



HOKKAIDO UNIVERSITY

Title	Homogeneous structures of 3-dimensional Riemannian manifold
Author(s)	大野, 優
Degree Grantor	北海道大学
Degree Name	博士(理学)
Dissertation Number	甲第16238号
Issue Date	2025-03-25
DOI	https://doi.org/10.14943/doctoral.k16238
Doc URL	https://hdl.handle.net/2115/95359
Type	doctoral thesis
File Information	Yu_Ohno.pdf



Doctoral Thesis

Homogeneous structures of 3-dimensional Riemannian
manifold
(3次元リーマン多様体の等質構造)

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March, 2025

Abstract

We determine all homogeneous Riemannian structures and homogeneous space representations on $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$ and 3-dimensional Riemannian Lie groups. This work together with previous results yields a complete classification of all homogeneous Riemannian structures and homogeneous space representations on 3-dimensional homogeneous Riemannian manifolds: $\mathbb{R}^3, \mathbb{H}^3, \mathbb{S}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$ and the 3-dimensional Riemannian Lie groups.

Introduction

A Riemannian manifold (M, g) is said to be **homogeneous** if there exists a Lie group G which acts transitively and isometrically on M . If (M, g) is homogeneous, then we can calculate a geometrical quantity at any point by calculating at one point and using the action of the Lie group G . Moreover, a homogeneous Riemannian manifold (M, g) can be realized as a coset manifold G/H , which is called a **homogeneous space representation**, where H is the isotropy subgroup at some point in M . Note that, for a homogeneous Riemannian manifold (M, g) , a homogeneous space representation is not uniquely determined. Indeed, the 3-sphere $\mathbb{S}^3$ with the standard metric has the homogeneous space representations $\mathbb{S}^3 \cong \text{SO}(4)/\text{SO}(3)$ and $\mathbb{S}^3 \cong \text{SU}(2)/\{Id\}$, where Id is the unit element of $\text{SU}(2)$. Hereafter the unit element of a Lie group is denoted by Id .

Generally, it is difficult to determine whether a given Riemannian manifold (M, g) is homogeneous or not. In 1958 Ambrose and Singer obtained a characterization of homogeneity of Riemannian manifolds in [2]. They proved that a Riemannian manifold (M, g) is locally homogeneous if and only if there exists a tensor field S of type $(1, 2)$ on M such that

$$\tilde{\nabla}g = 0, \quad \tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0, \quad (0.0.1)$$

where $\tilde{\nabla}$ is an affine connection on M defined by $\tilde{\nabla} = \nabla + S$, ∇ is a Levi-Civita connection of the metric g , and R is the curvature tensor field of ∇ . The tensor field S has been called a **homogeneous Riemannian structure** [22].

We recall that typical examples of homogeneous Riemannian manifolds are Riemannian symmetric spaces. It is known that locally symmetric Riemannian manifolds are characterized by E. Cartan as Riemannian manifolds whose curvature is constant along under parallel translation with respect to the Levi-Civita connection, that is, $\nabla R = 0$. Note that, from the equations (0.0.1), a Riemannian symmetric space has a homogeneous Riemannian structure tensor $S = 0$.

Other examples of homogeneous Riemannian manifolds are Riemannian Lie groups, which are given by a pair (G, g) of a Lie group G and a left invariant metric g of G . A Riemannian Lie group (G, g) is obviously homogeneous with a coset representation $G/\{Id\}$. In this case, the corresponding homogeneous Riemannian structure S is given by $S = \nabla^{(-)} - \nabla$, where $\nabla^{(-)}$ is Cartan-Schouten's $(-)$ -connection. Here the Cartan-Schouten's $(-)$ -connection of G is defined by

$$\nabla_X^{(-)}Y = 0, \quad X, Y \in \mathfrak{g},$$

where $\mathfrak{g}$ is the Lie algebra of the Lie group G .

To understand homogeneous Riemannian structures, we closely look at the proof of the theorem of Ambrose and Singer [2]. As a matter of fact the proof was constructive. The authors gave an explicit construction of a homogeneous Riemannian structure S from a Lie group G acting transitively and isometrically on M , and conversely they found a Lie group G acting transitively and isometrically on M from a homogeneous Riemannian structure S .

More precisely, any homogeneous Riemannian structure S corresponds to a **reductive decomposition** of the Lie algebra $\mathfrak{g}$ of a Lie group G . The reductive decomposition is the direct decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that $\mathfrak{m}$ is $\text{Ad}(H)$ -invariant, where $\mathfrak{h}$ is the Lie algebra of the isotropy subgroup H . A homogeneous manifold $M \cong G/H$ is called reductive if there exists a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. It is well-known that every homogeneous Riemannian manifold (M, g) is reductive (Proposition 2.2.4). If we have a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of a homogeneous Riemannian space (M, g) , we can define a metric connection $\tilde{\nabla}$ called the **cannonical connection**. Then the difference tensor $S = \tilde{\nabla} - \nabla$ is a homogeneous Riemannian structure. Conversely, let S be a homogeneous Riemannian structure on (M, g) , and let $\mathfrak{m}$ be a tangent space $T_o M$ at the origin $o \in M$ and $\mathfrak{h}$ the holonomy algebra of the connection $\tilde{\nabla} = S + \nabla$. Then we can define the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that $\tilde{\nabla}$ coincides with the cannonical connection defined from the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

In [2] Ambrose and Singer suggested the possibility of determining all homogeneous space representations by using homogeneous Riemannian structures. However, as pointed out by Tricerri and Vanhecke in [22], this suggestion gave rise to a difficult problem. For instance, the Euclidean plane $\mathbb{R}^2$ is represented by $\text{SE}(2)/\text{SO}(2)$, that is, the coset space representation of the special Euclidean motion group $\text{SE}(2)$ and the special orthogonal group $\text{SO}(2)$, as a Riemannian symmetric space with a homogeneous Riemannian structure $S = 0$. On the other hand, the homogeneous Riemannian space $\mathbb{R}^2 = \mathbb{R}^2/\{0\}$ also has a trivial homogeneous Riemannian structure $S = 0$. That is, there exists homogeneous space representations which cannot be obtained from the construction of the theorem of Ambrose and Singer. Therefore in general it is difficult to determine all homogeneous space representations by using homogeneous Riemannian structures. Here in order to determine all homogeneous space representations of a given homogeneous Riemannian manifold (M, g) , it is useful to divide the problem into two sub-problems as follows:

- (1) To determine all homogeneous Riemannian structures on (M, g) .
- (2) To determine all homogeneous space representations by using homogeneous Riemannian structures or other ways.

In this thesis we will discuss and solve the above problems about the 3-dimensional connected, simply connected and complete homogeneous Riemannian manifolds. First, in [21], Sekigawa showed that any 3-dimensional connected, simply connected and complete homogeneous Riemannian manifold is isometric to either a Riemannian symmetric space: $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{R}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, or a Riemannian group, *i.e.*, Lie group with left-invariant metric. Hereafter $\mathbb{S}^n$, $\mathbb{H}^n$ and $\mathbb{R}^n$ denote n -dimensional sphere, hyperbolic space and Euclidean space with canonical metrics, respectively.

The previous studies can be summerized as follows. Abe [1] determined all homogeneous Riemannian structures on three-dimensional space forms $\mathbb{S}^3$, $\mathbb{H}^3$ and $\mathbb{R}^3$. For Riemannian Lie groups, all homogeneous Riemannian structure tensors on the Heisenberg group and the Berger sphere were determined in [22] and [9], respectively. Recently, homogeneous Riemannian structures on non-symmetric three-dimensional Riemannian Lie groups were studied in [6]. From the results in [1] we can see that there exist the homogeneous space representations which cannot be obtained from homogeneous structures by the theorem of Ambrose and Singer on $\mathbb{R}^3$ or $\mathbb{H}^3$ (see section 4.2). Moreover it was not explicitly stated, but in [6] they classified homogeneous Riemannian structures on non-symmetric three dimensional Riemannian Lie groups under the assumption that homogeneous Riemannian structures are left invariant with respect to the Lie group structure, and the complete classifications of homogeneous Riemannian structures were not obtained.

The main results of this thesis are as follows:

- (1) We determined all homogeneous Riemannian structures on $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ (see Theorems 4.2.10, 4.2.11 or [18]).
- (2) We determined all homogeneous Riemannian structures on 3-dimensional Riemannian Lie groups without the assumption of left invariance (see Theorems 4.4.1, 4.4.2, 4.4.4, 4.4.5 or [13]). In particular, we find two Riemannian Lie groups that they are isometric but not isomorphic, see below for details.
- (3) We determined all homogeneous space representations of 3-dimensional connected, simply connected and complete homogeneous Riemannian manifolds.

These results allow us to fully determine the homogeneous Riemannian structures and the homogeneous space representations of three-dimensional homogeneous Riemannian manifolds, providing a complete solution to the problem posed by Ambrose and Singer. Here we outline the proof of each result. First, it is straightforward to determine all isometry groups and their isotropies of $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. Then we obtain all reductive decompositions corresponding to these homogeneous space representations, and we obtain the homogeneous Riemannian structures from these reductive decompositions.

Second, in order to determine all homogeneous Riemannian structures on 3-dimensional Riemannian Lie groups without the assumption of left invariance, we take a basis $\{e_1, e_2, e_3\}$ of its Lie algebra and let $\{\theta^1, \theta^2, \theta^3\}$ be its dual basis. From the condition $\tilde{\nabla}g = 0$, the homogeneous Riemannian structure S is represented by

$$S_b = \rho \otimes (\theta^1 \wedge \theta^2) + \sigma \otimes (\theta^2 \wedge \theta^3) + \tau \otimes (\theta^3 \wedge \theta^1),$$

where ρ, σ and τ are 1-forms. Hereafter we denote $S_b(X, Y, Z) = g(S_X Y, Z)$ for a homogeneous Riemannian structure S . From the condition $\tilde{\nabla}R = 0$ we can determine some of ρ, σ and τ . Here, for example, suppose that σ and τ are determined and ρ is not determined from $\tilde{\nabla}R = 0$. We define the functions ρ_1, ρ_2 and ρ_3 by $\rho = \rho_1\theta^1 + \rho_2\theta^2 + \rho_3\theta^3$, and then the condition $\tilde{\nabla}S = 0$ is differential equations with respect to the functions ρ_1, ρ_2 and ρ_3 and we

solve it. Note that the assumption of left invariance means that ρ_1, ρ_2 and ρ_3 are considered as constants.

In particular, we find that there exist a non-trivial homogeneous Riemannian structure and homogeneous space representation of the universal covering group $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ of the 3-dimensional special linear group $\mathrm{SL}(2, \mathbb{R})$. It should be noted that this homogeneous Riemannian structure S was not obtained in [5]. The corresponding homogeneous space representation is $G = L'/\{Id\}$. Here L' is a 3-dimensional non-unimodular Lie group whose Lie algebra is isomorphic to the Lie algebra $\mathfrak{ga}(1) \oplus \mathbb{R}$, where $\mathfrak{ga}(1)$ is the Lie algebra of the orientation preserving affine transformation group $\mathrm{GA}^+(1) = \mathrm{GL}^+(1, \mathbb{R}) \ltimes \mathbb{R}$ of the real line. The homogeneous Riemannian structure S is expressed as $S = \nabla^{(-)} - \nabla_{\sim}$, where $\nabla^{(-)}$ is a Cartan-Schouten's $(-)$ -connection for L' . It is important to note that $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ and $\mathrm{GA}^+(1) \times \mathbb{R}$ are not isomorphic as Lie groups, but are isometric as Riemannian manifolds.

Finally recalling that the theorem of Sekigawa [21], any three-dimensional connected, simply connected and complete homogeneous Riemannian manifold is isometric to either a Riemannian symmetric space; $\mathbb{S}^3, \mathbb{H}^3, \mathbb{R}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$, or a Riemannian group. Then we can determine all isometry groups acting simply transitively on these spaces. Note that there exist the homogeneous space representations of $\mathbb{H}^3$ and $\mathbb{R}^3$ which cannot be obtained from the homogeneous Riemannian structures of these spaces. In particular, the homogeneous Riemannian structures which are determined from these homogeneous space representations coincide with homogeneous Riemannian structures given by other homogeneous space representations.

This thesis is organized as follows. In chapter 1, we provide the necessary background and present some general facts on homogeneous spaces and Riemannian symmetric spaces. In chapter 2, we introduce notion of homogeneous Riemannian manifolds. In particular, we will be concerned with reductive decomposition. In chapter 3, we give a brief exposition of the theorem of Ambrose and Singer ([2]), and will look more closely at homogeneous Riemannian structures. Chapter 4 is devoted to the study of homogeneous Riemannian structures and homogeneous space representations of 3-dimensional manifolds. We will classify all homogeneous Riemannian structures and homogeneous space representations on 3-dimensional manifolds.

Acknowledgement

I would like to express my heartfelt gratitude to my supervisor, Professor Shimpei Kobayashi for his dedicated guidance and support over the past five years. His mentorship has been invaluable to my research and academic growth. I would also like to extend my sincere thanks to Professor Jun-ichi Inoguchi, for his generous assistance with my research. His insights and encouragement have greatly enriched my work. Without the cooperation and support of both professors, this doctoral thesis would not have been possible.

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Chapter 1

Preliminaries

In this chapter, we explain the basic concepts of this thesis, homogeneous spaces, affine connections, curvature tensors and symmetric spaces.

1.1 Homogeneous spaces

Let M be a C^∞ -manifold and G a Lie group. We give the definition of a smooth action on M by G .

Definition 1.1.1. An **action** of a Lie group G on a manifold M is a smooth map $\mu: G \times M \rightarrow M$ which satisfies the following two conditions:

- (1) $\mu(Id, x) = x$ for any $x \in M$, where Id is a unit element of G .
- (2) $\mu(g_1, \mu(g_2, x)) = \mu(g_1 \cdot g_2, x)$ for any $g_1, g_2 \in G$ and $x \in M$.

Hereafter, for simplicity, we will denote $\mu(g, x)$ by $g \cdot x$.

Definition 1.1.2. An action of a Lie group G on a manifold M is called **transitive** if for any $x, y \in M$ there exists $g \in G$ such that $g \cdot x = y$.

Definition 1.1.3. A manifold M is called a **homogeneous space** if there exists a Lie group which acts transitively on M .

Let M be a homogeneous space of G . Then we define the isotropy subgroup H at a point $p \in M$ by

$$H = \{g \in G \mid g \cdot p = p\}.$$

Let $\pi: G \rightarrow G/H$ be the natural projection and define a map $\phi: G \times (G/H) \rightarrow G/H$ by $\phi(g, hH) = ghH$. Since H is a closed subgroup of G , then there exists a differential structure on the coset space G/H such that π and ϕ are smooth maps.

Theorem 1.1.4 ([10], p.124 Proposition 4.3). *Let M be a homogeneous space of a Lie group G and H the isotropy subgroup at some point p . Moreover we define a function $\alpha: G/H \rightarrow M$ by $\alpha(gH) = g \cdot p$. Then the function α is a diffeomorphism.*

From this theorem, we obtain a diffeomorphism $M \cong G/H$, and we call the **coset representation** of a homogeneous space M .

Definition 1.1.5. An action of a Lie group G on a manifold M is called **free** if for any $x \in M$, it follows that $g \cdot x = h \cdot x$ if and only if $g = h$ where $g, h \in G$. An action is called **simply transitive** if it is transitive and free.

If a Lie group G acts on a manifold M simply transitively, the isotropy algebra at $p \in M$ is $H = \{Id\}$. Then we can identify G and M by the function $\alpha: G/\{Id\} \rightarrow M$ in Theorem 1.1.4.

Definition 1.1.6. An action of a Lie group G on a manifold M is called **effective** if for any $g, h \in G$ such that $g \neq h$ there exists $x \in M$ such that $g \cdot x \neq h \cdot x$.

1.2 Affine connections

We denote the set of all vector fields on M by $\mathcal{X}(M)$ and the set of all differentiable functions on M by $\mathcal{F}(M)$.

Definition 1.2.1. An **affine connection**

$$\nabla: \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M); \quad (X, Y) \mapsto \nabla_X Y$$

is a map satisfying the following four conditions;

- (1) $\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$,
- (2) $\nabla_{\phi X} Y = \phi \nabla_X Y$,
- (3) $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$,
- (4) $\nabla_X (\phi Y) = (X\phi)Y + \phi \nabla_X Y$.

Here $X, Y, X_1, X_2, Y_1, Y_2 \in \mathcal{X}(M)$ and $\phi \in \mathcal{F}(M)$.

For an affine connection ∇ , we can define covariant derivatives of tensor fields. Here, if K is an (r, s) -type tensor field, then $\nabla_X K$ is also an (r, s) -type tensor field. In this paper, the following two types of tensor fields will be used;

- (1) If K is a $(0, s)$ -type tensor field, $\nabla_X K$ is defined by

$$(\nabla_X K)(X_1, \dots, X_s) = X(K(X_1, \dots, X_s)) - \sum_{i=1}^s K(X_1, \dots, \nabla_X X_i, \dots, X_s) \quad (1.2.1)$$

for any vector fields $X, X_1, \dots, X_s \in \mathcal{X}(M)$.

(2) If K is a $(1, s)$ -type tensor field, $\nabla_X K$ is defined by

$$(\nabla_X K)(X_1, \dots, X_s) = \nabla_X(K(X_1, \dots, X_s)) - \sum_{i=1}^s K(X_1, \dots, \nabla_X X_i, \dots, X_s) \quad (1.2.2)$$

for any vector fields $X, X_1, \dots, X_s \in \mathcal{X}(M)$.

Hereafter, a tensor field K is said to be **parallel** if $\nabla_X K = 0$ for any $X \in \mathcal{X}(M)$.

1.3 Levi-Civita connections

Let (M, g) be a Riemannian manifold. Here, g is a positive definite symmetric $(0, 2)$ -type tensor field on M .

Definition 1.3.1. An affine connection ∇ on (M, g) is called **metrical** if $\nabla g = 0$, that is,

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \quad X, Y, Z \in \mathcal{X}(M). \quad (1.3.1)$$

Definition 1.3.2. The **torsion tensor field** T of an affine connection ∇ is a $(1, 2)$ -type tensor field defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad X, Y \in \mathcal{X}(M). \quad (1.3.2)$$

An affine connection is called **torsion-free** if $T = 0$ identically holds.

On a Riemannian manifold, there exists a unique metrical torsion-free affine connection ∇ , which is called the **Levi-Civita connection**.

In fact, if there exists a metrical torsion-free connection on (M, g) , from (1.3.1) and (1.3.2), it follows that

$$\begin{aligned} 2g(\nabla_X Y, Z) = & Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ & + g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X), \end{aligned} \quad (1.3.3)$$

and then a Levi-Civita connection is unique if it exists. Hereafter, the equation (1.3.3) is called **Koszul formula**.

Conversely define a connection ∇ by the equation (1.3.3), and then one can check ∇ is metrical and torsion-free, *i.e.*, ∇ is the Levi-Civita connection.

For simplicity, we denote the Levi-Civita connection on (M, g) by ∇ in the chapters 2, 3, and 4.

1.4 Curvatures

Definition 1.4.1. The **curvature tensor field** R of an affine connection ∇ is a $(1, 3)$ -type tensor field defined by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]}Z. \quad (1.4.1)$$

For a Riemannian manifold (M, g) , the curvature tensor field R of the Levi-Civita connection is called the **Riemannian curvature**.

The following theorem is well-known.

Theorem 1.4.2 ([19], p.75 Propositions 3.36 and 3.37). *The following identities hold:*

- (1) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$ (1st Bianchi identity),
- (2) $(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0$ (2nd Bianchi identity).

Proof. (1): With a direct computation, we can check

$$\begin{aligned} R(X, Y)Z + R(Y, Z)X + R(Z, X)Y &= \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z \\ &+ \nabla_Z \nabla_Y X - \nabla_Y \nabla_Z X + \nabla_{[Y, Z]} X + \nabla_X \nabla_Z Y - \nabla_Z \nabla_X Y + \nabla_{[Z, X]} Y \\ &= \nabla_Y [X, Z] + \nabla_Z [Y, X] + \nabla_X [Z, Y] - \nabla_{[Y, X]} Z - \nabla_{[Z, Y]} X - \nabla_{[X, Z]} Y \\ &= [Y, [X, Z]] + [Z, [Y, X]] + [X, [Z, Y]] = 0, \end{aligned}$$

(2): Since the objects involved are all tensors, it suffices to prove the equality at a point $p \in M$. Let take a geodesic frame $\{e_1, \dots, e_n\}$ around $p \in M$, that is, $(\nabla_{e_i} e_j)_p = 0$ for any indices i, j . Then we can compute

$$\begin{aligned} &(\nabla_{e_i} R)(e_j, e_k) + (\nabla_{e_j} R)(e_k, e_i) + (\nabla_{e_k} R)(e_i, e_j) \\ &= \nabla_{e_i} (R(e_j, e_k)) + \nabla_{e_j} (R(e_k, e_i)) + \nabla_{e_k} (R(e_i, e_j)) \\ &= -\nabla_{e_i} \nabla_{[e_j, e_k]} - \nabla_{e_j} \nabla_{[e_k, e_i]} - \nabla_{e_k} \nabla_{[e_i, e_j]} \\ &= -\nabla_{[e_i, [e_j, e_k]]} - \nabla_{[e_j, [e_k, e_i]]} - \nabla_{[e_k, [e_i, e_j]]} \\ &= 0. \end{aligned}$$

□

Definition 1.4.3. The **Ricci curvature tensor field** Ric of an affine connection ∇ on a Riemannian manifold (M, g) is a $(0, 2)$ -type tensor field defined by

$$\text{Ric}(X, Y) = \text{tr}_g \{ Z \mapsto R(Z, Y)X \}. \quad (1.4.2)$$

If ∇ is the Levi-Civita connection, its Ricci curvature tensor field Ric is symmetric. However, for an affine connection ∇ , Ric is not always symmetric.

Definition 1.4.4. A Riemannian manifold (M, g) is called an **Einstein space** if there exists a constant $k \in \mathbb{R}$ such that

$$\text{Ric} = kg.$$

Definition 1.4.5. The **scalar curvature** τ of a Riemannian manifold (M, g) is a function defined by

$$\tau = \text{tr}_g \text{Ric} = \sum_{i=1}^n \text{Ric}(e_i, e_i), \quad (1.4.3)$$

where $\{e_1, \dots, e_n\}$ is an orthonormal frame field on (M, g) and Ric is a Ricci curvature tensor field of the Levi-Civita connection.

Definition 1.4.6. Let A be a (0,2)-type tensor field. The **divergence** of A is a (0,1)-type tensor defined by

$$\begin{aligned}\operatorname{div} A(X) &= \operatorname{tr}_g\{(Y, Z) \mapsto (\nabla_Y A)(Z, X)\} \\ &= \sum_{i=1}^n (\nabla_{e_i} A)(e_i, X),\end{aligned}$$

where $\{e_1, \dots, e_n\}$ is an orthonormal frame field on (M, g) and ∇ is the Levi-Civita connection of a metric g .

Proposition 1.4.7 ([19], p.88 Corollary 3.54). *Let τ be the scalar curvature of (M, g) . Then $d\tau = 2 \operatorname{div} \operatorname{Ric}$.*

Proof. Let take a geodesic frame $\{e_1, \dots, e_n\}$. For any $X \in \mathcal{X}(M)$, we compute

$$\begin{aligned}d\tau(X) &= X(\tau) \\ &= X \left(\sum_{i=1}^n \operatorname{Ric}(e_i, e_i) \right) \\ &= \sum_{i=1}^n (\nabla_X \operatorname{Ric})(e_i, e_i) \\ &= \sum_{i=1}^n \sum_{j=1}^n g((\nabla_X R)(e_j, e_i)e_i, e_j),\end{aligned}$$

and

$$\begin{aligned}2 \operatorname{div} \operatorname{Ric}(X) &= 2 \sum_{i=1}^n (\nabla_{e_i} \operatorname{Ric})(e_i, X) \\ &= 2 \sum_{i=1}^n \sum_{j=1}^n g((\nabla_{e_i} R)(e_j, X)e_i, e_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n (-g((\nabla_{e_j} R)(e_i, X)e_i, e_j) - g((\nabla_{e_i} R)(X, e_j)e_i, e_j)),\end{aligned}$$

and then from 2nd Bianchi identity we obtain $d\tau = 2 \operatorname{div} \operatorname{Ric}$. □

1.5 Geodesics

Let (M, g) be a Riemannian manifold, and ∇ the Levi-Civita connection.

Definition 1.5.1. A curve $c: I \rightarrow M$ is called a **geodesic** if for all $t \in I$

$$\nabla_{\frac{d}{dt}} c' = 0.$$

Proposition 1.5.2 ([7], p.64 Proposition 2.7). *For any $p \in M$, there exists a neighborhood V of p , a number $\epsilon > 0$ and a smooth map $\gamma: (-2, -2) \times \mathcal{U} \rightarrow M$, where $\mathcal{U} = \{(q, w) \in TM \mid q \in V, w \in T_q M, \|w\| < \epsilon\}$, such that $t \mapsto \gamma(t) = \gamma(t, q, w)$ is the unique geodesic of M and it follows that $\gamma(0) = q$ and $\gamma'(0) = w$ for every $q \in V$ and $w \in T_q M$ with $\|w\| < \epsilon$.*

Here we denote $\|v\| = \sqrt{g_q(v, v)}$ for $T_q M$. Here we define the exponential map by using Proposition 1.5.2 in the following manner. Let $q \in M$ and $\mathcal{U} \subset TM$ an open set given in Proposition 1.5.2. Then the map $\exp: \mathcal{U} \rightarrow M$ given by

$$\exp(q, v) = \gamma(1, q, v) \quad (q, v) \in \mathcal{U}$$

is called the exponential map. We will denote $\exp_q(v) = \exp(q, v)$.

Proposition 1.5.3. *For any $t \in (-2, -2)$ and $(q, v) \in \mathcal{U}$, it follows that*

$$\exp_q(tv) = \gamma(t, q, v). \quad (1.5.1)$$

Proof. For fix $t \in \mathbb{R}$, we define $c(s) = \gamma(ts, q, v)$ for $s \in \mathbb{R}$ such that $ts \in (-2, -2)$. Then it follows that $c(0) = \gamma(0, q, v) = q$ and $c'(0) = t\gamma'(0, q, v) = tv$, and thus we obtain $c(s) = \gamma(s, q, tv)$, that is,

$$\gamma(ts, q, v) = \gamma(s, q, tv).$$

In particular, we can obtain (1.5.1) if $s = 1$ and $t \in (-2, -2)$. □

Proposition 1.5.4 ([7], p.65 Proposition 2.9). *For any $q \in M$, there exists an $\epsilon > 0$ such that $\exp_q: b_\epsilon(q) \rightarrow M$ is a diffeomorphism of $b_\epsilon(q)$ onto an open subset of M . Here we denote $b_\epsilon(q) = \{v \in T_q M \mid \|v\| < \epsilon\}$.*

Proof. From Proposition 1.5.3 for $v \in T_q M$ we compute

$$\begin{aligned} d(\exp_q)_0(v) &= \left. \frac{d}{dt}(\exp_q(tv)) \right|_0 \\ &= \left. \frac{d}{dt}(\gamma(t, q, v)) \right|_0 \\ &= v \end{aligned}$$

Therefore $d(\exp_q)_0$ is the identity of $T_q M$. Thus from the inverse function theorem it follows that $\exp_q$ is a local diffeomorphism on a neighborhood of $0 \in T_q M$. This completes the proof. □

We define $B_r(q) \subset M$ by

$$B_r(q) = \exp_q(b_r(q)).$$

Then from Proposition 1.5.4 we can take $r > 0$ such that $B_r(q)$ is an open neighborhood at p and $\exp_q$ induces a diffeomorphism from $b_r(p)$ to $B_r(p)$. This neighborhood $B_r(p)$ is called a **normal neighborhood**.

1.6 Riemannian symmetric spaces

Let (M, g) be a Riemannian manifold. At a point $p \in M$, take a normal neighborhood $\mathcal{U}$ and define a diffeomorphism s_p of $\mathcal{U}$ by

$$s_p = \exp_p \circ (-I_p) \circ \exp_p^{-1}. \quad (1.6.1)$$

Here I_p is an identity map on $T_p M$. Then the map s_p is called the local **geodesic symmetry**.

A Riemannian manifold (M, g) is said to be **locally symmetric** if for any $p \in M$, we can take a normal neighborhood $\mathcal{U}$ such that its local symmetry s_p is isometric on $\mathcal{U}$.

Local symmetry of Riemannian manifolds is characterized by the parallelism of the Riemannian curvature tensor R by E. Cartan.

Theorem 1.6.1 (Cartan, [10], p.201 Theorem 1.3). *A Riemannian manifold (M, g) is locally symmetric if and only if $\nabla R = 0$, where ∇ is the Levi-Civita connection of g .*

A Riemannian manifold is said to be a **Riemannian symmetric space** if at each point $p \in M$, there exists a unique isometry s_p such that $(ds_p)_p = -I_p$ and $s_p(p) = p$. Here s_p coincides with the geodesic symmetry on a normal neighborhood, then a Riemannian symmetric space is locally symmetric. Conversely the following holds.

Proposition 1.6.2 ([16], p.167 Theorem 2.1). *A complete, simply connected locally symmetric Riemannian manifold is a Riemannian symmetric space.*

Let (M, g) be a Riemannian symmetric space, and G the identity component of the full isometry group of (M, g) . Moreover we define a Lie group automorphism σ of G by

$$\sigma(a) = s_o a s_o, \quad a \in G,$$

where s_o is a global symmetry at the origin $o \in M$. Then it follows that σ is an involution of G , that is, $\sigma^2(a) = a$ for any $a \in G$. The isotropy subgroup H at o satisfies $G_o^\circ \subset H \subset G_\sigma$, where

$$G_\sigma = \{a \in G \mid \sigma(a) = a\},$$

and G_o° denotes its identity component. This fact can be obtain from the uniqueness of symmetry. Form $G_o^\circ \subset H \subset G_\sigma$ the isotropy subalgebra $\mathfrak{h}$ satisfies

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid d\sigma(X) = X\}.$$

Since $(d\sigma)^2 = I$, $d\sigma$ has eigenvalues ± 1 , and we obtain the eigenspace decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$

where

$$\mathfrak{m} = \{X \in \mathfrak{g} \mid d\sigma(X) = -X\}.$$

Since σ is an automorphism of G , it follows that $d\sigma([X, Y]) = [d\sigma(X), d\sigma(Y)]$ for any $X, Y \in \mathfrak{g}$, and then we can obtain

$$[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}. \quad (1.6.2)$$

We recall the following notion from O'Neill's textbook [19, p. 317]:

Definition 1.6.3. A triplet $(G/H, \sigma, \langle \cdot, \cdot \rangle)$ is called a **symmetric data** if

- (1) H is a closed subgroup of a connected Lie group G .
- (2) σ is an involutive Lie group automorphism of G satisfying $G^\circ \subset H \subset G_\sigma$.
- (3) $\langle \cdot, \cdot \rangle$ is an $\text{Ad}(H)$ -invariant inner product on the (-1) -eigenspace $\mathfrak{m}$ of $d\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$.

