1 and the stable vacuum of V_{eff} would slightly deviate. This is a consequence of the spontaneous breaking of the modular symmetry by the VEV $\langle\phi\rangle$ carrying non-vanishing modular weight. In what follows, we evaluate the resulting deviations of the vacuum from $\tau = \omega$. The stationary condition on the total potential $V_0 + nV_1$ reads

$$\frac{\partial}{\partial x}(V_0 + nV_1) = 0, \quad \frac{\partial}{\partial y}(V_0 + nV_1) = 0. \tag{3.80}$$

From eqs. (3.77) and (3.78), the one-loop potential $V_1(\tau)$ is approximated as

$$V_1(\tau) \simeq V_1(\omega) + \delta y \frac{\partial}{\partial y} V_1(\tau)|_{\tau=\omega} + \frac{1}{2}\left((\delta x)^2 \frac{\partial^2}{\partial x^2} + (\delta y)^2 \frac{\partial^2}{\partial y^2} \right) V_1(\tau)|_{\tau=\omega} + \mathcal{O}(|\delta\tau|^3), \tag{3.81}$$

where $\delta x := x + 1/2,\ \delta y := y - \sqrt{3}/2$. By substituting eq. (3.81) into eq. (3.80), we obtain following solutions:

$$\begin{aligned}\delta x_* &= 0,\\ \delta y_* &= -\frac{\partial_y V_1|_{\tau=\omega}}{2m_\tau^4/n + \partial_y^2 V_1|_{\tau=\omega}} + \mathcal{O}(|\delta\tau|^2).\end{aligned} \tag{3.82}$$

This shows that the shifted vacuum is still on the CP-conserving line $x = -\frac{1}{2}$. We have numerically illustrated the deviation δy_* as a function of $\langle\phi\rangle$ and m_0 where we have chosen $k_I = 2$ and $n/m_\tau^4 = 3 \times 10^5$. It can be seen that sufficiently large $\langle\phi\rangle$ and m_0 realize a phenomenologically favored deviation size $|\delta y_*| \sim 0.03$.

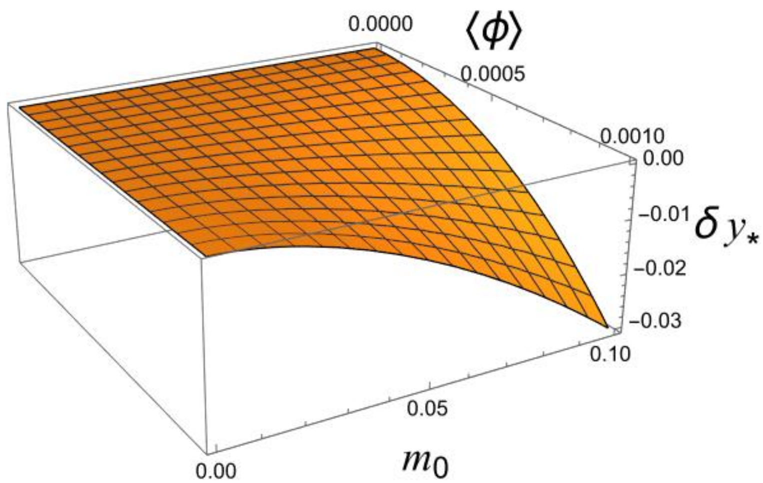

Figure 3.12: The deviation δy_* as a function of $\langle\phi\rangle$ and m_0. The parameters are chosen as $k_I = 2$ and $n/m_\tau^4 = 3 \times 10^5$.

Chapter 4

Spontaneous CP violation through moduli stabilization

In this chapter, we study spontaneous CP violation through moduli stabilization. In modular flavor models, VEVs of complex structure moduli are the sources of CP-violating phases in mixing matrices of quarks and leptons. Hence, we are motivated to understand the dynamics that can generate such CP-violating VEVs. This chapter is organized as follows. First, we review the idea that CP is a discrete gauge symmetry. This clarifies why CP needs to be broken spontaneously. Secondly, we review CP transformation in the framework of 4D modular flavor symmetric models. We identify the CP-breaking region in the moduli space. Thirdly, we show some useful properties of CP and modular invariant scalar potential in a model-independent way. Finally, we show spontaneous CP violation can be realized by the combined effects of three-form fluxes and matter-moduli couplings controlled by modular symmetries. We also discuss Kähler modulus stabilization.

4.1 CP transformation in extra-dimensional theories

This section reviews the 4D CP transformation from the perspective of extra-dimensional theories, providing motivations to study spontaneous CP violation. After discussing how to embed 4D CP in a 10D field theory, we explain that CP is a discrete gauge symmetry. Hence, it is natural to consider that CP is broken spontaneously.

4.1.1 4D CP from 10D perspective

Firstly, we discuss how 4D CP is embedded in underlying 10D theories following Refs. [4, 129–131]. As a typical example, we focus on 10D $\mathcal{N} = 1$ super Yang-Mills theory which can naturally emerge in the low energy effective field theory limit of Type I, Type IIB, and heterotic string theories.

In 10D super Yang-Mills theory, we have a 16-component Majorana-Weyl spinor, Ψ. Corresponding gamma matrices $\gamma_M^{(10)}$, $(M = 0, ..., 9)$ are given by,

$$\gamma_M^{(10)} = \begin{cases} \gamma_\mu \otimes \begin{pmatrix} 0 & \mathbf{1}_4 \\ \mathbf{1}_4 & 0 \end{pmatrix} & \text{for } M(=\mu) = 0, ..., 3, \\ i\mathbf{1}_4 \otimes \begin{pmatrix} 0 & -\alpha_i \\ \alpha_i & 0 \end{pmatrix} & \text{for } M(=i+3) = 4, ..., 6, \\ \gamma_5 \otimes \begin{pmatrix} 0 & \beta_i \\ \beta_i & 0 \end{pmatrix} & \text{for } M(=i+6) = 7, ..., 9, \end{cases} \tag{4.1}$$

where γ_μ and γ_5 denote the gamma matrices in the 4D Minkowski space-time and the 4D chirality matrix, respectively. For the 6D part, α_i, β_j denote 4×4 matrices satisfying $\{\alpha_i, \alpha_j\} = -2\delta_{ij}$, $[\alpha_i, \alpha_j] = -2\epsilon_{ijk}\alpha_k$, and $[\alpha_i, \beta_j] = 0$.[1] We have adopted the chiral basis where the 10D chirality matrix is diagonal,

$$\gamma_{\text{five}}^{(10)} = \mathbf{1}_4 \otimes \begin{pmatrix} \mathbf{1}_4 & 0 \\ 0 & -\mathbf{1}_4 \end{pmatrix}. \tag{4.2}$$

We can define a charge conjugation matrix $C^{(10)}$ satisfying $C^{(10)\dagger}\gamma_M^{(10)}C^{(10)} = -(\gamma_M^{(10)})^{\mathrm{T}}$ as

$$C^{(10)} = C \otimes \begin{pmatrix} 0 & \mathbf{1}_4 \\ \mathbf{1}_4 & 0 \end{pmatrix}, \tag{4.3}$$

where $C := i\gamma_0\gamma_2$ is the usual charge conjugation matrix in 4D theories.

Thus, a natural definition of charge conjugation in the 10D theory is

$$C^{(10)}: \ \Psi \to \Psi^C := C^{(10)}\bar{\Psi}^{\mathrm{T}} = (C\gamma_0 \otimes \mathbf{1}_8)\Psi^*. \tag{4.4}$$

We notice that 4D charge conjugation is embedded into the 10D charge conjugation. This is because the action of $C^{(10)}$ is trivial $\mathbf{1}_8$ for the extra-dimensional part whereas the ordinary 4D charge conjugation acts on the $\mathbb{R}^{1,3}$ part.

On the other hand, one must be more careful when defining a parity transformation in 10D field theories. A naive generalization, $\Psi \to \gamma_0^{(10)}\Psi$ does not work. This is because the 10D chirality of the spinor is flipped, making the theory incompatible with the Majorana-Weyl condition.[2] A consistent parity transformation is defined by

$$P^{(10)} : \Psi \to (\gamma_0 \otimes \mathbf{1}_8)\Psi =: \Gamma_0\Psi, \tag{4.5}$$

where the usual 4D chirality operator is embedded consistently. Since $[\Gamma_0, \gamma_{\text{five}}^{(10)}] = 0$, the 10D chirality of Ψ is preserved under $P^{(10)}$. To understand the action of $P^{(10)}$ on the space-time coordinates, let us study the transformation property of the following 10D vector,

$$X_M := \bar{\Psi}\gamma_M^{(10)}\Psi. \tag{4.6}$$

[1] Our basis are also along Ref. [132].

[2] Note that $\gamma_0^{(10)}$ satisfies $\{\gamma_{\text{five}}^{(10)}, \gamma_0^{(10)}\} = 0$.

One can straightforwardly check

$$P^{(10)} : X_M \to \begin{cases} X_M & \text{for } M = 4, 5, 6, \\ -X_M & \text{for } M = 7, 8, 9, \end{cases} \tag{4.7}$$

and 4D space-time directions are transformed as usual. Results in eq. (4.7) show that the 6D extra-space complex coordinates, $Z_a := X_{6+a} + iX_{3+a}$, $(a = 1, 2, 3)$ are complex conjugated

$$P^{(10)} : Z_a \to -Z_a^*, \tag{4.8}$$

under the parity. We notice that $P^{(10)}$ is embedded into the 10D proper Lorentz transformation. This is because the orientation of the 10D space-time is preserved since $P^{(10)}$ simultaneously reverses the orientations of both 4D space-time and 6D compact space. Note that there is arbitrariness in the 10D parity up to $SO(6)$ rotation in the compact dimension. In the following discussion, we adopt (4.8) for the parity transformation.

4.1.2 4D CP as a discrete gauge symmetry

Here, we review the idea that 4D CP is a discrete gauge symmetry and therefore cannot be broken explicitly both perturbatively and non-perturbatively [86, 87].

Discrete gauge symmetry is the discrete symmetry that emerges when a continuous local symmetry is spontaneously broken. For instance, a theory with $SO(3)$ gauge symmetry spontaneously breaks down to $O(2)$ when an adjoint field acquires a VEV proportional to $\mathrm{diag}(1, 1, -2)$. This $O(2)$ group contains a local continuous symmetry $SO(2)$, and a discrete part

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{4.9}$$

which reverses the $SO(2)$. This A corresponds to the discrete gauge symmetry, embedded in $SO(3)$.

Now, we explain how 4D CP can be identified as a discrete gauge symmetry. Firstly, to embed charge conjugation into a gauge symmetry G, all 4D fields need to be in the same irreducible representations as their complex conjugated ones. If an inner automorphism of G exchanges each element of the representation with its complex conjugated one, that is a realization of charge conjugation as a discrete gauge symmetry. It is known that such an embedding is possible only when the gauge symmetry G is E_8, E_7, $SO(2n+1)$, $SO(4n)$, $Sp(2n)$, G_2, F_4, or their product since only these Lie groups admit inner automorphism which flips the sign of all Cartan subalgebra [133]. We note that these Lie groups appear in superstring theories. Secondly, parity transformation $P^{(10)}$ in eq. (4.8) is also a discrete gauge transformation. This is because $P^{(10)}$ is embedded into the 10D proper Lorentz transformation and local coordinate transformation with positive Jacobian. Since both charge conjugation and parity are discrete gauge transformations, CP is as well. Thus, CP should be violated spontaneously, not explicitly.

4.2 CP in 4D modular flavor symmetric models

We briefly review CP symmetry in 4D modular flavor models along Ref. [72]. For concreteness, we focus on the genus $g = 1$ case, namely $\bar{\Gamma} = PSL(2, \mathbb{Z})$ symmetric model. For a consistent implementation of CP transformation with modular symmetry, there must exist $\gamma' \in \bar{\Gamma}$ for $\forall \gamma \in \bar{\Gamma}$, such that

$$CP^{-1} \circ \gamma \circ CP = \gamma' := u(\gamma), \tag{4.10}$$

is satisfied. Here, $CP^{-1} \circ \gamma \circ CP$ denotes the chain of transformations, $CP \to \gamma \to CP^{-1}$. Using the consistency condition (4.10), it was shown that CP acts on the complex structure modulus τ as

$$\tau \xrightarrow{\text{CP}} -\bar{\tau}, \tag{4.11}$$

within 4D modular flavor models [72].[3] (See also Refs. [134, 135].) Note that this is consistent with the earlier result (4.8). One can explicitly check that the action of CP on τ (4.11) is an outer automorphism of $\bar{\Gamma}$ as follows. Recalling the action of $\gamma \in \bar{\Gamma}$ on τ:

$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{\Gamma}, \tag{4.12}$$

we find

$$\tau \xrightarrow{CP} -\bar{\tau} \xrightarrow{\gamma} \frac{-a\bar{\tau} + b}{-c\bar{\tau} + d} \xrightarrow{CP^{-1}} \frac{a\tau - b}{-c\tau + d}. \tag{4.13}$$

Hence, the map $u(\gamma)$ corresponds to

$$u(\gamma) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \in \bar{\Gamma}. \tag{4.14}$$

This is homomorphic because for $\gamma_1, \gamma_2 \in \bar{\Gamma}$ we can verify $u(\gamma_2\gamma_1) = u(\gamma_2)u(\gamma_1)$. It is also bijective since $u(\gamma_1) = u(\gamma_2)$ is satisfied if and only if $\gamma_1 = \gamma_2$. Hence, $u(\gamma)$ is an automorphism. We can also check that there are no elements $h \in \bar{\Gamma}$ such that $u(\gamma) = h^{-1}\gamma h$, hence CP is indeed an outer automorphism.

Consequently, modular group $\bar{\Gamma}$ is generalized to extended modular group $\bar{\Gamma}^*$:

$$\bar{\Gamma}^* = \left\langle \tau \xrightarrow{S} -1/\tau,\ \tau \xrightarrow{T} \tau + 1,\ \tau \xrightarrow{\text{CP}} -\bar{\tau} \right\rangle, \tag{4.15}$$

which is mathematically understood as a semi-direct product $\bar{\Gamma}^* \simeq \bar{\Gamma} \rtimes \mathbb{Z}_2^{\text{CP}}$ [72, 134–137].

CP-conserving values of τ

In modular flavor symmetric models, CP is conserved when the VEV of τ is invariant under CP up to a modular transformation, i.e.,

$$-\bar{\tau} = \gamma\tau, \tag{4.16}$$

[3] $\bar{\tau}$ denotes the complex conjugate of τ.

for some $\gamma \in \bar{\Gamma}$ [72][4]. One can find all CP-conserving points as follows. We note that it is sufficient to consider VEVs in the fundamental domain $\mathcal{D}$ of $\bar{\Gamma}$ which is roughly given by

$$\mathcal{D} = \left\{ \tau \in \mathbb{C} : \ \mathrm{Im}(\tau) > 0, \ |\mathrm{Re}(\tau)| \leq \frac{1}{2}, \ |\tau| \geq 1 \right\}. \tag{4.17}$$

This is because points outside $\mathcal{D}$ are physically equivalent to those inside of $\mathcal{D}$ as they are related by modular transformation. When the real part of τ vanishes, i.e., $\mathrm{Re}(\tau) = 0$, CP is conserved because eq. (4.16) is satisfied for $\gamma = \mathbf{1}$. Next, let us focus on the case $\gamma \neq \mathbf{1}$. We notice that τ only on the boundary of $\mathcal{D}$ has the possibility of satisfying (4.16) for $\gamma \neq \mathbf{1}$. This is because if we act a modular transformation $\gamma \neq \mathbf{1}$ on τ in the interior of $\mathcal{D}$, it is mapped to the exterior of $\mathcal{D}$. We can check that the boundary of $\mathcal{D}$ is CP-conserving as follows. First, points on the arc $|\tau| = 1$ satisfy (4.16) when $\gamma = S$, cf. (2.73). Secondly, points on the line $\mathrm{Re}(\tau) = 1/2$ satisfy (4.16) when $\gamma = T$. Finally, for $\mathrm{Re}(\tau) = -1/2$, we choose $\gamma = T^{-1}$. To summarize, CP-conserving values of τ within $\mathcal{D}$ are the imaginary axis and the boundaries of the fundamental domain $\mathcal{D}$.

4.3 Properties of CP, modular invariant scalar potentials

This section studies some general properties of CP and/or modular invariant scalar potential V. We assume that V is a function of single complex structure modulus τ, ($\mathrm{Im}(\tau) > 0$) of a toroidal space. We will see that CP-conserving vacua are typically obtained when V is symmetric under the extended modular group. This implies that realizing spontaneous CP violation through moduli stabilization is a non-trivial task.

4.3.1 CP invariance

Here, we study the general property of a CP invariant scalar potential V. CP invariance implies

$$V(-s, t) = V(s, t), \tag{4.18}$$

where we have defined $s, t \in \mathbb{R}$ as

$$\tau = s + it. \tag{4.19}$$

From eq. (4.18), one obtains

$$\frac{\partial V}{\partial s}(s = 0, t) = 0. \tag{4.20}$$

Hence, one finds a local minimum or maximum of V on the CP-invariant line $\mathrm{Re}(\tau) = 0$.

[4] Extensions to CP in Siegel modular flavor models are discussed in [73].

4.3.2 CP and $\bar{\Gamma}$ invariance

Here, we study the general properties of a scalar potential that is invariant under the extended modular group, $\bar{\Gamma}^* \simeq \bar{\Gamma} \rtimes \mathbb{Z}_2^{\mathrm{CP}}$. We will see that on the boundaries of the fundamental domain $\mathcal{D}$, the first directional derivatives of V normal to the boundaries vanish. This implies that one typically finds a local minimum of V on the CP-conserving curves, i.e., boundaries of $\mathcal{D}$ and/or the imaginary axis $\mathrm{Re}(\tau) = 0$.

CP and T invariance

Owing to the CP and T invariance of V, we have

$$V(s,t) = V(s+1,t) = V(-s-1,t). \tag{4.21}$$

Hence, its first derivative satisfies

$$\frac{\partial V}{\partial s}(s,t) = -\frac{\partial V}{\partial s}(s',t), \tag{4.22}$$

where we have defined $s' := -s-1$. At $s = -1/2$, we find

$$\frac{\partial V}{\partial s}(s = -1/2, t) = 0. \tag{4.23}$$

Since V is invariant under the shift $s \to s+1$, we obtain

$$\frac{\partial V}{\partial s}(s,t) = 0, \quad s \equiv 0 \pmod{1/2}. \tag{4.24}$$

CP and S invariance

Owing to the CP and S invariance of V, we have

$$V(r,\theta) = V(1/r, -\theta+\pi) = V(1/r, \theta), \tag{4.25}$$

where $\tau =: re^{i\theta}$, $(r > 0,\ 0 < \theta < \pi)$. Hence, its first derivative satisfies

$$\frac{\partial V}{\partial r}(r,\theta) = -\frac{1}{r^2}\frac{\partial V}{\partial u}(u,\theta), \tag{4.26}$$

where $u := 1/r$. At $r = 1$, we find

$$\frac{\partial V}{\partial r}(1,\theta) = 0. \tag{4.27}$$

This shows that the first derivative of V along the radial direction r vanishes on the CP-conserving arc $|\tau| = 1$. Furthermore, if we recall the periodicity of V under the shift $\tau \to \tau+1$, one can generalize the result as

$$\frac{\partial V}{\partial r_n}(r_n = 1, \theta_n) = 0, \tag{4.28}$$

where $|\tau - n| =: r_n e^{i\theta_n}$, $(r_n > 0,\ 0 < \theta_n < \pi, n \in \mathbb{Z})$.

4.3.3 Comments

Our analyses show that the local minima of the scalar potential tend to stabilize on the imaginary axis $\mathrm{Re}(\tau) = 0$ and/or the boundaries of the fundamental domain $\mathcal{D}$ where CP is preserved. This means that the realization of spontaneous CP violation is not so straightforward. For instance, CP invariant Type IIB three-form flux compactification is studied in Ref. [88], and the corresponding CP symmetric scalar potential has been analyzed. The obtained VEVs of complex structure moduli are often CP-conserving as is implied by eq. (4.20). In other cases, flat directions are generated where CP-conserving and CP-breaking vacua are degenerate, still spontaneous CP violation is not realized. There are various studies of moduli stabilization in the presence of modular symmetries [78, 79, 82–84, 90, 91, 138]. Although moduli VEVs leading to the hierarchical flavor structures of quarks and leptons are obtained, spontaneous CP violation is hardly realized as is implied by eqs. (4.24) and (4.27). An exceptional case is shown in [82] where spontaneous CP violation is generated by engineering the modular invariant function constituting the scalar potential. In any case, it is intriguing to build a model that can spontaneously realize a sizable CP violation.

4.4 Type IIB flux compactification

In this section, we review CP symmetric flux compactification [88, 89]. Here and hereafter, we adopt the reduced Planck mass unit $M_{\mathrm{Pl}} = 1$.

4.4.1 Flux compactification with $T^6/\mathbb{Z}_2$ toroidal orientifold

We assume a simple set-up, Type IIB flux compactification on the factorizable $T^6/\mathbb{Z}_2 = (T_1^2 \times T_2^2 \times T_2^3)/\mathbb{Z}_2$ orientifold [139]. As its low-energy effective field theory, we study the 4D $\mathcal{N} = 1$ supergravity (SUGRA) action. We consider the following Kähler potential K_{moduli}:

$$K_{\mathrm{moduli}} = -\ln\left[-i(S - \bar{S})\right] - \ln\left[i(\tau_1 - \bar{\tau}_1)(\tau_2 - \bar{\tau}_2)(\tau_3 - \bar{\tau}_3)\right] - 2\ln\mathcal{V}, \tag{4.29}$$

where S, τ_i, $(i = 1, 2, 3)$, and $\mathcal{V}$ denote axio-dilaton, complex structure moduli, and volume (Kähler) modulus, respectively. When the background three-form flux G_3 is introduced over the compact space, the following superpotential is induced:

$$W_{\mathrm{flux}} = \frac{1}{l_s^2}\int G_3 \wedge \Omega, \tag{4.30}$$

known as the Gukov-Vafa-Witten superpotential [75]. Note that $l_s = 2\pi\sqrt{\alpha'}$ denotes the string length. The holomorphic three-form is given by

$$\Omega = dz_1 \wedge dz_2 \wedge dz_3, \tag{4.31}$$

where z_i, $(i=1,2,3)$ denote the complex coordinates on $T^6/\mathbb{Z}_2$. Note that moduli dependence comes into $W_{\rm flux}$ as $dz_i = dx^i + \tau_i dy^i, (i=1,2,3)$ where x^i, y^i denote the real coordinates. The three-form flux G_3 is written as

$$G_3 = F_3 - SH_3, \tag{4.32}$$

where F_3 and H_3 denote the three-form field strength in the Ramond-Ramond sector (R-R) and the Neveu-Schwarz sector (NS-NS), respectively. They can be expanded in terms of the basis α_I, β^J in $H^3(T^6,\mathbb{Z})$ satisfying $\int_{T^6} \alpha_I \wedge \beta^J = \delta_I^J$ as

$$\begin{aligned}\frac{1}{l_s^2}F_3 &= a^0\alpha_0 + a^i\alpha_i + b_i\beta^i + b_0\beta^0,\\ \frac{1}{l_s^2}H_3 &= c^0\alpha_0 + c^i\alpha_i + d_i\beta^i + d_0\beta^0,\end{aligned} \tag{4.33}$$

where $a^0, a^i, b_0, b_i, c^0, c^i, d_0, d_i$ are integer-valued flux quanta owing to the quantization condition:

$$\frac{1}{l_s^2}\int_{\Sigma_3} F_3 \in \mathbb{Z}, \quad \frac{1}{l_s^2}\int_{\Sigma_3} H_3 \in \mathbb{Z}, \tag{4.34}$$

for any 3-cycle Σ_3. The flux quanta take even and/or odd integers depending on the existence of exotic $O3'$-planes which may be introduced when discrete RR and/or NS are turned on [140,141]. The basis α_I, β^J are explicitly given by

$$\begin{aligned}\alpha_0 &= dx^1\wedge dx^2\wedge dx^3, & \alpha_1 &= dy^1\wedge dx^2\wedge dx^3,\\ \alpha_2 &= dy^2\wedge dx^3\wedge dx^1, & \alpha_3 &= dy^3\wedge dx^1\wedge dx^2,\\ \beta_0 &= dy^1\wedge dy^2\wedge dy^3, & \beta_1 &= -dx^1\wedge dy^2\wedge dy^3,\\ \beta_2 &= -dx^2\wedge dy^3\wedge dy^1, & \beta_3 &= -dx^3\wedge dy^1\wedge dy^2,\end{aligned} \tag{4.35}$$

where x^i, y^i correspond to real coordinates, i.e., $z_i = x^i + \tau_i y^i,\ (i=1,2,3)$.

