



HOKKAIDO UNIVERSITY

Title	A stability result for elliptic equations with singular nonlinearity and its applications to homogenization problems
Author(s)	Hara, Takanobu
Citation	Journal of mathematical analysis and applications, 528(1), 127509 https://doi.org/10.1016/j.jmaa.2023.127509
Issue Date	2023-12-01
Doc URL	https://hdl.handle.net/2115/95418
Rights	© <2023>. This manuscript version is made available under the CC-BY-NC-ND 4.0 license http://creativecommons.org/licenses/by-nc-nd/4.0/
Rights(URL)	https://creativecommons.org/licenses/by-nc-nd/4.0/
Type	journal article
File Information	sing.pdf



A stability result for elliptic equations with singular nonlinearity and its applications to homogenization problems

Takanobu Hara^{1,*}

*Graduate School of Information Science and Technology, Kita 14, Nishi 9, Kita-ku,
Sapporo, Hokkaido, 060-0814, Japan*

Abstract

We consider model semilinear elliptic equations of the type

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = \sigma u^{-\lambda}, & u > 0 \quad \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, $A \in L^\infty(\Omega)^{N \times N}$ is a coercive matrix, $0 < \lambda \leq 1$ and σ is a nonnegative function in $L_{loc}^1(\Omega)$, or more generally, nonnegative Radon measure on Ω . We discuss $H_0^1(\Omega)$ -stability of u under a minimal assumption on σ and apply the result to homogenization problems.

Keywords: Singular elliptic problem, Trace inequality, Potential theory, Homogenization

2020 MSC: 35J61, 35J25, 35B27.

1. Introduction and main results

Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 1$). We consider model elliptic differential equations of the type

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = \frac{\sigma}{u^\lambda}, & u > 0 \quad \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \quad (1.1)$$

where $H_0^1(\Omega)$ is the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|\nabla \cdot\|_{L^2(\Omega)}$, A is a coercive matrix-valued function in $M(\alpha, \beta, \Omega)$ (see Sect. 4 for details), σ is a nonnegative function in $L_{loc}^1(\Omega)$, or more generally, nonnegative (locally finite) Radon measure on Ω , and λ is a fixed positive constant. Results in this paper are for the case $0 < \lambda \leq 1$.

*Corresponding author

Email address: takanobu.hara.math@gmail.com (Takanobu Hara)

Elliptic boundary value problems of the type (1.1) are said to be singular from the difficulty to control near $u = 0$. After various studies aimed at applications to real problems (e.g., [21, 42, 8, 46]), Crandall, Rabinowitz and Tartar [12] established a general existence result of classical solutions. Although research intended to applications is still ongoing (e.g., [19, 18]), theoretical studies have been more active recently. In addition to the references given below, we refer to [48, 15, 13, 11, 38, 30, 22] for references on related problems. Traditionally, σ was assumed to be a continuous function in most studies. However, Boccardo and Orsina [7] studied L^p -integrable data and the corresponding generalized solutions. After their work, singular elliptic problems with measure valued coefficients have been studied extensively (e.g., [32, 43, 24]). Boccardo and Casado-Díaz [6] applied the results to H-convergence of elliptic operators. Extension to other types homogenization problems and further developments can be found in [23, 9, 17].

Let us observe known existence results of $H_0^1(\Omega)$ -solutions. For simplicity, we assume that $A(x) = I$. There are three type results. (i) Lazer and McKenna [35] proved that if σ is a positive Hölder continuous function on Ω and $\partial\Omega$ is sufficiently smooth, then there exists a weak solution to (1.1) if and only if $\lambda < 3$. Zhang and Cheng [53] extended the result to locally Hölder continuous σ satisfying $\sigma(x) \approx \text{dist}(x, \partial\Omega)^{-s}$. More generally, Díaz, Hernández and Rakotoson [16] considered

$$\sigma(x) = f(x) \text{dist}(x, \partial\Omega)^{-s}, \quad 0 < s \leq f(x) \in L^\infty(\Omega). \quad (1.2)$$

They proved that Eq. (1.1) has an $H_0^1(\Omega)$ -solution if and only if

$$s < (3 - \lambda)/2. \quad (1.3)$$

Note that σ in (1.2) is not integrable on Ω if $s \geq 1$. (ii) Lair and Shaker [33] proved that if $\lambda < 1$ and $0 \leq \sigma \in L^2(\Omega)$, then there exists an $H_0^1(\Omega)$ -solution to (1.1). Boccardo and Orsina [7] gave a more general result; if $\lambda \leq 1$ and

$$0 \leq \sigma \in L^m(\Omega), \quad m \geq \left(\frac{2^*}{1 - \lambda} \right)', \quad (1.4)$$

then there exists an $H_0^1(\Omega)$ -solution to (1.1), where p' is the Hölder conjugate of p and 2^* is the Sobolev conjugate of 2 (see, (3.3)). Further information can be found in [43]. (iii) Sun and Zhang [47] gave the following criterion for $\lambda > 1$: If $0 < \sigma \in L^1(\Omega)$, then there exists an $H_0^1(\Omega)$ -solution to (1.1) if and only if there exists $u_0 \in H_0^1(\Omega)$ such that

$$\int_{\Omega} |u_0|^{1-\lambda} \sigma(x) dx < \infty.$$

They also proved that there is no such a function if $\sigma \equiv 1$ and $\lambda \geq 3$ using the result by Lazer and McKenna.

When $0 < \lambda \leq 1$, the above existence results unify into one by an abstract framework in [28] (see, Theorem 4.6 below). For $0 < \lambda < 1$ and $A(x) = I$, we

consider the functional

$$J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{1-\lambda} \int_{\Omega} u_+^{1-\lambda} d\sigma.$$

Since J is convex on $\{u \in H_0^1(\Omega) : u \geq 0 \text{ a.e. in } \Omega\}$, if it is bounded from below, we can get a unique minimizer u by using the direct method. The Euler-Lagrange equation of J can not be derived directly because it is not differentiable, but we can show that u is a weak solution to (1.1) by making appropriate approximations (see, e.g., [15, 33, 7] for the precise meaning). The remaining problem, the boundedness of J , is equivalent to the validity of the $L^2(dx)$ - $L^{1-\lambda}(d\sigma)$ trace inequality

$$\|\varphi\|_{L^{1-\lambda}(\Omega; \sigma)} \leq C \|\nabla \varphi\|_{L^2(\Omega)}, \quad \forall \varphi \in C_c^\infty(\Omega). \quad (1.5)$$

As discussed later, (1.2)-(1.3) or (1.4) are sufficient for (1.5) and not necessary (see Sect. 3). For characterization of (1.5) and related results, we refer to ([39, Theorem 11.6.1] and [50, 51, 44, 27]. Similar arguments hold even if $A(x)$ is non-symmetric or $\lambda = 1$. In particular, the set $\mathcal{E}_\lambda$ of all measures σ such that (1.1) has an H_0^1 -solution has an explicit representation.

In this paper, we define a distance between measures in $\mathcal{E}_\lambda$ and discuss the stability of u with respect to it (Definition 3.1 and Theorem 4.7). Additionally, we apply the result to homogenization problems (Theorem 5.3). In conclusion, we will observe that weak solutions to (1.1) are stable for perturbations of A and σ . These results can be regarded as refinements of [6, Theorems 2.7 and 3.4]. The novelty of our results is that the class of σ has been extended to be necessary and sufficient for the existence of H_0^1 -solutions. To prove these results, we show that any measure in $\mathcal{E}_\lambda$ is approximated by smooth measures (Proposition 3.4) and that if $\sigma \in \mathcal{E}_\lambda$, then any weakly convergence sequence in $H_0^1(\Omega)$ has a subsequence that converges σ -a.e. (Corollary 2.4). These are well-known facts for L^p functions, but nontrivial for measures.