We obtain a symmetric data from a Riemannian symmetric space $M = G/H$. In fact, we have checked the conditions (1) and (2), and the condition (3) can be shown from the fact that the isotropy subgroup H is compact.

Conversely from a given symmetric data, we can obtain a Riemannian symmetric space $M = G/H$. For more details, we refer to [19].

Chapter 2

Homogeneous Riemannian manifolds

In this chapter, we introduce homogeneous Riemannian manifolds. In particular, we will prove that a homogeneous Riemannian manifold is reductive. Moreover, we will introduce naturally reductive homogeneous Riemannian manifolds.

2.1 Definitions and examples

Definition 2.1.1. A Riemannian manifold (M, g) is called a **homogeneous Riemannian manifold** if there exists a Lie group G of isometries which acts transitively on M . More generally, a Riemannian manifold (M, g) is called a **locally homogeneous Riemannian manifold** if for each $p, q \in M$ there exists a local isometry which sends p to q .

If (M, g) is a homogeneous Riemannian manifold, then we obtain $M \cong G/H$, where H is the isotropy group at a point $o \in M$, which will be called the **origin** of M . We call this isomorphism a **coset space representation** of a homogeneous Riemannian manifold.

Here we give two examples of homogeneous Riemannian manifolds.

Example 2.1.2 (Riemannian Lie groups). Let G be a Lie group and g a left invariant metric of G . Then G acts on G by itself isometrically and transitively. Therefore (G, g) is a homogeneous Riemannian manifold. A pair (G, g) is called a **Riemannian Lie group**. For a Riemannian Lie group (G, g) we can obtain a canonical coset representation $G \cong G/\{Id\}$, where Id is an unit element of G .

Example 2.1.3 (Riemannian symmetric spaces). If a Riemannian manifold (M, g) is a Riemannian symmetric space, then (M, g) is a homogeneous Riemannian manifold. In fact, since a Riemannian symmetric space (M, g) is complete, for arbitrary points $x, y \in M$ there exists a geodesic $\gamma(t)$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Then the symmetry s_m at $m = \gamma(\frac{1}{2})$ sends x to y isometrically, and thus the isometry group acts transitively on M .

2.2 Reductive decompositions

Next we give the definition of a reductive homogeneous space. Let $M \cong G/H$ be a homogeneous manifold, and $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H , respectively.

Definition 2.2.1. A homogeneous manifold $M \cong G/H$ is called **reductive** if there exists a linear subspace $\mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ such that

- (1) $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$,
- (2) $\mathfrak{m}$ is $\text{Ad}(H)$ -invariant.

Here $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is called a **reductive decomposition** and $\mathfrak{m}$ a **Lie subspace**.

Note that if H is connected, then the $\text{Ad}(H)$ -invariant condition is equivalent to

$$[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}.$$

Here we can identify the Lie subspace $\mathfrak{m}$ with the tangent space $T_o M$ at the origin $o \in M$. In fact, for $X \in \mathfrak{g}$, we define a vector field X^* on M by

$$X_p^* = \left. \frac{d}{dt} \exp(tX) \cdot p \right|_{t=0}, \quad (2.2.1)$$

then we can define the isomorphism $\tau: \mathfrak{m} \rightarrow T_o M$ by

$$\tau(X) = X_o^* \quad (X \in \mathfrak{m}). \quad (2.2.2)$$

Here $\exp: \mathfrak{g} \rightarrow G$ is the exponential map of Lie algebra. We show the following propositions.

Proposition 2.2.2. *Let $M = G/H$ be a homogeneous manifold and define a diffeomorphism $\phi_g: G/H \rightarrow G/H$ by*

$$\phi_g(hH) = ghH,$$

where $g \in G$. Then for any left invariant vector field $X \in \mathfrak{g}$ and any $g \in G$, we have

$$(\phi_g)_* X^* = (\text{Ad}_g(X))^*,$$

where $(\phi_g)_*$ denotes the differential of ϕ_g .

Proof. For any $p = hH \in G/H$, we compute

$$\begin{aligned} ((\phi_g)_* X^*)_{\phi_g(p)} &= (d\phi_g)_p(X_p^*) \\ &= \left. \frac{d}{dt} (g \exp(tX) hH) \right|_{t=0} \\ &= \left. \frac{d}{dt} (g \exp(tX) g^{-1} ghH) \right|_{t=0} \\ &= \left. \frac{d}{dt} (\exp(t \text{Ad}_g(X)) \cdot (ghH)) \right|_{t=0} \\ &= (\text{Ad}_g(X))^*_{\phi_g(p)}, \end{aligned}$$

and we obtain $(\phi_g)_* X^* = (\text{Ad}_g(X))^*$. □

Proposition 2.2.3. *The map $X \mapsto X^*$ is a Lie algebra anti-homomorphism, that is,*

$$[X^*, Y^*] = -[X, Y]^*, \quad X, Y \in \mathfrak{g}. \quad (2.2.3)$$

Proof. For any $p \in M$, we define a function $S_p: G \rightarrow M$ by

$$S_p(g) = \phi_g(p) = g \cdot p, \quad g \in G.$$

Then it follows that

$$X_p^* = (S_p)_*(X_e). \quad (2.2.4)$$

In fact, $a(t) = \exp tX$ is a curve in G such that $a(0) = e$ and $a'(0) = X_e$, and we can compute

$$\begin{aligned} (S_p)_*(X_e) &= \left. \frac{d}{dt}(S_p \circ a)(t) \right|_{t=0} \\ &= \left. \frac{d}{dt}(\exp tX) \cdot p \right|_{t=0} \\ &= X_p^*. \end{aligned}$$

Since $\phi_{\exp tX}(p)$ is a flow of the vector field X^* , from Proposition 2.2.2 and the equation (2.2.4), we compute

$$\begin{aligned} [X^*, Y^*]_p &= \lim_{t \rightarrow 0} \frac{Y_p^* - ((\phi_{\exp tX})_* Y^*)_p}{t} \\ &= \lim_{t \rightarrow 0} \frac{Y_p^* - (\text{Ad}_{\exp tX} Y)_p^*}{t} \\ &= \lim_{t \rightarrow 0} (S_p)_* \frac{Y_e - (\text{Ad}_{\exp tX} Y)_e}{t} \\ &= \lim_{t \rightarrow 0} (S_p)_* \frac{Y_e - ((R_{\exp(-tX)})_* Y)_e}{t} \\ &= -(S_p)_*[X, Y]_e \\ &= -[X, Y]_p^*, \end{aligned}$$

and we obtain the equation (2.2.3). □

For the Riemannian case, the following fact is well-known.

Proposition 2.2.4 ([5], Proposition 1.4.8). *Every homogeneous Riemannian manifold (M, g) is reductive.*

To prove this proposition, we introduce a Killing vector field.

Definition 2.2.5. Let (M, g) be a Riemannian manifold. A vector field X on M is called a **Killing vector field** if the Lie derivative of the metric vanishes; that is, $L_X g = 0$.

Here, $L_X g$ is (0,2) tensor field defined by

$$(L_X g)(Y, Z) = Xg(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]). \quad (2.2.5)$$

The following are important properties of Killing vector fields.

Proposition 2.2.6 ([19], p.251 Proposition 9.23). *A vector field is Killing if and only if the diffeomorphism Φ_t induced from the flow Φ of X are isometries.*

From this proposition, it follows that the vector field X^* defined by (2.2.1) is a Killing vector field. In fact, the flow of the vector field X^* is $\Phi_t(p) = \exp(tX) \cdot p$, and Φ_t is isomorphism by definition of homogeneous Riemannian manifold.

Definition 2.2.7. We define the **Kostant operator** A_X of a vector field X by

$$A_X = L_X - \nabla_X. \quad (2.2.6)$$

Here ∇ is the Levi-Civita connection.

In particular, it follows that

$$A_X Y = L_X Y - \nabla_X Y = -\nabla_Y X, \quad (2.2.7)$$

since $L_X Y = [X, Y]$ and ∇ is torsion free.

Proposition 2.2.8. *For any vector field X , the following statements are equivalent:*

- (1) X is a Killing vector field.
- (2) Kostant operator A_X is skew symmetric, that is,

$$g(A_X Y, Z) + g(Y, A_X Z) = 0, \quad Y, Z \in \mathcal{X}(M).$$

Proof. Since the connection ∇ is the Levi-Civita connection of g , ∇ is a metric connection, that is,

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \quad X, Y, Z \in \mathcal{X}(M). \quad (2.2.8)$$

Moreover, since the connection ∇ is torsion free, it follows that

$$\begin{aligned} \nabla_X Y &= \nabla_Y X + [X, Y], \\ \nabla_X Z &= \nabla_Z X + [X, Z]. \end{aligned}$$

Then from (2.2.8), we obtain

$$\begin{aligned} Xg(Y, Z) &= g(\nabla_Y X + [X, Y], Z) + g(Y, \nabla_Z X + [X, Z]) \\ &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) + g([X, Y], Z) + g(Y, [X, Z]), \end{aligned}$$

and we can obtain

$$(L_X g)(Y, Z) = -g(A_X Y, Z) - g(Y, A_X Z).$$

Thus $L_X g = 0$ if and only if A_X is skew symmetric. This completes the proof. $\square$

From the following proposition, we see that a Killing vector field X is uniquely determined over M by the value of X and A_X at any single point p .

Proposition 2.2.9 ([22], p.2 Proposition 1.2). *Let X and Y be two Killing vector fields on a connected Riemannian manifold (M, g) . Suppose that*

$$X_p = Y_p, \quad A_X|_p = A_Y|_p,$$

for some $p \in M$. Then $X = Y$.

Proof of Proposition 2.2.4. Let (M, g) be a homogeneous Riemannian manifold, and fix a point $o \in M$ as the origin. Then we have $M = G/H$, and let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H , respectively. We can suppose that the action of G on M is effective. Moreover, we define a vector field X^* on M by (2.2.1). Note that

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid X_o^* = 0\}.$$

Since X^* is Killing vector field, from the Proposition 2.2.8 the Kostant operator A_{X^*} is skew symmetric, and then $A_{X^*}|_o \in \mathfrak{so}(T_o M)$. Let B denote the Killing form of $\mathfrak{so}(T_o M)$. Then B is definite since g is positive definite. We define the symmetric bilinear form ϕ on $\mathfrak{g}$ by

$$\phi(X, Y) = -B(A_{X^*}|_o, A_{Y^*}|_o).$$

Then $\phi(X, X) = 0$ implies that $A_{X^*}|_o = 0$. Therefore, if $X \in \mathfrak{h}$ and $\phi(X, X) = 0$, we obtain $X_o^* = 0$ and $A_{X^*}|_o = 0$, and it follows that $X^* = 0$ from Proposition 2.2.9. Then we can obtain

$$\left. \frac{d}{dt} \exp(tX) \cdot p \right|_{t=s} = 0 \tag{2.2.9}$$

for any $p \in M$ and $s \in \mathbb{R}$. In fact, for any $s \in \mathbb{R}$ we compute

$$\begin{aligned} \left. \frac{d}{dt} \exp(tX) \cdot p \right|_{t=s} &= \left. \frac{d}{du} \exp((u+s)X) \cdot p \right|_{u=0} \\ &= \left. \frac{d}{du} \exp(uX) \cdot (\exp(sX) \cdot p) \right|_{u=0} \\ &= X_{\exp(sX) \cdot p}^*, \end{aligned}$$

and from $X^* = 0$ we obtain the equation (2.2.9). Here we put $u = t - s$. Then it follows that $\exp(tX) \cdot p = p$ for any $p \in M$ and $t \in \mathbb{R}$, and since G acts on M effectively, we obtain $X = 0$. Thus ϕ is definite on $\mathfrak{h}$. Then we can take the orthogonal complement $\mathfrak{m} = \mathfrak{h}^\perp$ with respect to ϕ , and it follows that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

It remains to prove that $\text{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$, and it is sufficient to show that ϕ is $\text{Ad}(H)$ -invariant. For any $h \in H$, $X, Y \in \mathfrak{g}$ and any orthonormal basis $\{e_1, e_2, \dots, e_n\}$ on $T_o M$, from Proposi-

tion 2.2.2, we can compute

$$\begin{aligned}
\phi(\text{Ad}_h(X), \text{Ad}_h(Y)) &= -B(A_{(\text{Ad}_h(X))^*}|_o, A_{(\text{Ad}_h(Y))^*}|_o) \\
&= -B(A_{(\phi_g)_*X^*}|_o, A_{(\phi_g)_*Y^*}|_o) \\
&= -\sum_{i=1}^n g(e_i, A_{(\phi_g)_*X^*}(A_{(\phi_g)_*Y^*}(e_i))) \\
&= \sum_{i=1}^n g(A_{(\phi_g)_*X^*}(e_i), A_{(\phi_g)_*Y^*}(e_i)) \\
&= \sum_{i=1}^n g(\nabla_{e_i}((\phi_g)_*X^*), \nabla_{e_i}((\phi_g)_*Y^*)) \\
&= \sum_{i=1}^n g(\nabla_{e_i}X^*, \nabla_{e_i}Y^*) \\
&= \phi(X, Y),
\end{aligned}$$

and thus we obtain

$$\phi(\text{Ad}_h(X), \text{Ad}_h(Y)) = \phi(X, Y).$$

This completes the proof. $\square$

2.3 Naturally reductive homogeneous spaces

In this section, we introduce the concept of **naturally reductive homogeneous spaces**, a significant subclass of reductive homogeneous Riemannian manifolds.

First let (M, g) be a homogeneous Riemannian manifold and $\mathfrak{m}$ a Lie subspace, and we give an expression for $(\nabla_{X^*}Y^*)_o$, where $X, Y \in \mathfrak{m}$.

Proposition 2.3.1. *The Levi-Civita connection ∇ at the origin o is given by:*

$$(\nabla_{X^*}Y^*)_o = -\frac{1}{2}([X, Y]_{\mathfrak{m}}^*)_o + U(X, Y)_o^*, \quad (2.3.1)$$

where $X, Y \in \mathfrak{m}$ and $U: \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$ is a symmetric bilinear form defined by

$$2\langle U(X, Y), Z \rangle = \langle [Z, X]_{\mathfrak{m}}, Y \rangle + \langle X, [Z, Y]_{\mathfrak{m}} \rangle. \quad (2.3.2)$$

Here $\langle \cdot, \cdot \rangle$ is an inner product on $\mathfrak{m}$ induced from g .

Proof. Since X^*, Y^*, Z^* are Killing vector fields, then by Definition 2.2.5 and (2.2.5), we obtain

$$\begin{aligned}
X^*\langle Y^*, Z^* \rangle &= \langle [X^*, Y^*], Z^* \rangle + \langle Y^*, [X^*, Z^*] \rangle, \\
Y^*\langle Z^*, X^* \rangle &= \langle [Y^*, Z^*], X^* \rangle + \langle Z^*, [Y^*, X^*] \rangle, \\
Z^*\langle X^*, Y^* \rangle &= \langle [Z^*, X^*], Y^* \rangle + \langle X^*, [Z^*, Y^*] \rangle.
\end{aligned}$$

Moreover, by the Koszul formula, it follows that

$$2\langle \nabla_{X^*} Y^*, Z^* \rangle = X^*(\langle Y^*, Z^* \rangle) + Y^*(\langle X^*, Z^* \rangle) - Z^*(\langle X^*, Y^* \rangle) \\ - \langle Y^*, [X^*, Z^*] \rangle - \langle X^*, [Y^*, Z^*] \rangle - \langle Z^*, [Y^*, X^*] \rangle,$$

and we obtain

$$2\langle \nabla_{X^*} Y^*, Z^* \rangle = \langle [X^*, Y^*], Z^* \rangle + \langle [X^*, Z^*], Y^* \rangle + \langle [Y^*, Z^*], X^* \rangle.$$

Thus, from Proposition 2.2.3, we have

$$2\langle \nabla_{X^*} Y^*, Z^* \rangle = -\langle [X, Y]^*, Z^* \rangle - \langle [X, Z]^*, Y^* \rangle - \langle [Y, Z]^*, X^* \rangle.$$

Here, by evaluating at o and using the fact that $X_o^* = (X_{\mathfrak{m}^*})_o$, we obtain

$$2\langle (\nabla_{X^*} Y^*)_o, Z_o^* \rangle = -\langle ([X, Y]_{\mathfrak{m}^*})_o, Z_o^* \rangle - \langle ([X, Z]_{\mathfrak{m}^*})_o, Y_o^* \rangle - \langle ([Y, Z]_{\mathfrak{m}^*})_o, X_o^* \rangle,$$

and consequently we have

$$(\nabla_{X^*} Y^*)_o = -\frac{1}{2}([X, Y]_{\mathfrak{m}^*})_o + U(X, Y)_o^*.$$

This completes the proof. $\square$

Definition 2.3.2 (Naturally reductive homogeneous space). A homogeneous Riemannian manifold $(M \cong G/H, g)$ is said to be **naturally reductive** if there exists a Lie subspace $\mathfrak{m}$ such that

$$\langle [X, Y]_{\mathfrak{m}}, Z \rangle + \langle Y, [X, Z]_{\mathfrak{m}} \rangle = 0, \quad X, Y, Z \in \mathfrak{m}, \quad (2.3.3)$$

that is, $U(X, Y) = 0$ for all $X, Y \in \mathfrak{m}$, where U is defined by (2.3.2). Here $\langle \cdot, \cdot \rangle$ is the inner product of $\mathfrak{m}$ induced from the metric g , and $[\cdot, \cdot]_{\mathfrak{m}}$ is the projection of $[\cdot, \cdot]$ on $\mathfrak{m}$.

A naturally reductive homogeneous Riemannian manifold is characterized by its geodesics.

Proposition 2.3.3 ([22], p.58 Theorem 6.2). *A homogeneous Riemannian manifold $(M \cong G/H, g)$ is naturally reductive if and only if there exists a Lie subspace $\mathfrak{m}$ such that every geodesic through $o \in M$ and tangent to $X \in \mathfrak{m} \cong T_o M$ is the orbit of the one-parameter subgroup $\{\exp(tX)\}$.*

Chapter 3

Ambrose-Singer connections and homogeneous Riemannian structures

In this chapter, we introduce an infinitesimal characterization of local homogeneity of Riemannian manifolds given by Ambrose and Singer, see [2] and [22] for more details.

3.1 The theorem of Ambrose and Singer

Theorem 3.1.1 ([2], [22, p.16 Theorem 1.11 and Theorem 1.12]). *A Riemannian manifold (M, g) is locally homogeneous if and only if there exists a tensor field S of type $(1, 2)$ on M such that*

$$\tilde{\nabla}g = 0, \quad \tilde{\nabla}R = 0, \quad \tilde{\nabla}S = 0, \quad (3.1.1)$$

where $\tilde{\nabla}$ is an affine connection on M defined by $\tilde{\nabla} = \nabla + S$, ∇ is a Levi-Civita connection of the metric g , and R is a Riemannian curvature tensor field.

Proof. We give a sketch of the proof. For details see [2] and [22].

Let $(M \cong G/H, g)$ be a homogeneous Riemannian manifold. From Proposition 2.2.4 there exists a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then we define an affine connection $\tilde{\nabla}$ by

$$(\tilde{\nabla}_{X^*} Y^*)_o = -([X, Y]_{\mathfrak{m}})^*_o, \quad X, Y \in \mathfrak{m}. \quad (3.1.2)$$

Note that it is sufficient to define $\tilde{\nabla}$ only at the origin o by homogeneity. This connection is called the **canonical connection** of a homogeneous Riemannian manifold $M \cong G/H$ with respect to $\mathfrak{m}$. One can see that the difference tensor field $S = \tilde{\nabla} - \nabla$ satisfies the equations (3.1.1) and S is a homogeneous Riemannian structure. Thus every homogeneous Riemannian manifold admits a homogeneous Riemannian structure. Moreover, for $u, v, w \in T_o(M)$, from (3.1.2) we have

$$2g(S_u v, w)_o = g([X, Y]^*, Z^*)_o - g([Y, Z]^*, X^*)_o + g([Z, X]^*, Y^*)_o, \quad (3.1.3)$$

where we take $X, Y, Z \in \mathfrak{m}$ such that $\tau(X) = u, \tau(Y) = v$, and $\tau(Z) = w$.

Conversely, let S be a homogeneous Riemannian structure on a homogeneous Riemannian manifold (M, g) . Fix a origin $o \in M$ and set $\mathfrak{m} = T_o M$. Let us denote by $\tilde{R}$ the curvature tensor field of the Ambrose-Singer connection $\tilde{\nabla} = S + \nabla$. Then the holonomy algebra $\mathfrak{h}$ of $\tilde{\nabla}$ is the Lie subalgebra of the Lie algebra $\mathfrak{so}(\mathfrak{m}, g_o)$ generated by the curvature operators $\tilde{R}(X, Y)$ with $X, Y \in \mathfrak{m}$. Now we can define a Lie bracket $[\cdot, \cdot]$ on the direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ by

$$\begin{aligned} [U, V] &= UV - VU, \\ [U, X] &= U(X), \\ [X, Y] &= -\tilde{R}(X, Y) - S_X Y + S_Y X, \end{aligned} \tag{3.1.4}$$

for all $X, Y \in \mathfrak{m}$ and $U, V \in \mathfrak{h}$.

Now let $\tilde{G}$ be the simply connected Lie group with Lie algebra $\mathfrak{g}$. Then M is a coset manifold $\tilde{G}/\tilde{H}$, where $\tilde{H}$ is a Lie subgroup of $\tilde{G}$ with Lie algebra $\mathfrak{h}$. Let Γ be the set of all elements in $\tilde{G}$ which act trivially on M . Then Γ is a discrete normal subgroup of $\tilde{G}$ and $G = \tilde{G}/\Gamma$ acts transitively and effectively on M as an isometry group. The isotropy subgroup H of G at o is $H = \tilde{H}/\Gamma$. Hence (M, g) is a homogeneous Riemannian space with coset space representation $M = G/H$. Note that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition since $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ from (3.1.4). $\square$

Remark 3.1.2 ([1],[14]). For the homogeneous Riemannian structure S defined from a reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, the Lie algebra of the isometry group and its reductive decomposition defined from S is given by $\tilde{\mathfrak{g}} = \mathfrak{m} \oplus [\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}}$.

Proposition 3.1.3 ([14], p.31 Proposition I.13). *Let M be a reductive homogeneous space and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ the corresponding reductive decomposition. Then the curvature $\tilde{R}$ of the canonical connection $\tilde{\nabla}$ satisfies*

$$(\tilde{R}(X, Y)Z)_o = -[[X, Y]_{\mathfrak{h}}, Z] \tag{3.1.5}$$

for any $X, Y, Z \in \mathfrak{m}$.

3.2 Homogeneous Riemannian structures

In this section, we will look more closely at homogeneous Riemannian structures. First we give two examples of homogeneous Riemannian structures.

Example 3.2.1 (Riemannian Lie groups). Let (G, g) be a Riemannian Lie group, which is defined in Example 2.1.2. Then (G, g) is homogeneous and we obtain a natural coset space representation $G \cong G/\{Id\}$. Let $\mathfrak{g}$ be the Lie algebra of the Lie group G , and then the isotropy algebra is $\{0\}$ and the tangent space at the unit element Id is identified with $\mathfrak{m} = \mathfrak{g}$.

Then we get the decomposition $\mathfrak{g} = \{0\} \oplus \mathfrak{g}$, which is obviously reductive. Here the canonical connection $\nabla^{(-)}$ of $G/\{Id\}$ is defined by

$$\nabla_X^{(-)}Y = 0, \quad X, Y \in \mathfrak{g}. \quad (3.2.1)$$

The canonical connection $\nabla^{(-)}$ is called the **Cartan-Schouten's $(-)$ -connection**. Then we obtain a homogeneous Riemannian structure $S = \nabla^{(-)} - \nabla$.

Example 3.2.2 (Riemannian symmetric spaces). Let (M, g) be a Riemannian symmetric space, then from Theorem 1.6.1 it follows that $\nabla R = 0$. Then if we set $S = 0$ and $\tilde{\nabla} = \nabla$, then S satisfies the equation (3.1.1). Thus $S = 0$ is a homogeneous Riemannian structure and ∇ is an Ambrose-Singer connection. Conversely, if a simply connected Riemannian manifold (M, g) has a homogeneous Riemannian structure $S = 0$, then (M, g) is symmetric.

To close this section, we define an isomorphism of homogeneous Riemannian structure tensors.

Definition 3.2.3 ([22], p.33 Definition 2.2). Let S be a homogeneous Riemannian structure tensor on (M, g) and S' be a homogeneous Riemannian structure tensor on (M', g') . Then S and S' are said to be **isomorphic** if there exists an isometry $\phi: (M, g) \rightarrow (M', g')$ such that

$$\phi_*(S_X Y) = S'_{\phi_* X} \phi_* Y$$

for all $X, Y \in \mathfrak{X}(M)$, where ϕ_* denotes the differential of ϕ .

Theorem 3.2.4 ([22], p.35 Theorem 2.4). *Let (M, g) be a homogeneous Riemannian space and G, G' connected Lie subgroups of the identity component of the full isometry group $\text{Isom}_o(M, g)$ acting transitively on M . Assume that the Lie algebras $\mathfrak{g}$ of G and $\mathfrak{g}'$ of G' has reductive decompositions $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$, respectively. Then the homogeneous Riemannian structures S determined by $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and S' determined by $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$ are isomorphic if and only if there exists a Lie algebra isomorphism $F: \mathfrak{g} \rightarrow \mathfrak{g}'$ such that*

$$F(\mathfrak{h}) = \mathfrak{h}', \quad F(\mathfrak{m}) = \mathfrak{m}'$$

and $F|_{\mathfrak{m}}$ is a linear isometry.

Hereafter, we denote the full isometry group of (M, g) by $\text{Isom}(M) = \text{Isom}(M, g)$ and the identity component of $\text{Isom}(M)$ by $\text{Isom}_o(M)$.

3.3 The eight classes of homogeneous Riemannian structures

For a homogeneous Riemannian structure S on a Riemannian manifold (M, g) , we denote by S_b the covariant tensor field metrically equivalent to S , that is,

$$S_b(X, Y, Z) = g(S_X Y, Z)$$

for all $X, Y, Z \in \mathcal{X}(M)$. The metrical condition $\tilde{\nabla}g = 0$ is rewritten as

$$S_b(X, Y, Z) + S_b(X, Z, Y) = 0 \quad (3.3.1)$$

for all vector fields X, Y and Z .

Tricerri and Vanhecke [22] obtained the following decompositions of all possible types of homogeneous Riemannian structures into eight classes (Table 3.1):

Classes	Defining conditions
Symmetric	$S = 0$
$\mathcal{T}_1$	$S_b(X, Y, Z) = g(X, Y)\omega(Z) - g(Z, X)\omega(Y)$ for some 1-form ω
$\mathcal{T}_2$	$\mathfrak{S}_{X,Y,Z} S_b(X, Y, Z) = 0$ and $c_{12}(S_b) = 0$
$\mathcal{T}_3$	$S_b(X, Y, Z) + S_b(Y, X, Z) = 0$
$\mathcal{T}_1 \oplus \mathcal{T}_2$	$\mathfrak{S}_{X,Y,Z} S_b(X, Y, Z) = 0$
$\mathcal{T}_1 \oplus \mathcal{T}_3$	$S_b(X, Y, Z) + S_b(Y, X, Z) = 2g(X, Y)\omega(Z) - g(Z, X)\omega(Y) - g(Y, Z)\omega(X)$ for some 1-form ω
$\mathcal{T}_2 \oplus \mathcal{T}_3$	$c_{12}(S_b) = 0$
$\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$	no conditions

Table 3.1: The eight classes

Here $\mathfrak{S}_{X,Y,Z} S_b$ denotes the cyclic sum of S_b , *i.e.*,

$$\mathfrak{S}_{X,Y,Z} S_b(X, Y, Z) = S_b(X, Y, Z) + S_b(Y, Z, X) + S_b(Z, X, Y),$$

and c_{12} denotes the contraction operator in $(1, 2)$ -entries:

$$c_{12}(S_b)(Z) = \sum_{i=1}^n S_b(e_i, e_i, Z),$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame field. In [22], Tricerri and Vanhecke characterized naturally reductive homogeneous Riemannian manifolds by homogeneous Riemannian structures as follows:

Theorem 3.3.1 ([22], p.57 Theorem 6.1). *Let (M, g) be a naturally reductive homogeneous Riemannian manifold, $\mathfrak{m}$ a corresponding Lie subspace and $\tilde{\nabla}$ the affine connection given by (3.1.2). Then $S = \tilde{\nabla} - \nabla$ is a homogeneous Riemannian structure which satisfies*

$$S_X X = 0, \quad \text{for any } X \in \mathcal{X}(M), \quad (3.3.2)$$

that is, $S_b \in \mathcal{T}_3$. Conversely, if a complete and simply connected Riemannian manifold (M, g) is equipped with a homogeneous Riemannian structure S which satisfies (3.3.2), then (M, g) is naturally reductive.

Proof. First let (M, g) be a naturally reductive homogeneous Riemannian manifold, and $\tilde{\nabla}$ be the affine connection given by (3.1.2). Then there exists a Lie subspace $\mathfrak{m}$ such that $U(X, Y) = 0$ for any $X, Y \in \mathfrak{m}$. Then, from (2.3.1), we obtain

$$(\nabla_{X^*} Y^*)_o = -\frac{1}{2}([X, Y]_{\mathfrak{m}}^*)_o,$$

and thus the homogeneous Riemannian structure $S = \tilde{\nabla} - \nabla$ is given by

$$\begin{aligned} (S_{X^*} Y^*)_o &= (\tilde{\nabla}_{X^*} Y^*)_o - (\nabla_{X^*} Y^*)_o \\ &= -([X, Y]_{\mathfrak{m}}^*)_o + \frac{1}{2}([X, Y]_{\mathfrak{m}}^*)_o \\ &= -\frac{1}{2}([X, Y]_{\mathfrak{m}}^*)_o. \end{aligned}$$

Therefore, it follows that $S_X X = 0$ for any vector field X .