The tadpole cancellation condition constrains the values of flux quanta. Firstly, we note that the background flux induces the $D3$ brane charge given by

$$n_{\rm flux} = \frac{1}{l_s^2}\int_{T^6} H_3\wedge F_3 = c^0 b_0 - d_0 a^0 + \sum_i (c^i b_i - d_i a^i). \tag{4.36}$$

The tadpole cancellation condition requires

$$\frac{1}{2}n_{\rm flux} + Q_{\rm local} = 0, \tag{4.37}$$

where $Q_{\rm local}$ denotes the sum of $D3$-brane charge contributions from the local sources, i.e., $D3$-branes and $O3$-planes. Each $D3$-brane contributes $+1$ charge and each $O3$-plane contributes $-1/4$. There are 64 $O3$-planes on $T^6/\mathbb{Z}_6$ orientifold, hence they contribute -16 to $Q_{\rm local}$. Since each exotic $O3'$-plane has a charge contribution $+1/4$. Thus, $Q_{\rm local}$ is given by

$$Q_{\rm local} = n_{\rm D3} - 16 + \frac{n_{\rm O3'}}{4}, \tag{4.38}$$

where n_{D3} and $n_{\mathrm{O3'}}$ denote the number of $D3$- and exotic $O3'$-planes, respectively. Here, we do not consider anti-$D3$-branes to preserve SUSY in the system. As a result, we obtain a constraint on flux quanta,

$$n_{\mathrm{flux}} = c^0 b_0 - d_0 a^0 + \sum_i (c^i b_i - d_i a^i) = 32 - 2n_{\mathrm{D3}} - n_{\mathrm{O3'}}/2 \leq 32. \tag{4.39}$$

Furthermore, n_{flux} is non-negative $n_{\mathrm{flux}} \geq 0$. This follows from the imaginary self-duality condition of the background three-form flux, i.e., $\star_6 G_3 = iG_3$, where $\star_6$ is the Hodge dual in the 6D extra dimension.

4.4.2 CP invariant flux compactifications

We construct a CP-invariant flux-induced potential following Refs. [88,89]. For this purpose, we first summarize the transformation rules of various quantities under the CP. We recall from (4.8) that CP transformation acts on the complex coordinates of $T^6/\mathbb{Z}_2$ as $z_i \to -\bar{z}_i$, $(i = 1, 2, 3)$. Hence, the holomorphic three-form is transformed as

$$\Omega \xrightarrow{\mathrm{CP}} -\bar{\Omega}. \tag{4.40}$$

Similarly, the complex structure moduli and axio-dilaton are also transformed as

$$\tau_i \xrightarrow{\mathrm{CP}} -\bar{\tau}_i, \quad S \xrightarrow{\mathrm{CP}} -\bar{S}, \tag{4.41}$$

under the CP. The inversion $z_i \to -\bar{z}_i$ corresponds to $x_i \to -x_i$ and $y_i \to y_i$ in real coordinates. Thus, the basis of $H^3(T^6, \mathbb{Z})$ are transformed as

$$\alpha_0 \xrightarrow{\mathrm{CP}} -\alpha_0, \quad \beta^0 \xrightarrow{\mathrm{CP}} \beta^0, \quad \alpha_i \xrightarrow{\mathrm{CP}} \alpha_i, \quad \beta^i \xrightarrow{\mathrm{CP}} -\beta^i. \tag{4.42}$$

To ensure CP invariance in our SUGRA model, the Kähler invariant function $G_{\mathrm{flux}} := K_{\mathrm{moduli}} + \ln |W_{\mathrm{flux}}|^2$ needs to be CP invariant. We notice that Kähler potential eq. (4.29) is manifestly CP invariant. On the other hand, superpotential (4.30) is required to behave as

$$W_{\mathrm{flux}} \xrightarrow{\mathrm{CP}} e^{i\gamma} \bar{W}_{\mathrm{flux}}, \tag{4.43}$$

where $\gamma \in \mathbb{R}$. In fact, γ is restricted $\gamma \equiv 0$ or $\pi \pmod{2\pi}$ as follows. Under the CP, orientation of the orientifold cover T^6 is inverted as $T^6 \xrightarrow{\mathrm{CP}} T^6_{(\mathrm{CP})}$. This is manifest in the sign difference between $\int_{T^6_{(\mathrm{CP})}} \alpha_I \wedge \beta^J = -\delta_I^J$ and $\int_{T^6} \alpha_I \wedge \beta^J = \delta_I^J$. By taking the sign flip into account, eqs. (4.40) and (4.43) lead to

$$G_3 \xrightarrow{\mathrm{CP}} e^{i\gamma} \bar{G}_3. \tag{4.44}$$

Eq. (4.44) is equivalent to demand

$$\begin{aligned} F_3 &\xrightarrow{\mathrm{CP}} e^{i\gamma} F_3, \\ H_3 &\xrightarrow{\mathrm{CP}} -e^{i\gamma} H_3. \end{aligned} \tag{4.45}$$

Since F_3 and H_3 are real-valued three-form flux, we obtain $\gamma \equiv 0, \text{ or } \pi \pmod{2\pi}$.

If we choose $\gamma \equiv \pi \pmod{2\pi}$, one finds that eq. (4.45) restricts the three-form flux to the following form:

$$\frac{1}{l_s^2} F_3 = a^0 \alpha_0 + b_i \beta^i, \tag{4.46}$$

$$\frac{1}{l_s^2} H_3 = c^i \alpha_i + d_0 \beta^0, \tag{4.47}$$

by noting eq. (4.42). Then, substitution of eqs. (4.46) and (4.47) into (4.30) gives us

$$W_{\text{flux}} = a^0 \tau_1 \tau_2 \tau_3 + c^1 S \tau_2 \tau_3 + c^2 S \tau_1 \tau_3 + c^3 S \tau_1 \tau_2 - \sum_{i=1}^{3} b_i \tau_i + d_0 S. \tag{4.48}$$

Since the superpotential (4.48) is an odd polynomial of moduli fields, we can directly confirm the CP transformation, $W_{\text{flux}} \xrightarrow{\text{CP}} -\bar{W}_{\text{flux}}$. In this paper, we understand that all of the flux quanta a^0, b_i, c^i, d_0 should be invariant under CP otherwise they explicitly break CP symmetry as spurions. As an example, let us explicitly check the CP invariance of the flux quantum $a^0 = \int_{T^6} F_3 \wedge \beta_0$. By acting the CP transformation, we find $\int_{T^6_{(\text{CP})}} (-F_3) \wedge \beta_0 = -a^0 \int_{T^6_{(\text{CP})}} \alpha^0 \wedge \beta_0 = a^0$ as we have expected.

On the other hand, if we choose $\gamma \equiv 0 \pmod{2\pi}$, three-form fluxes are restricted to

$$\frac{1}{l_s^2} F_3 = a^i \alpha_i + b_0 \beta^0, \tag{4.49}$$

$$\frac{1}{l_s^2} H_3 = c^0 \alpha_0 + d_i \beta^i. \tag{4.50}$$

As a result, we obtain

$$W_{\text{flux}} = -c^0 S \tau_1 \tau_2 \tau_3 - a^1 \tau_2 \tau_3 - a^2 \tau_1 \tau_3 - a^3 \tau_1 \tau_2 + \sum_{i=1}^{3} d_i S \tau_i - b_0. \tag{4.51}$$

Since the superpotential (4.51) is an even polynomial of moduli fields, we can directly confirm the CP transformation, $W_{\text{flux}} \xrightarrow{\text{CP}} \bar{W}_{\text{flux}}$. We note that flux quanta in eq. (4.51) are also CP invariant.

Flat directions

Potential minimum corresponding to the CP invariant superpotentials in eqs. (4.48) and (4.51) have been analyzed in Ref. [88]. The results show that flat directions can generally appear where CP-conserving and violating vacua are degenerate. Here, we show an illustrating example of eq. (4.48) by further restricting the flux quanta to

$$c^1 = c^2 = 0, \quad b_1 = b_2 = 0, \quad c^3 = -f a^0, \quad d_0 = f b_3, \tag{4.52}$$

following Refs. [142, 143]. We note that f is a rational number. As a result, the CP preserving superpotential eq. (4.48) is simplified to

$$W_{\text{flux}} = (\tau_3 - fS)\left[a^0\tau_1\tau_2 - b_3\right]. \tag{4.53}$$

The equations for SUSY Minkowski vacua are given by

$$\partial_{\tau_i} W_{\text{flux}} = 0, \quad \partial_S W_{\text{flux}} = 0, \quad W_{\text{flux}} = 0. \tag{4.54}$$

One can easily find the solution,

$$\tau_1\tau_2 = \frac{b_3}{a^0}, \quad \tau_3 = fS, \tag{4.55}$$

where CP conserving and violating vacua are degenerate. Hence, spontaneous CP violation is not realized in this example. Note that the D3-brane charge induced by the three-from fluxes is given by,

$$n_{\text{flux}} = -2fa^0b_3. \tag{4.56}$$

Hence, the flux quanta are constrained by the tadpole cancellation condition (4.39) and the imaginary self-duality condition as

$$0 \leq -fa^0b_3 \leq 16. \tag{4.57}$$

In addition, we have

$$\text{sgn}(f) = +1, \tag{4.58}$$

from (4.55) since we require $\text{Im}\tau_i > 0$ and $\text{Im}S > 0$ in our convention. Then, we automatically have $\text{sgn}(a^0b_3) = -1$.

Moduli masses

Let us evaluate the moduli masses at the supersymmetric vacuum eq. (4.55) to clarify the appearance of massive and massless flat directions. The squared masses can be evaluated by computing the second derivatives of the following SUGRA scalar potential:

$$V = e^{K_{\text{moduli}}} K^{A\bar{B}} D_A W_{\text{flux}} \overline{D_B W_{\text{flux}}}, \tag{4.59}$$

where $A, B \in \{\tau_i, S\}$. Here, $K^{A\bar{B}}$ denotes the inverse of the Kähler metric $K_{A\bar{B}} := \partial_A \partial_{\bar{B}} K_{\text{moduli}}$, i.e., $K^{A\bar{B}}K_{C\bar{B}} = \delta^A_C$. The covariant derivative of the superpotential is defined as

$$D_A W_{\text{flux}} = \partial_A W_{\text{flux}} + (\partial_A K_{\text{moduli}}) W_{\text{flux}}. \tag{4.60}$$

Since we have assumed the no-scale type Kähler potential $-2\ln\mathcal{V}$, the supergravity effect $-3|W_{\text{flux}}|^2$ is canceled. At the SUSY Minkowski vacuum, i.e., $\partial_A W_{\text{flux}} = W_{\text{flux}} = 0$, Hessian matrix of the scalar potential is given by

$$H = \begin{pmatrix} \partial_A \bar{\partial}_{\bar{B}} V & \partial_A \partial_B V \\ \bar{\partial}_{\bar{A}} \bar{\partial}_{\bar{B}} V & \bar{\partial}_{\bar{A}} \partial_B V \end{pmatrix} = \langle e^{K_{\text{moduli}}} \rangle \begin{pmatrix} m_{AC} K^{C\bar{D}} \bar{m}_{\bar{D}\bar{B}} & 0 \\ 0 & \bar{m}_{\bar{A}\bar{C}} K^{\bar{C}D} m_{DB} \end{pmatrix}, \tag{4.61}$$

where the supersymmetric mass m_{AB} is defined as

$$m_{AB} = \partial_A \partial_B W_{\text{flux}}. \tag{4.62}$$

When the superpotential is chosen as (4.53), non-vanishing components of the supersymmetric mass are explicitly given by

$$\begin{aligned} m_{\tau_1\tau_3} &= m_{\tau_3\tau_1} = a^0\langle\tau_2\rangle, \\ m_{\tau_1 S} &= m_{S\tau_1} = -fa^0\langle\tau_2\rangle, \\ m_{\tau_2\tau_3} &= m_{\tau_3\tau_2} = a^0\langle\tau_1\rangle, \\ m_{\tau_2 S} &= m_{S\tau_2} = -fa^0\langle\tau_1\rangle, \end{aligned} \tag{4.63}$$

where $\langle\tau_1\rangle$ and $\langle\tau_2\rangle$ denote possible VEVs of τ_1 and τ_2 satisfying (4.55). Then, the top-left 4×4 block of the Hessian matrix (4.61) can be written as

$$\begin{pmatrix} H_{\tau_1\bar\tau_1} & H_{\tau_1\bar\tau_2} & H_{\tau_1\bar\tau_3} & H_{\tau_1\bar S} \\ H_{\tau_2\bar\tau_1} & H_{\tau_2\bar\tau_2} & H_{\tau_2\bar\tau_3} & H_{\tau_2\bar S} \\ H_{\tau_3\bar\tau_1} & H_{\tau_3\bar\tau_2} & H_{\tau_3\bar\tau_3} & H_{\tau_3\bar S} \\ H_{S\bar\tau_1} & H_{S\bar\tau_2} & H_{S\bar\tau_3} & H_{S\bar S} \end{pmatrix} = \langle e^{K_{\text{moduli}}}\rangle \begin{pmatrix} |\langle\tau_2\rangle|^2\zeta & \langle\bar\tau_1\rangle\langle\tau_2\rangle\zeta & 0 & 0 \\ \langle\tau_1\rangle\langle\bar\tau_2\rangle\zeta & |\langle\tau_1\rangle|^2\zeta & 0 & 0 \\ 0 & 0 & (a^0)^2\eta & -f(a^0)^2\eta \\ 0 & 0 & -f(a^0)^2\eta & (fa^0)^2\eta \end{pmatrix}, \tag{4.64}$$

where

$$\begin{aligned} \zeta &= (a^0)^2[2\mathrm{Im}\langle\tau_3\rangle]^2 + (fa^0)^2[2\mathrm{Im}\langle S\rangle]^2, \\ \eta &= [2\mathrm{Im}\langle\tau_1\rangle]^2|\langle\tau_2\rangle|^2 + [2\mathrm{Im}\langle\tau_2\rangle]^2|\langle\tau_1\rangle|^2. \end{aligned} \tag{4.65}$$

We notice that the bottom-right 4×4 block of the Hessian matrix (4.61) is complex conjugate to (4.64). Diagonalization of (4.64) yields

$$\langle e^{K_{\text{moduli}}}\rangle \begin{pmatrix} (|\langle\tau_1\rangle|^2 + |\langle\tau_2\rangle|^2)\zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & [(a^0)^2 + (fa^0)^2]\eta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.66}$$

This shows that only half of the moduli fields become massive. Zero eigenvalues in (4.66) correspond to the massless flat directions.

4.5 Modular invariant matter contribution

In order to resolve the degeneracy of CP-conserving and violating vacua in flux compactifications, we introduce additional contributions. As one of the possibilities, we consider couplings between matter superfield X and a complex structure modulus τ in such a way as to respect the extended modular symmetry $\bar\Gamma^*$. For the sake of simplicity, we will first focus only on the matter contribution with a single complex structure modulus τ within 4D $\mathcal{N}=1$ SUGRA model following Refs. [84, 99]. For clear illustration, A_4 modular invariant models will be presented. We will see that spontaneous CP violation can hardly be realized solely by the matter contributions due to the control of the modular symmetry as reviewed in section 4.3. We will extend our model to multi-moduli cases in the next section where we study the combined effects of three-form flux and matter on moduli stabilization.

4D $\mathcal{N}=1$ SUGRA Lagrangian

Here, we present 4D $\mathcal{N}=1$ SUGRA Lagrangian corresponding to the matter contributions. We consider the following Kähler potential:

$$K = -\ln[-i(\tau-\bar{\tau})] + Z|X|^2 - 2\ln\mathcal{V}, \quad Z = (-i\tau - i\bar{\tau})^k, \tag{4.67}$$

where X denotes a matter chiral superfield and $\mathcal{V}$ is the volume of a compact space. We assume that it belongs to a trivial singlet representation $\mathbf{1}$ with modular weight $k \in \mathbb{Z}$ under $\bar{\Gamma}$,

$$X \to (c\tau + d)^k X, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{\Gamma}. \tag{4.68}$$

We consider the following superpotential

$$W_{\text{matter}} = \Lambda^2 Y(\tau) X, \tag{4.69}$$

where $Y(\tau)$ denotes a trivial singlet $\mathbf{1}$ modular form with modular weight $k_Y \in \mathbb{Z}$,

$$Y(\tau) \to (c\tau + d)^{k_Y} Y(\tau), \quad \gamma \in \bar{\Gamma}. \tag{4.70}$$

In eq. (4.69), Λ denotes a constant mass parameter characterizing the coupling between the modulus and the matter X [84, 99].[5] We note that for the $\bar{\Gamma}$ invariance of the Kähler invariant function $G_{\text{matter}} := e^K |W_{\text{matter}}|^2$, superpotential W_{matter} is required to carry modular weight -1. Hence, the following relation on modular weights needs to be satisfied

$$-1 = k + k_Y. \tag{4.71}$$

Next, let us comment on the CP invariance. As studied in Ref. [72], modulus, matter field, and modular form within modular flavor models transform under CP as

$$\tau \xrightarrow{\text{CP}} -\bar{\tau}, \quad X \xrightarrow{\text{CP}} \bar{X}, \quad Y(\tau) \xrightarrow{\text{CP}} e^{i\phi} \bar{Y}(\bar{\tau}), \tag{4.72}$$

where $\phi \in \mathbb{R}$ is a phase factor that depends on the normalization of the modular form $Y(\tau)$. Then, the superpotential behaves as $W_{\text{matter}} \xrightarrow{\text{CP}} e^{i\phi} \bar{W}_{\text{matter}}$ ensuring the CP invariance of the Kähler invariant function G_{matter}. To summarize, the matter contribution we have introduced respects extended modular invariance $\bar{\Gamma}^*$ if (4.71) is satisfied.

[5] Our superpotential W_{matter} can be generated as follows. Suppose a superpotential of the form $Y(\tau) Q\bar{Q} X$ where Q and $\bar{Q}$ are hidden-sector matter fields. Then, we have assumed the condensation $\langle Q\bar{Q} \rangle = \Lambda^2$.

Supersymmetric vacua

The equations for SUSY Minkowski vacua are given by

$$\partial_\tau W_{\text{matter}} = \partial_X W_{\text{matter}} = W_{\text{matter}} = 0. \tag{4.73}$$

One can straightforwardly find the solution:

$$\langle X \rangle = 0, \quad \langle Y(\tau) \rangle = 0, \tag{4.74}$$

when $\langle \partial_\tau Y \rangle \neq 0$. The complex structure modulus is stabilized at a zero-point of $Y(\tau)$.

SUGRA scalar potential

Let us compute the SUGRA scalar potential under the choice of no scale type Kähler potential eq. (4.67) and the superpotential eq. (4.69) to confirm the stability and masses of fields at (4.74). It is given by

$$V = e^K K^{I\bar{J}} D_I W_{\text{matter}} \overline{D_J W_{\text{matter}}}, \tag{4.75}$$

where the indices are $I, J \in \{\tau,\ X\}$. In the regime where $|X| \ll 1$, we can Taylor expand V in terms of X and $\bar{X}$ as:

$$\begin{aligned} V = &\frac{\Lambda^4}{\mathcal{V}^2}[2\mathrm{Im}(\tau)]^{k_Y}|Y(\tau)|^2 \\ &+ \frac{\Lambda^4}{\mathcal{V}^2}[2\mathrm{Im}(\tau)]^{-1}\left[3|Y(\tau)|^2 + |k_Y Y(\tau) + 2i\mathrm{Im}(\tau)\partial_\tau Y(\tau)|^2\right]|X|^2 + \mathcal{O}(|X|^4) \\ =:&\ V_0 + m_X^2|X|^2 + \mathcal{O}(|X|^4). \end{aligned} \tag{4.76}$$

We notice that both matter field X and modulus τ are massive at the SUSY preserving vacuum (4.74) only if $\langle \partial_\tau Y(\tau) \rangle \neq 0$. We also notice that the leading term V_0 and the mass term X are separately invariant under $\bar{\Gamma}^*$.

Examples with A_4 modular forms

As an illustrating example, we choose

$$Y(\tau) = Y_{\mathbf{1}}^{(4)}(\tau), \tag{4.77}$$

where $Y_{\mathbf{1}}^{(4)}(\tau)$ denotes a trivial singlet A_4 modular form carrying modular weight $k_Y = 4$. We show its definition in Appendix B.2. Fig. 4.1 shows a contour plot of $\log_{10}(V_0\mathcal{V}^2/\Lambda^4 + 1)$ where $V_0 := V|_{X=0}$. The red curves in the figure correspond to the CP conserving points. Consistent with the discussions in section 4.3, we observe that the equipotential lines are perpendicular to the CP-conserving curves. We find a minimum of the potential exactly at the $\mathbb{Z}_3$ fixed-point $\tau = \omega$. At $\tau = \omega$, one can check that the modular form is exactly zero and its first derivative is non-vanishing,

$$Y_{\mathbf{1}}^{(4)}(\omega) = 0, \quad \partial_\tau Y_{\mathbf{1}}^{(4)}\big|_{\tau=\omega} \simeq -6.04i. \tag{4.78}$$

Therefore, the matter field is stabilized at $\langle X\rangle = 0$ and we obtain a CP-conserving SUSY Minkowski vacuum.

As another illustration, let us choose

$$Y(\tau) = Y_{\mathbf{1}}^{(6)}(\tau), \tag{4.79}$$

where $Y_{\mathbf{1}}^{(6)}(\tau)$ denotes the trivial singlet A_4 modular form carrying modular weight $k_Y = 6$. We show its definition in Appendix B.2. Fig. 4.2 shows a similar contour plot of the scalar potential when $Y_{\mathbf{1}}^{(6)}(\tau)$ is chosen. We find a minimum of the potential exactly at the $\mathbb{Z}_2$ fixed-point $\tau = i$. At $\tau = i$, one can check that the modular form is exactly zero and its first derivative is non-vanishing,

$$Y_{\mathbf{1}}^{(6)}(i) = 0, \quad \partial_\tau Y_{\mathbf{1}}^{(6)}\big|_{\tau=i} \simeq -6.66i. \tag{4.80}$$

Therefore, the matter field is stabilized at $\langle X\rangle = 0$ and we obtain a CP-conserving SUSY Minkowski vacuum.

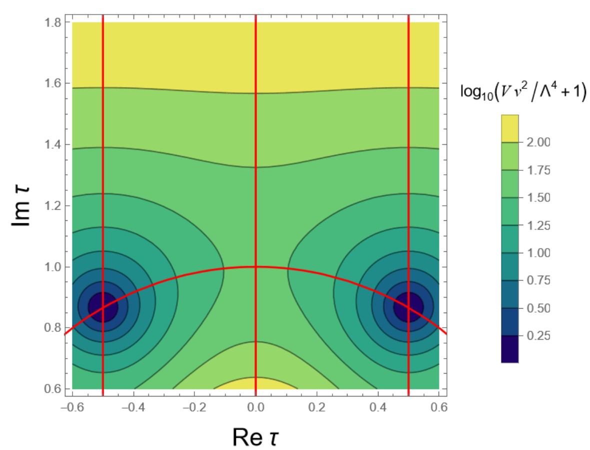

Figure 4.1: Contour plot of $\log_{10}\left(\frac{V_0\mathcal{V}^2}{\Lambda^4}+1\right)$ on τ-plane when A_4 trivial singlet modular form $Y(\tau) = Y_{\mathbf{1}}^{(4)}$ with weight $k_Y = 4$ is chosen. The red curves correspond to the CP-conserving region, i.e., $\mathrm{Re}(\tau) = 0, \pm 1/2$, and $|\tau| = 1$. The scalar potential $V_0 = V|_{X=0}$ has its minimum at $\tau = \omega$.

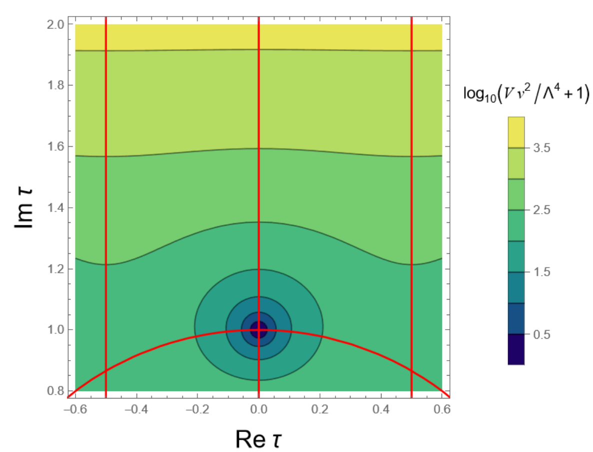

Figure 4.2: Contour plot of $\log_{10}\left(\frac{V_0\mathcal{V}^2}{\Lambda^4}+1\right)$ on τ-plane when A_4 trivial singlet modular form $Y(\tau) = Y_{\mathbf{1}}^{(6)}$ with weight $k_Y = 6$ is chosen. The definition of this figure is similar to that of Fig. 4.1. The scalar potential $V_0 = V|_{X=0}$ has its minimum at $\tau = i$

The results imply that spontaneous CP violation hardly occurs when we start with Lagrangian respecting the extended modular symmetry $\bar{\Gamma}^*$. Hence, we expect that some mechanisms that modify or break the modular symmetry will increase the possibility of obtaining CP violation.

