Proceedings of 50th Sapporo Symposium on Partial Differential Equations

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Proceedings of 50th Sapporo Symposium on Partial Differential Equations

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S.-I. Ei, Y. Giga, N. Hamamuki, S. Jimbo, K. Kita,
H. Kubo, H. Kuroda, S. Masaki, T. Ozawa, and K. Tsutaya

Sapporo, 2025

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Preface

This volume is intended as the proceedings of Sapporo Symposium on Partial Differential Equations, held on August 4 through August 6 in 2025 at Faculty of Science, Hokkaido University.

Sapporo Symposium on PDE has been held annually to present the latest developments on PDE with a broad spectrum of interests not limited to the methods of a particular school. Late Professor Taira Shirota started the symposium almost 50 years ago. Late Professor Kôji Kubota and late Professor Rentaro Agemi made a large contribution to its organization for many years.

We always thank their significant contribution to the progress of the Sapporo Symposium on PDE.

S.-I. Ei (Josai University)
Y. Giga (The University of Tokyo)
N. Hamamuki (Hokkaido University)
S. Jimbo (Hokkaido University)
K. Kita (Hokkaido University)
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S. Masaki (Hokkaido University)
T. Ozawa (Waseda University)
K. Tsutaya (Hirosaki University)

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Place	N308, The main Bldg. of Faculty of Science (Aug. 4th), 5-203, Faculty of Science Bldg. #5 (Aug. 5-6th)
Organizers	Nao Hamamuki, Kosuke Kita, Hideo Kubo, Hirotoshi Kuroda, Satoshi Masaki
Program Committee	Shin-Ichiro Ei, Yoshikazu Giga, Nao Hamamuki, Shuichi Jimbo, Kosuke Kita, Hideo Kubo, Hirotoshi Kuroda, Satoshi Masaki, Tohru Ozawa, Kimitoshi Tsutaya
URL	https://www.math.sci.hokudai.ac.jp/sympo/sapporo/program250804.html

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with arbitrary geometry

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Uniqueness and nondegeneracy of ground states for generalized Choquard equation

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July 18, 2025

1 Main Results

This talk is based on a joint work with Mirko Tarulli and George Venkov.

We study the uniqueness of the ground states for generalized Choquard equation

$$-\Delta u + u = I(|u|^p)|u|^{p-2}u \tag{1.1}$$

in the case $p \in (1 + 2/n, 2), n \geq 3$.

Here and below $I(f)$ is the Riesz potential defined by

$$I(f)(x) = (-\Delta)^{-1} f(x) = G_0 * f(x), \ G_0(|y|) = \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \frac{1}{|y|^{n-2}},$$

where $n \geq 3$ and $|\mathbb{S}^{n-1}| = n\pi^{n/2}/\Gamma(1 + n/2)$ being the surface measure of the unit sphere in $\mathbb{R}^n$.

The ground states for $p < 1 + 4/n$ (mass sub - critical case) can be obtained (see Theorem 2 in [5]) via the minimization problem

$$\mathcal{E}_\sigma = \inf_{u \in H^1, \ \|u\|_{L^2}^2 = \sigma} E_p(u). \tag{1.2}$$

Here and below

$$E_p(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{2p} D(|u|^p, |u|^p),$$

where

$$D(|u|^p, |u|^p) = \langle I(|u|^p), |u|^p \rangle_{L^2} = \left\| (-\Delta)^{-1/2} |u|^p \right\|_{L^2}^2 .$$

In the mass super - critical case, the ground states are minimizers (as in [4], [14]) Weinstein type functionals. For example one can take

$$W_p(u) = \frac{\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2}{\sqrt[p]{D(|u|^p, |u|^p)}}$$

and consider the minimization problem

$$\mathcal{W} = \inf_{u \in H^1_{rad} \setminus \{0\}} W_p(u).$$

This work continues the study of the uniqueness of ground states for the generalized Choquard equation, started in [6], [8], [7]. Especially, in [7] we considered the case $p \in (2, 1+4/(n-2), n \geq 3$ and we have the following local result.

Theorem 1.1. *Assume $n \geq 3$ and $2 \leq p < (n+4)/n$. Then one can find $\varepsilon > 0$ so that for any two radial positive minimizers Q_1, Q_2 of (1.2), satisfying*

$$\|Q_1 - Q_2\|_{H^1} \leq \varepsilon$$

we have $Q_1 = Q_2$.

The active study of the existence and qualitative behavior of the ground states Q is closely connected with stability/instability properties of the corresponding standing waves $U(t,x) = e^{i\omega t}u(x)$ that are solutions of the Cauchy problem for NLS

$$\begin{aligned} &i\partial_t U + \Delta U + I(|U|^p)|U|^{p-2}U = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ &U(0,x) = u(x). \end{aligned} \tag{1.3}$$

The study of the H^1-evolution dynamics of this Cauchy problem is motivated by the important question of orbital stability/instability properties of the standing waves. The existence of ground state is studied in [4], [14], [15], while [16] and [17] treat the decay and scattering properties of the ground states. A detailed classification result for linearized stability properties of the standing waves is obtained in [5]. Considering linearization of (1.3) around standing waves, one can apply the classification results from [5] and deduce that linearized stability holds for $p \in (1+2/n, 1+4/n)$, while linearized instability is fulfilled for $p \in [1+4/n, 1+4/(n-2))$.

