博士学位論文

Geometry of Surfaces with Singular Points and the Gauss-Bonnet Theorem

(特異点を持つ曲面の幾何学と Gauss-Bonnet の定理)

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Introduction

The classical Gauss-Bonnet theorem for a compact oriented 2-dimensional Riemannian manifold (M, g) with boundary ∂M is given as

$$\int_M K dA + \int_{\partial M} \kappa_g d\tau = 2\pi\chi(M), \tag{GB}$$

where K is the Gaussian curvature of the Riemannian metric g, dA is the area form of M, $\chi(M)$ is the Euler characteristic of M, κ_g is the geodesic curvature of ∂M, and $d\tau$ is the arc-length measure with respect to g (cf. [**UY17**, Theorem 14.2]). This formula establishes a profound connection between the intrinsic geometry of a 2-manifold and its topological properties, incorporating both the curvature of the manifold and the contribution from its boundary.

In this paper, we focus on surfaces with singular points. Singular surfaces arise naturally in geometry and physics (cf. [**AGZ12, IRRT16**]). A surface refers to a smooth mapping from a 2-manifold to a manifold (up to re-parametrizations of the source). A singular point of a surface means that the differential of the surface is not injective at the point (see Section 3 in Chapter 1). We establish Gauss-Bonnet type formulae for the following three classes of singular surfaces:

- Frontals and wave fronts, which are surfaces with a well-defined unit normal vector field (see Proposition 1.15).
- Smooth mappings between 2-manifolds (see Section 3.1 in Chapter 1).
- Surfaces that admit only cross caps as singular points (see Section 3.2 in Chapter 1).

Note that smooth mappings between 2-manifolds are treated as singular surfaces in our sense.

Our goal is to generalize Gauss-Bonnet formula (GB) to the above kinds of singular surfaces and to explore various applications for our generalized formulae. To analyze singular surfaces, we need specialized methods. By introducing appropriate geometric structures, such as coherent tangent bundles (see Definition 2.1), we aim to systematically extend the concepts of curvatures and the Euler characteristic to these singular surfaces. Also, we study positive semi-definite metrics, such as Whitney metrics, which admit intrinsic cross caps as singular points of a positive semi-definite metric (see Definition 3.8).

In the context of singular points of surfaces, $\mathcal{A}$-equivalence (see Section 3.1 in Chapter 1) allows us to classify singular surfaces into standard forms, simplifying their study. For instance, a smooth mapping germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is a *cuspidal edge* if f is $\mathcal{A}$-equivalent to

$$\mathrm{CED}: \left(\mathbb{R}^2, 0\right) \to \left(\mathbb{R}^3, 0\right), \quad (u, v) \mapsto \left(u, v^2, v^3\right)$$

at the origin 0 (see Example 1.25). Also, a smooth mapping germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is a *swallowtail* if f is $\mathcal{A}$-equivalent to

$$\mathrm{SW}: \left(\mathbb{R}^2, 0\right) \to \left(\mathbb{R}^3, 0\right), \quad (u, v) \mapsto \left(u, 3v^4 + uv^2, 4v^3 + 2uv\right)$$

at 0 (see Example 1.26). Cuspidal edges and swallowtails are generic singular points of wave fronts.

In [**LLR95**, Proposition 6] and [**Kos02**, Theorem 2], Gauss-Bonnet type formulae are derived for compact wave fronts without boundary that admit cuspidal edges and swallowtails as singular points of wave fronts. The Gauss-Bonnet type formulae in [**LLR95**] and [**Kos02**] are described by using only information on M, so it is expected that we can formulate wave fronts intrinsically. In [**SUY09**], coherent tangent bundles are introduced as a generalization of tangent bundles on Riemannian manifolds and limiting tangent bundles, where limiting tangent bundles are defined by planes orthogonal to the unit normal vector field of a wave front (see Definition 2.1 and Examples 2.4 and 2.6). Then the following assertion is obtained in [**SUY08**], and the Gauss-Bonnet type formulae in [**LLR95**] and [**Kos02**] are generalized to frontals that admit at most peaks as singular points (see Definitions 2.8 and 2.9).

THEOREM (Theorem B of [**SUY08**]). *Let $(M, \mathcal{E}, \langle\cdot,\cdot\rangle, D, \varphi)$ be a coherent tangent bundle over a compact oriented smooth 2-manifold M without boundary whose singular set $S(\varphi)$ of the smooth bundle homomorphism $\varphi : TM \to \mathcal{E}$ admits at most peaks. Then*

$$\begin{aligned} 2\pi\chi(M) &= \int_M K_\varphi dA_\varphi + 2\int_{S(\varphi)} \kappa_\varphi^s d\tau_\varphi, \\ \frac{1}{2\pi}\int_M K_\varphi d\widehat{A}_\varphi &= \chi\left(M_\varphi^+\right) - \chi\left(M_\varphi^-\right) + \#P^+(\varphi) - \#P^-(\varphi), \end{aligned}$$

where the notations that appear in the above equations are introduced in Chapter 2.

Then the theory of coherent tangent bundles and the Gauss-Bonnet type formulae in [**SUY08**] are generalized to the case of 2-manifolds with boundary in [**DZ20**] (see [**DZ20**, Theorem 2.20] and Theorem 2.45).

In [**SUY12**], simple proofs are given for the formula in [**Lev66**, Theorem 1], which relates the rotation index of the singular set of a smooth mapping from 2-manifolds into the plane $\mathbb{R}^2$ and the Euler characteristic of the source, and the formula in [**Qui78**, Theorem 1], which relates the degree of the mapping, the Euler characteristic and the number of cusps (see Example 1.18) as applications of the Gauss-Bonnet type formulae in [**SUY08**] to smooth mappings between 2-manifolds.

The first main result of this paper is the following two assertions, which generalize the formula in [**Lev66**] and the formula in [**Qui78**] to the case of smooth mappings between 2-manifolds with boundary that admit at most peaks as singular points of smooth mappings, by applying the Gauss-Bonnet type formulae in [**DZ20**].

THEOREM A (see Theorem 2.52). *Let M be a compact oriented 2-manifold with boundary ∂M and $f : M \to \mathbb{R}^2$ a smooth mapping into the plane $\mathbb{R}^2$ whose singular set $S(f)$ of f admits at most peaks, and suppose that $S(f) \backslash I(f)$ consists of smooth closed regular curves and does not intersect ∂M and that $I(f) \cap \partial M$ is empty, where $I(f)$ is the set of isolated singular points of f. Then there exists a unique orientation of the components $\{S_1, \dots, S_m\}$ of $S(f) \backslash I(f)$ and the components $\{\partial_1, \dots, \partial_n\}$ of ∂M satisfying*

$$\frac{\chi(M)}{2} = \sum_{i=1}^{m} R(S_i) + \frac{1}{2} \sum_{i=1}^{n} R(\partial_i),$$

where $R(S_i)$ is the rotation index of S_i and $R(\partial_i)$ is the rotation index of ∂_i.

THEOREM B (see Theorem 2.53). *Let M, N be two compact oriented 2-manifolds with boundary and $f : M \to N$ a smooth mapping satisfying $f^{-1}(\partial N) = \partial M$ whose singular set $S(f)$ of f admits at most peaks, and suppose that $S(f) \backslash I(f)$ is transversal to ∂M and that $I(f) \cap \partial M$ is empty. Then*

$$\begin{aligned} \deg(f)\,\chi(N) &= \chi\left(M_f^+\right) - \chi\left(M_f^-\right) + \#P_{\mathrm{Int}}^+(f) - \#P_{\mathrm{Int}}^-(f) \\ &\quad + \frac{\#(S(f) \cap \partial M)^+ - \#(S(f) \cap \partial M)^-}{2} + \frac{\#P_{\partial M_f^+} - \#P_{\partial M_f^-}}{2}, \end{aligned}$$

where the notations that appear in the above equations are introduced in Chapter 2.

In [SUY12], frontal bundles are introduced as a generalization of frontals. Frontal bundles are structures obtained by combining two types of coherent tangent bundles under a compatibility condition (see Section 6.2 in Chapter 2). They deduce four types of Gauss-Bonnet type formulae for frontal bundles over 2-manifolds without boundary and several formulae by combining these equations (see [SUY12, Theorems 3.12, 3.16, 3.18, 3.23]). For example, the formula in [SUY12, Theorem 3.12] is a generalization of the formula in [BW78], which relates the number of cusps of the Gauss mapping of a regular surface and the Euler characteristic of regions such that the Gaussian curvature is negative.

As the second main result of this paper, we derive Gauss-Bonnet type formulae for frontal bundles over 2-manifolds with boundary using the Gauss-Bonnet type formulae in [DZ20] (see Corollary 2.58), and by combining these equations, we derive the following four formulae. Note that the formulae in [SUY12] are for a wave front without boundary and its Gauss mapping, where one is an immersion and the other admits A_2-points and A_3-points (see Definitions 2.8). We generalize these formulae in [SUY12] to a frontal with boundary and its Gauss mapping, where both admit at most peaks as singular points of smooth mappings.

THEOREM C (see Theorems 2.59 and 2.60). *Let $(M, \mathcal{E}, \langle\cdot,\cdot\rangle, D, \varphi, \psi)$ be a frontal bundle over a compact oriented smooth 2-manifold M with boundary ∂M whose singular sets $S(\varphi)$ and $S(\psi)$ admit at most peaks, and suppose that $S(\varphi) \backslash I(\varphi)$ and $S(\psi) \backslash I(\psi)$ are transversal to ∂M and that $I(\varphi) \cap \partial M$ and $I(\psi) \cap \partial M$ are empty, where $I(\varphi)$ is the set* *of isolated singular points of* φ *and* $I(\psi)$ *is the set of isolated singular points of* ψ. *Then*

$$\begin{aligned}
2\chi\left(M_\psi^-\right) = \frac{1}{\pi}\Bigg(&\int_{S(\varphi)}\kappa_\varphi^s d\tau_\varphi - \int_{M_\varphi^-} K_\varphi d\widehat{A}_\varphi \\
&+\frac{1}{2}\int_{\partial M}\widetilde{\kappa}_\varphi^g d\tau_\varphi - \sum_{p\in(S(\varphi)\cap\partial M)^{\mathrm{null}}}\frac{2\alpha_\varphi^+(p)-\pi}{2}\Bigg) \\
&-\frac{1}{2\pi}\int_{\partial M}\widehat{\kappa}_\psi^g d\tau_\psi - \#P^\star(\psi) + \sum_{p\in P^\star(\psi)}\frac{\deg_{S(\psi)}(p)}{2} \\
&+\#P_{\mathrm{Int}}^+(\psi) - \#P_{\mathrm{Int}}^-(\psi) \\
&+\frac{\#(S(\psi)\cap\partial M)^+ - \#(S(\psi)\cap\partial M)^-}{2} \\
&+\frac{\#P_{\partial M_\psi^+} - \#P_{\partial M_\psi^-}}{2},
\end{aligned}$$

$$\begin{aligned}
2\chi\left(M_\varphi^-\right) = \frac{1}{\pi}\Bigg(&\int_{S(\psi)}\kappa_\psi^s d\tau_\psi - \int_{M_\psi^-} K_\psi d\widehat{A}_\psi \\
&+\frac{1}{2}\int_{\partial M}\widetilde{\kappa}_\psi^g d\tau_\psi - \sum_{p\in(S(\psi)\cap\partial M)^{\mathrm{null}}}\frac{2\alpha_\psi^+(p)-\pi}{2}\Bigg) \\
&-\frac{1}{2\pi}\int_{\partial M}\widehat{\kappa}_\varphi^g d\tau_\varphi - \#P^\star(\varphi) + \sum_{p\in P^\star(\varphi)}\frac{\deg_{S(\varphi)}(p)}{2} \\
&+\#P_{\mathrm{Int}}^+(\varphi) - \#P_{\mathrm{Int}}^-(\varphi) \\
&+\frac{\#(S(\varphi)\cap\partial M)^+ - \#(S(\varphi)\cap\partial M)^-}{2} \\
&+\frac{\#P_{\partial M_\varphi^+} - \#P_{\partial M_\varphi^-}}{2},
\end{aligned}$$

$$\begin{aligned}
&\int_{M_\varphi^-} K_\varphi d\widehat{A}_\varphi - \int_{S(\varphi)}\kappa_\varphi^s d\tau_\varphi - \frac{1}{2}\int_{\partial M}\widetilde{\kappa}_\varphi^g d\tau_\varphi + \sum_{p\in(S(\varphi)\cap\partial M)^{\mathrm{null}}}\frac{2\alpha_\varphi^+(p)-\pi}{2} \\
&= \int_{M_\psi^-} K_\psi d\widehat{A}_\psi - \int_{S(\psi)}\kappa_\psi^s d\tau_\psi - \frac{1}{2}\int_{\partial M}\widetilde{\kappa}_\psi^g d\tau_\psi + \sum_{p\in(S(\psi)\cap\partial M)^{\mathrm{null}}}\frac{2\alpha_\psi^+(p)-\pi}{2},
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial M}\widehat{\kappa}_\varphi^g d\tau_\varphi - \left(\chi\left(M_\varphi^+\right) - \chi\left(M_\varphi^-\right)\right) - \left(\#P_{\mathrm{Int}}^+(\varphi) - \#P_{\mathrm{Int}}^-(\varphi)\right) \\
&\quad - \frac{\#(S(\varphi)\cap\partial M)^+ - \#(S(\varphi)\cap\partial M)^-}{2} - \frac{\#P_{\partial M_\varphi^+} - \#P_{\partial M_\varphi^-}}{2}
\end{aligned}$$

$$= \frac{1}{2\pi}\int_{\partial M}\widehat{\kappa}^g_\psi d\tau_\psi - \left(\chi\left(M^+_\psi\right) - \chi\left(M^-_\psi\right)\right) - \left(\#P^+_{\mathrm{Int}}\left(\psi\right) - \#P^-_{\mathrm{Int}}\left(\psi\right)\right)$$
$$-\frac{\#\left(S\left(\psi\right)\cap\partial M\right)^+ - \#\left(S\left(\psi\right)\cap\partial M\right)^-}{2} - \frac{\#P_{\partial M^+_\psi} - \#P_{\partial M^-_\psi}}{2},$$

where the notations that appear in the above equations are introduced in Chapter 2.

Furthermore, we study geometry of cross caps, which are singular surfaces that are not frontals. A smooth mapping germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is a *cross cap* if f is $\mathcal{A}$-equivalent to

$$\mathrm{CR} : \left(\mathbb{R}^2, 0\right) \to \left(\mathbb{R}^3, 0\right), \quad (u, v) \mapsto \left(u, uv, v^2\right)$$

at the origin 0 (see Example 1.31). Since the Gauss mapping of CR is not extended to 0, a cross cap is not a frontal. In [Kui75, (7a)], a Gauss-Bonnet type formula is derived for compact surfaces without boundary that admit cross caps as singular points. In [HHNSUY15], intrinsic cross caps are introduced as singular points of positive semi-definite metrics, which are an intrinsic formulation of cross caps, and the following theorem is shown as a generalization of the Gauss-Bonnet type formula in [Kui75].

Theorem (Theorem 5.1 of [HHNSUY15]). *Let M be a compact oriented smooth 2-manifold without boundary and ds^2 a Whitney metric on M. Then*

$$\int_M KdA = 2\pi\chi\left(M\right).$$

As the final main result of this paper, we prove the boundedness of geodesic curvature measures of smooth regular curves through intrinsic cross caps (see Theorem 3.24). Then we prove the following assertion, which generalizes the Gauss-Bonnet type formula in [HHNSUY15] to the case of 2-manifolds with boundary.

Theorem D (see Theorem 3.29). *Let M be a compact oriented smooth 2-manifold with boundary ∂M and ds^2 a Whitney metric on M. Suppose that ∂M is transversal to the null space at each singular point of ds^2 on ∂M if the set of singular points of ds^2 on ∂M is non-empty. Then*

$$\int_M KdA + \int_{\partial M}\kappa_g d\tau = 2\pi\chi\left(M\right).$$

This paper consists as follows. Chapter 1 begins by introducing the concept of contact structures, which are essential for understanding geometry of frontals and wave fronts. Then we give concrete examples of singular points of smooth mappings between 2-manifolds and frontals. In Chapter 2, we develop differential geometry of frontals from the viewpoint of coherent tangent bundles. In particular, we delve into geometry of smooth regular curves passing through singular points of a frontal such as A_k-points and peaks. Also, we prove Gauss-Bonnet type formulae for frontals. Finally, we give applications of these formulae to smooth mappings between 2-manifolds and frontals. In particular, we prove Theorems A, B and C. Chapter 3 explores geometry of cross caps. The chapter begins by examining singular points of positive semi-definite metrics. Then we introduce intrinsic cross caps as singular points of positive semi-definite metrics and Bruce-West type coordinates which provide a framework for analyzing these singular points. Finally, we prove Theorem D.

All mappings and manifolds considered in this paper are differentiable of class C^∞ unless stated otherwise.

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CHAPTER 1

Singularities of frontals

1. Contact structures

In this section, we define contact structures on odd-dimensional manifolds in order to give a general definition of frontals and wave fronts.

Let M be a smooth $(2m+1)$-manifold, where m is a non-negative integer. A *hyperplane field* K is a correspondence from each point $p \in M$ to a $2m$-dimensional subspace K_p of the tangent space T_pM at p:

$$K : M \ni p \mapsto K_p \subseteq T_pM \quad \text{and} \quad \dim K_p = 2m.$$

A vector field X defined on an open subset $U \subseteq M$ is said to *belong to* K if the tangent vector $X_p \in T_pM$ belongs to K_p for each point $p \in U$. The hyperplane field K is *smooth* if for each point $p \in M$ there exist linearly independent vector fields $X_1, \ldots, X_{2m}$ defined on a neighborhood of p such that span$\{X_1, \ldots, X_{2m}\} = K$. Such a family $\{X_1, \ldots, X_{2m}\}$ is called a (*local*) *frame field* of K. We give a typical example of smooth hyperplane fields.

Example 1.1 (cf. Example 10.2.1 of [USY22]). Let θ be a non-zero 1-form on a smooth $(2m+1)$-manifold M. We show that

$$K : M \ni p \mapsto K_p \subseteq T_pM, \quad K_p := \ker \theta_p$$

is a smooth hyperplane field on M. We take a Riemannian metric g on M and a coordinate chart $(U; x^1, \ldots, x^{2m+1})$ centered at an arbitrary fixed point $p \in M$. If U is sufficiently small, there exists a vector field X without zeros on U such that $\theta(Y) = g(X, Y)$ for any vector field Y on U. In fact, suppose that Y, θ and g are written as

$$Y = Y^i \frac{\partial}{\partial x^i}, \quad \theta = \frac{\partial \theta}{\partial x^i} dx^i \quad \text{and} \quad g = g_{ij} dx^i dx^j,$$

respectively, where we use the Einstein summation convention. Then

$$X = \frac{\partial \theta}{\partial x^j} g^{ji} \frac{\partial}{\partial x^i}$$

is the desired vector field, where (g^{ij}) is the inverse of (g_{ij}). Note that (g^{ij}) exists because (g_{ij}) is regular. By choosing the order of $x^1, \ldots, x^{2m+1}$ if necessary, we may assume that $X, \partial/\partial x^2, \ldots, \partial/\partial x^{2m+1}$ are linearly independent on U. Applying Gram-Schmidt orthogonalization to these vector fields, we obtain an orthonormal frame field $\{X/|X|, e_2, \ldots, e_{2m+1}\}$ on U such that span$\left\{ e_2|_p, \ldots, e_{2m+1}|_p \right\} = K_p$ for each point $p \in U$, where $|X| = \sqrt{g(X, X)}$. Thus K is a smooth hyperplane field of M.

Let K be a smooth hyperplane field on a smooth $(2m+1)$-manifold M. A 1-form θ defined on an open subset $U \subseteq M$ is said to be *associated to* K if $K_p = \ker \theta_p$ for each point $p \in U$ (cf. Example 1.1). If such θ exists, θ is determined up to a non-zero functional multiple. The following assertion holds.

PROPOSITION 1.2 (cf. Proposition 10.2.3 of [USY22]). *Let K be a smooth hyperplane field on a smooth $(2m+1)$-manifold M. Then for each point $p \in M$ there exists a 1-form associated to K defined on a neighborhood of p.*

PROOF. For a frame field $\{X_1, \ldots, X_{2m}\}$ of K defined on an open subset $U \subseteq M$, we take a vector field Y defined on U such that $\operatorname{span}\left\{ X_1|_q, \ldots, X_{2m}|_q, Y_q \right\} = T_qM$ for each $q \in U$. We set $X_{2m+1} = Y$. Let $\xi^1, \ldots, \xi^{2m+1}$ be 1-forms defined on U such that

$$\xi^j(X_k) = \delta_k^j := \begin{cases} 1 & j = k, \\ 0 & j \neq k, \end{cases}$$

where δ_k^j is the Kronecker delta. Then $\theta = \xi^{2m+1}$ gives a 1-form associated to K on U. □

Now, we define a contact structure on an odd-dimensional manifold. A smooth hyperplane field K on a smooth $(2m+1)$-manifold M is called a *contact structure* (or a *contact hyperplane field*) if for each point $p \in M$ there exists a 1-form θ associated to K defined on a neighborhood of p satisfying

$$\theta \wedge \left(\bigwedge^m d\theta \right) \neq 0, \tag{1.1.1}$$

where $\bigwedge^m d\theta$ is the m-th exterior power of the exterior derivative $d\theta$ of θ. In this situation, the pair (M, K) or simply M is called a *contact manifold* and θ is called a (*local*) *contact form* on M. The condition (1.1.1) does not change by non-zero functional multiplication of θ. The following assertion holds.

LEMMA 1.3 (cf. Lemma 10.2.6 of [USY22]). *Let (M, K) be a contact manifold of dimension $(2m+1)$ and θ a contact form defined on M. Then for each point $p \in M$ the map $d\theta_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate alternating covariant 2-tensor on K_p.*

PROOF. We fix a point $p \in M$ arbitrarily. If $d\theta_p$ is degenerate on K_p, there exists a non-zero vector $w \in K_p$ such that $d\theta_p(v, w) = 0$ for any $v \in K_p$. We take vectors $u, v_1, \ldots, v_{2m-1} \in T_pM$ such that $u \notin K_p$, $v_1, \ldots, v_{2m-1} \in K_p$ and $\{u, v_1, \ldots, v_{2m-1}, w\}$ is a basis of T_pM. Since $\theta_p(v_1) = \cdots = \theta_p(v_{2m-1}) = \theta_p(w) = 0$, we have

$$\left(\theta \wedge \left(\bigwedge^m d\theta \right) \right)_p (u, v_1, \ldots, v_{2m-1}, w) = \theta_p(u) \cdot \left(\bigwedge^m d\theta \right)_p (v_1, \ldots, v_{2m-1}, w) = 0,$$

which means that $\left(\theta \wedge \left(\bigwedge^m d\theta\right)\right)_p = 0$. This contradicts the fact that M is a contact manifold. □

We give some concrete examples of contact structures.

EXAMPLE 1.4 (cf. Proposition 10.2.5 of [USY22]). Let θ be a 1-form defined on a smooth $(2m+1)$-manifold M satisfing the condition (1.1.1). If we set

$$K : M \ni p \mapsto K_p \subseteq T_pM, \quad K_p = \ker \theta_p,$$

the pair (M, K) is a contact manifold because K is a smooth hyperplane field on M by Example 1.1.

EXAMPLE 1.5 (cf. Example 10.2.7 of [USY22]). Let $(x, y, z) = (x^1, \ldots, x^m, y^1, \ldots, y^m, z)$ be coordinates of $\mathbb{R}^{2m+1}$ and set

$$\theta = dz + \sum_{i=1}^{m} y^i dx^i.$$

Then we have

$$\theta \wedge \left(\bigwedge^{m} d\theta \right) = m! dz \wedge dy^1 \wedge dx^1 \wedge \cdots \wedge dy^m \wedge dx^m,$$

which does not vanish at each point of $\mathbb{R}^{2m+1}$. Since $\ker \theta_{(x,z)} = \left\{ a^i \, E_i|_{(x,z)} \mid a^i \in \mathbb{R} \right\}$, where $E_i := a^i \partial/\partial x^i - y^i \partial/\partial z$, $i = 1, \ldots, m$, a smooth hyperplane field K on $\mathbb{R}^{2m+1}$ is given by

$$K_{(x,y,z)} := d \left(\pi_{\mathbb{R}^{2m+1}} \right)^{-1}_{(x,y,z)} \left(\ker \theta_{(x,z)} \right) = \left\{ a^i \, E_i|_{(x,y,z)} + b^i \, \frac{\partial}{\partial y^i}\bigg|_{(x,y,z)} \mid a^i, b^i \in \mathbb{R} \right\},$$

where $\pi_{\mathbb{R}^{2m+1}} : \mathbb{R}^{2m+1} \to \mathbb{R}^{m+1}$, $(x, y, z) \mapsto (x, z)$ is a canonical projection. Thus $(\mathbb{R}^{2m+1}, K)$ is a contact manifold and θ is a contact form on $\mathbb{R}^{2m+1}$.

EXAMPLE 1.6 (cf. Proposition 10.2.9 of [USY22]). Let M be a smooth $(m+1)$-manifold and $PT^*M := \coprod_{p \in M} PT_p^*M$ the projective cotangent bundle on M, where PT_p^*M is the projective space associated with the cotangent space T_p^*M at a point $p \in M$. Then PT^*M is a smooth $(2m+1)$-manifold and the canonical projection $\pi : T^*N \backslash \{0\} \to PT^*N$, $(p, \eta) \mapsto (p, [\eta])$ is a submersion. We show that a smooth hyperplane field K on PT^*M is given by

$$K_{(p,[\eta])} := d \left(\pi_{PT^*M} \right)^{-1}_{(p,[\eta])} \left(\ker \eta \right), \tag{1.1.2}$$

where $\pi_{PT^*M} : PT^*M \to M$, $(p, [\eta]) \mapsto p$ is the canonical projection (see Figure 1.1). We fix $(p, [\eta]) \in PT^*M$ arbitrarily. We take a coordinate chart $(U; x^0, x^1, \ldots, x^m)$ centered at p such that $\eta = dx^0|_p$. Writing $(x, y) = (x^0, x^1, \ldots, x^m, y^1, \ldots, y^m)$, we define a mapping $\varphi : U \times \mathbb{R}^m \to PT^*M|_U$ by

$$\varphi(x, y) = (x, [\xi]), \quad \xi := dx^0\big|_x + \sum_{i=1}^{m} y^i \, dx^i\big|_x.$$

Then we may regard (x, y) as local coordinates of PT^*M centered at $(p, [\eta])$ because φ is a diffeomorphism. Using these coordinates, a 1-form θ on $PT^*M|_U$ defined by

$$\theta = dx^0 + \sum_{i=1}^{m} y^i dx^i$$

satisfies the condition (1.1.1) (see Example 1.5). Since $\ker \xi = \{a^i \, X_i|_x \mid a^i \in \mathbb{R}\}$, where $X_i := -y^i \partial/\partial x^0 + \partial/\partial x^i,\ i = 1, \ldots, m$, we have

$$K_{(x,[\xi])} = d\left(\pi_{PT^*M}\right)^{-1}_{(x,[\xi])}(\ker \xi) = \left\{ a^i \, X_i|_{(x,[\xi])} + b^i \left.\frac{\partial}{\partial y^i}\right|_{(x,[\xi])} \mid a^i, b^i \in \mathbb{R} \right\}.$$

Then K is a smooth hyperplane field on PT^*M because $K_{(x,[\xi])}$ coincides with $\ker \theta_{(x,[\xi])}$ at each $(x, [\xi]) \in PT^*M$. Thus (PT^*M, K) is a contact manifold and θ is a contact form on PT^*M.

$$\begin{array}{ccc} T^*M \setminus \{0\} & \xrightarrow{\pi} & PT^*M \\ & & \downarrow{\scriptstyle \pi_{PT^*M}} \\ & & M \end{array} \qquad \begin{array}{ccc} T(T^*M) & \xrightarrow{d\pi} & T(PT^*M) \\ & & \downarrow{\scriptstyle d\pi_{PT^*M}} \\ & & TM \end{array}$$

FIGURE 1.1.

EXAMPLE 1.7 (cf. Example 10.2.10 of [USY22]). We fix a Riemannian metric g of a smooth $(2m+1)$-manifold M. Then the *unit cotangent bundle* U^*M is defined by

$$U^*M = \{(p, \eta) \mid \eta = g_p(v, \cdot),\ v \in U_pM\},$$

where U_pM is the fiber of the *unit tangent bundle* $UM := \{(p, v) \mid p \in M,\ v \in T_pM,\ |v| = 1\}$ at a point $p \in M$. The canonical projection $\Phi : U^*M \to PT^*M,\ (p, \eta) \mapsto (p, [\eta])$ is a two-to-one immersion. If K is the contact structure on PT^*M defined by (1.1.2), a smooth hyperplane field $\widetilde{K}$ on U^*M is given by

$$\widetilde{K}_{(p,\eta)} := d\Phi^{-1}_{(p,\eta)}\left(K_{(p,[\eta])}\right) \tag{1.1.3}$$

(see Figure 1.2). If θ is a 1-form associated to K on an open subset $W \subseteq PT^*M$, the pull-back $\Phi^*\theta$ is a 1-form associated to $\widetilde{K}$ on $\Phi^{-1}(W)$. Since θ satisfies the condition (1.1.1), $\Phi^*\theta$ is a contact form on U^*M. Thus $\widetilde{K}$ is a contact structure on U^*M, that is, $\left(U^*M, \widetilde{K}\right)$ is a contact manifold.

We show that there is a contact form associated with $\widetilde{K}$ globally defined on U^*M. We define a 1-form Θ on T^*M by $\Theta_{(p,\eta)} = (\pi^*_{T^*M}\eta)_{(p,\eta)}$ at each $(p, \eta) \in T^*M$, where $\pi_{T^*M} : T^*M \to M,\ (p, \eta) \mapsto p$ is the canonical projection. We fix a point $p \in M$ arbitrarily and take a coordinate chart $(U; x^0, \ldots, x^m)$ of M centered at p. Since each $(q, \xi) \in T^*M|_U$ is written as $(q, \xi) = \left(q, \xi_i \, dx^i|_q\right)$, $\xi_i \in \mathbb{R}$, a coordinate chart of T^*M is given by $\left(T^*M|_U; x^0, \ldots, x^m, \xi_0, \ldots, \xi_m\right)$. Using this coordinate chart, we write $\Theta = \xi_i dx^i$. Let $\widetilde{\theta}$ be the restriction of Θ to U^*M. Then $\widetilde{\theta}_{(q,\xi)} = \left(\Phi^* \pi^*_{PT^*M} \xi\right)_{(q,\xi)}$ for each $(q,\xi) \in U^*M$. By (1.1.2) and (1.1.3), we have

$$\begin{aligned} \ker \widetilde{\theta}_{(q,\xi)} &= \ker \left(\Phi^* \pi^*_{PT^*M} \xi\right)_{(q,\xi)} = d\Phi^{-1}_{(q,\xi)} \left(d \left(\pi_{PT^*M}\right)^{-1}_{(q,[\xi])} (\ker \xi) \right) \\ &= d\Phi^{-1}_{(q,\xi)} \left(K_{(q,[\xi])} \right) = \widetilde{K}_{(q,\xi)}. \end{aligned}$$

So $\widetilde{\theta}$ is a 1-form associated to $\widetilde{K}$. Since $\left(U^*M, \widetilde{K}\right)$ is a contact manifold, $\widetilde{\theta}$ satisfies the condition (1.1.1). In particular, $\widetilde{\theta}$ is a contact form globally defined on U^*M, which is called the *canonical contact form* on U^*M.

$$\begin{tikzcd} U^*M \arrow[r, "\Phi"] & PT^*M \arrow[d, "\pi_{PT^*M}"] \\ & M \end{tikzcd} \qquad \begin{tikzcd} T\left(U^*M\right) \arrow[r, "d\Phi"] & T\left(PT^*M\right) \arrow[d, "d\pi_{PT^*M}"] \\ & TM \end{tikzcd}$$

FIGURE 1.2.

An immersion (respectively, a diffeomorphism) $f : M \to M'$ between two same dimensional contact manifolds (M, K) and (M', K') is called a *local contact diffeomorphism* (respectively, *contact diffeomorphism*) if $df_p\left(K_p\right) = K'_{f(p)}$ for each point $p \in M$. We give a typical example of (local) contact diffeomorphism.

EXAMPLE 1.8 (cf. Proposition 10.2.12 of [USY22]). The canonical projection $\Phi : U^*M \to PT^*M$ is a local contact diffeomorphism by (1.1.3). Moreover, if M is connected, Φ is a double covering mapping.

The following assertion holds.

THEOREM 1.9 (cf. Theorem 10.2.13 of [USY22]). *Let M be a smooth $(2m+1)$-manifold and $\psi : M \to M$ a diffeomorphism. Then the mapping $\Psi : PT^*M \to PT^*M$ defined by*

$$\Psi\left(p, [\eta]\right) = \left(\psi(p), \left[\left(\psi^{-1}\right)^* \eta\right]\right)$$

is a contact diffeomorphism satisfying $\pi_{PT^*M} \circ \Psi = \psi \circ \pi_{PT^*M}$ (see Figure 1.3).

$$\begin{tikzcd} PT^*M \arrow[r, "\Psi"] \arrow[d, "\pi_{PT^*M}"'] & PT^*M \arrow[d, "\pi_{PT^*M}"] \\ M \arrow[r, "\psi"'] & M \end{tikzcd} \qquad \begin{tikzcd} T\left(PT^*M\right) \arrow[r, "d\Psi"] \arrow[d, "d\pi_{PT^*M}"'] & T\left(PT^*M\right) \arrow[d, "d\pi_{PT^*M}"] \\ TM \arrow[r, "d\psi"'] & TM \end{tikzcd}$$

FIGURE 1.3.

Proof. Since $\left(\psi^{-1}\right)^* \eta \in T^*_{\psi(p)}M$ for any $\eta \in T^*_p M$, Ψ is a diffeomorphism satisfying $\pi_{PT^*M} \circ \Psi = \psi \circ \pi_{PT^*M}$.