4.6 Flux and matter effects for spontaneous CP violation

This section studies the combined effects of three-form fluxes and matter contributions on spontaneous CP violation through moduli stabilization. Within 4D $\mathcal{N}=1$ SUGRA, we consider the following Kähler potential

$$K = K_{\text{moduli}} + K_{\text{matter}}, \tag{4.81}$$

and superpotential

$$W = W_{\text{flux}} + W_{\text{matter}}, \tag{4.82}$$

where we still use K_{moduli} and W_{flux} shown in eqs. (4.29) and (4.53), respectively. K_{matter} and W_{matter} will be specified later when we study concrete models. We assume the invariance of the matter contributions under the extended modular symmetries,

$$\left(\otimes_{i=1}^{3}\bar{\Gamma}_{\tau_i} \otimes \bar{\Gamma}_S\right) \rtimes \mathbb{Z}_2^{\text{CP}}, \tag{4.83}$$

as a generalization of the previous section. Here, $\bar{\Gamma}_{\tau_i}$ and $\bar{\Gamma}_S$ denote the $PSL(2,\mathbb{Z})$ symmetry associated with the complex structure moduli τ_i and axio-dilaton S, respectively[6]. We require the following CP transformation of W_{matter},

$$W_{\text{matter}} \xrightarrow{\text{CP}} -\bar{W}_{\text{matter}}. \tag{4.84}$$

Then, the full superpotential behaves as

$$W \xrightarrow{\text{CP}} -\bar{W}, \tag{4.85}$$

because W_{flux} in eq. (4.53) is transformed as $W_{\text{flux}} \xrightarrow{\text{CP}} -\bar{W}_{\text{flux}}$. Thus, Kähler invariant function $G := K + \ln|W|^2$ is CP invariant.

4.6.1 Model I

Here, we introduce single matter chiral super field X. We assume that X belongs to a trivial singlet representation and carries modular weights $k \in \mathbb{Z}$ under $\bar{\Gamma}_{\tau_1}$ and -1 under $\bar{\Gamma}_{\tau_2}, \bar{\Gamma}_{\tau_3}, \bar{\Gamma}_S$. The corresponding Kähler potential is given by

$$K_{\text{matter}} = Z|X|^2, \tag{4.86}$$

where

$$Z = (-i\tau_1 + i\bar{\tau}_1)^k(-i\tau_2 + i\bar{\tau}_2)^{-1}(-i\tau_3 + i\bar{\tau}_3)^{-1}(-iS + i\bar{S})^{-1}. \tag{4.87}$$

The matter superpotential is given by

$$W_{\text{matter}} = \Lambda^2 Y(\tau_1)X, \tag{4.88}$$

[6] For duality group, see for example Ref. [144] and references therein.

where $Y(\tau_1)$ is a trivial singlet modular form carrying modular weight k_Y under $\bar{\Gamma}_{\tau_1}$. To ensure the modular invariance of the matter sector, we require

$$k_Y = -(1+k). \tag{4.89}$$

In addition, we choose a normalization of the modular form $Y(\tau_1)$ to realize the following behavior under CP,

$$Y(\tau_1) \xrightarrow{\text{CP}} -\bar{Y}(\bar{\tau}_1). \tag{4.90}$$

This corresponds to the choice $\phi \equiv \pi \pmod{2\pi}$ in eq. (4.72). Hence, CP transformation of the matter superpotential follows eq. (4.84).

We solve the SUSY preserving Minkowski conditions:

$$\partial_{\tau_i} W = 0, \quad \partial_S W = 0, \quad \partial_X W = 0, \quad W = 0. \tag{4.91}$$

One can straightforwardly obtain the solution,

$$\tau_1\tau_2 = \frac{b_3}{a^0}, \quad \tau_3 = fS, \quad Y(\tau_1) = 0, \quad X = 0 \text{ (if } \partial_{\tau_1} Y \neq 0). \tag{4.92}$$

This shows that τ_1 is fixed at a zero-point of the modular form $Y(\tau_1)$. This leads to the stabilization of τ_2 via the relation $\tau_1\tau_2 = b_3/a^0$. On the other hand, there remain unfixed moduli corresponding to the flat direction $\tau_3 = fS$. In what follows, we show concrete models by choosing A_4 modular forms for $Y(\tau)$.

Weight 4 trivial singlet:

First, we choose

$$Y(\tau_1) = i\, Y_{\mathbf{1}}^{(4)}(\tau_1), \tag{4.93}$$

where $Y_{\mathbf{1}}^{(4)}(\tau_1)$ is a A_4 trivial singlet modular form carrying modular weight $k_Y = 4$. Eq. (B.9) in Appendix B.2 shows the definition of $Y_{\mathbf{1}}^{(4)}(\tau_1)$ whose normalization corresponds to $\phi \equiv 0 \pmod{2\pi}$ in eq. (4.72). Hence, we have added an overall factor i in (4.93) for making $Y(\tau_1)$ to behave as (4.90) under CP. According to eq. (4.92), the complex structure modulus τ_1 is stabilized at a zero point of $Y_{\mathbf{1}}^{(4)}(\tau_1)$, that is $\tau_1 = \omega$. Then, τ_2 is fixed at

$$\tau_2 = \frac{b_3}{a^0}\frac{1}{\omega}. \tag{4.94}$$

Certain choices of flux quanta lead to spontaneous CP violation by the VEV of τ_2. For instance, let us consider the following set of flux quanta

$$a^0 = 2, \quad b_3 = -3, \quad f = 1, \tag{4.95}$$

where the tadpole cancellation condition is satisfied, $n_{\text{flux}} = 12 \leq 32$. Then, τ_2 acquires a CP-breaking VEV at

$$\tau_2 = -\frac{3}{2\omega} = \frac{3}{4} + \frac{3\sqrt{3}}{4}i. \tag{4.96}$$

As another example, let us choose the following set of flux quanta

$$a^0 = 3, \quad b_3 = -4, \quad f = 1, \tag{4.97}$$

where the tadpole cancellation condition is satisfied, $n_{\text{flux}} = 24 \leq 32$. Then, τ_2 acquires a CP-breaking VEV at

$$\tau_2 = -\frac{4}{3\omega} = \frac{2}{3} + \frac{2i}{\sqrt{3}}. \tag{4.98}$$

Weight 6 trivial singlet:

Next, we set

$$Y(\tau_1) = i\ Y_{\mathbf{1}}^{(6)}(\tau_1), \tag{4.99}$$

where $Y_{\mathbf{1}}^{(6)}(\tau_1)$ is a A_4 trivial singlet modular form carrying modular weight $k_Y = 6$. Eq. (B.10) in Appendix B.2 shows the definition of $Y_{\mathbf{1}}^{(6)}(\tau_1)$ whose normalization corresponds to $\phi \equiv 0$ (mod 2π) in eq. (4.72). Hence, we have added an overall factor i in (4.93) for making $Y(\tau_1)$ to behave as (4.90) under CP.

Since $Y_{\mathbf{1}}^{(6)}(\tau_1)$ has a zero-point at $\tau_1 = i$, stabilization to $\tau_1 = i$ can occur. In this case, we obtain $\mathrm{Re}(\tau_2) = 0$ regardless of the assignments of flux quanta, hence CP is always conserved. In contrast, if τ_1 is stabilized at a different zero-point of $Y_{\mathbf{1}}^{(6)}(\tau_1)$ such as $\tau_1 = 1+i$, spontaneous CP violation can be realized. For instance, let us choose the following set of flux quanta

$$a^0 = 2, \quad b_3 = -5, \quad f = 1, \tag{4.100}$$

where the tadpole cancellation condition is satisfied, $n_{\text{flux}} = 20 \leq 32$. Then, τ_2 acquires a CP-breaking VEV at

$$\tau_2 = \frac{-5}{2(1+i)} = \frac{5}{4}(-1+i). \tag{4.101}$$

To summarize, CP violation can be obtained depending on the VEV of τ_1 and the flux quanta.

Discrete symmetry in the scalar potential

We discuss the relation between the spontaneous CP breaking and the discrete symmetry of the scalar potential. We explain how modification or partial breaking of the modular symmetry favor the CP violation. The scalar potential is given by

$$V = e^K K^{I\bar{J}} D_I W \overline{D_J W}, \tag{4.102}$$

where the indices denote $I, J \in \{\tau_i, S, X\}$. Restricting to $\langle X \rangle = 0$, cf. (4.92), the potential is reduced to

$$V\Big|_{X=0} = \frac{\Lambda^4}{\mathcal{V}^2}[2\mathrm{Im}(\tau_1)]^{k_Y}|Y_{\mathbf{1}}^{(k_Y)}(\tau_1)|^2 + \frac{2|\tau_3 - fS|^2\ |a^0\bar{\tau}_1\tau_2 - b_3|^2\ +\ 2|\tau_3 - f\bar{S}|^2\ |a^0\tau_1\tau_2 - b_3|^2}{\mathcal{V}^2[2\mathrm{Im}(\tau_1)][2\mathrm{Im}(\tau_2)][2\mathrm{Im}(\tau_3)][2\mathrm{Im}(S)]}, \tag{4.103}$$

where the first term corresponds to the matter contributions. The remaining terms are flux-induced contributions. By further restricting to the flat directions eq. (4.55) generated by the three-form flux, we are left with

$$V \to V_{\text{matter}} = \frac{\Lambda^4}{\mathcal{V}^2}[2\text{Im}(\tau_1)]^{k_Y}|Y_{\mathbf{1}}^{(k_Y)}(\tau_1)|^2. \tag{4.104}$$

Since this V_{matter} is invariant under the extended modular symmetry $\bar{\Gamma}^*_{\tau_1}$, the VEV of τ_1 tends to preserve CP as discussed in section 4.3.

Now, if we express V_{matter} in terms of τ_2 by the relation $\tau_1 = b_3/(a^0\tau_2)$, we obtain

$$V_{\text{matter}} = \frac{\Lambda^4}{\mathcal{V}^2}\left[\frac{|a^0|}{|b_3|}2\text{Im}(\tau_2)\right]^{k_Y}\left|Y_{\mathbf{1}}^{(k_Y)}\left(\frac{|a^0|}{|b_3|}\tau_2\right)\right|^2, \tag{4.105}$$

where we have used the invariance of V_{matter} under $\tau_1 \to -1/\tau_1 = (|a^0|/|b_3|)\tau_2$, i.e., $S \in \bar{\Gamma}_{\tau_1}$ transformation. We notice that eq. (4.105) is generally not symmetric under $\bar{\Gamma}_{\tau_2}$ unless $|a^0/b_3| = 1$. One can check that V_{matter} is invariant under the following set of transformations

$$\gamma \in \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2,\mathbb{Q})\middle|\begin{pmatrix} a & b \\ c & d\end{pmatrix} \equiv \begin{pmatrix} * & 0 \pmod{|b_3/a^0|} \\ 0 \pmod{|a^0/b_3|} & * \end{pmatrix}\right\}, \tag{4.106}$$

acting on τ_2 in the usual way

$$\tau_2 \xrightarrow{\gamma} \frac{a\tau_2 + b}{c\tau_2 + d}. \tag{4.107}$$

In eq. (4.106), the asterisk $*$ denotes an unspecified integer. We notice that (4.106) is identical to $SL(2,\mathbb{Z})_{\tau_2}$ when $|a^0/b_3| = 1$. In case of $|a^0/b_3| \neq 1$, the set of matrices in (4.106) form a discrete group which is isomorphic to $SL(2,\mathbb{Z})_{\tau_2}$. This can be checked as follows. Consider a matrix

$$P = \begin{pmatrix} \sqrt{|a^0|/|b_3|} & 0 \\ 0 & \sqrt{|b_3|/|a^0|}\end{pmatrix}, \tag{4.108}$$

where we have $\det P = 1$. Then, the following transformation maps an element γ in eq. (4.106) to an element in $SL(2,\mathbb{Z})_{\tau_2}$,

$$\gamma \to P\gamma P^{-1} \in SL(2,\mathbb{Z})_{\tau_2}. \tag{4.109}$$

It can be straightforwardly checked that eq. (4.109) is indeed an isomorphism. The discrete group contains the following two matrices which resemble T and S transformations in $SL(2,\mathbb{Z})$,

$$\hat{T} = \begin{pmatrix} 1 & |b_3|/|a^0| \\ 0 & 1\end{pmatrix}, \quad \hat{S} = \begin{pmatrix} 0 & |b_3|/|a^0| \\ -|a^0|/|b_3| & 0\end{pmatrix}. \tag{4.110}$$

Then, one can show that any scalar potential invariant under the $\hat{T}$ and $\hat{S}$ transformations satisfy the following properties,

$$\frac{\partial V_{\text{matter}}}{\partial s}(s,t) = 0, \quad s \equiv 0 \pmod{|b_3/(2a^0)|}, \tag{4.111}$$

where $\tau_2 = s + it$, and

$$\frac{\partial V_{\rm matter}}{\partial r}(r,\theta) = 0, \quad r = \frac{|b_3|}{|a^0|}, \tag{4.112}$$

where $\tau_2 = re^{i\theta}$. We notice that CP-breaking VEVs of τ_2 in the A_4 model given by eqs. (4.96), (4.98), and (4.101) are on the line $\mathrm{Re}(\tau_2) = \pm|b_3/(2a^0)|$ as eq. (4.111) implies. Moreover, (4.96) and (4.98) are also on the arc $|\tau_2| = |b_3|/|a^0|$ as eq. (4.112) implies. In fact, (4.96) and (4.98) correspond to $\hat{T}\hat{S}$ invariant fixed point. Similarly, (4.101) corresponds to $\hat{S}\hat{T}\hat{S}(\hat{S}\hat{T})^{-1}$ invariant fixed point.

We observed that there is a preferred condition on the values of flux quanta to obtain CP-breaking vacua. The condition is given by

$$b_3 \nmid a^0 \quad \text{and} \quad a^0 \nmid b_3, \tag{4.113}$$

where $b_3 \nmid a^0$ denotes a^0 is not divisible by b_3. This can be understood as follows. Recalling eq. (4.111), τ_2 tends to stabilize on the lines $\mathrm{Re}(\tau_2) \equiv 0 \pmod{|b_3/(2a^0)|}$. These do not coincide with the CP-conserving lines $\mathrm{Re}(\tau_2) \equiv 0 \pmod{1/2}$ if (4.113) is satisfied. All the CP-violating examples in model I satisfy (4.113).

On the other hand, if $|b_3/a^0| = n \in \mathbb{Z}$, eq. (4.111) implies that CP is hardly violated. This is because τ_2 tends to stabilize on the line $\mathrm{Re}(\tau_2) \equiv 0 \pmod{n/2}$ which is a subset of the CP-conserving region. Similarly, the case $|b_3/a^0| = 1/n,\ (n \in \mathbb{Z})$ tends to conserve CP. Recalling eqs. (4.111) and (4.112), τ_2 would stabilize at the intersecting points of $\mathrm{Re}(\tau_2) \equiv 0 \pmod{|1/(2n)|}$ and the arc $|\tau_2| = 1/n$. Such intersections are $\tau_2 \in \{i/n, \omega/n, -\omega^2/n\}$. In modular flavor symmetric models, τ_2 is equivalent to $\tau_2^{(S)} := -1/\tau_2 \in \{ni, -n\omega^2, n\omega\}$ which are CP-conserving.[7]

We have found that introducing odd-valued flux quanta is needed to meet the condition eq. (4.113) without violating the tadpole cancellation condition (4.57). We give a proof in Appendix C.3. This implies the existence of exotic $O3'$ planes favors the spontaneous CP violation in model I. If the tadpole cancellation condition is relaxed, all the flux quanta a^0, b_3, c^3 and d_0 can be even integers. For example, the condition (4.113) can be satisfied if we choose $(a^0, b_3, c^3, d_0) = (4, -6, -4, -6)$ with $f = 1$, although we have $n_{\rm flux} = 48$.

4.6.2 Model II

Here, we study the extension of the model I to stabilize all complex structure moduli $\tau_i, (i = 1, 2, 3)$ and axio-dilaton S. For this purpose, we introduce one more matter chiral superfield x which belongs to the trivial singlet representation of the modular group (4.83). We assume it

[7] We note that (4.113) is a preferred condition, namely not a necessary condition for CP violation. By making a very special set-up, CP violation can be realized without (4.113). For instance, choosing $|b_3/a^0| = 1/7$ and $\tau_1 = ST^2 i = -0.4 + 0.2i$, we obtain $\tau_2 \simeq 0.286 + 0.143i$. One can check that this cannot be mapped to a CP-conserving point by $\bar{\Gamma}_{\tau_2}$ modular transformation.

carries modular weights $l \in \mathbb{Z}$ under $\bar{\Gamma}_{\tau_3}$ and -1 under $\bar{\Gamma}_{\tau_1}, \bar{\Gamma}_{\tau_2}, \bar{\Gamma}_S$. Then, the Kähler potential for the matter sector is given by

$$K_{\rm matter} = Z|X|^2 + z|x|^2, \tag{4.114}$$

and the matter superpotential is given by

$$W_{\rm matter} = \Lambda^2(Y(\tau_1)X + \alpha\ y(\tau_3)x), \quad \alpha \in \mathbb{R}, \tag{4.115}$$

where we have defined

$$\begin{aligned} Z &:= (-i\tau_1 + i\bar{\tau}_1)^k(-i\tau_2 + i\bar{\tau}_2)^{-1}(-i\tau_3 + i\bar{\tau}_3)^{-1}(-iS + i\bar{S})^{-1}, \\ z &:= (-i\tau_1 + i\bar{\tau}_1)^{-1}(-i\tau_2 + i\bar{\tau}_2)^{-1}(-i\tau_3 + i\bar{\tau}_3)^{l}(-iS + i\bar{S})^{-1}. \end{aligned} \tag{4.116}$$

By requiring the discrete symmetry (4.83) to our model, both modular forms $Y(\tau_1)$ and $y(\tau_3)$ need to be trivial singlets. Furthermore, modular weight of $Y(\tau_1)$ is constrained to $k_Y = -(1+k)$ under $\bar{\Gamma}_{\tau_1}$ and that of $y(\tau_3)$ is constrained to $k_y = -(1+l)$ under $\bar{\Gamma}_{\tau_3}$. We have required the dimensionless parameter α in $W_{\rm matter}$ to be real for CP invariance [72].[8]

We solve the SUSY preserving Minkowski conditions:

$$\partial_{\tau_i}W = 0, \quad \partial_S W = 0, \quad \partial_X W = 0, \quad \partial_x W = 0, \quad W = 0. \tag{4.117}$$

The solution is given by

$$\begin{aligned} \tau_1\tau_2 = \frac{b_3}{a^0}, \quad \tau_3 = fS, \quad Y(\tau_1) = 0, \quad y(\tau_3) = 0, \\ X = 0 \ (\text{if } \partial_{\tau_1}Y \neq 0), \quad x = 0 \ (\text{if } \alpha\, \partial_{\tau_3}y \neq 0), \end{aligned} \tag{4.118}$$

showing that τ_1 and τ_3 are fixed at zero-points of $Y(\tau_1)$ and $y(\tau_3)$, respectively. As a result, the vacua degeneracy is resolved, stabilizing τ_2 and S. Matter fields are stabilized at $\langle X\rangle = \langle x\rangle = 0$ and become massive provided $\langle \partial_{\tau_1}Y\rangle \neq 0$ and $\alpha\langle\partial_{\tau_3}y\rangle \neq 0$.

Spontaneous CP violation in the weak gauge coupling regime

Let us study concrete examples using A_4 modular forms. We choose

$$Y(\tau_1) = i\ Y_{\mathbf{1}}^{(4)}(\tau_1), \quad y(\tau_3) = i\ Y_{\mathbf{1}}^{(6)}(\tau_3), \tag{4.119}$$

where $Y_{\mathbf{1}}^{(4)}$ and $Y_{\mathbf{1}}^{(6)}$ are A_4 trivial singlet modular forms carrying weight $k_Y = 4$ and $k_y = 6$, respectively. We have introduced an overall factor i in (4.119) for making $Y(\tau_1)$ and $y(\tau_3)$ to behave like (4.90) under CP. We can realize spontaneous CP violation in the weak coupling regime, i.e., $\mathrm{Im}(S) > 1$, under certain flux quanta. For instance, we choose

$$a^0 = 4, \quad b_3 = -6, \quad c^3 = -2, \quad d_0 = -3, \quad \left(f = \frac{1}{2}\right), \tag{4.120}$$

[8] We note that the results are unchanged even for $\alpha \in \mathbb{C}$ since the α appear in the scalar potential as $|\alpha|^2$.

where the tadpole cancellation condition is satisfied, $n_{\text{flux}} = 24 \leq 32$. As a result, all complex structure moduli and axio-dilaton are fixed at

$$\tau_1 = \omega, \quad \tau_2 = \frac{3}{4} + \frac{3\sqrt{3}}{4}i, \quad \tau_3 = i, \quad S = 2i. \tag{4.121}$$

This is a CP-breaking SUSY Minkowski vacuum. The source of CP violation is the VEV of τ_2. We have $\text{Im}(S) = 2$, hence weak coupling is simultaneously realized. This value is phenomenologically interesting as we will discuss in subsection 4.6.3. If we relax the tadpole cancellation condition, we can reproduce identical VEVs as eq. (4.121) with only even integer-valued flux quanta. For instance, the following set of flux quanta

$$(a^0, b_3, c^0, d_0) = (8, -12, -4, -6), \tag{4.122}$$

leads to the same VEVs as (4.121) although we have $n_{\text{flux}} = 96$.

So far, we have based our discussion on the flux-induced superpotential in the odd polynomial form eq. (4.48). We emphasize that the even polynomial version of the superpotential eq. (4.51) could have been chosen. Analogous analyses can be performed simply by replacing eq. (4.119) with $Y(\tau_1) = Y_{\mathbf{1}}^{(4)}(\tau_1)$ and $y(\tau_3) = Y_{\mathbf{1}}^{(6)}(\tau_3)$. Then, both the flux-induced superpotential W_{flux} and matter superpotential W_{matter} are transformed under CP as $W_{\text{flux}} \to \bar{W}_{\text{flux}}$ and $W_{\text{matter}} \to \bar{W}_{\text{matter}}$. Hence, the total superpotential $W = W_{\text{flux}} + W_{\text{matter}}$ behaves as $W \to \bar{W}$ ensuring the CP invariance of the Kähler invariant function.

Moduli masses

We evaluate moduli masses corresponding to the SUSY Minkowski vacuum in eq. (4.118). Non-vanishing components of the supersymmetric mass $m_{AB} = \partial_A \partial_B W$ are given by

$$\begin{aligned}
m_{\tau_1\tau_3} &= m_{\tau_3\tau_1} = a^0 \langle \tau_2 \rangle, \\
m_{\tau_1 S} &= m_{S\tau_1} = -f a^0 \langle \tau_2 \rangle, \\
m_{\tau_1 X} &= m_{X\tau_1} = \Lambda^2 \langle \partial_{\tau_1} Y(\tau_1) \rangle, \\
m_{\tau_2\tau_3} &= m_{\tau_3\tau_2} = a^0 \langle \tau_1 \rangle, \\
m_{\tau_2 S} &= m_{S\tau_2} = -f a^0 \langle \tau_1 \rangle, \\
m_{\tau_3 x} &= m_{x\tau_3} = \alpha^2 \Lambda^2 \langle \partial_{\tau_3} y(\tau_3) \rangle.
\end{aligned} \tag{4.123}$$

One finds the determinant of m_{AB}, $(A, B \in \{\tau_i, S, X, x\})$ as

$$\det(m_{AB}) = -(a^0)^2 f^2 \alpha^4 \Lambda^8 \langle \tau_1 \rangle^2 \langle \partial_{\tau_1} Y(\tau_1) \rangle^2 \langle \partial_{\tau_3} y(\tau_3) \rangle^2, \tag{4.124}$$

which is generally non-vanishing unless $\alpha = 0$. This shows that all complex structure moduli, axio-dilaton, and matter fields become massive. As we have seen in eq. (4.66), only half of the moduli acquired masses of order $\mathcal{O}(M_{\text{Pl}})$ owing to the three-form fluxes. Eq. (4.124) implies that remaining moduli and matter fields have acquired masses of order $\mathcal{O}(\Lambda^2/M_{\rm Pl})$ due to matter-moduli couplings if contributions from dimensionless constants are $\mathcal{O}(1)$.

At last, we note that Kähler (volume) modulus is still unfixed. We will show that Kähler modulus can be fixed without disturbing our result, i.e., spontaneous CP violation in the weak coupling regime.

4.6.3 Käher modulus stabilization

Here, we extend model II to stabilize Kähler modulus T which is related to the volume $\mathcal{V}$ by $\mathcal{V} \simeq [2\mathrm{Im}(T)]^{3/2}$. For this purpose, we add contributions from non-perturbative effects as

$$W = W_{\rm flux} + W_{\rm matter} + W_{\rm np}, \tag{4.125}$$

where

$$W_{\rm np} = \sum_m C_m e^{ia_m T + ib_m S}. \tag{4.126}$$

Here, C_m $(m = 1, 2, ...)$ have mass dimension three and we assume they are constants to simplify our discussion. One of the possible origins of the superpotential (4.126) is the D-brane instanton with $a_m, b_m = 2\pi$. Another origin is the gaugino condensation in a non-Abelian gauge theory on D7 branes. Along Ref. [81], we study the following simple form

$$W_{\rm np} = C_1\, e^{iaT} + C_2\, e^{ibS}, \tag{4.127}$$

in the following discussion. Note that $W_{\rm flux}$ and $W_{\rm matter}$ are given by (4.55) and (4.115), respectively.

