Doctoral Thesis

Picard-Fuchs equations of the generalized Dwork family

（一般化 Dwork 族の Picard-Fuchs 方程式）

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1 Introduction

The Dwork family is the one-parameter family of hypersurfaces of the projective space $\mathbb{P}^{n-1}$ given by the homogeneous equation

$$X_1^n + \cdots + X_n^n - n\lambda X_1 \cdots X_n = 0 \tag{1.1}$$

where λ is a parameter. The name "Dwork family" comes from the fact that Bernard Dwork preferred to use this family in his study on the Frobenius actions of algebraic families over a finite field (cf. [Kat09]).

In his paper [Kat09], Katz introduces a generalization of the Dwork family

$$\pi : \mathbb{X} \longrightarrow \mathbb{A}^1 - \mu_d$$

defined by the homogeneous equation

$$w_1 X_1^d + \cdots + w_n X_n^d - d\lambda X_1^{w_1} \cdots X_n^{w_n} = 0, \tag{1.2}$$

where $w_1, \ldots, w_n$ and d are positive integers such that $\sum_{i=1}^n w_i = d$ and $\gcd(w_1, \ldots, w_n) = 1$. The group $\Gamma_W := \{(\zeta_1, \ldots, \zeta_n) \in \mu_d \mid \zeta_1^{w_1} \cdots \zeta_n^{w_n} = 1\}$ acts on the equation (1.2) in a natural way (see §4.2), and this action induces the eigendecomposition of the primitive part of the ℓ-adic sheaf $R^{n-2}\pi_*\overline{\mathbb{Q}}_\ell$. Then he proves that each component is isomorphic to a hypergeometric sheaf.

Katz also discusses the Picard-Fuchs equations on the de Rham cohomology group. Thanks to the result on the ℓ-adic sheaves, it follows that each eigencomponent of the de Rham cohomology group is isomorphic to a hypergeometric $\mathcal{D}$-module. However, it is more delicate to determine the Picard-Fuchs equations (Remark 4.7). Katz computes the Picard-Fuchs equation of a holomorphic form which is invariant under the action by Γ_W ([Kat09, §8]). This is done by computing the period of integral along a certain homology cycle. However, it seems difficult to extend this argument to other cohomology classes. A general method to compute the Picard-Fuchs equation of a cohomology class is the Griffiths-Dwork method [CK, §5.3]. This usually works well only when the defining equation is provided individually (namely the degree and exponents of the equation are particular numbers), and this seems not useful in our setting. To the best knowledge of the author, the Picard-Fuchs equations of the family (1.2) have not been determined completely.

The aim of this paper is to determine the Picard-Fuchs equations of the family (1.2) for all components. The main result is Theorem 4.11 (or equivalently Theorem 4.14). For the proof, we use the technique of [Gäh, Lemma 2], while we employ Katz's results in [Kat09] in several places.

In §5, we give an application to p-adic cohomology. In his paper [Klo], Kloosterman computes the matrix $A(\lambda)$ describing the Frobenius action on the de Rham cohomology of certain families including the Dwork family without computing Picard-Fuchs equations (the method used in [Klo] is often called the deformation method, we refer the reader to [Ger] for an exposition in the term of the relative rigid cohomology). We give an alternative proof for computing $A(\lambda)$ using Picard-Fuchs equations.

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2 Hypergeometric $\mathcal{D}$-modules and hypergeometric $\overline{\mathbb{Q}}_\ell$-sheaves

2.1 Definition and properties of the hypergeometric $\mathcal{D}$-modules

Throughout this paper, we identify an $\mathcal{O}_S$-quasi coherent $\mathcal{D}_S$-module with global sections of it for a smooth affine variety S over a field of characteristic zero, where we note that a smooth affine variety is known to be D-affine [HTT, Definition 1.4.2, Proposition 1.4.3].

On the multiplication group $\mathbb{G}_{m,\mathbb{C}} = \operatorname{Spec}\mathbb{C}[t, t^{-1}]$ we denote the derivation d/dt by ∂_t and put $D_t := t\partial_t$. As mentioned above, we identify the sheaf $\mathcal{D}_{\mathbb{G}_{m,\mathbb{C}}}$ with $\mathbb{C}[t, t^{-1}, D_t]$.

Definition 2.1. *Let $m \geq 1$ be an integer and let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ be complex numbers. The differential operator*

$$\prod_{i=1}^{n}(D_t + \beta_i - 1) - t\prod_{i=1}^{n}(D_t + \alpha_i) \tag{2.1}$$

on $\mathbb{G}_{m,\mathbb{C}}$ is called a hypergeometric operator (of type (n,n)). α_i, β_i are called parameters and we denote the hypergeometric operator of parameters $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ as

$$Hyp\begin{pmatrix}\alpha_1, & \ldots, & \alpha_n \\ \beta_1, & \ldots, & \beta_n\end{pmatrix}; t\Big) \quad or\ Hyp(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n; t).$$

We call the left $\mathcal{D}$-module $\mathcal{D}_{\mathbb{G}_{m,\mathbb{C}}}/\mathcal{D}_{\mathbb{G}_{m,\mathbb{C}}}Hyp(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n;t)$ the hypergeometric $\mathcal{D}$-module and we denote it by

$$\mathcal{H}\begin{pmatrix}\alpha_1, & \ldots, & \alpha_n\\ \beta_1, & \ldots, & \beta_n\end{pmatrix} \text{ or } \mathcal{H}(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n).$$

We also use the notation by lists. For two lists of complex numbers $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n), \boldsymbol{\beta} = (\beta_1,\ldots,\beta_n)$ we write $\mathcal{H}(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n)$ as $\mathcal{H}(\boldsymbol{\alpha};\boldsymbol{\beta})$.

Remark 2.2. (1) The hypergeometric $\mathcal{D}$-module does not depend on the orderings of the parameters since differential operators of the form $D_t + \alpha$ $(\alpha \in \mathbb{C})$ are commutative with each other.

(2) Our definition of hypergeometric operators and hypergeometric $\mathcal{D}$-modules slightly differs from the definition in [Kat90, Chap. 3].

The hypergeometric $\mathcal{D}$-module $\mathcal{H}(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n)$ of type (n,n) has only singularities at $t = 0, 1, \infty$ and every points are regular. We use the same symbol for the restriction of $\mathcal{H}(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n)$ to $\mathbb{G}_{m,\mathbb{C}} - \{1\}$.

Definition 2.3. *For a hypergeometric differential operator*

$$Hyp(\alpha_1,\ldots,\alpha_n;\beta_1,\ldots,\beta_n;t) = \prod_{i=1}^{n}(D_t+\beta_i-1) - t\prod_{i=1}^{n}(D_t+\alpha_i)$$

whose parameters $(\alpha_1,\cdots,\alpha_r,\cdots,\alpha_n;\beta_1,\cdots,\beta_r,\cdots,\beta_n)$ are ordered so that $\alpha_i \equiv \beta_i \bmod \mathbb{Z}$ for all $i > r$ and $\alpha_i \not\equiv \beta_j \bmod \mathbb{Z}$ for all $i, j \leq r$, we define the cancel operation as

$$\textbf{Cancel } Hyp(\alpha_1,\cdots,\alpha_n;\beta_1,\cdots,\beta_n;t) := Hyp(\alpha_1,\cdots,\alpha_r;\beta_1,\cdots,\beta_r;t).$$

We also define

$$\textbf{Cancel } \mathcal{H}(\alpha_1,\cdots,\alpha_n;\beta_1,\cdots,\beta_n) := \mathcal{H}(\alpha_1,\cdots,\alpha_r;\beta_1,\cdots,\beta_r).$$

We say two lists $(\alpha_1,\ldots,\alpha_n)$ and $(\beta_1,\ldots,\beta_n)$ are *disjoint* if $\alpha_i \not\equiv \beta_j \bmod \mathbb{Z}$ for every i, j.

Theorem 2.4 ([Kat90, Corollary 3.2.1])**.** *The hypergeometric $\mathcal{D}$-module $\mathcal{H}(\boldsymbol{\alpha};\boldsymbol{\beta})$ given by two lists $\boldsymbol{\alpha},\boldsymbol{\beta}$ is irreducible if and only if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are disjoint.*

Corollary 2.5. Cancel $\mathcal{H}(\boldsymbol{\alpha};\boldsymbol{\beta})$ *is irreducible.*

The monodromy actions of hypergeometric differential equations are well-known.

Theorem 2.6 ([BH])**.** *Let $(\alpha_1,\ldots,\alpha_n), (\beta_1,\ldots,\beta_n)$ be disjoint lists of complex numbers.*

(1) *The local exponents of* $\mathcal{H}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$ *are*

$$\begin{aligned} 1-\beta_1, \ldots, 1-\beta_n \quad & (t=0) \\ \alpha_1, \ldots, \alpha_n \quad & (t=\infty) \\ 0, 1, \ldots, n-2, \sum_{i=1}^n \beta_i - \sum_{i=1}^n \alpha_i - 1 \quad & (t=1). \end{aligned}$$

(2) *The local monodromy of* $\mathcal{H}(\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n)$ *at* $t=1$ *is a pseudo-reflection i.e.* 1 *is an eigenvalue with multiplicity* $n-1$ *of the monodromy action at* $t=1$.

The hypergeometric $\mathcal{D}$-modules are "rigid", these can be characterized by local monodromies as follows.

Theorem 2.7 ([Kat90, Theorem 3.5.4])**.** *Let* $\mathcal{F}, \mathcal{G}$ *be* $\mathcal{O}$*-coherent irreducible* $\mathcal{D}$*-modules on* $\mathbb{G}_m - \{1\}$ *of the same rank* $n \geq 1$. *Then* $\mathcal{F}$ *and* $\mathcal{G}$ *are isomorphic if* $\mathcal{F}$ *and* $\mathcal{G}$ *satisfy the following conditions.*

(1) *The local monodromies at* 1 *of both* $\mathcal{F}$ *and* $\mathcal{G}$ *are pseudo-reflections.*

(2) $\mathcal{F}$ *and* $\mathcal{G}$ *have the same characteristic polynomials of local monodromies at* 0 *and* ∞.

For any differential operator $L = \sum_{i=0}^n f_i(t)\partial_t$ $(f_i(t) \in \mathbb{C}[t, t^{-1}])$, we define the *adjoint* of L by $L^* = \sum_{i=0}^n (-\partial_t)^i f_i(t)$.

The dualizing functor

$$\mathbb{D}M^\bullet := R\mathcal{H}om_{\mathcal{D}_X}(M^\bullet, \mathcal{D}_{\mathbb{G}_{m,\mathbb{C}}}) \otimes_{\mathcal{O}_{\mathbb{G}_{m,\mathbb{C}}}} \omega_{\mathbb{G}_{m,\mathbb{C}}}^{-1}[1]$$

where $\omega_{\mathbb{G}_{m,\mathbb{C}}}$ is the right $\mathcal{D}_{\mathbb{G}_{m,\mathbb{C}}}$-module $\Omega^1_{\mathbb{G}_{m,\mathbb{C}}/\mathbb{C}}$, transforms $\mathcal{D}_{\mathbb{G}_{m,\mathbb{C}}}/\mathcal{D}_{\mathbb{G}_{m,\mathbb{C}}}L$ into $\mathcal{D}_{\mathbb{G}_{m,\mathbb{C}}}/\mathcal{D}_{\mathbb{G}_{m,\mathbb{C}}}L^*$.

If we write $(-1-\alpha_1, \cdots, -1-\alpha_n)$ as $-1-\boldsymbol{\alpha}$, we can check that

$$\mathbb{D}\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \cong \mathcal{D}_{\mathbb{G}_{m,\mathbb{C}}}/\mathcal{D}_{\mathbb{G}_{m,\mathbb{C}}}Hyp(\boldsymbol{\alpha}, \boldsymbol{\beta})^* = \mathcal{H}(-1-\boldsymbol{\alpha}; -1-\boldsymbol{\beta})$$

by direct computation.