Let K be the contact structure on PT^*M defined by (1.1.2). We fix $(p, [\eta]) \in PT^*M$ arbitrarily. Then we have

$$\begin{aligned}
& d\left(\pi_{PT^*M}\right)_{\left(\psi(p),\left[\left(\psi^{-1}\right)^*\eta\right]\right)} \left(d\Psi_{(p,[\eta])}\left(K_{(p,[\eta])}\right)\right) \\
= \; & d\left(\pi_{PT^*M} \circ \Psi\right)_{(p,[\eta])} \left(d\left(\pi_{PT^*M}\right)^{-1}_{(p,[\eta])}(\ker \eta)\right) \\
= \; & d\left(\psi \circ \pi_{PT^*M}\right)_{(p,[\eta])} \left(d\left(\pi_{PT^*M}\right)^{-1}_{(p,[\eta])}(\ker \eta)\right) \\
= \; & d\psi_p \left(d\left(\pi_{PT^*M}\right)_{(p,[\eta])} \left(d\left(\pi_{PT^*M}\right)^{-1}_{(p,[\eta])}(\ker \eta)\right)\right) \\
= \; & d\psi_p(\ker \eta) = \ker\left(\left(\psi^{-1}\right)^* \eta\right)_{\psi(p)}.
\end{aligned}$$

So we have

$$d\Psi_{(p,[\eta])}\left(K_{(p,[\eta])}\right) \subseteq d\left(\pi_{PT^*M}\right)^{-1}_{\left(\psi(p),\left[\left(\psi^{-1}\right)^*\eta\right]\right)} \left(\ker\left(\left(\psi^{-1}\right)^* \eta\right)_{\psi(p)}\right) = K_{\left(\psi(p),\left[\left(\psi^{-1}\right)^*\eta\right]\right)},$$

which implies that Ψ is a contact diffeomorphism. □

2. Frontals and wave fronts

In this section, we define frontals and wave fronts in terms of projective cotangent bundles and unit cotangent bundles.

Let M be a smooth manifold and (N, K) a contact manifold. A smooth mapping $\varphi : M \to N$ is called *isotropic* if

$$d\varphi_p(T_p M) \subseteq K_{\varphi(p)} \tag{1.2.1}$$

for each point $p \in M$. If θ is a contact form on N, that is, $K_q = \ker \theta_q$ for each point $q \in N$, then the condition (1.2.1) holds if and only if the pull-back $\varphi^*\theta$ vanishes identically. The following assertion holds.

Proposition 1.10 (cf. Proposition 10.3.2 of [USY22]). *If $\varphi : M \to N$ is an isotropic map into a contact manifold N of dimension* $(2m + 1)$, $\dim M \leq m$ *holds.* (Such φ is called *Legendrian* when $\dim M = m$.)

Proof. Let θ be a contact form on a contact manifold (N, K). We fix a point $p \in M$ arbitrarily. Since $\varphi^*\theta$ vanishes identically,

$$0 = d\left(\varphi^*\theta\right)_p = \varphi^*\left(d\theta\right)_p,$$

which yields $d\theta_{\varphi(p)}(v, v) = 0$ for each $v \in d\varphi_p(T_p M)$. By Lemma 1.3, the restriction of $d\theta_{\varphi(p)}$ to $K_{\varphi(p)}$ is a non-degenerate alternating covariant 2-tensor, which yields

$$\dim T_p M \leq \frac{\dim K_{\varphi(p)}}{2} = m.$$

□

Now, we define frontals and wave fronts. Let M be a smooth m-manifold and N a smooth $(m+1)$-manifold. A smooth mapping $f : M \to N$ is called a *frontal* (respectively, *wave front*) if there exists an isotropic mapping (respectively, a Legendrian immersion) $L : M \to PT^*N$ satisfying $f = \pi_{PT^*N} \circ L$. Such L is called the *isotropic lift* (respectively, *Legendrian lift*) of f to PT^*N (see Figure 2.1). Also, f is called a *co-orientable frontal* (respectively, *co-orientable wave front*) if there exists an isotropic mapping (respectively, a Legendrian immersion) $L : M \to U^*N$ satisfying $f = \pi_{U^*N} \circ L$. Such L is called the *isotropic lift* (respectively, *Legendrian lift*) of f to U^*N, where $\pi_{U^*N} : U^*N \to N,\ (p, \eta) \mapsto p$ is the canonical projection (see Figure 2.2).

$$
\begin{tikzcd}
 & PT^*N \arrow[d, "\pi_{PT^*N}"] \\
M \arrow[ur, "L"] \arrow[r, "f"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
 & T(PT^*N) \arrow[d, "d\pi_{PT^*N}"] \\
TM \arrow[ur, "dL"] \arrow[r, "df"'] & TN
\end{tikzcd}
$$

FIGURE 2.1.

$$
\begin{tikzcd}
 & U^*N \arrow[d, "\pi_{U^*N}"] \\
M \arrow[ur, "L"] \arrow[r, "f"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
 & T(U^*N) \arrow[d, "d\pi_{U^*N}"] \\
TM \arrow[ur, "dL"] \arrow[r, "df"'] & TN
\end{tikzcd}
$$

FIGURE 2.2.

REMARK 1.11. In [**Ish18**, Definition 4.1], a further generalized definition of frontal defined above is introduced in terms of Grassmannians. In this sense, smooth mappings between smooth equi-dimensional manifolds are frontals (See [**Ish18**, Remark 4.2].)

Let $f : M \to N$ be a co-orientable frontal (respectively, co-orientable wave front) and $\Phi : U^*N \to PT^*N,\ (p, \eta) \mapsto (p, [\eta])$ the projection defined in Example 1.7. Then $\Phi \circ L : M \to PT^*N$ is a lift of f to PT^*N, which yields that f is a frontal (respectively, wave front). A frontal (or a wave front) is said to be *non-co-orientable* when it is not co-orientable.

REMARK 1.12 (cf. Remark 10.3.9 of [**USY22**]). If M is simply connected, any frontal $f : M \to N$ is co-orientable. If M is not simply connected and f is non-co-orientable, there exists a double covering $\widehat{M}$ of M such that f is lifted to a co-orientable frontal $\widehat{M} \to N$.

The following assertions hold by the definitions of frontals and wave fronts.

PROPOSITION 1.13 (cf. Proposition 10.3.5 of [**USY22**]). *A wave front from a smooth m-manifold M into a smooth $(m+1)$-manifold N is a frontal. Also, an immersion from M into N is a wave front.*

Proof. By the definitions of frontals and wave fronts. □

Proposition 1.14 (cf. Proposition 10.3.6 of [USY22]). *Let $f : M \to N$ be a frontal (respectively, wave front), and $\psi : M \to M$ and $\psi' : N \to N$ two diffeomorphisms. Then $\psi' \circ f \circ \psi$ is also a frontal (respectively, wave front).*

Proof. By the definitions of frontals and wave fronts, and Theorem 1.9 (see Figure 2.3). □

$$
\begin{tikzcd}
& & PT^*N \arrow[r, "\Psi'"] \arrow[d, "\pi_{PT^*N}"] & PT^*N \arrow[d, "\pi_{PT^*N}"] \\
M \arrow[r, "\psi"'] & M \arrow[r, "f"'] \arrow[ur, "L"] & N \arrow[r, "\psi'"'] & N
\end{tikzcd}
$$

$$
\begin{tikzcd}
& & T\left(PT^*N\right) \arrow[r, "d\Psi'"] \arrow[d, "d\pi_{PT^*N}"] & T\left(PT^*N\right) \arrow[d, "d\pi_{PT^*N}"] \\
TM \arrow[r, "d\psi"'] & TM \arrow[r, "df"'] \arrow[ur, "dL"] & TN \arrow[r, "d\psi'"'] & TN
\end{tikzcd}
$$

Figure 2.3.

Furthermore, if the target manifold is Riemannian, we obtain the following characterization of frontals and wave fronts.

Proposition 1.15 (cf. Proposition 10.3.10 of [USY22]). *Let M be a smooth m-manifold and (N, g) a Riemannian manifold of dimension $(m+1)$. A smooth mapping $f : M \to N$ is a frontal (respectively, wave front) if and only if for each point $p \in M$ there exist a neighborhood U_p of p and a smooth mapping (respectively, an immersion) $\nu_p : U_p \to UN$ satisfying*

1. *$\pi_{UN} \circ \nu_p (q) = f(q)$ for each point $q \in U_p$, where $\pi_{UN} : UN \to N,\ (q, v) \mapsto q$ is the canonical projection,*
2. *$g_{f(q)} \left(df_q (v), \nu_p (q)\right) = 0$ for each $(q, v) \in TM|_{U_p}$,*
3. *if $U_p \cap U_{p'} \neq \varnothing,\ p, p' \in M$, we have $\nu_p (q) = \nu_{p'} (q)$ or $\nu_p (q) = -\nu_{p'} (q)$ for each point $q \in U_p \cap U_{p'}$.*

(Such ν_p is called the *unit normal vector field* of f defined on U_p.) *In particular, a frontal (or a wave front) f is co-orientable if and only if there exists a unit normal vector field globally defined on M.*

Proof. We identify UN with U^*N by a smooth bundle isomorphism

$$\flat : UN \to U^*N, \quad (p, v) \mapsto (p, g_p (v, \cdot))$$

over M. Since U^*N has a canonical contact structure (see Example 1.7), UN has a contact structure induced by $\flat$.

Suppose that $f : M \to N$ is a frontal (respectively, wave front). Then there exists an isotropic mapping (respectively, a Legendrian immersion) $L : M \to PT^*N$ satisfying $\pi_{PT^*N} \circ L = f$. Since $\Phi : U^*N \to PT^*N,\ (q,\eta) \mapsto (q,[\eta])$ is a local contact diffeomorphism (see Example 1.8), there exist a neighborhood V of $f(p)$ and a local contact diffeomorphism $\Psi := \Phi^{-1} : PT^*N|_V \to U^*N$. By the continuous of f, we take a neighborhood U_p of p satisfying $f(U_p) \subseteq V$. We set $\widetilde{L} = \Psi \circ L : U_p \to U^*N$ and $\nu_p = \sharp \circ \widetilde{L} : U_p \to UN$, where $\sharp := \flat^{-1} : U^*N \to UN$. Then $\flat \circ \nu_p = \widetilde{L}$ and $\pi_{UN} \circ \nu_p(q) = \pi_{U^*N} \circ \widetilde{L}(q) = f(q)$ for each point $q \in U_p$. Let $\widetilde{\theta}$ be the canonical contact form on U^*N (see Example 1.7). Since $\widetilde{L}$ is isotropic (respectively, Legendrian), for each $(q,v) \in TM|_{U_p}$

$$\begin{aligned} 0 &= \left(\widetilde{L}^*\widetilde{\theta}\right)_q(v) = \widetilde{\theta}_{\widetilde{L}(q)}\left(d\widetilde{L}_q(v)\right) = \widetilde{L}_{f(q)}\left(d\left(\pi_{U^*N}\right)_{\widetilde{L}(q)}\left(d\widetilde{L}_q(v)\right)\right) \\ &= \widetilde{L}_{f(q)}\left(d\left(\pi_{U^*N} \circ \widetilde{L}\right)_q(v)\right) = \widetilde{L}_{f(q)}\left(df_q(v)\right) = \left(\flat \circ \nu_p\right)_{f(q)}\left(df_q(v)\right) \\ &= g_{f(q)}\left(\nu_p(q), df_q(v)\right), \end{aligned}$$

which means that ν_p satisfies the condition (2). Furthermore, since the local contact diffeomorphism $\Psi : PT^*N \to U^*N$ satisfying $\widetilde{L} = \Psi \circ L$ only has sign indeterminacy, the condition (3) holds.

Conversely, suppose that for each point $p \in M$ there exist a neighborhood U_p of p and a smooth mapping (respectively, an immersion) $\nu_p : U_p \to UN$ satisfying the conditions (1), (2) and (3). We set $\widetilde{L} = \flat \circ \nu_p : U_p \to U^*N$ and $L = \Phi \circ \widetilde{L} : U_p \to PT^*N$. Since L is independent of the choice of ν_p by the condition (3), L is smoothly extended to M. By the definition, $\pi_{U^*N} \circ \widetilde{L} = \pi_{PT^*N} \circ L = f$. By the above computations from the opposite direction and the condition (2), for each $(q,v) \in TM|_{U_p}$

$$\left(\widetilde{L}^*\widetilde{\theta}\right)_q(v) = g_{f(q)}\left(\nu_p(q), df_q(v)\right) = 0,$$

which means that $\widetilde{L}$ is isotropic (respectively, Legendrian), so L is isotropic (respectively, Legendrian). Thus f is a frontal (respectively, wave front). □

Corollary 1.16 (cf. Corollary 10.3.11 of [USY22]). *Let M be a smooth m-manifold. Then a smooth mapping $f : M \to \mathbb{R}^{m+1}$ into the Euclidean space $\mathbb{R}^{m+1}$ is a frontal (respectively, wave front) if and only if for each point $p \in M$ there exist a neighborhood U_p of p and a smooth mapping (respectively, an immersion) $(f, n_p) : U_p \to \mathbb{R}^{m+1} \times \mathbb{S}^m$, where $\mathbb{S}^m$ is the unit m-sphere, satisfying*

(1) *$\langle df_q(v), n_p(q)\rangle = 0$ for each $(q,v) \in TM|_{U_p}$, where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product,*
(2) *if $U_p \cap U_{p'} \neq \varnothing$, $p, p' \in M$, we have $n_p(q) = n_{p'}(q)$ or $n_p(q) = -n_{p'}(q)$ for each point $q \in U_p \cap U_{p'}$.*

(Such n_p is called the *Gauss mapping* of f defined on U_p.) *In particular, a frontal (or a wave front) f is co-orientable if and only if there exists the Gauss mapping globally defined on M.*

Proof. If $N = \mathbb{R}^{m+1}$ in Proposition 1.15, $U\mathbb{R}^{m+1}$ is identified with $\mathbb{R}^{m+1} \times \mathbb{S}^m$ and $\nu_p = (f, n_p)$. □

3. Examples of singular points of smooth mappings

In this section, we define regular points and singular points of smooth mappings, and give several typical examples.

Let M be a smooth manifold (possibly with boundary ∂M) and N a smooth manifold. A point $p \in M$ is called a *regular point* of a smooth mapping $f : M \to N$ if $\operatorname{rank}(df_p) = \min\{\dim M, \dim N\}$. On the other hand, p is called a *singular point* of f if $\operatorname{rank}(df_p) < \min\{\dim M, \dim N\}$. Note that if $p \in \partial M$, then f is extended to a C^∞-mapping beyond ∂M via a compatible coordinate neighborhood of p, and the above notions are well-defined independently of the choice of a compatible coordinate neighborhood of p. We denote by $S(f)$ the singular set of f.

3.1. Singular points of smooth mappings between 2-manifolds. We give some concrete examples of singular points of smooth mappings between 2-manifolds. Note that smooth mappings between 2-manifolds are frontals in the sense of [Ish18] (see Remark 1.11).

Before discussing specific types of singular points, we recall the $\mathcal{A}$-equivalence, which is fundamental for classifying singular points of smooth mappings. Two smooth mapping germs $f_1 : (\mathbb{R}^m, x_1) \to (\mathbb{R}^n, y_1)$ and $f_2 : (\mathbb{R}^m, x_2) \to (\mathbb{R}^n, y_2)$ are said to be *$\mathcal{A}$-equivalent* (or *right-left equivalent*) if there exist two diffeomorphism germs $\varphi : (\mathbb{R}^m, x_1) \to (\mathbb{R}^m, x_2)$ and $\Phi : (\mathbb{R}^n, y_1) \to (\mathbb{R}^n, y_2)$ satisfying $f_2 \circ \varphi = \Phi \circ f_1$. In other words, f_1 and f_2 are equivalent under coordinate changes in both the domain and the target.

$$\begin{tikzcd}
(\mathbb{R}^m, x_1) \arrow[r, "f_1"] \arrow[d, "\varphi"'] & (\mathbb{R}^n, y_1) \arrow[d, "\Phi"] \\
(\mathbb{R}^m, x_2) \arrow[r, "f_2"'] & (\mathbb{R}^n, y_2)
\end{tikzcd}$$

Example 1.17 (cf. 1:1 Theorem of [Rie87]). A smooth mapping germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is called a *fold* if f is $\mathcal{A}$-equivalent to

$$\mathrm{FO} : \left(\mathbb{R}^2, 0\right) \to \left(\mathbb{R}^2, 0\right), \quad (u, v) \mapsto \left(u, v^2\right)$$

at the origin 0. Then $S(\mathrm{FO}) = \{v = 0\}$ (see Figure 3.1).

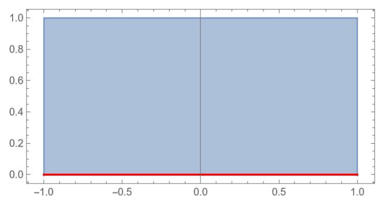

Figure 3.1. The red line represents the image of $S(\mathrm{FO})$.

EXAMPLE 1.18 (cf. 1:1 Theorem of [Rie87]). A smooth mapping germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is called a *cusp* (or a *pleat*) if f is $\mathcal{A}$-equivalent to

$$\mathrm{CU} : \left(\mathbb{R}^2, 0\right) \to \left(\mathbb{R}^2, 0\right), \quad (u, v) \mapsto \left(u, uv + v^3\right)$$

at 0. Then $S\,(\mathrm{CU}) = \{u + 3v^2 = 0\}$ (see Figure 3.2).

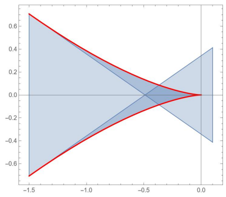

FIGURE 3.2. The red line represents the image of $S\,(\mathrm{CU})$.

EXAMPLE 1.19 (cf. 1:1 Theorem of [Rie87]). A smooth mapping germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is called a *lips* if f is $\mathcal{A}$-equivalent to

$$\mathrm{LI} : \left(\mathbb{R}^2, 0\right) \to \left(\mathbb{R}^2, 0\right), \quad (u, v) \mapsto \left(u, v^3 + u^2 v\right)$$

at 0. Then $S\,(\mathrm{LI}) = \{0\}$ (see Figure 3.3).

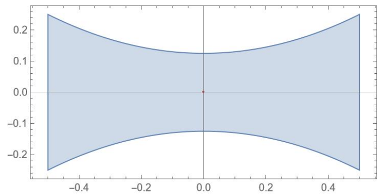

FIGURE 3.3. The red point represents the image of $S\,(\mathrm{LI})$.

EXAMPLE 1.20 (cf. 1:1 Theorem of [Rie87]). A smooth mapping germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ is called a *beaks* if f is $\mathcal{A}$-equivalent to

$$\mathrm{BE} : \left(\mathbb{R}^2, 0\right) \to \left(\mathbb{R}^2, 0\right), \quad (u, v) \mapsto \left(u, v^3 - u^2 v\right)$$

at 0. Then $S\,(\mathrm{BE}) = \{3v^2 - u^2 = 0\}$ (see Figure 3.4).

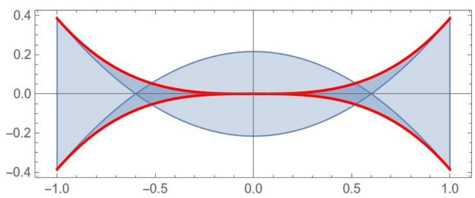

FIGURE 3.4. The red line represents the image of $S\,(\mathrm{BE})$.

Example 1.21 (cf. 1:1 Theorem of [Rie87]). A smooth mapping germ $f:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ is called a *Goose* if f is $\mathcal{A}$-equivalent to

$$\mathrm{GO}^+:\left(\mathbb{R}^2,0\right)\to\left(\mathbb{R}^2,0\right),\quad (u,v)\mapsto\left(u,v^3+u^3v\right),\ \text{or}$$
$$\mathrm{GO}^-:\left(\mathbb{R}^2,0\right)\to\left(\mathbb{R}^2,0\right),\quad (u,v)\mapsto\left(u,v^3-u^3v\right)$$

at 0. Then $S\left(\mathrm{GO}^+\right)=\{u^3+3v^2=0\}$ and $S\left(\mathrm{GO}^-\right)=\{u^3-3v^2=0\}$ (see Figure 3.5).

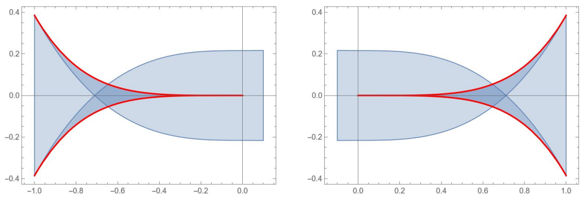

Figure 3.5. From left to right, the red lines represent the images of $S\left(\mathrm{GO}^+\right)$ and $S\left(\mathrm{GO}^-\right)$, respectively.

Example 1.22 (cf. 1:1 Theorem of [Rie87]). A smooth mapping germ $f:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ is called a *swallowtail* if f is $\mathcal{A}$-equivalent to

$$\mathrm{SW}:\left(\mathbb{R}^2,0\right)\to\left(\mathbb{R}^2,0\right),\quad (u,v)\mapsto\left(u,uv+v^4\right)$$

at 0. Then $S(\mathrm{SW})=\{u+4v^3=0\}$ (see Figure 3.6).

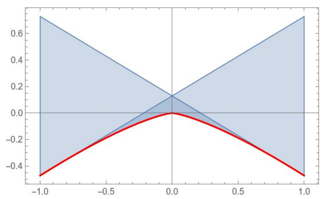

Figure 3.6. The red line represents the image of $S(\mathrm{SW})$.

Example 1.23 (cf. 1:1 Theorem of [Rie87]). A smooth mapping germ $f:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ is called a *butterfly* if f is $\mathcal{A}$-equivalent to

$$\mathrm{BU}^+:\left(\mathbb{R}^2,0\right)\to\left(\mathbb{R}^2,0\right),\quad (u,v)\mapsto\left(u,uv+v^5+v^7\right),\ \text{or}$$
$$\mathrm{BU}^-:\left(\mathbb{R}^2,0\right)\to\left(\mathbb{R}^2,0\right),\quad (u,v)\mapsto\left(u,uv+v^5-v^7\right)$$

at 0. Then $S\left(\mathrm{BU}^+\right)=\{u+7v^6+5v^4=0\}$ and $S\left(\mathrm{BU}^-\right)=\{u-7v^6+5v^4=0\}$ (see Figure 3.7).

Example 1.24 (cf. 1:1 Theorem of [Rie87]). A smooth mapping germ $f:(\mathbb{R}^2,0)\to(\mathbb{R}^2,0)$ is called a *gulls* if f is $\mathcal{A}$-equivalent to

$$\mathrm{GU}:\left(\mathbb{R}^2,0\right)\to\left(\mathbb{R}^2,0\right),\quad (u,v)\mapsto\left(u,uv^2+v^4+v^5\right)$$

at 0. Then $S(\mathrm{GU})=\{2uv+v^3(4+5v)=0\}$ (see Figure 3.8).

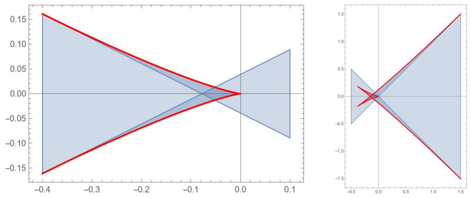

FIGURE 3.7. From left to right, the red lines represent the images of $S\left(\mathrm{BU}^{+}\right)$ and $S\left(\mathrm{BU}^{-}\right)$, respectively.

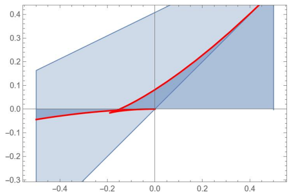

FIGURE 3.8. The red line represents the image of $S\,(\mathrm{GU})$.

3.2. Singular points of frontals and wave fronts. We give some concrete examples of singular points of frontals and wave fronts.

EXAMPLE 1.25 (cf. [**AGZ12**]). A smooth mapping germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is a *cuspidal edge* if f is $\mathcal{A}$-equivalent to

$$\mathrm{CED} : \left(\mathbb{R}^2, 0\right) \to \left(\mathbb{R}^3, 0\right), \quad (u, v) \mapsto \left(u, v^2, v^3\right)$$

at the origin 0. Since the Gauss mapping of CED is

$$n_{\mathrm{CED}}\,(u, v) = \frac{(0, -3v, 2)}{\sqrt{9v^2 + 4}}$$

and the pair $(\mathrm{CED}, n_{\mathrm{CED}}) : (\mathbb{R}^2, 0) \to \mathbb{R}^3 \times \mathbb{S}^2$ is Legendrian, CED is a wave front by Corollary 1.16. Thus cuspidal edges are wave fronts by Proposition 1.15. Then $S\,(\mathrm{CED}) = \{v = 0\}$ (see Figure 3.9). In fact, cuspidal edges are generic singular points of wave fronts (see [**AGZ12**]).

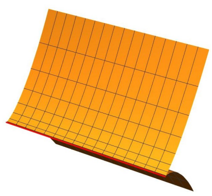

FIGURE 3.9. The red line represents the image of $S\,(\mathrm{CED})$.

EXAMPLE 1.26 (cf. [**AGZ12**]). A smooth mapping germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is a *swallowtail* if f is $\mathcal{A}$-equivalent to

$$\mathrm{SW} : \left(\mathbb{R}^2, 0\right) \to \left(\mathbb{R}^3, 0\right), \quad (u, v) \mapsto \left(u, 3v^4 + uv^2, 4v^3 + 2uv\right)$$

at 0. Since the Gauss mapping of SW is

$$n_{\mathrm{SW}}(u,v)=\frac{(-v^2,-1,v)}{\sqrt{v^4+v^2+1}}$$

and the pair $(\mathrm{SW}, n_{\mathrm{SW}})$ is Legendrian, SW is a wave front by Corollary 1.16. Thus swallowtails are wave fronts by Proposition 1.15. Then $S(\mathrm{SW})=\{u+6v^2=0\}$ (see Figure 3.10). In fact, swallowtails are also generic singular points of wave fronts (see [AGZ12]).

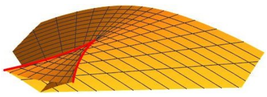

FIGURE 3.10. The red line represents the image of $S(\mathrm{SW})$.

EXAMPLE 1.27 (cf. [Giv90]). A smooth mapping germ $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ is a *cuspidal cross cap* (or a *folded umbrella*) if f is $\mathcal{A}$-equivalent to

$$\mathrm{CCR}:\left(\mathbb{R}^2,0\right)\to\left(\mathbb{R}^3,0\right),\quad (u,v)\mapsto\left(u,v^2,uv^3\right)$$

at 0. Since the Gauss mapping of CCR is

$$n_{\mathrm{CCR}}(u,v)=\frac{(-2v^3,-3uv,2)}{\sqrt{9u^2v^2+4v^6+4}}$$

and the pair $(\mathrm{CCR}, n_{\mathrm{CCR}})$ is not Legendrian, CCR is a frontal but not a wave front by Corollary 1.16. Thus cuspidal cross caps are frontals but not wave fronts by Proposition 1.15. Then $S(\mathrm{CCR})=\{v=0\}$ (see Figure 3.11). In fact, cuspidal cross caps are generic singular points of frontals (see [Giv90]).

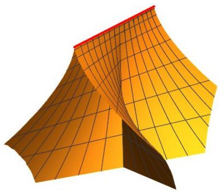

FIGURE 3.11. The red line represents the image of $S(\mathrm{CCR})$.

EXAMPLE 1.28 (cf. [IST10]). A smooth mapping germ $f:(\mathbb{R}^2,0)\to(\mathbb{R}^3,0)$ is a *cuspidal lips* if f is $\mathcal{A}$-equivalent to

$$\mathrm{CLI}:\left(\mathbb{R}^2,0\right)\to\left(\mathbb{R}^3,0\right),\quad (u,v)\mapsto\left(u,-2v^3-u^2v,3v^4+u^2v^2\right)$$

at 0. Since the Gauss mapping of CLI is

$$n_{\mathrm{CLI}}(u,v)=\frac{(-2uv^2,-2v,-1)}{\sqrt{4u^2v^4+4v^2+1}}$$

and the pair $(\mathrm{CLI}, n_{\mathrm{CLI}})$ is Legendrian, CLI is a wave front by Corollary 1.16. Thus cuspidal lips are wave fronts by Proposition 1.15. Then $S(\mathrm{CLI})=\{0\}$ (see Figure 3.12).

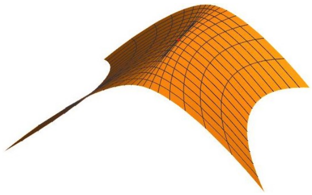

FIGURE 3.12. The red point represents the image of S (CLI).

EXAMPLE 1.29 (cf. [IST10]). A smooth mapping germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is a *cuspidal beaks* if f is $\mathcal{A}$-equivalent to

$$\mathrm{CBE} : \left(\mathbb{R}^2, 0\right) \to \left(\mathbb{R}^3, 0\right), \quad (u, v) \mapsto \left(u, -2v^3 + u^2 v, 3v^4 - u^2 v^2\right)$$

at 0. Since the Gauss mapping of CBE is

$$n_{\mathrm{CBE}}(u, v) = \frac{(-2uv^2, 2v, 1)}{\sqrt{4u^2v^4 + 4v^2 + 1}}$$

and the pair $(\mathrm{CBE}, n_{\mathrm{CBE}})$ is Legendrian, CBE is a wave front by Corollary 1.16. Thus cuspidal beaks are wave fronts by Proposition 1.15. Then $S(\mathrm{CBE}) = \{u^2 - 6v^2 = 0\}$ (see Figure 3.13).

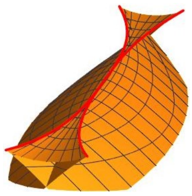

FIGURE 3.13. The red line represents the image of S (CBE).

EXAMPLE 1.30 (cf. [IS10]). A smooth mapping germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is a *cuspidal butterfly* if f is $\mathcal{A}$-equivalent to

$$\mathrm{CBU} : \left(\mathbb{R}^2, 0\right) \to \left(\mathbb{R}^3, 0\right), \quad (u, v) \mapsto \left(u, 5v^4 + 2uv, 4v^5 + uv^2 - u^2\right)$$

at 0. Since the Gauss mapping of CBU is

$$n_{\mathrm{CBU}}(u, v) = \frac{(2u + v^2, -v, 1)}{\sqrt{(2u + v^2)^2 + v^2 + 1}}$$

and the pair $(\mathrm{CBU}, n_{\mathrm{CBU}})$ is Legendrian, CBU is a wave front by Corollary 1.16. Thus cuspidal butterflies are wave fronts by Proposition 1.15. Then $S(\mathrm{CBU}) = \{u + 10v^3 = 0\}$ (see Figure 3.14).

EXAMPLE 1.31 (cf. [Whi43]). A smooth mapping germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ is a *cross cap* (or a *Whitney umbrella*) if f is $\mathcal{A}$-equivalent to

$$\mathrm{CR} : \left(\mathbb{R}^2, 0\right) \to \left(\mathbb{R}^3, 0\right), \quad (u, v) \mapsto \left(u, v^2, uv\right)$$

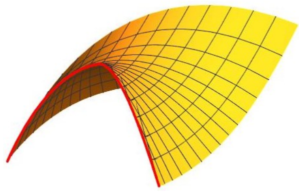

FIGURE 3.14. The red line represents the image of $S\,(\mathrm{CBU})$.

at 0. Since the Gauss mapping of CR is not extended to 0, CR is not a frontal by Corollary 1.16. Thus cross caps are not frontals by Proposition 1.15. Then $S\,(\mathrm{CR}) = \{0\}$ (see Figure 3.15). In fact, cross caps are generic singular points of smooth mappings from $\mathbb{R}^2$ to $\mathbb{R}^3$ (see [Whi44]).

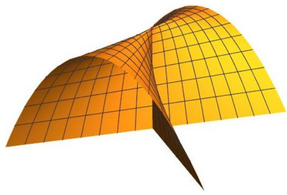

FIGURE 3.15. The red point represents the image of $S\,(\mathrm{CR})$.

CHAPTER 2

Geometry of frontals

1. Coherent tangent bundles

Coherent tangent bundles are introduced as generalizations of tangent bundles on Riemannian manifolds (see Example 2.4) and limiting tangent bundles (see Example 2.6). In this section, we define coherent tangent bundles and present some concrete examples.

Definition 2.1 (Definition 6.1 of [SUY09], Definition 2.1 of [DZ20]). A *coherent tangent bundle* (or an *abstract limiting tangent bundle*) over an oriented smooth 2-manifold M (possibly with boundary) is a 5-tuple $(M, \mathcal{E}, \langle\cdot,\cdot\rangle, D, \varphi)$, where

(1) $\mathcal{E}$ is an orientable smooth vector bundle of rank 2 over M with a metric $\langle\cdot,\cdot\rangle$,

(2) D is a metric connection on $(\mathcal{E}, \langle\cdot,\cdot\rangle)$, that is,

$$X\langle\xi,\eta\rangle = \langle D_X\xi,\eta\rangle + \langle\xi, D_X\eta\rangle$$

for any vector field X on M and any smooth sections ξ, η of $\mathcal{E}$,

(3) $\varphi : TM \to \mathcal{E}$ is a smooth bundle homomorphism over M satisfying

$$D_X\varphi(Y) - D_Y\varphi(X) = \varphi([X,Y]) \tag{2.1.1}$$

for any vector fields X, Y on M.

Remark 2.2. Note that we can define geometric objects associated with the tangent space at a boundary point of a manifold with boundary in the same way as a manifold without boundary. Let $\mathbb{H}^n$ be the closed n-dimensional upper half-space defined by

$$\mathbb{H}^n = \left\{\left(x^1,\ldots,x^n\right) \in \mathbb{R}^n \mid x^n \geq 0\right\}$$

and $\partial\mathbb{H}^n$ the boundary of $\mathbb{H}^n$. Then we can identify the tangent space $T_p\mathbb{H}^n$ with $T_p\mathbb{R}^n$ when $p \in \partial\mathbb{H}^n$ because the differential $d\iota_p : T_p\mathbb{H}^n \to T_p\mathbb{R}^n$ of the inclusion mapping $\iota : \mathbb{H}^n \to \mathbb{R}^n$ is an isomorphism for any point $p \in \partial\mathbb{H}^n$ (cf. [Lee13, Lemma 3.11]). From this assertion, we deduce that the tangent space T_pM at each point $p \in M$ in a smooth n-manifold M with boundary is an n-dimensional vector space (cf. [Lee13, Proposition 3.12]).

Remark 2.3. In [SUY12], coherent tangent bundles are defined for smooth manifolds of general dimension.

We give some concrete examples of coherent tangent bundles.

Example 2.4 ([SUY12], Example 9.1.3 of [USY22]). Let (M, g) be a Riemannian manifold of dimension 2 (possibly with boundary) and ∇ the Levi-Civita connection of g.