Supposing that the masses of the complex structure moduli, axio-dilaton, and matter fields are sufficiently larger than that of Kähler modulus, we can consider low energy regime where the Kähler modulus is the only dynamical degree of freedom. In this case, τ_i, S, X, x are set at their VEVs eq. (4.118). Then, we find eq. (4.127) is reduced to the effective superpotential given by

$$W_{\rm eff} = W_{\rm np}(\langle S\rangle, T). \tag{4.128}$$

If the VEV of axio-dilaton satisfies $\mathrm{Im}\langle S\rangle > 1$, one obtains the KKLT-type superpotential [145],

$$W_{\rm eff} \simeq \langle {\Lambda'}^3 e^{ibS}\rangle + C_1 e^{iaT}, \tag{4.129}$$

where the first term is an exponentially suppressed constant. Note that Λ' denotes a constant with mass dimension one, ${\Lambda'}^3 := C_2$. Then, the Kähler modulus is stabilized by the SUSY condition

$$D_T W_{\rm eff} = \partial_T W_{\rm eff} + (\partial_T K) W_{\rm eff} = 0. \tag{4.130}$$

The solution is roughly given by

$$a\ \mathrm{Im}\langle T\rangle \simeq \ln\left(C_1/w_0\right), \tag{4.131}$$

where $|w_0| = |\langle \Lambda'^3 e^{ibS} \rangle| \ll 1$ $(= M_{\rm Pl}^3)$. If we choose $|C_1| = 1$, $\Lambda' = 1$, $a = b = 2\pi$, and $\langle S \rangle = 2i$, we find $\mathrm{Im}\langle T \rangle \sim 2 > 1$.

Although we have succeeded in stabilizing the Kähler modulus, we have used approximation. First, we have set the VEVs of the complex structure moduli, axio-dilaton, and matter fields as eq. (4.118) in model II. Then, we have separately analyzed the Kähler modulus stabilization in the low-energy regime. This implies that there will be slight deviations in the VEVs of the fields if we simultaneously consider the dynamics of all fields taking into account three-form flux, moduli-matter couplings, and non-perturbative effects at once. Indeed, true vacuum is obtained by the SUSY condition:

$$D_I W = 0, \tag{4.132}$$

where $I \in \{\tau_i, S, X, x, T\}$ and W is given by eq. (4.125). The true vacuum can be written as

$$\begin{aligned}
\tau_i &= \langle \tau_i \rangle + \delta\tau_i, \\
S &= \langle S \rangle + \delta S, \\
X &= \langle X \rangle + \delta X = \delta X, \\
x &= \langle x \rangle + \delta x = \delta x, \\
T &= \langle T \rangle + \delta T,
\end{aligned} \tag{4.133}$$

where $\langle \tau_i \rangle$, $\langle S \rangle$, $\langle X \rangle$, and $\langle x \rangle$ denote VEVs in eq. (4.118). Also, $\langle T \rangle$ denotes the VEV satisfying eq. (4.130). These values are the reference point for the evaluation of the true vacuum. In what follows, we estimate the deviations up to linear order approximation based on the method of Refs. [81, 146, 147].

We begin with rewriting the SUSY condition eq. (4.132) in terms of the Kähler invariant function $G := K + \ln |W|^2$ as

$$G_A = \partial_A G = 0, \tag{4.134}$$

where the index A denotes both holomorphic and anti-holomorphic fields,

$$A = \{\tau_i, S, X, x, T, \bar{\tau}_i, \bar{S}, \bar{X}, \bar{x}, \bar{T}\}. \tag{4.135}$$

One finds the Taylor expansion of G_A about the reference point as

$$G_A = G_A|_{\langle\rangle} + \delta\phi^B G_{AB}|_{\langle\rangle} + \mathcal{O}((\delta\phi)^2), \tag{4.136}$$

where ϕ denotes the fields in eq. (4.135). The meaning of $|_{\langle\rangle}$ is to take values at the reference point, i.e., $\phi = \langle \phi \rangle$. Assuming $a, b > 1$, non-vanishing components of G_{AB} satisfy

$$G_{IJ}, G_{\bar{I}\bar{J}} \gg G_{\bar{I}J}, G_{I\bar{J}}. \tag{4.137}$$

This implies that the holomorphic and anti-holomorphic parts of G_{AB} and G^{AB} can be diagonalized independently at the leading approximation, where G^{AB} denotes the inverse matrix of G_{AB}. Hence, we obtain

$$\delta\phi^I = -G^{IJ}|_{\langle\rangle} G_J|_{\langle\rangle} + (\mathcal{O}(\delta\phi)^2). \tag{4.138}$$

Explicit computation leads to

$$\begin{aligned}
\delta\tau_1 &= \mathcal{O}(\varepsilon^2), \\
\delta\tau_2 &= -\frac{W_{\text{eff}}}{W_{\tau_2 S}} G_S\Big|_{\langle\rangle} + \mathcal{O}(\varepsilon^2), \\
\delta\tau_3 &= \mathcal{O}(\varepsilon^2), \\
\delta S &= -\frac{W_{\text{eff}}}{W_{\tau_2 S}} G_{\tau_2}\Big|_{\langle\rangle} + \mathcal{O}(\varepsilon^2), \\
\delta X &= \frac{W_{\text{eff}}}{W_{\tau_1 X}} \left(\frac{W_{\tau_1 S}}{W_{\tau_2 S}} G_{\tau_2} - G_{\tau_1} \right)\Big|_{\langle\rangle} + \mathcal{O}(\varepsilon^2), \\
\delta x &= \frac{W_{\text{eff}}}{W_{\tau_3 x}} \left(\frac{W_{\tau_2 \tau_3}}{W_{\tau_2 S}} G_S - G_{\tau_3} \right)\Big|_{\langle\rangle} + \mathcal{O}(\varepsilon^2), \\
\delta T &= -\frac{G_{ST}}{G_{TT}}\Big|_{\langle\rangle} \delta S,
\end{aligned} \tag{4.139}$$

where $\varepsilon = |W_{\text{eff}}|/(\Lambda^2 M_{\text{Pl}}) \ll 1$. The results show that deviations of all fields are suppressed by the exponentially small factor W_{eff}.

We note that the obtained VEVs correspond to an anti-de Sitter vacuum since the scalar potential is negative, $V = -3\langle e^K |W_{\text{eff}}|^2\rangle < 0$. To realize a de Sitter vacuum, we consider uplifting by SUSY breaking due to non-vanishing F-terms, D-terms and/or anti-branes. Since the masses of moduli and matter fields are sufficiently larger than that of the uplifting source, the corrections in the VEVs would be smaller than the above shifts.

In Appendix C.1, we present another model that can induce spontaneous CP violation and SUSY breaking. This leads to a de Sitter vacuum.

Implications on gauge coupling unification and strong CP problem

We comment on the phenomenological implications of the VEV, $\langle S\rangle \simeq 2i$ which is realized in (4.121).

Firstly, the value is interesting as it may lead to a realistic value of gauge coupling constant. Let us suppose that the gauge kinetic function corresponding to the standard gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$ in a supersymmetric standard model is given by

$$f_h = -iS. \tag{4.140}$$

Here, f_h is a unified holomorphic gauge coupling function at the UV cut-off scale. If $\langle S\rangle \simeq 2i$, we obtain $\alpha^{-1} = 8\pi \simeq 25$, where α is the fine-structure constant. We notice that this nicely coincides with the unified gauge coupling constant of the minimal supersymmetric standard model at the GUT scale, $M_{\text{GUT}} \sim 10^{16}$ GeV.

Secondly, we discuss implications on the strong CP problem[9]. The vanishing real part, $\langle \text{Re}S\rangle = 0$ leads to the vanishing QCD phase $\theta_0 = 0$ at tree level. However, owing to the

[9] See for example, Ref. [148] for a review.

correction $\Delta W := W_{\rm np}$, the VEV of S can slightly deviate from $S = 2i$ as we previously analyzed. According to eq. (4.139), the shift is approximately given by

$$\delta S \sim -i\frac{\Delta W}{W_{\tau_2 S}} \simeq \ -i\frac{m_{3/2}}{M_{\rm Pl}}e^{i\varphi}, \tag{4.141}$$

where $\langle \Delta W\rangle e^{-i\arg\langle W_{\tau_2 S}\rangle} = m_{3/2}M_{\rm Pl}^2 e^{i\varphi}$. Here, $m_{3/2}$ is the (real) gravitino mass. Thus, we obtain

$$|{\rm Re}(\delta S)| \approx |\sin\varphi|\ \frac{m_{3/2}}{M_{\rm Pl}}. \tag{4.142}$$

This is consistent with the experimental bound on the electric dipole moment of neutron if $m_{3/2}|\sin\varphi|/M_{\rm Pl} \lessapprox 10^{-10}$. Hence, the gravitino mass is bounded as $m_{3/2} \lessapprox 10^{-10}\times M_{\rm Pl}/|\sin\varphi|$. However, there can be other corrections to the strong QCD phase. If their effects are small as $m_{3/2}|\sin\varphi|/M_{\rm Pl}$, it is still consistent with experimental data. Nonetheless, the true QCD phase θ is corrected by quark masses M_q as

$$\theta = \theta_0 + \arg({\rm det}M_q). \tag{4.143}$$

In Refs. [149–151], the mass matrices leading to $\arg({\rm det}M_q) = 0$ are studied.[10] Combining our scenario with such mass matrices would be interesting as a solution for the strong QCD problem. Nevertheless, depending on other moduli, the gauge kinetic function would generally receive threshold corrections. The next challenge is understanding how to control corrections on θ_0 from other moduli.

[10]For other approaches relevant to the modular symmetry including the axion, see [138, 152–154].

Chapter 5

Summary

We first summarize chapters 2, 3, and 4 separately. Then, we present the concluding remarks of this dissertation as a whole.

5.1 Summary of magnetized torus and orbifold models

In chapter 2, we have discussed the magnetized torus and orbifold models based on 10D $\mathcal{N} = 1$ super Yang-Mills theory which is a low-energy effective field theory of superstring theory.

Firstly, we reviewed the dimensional reduction of the 10D super Yang-Mills theory under the magnetized extra dimension. 4D chiral theories can be realized due to the background magnetic fluxes. The generation size of chiral fermions is determined by the degeneracy of zero-mode wavefunctions in the extra dimension. The overlap integral of fermion zero-modes and the lightest bosonic modes over the compact space computes Yukawa coupling constants in the 4D theory. Hence, it is important to determine the degeneracy of zero-modes and their explicit wavefunctions to study the flavor structures of 4D effective field theory.

Secondly, we have reviewed magnetized T^2 as well as $T^2/\mathbb{Z}_N$, $(N = 2, 3, 4, 6)$ orbifold models. After presenting the explicit forms of zero-mode wavefunctions, we have clarified the conditions needed to realize three-generation models. For systematic analyses, we utilized $SL(2, \mathbb{Z})$ modular transformation properties of zero-mode wavefunctions.

Thirdly, we have studied extensions to the magnetized T^4 and $T^4/\mathbb{Z}_2$ models. We have shown the explicit form of zero-mode wavefunctions on magnetized T^4 in terms of the Riemann theta function. We have also taken into account the Scherk-Schwarz phases. The negative chirality zero-mode wavefunctions are discussed in Appendix A. Our negative chirality wavefunctions are reduced to the ones shown in Ref. [14] when the real part of the complex structure Ω is vanishing. We note that our result also accounts for the case $\mathrm{Re}\Omega \neq 0$. We have also found that there are a large number of three-generation models within both T^4 and $T^4/\mathbb{Z}_2$ models. In magnetized T^4 models, the degeneracy of zero-modes is equal to the determinant of the flux matrix $\mathbf{N}$. Hence, for the realization of three-generation models, we require $|\det\mathbf{N}| = 3$. In magnetized $T^4/\mathbb{Z}_2$ models, we have learned that the degeneracy of zero-modes is determined by the following four quantities: $\det\mathbf{N}$, Scherk-Schwarz phases, $\mathbb{Z}_2$ eigenvalue, and the mod 2 structure of $\mathbf{N}$. Note that the Scherk-Schwarz phases $\vec{\alpha}, \vec{\beta}$ have 16 different patterns on magnetized $T^4/\mathbb{Z}_2$. We have shown that three degenerated positive chirality zero-modes are produced only when $4 \leq \det\mathbf{N} \leq 8$.

Finally, we studied magnetized $T^4/\mathbb{Z}_N$, $(N = 3, 4, 5, 6, 8, 12)$ models. Zero-mode wavefunctions on these magnetized orbifolds are written by linear combinations of those on magnetized T^4 models. We performed systematic analyses of zero-mode numbers under vanishing Scherk-Schwarz phases by use of $Sp(4, \mathbb{Z})$ modular transformation. This is because $\mathbb{Z}_N$ twists defining the orbifolds can be described by modular transformations. Our results explicitly show the conditions on the magnetic fluxes needed to realize three generations of fermions in each $\mathbb{Z}_N$ twist sector. A large number of three-generation models can be produced in all of the $T^4/\mathbb{Z}_N$ models we have investigated. We comment that some of our $T^4/\mathbb{Z}_N$ models (e.g. $T^5/\mathbb{Z}_5$) may include tachyonic modes in the closed string sector when we consider embedding into superstring theory. This implies instability. Some moduli may develop their VEVs, but the topology of the space may be unchanged. If the manifold is stable, our results on $T^4/\mathbb{Z}_5$ could be realized, since the zero-mode number is determined by topological quantities. Such a study regarding the embedding into string theory is beyond the scope of this paper.

Future work

One of the most intriguing future works is to build concrete models based on the magnetized $T^4/\mathbb{Z}_N, T^6/\mathbb{Z}_N$ orbifold compactifications and to realize the quark and lepton masses and mixing angles similar to analyses on $T^2/\mathbb{Z}_2$ orbifold model [19–23]. For this purpose, we need to compute Yukawa coupling constants by the overlap integrals of zero-mode wavefunctions of various chiralities and Higgs modes over the compact space. We expect that generalization to the non-factorized tori and background flux will provide new insights into the flavor puzzle.

Another future work is the comprehensive analyses of the quantization of the Scherk-Schwarz phases on magnetized $T^4/\mathbb{Z}_N, T^6/\mathbb{Z}_N$ orbifolds. Furthermore, we can study the inclusion of the flux components p_{xx}, p_{yy}, c.f. (2.135). Under such a generalized setup, we would like to derive the explicit form of zero-mode wavefunctions as well as zero-mode numbers. Recently, a powerful way of zero-mode counting using the Atiyah-Segal-Singer fixed-point index theorem has been studied [41]. Nevertheless, studying the explicit form of zero-mode wavefunctions is necessary for the computation of mass ratios, mixing angles, and CP violation of quarks and leptons.

5.2 Summary of radiative modulus stabilization

In chapter 3, we have studied the modulus stabilization of the complex structure modulus τ by considering the 1-loop radiative correction. In modular flavor symmetric models, the VEV of τ is crucial in explaining the hierarchical flavor structures such as mass ratios and mixing angles of quarks and leptons. As an illustrating example, we have reviewed the $A_4 \times A_4 \times A_4$ modular flavor model with $\mathcal{N} = 1$ global SUSY where Yukawa coupling constants are written by modular forms of A_4's. We have confirmed that quark mass hierarchies and absolute values of CKM mixing matrix elements are successfully realized without fine-tuning constant parameters. The only continuous free parameter of the model is the VEV of the modulus τ. For the model, the VEV is chosen by hand close to the fixed point $\tau = \omega$ where $\mathbb{Z}_3$ residual symmetry exists. The slight deviation from the fixed point, i.e., $\tau = \omega + \varepsilon,\ (|\varepsilon| \ll 1)$ is essential for the natural realization of hierarchies. This comes from the fact that the values of modular forms $Y^{(k_Y)}(\tau)$ become hierarchical as $Y^{(k_Y)}(\omega + \varepsilon) \simeq \varepsilon^l, (l = 0, 1, 2)$ depending on their $\mathbb{Z}_3$ charges.

Thus, we need to understand the dynamics that can stabilize the modulus in the vicinity of $\tau = \omega$. We have focused particularly on whether the 1-loop correction can lead to a small deviation from the fixed point by assuming the "tree level" potential V_0 of a simple form (3.45). The possible origins of the "tree level" potential include three-form flux-induced potential in the Type IIB superstring theory or non-perturbative effects such as gaugino condensation.

In particular, we have studied the 1-loop effective potential within A_4 modular flavor symmetric global SUSY models. The 1-loop potential V_1 originates from the couplings between the modulus τ and matter supermultiplets Φ_I, $(I = 1, 2, \cdots, n)$. For concrete discussions, we have focused on the models in which A_4 modular forms have $k_Y = 8$ and belong to one of the singlet representations $\mathbf{r} = \mathbf{1}, \mathbf{1}', \mathbf{1}''$. The coupling we have introduced is modular symmetric if the modular weight of Φ_I is -4, i.e., $k_I = 4$. In our analyses, the detailed scenario of SUSY breaking is not specified for the generality. We have introduced soft SUSY-breaking scalar mass m_0 which is assumed to be independent of τ for simplicity.

We have found that the resulting 1-loop effective potential shows different behaviors depending on the choice of the representation $\mathbf{r} = \mathbf{1}, \mathbf{1}', \mathbf{1}''$ of the A_4 modular forms. In case trivial singlet $\mathbf{r} = \mathbf{1}$ is chosen, we have found that the 1-loop correction V_1 is flat near the fixed point $\tau = \omega$. Hence, the local minimum of the effective potential $V_{\text{eff}} = V_0 + nV_1$ does not deviate from $\tau = \omega$. On the other hand, in the case of $\mathbf{r} = \mathbf{1}', \mathbf{1}''$, we have found the possibilities to generate the phenomenologically preferred (small) deviation $\delta\tau$. When $\mathbf{r} = \mathbf{1}'$ is chosen, the number of flavors n contributing to the effective potential needs to be quite large for the 1-loop radiative correction to compete with the "tree level" potential V_0 if the modulus mass m_τ is as large as the compactification scale.[1] We have evaluated the required species number n to generate the deviations. Then, we have confirmed numerically that the desired slight deviations (e.g. $|\delta\tau| \sim 0.03$) can be realized. Note that our perturbative analyses are still valid even when the "tree level" potential and 1-loop contribution become comparable. This is because the "tree-level" potential we assumed has nothing to do with the "Yukawa couplings" of Φ_I. Nevertheless, one needs to be wary of the introduction of too large number of species as it can conflict with the perturbative description of (quantum) gravity. When the modulus mass is less than the compactification scale, a small number of species is sufficient.

When $\mathbf{r} = \mathbf{1}''$ is chosen, we have found that the (spontaneous) breaking of modular sym-

[1] A large number of massless modes appear at the compactification scale in generic string compactification [5].

metry, i.e., $k_I \neq 4$ makes V_1 to be non-stationary at $\tau = \omega$, which leads to a slight shift of the minimum from $\tau = \omega$. On the other hand, modular symmetric choice $k_I = 4$ cannot realize a small deviation of the minimum because the local minimum of V_1 is found at $\tau = \omega$.

Future work

One possible future direction is to go beyond the $SL(2,\mathbb{Z})$ modular symmetry. For instance, generalization to radiative stabilization in the presence of Siegel modular symmetry can be studied by introducing multi-moduli. Such extension is important because the 4D modular flavor models with Siegel modular symmetry have been actively studied recently where the moduli VEVs are crucial for the flavor structures. Furthermore, the radiative multi-moduli stabilization may be phenomenologically interesting since it may explain various hierarchies, not only masses of quarks and leptons but also the strong QCD phase, cosmological constant, etc. Recently, the possibility of radiative modulus stabilization as a solution to the strong QCD problem has been studied [85, 138].

While we have succeeded in generating a slight deviation of τ responsible for the hierarchical flavor structures, spontaneous CP violation was not induced. Hence, it is important to study the possibility of obtaining CP-breaking VEVs of moduli through radiative corrections although such extension was recently studied [85].

5.3 Summary of spontaneous CP violation through moduli stabilization

In chapter 4, we have studied spontaneous CP violation through moduli stabilization in the presence of symmetries under modular (sub)groups. To realize spontaneous CP violation, we have considered CP-invariant Type IIB three-form flux and modular invariant moduli-matter couplings.

Our study is motivated by the following background. Firstly, CP should be broken spontaneously because it is a discrete gauge symmetry as we reviewed in section 4.1. The 4D parity and charge conjugation are embedded in 10D proper Lorentz transformation and gauge transformation of the underlying theory, respectively. Secondly, in CP symmetric modular flavor models of quarks and leptons, the only source of CP violation is the VEV of complex structure modulus τ. As we reviewed in section 4.2, within $SL(2,\mathbb{Z})$ modular flavor models, CP is violated when τ is fixed at a point satisfying $-\bar{\tau} \neq \gamma\tau$ for any element $\gamma \in SL(2,\mathbb{Z})$. Hence, it is important to study the dynamics that lead to such CP-breaking vacua.

We have taken two steps to realize spontaneous CP violation. As a first step, we have considered CP symmetric flux-induced potential within Type IIB string theory on $T^6/\mathbb{Z}_2$ orientifold. There are three complex structure moduli $\tau_i, (i = 1, 2, 3)$, axio-dilaton S, and Kähler modulus $\mathcal{V}$. In general, flat directions can appear where CP-conserving and CP-breaking vacua are degenerate. Hence, we need additional contributions to resolve the degeneracy and possibly to obtain CP violation. As a second step, we have introduced matter contributions. We have assumed that the couplings between the moduli and matter fields respect the CP and modular symmetries following Refs. [84, 99]. Such couplings are written using modular forms. We have chosen A_4 trivial singlet modular forms as illustrating examples.

By combining the flux and matter contributions, we have found that the degeneracy of the vacua can be resolved. Furthermore, CP can be spontaneously broken under certain choices of flux quanta. In the model I, we introduced one matter field X coupled to τ_1. Then, CP-breaking SUSY Minkowski vacua induced by the VEV of τ_2. We have observed that if the flux quanta satisfy the condition such as (4.113), CP-breaking VEV is obtained quite easily. We studied the underlying reason by focusing on the discrete symmetry of the scalar potential. In fact, certain breakings or modifications of modular symmetries caused by the three-form fluxes favor the spontaneous CP violation. As explained in section 4.3, when the scalar potential V of a modulus τ respects the extended modular symmetry $SL(2,\mathbb{Z}) \rtimes \mathbb{Z}_2^{(\mathrm{CP})}$, its local minimum tends to appear on the imaginary axis $\mathrm{Re}(\tau) = 0$ and/or the boundaries of the fundamental domain $\mathcal{D}$ where CP is preserved. On the other hand, when the three-form flux quanta are introduced, such modular symmetries of V can be partially broken. When eq. (4.113) is satisfied, the residual discrete symmetry of V favors the spontaneous CP violation. As an extension, we studied model II where two matter fields coupled to moduli are introduced. Then, we have succeeded in realizing spontaneous CP violation in the weak coupling regime, i.e., $\mathrm{Im}(S) > 1$. In particular, we have obtained $\langle S \rangle = 2i$ which nicely corresponds to the unified gauge coupling constant of the minimal supersymmetric standard model at the GUT scale when the simple gauge kinetic function eq. (4.140) is assumed. The obtained VEVs correspond to SUSY preserving Minkowski vacua. However, the Kähler modulus $\mathcal{V}$ is still unfixed. Therefore, we have considered non-perturbative effects which would lead to Kähler modulus stabilization through a KKLT-like scenario. We have confirmed that spontaneous CP violation can be still realized even when such non-perturbative effects and up-lifting are included. One of the interesting outcomes is that the VEV of axio-dilaton can slightly shift from $\langle S \rangle = 2i$ and the tiny value of $\mathrm{Re}\langle S \rangle$ may lead to the solution of the strong QCD problem.

Future work

We have learned that spontaneous CP violation can be readily generated by the VEVs of complex structure moduli when the modular symmetries of the scalar potential are partially broken. Hence, it may be interesting to consider scenarios where modular symmetries become anomalous.[2]

[2]Modular symmetry anomalies within magnetized D-brane models are studied in Refs. [155, 156].

5.4 Concluding remarks

In this dissertation, we have studied the realization of flavor structures of quarks and leptons from effective field theories of superstring theory. The conditions needed to realize three-generation models based on magnetized toroidal orbifold compactifications were clarified. Through radiative moduli stabilization, we have learned the possibilities of reproducing moduli VEVs such that the 4D modular flavor symmetric models can naturally explain hierarchical mass ratios and mixing angles. By introducing couplings between matter fields and moduli, flat directions of the flux-induced CP symmetric scalar potential were lifted and spontaneous CP violation was realized.

However, we have not succeeded in the simultaneous realization of all of the flavor structures mentioned above. It is important to build a model that can derive all the flavor structures within a single concrete set-up for compactification. That is left for future work. For example, by computing Yukawa coupling constants based on magnetized $T^{2n}/\mathbb{Z}_N, (n = 2, 3)$ compactification, we hope to reproduce mass ratios and mixing matrices of quarks and leptons. This will allow us to identify phenomenologically preferred VEVs of moduli and/or Higgs fields. Then, our next task would be to derive those desirable VEVs through dynamics within $T^{2n}/\mathbb{Z}_N$ compactification.