Throughout the paper, we basically do not assume the smoothness of $\partial\Omega$. Such assumptions are added when observing specific examples of σ belonging to $\mathcal{E}_\lambda$. For example, bounded Lipschitz domains satisfy the conditions in Example 3.7.

Organization of the paper

In Sect. 2, we present auxiliary results from potential theory. In Sect. 3, we investigate some properties of σ satisfying (1.5). In Sect. 4, we prove a stability result for weak solutions to the singular elliptic problems. In Sect. 5, we apply the results in Sections 3 and 4 to homogenization problems.

Notation

Throughout the paper, Ω is a domain (connected open subset) in $\mathbb{R}^N$.

- $\mathbf{1}_E(x) :=$ the indicator function of a set E .

- $C_c^\infty(\Omega) :=$ the set of all infinitely-differentiable functions with compact support in Ω .
- $\mathcal{M}^+(\Omega) :=$ the set of all nonnegative Radon measures on Ω .
- $L^p(\Omega; \mu) :=$ the L^p space with respect to $\mu \in \mathcal{M}^+(\Omega)$.
- $f_+ := \max\{f, 0\}$ and $f_- := \max\{-f, 0\}$.

For simplicity, we write $L^p(\Omega; dx)$ as $L^p(\Omega)$, where dx is the Lebesgue measure. For measures μ and ν , we denote $\nu \leq \mu$ if $\mu - \nu$ is a nonnegative measure. For a sequence of extended real valued functions $\{f_j\}_{j=1}^\infty$, we denote $f_j \uparrow f$ if $f_{j+1} \geq f_j$ for all $j \geq 1$ and $\lim_{j \rightarrow \infty} f_j = f$. The letter C denotes various constants with and without indices.

2. Potential theory

We first recall some properties of Sobolev spaces from [34, 39, 29]. The Sobolev space $H^1(\Omega)$ is the set of all weakly differentiable functions u such that $\|u\|_{H^1(\Omega)}$ is finite, where

$$\|u\|_{H^1(\Omega)} := \left(\int_{\Omega} |u|^2 + |\nabla u|^2 dx \right)^{\frac{1}{2}}.$$

Below, we assume the boundedness of Ω . We denote by $H_0^1(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$. By the Poincaré inequality, we can take $\|\nabla \cdot\|_{L^2(\Omega)}$ as the norm of $H_0^1(\Omega)$. For simplicity, we denote by $(\cdot, \cdot)$ the inner product of $H_0^1(\Omega)$. Also, we denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Let $O \subset \mathbb{R}^N$ be open, and let $K \subset O$ be compact. We define the (*variational capacity*) $\text{cap}(K, O)$ of the condenser (K, O) by

$$\text{cap}(K, O) := \inf \left\{ \|\nabla u\|_{L^2(O)}^2 : u \geq 1 \text{ on } K, u \in C_c^\infty(O) \right\}.$$

For $E \subset O$, we define

$$\text{cap}(E, O) := \inf_{\substack{E \subset U \subset O \\ U: \text{open}}} \sup_{\substack{K \subset U \\ K: \text{compact}}} \text{cap}(K, O).$$

For $E \subset \mathbb{R}^N$, the *Sobolev capacity* of E is defined by

$$C(E) = \inf \int_{\mathbb{R}^n} (|u|^2 + |\nabla u|^2) dx,$$

where the infimum is taken over all $u \in H^1(\mathbb{R}^N)$ such that $u = 1$ in a neighborhood of E . If Ω is bounded and $E \subset \Omega$, $C(E) = 0$ if and only if $\text{cap}(E, \Omega) = 0$. We say that a property holds *quasieverywhere* (q.e.) if it holds except on a set of capacity zero. An extended real valued function u on Ω is called as *quasicontinuous* if for every $\epsilon > 0$ there exists an open set E such that $C(E) < \epsilon$ and

$u|_{\Omega \setminus E}$ is continuous. Every $u \in H_{\text{loc}}^1(\Omega)$ has a quasicontinuous representative $\tilde{u}$ such that $\tilde{u} = u$ a.e. Below, we denote $\tilde{u}$ by u again for simplicity of notation.

We denote by $\mathcal{M}_0^+(\Omega)$ the set of all Radon measures σ that are absolutely continuous with respect to the capacity, that is, for any Borel set E ,

$$\text{cap}(E, \Omega) = 0 \implies \sigma(E) = 0.$$

For simplicity, we use the following notations:

$$\begin{aligned} H^{-1}(\Omega)^+ &:= H^{-1}(\Omega) \cap \mathcal{M}^+(\Omega), \\ \mathcal{M}_{0,b}^+(\Omega) &:= \{\sigma \in \mathcal{M}_0^+(\Omega) : \sigma(\Omega) < \infty\}. \end{aligned}$$

Next, we introduce a class of smooth measures with compact support.

Definition 2.1. We define a class of smooth measures $S_c(\Omega)$ by

$$S_c(\Omega) := \left\{ \sigma \in \mathcal{M}^+(\Omega) : \sup_{\Omega} G_{\Omega} \sigma < \infty \text{ and } \text{supp } \sigma \Subset \Omega \right\},$$

where $G_{\Omega} \sigma(x) := \int_{\Omega} G_{\Omega}(x, y) d\sigma(y)$ and $G_{\Omega}(\cdot, \cdot)$ is the Green function for the Dirichlet Laplace operator.

The definition of $S_c(\Omega)$ is inspired by the theory of Dirichlet forms (see, e.g., [20]), but for the sake of later discussion, we assume the compactness of $\text{supp } \sigma$ unlike usual. Note that the Green function G_{Ω} exists since Ω is bounded. If $\sigma \in S_c(\Omega)$, then the energy of σ , $\int_{\Omega} G_{\Omega} \sigma d\sigma$ is finite. Therefore,

$$S_c(\Omega) \subset H^{-1}(\Omega)^+ \subset \mathcal{M}_0^+(\Omega).$$

The following decomposition theorem holds. See, [20, Theorem 2.2.4] or [27, Theorem 3.6].

Theorem 2.2. *Let $\sigma \in \mathcal{M}^+(\Omega)$. Then, $\sigma \in \mathcal{M}_0^+(\Omega)$ if and only if there exists an increasing sequence of compact sets $\{F_k\}_{k=1}^{\infty}$ such that $\sigma_k := \mathbf{1}_{F_k} \sigma \in S_c(\Omega)$ for all $k \geq 1$ and $\sigma(\Omega \setminus \bigcup_{k=1}^{\infty} F_k) = 0$.*

Lemma 2.3. *Assume that $\sigma \in S_c(\Omega)$. Then, the embedding*

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega; \sigma)$$

is continuous, and the embedding

$$H_0^1(\Omega) \hookrightarrow L^1(\Omega; \sigma) \tag{2.1}$$

is compact.