2.2 Convolution theorem of the hypergeometric $\mathcal{D}$-modules

Definition 2.8. *Let* $p_1, p_2 : \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}}$ *be the projections and let* $\pi : \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}}$ *be the multiplication map.*

We define the exterior product $\boxtimes$ *and multiplicative convolutions* $*$ *and* $*_!$ *as follows.*

$$\begin{aligned} M \boxtimes N &:= p_1^* M \otimes_{\mathcal{O}_{\mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}}} p_2^* N \\ M * N &:= R\pi_*(M \boxtimes N) \\ M *_! N &:= R\pi_!(M \boxtimes N) = \mathbb{D} \circ R\pi_* \circ \mathbb{D}(M \boxtimes N). \end{aligned}$$

For two lists $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n), \boldsymbol{\alpha'} = (\alpha'_1, \cdots, \alpha'_m)$, $(\boldsymbol{\alpha}, \boldsymbol{\alpha'})$ denotes the connected list $(\alpha_1, \cdots, \alpha_n, \alpha'_1, \cdots, \alpha'_m)$.

Theorem 2.9 ([Kat90, Theorem 5.3.2])**.** *Let $\boldsymbol{\alpha}, \boldsymbol{\beta}$ (resp. $\boldsymbol{\alpha}', \boldsymbol{\beta}'$) be lists of complex numbers of length n (resp. of length m) such that $(\boldsymbol{\alpha}, \boldsymbol{\alpha'})$ and $(\boldsymbol{\beta}, \boldsymbol{\beta'})$ are disjoint. Then*

$$\mathcal{H}(\boldsymbol{\alpha}; \boldsymbol{\beta}) * \mathcal{H}(\boldsymbol{\alpha}; \boldsymbol{\beta}) \cong \mathcal{H}((\boldsymbol{\alpha}, \boldsymbol{\alpha'}); (\boldsymbol{\beta}, \boldsymbol{\beta'})).$$

Proposition 2.10. *Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\alpha}', \boldsymbol{\beta}'$ be lists as in the previous theorem. Then*

$$\mathcal{H}(\boldsymbol{\alpha}; \boldsymbol{\beta}) * \mathcal{H}(\boldsymbol{\alpha}'; \boldsymbol{\beta}') \cong \mathcal{H}(\boldsymbol{\alpha}; \boldsymbol{\beta}) *_! \mathcal{H}(\boldsymbol{\alpha}'; \boldsymbol{\beta}').$$

Proof. From the result on constructible sheaves [HTT, Proposition 4.5.9. (iii)] and the Riemann-Hilbert correspondence, we have the isomorphism

$$\mathbb{D}(M \boxtimes N) \cong \mathbb{D}M \boxtimes \mathbb{D}N$$

for regular holonomic $\mathcal{D}$-modules M and N.

Therefore

$$\begin{aligned}
\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}) *_! \mathcal{H}(\boldsymbol{\alpha}', \boldsymbol{\beta}') &= \mathbb{D} \circ R\pi_* \circ \mathbb{D}(\mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \boxtimes \mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta})) \\
&\cong \mathbb{D}(\mathcal{H}(-1 - \boldsymbol{\alpha}; -1 - \boldsymbol{\beta}) * \mathcal{H}(-1 - \boldsymbol{\alpha}'; -1 - \boldsymbol{\beta}')) \\
&\cong \mathbb{D}\mathcal{H}((-1 - \boldsymbol{\alpha}, -1 - \boldsymbol{\alpha}'); (-1 - \boldsymbol{\beta}, -1 - \boldsymbol{\beta}')) \\
&\cong \mathcal{H}((\boldsymbol{\alpha}, \boldsymbol{\alpha}'); (\boldsymbol{\beta}, \boldsymbol{\beta}')) \\
&\cong \mathcal{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}) * \mathcal{H}(\boldsymbol{\alpha}', \boldsymbol{\beta}').
\end{aligned}$$

□

2.3 Hypergeometric $\overline{\mathbb{Q}}_\ell$-sheaves

We recall the definition of the hypergeometric sheaves (c.f. [Kat90, 8.17], [Kat09, §4]).

Fix an integer $N \geq 1$ and a prime number ℓ . Put

- $\Phi_N(X) :=$ the N-th cyclotomic polynomial,
- $R_N := \mathbb{Z}[1/N\ell][X]/(\Phi_N(X))$,
- $\mu_N(R_N) :=$ the group of N-th roots of unity in R_N.

Let $\chi : \mu_N(R_N) \to \overline{\mathbb{Q}}_\ell^\times$ be a character. The N-th power endomorphism on $\mathbb{G}_{m,R_N}$ induces the group homomorphism $\rho_N : \pi_1(\mathbb{G}_{m,R_N}) \to \mu_N(R_N)$. The rank one lisse sheaf $\mathcal{L}_\chi$ corresponding to the continuous character $\bar{\chi} \circ \rho_N$: $\pi_1(\mathbb{G}_{m,R_N}) \to \overline{\mathbb{Q}}_\ell^\times$ is called the Kummer sheaf.

For two $\overline{\mathbb{Q}}_\ell$-characters χ, ρ of $\mu_N(R_N)$ we define the lisse sheaf $\mathcal{H}(\chi;\rho)$ on $\mathbb{G}_{m,R_N} - \{1\}$ by

$$\mathcal{H}(\chi;\rho) := \mathcal{L}_\chi \otimes [t \mapsto 1-t]^* \mathcal{L}_{(\rho/\chi)}$$

where $[t \mapsto 1-t]^*$ means the pullback by the morphism

$$\begin{aligned} \mathbb{G}_{m,R_N} - \{1\} &\to \mathbb{G}_{m,R_N} \\ t &\mapsto 1-t. \end{aligned}$$

We regard $\mathcal{H}(\chi;\rho)$ as a $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathbb{G}_{m,R_N}$ by extending by zero.

We define the multiplicative convolution $*_!$ of constructible sheaves similarly to Definition 2.9.

Let $\boldsymbol{\chi} = (\chi_1, \ldots, \chi_n)$ and $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n)$ be (non-empty) lists of $\overline{\mathbb{Q}}_\ell^\times$-valued characters of $\mu_N(R_N)$. We say $\boldsymbol{\chi}$ and $\boldsymbol{\rho}$ are disjoint if $\chi_i \neq \rho_j$ for any i, j.

When $\boldsymbol{\chi}$ and $\boldsymbol{\rho}$ are disjoint, the object

$$\mathcal{H}(\chi_1, \rho_1)[1] *_! \cdots *_! \mathcal{H}(\chi_n, \rho_n)[1]$$

of $D_c^b(\mathbb{G}_{m,R_N}, \overline{\mathbb{Q}}_\ell)$ is of the form $\mathcal{F}[1]$ for some sheaf $\mathcal{F}$ on $\mathbb{G}_{m,R_N}$ [Kat90, 8.17.11]. We write this sheaf $\mathcal{F}$ as $\mathcal{H}(\chi_1, \ldots, \chi_n; \rho_1, \ldots, \rho_n)$ or $\mathcal{H}(\boldsymbol{\chi}; \boldsymbol{\rho})$.

Remark 2.11. $\mathcal{H}(\boldsymbol{\chi}; \boldsymbol{\rho})$ is depend on the orderings of characters opposite to that the canonical twist $\mathcal{H}^{can}(\boldsymbol{\chi}; \boldsymbol{\rho})$ which will be defined in §4.3, is not depend on the orderings of characters.

2.4 Comparison between hypergeometric $\mathcal{D}$-modules and hypergeometric $\overline{\mathbb{Q}}_\ell$-sheaves

Fix embeddings $R_N \hookrightarrow \overline{\mathbb{Q}}_\ell \overset{\iota}{\hookrightarrow} \mathbb{C}$.

For a scheme X of finite type over $\mathbb{C}$, we denote X^{an} the analytic space consist of rational points $X(\mathbb{C})$ and the usual topology, and let X_{cl} be the site given by local isomorphisms $U \to X^{an}$. Let us denote by the same symbol X^{an} the site of open sets of the analytic space X^{an}. Then the natural morphism $X^{an} \to X_{cl}$ gives an equivalence of topoi [SGA4, Exposé XI, 4].

For the étale site X_{et} there exists the natural morphism

$$\epsilon : X_{cl} \to X_{et}$$

and this induces the analytification functor

$$\epsilon^* : D_c^b(X_{et}, \overline{\mathbb{Q}}_\ell) \to D_c^b(X^{an}, \overline{\mathbb{Q}}_\ell)$$

(c.f. [BBD, §6]).

Assume that X is connected. Let $\mathcal{F}$ be a lisse sheaf corresponding to a continuous representation $\rho_{\mathcal{F}} : \pi_1(X) \to GL_n(\overline{\mathbb{Q}}_\ell)$. Then the analytification $\epsilon^*\mathcal{F}$ is the local system corresponding to the representation of the topological fundamental group

$$\pi_1^{top}(X^{an}) \to \widehat{\pi_1^{top}(X^{an})} \xrightarrow{\sim} \pi_1(X) \xrightarrow{\rho_{\mathcal{F}}} GL_n(\overline{\mathbb{Q}}_\ell).$$

In particular the analytification of the Kummer sheaf $\mathcal{L}_\chi$ on $\mathbb{G}_{m,\mathbb{C}}$ associated with a character $\chi : \mu_N(\mathbb{C}) \to \overline{\mathbb{Q}}_\ell^\times$ is the local system of the representation

$$\begin{aligned} \pi_1^{top}(\mathbb{G}_{m,\mathbb{C}}^{an}) = \mathbb{Z} &\to \overline{\mathbb{Q}}_\ell^\times \\ 1 &\mapsto \overline{\chi}(1). \end{aligned}$$

By the definition of $\mathcal{H}(\chi;\rho)$ and Theorem 2.6, Theorem 2.7 we have the following.

Proposition 2.12. *Let $p : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{m,R_N}$ be the projection along the embedding $R_N \hookrightarrow \mathbb{C}$ we have fixed. For two distinct $\overline{\mathbb{Q}}_\ell$-characters χ, ρ of $\mu(R_N)$, choose a rational number α (resp. β) such that*

$$\iota(\bar{\rho}(1)) = \exp(-2\pi\sqrt{-1}\alpha) \quad (\text{resp. } \iota(\bar{\chi}(1)) = \exp(2\pi\sqrt{-1}(1-\beta))).$$

Then $\epsilon^*p^*\mathcal{H}(\chi;\rho)\otimes_\iota\mathbb{C}$ is isomorphic to the local system of the hypergeometric $\mathcal{D}$-module $\mathcal{H}(\alpha;\beta)$.

Theorem 2.13. *Let $\boldsymbol{\chi} = (\chi_1, \dots, \chi_n)$ and $\boldsymbol{\rho} = (\rho_1, \dots, \rho_n)$ be disjoint lists of characters of $\mu_N(R_N)$. Choose lists of rational numbers $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$ as in the previous proposition.*

$\epsilon^*p^*\mathcal{H}(\boldsymbol{\chi};\boldsymbol{\rho}) \otimes_\iota \mathbb{C}$ is isomorphic to the local system of the hypergeometric $\mathcal{D}$-module $\mathcal{H}(\boldsymbol{\alpha};\boldsymbol{\beta})$.