Appendix A

More about magnetized T^4 and $T^4/\mathbb{Z}_N$ models

A.1 Negative chirality zero-mode wavefunctions

In this section, we present negative chirality zero-mode wavefunctions i.e., ψ_-^1, ψ_-^2 on magnetized T^4. For the sake of simplicity, we assume vanishing Scherk-Schwarz phases. The negative chirality components satisfy the following Dirac equation:

$$D_{z^1}\psi_-^2 + D_{z^2}\psi_-^1 = 0, \tag{A.1}$$
$$\bar{D}_{\bar{z}^2}\psi_-^2 - \bar{D}_{\bar{z}^1}\psi_-^1 = 0. \tag{A.2}$$

As it is clarified in Ref. [14], both ψ_-^1 and ψ_-^2 are non-vanishing when oblique fluxes are present, i.e., $F_{z^1\bar{z}^2} \neq 0$. Let us briefly explain. Suppose that only ψ_-^2 is non-zero. Then, we have $D_{z^1}\psi_-^2 = \bar{D}_{\bar{z}^2}\psi_-^2 = 0$, which immediately leads to $[D_{z^1}, \bar{D}_{\bar{z}^2}]\psi_-^2 = 0$. From eq. (2.153), we find $F_{z^1\bar{z}^2}\psi_-^2 = 0$. Hence, under the presence of non-vanishing oblique fluxes $F_{z^i\bar{z}^j} \neq 0$, we only find trivial solutions $\psi_-^2 = 0$. This suggests that the negative chirality components need to be excited simultaneously to obtain non-trivial zero-mode wavefunctions,

$$\psi_-^1 = \alpha\psi, \quad \psi_-^2 = \beta\psi, \tag{A.3}$$

where $\alpha, \beta \in \mathbb{C}$. Then eqs. (A.1) and (A.2) reduce to

$$\begin{aligned} (\beta D_{z^1} + \alpha D_{z^2})\psi = 0, \\ (\beta\bar{D}_{\bar{z}^2}\psi - \alpha\bar{D}_{\bar{z}^1})\psi = 0. \end{aligned} \tag{A.4}$$

To obtain non-trivial solution $\psi \neq 0$, the coefficients α and β need to satisfy,

$$-\alpha\beta F_{z^1\bar{z}^{\bar{1}}} - \alpha^2 F_{z^2\bar{z}^{\bar{1}}} + \beta^2 F_{z^1\bar{z}^{\bar{2}}} + \alpha\beta F_{z^2\bar{z}^{\bar{2}}} = 0. \tag{A.5}$$

This can be understood as follows. From eq. (A.4), one obtains

$$(\beta\bar{D}_{\bar{z}^2}\psi - \alpha\bar{D}_{\bar{z}^1})(\beta D_{z^1} + \alpha D_{z^2})\psi = (\beta D_{z^1} + \alpha D_{z^2})(\beta\bar{D}_{\bar{z}^2}\psi - \alpha\bar{D}_{\bar{z}^1})\psi, \tag{A.6}$$

which can be rewritten as

$$\left(\beta^2[D_{z^1}, \bar{D}_{\bar{z}^2}] - \alpha\beta[D_z^1, \bar{D}_{\bar{z}^1}] + \alpha\beta[D_{z^2}, \bar{D}_{\bar{z}^2}] - \alpha^2[D_{z^2}, \bar{D}_{\bar{z}^1}]\right)\psi = 0. \tag{A.7}$$

Using eq. (2.153), we find that eq. (A.5) is required for ψ to be non-trivial. Hence, we are led to solve eqs. (A.4), (A.5) simultaneously under the boundary conditions eq. (2.152).

For later convenience, let us rewrite eq. (A.4) as

$$-qF_{z^1\bar{z}^{\bar{1}}} - F_{z^2\bar{z}^{\bar{1}}} + q^2 F_{z^1\bar{z}^{\bar{2}}} + qF_{z^2\bar{z}^{\bar{2}}} = 0, \tag{A.8}$$

where we have defined $q := \beta/\alpha$. Since this is a quadratic equation for q, we generally find two solutions. In the next subsection, we show that only one of the solutions is valid by considering the Laplace equation.

A.1.1 Laplace equation of negative chirality modes

The negative chirality zero-modes ψ satisfy the following Laplace equation:

$$\Delta_{T^4}\psi \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \frac{2}{(2\pi R)^2} \begin{pmatrix} -F_{z^1\bar{z}^1} + F_{z^2\bar{z}^2} & -2F_{z^1\bar{z}^2} \\ -2F_{z^2\bar{z}^1} & F_{z^1\bar{z}^1} - F_{z^2\bar{z}^2} \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \psi, \tag{A.9}$$

where Δ_{T^4} is defined in eq. (2.154). Note that the Laplace operator is positive-semi definite on a compact space.

Choice of $q = \beta/\alpha$

To make our discussion simple, let us write the flux as

$$F = \pi[\mathbf{N}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}]_{ij}(idz^i \wedge d\bar{z}^j) \equiv \pi \begin{pmatrix} x & z \\ z & y \end{pmatrix}_{ij} (idz^i \wedge d\bar{z}^j). \tag{A.10}$$

where we assume $xy - z^2 < 0$. This is because if $xy - z^2 > 0$, negative chirality wavefunctions are not normalizable as we will see later, instead we obtain non-vanishing positive chirality zero-mode wavefunctions. There are two solutions q of eq.(A.8):

$$q_\pm = \frac{(x-y) \pm \sqrt{(x-y)^2 + 4z^2}}{2z}. \tag{A.11}$$

By the positive semi-definiteness of the Laplace operator in eq. (A.9), one realizes q_- is the right choice as follows. The Laplace operator acts on $(\beta, \alpha)^{\mathrm{T}}$ as

$$\Delta_{T^4} \leftrightarrow \frac{2}{(2\pi R)^2} \begin{pmatrix} -x+y & -2z \\ -2z & x-y \end{pmatrix}. \tag{A.12}$$

Its eigenvalues ξ_i and corresponding eigenvectors $\boldsymbol{x}_i$ are given by

$$\xi_1 = -\frac{2}{(2\pi R)^2}\sqrt{x^2-2xy+y^2+4z^2}, \quad \boldsymbol{x}_1 = \begin{pmatrix} -x+y-\sqrt{x^2-2xy+y^2+4z^2} \\ -2z \end{pmatrix}, \tag{A.13}$$

$$\xi_2 = \frac{2}{(2\pi R)^2}\sqrt{x^2-2xy+y^2+4z^2}, \quad \boldsymbol{x}_2 = \begin{pmatrix} -x+y+\sqrt{x^2-2xy+y^2+4z^2} \\ -2z \end{pmatrix}. \tag{A.14}$$

We should take ξ_2 because it is non-negative. Then, the corresponding value of q is

$$q = \frac{(\text{1st component of } \boldsymbol{x}_2)}{(\text{2nd component of } \boldsymbol{x}_2)} = \frac{x-y-\sqrt{(x-y)^2+4z^2}}{2z}. \tag{A.15}$$

A.1.2 Negative chirality zero-mode solution

Having determined $q \in \mathbb{R}$, we define $\hat{F}_{z^1\bar{z}^1}, \tilde{F}_{z^2\bar{z}^2} \in \mathbb{R}$ as

$$\begin{pmatrix} F_{z^1\bar{z}^1} & F_{z^1\bar{z}^2} \\ F_{z^2\bar{z}^1} & F_{z^2\bar{z}^2} \end{pmatrix} = \hat{F}_{z^1\bar{z}^1}\begin{pmatrix} 1 & -q \\ -q & q^2 \end{pmatrix} + \tilde{F}_{z^2\bar{z}^2}\begin{pmatrix} q^2 & q \\ q & 1 \end{pmatrix}. \tag{A.16}$$

One can verify that flux in (A.10) can be always written in the form eq. (A.16).

For convenience let us introduce 2×2 real matrices $\hat{\mathbf{N}}$ and $\tilde{\mathbf{N}}$ by

$$\pi[\hat{\mathbf{N}}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}] := \hat{F}_{z^1\bar{z}^1}\begin{pmatrix} 1 & -q \\ -q & q^2 \end{pmatrix}, \tag{A.17}$$

$$\pi[\tilde{\mathbf{N}}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}] := \tilde{F}_{z^2\bar{z}^2}\begin{pmatrix} q^2 & q \\ q & 1 \end{pmatrix}. \tag{A.18}$$

Then we can write

$$F = \pi[\hat{\mathbf{N}}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}]_{ij}(idz^i\wedge d\bar{z}^j) + \pi[\tilde{\mathbf{N}}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}]_{ij}(idz^i\wedge d\bar{z}^j). \tag{A.19}$$

Corresponding gauge potentials are given by

$$\begin{aligned} A_{z^i} &= -\frac{\pi i}{2}\left\{[\mathbf{N}\vec{\bar{z}}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}\right\}_i = -\frac{\pi i}{2}\left\{([\hat{\mathbf{N}}\vec{\bar{z}}]^{\mathrm{T}}+[\tilde{\mathbf{N}}\vec{\bar{z}}]^{\mathrm{T}})(\mathrm{Im}\Omega)^{-1}\right\}_i, \\ A_{\bar{z}^i} &= \frac{\pi i}{2}\left\{[\mathbf{N}\vec{z}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}\right\}_i = \frac{\pi i}{2}\left\{([\hat{\mathbf{N}}\vec{z}]^{\mathrm{T}}+[\tilde{\mathbf{N}}\vec{z}]^{\mathrm{T}})(\mathrm{Im}\Omega)^{-1}\right\}_i. \end{aligned} \tag{A.20}$$

The solution of eq. (A.4) under the boundary conditions eq. (2.152) is given by

$$\begin{aligned} \psi^{\vec{J}}_{\mathbf{N},\mathbf{M}} &= \mathcal{N}_{\vec{J}}\cdot f(\vec{z},\vec{\bar{z}})\cdot\hat{\Theta}(\vec{z},\vec{\bar{z}}), \\ f(\vec{z},\vec{\bar{z}}) &= e^{\pi i([\hat{\mathbf{N}}\vec{z}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\vec{z} - [\tilde{\mathbf{N}}\vec{\bar{z}}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\vec{\bar{z}})}, \\ \hat{\Theta}(\vec{z},\vec{\bar{z}}) &= \sum_{\vec{m}\in\mathbb{Z}^2} e^{\pi i[(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{M}i\mathrm{Im}\Omega)(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})]} e^{\pi i[(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{N}\mathrm{Re}\Omega)(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})]} \\ &\quad\cdot e^{2\pi i(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}[\hat{\mathbf{N}}\vec{z}]} e^{2\pi i(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}[\tilde{\mathbf{N}}\vec{\bar{z}}]}, \end{aligned} \tag{A.21}$$

where $\mathbf{M} := \hat{\mathbf{N}} - \tilde{\mathbf{N}}$. Here, $\vec{J}$ denotes integer points which lie inside the cell $\Lambda_{\mathbf{N}}$ spanned by

$$\mathbf{N}^{\mathrm{T}}\vec{e}_k, \qquad (k = 1, 2). \tag{A.22}$$

The normalization constant is denoted by $\mathcal{N}_{\vec{J}}$. By noting

$$\psi^{\vec{J}+\mathbf{N}^{\mathrm{T}}\vec{e}_k}_{\mathbf{N},\mathbf{M}} = \psi^{\vec{J}}_{\mathbf{N},\mathbf{M}}, \tag{A.23}$$

we find that the degeneracy of the zero-mode is $|\det\mathbf{N}|$. Our wavefunction (A.21) agrees with that presented in Ref. [14] when $\mathrm{Re}\Omega = 0$. We have generalized the result to the case $\mathrm{Re}\Omega \neq 0$.

We note that the negative chirality zero-mode wavefunction (A.21) is normalizable only if $\mathbf{M}(\mathrm{Im}\Omega)$ is positive definite, i.e., $\mathbf{M}(\mathrm{Im}\Omega) > 0$ which is equivalent to

$$\mathbf{M}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1} > 0. \tag{A.24}$$

Hence, we find

$$\det\left(\pi[\mathbf{M}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}]\right) = -\det\left(\pi[\mathbf{N}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}]\right) = -(xy - z^2) > 0, \tag{A.25}$$

which shows that $\mathrm{Im}(\mathbf{N}\Omega)$ is an indefinite matrix.

Is Dirac equation satisfied?

We confirm that eq. (A.21) indeed satisfies the Dirac equation. Firstly, let us check eq.(A.2). One finds

$$\bar{D}_{\bar{z}^k}\psi^{\vec{J}}_{\mathbf{N},\mathbf{M}} = \mathcal{N}_{\vec{J}}[(\bar{\partial}_{\bar{z}^k} f)\hat{\Theta} + (f\bar{\partial}_{\bar{z}^k}\hat{\Theta})] - iA_{\bar{z}^k}\psi^{\vec{J}}_{\mathbf{N},\mathbf{M}}, \tag{A.26}$$

where

$$\begin{aligned}\bar{\partial}_{\bar{z}^k} f(\vec{z},\bar{\vec{z}}) &= f(\vec{z},\bar{\vec{z}})\cdot\pi i\bar{\partial}_{\bar{z}^k}\left([\hat{\mathbf{N}}\vec{z}]^{\mathrm{T}}\cdot(\mathrm{Im}\Omega)^{-1}\cdot\mathrm{Im}\vec{z} - [\tilde{\mathbf{N}}\bar{\vec{z}}]^{\mathrm{T}}\cdot(\mathrm{Im}\Omega)^{-1}\cdot\mathrm{Im}\bar{\vec{z}}\right)\\ &= -\pi i f(\vec{z},\bar{\vec{z}})\left(\frac{1}{2i}([\hat{\mathbf{N}}\vec{z}]^{\mathrm{T}}\cdot(\mathrm{Im}\Omega)^{-1})_k + \frac{1}{2i}([\tilde{\mathbf{N}}\bar{\vec{z}}]^{\mathrm{T}}\cdot(\mathrm{Im}\Omega)^{-1})_k + [\tilde{\mathbf{N}}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\bar{\vec{z}}]_k\right),\end{aligned} \tag{A.27}$$

and

$$\begin{aligned}\bar{\partial}_{\bar{z}^k}\hat{\Theta}(\vec{z},\bar{\vec{z}}) &= \sum_{\vec{m}\in\mathbb{Z}^2} e^{\pi i[(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{M}i\mathrm{Im}\Omega)(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})]}e^{\pi i[(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{N}\mathrm{Re}\Omega)(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})]}\\ &\qquad\cdot\bar{\partial}_{\bar{z}^k}\left(e^{2\pi i(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}[\hat{\mathbf{N}}\vec{z}]}e^{2\pi i(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}[\tilde{\mathbf{N}}\bar{\vec{z}}]}\right)\\ &= \sum_{\vec{m}\in\mathbb{Z}^2} e^{\pi i[(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{M}i\mathrm{Im}\Omega)(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})]}e^{\pi i[(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{N}\mathrm{Re}\Omega)(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})]}\\ &\qquad\cdot e^{2\pi i(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}[\hat{\mathbf{N}}\vec{z}]}e^{2\pi i(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}[\tilde{\mathbf{N}}\bar{\vec{z}}]}\bar{\partial}_{\bar{z}^k}\left(2\pi i(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}[\tilde{\mathbf{N}}\bar{\vec{z}}]\right)\\ &= 2\pi i\sum_{\vec{m}\in\mathbb{Z}^2}(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})_i\tilde{\mathbf{N}}_{ik}e^{\pi i[(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{M}i\mathrm{Im}\Omega)(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})]}\\ &\qquad e^{\pi i[(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{N}\mathrm{Re}\Omega)(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})]}\cdot e^{2\pi i(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}[\hat{\mathbf{N}}\vec{z}]}e^{2\pi i(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}[\tilde{\mathbf{N}}\bar{\vec{z}}]}.\end{aligned} \tag{A.28}$$

Now, one can check

$$
\begin{aligned}
&\beta\bar{\partial}_{\bar{z}^2}\hat{\Theta} - \alpha\bar{\partial}_{\bar{z}^1}\hat{\Theta} \\
&= 2\pi i \sum_{\vec{m}\in\mathbb{Z}^2} (\vec{m} + \mathbf{N}^{-1\,\mathrm{T}}\vec{J})_i(\beta\tilde{\mathbf{N}}_{i2} - \alpha\tilde{\mathbf{N}}_{i1}) e^{\pi i[(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{M}i\mathrm{Im}\Omega)(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})]} \\
&\qquad e^{\pi i[(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{N}\mathrm{Re}\Omega)(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})]} \cdot e^{2\pi i(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})^{\mathrm{T}}[\hat{\mathbf{N}}\vec{z}]} e^{2\pi i(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})^{\mathrm{T}}[\tilde{\mathbf{N}}\bar{\vec{z}}]} \\
&= 2\pi i \sum_{\vec{m}\in\mathbb{Z}^2} (\vec{m} + \mathbf{N}^{-1\,\mathrm{T}}\vec{J})_k(\mathrm{Im}\Omega)_{kr}(\beta[\tilde{\mathbf{N}}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}]_{2r} - \alpha[\tilde{\mathbf{N}}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}]_{1r}) \\
&\qquad \cdot e^{\pi i[(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{M}i\mathrm{Im}\Omega)(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})]} e^{\pi i[(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{N}\mathrm{Re}\Omega)(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})]} \\
&\qquad \cdot e^{2\pi i(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})^{\mathrm{T}}[\hat{\mathbf{N}}\vec{z}]} e^{2\pi i(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})^{\mathrm{T}}[\tilde{\mathbf{N}}\bar{\vec{z}}]} \\
&= 0,
\end{aligned}
\tag{A.29}
$$

where we have used eq. (A.5). Then, we obtain

$$
\begin{aligned}
&\beta\bar{D}_{\bar{z}^2}\psi^{\vec{J}}_{\mathbf{N},\mathbf{M}} - \alpha\bar{D}_{\bar{z}^1}\psi^{\vec{J}}_{\mathbf{N},\mathbf{M}} \\
&= \beta\mathcal{N}_{\vec{J}}(\bar{\partial}_{\bar{z}^2}f)\hat{\Theta} - \beta i A_{\bar{z}^2}\psi^{\vec{J}}_{\mathbf{N},\mathbf{M}} - \alpha\mathcal{N}_{\vec{J}}(\bar{\partial}_{\bar{z}^1}f)\hat{\Theta} + \alpha i A_{\bar{z}^1}\psi^{\vec{J}}_{\mathbf{N},\mathbf{M}} \\
&= \mathcal{N}_{\vec{J}}\hat{\Theta}[\beta\bar{\partial}_{\bar{z}^2}f - \alpha\bar{\partial}_{\bar{z}^1}f] - i(\beta A_{\bar{z}^2} - \alpha A_{\bar{z}^1})\psi^{\vec{J}}_{\mathbf{N},\mathbf{M}} \\
&= -\frac{\pi}{2}\psi^{\vec{J}}_{\mathbf{N},\mathbf{M}}\Big[\beta\{([\hat{\mathbf{N}}\vec{z}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1})_2 + ([\tilde{\mathbf{N}}\bar{\vec{z}}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1})_2 + 2i[\tilde{\mathbf{N}}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\vec{z}]_2\} \\
&\quad - \alpha\{([\hat{\mathbf{N}}\vec{z}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1})_1 + ([\tilde{\mathbf{N}}\bar{\vec{z}}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1})_1 + 2i[\tilde{\mathbf{N}}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\vec{z}]_1\}\Big] - i(\beta A_{\bar{z}^2} - \alpha A_{\bar{z}^1})\psi^{\vec{J}}_{\mathbf{N},\mathbf{M}}.
\end{aligned}
\tag{A.30}
$$

By noticing $\beta[\tilde{\mathbf{N}}^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\vec{z}]_2 - \alpha[\tilde{\mathbf{N}}^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\vec{z}]_1 = 0$, one finds

$$
\begin{aligned}
&\beta\bar{D}_{\bar{z}^2}\psi^{\vec{J}}_{\mathbf{N},\mathbf{M}} - \alpha\bar{D}_{\bar{z}^1}\psi^{\vec{J}}_{\mathbf{N},\mathbf{M}} \\
&= -\frac{\pi}{2}\psi^{\vec{J}}_{\mathbf{N},\mathbf{M}}\left[\beta\{([\hat{\mathbf{N}}\vec{z}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1})_2 + ([\tilde{\mathbf{N}}\bar{\vec{z}}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1})_2\} - \alpha\{([\hat{\mathbf{N}}\vec{z}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1})_1 + ([\tilde{\mathbf{N}}\bar{\vec{z}}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1})_1\}\right] \\
&\quad + \frac{\pi}{2}\psi^{\vec{J}}_{\mathbf{N},\mathbf{M}}\left[\{([\hat{\mathbf{N}}\vec{z}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1})_2 + ([\tilde{\mathbf{N}}\vec{z}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1})_2\} - \alpha\{([\hat{\mathbf{N}}\vec{z}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1})_1 + ([\tilde{\mathbf{N}}\vec{z}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1})_1\}\right] \\
&= -\frac{\pi}{2}\psi^{\vec{J}}_{\mathbf{N},\mathbf{M}}\left[\beta([\tilde{\mathbf{N}}\bar{\vec{z}}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1})_2 - \alpha([\tilde{\mathbf{N}}\vec{z}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1})_1\right] \\
&\qquad + \frac{\pi}{2}\psi^{\vec{J}}_{\mathbf{N},\mathbf{M}}\left[\beta([\tilde{\mathbf{N}}\vec{z}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1})_2 - \alpha([\tilde{\mathbf{N}}\vec{z}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1})_1\right] \\
&= -\frac{\pi}{2}\psi^{\vec{J}}_{\mathbf{N},\mathbf{M}}\left[\bar{z}_j\left(\beta[\tilde{\mathbf{N}}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}]_{j2} - \alpha[\tilde{\mathbf{N}}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}]_{j1}\right) + z_j\left(\beta[\tilde{\mathbf{N}}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}]_{j2} - \alpha[\tilde{\mathbf{N}}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}]_{j1}\right)\right] \\
&= 0.
\end{aligned}
\tag{A.31}
$$

The last equality holds as both the first and the second terms become zero. Thus, we have verified eq.(A.2). Similarly, one can also check eq.(A.1).

Are boundary conditions satisfied?

Here, let's check that our solution (A.21) satisfies the boundary conditions eq. (2.152). First, we focus on the translation $\vec{z} \to \vec{z} + \vec{e}_k$. We find

$$\begin{aligned} f(\vec{z}+\vec{e}_k, \bar{\vec{z}}+\vec{e}_k) &= e^{\pi i([\hat{\mathbf{N}}\vec{e}_k]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\vec{z} - [\tilde{\mathbf{N}}\vec{e}_k]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\bar{\vec{z}})} f(\vec{z}, \bar{\vec{z}}) \\ &= e^{\pi i[\mathbf{N}\vec{e}_k]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\vec{z}} f(\vec{z}, \bar{\vec{z}}) \\ &= e^{i\chi_{\vec{e}_k}} f(\vec{z}, \bar{\vec{z}}). \end{aligned} \tag{A.32}$$

We can also check,

$$\begin{aligned} &\hat{\Theta}(\vec{z}+\vec{e}_k, \bar{\vec{z}}+\vec{e}_k) \\ &= \sum_{\vec{m}\in\mathbb{Z}^2} e^{\pi i[(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{M}i\mathrm{Im}\Omega)(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})]} e^{\pi i[(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{N}\mathrm{Re}\Omega)(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})]} \\ &\quad\cdot e^{2\pi i(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}[\hat{\mathbf{N}}(\vec{z}+\vec{e}_k)]} e^{2\pi i(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}[\tilde{\mathbf{N}}(\bar{\vec{z}}+\vec{e}_k)]} \\ &= \sum_{\vec{m}\in\mathbb{Z}^2} e^{\pi i[(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{M}i\mathrm{Im}\Omega)(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})]} e^{\pi i[(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{N}\mathrm{Re}\Omega)(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})]} e^{2\pi i(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}[\hat{\mathbf{N}}\vec{z}]} \\ &\quad\cdot e^{2\pi i(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}[\tilde{\mathbf{N}}\bar{\vec{z}}]} e^{2\pi i(\vec{m}+\mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}[\mathbf{N}\vec{e}_k]} \\ &= \hat{\Theta}(\vec{z}, \bar{\vec{z}}), \end{aligned} \tag{A.33}$$

where we have used $(\vec{m} + \mathbf{N}^{-1\mathrm{T}}\vec{J})^{\mathrm{T}}[\mathbf{N}\vec{e}_k] \in \mathbb{Z}$. Finally, we obtain

$$\psi^{\vec{J}}_{\mathbf{N},\mathbf{M}}(\vec{z}+\vec{e}_k, \bar{\vec{z}}+\vec{e}_k) = e^{i\chi_{\vec{e}_k}} \psi^{\vec{J}}_{\mathbf{N},\mathbf{M}}(\vec{z}, \bar{\vec{z}}), \tag{A.34}$$

and the boundary condition is satisfied.