Proof. By the Picone inequality (see [3]), we have

$$\int_{\Omega} \varphi^2 \frac{d\sigma}{G_{\Omega} \sigma} \leq \int_{\Omega} |\nabla \varphi|^2 dx, \quad \forall \varphi \in H_0^1(\Omega). \tag{2.2}$$

Thus, for $M := \|G_\Omega \sigma\|_{L^\infty(\Omega)}$, we have

$$\int_\Omega \varphi^2 d\sigma \leq M \int_\Omega \varphi^2 \frac{d\sigma}{G_\Omega \sigma} \leq M \int_\Omega |\nabla \varphi|^2 dx, \quad \forall \varphi \in H_0^1(\Omega).$$

Since $\sigma(\Omega)$ is finite, (2.1) is also valid. The compactness of (2.1) follows from a general result for trace inequalities. Since σ has compact support, we can take $\eta \in C_c^\infty(\Omega)$ such that $\eta \equiv 1$ on $\text{supp } \sigma$. Therefore,

$$\begin{aligned} \|u\|_{L^1(\Omega; \sigma)} &= \|\eta u\|_{L^1(\Omega; \sigma)} \\ &\leq (\sigma(\Omega)M)^{\frac{1}{2}} \|\nabla(\eta u)\|_{L^2(\Omega)} \\ &\leq C \|u\|_{H^1(\mathbb{R}^N)}, \quad \forall u \in H^1(\mathbb{R}^N). \end{aligned}$$

Then, [39, Theorem 11.9.2] yields the desired compactness. $\square$

Corollary 2.4. *Let $\sigma \in \mathcal{M}_0^+(\Omega)$. Let $\{u_j\} \subset H_0^1(\Omega)$, and let $u \in H_0^1(\Omega)$. Assume that $u_j \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Then, there exists a subsequence of $\{u_j\}$ such that $u_j \rightarrow u$ σ -a.e. in Ω .*

Proof. Let $\{F_k\}$ be a sequence of compact sets in Theorem 2.2. By the compact embedding result in Lemma 2.3, we can choose a subsequence of $\{u_j\}$ such that $u_j(x) \rightarrow u(x)$ for σ -a.e. $x \in F_1$. By a diagonalization argument, we can choose a subsequence of $\{u_j\}$ such that $u_j(x) \rightarrow u(x)$ for σ -a.e. $x \in \bigcup_{k=1}^\infty F_k$. This subsequence has the desired property. $\square$

3. Energy of measures

Continuing to recall potential theory, we define a distance of measures.

Definition 3.1. Let $0 \leq \lambda \leq 1$. We define a subset $\mathcal{E}_\lambda(\Omega)$ of $\mathcal{M}_0^+(\Omega)$ by

$$\mathcal{E}_\lambda(\Omega) := \left\{ \sigma \in \mathcal{M}_0^+(\Omega) : \|\sigma\|_\lambda < \infty \right\},$$

where

$$\|\sigma\|_\lambda := \sup \left\{ \int_\Omega |\varphi|^{1-\lambda} d\sigma : \varphi \in C_c^\infty(\Omega), \|\nabla \varphi\|_{L^2(\Omega)} \leq 1 \right\}$$

for $0 \leq \lambda < 1$, and

$$\|\sigma\|_1 := \sigma(\Omega).$$

For $\sigma, \nu \in \mathcal{E}_\lambda(\Omega)$ we define

$$\mathbf{d}_\lambda(\sigma, \nu) := \|\sigma - \nu\|_\lambda,$$

where $|\sigma - \nu|$ is the total variation of $\sigma - \nu$.

Remark 3.2. By definition, $\mathcal{E}_0(\Omega) = H^{-1}(\Omega)^+$ and $\mathcal{E}_1(\Omega) = \mathcal{M}_{0,b}^+(\Omega)$. Also, by Hölder's inequality,

$$\|\sigma\|_\lambda \leq \sigma(\Omega)^\lambda \|\sigma\|_{H^{-1}(\Omega)}^{1-\lambda}. \quad (3.1)$$

In particular,

$$S_c(\Omega) \subset \mathcal{E}_\lambda(\Omega) \subset \mathcal{M}_0^+(\Omega).$$

By definition and the inequality $|\sigma - \nu| \leq \sigma + \nu$,

$$\mathbf{d}_\lambda(\sigma, \nu) \leq \|\sigma + \nu\|_\lambda \leq \|\sigma\|_\lambda + \|\nu\|_\lambda < \infty, \quad \forall \sigma, \nu \in \mathcal{E}_\lambda.$$

Similarly, since $|\sigma - \nu| \leq |\sigma - \mu| + |\mu - \nu|$, we have

$$\mathbf{d}_\lambda(\sigma, \nu) \leq \mathbf{d}_\lambda(\sigma, \mu) + \mathbf{d}_\lambda(\mu, \nu), \quad \forall \sigma, \nu, \mu \in \mathcal{E}_\lambda.$$

Meanwhile, if $\mathbf{d}_\lambda(\sigma, \mu) = 0$, then $|\sigma - \nu|(K) = 0$ for any compact set $K \subset \Omega$, and thus, we have $\sigma = \nu$.

Remark 3.3. By [27, Theorem 1.3], the following two-sided estimate holds:

$$\|\sigma\|_\lambda \leq \left(\int_\Omega (G_\Omega \sigma)^{\frac{1-\lambda}{1+\lambda}} d\sigma \right)^{\frac{1+\lambda}{2}} \leq \frac{1}{(1-\lambda^2)^{\frac{1-\lambda}{2}}} \|\sigma\|_\lambda. \quad (3.2)$$

This inequality is valid even if $\lambda = 1$ in the sense that $\lim_{\lambda \rightarrow 1-} (1-\lambda^2)^{\frac{\lambda-1}{2}} = 1$.

Proposition 3.4. *Let $\sigma \in \mathcal{E}_\lambda(\Omega)$. (i) If $\{f_k\} \subset L^\infty(\Omega; \sigma)$ is a bounded sequence of nonnegative functions, and if $f_k \rightarrow f$ σ -a.e. in Ω , then, $\mathbf{d}_\lambda(f\sigma, f_k\sigma) \rightarrow 0$. (ii) There exists a sequence $\{\sigma_k\} \subset S_c(\Omega)$ such that $\sigma_k \uparrow \sigma$ and $\mathbf{d}_\lambda(\sigma, \sigma_k) \rightarrow 0$.*

Proof. Set $g_k = |f - f_k|$. By assumption, there is a constant M independent of k such that $g_k(x) \leq M$ for σ -a.e. $x \in \Omega$. Therefore,

$$0 \leq G_\Omega[g_k\sigma](x) \leq M G_\Omega\sigma(x), \quad \forall x \in \Omega.$$

By the dominated convergence theorem,

$$\int_\Omega G_\Omega[g_k\sigma]^{\frac{1-\lambda}{1+\lambda}} g_k d\sigma \rightarrow 0.$$

It follows from (3.2) that $\|\sigma - g_k\sigma\|_\lambda \rightarrow 0$. The latter claim follows from Theorem 2.2. $\square$

Let us observe examples of functions in $\mathcal{E}_\lambda(\Omega)$. For one-dimensional cases, we refer to the paper by Sinnamon and Stepanov [45].

Example 3.5. Suppose that $\Omega \subset \mathbb{R}^N$ is bounded. Set

$$2^* := \begin{cases} \frac{2N}{N-2} & N \geq 3, \\ \text{any finite constant} & N = 2, \\ +\infty & N = 1. \end{cases} \quad (3.3)$$

Let $0 \leq f \in L^m(\Omega)$, where m is a constant satisfying (1.4). By Sobolev's inequality and Hölder's inequality,

$$\begin{aligned} \int_{\Omega} |\varphi|^{1-\lambda} f \, dx &\leq C \|f\|_{L^m(\Omega)} \|\varphi\|_{L^{2^*}(\Omega)}^{1-\lambda} \\ &\leq C' \|f\|_{L^m(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)}^{1-\lambda}, \quad \forall \varphi \in H_0^1(\Omega). \end{aligned}$$

Therefore, $\|f\|_{\lambda} \leq C \|f\|_{L^m(\Omega)}$. In particular, if $f_k \rightarrow f$ strongly in $L^m(\Omega)$, then $\mathbf{d}_{\lambda}(f, f_k) \rightarrow 0$.