Proof. Since p^* and ϵ^* are compatible with the convolution $*_!$ [SGA5, Exposé VI, (2.2.3)], [BBD, 6.1.2. (C')], if we write the local system associated with $\mathcal{H}(\alpha_i;\beta_i)$ as $H(\alpha_i;\beta_i)$,

$$\begin{aligned} \epsilon^*p^*\mathcal{H}(\boldsymbol{\chi};\boldsymbol{\rho})[1] \otimes \mathbb{C} &\cong \epsilon^*p^*\mathcal{H}(\chi_1,\rho_1)[1] *_! \cdots *_! \epsilon^*p^*\mathcal{H}(\chi_n,\rho_n)[1] \otimes \mathbb{C} \\ &\cong H(\alpha_1,\beta_1)[1] *_! \cdots *_! H(\alpha_n,\beta_n)[1]. \end{aligned}$$

$\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are disjoint because $\boldsymbol{\chi}$ and $\boldsymbol{\rho}$ are disjoint. Hence we conclude the theorem from the Riemann-Hilbert correspondence and Theorem 2.9, Proposition 2.10. $\square$

3 Jacobian rings and Picard-Fuchs equations of hypersurfaces

In this section K denotes a field of characteristic zero.

3.1 Jacobian rings of hypersurfaces

Let Y be a nonsingular hypersurface of the projective space $\mathbb{P}_K^{n-1}$ given by a homogeneous polynomial $Q \in S := K[X_1, \ldots, X_n]$ of degree d. The graded ring

$$R_Q := S/J_Q, \quad J_Q := \left(\frac{\partial Q}{\partial X_1}, \ldots, \frac{\partial Q}{\partial X_n}\right)$$

is called the Jacobian ring. We will write the homogeneous part of degree N of a homogeneous ring R as R^N.

The natural morphism of complexes

$$\Omega^{n-1}_{\mathbb{P}^{n-1}}((n-p)Y)[-(n-1)] \to j_*\Omega^\bullet_{\mathbb{P}^{n-1}-Y}$$

induces the morphism

$$S^{pd-n} \cong H^0(\mathbb{P}^{n-1}, \Omega^{n-1}_{\mathbb{P}^{n-1}}((n-p)Y)) \longrightarrow H^{n-1}_{\mathrm{dR}}(\mathbb{P}^{n-1} - Y). \tag{3.1}$$

Theorem 3.1 ([Gri]). *Put $\Omega = \sum_{i=1}^n (-1)^{i+1} X_i dX_1 \wedge \cdots \wedge \widehat{dX_i} \wedge \cdots \wedge dX_n$.*

The map (3.1) induces the isomorphism

$$R_Q^{pd-n} \xrightarrow{\sim} \mathrm{Gr}_F^{n-p} H^{n-1}_{\mathrm{dR}}(\mathbb{P}^{n-1} - Y)$$
$$A \longmapsto \frac{A}{Q^p}\Omega$$

where F denotes the Hodge filtration on $H^{n-1}_{\mathrm{dR}}(\mathbb{P}^{n-1} - Y)$.

Furthermore, we have the relation

$$\frac{pA\frac{\partial Q}{\partial X_i}}{Q^{p+1}}\Omega = \frac{\frac{\partial A}{\partial X_i}}{Q^p}\Omega \tag{3.2}$$

as cohomology classes.

Theorem 3.2. *The residue map induces an isomorphism onto the primitive part*

$$\mathrm{Res} : \mathrm{Gr}_F^{n-p} H^{n-1}(\mathbb{P}^{n-1} - Y) \xrightarrow{\cong} H^{n-1-p,p-1}(Y)_{\mathrm{prim}}.$$

3.2 Picard-Fuchs equations of a family of projective hypersurfaces

Let S be a smooth connected affine variety over K and let $f : X \to S$ be a smooth K-morphism. Then $R^i f_* \Omega^\bullet_{X/S}$ is endowed a $\mathcal{D}_S$-module structure by the Gauss-Manin connection [KO]

$$\nabla^{\mathrm{GM}} : R^i f_* \Omega^\bullet_{X/S} \to R^i f_* \Omega^\bullet_{X/S} \otimes \Omega^1_{S/K}.$$

For $\omega \in H^i_{\mathrm{dR}}(X/S)$, a differential operator $P \in \mathcal{D}_S$ which satisfies

$$P\omega = 0$$

is called a Picard-Fuchs equation of ω.

Theorem 3.3. *Let $d \geq n \geq 3$ be integers.*

For $\mathbf{w}_1 = (w_{1j}), \dots, \mathbf{w}_m = (w_{mj}) \in \{(w_j)_{j=1,\dots,n} \in \mathbb{Z}^n_{\geq 0} \mid \sum_j w_j = d\}$, put

$$Q_\lambda := \sum_{i=1}^{m} \lambda_i X_1^{w_{i1}} \cdots X_n^{w_{in}} \in K[\lambda_1, \dots, \lambda_m][X_1, \dots, X_n].$$

Assume that the family of hypersurfaces of $\mathbb{P}^{n-1} = \operatorname{Proj} K[X_1, \dots, X_n]$ defined by Q_λ is smooth on an affine open subset $U \subset \mathbb{A}^m = \operatorname{Spec} K[\lambda_1, \dots, \lambda_m]$, we write this smooth family as X/U.

Let $v_1, \dots, v_n$ be positive integers such that $\sum_{i=1}^n v_i$ is divided by d and $\deg := (\sum_{i=1}^n v_i)/d \leq n$.

Then $\omega = \frac{X_1^{v_1-1} \cdots X_n^{v_n-1}}{Q_\lambda^{\deg}} \Omega \in H^{n-1}_{\mathrm{dR}}((\mathbb{P}^{n-1} - X)/U)$ satisfies the GKZ system given by the matrix

$$A = ({}^t\mathbf{w}_1, \dots, {}^t\mathbf{w}_m) = \begin{pmatrix} w_{11} & \cdots & w_{m1} \\ \vdots & \ddots & \vdots \\ w_{1n} & \cdots & w_{mn} \end{pmatrix}$$

and the parameter vector $\boldsymbol{\beta} = {}^t(v_1, \dots, v_n)$, that is

$$\begin{cases} ((\sum_{i=1}^m w_{ij} D_i) - v_j)\omega = 0 & (j = 1, \dots, n, \partial_i := \partial/\partial\lambda_i, D_i := \lambda_i \partial_i) \\ (\prod_{l_i>0} \partial_i^{l_i} - \prod_{l_i<0} \partial_i^{-l_i})\omega = 0 & ((l_i) \in \Lambda := \{(l_i) \in \mathbb{Z}^m \mid \sum_i l_i w_{ij} = 0, \forall j\}). \end{cases}$$

Proof. If we put $l^+ = \sum_{l_i>0} l_i$ and $l^- = -\sum_{l_i<0} l_i$ for an element $(l_i) \in \Lambda$, then $l^+ = l^-$ since $d \sum_{i=1}^m l_i = \sum_j \sum_i w_{ij} l_i = 0$.

Thus

$$\begin{aligned}
&\left(\prod_{l_i>0} \partial_i^{l_i} \right) \omega \\
&= (-1)^{l^+} \left(\prod_{k=0}^{l^+-1} (\deg + k) \right) \frac{X_1^{(\sum_{l_i>0} l_i w_{i1}) + v_1 - 1} \cdots X_n^{(\sum_{l_i>0} l_i w_{in}) + v_n - 1}}{Q_\lambda^{\deg + l^+}} \Omega \\
&= (-1)^{l^-} \left(\prod_{k=0}^{l^--1} (\deg + k) \right) \frac{X_1^{(-\sum_{l_i<0} l_i w_{i1}) + v_1 - 1} \cdots X_n^{(-\sum_{l_i<0} l_i w_{in}) + v_n - 1}}{Q_\lambda^{\deg + l^-}} \Omega \\
&= \left(\prod_{l_i<0} \partial_i^{-l_i} \right) \omega.
\end{aligned}$$

By the relation (3.2)

$$\begin{aligned}\left(\sum_{i=1}^{m} w_{ij} D_i\right)\omega &= -\deg \sum_{i=1}^{m} \frac{w_{ij}\lambda_i X_1^{w_{i1}+v_i-1}\cdots X_n^{w_{in}+v_n-1}}{Q_\lambda^{\deg+1}}\Omega \\ &= -\deg X_j \frac{\partial Q_\lambda}{\partial X_j}\frac{X_1^{v_1-1}\cdots X_n^{v_n-1}}{Q_\lambda^{\deg+1}}\Omega \\ &= -v_j \frac{X_1^{v_1-1}\cdots X_n^{v_n-1}}{Q_\lambda^{\deg}}\Omega \\ &= -v_j\omega. \end{aligned}$$

□

4 Picard-Fuchs equations of the generalized Dwork family

4.1 Definition of the generalized Dwork family

Fix integers $d \geq n \geq 3$ and a n-tuple of positive integers $W = (w_1, \ldots, w_n)$ such that

$$\sum_{i=1}^{n} w_i = d, \ \gcd(w_1, \ldots, w_n) = 1. \tag{4.1}$$

Put $d_W = lcm(w_1, \ldots, w_n)d$. Let $\Phi_{d_W}(T)$ be the d_W-th cyclotomic polynomial (the minimal polynomial of primitive d_W-th root of unity over $\mathbb{Q}$), and put

$$R_0 := \mathbb{Z}[1/d_W][T]/(\Phi_{d_W}(T)).$$

In the notation of §2.3, $R_0[1/\ell] = R_{d_W}$ for a prime number ℓ.

Let $\pi : \mathbb{X} \to \mathbb{A}^1_{R_0} := \operatorname{Spec} R_0[\lambda]$ be the one parameter family defined by the homogeneous equation

$$Q_\lambda := w_1 X_1^d + \cdots + w_n X_n^d - d\lambda X_1^{w_1} \cdots X_n^{w_n} = 0. \tag{4.2}$$

We call π the generalized Dwork family.

Lemma 4.1 ([Kat09, Lemma 2.1.]). *The morphism π is smooth on $U := \operatorname{Spec} R_0[\lambda, (1-\lambda^d)^{-1}]$.*

Fix a prime number ℓ and an embedding R_0 into $\overline{\mathbb{Q}}_\ell$. We define the primitive part $\mathrm{Prim}^{n-2}_{\overline{\mathbb{Q}}_\ell}$ on $U[1/\ell]$ as follows. When $n-2$ is odd, set $\mathrm{Prim}^{n-2}_{\overline{\mathbb{Q}}_\ell} = R^{n-2}\pi_*\overline{\mathbb{Q}}_\ell|_{U[1/\ell]}$. When $n-2$ is even, we set $\mathrm{Prim}^{n-2}_{\overline{\mathbb{Q}}_\ell}$ to be the subsheaf of $R^{n-2}\pi_*\overline{\mathbb{Q}}_\ell|_{U[1/\ell]}$ which vanishes by the cup product with $(n-2)/2$-fold product of the hyperplane class. We have a direct decomposition

$$R^{n-2}\pi_*\overline{\mathbb{Q}}_\ell|_{U[1/\ell]} = \mathrm{Prim}^{n-2}_{\overline{\mathbb{Q}}_\ell} \oplus \overline{\mathbb{Q}}_\ell(-(n-2)/2).$$

4.2 Group action and eigendecomposition

For a commutative ring A, let $\mu_d(A)$ denote the group of d-th root of unity in A. Put $\Gamma = (\mu_d(R_0))^n$ and

$$\Gamma_W := \{(\zeta_1, \ldots, \zeta_n) \in \Gamma \mid \zeta_1^{w_1} \cdots \zeta_n^{w_n} = 1\}.$$

Let Δ be the diagonal set of Γ_W. Then the finite abelian group Γ_W/Δ acts on the generalized Dwork family as

$$(X_1, \ldots, X_n, \lambda) \mapsto (\zeta_1 X_1, \ldots, \zeta_n X_n, \lambda).$$

The character group $D\Gamma = \mathrm{Hom}_{\mathrm{Group}}(\Gamma, R_0^\times)$ is identified with the group $(\mathbb{Z}/d\mathbb{Z})^n$ via the pairing

$$\begin{aligned} \Gamma \times (\mathbb{Z}/d\mathbb{Z})^n &\longrightarrow R_0^\times \\ (\zeta_1, \ldots, \zeta_n) \times (v_1, \ldots, v_n) &\mapsto \zeta_1^{v_1} \cdots \zeta_n^{v_n}. \end{aligned}$$