Next, we focus on the translation $\vec{z} \to \vec{z} + \Omega\vec{e}_k$. We find

$$\begin{aligned} &f(\vec{z}+\Omega\vec{e}_k, \bar{\vec{z}}+\bar{\Omega}\vec{e}_k) \\ &= e^{\pi i\left([\hat{\mathbf{N}}\vec{z}]^{\mathrm{T}}\vec{e}_k + [\hat{\mathbf{N}}\Omega\vec{e}_k]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}\mathrm{Im}(\vec{z}+\Omega\vec{e}_k)\right)} e^{\pi i\left([\tilde{\mathbf{N}}\bar{\vec{z}}]^{\mathrm{T}}\vec{e}_k + [\tilde{\mathbf{N}}\bar{\Omega}\vec{e}_k]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}\mathrm{Im}(\bar{\vec{z}}+\bar{\Omega}\vec{e}_k)\right)} f(\vec{z}, \bar{\vec{z}}) \\ &= e^{\pi i([\hat{\mathbf{N}}\vec{z}]^{\mathrm{T}}\vec{e}_k + [\tilde{\mathbf{N}}\bar{\vec{z}}]^{\mathrm{T}}\vec{e}_k)} e^{\pi i([\hat{\mathbf{N}}\Omega\vec{e}_k] + [\tilde{\mathbf{N}}\bar{\Omega}\vec{e}_k])^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}\mathrm{Im}(\vec{z}+\Omega\vec{e}_k)} f(\vec{z}, \bar{\vec{z}}). \end{aligned} \tag{A.35}$$

We can also check

$$
\begin{aligned}
&\hat{\Theta}(\vec{z}+\Omega\vec{e}_k,\bar{\vec{z}}+\bar{\Omega}\vec{e}_k)\\
&=\sum_{\vec{m}\in\mathbb{Z}^2} e^{\pi i[(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{M}i\mathrm{Im}\Omega)(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})]} e^{\pi i[(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{N}\mathrm{Re}\Omega)(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})]}\\
&\qquad\times e^{2\pi i(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})^{\mathrm{T}}[\hat{\mathbf{N}}(\vec{z}+\Omega\vec{e}_k)]} e^{2\pi i(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})^{\mathrm{T}}[\tilde{\mathbf{N}}(\bar{\vec{z}}+\bar{\Omega}\vec{e}_k)]}\\
&=\sum_{\vec{m}\in\mathbb{Z}^2} e^{\pi i[(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{M}i\mathrm{Im}\Omega)(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})]} e^{\pi i[(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{N}\mathrm{Re}\Omega)(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})]}\\
&\qquad\times e^{2\pi i(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})^{\mathrm{T}}[\hat{\mathbf{N}}\vec{z}]} e^{2\pi i(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})^{\mathrm{T}}[\tilde{\mathbf{N}}\bar{\vec{z}}]} e^{2\pi i(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{M}i\mathrm{Im}\Omega)\vec{e}_k} e^{2\pi i(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J})^{\mathrm{T}}(\mathbf{N}\mathrm{Re}\Omega)\vec{e}_k}\\
&=\sum_{\vec{m}\in\mathbb{Z}^2} e^{\pi i[(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J}+\vec{e}_k)^{\mathrm{T}}(\mathbf{M}i\mathrm{Im}\Omega)(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J}+\vec{e}_k)]} e^{\pi i[(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J}+\vec{e}_k)^{\mathrm{T}}(\mathbf{N}\mathrm{Re}\Omega)(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J}+\vec{e}_k)]}\\
&\qquad\times e^{2\pi i(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J}+\vec{e}_k)^{\mathrm{T}}[\hat{\mathbf{N}}\vec{z}]} e^{2\pi i(\vec{m}+\mathbf{N}^{-1\,\mathrm{T}}\vec{J}+\vec{e}_k)^{\mathrm{T}}[\tilde{\mathbf{N}}\bar{\vec{z}}]} e^{\pi\vec{e}_k^{\mathrm{T}}(\mathbf{M}\mathrm{Im}\Omega)\vec{e}_k} e^{-\pi i\vec{e}_k^{\mathrm{T}}(\mathbf{N}\mathrm{Re}\Omega)\vec{e}_k} e^{-2\pi i\vec{e}_k^{\mathrm{T}}[\hat{\mathbf{N}}\vec{z}]} e^{-2\pi i\vec{e}_k^{\mathrm{T}}[\tilde{\mathbf{N}}\bar{\vec{z}}]}\\
&= e^{\pi\vec{e}_k^{\mathrm{T}}(\mathbf{M}\mathrm{Im}\Omega)\vec{e}_k} e^{-\pi i\vec{e}_k^{\mathrm{T}}(\mathbf{N}\mathrm{Re}\Omega)\vec{e}_k} e^{-2\pi i\vec{e}_k^{\mathrm{T}}[\hat{\mathbf{N}}\vec{z}]} e^{-2\pi i\vec{e}_k^{\mathrm{T}}[\tilde{\mathbf{N}}\bar{\vec{z}}]}\cdot\hat{\Theta}(\vec{z},\bar{\vec{z}}).
\end{aligned}
\tag{A.36}
$$

Then, we find

$$
\begin{aligned}
&(f\ \hat{\Theta})(\vec{z}+\Omega\vec{e}_k,\bar{\vec{z}}+\bar{\Omega}\vec{e}_k)\\
&\quad= e^{\pi i([\hat{\mathbf{N}}\vec{z}]^{\mathrm{T}}\vec{e}_k+[\tilde{\mathbf{N}}\bar{\vec{z}}]^{\mathrm{T}}\vec{e}_k)} e^{\pi i([\hat{\mathbf{N}}\Omega\vec{e}_k]+[\tilde{\mathbf{N}}\bar{\Omega}\vec{e}_k])^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}\mathrm{Im}(\vec{z}+\Omega\vec{e}_k)}\\
&\qquad\cdot e^{\pi\vec{e}_k^{\mathrm{T}}(\mathbf{M}\mathrm{Im}\Omega)\vec{e}_k} e^{-\pi i\vec{e}_k^{\mathrm{T}}(\mathbf{N}\mathrm{Re}\Omega)\vec{e}_k} e^{-2\pi i\vec{e}_k^{\mathrm{T}}[\hat{\mathbf{N}}\vec{z}]} e^{-2\pi i\vec{e}_k^{\mathrm{T}}[\tilde{\mathbf{N}}\bar{\vec{z}}]}\cdot(f\ \hat{\Theta})(\vec{z},\bar{\vec{z}})\\
&\quad= e^{\pi i([\hat{\mathbf{N}}\vec{z}]^{\mathrm{T}}\vec{e}_k+[\tilde{\mathbf{N}}\bar{\vec{z}}]^{\mathrm{T}}\vec{e}_k)} e^{\pi i(\vec{e}_k^{\mathrm{T}}\mathbf{N}\mathrm{Re}\Omega+\vec{e}_k^{\mathrm{T}}\mathbf{M}i\mathrm{Im}\Omega)(\mathrm{Im}\Omega)^{-1}(\mathrm{Im}\vec{z}+\mathrm{Im}\Omega\vec{e}_k)}\\
&\qquad\cdot e^{\pi\vec{e}_k^{\mathrm{T}}(\mathbf{M}\mathrm{Im}\Omega)\vec{e}_k} e^{-\pi i\vec{e}_k^{\mathrm{T}}(\mathbf{N}\mathrm{Re}\Omega)\vec{e}_k} e^{-2\pi i\vec{e}_k^{\mathrm{T}}[\hat{\mathbf{N}}\vec{z}]} e^{-2\pi i\vec{e}_k^{\mathrm{T}}[\tilde{\mathbf{N}}\bar{\vec{z}}]}\cdot(f\ \hat{\Theta})(\vec{z},\bar{\vec{z}})\\
&\quad= e^{\pi i([\hat{\mathbf{N}}\vec{z}]^{\mathrm{T}}\vec{e}_k+[\tilde{\mathbf{N}}\bar{\vec{z}}]^{\mathrm{T}}\vec{e}_k)} e^{\pi i(\vec{e}_k^{\mathrm{T}}\mathbf{N}\mathrm{Re}\Omega+\vec{e}_k^{\mathrm{T}}\mathbf{M}i\mathrm{Im}\Omega)(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\vec{z}}\cdot e^{-2\pi i\vec{e}_k^{\mathrm{T}}[\hat{\mathbf{N}}\vec{z}]} e^{-2\pi i\vec{e}_k^{\mathrm{T}}[\tilde{\mathbf{N}}\bar{\vec{z}}]}\cdot(f\ \hat{\Theta})(\vec{z},\bar{\vec{z}})\\
&\quad= e^{-\pi i([\hat{\mathbf{N}}\vec{z}]^{\mathrm{T}}\vec{e}_k+[\tilde{\mathbf{N}}\bar{\vec{z}}]^{\mathrm{T}}\vec{e}_k)} e^{\pi i(\vec{e}_k^{\mathrm{T}}\mathbf{N}\mathrm{Re}\Omega+\vec{e}_k^{\mathrm{T}}\mathbf{M}i\mathrm{Im}\Omega)(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\vec{z}}\cdot(f\ \hat{\Theta})(\vec{z},\bar{\vec{z}})\\
&\quad= e^{-\pi i([\mathbf{N}\mathrm{Re}\vec{z}]^{\mathrm{T}}\vec{e}_k} e^{\pi i(\vec{e}_k^{\mathrm{T}}\mathbf{N}\mathrm{Re}\Omega)(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\vec{z}}\cdot(f\ \hat{\Theta})(\vec{z},\bar{\vec{z}})\\
&\quad= e^{\pi i\vec{e}_k^{\mathrm{T}}(-\mathbf{N}\mathrm{Re}\vec{z}+\mathbf{N}(\mathrm{Re}\Omega)(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\vec{z})}(f\ \hat{\Theta})(\vec{z},\bar{\vec{z}})\\
&\quad= e^{i\chi_{\Omega\vec{e}_k}}(f\ \hat{\Theta})(\vec{z},\bar{\vec{z}}).
\end{aligned}
\tag{A.37}
$$

Finally, we obtain

$$
\psi^{\vec{J}}_{\mathbf{N},\mathbf{M}}(\vec{z}+\Omega\vec{e}_k,\bar{\vec{z}}+\bar{\Omega}\vec{e}_k)=e^{i\chi_{\Omega\vec{e}_k}}\psi^{\vec{J}}_{\mathbf{N},\mathbf{M}}(\vec{z},\bar{\vec{z}}),
\tag{A.38}
$$

and the boundary condition is satisfied.

A.2 Modular transformation of zero-modes

We derive the modular transformation of the zero-mode wavefunctions on magnetized T^4 following Ref. [44]. For the derivation following properties of the Riemann theta function are useful. For $\vec{a'}, \vec{b'} \in \mathbb{Q}^2$, we have

$$\vartheta \begin{bmatrix} \vec{a}^{\,\mathrm{T}} \\ (\vec{b}+\vec{b'})^{\,\mathrm{T}} \end{bmatrix} (\vec{z},\Omega) = \vartheta \begin{bmatrix} \vec{a}^{\,\mathrm{T}} \\ \vec{b}^{\,\mathrm{T}} \end{bmatrix} (\vec{z}+\vec{b'},\Omega), \tag{A.39}$$

$$\vartheta \begin{bmatrix} (\vec{a}+\vec{a'})^{\,\mathrm{T}} \\ \vec{b}^{\,\mathrm{T}} \end{bmatrix} (\vec{z},\Omega) = e^{\pi i \vec{a'}^{\,\mathrm{T}}\cdot\Omega\cdot\vec{a'} + 2\pi i \vec{a'}^{\,\mathrm{T}}\cdot(\vec{z}+\vec{b})} \cdot \vartheta \begin{bmatrix} \vec{a}^{\,\mathrm{T}} \\ \vec{b}^{\,\mathrm{T}} \end{bmatrix} (\vec{z}+\Omega\vec{a'},\Omega). \tag{A.40}$$

For $\vec{k}, \vec{l} \in \mathbb{Z}^2$, we have

$$\vartheta \begin{bmatrix} (\vec{a}+\vec{k})^{\,\mathrm{T}} \\ (\vec{b}+\vec{l})^{\,\mathrm{T}} \end{bmatrix} (\vec{z},\Omega) = e^{2\pi i \vec{a}^{\mathrm{T}}\vec{l}} \cdot \vartheta \begin{bmatrix} \vec{a}^{\,\mathrm{T}} \\ \vec{b}^{\,\mathrm{T}} \end{bmatrix} (\vec{z},\Omega). \tag{A.41}$$

A.2.1 S transformation

Here, we derive eq.(2.235). The flux $\mathbf{N}$ is assumed to be a symmetric positive definite integer matrix. The starting point is the following property of the Riemann theta function [103],

$$\vartheta(-\Omega^{-1}\vec{z}, -\Omega^{-1}) = \sqrt{\det(\Omega/i)} \cdot e^{\pi i \vec{z}^{\mathrm{T}}\Omega^{-1}\vec{z}} \cdot \vartheta(\vec{z},\Omega). \tag{A.42}$$

The branch of the square root is chosen to give a positive value when Ω is purely imaginary. First, we shift the complex coordinates as $\vec{z} \to \vec{z} + \mathbf{N}^{-1}\vec{J}$. Using eqs. (A.39) and (A.40), we obtain

$$\vartheta \begin{bmatrix} \vec{J}^{\,\mathrm{T}}\mathbf{N}^{-1} \\ 0 \end{bmatrix} (-\Omega^{-1}\vec{z}, -\Omega^{-1}) = \sqrt{\det(\Omega/i)} \cdot e^{\pi i \vec{z}^{\mathrm{T}}\Omega^{-1}\vec{z}} \cdot \vartheta \begin{bmatrix} 0 \\ \vec{J}^{\,\mathrm{T}}\mathbf{N}^{-1} \end{bmatrix} (\vec{z},\Omega). \tag{A.43}$$

Second, we replace the complex structure moduli as $\Omega \to \mathbf{N}^{-1}\Omega$. This gives us

$$\vartheta \begin{bmatrix} \vec{J}^{\,\mathrm{T}}\mathbf{N}^{-1} \\ 0 \end{bmatrix} (-\Omega^{-1}\mathbf{N}\vec{z}, -\Omega^{-1}\mathbf{N}) = \sqrt{\det(\mathbf{N}^{-1}\Omega/i)} \cdot e^{\pi i \vec{z}^{\mathrm{T}}(\mathbf{N}^{-1}\Omega)^{-1}\vec{z}} \cdot \vartheta \begin{bmatrix} 0 \\ \vec{J}^{\,\mathrm{T}}\mathbf{N}^{-1} \end{bmatrix} (\vec{z},\mathbf{N}^{-1}\Omega). \tag{A.44}$$

We can rewrite the Riemann theta function on the right-hand side of eq. (A.44) as

$$\begin{aligned} \vartheta \begin{bmatrix} 0 \\ \vec{J}^{\,\mathrm{T}}\mathbf{N}^{-1} \end{bmatrix} (\vec{z},\mathbf{N}^{-1}\Omega) &= \sum_{\vec{l}\in\mathbb{Z}^2} e^{\pi i \vec{l}^{\mathrm{T}} N^{-1}\Omega\vec{l}} e^{2\pi i \vec{l}^{\mathrm{T}}\cdot(\vec{z}+\mathbf{N}^{-1}\vec{J})} \\ &= \sum_{\vec{K}\in\Lambda_{\mathbf{N}}} e^{2\pi i \vec{K}^{\mathrm{T}}\mathbf{N}^{-1}\vec{J}} \sum_{\vec{s}\in\mathbb{Z}^2} e^{\pi i (\mathbf{N}\vec{s}+\vec{K})^{\mathrm{T}}\cdot\mathbf{N}^{-1}\Omega\cdot(\mathbf{N}\vec{s}+\vec{K})} \cdot e^{2\pi i (\mathbf{N}\vec{s}+\vec{K})^{\mathrm{T}}\cdot\vec{z}} \\ &= \sum_{\vec{K}\in\Lambda_{\mathbf{N}}} e^{2\pi i \vec{J}^{\mathrm{T}}\mathbf{N}^{-1}\vec{K}} \cdot \vartheta \begin{bmatrix} \vec{K}^{\mathrm{T}}\mathbf{N}^{-1} \\ 0 \end{bmatrix} (\mathbf{N}\vec{z},\mathbf{N}\Omega). \end{aligned} \tag{A.45}$$

In the second equality, we modified the summation variables by $\vec{l} = \mathbf{N}\vec{s} + \vec{K}$, where $\vec{K}$'s are integer points inside the lattice $\Lambda_{\mathbf{N}}$ spanned by $\mathbf{N}\vec{e}_n$. Hence, we obtain

$$\begin{aligned}
&\vartheta \begin{bmatrix} \vec{J}^{\,\mathrm{T}}\mathbf{N}^{-1} \\ 0 \end{bmatrix} (\mathbf{N}(-\Omega^{-1}\vec{z}), \mathbf{N}(-\Omega^{-1})) \\
&= \sqrt{\det(\mathbf{N}^{-1}\Omega/i)} \cdot e^{\pi i \vec{z}^{\mathrm{T}}(\mathbf{N}^{-1}\Omega)^{-1}\vec{z}} \sum_{\vec{K}\in\Lambda_{\mathbf{N}}} e^{2\pi i \vec{J}^{\mathrm{T}}\mathbf{N}^{-1}\vec{K}} \cdot \vartheta \begin{bmatrix} \vec{K}^{\mathrm{T}}\mathbf{N}^{-1} \\ 0 \end{bmatrix} (\mathbf{N}\vec{z}, \mathbf{N}\Omega).
\end{aligned} \tag{A.46}$$

Thirdly, we consider the S transformation of the $\vec{z}$-dependent exponential factor in the zero-mode wavefunction, c.f. eq. (2.174). It is given by

$$e^{\pi i [\mathbf{N}\vec{z}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\vec{z}} \xrightarrow{S} e^{\pi i [-\mathbf{N}\Omega^{-1}\vec{z}]^{\mathrm{T}}(\mathrm{Im}(-\Omega^{-1}))^{-1}\mathrm{Im}(-\Omega^{-1}\vec{z})}. \tag{A.47}$$

Let us multiply $e^{\pi i \vec{z}^{\mathrm{T}}(\mathbf{N}^{-1}\Omega)^{-1}\vec{z}}$ which appear in the right-hand side of eq. (A.46). We can simplify the product of the exponential factors as

$$\begin{aligned}
&e^{\pi i [-\mathbf{N}\Omega^{-1}\vec{z}]^{\mathrm{T}}(\mathrm{Im}(-\Omega^{-1}))^{-1}\mathrm{Im}(-\Omega^{-1}\vec{z})} \cdot e^{\pi i \vec{z}^{\mathrm{T}}(\mathbf{N}^{-1}\Omega)^{-1}\vec{z}} \\
&= e^{-\pi i [\mathbf{N}\vec{z}]^{\mathrm{T}}\Omega^{-1}(\mathrm{Im}(-\Omega^{-1}))^{-1}\mathrm{Im}(-\Omega^{-1}\vec{z})} e^{\pi i [\mathbf{N}\vec{z}]^{\mathrm{T}}\Omega^{-1}\vec{z}} \\
&= e^{\pi i [\mathbf{N}\vec{z}]^{\mathrm{T}}\Omega^{-1}[-(\mathrm{Im}(-\Omega^{-1}))^{-1}\mathrm{Im}(-\Omega^{-1}\vec{z})+\vec{z}]} \\
&= e^{\pi i [\mathbf{N}\vec{z}]^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\vec{z}}.
\end{aligned} \tag{A.48}$$

Consequently, one straightforwardly obtains the S transformation of the zero-mode wavefunctions,

$$\psi^{\vec{J},\mathbf{N}}(-\Omega^{-1}\vec{z}, -\Omega^{-1}) = \sqrt{\det(\mathbf{N}^{-1}\Omega/i)} \sum_{\vec{K}\in\Lambda_{\mathbf{N}}} e^{2\pi i \vec{J}^{T}\cdot\mathbf{N}^{-1}\cdot\vec{K}} \psi^{\vec{K},\mathbf{N}}(\vec{z}, \Omega). \tag{A.49}$$

Here, we used the fact that the normalization constant eq. (2.179) is invariant under the S transformation.

A.2.2 T transformation

Here, we derive eq. (2.237). It can be checked that the Riemann theta function satisfies the following relation if all diagonal matrix elements of $\mathbf{N}B_i$ are even integers [103],

$$\vartheta(\mathbf{N}\vec{z}, \mathbf{N}(\Omega + B_i)) = \vartheta(\mathbf{N}\vec{z}, \Omega). \tag{A.50}$$

By shifting the complex coordinates as $\vec{z} \to \vec{z} + (\Omega + B_i){\mathbf{N}^{-1}}^{\mathrm{T}}\vec{J}$ and using eq.(A.40), we obtain

$$\vartheta \begin{bmatrix} \vec{J}^{\,\mathrm{T}}\mathbf{N}^{-1} \\ 0 \end{bmatrix} (\mathbf{N}\vec{z}, \mathbf{N}(\Omega + B_i)) = e^{-\pi i \vec{J}^{\,\mathrm{T}}\mathbf{N}^{-1}B_i\vec{J}} \vartheta \begin{bmatrix} \vec{J}^{\,\mathrm{T}}\mathbf{N}^{-1} \\ 0 \end{bmatrix} (\mathbf{N}\vec{z} + B_i\vec{J}, \mathbf{N}\Omega). \tag{A.51}$$

Next, by use of eqs. (A.39), (A.41), one finds

$$
\begin{aligned}
\vartheta \begin{bmatrix} \vec{J}^{\,\mathrm{T}}\mathbf{N}^{-1} \\ 0 \end{bmatrix} (\mathbf{N}\vec{z}, \mathbf{N}(\Omega + B_i)) &= e^{-\pi i \vec{J}^{\mathrm{T}}\mathbf{N}^{-1}B_i\vec{J}} \cdot \vartheta \begin{bmatrix} \vec{J}^{\,\mathrm{T}}\mathbf{N}^{-1} \\ \vec{J}^{\,\mathrm{T}}B_i \end{bmatrix} (\mathbf{N}\vec{z}, \mathbf{N}\Omega) \\
&= e^{\pi i \vec{J}^{\mathrm{T}}\mathbf{N}^{-1}B_i\vec{J}} \cdot \vartheta \begin{bmatrix} \vec{J}^{\,\mathrm{T}}\mathbf{N}^{-1} \\ 0 \end{bmatrix} (\mathbf{N}\vec{z}, \mathbf{N}\Omega).
\end{aligned}
\tag{A.52}
$$

The $\vec{z}$-dependent exponential factor $e^{\pi i [\mathbf{N}\vec{z}]^{\mathrm{T}} \cdot (\mathrm{Im}\Omega)^{-1} \cdot \mathrm{Im}\vec{z}}$ is invariant under the T_i transformation. Note also that the normalization constant eq. (2.179) is also unchanged. Hence, we obtain

$$
\psi_{\mathbf{N}}^{\vec{J}}(\vec{z}, \Omega + B_i) = e^{\pi i \vec{J}^{\mathrm{T}}(\mathbf{N}^{-1}B_i)\vec{J}} \psi_{\mathbf{N}}^{\vec{J}}(\vec{z}, \Omega). \tag{A.53}
$$

A.3 Scherk-Schwarz phases on magnetized $T^4/\mathbb{Z}_N$

On the magnetized $T^4/\mathbb{Z}_N$ models, the Scherk-Schwarz phases become quantized. As an illustrating example, we explicitly study the quantization of Scherk-Schwarz phases on the magnetized $T^4/\mathbb{Z}_3$ orbifold model analyzed in subsection 2.6.5. We will see that only certain values of Scherk-Schwarz phases are allowed depending on the even/odd of the matrix elements of the flux $\mathbf{N}$. This will clarify that the vanishing Scherk-Schwarz phases are allowed only if the diagonal elements of $\mathbf{N}$ are even integers, cf. (2.281) in the $T^4/\mathbb{Z}_3$ model. Our method for the analyses is along Ref. [26] where Scherk-Schwarz phases in magnetized $T^2/\mathbb{Z}_N, (N = 2, 3, 4, 6)$ models are studied.