Example 3.6. Assume that the following Hardy inequality holds:

$$\int_{\Omega} \frac{\varphi^2}{\delta^2} \, dx \leq C \int_{\Omega} |\nabla \varphi|^2 \, dx, \quad \forall \varphi \in C_c^\infty(\Omega), \quad (3.4)$$

where $\delta(x) := \text{dist}(x, \partial\Omega)$. Then, for any $0 \leq \lambda \leq 1$,

$$\begin{aligned} \int_{\Omega} |\varphi|^{1-\lambda} f \, dx &\leq \|f \delta^{1-\lambda}\|_{L^{\frac{2}{1+\lambda}}(\Omega)} \|\varphi\|_{L^{\frac{2}{1-\lambda}}(\Omega)}^{1-\lambda} \\ &= \|f\|_{L^{\frac{2}{1+\lambda}}(\Omega; w_{\lambda})} \left\| \frac{\varphi}{\delta} \right\|_{L^2(\Omega)}^{1-\lambda} \\ &\leq C^{1-\lambda} \|f\|_{L^{\frac{2}{1+\lambda}}(\Omega; w_{\lambda})} \|\nabla \varphi\|_{L^2(\Omega)}^{1-\lambda}, \quad \forall \varphi \in H_0^1(\Omega), \end{aligned}$$

where $w_{\lambda} := \delta^{\frac{2(1-\lambda)}{1+\lambda}}$. Thus, if $f_k \rightarrow f$ strongly in $L^{\frac{2}{1+\lambda}}(\Omega; w_{\lambda})$, then $\mathbf{d}_{\lambda}(f, f_k) \rightarrow 0$. We note that there is no inclusion between $L^{\frac{2}{1+\lambda}}(\Omega; w_{\lambda})$ and $L^1(\Omega)$ if $\lambda < 1$ (see, Example 3.7 below).

Ancona [4] proved (3.4) under the following capacity density condition.

$$\exists c > 0 \text{ s.t. } \forall x \in \partial\Omega, \forall r > 0, \frac{\text{cap}(\overline{B(x, r)} \setminus \Omega, B(x, 2r))}{\text{cap}(\overline{B(x, r)}, B(x, 2r))} \geq c. \quad (3.5)$$

We refer to [36, 52] and [31, Chapter 6] for other proofs and further references.

Example 3.7. Assume (3.4) and that there exists $\theta \in (0, 1)$ such that

$$\int_{\Omega} \delta^{-\theta} \, dx < \infty. \quad (3.6)$$

Then, $\delta^{-s} \in L^{\frac{2}{1+\lambda}}(\Omega; w_{\lambda})$ (and thus $\delta^{-s} \in \mathcal{E}_{\lambda}(\Omega)$) if $0 \leq \lambda \leq 1$ and

$$\lambda \leq \frac{2-2s+\theta}{2-\theta}.$$

Two sufficient conditions for (3.6) can be found in [26]. (i) If Ω is a bounded Lipschitz domain, then (3.6) holds if and only if $\theta < 1$. Therefore, $\delta^{-s} \in \mathcal{E}_{\lambda}(\Omega)$ if (1.3) holds. In particular, δ^{-1} is not integrable, but $\delta^{-1} \in \mathcal{E}_{\lambda}(\Omega)$ for any $0 \leq \lambda < 1$. (ii) The other condition is more complicated. A bounded domain Ω

is called *John domain* if there are $x_0 \in \Omega$ and $C_J \geq 1$ such that each $x \in \Omega$ can be connected to x_0 by a rectifiable curve $\gamma \subset \Omega$ with

$$l(\gamma(x, z)) \leq C_J \delta(z) \quad \forall z \in \gamma,$$

where $\gamma(x, z)$ is the subarc of γ from x to z and $l(\gamma(x, z))$ is the length of $\gamma(x, z)$. Trocenko [49] proved that if Ω is a bounded John domain, then (3.6) holds for some $\theta > 0$. For another more geometric proof, see [1, Lemma 4.4]. In this case, $\delta^{-\theta} \in \mathcal{E}_\lambda(\Omega)$ for all $0 \leq \lambda \leq 1$. We note that these conditions are independent of (3.5), because (3.5) is a condition for the outside of Ω .

4. Elliptic equations with singular nonlinearity

Fix constants $0 < \alpha \leq \beta < \infty$. We define the set of $N \times N$ -matrix-valued functions $M(\alpha, \beta, \Omega)$ by

$$M(\alpha, \beta, \Omega) := \left\{ A \in L^\infty(\Omega)^{N \times N} : \begin{array}{l} A(x)\xi \cdot \xi \geq \alpha|\xi|^2, \\ A^{-1}(x)\xi \cdot \xi \geq \beta^{-1}|\xi|^2, \end{array} \quad \forall \xi \in \mathbb{R}^N, \text{ a.e. } x \in \Omega \right\}.$$

If $A \in M(\alpha, \beta, \Omega)$, then the differential operator

$$\mathcal{L}u := -\operatorname{div}(A(x)\nabla u)$$

satisfies

$$|\langle \mathcal{L}u, v \rangle| \leq \beta \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}, \quad \forall u, v \in H_0^1(\Omega), \quad (4.1)$$

$$\langle \mathcal{L}u, u \rangle \geq \alpha \|\nabla u\|_{L^2(\Omega)}^2, \quad \forall u \in H_0^1(\Omega). \quad (4.2)$$

By the Lax-Milgram theorem, for any $f \in H^{-1}(\Omega)$, there exists a unique function $u \in H_0^1(\Omega)$ satisfying

$$\langle \mathcal{L}u, \varphi \rangle = \langle f, \varphi \rangle \quad \forall \varphi \in H_0^1(\Omega).$$

Assume also that f is a nonnegative distribution. Then, u is a supersolution to $\mathcal{L}u = 0$ in Ω , and its *lsc-regularization*

$$u^*(x) := \lim_{r \rightarrow 0} \operatorname{ess\,inf}_{B(x, r)} u$$

gives a quasicontinuous representative of u . For details and more information, including De Giorgi-Nash-Moser theory, we refer to [29]. We always use this representative below.

The pointwise estimate for Green functions of divergence form elliptic operators was established in [37] (see also [25] for non-symmetric A). The following two-sided estimate is one corollary of it.

Theorem 4.1 ([37, Corollary 7.1]). *Let $A \in M(\alpha, \beta, \Omega)$, and let K be a compact subset in Ω . Assume that σ is a nonnegative Radon measure in $H^{-1}(\Omega)$ such that $\text{supp } \sigma \subset K$. Let U be the weak solution to*

$$\begin{cases} -\text{div}(A(x)\nabla U) = \sigma & \text{in } \Omega, \\ U \in H_0^1(\Omega). \end{cases} \quad (4.3)$$

Then

$$\frac{1}{C}G_\Omega\sigma \leq U \leq CG_\Omega\sigma, \quad \text{on } K,$$

where C is a positive constant depending only on N, α, β, Ω and K .

Remark 4.2. Note that the constant C is depending on K . The boundary behavior of U is not comparable to $G_\Omega\sigma$ in general (see, e.g., [41]). However, if $\sigma \in S_c$, then U is bounded on Ω from Theorem 4.1 and the complete maximum principle.

We understand Eq. (1.1) by the following weak sense.