The character group $D(\Gamma_W/\Delta)$ of Γ_W/Δ is identified with the quotient of

$$(\mathbb{Z}/d\mathbb{Z})_0^n := \{V = (v_1, \ldots, v_n) \in (\mathbb{Z}/d\mathbb{Z})^n \mid \sum_{i=1}^n v_i = 0\}$$

by the subgroup $\langle W \rangle$ of $(\mathbb{Z}/d\mathbb{Z})_0^n$ generated by W, so that the pairing

$$\Gamma_W/\Delta \times (\mathbb{Z}/d\mathbb{Z})_0^n/\langle W \rangle \longrightarrow R_0^\times$$

is perfect. For a R_0-module M on which a finite abelian group $G = \Gamma_W/\Delta$ acts, let

$$M_\chi = \{m \in M \mid \forall g \in G, g \cdot m = \chi(g)m\},$$

so that we have the decomposition

$$M = \bigoplus_{\chi \in DG} M_\chi.$$

Let $\mathrm{Prim}^{n-2}(V \bmod W)_{\overline{\mathbb{Q}}_\ell}$ denote the eigensheaf of the primitive part $\mathrm{Prim}^{n-2}_{\overline{\mathbb{Q}}_\ell}$ for $V \in (\mathbb{Z}/d\mathbb{Z})_0^n/\langle W \rangle$. We have the decomposition

$$\mathrm{Prim}^{n-2}_{\overline{\mathbb{Q}}_\ell} = \bigoplus_{V \in (\mathbb{Z}/d\mathbb{Z})_0^n/\langle W \rangle} \mathrm{Prim}^{n-2}(V \bmod W)_{\overline{\mathbb{Q}}_\ell}.$$

4.3 Review of Katz's results

Here we review Katz's results on the generalized Dwork family in [Kat09], which we will use later.

We say that $V = (v_1, \dots, v_n) \in (\mathbb{Z}/d\mathbb{Z})_0^n$ is totally nonzero if and only if $v_i \neq 0 \pmod d$ for all i. For each i, let $\tilde{v}_i \in \{0, 1, \dots, d-1\}$ be the unique integer such that $\tilde{v}_i \equiv v_i \bmod d$. The degree of V is defined by

$$\deg V := \frac{1}{d} \sum_{i=1}^{n} \tilde{v}_i.$$

Lemma 4.2 ([Kat09, Lemma 3.1.]).

$$\operatorname{rank} \operatorname{Prim}^{n-2}(V \bmod W)_{\overline{\mathbb{Q}}_\ell} = \#\{r \in \mathbb{Z}/d\mathbb{Z} \mid V + rW \textit{ is totally nonzero}\}.$$

We consider the generalized Dwork family over $U - \{0\} = \mathbb{G}_m - \mu_d$, and write it by $\pi : \mathbb{X} \to \mathbb{G}_{m,R_0} - \mu_d$ for simplicity of notation. Let

$$[d] : \mathbb{G}_m - \mu_d \longrightarrow \mathbb{G}_m - \{1\} = \operatorname{Spec} R_0[t, (t(1-t))^{-1}]$$

be the $-d$-th power map, namely, the map corresponding to the ring morphism

$$\begin{aligned} R_0[t, (t(1-t))^{-1}] &\to R_0[\lambda, (\lambda(1-\lambda^d))^{-1}] \\ t &\mapsto \lambda^{-d}. \end{aligned}$$

Then there exists a descent family $\pi_{\mathrm{desc}} : \mathbb{Y} \to \mathbb{G}_m - \{1\}$ with respect to $[d]$,

Indeed, take integers $b = (b_1, \dots, b_n)$ such that $\sum_{i=1}^n b_i w_i = 1$, and we take new variables $Y_i := \lambda^{b_i} X_i$. Then the equation (4.2) becomes

$$Q_t := w_1 t^{-b_1} Y_1^d + \cdots + w_n t^{-b_n} Y_n^d - d Y_1^{w_1} \cdots Y_n^{w_n} = 0. \tag{4.3}$$

This gives rise to the descent family.

The group Γ_W/Δ also acts on the descent family, so that we have the eigensheaves $\operatorname{Prim}_{\mathrm{desc}}^{n-2}(V \bmod W)_{\overline{\mathbb{Q}}_\ell}$ of the primitive part of $R^{n-2}\pi_{\mathrm{desc},*}\overline{\mathbb{Q}}_\ell$.

Given an element $V = (v_1, \dots, v_n) \in (\mathbb{Z}/d\mathbb{Z})_0^n$, we define the (unordered) list of characters $\operatorname{List}(V, W) = \{\chi_1, \dots, \chi_d\}$ by the list of characters which is written as

$$\begin{aligned} \chi : \mu_{d_W}(R_0[1/\ell]) &\to \overline{\mathbb{Q}}_\ell^\times \\ \zeta &\mapsto \zeta^{(v_i d_W / w_i d) j} \end{aligned}$$

by some $1 \leq j \leq w_i$. We also put

$$\operatorname{List}(all, d) := \{\text{all characters of } \mu_{d_W}(R_0[1/\ell]) \text{ of order dividing } d\}.$$

We define the cancel operation of lists of characters as in Definition 2.3, i.e. for a pair of lists $\boldsymbol{\chi}, \boldsymbol{\rho}$ of characters, $\mathbf{Cancel}(\boldsymbol{\chi}; \boldsymbol{\rho})$ is the pair of lists obtained by removing all identical pair of characters.

Lemma 4.3 ([Kat09, Lemma 5.1])**.** *If* $\mathrm{Prim}^{n-2}_{\mathrm{desc}}(V \bmod W)_{\overline{\mathbb{Q}}_\ell}$ *is nonzero, then* $\mathrm{List}(-V, W)$ *and* $\mathrm{List}(all, d)$ *are not identical.*

For a maximal ideal $\mathcal{P}$ of $R_0[1/\ell]$ the residue field $k_{\mathcal{P}}$ is a finite field which contains d_W distinct d_W-th roots of unity because $k_{\mathcal{P}}$ is a $R_0[1/\ell]$-algebra. Given a character χ of $\mu_{d_W}(R_0[1/\ell])$ we use same symbol χ for the multiplicative character of the residue field $k_{\mathcal{P}}$ obtained by the composition with the surjective group homomorphism

$$\begin{aligned} k_{\mathcal{P}}^\times &\to \mu_{d_W}(k_{\mathcal{P}}) \xrightarrow{\sim} \mu_{d_W}(R_0[1/\ell]) \\ a &\mapsto a^{\# k_{\mathcal{P}}/d_W}. \end{aligned}$$

For two characters χ, ρ of $\mu_{d_W}(R_0[1/\ell])$ let us denote by $\Lambda_{\chi,\rho/\chi}$ the continuous character of $\pi_1(R_0[1/\ell])$ associated with the unramified character given by

$$\mathrm{Frob}_{\mathcal{P}} \mapsto -J(k_{\mathcal{P}}; \chi, \rho/\chi) := -\sum_{x \in k_{\mathcal{P}}^\times} \chi(x)(\rho/\chi)(1-x)$$

for each Frobenius element $\mathrm{Frob}_{\mathcal{P}}$.

The canonical field twist of the hypergeometric sheaf $\mathcal{H}(\chi;\rho)$ of two characters χ, ρ of $\mu_{d_W}(R_0[1/\ell])$ is defined as

$$\mathcal{H}^{can}(\chi;\rho) := \mathcal{H}(\chi;\rho) \otimes \Lambda_{\chi,\rho/\chi}.$$

Given disjoint lists of characters $\boldsymbol{\chi} = (\chi_1, \ldots, \chi_m), \boldsymbol{\rho} = (\rho_1, \ldots, \rho_m)$,

$$\mathcal{H}^{can}(\chi_1;\rho_1)[1] *_! \cdots *_! \mathcal{H}^{can}(\chi_m;\rho_m)[1]$$

is of the form $\mathcal{F}[1]$ for some sheaf $\mathcal{F}$ on $\mathbb{G}_{m,R_0[1/\ell]}$ and is not depend on the ordering of lists up to isomorphisms (c.f. [Kat09, §4]). We write this sheaf $\mathcal{F}$ as $\mathcal{H}^{can}(\boldsymbol{\chi};\boldsymbol{\rho})$.

From the above, we can make the following definition.

Definition 4.4. *Given an element* $V = (v_1, \ldots, v_n)$ *of* $(\mathbb{Z}/d\mathbb{Z})^n_0$ *such that* $\mathrm{Prim}^{n-2}_{\mathrm{desc}}(V \bmod W)_{\overline{\mathbb{Q}}_\ell}$ *is nonzero, we define*

$$\mathcal{H}_{V,W} := \mathcal{H}^{can}(\mathbf{Cancel}(\mathrm{List}(all, d); \mathrm{List}(-V, W))).$$

Lemma 4.5 ([Kat09, Lemma 6.2, Lemma 6.3.])**.** *Suppose that* $\mathrm{Prim}^{n-2}_{\mathrm{desc}}(V \bmod W)_{\overline{\mathbb{Q}}_\ell}$ *is nonzero. Then there exists a choice of* V *in the coset* $V \bmod W$ *and a continuous character*

$$\Lambda_{V,W} : \pi_1(\mathrm{Spec}\, R_0[1/\ell]) \to \overline{\mathbb{Q}}_\ell^\times$$

such that

$$\mathrm{Prim}^{n-2}_{\mathrm{desc}}(V \bmod W)_{\overline{\mathbb{Q}}_\ell} \cong \mathcal{H}_{V,W} \otimes \Lambda_{V,W}.$$

Fix embeddings $R_0[1/\ell] \hookrightarrow \overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$. Since $\mathcal{H}_{V,W}$ is constructed using the cancel operation, the analytification of $\mathcal{H}_{V,W}$ is the local system corresponding to an irreducible hypergeometric $\mathcal{D}$-module(Corollary 2.5, Theorem 2.13).

Lemma 4.6. *Let* $\mathrm{Prim}^{n-2}_{\mathrm{desc}}(V \bmod W)_{\overline{\mathbb{Q}}}$ *denote the eigencomponent of the primitive part of the local system* $R^{n-2}\pi^{an}_{\mathrm{desc},*}\overline{\mathbb{Q}}$ *on the complex analytic space* $\mathbb{G}^{an}_{m,\mathbb{C}} - \{1\}$*. Then it is the local system corresponding to an irreducible hypergeometric* $\mathcal{D}$*-module.*

Remark 4.7. Lemma 4.5 has the ambiguity in the choice of V in the coset, and the proof is not constructible. In particular, as long as the author sees, it seems impossible to obtain the Picard-Fuchs equation of a given section from Lemma 4.5.

4.4 Picard-Fuchs equations of the generalized Dwork family

In the rest of this section, we use the notation on $\mathcal{D}$-modules mentioned at the beginning of §2.1.

Let (n, d, W) be the data as in §4.1 and put $d_W = lcm(w_1, \dots, w_n)d$. Let K be a field of characteristic zero containing primitive d_W-th roots of unity, and let $\pi : \mathbb{X} \to \mathbb{A}^1$ be the generalized Dwork family over K defined by the equation (4.2). By Lemma 4.1, π is smooth on the affine variety

$$U := \operatorname{Spec} K[\lambda, (1-\lambda^d)^{-1}] \hookrightarrow \mathbb{A}^1_K. \tag{4.4}$$

We often write $\pi^{-1}(U) \to U$ by $\mathbb{X} \to U$ if there is no fear of confusion.

Let $\mathrm{Prim}^{n-2}_{\mathrm{dR}}$ be the primitive part of $H^{n-2}_{\mathrm{dR}}(\mathbb{X}/U)$ defined in the same way as in §4.1, and let

$$\mathrm{Prim}^{n-2}_{\mathrm{dR}} = \bigoplus_V \mathrm{Prim}^{n-2}_{\mathrm{dR}}(V \bmod W)$$

be the eigendecomposition by Γ_W/Δ. The residue map

$$\mathrm{Res} : H^{n-1}_{\mathrm{dR}}((\mathbb{P} - \mathbb{X})/U) \xrightarrow{\sim} \mathrm{Prim}^{n-2}_{\mathrm{dR}}$$

is horizontal and Γ_W/Δ-equivariant (e.g. [Kat09, §8]). By this isomorphism we identify both sides.