Then the tuple $(M, TM, g, \nabla, \mathrm{id}_{TM})$ is a coherent tangent bundle over M, which means that coherent tangent bundles are generalizations of tangent bundles on Riemannian manifolds.

Example 2.5 (Example 2.3 of [**SUY12**]). Let M be an oriented smooth 2-manifold (possibly with boundary), (N, g) an orientable Riemannian manifold of dimension 2 and $f : M \to N$ a smooth mapping. Then the metric g induces a metric on the pull-back $f^*(TN)$, the connection D induced by f of the Levi-Civita connection of g is a metric connection on $(f^*(TN), g)$ and the differential mapping $df : TM \to f^*(TN)$ satisfies the condition (2.1.1). Thus the tuple $(M, f^*(TN), g, D, df)$ is a coherent tangent bundle over M.

Example 2.6 (Example 1.3 of [**SUY08**], Example 2.4 of [**SUY12**]). Let M be an oriented 2-manifold M (possibly with boundary), (N, g) an orientable Riemannian manifold of dimension 3 and $f : M \to N$ a co-orientable frontal. Then there is a unit normal vector field ν of f globally defined on M by Proposition 1.15. The subbundle $\mathcal{E}_f$ of $f^*(TN)$ whose fiber $(\mathcal{E}_f)_p$ at each point $p \in M$ is defined by

$$(\mathcal{E}_f)_p = \left\{ v \in T_{f(p)}N \mid g_{f(p)}(v, \nu(p)) = 0 \right\}$$

is called the *limiting tangent bundle* of f. Then the metric g induces a metric on $\mathcal{E}_f$, the tangential part D of the connection on $f^*(TN)$ by Example 2.5 is a metric connection on $(\mathcal{E}_f, g)$ and the differential mapping $df : TM \to \mathcal{E}_f$ satisfies the condition (2.1.1). Thus the tuple $(M, \mathcal{E}_f, g, D, df)$ is a coherent tangent bundle over M, which means that coherent tangent bundles are generalizations of limiting tangent bundles of frontals.

2. Singular points of smooth bundle homomorphisms

In this section, we define singular points of smooth bundle homomorphisms of coherent tangent bundles. In the following, we fix a coherent tangent bundle $(M, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ over an oriented smooth 2-manifold M (possibly with boundary ∂M).

The pull-back ds^2_φ of the metric $\langle \cdot, \cdot \rangle$ by φ, that is, $ds^2_\varphi := \varphi^* \langle \cdot, \cdot \rangle$ is called the *first fundamental form* of φ (or the *φ-metric*).

A point $p \in M$ is called a *regular point* of φ if the linear mapping $\varphi_p : T_pM \to \mathcal{E}_p$ is a bijection, where $\mathcal{E}_p$ is the fiber of $\mathcal{E}$ at p, that is, the covariant 2-tensor $\left(ds^2_\varphi\right)_p : T_pM \times T_pM \to \mathbb{R}$ is positive-definite. On the other hand, p is called a *singular point* of φ if φ_p is not a bijection, that is, $\left(ds^2_\varphi\right)_p$ is degenerate. Note that if $p \in \partial M$, then φ is extended to a C^∞-mapping beyond ∂M via a compatible coordinate neighborhood of p, and the above notions are well-defined independently of the choice of a compatible coordinate neighborhood of p. We denote by $S(\varphi)$ the singular set of φ. If the coherent tangent bundle is induced by a smooth mappings between 2-manifolds (see Example 2.5) or a frontal (see Example 2.6) f, by the definition of singular points, $S(df) = S(f)$.

Since the vector bundle $\mathcal{E}$ is orientable, there exists a smooth non-vanishing section μ of the exterior power $\mathcal{E}^* \wedge \mathcal{E}^*$ of the dual bundle $\mathcal{E}^*$ of $\mathcal{E}$ satisfying $\mu(e_1, e_2) = \pm 1$ for any orthonormal frame $\{e_1, e_2\}$ for $\mathcal{E}$. An *orientation* of $\mathcal{E}$ is a choice of μ. Then a frame $\{e_1, e_2\}$

is called *positive* (respectively, *negative*) with respect to μ if $\mu(e_1, e_2) = 1$ (respectively, $\mu(e_1, e_2) = -1$). In the following, we fix an orientation μ of $\mathcal{E}$.

DEFINITION 2.7 (Definition 1.2 of [SUY08]). The *signed area form* $d\widehat{A}_\varphi$ and the (*unsigned*) *area form* dA_φ are defined by

$$d\widehat{A}_\varphi = \varphi^*\mu = \lambda_\varphi du \wedge dv, \quad dA_\varphi = |\lambda_\varphi|\, du \wedge dv \tag{2.2.1}$$

on a local coordinate chart $(U; u, v)$ of M, where

$$\lambda_\varphi := \mu(\varphi_u, \varphi_v) \quad \left(\varphi_u := \varphi\left(\frac{\partial}{\partial u}\right),\ \varphi_v := \varphi\left(\frac{\partial}{\partial v}\right)\right),$$

which is called the *signed area density function* with respect to φ on U. Note that the singular set of φ on U is expressed as

$$S(\varphi) \cap U = \{p \in U \mid \lambda_\varphi(p) = 0\}.$$

The 2-form $d\widehat{A}_\varphi$ is extended to a smooth 2-form globally defined on M because $d\widehat{A}_\varphi$ is independent of the choice of local coordinates on U. Also, dA_φ is extended to a continuous 2-form globally defined on M because dA_φ is independent of the choice of local coordinates compatible with the orientation of M on U. If $S(\varphi)$ is empty, $d\widehat{A}_\varphi$ and dA_φ coincide up to sign ambiguity. We set

$$\begin{aligned} M_\varphi^+ &= \left\{p \in M \backslash S(\varphi) \mid d\widehat{A}_\varphi(p) = dA_\varphi(p)\right\}, \\ M_\varphi^- &= \left\{p \in M \backslash S(\varphi) \mid d\widehat{A}_\varphi(p) = -dA_\varphi(p)\right\}. \end{aligned} \tag{2.2.2}$$

Let $\{e_1, e_2\}$ be a positive orthonormal frame with respect to μ for $\mathcal{E}$ defined on a domain $U \subseteq M$. Then there exists a unique 1-form ω defined on U, which is called the *connection form* with respect to $\{e_1, e_2\}$, satisfying

$$D_X e_1 = -\omega(X)\, e_2, \quad D_X e_2 = \omega(X)\, e_1 \tag{2.2.3}$$

for any vector field X defined on U. We see that the exterior derivative $d\omega$ is independent of the choice of positive frames for $\mathcal{E}$ on U, which yields that $d\omega$ is extended to a 2-form globally defined on M. By the definition of ω,

$$d\omega = K_\varphi d\widehat{A}_\varphi = \begin{cases} K_\varphi dA_\varphi & \text{on } M_\varphi^+, \\ -K_\varphi dA_\varphi & \text{on } M_\varphi^-, \end{cases} \tag{2.2.4}$$

where K_φ is the Gaussian curvature of ds_φ^2 on $M \backslash \Sigma_\varphi$.

Next, we define non-degenerate singular points. A point $p \in S(\varphi)$ is called *non-degenerate* if the exterior derivative $d\lambda_\varphi$ does not vanish at p. Note that this definition is independent of the choice of local coordinates. If we take a non-degenerate singular point p, by the implicit function theorem, there is a neighborhood U of p such that $S(\varphi) \cap U$ consists of a smooth regular curve $\gamma_\varphi : (-\varepsilon, \varepsilon) \to U$, $\gamma_\varphi(0) = p$, which is called a *singular curve*. Tangential directions of the singular curve are called *singular directions*. Also, we see that the rank of the linear mapping $\varphi_p : T_pM \to \mathcal{E}_p$ is one, that is, the kernel of φ_p is one dimensional. The kernel of φ_p is called the *null direction*. Furthermore, we can take a non-zero smooth vector field η_φ along γ_φ satisfying $\eta_\varphi(t) \in \ker \varphi_{\gamma_\varphi(t)}$ for each point $\gamma_\varphi(t) \in S(\varphi) \cap U$, which is called a *null vector field* along γ. Then, we can classify non-degenerate singular points as follows.

DEFINITION 2.8 (Definition 1.4 of [**SUY08**], [**MSUY16**], Definition 2.1 of [**Saj18**]). We set

$$\delta_\varphi(t) = \dot{\gamma}_\varphi(t) \wedge \eta_\varphi(t),$$

where $\dot{\gamma}_\varphi(t) := d\gamma_\varphi/dt\,(t)$ and $\wedge$ is the exterior product on TM. A non-degenerate singular point p is called an *A_2-point* (or a *singular point of the first kind*) if $\dot{\gamma}_\varphi(0)$ and $\eta_\varphi(0)$ are linearly independent, that is, $\delta_\varphi(0) \neq 0$. Also, p is called an *A_k-point* (or a *singular point of the k-th kind*), $k \geq 3$, if

$$\delta_\varphi(0) = \dot{\delta}_\varphi(0) = \cdots = \delta_\varphi^{(k-3)}(0) = 0 \quad \text{and} \quad \delta_\varphi^{(k-2)}(0) \neq 0.$$

Next, we define peaks, which are a generalization of A_k-points, $k \geq 3$, and degenerate singular points such as (cuspidal) lips (see Examples 1.19 and 1.28) and (cuspidal) beaks (see Examples 1.20 and 1.29).

DEFINITION 2.9 (Definition 1.10 of [**SUY09**], Definition 2.1 of [**SUY08**]). A point $p \in S(\varphi)$ which is not an A_2-point is called a *peak* if there exists a neighborhood U of p such that

(1) there are no singular points other than A_2-points on $U \setminus \{p\}$,
(2) the rank of the linear mapping $\varphi_p : T_pM \to \mathcal{E}_p$ is one,
(3) the set $S(\varphi) \cap U$ consists of finitely many smooth regular curves starting from p. Note that p is isolated if such a set of smooth regular curves is empty.

A peak is called a *non-degenerate peak* if it is non-degenerate. Note that A_k-points, $k \geq 3$, are non-degenerate peaks. The set $S(\varphi)$ is said to *admit at most peaks* if it consists of A_2-points and peaks.

3. Differential geometry of curves through singular points

In this section, we consider differential geometry of regular curves passing through singular points of bundle homomorphisms of coherent tangent bundles. In the following, we fix a coherent tangent bundle $(M, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ over an oriented smooth 2-manifold M (possibly with boundary ∂M) whose singular set $S(\varphi)$ admits at most peaks.

First, we introduce g-coordinates centered at a point $p \in S(\varphi)$. Take a Riemannian metric g on M. Then there exists a smooth bundle homomorphism $I : TM \to TM$ over M satisfying

$$ds_\varphi^2(X, X) = g(I(X), X)$$

for any vector field X on M. Since $S(\varphi)$ admits at most peaks, the kernel of the linear mapping $\varphi_p : T_pM \to \mathcal{E}_p$ is one dimensional, which yields that one of the eigenvalues of the linear mapping $I_p : T_pM \to T_pM$ vanishes. Thus there exists a neighborhood V of p such that the linear mapping $I_q : T_qM \to T_qM$ has two distinct eigenvalues $\lambda_1(q), \lambda_2(q)$ satisfying $0 \leq \lambda_1(q) < \lambda_2(q)$ for each point $q \in V$. Since the eigenvectors of $\lambda_1(q), \lambda_2(q)$ depend smoothly on q, there exists a local coordinate chart $(U; u, v) \subseteq V$ centered at p such that u-directions give λ_1-eigendirections of I on U and v-directions give λ_2-eigendirections of I on U. We call such (u,v) *g-coordinates* centered at p.

Next, we consider the existence of $\mathcal{E}$-initial vectors at singular points.

PROPOSITION 2.10 (Proposition 2.6 of [SUY08]). *Let $\gamma : [0,\varepsilon) \to M$ be a smooth regular curve starting from a point $p \in S(\varphi)$ such that $\dot{\gamma}(0)$ is not contained in the null direction at p or γ is a singular curve, where $\varepsilon > 0$ is a number such that $\gamma((0,\varepsilon))$ does not contain peaks. Then there exists a vector*

$$\Phi_\gamma := \lim_{t\to 0+0} \frac{\varphi(\dot{\gamma}(t))}{|\varphi(\dot{\gamma}(t))|} \in \mathcal{E}_p,$$

where $|\xi| := \sqrt{\langle \xi, \xi\rangle}$. We call Φ_γ the *$\mathcal{E}$-initial vector* of γ at p.

PROOF. The assertion is obvious if $\dot{\gamma}(0)$ is not contained in the null direction at p.

We assume that γ is a singular curve such that $\dot{\gamma}(0)$ is contained in the null direction at p. Note that p is a peak. Take g-coordinates (u,v) centered at p and write $\gamma(t) = (u(t), v(t))$. Since γ is a singular curve, we have $\varphi_u(\gamma(t)) = 0$, which yields $\varphi(\dot{\gamma}(t)) = \dot{v}(t)\,\varphi_v(\gamma(t))$. Since $\varphi_v(p) \neq 0$ by the definition of g-coordinates,

$$\lim_{t\to 0+0} \frac{\varphi(\dot{\gamma}(t))}{|\varphi(\dot{\gamma}(t))|} = \left(\lim_{t\to 0+0} \operatorname{sgn}(\dot{v}(t))\right) \frac{\varphi_v(p)}{|\varphi_v(p)|}.$$

Since $\gamma(t),\ t \in (0,\varepsilon)$, is an A_2-point, we have $\dot{v}(t) \neq 0,\ t \in (0,\varepsilon)$, which yields that $\operatorname{sgn}(\dot{v}(t)),\ t \in (0,\varepsilon)$, never changes and $\lim_{t\to 0+0} \operatorname{sgn}(\dot{v}(t))$ exists. □

PROPOSITION 2.11 (Propositions 2.8 and 2.9 of [DZ20]). *Let $\gamma : [0,\varepsilon) \to M$ be a smooth regular curve starting from an A_2-point $p \in S(\varphi)$ such that $S(\varphi) \cap \gamma([0,\varepsilon)) = \{p\}$ and $\dot{\gamma}(0)$ is contained in the null direction at p, where $\varepsilon > 0$ is a number such that $\gamma((0,\varepsilon))$ consists of regular points. Then the $\mathcal{E}$-initial vector Φ_γ exists and*

$$\Phi_\gamma = \frac{D_t(\varphi \circ \dot{\gamma})(0)}{|D_t(\varphi \circ \dot{\gamma})(0)|} \in \mathcal{E}_p. \tag{2.3.1}$$

Also, if we set $\overline{\gamma}(t) = \gamma(-t)$,

$$\Phi_\gamma = \Phi_{\overline{\gamma}}. \tag{2.3.2}$$

PROOF. Take g-coordinates (u,v) centered at p. Let $\sigma = \sigma(t),\ t \in [0,\varepsilon)$, be a singular curve starting from p. Since $\lambda_\varphi(\sigma(t)) = 0$,

$$d\lambda_\varphi(\dot{\sigma}(0)) = \left.\frac{d}{dt}\right|_{t=0} \lambda_\varphi(\sigma(t)) = 0.$$

Since $\dot{\gamma}(0)$ and $\dot{\sigma}(0)$ are linearly independent and $d\lambda_\varphi(p) \neq 0$,

$$d\lambda_\varphi(\dot{\gamma}(0)) = \left.\frac{d}{dt}\right|_{t=0} \lambda_\varphi(\gamma(t)) \neq 0. \tag{2.3.3}$$

Since $\lambda_\varphi(\gamma(t)) = \mu(\varphi_u(\gamma(t)), \varphi_v(\gamma(t)))$ and $\varphi_u(p) = 0$,

$$\left.\frac{d}{dt}\right|_{t=0} \lambda_\varphi(\gamma(t)) = \mu(D_t(\varphi_u \circ \gamma)(0), \varphi_v(p)), \tag{2.3.4}$$

where μ is an orientation of $\mathcal{E}$. By (2.3.3) and (2.3.4), $D_t(\varphi_u \circ \gamma)(0)$ and $\varphi_v(p)$ are linearly independent.

Since $\dot\gamma(0)$ is contained in the null direction at p, that is, the u-axis, by the division lemma, there exist a number $\varepsilon_1 > 0$, $\varepsilon_1 \le \varepsilon$, and smooth functions $g, h : [0, \varepsilon_1) \to \mathbb{R}$, $g(0) \neq 0$, such that $\dot\gamma(t)$, $t \in [0, \varepsilon_1)$, is written as

$$\dot\gamma(t) = (g(t) + t\dot g(t)) \left.\frac{\partial}{\partial u}\right|_{\gamma(t)} + t\left(2h(t) + t\dot h(t)\right) \left.\frac{\partial}{\partial v}\right|_{\gamma(t)}.$$

Since $\varphi_u(p) = 0$ and $D_t(\varphi_u \circ \gamma)(0) \neq 0$, by the division lemma, there exist a number $\varepsilon_2 > 0$, $\varepsilon_2 \le \varepsilon$, and a smooth section ξ, $\xi(0) \neq 0$, of $\mathcal{E}$ along γ such that $\varphi_u(\gamma(t))$, $t \in [0, \varepsilon_2)$, is written as

$$\varphi_u(\gamma(t)) = t\xi(t).$$

Then for any $t \in (0, \min\{\varepsilon_1, \varepsilon_2\})$,

$$\begin{aligned}\frac{\varphi(\dot\gamma(t))}{|\varphi(\dot\gamma(t))|} &= \frac{t\left((g(t) + t\dot g(t))\xi(t) + \left(2h(t) + t\dot h(t)\right)\varphi_v(\gamma(t))\right)}{t\left|(g(t) + t\dot g(t))\xi(t) + \left(2h(t) + t\dot h(t)\right)\varphi_v(\gamma(t))\right|} \\ &\to \frac{g(0)\xi(0) + 2h(0)\varphi_v(p)}{|g(0)\xi(0) + 2h(0)\varphi_v(p)|} \quad (t \to 0+0).\end{aligned}$$

Note that $g(0)\xi(0) + 2h(0)\varphi_v(p)$ is non-zero because $\xi(0) = D_t(\varphi_u \circ \gamma)(0)$ and $\varphi_v(p)$ are linearly independent and $g(0) \neq 0$.

Since $D_t(\varphi \circ \dot\gamma)(0) = g(0)\xi(0) + 2h(0)\varphi_v(p)$, we have the equality (2.3.1).

By the definition, we have $\dot{\bar\gamma}(t) = -\dot\gamma(-t)$. Then

$$D_t\left(\varphi \circ \dot{\bar\gamma}\right)(t) = D_{-t}(-\varphi \circ \dot\gamma)(-t) = D_t(\varphi \circ \dot\gamma)(-t),$$

which yields that we have the equality (2.3.2). □

Next, we define singular sectors and their interior angles. Take a point $p \in S(\varphi)$ and a sufficiently small neighborhood U of p. Let $\sigma_1, \sigma_2 : [0, \varepsilon) \to U$ be two smooth regular curves starting from p such that σ_1 and σ_2 are singular curves, or σ_1 is a singular curve and the image of σ_2 is contained in ∂M, where $\varepsilon > 0$ is a number such that $\gamma((0, \varepsilon))$ does not contain peaks. A domain $\Omega \subseteq U$ is called a *singular sector* at p if

(1) the boundary of Ω consists of σ_1, σ_2 and a part of the boundary of U, and
(2) there are no singular points in Ω.

Note that we call the domain $\Omega = U \setminus \{p\}$ a singular sector if p is isolated. If Ω is a singular sector at p, we have $\Omega \subseteq M_\varphi^+$ or $\Omega \subseteq M_\varphi^-$. If $\Omega \subseteq M_\varphi^+$ (respectively, $\Omega \subseteq M_\varphi^-$), it is called *positive* (respectively, *negative*). If there are two or more singular sectors at each peak in $S(\varphi) \setminus \partial M$, the number of positive sectors is equal to the number of negative sectors.

Definition 2.12 (Definition 2.7 of [SUY08]). Let $\gamma_1, \gamma_2 : [0, \varepsilon) \to M$ be two smooth regular curves starting from a point $p \in S(\varphi)$ such that the $\mathcal{E}$-initial vectors $\Phi_{\gamma_1}, \Phi_{\gamma_2}$ exist, where $\varepsilon > 0$ is a number as in Propositions 2.10 and 2.11. The angle $\arccos(\langle \Phi_{\gamma_1}, \Phi_{\gamma_2} \rangle) \in [0, \pi]$ is called the *angle between the $\mathcal{E}$-initial vectors* of γ_1, γ_2.

Let U be a sufficiently small neighborhood of a point $p \in S(\varphi)$ and $\Omega \subseteq U$ a singular sector bounded by two smooth regular curves $\sigma_0, \sigma_1 : [0, \varepsilon) \to U$ starting from p such that the $\mathcal{E}$-initial vectors $\Phi_{\sigma_0}, \Phi_{\sigma_1}$ exist, where $\varepsilon > 0$ is a number as in Propositions 2.10 and 2.11. Then we take a positive integer n and a sequence $\sigma_0 = \gamma_0, \gamma_1, \ldots, \gamma_{n-1}, \gamma_n = \sigma_1 : [0, \varepsilon) \to U$ of smooth regular curves starting from p such that

(1) $\gamma_1, \ldots, \gamma_{n-1}$ satisfy the assumptions of Proposition 2.10 or 2.11,
(2) $\gamma_0, \ldots, \gamma_n$ do not intersect each other in Ω,
(3) there exists a domain $\omega_j \subseteq \Omega$ bounded by γ_{j-1} and γ_j which does not intersect γ_k for each $k \neq j-1, j$,
(4) the interior angle of $\dot{\gamma}_{j-1}(0)$ and $\dot{\gamma}_j(0)$ with respect to a Riemannian metric on M is less than π for each $j = 1, \ldots, n$,
(5) if $n \geq 2$, $\{\dot{\gamma}_{j-1}(0), \dot{\gamma}_j(0)\}$ forms a positively oriented basis for T_pM for each $j = 1, \ldots, n$.

Note that we take a sequence $\gamma_0, \gamma_1, \gamma_2 : [0, \varepsilon) \to U$ satisfying the above conditions if $p \in S(\varphi) \setminus \partial M$ is isolated, where $\gamma_3 := \gamma_0$. Then the *interior angle of the singular sector* Ω is defined by

$$\sum_{j=1}^{n} \arccos\left(\langle \Phi_{\gamma_{j-1}}, \Phi_{\gamma_j} \rangle\right).$$

Next, we define the admissibility of regular curves in order to calculate interior angles of singular sectors.

Definition 2.13 (Definition 2.12 of [SUY08], Definition 3.1 of [DZ20]). A smooth regular curve $\sigma : [a, b] \to M$ is called *admissible* if it satisfies one of

(1) $\sigma([a, b])$ consists of regular points,
(2) $\sigma((a, b])$ consists of regular points, $\sigma(a) \in S(\varphi)$ and $\dot{\sigma}(a)$ is not contained in the null direction at $\sigma(a)$,
(3) σ is a singular curve such that $\sigma((a, b])$ consists of A_2-points,
(4) $\sigma((a, b])$ consists of regular points, $\sigma(a) \in \partial M$ is an A_2-point, and $\dot{\sigma}(a)$ is contained in the null direction and is not contained in the singular direction at $\sigma(a)$.

Note that admissible curves have $\mathcal{E}$-initial vectors by Propositions 2.10 and 2.11. Then we have the following assertion.

Proposition 2.14 (Proposition 2.13 of [SUY08]). *There exist g-coordinates (u, v) centered at a point $p \in S(\varphi)$ such that any admissible curve $\gamma = \gamma(t)$, $t \in [0, \varepsilon)$, starting from p satisfying the condition* (2) *or* (3) *in Definition 2.13 has tangential vectors not contained in u-directions when $t \in (0, \varepsilon)$, where $\varepsilon > 0$ is a number as in Proposition 2.10. In particular, $\gamma(t)$, $t \in (0, \varepsilon)$, never meets the u-axis.*

Proof. Take g-coordinates (u, v) centered at p.

For any admissible curve $\gamma = \gamma(t)$, $t \in [0, \varepsilon)$, starting from p satisfying the condition (2) in Definition 2.13, $\dot{\gamma}(t)$, $t \in [0, \varepsilon)$, is not contained in u-directions because $\dot{\gamma}(0)$ is not contained in the u-direction at p.

Fix a singular curve $\gamma = \gamma(t)$, $t \in [0, \varepsilon)$, starting from p satisfying the condition (3) in Definition 2.13. Since the number of singular curves starting from p is finite (see Definition 2.9), it is sufficient to show that $\dot{\gamma}(t)$, $t \in (0, \varepsilon)$, is not contained in u-directions. If $\dot{\gamma}(0)$ is not contained in the u-direction at p, it is obvious. We assume that $\dot{\gamma}(0)$ is contained in the u-direction at p. Then there exists a number $\varepsilon' > 0$, $\varepsilon' \leq \varepsilon$, such that γ is expressed as $\gamma(u) = (u, F(u))$, $u \in [0, \varepsilon')$. If there exists a number $c \in (0, \varepsilon')$ satisfying $dF/du(c) = 0$, we have a contradiction to the condition (3) in Definition 2.13 because $\partial/\partial u$ is contained in the u-direction at $\gamma(c)$. $\square$

By Proposition 2.14, we divide the set of admissible curves starting from a point $p \in S(\varphi)$ satisfying the condition (2) or (3) in Definition 2.13 into the following two classes: admissible curves which lie in the upper half-plane of g-coordinates are called *upper admissible curves* and admissible curves which lie in the lower half-plane are called *lower admissible curves*. Then we have the following assertion.

Proposition 2.15 (Proposition 2.15 of [SUY08]). *Let $\gamma_1, \gamma_2 : [0, \varepsilon) \to M$ be two admissible curves starting from a point $p \in S(\varphi)$ satisfying the condition (2) or (3) in Definition 2.13, where $\varepsilon > 0$ is a number as in Proposition 2.10. Then $\{\gamma_1, \gamma_2\}$ are in the same class if and only if $\Phi_{\gamma_1} = \Phi_{\gamma_2}$. Also, $\{\gamma_1, \gamma_2\}$ are in distinct classes if and only if $\Phi_{\gamma_1} = -\Phi_{\gamma_2}$.*

Proof. Take g-coordinates (u, v) centered at p. By the proof of Proposition 2.10,

$$\Phi_{\gamma_j} = \operatorname{sgn}(\dot{v}_j(0)) \frac{\varphi_v(p)}{|\varphi_v(p)|}$$

if $\dot{\gamma}_j(0)$ is not contained in the u-direction at p, where $\gamma_j(t) := (u_j(t), v_j(t))$. On the other hand, by the proof of Proposition 2.10,

$$\Phi_{\gamma_j} = \left(\lim_{t \to 0+0} \operatorname{sgn}(\dot{v}_j(t))\right) \frac{\varphi_v(p)}{|\varphi_v(p)|}$$

if $\dot{\gamma}_j(0)$ is contained in the u-direction at p. These two formulae prove the assertion. $\square$

Now, we calculate interior angles of singular sectors.

Theorem 2.16 (Theorem A of [SUY08]). *Let $(M, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ be a coherent tangent bundle over an oriented smooth 2-manifold M with boundary ∂M whose singular set $S(\varphi)$ admits at most peaks.*

(1) *If a point $p \in S(\varphi) \setminus \partial M$ is an A_2-point,*

$$\alpha_\varphi^+(p) = \alpha_\varphi^-(p) = \pi,$$

where $\alpha_\varphi^+(p)$ is the sum of interior angles of positive singular sectors at p and $\alpha_\varphi^-(p)$ is the sum of interior angles of negative singular sectors at p.

(2) *If a point $p \in S(\varphi) \setminus \partial M$ is a peak,*

$$\alpha_\varphi^+(p) + \alpha_\varphi^-(p) = 2\pi, \quad \alpha_\varphi^+(p) - \alpha_\varphi^-(p) \in \{2\pi, 0, -2\pi\}.$$

PROOF. Admissible curves starting from a point $p \in S(\varphi) \setminus \partial M$ satisfying the condition (2) or (3) in Definition 2.13 divide a neighborhood of p into singular sectors consisting of subsets of M_φ^+ or M_φ^-. By Proposition 2.15, there is no contribution of interior angles of singular sectors unless they contain the u-axis of g-coordinates (u, v) centered at p, that is, the two singular sectors containing the u-axis have interior angle π, which yields that

$$\alpha_\varphi^+(p) = \alpha_\varphi^-(p) = \pi$$

if p is an A_2-point and

$$\alpha_\varphi^+(p), \alpha_\varphi^-(p) \in \{0, \pi, 2\pi\}$$

if p is a peak. Thus we get the assertion. □

Based on Theorem 2.16, we give the following definition.

DEFINITION 2.17 ([SUY09], Definition 2.9 of [SUY08]). A peak $p \in S(\varphi) \setminus \partial M$ is called *positive* (respectively, *null*, *negative*) if $\alpha_\varphi^+(p) - \alpha_\varphi^-(p) = 2\pi$ (respectively, $\alpha_\varphi^+(p) - \alpha_\varphi^-(p) = 0$, $\alpha_\varphi^+(p) - \alpha_\varphi^-(p) = -2\pi$). We denote by $P_{\mathrm{Int}}^+(\varphi)$ the set of positive peaks in $S(\varphi) \setminus \partial M$ and $P_{\mathrm{Int}}^-(\varphi)$ the set of negative peaks in $S(\varphi) \setminus \partial M$.

THEOREM 2.18 (Theorem 2.13 of [DZ20]). *Let $(M, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ be a coherent tangent bundle over an oriented smooth 2-manifold M with boundary ∂M whose singular set $S(\varphi)$ admits at most peaks. Suppose that the singular direction at a point $p \in S(\varphi) \cap \partial M$ is transversal to $T_p\partial M$. (Note that p is not isolated.)*

(1) *If the null direction at p is transversal to $T_p\partial M$*

$$\alpha_\varphi^+(p) + \alpha_\varphi^-(p) = \pi, \quad \alpha_\varphi^+(p) - \alpha_\varphi^-(p) \in \{-\pi, \pi\}.$$

(2) *If the null direction at p is not transversal to $T_p\partial M$*

$$\alpha_\varphi^+(p) = \alpha_\varphi^-(p).$$

PROOF. If the null direction at p is transversal to $T_p\partial M$, the assertion follows from Proposition 2.15. On the other hand, if the null direction at p is not transversal to $T_p\partial M$, the assertion follows from Proposition 2.11. □

Based on Theorem 2.18, we give the following definition.

DEFINITION 2.19 (Definition 2.15 of [DZ20]). A non-isolated singular point $p \in \partial M$ is called *positive* (respectively, *null*, *negative*) if $\alpha_\varphi^+(p) - \alpha_\varphi^-(p) = \pi$ (respectively, $\alpha_\varphi^+(p) - \alpha_\varphi^-(p) = 0$, $\alpha_\varphi^+(p) - \alpha_\varphi^-(p) = -\pi$). We denote by $(S(\varphi) \cap \partial M)^+$ the set of non-isolated positive singular points in $S(\varphi) \cap \partial M$, $(S(\varphi) \cap \partial M)^{\mathrm{null}}$ the set of non-isolated null singular points in $S(\varphi) \cap \partial M$ and $(S(\varphi) \cap \partial M)^-$ the set of non-isolated negative singular points in $S(\varphi) \cap \partial M$.

Next, we define the singular curvature for a singular curve consisting of A_2-points. Let $\gamma_\varphi = \gamma_\varphi(t)$ be a singular curve consisting of A_2-points of φ. Since $\dot{\gamma}_\varphi(t)$ is not contained in the null direction at each point $\gamma_\varphi(t)$, we have $\varphi(\dot{\gamma}_\varphi(t)) \neq 0$. Take a null vector field η_φ along γ_φ such that $\{\dot{\gamma}_\varphi(t), \eta_\varphi(t)\}$ is a positively oriented basis for $T_{\gamma_\varphi(t)}M$, and a smooth section n_φ of $\mathcal{E}$ along γ_φ such that $\{\varphi(\dot{\gamma}_\varphi(t)) / |\varphi(\dot{\gamma}_\varphi(t))|, n_\varphi(t)\}$ is a positive orthonormal basis for $\mathcal{E}_{\gamma_\varphi(t)}$ with respect to the orientation μ of $\mathcal{E}$, which is called the *$\mathcal{E}$-conormal vector field* along γ_φ. The *$\mathcal{E}$-geodesic curvature* $\widehat{\kappa}^g_\varphi$ of γ_φ is defined by

$$\widehat{\kappa}^g_\varphi(t) = \frac{\langle D_t(\varphi \circ \dot{\gamma}_\varphi)(t), n_\varphi(t)\rangle}{|\varphi(\dot{\gamma}_\varphi(t))|^2} = \frac{\mu(\varphi(\dot{\gamma}_\varphi(t)), D_t(\varphi \circ \dot{\gamma}_\varphi)(t))}{|\varphi(\dot{\gamma}_\varphi(t))|^3}, \tag{2.3.5}$$

and the *singular curvature* κ^s_φ of γ_φ is defined by

$$\kappa^s_\varphi(t) = \operatorname{sgn}(\eta_\varphi \lambda_\varphi(t))\, \widehat{\kappa}^g_\varphi(t), \tag{2.3.6}$$

where λ_φ is the signed area density function with respect to φ. Also, for an admissible curve γ (see Definition 2.13), we defined the *geometric curvature* $\widetilde{\kappa}^g_\varphi$ by

$$\widetilde{\kappa}^g_\varphi(t) = \begin{cases} \widehat{\kappa}^g_\varphi(t) & \text{if } \gamma(t) \in M^+_\varphi, \\ -\widehat{\kappa}^g_\varphi(t) & \text{if } \gamma(t) \in M^-_\varphi, \\ \kappa^s_\varphi(t) & \text{if } \gamma(t) \in S(\varphi). \end{cases}$$

Remark 2.20. The singular curvature (2.3.6) is defined in [SUY08], which is an intrinsic version of the singular curvature defined originally in [SUY09].

The following propositions hold for the singular curvature.