Definition 4.3. Let $A \in M(\alpha, \beta, \Omega)$, $0 < \lambda \leq 1$ and $\sigma \in \mathcal{M}_0^+(\Omega) \setminus \{0\}$. We say that a function $u \in H_0^1(\Omega)$ is a (finite energy) weak solution to (1.1) if $u > 0$ in Ω and

$$\int_{\Omega} A \nabla u \cdot \nabla \varphi \, dx = \int_{\Omega} \frac{\varphi}{u^\lambda} \, d\sigma \quad \forall \varphi \in C_c^\infty(\Omega).$$

Remark 4.4. From the structure of the lower order term, we may assume that σ is absolutely continuous with respect to the capacity without loss of generality. For details, see [7, Theorem 3.4] and [43, Remark 2.2].

Remark 4.5. If there exists a weak solution u to (1.1), then the embedding $H_0^1(\Omega) \hookrightarrow L^1(\Omega; \sigma/u^\lambda)$ is continuous. Thus, it follows from the completeness of $L^1(\Omega; \sigma/u^\lambda)$ that every $\varphi \in H_0^1(\Omega)$ is integrable with respect to σ/u^λ . If v is a weak solution to

$$\begin{cases} -\text{div}(A(x)\nabla v) = \frac{\nu}{v^\lambda}, & v > 0 \quad \text{in } \Omega, \\ v \in H_0^1(\Omega), \end{cases} \quad (4.4)$$

and if $\sigma \leq \nu$, then

$$\int_{\Omega} A \nabla(u-v) \cdot \nabla(u-v)_+ \, dx = \int_{\Omega} \left(\frac{f}{u^\lambda} - \frac{1}{v^\lambda} \right) (u-v)_+ \, d\nu \leq 0,$$

where $f = \frac{d\sigma}{d\nu}$ is the Radon-Nikodým derivative of σ with respect to ν . By (4.2), $\nabla(u-v)_+ = 0$ a.e. in Ω . Consequently, $u \leq v$ in $H_0^1(\Omega)$. In particular, if there exists a weak solution to (1.1), it is unique in $H_0^1(\Omega)$.

Below, we always assume that $\sigma \in \mathcal{E}_\lambda(\Omega) \setminus \{0\}$. This assumption is minimal in the following sense.

Theorem 4.6. Let $A \in M(\alpha, \beta, \Omega)$, $0 < \lambda \leq 1$ and $\sigma \in \mathcal{M}_0^+(\Omega) \setminus \{0\}$. Then, there exists a unique weak solution $u \in H_0^1(\Omega)$ to (1.1) if and only if $\sigma \in \mathcal{E}_\lambda(\Omega) \setminus \{0\}$. Moreover,

$$\frac{1}{\beta^{\frac{1}{1+\lambda}}} \|\sigma\|_\lambda^{\frac{1}{1+\lambda}} \leq \|\nabla u\|_{L^2(\Omega)} \leq \frac{1}{\alpha^{\frac{1}{1+\lambda}}} \|\sigma\|_\lambda^{\frac{1}{1+\lambda}}. \quad (4.5)$$

Assume also that $\sigma \in S_c$. Then, u is bounded in Ω and

$$u(x) \leq (1 + \lambda)^{\frac{1}{1+\lambda}} U(x)^{\frac{1}{1+\lambda}}, \quad \forall x \in \Omega, \quad (4.6)$$

where U is the weak solution to (4.3).

Proof. See the proof of [28, Theorem 1.2 and Corollary 6.5]. $\square$

Theorem 4.7. Let $\sigma, \nu \in \mathcal{E}_\lambda(\Omega) \setminus \{0\}$. Let u be the weak solution to (1.1), and let v be the weak solution to (4.4). Then,

$$\|\nabla(u - v)\|_{L^2(\Omega)} \leq \frac{1}{\alpha^{\frac{1}{1+\lambda}}} \mathbf{d}_\lambda(\sigma, \nu)^{\frac{1}{1+\lambda}}.$$

Proof. Let $0 < \lambda < 1$. By assumption,

$$\int_\Omega A \nabla u \cdot \nabla(u - v) dx = \int_\Omega (u - v)_+ \frac{1}{u^\lambda} d\sigma - \int_\Omega (u - v)_- \frac{1}{u^\lambda} d\sigma$$

and

$$\int_\Omega A \nabla v \cdot \nabla(u - v) dx = \int_\Omega (u - v)_+ \frac{1}{v^\lambda} d\nu - \int_\Omega (u - v)_- \frac{1}{v^\lambda} d\nu.$$

If $u > v$, then $\frac{1}{u^\lambda} < \frac{1}{v^\lambda}$. Thus,

$$\begin{aligned} \int_\Omega (u - v)_+ \frac{1}{u^\lambda} d\sigma - \int_\Omega (u - v)_+ \frac{1}{v^\lambda} d\nu &\leq \int_\Omega (u - v)_+ \frac{1}{u^\lambda} d(\sigma - \nu) \\ &\leq \int_\Omega (u - v)_+^{1-\lambda} d(\sigma - \nu)_+, \end{aligned}$$

where $(\sigma - \nu)_+$ is the positive part of $\sigma - \nu$. Similarly,

$$\int_\Omega (u - v)_- \frac{1}{v^\lambda} d\nu - \int_\Omega (u - v)_- \frac{1}{u^\lambda} d\sigma \leq \int_\Omega (u - v)_-^{1-\lambda} d(\nu - \sigma)_+.$$

Combining the four inequalities and using (4.2), we obtain,

$$\begin{aligned} \alpha \int_\Omega |\nabla(u - v)|^2 dx &\leq \int_\Omega |u - v|^{1-\lambda} d|\sigma - \nu| \\ &\leq \|\sigma - \nu\|_\lambda \|\nabla|u - v|\|_{L^2(\Omega)}^{1-\lambda}. \end{aligned}$$

By the inequality $|\nabla| \cdot || \leq |\nabla \cdot |$, we arrive at the desired estimate. The same argument is valid even if $\lambda = 1$. $\square$

5. Homogenization

Let us recall the definition of H-convergence. The concept of H-convergence was introduced by Murat and Tartar in the late 1970s ([40, 2]).

Definition 5.1. Let $\{A_\epsilon\} \subset M(\alpha, \beta, \Omega)$, and let $A_0 \in L^\infty(\Omega)^{N \times N}$. We say that A_ϵ *H-converges* to A_0 ($A_\epsilon \xrightarrow{H} A_0$) if for every $f \in H^{-1}(\Omega)$, the sequence $\{u_\epsilon\}$ of weak solutions to

$$\begin{cases} -\operatorname{div}(A_\epsilon(x)\nabla u_\epsilon) = f & \text{in } \Omega, \\ u_\epsilon \in H_0^1(\Omega), \end{cases}$$

satisfies

$$\begin{aligned} u_\epsilon &\rightharpoonup u_0 \text{ weakly in } H_0^1(\Omega), \\ A_\epsilon \nabla u_\epsilon &\rightharpoonup A_0 \nabla u_0 \text{ weakly in } L^2(\Omega)^N, \end{aligned}$$

where u_0 is the weak solution to

$$\begin{cases} -\operatorname{div}(A_0(x)\nabla u_0) = f & \text{in } \Omega, \\ u_0 \in H_0^1(\Omega). \end{cases} \quad (5.1)$$

If $\{A_\epsilon\} \subset M(\alpha, \beta, \Omega)$ and $A_\epsilon \xrightarrow{H} A_0$, then A_0 belongs to $M(\alpha, \beta, \Omega)$ again. If $A_\epsilon \rightarrow A_0$ a.e. in Ω , then $A_\epsilon \xrightarrow{H} A_0$. Many nontrivial examples can be found in periodic homogenization (e.g., [10]).