Theorem 4.8. *Let* $V = (v_1, \dots, v_n) \in (\mathbb{Z}/d\mathbb{Z})^n_0$ *be a totally nonzero element. Let* $D_\lambda = \lambda\frac{d}{d\lambda} = \lambda\nabla^{\mathrm{GM}}_{d/d\lambda}$*. Put*

$$\omega_V := \mathrm{Res}\left(\frac{X_1^{\tilde{v}_1-1}\cdots X_n^{\tilde{v}_n-1}}{Q_\lambda^{\deg V}}\Omega\right) \in \mathrm{Prim}^{n-2}_{\mathrm{dR}}(V \bmod W), \tag{4.5}$$

$$P'(V,W) := \prod_{k=0}^{d-1}(D_\lambda - k) - \lambda^d \prod_{i=1}^{n}\prod_{j=0}^{w_i-1}\left(D_\lambda + \frac{\tilde{v}_i + jd}{w_i}\right) \tag{4.6}$$

where $\tilde{v}_i \in \{0, 1, \ldots, d-1\}$ *denotes the unique integer such that* $\tilde{v}_i \equiv v_i \bmod d$*. Then* $P'(V, W)\omega_V = 0$*.*

Proof. Let $\mu = (\mu_1, \ldots, \mu_{n+1})$ be parameters and let

$$\begin{aligned} Q_\mu &= -d\mu_1 X_1^d - \cdots - d\mu_n X_n^d - d\mu_{n+1} X_1^{w_1} \cdots X_n^{w_n}, \\ \omega_\mu &= \frac{X_1^{\tilde{v}_1 - 1} \cdots X_n^{\tilde{v}_n - 1}}{Q_\mu^{\deg V}} \Omega, \\ D_i &= \mu_i \frac{\partial}{\partial \mu_i}. \end{aligned}$$

By Theorem 3.3, ω_μ satisfies the GKZ system given by

$$A = (a_{ij}) = \begin{pmatrix} d & \cdots & 0 & w_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & d & w_n \end{pmatrix} \text{ and } \boldsymbol{\beta} = \begin{pmatrix} \tilde{v_1} \\ \vdots \\ \tilde{v_n} \end{pmatrix}$$

i.e. ω_μ satisfies the system

$$\begin{cases} \left(\dfrac{\partial}{\partial \mu_{n+1}} \right)^d \omega_\mu = \displaystyle\prod_{i=1}^n \left(\dfrac{\partial}{\partial \mu_i} \right)^{w_i} \omega_\mu & (4.7) \\ \left(\displaystyle\sum_{j=1}^{n+1} a_{ij} D_j + \tilde{v}_i \right) \omega_\mu = 0 \quad (i = 1, \ldots, n). & (4.8) \end{cases}$$

From (4.7), note that $(\frac{\partial}{\partial \mu_i})^m = \left(\mu_i^{-1} D_i\right)^m = \mu_i^{-m} \prod_{j=0}^{m-1} (D_i - j)$, we have

$$\mu_{n+1}^{-d} \prod_{k=0}^{d-1} (D_{n+1} - k)\omega_\mu = \prod_{i=1}^n \mu_i^{-w_i} \prod_{j=0}^{w_i - 1} (D_i - j)\omega_\mu.$$

Moreover by the equation (4.8) the right hand side is equal to

$$\begin{aligned} &\prod_{i=1}^n \mu_i^{-w_i} \prod_{j=0}^{w_i - 1} \left(-\frac{w_i}{d} D_{n+1} - \frac{\tilde{v}_i}{d} - j \right) \omega_\mu \\ &= \prod_{i=1}^n \left(-\frac{d}{w_i} \mu_i \right)^{-w_i} \prod_{j=0}^{w_i - 1} \left(D_{n+1} + \frac{\tilde{v}_i + jd}{w_i} \right) \omega_\mu. \end{aligned}$$

By putting $\mu = (-w_1/d, \ldots, -w_n/d, \lambda)$ we obtain the desired equation. $\square$

Fix $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{Z}^n$ such that $\sum_{i=1}^n b_i w_i = 1$, and let $\mathbb{Y}/\mathbb{G}_m - \{1\}$ be the descent family defined by the equation (4.3).

Proposition 4.9. *Let $V \in (\mathbb{Z}/d\mathbb{Z})_0^n/\langle W\rangle$ be a totally nonzero element. Put $|\mathbf{b}\tilde{\mathbf{v}}| := \sum_{i=1}^n b_i\tilde{v}_i$. Then*

$$\omega_{V,t} := \mathrm{Res}\left(\frac{Y_1^{\tilde{v}_1-1}\cdots Y_n^{\tilde{v}_n-1}}{Q_t^{\deg V}}\Omega\right) \in \mathrm{Prim}^{n-2}_{\mathrm{dR,desc}}(V \bmod W)$$

is annihilated by the hypergeometric operator

$$\begin{aligned}
&Hyp'(V,W,b)\\
&:= Hyp\left(\begin{matrix} \frac{|\mathbf{b}\tilde{\mathbf{v}}|}{d}, & \frac{1}{d}+\frac{|\mathbf{b}\tilde{\mathbf{v}}|}{d}, & \ldots, & \frac{d-1}{d}+\frac{|\mathbf{b}\tilde{\mathbf{v}}|}{d}\\ \frac{w_1d-\tilde{v}_1}{w_1d}+\frac{|\mathbf{b}\tilde{\mathbf{v}}|}{d}, & \frac{(w_1-1)d-\tilde{v}_1}{w_1d}+\frac{|\mathbf{b}\tilde{\mathbf{v}}|}{d}, & \ldots, & \frac{d-\tilde{v}_n}{w_nd}+\frac{|\mathbf{b}\tilde{\mathbf{v}}|}{d}\end{matrix}; t\right)\\
&=\prod_{i=1}^n\prod_{j=0}^{w_i-1}\left(D_t+\frac{(w_i-j)d-\tilde{v}_i}{w_id}+\frac{|\mathbf{b}\tilde{\mathbf{v}}|}{d}-1\right)-t\prod_{k=0}^{d-1}\left(D_t+\frac{k}{d}+\frac{|\mathbf{b}\tilde{\mathbf{v}}|}{d}\right).
\end{aligned}$$

Proof. Since the equation (4.3) is obtained by putting $Y_i = \lambda^{b_i}X_i$ and $t = \lambda^{-d}$, the section

$$\begin{aligned}
\lambda^{|\mathbf{b}\tilde{\mathbf{v}}|}\omega_V =&\frac{(\lambda^{b_1}X_1)^{\tilde{v}_1-1}\cdots(\lambda^{b_n}X_n)^{\tilde{v}_n-1}}{Q_\lambda^{\deg V}}\\
&\times\sum_{i=1}^n(-1)^{n+1}\lambda^{b_i}X_id(\lambda^{b_1}X_1)\cdots\widehat{d(\lambda^{b_i}X_i)}\cdots d(\lambda^{b_n}X_n)
\end{aligned}$$

of $\mathrm{Prim}^{n-2}_{\mathrm{dR}}(V,W)|_{U-\{0\}}$ descents to $\omega_{V,t}$. From Theorem 4.8 we have

$$\begin{aligned}
0 &= (P'(V,W)\lambda^{-|\mathbf{b}\tilde{\mathbf{v}}|})\lambda^{|\mathbf{b}\tilde{\mathbf{v}}|}\omega_V\\
&= \lambda^{-|\mathbf{b}\tilde{\mathbf{v}}|}\left(\prod_{k=0}^{d-1}(D_\lambda-k-|\mathbf{b}\tilde{\mathbf{v}}|)-\lambda^d\prod_{i=1}^n\prod_{j=0}^{w_i-1}\left(D_\lambda+\frac{\tilde{v}_i+jd}{w_i}-|\mathbf{b}\tilde{\mathbf{v}}|\right)\right)\lambda^{|\mathbf{b}\tilde{\mathbf{v}}|}\omega_V.
\end{aligned}$$

Since the action of D_λ on $\mathrm{Prim}^{n-2}_{\mathrm{dR}}(V,W)|_{U-\{0\}}$ corresponds to $-dD_t$, by variable transforming and dividing by a power of $-d$, we can see that $\omega_{V,t}$ is annihilated by $Hyp'(V,W,b)$. □

Lemma 4.10. *Let $0 \le j \le w_i-1, 0 \le k \le d-1$ and $1 \le \tilde{v}_i \le d-1$ be integers. Then, the condition*

$$\frac{(w_i-j)d-\tilde{v}_i}{w_id} \equiv \frac{k}{d} \mod \mathbb{Z}$$

implies

$$\frac{(w_i-j)d-\tilde{v}_i}{w_id} = \frac{k}{d}$$

In particular, the cancel operation removes pairs of identical parameters from $Hyp'(V,W,b)$ in Proposition 4.9.

Proof. This is immediate as

$$\frac{(w_i-j)d-\tilde{v}_i}{w_i d}-\frac{k}{d}=1-\frac{jd+\tilde{v}_i+w_i k}{w_i d}<1,$$

$$\frac{(w_i-j)d-\tilde{v}_i}{w_i d}-\frac{k}{d}=\frac{d-k}{d}-\frac{\tilde{v}_i+jd}{w_i d}\geq\frac{1}{d}-\frac{w_i d-1}{w_i d}>-1.$$

□

Theorem 4.11. *Under the setting of Proposition 4.9, $\omega_{V,t}$ annihilated by*

$$Hyp(V,W,b):=\textbf{Cancel } Hyp'(V,W,b).$$

Proof. Suppose that $\omega\in\mathrm{Prim}^{n-2}_{\mathrm{dR,desc}}(V \bmod W)$ is annihilated by the operator D_t-c for some $c\in\mathbb{Q}$. If $\omega\neq 0$, the submodule generated by ω is isomorphic to $\mathcal{D}/\mathcal{D}(D_t-c)$. Since $\mathrm{Prim}^{n-2}_{\mathrm{dR,desc}}(V \bmod W)$ is irreducible by Lemma 4.6, we have $\mathrm{Prim}^{n-2}_{\mathrm{dR,desc}}(V \bmod W)\cong\mathcal{D}/\mathcal{D}(D_t-c)$, and hence $\mathcal{D}/\mathcal{D}(D_t-c)\cong\mathcal{D}/\mathcal{D}Hyp(\alpha;\beta;t)$ whose parameters satisfy the condition $\alpha-\beta\notin\mathbb{Z}$. This is impossible. This shows $\omega=0$.

Write

$$Hyp'(V,W,b)=Hyp(\alpha_1,\ldots,\alpha_r,\ldots,\alpha_d;\beta_1,\ldots,\beta_r,\ldots,\beta_d;t)$$

so that $\alpha_i=\beta_i$ for each $i>r$. Since

$$\begin{aligned}&Hyp(\alpha_1,\ldots,\alpha_d;\beta_1,\ldots,\beta_d;t)\\=&(D_t+\beta_i-1)Hyp(\alpha_1,\ldots,\widehat{\alpha_i},\ldots,\alpha_d;\beta_1,\ldots,\widehat{\beta_i},\ldots,\beta_d;t)\end{aligned}$$

for any $i>r$, the element

$$Hyp(\alpha_1,\ldots,\widehat{\alpha_i},\ldots,\alpha_d;\beta_1,\ldots,\widehat{\beta_i},\ldots,\beta_d;t)\omega_{V,t}\in\mathrm{Prim}^{n-2}_{\mathrm{dR,desc}}(V \bmod W)$$

is annihilated by $(D_t+\beta_i-1)$, and hence it vanishes. Repeating this argument, we have that $Hyp(V,W,b)\omega_{V,t}=0$ by Lemma 4.10. □

Example 4.12. In the case of $V=W$, $|\mathbf{b}\tilde{\mathbf{v}}|=\sum_{i=1}^n b_i\tilde{v}_i=\sum_{i=1}^n b_i w_i=1$. Then the section

$$\omega_{W,t}=\mathrm{Res}\left(\frac{Y_1^{w_1-1}\cdots Y_n^{w_n-1}}{Q_t}\Omega\right)$$

is annihilated by

$$\textbf{Cancel } Hyp\begin{pmatrix}\frac{1}{d}, & \frac{2}{d}, & \ldots, & 1 \\ \frac{w_1}{w_1}, & \frac{(w_1-1)}{w_1}, & \ldots, & \frac{1}{w_n}\end{pmatrix};t\Bigg).$$

This result corresponds to Lemma 8.3 of [Kat09].