As a consequence, it satisfies the Pohozaev identity

$$\frac{\|\nabla u\|^2}{np - n - 2} = \frac{D(|u|^p, |u|^p)}{2p}.$$

Summarizing, we have the following relations

$$\frac{\|u\|^2}{\beta} = \frac{\|\nabla u\|^2}{\gamma} = \frac{D(|u|^p, |u|^p)}{p} = k_{\mathcal{W}},$$

where

$$\beta = \frac{n+2-p(n-2)}{2}, \; \gamma = \frac{np-n-2}{2} = p - \beta$$

and

$$k_{\mathcal{W}} = \frac{1}{p}\mathcal{W}^{p/(p-1)}.$$

The main novelty in the case $p \in (1+2/n, 2)$ is the uniqueness of the ground state without any smallness assumption.Our first result is the following.

Theorem 1.2. *Assume $n \geq 3$ and $(n+2)/n < p < 2$. Then for any two positive minimizers $Q_1, Q_2 \in H^1$ of (1.2), satisfying the normalization condition*

$$\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 = D(|u|^p, |u|^p),$$

we have $Q_1 = Q_2$.

Some observations are listed below.

- Our approach follows the idea from [3], [11] based on the substitution $v(u) = |u|^2$ and the fact that the Weinstein functional becomes convex with respect to v. This fact enables us to avoid radiality assumption on the the minimizer and use only its positiveness.
- There are results on classification of positive solutions of local and non local elliptic equations.. Starting with seminal work [10], treating nonlinearity of local type, there are some works treating non - local nonlinearities (see [2], [13]). The result of [13] guarantee that any positive solution of (1.1) is radial provided $p \geq 2$. We do not use a result of type "positiveness implies radiality." However our uniqueness result shows that any positive minimizer of (1.2) is readial.
- The positiveness assumption in Theorem 1.2 is crucial. Note that minimization over odd function in H^1 gives nodal solutions discussed in [9]. They have cylindrical symmetry with respect to a fixed direction.
- Note that $1 + 4/n < 2$ for $n \geq 5$, so we need the Weinstein functional to guarantee the existence of ground states for $p \in [1 + 4/n, 2)$.

Another important question is the non degeneracy of the ground state. A weaker form of degeneracy of the ground state means that the kernel of the operator

$$L_+ = -\Delta + 1 - pI(Q^{p-1}\cdot)Q^{p-1} - (p-1)I(Q^p)Q^{p-2}$$

is non trivial on H^1_{rad} . Here and below $Q(|x|) \in H^1_{rad}(\mathbb{R}^n)$ is a radial positive solution of (1.1), so that setting $A(|x|) = (-\Delta)^{-1}Q^p(|x|)$ and $r = |x|$ we have the following ordinary differential system

$$\begin{aligned} &-\partial_r^2 Q(r) - \frac{n-1}{r}\partial_r Q(r) + Q(r) = A(r)Q(r)^{p-1} \\ &-\partial_r^2 A(r) - \frac{n-1}{r}\partial_r A(r) = Q^p(r). \end{aligned}$$

The operator L_+ becomes

$$L_+ = -\partial_r^2 - \frac{n-1}{r}\partial_r + 1 - pI(Q^{p-1}\cdot)Q^{p-1} - (p-1)AQ^{p-2}. \tag{1.4}$$

Our next result treats the triviality of the kernel of L_+ in $H^1_{rad}(\mathbb{R}^n)$. To be more precise if $h \in H^1_{rad}(\mathbb{R}^n) \cap \mathrm{Ker} L_+$, then we can have stronger regularity properties

$$h \in H^s_q,\ s \in [0, p+1),\ 1 < q < \infty.$$

Any radial solution h of the equation $L_+ h = 0$ is a solution of the ordinary differential equation (1.4). Then the couple of h and $B = (-\Delta)^{-1} Q^{p-1} h$ satisfies the system of nonlinear second-order differential equations

$$\begin{aligned} h''(r) + \frac{n-1}{r} h'(r) &= h(r) - pBQ^{p-1} - (p-1)AQ^{p-2}h, \\ B''(r) + \frac{n-1}{r} B'(r) &= -Q^{p-1}h. \end{aligned} \tag{1.5}$$

Our next result justifies the triviality of the kernel in H^1_{rad}.

Theorem 1.3. *Let $n \geq 3$ and $(n+2)/n < p < 2$. There is no classical non-trivial solution (h, B) to the ODE problem* (1.5).

Here we give some remarks and additional information connected with this Theorem.

- In the case $n \geq 3$, and $2 \leq p < (n+4)/n$ we have at most one solution to the system (1.5). This fact plays crucial role in establishing local uniqueness result of Theorem 1.1.