Remark 5.2. Assume that $A_\epsilon \xrightarrow{H} A_0$ and $f_\epsilon \rightarrow f$ strongly in $H^{-1}(\Omega)$. Let $\{u_\epsilon\}$ be the sequence of weak solutions to

$$\begin{cases} -\operatorname{div}(A_\epsilon(x)\nabla u_\epsilon) = f_\epsilon & \text{in } \Omega, \\ u_\epsilon \in H_0^1(\Omega). \end{cases}$$

Then, $u_\epsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$, where u_0 is the weak solution to (5.1). The strong convergence of f_ϵ can not be replaced by the weak convergence generally. Further information can be found in [10, Section 3.1.5].

Our main result is as follows.

Theorem 5.3. Let $\{A_\epsilon\} \subset M(\alpha, \beta, \Omega)$, $0 < \lambda \leq 1$ and $\{\sigma_\epsilon\} \subset \mathcal{E}_\lambda(\Omega) \setminus \{0\}$. Let $\{u_\epsilon\} \subset H_0^1(\Omega)$ be the sequence of weak solutions to

$$\begin{cases} -\operatorname{div}(A_\epsilon(x)\nabla u_\epsilon) = \frac{\sigma_\epsilon}{u_\epsilon^\lambda}, & u_\epsilon > 0 \quad \text{in } \Omega, \\ u_\epsilon \in H_0^1(\Omega). \end{cases} \quad (5.2)$$

Assume that $A_\epsilon \xrightarrow{H} A_0$ and that there exists $\sigma \in \mathcal{E}_\lambda(\Omega) \setminus \{0\}$ such that $\mathbf{d}_\lambda(\sigma, \sigma_\epsilon) \rightarrow 0$. Then, $u_\epsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$, where u_0 is the weak solution to

$$\begin{cases} -\operatorname{div}(A_0(x)\nabla u_0) = \frac{\sigma}{u_0^\lambda}, & u_0 > 0 \quad \text{in } \Omega, \\ u_0 \in H_0^1(\Omega). \end{cases} \quad (5.3)$$

Moreover, $\sigma_\epsilon/u_\epsilon^\lambda \rightarrow \sigma/u_0^\lambda$ strongly in $H^{-1}(\Omega)$.

Remark 5.4. If $A_\epsilon \rightarrow A_0$ a.e. in Ω , then Theorem 5.3 yields the strong convergence of u_ϵ in $H_0^1(\Omega)$ (see, e.g., [2, Theorem 1.2.20]).

H-convergence can be rewritten as Γ -convergence by defining an appropriate functional, and stability of Γ -convergence with respect to continuous perturbations is well-known (see, [14, 5]). However, our lower-order term does not admit this framework. Hence, following [6], we reduce the problem and use elliptic regularity.

Lemma 5.5. Let $\{A_\epsilon\} \subset M(\alpha, \beta, \Omega)$, $0 < \lambda \leq 1$ and $\sigma \in S_c(\Omega) \setminus \{0\}$. Let $\{u_\epsilon\} \subset H_0^1(\Omega)$ be the sequence of weak solutions to

$$\begin{cases} -\operatorname{div}(A_\epsilon(x)\nabla u_\epsilon) = \frac{\sigma}{u_\epsilon^\lambda}, & u_\epsilon > 0 \quad \text{in } \Omega, \\ u_\epsilon \in H_0^1(\Omega). \end{cases} \quad (5.4)$$

Assume also that $A_\epsilon \xrightarrow{H} A_0$. Then, $u_\epsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$, where u_0 is the weak solution to (5.3). Moreover, $\sigma/u_\epsilon^\lambda \rightarrow \sigma/u_0^\lambda$ strongly in $H^{-1}(\Omega)$.

Proof. By assumption and Theorem 4.6, there exists a constant C independent of ϵ such that $\|\nabla u_\epsilon\|_{L^2(\Omega)} \leq C$. Denote by $\{u_\epsilon\}$ any subsequence of $\{u_\epsilon\}$ again. By weak compactness and Lemma 2.3, we can choose a subsequence of $\{u_\epsilon\}$ such that $u_\epsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$ and $u_\epsilon \rightarrow u_0$ σ -a.e. in Ω . Since $\sigma \in S_c(\Omega)$, by Theorems 4.1 and 4.6,

$$\|u_\epsilon\|_{L^\infty(\Omega; \sigma)} \leq C \|G_\Omega \sigma\|_{L^\infty(\Omega; \sigma)}^{\frac{1}{1+\lambda}} (= M).$$

Note that the right-hand side is independent of ϵ . Since

$$-\operatorname{div}(A_\epsilon(x)\nabla u_\epsilon) \geq \frac{\sigma}{M^\lambda} > 0 \quad \text{in } \Omega,$$

using Theorem 4.1 again, we obtain

$$u_\epsilon(x) \geq \frac{1}{C} G_\Omega \sigma(x) \geq m, \quad \forall x \in \operatorname{supp} \sigma,$$

where m is a positive constant independent of ϵ . By the dominated convergence theorem,

$$\frac{1}{u_\epsilon^\lambda} \rightarrow \frac{1}{u_0^\lambda} \quad \text{strongly in } L^2(\Omega; \sigma).$$

Meanwhile, by Lemma 2.3 and Hölder's inequality,

$$\left| \int_{\Omega} \varphi \left(\frac{1}{u_{\epsilon}^{\lambda}} - \frac{1}{u_0^{\lambda}} \right) d\sigma \right| \leq C \left\| \frac{1}{u_{\epsilon}^{\lambda}} - \frac{1}{u_0^{\lambda}} \right\|_{L^2(\Omega; \sigma)} \|\nabla \varphi\|_{L^2(\Omega)}, \quad \forall \varphi \in H_0^1(\Omega).$$

Thus, $\sigma/u_{\epsilon}^{\lambda} \rightarrow \sigma/u_0^{\lambda}$ strongly in $H^{-1}(\Omega)$. It follows from Remark 5.2 that u_0 satisfies (5.3). By the uniqueness of weak solutions, the entire sequence $\{u_{\epsilon}\}$ converges to u_0 weakly in $H_0^1(\Omega)$. $\square$

Lemma 5.6. *The results in Lemma 5.5 are still valid for $\sigma \in \mathcal{E}_{\lambda}(\Omega) \setminus \{0\}$.*

Proof. By Theorem 4.6, $\{u_{\epsilon}\}$ is bounded in $H_0^1(\Omega)$. Moreover, there exists a unique weak solution u_0 to (5.3). Using Theorem 2.2, we take a sequence $\{\sigma_k\} \subset S_c(\Omega) \setminus \{0\}$ such that $\sigma_k \uparrow \sigma$. For $\epsilon > 0$ and $\epsilon = 0$, we consider the following approximating problems:

$$\begin{cases} -\operatorname{div}(A_{\epsilon}(x)\nabla u_{\epsilon}^k) = \frac{\sigma_k}{(u_{\epsilon}^k)^{\lambda}}, & u_{\epsilon}^k > 0 \quad \text{in } \Omega, \\ u_{\epsilon}^k \in H_0^1(\Omega). \end{cases} \quad (5.5)$$