Proposition 2.21 (Proposition 1.7 of [SUY08]). *The singular curvature is independent of the orientation of M, the orientation of $\mathcal{E}$ and the parameter of the singular curve.*

Proof. If the orientation of M reverses, λ_φ and η_φ change sign, which yields that the value of κ^s_φ does not change. If the orientation of $\mathcal{E}$ reverses, λ_φ and n_φ change sign, which yields that the value of κ^s_φ does not change. If we apply the parameter transformation $t = t(s)$, $dt/ds(s) \neq 0$, to the singular curve $\gamma_\varphi(t)$, by (2.3.5) and (2.3.6),

$$\kappa^s_\varphi(s) = \operatorname{sgn}(\eta_\varphi \lambda_\varphi(t)) \operatorname{sgn}\left(\frac{dt}{ds}(s)\right) \frac{\mu(\varphi(\dot{\gamma}_\varphi(t)), D_t(\varphi \circ \dot{\gamma}_\varphi)(t))}{|\varphi(\dot{\gamma}_\varphi(t))|^3}$$

If $dt/ds(s) > 0$, η_φ does not change sign, which yields that the value of κ^s_φ does not change. On the other hand, if $dt/ds(s) < 0$, η_φ changes sign, which yields that the value of κ^s_φ does not change. □

Proposition 2.22 (Proposition 2.11 of [SUY08]). *Let $\gamma_\varphi : [0, \varepsilon) \to M$ be a singular curve starting from a peak $p \in S(\varphi)$, where $\varepsilon > 0$ is a number such that $\gamma((0, \varepsilon))$ does not contain peaks. Then the singular curvature measure $\kappa^s_\varphi d\tau_\varphi$ defined on $(0, \varepsilon)$ is continuously extended on $[0, \varepsilon)$, where $d\tau_\varphi$ is the arc-length measure with respect to the first fundamental form ds^2_φ, that is, $d\tau_\varphi = |\varphi(\dot{\gamma}_\varphi(t))|\, dt$.*

PROOF. Take g-coordinates (u,v) centered at p. Then

$$\varphi_u(\gamma_\varphi(t)) = 0, \quad D_t(\varphi_u \circ \gamma_\varphi)(t) = 0,$$

which yields

$$\varphi(\dot\gamma_\varphi(t)) = \dot v(t)\,\varphi_v(\gamma_\varphi(t)), \quad D_t(\varphi \circ \dot\gamma)(t) = \ddot v(t)\,\varphi_v(\gamma_\varphi(t)) + \dot v(t)\,D_t(\varphi_v \circ \gamma_\varphi)(t),$$

where $\gamma_\varphi(t) := (u(t), v(t))$. Hence by (2.3.5) and (2.3.6),

$$\kappa^s_\varphi(t) = \operatorname{sgn}\left((\lambda_\varphi)_u(\gamma_\varphi(t))\right) \frac{\mu(\varphi_v(\gamma_\varphi(t)), D_t(\varphi_v \circ \gamma_\varphi)(t))}{|\dot v(t)|\,|\varphi_v(\gamma_\varphi(t))|^3}$$

for any $t \in (0,\varepsilon)$. Since $d\tau_\varphi = |\dot v(t)|\,|\varphi_v(\gamma_\varphi(t))|\,dt$ and $\varphi_v(p) \neq 0$,

$$\kappa^s_\varphi(t)\,d\tau_\varphi = \operatorname{sgn}\left((\lambda_\varphi)_u(\gamma_\varphi(t))\right) \frac{\mu(\varphi_v(\gamma_\varphi(t)), D_t(\varphi_v \circ \gamma_\varphi)(t))}{|\varphi_v(\gamma_\varphi(t))|^2} dt,$$

which means that $\kappa^s_\varphi d\tau_\varphi$ defined on $(0,\varepsilon)$ is continuously extended on $[0,\varepsilon)$. □

The following proposition holds for the $\mathcal{E}$-geodesic curvature.

PROPOSITION 2.23 (Proposition 2.19 of [DZ20]). *Let $\gamma : [0,\varepsilon) \to M$ be a smooth regular curve starting from an A_2-point $p \in S(\varphi)$ such that $S(\varphi) \cap \gamma([0,\varepsilon)) = \{p\}$ and $\dot\gamma(0)$ is contained in the null direction at p, where $\varepsilon > 0$ is a number such that $\gamma((0,\varepsilon))$ consists of regular points. Then the $\mathcal{E}$-geodesic curvature measure $\widehat{\kappa}^g_\varphi d\tau_\varphi$ defined on $(0,\varepsilon)$ is continuously extended on $[0,\varepsilon)$.*

PROOF. By the proof of Proposition 2.11, there exist a number $\varepsilon' > 0$, $\varepsilon' \leq \varepsilon$, and a smooth section ζ, $\zeta(0) \neq 0$, of $\mathcal{E}$ along γ such that $\varphi(\gamma(t))$, $t \in [0,\widehat{\varepsilon})$, is written as $\varphi(\dot\gamma(t)) = t\zeta(t)$. Hence by (2.3.5),

$$\widehat{\kappa}^g_\varphi(t) = \frac{\mu(\zeta(t), D_t\zeta(t))}{t\,|\zeta(t)|^3}$$

for any $t \in (0,\widehat{\varepsilon})$. Since $d\tau_\varphi = t\,|\zeta(t)|\,dt$ and $\zeta(0) \neq 0$,

$$\widehat{\kappa}^g_\varphi(t)\,d\tau_\varphi = \frac{\mu(\zeta(t), D_t\zeta(t))}{|\zeta(t)|^2} dt,$$

which means that $\widehat{\kappa}^g_\varphi d\tau_\varphi$ defined on $(0,\varepsilon)$ is continuously extended on $[0,\varepsilon)$ because $\gamma((0,\varepsilon))$ consists of regular points. □

4. Examples of A_k-points and peaks

Here, we give concrete examples of A_k-points and peaks. First, we give concrete examples for smooth mappings between 2-manifolds.

EXAMPLE 2.24. Since $\mathrm{FO}_u = (1,0), \mathrm{FO}_v = (0,2v)$ by Example 1.17, the null vector field along $S(\mathrm{FO}) = \{v = 0\}$ is $\partial/\partial v$, which yields that singular points are A_2-points (see Figure 4.1).

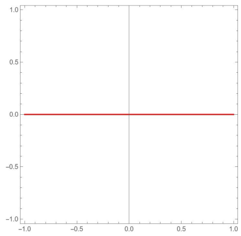

FIGURE 4.1. The red line represents $S(\mathrm{FO})$.

EXAMPLE 2.25. Since $\mathrm{CU}_u = (1, v), \mathrm{CU}_v = (0, u + 3v^2)$ by Example 1.18, the null vector field is $\partial/\partial v$ along $S(\mathrm{CU}) = \{u + 3v^2 = 0\}$, which yields that singular points other than the origin are A_2-points and the origin is an A_3-point (see Figure 4.2). In particular, the origin is a peak of CU, whose sufficiently small neighborhood is divided into two singular sectors.

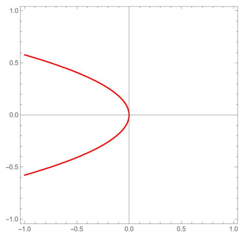

FIGURE 4.2. The red line represents $S(\mathrm{CU})$.

EXAMPLE 2.26. Since $\mathrm{LI}_u = (1, 2uv), \mathrm{LI}_v = (0, u^2 + 3v^2)$ by Example 1.19, the null vector at $S(\mathrm{LI}) = \{0\}$ is $\partial/\partial v|_0$, which yields that the origin is a peak of LI.

EXAMPLE 2.27. Since $\mathrm{BE}_u = (1, -2uv), \mathrm{BE}_v = (0, 3v^2 - u^2)$ by Example 1.20, the null vector field along $S(\mathrm{BE}) = \{3v^2 - u^2 = 0\}$ is $\partial/\partial v$, which yields that singular points other than the origin are A_2-points and the origin is a degenerate singular point (see Figure 4.3). In particular, the origin is a peak of BE, whose sufficiently small neighborhood is divided into four singular sectors.

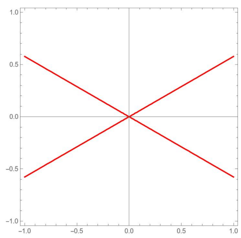

FIGURE 4.3. The red line represents $S(\mathrm{BE})$.

Example 2.28. Since $\mathrm{GO}^+_u = (1, 3u^2v), \mathrm{GO}^+_v = (0, u^3 + 3v^2)$ by Example 1.21, the null vector field along $S(\mathrm{GO}^+) = \{u^3 + 3v^2 = 0\}$ is $\partial/\partial v$, which yields that singular points other than the origin are A_2-points and the origin is a degenerate singular point (see the left of Figure 4.4). In particular, the origin is a peak of GO^+. Similarly, since $\mathrm{GO}^-_u = (1, -3u^2v), \mathrm{GO}^-_v = (0, 3v^2 - u^3)$, the null vector field along $S(\mathrm{GO}^-) = \{3v^2 - u^3 = 0\}$ is $\partial/\partial v$, which yields that singular points other than the origin are A_2-points and the origin is a degenerate singular point (see the right of Figure 4.4). In particular, the origin is a peak of GO^-.

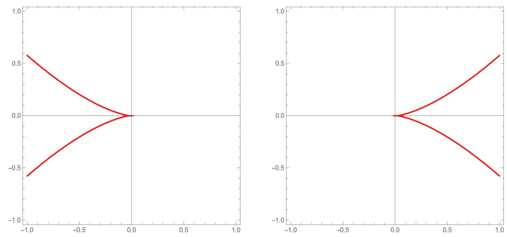

Figure 4.4. From left to right, the red lines represent $S(\mathrm{GO}^+)$ and $S(\mathrm{GO}^-)$, respectively.

Example 2.29. Since $\mathrm{SW}_u = (1, v), \mathrm{SW}_v = (0, u + 4v^3)$ by Example 1.22, the null vector field along $S(\mathrm{SW}) = \{u + 4v^3 = 0\}$ is $\partial/\partial v$, which yields that singular points other than the origin are A_2-points and the origin is an A_4-point (see Figure 4.5). In particular, the origin is a peak of SW, whose sufficiently small neighborhood is divided into two singular sectors.

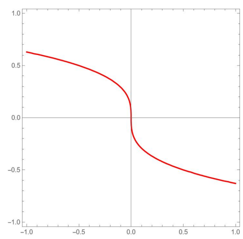

Figure 4.5. The red line represents $S(\mathrm{SW})$.

Example 2.30. Since $\mathrm{BU}^+_u = (1, v), \mathrm{BU}^+_v = (0, u + 7v^6 + 5v^4)$ by Example 1.23, the null vector field along $S(\mathrm{BU}^+) = \{u + 7v^6 + 5v^4 = 0\}$ is $\partial/\partial v$, which yields that singular points other than the origin are A_2-points and the origin is an A_5-point (see the left of Figure 4.6). In particular, the origin is a peak of BU^+, whose sufficiently small neighborhood is divided into two singular sectors. Similarly, since $\mathrm{BU}^-_u = (1, v), \mathrm{BU}^-_v = (0, u - 7v^6 + 5v^4)$, the null vector field along $S(\mathrm{BU}^-) = \{u - 7v^6 + 5v^4 = 0\}$ is $\partial/\partial v$, which yields that singular points other than the origin are A_2-points and the origin is an A_5-point (see the right of Figure 4.6). In particular, the origin is a peak of BU^-, whose sufficiently small neighborhood is divided into two singular sectors.

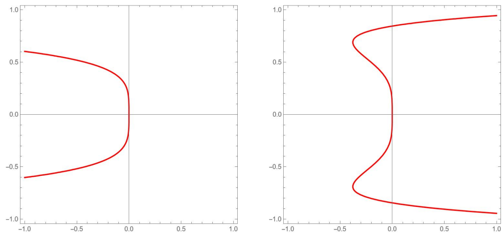

FIGURE 4.6. From left to right, the red lines represent $S\left(\mathrm{BU}^{+}\right)$ and $S\left(\mathrm{BU}^{-}\right)$, respectively.

EXAMPLE 2.31. Since $\mathrm{GU}_u = (1, v^2), \mathrm{GU}_v = (0, 2uv + (5v+4)\, v^3)$ by Example 1.24, the null vector field along $S\,(\mathrm{GU}) = \{2uv + (5v+4)\, v^3 = 0\}$ is $\partial/\partial v$, which yields that singular points other than the origin are A_2-points and the origin is a degenerate singular point (see Figure 4.7). In particular, the origin is a peak of GU, whose sufficiently small neighborhood is divided into four singular sectors.

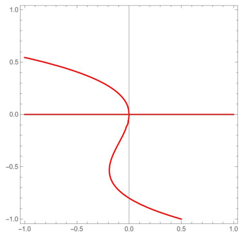

FIGURE 4.7. The red line represents $S\,(\mathrm{GU})$.

Next, we give concrete examples of A_k-points and peaks of frontals and wave fronts.

EXAMPLE 2.32. Since $\mathrm{CED}_u = (1,0,0), \mathrm{CED}_v = (0, 2v, 3v^2)$ by Example 1.25, the null vector field along $S\,(\mathrm{CED}) = \{v = 0\}$ is $\partial/\partial v$, which yields that singular points are A_2-points (see Figure 4.8).

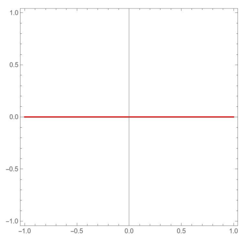

FIGURE 4.8. The red line represents $S\,(\mathrm{CED})$.

EXAMPLE 2.33. Since $\mathrm{SW}_u = (1, v^2, 2v), \mathrm{SW}_v = (0, 2v\,(u+6v^2), 2\,(u+6v^2))$ by Example 1.26, the null vector field along $S\,(\mathrm{SW}) = \{u + 6v^2 = 0\}$ is $\partial/\partial v$, which yields that singular points other than the origin are A_2-points and the origin is an A_3-point (see Figure 4.9). In particular, the origin is a peak of SW, whose sufficiently small neighborhood is divided into two singular sectors.

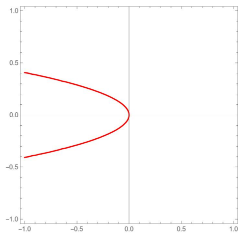

FIGURE 4.9. The red line represents $S\,(\mathrm{SW})$.

EXAMPLE 2.34. Since $\mathrm{CCR}_u = (1, 0, v^3), \mathrm{CCR}_v = (0, 2v, 3uv^2)$ by Example 1.27, the null vector field along $S\,(\mathrm{CCR}) = \{v = 0\}$ is $\partial/\partial v$, which yields that singular points are A_2-points (see Figure 4.10).

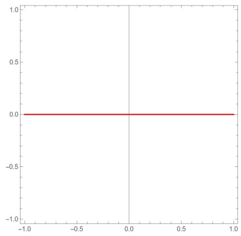

FIGURE 4.10. The red line represents $S\,(\mathrm{CCR})$.

EXAMPLE 2.35. Since $\mathrm{CLI}_u = (1, -2uv, 2uv^2), \mathrm{CLI}_v = (0, -u^2 - 6v^2, 2v\,(u^2 + 6v^2))$ by Example 1.28, the null vector at $S\,(\mathrm{CLI}) = \{0\}$ is $\partial/\partial v|_0$, which yields that the origin is a peak of CLI.

EXAMPLE 2.36. Since $\mathrm{CBE}_u = (1, 2uv, -2uv^2), \mathrm{CBE}_v = (0, u^2 - 6v^2, -2v\,(u^2 - 6v^2))$ by Example 1.29, the null vector field along $S\,(\mathrm{CBE}) = \{u^2 - 6v^2 = 0\}$ is $\partial/\partial v$, which yields that singular points other than the origin are A_2-points and the origin is a degenerate singular point (see Figure 4.11). In particular, the origin is a peak of CBE, whose sufficiently small neighborhood is divided into four singular sectors.

EXAMPLE 2.37. Since $\mathrm{CBU}_u = (1, 2v, v^2 - 2u), \mathrm{CBU}_v = (0, 2\,(u + 10v^3), 2v\,(u + 10v^3))$ by Example 1.30, the null vector field along $S\,(\mathrm{CBU}) = \{u + 10v^3 = 0\}$ is $\partial/\partial v$, which yields that singular points other than the origin are A_2-points and the origin is an A_4-point (see Figure 4.12). In particular, the origin is a peak of CBU, whose sufficiently small neighborhood is divided into two singular sectors.

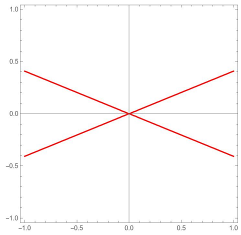

FIGURE 4.11. The red line represents $S\,(\mathrm{CBE})$.

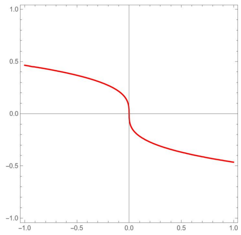

FIGURE 4.12. The red line represents $S\,(\mathrm{CBU})$.

5. Gauss-Bonnet type theorems for frontals

In this section, we present Gauss-Bonnet type theorems for coherent tangent bundles and give their proofs.

In the following, we fix a coherent tangent bundle $(M, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ over a compact oriented smooth 2-manifold M with boundary ∂M whose singular set $S\,(\varphi)$ admits at most peaks, and suppose that $S\,(\varphi) \setminus I\,(\varphi)$ is transversal to ∂M and that $I\,(\varphi) \cap \partial M$ is empty, where $I\,(\varphi)$ is the set of isolated singular points.

First, we define the admissibility of triangles with admissible curves as their edges.

DEFINITION 2.38 (Definition 3.1 of [SUY08], Definition 3.3 of [DZ20]). Let U be a domain of M, $\overline{T} \subseteq U$ the closure of a simply connected domain T which is bounded by three admissible curves $\gamma_1, \gamma_2, \gamma_3$, and A (respectively, B, C) the boundary point of T which is the intersection of γ_2 and γ_3 (respectively, γ_3 and γ_1, γ_1 and γ_2). The closure $\overline{T}$ is called *admissible* if it satisfies

(1) $\overline{T}$ admits at most one peak on $\{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ and at most one singular curve on $\{\gamma_1, \gamma_2, \gamma_3\}$,
(2) there is no singular points in T,
(3) the three interior angles at A, B and C with respect to a Riemannian metric on M are all less than π.

Then we denote $\triangle\mathrm{ABC} := \overline{T}$, $\widehat{\mathrm{BC}} := \gamma_1$, $\widehat{\mathrm{CA}} := \gamma_2$ and $\widehat{\mathrm{AB}} := \gamma_3$, and call $\{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ the *vertices* of $\triangle\mathrm{ABC}$. Also, we denote by $\angle\mathrm{A}_\varphi$ (respectively, $\angle\mathrm{B}_\varphi$, $\angle\mathrm{C}_\varphi$) the interior angle with respect to the first fundamental form ds^2_φ of the piecewise smooth boundary of $\triangle\mathrm{ABC}$ at A (respectively, B, C). We give the boundary $\partial\triangle\mathrm{ABC}$ the orientation compatible with the orientation of M.

Then we have the following local Gauss-Bonnet type theorems for admissible triangles.

THEOREM 2.39 (Theorem 3.3 of [SUY08], Proposition 3.4 of [DZ20]). *Let $(M, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi)$ be a coherent tangent bundle over an oriented smooth 2-manifold M (possibly with boundary) whose singular set admits at most peaks, and $\triangle\mathrm{ABC}$ an admissible triangle on M. Then*

$$\angle \mathrm{A}_\varphi + \angle \mathrm{B}_\varphi + \angle \mathrm{C}_\varphi - \pi = \int_{\partial\triangle\mathrm{ABC}} \widetilde{\kappa}^g_\varphi d\tau_\varphi + \int_{\triangle\mathrm{ABC}} K_\varphi dA_\varphi. \tag{2.5.1}$$

It is sufficient to prove local Gauss-Bonnet type theorems for five types of admissible triangles in order to prove this theorem (see Lemmas Lemmas 2.40, 2.41, 2.42, 2.43 and 2.44).

LEMMA 2.40 (Lemma 3.4 of [SUY08]). *If an admissible triangle $\triangle\mathrm{ABC}$ lies in M^+_φ or M^-_φ, we have the equation* (2.5.1).

PROOF. Since M^+_φ and M^-_φ consist of regular points, ds^2_φ is positive-definite on M^+_φ and M^-_φ, which yields that we have the assertion by the classical local Gauss-Bonnet theorem. □

LEMMA 2.41 (Lemma 3.5 of [SUY08]). *Let $\triangle\mathrm{ABC}$ be an admissible triangle such that A is a singular point, the tangential vectors of $\overset{\frown}{\mathrm{AB}}$ and $\overset{\frown}{\mathrm{AC}}$ at A are not contained in the null direction at A, and $\triangle\mathrm{ABC}\setminus\{\mathrm{A}\}$ lies in M^+_φ or M^-_φ. Suppose that $\overset{\frown}{\mathrm{BA}}$ and $\overset{\frown}{\mathrm{BC}}$ are transversal at B. Then we have the equation* (2.5.1).

PROOF. We assume that $\triangle\mathrm{ABC}\setminus\{\mathrm{A}\}$ lies in M^+_φ. Take a short extension of $\overset{\frown}{\mathrm{BA}}$ beyond A and rotate it around B with respect to the canonical metric $du^2 + dv^2$ on the uv-plane. Then we obtain a smooth one-parameter family of admissible curves starting from B. Since $\overset{\frown}{\mathrm{BA}}$ and $\overset{\frown}{\mathrm{BC}}$ are transversal at B, restricting the image of this family on $\triangle\mathrm{ABC}$, we obtain a family of admissible curves $\{\gamma_\varepsilon : [0,1] \to \triangle\mathrm{ABC}\}_{\varepsilon\in[0,1]}$ such that

(1) γ_0 parametrizes $\overset{\frown}{\mathrm{AB}}$ satisfying $\gamma_0(0) = \mathrm{A}$ and $\gamma_0(1) = \mathrm{B}$,
(2) $\gamma_\varepsilon(1) = \mathrm{B}$ for any $\varepsilon \in [0,1]$,
(3) the image of the correspondence $\sigma : \varepsilon \mapsto \gamma_\varepsilon(0)$ is contained in $\overset{\frown}{\mathrm{AC}}$, where $\mathrm{A}_\varepsilon := \gamma_\varepsilon(0)$ and $\mathrm{A}_0 := \mathrm{A}$.

Then each $\triangle\mathrm{A}_\varepsilon\mathrm{BC}$, $\varepsilon \in (0,1]$, is admissible because it lies in M^+_φ, which yields that by Lemma 2.40,

$$\angle(\mathrm{A}_\varepsilon)_\varphi + \angle(\mathrm{A}_\varepsilon\mathrm{BC})_\varphi + \angle \mathrm{C}_\varphi - \pi = \int_{\partial\triangle\mathrm{A}_\varepsilon\mathrm{BC}} \widehat{\kappa}^g_\varphi d\tau_\varphi + \int_{\triangle\mathrm{A}_\varepsilon\mathrm{BC}} K_\varphi dA_\varphi.$$

If we take the limit as $\varepsilon \to 0+0$,

$$\lim_{\varepsilon\to 0+0} \angle(\mathrm{A}_\varepsilon)_\varphi + \angle \mathrm{B}_\varphi + \angle \mathrm{C}_\varphi - \pi = \int_{\partial\triangle\mathrm{ABC}} \widetilde{\kappa}^g_\varphi d\tau_\varphi + \int_{\triangle\mathrm{ABC}} K_\varphi dA_\varphi.$$

Note that $\widetilde{\kappa}^g_\varphi d\tau_\varphi$ is bounded on $\overset{\frown}{\mathrm{AB}}$ and $\overset{\frown}{\mathrm{CA}}$ because $\triangle\mathrm{ABC}$ is admissible. Also, by Propositions 2.10 and 2.15,

$$\begin{aligned}\angle \mathrm{A}_\varphi &:= \lim_{\varepsilon\to 0+0} \angle\left(\mathrm{A}_\varepsilon\right)_\varphi = \lim_{\varepsilon\to 0+0} \arccos\left(\left\langle \frac{\varphi\left(\dot\gamma_\varepsilon\left(0\right)\right)}{\left|\varphi\left(\dot\gamma_\varepsilon\left(0\right)\right)\right|}, \frac{\varphi\left(d\sigma/d\varepsilon\left(\varepsilon\right)\right)}{\left|\varphi\left(d\sigma/d\varepsilon\left(\varepsilon\right)\right)\right|}\right\rangle\right) \\ &= \arccos\left(\left\langle \Phi_{\gamma_0}, \Phi_\sigma\right\rangle\right) = \begin{cases} \pi & \text{if the } u\text{-axis separates } \overset{\frown}{\mathrm{AB}} \text{ and } \overset{\frown}{\mathrm{AC}}, \\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

where (u, v) is g-coodinates centered at A. This completes the proof. □

Lemma 2.42 (Lemma 3.6 of [SUY08]). *Let $\triangle\mathrm{ABC}$ be an admissible triangle such that $\overset{\frown}{\mathrm{AB}}$ consists of A_2-points, the tangential vector of $\overset{\frown}{\mathrm{AC}}$ at* A *is not contained in the null direction at* A*, the tangential vector of $\overset{\frown}{\mathrm{BC}}$ at* B *is not contained in the null direction at* B*, and $\triangle\mathrm{ABC}\backslash\overset{\frown}{\mathrm{AB}}$ lies in M^+_φ or M^-_φ. Suppose that $\overset{\frown}{\mathrm{BA}}$ and $\overset{\frown}{\mathrm{BC}}$ are transversal at B. Then we have the equation* (2.5.1).

Proof. We assume that $\triangle\mathrm{ABC}\backslash\overset{\frown}{\mathrm{AB}}$ lies in M^+_φ. By the same method as in Lemma 2.41, we obtain a family of admissible curves $\{\gamma_\varepsilon : [0,1] \to \triangle\mathrm{ABC}\}_{\varepsilon\in[0,1]}$ such that

(1) γ_0 parametrizes $\overset{\frown}{\mathrm{AB}}$ satisfying $\gamma_0\left(0\right) = \mathrm{A}$ and $\gamma_0\left(1\right) = \mathrm{B}$,
(2) $\gamma_\varepsilon\left(1\right) = \mathrm{B}$ for any $\varepsilon \in [0,1]$,
(3) the image of the correspondence $\varepsilon \mapsto \gamma_\varepsilon\left(0\right)$ is contained in $\overset{\frown}{\mathrm{AC}}$, where $\mathrm{A}_\varepsilon := \gamma_\varepsilon\left(0\right)$ and $\mathrm{A}_0 := \mathrm{A}$.

Then each $\triangle\mathrm{A}_\varepsilon\mathrm{BC}$, $\varepsilon \in (0,1]$, is admissible, which yields that by Lemma 2.41,

$$\angle\left(\mathrm{A}_\varepsilon\right)_\varphi + \angle\left(\mathrm{A}_\varepsilon\mathrm{BC}\right)_\varphi + \angle\mathrm{C}_\varphi - \pi = \int_{\partial\triangle\mathrm{A}_\varepsilon\mathrm{BC}} \widetilde{\kappa}^g_\varphi d\tau_\varphi + \int_{\triangle\mathrm{A}_\varepsilon\mathrm{BC}} K_\varphi dA_\varphi.$$

If we take the limit as $\varepsilon \to 0+0$, we have the assertion by the same argument as in the proof of Lemma 2.41. □

Lemma 2.43 (Lemma 3.7 of [SUY08]). *Let $\triangle\mathrm{ABC}$ be an admissible triangle such that* A *is a peak, $\overset{\frown}{\mathrm{AB}}\backslash\{\mathrm{A}\}$ consists of A_2-points, the tangential vector of $\overset{\frown}{\mathrm{AC}}$ at* A *is not contained in the null direction at* A*, the tangential vector of $\overset{\frown}{\mathrm{BC}}$ at* B *is not contained in the null direction at* B*, and $\triangle\mathrm{ABC}\backslash\overset{\frown}{\mathrm{AB}}$ lies in M^+_φ or M^-_φ. Suppose that $\overset{\frown}{\mathrm{CB}}$ and $\overset{\frown}{\mathrm{CA}}$ are transversal at* C*. Then we have the equation* (2.5.1).

Proof. We assume that $\triangle\mathrm{ABC}\backslash\overset{\frown}{\mathrm{AB}}$ lies in M^+_φ. Take a short extension of $\overset{\frown}{\mathrm{CA}}$ beyond A and rotate it around C with respect to $du^2 + dv^2$ on the uv-plane. Then we get a smooth one-parameter family of admissible curves starting from C. Since $\overset{\frown}{\mathrm{CB}}$ and $\overset{\frown}{\mathrm{CA}}$ are transversal at C, restricting the image of this family on $\triangle\mathrm{ABC}$, we obtain a family of admissible curves $\{\gamma_\varepsilon : [0,1] \to \triangle\mathrm{ABC}\}_{\varepsilon\in[0,1]}$ such that

(1) γ_0 parametrizes $\widehat{\mathrm{AC}}$ satisfying $\gamma_0(0) = \mathrm{A}$ and $\gamma_0(1) = \mathrm{C}$,
(2) $\gamma_\varepsilon(1) = \mathrm{C}$ for any $\varepsilon \in [0, 1]$,
(3) the image of the correspondence $\sigma : \varepsilon \mapsto \gamma_\varepsilon(0)$ is contained in $\widehat{\mathrm{AB}}$, where $\mathrm{A}_\varepsilon := \gamma_\varepsilon(0)$ and $\mathrm{A}_0 := \mathrm{A}$.

Since the tangential vector of $\widehat{\mathrm{AC}}$ at A is not contained in the null direction at A, the tangential vector of $\widehat{\mathrm{A}_\varepsilon\mathrm{C}}$ at A_ε is not contained in the null direction at A_ε for sufficiently small $\varepsilon > 0$, which means that $\triangle\mathrm{A}_\varepsilon\mathrm{BC}$ is admissible. By Lemma 2.42,

$$\angle(\mathrm{A}_\varepsilon)_\varphi + \angle\mathrm{B}_\varphi + \angle(\mathrm{BCA}_\varepsilon)_\varphi - \pi = \int_{\partial\triangle\mathrm{A}_\varepsilon\mathrm{BC}} \widetilde{\kappa}^g_\varphi d\tau_\varphi + \int_{\triangle\mathrm{A}_\varepsilon\mathrm{BC}} K_\varphi dA_\varphi.$$

By Proposition 2.22 and the Lebesgue convergence theorem,

$$\int_{\partial\triangle\mathrm{ABC}} \widetilde{\kappa}^g_\varphi d\tau_\varphi := \lim_{\varepsilon\to 0+0} \int_{\partial\triangle\mathrm{A}_\varepsilon\mathrm{BC}} \widetilde{\kappa}^g_\varphi d\tau_\varphi$$

exists. If we take g-coordinates (u, v) centered at A, $\angle(\mathrm{A}_\varepsilon)_\varphi$ is common in each $\varepsilon \in (0, 1]$ because the property that the null direction $\partial/\partial u$ at $\sigma(\varepsilon)$ separates $\widehat{\mathrm{A}_\varepsilon\mathrm{B}}$ and $\widehat{\mathrm{A}_\varepsilon\mathrm{C}}$ or not is common in ε by Proposition 2.15, which yields that

$$\angle\mathrm{A}_\varphi := \lim_{\varepsilon\to 0+0} \angle(\mathrm{A}_\varepsilon)_\varphi = \begin{cases} \pi & \text{if the } u\text{-axis separates } \widehat{\mathrm{AB}} \text{ and } \widehat{\mathrm{AC}}, \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof. □

Lemma 2.44 (Proposition 3.4 of [DZ20]). *Let $\triangle\mathrm{ABC}$ be an admissible triangle such that* A *is an A_2-point, the tangential vector of* $\widehat{\mathrm{AC}}$ *at* A *is contained in the null direction at* A *and not contained in the singular direction at* A, *and $\triangle\mathrm{ABC}\setminus\{\mathrm{A}\}$ lies in M^+_φ or M^-_φ. Suppose that* $\widehat{\mathrm{BA}}$ *and* $\widehat{\mathrm{BC}}$ *are transversal at* B. *Then we have the equation* (2.5.1).

Proof. We may assume that $\triangle\mathrm{ABC}\setminus\{\mathrm{A}\}$ lies in M^+_φ. By the same method as in Lemma 2.41, we obtain a family of admissible curves $\{\gamma_\varepsilon : [0, 1] \to \triangle\mathrm{ABC}\}_{\varepsilon\in[0,1]}$ such that

(1) γ_0 parametrizes $\widehat{\mathrm{AB}}$ satisfying $\gamma_0(0) = \mathrm{A}$ and $\gamma_0(1) = \mathrm{B}$,
(2) $\gamma_\varepsilon(1) = \mathrm{B}$ for any $\varepsilon \in [0, 1]$,
(3) the image of the correspondence $\varepsilon \mapsto \gamma_\varepsilon(0)$ is contained in $\widehat{\mathrm{AC}}$, where $\mathrm{A}_\varepsilon := \gamma_\varepsilon(0)$ and $\mathrm{A}_0 := \mathrm{A}$.

Since each $\triangle\mathrm{A}_\varepsilon\mathrm{BC}$, $\varepsilon \in (0, 1]$, is admissible, by Lemma 2.40,

$$\angle(\mathrm{A}_\varepsilon)_\varphi + \angle(\mathrm{A}_\varepsilon\mathrm{BC})_\varphi + \angle\mathrm{C}_\varphi - \pi = \int_{\partial\triangle\mathrm{A}_\varepsilon\mathrm{BC}} \widehat{\kappa}^g_\varphi d\tau_\varphi + \int_{\triangle\mathrm{A}_\varepsilon\mathrm{BC}} K_\varphi dA_\varphi.$$

By Proposition 2.23,

$$\int_{\partial\triangle \mathrm{ABC}} \widetilde{\kappa}^g_\varphi d\tau_\varphi := \lim_{\varepsilon\to 0+0} \int_{\partial\triangle \mathrm{A}_\varepsilon \mathrm{BC}} \widehat{\kappa}^g_\varphi d\tau_\varphi$$

exists. Also, by Propositions 2.10 and 2.11,

$$\begin{aligned} \angle \mathrm{A}_\varphi &:= \lim_{\varepsilon\to 0+0} \angle \left(\mathrm{A}_\varepsilon\right)_\varphi = \lim_{\varepsilon\to 0+0} \arccos\left(\left\langle \frac{\varphi\left(\dot{\gamma}_\varepsilon(0)\right)}{\left|\varphi\left(\dot{\gamma}_\varepsilon(0)\right)\right|}, \frac{\varphi\left(d\sigma/d\varepsilon\left(\varepsilon\right)\right)}{\left|\varphi\left(d\sigma/d\varepsilon\left(\varepsilon\right)\right)\right|}\right\rangle\right) \\ &= \arccos\left(\left\langle \Phi_{\gamma_0}, \Phi_\sigma\right\rangle\right). \end{aligned}$$

This completes the proof. □

Proof of Theorem 2.39. By Lemmas 2.40, 2.41, 2.42, 2.43 and 2.44. □

Next, we prove global Gauss-Bonnet type theorems by using the local Gauss-Bonnet type theorem in Theorem 2.39. The global Gauss-Bonnet type theorems are as follows.