- Our approach is based on use of appropriate asymptotic classes and asymptotic expansions of the ground state Q and the corresponding electric potential A as well the precise asymptotic of h, B. In fact, we obtain the following expansions at $|x| \to \infty$

$$\begin{aligned} A(|x|) &= A_0 |x|^{-n+2}(1 + o(1)), \quad A_0 = c_n \int_{\mathbb{R}^n} Q(|x|)^p dx, \\ Q(|x|) &= Q_0 |x|^{-n+2}\left(1 + o(1)\right),\ Q_0 = A_0^{1/(2-p)}, \end{aligned} \tag{1.6}$$

$$B(|x|) = B_0 |x|^{-n+2}\left(1 + o(1)\right), B_0 = c_n \int_{\mathbb{R}^n} Q(|x|)^{p-1} h(|x|) dx, \tag{1.7}$$

$$h(|x|) = h_0 |x|^{-(n-2)/(2-p)}\left(1 + o(1)\right), \quad h_0 = \frac{B_0 Q_0^{p-1}}{2-p}. \tag{1.8}$$

- We follow the approach in [7] and introduce the quantities

$$\begin{aligned} \xi_B(r) &= -\int_r^\infty \tau^{n-1} B(\tau) Q^p(\tau) d\tau, \\ \xi_h(r) &= -\int_r^\infty \tau^{n-1} A(\tau) Q^{p-1}(\tau) h(\tau) d\tau. \end{aligned}$$

  They are bounded solutions to a system of ODE. Careful analysis of the equation satisfied by ξ_B leads to the crucial conclusion $B_0 = 0$. Then we have $h_0 = 0$ and the asymptotic expansions (1.6), (1.7), (1.8) enables us to use Gronwall type lemma (as it was done in [6]) and arrive at $h(|x|) = 0$.

- The result of Theorem 1.3 means that

$$\mathrm{Ker}L_+ \cap H^1_{rad}$$

is trivial and this can be interpreted as a weaker non degeneracy.

Our next step is to prove the non degeneracy property (as this notion was introduced in [19]).

Theorem 1.4. *Assume $n \geq 3$ and $(n+2)/n < p < 2$. Then*

$$\mathrm{Ker}L_+ = \mathrm{Span}\left\{\partial_j Q(|x|), j = 1, \cdots, n\right\}.$$

Let us explain key points in the proof.

- Using a decomposition in spherical harmonics we consider the operators $L_{+,(\ell)}$ that are the projections of the operator L_+ on spherical harmonics of order ℓ. We use the following representation of this operator

$$\begin{aligned}\left(L_{+,(\ell)}\eta\right)(r) &= -\eta''(r) - \frac{n-1}{r}\eta'(r)\\ &+\frac{\ell(\ell+n-2)}{r^2}\eta(r) + V(r)\eta(r) + \left(W_{(\ell)}\eta\right)(r)\end{aligned}$$

with

$$V(r) = -(p-1)I(Q^p)Q^{p-2}$$

and

$$\begin{aligned}&\left(W_{(\ell)}\eta\right)(r)\\ &= -\frac{2p(n-2)\pi^{n/2}}{(2\ell+n-2)\Gamma(n/2)}\left(Q(r)\right)^{p-1}\int_0^\infty \frac{r_<^\ell}{r_>^{\ell+n-2}}(Q(s))^{p-1}\eta(s)s^{n-1}ds,\end{aligned}$$

where $r_< = \min(r,s)$ and $r_> = \max(r,s)$. The weak form of non degeneracy obtained in Theorem 1.3 enables us to consider only the case $\ell \geq 1$. The key point in the proof is that each $L_{+,(\ell)}$ fulfills a Perron-Frobenius property.

- The proof of Perron - Frobenius lemma follows the approach in [12]. We need appropriate generalizations of the multipole expansions and heat propagator decomposition for the case of $n \geq 3$.

An important corollary of Theorem 1.4 is the coercivity property of the operator L_+, which arises in the stability and blow-up analysis of solitary waves for the corresponding time-dependent equation (1.3). More precisely, we have the following result.

Theorem 1.5. *Assume*

$$1 + \frac{2}{n} < p < 2.$$

Then we have

$$\langle L_+ h, h\rangle_{L^2} \geq C\|h\|^2_{H^1},\ \forall\ h \in \mathcal{H}_Q,$$

where

$$\mathcal{H}_Q = \left\{h \in H^1,\ h \perp L_+(Q),\ h \perp \mathrm{Ker}L_+\right\}.$$

2 Global Uniqueness

The goal of the section is to establish the global uniqueness of the ground state stated in Theorem 1.2.

Starting with the pth power of the Weinstein functional

$$W_p(u)^p = \frac{\left(\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2\right)^p}{D(|u|^p, |u|^p)}$$

with $u > 0$ we assume $(n+2)/n < p < 2$.

Any H^1 non-negative minimizer $u \neq 0$ of this functional satisfies (after renormalization) the equation (1.1). Applying the strong maximum principle for weak solutions (see for example Theorem 1 in [18]), we see that for any compact set $K \subset \mathbb{R}^n$ there exists a constant $\gamma_K > 0$ so that

$$u(x) \geq \gamma_K, \quad \text{a.e. in } K.$$

Then, utilizing an exhaustion of $\mathbb{R}^n$ by compact sets, and noting that every point $x \in \mathbb{R}^n$ belongs to some K, up to a set of measure zero, we infer that

$$u(x) > 0 \quad \text{a.e. in } \mathbb{R}^n.$$

Following [1], [11], we introduce the substitution

$$v(u) = |u|^2$$

and study the minimization problem for the induced functional

$$\Phi_p(v) = \frac{\left(\int \frac{\|\nabla v(x)|^2}{v(x)} dx + \int v(x) dx\right)^p}{D(v^{p/2}, v^{p/2})},$$

defined on the convex domain

$$\mathcal{D} = \{v = |u|^2; u \in H^1(\mathbb{R}^n), u(x) > 0 \ \ a.e.\}.$$

Then we can verify

$$W_p(u)^p = \Phi_p(|u|^2)$$

and $\Phi_p(v)$ is strictly convex on $\mathcal{D}$. It is well known (see [11], [1]) that the functional

$$\mathbb{K}(v) = \int \frac{|\nabla v(x)|^2}{v(x)} dx$$

is convex, i.e.