Conversely, we suppose that there exists a homogeneous Riemannian structure S such that $S_X X = 0$ for any $X \in \mathcal{X}(M)$. It is sufficient to show that $U(X, Y) = 0$ for any $X, Y \in \mathfrak{m}$. From (3.1.4) and (3.3.1), we compute

$$\begin{aligned} 2\langle U(X, Y), Z \rangle &= \langle [Z, X]_{\mathfrak{m}}, Y \rangle + \langle X, [Z, Y]_{\mathfrak{m}} \rangle \\ &= \langle -S_Z X + S_X Z, Y \rangle + \langle X, -S_Z Y + S_Y Z \rangle \\ &= -\langle S_X Y + S_Y X, Z \rangle, \end{aligned}$$

and since $S_X Y + S_Y X = 0$, we obtain $U(X, Y) = 0$. This completes the proof. $\square$

3.4 The 2-dimensional Riemannian manifolds

In this section, we give the classification of homogeneous Riemannian structures on 2-dimensional Riemannian manifolds according to [22]. Moreover, we also give the classification of their homogeneous representations.

Let (M, g) be a connected, simply connected and complete 2-dimensional Riemannian manifold. Moreover, we suppose that (M, g) has a homogeneous Riemannian structure $S \neq 0$. Then $S \in \mathcal{T}_1$ since $\dim M = 2$ (see [22], Theorem 3.1), that is, there exists a 1-form ω such that

$$S_b(X, Y, Z) = g(X, Y)\omega(Z) - g(Z, X)\omega(Y)$$

holds for any $X, Y, Z \in \mathcal{X}(M)$. We now define a vector field ξ by $g(\xi, Z) = -\omega(Z)$, and it follows that

$$S_X Y = -g(X, Y)\xi + g(\xi, Y)X. \quad (3.4.1)$$

Then the following statement has been proved.

Theorem 3.4.1 ([22]). *Let (M, g) be a connected, simply connected and complete 2-dimensional Riemannian manifold. Then (M, g) admits a homogeneous Riemannian structure $S \neq 0$ if and only if (M, g) is isometric to the hyperbolic plane $\mathbb{H}^2$.*

Proof. Let $\tilde{\nabla} = S + \nabla$ where S is the nonzero homogeneous Riemannian structure given by (3.4.1). From $\tilde{\nabla}g = 0$ and (3.4.1), we compute

$$\begin{aligned}
(\tilde{\nabla}_X S)(Y, Z) &= \tilde{\nabla}_X(S_Y Z) - S_{\tilde{\nabla}_X Y} Z - S_Y \tilde{\nabla}_X Z \\
&= \tilde{\nabla}_X(-g(Y, Z)\xi + g(\xi, Z)Y) \\
&\quad + g(\tilde{\nabla}_X Y, Z)\xi - g(\xi, Z)\tilde{\nabla}_X Y + g(Y, \tilde{\nabla}_X Z)\xi - g(\xi, \tilde{\nabla}_X Z)Y \\
&= -g(\tilde{\nabla}_X Y, Z)\xi - g(Y, \tilde{\nabla}_X Z)\xi - g(Y, Z)\tilde{\nabla}_X \xi \\
&\quad + g(\tilde{\nabla}_X \xi, Z)Y + g(\xi, \tilde{\nabla}_X Z)Y + g(\xi, Z)\tilde{\nabla}_X Y \\
&\quad + g(\tilde{\nabla}_X Y, Z)\xi - g(\xi, Z)\tilde{\nabla}_X Y + g(Y, \tilde{\nabla}_X Z)\xi - g(\xi, \tilde{\nabla}_X Z)Y \\
&= -g(Y, Z)\tilde{\nabla}_X \xi + g(\tilde{\nabla}_X \xi, Z)Y
\end{aligned}$$

for any vector field $X, Y, Z \in \mathcal{X}(M)$, and then $\tilde{\nabla}S = 0$ is equivalent to $\tilde{\nabla}\xi = 0$. Here it follows that

$$\begin{aligned}
0 &= 2g(\tilde{\nabla}_X \xi, \xi) \\
&= Xg(\xi, \xi)
\end{aligned}$$

for any vector field $X \in \mathcal{X}(M)$, and then $g(\xi, \xi)$ is nonzero constant since $S \neq 0$ and we set $g(\xi, \xi) = c^2 > 0$. Moreover, from the equation (3.4.1) and $\tilde{\nabla}\xi = 0$ we compute

$$\begin{aligned}
0 &= \tilde{\nabla}_X \xi \\
&= S_X \xi + \nabla_X \xi \\
&= -g(X, \xi)\xi + g(\xi, \xi)X + \nabla_X \xi,
\end{aligned}$$

then we obtain

$$\nabla_X \xi = g(X, \xi)\xi - g(\xi, \xi)X. \quad (3.4.2)$$

Moreover from the equation (3.4.2) and $T(X, Y) = 0$, we compute

$$\begin{aligned}
R(X, Y)\xi &= \nabla_X(\nabla_Y \xi) - \nabla_Y(\nabla_X \xi) - \nabla_{[X, Y]}\xi \\
&= \nabla_X(g(Y, \xi)\xi - c^2 Y) - \nabla_Y(g(X, \xi)\xi - c^2 X) - (g([X, Y], \xi)\xi - c^2[X, Y]) \\
&= g(\nabla_X Y - \nabla_Y X - [X, Y], \xi)\xi - c^2(\nabla_X Y - \nabla_Y X - [X, Y]) \\
&\quad + g(Y, \nabla_X \xi)\xi - g(X, \nabla_Y \xi)\xi + g(Y, \xi)\nabla_X \xi - g(X, \xi)\nabla_Y \xi \\
&= g(Y, \nabla_X \xi)\xi - g(X, \nabla_Y \xi)\xi + g(Y, \xi)\nabla_X \xi - g(X, \xi)\nabla_Y \xi \\
&= -c^2\{g(Y, \xi)X - g(X, \xi)Y\},
\end{aligned}$$

that is, it follows that

$$R(X, Y)\xi = -c^2\{g(Y, \xi)X - g(X, \xi)Y\} \quad X, Y \in \mathcal{X}(M), \quad (3.4.3)$$

and therefore (M, g) has a constant negative curvature $-c^2$. Then since (M, g) is connected, simply connected and complete by assumption, (M, g) is isometric to $\mathbb{H}^2$.

Conversely, we realize the hyperbolic plane $\mathbb{H}^2$ as upper plane $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$, with the metric

$$g = \frac{c^2(dx^2 + dy^2)}{y^2},$$

where c is a constant. Then the Levi-Civita connection ∇ of g is given by

$$\nabla_{\partial_x} \partial_x = \frac{1}{y} \partial_y, \quad \nabla_{\partial_x} \partial_y = \nabla_{\partial_y} \partial_x = -\frac{1}{y} \partial_x, \quad \nabla_{\partial_y} \partial_y = -\frac{1}{y} \partial_y, \quad (3.4.4)$$

where $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$. Here we put

$$\xi = \frac{1}{c^2} y \partial_y,$$

and define a tensor field S by (3.4.1). Then we can check the equation (3.4.2), and hence $\tilde{\nabla} S = 0$. In fact, for any vector field $X = X_1 \partial_x + X_2 \partial_y$, from (3.4.4) we can compute

$$\begin{aligned} \nabla_X \xi &= \nabla_{(X_1 \partial_x + X_2 \partial_y)} \left(\frac{1}{c^2} y \partial_y \right) \\ &= X_1 \nabla_{\partial_x} \left(\frac{1}{c^2} y \partial_y \right) + X_2 \nabla_{\partial_y} \left(\frac{1}{c^2} y \partial_y \right) \\ &= \frac{X_1 y}{c^2} \nabla_{\partial_x} \partial_y + \frac{X_2}{c^2} \partial_y + \frac{X_2 y}{c^2} \nabla_{\partial_y} \partial_y \\ &= \frac{X_2}{y} \xi - \frac{1}{c^2} (X_1 \partial_x + X_2 \partial_y) \\ &= \frac{X_2}{y} \xi - \frac{1}{c^2} X \\ &= g(X, \xi) \xi - g(\xi, \xi) X, \end{aligned}$$

and we obtain the equation (3.4.2). Moreover, $\tilde{\nabla} R = 0$ is automatically satisfied since $\mathbb{H}^2$ has a constant curvature. Thus S is a nonzero homogeneous Riemannian structure on $\mathbb{H}^2$. This completes the proof. $\square$

We then obtain the following corollary from the equation (3.4.3).

Corollary 3.4.2. *The only homogeneous Riemannian structure on the Riemannian symmetric spaces $\mathbb{R}^2$ or $\mathbb{S}^2$ is given by $S = 0$.*

All homogeneous Riemannian structures on $\mathbb{H}^2$ is determined in [22] as follows. In order to determine them, we have to found all solutions ξ of the equation (3.4.2). We choose an orthonormal frame $\{E_1, E_2\}$ on $\mathbb{H}^2$ by

$$E_1 = \frac{y}{c} \partial_y, \quad E_2 = \frac{y}{c} \partial_x, \quad (3.4.5)$$

and let $\xi = \xi^1 E_1 + \xi^2 E_2$ where ξ^1 and ξ^2 are functions. From the equations (3.4.4) we obtain

$$\nabla_{E_1} E_1 = \nabla_{E_1} E_2 = 0, \quad \nabla_{E_2} E_1 = -\frac{1}{c} E_2, \quad \nabla_{E_2} E_2 = \frac{1}{c} E_1. \quad (3.4.6)$$

Then we compute

$$\begin{aligned} \nabla_{E_1} \xi &= E_1(\xi^1) E_1 + \xi^1 \nabla_{E_1} E_1 + E_1(\xi^2) E_2 + \xi^2 \nabla_{E_1} E_2 \\ &= E_1(\xi^1) E_1 + E_1(\xi^2) E_2, \\ \nabla_{E_2} \xi &= E_2(\xi^1) E_1 + \xi^1 \nabla_{E_2} E_1 + E_2(\xi^2) E_2 + \xi^2 \nabla_{E_2} E_2 \\ &= E_2(\xi^1) E_1 - \xi^1 \frac{1}{c} E_2 + E_2(\xi^2) E_2 + \xi^2 \frac{1}{c} E_1, \end{aligned}$$

and the equation (3.4.2) is equivalent to

$$E_1(\xi^1) = -(\xi^2)^2, \quad E_1(\xi^2) = \xi^1 \xi^2, \quad E_2(\xi^1) = \xi^2 \left(\xi^1 - \frac{1}{c} \right), \quad E_2(\xi^2) = \xi^1 \left(\frac{1}{c} - \xi^1 \right). \quad (3.4.7)$$

The integrability condition of this differential system is

$$(XY - YX - [X, Y])\xi^i = 0, \quad (i = 1, 2), \quad (3.4.8)$$

for any vector fields X, Y . From the differential system (3.4.7) we can compute

$$\begin{aligned} (E_1 E_2 - E_2 E_1) \xi^1 &= -\xi^2 \left((\xi^1)^2 + (\xi^2)^2 - \xi^1 \frac{1}{c} \right), \\ (E_1 E_2 - E_2 E_1) \xi^2 &= \xi^1 \left((\xi^1)^2 + (\xi^2)^2 - \xi^1 \frac{1}{c} \right), \end{aligned}$$

on the other hand, from (3.4.5) we obtain

$$[E_1, E_2] = \frac{1}{c} E_2, \quad (3.4.9)$$

and the integrability condition (3.4.8) is equivalent to

$$\xi^2 \left((\xi^1)^2 + (\xi^2)^2 - \frac{1}{c^2} \right) = 0, \quad (3.4.10)$$

$$\xi^1 \left((\xi^1)^2 + (\xi^2)^2 - \frac{1}{c^2} \right) = 0. \quad (3.4.11)$$

Then we obtain $\xi^1 = \xi^2 = 0$ or

$$(\xi^1)^2 + (\xi^2)^2 - \frac{1}{c^2} = 0. \quad (3.4.12)$$

If $\xi^1 = \xi^2 = 0$, it follows $S = 0$. Then we consider the case $(\xi^1)^2 + (\xi^2)^2 - \frac{1}{c^2} = 0$. According to [22], the solution of the differential system (3.4.7) are

$$\xi^1 = \frac{1}{c}, \quad \xi^2 = 0, \quad (3.4.13)$$

and

$$\begin{aligned}\xi^1 &= \frac{1}{c} \frac{(\lambda - x)^2 - y^2}{(\lambda - x)^2 + y^2}, \\ \xi^2 &= \frac{1}{c} \frac{(\lambda - x)y}{(\lambda - x)^2 + y^2},\end{aligned}\tag{3.4.14}$$

where λ is an arbitrary constant. Thus we may conclude that there are an infinite number of homogeneous Riemannian structures on $\mathbb{H}^2$, however, the following theorem holds.

Theorem 3.4.3 ([22]). *Up to isomorphism of homogeneous Riemannian structures, there are only two homogeneous Riemannian structures on the hyperbolic plane $\mathbb{H}^2$:*

$$\begin{aligned}S^1 &= 0, \\ S_X^2 Y &= -g(X, Y)\eta + g(\eta, Y)X,\end{aligned}$$

where η is the vector field given by $\eta = \frac{1}{c^2}y\partial_y$ and $X, Y \in \mathcal{X}(\mathbb{H}^2)$.

Proof. We will prove that the homogeneous Riemannian structure defined by (3.4.14) is isomorphic to the homogeneous Riemannian structure S^2 .

Let S^ξ be the homogeneous Riemannian manifold given by (3.4.1) where ξ is non-zero solution of the equation (3.4.2). Then the homogeneous Riemannian structure S^ξ is given by

$$S_{E_1}^\xi E_1 = -\xi^2 E_2, \quad S_{E_1}^\xi E_2 = \xi^2 E_1, \quad S_{E_2}^\xi E_1 = -\xi^1 E_1, \quad S_{E_2}^\xi E_2 = \xi^1 E_2,$$

and the Ambrose-Singer connection $\tilde{\nabla}^\xi = \nabla + S^\xi$ is given by

$$\tilde{\nabla}_{E_1}^\xi E_1 = -\xi^2 E_2, \quad \tilde{\nabla}_{E_1}^\xi E_2 = \xi^2 E_1, \quad \tilde{\nabla}_{E_2}^\xi E_1 = \left(\xi^1 - \frac{1}{c}\right) E_2, \quad \tilde{\nabla}_{E_2}^\xi E_2 = -\left(\xi^1 - \frac{1}{c}\right) E_1,$$

and the curvature tensor $\tilde{R}^\xi$ of the Ambrose-Singer connection is computed by

$$\tilde{R}^\xi(E_1, E_2)E_1 = -\left((\xi^1)^2 + (\xi^2)^2 - \frac{1}{c^2}\right) E_2.\tag{3.4.15}$$

Here from the condition (3.4.12) we obtain $\tilde{R}^\xi = 0$. Then the holonomy algebra $\mathfrak{h}^\xi$ of $\tilde{\nabla}^\xi$ is $\mathfrak{h}^\xi = \{0\}$. Let G^ξ be the Lie group acting isometrically and transitively on $\mathbb{H}^2$ corresponding to the homogeneous Riemannian structure S^ξ and its reductive decomposition $\mathfrak{g}^\xi = \mathfrak{h}^\xi \oplus \mathfrak{m}^\xi = \{0\} \oplus \mathfrak{m}^\xi$. Here we identify $\mathfrak{m}^\xi$ with $T_o\mathbb{H}^2 = \langle E_1, E_2 \rangle$ where $o = (0, 1)$. Then $\{E_1, E_2\}$ is also basis of $\mathfrak{g}^\xi$ and from (3.1.4) the Lie bracket $[\cdot, \cdot]_{\mathfrak{g}^\xi}$ of $\mathfrak{g}^\xi$ is defined by

$$\begin{aligned}[E_1, E_2]_{\mathfrak{g}^\xi} &= -S_{E_1}^\xi E_2 + S_{E_2}^\xi E_1 - \tilde{R}^\xi(E_1, E_2) \\ &= -\xi^2 E_1 + \xi^1 E_2.\end{aligned}$$

Let us change the orthonormal basis $\{E_1, E_2\}$ to

$$\tilde{E}_1 = \frac{1}{c}(\xi^1 E_1 + \xi^2 E_2), \quad \tilde{E}_2 = \frac{1}{c}(-\xi^2 E_1 + \xi^1 E_2),$$

and we obtain

$$[\tilde{E}_1, \tilde{E}_2]_{\mathfrak{g}^\xi} = \frac{1}{c} \tilde{E}_2,$$

that is, $\mathfrak{g}^\xi$ and $\mathfrak{g}^{\xi'}$ are isomorphic for any solutions ξ and ξ' of (3.4.2). Then from the Theorem 3.2.4 the homogeneous Riemannian structure S^ξ is isomorphic to S^2 . Note that $S^\xi = S^2$ if $\xi^1 = \frac{1}{c}$ and $\xi^2 = 0$. This completes the proof. $\square$

The homogeneous Riemannian structure $S^1 = 0$ corresponds to the coset representation of $\mathbb{H}^2$ as a Riemannian symmetric space. The special linear group $\mathrm{SL}(2, \mathbb{R})$ acts isometrically and transitively on $\mathbb{H}^2$ via the linear fractional action:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d},$$

where $z = x + iy$ and $(x, y) \in \mathbb{H}^2$. The isotropy subgroup at i is the rotation group $\mathrm{SO}(2)$. The natural projection $\pi : \mathrm{SL}(2, \mathbb{R}) \rightarrow \mathbb{H}^2$ is given explicitly by

$$\pi \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \frac{(ac + bd) + i}{c^2 + d^2}.$$

The Iwasawa decomposition of $\mathrm{SL}(2, \mathbb{R})$ is carried out as $\mathrm{SL}(2, \mathbb{R}) = NAK$, where

$$\begin{aligned} N &= \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\} \cong (\mathbb{R}, +), \\ A &= \left\{ \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \mid y > 0 \right\} \cong \mathrm{SO}^+(1, 1), \\ K &= \left\{ \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \mid \phi \in \mathbb{R}/2\pi\mathbb{Z} \right\} = \mathrm{SO}(2), \end{aligned}$$

We refer (x, y, ϕ) as a coordinate system of $\mathrm{SL}(2, \mathbb{R})$. We can take an orthonormal basis

$$\bar{e}_0 = \frac{1}{c} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \bar{e}_1 = \frac{1}{c} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \bar{e}_2 = \frac{1}{c} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. The tangent space $T_i \mathbb{H}^2$ is identified with the linear subspace $\mathfrak{p}$ spanned by $\bar{e}_1$ and $\bar{e}_2$. On the other hand, the Lie algebra $\mathfrak{k}$ of the isotropy subgroup $\mathrm{SO}(2)$ is spanned by $\bar{e}_0$. One can see that $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. Hence $\mathbb{H}^2 = \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ is a Riemannian symmetric space. Then the homogeneous space representation corresponding to S^1 is

$$\mathbb{H}^2 \cong \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2). \quad (3.4.16)$$

The homogeneous Riemannian structure S^2 corresponds to the homogeneous space representation of $\mathbb{H}^2$ as a solvable Lie group. Here, the upper half plane model $\mathbb{H}^2$ is identified with the solvable Lie group

$$\mathcal{S} = NA = \left\{ \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \mid (x, y) \in \mathbb{H}^2 \right\}.$$

The Lie algebra $\mathfrak{s}$ of $\mathcal{S}$ is spanned by $\bar{e}_0 + \bar{e}_1$ and $\bar{e}_2$. The tangent space $T_i\mathbb{H}^2$ is identified with the Lie algebra $\mathfrak{s}$ under the isomorphism:

$$y \frac{\partial}{\partial x} \Big|_i \mapsto \bar{e}_0 + \bar{e}_1, \quad y \frac{\partial}{\partial y} \Big|_i \mapsto \bar{e}_2.$$

Hence $\mathbb{H}^2$ is regarded as a solvable Lie group with respect to the multiplication

$$(x_1, y_1)(x_2, y_2) = (x_1 + y_1 x_2, y_1 y_2).$$

It should be remarked that $\mathfrak{sl}(2, \mathbb{R})$ is decomposed as $\mathfrak{sl}(2, \mathbb{R}) = \mathfrak{k} + \mathfrak{s}$. However this decomposition is *not* reductive.

The solvable Lie group $\mathcal{S}$ is isomorphic to the identity component $\mathrm{GA}^+(1) = \mathrm{GL}^+(1, \mathbb{R}) \ltimes \mathbb{R}$ of the affine transformation group $\mathrm{GA}(1) = \mathrm{GL}(1, \mathbb{R}) \ltimes \mathbb{R}$ of the real line. Indeed,

$$\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \mapsto \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$$

gives the Lie group isomorphism from $\mathcal{S}$ onto $\mathrm{GA}^+(1)$. The orthonormal basis $\{\bar{e}_1, \bar{e}_2\}$ of $\mathfrak{s}$ corresponds to the orthonormal basis

$$\bar{e}_1 = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, \quad \bar{e}_2 = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}$$

of the Lie algebra $\mathfrak{ga}(1)$. Here, the homogeneous space representation corresponding to S^2 is

$$\mathbb{H}^2 \cong \mathrm{GA}^+(1)/\{Id\}. \quad (3.4.17)$$

Then there exist two homogeneous representations (3.4.16) and (3.4.17) of $\mathbb{H}^2$.

Moreover, there exist two homogeneous representations of $\mathbb{R}^2$:

$$\mathbb{R}^2 \cong \mathrm{SE}(2)/\mathrm{SO}(2), \quad (3.4.18)$$

$$\cong \mathbb{R}^2/\{0\}. \quad (3.4.19)$$

Here, $\mathbb{R}^2$ on the right side of the second equation is considered as an abelian Lie group. There exists only one homogeneous space representation of $\mathbb{S}^2$:

$$\mathbb{S}^2 \cong \mathrm{SO}(3)/\mathrm{SO}(2). \quad (3.4.20)$$

Chapter 4

Homogeneous structures of the 3-dimensional Riemannian manifolds

In this chapter we will give a complete classification of all homogeneous Riemannian structure tensors on 3-dimensional Riemannian manifolds. Moreover, we will obtain a classification of homogeneous representations of them.

4.1 The theorem of Sekigawa

As a preparation, we give the following lemma.

Lemma 4.1.1 ([5], p.117). *Let (M, g) be a simply connected homogeneous Riemannian n -manifold, and $\{e_1, e_2, \dots, e_n\}$ an orthonormal frame field, such that*

$$\nabla_{e_i} e_j = \sum_{k=1}^n c_{ij}^k e_k, \quad (4.1.1)$$

with c_{ij}^k constant for all indices i, j and k . Then (M, g) is isometric to a Lie group with left invariant metric g .

Sekigawa classified 3-dimensional connected, simply connected, complete homogeneous Riemannian manifolds by the following theorem. First, note that the curvature tensor field R of a 3-dimensional Riemannian manifold (M, g) is completely determined by its Ricci curvature tensor field Ric . In fact, if $\dim M = 3$, it follows that

$$R(X, Y) = R^1 X \wedge Y + X \wedge R^1 Y - \left(\frac{\tau}{2}\right) X \wedge Y, \quad X, Y \in \mathcal{X}(M), \quad (4.1.2)$$

where R^1 is the endomorphism of $\mathcal{X}(M)$ defined by

$$g(R^1 X, Y) = \text{Ric}(X, Y), \quad X, Y \in \mathcal{X}(M), \quad (4.1.3)$$

and

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y, \quad X, Y, Z \in \mathcal{X}(M), \quad (4.1.4)$$

see p.89 of [21]. One can see that (M, g) has constant curvature c if and only if

$$R(X, Y)Z = c(X \wedge Y)Z \quad (4.1.5)$$

for all vector fields $X, Y, Z \in \mathcal{X}(M)$. Moreover, (M, g) is an Einstein space if and only if there exists a constant k such that $R^1 X = kX$ for any vector field $X \in \mathcal{X}(M)$. Then if $\dim M = 3$, from (4.1.2) and (4.1.5), the Riemannian manifold (M, g) is an Einstein space if and only if (M, g) has constant curvature.

Theorem 4.1.2 ([21], p.88 Theorem B). *A 3-dimensional connected, simply connected, complete homogeneous Riemannian manifold (M, g) is isometric to either a symmetric space: $\mathbb{S}^3, \mathbb{H}^3, \mathbb{R}^3, \mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$ or to some 3-dimensional Riemannian Lie group.*

Proof. We will prove that if (M, g) is not symmetric, that is, $\nabla R \neq 0$, then there exists an orthonormal frame field $\{e_1, e_2, e_3\}$ such that

$$\nabla_{e_i} e_j = \sum_{k=1}^n c_{ij}^k e_k,$$

where c_{ij}^k is constant for any indices $i, j, k \in \{1, 2, 3\}$.

First we take an orthonormal frame field $\{e_1, e_2, e_3\}$ such that

$$\text{Ric}(e_i, e_j) = \delta_{ij} r_i \quad (4.1.6)$$

for any indices i, j , where r_1, r_2, r_3 are constants and δ_{ij} is the Kronecker delta. Let c_{ij}^k be a function such that

$$\nabla_{e_i} e_j = \sum_{k=1}^n c_{ij}^k e_k.$$

Note that from $\nabla g = 0$ we can obtain

$$c_{ij}^k + c_{ik}^j = 0,$$

and in particular it follows that

$$c_{ij}^j = 0.$$

Then, $(\nabla_{e_i} \text{Ric})(e_j, e_k)$ is also constants and we compute

$$\begin{aligned} (\nabla_{e_i} \text{Ric})(e_j, e_k) &= -\text{Ric}(\nabla_{e_i} e_j, e_k) - \text{Ric}(e_j, \nabla_{e_i} e_k) \\ &= -c_{ij}^k r_k - c_{ik}^j r_j \\ &= -c_{ij}^k (r_k - r_j). \end{aligned} \quad (4.1.7)$$

We consider separately the following three cases depending on $\{r_1, r_2, r_3\}$.

- (1) If $r_1 = r_2 = r_3$, then (M, g) is a 3-dimensional Einstein space, and hence a space of constant curvature $\mathbb{S}^3, \mathbb{R}^3$ or $\mathbb{H}^3$. In particular, (M, g) is symmetric Riemannian space.
- (2) If r_1, r_2, r_3 are distinct, then from (4.1.7), c_{ij}^k is constant if $j \neq k$. Moreover, it follows that $c_{ij}^j = 0$, and then c_{ij}^k is constant for any indices i, j, k .
- (3) If $r_1 = r_2 \neq r_3$, then from (4.1.7), c_{i1}^3 and c_{i2}^3 are constants for any index i . We will prove that there exists an orthonormal frame field $\{e_1, e_2, e_3\}$ such that c_{i1}^2 is constant if (M, g) is not symmetric.

First we check $c_{31}^2 = 0$. Let τ be a scalar curvature, that is,

$$\tau = \text{tr}_g \text{Ric} = \sum_{i=1}^3 \text{Ric}(e_i, e_i).$$

Since (M, g) is homogeneous, its scalar curvature τ is constant. Moreover, from Proposition 1.4.7 it follows that

$$d\tau = 2\text{div Ric},$$

and then

$$0 = d\tau(e_i) = 2 \sum_{j=1}^3 (\nabla_{e_j} \text{Ric})(e_i, e_j) \quad (4.1.8)$$

for $i = 1, 2, 3$. Writing (4.1.8) for $i = 1, 2, 3$, from (4.1.7), we obtain

$$(\nabla_{e_3} \text{Ric})(e_1, e_3) = (\nabla_{e_3} \text{Ric})(e_2, e_3) = (\nabla_{e_1} \text{Ric})(e_3, e_1) + (\nabla_{e_2} \text{Ric})(e_3, e_2) = 0,$$

that is,

$$c_{31}^3 = c_{32}^3 = c_{13}^1 + c_{23}^2 = 0. \quad (4.1.9)$$

In particular, since $c_{31}^3 = c_{32}^3 = 0$, it follows that $\nabla_{e_3} e_3 = 0$, and the integral curves of e_3 are geodesics. Therefore, we can choose $\{e_1, e_2, e_3\}$ such that $\nabla_{e_3} e_i = 0$, that is, $c_{31}^2 = 0$.

Then we check c_{11}^2 and c_{21}^2 are constant. Here, $(\nabla_{e_l}(\nabla_{e_i} \text{Ric}))(e_j, e_k)$ is also constant for any indices i, j, k, l , and we compute

$$\begin{aligned} (\nabla_{e_l}(\nabla_{e_i} \text{Ric}))(e_j, e_k) &= -(\nabla_{\nabla_{e_l} e_i} \text{Ric})(e_j, e_k) \\ &\quad - (\nabla_{e_i} \text{Ric})(\nabla_{e_l} e_j, e_k) - (\nabla_{e_i} \text{Ric})(e_j, \nabla_{e_l} e_k) \\ &= - \sum_{m=1}^3 (c_{li}^m (\nabla_{e_m} \text{Ric})(e_j, e_k) \\ &\quad + c_{lj}^m (\nabla_{e_i} \text{Ric})(e_m, e_k) + c_{lk}^m (\nabla_{e_i} \text{Ric})(e_j, e_m)). \end{aligned} \quad (4.1.10)$$

Inserting $(i, j, k) = (1, 2, 3)$ into the equation (4.1.10), we can check that

$$c_{l1}^2 ((\nabla_{e_2} \text{Ric})(e_2, e_3) - (\nabla_{e_1} \text{Ric})(e_1, e_3))$$

is constant. Then c_{l1}^2 is constant if $(\nabla_{e_2} \text{Ric})(e_2, e_3) - (\nabla_{e_1} \text{Ric})(e_1, e_3) \neq 0$. Here we compute

$$(\nabla_{e_2} \text{Ric})(e_2, e_3) - (\nabla_{e_1} \text{Ric})(e_1, e_3) = -c_{22}^3(r_2 - r_3) + c_{11}^3(r_1 - r_3),$$

and since $r_2 - r_3 = r_1 - r_3 \neq 0$, c_{l1}^2 is constant if $c_{11}^3 \neq c_{22}^3$. Moreover, inserting $(i, j, k) = (2, 2, 3)$ into (4.1.10), we can check that

$$c_{l1}^2((\nabla_{e_1} \text{Ric})(e_2, e_3) + (\nabla_{e_2} \text{Ric})(e_1, e_3))$$

is constant. Then c_{l1}^2 is constant if $(\nabla_{e_1} \text{Ric})(e_2, e_3) + (\nabla_{e_2} \text{Ric})(e_1, e_3) \neq 0$. Here we compute

$$(\nabla_{e_1} \text{Ric})(e_2, e_3) + (\nabla_{e_2} \text{Ric})(e_1, e_3) = -c_{12}^3(r_2 - r_3) - c_{21}^3(r_1 - r_3),$$

and since $r_2 - r_3 = r_1 - r_3 \neq 0$, c_{l1}^2 is constant if $c_{12}^3 \neq -c_{21}^3$.