We turn to the family $\mathbb{X}/U$. Let V, W be as before, and $\tilde{v}_i \in \{0, 1, \dots, d-1\}$ denotes the unique integer such that $\tilde{v}_i \equiv v_i \bmod d$. Let

$$I(V,W) := \mathbb{Z} \cap \bigcup_{i=1}^{n} \left\{ d - \frac{\tilde{v}_i}{w_i}, d - \frac{\tilde{v}_i + d}{w_i}, \dots, d - \frac{\tilde{v}_i + (w_i - 1)d}{w_i} \right\}.$$

If j is an integer such that $0 \le j \le w_i - 1$, then

$$\frac{\tilde{v}_i + dj}{w_i} < \frac{d + d(w_i - 1)}{w_i} = d,$$

and hence $I(V,W)$ is a subset of $\{0, 1, \dots, d-1\}$. Let $P'(V,W)$ be the differential operator (4.6). If $k \in I(V,W)$, then $P'(V,W)$ is factorized by $(D_\lambda - k)$ from left. Let $P(V,W)$ denote the differential equation obtained by removing all such factors from $P'(V,W)$, namely

$$P'(V,W) = \prod_{k \in I(V,W)} (D_\lambda - k) P(V,W).$$

Lemma 4.13. *Assume that $V \in (\mathbb{Z}/dZ)_0^n$ is totally nonzero. Put*

$$J(V,W) := \{r \in \{0, 1, \dots, d-1\} \mid \tilde{v}_i + rw_i \equiv 0 \bmod d \textit{ for some } i\}.$$

Then $I(V,W) = J(V,W)$.

Proof. Obviously $I(V,W) \subset J(V,W)$. For an element $r \in J(V,W)$, let s be the integer such that $\tilde{v}_i + rw_i = sd$. This satisfies $1 \le s \le w_i$ as $sd = \tilde{v}_i + rw_i < d + dw_i = d(1 + w_i)$. Hence we have

$$r = \frac{-\tilde{v}_i + sd}{w_i} = d - \frac{\tilde{v}_i + (w_i - s)d}{w_i} \in I(V,W).$$

This shows $I(V,W) \supset J(V,W)$. $\square$

Theorem 4.14. *Under the setting of Theorem 4.8, we have*

$$P(V,W)\omega_V = 0.$$

Proof. We notice that

$$\lambda^{|\mathbf{b}\tilde{\mathbf{v}}|} P'(V,W) \lambda^{-|\mathbf{b}\tilde{\mathbf{v}}|} = (\text{constant}) \times Hyp'(V,W,b),$$

and the cancel operation for $Hyp'(V,W,b)$ corresponds to the cancel operation for $P'(V,W)$ by Lemma 4.13. Therefore the assertion follows from Theorem 4.11 together with the fact that $\omega_{V,t} = \lambda^{|\mathbf{b}\tilde{\mathbf{v}}|}\omega_V$. $\square$

4.5 $\mathcal{D}$-module structure of $\mathrm{Prim}_{\mathrm{dR}}^{n-2}$.

Lemma 4.15. *Assume that V is totally nonzero.*

(1) *The order of $Hyp(V,W,b)$ coincides with the rank of $\mathrm{Prim}_{\mathrm{dR,desc}}^{n-2}(V \bmod W)$.*

(2) *The $\mathcal{D}_{\mathbb{G}_m-\{1\}}$-module $\mathcal{D}_{\mathbb{G}_m-\{1\}}/\mathcal{D}_{\mathbb{G}_m-\{1\}}Hyp(V,W,b)$ is irreducible.*

Proof. (1) By Lemma 4.2 and Lemma 4.13, we have

$$\begin{aligned}\text{rank } \mathrm{Prim}_{\mathrm{dR,desc}}^{n-2}(V \bmod W) &= \text{rank } \mathrm{Prim}_{\mathrm{dR}}^{n-2}(V \bmod W)\\ &= d - \#J(V,W)\\ &= d - \#I(V,W)\\ &= \text{The order of } P(V,W)\\ &= \text{The order of } Hyp(V,W,b)\end{aligned}$$

as required.

(2) This is an immediate consequence of Corollary 2.5.

□

Given a rational number γ we define the ring automorphism B_γ of $\mathcal{D}_{\mathbb{G}_m}$ (or $\mathcal{D}_{\mathbb{G}_{m-\{1\}}}$) as

$$\begin{aligned}B_\gamma : t &\mapsto t\\ D_t &\mapsto D_t + \gamma.\end{aligned}$$

Let us denote by $t^\gamma\mathcal{O}_{\mathbb{G}_m}$ the $\mathcal{D}_{\mathbb{G}_m}$-module obtained by twisting $\mathcal{O}_{\mathbb{G}_m}$ by B_γ, i.e. $t^\gamma\mathcal{O}_{\mathbb{G}_m} := \mathcal{O}_{\mathbb{G}_m} \otimes_{B_\gamma} \mathcal{D}_{\mathbb{G}_m}$. Because the action of D_t on a $\mathcal{D}$-module is translated to the action of $\mathcal{D}+\gamma$ by twisting,

$$\mathcal{H}(\alpha_1,\ldots,\alpha_m;\beta_1,\ldots,\beta_m)\otimes t^\gamma\mathcal{O} \cong \mathcal{H}(\alpha_1+\gamma,\ldots,\alpha_m+\gamma;\beta_1+\gamma,\ldots,\beta_m+\gamma).$$

We remark that, if $\gamma \in \frac{1}{d}\mathbb{Z}$, $[d]^*t^\gamma\mathcal{O}_{\mathbb{G}_m-\{1\}} = \lambda^{d\gamma}\mathcal{O}_{U-\{0\}} \cong \mathcal{O}_{U-\{0\}}$.

The following theorem is a generalization of Theorem 8.4 and Corollary 8.5 of [Kat09].

Theorem 4.16. *Assume that $V \in (\mathbb{Z}/d\mathbb{Z})_0^n$ is totally nonzero. We define*

$$\begin{aligned}Hyp(V,W) &:= \textbf{Cancel}\,Hyp\begin{pmatrix}0, & \frac{1}{d}, & \ldots, & \frac{d-1}{d} \\ \frac{w_1d-\tilde{v}_1}{w_1d}, & \frac{(w_1-1)d-\tilde{v}_1}{w_1d}, & \ldots, & \frac{d-\tilde{v}_n}{w_nd}\end{pmatrix}; t\\ \mathcal{H}_{V,W}^{\mathcal{D}} &:= \mathcal{D}_{\mathbb{G}_m-\{1\}}/\mathcal{D}_{\mathbb{G}_m-\{1\}}Hyp(V,W).\end{aligned}$$

Then there exist isomorphisms of $\mathcal{D}$-modules

$$\begin{aligned}\mathrm{Prim}^{n-2}_{\mathrm{dR,desc}}(V \bmod W) &\cong \mathcal{H}^{\mathcal{D}}_{V,W} \otimes t^{|\mathrm{bv}|/d}\mathcal{O}_{\mathbb{G}_m-\{1\}},\\ \mathrm{Prim}^{n-2}_{\mathrm{dR}}(V \bmod W)|_{U-\{0\}} &\cong [d]^*\mathcal{H}^{\mathcal{D}}_{V,W},\\ \mathrm{Prim}^{n-2}_{\mathrm{dR}}(V \bmod W) &\cong j_{!*}[d]^*\mathcal{H}^{\mathcal{D}}_{V,W}.\end{aligned}$$

where $j_{!*}$ is the minimal extension by the inclusion $j : U - \{0\} \to U$.

Proof. It is enough to show the first isomorphism, because the 2nd and 3rd isomorphisms can be derived from it. Consider the morphism

$$\begin{aligned}\mathcal{H}^{\mathcal{D}}_{V,W} \otimes t^{|\mathrm{bv}|/d}\mathcal{O} \cong \mathcal{D}/\mathcal{D}Hyp(V,W,b) &\longrightarrow \mathrm{Prim}^{n-2}_{\mathrm{dR,desc}}(V \bmod W)\\ P &\longmapsto P\omega_{V,t}.\end{aligned}$$

Since $\mathcal{D}/\mathcal{D}Hyp(V,W,b)$ is irreducible, this morphism is injective. Moreover we know that both sides are locally free $\mathcal{O}_{\mathbb{G}_m-\{1\}}$-modules with the same rank by Lemma 4.15, hence the morphism is surjective. □

5 Application to p-adic cohomology theory

5.1 Differential modules and Matrix calculus

A commutative ring R is called a differential ring if it is equipped with an additive map $d : R \to R$ satisfying the Leibniz rule

$$d(ab) = ad(b) + bd(a), \quad (a, b \in R).$$

The map d is called a derivation. For a matrix $A = (a_{ij})$ with $a_{ij} \in R$, we write $d(A) = (d(a_{ij}))$. A differential module over a differential ring (R, d) is a R-module M equipped with an additive map $D : M \to M$ satisfying

$$D(am) = aD(m) + d(a)m, \quad (a \in R,\ m \in M).$$

The map D is called a differential operator on M.

Let (M, D) be a finite free differential module of rank n over a differential ring (R, d). Let $e_1, \ldots, e_n$ be a basis of M. Define the matrix of action of D with respect to the basis $e_1, \ldots, e_n$ to be the $n \times n$ matrix $C = (c_{ij})$ with $c_{ij} \in R$ given by

$$D(e_j) = \sum_{i=1}^{n} c_{ij} e_i.$$

Proposition 5.1. *Let (M, D) be a finite free differential module of rank n over a differential ring (R, d).*

(1) *For an invertible matrix $A = (a_{ij})$ with $a_{ij} \in R$, we have*

$$d(A^{-1}) = -A^{-1}d(A)A^{-1}. \tag{5.1}$$

(2) *Let* $C, \widetilde{C}$ *be the matrices of the action of* D *with respect to bases* $e_1, \ldots, e_n$ *and* $\tilde{e}_1, \ldots, \tilde{e}_n$ *of* M *respectively. Then*

$$\tilde{C} = B^{-1}CB + B^{-1}d(B), \tag{5.2}$$

where B *is the change-of-basis matrix from* $e_1, \ldots, e_n$ *to* $\tilde{e}_1, \ldots, \tilde{e}_n$.

(3) *Let* $C, \widetilde{C}$ *and* B *be as above and let* X *be a* $n \times n$*-matrix. Then*

$$d(X) - XC = O \iff d(XB) - (XB)\widetilde{C} = O.$$

Proof. Straightforward. □

5.2 Deformation matrix of Dwork families

We compute the Frobenius matrix on the Dwork families by the deformation method. We refer the reader to [Ger], [Ked] for general references of deformation method.