$$\mathbb{K}(\lambda v_1 + (1-\lambda) v_2) \leq \lambda \mathbb{K}(v_1) + (1-\lambda)\mathbb{K}(v_2), \lambda \in [0,1].$$

for $v_1, v_2 \in \mathcal{D}$. We shall prove the following

Lemma 2.1. *For $1 < p < 2$ the functional $D(v^{p/2}, v^{p/2})$ is strictly concave over $\mathcal{D}$.*

Proof. We have to show that

$$D((\lambda v_1 + (1-\lambda)v_2)^\alpha, (\lambda v_1 + (1-\lambda)v_2)^\alpha) > \\ > \lambda D(v_1^\alpha, v_1^\alpha) + (1-\lambda)D(v_2^\alpha, v_2^\alpha)$$

for $1/2 < \alpha = p/2 < 1$ and $v_1, v_2 \in \mathcal{D} \setminus \{0\}$. Since

$$D(v^\alpha, v^\alpha) = \int\int v(x)^\alpha v(y)^\alpha k(x-y)dxdy$$

it is sufficient to check the inequality

$$(\lambda v_1(x) + (1-\lambda)v_2(x))^\alpha (\lambda v_1(y) + (1-\lambda)v_2(y))^\alpha > \\ \lambda v_1(x)^\alpha v_1(y)^\alpha + (1-\lambda)v_2(x)^\alpha v_2(y)^\alpha.$$

for $v_1(x)v_2(x)v_1(y)v_2(y) > 0$ and $v_1 \neq v_2 \in \mathcal{D}$. This inequality follows from the concavity of the following function

$$\Phi(\xi) = \xi_1^\alpha \xi_2^\alpha$$

of two variables $\xi = (\xi_1, \xi_2)$. The Hessian of the matrix is

$$H(\xi) = (\nabla^2_{\xi_j \xi_k} \Phi(\xi))_{j,k=1,2} = \Phi(\xi) \begin{pmatrix} \alpha(\alpha-1)\xi_1^{-2} & \alpha^2 \xi_1^{-1}\xi_2^{-1} \\ \alpha^2 \xi_1^{-1}\xi_2^{-1} & \alpha(\alpha-1)\xi_2^{-2} \end{pmatrix}.$$

The matrix $H(\xi)$ is strictly negative for $\xi_1 > 0, \xi_2 > 0$ if $(\alpha - 1) < 0$ and

$$\det H(x) = -\Phi(\xi)^2(\xi_1\xi_2)^{-2}\alpha^2(2\alpha - 1) < 0.$$

This conditions are verified due to the assumption

$$\alpha = \frac{p}{2} \in (1/2, 1).$$

□

Now we complete

Proof of Theorem 1.1. The minimization problem (1.2) after the substitution $v = |u|^2$ is reduced to the minimization problem

$$\inf_{v \in \mathcal{D},\ v \neq 0} \Phi_p(v)$$

and the uniqueness follows from the fact that $\Phi_p(v)$ is strictly convex. □

References

[1] Rafael Benguria, Haïm Brézis, and Elliott H. Lieb. The Thomas-Fermi-von Weizsäcker theory of atoms and molecules. *Comm. Math. Phys.*, 79(2):167–180, 1981.

[2] Wenxiong Chen, Congming Li, and Biao Ou. Classification of solutions for an integral equation. *Comm. Pure Appl. Math.*, 59(3):330–343, 2006.

[3] J. Fröhlich, S. Gustafson, B. L. G. Jonsson, and I. M. Sigal. Solitary wave dynamics in an external potential. *Comm. Math. Phys.*, 250(3):613–642, 2004.

[4] Hristo Genev and George Venkov. Soliton and blow-up solutions to the time-dependent Schrödinger-Hartree equation. *Discrete Contin. Dyn. Syst. Ser. S*, 5(5):903–923, 2012.

[5] Vladimir Georgiev and Atanas Stefanov. On the classification of the spectrally stable standing waves of the Hartree problem. *Phys. D*, 370:29–39, 2018.

[6] Vladimir Georgiev, Mirko Tarulli, and George Venkov. Existence and uniqueness of ground states for p-Choquard model. *Nonlinear Anal.*, 179:131–145, 2019.

[7] Vladimir Georgiev, Mirko Tarulli, and George Venkov. Local uniqueness of ground states for the generalized Choquard equation. *Calc. Var. Partial Differential Equations*, 63(5):Paper No. 135, 39, 2024.

[8] Vladimir Georgiev and George Venkov. On uniqueness for the generalized Choquard equation. In *Advances in harmonic analysis and partial differential equations*, Trends Math., pages 263–278. Birkhäuser/Springer, Cham, [2020] ©2020.