Thus we consider the case $c_{11}^3 = c_{22}^3$ and $c_{12}^3 = -c_{21}^3$. From (4.1.9), we obtain $c_{11}^3 = c_{22}^3 = 0$. We put $\phi = c_{11}^2$, $\psi = c_{22}^1$ and $\alpha = c_{12}^3$. Note that ϕ and ψ are functions and α is a constant. Then ∇Ric is completely determined by the only possibly non zero components

$$(\nabla_{e_1} \text{Ric})(e_2, e_3) = -(\nabla_{e_2} \text{Ric})(e_1, e_3) = \alpha(r_1 - r_3).$$

If $\alpha = 0$, then $\nabla \text{Ric} = 0$ and (M, g) is symmetric, and then we suppose that $\alpha \neq 0$. Consider the differential systems

$$\begin{cases} e_1 \eta = \phi, \\ e_2 \eta = -\psi, \\ e_3 \eta = -\frac{r_1}{2\alpha}, \end{cases} \quad (4.1.11)$$

where η is some function. Then we can check that

$$[e_i, e_j] \eta = e_i(e_j \eta) - e_j(e_i \eta),$$

for all indices i, j . Hence the differential systems (4.1.11) admits a unique solution under an initial condition $\eta_0 = \eta(p)$, where $p \in M$. For the solution η of (4.1.11), we can construct a new orthonormal frame $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$, defined by

$$\tilde{e}_1 = (\cos \eta) e_1 - (\sin \eta) e_2, \quad \tilde{e}_2 = (\sin \eta) e_1 + (\cos \eta) e_2, \quad \tilde{e}_3 = e_3, \quad (4.1.12)$$

and we can check

$$\nabla_{\tilde{e}_i} \tilde{e}_j = \sum_{k=1}^3 \tilde{c}_{ij}^k e_k,$$

where $\tilde{c}_{ij}^k$ is constant for all indices i, j, k . This completes the proof.

□

Thus, to classify all homogeneous Riemannian structure tensors on 3-dimensional homogeneous Riemannian manifolds, we only need to consider two cases: Riemannian symmetric spaces and Riemannian Lie groups.

4.2 Homogeneous structures on 3-dimensional Riemannian symmetric spaces.

4.2.1 Homogeneous structures on $\mathbb{S}^3$

First we consider the case (M, g) is a space form. In this case, homogeneous Riemannian structures on $\mathbb{S}^3, \mathbb{H}^3, \mathbb{R}^3$ are classified by Abe [1].

Theorem 4.2.1 ([1], Theorem 2.1). *The 3-dimensional sphere $(\mathbb{S}^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \sum_{i=1}^4 x_i^2 = 1\}, g)$ admits two types of homogeneous Riemannian structures up to isomorphism:*

- (1) $(S_1^\lambda)_b = \lambda dV_{\mathbb{S}^3} \quad (\lambda \geq 0),$
- (2) $(S_2^\lambda)_b = dV_{\mathbb{S}^3} + \lambda \theta^1 \otimes (\theta^2 \wedge \theta^3) \quad (\lambda \neq 0),$

where $dV_{\mathbb{S}^3}$ is the volume form of $\mathbb{S}^3$ and

$$\begin{aligned}\theta^1|_q &= -x_2 dx_1 + x_1 dx_2 + x_4 dx_3 - x_3 dx_4, \\ \theta^2|_q &= -x_3 dx_1 - x_4 dx_2 + x_1 dx_3 + x_2 dx_4, \\ \theta^3|_q &= -x_4 dx_1 + x_3 dx_2 - x_2 dx_3 + x_1 dx_4,\end{aligned}$$

at $q = (x_1, x_2, x_3, x_4) \in \mathbb{S}^3$.

The corresponding coset space representation of S_1^λ is

$$\mathbb{S}^3 \cong \text{SO}(4)/\text{SO}(3), \quad (\lambda \neq 1, \lambda \geq 0), \quad (4.2.1)$$

$$\cong \text{SU}(2)/\{Id\}, \quad (\lambda = 1), \quad (4.2.2)$$

and the corresponding coset space representation of S_2^λ is

$$\mathbb{S}^3 \cong (\text{SU}(2) \times \text{SO}(2))/\text{SO}(2). \quad (4.2.3)$$

Proof. The identity component $\text{Isom}_o(\mathbb{S}^3)$ of the full isometry group of $\mathbb{S}^3$ is $\text{Isom}_o(\mathbb{S}^3) = \text{SO}(4)$ with Lie algebra $\mathfrak{so}(4)$. We denote the isotropy group of $\text{SO}(4)$ at the origin $o = (1, 0, 0, 0) \in \mathbb{S}^3$ by H , which is isomorphic to $\text{SO}(3)$. We take an basis $\{w_1, \dots, w_6\}$ of $\mathfrak{so}(4)$ given by

$$\begin{aligned}w_i &= E_{1,i+1} - E_{i+1,1} \quad (i = 1, 2, 3), \\ w_4 &= E_{4,3} - E_{3,4}, \\ w_5 &= E_{2,4} - E_{4,2}, \\ w_6 &= E_{3,2} - E_{2,3},\end{aligned}$$

where $E_{i,j}$ is the matrix unit of 4×4 . Moreover we set

$$u_i = w_{i+3} - w_i, \quad v_i = w_{i+3} + w_i,$$

for $i = 1, 2, 3$. Then we obtain

$$\begin{aligned} [u_1, u_2] &= 2u_3, & [u_2, u_3] &= 2u_1, & [u_3, u_1] &= 2u_2, \\ [v_1, v_2] &= 2v_3, & [v_2, v_3] &= 2v_1, & [v_3, v_1] &= 2v_2, \end{aligned}$$

and $[u_i, v_j] = 0$ for $i, j = 1, 2, 3$. Thus we obtain the direct sum as Lie algebra $\mathfrak{so}(4) = \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2$ with $\mathfrak{su}(2)_1 = \langle \{v_1, v_2, v_3\} \rangle$ and $\mathfrak{su}(2)_2 = \langle \{u_1, u_2, u_3\} \rangle$. Note that the Lie algebra of the isotropy H is $\mathfrak{h} = \mathfrak{su}(2)_1 = \langle \{v_1, v_2, v_3\} \rangle$. Here, from (2.2.1) and (2.2.2), we obtain

$$\tau(v_1) = \frac{\partial}{\partial x_2} \Big|_o, \quad \tau(v_2) = \frac{\partial}{\partial x_3} \Big|_o, \quad \tau(v_3) = \frac{\partial}{\partial x_4} \Big|_o. \quad (4.2.4)$$

Let G be a connected subgroup of $\text{Isom}_o(\mathbb{S}^3) = \text{SO}(4)$ acting transitively on $\mathbb{S}^3$. Note that there are no subgroups of dimension 2 in $\text{SO}(3)$, since $\text{SO}(3)$ is simple. Then it follows that $\dim G \neq 5$. In fact, if $\dim G = 5$, the isotropy subgroup of G has dimension 2, which is a subgroup of $\text{SO}(3)$. Then the possible subgroups G are as follows:

- (1) $\dim G = 6$,
- (2) $\dim G = 4$ and $\dim(G \cap H) = 1$,
- (3) $\dim G = 3$ and $\dim(G \cap H) = 0$.

Case (1): In this case, $G = \text{SO}(4)$ and its Lie algebra is $\mathfrak{g} = \mathfrak{so}(4)$. Let $\mathfrak{m}$ be a complementary subspace of $\mathfrak{h}$ in $\mathfrak{g}$. Since $\mathfrak{h} \cap \mathfrak{su}(2)_1 = \{0\}$, we can denote $\mathfrak{m} = \langle \{v_1 + \phi(v_1), v_2 + \phi(v_2), v_3 + \phi(v_3)\} \rangle$, where $\phi: \mathfrak{su}(2)_1 \rightarrow \mathfrak{h}$ is a linear map. If $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition, then from $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ we obtain

$$\mathfrak{m} = \mathfrak{m}^\lambda = \langle \{v_1 + (\lambda - 1)w_4, v_2 + (\lambda - 1)w_5, v_3 + (\lambda - 1)w_6\} \rangle, \quad (\lambda \in \mathbb{R}). \quad (4.2.5)$$

From (3.1.3) and (4.2.4), the homogeneous Riemannian structure S_1^λ defined by the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}^\lambda$ satisfies

$$\left\{ \begin{array}{lll} (S_1^\lambda)_{\frac{\partial}{\partial x^1} \frac{\partial}{\partial x^1}} \Big|_o = 0, & (S_1^\lambda)_{\frac{\partial}{\partial x^1} \frac{\partial}{\partial x^2}} \Big|_o = \lambda \frac{\partial}{\partial x^3} \Big|_o, & (S_1^\lambda)_{\frac{\partial}{\partial x^1} \frac{\partial}{\partial x^3}} \Big|_o = -\lambda \frac{\partial}{\partial x^2} \Big|_o, \\ (S_1^\lambda)_{\frac{\partial}{\partial x^2} \frac{\partial}{\partial x^1}} \Big|_o = -\lambda \frac{\partial}{\partial x^3} \Big|_o, & (S_1^\lambda)_{\frac{\partial}{\partial x^2} \frac{\partial}{\partial x^2}} \Big|_o = 0, & (S_1^\lambda)_{\frac{\partial}{\partial x^2} \frac{\partial}{\partial x^3}} \Big|_o = \lambda \frac{\partial}{\partial x^1} \Big|_o, \\ (S_1^\lambda)_{\frac{\partial}{\partial x^3} \frac{\partial}{\partial x^1}} \Big|_o = \lambda \frac{\partial}{\partial x^2} \Big|_o, & (S_1^\lambda)_{\frac{\partial}{\partial x^3} \frac{\partial}{\partial x^2}} \Big|_o = -\lambda \frac{\partial}{\partial x^1} \Big|_o, & (S_1^\lambda)_{\frac{\partial}{\partial x^3} \frac{\partial}{\partial x^3}} \Big|_o = 0. \end{array} \right.$$

Therefore, we obtain

$$(S_1^\lambda)_b|_o = \lambda dV_{\mathbb{S}^3}|_o,$$

where $dV_{\mathbb{S}^3}$ denotes the volume form of $\mathbb{S}^3$. Since S_1^λ and $dV_{\mathbb{S}^3}$ are $\text{SO}(4)$ -invariant, we have

$$(S_1^\lambda)_b = \lambda dV_{\mathbb{S}^3}.$$

In addition, for an isometry ϕ , it follows that $\phi^*S_1^\lambda = S_1^\lambda$ if ϕ preserves the orientation, and $\phi^*S_1^\lambda = -S_1^\lambda$ if ϕ changes the orientation. Therefore, S_1^λ and S_1^μ are isomorphic if and only if $\lambda = \pm\mu$. This gives the homogeneous Riemannian structures of (1).

Case (2): In this case, $\dim \mathfrak{g} = 4$ and $\dim(\mathfrak{g} \cap \mathfrak{h}) = 1$. According to Abe [1], for any reductive decomposition, the corresponding homogeneous Riemannian structure is isomorphic to the homogeneous Riemannian structure defined from

$$\mathfrak{g} = \mathfrak{su}(2)_1 \oplus \langle \{w_4\} \rangle = \langle \{v_1 + \lambda w_4, v_2, v_3\} \rangle \oplus \langle \{w_4\} \rangle, \quad (4.2.6)$$

where $\lambda \in \mathbb{R}$. Here $\mathfrak{m}(\lambda) = \langle \{v_1 + \lambda w_4, v_2, v_3\} \rangle$ is the Lie subspace. Then from (3.1.3) and (4.2.4), the homogeneous Riemannian structure S_2^λ defined by the reductive decomposition satisfies

$$(S_2^\lambda)_b(X, Y, Z)|_o = (dV_{\mathbb{S}^3} + \lambda dx_2 \otimes (dx_3 \wedge dx_4))_o(X, Y, Z),$$

at the origin $o = (1, 0, 0, 0) \in \mathbb{S}^3$. Then since both S_2^λ and $dV_{\mathbb{S}^3}$ are $SU(2)$ -invariant, we have

$$(S_2^\lambda)_b(X, Y, Z)|_p = (dV_{\mathbb{S}^3} + \lambda \phi^* dx_2 \otimes (\phi^* dx_3 \wedge \phi^* dx_4))_p(X, Y, Z),$$

at any point $p \in \mathbb{S}^3$, where $o = \phi(p)$, $\phi \in SU(2)$ and ϕ^* is the pull back of ϕ . Moreover, it follows that

$$\begin{aligned} \phi^* dx_2|_q &= -x_2 dx_1 + x_1 dx_2 + x_4 dx_3 - x_3 dx_4, \\ \phi^* dx_3|_q &= -x_3 dx_1 - x_4 dx_2 + x_1 dx_3 + x_2 dx_4, \\ \phi^* dx_4|_q &= -x_4 dx_1 + x_3 dx_2 - x_2 dx_3 + x_1 dx_4. \end{aligned}$$

Case (3): In this case, $\dim \mathfrak{g} = 3$ and $\dim(\mathfrak{g} \cap \mathfrak{h}) = 1$. According to Abe [1], it follows that $\mathfrak{g} = \mathfrak{su}(2)$. Then the reductive decomposition is $\mathfrak{g} = \{0\} \oplus \mathfrak{su}(2)$. The corresponding homogeneous Riemannian structure is $S = \nabla^{(-)} - \nabla$, where $\nabla^{(-)}$ is the Cartan-Schouten's $(-)$ -connection, and we obtain

$$S = S_1^1 = dV_{\mathbb{S}^3}.$$

This completes the proof. □

4.2.2 Homogeneous structures on $\mathbb{H}^3$

To classify all homogeneous Riemannian structures on $\mathbb{H}^3$, we give definitions and theorems proved by Azencott and Wilson.

Theorem 4.2.2 ([3], p.328 Corollary 2.6). *Let M be a simply connected homogeneous Riemannian space of non-positive curvature and G a connected Lie subgroup of $\text{Isom}_o(M)$ acting transitively on M . Then there exists a solvable Lie subgroup $\mathcal{S}$ of G acting simply and transitively on M .*

Definition 4.2.3 ([4], p.44 Definition 6.2). Let (M, g) be a simply connected homogeneous Riemannian space of non-positive curvature satisfying the assumption of Theorem 4.2.2. Assume that M has no Euclidean factor in its de Rham decomposition. Then a connected Lie subgroup $\mathcal{S}'$ of $\text{Isom}_o(M)$ is said to be a **modification** of $\mathcal{S}$ if $\mathcal{S}'$ acts simply transitively on M and the Lie algebra $\mathfrak{s}'$ of $\mathcal{S}'$ is in the normalizer $\mathfrak{n}(\mathfrak{s})$. Here $\mathfrak{s}$ is the Lie algebra of $\mathcal{S}$.

Here the **normalizer** $\mathfrak{n}(\mathfrak{h})$ of a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is defined by

$$\mathfrak{n}(\mathfrak{h}) = \{X \in \mathfrak{g} \mid [X, Y] \in \mathfrak{h} \text{ for any } Y \in \mathfrak{h}\}. \quad (4.2.7)$$

Definition 4.2.4 ([4], p.45 Definition 6.4). Let (M, g) be a simply connected homogeneous Riemannian space of non-positive curvature satisfying the assumption of Theorem 4.2.2. A solvable Lie subgroup $\mathcal{S}$ of $\text{Isom}_o(M)$ is said to be in **standard position** if for some point $o \in M$, $B(V, U) = 0$ for all $V \in [\mathfrak{s}, \mathfrak{s}]^\perp$ and $U \in \mathfrak{h}$, where B is the Killing form of $\mathfrak{iso}(M)$ and $\mathfrak{h}$ is the isotropy algebra at o . The linear subspace $[\mathfrak{s}, \mathfrak{s}]^\perp$ is the orthogonal complement of $[\mathfrak{s}, \mathfrak{s}]$ in $\mathfrak{s}$ with respect to the inner product on $\mathfrak{s}$ induced from the metric g . The Lie algebra $\mathfrak{s}$ is also called in standard position.

Theorem 4.2.5 ([4], p.48 Theorem 6.7). *Let (M, g) be a simply connected homogeneous Riemannian space of non-positive curvature satisfying the assumption of Theorem 4.2.2 and has no Euclidean factor in its de Rham decomposition. Then for any connected Lie subgroup $\mathcal{S}'$ of $\text{Isom}_o(M)$ acting simply transitively on M , there exists a unique modification $\mathcal{S}$ of $\mathcal{S}'$ which is in standard position.*

Theorem 4.2.6 ([4], p.54 Corollary 6.10). *Let (M, g) be a simply connected homogeneous Riemannian space of non-positive curvature satisfying the assumption of Theorem 4.2.2 and has no Euclidean factor in its de Rham decomposition. Then for any connected Lie subgroups $\mathcal{S}$ and $\mathcal{S}'$ of $\text{Isom}_o(M)$ acting simply transitively on M and are in standard position. Then there exists an element $a \in \text{Isom}_o(M)$ such that $a\mathcal{S}a^{-1} = \mathcal{S}'$.*

Conversely, if $\mathcal{S}$ is in standard position, then for any $a \in \text{Isom}_o(M)$, $a\mathcal{S}a^{-1}$ is in standard position.

Let M be a simply connected homogeneous Riemannian space of non-positive curvature satisfying the assumption of Theorem 4.2.2 and has no Euclidean factor in its de Rham decomposition. Take a (solvable) Lie subgroup $\mathcal{S} \subset \text{Isom}_o(M)$ which is in standard position. Then for any $a \in \text{Isom}_o(M)$, the Lie algebra of $a\mathcal{S}a^{-1}$ is $\text{Ad}(a)\mathfrak{s}$ and isomorphic to $\mathfrak{s}$ as a Lie algebra. Moreover $\mathfrak{s}$ and $\text{Ad}(a)\mathfrak{s}$ are isometric relative to the inner product induced from $(T_o M, g_o)$ and $(T_{a \cdot o} M, g_{a \cdot o})$. In fact, from Proposition 2.2.2 we can check that

$$\begin{aligned} g_{a \cdot o}((\text{Ad}(a)X)^*, (\text{Ad}(a)Y)^*) &= g_{\phi_a(o)}((\phi_a)_* X^*, (\phi_a)_* Y^*) \\ &= g_o(X^*, Y^*), \end{aligned}$$

since ϕ_a is isometry of (M, g) . Thus the homogeneous Riemannian structures S determined by $\mathcal{S} = \mathcal{S}/\{\mathbf{e}\}$ and S' determined by $\mathcal{S}' = (\text{Ad}(a)\mathcal{S})/\{\mathbf{e}\}$ are isomorphic. Furthermore if $\mathfrak{s}'$ is a modification of $\mathfrak{s}$, then $\text{Ad}(a)\mathfrak{s}'$ is a modification of $\text{Ad}(a)\mathfrak{s}$, since $\mathfrak{n}(\text{Ad}(a)\mathfrak{s}) = \text{Ad}(a)\mathfrak{n}(\mathfrak{s})$.

Therefore the homogeneous Riemannian structure defined from the Lie algebra of any subgroup of $\text{Isom}_o(M)$ acting simply transitively on M is isomorphic to that defined from the Lie algebra of a subgroup of $\text{Isom}_o(M)$ which is a modification of a fixed subgroup $\mathcal{S}_0$ being in standard position. In other words, it is sufficient to determine homogeneous Riemannian structures of a fixed subgroup $\mathcal{S}_0$ being in standard position and its modifications.

Theorem 4.2.7 ([1], p.7 Theorem 3.1). *The 3-dimensional hyperbolic space $(\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}, g)$ admits two types of homogeneous Riemannian structures up to isomorphisms:*

$$(1) (S_1^\lambda)_b = \lambda dV_{\mathbb{H}^3} \quad (\lambda \geq 0),$$

$$(2) (S_2^\lambda)_X Y = -\lambda g(X, \xi) \{g(Y, \eta)\zeta - g(Y, \zeta)\eta\} - g(X, Y)\xi + g(\xi, Y)X \quad (\lambda \geq 0),$$

for $X, Y \in \mathcal{X}(\mathbb{H}^3)$, where $dV_{\mathbb{H}^3}$ is the volume form of $\mathbb{H}^3$ and $\eta = z \frac{\partial}{\partial x}, \zeta = z \frac{\partial}{\partial y}, \xi = z \frac{\partial}{\partial z}$.

The corresponding coset space representation of S_1^λ is

$$\mathbb{H}^3 \cong \text{SL}(2, \mathbb{C}) / \text{SU}(2), \quad (4.2.8)$$

and the corresponding coset space representation of S_2^λ is

$$\mathbb{H}^3 \cong G^\lambda / \{Id\}, \quad (4.2.9)$$

where G^λ is a 3-dimensional Lie group

$$G^\lambda = \left\{ \begin{pmatrix} e^{t(1+\lambda i)} & w \\ 0 & e^{-t(1+\lambda i)} \end{pmatrix} \mid t \in \mathbb{R}, w \in \mathbb{C} \right\}. \quad (4.2.10)$$

Moreover $\mathbb{H}^3$ admits three types of homogeneous space representations: (4.2.8), (4.2.9) and

$$\mathbb{H}^3 \cong (\text{B}^+(2, \mathbb{C}) \cdot \text{U}(1)) / \text{U}(1), \quad (4.2.11)$$

where $\text{B}^+(2, \mathbb{C})$ is a 3-dimensional Lie group defined by

$$\text{B}^+(2, \mathbb{C}) = \left\{ \begin{pmatrix} \sqrt{z} & (x + y\sqrt{-1})/\sqrt{z} \\ 0 & 1/\sqrt{z} \end{pmatrix} \mid x, y \in \mathbb{R}, z > 0 \right\}.$$

From the homogeneous representation (4.2.11), the homogeneous Riemannian structure S_2^λ is obtained.

Proof. First we realize $\mathbb{H}^3$ as a Hermitian matrix model. Let us denote by $\text{Her}(2, \mathbb{C})$ the space of all Hermitian 2×2 matrices. Then $\text{Her}(2, \mathbb{C})$ is parametrized explicitly as follows:

$$\text{Her}(2, \mathbb{C}) = \left\{ X = \begin{pmatrix} x_0 + x_1 & x_3 - \sqrt{-1}x_2 \\ x_3 + \sqrt{-1}x_2 & x_0 - x_1 \end{pmatrix} \mid x_0, x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

We define a basis $\{e_0, e_1, e_2, e_3\}$ of $\text{Her}(2, \mathbb{C})$ by

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let us introduce a scalar product on $\text{Her}(2, \mathbb{C})$ by

$$\langle X, X \rangle = -\det X.$$

Then $\text{Her}(2, \mathbb{C})$ is identified with Minkowski 4-space $\mathbb{E}^{1,3}$ with Lorentzian metric $-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$. The hyperbolic 3-space $\mathbb{H}^3$ is realized as a hyperquadric

$$\{X \in \text{Her}(2, \mathbb{C}) \mid \det X = 1, \text{tr } X > 0\}.$$

The special linear group $\text{SL}(2, \mathbb{C})$ acts isometrically on $\text{Her}(2, \mathbb{C})$ via the action

$$\text{SL}(2, \mathbb{C}) \times \text{Her}(2, \mathbb{C}) \rightarrow \text{Her}(2, \mathbb{C}); (g, X) \mapsto gXg^*.$$

The isotropy subgroup of this action at the identity matrix is $\text{SU}(2)$. The isotropy algebra is spanned by $\{\sqrt{-1}e_1, \sqrt{-1}e_2, \sqrt{-1}e_3\}$. The tangent space $T_{e_0}\mathbb{H}^3$ is identified with the Lie subspace

$$\mathfrak{m}_0 = \text{Her}(2, \mathbb{C}) \cap \mathfrak{sl}_2\mathbb{C} = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3.$$

The representation $\text{SL}(2, \mathbb{C})/\text{SU}(2)$ is a Riemannian symmetric space with Lie subspace $\mathfrak{m}_0$. In particular, we obtain $\text{Isom}_o(\mathbb{H}^3) = \text{SL}(2, \mathbb{C})$. Note that the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ is represented by

$$\mathfrak{sl}_2\mathbb{C} = \mathfrak{m}_0^{\mathbb{C}} = \mathfrak{m}_0 \oplus \sqrt{-1}\mathfrak{m}_0, \quad \mathfrak{su}(2) = \sqrt{-1}\mathfrak{m}_0.$$

The Iwasawa decomposition of $\text{SL}(2, \mathbb{C})$ is given by

$$\text{SL}(2, \mathbb{C}) = NA \cdot \text{SU}(2),$$

where

$$N = \left\{ \begin{pmatrix} 1 & x + y\sqrt{-1} \\ 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{R} \right\}, \quad A = \left\{ \begin{pmatrix} \sqrt{z} & 0 \\ 0 & 1/\sqrt{z} \end{pmatrix} \mid z > 0 \right\}.$$

The solvable part $B^+(2, \mathbb{C}) = NA$ is parametrized as

$$B^+(2, \mathbb{C}) = \left\{ \begin{pmatrix} \sqrt{z} & (x + y\sqrt{-1})/\sqrt{z} \\ 0 & 1/\sqrt{z} \end{pmatrix} \mid x, y \in \mathbb{R}, z > 0 \right\}.$$

The multiplication law is described as

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + z_1x_2, y_1 + z_1y_2, z_1z_2).$$

The Maurer-Cartan form of $B^+(2, \mathbb{C})$ is computed as

$$\frac{dx}{z} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{dy}{z} \begin{pmatrix} 0 & \sqrt{-1} \\ 0 & 0 \end{pmatrix} + \frac{dz}{z} \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

Thus the Lie algebra $\mathfrak{b}(2, \mathbb{C})$ of $B^+(2, \mathbb{C})$ is spanned by

$$\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{-1} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \right\}.$$

Let us introduce a left invariant metric so that this basis is orthonormal with respect to it. Then $B^+(2, \mathbb{C})$ is isometric to the upper half space model of $\mathbb{H}^3$. It should be remarked that inner product of $\mathfrak{b}(2, \mathbb{C})$ is different from the one induced from the normalized Killing metric. Indeed, with respect to the normalized Killing metric,

$$\left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 0 & \sqrt{-1} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{-1} \\ 0 & 0 \end{pmatrix} \right\rangle = 0,$$

$$\left\langle \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \right\rangle = \frac{1}{4}.$$

Let G be a connected subgroup of $\text{Isom}_o(\mathbb{H}^3) = \text{SL}(2, \mathbb{C})$ acting transitively on $\mathbb{H}^3$. Since the isotropy subgroup $H = \text{SU}(2)$ is simple, there exist three cases as in $\mathbb{S}^3$:

- (1) $\dim G = 6$,
- (2) $\dim G = 4$ and $\dim(G \cap H) = 1$,
- (3) $\dim G = 3$ and $\dim(G \cap H) = 0$.

Case (1): In this case, $G = \text{SL}(2, \mathbb{C})$ and its Lie algebra is $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Let $\mathfrak{m}$ be a Lie subspace of $\mathfrak{su}(2)$ in $\mathfrak{g}$. As in $\mathbb{S}^3$, from $[\mathfrak{m}, \mathfrak{su}(2)] \subset \mathfrak{m}$, we can obtain

$$\mathfrak{m} = \mathfrak{m}^\lambda = \langle \{e_1 + \lambda(\sqrt{-1})e_1, e_2 + \lambda(\sqrt{-1})e_2, e_3 + \lambda(\sqrt{-1})e_3\} \rangle,$$

in other words, $\mathfrak{m}^\lambda = (1 + \lambda\sqrt{-1})\mathfrak{m}_0$. Then the corresponding homogeneous Riemannian structure is $(S_1^\lambda)_b = \lambda dV_{\mathbb{H}^3}$.

Case (3): In this case, from Theorem 4.2.2 it follows that the isometry group G is a solvable subgroup of $\text{SL}(2, \mathbb{C})$. Note that $\dim \mathfrak{g} = 3$ and its isotropy is $\mathfrak{g} \cap \mathfrak{su}(2) = \{0\}$. Then the Lie algebra $\mathfrak{b}(2, \mathbb{C})$ is in standard position. In fact, it follows that

$$[\mathfrak{b}(2, \mathbb{C}), \mathfrak{b}(2, \mathbb{C})] = \left\langle \left\{ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \right\} \right\rangle,$$

and then we obtain

$$[\mathfrak{b}(2, \mathbb{C}), \mathfrak{b}(2, \mathbb{C})]^\perp = \left\langle \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{-1} \\ 0 & 0 \end{pmatrix} \right\} \right\rangle.$$

Then it follows that $B(U, V) = 0$ for any $U \in [\mathfrak{b}(2, \mathbb{C}), \mathfrak{b}(2, \mathbb{C})]^\perp$ and $V \in \mathfrak{su}(2)$.