Let $p > 0$ be a prime number. Let $W = W(\overline{\mathbb{F}}_p)$ be the ring of Witt vectors of the algebraic closure $\overline{\mathbb{F}}_p$, and K the fractional field. Let $W[X_1, \ldots, X_m]^\dagger$ be the ring of power series $\sum a_\alpha X^\alpha \in W[[X_1, \ldots, X_m]]$ such that $|a_\alpha| r^{|\alpha|} \to 0$ as $|\alpha| \to \infty$ for some $r > 1$. For a W-algebra B of finite type, the weak completion $B^\dagger$ is defined to be $W[X_1, \ldots, X_m]^\dagger / IW[X_1, \ldots, X_m]^\dagger$ for a presentation $B \cong W[X_1, \ldots, X_m]/I$. We write $B_K^\dagger := B^\dagger \otimes K$.

Fix integers $d \geq n \geq 3$ and $w_1, \ldots, w_n \geq 1$ such that the condition (4.1) is satisfied and assume that $d, w_1, \ldots, w_n$ are prime to p. Consider the generalized Dwork family $\mathbb{X} \to U = \operatorname{Spec} W[\lambda, (1-\lambda^d)^{-1}]$ defined by the equation (4.2), and the family of complements $(\mathbb{P} - \mathbb{X})/U$. Put $B = \Gamma(U, \mathcal{O}_U)$. Let $\sigma : B^\dagger \to B^\dagger$ be the continuous F_W-linear ring homomorphism such that $\lambda \mapsto \lambda^p$ where F_W is the p-th Frobenius on W. Let $H^*_{\mathrm{rig}}((\mathbb{P} - \mathbb{X})_{\overline{\mathbb{F}}_p} / U_{\overline{\mathbb{F}}_p})$ be the rigid cohomology group. This is a $B_K^\dagger$-module, and it is endowed the following structure,

- the integrable connection

$$\nabla^{\mathrm{rig}} : H^i_{\mathrm{rig}}((\mathbb{P} - \mathbb{X})_{\overline{\mathbb{F}}_p} / U_{\overline{\mathbb{F}}_p}) \to H^i_{\mathrm{rig}}((\mathbb{P} - \mathbb{X})_{\overline{\mathbb{F}}_p} / U_{\overline{\mathbb{F}}_p}) \otimes \Omega^1_{B_K^\dagger}$$

  induced from the Gauss-Manin connection under the comparison

$$H^i_{\mathrm{rig}}((\mathbb{P} - \mathbb{X})_{\overline{\mathbb{F}}_p} / U_{\overline{\mathbb{F}}_p}) \cong B_K^\dagger \otimes_B H^i_{\mathrm{dR}}((\mathbb{P} - \mathbb{X})_K / U_K) \tag{5.3}$$

  with the de Rham cohomology group,

- an isomorphism

$$\Phi : \sigma^* H^i_{\mathrm{rig}}((\mathbb{P} - \mathbb{X})_{\overline{\mathbb{F}}_p} / U_{\overline{\mathbb{F}}_p}) \xrightarrow{\sim} H^i_{\mathrm{rig}}((\mathbb{P} - \mathbb{X})_{\overline{\mathbb{F}}_p} / U_{\overline{\mathbb{F}}_p}) \tag{5.4}$$

  of $B_K^\dagger$-modules which commutes with ∇^{rig}.

Let $\omega_1, \ldots, \omega_r$ be a basis of $H^{n-1}_{\mathrm{rig}}((\mathbb{P}-\mathbb{X})_{\overline{\mathbb{F}}_p}/U_{\overline{\mathbb{F}}_p})$. Let $C(\lambda) = (c_{ij}(\lambda))$ be the matrix of the action of $\frac{d}{d\lambda} := \nabla^{\mathrm{rig}}_{d/d\lambda}$, and let $F(\lambda) = (f_{ij}(\lambda))$ be the matrix of the action of Φ,

$$\frac{d}{d\lambda}\omega_j = \sum_{i=1}^{r} c_{ij}(\lambda)\omega_i, \quad \Phi\omega_j = \sum_{i=1}^{r} f_{ij}(\lambda)\omega_i.$$

The commutativity of ∇^{rig} and Φ means that $F(\lambda)$ satisfies a differential equation

$$\frac{d}{d\lambda}F(\lambda) + C(\lambda)F(\lambda) = p\lambda^{p-1}F(\lambda)C(\lambda^p). \tag{5.5}$$

Let $A(\lambda)$ be the unique solution of the differential equation

$$\frac{d}{d\lambda}A(\lambda) = A(\lambda)C(\lambda) \tag{5.6}$$

with the initial condition

$$A(0) = I. \tag{5.7}$$

The matrix $A(\lambda)$ is called the deformation matrix with respect to the basis $\omega_1, \ldots, \omega_r$. Straightforward calculation shows that $X(\lambda) := A(\lambda)F(\lambda)A(\lambda^p)^{-1}$ satisfies $\frac{d}{d\lambda}X(\lambda) = O$, so that $X(\lambda)$ is a constant matrix, namely $X = F(0)$,

$$F(\lambda) = A(\lambda)^{-1}F(0)A(\lambda^p). \tag{5.8}$$

Recall the decomposition

$$H^{n-1}_{\mathrm{dR}}((\mathbb{P}-\mathbb{X})_K/U_K) \cong \mathrm{Prim}^{n-2}_{\mathrm{dR}} = \bigoplus_V \mathrm{Prim}^{n-2}_{\mathrm{dR}}(V \bmod W)$$

from §4. Suppose that $\mathrm{Prim}^{n-2}_{\mathrm{dR}}(V \bmod W)$ has a basis of cyclic vectors $\omega_V, \frac{d}{d\lambda}\omega_V, \ldots, \frac{d^{r-1}}{d\lambda^{r-1}}\omega_V$. Let $C(\lambda)_V$ denote the matrix of the action of $\frac{d}{d\lambda}$ on $\mathrm{Prim}^{n-2}(V \bmod W)_{\mathrm{dR}}$, which is of the form

$$C(\lambda)_V = \begin{pmatrix} 0 & \cdots & 0 & -c_0(\lambda) \\ 1 & \cdots & 0 & -c_1(\lambda) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -c_{r-1}(\lambda) \end{pmatrix}$$

where $c_1(\lambda), \ldots, c_r(\lambda)$ are the coefficients of the Picard-Fuchs equation

$$\frac{d^r}{d\lambda^r}\omega_V + c_{r-1}(\lambda)\frac{d^{r-1}}{d\lambda^{r-1}}\omega_V + \cdots + c_0(\lambda)\omega_V = 0. \tag{5.9}$$

Moreover the deformation matrix $A(\lambda)_V$ with respect to the basis $\{\frac{d^i}{d\lambda^i}\omega_V\}_i$ is given as follows,

$$\frac{d}{d\lambda}A(\lambda)_V = A(\lambda)_V C(\lambda)_V. \tag{5.10}$$

Let $\{w_0(\lambda)_V, \ldots, w_{r-1}(\lambda)_V\}$ be a fundamental set of solutions of the Picard-Fuchs equation (5.9) around $\lambda = 0$. Then the Wronskian matrix

$$W(\lambda)_V = \begin{pmatrix} w_0(\lambda)_V & \frac{d}{d\lambda}w_0(\lambda)_V & \cdots & \frac{d^{r-1}}{d\lambda^{r-1}}w_0(\lambda)_V \\ \vdots & \vdots & \cdots & \vdots \\ w_{r-1}(\lambda)_V & \frac{d}{d\lambda}w_{r-1}(\lambda)_V & \cdots & \frac{d^{r-1}}{d\lambda^{r-1}}w_{r-1}(\lambda)_V \end{pmatrix} \tag{5.11}$$

is a solution of the differential equation (5.10), so that one has

$$A(\lambda)_V = W(0)_V^{-1} W(\lambda)_V \tag{5.12}$$

(note that the matrix $W(0)_V$ is invertible as $w_0(\lambda)_V, \ldots, w_{r-1}(\lambda)_V$ are linearly independent over K).

Put $V^{(1)} = (p^{-1}v_1, \ldots, p^{-1}v_n) \in (\mathbb{Z}/d\mathbb{Z})_0^n$. Since the Frobenius Φ commutes with the action of Γ_W/Δ, we have

$$\Phi(\mathrm{Prim}_{\mathrm{dR}}^{n-2}(V^{(1)} \bmod W)) \subset B_K^\dagger \otimes \mathrm{Prim}_{\mathrm{dR}}^{n-2}(V \bmod W).$$

Thanks to the isomorphism (5.4), $\mathrm{Prim}_{\mathrm{dR}}^{n-2}(V^{(1)} \bmod W)$ also has a basis of cyclic vectors $\{\frac{d^i}{d\lambda^i}\omega_{V^{(1)}}\}_i$. Let $F(\lambda)_V$ be the matrix of Φ with respect to the bases $\{\frac{d^i}{d\lambda^i}\omega_{V^{(1)}}\}_i$ and $\{\frac{d^j}{d\lambda^j}\omega_V\}_j$. It follows from (5.8) that we have

$$F(\lambda)_V = A(\lambda)_V^{-1} F(0)_V A(\lambda^p)_{V^{(1)}}. \tag{5.13}$$

Suppose that one can take ω_V to be the differential form (4.5) in the above setting. Then the Picard-Fuchs equation is given by the hypergeometric equation $P(V, W)$ by Theorem 4.14. Write

$$P(V, W) = \prod_{i=0}^{r-1}(D_\lambda - k_i) - \lambda^d \prod_{j=0}^{r-1}(D_\lambda + \alpha_j),$$

where we note that $k_0, \ldots, k_r$ are pairwise distinct integers such that $0 \leq k_i < d$ and $\alpha_j \in \{(\tilde{v}_k + ld)/w_k \mid 1 \leq k \leq n,\ 0 \leq l < w_k\}$. We can obtain a fundamental set $\{w_i(\lambda)_V\}_i$ of solutions by the hypergeometric series (cf. [NIST, 16.8.6]), namely we set

$$w_i(\lambda)_V := \lambda^{k_i} {}_rF_{r-1}\left(\begin{matrix} \frac{\alpha_0 + k_i}{d}, \ldots, \frac{\alpha_{r-1}+k_i}{d} \\ 1 + \frac{-k_0+k_i}{d}, \ldots, \widehat{1 + \frac{-k_i+k_i}{d}}, \ldots, 1 + \frac{-k_{r-1}+k_i}{d} \end{matrix} ; \lambda^d \right) \tag{5.14}$$

for $i = 0, 1, \ldots, r-1$, and let $W(\lambda)_V$ be defined by (5.11). Then the deformation matrix $A(\lambda)_V$ is given by (5.12).

Summing up the above, we have the following theorem.

Theorem 5.2. *Let ω_V be the differential form* (4.5)*. Let $\{w_i(\lambda)_V\}_i$ be the hypergeometric series* (5.14)*. Suppose that $\mathrm{Prim}^{n-2}_{\mathrm{dR}}(V \bmod W)$ has a basis $\{\frac{d^i}{d\lambda^i}\omega_V\}_i$. Then the deformation matrix $A(\lambda)_V$ satisfies*

$$A(\lambda)_V = W(0)_V^{-1} W(\lambda)_V$$

where $W(\lambda)_V$ is the Wronskian matrix (5.11)*. Let $\{\omega_{V^{(1)}}\}_i$ and $\{w_i(\lambda)_{V^{(1)}}\}_i$ be defined in the same way from $V^{(1)}$. Then the same thing holds for $A(\lambda)_{V^{(1)}}$. Let $F(\lambda)_V$ be the matrix of the Frobenius*

$$\Phi : B_K^\dagger \otimes \mathrm{Prim}^{n-2}_{\mathrm{dR}}(V^{(1)} \bmod W) \longrightarrow B_K^\dagger \otimes \mathrm{Prim}^{n-2}_{\mathrm{dR}}(V \bmod W).$$

with respect to the bases $\{\frac{d^i}{d\lambda^i}\omega_{V^{(1)}}\}_i$ and $\{\frac{d^j}{d\lambda^j}\omega_V\}_j$. Then it satisfies

$$F(\lambda)_V = A(\lambda)_V^{-1} F(0)_V A(\lambda^p)_{V^{(1)}}.$$

The matrix $F(0)_V$ is the Frobenius on the Fermat variety, so it is more or less computable.