[9] Marco Ghimenti and Jean Van Schaftingen. Nodal solutions for the Choquard equation. *J. Funct. Anal.*, 271(1):107–135, 2016.

[10] B. Gidas, Wei Ming Ni, and L. Nirenberg. Symmetry and related properties via the maximum principle. *Comm. Math. Phys.*, 68(3):209–243, 1979.

[11] Bernd Kawohl and Stefan Krömer. Uniqueness and symmetry of minimizers of Hartree type equations with external Coulomb potential. *Adv. Calc. Var.*, 5(4):427–432, 2012.

[12] Enno Lenzmann. Uniqueness of ground states for pseudorelativistic Hartree equations. *Anal. PDE*, 2(1):1–27, 2009.

[13] Li Ma and Lin Zhao. Classification of positive solitary solutions of the nonlinear Choquard equation. *Arch. Ration. Mech. Anal.*, 195(2):455–467, 2010.

[14] Vitaly Moroz and Jean Van Schaftingen. Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics. *J. Funct. Anal.*, 265(2):153–184, 2013.

[15] Vitaly Moroz and Jean Van Schaftingen. Existence of groundstates for a class of nonlinear Choquard equations. *Trans. Amer. Math. Soc.*, 367(9):6557–6579, 2015.

[16] M. Tarulli and G. Venkov. Decay and scattering in energy space for the solution of weakly coupled Schrödinger-Choquard and Hartree-Fock equations. *J. Evol. Equ.*, 21(2):1149–1178, 2021.

[17] Mirko Tarulli and George Venkov. Decay in energy space for the solution of fourth-order Hartree-Fock equations with general non-local interactions. *J. Math. Anal. Appl.*, 516(2):Paper No. 126533, 33, 2022.

[18] J. L. Vázquez. A strong maximum principle for some quasilinear elliptic equations. *Appl. Math. Optim.*, 12(3):191–202, 1984.

[19] Michael I. Weinstein. Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.*, 16(3):472–491, 1985.

GLOBAL SOLUTIONS TO SYSTEMS OF NONLINEAR WAVE EQUATIONS UNDER A KIND OF WEAK NULL CONDITION

SOICHIRO KATAYAMA

1. Introduction

We consider the Cauchy problem for systems of nonlinear wave equations of the following type:

$$\begin{cases} \Box u = F(u, \partial u, \partial^2 u), & (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u(0,x) = \varepsilon f(x), (\partial_t u)(0,x) = \varepsilon g(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $\Box = \partial_t^2 - \Delta$, $u = (u_1, \ldots, u_N)$, $\partial u = (\partial_a u_j)_{1 \le j \le N, 0 \le a \le d}$, and $\partial^2 u = (\partial_a \partial_b u_j)_{1 \le j \le N, 0 \le a,b \le d}$ with $\partial_0 = \partial_t$ and $\partial_k = \partial_{x_k}$ for $1 \le k \le d$. For simplicity, each component of $F(\lambda, \mu, \nu) = \bigl(F_j(\lambda, \mu, \nu)\bigr)_{1 \le j \le N}$ is assumed to be a homogeneous polynomial of degree $p(\ge 2)$ in $(\lambda, \mu, \nu) \in \mathbb{R} \times \mathbb{R}^{N(1+d)} \times \mathbb{R}^{N(1+d)^2}$. We suppose that $f, g \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^N)$ and that ε is a small and positive parameter.

We are interested in the case where derivatives of u are dominant in the nonlinearity, and we always assume that

$$F(\lambda, 0, 0) = 0. \tag{1.2}$$

We also suppose that F is quasi-linear and diagonal, that is to say that each F_j has the form

$$F_j(u, \partial u, \partial^2 u) = \sum_{a,b=0}^{d} G_j^{ab}(u, \partial u) \partial_a \partial_b u_j + H_j(u, \partial u) \tag{1.3}$$

with some homogeneous polynomials $G_j^{ab}(\lambda, \mu)$ of degree $p-1$ and $H_j(\lambda, \mu)$ of degree p, where $H_j(\lambda, 0) = 0$ by (1.2). For simplicity, we assume that $G_j^{00}(\lambda, \mu) \equiv 0$.

We say that ϕ is a free solution if it satisfies $\Box \phi = 0$. If the solution u to (1.1) behaves like free solutions with C_0^∞-data, then we have

$$\|F(u, \partial u, \partial^2 u)(t)\|_{L^2(\mathbb{R}^d)} \le C \varepsilon^p (1+t)^{-\frac{(d-1)(p-1)}{2}},$$

and, taking the energy inequality into account, we may expect the small data global existsnce when $(d-1)(p-1) > 1$, as $\|F(u, \partial u, \partial^2 u)(t)\|_{L^2(\mathbb{R}^d)}$ is integrable with respect to t over the interval $[0, \infty)$. Indeed, it is known that this expectation is true. Hence we are interested in the critical case $(d-1)(p-1) = 1$, which happens for $(d,p) = (3,2)$ and $(d,p) = (2,3)$. In the critical case, it is known that there is some nonliearity F of the critical power and some initial data $(\varepsilon f, \varepsilon g)$, for which the solution blows up in finite time no matter how small $\varepsilon > 0$ is. On the other hand, there is also some nonlinearity F of the critical power for which we have the small data global existence, and we would like to clarify the nonlinearity for which the small data global existence holds. In the critical cases $(d,p) = (3,2)$ and $(d,p) = (2,3)$, the following formal observation is known to be useful for this purpose. In what follows, we write $r = |x|$. For a free solution ϕ, it is known that $\partial\phi$ decays faster away from the light cone $t = r$, and that $(\partial_t + \partial_r)\phi$ enjoys a faster uniform decay estimate faster than that for general $\partial\phi$. Suppose that the solution u to the nonlinear system (1.1) also has these properties. Writing $x = r\omega$ with $\omega = x/|x|$, and writing $\sigma = r - t$, we put