Here the normalization $\mathfrak{n}(\mathfrak{b}(2, \mathbb{C}))$ is

$$\mathfrak{n}(\mathfrak{b}(2, \mathbb{C})) = \left\{ \begin{pmatrix} z & w \\ 0 & -z \end{pmatrix} \middle| z, w \in \mathbb{C} \right\}. \quad (4.2.12)$$

Let $\mathfrak{s}$ be a modification of $\mathfrak{b}(2, \mathbb{C})$. Then since $\mathfrak{s} \subset \mathfrak{n}(\mathfrak{b}(2, \mathbb{C}))$, $\dim \mathfrak{s} = 3$ and $\mathfrak{s} \cap \mathfrak{su}(2) = \{0\}$, for some $\lambda \in \mathbb{R}$ it follows that

$$\mathfrak{s} = \mathfrak{s}^\lambda = \left\{ \begin{pmatrix} t(1 + \lambda\sqrt{-1}) & w \\ 0 & -t(1 + \lambda\sqrt{-1}) \end{pmatrix} \middle| t \in \mathbb{R}, w \in \mathbb{C} \right\}. \quad (4.2.13)$$

Note that $\mathfrak{b}(2, \mathbb{C}) = \mathfrak{s}^0$. Here the corresponding Lie group $\mathcal{S}^\lambda$ is given by

$$\mathcal{S}^\lambda = \left\{ \begin{pmatrix} \exp t(1 + \lambda\sqrt{-1}) & w \\ 0 & \exp(-t(1 + \lambda\sqrt{-1})) \end{pmatrix} \middle| t \in \mathbb{R}, w \in \mathbb{C} \right\},$$

and then we obtain the coset representation $\mathbb{H}^3 = \mathcal{S}^\lambda / \{Id\}$. By a diffeomorphism $\Phi: \mathbb{H}^3 \rightarrow \mathcal{S}^\lambda$ such that

$$\Phi(x, y, z) = \begin{pmatrix} e^{(\log z)(1 + \lambda\sqrt{-1})} & x + y\sqrt{-1} \\ 0 & e^{-(\log z)(1 + \lambda\sqrt{-1})} \end{pmatrix},$$

we can identify $\mathbb{H}^3$ and $\mathcal{S}^\lambda$. Then we take a left invariant orthonormal basis $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ on $\mathbb{H}^3$ defined by

$$\begin{aligned} \tilde{e}_1 &= z \frac{\partial}{\partial z}, & \tilde{e}_2 &= z \cos(\lambda \log z) \frac{\partial}{\partial x} + z \sin(\lambda \log z) \frac{\partial}{\partial y}, \\ \tilde{e}_3 &= -z \sin(\lambda \log z) \frac{\partial}{\partial x} + z \cos(\lambda \log z) \frac{\partial}{\partial y}, \end{aligned} \quad (4.2.14)$$

and we can obtain

$$[\tilde{e}_1, \tilde{e}_2] = \tilde{e}_2 + \lambda \tilde{e}_3, \quad [\tilde{e}_1, \tilde{e}_3] = \tilde{e}_2 + \lambda \tilde{e}_3, \quad [\tilde{e}_2, \tilde{e}_3] = 0. \quad (4.2.15)$$

The Levi-Civita connection of $\mathcal{S}^\lambda$ is given by

$$\begin{aligned} \nabla_{\tilde{e}_1} \tilde{e}_1 &= 0, & \nabla_{\tilde{e}_1} \tilde{e}_2 &= \lambda \tilde{e}_3, & \nabla_{\tilde{e}_1} \tilde{e}_3 &= -\lambda \tilde{e}_2, \\ \nabla_{\tilde{e}_2} \tilde{e}_1 &= -\tilde{e}_2, & \nabla_{\tilde{e}_2} \tilde{e}_2 &= \tilde{e}_1, & \nabla_{\tilde{e}_2} \tilde{e}_3 &= 0, \\ \nabla_{\tilde{e}_3} \tilde{e}_1 &= -\tilde{e}_3, & \nabla_{\tilde{e}_3} \tilde{e}_2 &= 0, & \nabla_{\tilde{e}_3} \tilde{e}_3 &= \tilde{e}_1. \end{aligned} \quad (4.2.16)$$

The corresponding Ambrose-Singer connection is the Cartan-Schouten's $(-)$ -connection $\nabla^{(-)}$ of $\mathcal{S}^\lambda$, and we obtain the homogeneous Riemannian structure $S = \nabla^{(-)} - \nabla$. Then from (4.2.14), (4.2.15) and (4.2.16) we can obtain

$$(S_2^\lambda)_X Y = -\lambda g(X, \xi) \{g(Y, \eta) \zeta - g(Y, \zeta) \eta\} - g(X, Y) \xi + g(\xi, Y) X \quad (\lambda \geq 0),$$

where $\eta = z \frac{\partial}{\partial x}$, $\zeta = z \frac{\partial}{\partial y}$, $\xi = z \frac{\partial}{\partial z}$.

Case (2): Abe proved the following fact.

Proposition 4.2.8 ([1], p.12 Lemma 3.10). *Let G be a 4-dimensional connected subgroup of $SL(2, \mathbb{C})$ acting transitively on $\mathbb{H}^3$, then there exists a solvable subgroup $\mathcal{S}$ of G and G is the normalizer $N(\mathcal{S})$ of $\mathcal{S}$. Moreover there exists $a \in \text{Isom}_o(\mathbb{H}^3)$ such that $G = N(aB^+(2, \mathbb{C})a^{-1})$.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G . From Theorem 4.2.2 there exists a solvable Lie subalgebra $\mathfrak{s} \subset \mathfrak{g}$ such that $\dim \mathfrak{s} = 3$ and $\mathfrak{su}(2) \cap \mathfrak{s} = \{0\}$. Moreover, from the discussion in Case (2), the solvable Lie subalgebra $\mathfrak{s}$ is represented as $\mathfrak{s} = \text{Ad}(a)\mathfrak{s}^\lambda$, where $a \in \text{Isom}_o(\mathbb{H}^3)$ and $\mathfrak{s}^\lambda$ is the Lie algebra defined by (4.2.13). Then from $\dim \mathfrak{g} = 4$ and $\dim(\mathfrak{g} \cap \mathfrak{su}(2)) = 1$, there exists $v \in \mathfrak{su}(2)$ such that $\mathfrak{g} = \text{Ad}(a)\mathfrak{s}^\lambda \oplus \langle \{v\} \rangle$. It follows that $\text{Ad}(a^{-1})\mathfrak{g} = \mathfrak{s}^\lambda \oplus \langle \{\text{Ad}(a^{-1})v\} \rangle$. Let $\text{Ad}(a^{-1})v = \begin{pmatrix} k & b \\ c & -k \end{pmatrix}$. Then for any element $\begin{pmatrix} \alpha & w \\ 0 & -\alpha \end{pmatrix} \in \mathfrak{s}^\lambda$, we compute

$$\left[\begin{pmatrix} k & b \\ c & -k \end{pmatrix}, \begin{pmatrix} \alpha & w \\ 0 & -\alpha \end{pmatrix} \right] = \begin{pmatrix} cw & 2(\alpha b - kw) \\ -2\alpha c & -cw \end{pmatrix} \in \text{Ad}(a^{-1})\mathfrak{g} = \mathfrak{s}^\lambda \oplus \langle \{\text{Ad}(a^{-1})v\} \rangle,$$

and there exists $\begin{pmatrix} \beta & z \\ 0 & -\beta \end{pmatrix} \in \mathfrak{s}^\lambda$ and $s \in \mathbb{R}$ such that

$$\begin{pmatrix} cw & 2(\alpha b - kw) \\ -2\alpha c & -cw \end{pmatrix} = \begin{pmatrix} \beta & z \\ 0 & -\beta \end{pmatrix} + s \begin{pmatrix} k & b \\ c & -k \end{pmatrix}. \quad (4.2.17)$$

If $c = 0$, it follows that $\text{Ad}(a^{-1})v \in \mathfrak{n}(\mathfrak{s}^\lambda) = \mathfrak{n}(\mathfrak{b}(2, \mathbb{C}))$, and we obtain $\text{Ad}(a^{-1})\mathfrak{g} \subset \mathfrak{n}(\mathfrak{b}(2, \mathbb{C}))$. Moreover since $\dim \text{Ad}(a^{-1})\mathfrak{g} = \dim \mathfrak{n}(\mathfrak{b}(2, \mathbb{C})) = 4$, we obtain $\text{Ad}(a^{-1})\mathfrak{g} = \mathfrak{n}(\mathfrak{b}(2, \mathbb{C}))$, that is, $\mathfrak{g} = \mathfrak{n}(\text{Ad}(a)\mathfrak{b}(2, \mathbb{C}))$.

Then we suppose that $c \neq 0$. Then from (4.2.17) it follows that $-2\alpha c = sc$ and then $-2\alpha = s$.

If $\lambda \neq 0$, then $\alpha, \beta \notin \mathbb{R}$ and this contradicts $s \in \mathbb{R}$ since $-2\alpha = s$.

If $\lambda = 0$, then $\alpha, \beta \in \mathbb{R}$. From (4.2.17) we can obtain $cw = \beta + sk$. If we take $w = 0$ and $s = -2\alpha \neq 0$, then $k \in \mathbb{R}$. However, if we take $w = \bar{c}\sqrt{-1}$ and $s \neq 0$, it follows that $k \notin \mathbb{R}$. This contradicts.

From these it follows that $c = 0$, and thus $\mathfrak{g} = \mathfrak{n}(\text{Ad}(a)\mathfrak{b}(2, \mathbb{C}))$. This completes the proof. $\square$

Let $\mathfrak{g}$ be the Lie algebra of G . Then from Proposition 4.2.8 it follows that $\mathfrak{g} = \text{Ad}(a)\mathfrak{n}(\mathfrak{b}(2, \mathbb{C}))$. Then the isotropy algebra is $\text{Ad}(a)\mathfrak{n}(\mathfrak{b}(2, \mathbb{C})) \cap \mathfrak{su}(2)$. Let $\mathfrak{m}$ be the Lie subspace, then we can obtain $\mathfrak{m} = \text{Ad}(a)\mathfrak{s}^\lambda$. From $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}$ and Proposition 3.1.3, the curvature $\tilde{R}$ of the corresponding Ambrose-Singer connection $\tilde{\nabla}$ is equal to 0. Then the corresponding homogeneous Riemannian structure is equal to the homogeneous Riemannian structure defined from $\mathbb{H}^3 = \mathcal{S}^\lambda / \{Id\}$. Moreover, comparing with [8, p.567], we obtain a homogeneous space representation $\mathbb{H}^3 \cong (\text{B}^+(2, \mathbb{C}) \cdot \text{U}(1)) / \text{U}(1)$. This completes the proof. $\square$

4.2.3 Homogeneous structures on $\mathbb{R}^3$

Theorem 4.2.9 ([1], p.13 Theorem 4.1). *The 3-dimensional Euclidean space $(\mathbb{R}^3, g)$ admits two types of homogeneous Riemannian structures up to isomorphisms:*

$$(1) (S_1^\lambda)_\flat = \lambda dV_{\mathbb{R}^3} \quad (\lambda \geq 0),$$

$$(2) \quad (S_2^\lambda)_X Y = \lambda g \left(X, \frac{\partial}{\partial x} \right) \left\{ g \left(Y, \frac{\partial}{\partial y} \right) \frac{\partial}{\partial z} - g \left(Y, \frac{\partial}{\partial z} \right) \frac{\partial}{\partial y} \right\} \quad (\lambda > 0),$$

for $X, Y \in \mathcal{X}(\mathbb{R}^3)$, where $dV_{\mathbb{R}^3}$ is the volume form of $\mathbb{R}^3$.

The corresponding coset space representation of S_1^λ is

$$\mathbb{R}^3 \cong \text{SE}(3)/\text{SO}(3), \quad (4.2.18)$$

and the corresponding coset space representation of S_2^λ is

$$\mathbb{R}^3 \cong G'^\lambda / \{Id\}, \quad (4.2.19)$$

where G'^λ is a 3-dimensional Lie group

$$G'^\lambda = \left\{ \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & \cos(\lambda x) & -\sin(\lambda x) & y \\ 0 & \sin(\lambda x) & \cos(\lambda x) & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}. \quad (4.2.20)$$

Moreover, $\mathbb{R}^3$ admits three types of homogeneous space representations: (4.2.18), (4.2.19) and

$$\mathbb{R}^3 \cong \text{SE}(2) \times \mathbb{R}/\text{SO}(2). \quad (4.2.21)$$

From the homogeneous representation (4.2.21), the homogeneous Riemannian structure S_2^λ is obtained.

Proof. The identity component $\text{Isom}_o(\mathbb{R}^3)$ of the full isometry group of $\mathbb{R}^3$ is $\text{Isom}_o(\mathbb{R}^3) = \text{SE}(3) = \text{SO}(3) \ltimes \mathbb{R}^3$. The isotropy subgroup of $\text{SE}(3)$ at the origin $o \in \mathbb{R}^3$ is $\text{SO}(3)$. The Lie algebra $\mathfrak{se}(3)$ of the Lie group $\text{SE}(3)$ is represented by

$$\mathfrak{se}(3) = \left\{ \begin{pmatrix} 0 & -w & v & x \\ w & 0 & -u & y \\ -v & u & 0 & z \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

We take a basis $\{u_1, u_2, u_3, e_1, e_2, e_3\}$ of $\mathfrak{se}(3)$ by

$$u_1 = E_{32} - E_{23}, \quad u_2 = E_{13} - E_{31}, \quad u_3 = E_{21} - E_{12}, \quad e_1 = E_{14}, \quad e_2 = E_{24}, \quad e_3 = E_{34}.$$

Then we have

$$\begin{aligned} [u_1, u_2] &= u_3, & [u_2, u_3] &= u_1, & [u_3, u_1] &= u_2, \\ [u_1, e_2] &= e_3, & [u_1, e_3] &= -e_2, \\ [u_2, e_3] &= e_1, & [u_2, e_1] &= -e_3, \\ [u_3, e_1] &= e_2, & [u_3, e_2] &= -e_1, \\ [u_i, e_i] &= 0, & [e_i, e_j] &= 0, \end{aligned} \quad (4.2.22)$$

and from (2.2.1) we compute

$$e_1^*|_o = \frac{\partial}{\partial x}, \quad e_2^*|_o = \frac{\partial}{\partial y}, \quad e_3^*|_o = \frac{\partial}{\partial z},$$

at the origin $o = (0, 0, 0) \in \mathbb{R}^3$. Here the isotropy algebra is $\mathfrak{so}(3) = \langle \{u_1, u_2, u_3\} \rangle$.

Let G be a connected subgroup of $\text{Isom}_o(\mathbb{R}^3) = \text{SE}(3)$ acting transitively on $\mathbb{R}^3$. Since the isotropy subgroup $H = \text{SO}(3)$ is simple, there exist three cases as in $\mathbb{S}^3$ and $\mathbb{H}^3$:

- (1) $\dim G = 6$,
- (2) $\dim G = 4$ and $\dim(G \cap H) = 1$,
- (3) $\dim G = 3$ and $\dim(G \cap H) = 0$.

Case (1): In this case, as in the case of $\mathbb{S}^3$ and $\mathbb{H}^3$, from $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ the Lie subspace is given by $\mathfrak{m} = \mathfrak{m}^\lambda = \langle \{e_1 + \lambda u_1, e_2 + \lambda u_2, e_3 + \lambda u_3\} \rangle$, and the homogeneous Riemannian structures defined from $\mathfrak{g} = \mathfrak{m}^\lambda \oplus \mathfrak{h}$ is $(S_1^\lambda)_b = \lambda dV_{\mathbb{R}^3}$.

Case (3): In this case, as in the case of $\mathbb{H}^3$, from Theorem 4.2.2 it follows that the isometry group G is a solvable subgroup of $\text{SE}(3)$, and its Lie algebra $\mathfrak{g}$ satisfies $\dim \mathfrak{g} = 3$ and $\dim \mathfrak{g} \cap \mathfrak{se}(3) = 0$. From $\mathfrak{g} \in \mathfrak{se}(3) = \mathfrak{so}(3) \oplus \mathbb{R}^3$ and $\mathfrak{g} \cap \mathfrak{so}(3) = \{0\}$, we can denote $\mathfrak{g} = \langle \{e_1 + f(e_1), e_2 + f(e_2), e_3 + f(e_3)\} \rangle$ where $f : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ is a linear map. Since the Lie algebra $\mathfrak{g}$ is solvable, it follows that $[f(e_1), f(e_2)] = [f(e_2), f(e_3)] = [f(e_3), f(e_1)] = 0$, and then we have $f(e_2) = sf(e_1)$ and $f(e_3) = tf(e_1)$. Moreover by linear transformation the Lie algebra $\mathfrak{g}$ is expressed as $\langle \tilde{e}_1 + u, \tilde{e}_2, \tilde{e}_3 \rangle$, where $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ is an orthonormal basis of $\mathbb{R}^3$ and $u \in \mathfrak{so}(3)$. If we choose a basis $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ such that satisfies the relation (4.2.22), it follows that $\mathfrak{g} = \langle \lambda \tilde{u}_1 + \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \rangle$, which is isomorphic to the Lie algebra $\mathfrak{g}^\lambda = \langle \lambda u_1 + e_1, e_2, e_3 \rangle$. Moreover, the corresponding Lie group G'^λ is given by (4.2.9). According to Abe, it follows that the corresponding homogeneous Riemannian structure S_2^λ is given by

$$(S_2^\lambda)_X Y = \lambda g \left(X, \frac{\partial}{\partial x} \right) \left\{ g \left(Y, \frac{\partial}{\partial y} \right) \frac{\partial}{\partial z} - g \left(Y, \frac{\partial}{\partial z} \right) \frac{\partial}{\partial y} \right\}.$$

The corresponding Ambrose-Singer connection $\tilde{\nabla} = \nabla - S_2^\lambda$ is Cartan-Schouten's $(-)$ -connection of G'^λ . In fact, we take left invariant orthonormal frame field $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ defined by

$$\tilde{e}_1 = \frac{\partial}{\partial x}, \quad \tilde{e}_2 = \cos(\lambda x) \frac{\partial}{\partial y} + \sin(\lambda x) \frac{\partial}{\partial z}, \quad \tilde{e}_3 = -\sin(\lambda x) \frac{\partial}{\partial y} + \cos(\lambda x) \frac{\partial}{\partial z}. \quad (4.2.23)$$

The Levi-Civita connection of G^λ is given by

$$\begin{aligned} \nabla_{\tilde{e}_1} \tilde{e}_1 &= 0, & \nabla_{\tilde{e}_1} \tilde{e}_2 &= \lambda \tilde{e}_3, & \nabla_{\tilde{e}_1} \tilde{e}_3 &= -\lambda \tilde{e}_2, \\ \nabla_{\tilde{e}_2} \tilde{e}_1 &= 0, & \nabla_{\tilde{e}_2} \tilde{e}_2 &= 0, & \nabla_{\tilde{e}_2} \tilde{e}_3 &= 0, \\ \nabla_{\tilde{e}_3} \tilde{e}_1 &= 0, & \nabla_{\tilde{e}_3} \tilde{e}_2 &= 0, & \nabla_{\tilde{e}_3} \tilde{e}_3 &= 0. \end{aligned} \quad (4.2.24)$$

Moreover we can obtain $\tilde{\nabla}_{\tilde{e}_i}\tilde{e}_j = \nabla_{\tilde{e}_i}\tilde{e}_j - (S_2^\lambda)_{\tilde{e}_i}\tilde{e}_j = 0$, the Ambrose-Singer connection $\tilde{\nabla}$ is the Cartan-Schouten's $(-)$ -connection of G'^λ .

Case (2): In this case, from Theorem 4.2.2, the Lie algebra $\mathfrak{g}$ contains a Lie algebra which is isomorphic to $\mathfrak{g}^\lambda$. Then $\mathfrak{g}$ is expressed as $\mathfrak{g} = \langle \lambda\tilde{u}_1 + \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \rangle \oplus \langle u \rangle$. Moreover since $\mathfrak{g}|_{\mathfrak{so}(3)}$ is a subalgebra of $\mathfrak{so}(3)$ and there are no subalgebras of dimension 2, it follows that $\mathfrak{g} = \langle \lambda\tilde{u}_1 + \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \rangle \oplus \langle \tilde{u}_1 \rangle$. Conversely from $[\mathfrak{so}(3), \mathfrak{m}] \subset \mathfrak{m}$ the Lie subspace is $\mathfrak{m} = \mathfrak{m}^\lambda = \langle \lambda\tilde{u}_1 + \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \rangle = \mathfrak{g}^\lambda$. Hence $[\mathfrak{m}^\lambda, \mathfrak{m}^\lambda] \subset \mathfrak{m}^\lambda$, the corresponding homogeneous Riemannian structure is S_2^λ . Moreover it follows that $\mathfrak{g} = \mathfrak{se}(2) \oplus \mathbb{R}$, and then we obtain the homogeneous space representation $\mathbb{R}^3 = (\text{SE}(2) \times \mathbb{R})/\text{SO}(2)$. This completes the proof. $\square$

4.2.4 Homogeneous structures on $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$

Next we classify homogeneous Riemannian structures on $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. First we consider $\mathbb{S}^2 \times \mathbb{R}$. Let us realize the Riemannian product $\mathbb{S}^2 \times \mathbb{R}$ as a hyperquadric

$$\mathbb{S}^2 \times \mathbb{R} = \{(x^1, x^2, x^3, y) \in \mathbb{R}^4 \mid (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$$

of the Euclidean 4-space $\mathbb{R}^4$. The metric induced from $\mathbb{R}^4$ coincides with the product metric $g = g_{\mathbb{S}^2} + dy^2$.

Theorem 4.2.10 ([18], Theorem 1). *The Riemannian product $\mathbb{S}^2 \times \mathbb{R}$ admits only one type of homogeneous Riemannian structure tensors up to isomorphism:*

$$(S^\lambda)_b = \lambda(dy \otimes dV_{\mathbb{S}^2}) \quad (\lambda \geq 0),$$

where $dV_{\mathbb{S}^2}$ is the area form of $\mathbb{S}^2$ and y is the coordinate of $\mathbb{R}$. The corresponding coset space representation of each S^λ is $\text{SO}(3) \times \mathbb{R}/\text{SO}(2)$.

Proof. The identity component $\text{Isom}_o(\mathbb{S}^2 \times \mathbb{R})$ of the full isometry group of $\mathbb{S}^2 \times \mathbb{R}$ is $\text{Isom}_0(\mathbb{S}^2 \times \mathbb{R}) = \text{SO}(3) \times \mathbb{R}$ with Lie algebra $\mathfrak{so}(3) \oplus \mathbb{R}$. Under the hyperquadric model of $\mathbb{S}^2 \times \mathbb{R}$, the isometry group $\text{SO}(3) \times \mathbb{R}$ is identified with

$$\left\{ \left(\begin{array}{cc} A & 0 \\ 0 & e^t \end{array} \right) \mid a \in \text{SO}(3), t \in \mathbb{R} \right\} \subset \text{GL}(4, \mathbb{R}).$$

We denote the isotropy subgroup of $\text{SO}(3) \times \mathbb{R}$ at the origin $o = (1, 0, 0, 0) \in \mathbb{S}^2 \times \mathbb{R}$ by H , which is isomorphic to $\text{SO}(2)$. Since there are no subgroups of dimension 2 in $\text{SO}(3)$, the connected subgroup G of $\text{Isom}_o(\mathbb{S}^2 \times \mathbb{R})$ acting transitively on $\mathbb{S}^2 \times \mathbb{R}$ is $\text{SO}(3) \times \mathbb{R}$. We take an orthonormal basis $\{u_1, u_2, u_3, e\}$ of $\mathfrak{so}(3) \oplus \mathbb{R}$ given by

$$u_1 = E_{23} - E_{32}, \quad u_2 = E_{12} - E_{21}, \quad u_3 = E_{31} - E_{13}, \quad e = E_{44}, \quad (4.2.25)$$

where E_{ij} is the matrix unit of 4×4 matrix. Then, we have

$$[u_1, u_2] = u_3, \quad [u_2, u_3] = u_1, \quad [u_3, u_1] = u_2, \quad [u_i, e] = 0 \quad (i = 1, 2, 3). \quad (4.2.26)$$

Here, the Lie algebra of H is $\mathfrak{h} = \langle \{u_1\} \rangle$. Moreover, from (2.2.1), (2.2.2) and (4.2.25), we have

$$\tau(u_2) = \frac{\partial}{\partial x^2} \Big|_o, \quad \tau(u_3) = -\frac{\partial}{\partial x^3} \Big|_o, \quad \tau(e) = \frac{\partial}{\partial y} \Big|_o. \quad (4.2.27)$$

Let $\mathfrak{m}$ be a Lie subspace of $\mathfrak{h}$ in $\mathfrak{so}(3) \oplus \mathbb{R}$, then $\mathfrak{m}$ is expressed as $\langle \{u_2 + \lambda_2 u_1, u_3 + \lambda_3 u_1, e + \lambda u_1\} \rangle$ for $\lambda_2, \lambda_3, \lambda \in \mathbb{R}$. Moreover, from $[\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}$, it follows that $\lambda_2 = \lambda_3 = 0$, and therefore $\mathfrak{m}$ is given by

$$\mathfrak{m} = \mathfrak{m}^\lambda = \langle \{u_2, u_3, e + \lambda u_1\} \rangle \quad (\lambda \in \mathbb{R}).$$

Then, from (3.1.3), (4.2.26) and (4.2.27), the homogeneous Riemannian structure S^λ defined from the reductive decomposition $\mathfrak{so}(3) \oplus \mathbb{R} = \mathfrak{m}^\lambda \oplus \mathfrak{h}$ satisfies

$$\begin{cases} S^\lambda_{\frac{\partial}{\partial x^2} \frac{\partial}{\partial x^2}} \Big|_o = 0, & S^\lambda_{\frac{\partial}{\partial x^2} \frac{\partial}{\partial x^3}} \Big|_o = 0, & S^\lambda_{\frac{\partial}{\partial x^2} \frac{\partial}{\partial y}} \Big|_o = 0, \\ S^\lambda_{\frac{\partial}{\partial x^3} \frac{\partial}{\partial x^2}} \Big|_o = 0, & S^\lambda_{\frac{\partial}{\partial x^3} \frac{\partial}{\partial x^3}} \Big|_o = 0, & S^\lambda_{\frac{\partial}{\partial x^3} \frac{\partial}{\partial y}} \Big|_o = 0, \\ S^\lambda_{\frac{\partial}{\partial y} \frac{\partial}{\partial x^2}} \Big|_o = \lambda \frac{\partial}{\partial x^3} \Big|_o, & S^\lambda_{\frac{\partial}{\partial y} \frac{\partial}{\partial x^3}} \Big|_o = -\lambda \frac{\partial}{\partial x^2} \Big|_o, & S^\lambda_{\frac{\partial}{\partial y} \frac{\partial}{\partial y}} \Big|_o = 0. \end{cases}$$

Therefore, we obtain

$$(S^\lambda)_\flat|_o = \lambda(dy \otimes (dx^2 \wedge dx^3))|_o = \lambda(dy \otimes dV_{\mathbb{S}^2})|_o,$$

where $dV_{\mathbb{S}^2}$ denotes the area form of $\mathbb{S}^2$. Since S^λ and $dy \otimes dV_{\mathbb{S}^2}$ are $(\text{SO}(3) \times \mathbb{R})$ -invariant, we have

$$(S^\lambda)_\flat = \lambda(dy \otimes dV_{\mathbb{S}^2}).$$

In addition, for an isometry ϕ , it follows that $\phi^* S^\lambda = S^\lambda$ if ϕ preserves the orientation, and $\phi^* S^\lambda = -S^\lambda$ if ϕ changes the orientation. Therefore, S^λ and S^μ are isomorphic if and only if $\lambda = \pm\mu$. This completes the proof. $\square$

Next we determine homogeneous Riemannian structures on $\mathbb{H}^2 \times \mathbb{R}$. Let us realize the Riemannian product $\mathbb{H}^2 \times \mathbb{R}$ as

$$\mathbb{H}^2 \times \mathbb{R} = \{(w = x + yi, z) \in \mathbb{C} \times \mathbb{R} \mid y > 0\}$$

with the product metric

$$g = g_{\mathbb{H}^2} + dz^2 = \frac{1}{y^2}(dx^2 + dy^2) + dz^2.$$

Theorem 4.2.11 ([18], Theorem 2). *The Riemannian product $\mathbb{H}^2 \times \mathbb{R}$ of the upper half plane model $\mathbb{H}^2$ of the hyperbolic plane with Poincare metric $(dx^2 + dy^2)/y^2$ and the real line $(\mathbb{R}, dz^2)$ admits two types of homogeneous Riemannian structure tensors up to isomorphism:*

$$\begin{aligned}(S^\lambda)_b &= \lambda \theta^3 \otimes (\theta^1 \wedge \theta^2) = \lambda (\theta_3 \otimes dV_{\mathbb{H}^2}) \quad (\lambda \geq 0), \\ S_b &= \theta^1 \otimes (\theta^1 \wedge \theta^2) = \theta^1 \otimes dV_{\mathbb{H}^2},\end{aligned}$$

where $\theta^1 = \frac{dx}{y}$, $\theta^2 = \frac{dy}{y}$, $\theta^3 = dz$, and $dV_{\mathbb{H}^2}$ is the area form of $\mathbb{H}^2$. The corresponding coset space representations of S^λ and S are $\mathbb{H}^2 \times \mathbb{R} = \text{SL}(2, \mathbb{R}) \times \mathbb{R} / \text{SO}(2)$ and $\mathbb{H}^2 \times \mathbb{R} = \text{GA}^+(1) \times \mathbb{R} / \{Id\}$, respectively.

Proof. The identity component $\text{Isom}_o(\mathbb{H}^2 \times \mathbb{R})$ of the full isometry group of $\mathbb{H}^2 \times \mathbb{R}$ is $\text{Isom}_o(\mathbb{H}^2 \times \mathbb{R}) = \text{SL}(2, \mathbb{R}) \times \mathbb{R}$ with Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$. The isometric action of $\text{SL}(2, \mathbb{R}) \times \mathbb{R}$ on $\mathbb{H}^2 \times \mathbb{R}$ is defined by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, s \right) \cdot (w, z) = \left(\frac{aw + b}{cw + d}, z + s \right).$$

Then the isotropy subgroup H of $\text{SL}(2, \mathbb{R}) \times \mathbb{R}$ at the origin $o = (i, 0)$ is $\text{SO}(2)$. Let G be a connected subgroup of $\text{SL}(2, \mathbb{R}) \times \mathbb{R}$ acting transitively on $\mathbb{H}^2 \times \mathbb{R}$, and then $\dim G = 4$ or 3 .