Theorem 2.45 (Theorem B of [SUY08] if $\partial M = \varnothing$, Theorem 2.20 of [DZ20] if $\partial M \neq \varnothing$). *Let $(M, \mathcal{E}, \langle\cdot,\cdot\rangle, D, \varphi)$ be a coherent tangent bundle over a compact oriented smooth 2-manifold M with boundary ∂M whose singular set $S(\varphi)$ admits at most peaks, and suppose that $S(\varphi)\backslash I(\varphi)$ is transversal to ∂M and that $I(\varphi)\cap\partial M$ is empty. Then*

$$\begin{aligned} 2\pi\chi(M) &= \int_M K_\varphi dA_\varphi + 2\int_{S(\varphi)} \kappa^s_\varphi d\tau_\varphi + \int_{\partial M} \widetilde{\kappa}^g_\varphi d\tau_\varphi \\ &\quad - \sum_{p\in (S(\varphi)\cap\partial M)^{\mathrm{null}}} \left(2\alpha^+_\varphi(p) - \pi\right), \end{aligned} \tag{2.5.2}$$

$$\begin{aligned} \int_M K_\varphi d\widehat{A}_\varphi + \int_{\partial M} \widehat{\kappa}^g_\varphi d\tau_\varphi &= 2\pi\left(\chi\left(M^+_\varphi\right) - \chi\left(M^-_\varphi\right)\right) \\ &\quad + 2\pi\left(\#P^+_{\mathrm{Int}}(\varphi) - \#P^-_{\mathrm{Int}}(\varphi)\right) \\ &\quad + \pi\left(\#\left(S(\varphi)\cap\partial M\right)^+ - \#\left(S(\varphi)\cap\partial M\right)^-\right) \\ &\quad + \pi\left(\#P_{\partial M^+_\varphi} - \#P_{\partial M^-_\varphi}\right), \end{aligned} \tag{2.5.3}$$

where $P_{\partial M^+_\varphi}$ is the set of peaks in the positive boundary with respect to φ and $P_{\partial M^-_\varphi}$ is the set of peaks in the negative boundary with respect to φ (see Definition 2.47).

In the following, we prove global Gauss-Bonnet type theorems. Triangulate M by admissible triangles such that each point in the set $P^\star(\varphi) := P_{\mathrm{Int}}(\varphi)\cup(S(\varphi)\cap\partial M)$ is a vertex, where $P_{\mathrm{Int}}(\varphi)$ is the set of peaks in $S(\varphi)\backslash\partial M$. We denote by F the set of faces, E the set of edges and V the set of vertices in the given triangulation.

Lemma 2.46 (Lemma 3.6 of [DZ20]). *The equation*

$$\begin{aligned} \#\left\{v\in V \mid v\in \left(M^+_\varphi\right)^\circ\right\} &= \chi\left(\overline{M^+_\varphi}\right) + \frac{\#\left\{f\in F \mid f\subseteq \overline{M^+_\varphi}\right\}}{2} \\ &\quad + \frac{\#\left\{e\in E \mid e\subseteq \partial M^+_\varphi\right\}}{2} \end{aligned} \tag{2.5.4}$$

$$-\#\left\{v\in V\mid v\in\partial M_{\varphi}^{+}\backslash P^{\star}(\varphi)\right\}$$
$$-\#I_{\mathrm{Int}}^{+}(\varphi)-\#NP_{\mathrm{Int}}(\varphi)-\#\left(S(\varphi)\cap\partial M\right)$$

holds, where A° is the interior of a topologial space A, $\overline{A}$ is the closure of A, $I_{\mathrm{Int}}^{+}(\varphi)$ is the set of positive isolated singular points in $S(\varphi)\backslash\partial M$ and $NP_{\mathrm{Int}}(\varphi)$ is the set of non-degenerate peaks in $S(\varphi)\backslash\partial M$.

PROOF. By the definition of the Euler characteristic,

$$\#\left\{v\in V\mid v\in\overline{M_{\varphi}^{+}}\right\}=\chi\left(\overline{M_{\varphi}^{+}}\right)+\#\left\{e\in E\mid e\subseteq\overline{M_{\varphi}^{+}}\right\}-\#\left\{f\in F\mid f\subseteq\overline{M_{\varphi}^{+}}\right\}. \tag{2.5.5}$$

Furthermore,

$$\#\left\{e\in E\mid e\subseteq\overline{M_{\varphi}^{+}}\right\}=\frac{3\#\left\{f\in F\mid f\subseteq\overline{M_{\varphi}^{+}}\right\}}{2}+\frac{\#\left\{e\in E\mid e\subseteq\partial M_{\varphi}^{+}\right\}}{2} \tag{2.5.6}$$

and

$$\begin{aligned}\#\left\{v\in V\mid v\in\overline{M_{\varphi}^{+}}\right\} &= \#\left\{v\in V\mid v\in\left(M_{\varphi}^{+}\right)^{\circ}\right\}\\ &\quad+\#\left\{v\in V\mid v\in\partial M_{\varphi}^{+}\backslash P^{\star}(\varphi)\right\}\\ &\quad+\#I_{\mathrm{Int}}^{+}(\varphi)+\#NP_{\mathrm{Int}}(\varphi)+\#\left(S(\varphi)\cap\partial M\right).\end{aligned} \tag{2.5.7}$$

Combining together (2.5.5), (2.5.6) and (2.5.7), we get the equation (2.5.4). □

Then by Lemma 2.46 and Theorems 2.16 and 2.18,

$$\begin{aligned}S_{\varphi}^{+} &:= \sum_{\triangle\mathrm{ABC}\subseteq\overline{M_{\varphi}^{+}}}\left(\angle\mathrm{A}_{\varphi}+\angle\mathrm{B}_{\varphi}+\angle\mathrm{C}_{\varphi}-\pi\right)\\
&= 2\pi\#\left\{v\in V\mid v\in\left(M_{\varphi}^{+}\right)^{\circ}\right\}+\pi\#\left\{v\in V\mid v\in\partial M_{\varphi}^{+}\backslash P^{\star}(\varphi)\right\}\\
&\quad+2\pi\#I_{\mathrm{Int}}^{+}(\varphi)+\sum_{p\in NP_{\mathrm{Int}}(\varphi)}\alpha_{\varphi}^{+}(p)+\sum_{p\in S(\varphi)\cap\partial M}\alpha_{\varphi}^{+}(p)-\pi\#\left\{f\in F\mid f\subseteq\overline{M_{\varphi}^{+}}\right\},\\
&= 2\pi\chi\left(\overline{M_{\varphi}^{+}}\right)+\pi\#\left\{e\in E\mid e\subseteq\partial M_{\varphi}^{+}\right\}-\pi\#\left\{v\in V\mid v\in\partial M_{\varphi}^{+}\backslash P^{\star}(\varphi)\right\}\\
&\quad+\sum_{p\in NP_{\mathrm{Int}}(\varphi)}\alpha_{\varphi}^{+}(p)+\sum_{p\in S(\varphi)\cap\partial M}\alpha_{\varphi}^{+}(p)-2\pi\#NP_{\mathrm{Int}}(\varphi)-2\pi\#\left(S(\varphi)\cap\partial M\right)\\
&= 2\pi\chi\left(\overline{M_{\varphi}^{+}}\right)+\frac{\pi}{2}\sum_{p\in\partial M_{\varphi}^{+}}\deg_{\partial M_{\varphi}^{+}}(p)-\pi\#\left\{v\in V\mid v\in\partial M_{\varphi}^{+}\backslash P^{\star}(\varphi)\right\}\\
&\quad+\sum_{p\in NP_{\mathrm{Int}}(\varphi)}\alpha_{\varphi}^{+}(p)+\sum_{p\in S(\varphi)\cap\partial M}\alpha_{\varphi}^{+}(p)-2\pi\#P^{\star}(\varphi)+2\pi\#I_{\mathrm{Int}}(\varphi),\end{aligned}$$

where $\deg_{\partial M_{\varphi}^{+}}(p):=\#\left\{e\in E\mid p\in e\subseteq\partial M_{\varphi}^{+}\right\}$. Since ∂M_{φ}^{+} is an Eulerian graph, we set

$$m_{\varphi}^{+}(p)=\frac{\deg_{\partial M_{\varphi}^{+}}(p)}{2}$$

because $\deg_{\partial M_\varphi^+}(p)$ is even. Since $\deg_{\partial M_\varphi^+}(p) = 2$ if $p \in \partial M_\varphi^+ \backslash P^\star(\varphi)$ and $\deg_{\partial M_\varphi^+}(p) = 0$ if $p \in I_{\mathrm{Int}}^+(\varphi)$,

$$\begin{aligned}\frac{\pi}{2}\sum_{p\in\partial M_\varphi^+}\deg_{\partial M_\varphi^+}(p) &= \pi\#\left\{v\in V \mid v\in\partial M_\varphi^+\backslash P^\star(\varphi)\right\}\\ &\quad + \sum_{p\in NP_{\mathrm{Int}}(\varphi)}\pi m_\varphi^+(p) + \sum_{p\in S(\varphi)\cap\partial M}\pi m_\varphi^+(p).\end{aligned}$$

Hence

$$\begin{aligned}S_\varphi^+ &= 2\pi\chi\left(\overline{M_\varphi^+}\right) + \sum_{p\in NP_{\mathrm{Int}}(\varphi)}\left(\alpha_\varphi^+(p) + \pi m_\varphi^+(p)\right)\\ &\quad + \sum_{p\in S(\varphi)\cap\partial M}\left(\alpha_\varphi^+(p) + \pi m_\varphi^+(p)\right) - 2\pi\#P^\star(\varphi) + 2\pi\#I_{\mathrm{Int}}(\varphi).\end{aligned}\tag{2.5.8}$$

Similarly,

$$\begin{aligned}S_\varphi^- &:= \sum_{\triangle\mathrm{ABC}\subseteq\overline{M_\varphi^-}}\left(\angle\mathrm{A}_\varphi + \angle\mathrm{B}_\varphi + \angle\mathrm{C}_\varphi - \pi\right)\\ &= 2\pi\chi\left(\overline{M_\varphi^-}\right) + \sum_{p\in NP_{\mathrm{Int}}(\varphi)}\left(\alpha_\varphi^-(p) + \pi m_\varphi^-(p)\right)\\ &\quad + \sum_{p\in S(\varphi)\cap\partial M}\left(\alpha_\varphi^-(p) + \pi m_\varphi^-(p)\right) - 2\pi\#P^\star(\varphi) + 2\pi\#I_{\mathrm{Int}}(\varphi),\end{aligned}\tag{2.5.9}$$

where $m_\varphi^-(p) := \deg_{\partial M_\varphi^-}(p)/2$.

Here, we classify peaks on the boundary ∂M into the following two types.

DEFINITION 2.47 (Definition 2.17 of [DZ20]). We say that a peak $p \in \partial M$ is in the *positive boundary* (respectively, *negative boundary*) with respect to φ if there exists a neighborhood U of p satisfying $(U\backslash\{p\})\cap\partial M \subseteq M_\varphi^+$ (respectively, $(U\backslash\{p\})\cap\partial M \subseteq M_\varphi^-$).

Then

$$m_\varphi^+(p) + m_\varphi^-(p) = \begin{cases}\deg_{S(\varphi)}(p) & \text{if } p\in S(\varphi)\backslash\partial M,\\ \deg_{S(\varphi)}(p) + 1 & \text{if } p\in S(\varphi)\cap\partial M,\end{cases}\tag{2.5.10}$$

$$m_\varphi^+(p) - m_\varphi^-(p) = \begin{cases}1 & \text{if } p\in P_{\partial M_\varphi^+},\\ -1 & \text{if } p\in P_{\partial M_\varphi^-},\\ 0 & \text{otherwise.}\end{cases}\tag{2.5.11}$$

Next, we give the formula for the Euler characteristic of $S(\varphi)$.

LEMMA 2.48 (Lemma 3.7 of [DZ20]). *The equation*

$$\chi(S(\varphi)) = \#P^\star(\varphi) - \frac{1}{2}\sum_{p\in P^\star(\varphi)}\left(m_\varphi^+(p) + m_\varphi^-(p)\right) + \frac{\#(S(\varphi)\cap\partial M)}{2}$$

holds.

Proof. By the definition of the Euler characteristic and the equation (2.5.10),

$$\begin{aligned}
\chi(S(\varphi)) &= \#\{v\in V \mid v\in S(\varphi)\} - \#\{e\in E \mid e\subseteq S(\varphi)\}\\
&= \#P^{\star}(\varphi) + \#(S(\varphi)\cap\partial M)\\
&\quad -\frac{1}{2}\sum_{p\in P_{\mathrm{Int}}(\varphi)}\deg_{S(\varphi)}(p) - \frac{1}{2}\sum_{p\in S(\varphi)\cap\partial M}\deg_{S(\varphi)}(p)\\
&= \#P^{\star}(\varphi) - \frac{1}{2}\sum_{p\in P^{\star}(\varphi)}\left(m_{\varphi}^{+}(p)+m_{\varphi}^{-}(p)\right) + \frac{\#(S(\varphi)\cap\partial M)}{2}.
\end{aligned}$$

□

Next, we give the formula for the sum and difference of S_{φ}^{+} and S_{φ}^{-}.

Lemma 2.49 (Lemma 3.8 of [DZ20]). *The equation*

$$S_{\varphi}^{+}+S_{\varphi}^{-} = 2\pi\chi(M) + \sum_{p\in(S(\varphi)\cap\partial M)^{\mathrm{null}}}\left(2\alpha_{\varphi}^{+}(p)-\pi\right)$$

holds.

Proof. By $\chi\left(\overline{M_{\varphi}^{+}}\right)+\chi\left(\overline{M_{\varphi}^{-}}\right)=\chi(M)+\chi(S(\varphi))-\#I_{\mathrm{Int}}(\varphi)$, the equations (2.5.8), (2.5.9) and (2.5.10), Lemma 2.48 and Theorems 2.16 and 2.18,

$$\begin{aligned}
S_{\varphi}^{+}+S_{\varphi}^{-} &= 2\pi\left(\chi\left(\overline{M_{\varphi}^{+}}\right)+\chi\left(\overline{M_{\varphi}^{-}}\right)\right) - 4\pi\#P^{\star}(\varphi) + 4\pi\#I_{\mathrm{Int}}(\varphi)\\
&\quad + \sum_{p\in NP_{\mathrm{Int}}(\varphi)}\left(\alpha_{\varphi}^{+}(p)+\alpha_{\varphi}^{-}(p)\right) + \pi\sum_{p\in NP_{\mathrm{Int}}(\varphi)}\left(m_{\varphi}^{+}(p)+m_{\varphi}^{-}(p)\right)\\
&\quad + \sum_{p\in S(\varphi)\cap\partial M}\left(\alpha_{\varphi}^{+}(p)+\alpha_{\varphi}^{-}(p)\right) + \pi\sum_{p\in S(\varphi)\cap\partial M}\left(m_{\varphi}^{+}(p)+m_{\varphi}^{-}(p)\right)\\
&= 2\pi\chi(M) + 2\pi\chi(S(\varphi)) - 4\pi\#P^{\star}(\varphi) + 2\pi\#I_{\mathrm{Int}}(\varphi)\\
&\quad + \sum_{p\in NP_{\mathrm{Int}}(\varphi)}\left(\alpha_{\varphi}^{+}(p)+\alpha_{\varphi}^{-}(p)\right) + \sum_{p\in S(\varphi)\cap\partial M}\left(\alpha_{\varphi}^{+}(p)+\alpha_{\varphi}^{-}(p)\right)\\
&\quad +\pi\sum_{p\in P^{\star}(\varphi)}\left(m_{\varphi}^{+}(p)+m_{\varphi}^{-}(p)\right)\\
&= 2\pi\chi(M) - \pi\sum_{p\in P^{\star}(\varphi)}\left(m_{\varphi}^{+}(p)+m_{\varphi}^{-}(p)\right) + \pi\#(S(\varphi)\cap\partial M)\\
&\quad -2\pi\#P^{\star}(\varphi) + 2\pi\#I_{\mathrm{Int}}(\varphi) + \sum_{p\in NP_{\mathrm{Int}}(\varphi)}\left(\alpha_{\varphi}^{+}(p)+\alpha_{\varphi}^{-}(p)\right)\\
&\quad + \sum_{p\in S(\varphi)\cap\partial M}\left(\alpha_{\varphi}^{+}(p)+\alpha_{\varphi}^{-}(p)\right) + \pi\sum_{p\in P^{\star}(\varphi)}\left(m_{\varphi}^{+}(p)+m_{\varphi}^{-}(p)\right)\\
&= 2\pi\chi(M) - 2\pi\#P_{\mathrm{Int}}(\varphi) + 2\pi\#I_{\mathrm{Int}}(\varphi) + 2\pi\#NP_{\mathrm{Int}}(\varphi)\\
&\quad -\pi\#(S(\varphi)\cap\partial M)^{+} - \pi\#(S(\varphi)\cap\partial M)^{\mathrm{null}} - \pi\#(S(\varphi)\cap\partial M)^{-}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{p\in(S(\varphi)\cap\partial M)^{+}}\left(\alpha_{\varphi}^{+}(p)+\alpha_{\varphi}^{-}(p)\right)+\sum_{p\in(S(\varphi)\cap\partial M)^{\mathrm{null}}}\left(\alpha_{\varphi}^{+}(p)+\alpha_{\varphi}^{-}(p)\right)\\
&+\sum_{p\in(S(\varphi)\cap\partial M)^{-}}\left(\alpha_{\varphi}^{+}(p)+\alpha_{\varphi}^{-}(p)\right)\\
=\;&2\pi\chi(M)+\sum_{p\in(S(\varphi)\cap\partial M)^{\mathrm{null}}}\left(2\alpha_{\varphi}^{+}(p)-\pi\right).
\end{aligned}
$$

□

Lemma 2.50 (Lemma 3.9 of [DZ20]). *The equation*

$$
\begin{aligned}
S_{\varphi}^{+}-S_{\varphi}^{-} \;=\;& 2\pi\left(\chi\left(M_{\varphi}^{+}\right)-\chi\left(M_{\varphi}^{-}\right)\right)+2\pi\left(\#P_{\mathrm{Int}}^{+}(\varphi)-\#P_{\mathrm{Int}}^{-}(\varphi)\right)\\
&+\pi\left(\#\left(S(\varphi)\cap\partial M\right)^{+}-\#\left(S(\varphi)\cap\partial M\right)^{-}\right)+\pi\left(\#P_{\partial M_{\varphi}^{+}}-\#P_{\partial M_{\varphi}^{-}}\right)
\end{aligned}
$$

holds.

Proof. By $\chi\left(\overline{M_{\varphi}^{+}}\right)-\chi\left(\overline{M_{\varphi}^{-}}\right)=\chi\left(M_{\varphi}^{+}\right)-\chi\left(M_{\varphi}^{-}\right)+\#I_{\mathrm{Int}}^{+}(\varphi)-\#I_{\mathrm{Int}}^{-}(\varphi)$, the equations (2.5.8), (2.5.9) and (2.5.11) and Theorems 2.16 and 2.18,

$$
\begin{aligned}
S_{\varphi}^{+}-S_{\varphi}^{-} \;=\;& 2\pi\left(\chi\left(\overline{M_{\varphi}^{+}}\right)-\chi\left(\overline{M_{\varphi}^{-}}\right)\right)+\sum_{p\in NP_{\mathrm{Int}}(\varphi)}\left(\alpha_{\varphi}^{+}(p)-\alpha_{\varphi}^{-}(p)\right)\\
&+\pi\sum_{p\in NP_{\mathrm{Int}}(\varphi)}\left(m_{\varphi}^{+}(p)-m_{\varphi}^{-}(p)\right)+\sum_{p\in S(\varphi)\cap\partial M}\left(\alpha_{\varphi}^{+}(p)-\alpha_{\varphi}^{-}(p)\right)\\
&+\pi\sum_{p\in S(\varphi)\cap\partial M}\left(m_{\varphi}^{+}(p)-m_{\varphi}^{-}(p)\right)\\
=\;& 2\pi\left(\chi\left(M_{\varphi}^{+}\right)-\chi\left(M_{\varphi}^{-}\right)\right)+2\pi\left(\#I_{\mathrm{Int}}^{+}(\varphi)-\#I_{\mathrm{Int}}^{-}(\varphi)\right)\\
&+\sum_{p\in NP_{\mathrm{Int}}(\varphi)}\left(\alpha_{\varphi}^{+}(p)-\alpha_{\varphi}^{-}(p)\right)+\pi\sum_{p\in NP_{\mathrm{Int}}(\varphi)}\left(m_{\varphi}^{+}(p)-m_{\varphi}^{-}(p)\right)\\
&+\sum_{p\in S(\varphi)\cap\partial M}\left(\alpha_{\varphi}^{+}(p)-\alpha_{\varphi}^{-}(p)\right)+\pi\sum_{p\in S(\varphi)\cap\partial M}\left(m_{\varphi}^{+}(p)-m_{\varphi}^{-}(p)\right)\\
=\;& 2\pi\left(\chi\left(M_{\varphi}^{+}\right)-\chi\left(M_{\varphi}^{-}\right)\right)+2\pi\left(\#I_{\mathrm{Int}}^{+}(\varphi)-\#I_{\mathrm{Int}}^{-}(\varphi)\right)\\
&+2\pi\left(\#NP_{\mathrm{Int}}^{+}(\varphi)-\#NP_{\mathrm{Int}}^{-}(\varphi)\right)\\
&+\pi\left(\#\left(S(\varphi)\cap\partial M\right)^{+}-\#\left(S(\varphi)\cap\partial M\right)^{-}\right)+\pi\left(\#P_{\partial M_{\varphi}^{+}}-\#P_{\partial M_{\varphi}^{-}}\right)\\
=\;& 2\pi\left(\chi\left(M_{\varphi}^{+}\right)-\chi\left(M_{\varphi}^{-}\right)\right)+2\pi\left(\#P_{\mathrm{Int}}^{+}(\varphi)-\#P_{\mathrm{Int}}^{-}(\varphi)\right)\\
&+\pi\left(\#\left(S(\varphi)\cap\partial M\right)^{+}-\#\left(S(\varphi)\cap\partial M\right)^{-}\right)+\pi\left(\#P_{\partial M_{\varphi}^{+}}-\#P_{\partial M_{\varphi}^{-}}\right).
\end{aligned}
$$

□

Proof of Theorem 2.45. Since the integration of the geometric curvature on curves which are not contained in $S(\varphi)\cup\partial M$ is canceled by opposite integrations, by Theorem

2.39,

$$\begin{aligned} S_\varphi^+ &= \int_{M_\varphi^+} K_\varphi dA_\varphi + \int_{\partial M_\varphi^+} \widetilde{\kappa}_\varphi^g d\tau_\varphi = \int_{M_\varphi^+} K_\varphi dA_\varphi + \int_{S(\varphi)} \kappa_\varphi^s d\tau_\varphi + \int_{\partial M \cap M_\varphi^+} \widehat{\kappa}_\varphi^g d\tau_\varphi, \\ S_\varphi^- &= \int_{M_\varphi^-} K_\varphi dA_\varphi + \int_{\partial M_\varphi^-} \widetilde{\kappa}_\varphi^g d\tau_\varphi = \int_{M_\varphi^-} K_\varphi dA_\varphi + \int_{S(\varphi)} \kappa_\varphi^s d\tau_\varphi - \int_{\partial M \cap M_\varphi^-} \widehat{\kappa}_\varphi^g d\tau_\varphi. \end{aligned}$$

Since the singular curvature is independent of the orientation of the singular curve by Proposition 2.21,

$$S_\varphi^+ + S_\varphi^- = \int_M K_\varphi dA_\varphi + 2\int_{\Sigma_\varphi} \kappa_\varphi^s d\tau_\varphi + \int_{\partial M} \widetilde{\kappa}_\varphi^g d\tau_\varphi, \tag{2.5.12}$$

$$S_\varphi^+ - S_\varphi^- = \int_M K_\varphi d\widehat{A}_\varphi + \int_{\partial M} \widehat{\kappa}_\varphi^g d\tau_\varphi. \tag{2.5.13}$$

Therefore, by Lemmas 2.49 and 2.50 and the equations (2.5.12) and (2.5.13), we obtain the equations (2.5.2) and (2.5.3). □

REMARK 2.51. If the boundary ∂M is empty,

$$2\pi\chi(M) = \int_M K_\varphi dA_\varphi + 2\int_{S(\varphi)} \kappa_\varphi^s d\tau_\varphi, \tag{2.5.14}$$

$$\frac{1}{2\pi}\int_M K_\varphi d\widehat{A}_\varphi = \chi\left(M_\varphi^+\right) - \chi\left(M_\varphi^-\right) + \#P^+(\varphi) - \#P^-(\varphi), \tag{2.5.15}$$

where $P^+(\varphi)$ is the set of positive peaks in $S(\varphi)$ and $P^-(\varphi)$ is the set of negative peaks in $S(\varphi)$. If the coherent tangent bundle $\mathcal{E}$ is induced from a wave front whose singular set admits cuspidal edges and swallowtails (see Example 2.6), the equation (2.5.14) is the same as the equation in [Kos02, Theorem 2] and the equation (2.5.15) is the same as the equation in [LLR95, Proposition 6] and [Kos02, Theorem 2].

6. Applications of Gauss-Bonnet type theorems

In this section, we give applications of the Gauss-Bonnet type theorems in Theorem 2.45 to smooth mappings between smooth 2-manifolds and to frontals.

6.1. Applications to smooth mappings between smooth 2-manifolds. First, we give an application of the equation (2.5.2) to smooth mappings between smooth 2-manifolds.

THEOREM 2.52 (Theorem 1 of [Lev66], Theorem of [Lev95], [Has23b]). *Let M be a compact oriented 2-manifold with boundary ∂M and $f : M \to \mathbb{R}^2$ a smooth mapping into the plane $\mathbb{R}^2$ whose singular set $S(f)$ of f admits at most peaks, and suppose that $S(f)\setminus I(f)$ consists of smooth closed regular curves and does not intersect ∂M and that $I(f)\cap\partial M$ is empty, where $I(f)$ is the set of isolated singular points of f. Then there exists* *a unique orientation of the components* $\{S_1, \dots, S_m\}$ *of* $S(f) \backslash I(f)$ *and the components* $\{\partial_1, \dots, \partial_n\}$ *of* ∂M *satisfying*

$$\frac{\chi(M)}{2} = \sum_{i=1}^{m} R(S_i) + \frac{1}{2}\sum_{i=1}^{n} R(\partial_i), \tag{2.6.1}$$

where $R(S_i)$ *is the rotation index of* S_i *and* $R(\partial_i)$ *is the rotation index of* ∂_i*.*

Proof. We equip $\mathbb{R}^2$ with the Euclidean metric $\overline{g}$ and construct a coherent tangent bundle $(M, f^*(T\mathbb{R}^2), \overline{g}, D, df)$ over M (see Example 2.5). Since the Gaussian curvature of $\overline{g}$ is zero and $S(f) \backslash I(f)$ does not intersect ∂M, the equation (2.5.2) is reduced to

$$2\pi\chi(M) = 2\int_{S(f)} \kappa_f^s d\tau_f + \int_{\partial M} \widetilde{\kappa}_f^g d\tau_f, \tag{2.6.2}$$

where $\kappa_f^s d\tau_f$ is the singular curvature measure with respect to df and $\widetilde{\kappa}_f^g d\tau_f$ is the geometric curvature measure with respect to df. Since the integral of the curvature of the image $f(S_i)$ as a planar curve is equal to the integral of the singular curvature of S_i by [SUY12, Proposition 3.1],

$$\int_{S(f)} \kappa_f^s d\tau_f = \int_{S(f)\backslash I(f)} \kappa_f^s d\tau_f = 2\pi \sum_{i=1}^{m} R(S_i). \tag{2.6.3}$$

Also, since the integral of the curvature of the image $f(\partial_i)$ as a planar curve is equal to the integral of the geometric curvature of ∂_i,

$$\int_{\partial M} \widetilde{\kappa}_f^g d\tau_f = 2\pi \sum_{i=1}^{n} R(\partial_i). \tag{2.6.4}$$

Hence we obtain the equation (2.6.1) by substituting the equations (2.6.3) and (2.6.4) into the equation (2.6.2). □

Next, we give an application of the equation (2.5.3) to smooth mappings between surfaces.

Let M, N be two oriented 2-manifolds without boundary and $f : M \to N$ a smooth mapping whose singular set $S(f)$ admits at most peaks. Take a 1-dimensional embedded submanifold S of N such that f is transversal to S, that is, the vector spaces $T_{f(p)}S$ and $df_p(T_pM)$ together span $T_{f(p)}N$ for each point $p \in f^{-1}(S)$. By regarding S as the boundary ∂N, we consider N as a smooth 2-manifold with boundary. Then the inverse $f^{-1}(\partial N)$ is a 1-dimensional embedded submanifold of M (cf. [Lee13, Theorem 6.30]). By regarding $f^{-1}(\partial N)$ as the boundary ∂M, we consider M as a smooth 2-manifold with boundary. We suppose that $S(f) \backslash I(f)$ is transversal to ∂M and that $I(f) \cap \partial M$ is empty.

We equip N with a Riemannian metric g and construct a coherent tangent bundle $(M, f^*(TN), g, D, df)$ over M (see Example 2.5). Then the equation (2.5.3) holds. By [Lee13, Theorem 17.35] and the classical Gauss-Bonnet theorem on N,

$$\int_M K_f d\widehat{A}_f \;=\; \int_M f^*(K_N dA_N) = \deg(f)\int_N K_N dA_N \tag{2.6.5}$$

$$= \deg(f)\left(2\pi\chi(N) - \int_{\partial N} \kappa_N^g d\tau_N\right),$$

where K_N is the Gaussian curvature of g, dA_N is the area form of g, $\deg(f)$ is the degree of f and $\kappa_N^g d\tau_N$ is a geodesic curvature measure on ∂N. On the other hand, by the property $f^{-1}(\partial N) = \partial M$ and [DFN85, Theorem 13.2.1],

$$\int_{\partial M} \widehat{\kappa}_f^g d\tau_f = \deg(f|_{\partial M}) \int_{\partial N} \kappa_N^g d\tau_N = \deg(f) \int_{\partial N} \kappa_N^g d\tau_N, \tag{2.6.6}$$

where $\widehat{\kappa}_f^g d\tau_f$ is the $f^*(TN)$-geodesic curvature measure with respect to df. Hence the following assertion holds by substituting the equations (2.6.5) and (2.6.6) into the equation (2.5.3).

Theorem 2.53 (Theorem 1 of [Qui78], Theorem 1.1 of [FI87], Proposition 4.1 of [DZ20], [Has23b]). *Let M, N be two compact oriented 2-manifolds with boundary and $f : M \to N$ a smooth mapping satisfying $f^{-1}(\partial N) = \partial M$ whose singular set $S(f)$ of f admits at most peaks, and suppose that $S(f) \setminus I(f)$ is transversal to ∂M and that $I(f) \cap \partial M$ is empty. Then*

$$\begin{aligned}
\deg(f)\chi(N) &= \chi\left(M_f^+\right) - \chi\left(M_f^-\right) + \#P_{\mathrm{Int}}^+(f) - \#P_{\mathrm{Int}}^-(f) \\
&\quad + \frac{\#(S(f)\cap\partial M)^+ - \#(S(f)\cap\partial M)^-}{2} + \frac{\#P_{\partial M_f^+} - \#P_{\partial M_f^-}}{2}.
\end{aligned}$$

6.2. Applications to frontals. First, we define frontal bundles in order to apply the Gauss-Bonnet type theorems in Theorem 2.45 to frontals. Let $(M, \mathcal{E}, \langle\cdot,\cdot\rangle, D, \varphi)$ be a coherent tangent bundle over an oriented smooth 2-manifold M (possibly with boundary). The 6-tuple $(M, \mathcal{E}, \langle\cdot,\cdot\rangle, D, \varphi, \psi)$ is called a *frontal bundle* over M if there exists another smooth bundle homomorphism $\psi : TM \to \mathcal{E}$ over M such that the tuple $(M, \mathcal{E}, \langle\cdot,\cdot\rangle, D, \psi)$ is also a coherent tangent bundle over M and the pair (φ, ψ) satisfies a compatibility condition

$$\langle\varphi(X), \psi(Y)\rangle = \langle\varphi(Y), \psi(X)\rangle \tag{2.6.7}$$

for any vector fields X, Y on M. Then the mapping φ is called the *first homomorphism* and ψ is called the *second homomorphism*. We define smooth symmetric covariant 2-tensor fields I, II, III on M by

$$\begin{aligned}
I(X,Y) &= ds_\varphi^2(X,Y) = \langle\varphi(X), \varphi(Y)\rangle, \\
II(X,Y) &= -\langle\varphi(X), \psi(Y)\rangle, \\
III(X,Y) &= ds_\psi^2(X,Y) = \langle\psi(X), \psi(Y)\rangle
\end{aligned}$$

for any vector fields X, Y on M, respectively (see (2.6.7)). We call I the *first fundamental form*, II the *second fundamental form* and III the *third fundamental form*.

In addition, we define front bundles.

DEFINITION 2.54 (Definition 2.13 of [SUY12]). A frontal bundle $(M, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi, \psi)$ is a *front bundle* if

$$\ker(\varphi_p) \cap \ker(\psi_p) = \{0\} \tag{2.6.8}$$

for each point $p \in M$.

We give a typical example of frontal bundles (front bundles).

EXAMPLE 2.55 (Example 2.14 of [SUY12], Proposition 2.4 of [SUY11]). Let M be an oriented smooth 2-manifold (possibly with boundary), $(N(c), g)$ an orientable 3-dimensional Riemannian manifold of constant sectional curvature c and $f : M \to N$ a co-orientable frontal. Then there exists a unit normal vector field ν of f globally defined on M by Proposition 1.15. Since the coherent tangent bundle $(M, \mathcal{E}_f, g, D, df)$ as in Example 2.6 is orthogonal to ν, we have a smooth bundle homomorphism

$$\nabla\nu : TM \to \mathcal{E}_f, \quad \nabla\nu(X) := \nabla_X \nu$$

over M, where ∇ is the induced connection of the Levi-Civita connection of g by f.