$$U(t, \sigma, \omega) = r^{\frac{d-1}{2}} u(t, r\omega)\Big|_{r=t+\sigma}. \tag{1.4}$$

We write $\phi \sim \psi$ if $|\phi - \psi|$ decays faster than ψ as $t \to \infty$ in a region where $|r - t|$ is small compared to t. By the above assumption and the finite speed of propagation, we have

$$r^{\frac{d-1}{2}} \partial_a u(t, r\omega) \sim \omega_a (\partial_\sigma U)(t, r - t, \omega), \tag{1.5}$$

$$r^{\frac{d-1}{2}} \partial_a \partial_b u(t, r\omega) \sim \omega_a \omega_b (\partial_\sigma^2 U)(t, r - t, \omega) \tag{1.6}$$

for $0 \le a, b \le d$, where $\omega = (\omega_1, \ldots, \omega_d)$ and $\omega_0 = -1$. Recalling that we are considering the critical case $(d - 1)(p - 1)/2 = 1$, we obtain

$$\begin{aligned} F(u, \partial u, \partial^2 u) &= r^{-1-\frac{d-1}{2}} F(r^{\frac{d-1}{2}} u, r^{\frac{d-1}{2}} \partial u, r^{\frac{d-1}{2}} \partial^2 u) \\ &\sim t^{-1} r^{-\frac{d-1}{2}} {}^\star F(\omega, U, \partial_\sigma U, \partial_\sigma^2 U)|_{\sigma=r-t}, \end{aligned} \tag{1.7}$$

where, writing $F(u, \partial u, \partial^2 u) = F\big(u, (\partial_a u)_{0\le a\le d}, (\partial_a \partial_b u)_{0\le a,b\le d}\big)$, the reduced nonlinearity ${}^\star F$ is defined by

$${}^\star F(\omega, X, Y, Z) = F\big(X, (\omega_a Y)_{0\le a\le d}, (\omega_a \omega_b Z)_{0\le a,b\le d}\big) \tag{1.8}$$

for $\omega = (\omega_1, \ldots, \omega_d)$ with $\omega_0 = -1$, and $X, Y, Z \in \mathbb{R}^N$. Let $\mathbb{S}^{d-1}$ denote the $(d-1)$-dimensional unit sphere. From (1.7), we can expect that F decays faster than the general nonlinearity of the critical degree, if

$${}^\star F(\omega, X, Y, Z) \equiv 0, \quad \omega \in \mathbb{S}^{d-1}, \ X, Y, Z \in \mathbb{R}^N, \tag{1.9}$$

which is called the *null condition*. Indeed, Klainerman [17] and Christodoulou [3] showed that the null condition implies the small data global existence for $(d, p) = (3, 2)$. The small data global existence under the null condition for $(d, p) = (2, 3)$ was proved by Godin [4], Hoshiga [6], and the author [8, 9] (see Alinhac [1] and the author [11] for the subcritical case $(d, p) = (2, 2)$). The null condition is not a necessary condition for the small data global existence as the following two typical examples show. The first example is for $(d, p) = (3, 2)$ and $(d, p) = (2, 3)$:

$$\begin{cases} \Box u_1 = (\partial_t u_2)^{p-1} (\partial_t u_1), \\ \Box u_2 = 0. \end{cases} \tag{1.10}$$

It is trivial that we have the global existence of solutions to (1.10) since we can first solve u_2 and then u_1; however, the null condition is not satisfied. The second example is only for the case $(d, p) = (2, 3)$:

$$\Box u = -(\partial_t u)^3. \tag{1.11}$$

The global existence is less trivial than (1.10), but using the conservation of

$$\frac{1}{2} \int_{\mathbb{R}^2} |\partial u(t, x)|^2 \, dx + \int_0^t \int_{\mathbb{R}^2} |\partial_t u(\tau, x)|^4 \, dx \, d\tau,$$

we can show the global existence of solutions, though the null condition is violated.