First, we consider the case $\dim G = 4$. This means that $G = \text{SL}(2, \mathbb{R}) \times \mathbb{R}$. We identify the Lie group $\text{SL}(2, \mathbb{R}) \times \mathbb{R}$ with the linear Lie group

$$\text{SL}(2, \mathbb{R}) \times \mathbb{R} = \left\{ \begin{pmatrix} A & 0 \\ 0 & e^t \end{pmatrix} \middle| A \in \text{SL}(2, \mathbb{R}), t \in \mathbb{R} \right\} \subset \text{GL}(3, \mathbb{R}).$$

Moreover, we take an orthonormal basis $\{v_1, v_2, v_3, e\}$ of $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$ given by

$$v_1 = \frac{1}{2}(E_{21} - E_{12}), \quad v_2 = \frac{1}{2}(E_{12} + E_{21}), \quad v_3 = \frac{1}{2}(E_{22} - E_{11}), \quad e = E_{33} \quad (4.2.28)$$

where E_{ij} is the matrix unit of 3×3 matrix. Then, we have

$$[v_1, v_2] = v_3, \quad [v_2, v_3] = -v_1, \quad [v_3, v_1] = v_2, \quad [v_i, e] = 0 \quad (i = 1, 2, 3).$$

Here, the Lie algebra of H is $\mathfrak{h} = \langle \{v_1\} \rangle$. From (2.2.1), (2.2.2) and (4.2.28), we have

$$\tau(v_2) = \frac{\partial}{\partial x} \Big|_o, \quad \tau(v_3) = -\frac{\partial}{\partial y} \Big|_o, \quad \tau(e) = \frac{\partial}{\partial z} \Big|_o.$$

Let $\mathfrak{m}$ be a Lie subspace of $\mathfrak{h}$ in $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$. As in the case of $\mathbb{S}^2 \times \mathbb{R}$, from $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, the Lie subspace $\mathfrak{m}$ is given by

$$\mathfrak{m} = \mathfrak{m}^\lambda = \langle \{v_2, v_3, e + \lambda v_1\} \rangle \quad (\lambda \in \mathbb{R}),$$

and the homogeneous Riemannian structure tensor S^λ determined by the reductive decomposition $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R} = \mathfrak{m}^\lambda \oplus \mathfrak{h}$ satisfies

$$(S^\lambda)_b = \lambda(dz \otimes dV_{\mathbb{H}^2}).$$

For the same reason as in $\mathbb{S}^2 \times \mathbb{R}$, S^λ and S^μ are isomorphic if and only if $\lambda = \pm\mu$.

Next, we consider the case $\dim G = 3$. In this case, the coset space representation is $\mathbb{H}^2 \times \mathbb{R} = G/\{e\}$, and then $\mathbb{H}^2 \times \mathbb{R}$ is identified with a Lie group G . Thus, G is isomorphic to the solvable Lie group

$$\mathbb{H}^2 \times \mathbb{R} = \text{GA}^+(1) \times \mathbb{R} = \left\{ \begin{pmatrix} y & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^z \end{pmatrix} \middle| x, y, z \in \mathbb{R}, y > 0 \right\},$$

with the left invariant metric

$$g = \frac{1}{y^2}(dx^2 + dy^2) + dz^2.$$

We define a left invariant orthonormal vector field $\{e_1, e_2, e_3\}$ as

$$e_1 = y \frac{\partial}{\partial x}, \quad e_2 = y \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}. \quad (4.2.29)$$

Then we have

$$[e_1, e_2] = -e_1, \quad [e_1, e_3] = [e_2, e_3] = 0. \quad (4.2.30)$$

Moreover, from (3.1.3) and (4.2.30), the homogeneous Riemannian structure tensor S determined by the reductive decomposition $\mathfrak{g} = \{0\} \oplus \mathfrak{g}$ satisfies

$$\begin{cases} S_{e_1}e_1 = e_2, & S_{e_1}e_2 = -e_1, & S_{e_1}e_3 = 0, \\ S_{e_2}e_1 = 0, & S_{e_2}e_2 = 0, & S_{e_2}e_3 = 0, \\ S_{e_3}e_1 = 0, & S_{e_3}e_2 = 0, & S_{e_3}e_3 = 0. \end{cases} \quad (4.2.31)$$

Thus, from (4.2.29) and (4.2.31), we obtain

$$S_b = \theta^1 \otimes (\theta^1 \wedge \theta^2),$$

where $\theta^1 = \frac{dx}{y}$, $\theta^2 = \frac{dy}{y}$. This completes the proof. $\square$

4.3 The classification of 3-dimensional Lie groups

In this section, we introduce the classification of 3-dimensional Riemannian Lie group given by Milnor (see [17]).

4.3.1 Unimodularity

Let G be a Lie group with a Lie algebra $\mathfrak{g}$ and a left invariant Riemannian metric g . We denote by $\langle \cdot, \cdot \rangle = g_{Id}$ the inner product of $\mathfrak{g}$ induced from g . Then the Levi-Civita connection ∇ of g is described by the *Koszul formula*:

$$2\langle \nabla_X Y, Z \rangle = -\langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle, \quad X, Y, Z \in \mathfrak{g}.$$

In particular, choosing an orthonormal basis $\{e_1, \dots, e_n\}$ and setting $\alpha_{ijk} = \langle [e_i, e_j], e_k \rangle$ it follows that

$$\nabla_{e_i} e_j = \sum_{k=1}^n \frac{1}{2} (\alpha_{ijk} - \alpha_{jki} + \alpha_{kij}) e_k. \quad (4.3.1)$$

A Lie group G is said to be **unimodular** if its left invariant Haar measure is also right invariant.

Lemma 4.3.1 ([17], Lemma 6.3). *A connected Lie group is unimodular if and only if the linear transformation $\text{ad}(X)$ has trace zero for every $X \in \mathfrak{g}$.*

Here $\text{ad}: \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ is a homomorphism defined by

$$\text{ad}(X)Y = [X, Y].$$

Milnor gave an infinitesimal reformulation of unimodularity for 3-dimensional Lie groups in [17]. We recall it briefly here.

Let $\mathfrak{g}$ be a 3-dimensional oriented Lie algebra with an inner product $\langle \cdot, \cdot \rangle$. Denote by $\times$ the *vector product operation* of the oriented inner product space $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$. The vector product operation is a skew-symmetric bilinear map $\times: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ which is uniquely determined by the following conditions:

- (i) $\langle X, X \times Y \rangle = \langle Y, X \times Y \rangle = 0$,
- (ii) $\langle X \times Y, X \times Y \rangle = \langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2$,
- (iii) if X and Y are linearly independent, then $\det(X, Y, X \times Y) > 0$

for all $X, Y \in \mathfrak{g}$. On the other hand, the Lie-bracket $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ is a skew-symmetric bilinear map. Comparing these two operations, we get a linear endomorphism $L_{\mathfrak{g}}$ which is uniquely determined by the formula

$$[X, Y] = L_{\mathfrak{g}}(X \times Y), \quad X, Y \in \mathfrak{g}.$$

Now let G be an oriented 3-dimensional Lie group equipped with a left invariant Riemannian metric. Then the metric induces an inner product on the Lie algebra $\mathfrak{g}$. With respect to the orientation on $\mathfrak{g}$ induced from G , the endomorphism field $L_{\mathfrak{g}}$ is uniquely determined. The unimodularity of G is characterized as follows.

Proposition 4.3.2 ([17], Lemma 4.1). *Let G be an oriented 3-dimensional Lie group with a left invariant Riemannian metric g . Then G is unimodular if and only if the endomorphism $L_{\mathfrak{g}}$ is self-adjoint with respect to the metric.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of the Lie group G , and $\{e_1, e_2, e_3\}$ an oriented orthonormal basis of $\mathfrak{g}$. Then we define a linear transformation $L: \mathfrak{g} \rightarrow \mathfrak{g}$ by

$$L(e_1) = [e_2, e_3], \quad L(e_2) = [e_3, e_1], \quad L(e_3) = [e_1, e_2].$$

Then it follows that $L(e_i \times e_j) = [e_i, e_j]$ for all i, j , hence $L(x \times y) = [x, y]$ for all $x, y \in \mathfrak{g}$. Let α_{ij} be

$$L(e_i) = \sum_{j=1}^3 \alpha_{ij} e_j,$$

and we can check that

$$\begin{aligned} \text{tr ad}(e_1) &= -\alpha_{23} + \alpha_{32}, \\ \text{tr ad}(e_2) &= -\alpha_{31} + \alpha_{13}, \\ \text{tr ad}(e_3) &= -\alpha_{12} + \alpha_{21}. \end{aligned}$$

Thus $\mathfrak{g}$ is unimodular if and only if the representation matrix (α_{ij}) is skew symmetric, in other words, if and only if the linear transformation L is self adjoint. $\square$

4.3.2 3-dimensional unimodular Lie groups

Let G be a 3-dimensional unimodular Lie group with a left invariant metric g . Then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{g}$ such that

$$[e_1, e_2] = \lambda_3 e_3, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad (4.3.2)$$

where $\lambda_i \in \mathbb{R}$. Three-dimensional unimodular Lie groups are classified by Milnor as follows.

Signature of $(\lambda_1, \lambda_2, \lambda_3)$	Property	Simply connected Lie group
$(+, +, +)$	simple, compact	$\text{SU}(2)$
$(+, +, -)$	simple, noncompact	$\widetilde{\text{SL}}(2, \mathbb{R})$
$(+, +, 0)$	solvable	$\widetilde{\text{SE}}(2)$
$(+, -, 0)$	solvable	$\text{SE}(1, 1)$
$(+, 0, 0)$	nilpotent	$\text{Nil}(3)$
$(0, 0, 0)$	Abelian	$\mathbb{R}^3$

Table 4.1: Classification of 3-dimensional unimodular Lie groups

To describe the Levi-Civita connection ∇ of G , we define numbers c_1, c_2, c_3 by the formula

$$c_i = \frac{\lambda_1 + \lambda_2 + \lambda_3}{2} - \lambda_i, \quad i \in \{1, 2, 3\}. \quad (4.3.3)$$

From (4.3.1) and (4.3.2), we can obtain the following proposition.

Proposition 4.3.3 ([17], p.319). *The Levi-Civita connection ∇ of the left invariant metric g is given by*

$$\begin{aligned}\nabla_{e_1}e_1 &= 0, & \nabla_{e_1}e_2 &= c_1e_3, & \nabla_{e_1}e_3 &= -c_1e_2, \\ \nabla_{e_2}e_1 &= -c_2e_3, & \nabla_{e_2}e_2 &= 0, & \nabla_{e_2}e_3 &= c_2e_1, \\ \nabla_{e_3}e_1 &= c_3e_2, & \nabla_{e_3}e_2 &= -c_3e_1, & \nabla_{e_3}e_3 &= 0.\end{aligned}$$

Theorem 4.3.4 ([17], Theorem 4.3). *Let $\{e_1, e_2, e_3\}$ be the orthonormal basis, chosen as above, and then the Riemannian curvature R is given by*

$$\begin{aligned}R(e_1, e_2)e_1 &= (c_1c_2 - \lambda_3c_3)e_2, & R(e_1, e_2)e_2 &= -(c_1c_2 - \lambda_3c_3)e_1, \\ R(e_2, e_3)e_2 &= (c_2c_3 - \lambda_1c_1)e_3, & R(e_2, e_3)e_3 &= -(c_2c_3 - \lambda_1c_1)e_2, \\ R(e_1, e_3)e_1 &= (c_3c_1 - \lambda_2c_2)e_3, & R(e_1, e_3)e_3 &= -(c_3c_1 - \lambda_2c_2)e_1.\end{aligned}$$

The basis $\{e_1, e_2, e_3\}$ diagonalizes the Ricci curvatures given by

$$\text{Ric}(e_1, e_1) = 2c_2c_3, \quad \text{Ric}(e_2, e_2) = 2c_1c_3, \quad \text{Ric}(e_3, e_3) = 2c_1c_2.$$

We define numbers r_1, r_2, r_3 by

$$r_1 = \text{Ric}(e_1, e_1), \quad r_2 = \text{Ric}(e_2, e_2), \quad r_3 = \text{Ric}(e_3, e_3).$$

Note that r_1, r_2 and r_3 are the eigenvalues of the endomorphism R^1 defined by the equation (4.1.3).

4.3.3 3-dimensional non-unimodular Lie groups

Let G be a non-unimodular Lie group with a left invariant metric. Then the unimodular kernel $\mathfrak{u} = \{x \in \mathfrak{g} \mid \text{trace ad}(x) = 0\}$ of the Lie algebra $\mathfrak{g}$ is 2-dimensional, unimodular and commutative. On the Lie algebra $\mathfrak{g}$, we can take an orthonormal basis $\{e_1, e_2, e_3\}$ such that

- (1) $\langle e_1, x \rangle = 0$ for any $x \in \mathfrak{u}$,
- (2) $\langle [e_1, e_2], [e_1, e_3] \rangle = 0$.

Then the bracket product can be expressed as

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 + \delta e_3, \quad [e_2, e_3] = 0, \quad (4.3.4)$$

where $\alpha + \delta \neq 0$ and $\alpha\gamma + \beta\delta = 0$. We denote by A the 2×2 matrix such that

$$A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}.$$

For simplicity, we make a scale change of the metric if necessary so that $\alpha + \delta = 2$. Then we set

$$\alpha = 1 + \xi, \quad \beta = (1 + \xi)\eta, \quad \gamma = -(1 - \xi)\eta, \quad \delta = 1 - \xi.$$

Then the bracket product is rewritten as

$$[e_1, e_2] = (1 + \xi)e_2 + (1 + \xi)\eta e_3, \quad [e_1, e_3] = -(1 - \xi)\eta e_2 + (1 - \xi)e_3, \quad [e_2, e_3] = 0. \quad (4.3.5)$$

By changing the sign of e_1, e_2 and e_3 , we assume that $\xi, \eta \geq 0$.

From (4.3.1) and (4.3.5), we can obtain following proposition.

Proposition 4.3.5. *The Levi-Civita connection ∇ of the left invariant metric g is given by*

$$\begin{aligned} \nabla_{e_1} e_1 &= 0, & \nabla_{e_1} e_2 &= \eta e_3, & \nabla_{e_1} e_3 &= -\eta e_2, \\ \nabla_{e_2} e_1 &= -(1 + \xi)e_2 - \xi\eta e_3, & \nabla_{e_2} e_2 &= (1 + \xi)e_1, & \nabla_{e_2} e_3 &= \xi\eta e_1, \\ \nabla_{e_3} e_1 &= -\xi\eta e_2 - (1 - \xi)e_3, & \nabla_{e_3} e_2 &= \xi\eta e_1, & \nabla_{e_3} e_3 &= (1 - \xi)e_1. \end{aligned}$$

Theorem 4.3.6. *The orthonormal basis $\{e_1, e_2, e_3\}$, chosen as above, the Riemannian curvature R is given by*

$$\begin{aligned} R(e_1, e_2)e_1 &= \{\xi\eta^2 + (1 + \xi)^2 + \xi\eta^2(1 + \xi)\}e_2, \\ R(e_1, e_2)e_2 &= -\{\xi\eta^2 + (1 + \xi)^2 + \xi\eta^2(1 + \xi)\}e_1, \\ R(e_2, e_3)e_2 &= \{1 - \xi^2(1 + \eta^2)\}e_3, \\ R(e_2, e_3)e_3 &= -\{1 - \xi^2(1 + \eta^2)\}e_2, \\ R(e_1, e_3)e_1 &= -\{\xi\eta^2 - (1 - \xi)^2 + \xi\eta^2(1 - \xi)\}e_3, \\ R(e_1, e_3)e_3 &= \{\xi\eta^2 - (1 - \xi)^2 + \xi\eta^2(1 - \xi)\}e_1. \end{aligned}$$

The basis $\{e_1, e_2, e_3\}$ diagonalizes the Ricci curvatures given by

$$\begin{aligned} r_1 &= \text{Ric}(e_1, e_1) = -2\{1 + \xi^2(1 + \eta^2)\}, \\ r_2 &= \text{Ric}(e_2, e_2) = -2\{1 + \xi(1 + \eta^2)\}, \\ r_3 &= \text{Ric}(e_3, e_3) = -2\{1 - \xi(1 + \eta^2)\}. \end{aligned}$$

Non-unimodular Lie algebras $\mathfrak{g} = \mathfrak{g}(\xi, \eta)$ are classified by the *Milnor invariant* $\mathcal{D} := \det A = (1 - \xi^2)(1 + \eta^2)$. More precisely we have the following fact (cf [17]).

Proposition 4.3.7 ([17], Lemma 4.10). *For any pair of (ξ, η) and (ξ', η') which are not $(0, 0)$, two Lie algebras $\mathfrak{g}(\xi, \eta)$ and $\mathfrak{g}(\xi', \eta')$ are isomorphic if and only if their Milnor invariants $\mathcal{D}$ and $\mathcal{D}'$ agree.*

The characteristic equation for the matrix A is $\lambda^2 - 2\lambda + \mathcal{D} = 0$. Thus the eigenvalues of A are $1 \pm \sqrt{1 - \mathcal{D}}$. Roughly speaking, the moduli space of non-unimodular Lie algebras $\mathfrak{g}(\xi, \eta)$ may be decomposed into three kinds of types which are determined by the conditions $\mathcal{D} > 1$, $\mathcal{D} = 1$ and $\mathcal{D} < 1$.

The simply connected Lie group $\tilde{G} = \tilde{G}(\xi, \eta)$ corresponding to the non-unimodular Lie algebra $\mathfrak{g}(\xi, \eta)$ is given explicitly by [11]:

$$\tilde{G}(\xi, \eta) = \left\{ \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & \alpha_{11}(x) & \alpha_{12}(x) & y \\ 0 & \alpha_{21}(x) & \alpha_{22}(x) & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\},$$

where

$$\begin{pmatrix} \alpha_{11}(x) & \alpha_{12}(x) \\ \alpha_{21}(x) & \alpha_{22}(x) \end{pmatrix} = \exp \left[x \begin{pmatrix} 1 + \xi & -(1 - \xi)\eta \\ (1 + \xi)\eta & 1 - \xi \end{pmatrix} \right].$$

This shows that $\tilde{G}(\xi, \eta)$ is the semi-direct product $\mathbb{R} \ltimes_{e^{xA}} \mathbb{R}^2$ with multiplication

$$(x, y, z) \cdot (x', y', z') = (x + x', y + \alpha_{11}(x)y' + \alpha_{12}(x)z', z + \alpha_{21}(x)y' + \alpha_{22}(x)z').$$

The Lie algebra of $\tilde{G}(\xi, \eta)$ is spanned by the basis

$$e_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 + \xi & -(1 - \xi)\eta & 0 \\ 0 & (1 + \xi)\eta & 1 - \xi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This basis satisfies the commutation relation

$$[e_1, e_2] = (1 + \xi)(e_2 + \eta e_3), \quad [e_2, e_3] = 0, \quad [e_3, e_1] = (1 - \xi)(\eta e_2 - e_3).$$

Thus the Lie algebra of $\tilde{G}(\xi, \eta)$ is the non-unimodular Lie algebra $\mathfrak{g} = \mathfrak{g}(\xi, \eta)$. The left invariant vector fields corresponding to e_1, e_2 and e_3 are

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \alpha_{11}(x) \frac{\partial}{\partial y} + \alpha_{12}(x) \frac{\partial}{\partial z}, \quad e_3 = \alpha_{21}(x) \frac{\partial}{\partial y} + \alpha_{22}(x) \frac{\partial}{\partial z}.$$

Example 4.3.8 ($\xi = 0, \mathcal{D} \geq 1$). The simply connected Lie group $\tilde{G}(0, \eta)$ is isometric to the hyperbolic 3-space $\mathbb{H}^3(-1)$ of curvature -1 and given explicitly by

$$\tilde{G}(0, \eta) = \left\{ \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & e^x \cos(\eta x) & -e^x \sin(\eta x) & y \\ 0 & e^x \sin(\eta x) & e^x \cos(\eta x) & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}.$$

The left invariant metric is $dx^2 + e^{-2x}(dy^2 + dz^2)$. Thus $\tilde{G}(0, \eta)$ is the warped product model of $\mathbb{H}^3(-1)$. Indeed, by setting $w = e^x$, then the left invariant metric of $\tilde{G}$ is rewritten as the Poincaré metric

$$\frac{dy^2 + dz^2 + dw^2}{w^2}.$$

The Milnor invariant of $\tilde{G}(0, \eta)$ is $\mathcal{D} = 1 + \eta^2 \geq 1$.

Example 4.3.9 ($\eta = 0, \mathcal{D} \leq 1$). For each $\xi \geq 0$, $\tilde{G}(\xi, 0)$ is given by:

$$\tilde{G}(\xi, 0) = \left\{ \left(\begin{array}{cccc} 1 & 0 & 0 & x \\ 0 & e^{(1+\xi)x} & 0 & y \\ 0 & 0 & e^{(1-\xi)x} & z \\ 0 & 0 & 0 & 1 \end{array} \right) \middle| x, y, z \in \mathbb{R} \right\}.$$

The left invariant Riemannian metric is given explicitly by

$$dx^2 + e^{-2(1+\xi)x} dy^2 + e^{-2(1-\xi)x} dz^2.$$

The Milnor invariant is $\mathcal{D} = 1 - \xi^2 \leq 1$.

The Lie group $\tilde{G}(\xi, 0)$ is realized as a warped product $\mathbb{H}^2(-(1+\xi)^2) \times_{f_\xi} \mathbb{R}$ of the hyperbolic plane $\mathbb{H}^2(-(1+\xi)^2)$ of curvature $-(1+\xi)^2$ and the real line $\mathbb{R}$ with warping function $f_\xi(x) = e^{(\xi-1)x}$. In fact, via the coordinate change $(u, v) = ((1+\xi)y, e^{(1+\xi)x})$, the metric is rewritten as

$$\frac{du^2 + dv^2}{(1+\xi)^2 v^2} + f_\xi(u, v)^2 dz^2, \quad f_\xi(u, v) = \exp\left(\frac{\xi-1}{\xi+1} \log v\right).$$

Here we observe locally symmetric examples:

- If $\xi = 0$ then $\tilde{G}(0, 0)$ is a warped product model of hyperbolic 3-space $\mathbb{H}^3(-1)$.
- If $\xi = 1$ then $\tilde{G}(1, 0)$ is isometric to the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$.

Example 4.3.10 ($\xi = 1, \mathcal{D} = 0$). Assume that $\xi = 1$. Then $\tilde{G}(1, \eta)$ is given explicitly by

$$\tilde{G}(1, \eta) = \left\{ \left(\begin{array}{cccc} 1 & 0 & 0 & x \\ 0 & e^{2x} & 0 & y \\ 0 & \eta(e^{2x} - 1) & 1 & z \\ 0 & 0 & 0 & 1 \end{array} \right) \middle| x, y, z \in \mathbb{R} \right\}.$$

The left invariant metric is

$$dx^2 + \{e^{-4x} + \eta^2(1 - e^{-2x})^2\} dy^2 - 2\eta(1 - e^{-2x}) dy dz + dz^2.$$

The non-unimodular Lie group $\tilde{G}(1, \eta)$ has the sectional curvatures

$$K_{12} = -3\eta^2 - 4, \quad K_{13} = K_{23} = \eta^2,$$

where K_{ij} ($i \neq j$) denote the sectional curvatures of the planes spanned by vectors e_i and e_j . The principal Ricci curvatures are

$$r_1 = r_2 = -2(\eta^2 + 2), \quad r_3 = 2\eta^2 \geq 0.$$

Hence Ric has the form

$$\text{Ric} = -2(\eta^2 + 2)g + 4(\eta^2 + 1)\theta^3 \otimes \theta^3.$$

One can check that $\tilde{G}(1, \eta)$ is isometric to the so-called *Bianchi-Cartan-Vranceanu space* $M^3(-4, \eta)$ with 4-dimensional isometry group and isotropy subgroup $\text{SO}(2)$:

$$M^3(-4, \eta) = (\{(u, v, w) \in \mathbb{R}^3 \mid u^2 + v^2 < 1\}, g_\eta),$$

with metric

$$g_\eta = \frac{du^2 + dv^2}{(1 - u^2 - v^2)^2} + \left(dw + \frac{\eta(vdu - u dv)}{1 - u^2 - v^2} \right)^2, \quad \eta \geq 0.$$

The family $\{\tilde{G}(1, \eta)\}_{\eta \geq 0}$ is characterized by the condition $\mathcal{D} = 0$. In particular $M^3(-4, \eta)$ with *positive* η is isometric to the universal covering $\widetilde{\text{SL}}(2, \mathbb{R})$ of the special linear group equipped with the above metric, but *not isomorphic* to $\widetilde{\text{SL}}(2, \mathbb{R})$ as Lie groups. We here note that $\widetilde{\text{SL}}(2, \mathbb{R})$ is a *unimodular* Lie group, while $\tilde{G}(1, \eta)$ is *non-unimodular*.

Let us note that the non-unimodular Lie algebra $\mathfrak{g}(1, \eta)$ is isomorphic to the direct sum $\mathfrak{ga}(1) \oplus \mathbb{R}$, where $\mathfrak{ga}(1)$ is the Lie algebra of the Lie group $\text{GA}^+(1)$. Here the direct sum $\mathfrak{ga}(1) \oplus \mathbb{R}$ is given explicitly by

$$\left\{ \begin{pmatrix} u & v & 0 \\ 0 & 0 & 0 \\ 0 & 0 & w \end{pmatrix} \middle| u, v, w \in \mathbb{R} \right\}$$

and it is spanned by the basis

$$\bar{e}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{e}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{e}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the bracket product is expressed as

$$[\bar{e}_1, \bar{e}_2] = -\bar{e}_1, \quad [\bar{e}_2, \bar{e}_3] = [\bar{e}_3, \bar{e}_1] = 0.$$

One can confirm that

$$\bar{e}_1 \mapsto 2e_2 + 2\eta e_3, \quad \bar{e}_2 = \frac{1}{2}e_1, \quad \bar{e}_3 = e_3$$

gives a Lie algebra isomorphism from $\mathfrak{ga}(1) \oplus \mathbb{R}$ to $\mathfrak{g}(1, \eta)$.

A direct computation of ∇R yields the following result.

Proposition 4.3.11. *Let (G, g) be a 3-dimensional non-unimodular Lie group and use the notations introduced above. Then G is locally symmetric if and only if $\xi = 0$ or $(\xi, \eta) = (1, 0)$.*

4.4 Homogeneous structures on 3-dimensional Lie groups

4.4.1 Homogeneous structures on unimodular Lie groups

In this section, we classify homogeneous Riemannian structures on 3-dimensional unimodular Lie groups. Let $\tilde{\nabla}$ be an Ambrose-Singer connection, that is,

$$\tilde{\nabla} g = 0, \quad \tilde{\nabla} R = 0, \quad \tilde{\nabla} S = 0,$$

where $S = \tilde{\nabla} - \nabla$ and ∇ is a Levi-Civita connection of g .

$$Xg(Y, Z) = g(\tilde{\nabla}_X Y, Z) + g(Y, \tilde{\nabla}_X Z). \quad (4.4.1)$$

Moreover, since ∇ is the Levi-Civita connection, it follows that

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \quad (4.4.2)$$

From (4.4.1) and (4.4.2), we obtain that

$$0 = S_b(X, Y, Z) + S_b(X, Z, Y). \quad (4.4.3)$$

Let $\{\theta^1, \theta^2, \theta^3\}$ be a dual basis of $\{e_1, e_2, e_3\}$ and define 1-forms ρ, σ, τ by

$$\rho(X) = S_b(X, e_1, e_2), \quad \sigma(X) = S_b(X, e_2, e_3), \quad \tau(X) = S_b(X, e_3, e_1). \quad (4.4.4)$$

Then S is expressed as

$$S_b = 2\rho \otimes (\theta^1 \wedge \theta^2) + 2\sigma \otimes (\theta^2 \wedge \theta^3) + 2\tau \otimes (\theta^3 \wedge \theta^1). \quad (4.4.5)$$

Next we consider the condition $\tilde{\nabla}R = 0$. Since G is 3-dimensional, the condition $\tilde{\nabla}R = 0$ is equivalent to $\tilde{\nabla}\text{Ric} = 0$. Moreover, it is equivalent to

$$(\nabla_X \text{Ric})(Y, Z) = \text{Ric}(S_X Y, Z) + \text{Ric}(Y, S_X Z).$$

From Proposition 4.3.3 and Theorem 4.3.4, we compute

$$\begin{aligned} (\nabla_X \text{Ric})(e_1, e_2) &= -\text{Ric}(\nabla_X e_1, e_2) - \text{Ric}(e_1, \nabla_X e_2) \\ &= -c_3 \theta^3(X) r_2 + c_3 \theta^3(X) r_1 \\ &= c_3 \theta^3(X) (r_1 - r_2), \\ (\nabla_X \text{Ric})(e_2, e_3) &= -\text{Ric}(\nabla_X e_2, e_3) - \text{Ric}(e_2, \nabla_X e_3) \\ &= c_1 \theta^1(X) (r_2 - r_3), \\ (\nabla_X \text{Ric})(e_3, e_1) &= -\text{Ric}(\nabla_X e_3, e_1) - \text{Ric}(e_3, \nabla_X e_1) \\ &= c_2 \theta^2(X) (r_3 - r_1), \end{aligned}$$

and

$$\begin{aligned} \text{Ric}(S_X e_1, e_2) + \text{Ric}(e_1, S_X e_2) &= -\rho(X) (r_1 - r_2), \\ \text{Ric}(S_X e_2, e_3) + \text{Ric}(e_2, S_X e_3) &= -\sigma(X) (r_2 - r_3), \\ \text{Ric}(S_X e_3, e_1) + \text{Ric}(e_3, S_X e_1) &= -\tau(X) (r_3 - r_1). \end{aligned}$$

Then the condition $\tilde{\nabla}R = 0$ is equivalent to

$$\begin{cases} (r_1 - r_2)(c_3 \theta^3 + \rho) = 0, \\ (r_2 - r_3)(c_1 \theta^1 + \sigma) = 0, \\ (r_3 - r_1)(c_2 \theta^2 + \tau) = 0. \end{cases} \quad (4.4.6)$$

Note that

$$\begin{cases} r_1 - r_2 = -2c_3(\lambda_1 - \lambda_2), \\ r_2 - r_3 = -2c_1(\lambda_2 - \lambda_3), \\ r_3 - r_1 = -2c_2(\lambda_3 - \lambda_1). \end{cases} \quad (4.4.7)$$

First, we consider the case that λ_1, λ_2 and λ_3 are distinct.