Fix $\varphi \in H_0^1(\Omega)$. By Theorem 4.7,

$$\begin{aligned} |(u_{\epsilon} - u_0, \varphi)| &= |(u_{\epsilon} - u_{\epsilon}^k, \varphi)| + |(u_{\epsilon}^k - u_0^k, \varphi)| + |(u_0^k - u_0, \varphi)| \\ &\leq \frac{2}{\alpha^{\frac{1}{1+\lambda}}} \mathbf{d}_{\lambda}(\sigma, \sigma_k)^{\frac{1}{1+\lambda}} \|\nabla \varphi\|_{L^2(\Omega)} + |(u_{\epsilon}^k - u_0^k, \varphi)|. \end{aligned}$$

By Lemma 5.5, the latter term goes to zero as $\epsilon \rightarrow 0$. Thus,

$$\limsup_{\epsilon \rightarrow 0} |(u_{\epsilon} - u_0, \varphi)| \leq \frac{2}{\alpha^{\frac{1}{1+\lambda}}} \mathbf{d}_{\lambda}(\sigma, \sigma_k)^{\frac{1}{1+\lambda}} \|\nabla \varphi\|_{L^2(\Omega)}.$$

By Proposition 3.4, the right-hand side goes to zero as $k \rightarrow \infty$. Therefore, $u_{\epsilon} \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$. Similarly, by Theorem 4.7 and (4.1),

$$\begin{aligned} \|\mathcal{L}_{\epsilon} u_{\epsilon} - \mathcal{L}_0 u_0\|_{H^{-1}(\Omega)} &\leq \|\mathcal{L}_{\epsilon} u_{\epsilon} - \mathcal{L}_{\epsilon} u_{\epsilon}^k\|_{H^{-1}(\Omega)} + \|\mathcal{L}_{\epsilon} u_{\epsilon}^k - \mathcal{L}_0 u_0^k\|_{H^{-1}(\Omega)} \\ &\quad + \|\mathcal{L}_0 u_0^k - \mathcal{L}_0 u_0\|_{H^{-1}(\Omega)} \\ &\leq \frac{2\beta}{\alpha^{\frac{1}{1+\lambda}}} \mathbf{d}_{\lambda}(\sigma, \sigma_k)^{\frac{1}{1+\lambda}} + \|\mathcal{L}_{\epsilon} u_{\epsilon}^k - \mathcal{L}_0 u_0^k\|_{H^{-1}(\Omega)}. \end{aligned}$$

By Lemma 5.5,

$$\|\mathcal{L}_{\epsilon} u_{\epsilon}^k - \mathcal{L}_0 u_0^k\|_{H^{-1}(\Omega)} = \left\| \frac{\sigma_k}{(u_{\epsilon}^k)^{\lambda}} - \frac{\sigma_k}{(u_0^k)^{\lambda}} \right\|_{H^{-1}(\Omega)} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Thus,

$$\limsup_{\epsilon \rightarrow 0} \|\mathcal{L}_{\epsilon} u_{\epsilon} - \mathcal{L}_0 u_0\|_{H^{-1}(\Omega)} \leq \frac{2\beta}{\alpha^{\frac{1}{1+\lambda}}} \mathbf{d}_{\lambda}(\sigma, \sigma_k)^{\frac{1}{1+\lambda}}.$$

Taking the limit $k \rightarrow \infty$, we find that $\mathcal{L}_{\epsilon} u_{\epsilon} \rightarrow \mathcal{L}_0 u_0$ strongly in $H^{-1}(\Omega)$. $\square$

Proof of Theorem 5.3. Let $\{v_\epsilon\}$ be the sequence of weak solutions to (5.4). Fix $\varphi \in H_0^1(\Omega)$. By Theorem 4.7 and Lemma 5.6,

$$\begin{aligned} |(u_\epsilon - u_0, \varphi)| &\leq |(u_\epsilon - v_\epsilon, \varphi)| + |(v_\epsilon - u_0, \varphi)| \\ &\leq \frac{1}{\alpha^{\frac{1}{1+\lambda}}} \mathbf{d}_\lambda(\sigma, \sigma_\epsilon)^{\frac{1}{1+\lambda}} \|\nabla \varphi\|_{L^2(\Omega)} + |(v_\epsilon - u_0, \varphi)| \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \end{aligned}$$

Therefore, $u_\epsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$. Similarly,

$$\begin{aligned} \|\mathcal{L}_\epsilon u_\epsilon - \mathcal{L}_0 u_0\|_{H^{-1}(\Omega)} &\leq \|\mathcal{L}_\epsilon u_\epsilon - \mathcal{L}_\epsilon v_\epsilon\|_{H^{-1}(\Omega)} + \|\mathcal{L}_\epsilon v_\epsilon - \mathcal{L}_0 u_0\|_{H^{-1}(\Omega)} \\ &\leq \frac{\beta}{\alpha^{\frac{1}{1+\lambda}}} \mathbf{d}_\lambda(\sigma, \sigma_\epsilon)^{\frac{1}{1+\lambda}} + \|\mathcal{L}_\epsilon v_\epsilon - \mathcal{L}_0 u_0\|_{H^{-1}(\Omega)} \rightarrow 0. \end{aligned}$$

This completes the proof. $\square$

Acknowledgments

This work was supported by JST CREST Grant Number JPMJCR18K3 and JSPS KAKENHI Grant Number JP17H01092. The author would like to thank Professor Hiroaki Aikawa and Professor Jun Masamune for useful discussions.

References

- [1] H. Aikawa and N. Shanmugalingam. Carleson-type estimates for p -harmonic functions and the conformal Martin boundary of John domains in metric measure spaces. *Michigan Math. J.*, 53(1):165–188, 2005.
- [2] G. Allaire. *Shape optimization by the homogenization method*, volume 146 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2002.
- [3] W. Allegretto and Y. X. Huang. A Picone’s identity for the p -Laplacian and applications. *Nonlinear Anal.*, 32(7):819–830, 1998.
- [4] A. Ancona. On strong barriers and an inequality of Hardy for domains in $\mathbf{R}^n$. *J. London Math. Soc. (2)*, 34(2):274–290, 1986.
- [5] N. Ansini, G. Dal Maso, and C. I. Zeppieri. Γ -convergence and H -convergence of linear elliptic operators. *J. Math. Pures Appl. (9)*, 99(3):321–329, 2013.
- [6] L. Boccardo and J. Casado-Díaz. Some properties of solutions of some semilinear elliptic singular problems and applications to the G -convergence. *Asymptot. Anal.*, 86(1):1–15, 2014.
- [7] L. Boccardo and L. Orsina. Semilinear elliptic equations with singular nonlinearities. *Calc. Var. Partial Differential Equations*, 37(3-4):363–380, 2010.