In order to go beyond the null condition to explore weaker sufficient conditions for the small data global existence, we have to investigate not only the main part of the nonlinearity F, but also that of the system (1.1) itself. Let the solution u to (1.1) has the similar properties to the free solution as before. Then we have

$$r^{\frac{d-1}{2}}\Box u(t, r\omega) \sim -2(\partial_t\partial_\sigma U)(t, r-t, \omega),$$

which, combined with (1.1) and (1.7), gives

$$(\partial_t\partial_\sigma U) \sim -\frac{1}{2t}{}^\star F(\omega, U, \partial_\sigma U, \partial_\sigma^2 U). \tag{1.12}$$

Neglecting the implicit "error" terms on the right-hand side, and writing the unknown in a different letter $A = A(t, \sigma, \omega)$, we arrive at the *reduced system*

$$\partial_t\partial_\sigma A = -\frac{1}{2t}{}^\star F(\omega, A, \partial_\sigma A, \partial_\sigma^2 A), \quad (t, \sigma, \omega) \in (1, \infty) \times \mathbb{R} \times \mathbb{S}^{d-1}. \tag{1.13}$$

We put the initial condition at $t = 1$, say. To be more specific, we put

$$A(1, \sigma, \omega) = \varepsilon A^0(\sigma, \omega). \tag{1.14}$$

It is natural to assume a solution A to (1.13), as well as A^0 in (1.14), vanishes for $\sigma \geq R$ with some $R > 0$, because if the support of f, g is contained in a ball of radius R, then $u(t, x) = 0$ for $r \geq t + R$, which is $\sigma = r - t \geq R$, by the finite speed of propagation. Hence when we consider (1.13) and (1.14), we also impose the boundary condition that $A(t, \sigma, \omega)$ vanishes for large σ.

It is likely that the small data global existence and the small data blow-up of solutions to (1.1) is closely related to those for (1.13)–(1.14). Also it can be expected that the asymptotics for $r^{\frac{d-1}{2}}\partial_a u(t, r\omega)$ is given by $\omega_a(\partial_\sigma A)(t, r-t, \omega)$ if $A(t, \sigma, \omega)$ is a solution to (1.13)–(1.14) with appropriately chosen A^0. In their study of the Einstein equation, which can be written as a system of quasi-linear wave equations in the wave coordinates, Lindblad-Rodnianski [19] introduced the *weak null condition*, which means that the following holds: For A^0 vanishing for $\sigma \geq R$ with some $R > 0$ and decaying sufficiently fast as $\sigma \to -\infty$, the problem (1.13)–(1.14) admits a global solution A having at most polynomial growth of small power as $t \to \infty$, provided that $\varepsilon > 0$ is sufficiently small. They proved that the quasi-linear system of wave equations coming from the Einstein equation satisfies the weak null condition in [19], and obtained the small data global existence for this system in [20]. However, it is not known so far if the weak null condition implies the small data global existence to (1.1) in general.

2. Global Existence under weaker conditions than the null condition.

In this section, we will introduse some sufficient conditions for the small data global existence, which are in between the null condition and the weak null condition.

The first one is connected with the example (1.10). First we introduce some notation: Given $N_{\rm b} \in \{1, \ldots, N-1\}$, we put $N_{\rm g} = N - N_{\rm b}$, and we split $A = (A_i)_{1\leq i\leq N}$ into

$$A^{\rm b} = (A_i^{\rm b})_{1\leq i\leq N_{\rm b}} := (A_i)_{1\leq i\leq N_{\rm b}}, A^{\rm g} = (A_i^{\rm g})_{N_{\rm b}+1\leq i\leq N} := (A_i)_{N_{\rm b}+1\leq i\leq N},$$

so that we have $A = (A^{\rm b}, A^{\rm g})$. We use the same notation for other functions and vectors of size N: For example, we write $u = (u^{\rm b}, u^{\rm g})$, $F = (F^{\rm b}, F^{\rm g})$, ${}^\star F =$

$({}^{\star}F^{\mathrm{b}}, {}^{\star}F^{\mathrm{g}})$, $X = (X^{\mathrm{b}}, X^{\mathrm{g}})$, and so on. For $m \in \mathbb{N}$, $\mathbb{R}^m$ vectors are sometimes identified with $m \times 1$-matrices.

Definition 2.1. We say that Condition A is satisfied if there are some $N_{\mathrm{b}} \in \{1, \dots, N-1\}$, an $N_{\mathrm{b}} \times N_{\mathrm{b}}$-matrix valued function $\mathcal{G}(\omega, X^{\mathrm{g}}, Y^{\mathrm{g}})$ and an $N_b \times N$-matrix valued function $\mathcal{H}(\omega, X^{\mathrm{g}}, Y^{\mathrm{g}})$ such that

$$
\begin{aligned}
&{}^{\star}F^{\mathrm{b}}(\omega, X, Y, Z) = \mathcal{G}(\omega, X^{\mathrm{g}}, Y^{\mathrm{g}})Z^{\mathrm{b}} + \mathcal{H}(\omega, X^{\mathrm{g}}, Y^{\mathrm{g}})Y,\\
&{}^{\star}F^{\mathrm{g}}(\omega, X^{\mathrm{g}}, U^{\mathrm{g}}) = 0.
\end{aligned}
$$

An essentially equivalent condition to the above was first introduced by Alinhac [2] for semilinear systems in three space dimensions in a different expression, and the above expression for the semilinear case was introduced by the author [10]. Hidano-Yokoyama [5] extended it to quasi-linear systems of the form $\Box u = F(\partial u, \partial^2 u)$ in three space dimensions, and the author [12, 13] obtained it for quasi-linear systems of the form $\Box u = F(u, \partial u, \partial^2 u)$ in two and three space dimensions.