We show that the tuple $(M, \mathcal{E}_f, g, D, df, \nabla\nu)$ is a frontal bundle over M. Let R^c be the curvature tensor field of g. Since $\nabla_X \xi = D_X \xi - g(\nabla_X \nu, \xi)\nu$ for any vector field X on M and any smooth section ξ of $\mathcal{E}_f$,

$$\begin{aligned} R^c(X, Y)\xi &= R^D(X, Y)\xi - g(\nabla_Y \nu, \xi)\nabla_X \nu + g(\nabla_X \nu, \xi)\nabla_Y \nu \\ &\quad - \left(g(D_X \nabla_Y \nu, \xi) - g(D_Y \nabla_X \nu, \xi) - g\left(\nabla_{[X,Y]}\nu, \xi\right)\right)\nu, \end{aligned}$$

where R^D is the curvature tensor field of D, that is, $R^D(X, Y)\xi := D_X D_Y \xi - D_Y D_X \xi - D_{[X,Y]}\xi$. Since $N(c)$ has a constant sectional curvature c, taking the normal component,

$$g\left(D_X \nabla_Y \nu - D_Y \nabla_X \nu - \nabla_{[X,Y]}\nu, \xi\right) = 0,$$

which means that $(M, \mathcal{E}_f, g, D, \nabla\nu)$ is a coherent tangent bundle. Moreover, by the property of the coherent tangent bundle $(M, \mathcal{E}_f, g, D, df)$,

$$\begin{aligned} g(df(X), \nabla\nu(Y)) &= -g(\nabla_Y df(X), \nu) \\ &= -g(\nabla_X df(Y) - df([X, Y]), \nu) \\ &= -g(\nabla_X df(Y), \nu) \\ &= g(df(Y), \nabla\nu(X)), \end{aligned}$$

which means that the tuple $(M, \mathcal{E}_f, \langle \cdot, \cdot \rangle, D, df, \nabla\nu)$ is a frontal bundle.

By the condition (2.6.8), this frontal bundle is a front bundle if and only if f is a wave front, which is equivalent to $I + III$ being positive definite, that is, $(I + III)(v, v) \geq 0$ for each $v \in TM$, with equality if and only if $v = 0$.

Next, we define the extrinsic curvature of frontal bundles.

DEFINITION 2.56 (Definition 2.15 of [SUY12]). Let $(M, \mathcal{E}, \langle \cdot, \cdot \rangle, D, \varphi, \psi)$ be a frontal bundle. We define the *extrinsic curvature* K_φ^{ext} at each regular point $p \in M$ of φ by

$$K_\varphi^{\text{ext}}(v, w) = \frac{II(v, v)\, II(w, w) - II(v, w)^2}{I(v, v)\, I(w, w) - I(v, w)^2},$$

where $\{v, w\}$ is a basis for T_pM. Similarly, we define the *extrinsic curvature* K_ψ^{ext} at each regular point $p \in M$ of ψ by

$$K_\psi^{\mathrm{ext}}(v, w) = \frac{I\!I(v,v)\, I\!I(w,w) - I\!I(v,w)^2}{I\!I\!I(v,v)\, I\!I\!I(w,w) - I\!I\!I(v,w)^2}.$$

If we consider a frontal bundle $(M, \mathcal{E}_f, g, D, df, \nabla\nu)$ as in Example 2.55,

$$K_M = K_f^{\mathrm{ext}} + c$$

by the Gauss equation (cf. [KN69, Propositions 4.1 and 4.5]), where K_M is the sectional curvature at a regular point $p \in M$ of f and K_f^{ext} is the extrinsic curvature at a regular point $p \in M$ of f.

If we take local coordinates (u, v) compatible with the orientation of M and denote by $\widehat{I}$ (respectively, $\widehat{I\!I}$, $\widehat{I\!I\!I}$) the representation matrix of I (respectively, $I\!I$, $I\!I\!I$) with respect to $\{\partial/\partial u, \partial/\partial v\}$, that is,

$$\begin{aligned} \widehat{I} &= \begin{pmatrix} \langle\varphi_u, \varphi_u\rangle & \langle\varphi_u, \varphi_v\rangle \\ \langle\varphi_v, \varphi_u\rangle & \langle\varphi_v, \varphi_v\rangle \end{pmatrix}, \\ \widehat{I\!I} &= \begin{pmatrix} -\langle\varphi_u, \psi_u\rangle & -\langle\varphi_u, \psi_v\rangle \\ -\langle\varphi_v, \psi_u\rangle & -\langle\varphi_v, \psi_v\rangle \end{pmatrix}, \\ \widehat{I\!I\!I} &= \begin{pmatrix} \langle\psi_u, \psi_u\rangle & \langle\psi_u, \psi_v\rangle \\ \langle\psi_v, \psi_u\rangle & \langle\psi_v, \psi_v\rangle \end{pmatrix}, \end{aligned} \tag{2.6.9}$$

by Definition 2.56, we have

$$K_\varphi^{\mathrm{ext}} = \frac{\det \widehat{I\!I}}{\det \widehat{I}}, \quad K_\psi^{\mathrm{ext}} = \frac{\det \widehat{I\!I}}{\det \widehat{I\!I\!I}} \tag{2.6.10}$$

on (u, v). We prove the following assertion by using these representation matrices of the fundamental forms.

LEMMA 2.57 (Lemma 3.10 of [SUY12]). *Let $(M, \mathcal{E}, \langle\cdot,\cdot\rangle, D, \varphi, \psi)$ be a frontal bundle. Then*

(1) $K_\varphi d\widehat{A}_\varphi = K_\psi d\widehat{A}_\psi$,
(2) $K_\varphi^{\mathrm{ext}} d\widehat{A}_\varphi = d\widehat{A}_\psi$ *and* $K_\psi^{\mathrm{ext}} d\widehat{A}_\psi = d\widehat{A}_\varphi$,
(3) $\left|K_\varphi^{\mathrm{ext}}\right| dA_\varphi = dA_\psi$ *and* $\left|K_\psi^{\mathrm{ext}}\right| dA_\psi = dA_\varphi$,
(4) $K_\varphi^{\mathrm{ext}} K_\psi^{\mathrm{ext}} = 1$.

PROOF. Take local coordinates (u, v) compatible with the orientation μ of M, a positively oriented orthonormal frame $\{e_1, e_2\}$ with respect to the orientation of $\mathcal{E}$ and the connection form ω with respect to $\{e_1, e_2\}$.

The 2-form $K_\varphi d\widehat{A}_\varphi$ is determined by just the connection D by the equation (2.2.4), which yields that we have the equation (1).

Take 2×2-matrix valued functions G_φ and G_ψ as

$$(\varphi_u, \varphi_v) = (e_1, e_2)\, G_\varphi, \quad (\psi_u, \psi_v) = (e_1, e_2)\, G_\psi. \tag{2.6.11}$$

Then, by the equation (2.2.1),

$$d\widehat{A}_\varphi = \lambda_\varphi du \wedge dv = \mu(\varphi_u, \varphi_v)\, du \wedge dv = (\det G_\varphi)\, du \wedge dv, \tag{2.6.12}$$
$$d\widehat{A}_\psi = \lambda_\psi du \wedge dv = \mu(\psi_u, \psi_v)\, du \wedge dv = (\det G_\psi)\, du \wedge dv.$$

In particular,

$$dA_\varphi = |\det G_\varphi|\, du \wedge dv, \quad dA_\psi = |\det G_\psi|\, du \wedge dv.$$

Also, the matrices $\widehat{I}$, $\widehat{II}$ and $\widehat{III}$ as in (2.6.9) are written as

$$\widehat{I} = G_\varphi^T G_\varphi, \quad \widehat{II} = -G_\varphi^T G_\psi = -G_\psi^T G_\varphi, \quad \widehat{III} = G_\psi^T G_\psi,$$

respectively, where A^T denotes the transposition of A. Thus, by the equation (2.6.10),

$$K_\varphi^{\mathrm{ext}} = \frac{\det G_\psi}{\det G_\varphi}, \quad K_\psi^{\mathrm{ext}} = \frac{\det G_\varphi}{\det G_\psi}, \tag{2.6.13}$$

which yields that we have the equations (2), (3) and (4). $\square$

Next, we give Gauss-Bonnet type theorems for frontal bundles.

Corollary 2.58 ([SUY12] if $\partial M = \varnothing$, [Has23a] if $\partial M \neq \varnothing$). *Let $(M, \mathcal{E}, \langle\cdot,\cdot\rangle, D, \varphi, \psi)$ be a frontal bundle over a compact oriented smooth 2-manifold M with boundary ∂M whose singular sets $S(\varphi)$ and $S(\psi)$ admit at most peaks, and suppose that $S(\varphi) \setminus I(\varphi)$ and $S(\psi) \setminus I(\psi)$ are transversal to ∂M and that $I(\varphi) \cap \partial M$ and $I(\psi) \cap \partial M$ are empty. Then*

$$2\pi\chi(M) = \int_M K_\varphi dA_\varphi + 2\int_{S(\varphi)} \kappa_\varphi^s d\tau_\varphi + \int_{\partial M} \widetilde{\kappa}_\varphi^g d\tau_\varphi - \sum_{p \in (S(\varphi)\cap\partial M)^{\mathrm{null}}} \left(2\alpha_\varphi^+(p) - \pi\right), \tag{2.6.14}$$

$$\int_M K_\varphi d\widehat{A}_\varphi + \int_{\partial M} \widehat{\kappa}_\varphi^g d\tau_\varphi = 2\pi\left(\chi\left(M_\varphi^+\right) - \chi\left(M_\varphi^-\right)\right) + 2\pi\left(\#P_{\mathrm{Int}}^+(\varphi) - \#P_{\mathrm{Int}}^-(\varphi)\right) + \pi\left(\#(S(\varphi)\cap\partial M)^+ - \#(S(\varphi)\cap\partial M)^-\right) + \pi\left(\#P_{\partial M_\varphi^+} - \#P_{\partial M_\varphi^-}\right), \tag{2.6.15}$$

$$2\pi\chi(M) = \int_M K_\psi dA_\psi + 2\int_{S(\psi)} \kappa_\psi^s d\tau_\psi + \int_{\partial M} \widetilde{\kappa}_\psi^g d\tau_\psi - \sum_{p \in (S(\psi)\cap\partial M)^{\mathrm{null}}} \left(2\alpha_\psi^+(p) - \pi\right), \tag{2.6.16}$$

$$\int_M K_\psi d\widehat{A}_\psi + \int_{\partial M} \widehat{\kappa}_\psi^g d\tau_\psi = 2\pi\left(\chi\left(M_\psi^+\right) - \chi\left(M_\psi^-\right)\right) + 2\pi\left(\#P_{\mathrm{Int}}^+(\psi) - \#P_{\mathrm{Int}}^-(\psi)\right) + \pi\left(\#(S(\psi)\cap\partial M)^+ - \#(S(\psi)\cap\partial M)^-\right) \tag{2.6.17}$$

$$+\pi\left(\#P_{\partial M_\psi^+}-\#P_{\partial M_\psi^-}\right).$$

PROOF. By applying the Gauss-Bonnet type theorems in Theorem 2.45 to the mappings φ and ψ. □

We derive four new types of equalities by appropriately combining these Gauss-Bonnet type theorems.

THEOREM 2.59 ([Has23a]). *If $(M, \mathcal{E}, \langle\cdot,\cdot\rangle, D, \varphi, \psi)$ is a frontal bundle as in Corollary 2.58,*

$$\begin{aligned}(2.6.18)\qquad 2\chi\left(M_\psi^-\right) = {} & \frac{1}{\pi}\left(\int_{S(\varphi)}\kappa_\varphi^s d\tau_\varphi - \int_{M_\varphi^-} K_\varphi d\widehat{A}_\varphi\right.\\ & \left.+\frac{1}{2}\int_{\partial M}\widetilde{\kappa}_\varphi^g d\tau_\varphi - \sum_{p\in(S(\varphi)\cap\partial M)^{\mathrm{null}}}\frac{2\alpha_\varphi^+(p)-\pi}{2}\right)\\ & -\frac{1}{2\pi}\int_{\partial M}\widehat{\kappa}_\psi^g d\tau_\psi - \#P^\star(\psi) + \sum_{p\in P^\star(\psi)}\frac{\deg_{S(\psi)}(p)}{2}\\ & +\#P_{\mathrm{Int}}^+(\psi) - \#P_{\mathrm{Int}}^-(\psi)\\ & +\frac{\#(S(\psi)\cap\partial M)^+ - \#(S(\psi)\cap\partial M)^-}{2}\\ & +\frac{\#P_{\partial M_\psi^+}-\#P_{\partial M_\psi^-}}{2},\end{aligned}$$

$$\begin{aligned}(2.6.19)\qquad 2\chi\left(M_\varphi^-\right) = {} & \frac{1}{\pi}\left(\int_{S(\psi)}\kappa_\psi^s d\tau_\psi - \int_{M_\psi^-} K_\psi d\widehat{A}_\psi\right.\\ & \left.+\frac{1}{2}\int_{\partial M}\widetilde{\kappa}_\psi^g d\tau_\psi - \sum_{p\in(S(\psi)\cap\partial M)^{\mathrm{null}}}\frac{2\alpha_\psi^+(p)-\pi}{2}\right)\\ & -\frac{1}{2\pi}\int_{\partial M}\widehat{\kappa}_\varphi^g d\tau_\varphi - \#P^\star(\varphi) + \sum_{p\in P^\star(\varphi)}\frac{\deg_{S(\varphi)}(p)}{2}\\ & +\#P_{\mathrm{Int}}^+(\varphi) - \#P_{\mathrm{Int}}^-(\varphi)\\ & +\frac{\#(S(\varphi)\cap\partial M)^+ - \#(S(\varphi)\cap\partial M)^-}{2}\\ & +\frac{\#P_{\partial M_\varphi^+}-\#P_{\partial M_\varphi^-}}{2}.\end{aligned}$$

In particular, if all peaks of the mappings φ and ψ are non-degenerate,

$$
\begin{aligned}
2\chi\left(M_{\psi}^{-}\right) &= \frac{1}{\pi}\left(\int_{S(\varphi)} \kappa_{\varphi}^{s} d\tau_{\varphi} - \int_{M_{\varphi}^{-}} K_{\varphi} d\widehat{A}_{\varphi} + \frac{1}{2}\int_{\partial M} \widetilde{\kappa}_{\varphi}^{g} d\tau_{\varphi} - \sum_{p\in(S(\varphi)\cap\partial M)^{\mathrm{null}}} \alpha_{\varphi}^{+}(p)\right) \\
&\quad -\frac{1}{2\pi}\int_{\partial M} \widehat{\kappa}_{\psi}^{g} d\tau_{\psi} + \#P_{\mathrm{Int}}^{+}(\psi) - \#P_{\mathrm{Int}}^{-}(\psi) - \#\left(S(\psi)\cap\partial M\right)^{-}, \\
2\chi\left(M_{\varphi}^{-}\right) &= \frac{1}{\pi}\left(\int_{S(\psi)} \kappa_{\psi}^{s} d\tau_{\psi} - \int_{M_{\psi}^{-}} K_{\psi} d\widehat{A}_{\psi} + \frac{1}{2}\int_{\partial M} \widetilde{\kappa}_{\psi}^{g} d\tau_{\psi} - \sum_{p\in(S(\psi)\cap\partial M)^{\mathrm{null}}} \alpha_{\psi}^{+}(p)\right) \\
&\quad -\frac{1}{2\pi}\int_{\partial M} \widehat{\kappa}_{\varphi}^{g} d\tau_{\varphi} + \#P_{\mathrm{Int}}^{+}(\varphi) - \#P_{\mathrm{Int}}^{-}(\varphi) - \#\left(S(\varphi)\cap\partial M\right)^{-}.
\end{aligned}
$$

Proof. First, we prove the equation (2.6.18). By Lemma 2.48 and the equation (2.5.10),

$$
\begin{aligned}
\chi(M) &= \chi\left(M_{\psi}^{+}\right) + \chi\left(M_{\psi}^{-}\right) + \chi\left(S(\psi)\right) \\
&= \chi\left(M_{\psi}^{+}\right) + \chi\left(M_{\psi}^{-}\right) + \#P^{\star}(\psi) \\
&\quad -\frac{1}{2}\sum_{p\in P^{\star}(\psi)} \left(m_{\psi}^{+}(p) + m_{\psi}^{-}(p)\right) + \frac{\#\left(S(\psi)\cap\partial M\right)}{2} \\
&= \chi\left(M_{\psi}^{+}\right) + \chi\left(M_{\psi}^{-}\right) + \#P^{\star}(\psi) - \frac{1}{2}\sum_{p\in P^{\star}(\psi)} \deg_{S(\psi)}(p).
\end{aligned}
\tag{2.6.20}
$$

By the equation (1) in Lemma 2.57 and the equations (2.6.17) and (2.6.20),

$$
\begin{aligned}
\int_{M} K_{\varphi} dA_{\varphi} &= \int_{M} K_{\varphi} d\widehat{A}_{\varphi} - 2\int_{M_{\varphi}^{-}} K_{\varphi} d\widehat{A}_{\varphi} \\
&= \int_{M} K_{\psi} d\widehat{A}_{\psi} - 2\int_{M_{\varphi}^{-}} K_{\varphi} d\widehat{A}_{\varphi} \\
&= -2\int_{M_{\varphi}^{-}} K_{\varphi} d\widehat{A}_{\varphi} - \int_{\partial M} \widehat{\kappa}_{\psi}^{g} d\tau_{\psi} + 2\pi\left(\chi\left(M_{\psi}^{+}\right) - \chi\left(M_{\psi}^{-}\right)\right) \\
&\quad +2\pi\left(\#P_{\mathrm{Int}}^{+}(\psi) - \#P_{\mathrm{Int}}^{-}(\psi)\right) \\
&\quad +\pi\left(\#\left(S(\psi)\cap\partial M\right)^{+} - \#\left(S(\psi)\cap\partial M\right)^{-}\right) \\
&\quad +\pi\left(\#P_{\partial M_{\psi}^{+}} - \#P_{\partial M_{\psi}^{-}}\right) \\
&= -2\int_{M_{\varphi}^{-}} K_{\varphi} d\widehat{A}_{\varphi} - \int_{\partial M} \widehat{\kappa}_{\psi}^{g} d\tau_{\psi} + 2\pi\chi(M) - 4\pi\chi\left(M_{\psi}^{-}\right) \\
&\quad -2\pi\#P^{\star}(\psi) + \pi\sum_{p\in P^{\star}(\psi)} \deg_{S(\psi)}(p) \\
&\quad +2\pi\left(\#P_{\mathrm{Int}}^{+}(\psi) - \#P_{\mathrm{Int}}^{-}(\psi)\right)
\end{aligned}
\tag{2.6.21}
$$

$$+\pi\left(\#\left(S\left(\psi\right)\cap\partial M\right)^{+}-\#\left(S\left(\psi\right)\cap\partial M\right)^{-}\right)$$
$$+\pi\left(\#P_{\partial M_{\psi}^{+}}-\#P_{\partial M_{\psi}^{-}}\right).$$

By the equations (2.6.14) and (2.6.21),

$$\begin{aligned}
2\pi\chi\left(M\right) \;=\; & -2\int_{M_{\varphi}^{-}}K_{\varphi}d\widehat{A}_{\varphi}+2\int_{S(\varphi)}\kappa_{\varphi}^{s}d\tau_{\varphi}+\int_{\partial M}\widetilde{\kappa}_{\varphi}^{g}d\tau_{\varphi} \\
& -\sum_{p\in\left(S(\varphi)\cap\partial M\right)^{\mathrm{null}}}\left(2\alpha_{\varphi}^{+}\left(p\right)-\pi\right) \\
& -\int_{\partial M}\widehat{\kappa}_{\psi}^{g}d\tau_{\psi}+2\pi\chi\left(M\right)-4\pi\chi\left(M_{\psi}^{-}\right) \\
& -2\pi\#P^{\star}\left(\psi\right)+\pi\sum_{p\in P^{\star}(\psi)}\deg_{S(\psi)}\left(p\right) \\
& +2\pi\left(\#P_{\mathrm{Int}}^{+}\left(\psi\right)-\#P_{\mathrm{Int}}^{-}\left(\psi\right)\right) \\
& +\pi\left(\#\left(S\left(\psi\right)\cap\partial M\right)^{+}-\#\left(S\left(\psi\right)\cap\partial M\right)^{-}\right) \\
& +\pi\left(\#P_{\partial M_{\psi}^{+}}-\#P_{\partial M_{\psi}^{-}}\right).
\end{aligned}\tag{2.6.22}$$

By eliminating $2\pi\chi\left(M\right)$ on both sides of the equation (2.6.22), the equation (2.6.18) holds.

On the other hand, the equation (2.6.19) is obtained by exchanging the roles of the mappings φ and ψ in the above proof.

Note that $\deg_{S(\varphi)}\left(p\right)=2$ if $p\in P_{\mathrm{Int}}\left(\varphi\right)$ is non-degenerate and $\deg_{S(\varphi)}\left(p\right)=1$ if $p\in S\left(\varphi\right)\cap\partial M$ is non-degenerate. $\square$

THEOREM 2.60 ([Has23a]). *If $\left(M,\mathcal{E},\langle\cdot,\cdot\rangle,D,\varphi,\psi\right)$ is a frontal bundle as in Corollary 2.58,*

$$\begin{aligned}
& \int_{M_{\varphi}^{-}}K_{\varphi}d\widehat{A}_{\varphi}-\int_{S(\varphi)}\kappa_{\varphi}^{s}d\tau_{\varphi}-\frac{1}{2}\int_{\partial M}\widetilde{\kappa}_{\varphi}^{g}d\tau_{\varphi}+\sum_{p\in\left(S(\varphi)\cap\partial M\right)^{\mathrm{null}}}\frac{2\alpha_{\varphi}^{+}\left(p\right)-\pi}{2} \\
= & \int_{M_{\psi}^{-}}K_{\psi}d\widehat{A}_{\psi}-\int_{S(\psi)}\kappa_{\psi}^{s}d\tau_{\psi}-\frac{1}{2}\int_{\partial M}\widetilde{\kappa}_{\psi}^{g}d\tau_{\psi}+\sum_{p\in\left(S(\psi)\cap\partial M\right)^{\mathrm{null}}}\frac{2\alpha_{\psi}^{+}\left(p\right)-\pi}{2},
\end{aligned}\tag{2.6.23}$$

$$\begin{aligned}
& \frac{1}{2\pi}\int_{\partial M}\widehat{\kappa}_{\varphi}^{g}d\tau_{\varphi}-\left(\chi\left(M_{\varphi}^{+}\right)-\chi\left(M_{\varphi}^{-}\right)\right)-\left(\#P_{\mathrm{Int}}^{+}\left(\varphi\right)-\#P_{\mathrm{Int}}^{-}\left(\varphi\right)\right) \\
& -\frac{\#\left(S\left(\varphi\right)\cap\partial M\right)^{+}-\#\left(S\left(\varphi\right)\cap\partial M\right)^{-}}{2}-\frac{\#P_{\partial M_{\varphi}^{+}}-\#P_{\partial M_{\varphi}^{-}}}{2} \\
= & \frac{1}{2\pi}\int_{\partial M}\widehat{\kappa}_{\psi}^{g}d\tau_{\psi}-\left(\chi\left(M_{\psi}^{+}\right)-\chi\left(M_{\psi}^{-}\right)\right)-\left(\#P_{\mathrm{Int}}^{+}\left(\psi\right)-\#P_{\mathrm{Int}}^{-}\left(\psi\right)\right) \\
& -\frac{\#\left(S\left(\psi\right)\cap\partial M\right)^{+}-\#\left(S\left(\psi\right)\cap\partial M\right)^{-}}{2}-\frac{\#P_{\partial M_{\psi}^{+}}-\#P_{\partial M_{\psi}^{-}}}{2}.
\end{aligned}\tag{2.6.24}$$

PROOF. By the equations (2.6.14), (2.6.16), $\int_M K_\varphi dA_\varphi = \int_M K_\varphi d\widehat{A}_\varphi - 2\int_{M_\varphi^-} K_\varphi d\widehat{A}_\varphi$ and $\int_M K_\psi dA_\psi = \int_M K_\psi d\widehat{A}_\psi - 2\int_{M_\psi^-} K_\psi d\widehat{A}_\psi$,

$$\begin{aligned}\int_M K_\varphi d\widehat{A}_\varphi &= 2\pi\chi(M) + 2\int_{M_\varphi^-} K_\varphi d\widehat{A}_\varphi - 2\int_{S(\varphi)} \kappa_\varphi^s d\tau_\varphi - \int_{\partial M} \widetilde{\kappa}_\varphi^g d\tau_\varphi \\ &\quad + \sum_{p\in(S(\varphi)\cap\partial M)^{\mathrm{null}}} \left(2\alpha_\varphi^+(p) - \pi\right),\end{aligned} \tag{2.6.25}$$

$$\begin{aligned}\int_M K_\psi d\widehat{A}_\psi &= 2\pi\chi(M) + 2\int_{M_\psi^-} K_\psi d\widehat{A}_\psi - 2\int_{S(\psi)} \kappa_\psi^s d\tau_\psi - \int_{\partial M} \widetilde{\kappa}_\psi^g d\tau_\psi \\ &\quad + \sum_{p\in(S(\psi)\cap\partial M)^{\mathrm{null}}} \left(2\alpha_\psi^+(p) - \pi\right).\end{aligned} \tag{2.6.26}$$

By the equation (1) of Lemma 2.57, the right-hand sides of (2.6.25) and (2.6.26) are equal, which yields the equation (2.6.23).

On the other hand, the equation (2.6.24) is obtained by the equations (2.6.15) and (2.6.17) and the equation (1) of Lemma 2.57. □

Next, we give special versions of Theorems 2.59 and 2.60.

COROLLARY 2.61 ([Has23a]). *Let $(M, \mathcal{E}, \langle\cdot,\cdot\rangle, D, \varphi, \psi)$ be a frontal bundle as in Corollary 2.58.*

(1) *If $S(\varphi) \neq \varnothing$, $S(\psi) \neq \varnothing$ and $\partial M = \varnothing$,*

$$\begin{aligned}2\chi\left(M_\psi^-\right) &= \frac{1}{\pi}\left(\int_{S(\varphi)} \kappa_\varphi^s d\tau_\varphi - \int_{M_\varphi^-} K_\varphi d\widehat{A}_\varphi\right) - \#P(\psi) + \sum_{p\in P(\psi)} \frac{\deg_{S(\psi)}(p)}{2} \\ &\quad + \#P^+(\psi) - \#P^-(\psi),\end{aligned}$$

$$\begin{aligned}2\chi\left(M_\varphi^-\right) &= \frac{1}{\pi}\left(\int_{S(\psi)} \kappa_\psi^s d\tau_\psi - \int_{M_\psi^-} K_\psi d\widehat{A}_\psi\right) - \#P(\varphi) + \sum_{p\in P(\varphi)} \frac{\deg_{S(\varphi)}(p)}{2} \\ &\quad + \#P^+(\varphi) - \#P^-(\varphi),\end{aligned}$$

$$\begin{aligned}&\int_{M_\varphi^-} K_\varphi d\widehat{A}_\varphi - \int_{S(\varphi)} \kappa_\varphi^s d\tau_\varphi \\ &\qquad = \int_{M_\psi^-} K_\psi d\widehat{A}_\psi - \int_{S(\psi)} \kappa_\psi^s d\tau_\psi,\end{aligned}$$

$$\begin{aligned}&\chi\left(M_\varphi^+\right) - \chi\left(M_\varphi^-\right) + \#P^+(\varphi) - \#P^-(\varphi) \\ &\qquad = \chi\left(M_\psi^+\right) - \chi\left(M_\psi^-\right) + \#P^+(\psi) - \#P^-(\psi).\end{aligned}$$

In particular, if all peaks of the mappings φ and ψ are non-degenerate,

$$2\chi\left(M_\psi^-\right) = \frac{1}{\pi}\left(\int_{S(\varphi)} \kappa_\varphi^s d\tau_\varphi - \int_{M_\varphi^-} K_\varphi d\widehat{A}_\varphi\right) + \#P^+(\psi) - \#P^-(\psi),$$

$$2\chi\left(M_{\varphi}^{-}\right)=\frac{1}{\pi}\left(\int_{S(\psi)}\kappa_{\psi}^{s}d\tau_{\psi}-\int_{M_{\psi}^{-}}K_{\psi}d\widehat{A}_{\psi}\right)+\#P^{+}\left(\varphi\right)-\#P^{-}\left(\varphi\right),$$

$$\begin{aligned}&\int_{M_{\varphi}^{-}}K_{\varphi}d\widehat{A}_{\varphi}-\int_{S(\varphi)}\kappa_{\varphi}^{s}d\tau_{\varphi}\\&\qquad=\int_{M_{\psi}^{-}}K_{\psi}d\widehat{A}_{\psi}-\int_{S(\psi)}\kappa_{\psi}^{s}d\tau_{\psi},\end{aligned}$$

$$\begin{aligned}&\chi\left(M_{\varphi}^{+}\right)-\chi\left(M_{\varphi}^{-}\right)+\#P^{+}\left(\varphi\right)-\#P^{-}\left(\varphi\right)\\&\qquad=\chi\left(M_{\psi}^{+}\right)-\chi\left(M_{\psi}^{-}\right)+\#P^{+}\left(\psi\right)-\#P^{-}\left(\psi\right).\end{aligned}$$

(2) *If $S(\varphi)\neq\varnothing$, $S(\psi)=\varnothing$ and $\partial M\neq\varnothing$,*

$$\begin{aligned}2\chi\left(M_{\varphi}^{-}\right) &= \frac{1}{2\pi}\int_{\partial M}\widetilde{\kappa}_{\psi}^{g}d\tau_{\psi}-\frac{1}{2\pi}\int_{\partial M}\widehat{\kappa}_{\varphi}^{g}d\tau_{\varphi}-\#P^{\star}\left(\varphi\right)\\&\quad+\sum_{p\in P^{\star}(\varphi)}\frac{\deg_{S(\varphi)}\left(p\right)}{2}+\#P_{\mathrm{Int}}^{+}\left(\varphi\right)-\#P_{\mathrm{Int}}^{-}\left(\varphi\right)\\&\quad+\frac{\#\left(S\left(\varphi\right)\cap\partial M\right)^{+}-\#\left(S\left(\varphi\right)\cap\partial M\right)^{-}}{2}+\frac{\#P_{\partial M_{\varphi}^{+}}-\#P_{\partial M_{\varphi}^{-}}}{2},\\
\int_{\partial M}\widehat{\kappa}_{\psi}^{g}d\tau_{\psi} &= 2\int_{S(\varphi)}\kappa_{\varphi}^{s}d\tau_{\varphi}-2\int_{M_{\varphi}^{-}}K_{\varphi}d\widehat{A}_{\varphi}+\int_{\partial M}\widetilde{\kappa}_{\varphi}^{g}d\tau_{\varphi}\\&\quad-\sum_{p\in(S(\varphi)\cap\partial M)^{\mathrm{null}}}\left(2\alpha_{\varphi}^{+}\left(p\right)-\pi\right).\end{aligned}$$

In particular, if all peaks of the mapping φ are non-degenerate,

$$\begin{aligned}2\chi\left(M_{\varphi}^{-}\right) &= \frac{1}{2\pi}\int_{\partial M}\widetilde{\kappa}_{\psi}^{g}d\tau_{\psi}-\frac{1}{2\pi}\int_{\partial M}\widehat{\kappa}_{\varphi}^{g}d\tau_{\varphi}\\&\quad+\#P_{\mathrm{Int}}^{+}\left(\varphi\right)-\#P_{\mathrm{Int}}^{-}\left(\varphi\right)\\&\quad+\#\left(S\left(\varphi\right)\cap\partial M\right)^{+}+\frac{\#\left(S\left(\varphi\right)\cap\partial M\right)^{\mathrm{null}}}{2},\\
\int_{\partial M}\widehat{\kappa}_{\psi}^{g}d\tau_{\psi} &= 2\int_{S(\varphi)}\kappa_{\varphi}^{s}d\tau_{\varphi}-2\int_{M_{\varphi}^{-}}K_{\varphi}d\widehat{A}_{\varphi}+\int_{\partial M}\widetilde{\kappa}_{\varphi}^{g}d\tau_{\varphi}\\&\quad-\sum_{p\in(S(\varphi)\cap\partial M)^{\mathrm{null}}}\left(2\alpha_{\varphi}^{+}\left(p\right)-\pi\right).\end{aligned}$$

(3) *If $S(\varphi)=\varnothing$, $S(\psi)\neq\varnothing$ and $\partial M\neq\varnothing$,*

$$\begin{aligned}2\chi\left(M_{\psi}^{-}\right) &= \frac{1}{2\pi}\int_{\partial M}\widetilde{\kappa}_{\varphi}^{g}d\tau_{\varphi}-\frac{1}{2\pi}\int_{\partial M}\widehat{\kappa}_{\psi}^{g}d\tau_{\psi}-\#P^{\star}\left(\psi\right)\\&\quad+\sum_{p\in P^{\star}(\psi)}\frac{\deg_{S(\psi)}\left(p\right)}{2}+\#P_{\mathrm{Int}}^{+}\left(\psi\right)-\#P_{\mathrm{Int}}^{-}\left(\psi\right)\end{aligned}$$

$$+\frac{\#\left(S\left(\psi\right)\cap\partial M\right)^{+}-\#\left(S\left(\psi\right)\cap\partial M\right)^{-}}{2}+\frac{\#P_{\partial M_{\psi}^{+}}-\#P_{\partial M_{\psi}^{-}}}{2},$$

$$\begin{aligned}\int_{\partial M}\widehat{\kappa}_{\varphi}^{g}d\tau_{\varphi} &= 2\int_{S(\psi)}\kappa_{\psi}^{s}d\tau_{\psi}-2\int_{M_{\psi}^{-}}K_{\psi}d\widehat{A}_{\psi}+\int_{\partial M}\widetilde{\kappa}_{\psi}^{g}d\tau_{\psi}\\ &\quad -\sum_{p\in(S(\psi)\cap\partial M)^{\mathrm{null}}}\left(2\alpha_{\psi}^{+}\left(p\right)-\pi\right).\end{aligned}$$

In particular, if all peaks of the mapping ψ are non-degenerate,

$$\begin{aligned}2\chi\left(M_{\psi}^{-}\right) &= \frac{1}{2\pi}\int_{\partial M}\widetilde{\kappa}_{\varphi}^{g}d\tau_{\varphi}-\frac{1}{2\pi}\int_{\partial M}\widehat{\kappa}_{\psi}^{g}d\tau_{\psi}\\ &\quad +\#P_{\mathrm{Int}}^{+}\left(\psi\right)-\#P_{\mathrm{Int}}^{-}\left(\psi\right)\\ &\quad +\#\left(S\left(\psi\right)\cap\partial M\right)^{+}+\frac{\#\left(S\left(\psi\right)\cap\partial M\right)^{\mathrm{null}}}{2},\\ \int_{\partial M}\widehat{\kappa}_{\varphi}^{g}d\tau_{\varphi} &= 2\int_{S(\psi)}\kappa_{\psi}^{s}d\tau_{\psi}-2\int_{M_{\psi}^{-}}K_{\psi}d\widehat{A}_{\psi}+\int_{\partial M}\widetilde{\kappa}_{\psi}^{g}d\tau_{\psi}\\ &\quad -\sum_{p\in(S(\psi)\cap\partial M)^{\mathrm{null}}}\left(2\alpha_{\psi}^{+}\left(p\right)-\pi\right).\end{aligned}$$

(4) *If $S\left(\varphi\right)\neq\varnothing$, $S\left(\psi\right)=\varnothing$ and $\partial M=\varnothing$,*

$$\begin{aligned}2\chi\left(M_{\varphi}^{-}\right) &= \sum_{p\in P(\varphi)}\frac{\deg_{S(\varphi)}\left(p\right)}{2}-\#P\left(\varphi\right)+\#P^{+}\left(\varphi\right)-\#P^{-}\left(\varphi\right),\\ \int_{M_{\varphi}^{-}}K_{\varphi}d\widehat{A}_{\varphi} &= \int_{S(\varphi)}\kappa_{\varphi}^{s}d\tau_{\varphi}.\end{aligned}$$

In particular, if all peaks of the mapping φ are non-degenerate,

$$2\chi\left(M_{\varphi}^{-}\right) = \#P^{+}\left(\varphi\right)-\#P^{-}\left(\varphi\right), \tag{2.6.27}$$

$$\int_{M_{\varphi}^{-}}K_{\varphi}d\widehat{A}_{\varphi} = \int_{S(\varphi)}\kappa_{\varphi}^{s}d\tau_{\varphi}. \tag{2.6.28}$$

(5) *If $S\left(\varphi\right)=\varnothing$, $S\left(\psi\right)\neq\varnothing$ and $\partial M=\varnothing$,*

$$\begin{aligned}2\chi\left(M_{\psi}^{-}\right) &= \sum_{p\in P^{\star}(\psi)}\frac{\deg_{S(\psi)}\left(p\right)}{2}-\#P^{\star}\left(\psi\right)+\#P^{+}\left(\psi\right)-\#P^{-}\left(\psi\right),\\ \int_{M_{\psi}^{-}}K_{\psi}d\widehat{A}_{\psi} &= \int_{S(\psi)}\kappa_{\psi}^{s}d\tau_{\psi}.\end{aligned}$$

In particular, if all peaks of the mapping ψ are non-degenerate,

$$2\chi\left(M_{\psi}^{-}\right) = \#P^{+}\left(\psi\right)-\#P^{-}\left(\psi\right), \tag{2.6.29}$$

$$\int_{M_\psi^-} K_\psi d\widehat{A}_\psi = \int_{S(\psi)} \kappa_\psi^s d\tau_\psi. \tag{2.6.30}$$

(6) *If $S(\varphi) = \varnothing$, $S(\psi) = \varnothing$ and $\partial M \neq \varnothing$,*

$$\frac{1}{2\pi}\int_{\partial M} \widehat{\kappa}_\varphi^g d\tau_\varphi = \frac{1}{2\pi}\int_{\partial M} \widehat{\kappa}_\psi^g d\tau_\psi.$$

REMARK 2.62. If the frontal bundle $\mathcal{E}$ is as in Example 2.55, the equation (2.6.27) is a generalization of the equation in [SUY12, Theorem 3.16], the equation (2.6.28) is a generalization of the equation in [SUY12, Theorem 3.23], the equation (2.6.29) is a generalization of the equation in [BW78] and [SUY12, Theorem 3.12] and the equation (2.6.30) is a generalization of the equation in [SUY12, Theorem 3.18].