The computation of the deformation matrices for the Dwork families was first given by Kloosterman [Klo]. He uses another basis $\{\omega_{V_1}, \dots, \omega_{V_r}\}$ of $\mathrm{Prim}^{n-2}_{\mathrm{dR}}$, where $V_1, \dots, V_r$ are all elements of $(\mathbb{Z}/d\mathbb{Z})^n_0$ which are totally nonzero, and he computes the deformation matrices without using the Picard-Fuchs equations.

Remark 5.3. In Theorem 5.2, the condition

$$\text{“}\mathrm{Prim}^{n-2}_{\mathrm{dR}}(V \bmod W) \text{ is spanned by } \left\{\frac{d^i}{d\lambda^i}\omega_V\right\}_i\text{”}$$

does *not* hold in general. It is true that $\mathrm{Prim}^{n-2}_{\mathrm{dR}}(V \bmod W) \otimes_{K[\lambda]} K[\lambda^{-1}]$ is spanned by $\{\frac{d^i}{d\lambda^i}\omega_V\}_i$ by Theorem 4.16, while one cannot remove “$\otimes K[\lambda^{-1}]$” in general. If the condition fails, we need an additional argument to compute $A(\lambda)$. This is illustrated in §5.3.

5.3 Example ([Dwo68, §6 (j)], [Klo, Example 5.7])

We compute the deformation matrix of the Dwork pencil of quartic K3 surfaces. Let p be an odd prime and let $\mathbb{X}/U$ be the Dwork family given by the data $(n, d, W) = (4, 4, (1, 1, 1, 1))$ over $W(\overline{\mathbb{F}}_p)$, i.e. the family defined by the equation

$$X_1^4 + X_2^4 + X_3^4 + X_4^4 - 4\lambda X_1 X_2 X_3 X_4 = 0.$$

In this case, $\{(1, 1, 1, 1), (1, 2, 2, 3), (1, 1, 3, 3)\}$ is a complete system of representatives of $(\mathbb{Z}/4\mathbb{Z})^4_0/\langle W \rangle$.

Let $V_1 = (1,2,2,3) \in (\mathbb{Z}/4\mathbb{Z})_0^4$. By Theorem 4.8 the Picard-Fuchs equation of ω_{V_1} is

$$\begin{aligned} P(V_1, W) &= D_\lambda \cancel{(D_\lambda - 1)} \cancel{(D_\lambda - 2)} \cancel{(D_\lambda - 3)} \\ &\quad - \lambda^4 \cancel{(D_\lambda + 1)} (D_\lambda + 2) \cancel{(D_\lambda + 2)} \cancel{(D_\lambda + 3)} \\ &= D_\lambda - \lambda^4 (D_\lambda + 2). \end{aligned}$$

Therefore ${}_1F_0\left(\begin{matrix} \frac{1}{2} \\ - \end{matrix}; \lambda^4\right)$ is the solution of this equation with the initial condition ${}_1F_0\left(\begin{matrix} \frac{1}{2} \\ - \end{matrix}; 0\right) = 1$. Thus $A(\lambda)_{V_1} = {}_1F_0\left(\begin{matrix} \frac{1}{2} \\ - \end{matrix}; \lambda^4\right)$ by Theorem 5.2.

Let $V_2 = (1,1,1,1) \in (\mathbb{Z}/4\mathbb{Z})_0^4$. The Picard-Fuchs equation of ω_{V_2} is

$$\begin{aligned} P(V_2, W) &= D_\lambda (D_\lambda - 1)(D_\lambda - 2) \cancel{(D_\lambda - 3)} \\ &\quad - \lambda^4 \cancel{(D_\lambda + 1)} (D_\lambda + 1)(D_\lambda + 1)(D_\lambda + 1) \\ &= D_\lambda (D_\lambda - 1)(D_\lambda - 2) - \lambda^4 (D_\lambda + 1)^3. \end{aligned}$$

For this hypergeometric differential equation,

$$\begin{aligned} & w_0(\lambda)_{V_2} = {}_3F_2\left(\begin{matrix} \frac{1}{4}\ \frac{1}{4}\ \frac{1}{4} \\ \frac{1}{2}\ \frac{3}{4} \end{matrix}; \lambda^4\right), w_1(\lambda)_{V_2} = \lambda {}_3F_2\left(\begin{matrix} \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2} \\ \frac{3}{4}\ \frac{5}{4} \end{matrix}; \lambda^4\right), \\ & w_2(\lambda)_{V_2} = 8\lambda^2 {}_3F_2\left(\begin{matrix} \frac{3}{4}\ \frac{3}{4}\ \frac{3}{4} \\ \frac{5}{4}\ \frac{3}{2} \end{matrix}; \lambda^4\right) \end{aligned}$$

form a fundamental set of solutions. Then

$$W(\lambda)_{V_2} = \begin{pmatrix} {}_3F_2\left(\begin{matrix} \frac{1}{4}\ \frac{1}{4}\ \frac{1}{4} \\ \frac{1}{2}\ \frac{3}{4} \end{matrix}; \lambda^4\right) & \frac{\lambda^3}{6} {}_3F_2\left(\begin{matrix} \frac{5}{4}\ \frac{5}{4}\ \frac{5}{4} \\ \frac{3}{2}\ \frac{7}{4} \end{matrix}; \lambda^4\right) & \frac{\lambda^2}{2} {}_3F_2\left(\begin{matrix} \frac{5}{4}\ \frac{5}{4}\ \frac{5}{4} \\ \frac{3}{2}\ \frac{3}{4} \end{matrix}; \lambda^4\right) \\ \lambda {}_3F_2\left(\begin{matrix} \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2} \\ \frac{3}{4}\ \frac{5}{4} \end{matrix}; \lambda^4\right) & {}_3F_2\left(\begin{matrix} \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2} \\ \frac{3}{4}\ \frac{1}{4} \end{matrix}; \lambda^4\right) & \frac{8}{3}\lambda {}_3F_2\left(\begin{matrix} \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2} \\ \frac{7}{4}\ \frac{5}{4} \end{matrix}; \lambda^4\right) \\ 8\lambda^2 {}_3F_2\left(\begin{matrix} \frac{3}{4}\ \frac{3}{4}\ \frac{3}{4} \\ \frac{5}{4}\ \frac{3}{2} \end{matrix}; \lambda^4\right) & \lambda {}_3F_2\left(\begin{matrix} \frac{3}{4}\ \frac{3}{4}\ \frac{3}{4} \\ \frac{5}{4}\ \frac{1}{2} \end{matrix}; \lambda^4\right) & {}_3F_2\left(\begin{matrix} \frac{3}{4}\ \frac{3}{4}\ \frac{3}{4} \\ \frac{1}{4}\ \frac{1}{2} \end{matrix}; \lambda^4\right) \end{pmatrix}$$

satisfies the initial condition $W(0)_{V_2} = I$ and hence $A(\lambda)_{V_2} = W(\lambda)_{V_2}$ by Theorem 5.2.

For $V_3 = (1,1,3,3)$, the Picard-Fuchs equation of ω_{V_3} is

$$\begin{aligned} P(V_3, W) &= D_\lambda \cancel{(D_\lambda - 1)} (D_\lambda - 2) \cancel{(D_\lambda - 3)} \\ &\quad - \lambda^4 \cancel{(D_\lambda + 1)} (D_\lambda + 1) \cancel{(D_\lambda + 3)} (D_\lambda + 3) \\ &= D_\lambda (D_\lambda - 2) - \lambda^4 (D_\lambda + 1)(D_\lambda + 3) \\ &= \lambda^2 (1 - \lambda^4) \frac{d^2}{d\lambda^2} - (1 + 5\lambda) \frac{d}{d\lambda} - 3\lambda^4 \frac{d}{d\lambda}. \end{aligned}$$

The fundamental solutions of this equation are

$$w_0(\lambda)_{V_3} = {}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ \frac{1}{2}\end{matrix};\lambda^4\right), w_1(\lambda)_{V_3} = \lambda^2{}_2F_1\left(\begin{matrix}\frac{3}{4}\ \frac{5}{4}\\ \frac{3}{2}\end{matrix};\lambda^4\right)$$

and the Wronskian is

$$W(\lambda)_{V_3} = \begin{pmatrix} {}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ \frac{1}{2}\end{matrix};\lambda^4\right) & \frac{3}{2}\lambda^3{}_2F_1\left(\begin{matrix}\frac{5}{4}\ \frac{7}{4}\\ \frac{3}{2}\end{matrix};\lambda^4\right) \\ \lambda^2{}_2F_1\left(\begin{matrix}\frac{3}{4}\ \frac{5}{4}\\ \frac{3}{2}\end{matrix};\lambda^4\right) & 2\lambda\,{}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ \frac{1}{2}\end{matrix};\lambda^4\right) \end{pmatrix}.$$

However, this is the case in which ω_{V_3} and $\frac{d}{d\lambda}\omega_{V_3}$ do not form a basis, so we cannot apply Theorem 5.2 directly.

The eigenspace $\mathrm{Prim}^{n-2}_{\mathrm{dR}}(V_3 \bmod W)$ is characterized as Deligne's canonical extension of $\mathrm{Prim}^{n-2}_{\mathrm{dR}}(V_3 \bmod W)|_{U_K-\{0\}}$ ([Del, II. §5]). In this case, it is the coherent $\mathcal{O}_{U_K}$-submodule of $\mathrm{Prim}^{n-2}_{\mathrm{dR}}(V_3 \bmod W)|_{U_K-\{0\}}$ stable under D_λ such that $(D_\lambda \bmod \lambda)$ is nilpotent. We can compute it from the Picard-Fuchs equation $P(V_3, W)$. Put $e_1 = \omega_{V_3}$ and $e_2 = \frac{1}{2\lambda}\frac{d}{d\lambda}\omega_{V_3}$, then we have that

$$\begin{aligned} D_\lambda(e_1) &= 2\lambda^2\, e_2 \\ D_\lambda(e_2) &= \frac{3\lambda^2}{2(1-\lambda^4)}e_1 + \frac{6\lambda^4}{1-\lambda^4}e_2, \end{aligned}$$

and hence

$$\mathcal{O}_{U_K}e_1 \oplus \mathcal{O}_{U_K}e_2 = \mathrm{Prim}^{n-2}_{\mathrm{dR}}(V_3 \bmod W).$$

Since the change-of-basis matrix from $\{\omega_{V_3}, \frac{d}{d\lambda}\omega_{V_3}\}$ to $\{\omega_{V_3}, \frac{1}{2\lambda}\frac{d}{d\lambda}\omega_{V_3}\}$ is

$$B(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2\lambda} \end{pmatrix}$$

on $U_K - \{0\}$, the matrix

$$W(\lambda)_{V_3}\,B(\lambda) = \begin{pmatrix} {}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ \frac{1}{2}\end{matrix};\lambda^4\right) & \frac{3}{4}\lambda^2{}_2F_1\left(\begin{matrix}\frac{5}{4}\ \frac{7}{4}\\ \frac{3}{2}\end{matrix};\lambda^4\right) \\ \lambda^2{}_2F_1\left(\begin{matrix}\frac{3}{4}\ \frac{5}{4}\\ \frac{3}{2}\end{matrix};\lambda^4\right) & {}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ \frac{1}{2}\end{matrix};\lambda^4\right) \end{pmatrix} \tag{5.15}$$

is the solution of (5.6) for the basis $\{\omega_{V_3}, \frac{1}{2\lambda}\frac{d}{d\lambda}\omega_{V_3}\}$ by Proposition 5.1. Obviously $W(\lambda)_{V_3}\,B(\lambda)|_{\lambda=0} = I$, that concludes $A(\lambda)_{V_3} = W(\lambda)_{V_3}\,B(\lambda)$.

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