A.3.1 Boundary conditions of wavefunctions on $T^4/\mathbb{Z}_N$

We begin with a general discussion on the magnetized $T^4/\mathbb{Z}_N$ whose complex coordinates $\vec{z}$ are identified as $\vec{z} \sim \vec{z} + \vec{e}_k$, $\vec{z} \sim \vec{z} + \Omega\vec{e}_k$, and $\vec{z} \sim \hat{\Omega}\vec{z}$. We use $\hat{\Omega}$ to denote the twist matrix which can be in general different from the complex structure moduli Ω, cf. (2.317) and (2.318). We define the transformation function $U_{m_j\vec{e}_j + n_k\Omega\vec{e}_k}(\vec{z})$ acting on the 4-component spinor $\Psi(\vec{z})$ by the relation:

$$
\Psi(\vec{z} + m_k\vec{e}_k + n_k\Omega\vec{e}_k) = U_{m_k\vec{e}_k + n_k\Omega\vec{e}_k}(\vec{z})\Psi(\vec{z}), \tag{A.54}
$$

where $m_k\vec{e}_k := m_1\vec{e}_1 + m_2\vec{e}_2$, $n_k\Omega\vec{e}_k := n_1\Omega\vec{e}_1 + n_2\Omega\vec{e}_2$, and $m_k, n_k \in \mathbb{Z}$. In particular, we have

$$
\Psi(\vec{z} + \vec{e}_k) = U_{\vec{e}_k}(\vec{z})\Psi(\vec{z}), \tag{A.55}
$$

$$
\Psi(\vec{z} + \Omega\vec{e}_k) = U_{\Omega\vec{e}_k}(\vec{z})\Psi(\vec{z}). \tag{A.56}
$$

Under the $\mathbb{Z}_N$ twist, we have

$$
\Psi(\hat{\Omega}\vec{z}) = \mathcal{S}V(\vec{z})\Psi(\vec{z}), \tag{A.57}
$$

where $\mathcal{S}$ denotes the spinor representation corresponding to the $\mathbb{Z}_N$ twist, i.e., $\mathcal{S}^{-1}\Gamma^{z_i}\mathcal{S} = \hat{\Omega}^i{}_j\Gamma^{z_j}$. $V(\vec{z})$ is the transformation function given by

$$V(\vec{z}) = e^{2\pi i\lambda}, \quad \lambda \in \mathbb{R}, \tag{A.58}$$

which corresponds to the $\mathbb{Z}_N$ charge of $\Psi(\vec{z})$. Then, the following relation must be satisfied,

$$U_{m_k\vec{e}_k+n_k\Omega\vec{e}_k}(\vec{z}\,) = U_{\hat{\Omega}^l(m_k\vec{e}_k+n_k\Omega\vec{e}_k)}(\hat{\Omega}^l\vec{z}\,), \tag{A.59}$$

where $l \in \mathbb{Z}$. This can be verified as follows. We begin with

$$\Psi(\hat{\Omega}^l(\vec{z}+m_k\vec{e}_k+n_k\Omega\vec{e}_k)) = U_{\hat{\Omega}^l(m_k\vec{e}_k+n_k\Omega\vec{e}_k)}(\hat{\Omega}^l\vec{z})\Psi(\hat{\Omega}^l\vec{z}), \tag{A.60}$$

The left-hand side of (A.60) can be rewritten as

$$\begin{aligned} l.h.s. = \mathcal{S}V(\hat{\Omega}^{l-1}(\vec{z}+m_k\vec{e}_k+n_k\Omega\vec{e}_k)) \cdot \mathcal{S}V(\hat{\Omega}^{l-2}(\vec{z}+m_k\vec{e}_k+n_k\Omega\vec{e}_k)) \\ \cdots \mathcal{S}V(\hat{\Omega}(\vec{z}+m_k\vec{e}_k+n_k\Omega\vec{e}_k)) \cdot U_{m_k\vec{e}_k+n_k\Omega\vec{e}_k}(\vec{z})\Psi(\vec{z}), \end{aligned} \tag{A.61}$$

by using eqs. (A.54) and (A.57). On the other hand, the right-hand side of (A.60) can be rewritten as

$$r.h.s. = U_{\hat{\Omega}^l(m_k\vec{e}_k+n_k\Omega\vec{e}_k)}(\hat{\Omega}^l\vec{z}) \cdot \mathcal{S}V(\hat{\Omega}^{l-1}\vec{z}) \cdots \mathcal{S}V(\hat{\Omega}\vec{z})\Psi(\vec{z}). \tag{A.62}$$

Noting that $V(\vec{z})$ is given by (A.58), we obtain (A.59).

A.3.2 Scherk-Schwarz phases in magnetized $T^4/\mathbb{Z}_3$

In the $T^4/\mathbb{Z}_3$ orbifold model, the complex structure moduli Ω and the twist matrix $\hat{\Omega}$ are fixed to

$$\Omega = \hat{\Omega} = \omega\mathbf{1}_2, \tag{A.63}$$

where $\omega := e^{2\pi i/3}$. From eq. (A.59), the following four relations can be obtained,

$$U_{\vec{e}_1}(\vec{z}) = U_{\hat{\Omega}\vec{e}_1}(\hat{\Omega}\vec{z}), \tag{A.64}$$
$$U_{\vec{e}_2}(\vec{z}) = U_{\hat{\Omega}\vec{e}_2}(\hat{\Omega}\vec{z}), \tag{A.65}$$
$$U_{\Omega\vec{e}_1}(\vec{z}) = U_{\hat{\Omega}\Omega\vec{e}_1}(\hat{\Omega}\vec{z}), \tag{A.66}$$
$$U_{\Omega\vec{e}_2}(\vec{z}) = U_{\hat{\Omega}\Omega\vec{e}_2}(\hat{\Omega}\vec{z}). \tag{A.67}$$

According to (2.152), the transformation functions are explicitly given by

$$U_{\vec{e}_k}(\vec{z}) = e^{2\pi i\alpha_k}e^{i\chi_{\vec{e}_k}(\vec{z})}, \tag{A.68}$$
$$U_{\Omega\vec{e}_k}(\vec{z}) = e^{2\pi i\beta_k}e^{i\chi_{\Omega\vec{e}_k}(\vec{z})}, \tag{A.69}$$

where $\alpha_k, \beta_k,\ (k=1,2)$ denote the Scherk-Schwarz phases. Note that $\chi_{\vec{e}_k}$ and $\chi_{\Omega\vec{e}_k}$ are written as

$$\chi_{\vec{e}_k}(\vec{z}) = \pi[\mathbf{N}^{\mathrm{T}}(\mathrm{Im}\Omega)^{-1}\mathrm{Im}(\vec{z})]_k = \pi\frac{2}{\sqrt{3}}\begin{pmatrix} n_1\mathrm{Im}z_1 + m\mathrm{Im}z_2 \\ m\mathrm{Im}z_1 + n_2\mathrm{Im}z_2 \end{pmatrix}_k, \tag{A.70}$$

$$\chi_{\Omega\vec{e}_k}(\vec{z}) = \pi\mathrm{Im}[\mathbf{N}\bar{\Omega}(\mathrm{Im}\Omega)^{-1}(\vec{z})]_k = \pi\frac{2}{\sqrt{3}}\mathrm{Im}\left[\bar{\omega}\begin{pmatrix} n_1z_1 + mz_2 \\ mz_1 + n_2z_2 \end{pmatrix}\right]_k, \tag{A.71}$$

where the integer flux matrix $\mathbf{N}$ is parametrized as

$$\mathbf{N} = \begin{pmatrix} n_1 & m \\ m & n_2 \end{pmatrix}. \tag{A.72}$$

From eqs. (A.64) and (A.65), one obtains

$$e^{2\pi i\alpha_1} = e^{2\pi i\beta_1}, \tag{A.73}$$
$$e^{2\pi i\alpha_2} = e^{2\pi i\beta_2}. \tag{A.74}$$

Hence, we find $\alpha_k \equiv \beta_k \pmod 1$ for $k=1,2$. From eqs. (A.66) and (A.65), one finds

$$e^{2\pi i\alpha_1} = e^{\pi i n_1}e^{-2\pi i(\alpha_1+\beta_1)}, \tag{A.75}$$
$$e^{2\pi i\alpha_2} = e^{\pi i n_2}e^{-2\pi i(\alpha_2+\beta_2)}. \tag{A.76}$$

Consequently, the Scherk-Schwarz phases are quantized depending on the even/odd of the diagonal elements of the flux $\mathbf{N}$. When $n_k \equiv 0 \pmod 2$, the allowed Scherk-Schwarz phases are given by

$$(\alpha_k, \beta_k) = (0,0),\ (1/3,1/3),\ (2/3,2/3), \qquad \mathrm{mod}\ 1. \tag{A.77}$$

When $n_k \equiv 1 \pmod 2$, the allowed Scherk-Schwarz phases are given by

$$(\alpha_k, \beta_k) = (1/6,1/6),\ (1/2,1/2),\ (5/6,5/6), \qquad \mathrm{mod}\ 1. \tag{A.78}$$

Hence, vanishing Scherk-Schwarz phases, i.e., $\vec{\alpha} = \vec{\beta} = \vec{0}$ are possible only when $n_1 \equiv n_2 \equiv 0 \pmod 2$. Similarly, one can analyze the Scherk-Schwarz phases in other magnetized $T^4/\mathbb{Z}_N$ orbifolds.

A.4 Details on the existence of N

For the discussion about the existence of the flux matrix $\mathbf{N}$, we use Kronecker symbol which is defined as follows [158, 159]. We denote arbitrary non-zero integer by n, with prime factorization,

$$n = u \cdot 2^{e_0}p_1^{e_1}\cdots p_r^{e_r}, \tag{A.79}$$

where u is a unit (± 1), and $p_i \, (i = 1, ..., r)$ denotes odd primes. Let a be an integer. Then the Kronecker symbol $\left(\frac{a}{n}\right)_K$ is defined as

$$\left(\frac{a}{n}\right)_K := \left(\frac{a}{u}\right)\left(\frac{a}{2}\right)^{e_0} \prod_{i=1}^{r} \left(\frac{a}{p_i}\right)_L^{e_i}. \tag{A.80}$$

Let us explain the meaning of eq. (A.80). The factor $\left(\frac{a}{p}\right)_L$ denotes the Legendre symbol which is defined for an odd prime p and an integer a as,

$$\left(\frac{a}{p}\right)_L := \begin{cases} +1: \text{ if } a \text{ is a quadratic residue modulo } p \text{ and } a \not\equiv 0 \pmod{p}, \\ -1: \text{ if } a \text{ is not a quadratic residue modulo } p, \\ 0: \text{if } a \equiv 0 \pmod{p}. \end{cases} \tag{A.81}$$

The factor $\left(\frac{a}{u}\right)$ is equal to 1 when $u = 1$. When $u = -1$, we define it by

$$\left(\frac{a}{-1}\right) = \begin{cases} -1: & \text{if } a < 0, \\ +1: & \text{if } a > 0. \end{cases} \tag{A.82}$$

Finally, $\left(\frac{a}{2}\right)$ is defined as

$$\left(\frac{a}{2}\right) = (-1)^{\frac{a^2-1}{8}} = \begin{cases} 0: & \text{if } a \text{ is even}, \\ +1: & \text{if } a \equiv \pm 1 \pmod{8}, \\ -1: & \text{if } a \equiv \pm 3 \pmod{8}. \end{cases} \tag{A.83}$$

It is worth noting that the Kronecker symbol has the following property [159]. For $a \not\equiv 3 \pmod{4}$, $a \neq 0$,

$$\left(\frac{a}{m}\right) = \left(\frac{a}{n}\right), \tag{A.84}$$

holds whenever

$$m \equiv n \mod \begin{cases} 4|a|: & \text{if } a \equiv 2 \pmod{4}, \\ |a|: & \text{otherwise}. \end{cases} \tag{A.85}$$

When n is a positive odd integer, the Kronecker symbol is identical to the Jacobi symbol:

$$\left(\frac{a}{n}\right)_J := \left(\frac{a}{p_1}\right)_L^{e_1} \left(\frac{a}{p_2}\right)_L^{e_2} \cdots \left(\frac{a}{p_r}\right)_L^{e_r}. \tag{A.86}$$

One of the important facts about the Jacobi symbol is that

- If $\left(\frac{a}{n}\right)_J = -1$, a is not a quadratic residue modulo n.

A.4.1 $T^4/\mathbb{Z}_4$

Here, we show that there is no symmetric positive definite 2×2 integer matrix $\mathbf{N}$ such that $n_i \equiv 3 \pmod 4$, $(i=1,\text{or},\ 2)$ when $\det\mathbf{N}=4,16$.

First, we consider the case $\det\mathbf{N}=4$. Suppose that there is a symmetric positive definite integer matrix $\mathbf{N}$ given by

$$\mathbf{N}=\begin{pmatrix} n_1 & m \\ m & 3+4l \end{pmatrix}, \quad l\in\mathbb{Z}, \tag{A.87}$$

where we imposed $n_2 \equiv 3 \pmod 4$. Then, we obtain

$$4=\det\mathbf{N}=n_1(3+4l)-m^2, \tag{A.88}$$

which immediately gives us

$$m^2 \equiv -4 \pmod{(3+4l)}. \tag{A.89}$$

There is no integer solution to eq. (A.89) by the following logic. We find

$$\left(\frac{-4}{3+4l}\right)_K=\left(\frac{-4}{3}\right)_K=-1, \tag{A.90}$$

where (A.84) is applied in the first equality. Since $(-1+4l)$ is a positive odd integer, the Kronecker symbol is reduced to the Jacobi symbol,

$$\left(\frac{-4}{3+4l}\right)_J=-1. \tag{A.91}$$

This shows that -4 is not a quadratic residue modulo $3+4l$. Hence, (A.89) is never satisfied. We reached a contradiction. Thus, there is no positive definite flux matrix $\mathbf{N}$ such that $n_i \equiv 3 \pmod 4$, $(i = 1\,\text{or}\,2)$ when $\det\mathbf{N}=4$. Although we have explicitly discussed only the case $n_2 \equiv 3 \pmod 4$, same argument can be applied to the case $n_1 \equiv 3 \pmod 4$.

By the same method, one can show there is no positive definite flux matrix $\mathbf{N}$ such that $n_i \equiv 3 \pmod 4$, $(i=1,\text{or},\ 2)$ when $\det\mathbf{N}=16$.

A.4.2 $T^4/\mathbb{Z}_3$

Here, we show that there is no symmetric positive definite 2×2 integer matrix $\mathbf{N}$ such that $n'_i \equiv 2 \pmod 3$, $(i=1,\text{or},\ 2)$ when $\det\mathbf{N}=3,27$.

First, we consider the case $\det\mathbf{N}=3$. Suppose that there is a symmetric positive definite integer matrix $\mathbf{N}$ given by

$$\mathbf{N}=\begin{pmatrix} 2n'_1 & m \\ m & 2(3l+2) \end{pmatrix}, \quad l\in\mathbb{Z}, \tag{A.92}$$

where we imposed $n'_2 \equiv 2 \pmod 3$. Then, we obtain

$$3=\det\mathbf{N}=4n'_1(3l+2)-m^2, \tag{A.93}$$

which immediately gives us

$$m^2 \equiv -3 \pmod{(12l+8)}. \tag{A.94}$$

There is no integer solution to eq. (A.94) by the following logic. We consider the prime factorization of $12l+8$,

$$12l+8 = 2^{e_0} p_1^{e_1} \cdots p_r^{e_r}, \tag{A.95}$$

where p_i, $(i = 1, ..., r)$ denotes odd primes. We notice e_0 must be 2 otherwise we have a contradiction.[1] On the other hand, by applying eq. (A.84) we find

$$\left(\frac{-3}{p_1^{e_1} \cdots p_r^{e_r}}\right)_K = \left(\frac{-3}{3l+2}\right)_K = \left(\frac{-3}{2}\right)_K = -1, \tag{A.96}$$

Since $(3l+2) = p_1^{e_1} \cdots p_r^{e_r}$ is a positive odd integer, the Kronecker symbol is reduced to the Jacobi symbol,

$$\left(\frac{-3}{p_1^{e_1} \cdots p_r^{e_r}}\right)_J = -1. \tag{A.97}$$

When the Jacobi symbol is -1, we must have at least one $p_l^{e_l}, (1 \le l \le r)$ such that -3 is its non-quadratic residue, i.e.,

$$\left(\frac{-3}{p_l^{e_l}}\right)_J = -1. \tag{A.98}$$

Now, we note the following basic fact about quadratic residue. Consider two integers $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. Suppose that n is divisible by p^k where p is a prime number and $k \in \mathbb{Z}^+$. Then, we can check the following property: [2]

- a is a quadratic residue modulo $n \leftrightarrow a$ is a quadratic residue modulo p^k, i.e., every prime power dividing n.

Then, we can see that eq. (A.94) is never satisfied. Thus, we reached a contradiction. To conclude there is no positive definite flux matrix $\mathbf{N}$ of the form (A.92) when $\det\mathbf{N} = 3$.[3] By the same method, one can show there is no positive definite flux matrix $\mathbf{N}$ of the form (A.92) when $\det\mathbf{N} = 27$.

[1] Note that $-3 \equiv 5 \pmod 8$ is not a quadratic residue modulo 8.

[2] $(\rightarrow)$: $m^2 - a = ns = p_0^{e_0} p_1^{e_1} \cdots p_r^{e_r} s$ for some $s \in \mathbb{Z}$. Here, $p_i, (i = 0, \cdots)$ denotes prime numbers including 2. By taking modulo $p_j^{\alpha_j}, (0 \le j \le r, 1 \le \alpha_j \le e_j)$, we obtain us $m^2 \equiv a \pmod{p_j^{\alpha_j}}$.
$(\leftarrow)$: We have $m^2 \equiv a \pmod{p_j^{e_j}}, (0 \le j \le r)$. Then, $m^2 - a = s_j p_j^{e_j}$, $s_j \in \mathbb{Z}$ for every $j (= 0, ..., r)$. If $i \ne j$, $\gcd(p_i^{e_i}, p_j^{e_j}) = 1$ holds. Thus, $m^2 - a = p_0^{e_0} \cdots p_r^{e_r} s$ for some $s \in \mathbb{Z}$.

[3] Although we have explicitly discussed only the case $n_2' \equiv 2 \pmod 3$, same argument can be applied to the case $n_1' \equiv 2 \pmod 3$.

Appendix B

A_4 group and more about radiative modulus stabilization

Here, we review the non-Abelian discrete group A_4 and its modular forms. We also show the derivation of eq. (3.68).

B.1 Group theoretical aspects of A_4

The A_4 group is generated by two generators S and T which satisfy the following algebraic relations:

$$S^2 = (ST)^3 = T^3 = \mathbb{I}. \tag{B.1}$$

There are four irreducible representations in A_4. They are are three singlets $\mathbf{1}$, $\mathbf{1}'$ and $\mathbf{1}''$ and one triplet $\mathbf{3}$. Representation matrices of the generators in each representation are given by

$$\begin{aligned}
&\mathbf{1} \quad \rho(S) = 1,\ \rho(T) = 1, \\
&\mathbf{1}' \quad \rho(S) = 1,\ \rho(T) = \omega, \\
&\mathbf{1}'' \quad \rho(S) = 1,\ \rho(T) = \omega^2, \\
&\mathbf{3} \quad \rho(S) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad \rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},
\end{aligned} \tag{B.2}$$

where we have chosen the T-diagonal basis and $\omega = e^{2\pi i/3}$. One can consider tensor products of these representations and they can be decomposed into irreducible ones as shown in Table B.1. Further details can be found at [115, 116].

Tensor product	Clebsch-Gordan decomposition
$\mathbf{1}'' \otimes \mathbf{1}'' = \mathbf{1}'$ $\mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}'' \quad (a^1 b^1)$ $\mathbf{1}'' \otimes \mathbf{1}' = \mathbf{1}$	$a^1 b^1$
$\mathbf{1}'' \otimes \mathbf{3} = \mathbf{3} \quad (a^1 b^i)$	$\begin{pmatrix} a^1b^3 \\ a^1b^1 \\ a^1b^2 \end{pmatrix}$
$\mathbf{1}' \otimes \mathbf{3} = \mathbf{3} \quad (a^1 b^i)$	$\begin{pmatrix} a^1b^2 \\ a^1b^3 \\ a^1b^1 \end{pmatrix}$
$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'' \oplus \mathbf{3} \oplus \mathbf{3}$ $(a^i b^j)$	$(a^1b^1+a^2b^3+a^3b^2)$ $\oplus\, (a^1b^2+a^2b^1+a^3b^3)$ $\oplus\, (a^1b^3+a^2b^2+a^3b^1)$ $\oplus \frac{1}{3}\begin{pmatrix} 2a^1b^1-a^2b^3-a^3b^2 \\ -a^1b^2-a^2b^1+2a^3b^3 \\ -a^1b^3+2a^2b^2-a^3b^1 \end{pmatrix}$ $\oplus \frac{1}{2}\begin{pmatrix} a^2b^3-a^3b^2 \\ a^1b^2-a^2b^1 \\ -a^1b^3+a^3b^1 \end{pmatrix}$

Table B.1: Clebsch-Gordan decomposition for tensor products of A_4 irreducible representations.

B.2 Modular forms of A_4

Here we briefly review the modular forms of A_4. We can construct them from the Dedekind eta function and its first derivative:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i \tau}, \tag{B.3}$$

$$\eta'(\tau) := \frac{d}{d\tau} \eta(\tau), \tag{B.4}$$

where $\tau \in \mathbb{C}$ with positive imaginary part, $\mathrm{Im}(\tau) > 0$. A_4 triplet $\mathbf{3}$ modular forms of weight 2 is explicitly given in Ref. [48] as

$$Y_{\mathbf{3}}^{(2)}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}, \tag{B.5}$$

where

$$Y_1(\tau) = \frac{i}{2\pi} \left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)} \right), \tag{B.6}$$

$$Y_2(\tau) = \frac{-i}{\pi} \left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2 \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} \right), \tag{B.7}$$

$$Y_3(\tau) = \frac{-i}{\pi} \left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2 \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} \right). \tag{B.8}$$

From the tensor products of $Y_i(\tau)$, higher-weight modular forms of A_4 can be constructed. Weight 4 singlet modular forms are given by

$$Y_{\mathbf{1}}^{(4)}(\tau) = Y_1^2 + 2Y_2Y_3, \quad Y_{\mathbf{1}'}^{(4)} = Y_3^2 + 2Y_1Y_2, \quad Y_{\mathbf{1}''}^{(4)} = Y_2^2 + 2Y_1Y_3 = 0. \tag{B.9}$$

Weight 6 singlet modular form is given by

$$Y_{\mathbf{1}}^{(6)}(\tau) = Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1Y_2Y_3, \tag{B.10}$$

respectively. The q-expansions of trivial singlets are given by [84]

$$Y_{\mathbf{1}}^{(4)}(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + \mathcal{O}(q^4), \tag{B.11}$$
$$Y_{\mathbf{1}}^{(6)}(\tau) = 1 - 504q - 16632q^2 - 122976q^3 + \mathcal{O}(q^4). \tag{B.12}$$

Singlet modular forms of weight 8 can also be constructed as

$$\begin{aligned} Y_{\mathbf{1}}^{(8)} &= (Y_1^2 + 2Y_2Y_3)^2, \\ Y_{\mathbf{1}'}^{(8)} &= (Y_1^2 + 2Y_2Y_3)(Y_3^2 + 2Y_1Y_2), \\ Y_{\mathbf{1}''}^{(8)} &= (Y_3^2 + 2Y_1Y_2)^2. \end{aligned} \tag{B.13}$$

B.3 Approximation

Here, we show the derivation of the approximated form of V_1 in eq. (3.68). From eq. (3.61), we can expand as

$$(M^2 + m_0^2)^2 = (\xi^2|\delta\tau|^2 + m_0^2)\left[(\xi^2|\delta\tau|^2 + m_0^2) + 2\xi^2|\delta\tau|^2\delta y\left(\frac{4k_I}{\sqrt{3}} - 6\sqrt{3}\right)\right] + \mathcal{O}(\epsilon^6), \tag{B.14}$$

where $\epsilon \sim |\delta\tau|$. Similarly, we obtain

$$\begin{aligned} \ln\left(\frac{M^2 + m_0^2}{\sqrt{e}\Lambda^2}\right) &= \ln\left[\frac{(\xi^2|\delta\tau|^2 + m_0^2) + \xi^2|\delta\tau|^2\delta y\left(\frac{4k_I}{\sqrt{3}} - 6\sqrt{3}\right) + \mathcal{O}(\epsilon^4)}{\sqrt{e}\Lambda^2}\right] \\ &= \ln\left(\frac{\xi^2|\delta\tau|^2 + m_0^2}{\sqrt{e}\Lambda^2}\right) + \ln\left[1 + \frac{\xi^2|\delta\tau|^2\delta y\left(\frac{4k_I}{\sqrt{3}} - 6\sqrt{3}\right) + \mathcal{O}(\epsilon^4)}{\xi^2|\delta\tau|^2 + m_0^2}\right] \\ &= \ln\left(\frac{\xi^2|\delta\tau|^2 + m_0^2}{\sqrt{e}\Lambda^2}\right) + \frac{\xi^2|\delta\tau|^2\delta y\left(\frac{4k_I}{\sqrt{3}} - 6\sqrt{3}\right)}{\xi^2|\delta\tau|^2 + m_0^2} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{B.15}$$

where we assume $\epsilon^2 := \max(\xi^2|\delta\tau|^2, m_0^2)/\Lambda^2 \leq 0.01$. Thus, we find

$$
\begin{aligned}
&(M^2+m_0^2)^2 \ln\left(\frac{M^2+m_0^2}{\sqrt{e}\Lambda^2}\right) \\
&= (\xi^2|\delta\tau|^2+m_0^2)\left[(\xi^2|\delta\tau|^2+m_0^2) + 2\xi^2|\delta\tau|^2\left\{\delta y\left(\frac{4k_I}{\sqrt{3}} - 6\sqrt{3}\right) + \mathcal{O}(\epsilon^2)\right\}\right] \ln\left[\frac{\xi^2|\delta\tau|^2+m_0^2}{\sqrt{e}\Lambda^2}\right] \\
&\qquad + (\xi^2|\delta\tau|^2+m_0^2)\xi^2|\delta\tau|^2\delta y\left(\frac{4k_I}{\sqrt{3}} - 6\sqrt{3}\right) + \mathcal{O}(\epsilon^6) \\
&\simeq (\xi^2|\delta\tau|^2+m_0^2)\left[(\xi^2|\delta\tau|^2+m_0^2)\ln\left(\frac{\xi^2|\delta\tau|^2+m_0^2}{\sqrt{e}\Lambda^2}\right) + 2\xi^2|\delta\tau|^2\delta y\left(\frac{4k_I-18}{\sqrt{3}}\right)\ln\left(\frac{\xi^2|\delta\tau|^2+m_0^2}{\Lambda^2}\right)\right].
\end{aligned}
\tag{B.16}
$$

This leads to eq. (3.68).