Next, we consider the case of bounded extrinsic curvature. We prove the following assertion.

LEMMA 2.63 (Lemma 3.25 of [SUY12]). *Let M be an oriented smooth 2-manifold (possibly with boundary) and N an orientable 3-dimensional Riemannian manifold. Let $f : M \to N$ be a co-orientable wave front and ν the unit normal vector field of f globally defined on M. Suppose that there exists a neighborhood U of an A_2-point $p \in S(f)$ such that $\log |K_f^{\mathrm{ext}}|$ is bounded on $U \backslash S(f)$.*

(1) $S(f) \cap U = S(\nu) \cap U$,

(2) $M_f^+ = M_\nu^\epsilon$, $\kappa_f^s d\tau_f = \epsilon \kappa_\nu^s d\tau_\nu$, *where* $\epsilon := \operatorname{sgn}\left(K_f^{\mathrm{ext}}\big|_U\right)$.

PROOF. Take a front bundle $(M, \mathcal{E}_f, g, D, df, \nabla\nu)$ as in Example 2.55. Fix a positively oriented orthonormal frame $\{e_1, e_2\}$ with respect to an orientation of $\mathcal{E}_f$ and take matrix valued functions $G_f := G_{df}$ and $G_\nu := G_{\nabla\nu}$ as in (2.6.11). Since K_f^{ext} is non-zero bounded on $U\backslash S(f)$, $\det G_\nu = 0$ on $S(f) \cap U = \{\det G_f = 0\}$ by the equation (2.6.13), which yields $S(f) \subseteq S(\nu)$. On the other hand, $K_\nu^{\mathrm{ext}} = 1/K_f^{\mathrm{ext}}$ is also bounded by the equation (4) of Lemma 2.57 and the assumption, which yields $S(\nu) \subseteq S(f)$ by exchanging the roles of df and $\nabla\nu$ in the above proof. Hence the equation (1) holds.

Parameterize the set $S(f) \cap U = S(\nu) \cap U$ by a smooth regular curve $\gamma(t)$. Since $p \in S(f)$ is an A_2-point, $df(\dot\gamma(t))$ does not vanish near p. So we retake a neighborhood U such that $df(\dot\gamma(t))$ does not vanish on U. Take a smooth function θ_f satisfying

$$df(\dot\gamma(t)) = |df(\dot\gamma(t))| \left(\cos\theta_f(t)\, e_1(\gamma(t)) + \sin\theta_f(t)\, e_2(\gamma(t))\right). \tag{2.6.31}$$

Then the $\mathcal{E}_f$-conormal vector field n_f along γ is

$$n_f(t) = -\sin\theta_f(t)\, e_1(\gamma(t)) + \cos\theta_f(t)\, e_2(\gamma(t)).$$

Let ω be the connection form of D with respect to $\{e_1, e_2\}$ as in (2.2.3). Then

$$D_t n_f(t) = \frac{\left(\omega(\dot\gamma(t)) - \dot\theta_f(t)\right) df(\dot\gamma(t))}{|df(\dot\gamma(t))|}.$$

By the definition (2.3.6) of the singular curvature,

$$\kappa_f^s(t) := \kappa_{df}^s(t) = -\operatorname{sgn}(\eta_f \lambda_f(t)) \frac{\omega(\dot{\gamma}(t)) - \dot{\theta}_f(t)}{|df(\dot{\gamma}(t))|},$$

where $\lambda_f := \lambda_{df}$ and $\eta_f := \eta_{df}$. Then

$$\kappa_f^s(t)\, d\tau_f = \kappa_f^s(t)\, |df(\dot{\gamma}(t))|\, dt = -\operatorname{sgn}(\eta_f \lambda_f(t)) \Big(\omega(\dot{\gamma}(t)) - \dot{\theta}_f(t)\Big)\, dt. \tag{2.6.32}$$

Similarly, if we set

$$\nabla_t \nu = |\nabla_t \nu| \left(\cos\theta_\nu(t)\, e_1(\gamma(t)) + \sin\theta_\nu(t)\, e_2(\gamma(t))\right), \tag{2.6.33}$$

we have

$$\kappa_\nu^s(t)\, d\tau_\nu = -\operatorname{sgn}(\eta_\nu \lambda_\nu(t)) \Big(\omega(\dot{\gamma}(t)) - \dot{\theta}_\nu(t)\Big)\, dt, \tag{2.6.34}$$

where $\lambda_\nu := \lambda_{\nabla\nu}$ and $\eta_\nu := \eta_{\nabla\nu}$. Since $\lambda_f(\gamma(t)) = \lambda_\nu(\gamma(t)) = 0$, by the equations (2.6.12) and (2.6.13),

$$\operatorname{sgn}(\eta_f \lambda_f(t)) = \operatorname{sgn}(\eta_\nu \lambda_f(t)) = \operatorname{sgn}\left(K_f^{\mathrm{ext}}\big|_U\right) \operatorname{sgn}(\eta_\nu \lambda_\nu(t)). \tag{2.6.35}$$

Note that the second fundamental form II vanishes along γ because K_f^{ext} is bounded on $U \backslash S(f)$ (see [SUY09, Theorem 3.1]). By the equations (2.6.31) and (2.6.33),

$$0 = II(\dot{\gamma}(t), \dot{\gamma}(t)) = -\langle df(\dot{\gamma}(t)), \nabla_t \nu\rangle = -|df(\dot{\gamma}(t))|\, |\nabla_t \nu| \cos(\theta_f(t) - \theta_\nu(t)),$$

which means that $\theta_f(t) - \theta_\nu(t)$ is constant. Hence

$$\dot{\theta}_f(t) = \dot{\theta}_\nu(t). \tag{2.6.36}$$

By the equations (2.6.32), (2.6.34), (2.6.35) and (2.6.36), we have the equation (2). $\square$

Then the following assertion holds by the Gauss-Bonnet type theorems in Corollary 2.58.

THEOREM 2.64 ([Has23a]). *Let M be a compact oriented smooth 2-manifold with boundary ∂M and $N(c)$ an oriented 3-dimensional Riemannian manifold of constant sectional curvature c. Let $f : M \to N(c)$ be a co-oriented wave front whose singular set $S(f)$ admits at most peaks such that $\log\left|K_f^{\mathrm{ext}}\right|$ is bounded on $M \backslash S(f)$ and ν the unit normal vector field of f globally defined on M whose singular set $S(\nu)$ admits at most peaks. Suppose that $S(f) \backslash I(f)$ and $S(\nu) \backslash I(\nu)$ are transversal to ∂M and that $I(f) \cap \partial M$ and $I(\nu) \cap \partial M$ are empty.*

(1) *If $K_f^{\mathrm{ext}} > 0$,*

$$\sum_{p \in (S(f) \cap \partial M)^{\mathrm{null}}} \frac{\alpha_f^+(p)}{\pi} - \sum_{p \in (S(\nu) \cap \partial M)^{\mathrm{null}}} \frac{\alpha_\nu^+(p)}{\pi} \tag{2.6.37}$$

$$= \frac{\#(S(f) \cap \partial M)^{\mathrm{null}} - \#(S(\nu) \cap \partial M)^{\mathrm{null}}}{2},$$

$$\#P_{\mathrm{Int}}^+(f) - \#P_{\mathrm{Int}}^-(f) \tag{2.6.38}$$

$$
\begin{aligned}
&+\frac{\#\left(S\left(f\right)\cap\partial M\right)^{+}-\#\left(S\left(f\right)\cap\partial M\right)^{-}}{2}+\frac{\#P_{\partial M_f^+}-\#P_{\partial M_f^-}}{2}\\
=\ &\#P_{\mathrm{Int}}^{+}\left(\nu\right)-\#P_{\mathrm{Int}}^{-}\left(\nu\right)\\
&+\frac{\#\left(S\left(\nu\right)\cap\partial M\right)^{+}-\#\left(S\left(\nu\right)\cap\partial M\right)^{-}}{2}+\frac{\#P_{\partial M_\nu^+}-\#P_{\partial M_\nu^-}}{2},
\end{aligned}
$$

(2) *If* $K_f^{\mathrm{ext}}<0$,

$$
\chi\left(M\right)+\sum_{p\in\left(S(f)\cap\partial M\right)^{\mathrm{null}}}\frac{2\alpha_f^{+}\left(p\right)-\pi}{4\pi}+\sum_{p\in\left(S(\nu)\cap\partial M\right)^{\mathrm{null}}}\frac{2\alpha_\nu^{+}\left(p\right)-\pi}{4\pi}=0, \tag{2.6.39}
$$

$$
\begin{aligned}
&2\left(\chi\left(M_f^+\right)-\chi\left(M_f^-\right)\right)+\#P_{\mathrm{Int}}^{+}\left(f\right)-\#P_{\mathrm{Int}}^{-}\left(f\right)\\
&+\frac{\#\left(S\left(f\right)\cap\partial M\right)^{+}-\#\left(S\left(f\right)\cap\partial M\right)^{-}}{2}+\frac{\#P_{\partial M_f^+}-\#P_{\partial M_f^-}}{2}\\
=\ &\#P_{\mathrm{Int}}^{+}\left(\nu\right)-\#P_{\mathrm{Int}}^{-}\left(\nu\right)\\
&+\frac{\#\left(S\left(\nu\right)\cap\partial M\right)^{+}-\#\left(S\left(\nu\right)\cap\partial M\right)^{-}}{2}+\frac{\#P_{\partial M_\nu^+}-\#P_{\partial M_\nu^-}}{2}.
\end{aligned}
\tag{2.6.40}
$$

PROOF. If we construct the front bundle $(M,\mathcal{E}_f,\langle\cdot,\cdot\rangle,D,df,\nabla\nu)$ as in Example 2.55, we have the equations (2.6.23) and (2.6.24).

(1) By Lemmas 2.57 and 2.63 and (6) of Corollary 2.61, the equations (2.6.23) and (2.6.24) are reduced as the equations (2.6.37) and (2.6.38).

(2) By Lemmas 2.57 and 2.63 and (6) of Corollary 2.61, the equations (2.6.23) and (2.6.24) are reduced as

$$
\begin{aligned}
&\sum_{p\in\left(S(f)\cap\partial M\right)^{\mathrm{null}}}\frac{2\alpha_f^{+}\left(p\right)-\pi}{2}-\left(\int_M K_f dA_f+2\int_{S(f)}\kappa_f^s d\tau_f+\int_{\partial M}\widetilde{\kappa}_f^g d\tau_f\right)\\
=\ &\sum_{p\in\left(S(\nu)\cap\partial M\right)^{\mathrm{null}}}\frac{2\alpha_\nu^{+}\left(p\right)-\pi}{2},
\end{aligned}
\tag{2.6.41}
$$

and the equation (2.6.40), respectively. By the equations (2.6.14) and (2.6.41), we have the equation (2.6.39). □

COROLLARY 2.65 (Theorems 3.28, 3.30 of [SUY12]). *Let* $f:M\to N\left(c\right)$ *be a co-oriented wave front as in Theorem* 2.64.

(1) *If* $K_f^{\mathrm{ext}}>0$ *and* $\partial M=\varnothing$,

$$
\#P^{+}\left(f\right)-\#P^{-}\left(f\right)=\#P^{+}\left(\nu\right)-\#P^{-}\left(\nu\right).
$$

(2) *If* $K_f^{\mathrm{ext}}<0$ *and* $\partial M=\varnothing$,

$$
\chi\left(M\right)=0,\quad 2\left(\chi\left(M_f^+\right)-\chi\left(M_f^-\right)\right)+\#P^{+}\left(f\right)-\#P^{-}\left(f\right)=\#P_{\mathrm{Int}}^{+}\left(\nu\right)-\#P_{\mathrm{Int}}^{-}\left(\nu\right).
$$

CHAPTER 3

Geometry of cross caps

1. Singular points of positive semi-definite metrics

In this section, we define singular points of positive semi-definite metrics and give some properties.

Let M be a smooth 2-manifold (possibly with boundary ∂M) and ds^2 a positive semi-definite metric on M. A point $p \in M$ is a *regular point* of ds^2 if the covariant 2-tensor $\left(ds^2\right)_p : T_pM \times T_pM \to \mathbb{R}$ is positive-definite. On the other hand, p is a *singular point* of ds^2 if $\left(ds^2\right)_p$ is degenerate. Note that if $p \in \partial M$, then the metric ds^2 is extended to a C^∞-metric beyond ∂M via a compatible coordinate neighborhood of p, and the above notions are well-defined independently of the choice of a compatible coordinate neighborhood of p.

Next, we define the Kossowski pseudo-connection. We denote by $\mathfrak{X}(M)$ the set of smooth vector fields on M and by $C^\infty(M)$ the set of $\mathbb{R}$-valued smooth functions on M. We define a smooth mapping $\Gamma : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ by

$$\begin{aligned}\Gamma(X,Y,Z) = {} & \frac{1}{2}\left(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \right.\\ & \left. + \langle [X,Y],Z\rangle - \langle [Y,Z],X\rangle + \langle [Z,X],Y\rangle\right),\end{aligned} \tag{3.1.1}$$

which is called the *Kossowski pseudo-connection*, where $\langle X,Y\rangle := d\sigma^2(X,Y)$.

If ds^2 is positive definite,

$$\Gamma(X,Y,Z) = \langle \nabla_X Y, Z\rangle,$$

where ∇ is the Levi-Civita connection of ds^2. Also, the following two equations hold:

$$X\langle Y,Z\rangle = \Gamma(X,Y,Z) + \Gamma(X,Z,Y), \tag{3.1.2}$$

$$\Gamma(X,Y,Z) - \Gamma(Y,X,Z) = \langle [X,Y],Z\rangle. \tag{3.1.3}$$

The equation (3.1.2) corresponds to the condition that ∇ is a metric connection and the equation (3.1.3) corresponds to the condition that ∇ is torsion free.

The following assertion holds.

Proposition 3.1 ([Kos04]). *The mapping*

$$T_pM \times T_pM \to \mathbb{R}, \quad (v_1, v_2) \mapsto \Gamma(V_1, Y, V_2)(p)$$

is a well-defined bi-linear mapping for each point $p \in M$ and each $Y \in \mathfrak{X}(M)$, where $V_j \in \mathfrak{X}(M)$, $j = 1, 2$, satisfying $v_j = V_j(p)$.

Proof. By direct calculation. □

The subspace
$$\mathcal{N}_p := \{v \in T_pM \mid \langle v, w\rangle = 0 \text{ for all } w \in T_pM\}$$
is called the *null space* or the *radical* for each point $p \in M$. A nonzero vector which belongs to $\mathcal{N}_p$ is called a *null vector* at p. Also, a singular point p of ds^2 is called of *rank one* if $\mathcal{N}_p$ is a 1-dimensional subspace of T_pM. The following assertion holds.

LEMMA 3.2 ((2) of [Kos04]). *The Kossowski pseudo-connection* Γ *induces a tri-linear mapping*
$$\widehat{\Gamma}_p : T_pM \times T_pM \times \mathcal{N}_p \to \mathbb{R}, \quad (v_1, v_2, v_3) \mapsto \Gamma(V_1, V_2, V_3)(p),$$
where $V_j \in \mathfrak{X}(M)$, $j = 1, 2, 3$, *satisfying* $v_j = V_j(p)$.

PROOF. By the equation (3.1.1) and the fact that $V_3(p) \in \mathcal{N}_p$,
$$\begin{aligned} 2\Gamma(V_1, fV_2, V_3) &= V_1\langle fV_2, V_3\rangle + fV_2\langle V_3, V_1\rangle - V_3\langle V_1, fV_2\rangle \\ &\quad + \langle [V_1, fV_2], V_3\rangle - \langle [fV_2, V_3], V_1\rangle + \langle [V_3, V_1], fV_2\rangle \\ &= 2f\Gamma(V_1, V_2, V_3) + 2(V_1 f)\langle V_2, V_3\rangle \\ &= 2f\Gamma(V_1, V_2, V_3) \end{aligned}$$
for each point $p \in M$. □

Next, we define the admissibility of singular points of positive semi-definite metrics.

DEFINITION 3.3 ([Kos04], Definition 2.3 of [HHNSUY15]). A singular point p of a positive semi-definite metric ds^2 is called *admissible* if $\widehat{\Gamma}_p$ in Lemma 3.2 vanishes. The metric ds^2 is called *admissible* or $\widehat{\Gamma}$-*flat* if each singular point of ds^2 is admissible.

Then, the following assertion shows that the induced metric of a smooth mapping is admissible.

PROPOSITION 3.4 (Proposition 2.4 of [HHNSUY15]). *If* $f : M \to \mathbb{R}^3$ *is a smooth mapping, the induced metric* ds^2 $(:= df \cdot df)$ *by* f *on* M *is admissible.*

PROOF. Let D be the Levi-Civita connection of the canonical metric of $\mathbb{R}^3$. Then the Kossowski pseudo-connection of ds^2 is given by
$$\Gamma(X, Y, Z) = D_X df(Y) \cdot df(Z),$$
which vanishes if $df_p(Z_p) = 0$, that is, $Z_p \in \mathcal{N}_p$. □

Next, we define adjusted coordinates at singular points.

DEFINITION 3.5 (Definition 2.5 of [HHNSUY15]). Local coordinates (u, v) centered at a singular point $p \in M$ is called *adjusted* if $\partial_v(p)$ belongs to $\mathcal{N}_p$, where $\partial_v := \partial/\partial v$.

Note that we can take local coordinates which are adjusted at a rank one singular point by a suitable affine transformation in the uv-plane. The following assertions hold.

Lemma 3.6 (Lemma 2.6 of [HHNSUY15]). *Let (u, v) and (ξ, η) be two local coordinates centered at a rank one singular point p, and suppose that (u, v) is adjusted at p. Then (ξ, η) is also adjusted at p if and only if*

$$u_\eta (0, 0) = 0. \tag{3.1.4}$$

Proof. By the chain rule, $\partial_\eta = u_\eta \partial_u + v_\eta \partial_v$. If $\partial_v (p) \in \mathcal{N}_p$, then $\partial_\eta (p) \in \mathcal{N}_p$ if and only if u_η vanishes at p. □

Proposition 3.7 (Proposition 2.7 of [HHNSUY15]). *Let (u, v) be adjusted local coordinates centered at a rank one singular point p of a positive semi-definite metric ds^2. If p is admissible,*

$$E_v (0, 0) = 2F_u (0, 0), \quad G_u (0, 0) = G_v (0, 0) = 0, \tag{3.1.5}$$

where $ds^2 = Edu^2 + 2Fdudv + Gdv^2$. Conversely, if there exist adjusted local coordinates (u, v) centered at p satisfying the condition (3.1.5), then p is admissible.

Proof. Since $[\partial_u, \partial_v]$ vanishes at p and $\partial_v (p) \in \mathcal{N}_p$, the equation (3.1.1) yields that

$$\begin{aligned}
2\Gamma (\partial_u, \partial_u, \partial_v) &= \partial_u \langle \partial_u, \partial_v \rangle + \partial_u \langle \partial_v, \partial_u \rangle - \partial_v \langle \partial_u, \partial_v \rangle = 2F_u - E_v, \\
2\Gamma (\partial_u, \partial_v, \partial_v) &= \partial_u \langle \partial_v, \partial_v \rangle + \partial_v \langle \partial_v, \partial_u \rangle - \partial_v \langle \partial_u, \partial_v \rangle = G_u, \\
2\Gamma (\partial_v, \partial_v, \partial_v) &= \partial_v \langle \partial_v, \partial_v \rangle + \partial_v \langle \partial_v, \partial_v \rangle - \partial_v \langle \partial_v, \partial_v \rangle = G_v
\end{aligned}$$

at the origin. Thus $\widehat{\Gamma}_p$ in Lemma 3.2 vanishes if and only if the condition (3.1.5) holds at p. □

2. Intrinsic cross caps and Bruce-West type coordinates

In this section, we define intrinsic cross caps, which are an intrinsic formulation of cross caps (see Example 1.31), and give some properties. In particular, we give the existence theorem of Bruce-West type coordinates, which are coordinates centered at intrinsic cross caps, and prove the theorem.

Definition 3.8 (Definition 4.1 of [HHNSUY15]). An admissible singular point p of a positive semi-definite metric ds^2 is an *intrinsic cross cap* if we can take adjusted local coordinates (u, v) centered at p such that the Hessian

$$H_\lambda (u, v) := \det \begin{pmatrix} \lambda_{uu} & \lambda_{uv} \\ \lambda_{uv} & \lambda_{vv} \end{pmatrix}, \quad \lambda := EG - F^2 \, (\geq 0)$$

does not vanish at the origin, where $ds^2 = Edu^2 + 2Fdudv + Gdv^2$. The metric ds^2 is called a *Whitney metric* if ds^2 admits only intrinsic cross caps as singular points.

We check that the definition of intrinsic cross caps is independent of the choice of adjusted coordinates. By Definition 3.8, an intrinsic cross cap p is a non-degenerate critical point of λ, that is, p is a minimal point of λ, which means that p is an isolated singular point of ds^2. Hence,

$$\lambda_u (0, 0) = \lambda_v (0, 0) = 0 \quad \text{and} \quad \lambda (u, v) > 0 \text{ on } U \setminus \{p\}$$

for a sufficiently small neighborhood U of p. Take two adjusted local coordinates (u,v) and (ξ,η) centered at p. Then

$$\widetilde{\lambda} = J^2\lambda, \quad J := \det\begin{pmatrix} u_\xi & u_\eta \\ v_\xi & v_\eta \end{pmatrix},$$

where $\widetilde{\lambda} := \widetilde{E}\widetilde{G} - \widetilde{F}^2$ and $ds^2 = \widetilde{E}d\xi^2 + 2\widetilde{F}d\xi d\eta + \widetilde{G}d\eta^2$, which yields that

$$H_{\widetilde{\lambda}}(0,0) = J(0,0)^6 H_\lambda(0,0). \tag{3.2.1}$$

Thus the definition of intrinsic cross caps is independent of the choice of adjusted local coordinates.

The following assertion shows that intrinsic cross caps are rank one singular points.

Proposition 3.9 (Proposition 4.3 of [**HHNSUY15**]). *An intrinsic cross cap $p \in M$ is an isolated singular point where the null space $\mathcal{N}_p$ is a 1-dimensional linear subspace of T_pM.*

Proof. We suppose that $\dim \mathcal{N}_p = 2$. Then

$$E(0,0) = F(0,0) = G(0,0) = 0,$$

where $ds^2 = Edu^2 + 2Fdudv + Gdv^2$. Since ds^2 is positive semi-definite, $E, G \geq 0$, which yields that p is a critical point of the functions E, G. Thus,

$$E_u(0,0) = E_v(0,0) = G_u(0,0) = G_v(0,0) = 0.$$

Then,

$$H_\lambda(0,0) = 4\det\begin{pmatrix} F_u(0,0)^2 & F_u(0,0)F_v(0,0) \\ F_u(0,0)F_v(0,0) & F_v(0,0)^2 \end{pmatrix} = 0,$$

which contradicts that p is an intrinsic cross cap. □

Next, we prove the following assertion to prove that intrinsic cross caps are an intrinsic formulation of cross caps.

Proposition 3.10 (Proposition 4.4 of [**HHNSUY15**]). *Let $p \in M$ be an intrinsic cross cap and (u,v) local coordinates centered at p as in Definition 3.8. Then*

$$H_\lambda(0,0) = 4E(0,0)\Delta(0,0), \quad \Delta := \det\begin{pmatrix} E & F_u & F_v \\ F_u & \frac{G_{uu}}{2} & \frac{G_{uv}}{2} \\ F_v & \frac{G_{uv}}{2} & \frac{G_{vv}}{2} \end{pmatrix}, \tag{3.2.2}$$

where $ds^2 = Edu^2 + 2Fdudv + Gdv^2$.

Proof. By the condition (3.1.5),

$$F(0,0) = G(0,0) = G_u(0,0) = G_v(0,0) = 0.$$

Differentiating $\lambda = EG - F^2$ twice,

$$\begin{aligned} \lambda_{uu}(0,0) &= E(0,0)G_{uu}(0,0) - 2F_u(0,0)^2, \\ \lambda_{uv}(0,0) &= E(0,0)G_{uv}(0,0) - 2F_u(0,0)F_v(0,0), \end{aligned}$$

$$\lambda_{vv}(0,0) \quad = \quad E(0,0)\,G_{vv}(0,0) - 2F_v(0,0)^2\,,$$

which yields that the equation (3.2.2) holds. □

Then the following shows that intrinsic cross caps are an intrinsic formulation of cross caps.

COROLLARY 3.11 (Corollary 4.5 of [HHNSUY15]). *If $p \in M$ is a cross cap singular point of a smooth mapping $f : M \to \mathbb{R}^3$, then p is an intrinsic cross cap of the induced metric $ds^2 := df \cdot df$.*

PROOF. By Proposition 3.4, p is admissible. Take local coordinates (u, v) centered at p satisfying $f_v(0,0) = 0$, which means that (u, v) are local coordinates adjusted at p. Then the three vectors $f_u(0,0)\,, f_{uv}(0,0)\,, f_{vv}(0,0)$ are linearly independent by a criterion of cross caps in [Whi43], which yields that

$$\begin{pmatrix} f_u(0,0) \\ f_{uv}(0,0) \\ f_{vv}(0,0) \end{pmatrix} (f_u(0,0)\,, f_{uv}(0,0)\,, f_{vv}(0,0)) = \begin{pmatrix} E(0,0) & F_u(0,0) & F_v(0,0) \\ F_u(0,0) & \frac{G_{uu}(0,0)}{2} & \frac{G_{uv}(0,0)}{2} \\ F_v(0,0) & \frac{G_{uv}(0,0)}{2} & \frac{G_{vv}(0,0)}{2} \end{pmatrix} \tag{3.2.3}$$

is regular, where $E := f_u \cdot f_u, F := f_u \cdot f_v, G := f_v \cdot f_v$. By taking the determinant of the equation (3.2.3), the conclusion follows from Proposition 3.10. □

Next, we introduce one invariant for intrinsic cross caps.

PROPOSITION 3.12 (Proposition 4.6 of [HHNSUY15]). *Let $p \in M$ be an intrinsic cross cap of a positive semi-definite metric ds^2 and (u, v) adjusted local coordinates centered at p. Set*

$$\alpha = \frac{\lambda_{vv}(0,0)}{2} \left(= \frac{E(0,0)\,G_{vv}(0,0)}{2} - F_v(0,0)^2 \right),$$

where $ds^2 = Edu^2 + 2Fdudv + Gdv^2$. Then

$$\alpha_{02} := \frac{\sqrt{E(0,0)}\alpha^{\frac{3}{2}}}{\Delta(0,0)} \tag{3.2.4}$$

is a positive value and does not depend on the choice of adjusted local coordinates centered at p, where Δ is defined as in Proposition 3.10.

PROOF. Since p is an intrinsic cross cap and is a minimal point of λ, we have $H_\lambda(0,0) > 0$ and $\alpha > 0$. By Proposition 3.10,

$$4E(0,0)\,\Delta(0,0) = \lambda_{uu}(0,0)\,\lambda_{vv}(0,0) - \lambda_{uv}(0,0)^2 > 0.$$

Hence we have the inequality $\alpha_{02} > 0$.

Let (ξ, η) be other adjusted local coordinates centered at p and set $\widetilde{\lambda} = \widetilde{E}\widetilde{G} - \widetilde{F}^2$. By the equation (3.1.5),

$$\widetilde{E}(0,0) = u_\xi(0,0)^2\,E(0,0)\,.$$

By the equations (3.1.4) and (3.1.5),

$$\widetilde{\lambda}_{\eta\eta}(0,0) = v_\eta(0,0)^2\,\widetilde{\lambda}_{vv}(0,0) = u_\xi(0,0)^2\,v_\eta(0,0)^4\,\lambda_{vv}(0,0)\,.$$

Hence,

$$\left(\widetilde{E}(0,0)\,\widetilde{\lambda}_{\eta\eta}(0,0)\right)^{3/2} = \left(u_\xi(0,0)\,v_\eta(0,0)\right)^6 \left(E(0,0)\,\lambda_{vv}(0,0)\right)^{3/2}. \tag{3.2.5}$$

On the other hand, by the equation (3.2.1),

$$H_{\widetilde{\lambda}}(0,0) = \left(u_\xi(0,0)\,v_\eta(0,0)\right)^6 H_\lambda(0,0). \tag{3.2.6}$$

By the equations (3.2.5) and (3.2.6),

$$\frac{\left(\widetilde{E}(0,0)\,\widetilde{\lambda}_{\eta\eta}(0,0)\right)^{3/2}}{H_{\widetilde{\lambda}}(0,0)} = \frac{\left(E(0,0)\,\lambda_{vv}(0,0)\right)^{3/2}}{H_\lambda(0,0)},$$

which yields the coordinate invariance of α_{02}. □

Next, we define adapted coordinates centered at intrinsic cross caps.

DEFINITION 3.13 (Definition 4.7 of [HHNSUY15]). Let p be an intrinsic cross cap of a positive semi-definite metric ds^2. Local coordinates (u,v) adjusted at p is called *adapted* if

$$E(0,0) = 1,\; E_u(0,0) = E_v(0,0) = F_u(0,0) = F_v(0,0) = G_u(0,0) = G_v(0,0) = 0, \tag{3.2.7}$$

where $ds^2 = Edu^2 + 2Fdudv + Gdv^2$.

The following assertion shows that there exist adapted local coordinates centered at intrinsic cross caps.

PROPOSITION 3.14 (Proposition 4.8 of [HHNSUY15]). *There exist adapted coordinates centered at an intrinsic cross cap p. Moreover, if (u,v) and (ξ,η) are two adapted coordinates centered at p,*

$$u_\xi(0,0) = \pm 1, \quad u_\eta(0,0) = u_{\xi\xi}(0,0) = u_{\xi\eta}(0,0) = u_{\eta\eta}(0,0) = 0. \tag{3.2.8}$$

Conversely, if (u,v) is adapted at p and (ξ,η) adjusted at p satisfies the equation (3.2.8), then (ξ,η) is also adapted at p.

PROOF. Let (u,v) be local coordinates adjusted at p. If we set

$$u(\xi,\eta) = \frac{1}{\sqrt{E(0,0)}}\xi - \frac{E_u(0,0)}{4E(0,0)^2}\xi^2 - \frac{E_v(0,0)}{2E(0,0)^{3/2}}\xi\eta - \frac{F_v(0,0)}{2E(0,0)}\eta^2, \quad v(\xi,\eta) = \eta,$$

then (ξ,η) is adapted. The second assertion follows by direct calculation. □

Next, we introduce Bruce-West type coordinates, which play a useful role in the next section.