Theorem 4.4.1 ([13], Theorem 2). *The only Ambrose-Singer connection on a 3-dimensional unimodular Lie group G with distinct structure constants is $\tilde{\nabla}_X Y = 0$ for any $X, Y \in \mathfrak{g}$, and the Ambrose-Singer $\tilde{\nabla}$ coincides with Cartan-Schouten's $(-)$ -connection.*

Proof. First, suppose that $c_1 c_2 c_3 \neq 0$. From (4.4.6) and (4.4.7), since λ_1, λ_2 and λ_3 are distinct, it follows that

$$\rho = -c_3 \theta^3, \quad \sigma = -c_1 \theta^1, \quad \tau = -c_2 \theta^2.$$

Then, from (4.4.4) and (4.4.5), it follows that

$$S_b(X, Y, Z) = -2c_3 \theta^3(X)(\theta^1 \wedge \theta^2)(Y, Z) - 2c_1 \theta^1(X)(\theta^2 \wedge \theta^3)(Y, Z) - 2c_2 \theta^2(X)(\theta^3 \wedge \theta^1)(Y, Z).$$

Moreover, since $S = \tilde{\nabla} - \nabla$ and $S_b(X, Y, Z) = g(S_X Y, Z)$, we obtain

$$\tilde{\nabla}_{e_i} e_j = 0, \quad (i, j \in \{1, 2, 3\}). \quad (4.4.8)$$

Next, suppose that $c_1 c_2 c_3 = 0$. Without loss of generality, we can suppose that $c_3 = 0$. Then, if $c_1 = 0$, it follows that $\lambda_1 = \lambda_3$ and $\lambda_2 = 0$, and this contradicts the assumption that λ_1, λ_2 and λ_3 are distinct, and we obtain $c_1 \neq 0$. Similarly, we obtain $c_2 \neq 0$. Therefore, from (4.4.6) and (4.4.7), we obtain

$$\sigma = -c_1 \theta^1, \quad \tau = -c_2 \theta^2. \quad (4.4.9)$$

Finally, we consider the condition $\tilde{\nabla} S = 0$. We define functions ρ_1, ρ_2, ρ_3 by $\rho = \rho_1 \theta^1 + \rho_2 \theta^2 + \rho_3 \theta^3$. From (4.4.9) and $S = \tilde{\nabla} - \nabla$, we compute

$$\begin{aligned} \tilde{\nabla}_{e_1} e_1 &= \rho_1 e_2, & \tilde{\nabla}_{e_1} e_2 &= -\rho_1 e_1, & \tilde{\nabla}_{e_1} e_3 &= 0, \\ \tilde{\nabla}_{e_2} e_1 &= \rho_2 e_2, & \tilde{\nabla}_{e_2} e_2 &= -\rho_2 e_1, & \tilde{\nabla}_{e_2} e_3 &= 0, \\ \tilde{\nabla}_{e_3} e_1 &= \rho_3 e_2, & \tilde{\nabla}_{e_3} e_2 &= -\rho_3 e_1, & \tilde{\nabla}_{e_3} e_3 &= 0. \end{aligned}$$

From this and the assumption $\tilde{\nabla} S = 0$, we compute

$$\begin{aligned} 0 &= (\tilde{\nabla}_{e_1} S)(e_2, e_2, e_3) \\ &= -S(\tilde{\nabla}_{e_1} e_2, e_2, e_3) - S(e_2, \tilde{\nabla}_{e_1} e_2, e_3) - S(e_2, e_2, \tilde{\nabla}_{e_1} e_3) \\ &= \rho_1 S(e_1, e_2, e_3) + \rho_1 S(e_2, e_1, e_3) \\ &= \rho_1(c_1 - c_2) \\ &= -\rho_1(\lambda_1 - \lambda_2), \end{aligned}$$

and we obtain $\rho_1 = 0$. Similarly, we can check that $\rho_2 = \rho_3 = 0$ by computing $(\tilde{\nabla}_{e_2} S)(e_2, e_2, e_3)$ and $(\tilde{\nabla}_{e_3} S)(e_2, e_2, e_3)$. Therefore, if $c_3 = 0$, we can obtain

$$S_b(X, Y, Z) = -2c_1\theta^1(X)(\theta^2 \wedge \theta^3)(Y, Z) - 2c_2\theta^2(X)(\theta^3 \wedge \theta^1)(Y, Z),$$

and it follows that

$$\tilde{\nabla}_{e_i} e_j = 0, \quad (i, j \in \{1, 2, 3\}). \quad (4.4.10)$$

This completes the proof. $\square$

Next, we consider the case that $\lambda_1 = \lambda_2 \neq \lambda_3 \neq 0$. Let λ be $\lambda = \lambda_1 = \lambda_2$.

Theorem 4.4.2 ([12], Theorem 8.1). (1) *If $\lambda \cdot \lambda_3 \geq 0$, all homogeneous Riemannian structures S on (G, g) are given by*

$$S_b^{(t)} = 2t\theta^3 \otimes (\theta^1 \wedge \theta^2) - \lambda_3\theta^1 \otimes (\theta^2 \wedge \theta^3) - \lambda_3\theta^2 \otimes (\theta^3 \wedge \theta^1), \quad (4.4.11)$$

where t is an arbitrary constant.

(2) *If $\lambda \cdot \lambda_3 < 0$, all homogeneous Riemannian structures S on (G, g) are following two types;*

$$S_b^{(t)} = 2t\theta^3 \otimes (\theta^1 \wedge \theta^2) - \lambda_3\theta^1 \otimes (\theta^2 \wedge \theta^3) - \lambda_3\theta^2 \otimes (\theta^3 \wedge \theta^1), \quad (4.4.12)$$

$$S_b^{(\phi)} = 2\rho \otimes (\theta^1 \wedge \theta^2) - \lambda_3\theta^1 \otimes (\theta^2 \wedge \theta^3) - \lambda_3\theta^2 \otimes (\theta^3 \wedge \theta^1), \quad (4.4.13)$$

where t is an arbitrary constant and

$$\rho = \sqrt{-\lambda\lambda_3} \cos \phi \theta^1 + \sqrt{-\lambda\lambda_3} \sin \phi \theta^2 + \frac{\lambda_3}{2} \theta^3, \quad (4.4.14)$$

for some function ϕ .

Proof. From the assumption $\lambda_1 = \lambda_2 \neq \lambda_3 \neq 0$, it follows that

$$c_1 = c_2 = \frac{\lambda_3}{2} \neq 0.$$

Thus from the conditions (4.4.6) and (4.4.7), that is, $\tilde{\nabla} R = 0$, we obtain

$$\sigma = -c_1\theta^1, \quad \tau = -c_2\theta^2, \quad (4.4.15)$$

and the homogeneous Riemannian structure S has the form

$$S_b = 2\rho \otimes (\theta^1 \wedge \theta^2) - \lambda_3\theta^1 \otimes (\theta^2 \wedge \theta^3) - \lambda_3\theta^2 \otimes (\theta^3 \wedge \theta^1). \quad (4.4.16)$$

We consider the condition $\tilde{\nabla} S = 0$. From (4.4.15) and $S = \tilde{\nabla} - \nabla$, we compute

$$\begin{aligned} \tilde{\nabla}_{e_1} e_1 &= \rho_1 e_2, & \tilde{\nabla}_{e_1} e_2 &= -\rho_1 e_1, & \tilde{\nabla}_{e_1} e_3 &= 0, \\ \tilde{\nabla}_{e_2} e_1 &= \rho_2 e_2, & \tilde{\nabla}_{e_2} e_2 &= -\rho_2 e_1, & \tilde{\nabla}_{e_2} e_3 &= 0, \\ \tilde{\nabla}_{e_3} e_1 &= \rho_3 e_2, & \tilde{\nabla}_{e_3} e_2 &= -\rho_3 e_1, & \tilde{\nabla}_{e_3} e_3 &= 0. \end{aligned}$$

From these equations we also compute

$$\begin{aligned}\tilde{\nabla}_X \theta^1 &= (\rho(X) + c_3 \theta^3(X)) \theta^2, \\ \tilde{\nabla}_X \theta^2 &= -(\rho(X) + c_3 \theta^3(X)) \theta^1, \\ \tilde{\nabla}_X \theta^3 &= 0.\end{aligned}$$

By using these equations, we obtain

$$\tilde{\nabla}_X S_b = 2(\tilde{\nabla}_X \rho) \otimes (\theta^1 \wedge \theta^2). \quad (4.4.17)$$

Hence $\tilde{\nabla} S = 0$ if and only if $\tilde{\nabla} \rho = 0$.

By using $\rho = \rho_1 \theta^1 + \rho_2 \theta^2 + \rho_3 \theta^3$, $\tilde{\nabla}_X \rho$ is computed as

$$\begin{aligned}\tilde{\nabla}_X \rho &= \{(d\rho_1)(X) - \rho_2(\rho(X) + c_3 \theta^3(X))\} \theta^1 \\ &\quad + \{(d\rho_2)(X) + \rho_1(\rho(X) + c_3 \theta^3(X))\} \theta^2 + (d\rho_3)(X) \theta^3.\end{aligned}$$

Thus the parallelism of S with respect to $\tilde{\nabla}$ is equivalent to the differential system:

$$d\rho_1 = \rho_2(\rho + c_3 \theta^3), \quad d\rho_2 = -\rho_1(\rho + c_3 \theta^3), \quad d\rho_3 = 0. \quad (4.4.18)$$

Hence ρ_3 is a constant, and we put $\rho_3 = t$.

Let us investigate the solutions $\{\rho_1, \rho_2\}$ to this differential system. First, if both ρ_1 and ρ_2 are identically 0, then the pair $\{\rho_1, \rho_2\}$ is clearly a solution to the differential system. Then we suppose that $\{\rho_1, \rho_2\}$ are not identically 0. The integrability condition is

$$(XY - YX - [X, Y])\rho_i = 0, \quad (i = 1, 2), \quad (4.4.19)$$

for any $X, Y \in \mathfrak{g}$. From (4.4.18), the integrability condition (4.4.19) is equivalent to

$$\begin{aligned}\rho_1(X(\rho(Y)) - Y(\rho(X)) - (\rho + c_3 \theta^3)([X, Y])) &= 0, \\ \rho_2(X(\rho(Y)) - Y(\rho(X)) - (\rho + c_3 \theta^3)([X, Y])) &= 0.\end{aligned}$$

Then for any $p \in G$, it follows that $\rho_1(p) = \rho_2(p) = 0$ or

$$(X(\rho(Y)) - Y(\rho(X)) - (\rho + c_3 \theta^3)([X, Y]))(p) = 0. \quad (4.4.20)$$

By inserting $(X, Y) = (e_1, e_2), (e_1, e_3), (e_2, e_3)$ into the equation (4.4.20), from (4.3.2) and (4.4.18), we obtain

$$\begin{cases} \rho_1(p)^2 + \rho_2(p)^2 = -\lambda \lambda_3, \\ \rho_1(p)(-t + \frac{\lambda_3}{2}) = 0, \\ \rho_2(p)(-t + \frac{\lambda_3}{2}) = 0. \end{cases} \quad (4.4.21)$$

From the second and third equations of (4.4.21), if $t \neq \frac{\lambda_3}{2}$, it follows that $\rho_1 = \rho_2 = 0$, and thus we have $t = \frac{\lambda_3}{2}$. We consider two cases: (1) $\lambda \lambda_3 \geq 0$ or (2) $\lambda \lambda_3 < 0$.

- (1) If $\lambda\lambda_3 \geq 0$, from the first equation of (4.4.21), for any $p \in G$ it follows that $\rho_1(p) = \rho_2(p) = 0$ or

$$\rho_1(p)^2 + \rho_2(p)^2 = -\lambda\lambda_3 \leq 0.$$

Hence $\rho_1 = \rho_2 = 0$, and this is the solution (4.4.11).

- (2) If $\lambda\lambda_3 < 0$, for any $p \in G$, it follows that $\rho_1(p) = \rho_2(p) = 0$ or

$$\rho_1(p)^2 + \rho_2(p)^2 = -\lambda\lambda_3 > 0.$$

Then from the continuity of ρ_1 and ρ_2 , it follows that

$$\rho_1^2 + \rho_2^2 = -\lambda\lambda_3.$$

Moreover, from the smoothness of ρ_1 and ρ_2 , these are expressed

$$\rho_1 = \sqrt{-\lambda\lambda_3} \cos \phi, \quad \rho_2 = \sqrt{-\lambda\lambda_3} \sin \phi, \quad (4.4.22)$$

for some function ϕ . Inserting these formulas into the system (4.4.18), we obtain

$$d\phi = - \left(\sqrt{-\lambda\lambda_3} \cos \phi \theta^1 + \sqrt{-\lambda\lambda_3} \sin \phi \theta^2 + \lambda \theta^3 \right). \quad (4.4.23)$$

Conversely let ϕ be a solution to (4.4.23). Define two functions ρ_1 and ρ_2 by (4.4.22), then $\{\rho_1, \rho_2\}$ is the solution of the system (4.4.18). This completes the proof. □

Next we give the coset representation corresponding to homogeneous Riemannian structures $S^{(t)}$ and S . Suppose that G is simply connected.

Theorem 4.4.3 ([12], Theorem 8.1). (1) *If $\lambda\lambda_3 > 0$, that is, $G \cong \text{SU}(2)$, the corresponding coset space representation of $S^{(t)}$ is*

$$\text{SU}(2) \cong (\text{SU}(2) \times \text{SO}(2)) / \text{SO}(2) \quad (t \neq -c_3), \quad (4.4.24)$$

$$\cong \text{SU}(2) / \{Id\} \quad (t = -c_3). \quad (4.4.25)$$

- (2) *If $\lambda\lambda_3 = 0$ (i.e. $\lambda = 0$), that is, $G \cong \text{Nil}(3)$, the corresponding coset space representation of $S^{(t)}$ is*

$$\text{Nil}(3) \cong (\text{Nil}(3) \ltimes \text{SO}(2)) / \text{SO}(2) \quad (t \neq -c_3), \quad (4.4.26)$$

$$\cong \text{Nil}(3) / \{Id\} \quad (t = -c_3). \quad (4.4.27)$$

- (3) *If $\lambda\lambda_3 < 0$, that is, $G \cong \widetilde{\text{SL}}(2, \mathbb{R})$, the corresponding coset space representation of $S^{(t)}$ is*

$$\widetilde{\text{SL}}(2, \mathbb{R}) \cong (\widetilde{\text{SL}}(2, \mathbb{R}) \times \text{SO}(2)) / \text{SO}(2) \quad (t \neq -c_3), \quad (4.4.28)$$

$$\cong \widetilde{\text{SL}}(2, \mathbb{R}) / \{Id\} \quad (t = -c_3), \quad (4.4.29)$$

and the corresponding coset space representation of S is

$$\widetilde{\text{SL}}(2, \mathbb{R}) \cong (\text{GA}^+(1) \times \mathbb{R}) / \{Id\}. \quad (4.4.30)$$

Proof. First we consider the homogeneous Riemannian structure $S^{(t)}$. From (4.4.11) and (4.4.12), we compute

$$\begin{aligned} S_{e_1}^{(t)}e_1 &= 0, & S_{e_1}^{(t)}e_2 &= -\frac{\lambda_3}{2}e_3, & S_{e_1}^{(t)}e_3 &= \frac{\lambda_3}{2}e_2, \\ S_{e_2}^{(t)}e_1 &= \frac{\lambda_3}{2}e_3, & S_{e_2}^{(t)}e_2 &= 0, & S_{e_2}^{(t)}e_3 &= -\frac{\lambda_3}{2}e_1, \\ S_{e_3}^{(t)}e_1 &= te_2, & S_{e_3}^{(t)}e_2 &= -te_1, & S_{e_3}^{(t)}e_3 &= 0. \end{aligned}$$

Let $\tilde{\nabla}^{(t)}$ be the Ambrose-Singer connection, that is, $\tilde{\nabla}^{(t)} = \nabla + S^{(t)}$. Then from Proposition 4.3.3 we obtain

$$\begin{aligned} \tilde{\nabla}_{e_1}^{(t)}e_1 &= 0, & \tilde{\nabla}_{e_1}^{(t)}e_2 &= 0, & \tilde{\nabla}_{e_1}^{(t)}e_3 &= 0, \\ \tilde{\nabla}_{e_2}^{(t)}e_1 &= 0, & \tilde{\nabla}_{e_2}^{(t)}e_2 &= 0, & \tilde{\nabla}_{e_2}^{(t)}e_3 &= 0, \\ \tilde{\nabla}_{e_3}^{(t)}e_1 &= (t + c_3)e_2, & \tilde{\nabla}_{e_3}^{(t)}e_2 &= -(t + c_3)e_1, & \tilde{\nabla}_{e_3}^{(t)}e_3 &= 0. \end{aligned}$$

The curvature $\tilde{R}^{(t)}$ of the Ambrose-Singer connection $\tilde{\nabla}^{(t)}$ is given by

$$\begin{aligned} \tilde{R}^{(t)}(e_1, e_2)e_1 &= -\lambda_3(t + c_3)e_2, & \tilde{R}^{(t)}(e_1, e_2)e_2 &= \lambda_3(t + c_3)e_1, & \tilde{R}^{(t)}(e_1, e_2)e_3 &= 0, \\ \tilde{R}^{(t)}(e_2, e_3)e_1 &= 0, & \tilde{R}^{(t)}(e_2, e_3)e_2 &= 0, & \tilde{R}^{(t)}(e_2, e_3)e_3 &= 0, \\ \tilde{R}^{(t)}(e_3, e_1)e_1 &= 0, & \tilde{R}^{(t)}(e_3, e_1)e_2 &= 0, & \tilde{R}^{(t)}(e_3, e_1)e_3 &= 0. \end{aligned}$$

Let us determine the Lie group $L^{(t)}$ acting isometrically and transitively on (G, g) corresponding to the homogeneous Riemannian structure $S^{(t)}$ and its reductive decomposition $\mathfrak{l}^{(t)} = \mathfrak{h}^{(t)} \oplus \mathfrak{m}$. Here $\mathfrak{m}$ is identified with the Lie algebra $\mathfrak{g} = \langle e_1, e_2, e_3 \rangle$ as a real vector space equipped with an inner product.

If $t + c_3 = 0$, then $\tilde{\nabla}_X^{(t)}Y = \tilde{\nabla}_X^{(-c_3)}Y = 0$ for any $X, Y \in \mathfrak{g}$, and the corresponding coset space representation to the homogeneous Riemannian structure $S^{(t)} = S^{(-c_3)}$ is $G \cong G/\{Id\}$. Hence we obtain the coset representations (4.4.25), (4.4.27) and (4.4.29).

Next we suppose $t + c_3 \neq 0$. Let $e_4^{(t)} = \tilde{R}^{(t)}(e_1, e_2)$, then the holonomy algebra $\mathfrak{h}^{(t)}$ of the Ambrose-Singer connection $\tilde{\nabla}^{(t)}$ is spanned by $e_4^{(t)}$. Thus we have $\mathfrak{l}^{(t)} = \mathfrak{h}^{(t)} \oplus \mathfrak{m} =$

$\langle e_4^{(t)} \rangle \oplus \langle e_1, e_2, e_3 \rangle$, and the Lie bracket $[\cdot, \cdot]_{\mathfrak{l}^{(t)}}$ of $\mathfrak{l}^{(t)}$ is defined by

$$\begin{aligned}
[e_1, e_2]_{\mathfrak{l}^{(t)}} &= -S_{e_1}^{(t)} e_2 + S_{e_2}^{(t)} e_1 - \tilde{R}^{(t)}(e_1, e_2), \\
&= \lambda_3 e_3 - e_4^{(t)}, \\
[e_2, e_3]_{\mathfrak{l}^{(t)}} &= -S_{e_2}^{(t)} e_3 + S_{e_3}^{(t)} e_2 - \tilde{R}^{(t)}(e_2, e_3), \\
&= \left(\frac{\lambda_3}{2} - t \right) e_1, \\
[e_3, e_1]_{\mathfrak{l}^{(t)}} &= -S_{e_3}^{(t)} e_1 + S_{e_1}^{(t)} e_3 - \tilde{R}^{(t)}(e_3, e_1), \\
&= \left(\frac{\lambda_3}{2} - t \right) e_2, \\
[e_1, e_4^{(t)}]_{\mathfrak{l}^{(t)}} &= -e_4^{(t)}(e_1), \\
&= -\tilde{R}^{(t)}(e_1, e_2) e_1, \\
&= \lambda_3(t + c_3) e_2, \\
[e_2, e_4^{(t)}]_{\mathfrak{l}^{(t)}} &= -e_4^{(t)}(e_2), \\
&= -\tilde{R}^{(t)}(e_1, e_2) e_2, \\
&= -\lambda_3(t + c_3) e_1, \\
[e_3, e_4^{(t)}]_{\mathfrak{l}^{(t)}} &= 0.
\end{aligned}$$

Next we determine the Lie algebra $\mathfrak{l}^{(t)}$.

(1) When $\lambda \neq 0$, let us change the basis $\{e_1, e_2, e_3, e_4^{(t)}\}$ to

$$\tilde{e}_1 = e_1, \quad \tilde{e}_2 = e_2, \quad \tilde{e}_3 = e_3 - \frac{1}{\lambda_3} e_4^{(t)}, \quad \tilde{e}_4^{(t)} = e_3 - \frac{1}{\lambda_3} \frac{t - \frac{\lambda_3}{2}}{t + c_3} e_4^{(t)}.$$

Note that $\tilde{e}_3$ and $\tilde{e}_4^{(t)}$ are linearly independent from $\lambda \neq 0$ and $c_3 = \frac{\lambda_1 + \lambda_2 - \lambda_3}{2} = \lambda - \frac{\lambda_3}{2}$.

Then we obtain

$$\begin{aligned}
[\tilde{e}_1, \tilde{e}_2]_{\mathfrak{l}^{(t)}} &= \lambda_3 \tilde{e}_3, \\
[\tilde{e}_2, \tilde{e}_3]_{\mathfrak{l}^{(t)}} &= \left[e_2, e_3 - \frac{1}{\lambda_3} e_4^{(t)} \right]_{\mathfrak{l}^{(t)}}, \\
&= \left(\frac{\lambda_3}{2} - t \right) e_1 - \frac{1}{\lambda_3} (-\lambda_3(t + c_3)) e_1, \\
&= \lambda e_1 = \lambda \tilde{e}_1, \\
[\tilde{e}_3, \tilde{e}_1]_{\mathfrak{l}^{(t)}} &= \left[e_3 - \frac{1}{\lambda_3} e_4^{(t)} \right]_{\mathfrak{l}^{(t)}}, \\
&= \left(\frac{\lambda_3}{2} - t \right) e_2 + \frac{1}{\lambda_3} \lambda_3(t + c_3) e_2, \\
&= \lambda e_2 = \lambda \tilde{e}_2, \\
[\tilde{e}_1, \tilde{e}_4^{(t)}]_{\mathfrak{l}^{(t)}} &= \left[e_1, e_3 - \frac{1}{\lambda_3} \frac{t - \frac{\lambda_3}{2}}{t + c_3} e_4^{(t)} \right]_{\mathfrak{l}^{(t)}}, \\
&= - \left(\frac{\lambda_3}{2} - t \right) e_2 - \frac{1}{\lambda_3} \frac{t - \frac{\lambda_3}{2}}{t + c_3} \lambda_3(t + c_3) e_2, \\
&= 0, \\
[\tilde{e}_2, \tilde{e}_4^{(t)}]_{\mathfrak{l}^{(t)}} &= \left[e_2, e_3 - \frac{1}{\lambda_3} \frac{t - \frac{\lambda_3}{2}}{t + c_3} e_4^{(t)} \right]_{\mathfrak{l}^{(t)}}, \\
&= \left(\frac{\lambda_3}{2} - t \right) e_1 + \frac{1}{\lambda_3} \frac{t - \frac{\lambda_3}{2}}{t + c_3} \lambda_3(t + c_3) e_1 \\
&= 0, \\
[\tilde{e}_3, \tilde{e}_4^{(t)}]_{\mathfrak{l}^{(t)}} &= 0.
\end{aligned}$$

Hence the Lie algebra $\mathfrak{l}^{(t)}$ is the direct product of the Lie algebra $\mathfrak{so}(2) = \langle \tilde{e}_4^{(t)} \rangle$ and $\mathfrak{g} = \langle \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \rangle$. Here from Table 4.1, if $\lambda\lambda_3 > 0$, $\mathfrak{g} \cong \mathfrak{su}(2)$ and if $\lambda\lambda_3 < 0$, $\mathfrak{g} \cong \mathfrak{sl}(2, \mathbb{R})$. Thus we obtain the coset representations (4.4.24) and (4.4.28).

(2) When $\lambda = 0$, let us change the basis $\{e_1, e_2, e_3, e_4^{(t)}\}$ to

$$\tilde{e}_1 = e_1, \quad \tilde{e}_2 = e_2, \quad \tilde{e}_3 = e_3 - \frac{1}{\lambda_3} e_4^{(t)}, \quad \tilde{e}_4^{(t)} = \frac{1}{t + c_3} e_4^{(t)}.$$

Then we obtain

$$\begin{aligned}
[\tilde{e}_1, \tilde{e}_2]_{\mathfrak{l}^{(t)}} &= \lambda_3 \tilde{e}_3, & [\tilde{e}_2, \tilde{e}_3]_{\mathfrak{l}^{(t)}} &= 0, & [\tilde{e}_3, \tilde{e}_1]_{\mathfrak{l}^{(t)}} &= 0, \\
[\tilde{e}_1, \tilde{e}_4^{(t)}]_{\mathfrak{l}^{(t)}} &= \lambda_3 \tilde{e}_2, & [\tilde{e}_2, \tilde{e}_4^{(t)}]_{\mathfrak{l}^{(t)}} &= -\lambda_3 \tilde{e}_1, & [\tilde{e}_3, \tilde{e}_4^{(t)}]_{\mathfrak{l}^{(t)}} &= 0.
\end{aligned}$$

Hence the Lie algebra $\mathfrak{l}^{(t)}$ is the semi-direct product of the Lie algebra $\mathfrak{so}(2) = \langle \tilde{e}_4^{(t)} \rangle$ and $\mathfrak{g} = \mathfrak{nil}_3 = \langle \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \rangle$. Thus we obtain the coset representation (4.4.26).

Finally we consider the homogeneous Riemannian structure S . From (4.4.13) and (4.4.23), we compute

$$\begin{aligned} S_{e_1}e_1 &= \rho_1 e_2, & S_{e_1}e_2 &= -\rho_1 e_1 - \frac{\lambda_3}{2}e_3, & S_{e_1}e_3 &= \frac{\lambda_3}{2}e_2, \\ S_{e_2}e_1 &= \rho_2 e_2 + \frac{\lambda_3}{2}e_3, & S_{e_2}e_2 &= -\rho_2 e_1, & S_{e_2}e_3 &= -\frac{\lambda_3}{2}e_1, \\ S_{e_3}e_1 &= \frac{\lambda_3}{2}e_2, & S_{e_3}e_2 &= -\frac{\lambda_3}{2}e_1, & S_{e_3}e_3 &= 0, \end{aligned}$$

where ρ_1 and ρ_2 are functions defined by (4.4.22) and (4.4.23). Then from the Proposition 4.3.3 the Ambrose-Singer connection $\tilde{\nabla}$ of the homogeneous Riemannian structure S is given by

$$\begin{aligned} \tilde{\nabla}_{e_1}e_1 &= \rho_1 e_2, & \tilde{\nabla}_{e_1}e_2 &= -\rho_1 e_1, & \tilde{\nabla}_{e_1}e_3 &= 0, \\ \tilde{\nabla}_{e_2}e_1 &= \rho_2 e_2, & \tilde{\nabla}_{e_2}e_2 &= -\rho_2 e_1, & \tilde{\nabla}_{e_2}e_3 &= 0, \\ \tilde{\nabla}_{e_3}e_1 &= \lambda e_2, & \tilde{\nabla}_{e_3}e_2 &= -\lambda e_1, & \tilde{\nabla}_{e_3}e_3 &= 0. \end{aligned}$$

From these equations, we can find that $\tilde{\nabla}$ is flat. In fact, the curvature tensor $\tilde{R}$ of the Ambrose-Singer connection $\tilde{\nabla}$ is calculated by

$$\begin{aligned} \tilde{R}(e_1, e_2)e_1 &= \tilde{\nabla}_{e_1}(\tilde{\nabla}_{e_2}e_1) - \tilde{\nabla}_{e_2}(\tilde{\nabla}_{e_1}e_1) - \tilde{\nabla}_{[e_1, e_2]}e_1 \\ &= \tilde{\nabla}_{e_1}(\rho_2 e_2) - \tilde{\nabla}_{e_2}(\rho_1 e_2) - \lambda_3 \tilde{\nabla}_{e_3}e_1 \\ &= (d\rho_2)(e_1)e_2 + \rho_2 \tilde{\nabla}_{e_1}e_2 - (d\rho_1)(e_2)e_2 - \rho_1 \tilde{\nabla}_{e_2}e_2 - \lambda_3 \tilde{\nabla}_{e_3}e_1 \\ &= -\rho_1^2 e_2 - \rho_1 \rho_2 e_1 - \rho_2^2 e_2 + \rho_1 \rho_2 e_1 - \lambda \lambda_3 e_2 \\ &= -(\rho_1^2 + \rho_2^2 + \lambda \lambda_3) e_2 = 0, \\ \tilde{R}(e_2, e_3)e_3 &= \tilde{R}(e_3, e_1)e_3 = 0, \end{aligned}$$

and we obtain $\tilde{R} = 0$.