Appendix C

More about spontaneous CP violation

C.1 Model III

In this section, we study another model that can lead to spontaneous CP violation and SUSY breaking. We introduce two matter chiral superfields X_1 and X_2 which transform as trivial singlets under modular transformation. We assume X_1 carries modular weights $k_1 \in \mathbb{Z}$ under $\bar{\Gamma}_{\tau_1}$ and -1 under $\bar{\Gamma}_{\tau_2}, \bar{\Gamma}_{\tau_3}, \bar{\Gamma}_S$. Similarly, X_2 carries modular weights $k_2 \in \mathbb{Z}$ under $\bar{\Gamma}_{\tau_2}$ and -1 under $\bar{\Gamma}_{\tau_1}, \bar{\Gamma}_{\tau_3}, \bar{\Gamma}_S$. For the matter sector, Kähler potential is given by

$$K_{\text{matter}} = Z_1|X_1|^2 + Z_2|X_2|^2. \tag{C.1}$$

and super potential is given by

$$W_{\text{matter}} = \Lambda^2(Y_1(\tau_1)X_1 + \alpha Y_2(\tau_2)X_2), \quad \alpha \in \mathbb{R}, \tag{C.2}$$

where we have defined Z_1 and Z_2 as

$$\begin{aligned} Z_1 &= (-i\tau_1 + i\bar{\tau}_1)^{k_1}(-i\tau_2 + i\bar{\tau}_2)^{-1}(-i\tau_3 + i\bar{\tau}_3)^{-1}(-iS + i\bar{S})^{-1}, \\ Z_2 &= (-i\tau_1 + i\bar{\tau}_1)^{-1}(-i\tau_2 + i\bar{\tau}_2)^{k_2}(-i\tau_3 + i\bar{\tau}_3)^{-1}(-iS + i\bar{S})^{-1}. \end{aligned} \tag{C.3}$$

By requiring the discrete symmetry (4.83) to the matter contributions, both modular forms $Y_1(\tau_1)$ and $Y_2(\tau_2)$ need to be trivial singlets. Furthermore, modular weight of $Y_1(\tau_1)$ is constrained to $k_{Y1} = -(1 + k_1)$ under $\bar{\Gamma}_{\tau_1}$ and that of $Y_2(\tau_2)$ is constrained to $k_{Y2} = -(1 + k_2)$ under $\bar{\Gamma}_{\tau_2}$.

C.1.1 Potential minimum

Let us consider the SUSY condition:

$$\partial_{\tau_i} W = 0, \quad \partial_S W = 0, \quad \partial_{X_1} W = 0, \quad \partial_{X_2} W = 0, \quad W = 0, \tag{C.4}$$

where $W = W_{\text{flux}} + W_{\text{matter}}$ and W_{flux} is given by eq. (4.53). In general, we have no solution to eq. (C.4). One finds that $\partial_{\tau_3} W = 0$ leads to

$$\tau_1 \tau_2 = \frac{b_3}{a^0}. \tag{C.5}$$

Meanwhile, $\partial_{X_1} W = 0$ and $\partial_{X_2} W = 0$ lead to

$$Y_1(\tau_1) = 0, \quad Y_2(\tau_2) = 0. \tag{C.6}$$

Clearly, (C.5) and (C.6) are not generally compatible. This shows that SUSY is spontaneously broken. This leads us to evaluate the SUGRA scalar potential directly.

Scalar potential

We analyze the SUGRA scalar potential of the no-scale type. In the regime $|X_1|, |X_2| \ll 1$, the scalar potential can be expanded as

$$V = \sum_{p_1=0}^{\infty} \sum_{q_1=0}^{\infty} \sum_{p_2=0}^{\infty} \sum_{q_2=0}^{\infty} V_{(p_1,p_2|q_1,q_2)} X_1^{p_1} \bar{X}_1^{q_1} X_2^{p_2} \bar{X}_2^{q_2}. \tag{C.7}$$

The leading order term is explicitly written as

$$\begin{aligned} V_{(0,0|0,0)} = & \frac{\Lambda^4}{\mathcal{V}^2} \left([2\mathrm{Im}(\tau_1)]^{k_{Y1}} |Y_1(\tau_1)|^2 + \alpha^2 [2\mathrm{Im}(\tau_2)]^{k_{Y2}} |Y_2(\tau_2)|^2 \right) \\ & + \frac{2|\tau_3 - fS|^2 \, |a^0 \bar{\tau}_1 \tau_2 - b_3|^2 + 2|\tau_3 - f\bar{S}|^2 \, |a^0 \tau_1 \tau_2 - b_3|^2}{\mathcal{V}^2 [2\mathrm{Im}(\tau_1)][2\mathrm{Im}(\tau_2)][2\mathrm{Im}(\tau_3)][2\mathrm{Im}(S)]}, \end{aligned} \tag{C.8}$$

where the first line corresponds to the matter contributions, while the second line is due to the three-form fluxes.

In what follows, we focus on the case $\Lambda \ll 1 (= M_{\text{Pl}})$ to continue explicit analyses. In this case, the effects of three-form fluxes dominate the potential. Then, even the matter effects would only cause negligibly small deviations from the flat directions:

$$\tau_1 \tau_2 = \frac{b_3}{a^0}, \quad \tau_3 = fS, \tag{C.9}$$

generated by the flux. Hence, $V_{(0,0|0,0)}$ is essentially given by

$$V_{\text{matter}} = \frac{\Lambda^4}{\mathcal{V}^2} \left([2\mathrm{Im}(\tau_1)]^{k_{Y1}} |Y_1(\tau_1)|^2 + \alpha^2 [2\mathrm{Im}(\tau_2)]^{k_{Y2}} |Y_2(\tau_2)|^2 \right), \tag{C.10}$$

with constraints (C.9). If we express V_{matter} in terms of $\tau_1, \bar{\tau}_1$, we obtain

$$V_{\text{matter}} = \frac{\Lambda^4}{\mathcal{V}^2} \left([2\mathrm{Im}(\tau_1)]^{k_{Y1}} |Y_1(\tau_1)|^2 + \alpha^2 \left[\frac{|a^0|}{|b_3|} 2\mathrm{Im}(\tau_1) \right]^{k_{Y2}} \left| Y_2 \left(\frac{|a^0|}{|b_3|} \tau_1 \right) \right|^2 \right). \tag{C.11}$$

This is not invariant under $SL(2,\mathbb{Z})_{\tau_1}$ in general. Indeed, V_{matter} is symmetric under the following discrete group:

$$\Gamma_0(\tilde{a}^0)\cap\Gamma^0(\tilde{b}_3)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})\middle|\begin{pmatrix}a & b\\ c & d\end{pmatrix}\equiv\begin{pmatrix}* & 0\ (\mathrm{mod}\ \tilde{b}_3)\\ 0\ (\mathrm{mod}\ \tilde{a}^0) & *\end{pmatrix}\right\}, \tag{C.12}$$

where $*$ denotes unspecified integers and $\tilde{a}^0,\tilde{b}_3\in\mathbb{Z}^+$ are defined as

$$\tilde{a}^0=\frac{|a^0|}{\gcd(|a^0|,|b_3|)},\quad \tilde{b}_3=\frac{|b_3|}{\gcd(|a^0|,|b_3|)}. \tag{C.13}$$

We explain how the discrete group (C.12) is obtained. We notice the second term on the right-hand side of eq. (C.11) is symmetric under the scale transformed modular group in eq. (4.106). On the other hand, the first term on the right-hand side of eq. (C.11) is $SL(2,\mathbb{Z})_{\tau_1}$ invariant. Therefore, symmetry of V_{matter} is the intersection of (4.106) and $SL(2,\mathbb{Z})_{\tau_1}$. Suppose that γ' is an element that belongs to the intersection. We note that arbitrary element γ' of the group (4.106) can be written as

$$\gamma'=\begin{pmatrix}a' & b'(\tilde{b}_3/\tilde{a}^0)\\ c'(\tilde{a}^0/\tilde{b}_3) & d'\end{pmatrix}, \tag{C.14}$$

where $a',b',c',d'\in\mathbb{Z}$ satisfying $a'd'-b'c'=1$. By the requirement $\gamma'\in SL(2,\mathbb{Z})_{\tau_1}$, all matrix elements of γ' need to be integers. Hence, b',c' can be written as $b'=\tilde{a}^0B, c'=\tilde{b}_3C$ where $B,C\in\mathbb{Z}$. Then, we find

$$\gamma'=\begin{pmatrix}a' & \tilde{b}_3B\\ \tilde{a}^0C & d'\end{pmatrix}\in\Gamma_0(\tilde{a}^0)\cap\Gamma^0(\tilde{b}_3). \tag{C.15}$$

On the other hand, one can also check that arbitrary element of (C.12) belongs to the intersection of (4.106) and $SL(2,\mathbb{Z})_{\tau_1}$. Thus, we understand that the symmetry of V_{matter} is identified as (C.12).

C.1.2 Examples with A_4 modular forms

Let us study concrete examples using A_4 modular forms. We choose

$$Y_1(\tau_1)=i\,Y_{\mathbf{1}}^{(4)}(\tau_1),\quad Y_2(\tau_2)=i\,Y_{\mathbf{1}}^{(6)}(\tau_2), \tag{C.16}$$

where $Y_{\mathbf{1}}^{(4)}$ and $Y_{\mathbf{1}}^{(6)}$ are A_4 trivial singlet modular forms carrying weight $k_{Y1}=4$ and $k_{Y2}=6$, respectively. We have introduced an overall factor i in (C.16) for making $Y_1(\tau_1)$ and $Y_2(\tau_2)$ to behave like (4.90) under CP. By assigning flux quanta, we have numerically analyzed potential minima. We have observed that spontaneous CP violation can be induced by the VEVs of both τ_1 and τ_2 quite easily when the flux quanta satisfy $b_3\nmid a^0$ and $a^0\nmid b_3$.

Example (CP violation)

As an illustrating example, we choose the following flux quanta:

$$a^0 = 2, \quad b_3 = -3, \quad f = 1, \tag{C.17}$$

where the tadpole cancellation condition is satisfied, $n_{\rm flux} = 12 \leq 32$. Then, the scalar potential $V_{\rm matter}$ is explicitly written as

$$V_{\rm matter} = \frac{\Lambda^4}{\mathcal{V}^2}\left([2{\rm Im}(\tau_1)]^4 |Y_{\mathbf{1}}^{(4)}(\tau_1)|^2 + \alpha^2 \left[\frac{4{\rm Im}(\tau_1)}{3}\right]^6 \left|Y_{\mathbf{1}}^{(6)}\left(\frac{2}{3}\tau_1\right)\right|^2\right). \tag{C.18}$$

This is invariant under $\Gamma_0(2) \cap \Gamma^0(3)$ acting on τ_1. We note that the first term on the right-hand side of (C.18) has a zero point at $\tau_1 = \omega$, while the second term is zero at $\tau_1 = 3i/2$. In Fig. C.1, we have shown a contour plot of $\log_{10}(V_{\rm matter}\mathcal{V}^2/\Lambda^4)$ on the τ_1-plane when $\alpha = 0.1$. We have numerically found a local minimum at

$$\langle \tau_1 \rangle \simeq -0.529 + 0.867i, \tag{C.19}$$

where CP is violated. This is a de-Siter vacuum since $(V_{\rm matter}\mathcal{V}^2)/\Lambda^4 \simeq 2.82$. In a modular flavor symmetric model with $SL(2,\mathbb{Z})$ symmetry, VEVs which are related by $\gamma \in SL(2,\mathbb{Z})$ are physically equivalent. Noting this fact, let us consider $T^{-1}S$-transformation which maps $\langle \tau_1 \rangle$ to

$$T^{-1}S\langle \tau_1 \rangle \simeq -0.487 + 0.841i, \tag{C.20}$$

which is inside the fundamental domain $\mathcal{D}$. We notice that the mapped value is close to the $\mathbb{Z}_3$ fixed point $\tau_1 = \omega$. The deviation from $\tau_1 = \omega$ is given by

$$\Delta\tau_1 = T^{-1}S\langle \tau_1 \rangle - \omega \simeq 0.0286 e^{-1.96i}. \tag{C.21}$$

Due to the constraint eq. (C.9), τ_2 is simultaneously fixed at

$$\langle \tau_2 \rangle = -\frac{3}{2\langle \tau_1 \rangle} \simeq 0.770 + 1.26i, \tag{C.22}$$

where CP is violated. We can map $\langle \tau_2 \rangle$ to the interior of the fundamental domain $\mathcal{D}$ by T^{-1} transformation as,

$$T^{-1}\langle \tau_2 \rangle \simeq -0.230 + 1.26i. \tag{C.23}$$

The obtained VEVs are phenomenologically attractive in modular flavor symmetric models. Firstly, hierarchical mass ratios and mixing angles would be realized without fine-tuning. That is possible when a complex structure modulus τ is stabilized in the vicinity of a fixed point [60–68, 70, 71, 74] as in the $A_4 \times A_4 \times A_4$ model in section 3.1. The preferred size of the deviation is typically $|\langle \tau \rangle - \tau_*| \sim \mathcal{O}(0.01 - 0.1)$ where τ_* denotes a finite fixed point. We have successfully reproduced such a slight shift by the VEV of τ_1. Secondly, sizable CP violation would be simultaneously realized. This is because the VEV of another modulus τ_2 breaks CP. Note that modular flavor models with single-modulus can hardly reproduce both sufficient CP violation and hierarchical flavor structures as discussed in Refs. [62–68, 70, 71, 74]. Such difficulty can be lifted by introducing multi-moduli and we have stabilized them at preferable VEVs.

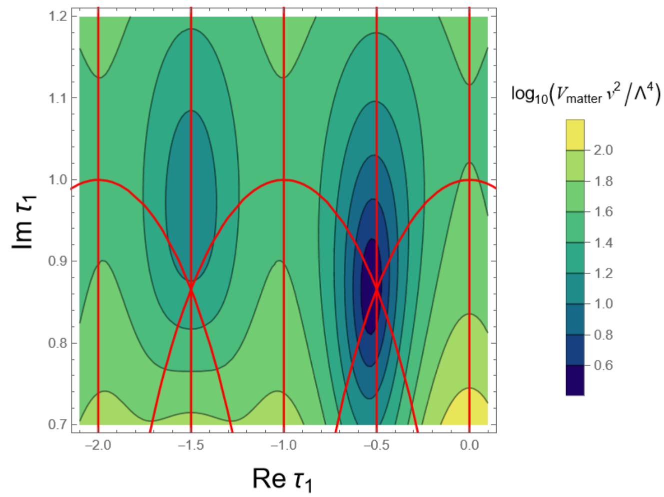

Figure C.1: Contour plot of $\log_{10}(V_{\rm matter}\mathcal{V}^2/\Lambda^4)$ on the τ_1-plane where we have chosen $\alpha = 0.1$. The matter scalar potential $V_{\rm matter}$ is shown in eq. (C.18). The red curves denote the CP-conserving region in the moduli space. Spontaneous CP violation is realized because the potential minimum is not on the curves.

Example (No CP violation)

As another example, we choose the following flux quanta:

$$a^0 = 2, \quad b_3 = -4, \quad f = 1, \tag{C.24}$$

satisfying the tadpole cancellation condition $n_{\rm flux} = 16 \leq 32$. In this case, the scalar potential $V_{\rm matter}$ can be written as

$$V_{\rm matter} = \frac{\Lambda^4}{\mathcal{V}^2}\left([2{\rm Im}(\tau_1)]^4 |Y_{\mathbf{1}}^{(4)}(\tau_1)|^2 + \alpha^2\,[{\rm Im}(\tau_1)]^6 \left|Y_{\mathbf{1}}^{(6)}\left(\frac{1}{2}\tau_1\right)\right|^2\right), \tag{C.25}$$

which is invariant under $\Gamma^0(2)$ acting on τ_1. We note that the first term on the right-hand side of (C.25) has a zero point at $\tau_1 = \omega$, while the second term is zero at $\tau_1 = 2i$. In Fig. C.2, we have shown a contour plot of $\log_{10}(V_{\rm matter}\mathcal{V}^2/\Lambda^4)$ on the τ_1-plane when $\alpha = 0.1$. We have numerically found a local minimum at

$$\langle \tau_1 \rangle \simeq -0.591 + 0.912i, \tag{C.26}$$

which is CP-preserving. This is because the VEV (C.26) is exactly on the CP-conserving arc, $|\tau_1 + 1| = 1$ due to the residual symmetry $\Gamma^0(2)$ whose details are explained in Appendix C.2. Due to the constraint eq. (C.9), τ_2 is fixed at

$$\langle \tau_2 \rangle = -\frac{2}{\langle \tau_1 \rangle} \simeq 1.00 + 1.54i, \tag{C.27}$$

which is CP-conserving. When τ_1 is exactly on the CP-conserving arc, i.e., $\tau_1 = -1 + e^{i\theta}$, $(0 < \theta < \pi)$, we find $\mathrm{Re}(\tau_2) = -\mathrm{Re}(2/\tau_1) = 1$ for any value of θ. This implies that CP is hardly broken even when the value of α is modified.

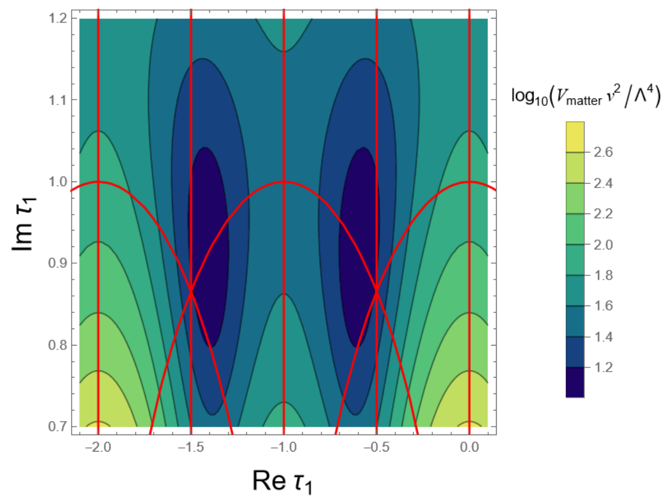

Figure C.2: Contour plot of $\log_{10}(V_{\mathrm{matter}}\mathcal{V}^2/\Lambda^4)$ on the τ_1-plane where we have chosen $\alpha = 0.1$ and $b_3/a^0 = -2$. The red curves denote the CP-conserving region in the moduli space. CP is conserved because the potential minima are precisely lie on the curves.

In our model, we have assumed $\Lambda \ll M_{\mathrm{Pl}}$ so that deviations from the flat directions (C.9) are negligibly small. However, in the regime where Λ becomes comparable to M_{Pl}, such deviations are no longer negligible and need to be treated carefully. A comprehensive study in that direction is beyond the scope of this paper.

C.2 Properties of CP and $\Gamma_0(N), \Gamma^0(N)$ invariant scalar potential

We extend the discussion in section 4.3 to cases when the scalar potential is invariant under the CP and the congruence subgroups such as $\Gamma_0(N)$ and $\Gamma^0(N)$ where $N \in \mathbb{Z}^+$. Here, $\Gamma_0(N)$ and $\Gamma^0(N)$ are defined as

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 \ (\mathrm{mod} N) & * \end{pmatrix} \right\}, \tag{C.28}$$

and

$$\Gamma^0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & 0 \ (\mathrm{mod} N) \\ * & * \end{pmatrix} \right\}, \tag{C.29}$$

respectively. We use the asterisk $*$ to denote unspecified integers.

C.2.1 CP and $\Gamma_0(N)$ invariance

We show some general properties of CP and $\Gamma_0(N)$ invariant scalar potential V. Let us consider the following two elements in $\Gamma_0(N)$,

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad U^N = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}. \tag{C.30}$$

From the CP and T invariance, we immediately find

$$\frac{\partial V}{\partial s}(s,t) = 0, \quad s \equiv 0 \pmod{1/2}, \tag{C.31}$$

where $\tau =: s + it$. Next, we note that U^N transformation maps τ as

$$\tau \xrightarrow{U^N} \tau' = \frac{\tau}{N\tau + 1}. \tag{C.32}$$

Consecutive actions of U^N and CP on τ is given by

$$\tau \xrightarrow{U^N} \frac{1}{N} - \frac{1}{N^2}\frac{e^{-i\theta}}{r} \xrightarrow{CP} -\left(\frac{1}{N} - \frac{1}{N^2}\frac{e^{-i\theta}}{r}\right)^* = -\frac{1}{N} + \frac{1}{N^2}\frac{e^{i\theta}}{r}, \tag{C.33}$$

where r, θ are defined by $|\tau - 1/N| =: re^{i\theta}$. Owing to the symmetry of V, we find

$$V(r,\theta) = V\left(\frac{1}{N^2 r}, \theta\right). \tag{C.34}$$

Then, the first derivative of V with respect to r satisfies

$$\frac{\partial V}{\partial r}\left(\frac{1}{N}, \theta\right) = 0, \tag{C.35}$$

at $r = 1/N$. Note that the arc $|\tau + \frac{1}{N}| = \frac{1}{N}$ is CP-conserving in $SL(2,\mathbb{Z})$ modular flavor symmetric models. This is because $-\bar{\tau} = \gamma\tau$ holds under the choice $\gamma = ST^{-N}S$.

Thus, the scalar potential is stationary at the intersecting points of the CP-conserving arc, i.e., $|\tau - \frac{1}{N}| = \frac{1}{N}$ and $\mathrm{Re}(\tau) \equiv 0 \pmod{1/2}$. As we will see later, these points include the fixed points of $\Gamma_0(N)$. In what follows, we discuss concrete examples, $\Gamma_0(2)$ and $\Gamma_0(3)$.[1]

CP and $\Gamma_0(2)$ invariance

We choose the following two generators for $\Gamma_0(2)$ [161]:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U^2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}. \tag{C.36}$$

From the previous discussion, $\tau_* = \frac{1}{2}(1+i)$ is a stationary point of V. In fact, this corresponds to the fixed point of U^2T^{-1} transformation. Since the order of U^2T^{-1} is 4, i.e., $(U^2T^{-1})^2 = -\mathbf{1}$, the fixed point τ_* preserves the $\mathbb{Z}_4$ residual symmetry.

[1] Such a modular subgroup can emerge in the low-energy effective field theory of superstring theory as recently pointed out in Refs. [157, 160].

CP and $\Gamma_0(3)$ invariance

We choose the following two generators for $\Gamma_0(3)$ [161]:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad -U^3 = \begin{pmatrix} -1 & 0 \\ -3 & -1 \end{pmatrix}. \tag{C.37}$$

From the previous discussion, $\tau_* = \frac{1}{2} + \frac{i}{2\sqrt{3}}$ is a stationary point of V. In fact, this corresponds to the fixed point of $-U^3T^{-1}$ transformation. Since the order of $-U^3T^{-1}$ is 6, i.e., $(-U^3T^{-1})^3 = -\mathbf{1}$, the fixed point τ_* preserves the $\mathbb{Z}_6$ residual symmetry.

C.2.2 CP and $\Gamma^0(N)$ invariance

Similarly, we can analyze the general properties of CP and $\Gamma^0(N)$ invariant scalar potential V. Let us consider the following two elements in $\Gamma^0(N)$,

$$U = \left(\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right), \qquad T^N = \left(\begin{array}{cc} 1 & N \\ 0 & 1 \end{array} \right). \tag{C.38}$$

From the CP and U invariance of V, one finds

$$\frac{\partial V}{\partial r}(r = 1, \theta) = 0, \tag{C.39}$$

where we have defined $|\tau - 1| =: re^{i\theta}$. From the CP and T^N invariance of V, one finds

$$\frac{\partial V}{\partial s}(s, t) = 0, \quad s \equiv 0 \pmod{N/2}, \tag{C.40}$$

where $\tau =: s + it$.

C.3 Condition eq. (4.113) and odd-valued flux quanta

Here, we prove at least one odd integer-valued flux quantum needs to be assumed to satisfy the condition (4.113) under the tadpole cancellation condition (4.57). Let us consider proof by contradiction. Suppose that there exist integers m, n defined by $|a^0| = 2n,\ |b_3| = 2m$ which simultaneously satisfy the following three conditions:

1. $m \nmid n$ and $n \nmid m$. (cf. (4.113))

2. $fmn \leq 4$. (Tadpole cancellation condition, cf. (4.57))

3. $nf \in \mathbb{Z},\ mf \in \mathbb{Z}$. (All flux quanta are even integers, cf. (4.52))

From the first condition, we obtain

$$mn \geq 6. \tag{C.41}$$

Taking into account the second condition, we find

$$f \leq \frac{2}{3}. \tag{C.42}$$

Next, from the first and third conditions there exist positive integers $p, q \in \mathbb{Z}^+$ such that $nf = p,\ mf = q$ with requirements $p \nmid q$ and $q \nmid p$. Thus, we find

$$pq = nmf^2 \leq 4f \leq \frac{8}{3} = 2.66\cdots, \tag{C.43}$$

where we have applied (C.42). This shows that pq is either 1 or 2. Hence, possible choices are given by

$$(p, q) = (1, 1), (1, 2), (2, 1). \tag{C.44}$$

This is in contradiction to the requirement, i.e., $p \nmid q$ and $q \nmid p$. Therefore, we conclude that condition (4.113) and the tadpole cancellation condition (4.57) cannot be satisfied simultaneously if we only allow even integer-valued flux quanta.

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