DEFINITION 3.15 (Definition 4.9 of [HHNSUY15]). Adapted coordinates (u,v) centered at an intrinsic cross cap of a positive semi-definite metric ds^2 are called *Bruce-West type coordinates of the second-order* if there exist two real numbers α_{20} and α_{11} satisfying

$$E(u,v) = 1 + (\alpha_{20})^2 u^2 + 2\alpha_{11}\alpha_{20}uv + \left(1 + (\alpha_{11})^2\right) v^2 + O(u,v)^3, \tag{3.2.9}$$

$$F(u,v) = \alpha_{11}\alpha_{20}u^2 + \left(1+(\alpha_{11})^2+\alpha_{02}\alpha_{20}\right)uv + \alpha_{02}\alpha_{11}v^2 + O(u,v)^3, \tag{3.2.10}$$
$$G(u,v) = \left(1+(\alpha_{11})^2\right)u^2 + 2\alpha_{02}\alpha_{11}uv + (\alpha_{02})^2v^2 + O(u,v)^3, \tag{3.2.11}$$

where $ds^2 = Edu^2 + 2Fdudv + Gdv^2$, α_{02} is given by the equation (3.2.4) and $O(u,v)^3$ is the term whose order is greater than or equal to 3 with respect to (u,v).

In the following, we prepare to prove that there exist Bruce-West type coordinates. First, we introduce adapted coordinates adjusted in the first-level.

Definition 3.16 (Definition 4.12 of [HHNSUY15]). Adapted coordinates (u,v) centered at an intrinsic cross cap of a positive semi-definite metric ds^2 is said to be *adjusted in the first-level* if

$$G_{vv}(0,0) = 2(\alpha_{02})^2, \tag{3.2.12}$$

where $ds^2 = Edu^2 + 2Fdudv + Gdv^2$.

By the equation (3.2.11), West type coordinates of the second-order are adjusted in the first-level. The following assertion shows that there exist adapted coordinates adjusted in the first-level.

Lemma 3.17 (Lemma 4.13 of [HHNSUY15]). *There exist adapted coordinates (u,v) adjusted in the first-level centered at an intrinsic cross cap p. Moreover, under the assumption that (u,v) and (ξ,η) are two adapted coordinates centered at p and (u,v) are adjusted in the first-level, (ξ,η) are adjusted in the first-level if and only if*

$$v_\eta(0,0) = \pm 1. \tag{3.2.13}$$

Proof. If (u,v) are adapted coordinates centered at p, the coordinates (ξ,η) given by

$$u(\xi,\eta) = \xi, \quad v(\xi,\eta) = \frac{\sqrt[4]{2}\sqrt{\alpha_{02}}}{\sqrt[4]{G_{vv}(0,0)}}\eta$$

have the desired property. The second assertion follows by direct calculation. $\square$

Next, we introduce adapted coordinates adjusted in the second-level.

Definition 3.18 (Definition 4.15 of [HHNSUY15]). Adapted coordinates (u,v) adjusted in the first-level centered at an intrinsic cross cap of a positive semi-definite metric ds^2 is said to be *adjusted in the second-level* if

$$\det\begin{pmatrix} F_{uu}(0,0) - \frac{E_{uv}(0,0)}{2} & G_{uv}(0,0) \\ F_{uv}(0,0) - \frac{E_{vv}(0,0)}{2} & G_{vv}(0,0) \end{pmatrix} = 0, \tag{3.2.14}$$

where $ds^2 = Edu^2 + 2Fdudv + Gdv^2$.

By the equations (3.2.9), (3.2.10) and (3.2.11), Bruce-West type coordinates of the second-order are adjusted in the second-level. The following assertion shows that there exist adapted coordinates adjusted in the second-level.

LEMMA 3.19 (Lemma 4.16 of [HHNSUY15]). *There exist adapted coordinates (u, v) adjusted in the second-level centered at an intrinsic cross cap p. Moreover, under the assumption that (u, v) and (ξ, η) are two adapted coordinates centered at p and (u, v) are adjusted in the second-level, (ξ, η) are adjusted in the second-level if and only if*

$$v_\eta (0,0) = \pm 1, \quad v_\xi (0,0) = 0. \tag{3.2.15}$$

PROOF. Let (u, v) and (ξ, η) be two adapted coordinates adjusted in the first-level centered at p. For the sake of simplicity, we consider the case of $u_\xi (0,0) = v_\eta (0,0) = 1$ (see the equations (3.2.8) and (3.2.13)). By the condition (3.2.8),

$$\begin{aligned} &\det \begin{pmatrix} \widetilde{F}_{\xi\xi}(0,0) - \frac{\widetilde{E}_{\xi\eta}(0,0)}{2} & \widetilde{G}_{\xi\eta}(0,0) \\ \widetilde{F}_{\xi\eta}(0,0) - \frac{\widetilde{E}_{\eta\eta}(0,0)}{2} & \widetilde{G}_{\eta\eta}(0,0) \end{pmatrix} \\ = \; &\det \begin{pmatrix} F_{uu}(0,0) - \frac{E_{uv}(0,0)}{2} & G_{uv}(0,0) \\ F_{uv}(0,0) - \frac{E_{vv}(0,0)}{2} & G_{vv}(0,0) \end{pmatrix} + v_\xi (0,0) \det \begin{pmatrix} G_{uu}(0,0) & G_{uv}(0,0) \\ G_{uv}(0,0) & G_{vv}(0,0) \end{pmatrix}, \end{aligned}$$

where $ds^2 = \widetilde{E} d\xi^2 + 2\widetilde{F} d\xi d\eta + \widetilde{G} d\eta^2$. Since (u, v) is adapted, $E(0,0) = 1, F_u(0,0) = F_v(0,0) = 0$, which yields that

$$0 \neq H_\lambda (0,0) = 4\Delta (0,0) = \det \begin{pmatrix} G_{uu}(0,0) & G_{uv}(0,0) \\ G_{uv}(0,0) & G_{vv}(0,0) \end{pmatrix} \tag{3.2.16}$$

by the equation (3.2.2). Thus the coordinates (ξ, η) given by

$$u(\xi,\eta) = \xi, \quad v(\xi,\eta) = \eta - \frac{\det \begin{pmatrix} F_{uu}(0,0) - \frac{E_{uv}(0,0)}{2} & G_{uv}(0,0) \\ F_{uv}(0,0) - \frac{E_{vv}(0,0)}{2} & G_{vv}(0,0) \end{pmatrix}}{\det \begin{pmatrix} G_{uu}(0,0) & G_{uv}(0,0) \\ G_{uv}(0,0) & G_{vv}(0,0) \end{pmatrix}} \xi$$

have the desired property. The second assertion follows by direct calculation. □

Let (u, v) be adapted coordinates adjusted in the second-level centered at an intrinsic cross cap. Since $E(0,0) = 1$ and $F_v(0,0) = 0$ by Definition 3.13, we have $\alpha = G_{vv}(0,0)/2$. By the equation (3.2.13), we have $\alpha = (\alpha_{02})^2$. Thus,

$$\Delta (0,0) = \frac{\alpha^{3/2}}{\alpha_{02}} = (\alpha_{02})^2 = \frac{G_{vv}(0,0)}{2}. \tag{3.2.17}$$

By the equation (3.2.16),

$$4(\alpha_{02})^2 = G_{uu}(0,0)\, G_{vv}(0,0) - G_{uv}(0,0)^2. \tag{3.2.18}$$

We define a constant α_{11} so that

$$G_{uv}(0,0) = 2\alpha_{11}\alpha_{02}. \tag{3.2.19}$$

By the equations (3.2.17), (3.2.18) and (3.2.19),

$$G_{uu}(0,0) = 2\left(1 + (\alpha_{11})^2\right). \tag{3.2.20}$$

We define a constant α_{20} so that

$$F_{uv}(0,0) - \frac{E_{vv}(0,0)}{2} = \alpha_{20}\alpha_{02}. \tag{3.2.21}$$

By the equations (3.2.14), (3.2.19), (3.2.21) and (3.2.12),

$$\begin{aligned} 0 &= \det \begin{pmatrix} F_{uu}(0,0) - \frac{E_{uv}(0,0)}{2} & G_{uv}(0,0) \\ F_{uv}(0,0) - \frac{E_{vv}(0,0)}{2} & G_{vv}(0,0) \end{pmatrix} \\ &= \det \begin{pmatrix} F_{uu}(0,0) - \frac{E_{uv}(0,0)}{2} & 2\alpha_{11}\alpha_{02} \\ \alpha_{20}\alpha_{02} & 2(\alpha_{02})^2 \end{pmatrix} \\ &= 2(\alpha_{02})^2 \det \begin{pmatrix} F_{uu}(0,0) - \frac{E_{uv}(0,0)}{2} & \alpha_{11} \\ \alpha_{20} & 1 \end{pmatrix}, \end{aligned}$$

which yields that

$$F_{uu}(0,0) - \frac{E_{uv}(0,0)}{2} = \alpha_{11}\alpha_{20} \tag{3.2.22}$$

because α_{02} is a positive number.

The following assertion shows that α_{20} and α_{11} are invariants for intrinsic cross caps.

THEOREM 3.20 (Theorem 4.17 of [HHNSUY15]). *The two constants α_{20} and $|\alpha_{11}|$ are independent of the choice of adapted coordinates adjusted in the second-level centered at an intrinsic cross cap $p \in M$. Moreover, if the manifold M is oriented, α_{11} is independent of the choice of oriented adapted coordinates adjusted in the second-level centered at p.*

PROOF. The coordinate independence of α_{20} and $|\alpha_{11}|$ are proved by the equations (3.2.8), (3.2.15), (3.2.19) and (3.2.21). Let (ξ, η) be other adapted coordinates adjusted in the second-level. Then

$$\widetilde{G}_{\xi\eta}(0,0) = u_\xi(0,0)\, v_\eta(0,0)\, G_{uv}(0,0).$$

If M is oriented, an orientation preserving coordinate change between adapted coordinates adjusted in the second-level satisfies $u_\xi(0,0)\, v_\eta(0,0) = 1$, which yields that $G_{uv}(0,0)$ is independent of such a coordinate change, and the equation (3.2.19) implies the desired coordinate invariance of α_{11}. $\square$

We prove the following assertion as a corollary of Proposition 3.12 and Theorem 3.20.

COROLLARY 3.21 (Corollary 4.18 of [HHNSUY15]). *Let (u, v) be oriented adapted coordinates adjusted in the second-level centered at an intrinsic cross cap of a positive semi-definite metric ds^2 and set $u(r,\theta) = r\cos\theta, v(r,\theta) = r\sin\theta$. Then*

$$\lim_{r \to 0+0} r^2 K(r,\theta) = \frac{\alpha_{02}\left(\alpha_{20}\cos^2\theta - \alpha_{02}\sin^2\theta\right)}{\left(\cos^2\theta + (\alpha_{11}\cos\theta + \alpha_{02}\sin\theta)^2\right)^2}, \tag{3.2.23}$$

where K is the Gaussian curvature of ds^2 and $K(r,\theta) := K(u(r,\theta), v(r,\theta))$.

PROOF. By the equations (3.2.8), (3.2.22), (3.2.21), (3.2.20), (3.2.19) and (3.2.12), we can write

$$
\begin{aligned}
E(u,v) &= 1+e_{20}u^2+2e_{11}uv+e_{02}v^2+O(u,v)^3,\\
F(u,v) &= \frac{1}{2}(\alpha_{11}\alpha_{20}+e_{11})u^2+(e_{02}+\alpha_{02}\alpha_{20})uv+f_{02}v^2+O(u,v)^3,\\
G(u,v) &= \left(1+(\alpha_{11})^2\right)u^2+2\alpha_{02}\alpha_{11}uv+(\alpha_{02})^2v^2+O(u,v)^3,
\end{aligned}
$$

where e_{20}, e_{11}, e_{02} and f_{02} are constants. Substituting them into the Gauss equation

$$
\begin{aligned}
K &= \frac{E\left(E_vG_v-2F_uG_v+(G_u)^2\right)}{4(EG-F^2)^2}+\frac{F\left(E_uG_v-E_vG_u-2E_vF_v-2F_uG_u+4F_uF_v\right)}{4(EG-F^2)^2}\\
&\quad+\frac{G\left(E_uG_u-2E_uF_v+(E_v)^2\right)}{4(EG-F^2)^2}-\frac{E_{vv}-2F_{uv}+G_{uu}}{2(EG-F^2)},
\end{aligned}
$$

we see that the limit does not depend on the constants e_{20}, e_{11}, e_{02} and f_{02} and get the equation (3.2.23) (cf. [UY17, Theorem 11.2]). □

REMARK 3.22 (Remark 4.21 of [HHNSUY15]). Let $(U;u,v)$ be an oriented adapted coordinate chart adjusted in the second-level centered at an intrinsic cross cap of a positive semi-definite metric ds^2 and set

$$dA=\sqrt{EG-F^2}\,du\wedge dv,$$

where $ds^2=Edu^2+2Fdudv+Gdv^2$. Then KdA gives a smooth 2-form with respect to the polar coordinates (r,θ) by the equation (3.2.23), where $u(r,\theta):=r\cos\theta$ and $v(r,\theta):=r\sin\theta$, which yields that the integral $\int_U KdA$ is well-defined.

At the end of this section, we prove the existence of Bruce-West type coordinates.

THEOREM 3.23 (Theorem 4.11 of [HHNSUY15]). *There exist Bruce-West type coordinates of the second-order centered at an intrinsic cross cap.*

PROOF. Let (u,v) be adapted coordinates adjusted in the second-level centered at an intrinsic cross cap p. We define local coordinates (ξ,η) centered at p by

$$
\begin{aligned}
u(\xi,\eta) &= \xi+\frac{2(\alpha_{20})^2-E_{uu}(0,0)}{12}\xi^3+\frac{2\alpha_{11}\alpha_{20}-E_{uv}(0,0)}{4}\xi^2\eta\\
&\quad+\frac{2+2(\alpha_{11})^2-E_{vv}(0,0)}{4}\xi\eta^2+\frac{2\alpha_{02}\alpha_{11}-F_{vv}(0,0)}{6}\eta^3,\\
v(\xi,\eta) &= \eta.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\widetilde{E}_{\xi\xi}(0,0)=2(\alpha_{20})^2, \quad \widetilde{E}_{\xi\eta}(0,0)=2\alpha_{11}\alpha_{20},\\
&\widetilde{E}_{\eta\eta}(0,0)=2\left(1+(\alpha_{11})^2\right), \quad \widetilde{F}_{\eta\eta}(0,0)=2\alpha_{02}\alpha_{11}.
\end{aligned}
$$

Then, $\widetilde{F}_{\xi\xi}$ and $\widetilde{F}_{\xi\eta}$ are determined by the equations (3.2.22) and (3.2.21). □

3. A Gauss-Bonnet type theorem for Whitney metrics

In this section, we prove a Gauss-Bonnet type theorem for Whitney metrics, which are positive semi-definite metrics that admit only intrinsic cross caps as singular points.

First, we prove the boundedness of the geodesic curvature measure of a smooth regular curve passing through an intrinsic cross cap.

THEOREM 3.24 (Theorem 1.2 of [**Has24**]). *Let M be a smooth 2-manifold (possibly with boundary) and ds^2 a positive semi-definite metric on M. Let $\gamma : [0, \varepsilon) \to M$ a smooth regular curve starting from an intrinsic cross cap, where $\varepsilon > 0$ is a number such that $\gamma((0, \varepsilon))$ consists of regular points of ds^2. Then the geodesic curvature measure $\kappa_g d\tau$ defined on $(0, \varepsilon)$ is continuously extended on $[0, \varepsilon)$, where $d\tau$ is the arc-length measure with respect to ds^2, that is, $d\tau = |\dot{\gamma}(t)|\, dt$, where $|\dot{\gamma}(t)| = \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle}$. In particular, the integral $\int_\gamma \kappa_g d\tau$ is well-defined.*

PROOF. Take Bruce-West type coordinates of the second order (u, v) centered at an intrinsic cross cap $p = \gamma(0)$ and write $\gamma(t) = (u(t), v(t))$.

First, we express the geodesic curvature κ_g of $\gamma(t)$, $t \in (0, \varepsilon)$, using the coefficients of $ds^2 = Edu^2 + 2Fdudv + Gdv^2$. The tangential vector field $\dot{\gamma}$ along γ, the unit normal vector field n along γ such that $\{\dot{\gamma}(t) / |\dot{\gamma}(t)|, n(t)\}$ is a positively oriented orthonormal basis for $T_{\gamma(t)}M$, and the covariant derivative $\nabla_{\dot{\gamma}}\dot{\gamma}$ along γ can be written by

$$\dot{\gamma}(t) = \dot{u}(t) \left.\frac{\partial}{\partial u}\right|_{\gamma(t)} + \dot{v}(t) \left.\frac{\partial}{\partial v}\right|_{\gamma(t)}, \tag{3.3.1}$$

$$\begin{aligned} n(t) = {} & \frac{-(\dot{u}(t) F + \dot{v}(t) G)}{\sqrt{EG - F^2}\sqrt{\dot{u}(t)^2 E + 2\dot{u}(t)\dot{v}(t) F + \dot{v}(t)^2 G}} \left.\frac{\partial}{\partial u}\right|_{\gamma(t)} \\ & + \frac{\dot{u}(t) E + \dot{v}(t) F}{\sqrt{EG - F^2}\sqrt{\dot{u}(t)^2 E + 2\dot{u}(t)\dot{v}(t) F + \dot{v}(t)^2 G}} \left.\frac{\partial}{\partial v}\right|_{\gamma(t)}, \end{aligned} \tag{3.3.2}$$

$$\begin{aligned} \nabla_{\dot{\gamma}}\dot{\gamma}(t) = {} & \left\{ \ddot{u}(t) + \frac{\dot{u}(t)^2 (GE_u - 2FF_u + FE_v)}{2(EG - F^2)} \right. \\ & + \frac{\dot{u}(t)\dot{v}(t)(GE_v - FG_u)}{EG - F^2} \\ & \left. + \frac{\dot{v}(t)^2 (2GF_v - GG_u - FG_v)}{2(EG - F^2)} \right\} \left.\frac{\partial}{\partial u}\right|_{\gamma(t)} \\ & + \left\{ \ddot{v}(t) + \frac{\dot{u}(t)^2 (2EF_u - EE_v - FE_u)}{2(EG - F^2)} \right. \\ & + \frac{\dot{u}(t)\dot{v}(t)(EG_u - FE_v)}{EG - F^2} \end{aligned} \tag{3.3.3}$$

$$+\frac{\dot{v}(t)^2(EG_v-2FF_v+FG_u)}{2(EG-F^2)}\Bigg\}\left.\frac{\partial}{\partial v}\right|_{\gamma(t)},$$

respectively, where $E=E(\gamma(t)), F=F(\gamma(t)), G=G(\gamma(t)), E_u=E_u(\gamma(t)), E_v=E_v(\gamma(t)), F_u=F_u(\gamma(t)), F_v=F_v(\gamma(t)), G_u=G_u(\gamma(t)), G_v=G_v(\gamma(t))$, and ∇ is the Levi-Civita connection on the set of regular points of ds^2. Note that we used the expressions of Christoffel's symbols

$$\begin{aligned}\Gamma^1_{11} &= \frac{GE_u-2FF_u+FE_v}{2(EG-F^2)}, \quad \Gamma^2_{11}=\frac{2EF_u-EE_v-FE_u}{2(EG-F^2)},\\ \Gamma^1_{12} &= \Gamma^1_{21}=\frac{GE_v-FG_u}{2(EG-F^2)}, \quad \Gamma^2_{12}=\Gamma^2_{21}=\frac{EG_u-FE_v}{2(EG-F^2)},\\ \Gamma^1_{22} &= \frac{2GF_v-GG_u-FG_v}{2(EG-F^2)}, \quad \Gamma^2_{22}=\frac{EG_v-2FF_v+FG_u}{2(EG-F^2)}\end{aligned}$$

to obtain the expression (3.3.3) (cf. [UY17, (10.6)]). By the equations (3.3.1), (3.3.2) and (3.3.3), the geodesic curvature $\kappa_g(t)$ of $\gamma(t)$, $t\in(0,\varepsilon)$, is given by

$$\begin{aligned}\kappa_g(t) &:= \frac{\langle\nabla_{\dot\gamma}\dot\gamma(t), n(t)\rangle}{|\dot\gamma(t)|^2}\\ &= \frac{(\dot u(t)\ddot v(t)-\dot v(t)\ddot u(t))\sqrt{EG-F^2}}{\left(\dot u(t)^2E+2\dot u(t)\dot v(t)F+\dot v(t)^2G\right)^{3/2}}\\ &\quad+\frac{\dot u(t)^3(2EF_u-EE_v-FE_u)}{2\sqrt{EG-F^2}\left(\dot u(t)^2E+2\dot u(t)\dot v(t)F+\dot v(t)^2G\right)^{3/2}}\\ &\quad-\frac{\dot v(t)^3(2GF_v-GG_u-FG_v)}{2\sqrt{EG-F^2}\left(\dot u(t)^2E+2\dot u(t)\dot v(t)F+\dot v(t)^2G\right)^{3/2}}\\ &\quad+\frac{\dot u(t)^2\dot v(t)(2EG_u-3FE_v-GE_u+2FF_u)}{2\sqrt{EG-F^2}\left(\dot u(t)^2E+2\dot u(t)\dot v(t)F+\dot v(t)^2G\right)^{3/2}}\\ &\quad-\frac{\dot u(t)\dot v(t)^2(2GE_v-3FG_u-EG_v+2FF_v)}{2\sqrt{EG-F^2}\left(\dot u(t)^2E+2\dot u(t)\dot v(t)F+\dot v(t)^2G\right)^{3/2}}.\end{aligned} \tag{3.3.4}$$

Now, we prove that the geodesic curvature measure is continuously extended to an intrinsic cross cap by using the equation (3.3.4).

First, consider the case where the curve γ is transversal to the null space at p, that is, the v-axis. Then we can parameterize γ as

$$\gamma(t)=(t,v(t)),\quad v(0)=0. \tag{3.3.5}$$

Computing E, F, G and their partial derivatives with respect to the equation (3.3.5) and substituting these expressions into the equation (3.3.4), we obtain

$$\kappa_g(t) = \frac{(\alpha_{11} + \alpha_{02}\dot{v}(0))(\alpha_{20} + \dot{v}(0)(2\alpha_{11} + \alpha_{02}\dot{v}(0))) + 2\dot{v}(0)}{\sqrt{(\alpha_{11} + \alpha_{02}\dot{v}(0))^2 + 1}} + O(t)$$

for $t > 0$. Thus κ_g is a continuous function on $[0, \varepsilon)$. In particular, $\kappa_g d\tau$ is a continuous 1-form on $[0, \varepsilon)$ because

$$d\tau = \sqrt{\dot{u}(t)^2 E + 2\dot{u}(t)\dot{v}(t)F + \dot{v}(t)^2 G}dt = (1 + O(t))\,dt.$$

Next, consider the case where γ is tangential to the null space at p. Then we can parameterize γ as

$$\gamma(t) = (u(t), t), \quad u(0) = \dot{u}(0) = 0. \tag{3.3.6}$$

Computing E, F, G and their partial derivatives with respect to the equation (3.3.6) and substituting these expressions into the equation (3.3.4), we obtain

$$\begin{aligned} \kappa_g(t) &= \frac{3\alpha_{11}\ddot{u}(0)^2 - \alpha_{02}u^{(3)}(0)}{2t\left((\alpha_{02})^2 + \ddot{u}(0)^2\right)^{3/2}} - \frac{\alpha_{02}u^{(4)}(0)}{3\left((\alpha_{02})^2 + \ddot{u}(0)^2\right)^{3/2}} \\ &\quad + \frac{\ddot{u}(0)\begin{pmatrix} 9(\alpha_{02})^2 u^{(3)}(0)^2 \\ +\alpha_{02}\alpha_{11}u^{(3)}(0)\left(43(\alpha_{02})^2 - 11\ddot{u}(0)^2\right) \\ +3\begin{pmatrix} -6(\alpha_{02})^4 \\ +\left(-27(\alpha_{11})^2 + 4\alpha_{02}\alpha_{20} - 3\right)(\alpha_{02})^2\ddot{u}(0)^2 \\ +(4\alpha_{02}\alpha_{20} + 3)\ddot{u}(0)^4 \end{pmatrix} \end{pmatrix}}{12\alpha_{02}\left((\alpha_{02})^2 + \ddot{u}(0)^2\right)^{5/2}} \\ &\quad + O(t) \end{aligned} \tag{3.3.7}$$

for $t > 0$. Since $d\tau = \left(t\sqrt{(\alpha_{02})^2 + \ddot{u}(0)^2} + O(t)^2\right)dt$, the geodesic curvature measure $\kappa_g d\tau$ is given by

$$\kappa_g(t)\,d\tau = \left(\frac{3\alpha_{11}\ddot{u}(0)^2 - \alpha_{02}u^{(3)}(0)}{2\left((\alpha_{02})^2 + \ddot{u}(0)^2\right)} + O(t)\right)dt$$

for $t > 0$. Recall that α_{02} is a positive number because (u, v) are Bruce-West type coordinates of the second order. Hence $\kappa_g d\tau$ is a continuous 1-form on $[0, \varepsilon)$. □

By the equation (3.3.7), the following assertion holds.

Corollary 3.25 (Corollary 2.3 of [Has24]). *If a smooth regular curve $\gamma : [0, \varepsilon) \to M$ starting from an intrinsic cross cap $p \in M$ is tangential to the null space at p, the geodesic curvature of γ defined on $(0, \varepsilon)$ is continuously extended on $[0, \varepsilon)$ if and only if γ is parameterized by*

$$\gamma(t) = (u(t), t), \quad u(0) = \dot{u}(0) = 0, \quad 3\alpha_{11}\ddot{u}(0)^2 - \alpha_{02}u^{(3)}(0) = 0 \tag{3.3.8}$$

with respect to Bruce-West type coordinates of the second order (u, v) centered at p.

REMARK 3.26. The formula for the geodesic curvature measure with respect to polar coordinates centered at an intrinsic cross cap for setting $\gamma(t) = (t, 0)$ in the equation (3.3.5) is derived in the proof of [HHNSUY15, Theorem 5.1].

Here, we give concrete examples to illustrate Theorem 3.24 and Corollary 3.25.

EXAMPLE 3.27 (Example 3.1 of [Has24]). Let $\mathrm{CR} : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ be a cross cap defined in Example 1.31. Then the local coordinates of the domain of CR are Bruce-West type coordinates of the second order and the null space at the origin is the v-axis. Consider the following curves γ_i, $i = 1, 2, 3, 4$, in the domain of CR:

$$\gamma_1(t) = (t, t), \quad \gamma_2(t) = \left(\frac{t^2}{2}, t\right), \quad \gamma_3(t) = \left(\frac{t^3}{6}, t\right), \quad \gamma_4(t) = \left(\frac{t^4}{24}, t\right).$$

The curve γ_1 is transversal to the v-axis and γ_i, $i = 2, 3, 4$, are tangential to the v-axis. The images of these curves $\widehat{\gamma}_i = \mathrm{CR} \circ \gamma_i$ are expressed as

$$\widehat{\gamma}_1(t) = \left(t, t^2, t^2\right), \ \widehat{\gamma}_2(t) = \left(\frac{t^2}{2}, \frac{t^3}{2}, t^2\right), \ \widehat{\gamma}_3(t) = \left(\frac{t^3}{6}, \frac{t^4}{6}, t^2\right), \ \widehat{\gamma}_4(t) = \left(\frac{t^4}{24}, \frac{t^5}{24}, t^2\right),$$

respectively (see Figure 3.1). The geodesic curvatures κ_g^i of $\widehat{\gamma}_i$ are calculated as

$$\begin{aligned}
\kappa_g^1(t) &= \frac{6}{\sqrt{4t^2+5}\,(8t^2+1)^{3/2}} \to \frac{6}{\sqrt{5}} \quad (t \to 0),\\
\kappa_g^2(t) &= \frac{-\operatorname{sgn}(t)\,84}{(9t^2+20)^{3/2}\sqrt{17t^2+16}} \to \mp\frac{21}{40\sqrt{5}} \quad (t \to 0 \pm 0),\\
\kappa_g^3(t) &= \frac{72\,(t^4-96t^2-36)}{t\sqrt{t^4+144t^2+144}\,(16t^4+9t^2+144)^{3/2}} \to -\infty \quad (t \to 0),\\
\kappa_g^4(t) &= \frac{\operatorname{sgn}(t)\,96\,(5t^6-8640t^2-4608)}{\sqrt{t^6+2304t^2+2304}\,(25t^6+16t^4+2304)^{3/2}} \to \mp\frac{1}{12} \quad (t \to 0 \pm 0),
\end{aligned}$$

respectively. Note that $\kappa_g^3(t)$ diverges to $-\infty$ as $t \to 0$. In fact, γ_3 does not satisfy the condition (3.3.8). Also, the geodesic curvature measures $\kappa_g^i d\tau_i$ of $\widehat{\gamma}_i$ are computed as

$$\begin{aligned}
\kappa_g^1(t)\,d\tau_1 &= \frac{6dt}{\sqrt{4t^2+5}\,(8t^2+1)} \to \frac{6dt}{\sqrt{5}} \quad (t \to 0),\\
\kappa_g^2(t)\,d\tau_2 &= \frac{-\operatorname{sgn}(t)\,42tdt}{(9t^2+20)\sqrt{17t^2+16}} \to 0dt \quad (t \to 0),\\
\kappa_g^3(t)\,d\tau_3 &= \frac{12\,(t^4-96t^2-36)\,dt}{\sqrt{t^4+144t^2+144}\,(16t^4+9t^2+144)} \to -\frac{dt}{4} \quad (t \to 0),\\
\kappa_g^4(t)\,d\tau_4 &= \frac{\operatorname{sgn}(t)\,4t\,(5t^6-8640t^2-4608)\,dt}{\sqrt{t^6+2304t^2+2304}\,(25t^6+16t^4+2304)} \to 0dt \quad (t \to 0),
\end{aligned}$$

respectively. Hence the 1-forms $\kappa_g^i d\tau_i$ are continuous at the origin.

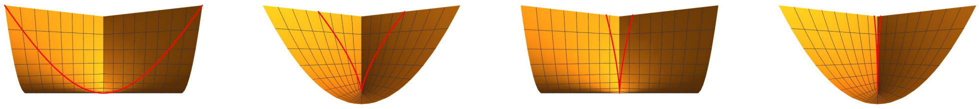

FIGURE 3.1. From left to right, the red lines represent $\widehat{\gamma}_i$, $i = 1, 2, 3, 4$, respectively.

REMARK 3.28. Consider a sufficiently small perturbation $\Gamma_1 : [0, \varepsilon) \times (-\delta, \delta) \to M$ of the curve $\gamma_1 : [0, \varepsilon) \to M$ in Example 3.27 given by

$$\Gamma_1(t, s) = (t, t + s).$$

Since the geodesic curvature measure $\kappa_g d\tau$ of Γ_1 is calculated as

$$\kappa_g(t, s)\, d\tau = \frac{(6t - 4s(s + t - 1)(s + t + 1))\, dt}{(5s^2 + 12st + 8t^2 + 1)\sqrt{4(s + t)^4 + 4(s + t)^2 + t^2}}$$

for $t > 0$, we have

$$\lim_{t \to 0+0}\left(\lim_{s \to 0} \kappa_g(t, s)\, d\tau\right) = \lim_{t \to 0+0}\left(\frac{6dt}{\sqrt{4t^2 + 5}\,(8t^2 + 1)}\right) = \frac{6dt}{\sqrt{5}}.$$

On the other hand, if $s > 0$,

$$\lim_{s \to 0+0}\left(\lim_{t \to 0+0} \kappa_g d\tau\right) = \lim_{s \to 0+0}\left(-\frac{2(s^2 - 1)\, dt}{\sqrt{s^2 + 1}\,(5s^2 + 1)}\right) = -2dt.$$

Also, if $s < 0$,

$$\lim_{s \to 0-0}\left(\lim_{t \to 0+0} \kappa_g d\tau\right) = \lim_{s \to 0-0}\left(\frac{2(s^2 - 1)\, dt}{\sqrt{s^2 + 1}\,(5s^2 + 1)}\right) = 2dt.$$

Hence the geodesic curvature measure defined on $(0, \varepsilon) \times (-\delta, \delta)$ is not necessarily continuously extended to $[0, \varepsilon) \times (-\delta, \delta)$ if we perturb a curve through an intrinsic cross cap.

Finally, we prove a Gauss-Bonnet type theorem for Whitney metrics.

THEOREM 3.29 (Theorem 5.1 of [HHNSUY15] if $\partial M = \varnothing$, Corollary 1.3 of [Has24] if $\partial M \neq \varnothing$). *Let M be a compact oriented smooth 2-manifold with boundary ∂M and ds^2 a Whitney metric on M. Suppose that ∂M is transversal to the null space at each singular point of ds^2 on ∂M if the set of singular points of ds^2 on ∂M is non-empty. Then*

$$\int_M K dA + \int_{\partial M} \kappa_g d\tau = 2\pi\chi(M). \tag{3.3.9}$$

PROOF. First, consider the case where the set of singular points of the Whitney metric ds^2 on the boundary ∂M is empty. Triangulating M so that each singular point of ds^2 is in the interior of triangles, a local Gauss-Bonnet type formula

$$\int_{\triangle \mathrm{ABC}} K dA + \int_{\partial \triangle \mathrm{ABC}} \kappa_g d\tau = \angle \mathrm{A} + \angle \mathrm{B} + \angle \mathrm{C} - \pi \tag{3.3.10}$$

holds for each triangle $\triangle$ABC by Corollary 3.21. By taking the sum of the equations (3.3.10) for each triangle, we obtain the equation (3.3.9).

Next, consider the case where the set of singular points on ∂M is non-empty. Take a positively oriented Bruce-West type coordinate chart of the second order $(U; u, v)$ centered at a singular point p on ∂M. Let $D(r)$ be a closed disc contained in U with center p and radius $r > 0$, and $\{\mathrm{P}_1, \mathrm{P}_2\}$ the intersection of the boundary $\partial D(r)$ with ∂M. Triangulating $D(r)$ so that p is a vertex and edges starting from p are transversal to the null space at p, the local Gauss-Bonnet type formula (3.3.10) holds for each triangle $\triangle$ABC by Corollary 3.21 and Theorem 3.24. Note that even if the vertices A, B and C are singular points, the angles $\angle$A, $\angle$B and $\angle$C on the right-hand side of the equation (3.3.10) are well-defined because there exists an initial vector of edges starting from A, B and C, respectively by the assumption. By taking the sum of the equations (3.3.10) for each triangle, we obtain

$$\int_{D(r)} K dA + \int_{\partial D(r)} \kappa_g d\tau = 2\pi - \sum_{i=1,2} (\pi - \angle \mathrm{P}_i).$$

This completes the proof of Theorem 3.29. $\square$

REMARK 3.30. If the boundary ∂M is empty,

$$\int_M K dA = 2\pi \chi(M). \tag{3.3.11}$$

If the Whitney metric ds^2 is the induced metric of a cross cap (see Corollary 3.2.3), the equation (3.3.11) is the same as the equation [Kui75, (7a)].

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