Let us determine the Lie group L acting isometrically and transitively on $(G, g) = (\text{SL}(2, \mathbb{R}), g)$ corresponding to the homogeneous Riemannian structure S and its reductive decomposition $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{m}$. Since $\tilde{\nabla}$ is flat, the holonomy algebra $\mathfrak{h} = \{0\}$. The vector space $\mathfrak{m}$ is identified with the tangent space of $G = \text{SL}(2, \mathbb{R})$ at the origin o . Let us choose the identity e as the origin, and then $\mathfrak{m} \cong \mathfrak{g}$. Hence we obtain $\mathfrak{l} \cong \mathfrak{h} \oplus \mathfrak{m} \cong \{0\} \oplus \mathfrak{g} \cong \mathfrak{g}$. Then $\{e_1, e_2, e_3\}$ is also basis of $\mathfrak{l}$, and the Lie bracket $[\cdot, \cdot]_{\mathfrak{l}}$ of $\mathfrak{l}$ is defined by

$$\begin{aligned} [e_1, e_2]_{\mathfrak{l}} &= (-S_{e_1}e_2 + S_{e_2}e_1)_e = \rho_1(e)e_1 + \rho_2(e)e_2 + \lambda_3 e_3, \\ [e_2, e_3]_{\mathfrak{l}} &= (-S_{e_2}e_3 + S_{e_3}e_2)_e = 0, \\ [e_3, e_1]_{\mathfrak{l}} &= (-S_{e_3}e_1 + S_{e_1}e_3)_e = 0. \end{aligned}$$

Let us change the basis $\{e_1, e_2, e_3\}$ to

$$\tilde{e}_1 = \frac{1}{\sqrt{-\lambda \lambda_3}}(\rho_2 e_1 - \rho_1 e_2), \quad \tilde{e}_2 = \frac{1}{\sqrt{-\lambda \lambda_3}}(\rho_1 e_1 + \rho_2 e_2), \quad \tilde{e}_3 = e_3,$$

and we obtain

$$\begin{aligned} [\tilde{e}_1, \tilde{e}_2]_{\mathfrak{l}} &= \sqrt{-\lambda\lambda_3}\tilde{e}_2 + \lambda_3\tilde{e}_3, \\ [\tilde{e}_2, \tilde{e}_3]_{\mathfrak{l}} &= 0, \\ [\tilde{e}_3, \tilde{e}_1]_{\mathfrak{l}} &= 0. \end{aligned}$$

Thus the Lie algebra $\mathfrak{l}$ is isomorphic to $\mathfrak{ga}(1) \oplus \mathbb{R}$, and $L \cong (\mathrm{GA}^+(1) \times \mathbb{R})/\{Id\}$, which is the coset representation (4.4.30). This completes the proof. $\square$

4.4.2 Homogeneous structures on non-unimodular Lie groups

In this section, we classify homogeneous Riemannian structures on 3-dimensional non-unimodular Lie groups. Note that we omit the same discussion as in the section 4.4.1. Let $\tilde{\nabla}$ be an Ambrose-Singer connection and $S = \tilde{\nabla} - \nabla$. As in the section 4.4.1, from $\tilde{\nabla}g = 0$, S is expressed as

$$S_{\flat} = 2\rho \otimes (\theta^1 \wedge \theta^2) + 2\sigma \otimes (\theta^2 \wedge \theta^3) + 2\tau \otimes (\theta^3 \wedge \theta^1), \quad (4.4.31)$$

where ρ, σ and τ are 1-forms defined by (4.4.4). From Theorem 4.3.6, the condition $\tilde{\nabla}R = 0$ is equivalent to

$$\begin{cases} 0 = (r_1 - r_2)(\omega_1^2 + \rho) = \{-2(1 + \eta^2)\xi(\xi - 1)\}(\omega_1^2 + \rho), \\ 0 = (r_2 - r_3)(\omega_2^3 + \sigma) = \{-4(1 + \eta^2)\xi\}(\omega_2^3 + \sigma), \\ 0 = (r_3 - r_1)(\omega_3^1 + \tau) = \{2(1 + \eta^2)\xi(\xi + 1)\}(\omega_3^1 + \tau), \end{cases} \quad (4.4.32)$$

where ω_i^j is a 1-form defined by $\omega_i^j(X) = g(\nabla_X e_i, e_j)$. From the equations (4.4.32), the following results are easily proved.

Theorem 4.4.4 ([13], Theorem 3). *The only Ambrose-Singer connection on a 3-dimensional non-unimodular Lie group G with $\xi \neq 0, 1$ is $\tilde{\nabla}_X Y = 0$ for any $X, Y \in \mathfrak{g}$, and the Ambrose-Singer $\tilde{\nabla}$ coincides with Cartan-Schouten's $(-)$ -connection.*

Thus we consider the case $\xi = 0$ or $\xi = 1$.

If $\xi = 0$, then (G, g) is isomorphic as Lie group and isometric to $\mathbb{H}^3$. Moreover, if $\xi = 1$ and $\eta = 0$, then (G, g) is isomorphic as Lie group and isometric to $\mathbb{H}^2 \times \mathbb{R}$. Thus we suppose that $\xi = 1$ and $\eta \neq 0$, that is, $G \cong \mathrm{GA}^+(1) \times \mathbb{R}$.

Theorem 4.4.5 ([13], Theorem 3). *The homogeneous Riemannian structures S on $(G \cong \mathrm{GA}^+(1) \times \mathbb{R}, g)$ are following three types;*

$$S_{\flat}^{(t)} = 2t\theta^3 \otimes (\theta^1 \wedge \theta^2) - 2\eta\theta^1 \otimes (\theta^2 \wedge \theta^3) - 2\eta\theta^2 \otimes (\theta^3 \wedge \theta^1), \quad (4.4.33)$$

$$S_{\flat}^{(\phi)} = 2(\rho \otimes (\theta^1 \wedge \theta^2) - \eta\theta^1 \otimes (\theta^2 \wedge \theta^3) - \eta\theta^2 \otimes (\theta^3 \wedge \theta^1)), \quad (4.4.34)$$

where t is an arbitrary constant and

$$\rho = 2 \cos \phi \theta^1 + 2 \sin \phi \theta^2 + \eta \theta^3 \quad (4.4.35)$$

for a function ϕ such that

$$d\phi = 2 \cos \phi \theta^1 + 2(\sin \phi - 1)\theta^2. \quad (4.4.36)$$

Epecially, the constant function $\phi = \frac{\pi}{2}$ satisfies (4.4.36), and then the corresponding Ambrose-Singer connection is $\tilde{\nabla}_X Y = 0$ for any $X, Y \in \mathfrak{g}$.

Proof. From Proposition 4.3.5 and the assumption $\xi = 1$, we obtain

$$\begin{aligned} \nabla_{e_1} e_1 &= 0, & \nabla_{e_1} e_2 &= \eta e_3, & \nabla_{e_1} e_3 &= -\eta e_2, \\ \nabla_{e_2} e_1 &= -2e_2 - \eta e_3, & \nabla_{e_2} e_2 &= 2e_1, & \nabla_{e_2} e_3 &= \eta e_1, \\ \nabla_{e_3} e_1 &= -\eta e_2, & \nabla_{e_3} e_2 &= \eta e_1, & \nabla_{e_3} e_3 &= 0. \end{aligned} \quad (4.4.37)$$

and

$$\omega_1^2 = -2\theta^2 - \eta\theta^3, \quad \omega_2^3 = \eta\theta^1, \quad \omega_3^1 = \eta\theta^2. \quad (4.4.38)$$

Then, from (4.4.32) and (4.4.38), it follows that

$$\sigma = -\omega_2^3 = -\eta\theta^1, \quad \tau = -\omega_3^1 = -\eta\theta^2, \quad (4.4.39)$$

and the homogeneous Riemannian structure S has the form

$$S_b = 2(\rho \otimes (\theta^1 \wedge \theta^2) - \eta\theta^1 \otimes (\theta^2 \wedge \theta^3) - \eta\theta^2 \otimes (\theta^3 \wedge \theta^1)). \quad (4.4.40)$$

We consider the condition $\tilde{\nabla} S = 0$. From (4.4.37), (4.4.40) and $S = \tilde{\nabla} - \nabla$, we obtain

$$\begin{aligned} \tilde{\nabla}_{e_1} e_1 &= \rho_1 e_2, & \tilde{\nabla}_{e_1} e_2 &= -\rho_1 e_1, & \tilde{\nabla}_{e_1} e_3 &= 0, \\ \tilde{\nabla}_{e_2} e_1 &= (\rho_2 - 2)e_3, & \tilde{\nabla}_{e_2} e_2 &= -(\rho_2 - 2)e_1, & \tilde{\nabla}_{e_2} e_3 &= 0, \\ \tilde{\nabla}_{e_3} e_1 &= (\rho_3 - \eta)e_2, & \tilde{\nabla}_{e_3} e_2 &= -(\rho_3 - \eta)e_1, & \tilde{\nabla}_{e_3} e_3 &= 0. \end{aligned} \quad (4.4.41)$$

From these equations we also compute

$$\begin{aligned} \tilde{\nabla}_X \theta^1 &= (\rho(X) - 2\theta^2(X) - \eta\theta^3(X))\theta^2, \\ \tilde{\nabla}_X \theta^2 &= -(\rho(X) - 2\theta^2(X) - \eta\theta^3(X))\theta^1, \\ \tilde{\nabla}_X \theta^3 &= 0. \end{aligned}$$

Then the condition $\tilde{\nabla} S = 0$ is equivalent to $\tilde{\nabla} \rho = 0$, that is,

$$\begin{aligned} 0 = \tilde{\nabla}_X \rho &= \{(d\rho_1)(X) - \rho_2(\rho(X) - 2\theta^2(X) - \eta\theta^3(X))\}\theta^1 \\ &\quad + \{(d\rho_2)(X) + \rho_1(\rho(X) - 2\theta^2(X) - \eta\theta^3(X))\}\theta^2 + (d\rho_3)(X)\theta^3. \end{aligned}$$

Thus the parallelism of S with respect to $\tilde{\nabla}$ is equivalent to the differential system;

$$d\rho_1 = \rho_2(\rho - 2\theta^2 - \eta\theta^3), \quad d\rho_2 = -\rho_1(\rho - 2\theta^2 - \eta\theta^3), \quad d\rho_3 = 0. \quad (4.4.42)$$

Hence ρ_3 is a constant, and we put $\rho_3 = t$. The integrability condition of (4.4.42) is

$$\begin{aligned}\rho_1(X(\rho(Y)) - Y(\rho(X)) - (\rho - 2\theta^2 - \eta\theta^3)) &= 0, \\ \rho_2(X(\rho(Y)) - Y(\rho(X)) - (\rho - 2\theta^2 - \eta\theta^3)) &= 0.\end{aligned}$$

Then for any $p \in G$, it follows that $\rho_1(p) = \rho_2(p) = 0$ or

$$(X(\rho(Y)) - Y(\rho(X)) - (\rho - 2\theta^2 - \eta\theta^3))(p) = 0. \quad (4.4.43)$$

By inserting $(X, Y) = (e_1, e_2), (e_1, e_3), (e_2, e_3)$ into the equation (4.4.43), from (4.3.5) and (4.4.42), we can obtain

$$\begin{cases} \rho_1(p)^2 + \rho_2(p)^2 = 4 - 2\eta(t - \eta), \\ \rho_1(p)(t - \eta) = 0, \\ \rho_2(p)(t - \eta) = 0. \end{cases} \quad (4.4.44)$$

From the second and third equations of (4.4.44), if $t \neq \eta$, it follows that $\rho_1 = \rho_2 = 0$, and thus we obtain $S^{(t)}$. If $t = \eta$, for any $p \in G$, it follows that $\rho_1(p) = \rho_2(p) = 0$ or

$$\rho_1(p)^2 + \rho_2(p)^2 = 4.$$

Then from the smoothness of ρ_1 and ρ_2 , it follows that

$$\rho_1 = 2 \cos \phi, \quad \rho_2 = 2 \sin \phi, \quad (4.4.45)$$

for some function ϕ . Inserting these formulas into the system (4.4.42), we obtain the equation (4.4.36). This completes the proof. $\square$

Next we give the coset representation corresponding to homogeneous Riemannian structures $S^{(t)}$ and $S^{(\phi)}$. Here we denote $S^{(0)} = S^{(\phi)}$ for $\phi = \frac{\pi}{2}$.

Theorem 4.4.6 ([13], Theorem 3). *The corresponding coset space representation of $S^{(t)}$ is*

$$\text{GA}^+(1) \times \mathbb{R} \cong L \times \text{SO}(2) / \text{SO}(2) \quad \left(t \neq \eta + \frac{2}{\eta} \right), \quad (4.4.46)$$

$$\cong L / \{Id\} \quad \left(t = \eta + \frac{2}{\eta} \right), \quad (4.4.47)$$

where L is a Lie group acting transitively and isometrically on $\text{GA}^+(1) \times \mathbb{R}$ and its Lie algebra is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, and the corresponding coset space representation of $S^{(\phi)}$ is

$$\text{GA}^+(1) \times \mathbb{R} \cong \text{GA}^+(1) \times \mathbb{R} / \{Id\}. \quad (4.4.48)$$

Moreover, for any function ϕ satisfying the equation (4.4.36), the homogeneous Riemannian structure $S^{(\phi)}$ is isomorphic to $S^{(0)}$.

Proof. First we consider the homogeneous Riemannian structure $S^{(t)}$. From (4.4.33), we compute

$$\begin{aligned} S_{e_1}^{(t)}e_1 &= 0, & S_{e_1}^{(t)}e_2 &= -\eta e_3, & S_{e_1}^{(t)}e_3 &= \eta e_2, \\ S_{e_2}^{(t)}e_1 &= \eta e_3, & S_{e_2}^{(t)}e_2 &= 0, & S_{e_2}^{(t)}e_3 &= -\eta e_1, \\ S_{e_3}^{(t)}e_1 &= t e_2, & S_{e_3}^{(t)}e_2 &= -t e_1, & S_{e_3}^{(t)}e_3 &= 0. \end{aligned}$$

Let $\tilde{\nabla}^{(t)}$ be the Ambrose-Singer connection corresponding to $S^{(t)}$, then fom Proposition 4.3.5 we obtain

$$\begin{aligned} \tilde{\nabla}_{e_1}^{(t)}e_1 &= 0, & \tilde{\nabla}_{e_1}^{(t)}e_2 &= 0, & \tilde{\nabla}_{e_1}^{(t)}e_3 &= 0, \\ \tilde{\nabla}_{e_2}^{(t)}e_1 &= -2e_2, & \tilde{\nabla}_{e_2}^{(t)}e_2 &= 2e_1, & \tilde{\nabla}_{e_2}^{(t)}e_3 &= 0, \\ \tilde{\nabla}_{e_3}^{(t)}e_1 &= (t - \eta)e_2, & \tilde{\nabla}_{e_3}^{(t)}e_2 &= -(t - \eta)e_1, & \tilde{\nabla}_{e_3}^{(t)}e_3 &= 0. \end{aligned}$$

The curvature $\tilde{R}^{(t)}$ of the Ambrose-Singer connection $\tilde{\nabla}^{(t)}$ is given by

$$\begin{aligned} \tilde{R}^{(t)}(e_1, e_2)e_1 &= (4 - 2\eta(t - \eta))e_2, & \tilde{R}^{(t)}(e_1, e_2)e_2 &= -(4 - 2\eta(t - \eta))e_1, & \tilde{R}^{(t)}(e_1, e_2)e_3 &= 0, \\ \tilde{R}^{(t)}(e_2, e_3)e_1 &= 0, & \tilde{R}^{(t)}(e_2, e_3)e_2 &= 0, & \tilde{R}^{(t)}(e_2, e_3)e_3 &= 0, \\ \tilde{R}^{(t)}(e_3, e_1)e_1 &= 0, & \tilde{R}^{(t)}(e_3, e_1)e_2 &= 0, & \tilde{R}^{(t)}(e_3, e_1)e_3 &= 0. \end{aligned}$$

Let us determine the reductive decomposition $\mathfrak{l}^{(t)} = \mathfrak{h}^{(t)} \oplus \mathfrak{m}$ corresponding to the homogeneous Riemannian structure $S^{(t)}$. If $t = \eta + \frac{2}{\eta}$, then $\tilde{R}^{(t)} = \tilde{R}^{(\eta + \frac{2}{\eta})} = 0$ and the holonomy algebra $\mathfrak{h}^{(\eta + \frac{2}{\eta})} = \{0\}$. Then the Lie bracket $[\cdot, \cdot]_{\mathfrak{l}^{(\eta + \frac{2}{\eta})}}$ is defined by

$$\begin{aligned} [e_1, e_2]_{\mathfrak{l}^{(\eta + \frac{2}{\eta})}} &= (-S^{(\eta + \frac{2}{\eta})}_{e_1}e_2 + S^{(\eta + \frac{2}{\eta})}_{e_2}e_1)_e = 2\eta e_3, \\ [e_2, e_3]_{\mathfrak{l}^{(\eta + \frac{2}{\eta})}} &= (-S^{(\eta + \frac{2}{\eta})}_{e_2}e_3 + S^{(\eta + \frac{2}{\eta})}_{e_3}e_2)_e = -\frac{2}{\eta}e_1, \\ [e_3, e_1]_{\mathfrak{l}^{(\eta + \frac{2}{\eta})}} &= (-S^{(\eta + \frac{2}{\eta})}_{e_3}e_1 + S^{(\eta + \frac{2}{\eta})}_{e_1}e_3)_e = -\frac{2}{\eta}e_2. \end{aligned}$$

Thus the signature of the structure constants is $(+, -, -)$, and we obtain $L^{(\eta + \frac{2}{\eta})} \cong \text{SL}(2, \mathbb{R})$ from the Table 4.1 and the coset representation (4.4.47).

Next we consider the case $t \neq \eta + \frac{2}{\eta}$. Let $e_4^{(t)} = \tilde{R}^{(t)}(e_1, e_2)$, then we have $\mathfrak{l}^{(t)} = \mathfrak{h}^{(t)} \oplus \mathfrak{m} =$

$\langle e_1, e_2, e_3 \rangle$ and the Lie bracket $[\cdot, \cdot]_{\mathfrak{l}^{(t)}}$ of $\mathfrak{l}^{(t)}$ is defined by

$$\begin{aligned}
[e_1, e_2]_{\mathfrak{l}^{(t)}} &= -S_{e_1}^{(t)}e_2 + S_{e_2}^{(t)}e_1 - \tilde{R}^{(t)}(e_1, e_2), \\
&= 2\eta e_3 - e_4, \\
[e_2, e_3]_{\mathfrak{l}^{(t)}} &= -S_{e_2}^{(t)}e_3 + S_{e_3}^{(t)}e_2 - \tilde{R}^{(t)}(e_2, e_3), \\
&= (\eta - t)e_1, \\
[e_3, e_1]_{\mathfrak{l}^{(t)}} &= -S_{e_3}^{(t)}e_1 + S_{e_1}^{(t)}e_3 - \tilde{R}^{(t)}(e_3, e_1), \\
&= (\eta - t)e_2, \\
[e_1, e_4^{(t)}]_{\mathfrak{l}^{(t)}} &= -e_4^{(t)}(e_1), \\
&= -\tilde{R}^{(t)}(e_1, e_2)e_1, \\
&= -(4 + 2\eta(\eta - t))e_2, \\
[e_2, e_4^{(t)}]_{\mathfrak{l}^{(t)}} &= -e_4^{(t)}(e_2), \\
&= -\tilde{R}^{(t)}(e_1, e_2)e_2, \\
&= (4 + 2\eta(\eta - t))e_1.
\end{aligned}$$

Let us change the basis $\{e_1, e_2, e_3, e_4^{(t)}\}$ to

$$\begin{aligned}
\tilde{e}_1 &= \frac{1}{2}e_1, \quad \tilde{e}_2 = \frac{1}{2}, \quad \tilde{e}_3 = \frac{1}{4}(2\eta e_3 - e_4), \\
\tilde{e}_4^{(t)} &= \frac{1}{4} \left(2\eta e_3 - \frac{2\eta(\eta - t)}{4 + 2\eta(\eta - 2)}e_4 \right).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
[\tilde{e}_1, \tilde{e}_2]_{\mathfrak{l}^{(t)}} &= \tilde{e}_3, \quad [\tilde{e}_2, \tilde{e}_3]_{\mathfrak{l}^{(t)}} = -\tilde{e}_1, \quad [\tilde{e}_3, \tilde{e}_1]_{\mathfrak{l}^{(t)}} = -\tilde{e}_2, \\
[\tilde{e}_1, \tilde{e}_4^{(t)}]_{\mathfrak{l}^{(t)}} &= [\tilde{e}_2, \tilde{e}_4^{(t)}]_{\mathfrak{l}^{(t)}} = [\tilde{e}_3, \tilde{e}_4^{(t)}]_{\mathfrak{l}^{(t)}} = 0.
\end{aligned}$$

Hence the Lie algebra $\mathfrak{l}^{(t)}$ is the direct sum of the Lie algebra $\mathfrak{so}(2) = \langle \tilde{e}_4^{(t)} \rangle$ and $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) = \langle e_1, e_2, e_3 \rangle$. Thus we obtain the coset representation (4.4.46).

Finally we consider the homogeneous Riemannian structure $S^{(\phi)}$. From (4.4.34) and (4.4.36) we compute

$$\begin{aligned}
S_{e_1}e_1 &= \rho_1 e_2, & S_{e_1}e_2 &= -\rho_1 e_2 - \eta e_3, & S_{e_1}e_3 &= \eta e_2, \\
S_{e_2}e_1 &= \rho_2 e_2 + \eta e_3, & S_{e_2}e_2 &= -\rho_2 e_1, & S_{e_2}e_3 &= -\eta e_1, \\
S_{e_3}e_1 &= \eta e_2, & S_{e_3}e_2 &= -\eta e_1, & S_{e_3}e_3 &= 0,
\end{aligned}$$

where ρ_1 and ρ_2 are functions defined by (4.4.36) and (4.4.45). Then from the Proposition 4.3.5 the Ambrose-singer connection $\tilde{\nabla}^{(\phi)}$ of the homogeneous Riemannian structure $S^{(\phi)}$ is given by

$$\begin{aligned}
\tilde{\nabla}_{e_1}^{(\phi)}e_1 &= \rho_1 e_2, & \tilde{\nabla}_{e_1}^{(\phi)}e_2 &= -\rho_1 e_1, & \tilde{\nabla}_{e_1}^{(\phi)}e_3 &= 0, \\
\tilde{\nabla}_{e_2}^{(\phi)}e_1 &= (\rho_2 - 2)e_2, & \tilde{\nabla}_{e_2}^{(\phi)}e_2 &= -(\rho_2 - 2)e_1, & \tilde{\nabla}_{e_2}^{(\phi)}e_3 &= 0, \\
\tilde{\nabla}_{e_3}^{(\phi)}e_1 &= 0, & \tilde{\nabla}_{e_3}^{(\phi)}e_2 &= 0, & \tilde{\nabla}_{e_3}^{(\phi)}e_3 &= 0.
\end{aligned}$$

From these equations, we can find $\tilde{\nabla}^{(\phi)}$ is flat. In fact, the curvature tensor $\tilde{R}$ of the Ambrose-Singer connection $\tilde{\nabla}^{(\phi)}$ is calculated by

$$\begin{aligned}
\tilde{R}(e_1, e_2)e_1 &= \tilde{\nabla}_{e_1}^{(\phi)}(\tilde{\nabla}_{e_2}^{(\phi)}e_2) - \tilde{\nabla}_{e_2}^{(\phi)}(\tilde{\nabla}_{e_1}^{(\phi)}e_1) - \tilde{\nabla}_{[e_1, e_2]}^{(\phi)}e_1, \\
&= \tilde{\nabla}_{e_1}^{(\phi)}((\rho_2 - 2)e_2) - \tilde{\nabla}_{e_2}^{(\phi)}(\rho_1 e_2) - 2\tilde{\nabla}_{e_2}^{(\phi)}e_1 - 2\eta\tilde{\nabla}_{e_3}^{(\phi)}e_1, \\
&= (d\rho_2)(e_1)e_2 + (\rho_2 - 2)\tilde{\nabla}_{e_1}^{(\phi)}e_2 - (d\rho_1)(e_2)e_2 - \rho_1\tilde{\nabla}_{e_2}^{(\phi)}e_2 - 2(\rho_2 - 2)e_2, \\
&= -(\rho_1^2 + \rho_2^2 - 4)e_2 = 0, \\
\tilde{R}(e_2, e_3)e_2 &= \tilde{R}(e_3, e_1)e_3 = 0,
\end{aligned}$$

and we obtain $\tilde{R} = 0$. Then the holonomy algebra $\mathfrak{h}^{(\phi)} = 0$, and the Lie bracket $[\cdot, \cdot]_{\mathfrak{l}^{(\phi)}}$ is defined by

$$[e_1, e_2]_{\mathfrak{l}^{(\phi)}} = \rho_1 e_2 + \rho_2 e_1 + 2\eta e_3, \quad [e_2, e_3]_{\mathfrak{l}^{(\phi)}} = [e_3, e_1]_{\mathfrak{l}^{(\phi)}} = 0.$$

Let us change the basis $\{e_1, e_2, e_3\}$ to

$$\tilde{e}_1 = \rho_1 e_1 - \rho_2 e_2, \quad \tilde{e}_2 = \rho_2 e_1 + \rho_1 e_2, \quad \tilde{e}_3 = e_3,$$

and we obtain

$$[e_1, e_2]_{\mathfrak{l}^{(\phi)}} = 2\tilde{e}_2 + 4\eta\tilde{e}_3, \quad [e_2, e_3]_{\mathfrak{l}^{(\phi)}} = [e_3, e_1]_{\mathfrak{l}^{(\phi)}} = 0.$$

Thus the Lie algebra $\mathfrak{l}^{(\phi)}$ is isomorphic to $\mathfrak{ga}(1) \oplus \mathbb{R}$ and we obtain the coset representation (4.4.48). This completes the proof. $\square$

4.5 The classifications of homogeneous representations

At the end of this thesis we give the tables of homogeneous space representations of 3-dimensional Riemannian manifolds by using the results in section 4.2 and section 4.3.

	$\dim G$	G/H	Homogeneous structures
$\mathbb{S}^3$	6	$\mathrm{SO}(4)/\mathrm{SO}(3)$	$\lambda dV_{\mathbb{S}^3} \ (\lambda \neq 1, \lambda \geq 0)$
	4	$(\mathrm{SU}(2) \times \mathrm{SO}(2))/\mathrm{SO}(2)$	$dV_{\mathbb{S}^3} + \lambda \theta^1 \otimes (\theta^2 \wedge \theta^3)$
	3	$\mathrm{SU}(2)/\{Id\}$	$dV_{\mathbb{S}^3}$
$\mathbb{H}^3$	6	$\mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2)$	$\lambda dV_{\mathbb{S}^3} \ (\lambda \geq 0)$
	4	$\mathrm{B}^+(2, \mathbb{C}) \cdot \mathrm{U}(1)/\mathrm{U}(1)$	$S_2^\lambda \ (\lambda \geq 0)$
	3	$G^\lambda/\{Id\}$	$S_2^\lambda \ (\lambda \geq 0)$
$\mathbb{R}^3$	6	$\mathrm{SE}(3)/\mathrm{SO}(3)$	$\lambda dV_{\mathbb{R}^3} \ (\lambda \geq 0)$
	4	$(\mathrm{SE}(2) \times \mathbb{R})/\mathrm{SO}(2)$	$S_2^\lambda \ (\lambda \geq 0)$
	3	$G'^\lambda/\{Id\}$	$S_2^\lambda \ (\lambda \geq 0)$
$\mathbb{S}^2 \times \mathbb{R}$	4	$(\mathrm{SO}(3) \times \mathbb{R})/\mathrm{SO}(2)$	$\lambda(dy \otimes dV_{\mathbb{S}^2})(\lambda \geq 0)$
$\mathbb{H}^2 \times \mathbb{R}$	4	$(\mathrm{SL}(2, \mathbb{R}) \times \mathbb{R})/\mathrm{SO}(2)$	$\lambda(\theta^3 \otimes dV_{\mathbb{H}^2})(\lambda \geq 0)$
	3	$(\mathbb{H}^2 \times \mathbb{R})/\{Id\}$	$\theta^1 \otimes dV_{\mathbb{H}^2}$

Table 4.2: Homogeneous structures of 3-dimensional symmetric spaces

	Metric	$\dim G$	G/H	Homogeneous structures
$\mathrm{SU}(2)$	$\lambda_1 = \lambda_2 \neq \lambda_3$	4	$(\mathrm{SU}(2) \times \mathrm{SO}(2))/\mathrm{SO}(2)$	$S^{(t)} \ (t \neq -c_3)$
	$\lambda_1 = \lambda_2 \neq \lambda_3$	3	$\mathrm{SU}(2)/\{Id\}$	$S^{(-c_3)} = \nabla^{(-)} - \nabla$
	$\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$	3	$\mathrm{SU}(2)/\{Id\}$	$S^{(-c_3)} = \nabla^{(-)} - \nabla$
$\mathrm{SL}(2, \mathbb{R})$	$\lambda_1 = \lambda_2$	4	$(\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(2))/\mathrm{SO}(2)$	$S^{(t)} \ (t \neq -c_3)$
	$\lambda_1 = \lambda_2$	3	$\mathrm{SL}(2, \mathbb{R})/\{Id\}$	$S^{(-c_3)} = \nabla^{(-)} - \nabla$
	$\lambda_1 = \lambda_2$	3	$(\mathrm{GA}^+(1) \times \mathbb{R})/\{Id\}$	$S^{(\phi)}$
	$\lambda_1 \neq \lambda_2$	3	$\mathrm{SL}(2, \mathbb{R})/\{Id\}$	$S^{(-c_3)} = \nabla^{(-)} - \nabla$
$\mathrm{E}(2)$	$\lambda_1 \neq \lambda_2$	3	$\mathrm{E}(2)/\{Id\}$	$\nabla^{(-)} - \nabla$
$\mathrm{E}(1, 1)$		3	$\mathrm{E}(1, 1)/\{Id\}$	$\nabla^{(-)} - \nabla$
$\mathrm{Nil}(3)$		4	$(\mathrm{Nil}(3) \ltimes \mathrm{SO}(2))/\mathrm{SO}(2)$	$S^{(t)} \ (t \neq -c_3)$
		3	$\mathrm{Nil}(3)/\{Id\}$	$S^{(-c_3)} = \nabla^{(-)} - \nabla$

Table 4.3: Homogeneous Riemannian structures of 3-dimensional unimodular Lie groups

	Metric	$\dim G$	G/H	Homogeneous structures
$\mathrm{GA}^+(1) \times \mathbb{R}$	$\xi = 1, \eta \neq 0$	4	$(\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(2))/\mathrm{SO}(2)$	$S^{(t)} \ (t \neq \eta + \frac{2}{\eta})$
	$\xi = 1, \eta \neq 0$	3	$(\mathrm{GA}^+(1) \times \mathbb{R})/\{Id\}$	S^ϕ
	$\xi = 1, \eta \neq 0$	3	$\mathrm{SL}(2, \mathbb{R})/\{Id\}$	$S^{(\eta + \frac{2}{\eta})} = \nabla^{(-)} - \nabla$
$G(\xi, \eta)$	$\xi \neq 0, 1$	3	$G(\xi, \eta)/\{Id\}$	$\nabla^{(-)} - \nabla$

Table 4.4: Homogeneous Riemannian structures of 3-dimensional unimodular Lie groups

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