- [8] A. J. Callegari and M. B. Friedman. An analytical solution of a nonlinear, singular boundary value problem in the theory of viscous fluids. *J. Math. Anal. Appl.*, 21:510–529, 1968.
- [9] J. Carmona and P. J. Martínez-Aparicio. Homogenization of singular quasi-linear elliptic problems with natural growth in a domain with many small holes. *Discrete Contin. Dyn. Syst.*, 37(1):15–31, 2017.
- [10] D. Cioranescu, A. Damlamian, and G. Griso. *The periodic unfolding method*, volume 3 of *Series in Contemporary Mathematics*. Springer, Singapore, 2018. Theory and applications to partial differential problems.
- [11] M. M. Coclite. Positive solutions of a Hammerstein integral equation with a singular nonlinear term. *Topol. Methods Nonlinear Anal.*, 15(2):235–250, 2000.
- [12] M. G. Crandall, P. H. Rabinowitz, and L. Tartar. On a Dirichlet problem with a singular nonlinearity. *Comm. Partial Differential Equations*, 2(2):193–222, 1977.
- [13] S. Cui. Existence and nonexistence of positive solutions for singular semilinear elliptic boundary value problems. *Nonlinear Anal.*, 41(1-2, Ser. A: Theory Methods):149–176, 2000.
- [14] G. Dal Maso. *An introduction to Γ -convergence*, volume 8 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 1993.
- [15] M. A. del Pino. A global estimate for the gradient in a singular elliptic boundary value problem. *Proc. Roy. Soc. Edinburgh Sect. A*, 122(3-4):341–352, 1992.
- [16] J. I. Díaz, J. Hernández, and J. M. Rakotoson. On very weak positive solutions to some semilinear elliptic problems with simultaneous singular nonlinear and spatial dependence terms. *Milan J. Math.*, 79(1):233–245, 2011.
- [17] P. Donato, S. Monsurrò, and F. Raimondi. Homogenization of a class of singular elliptic problems in perforated domains. *Nonlinear Anal.*, 173:180–208, 2018.
- [18] R. Durastanti and L. Giacomelli. Spreading equilibria under mildly singular potentials: pancakes versus droplets. *J. Nonlinear Sci.*, 32(5):Paper No. 71, 61, 2022.
- [19] N. El Berdan, J. I. Díaz, and J. M. Rakotoson. The uniform Hopf inequality for discontinuous coefficients and optimal regularity in BMO for singular problems. *J. Math. Anal. Appl.*, 437(1):350–379, 2016.

- [20] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, extended edition, 2011.
- [21] W. Fulks and J. S. Maybee. A singular non-linear equation. *Osaka Math. J.*, 12:1–19, 1960.
- [22] M. Ghergu and V. D. Rădulescu. *Nonlinear PDEs*. Springer Monographs in Mathematics. Springer, Heidelberg, 2012. Mathematical models in biology, chemistry and population genetics, With a foreword by Viorel Barbu.
- [23] D. Giachetti, P. J. Martínez-Aparicio, and F. Murat. A semilinear elliptic equation with a mild singularity at $u = 0$: existence and homogenization. *J. Math. Pures Appl. (9)*, 107(1):41–77, 2017.
- [24] D. Giachetti, P. J. Martínez-Aparicio, and F. Murat. Definition, existence, stability and uniqueness of the solution to a semilinear elliptic problem with a strong singularity at $u = 0$. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 18(4):1395–1442, 2018.
- [25] M. Grüter and K.-O. Widman. The Green function for uniformly elliptic equations. *Manuscripta Math.*, 37(3):303–342, 1982.
- [26] P. Hajłasz and P. Koskela. Isoperimetric inequalities and imbedding theorems in irregular domains. *J. London Math. Soc. (2)*, 58(2):425–450, 1998.
- [27] T. Hara. Quasilinear elliptic equations with sub-natural growth terms in bounded domains. *NoDEA Nonlinear Differential Equations Appl.*, 28(6):Paper No. 62, 2021.
- [28] T. Hara. Trace inequalities of the Sobolev type and nonlinear Dirichlet problems. *Calc. Var. Partial Differential Equations*, 61(6):Paper No. 216, 2022.
- [29] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations*. Dover Publications, Inc., Mineola, NY, 2006. Unabridged republication of the 1993 original.
- [30] J. Hernández, F. J. Mancebo, and J. M. Vega. Positive solutions for singular nonlinear elliptic equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 137(1):41–62, 2007.
- [31] J. Kinnunen, J. Lehrbäck, and A. Vähäkangas. *Maximal function methods for Sobolev spaces*, volume 257 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2021.
- [32] T. Klimsiak. Semilinear elliptic equations with Dirichlet operator and singular nonlinearities. *J. Funct. Anal.*, 272(3):929–975, 2017.
- [33] A. V. Lair and A. W. Shaker. Classical and weak solutions of a singular semilinear elliptic problem. *J. Math. Anal. Appl.*, 211(2):371–385, 1997.

- [34] N. S. Landkof. *Foundations of modern potential theory*. Die Grundlehren der mathematischen Wissenschaften, Band 180. Springer-Verlag, New York-Heidelberg, 1972. Translated from the Russian by A. P. Doohovskoy.
- [35] A. C. Lazer and P. J. McKenna. On a singular nonlinear elliptic boundary-value problem. *Proc. Amer. Math. Soc.*, 111(3):721–730, 1991.
- [36] J. L. Lewis. Uniformly fat sets. *Trans. Amer. Math. Soc.*, 308(1):177–196, 1988.
- [37] W. Littman, G. Stampacchia, and H. F. Weinberger. Regular points for elliptic equations with discontinuous coefficients. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 17:43–77, 1963.
- [38] H. Mâagli and M. Zribi. Existence and estimates of solutions for singular nonlinear elliptic problems. *J. Math. Anal. Appl.*, 263(2):522–542, 2001.
- [39] V. Maz'ya. *Sobolev spaces with applications to elliptic partial differential equations*, volume 342 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, augmented edition, 2011.
- [40] F. Murat and L. Tartar. H -convergence. In *Topics in the mathematical modelling of composite materials*, volume 31 of *Progr. Nonlinear Differential Equations Appl.*, pages 21–43. Birkhäuser Boston, Boston, MA, 1997.
- [41] A. I. Nazarov. A centennial of the Zaremba-Hopf-Oleinik lemma. *SIAM J. Math. Anal.*, 44(1):437–453, 2012.
- [42] P. Nowosad. On the integral equation $\kappa f = 1/f$ arising in a problem in communication. *J. Math. Anal. Appl.*, 14:484–492, 1966.
- [43] F. Oliva and F. Petitta. Finite and infinite energy solutions of singular elliptic problems: existence and uniqueness. *J. Differential Equations*, 264(1):311–340, 2018.
- [44] A. Seesanea and I. E. Verbitsky. Solutions to sublinear elliptic equations with finite generalized energy. *Calc. Var. Partial Differential Equations*, 58(1):Art. 6, 21, 2019.
- [45] G. Sinnamon and V. D. Stepanov. The weighted Hardy inequality: new proofs and the case $p = 1$. *J. London Math. Soc. (2)*, 54(1):89–101, 1996.
- [46] C. A. Stuart. Existence and approximation of solutions of non-linear elliptic equations. *Math. Z.*, 147(1):53–63, 1976.
- [47] Y. Sun and D. Zhang. The role of the power 3 for elliptic equations with negative exponents. *Calc. Var. Partial Differential Equations*, 49(3-4):909–922, 2014.

- [48] S. D. Taliaferro. A nonlinear singular boundary value problem. *Nonlinear Anal.*, 3(6):897–904, 1979.
- [49] D. A. Trocenko. Properties of regions with a nonsmooth boundary. *Sibirsk. Mat. Zh.*, 22(4):221–224, 232, 1981.
- [50] I. E. Verbitsky. Nonlinear potentials and trace inequalities. In *The Maz'ya anniversary collection, Vol. 2 (Rostock, 1998)*, volume 110 of *Oper. Theory Adv. Appl.*, pages 323–343. Birkhäuser, Basel, 1999.
- [51] I. E. Verbitsky. Pointwise estimates of solutions and existence criteria for sublinear elliptic equations. *Mat. Fiz. Komp'yut. Model.*, (3(40)):18–33, 2017.
- [52] A. Wannebo. Hardy inequalities. *Proc. Amer. Math. Soc.*, 109(1):85–95, 1990.
- [53] Z. Zhang and J. Cheng. Existence and optimal estimates of solutions for singular nonlinear Dirichlet problems. *Nonlinear Anal.*, 57(3):473–484, 2004.