The null condition implies Condition A by choosing $\mathcal{G} \equiv 0$ and $\mathcal{H} \equiv 0$. Under Condition A, the reduced system is of the form

$$
\begin{cases}
\partial_t \partial_\sigma A^{\mathrm{b}} = \mathcal{G}(\omega, A^{\mathrm{g}}, \partial_\sigma A^{\mathrm{g}})\partial_\sigma^2 A^{\mathrm{b}} + \mathcal{H}(\omega, A^{\mathrm{g}}, \partial_\sigma A^{\mathrm{g}})\partial_\sigma A,\\
\partial_t \partial_\sigma A^{\mathrm{g}} = 0,
\end{cases}
$$

and we can solve A^{g} first, and then the first equation is a linear hyperbolic system for the unknown $\partial_\sigma A^{\mathrm{b}}$, and we can obtain a global solution with at most polynomial growth of the small power. Namely, Condition A implies the weak null condition.

Theorem 2.2 ([12, 13])**.** *Let F satisfy* (1.2) *and* (1.3)*, and let $(d, p) = (3, 2)$ or $(2, 3)$. If Condition A is satisfied, then we have the small data global existence for* (1.1)*, that is to say, for any $f, g \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^N)$, there is a positive constant ε_0 such that* (1.1) *admits a global solution $u \in C^\infty\big([0, \infty) \times \mathbb{R}^d; \mathbb{R}^N\big)$ for any $\varepsilon \in (0, \varepsilon_0]$.*

A typical and trivial example satisfying Condition A is (1.10). A less trivial example is

$$
\begin{cases}
\Box u_1 = u_2^{p-1}\Delta u_1 + (\partial_t u_2)^{p-1}(\partial_t u_1) + (\partial_1 u_2)^p,\\
\Box u_2 = u_1^{p-2}\{(\partial_2 u_2)(\partial_1^2 u_2) - (\partial_1 u_2)(\partial_1\partial_2 u_2) + (\partial_t u_1)^2 - |\nabla_x u_1|^2\}
\end{cases}
\tag{2.1}
$$

for $(d, p) = (3, 2)$ and $(d, p) = (2, 3)$.

Next we turn our attention to conditions related to (1.11). In these conditions, we only treat the semilinear case where $F = F(\partial u)$, and we write ${}^{\star}F(\omega, Y)$ instead of ${}^{\star}F(\omega, X, Y, Z)$, as X and Z correspond to u and $\partial^2 u$, respectively.

If we further restrict our consideration to the single equation with $(d, p) = (2, 3)$, then ${}^{\star}F$ has the form ${}^{\star}F(\omega, Y) = P(\omega)Y^3$ with P being a polynomial in ω of degree at most 3, and Agemi conjectured that, for single semilinear wave equations $\Box u = F(\partial u)$ with $(d, p) = (2, 3)$, the condition $P(\omega) \geq 0$ would imply the small data global existence. This conjecture was later proved to be true by Hoshiga [7] and Kubo [18] independently. The Agemi condition was generalized for systems with $(d, p) = (3, 2)$ and $(d, p) = (2, 3)$ by Katayama-Matoba-Sunagawa [15] and Katayama-Matsumura-Sunagawa [16] as follows.

Definition 2.3. Let $F = F(\partial u)$, and let $(d, p) = (3, 2)$ or $(2, 3)$. We say that Condition B is satisfied if there is a positive-definite and real-symmetric matrix valued continuous function $\mathcal{H}(\omega)$ of size N such that

$$\big\langle Y, \mathcal{H}(\omega)F^{\mathrm{red}}(\omega, Y)\big\rangle_{\mathbb{R}^N} \geq 0, \quad \omega \in \mathbb{S}^{d-1},\ Y \in \mathbb{R}^N. \tag{2.2}$$

We note that, when $(d,p) = (3,2)$, the above condition (2.2) is equivalent to

$$\big\langle Y, \mathcal{H}(\omega)F^{\mathrm{red}}(\omega, Y)\big\rangle_{\mathbb{R}^N} = 0, \quad \omega \in \mathbb{S}^{2},\ Y \in \mathbb{R}^N, \tag{2.3}$$

because the left-hand side of (2.2) is cubic in Y. As $F = F(\partial u)$, the reduced system looks like

$$\partial_t(\partial_\sigma A) = -\frac{1}{2t}{}^\star F(\omega, \partial_\sigma A), \tag{2.4}$$

which can be seen as a system of ordinary differential equations of the unknown $B(t,\sigma,\omega) := (\partial_\sigma A)(t,\sigma,\omega)$ with parameters σ and ω. It is trivial that the null condition ${}^\star F(\omega, Y) \equiv 0$ implies Condition B. If Condition B is satisfied, then, writing $B = \partial_\sigma A$ for A satisfying (2.4) with $A(1,\sigma,\omega) = \varepsilon A^0(\sigma,\omega)$, we have

$$\partial_t \langle B, \mathcal{H}B\rangle_{\mathbb{R}^N} = 2\langle B, \mathcal{H}\partial_t B\rangle_{\mathbb{R}^N} = -\frac{1}{t}\langle B, \mathcal{H}{}^\star F(\omega, B)\rangle_{\mathbb{R}^N} \leq 0, \tag{2.5}$$

which leads to

$$|B(t,\sigma,\omega)| \leq C_0\varepsilon|\partial_\sigma A^0(\sigma,\omega)| \tag{2.6}$$

with a positive constant C_0 which is independent of (t,σ,ω), because $|B|$ is equivalent to $\langle B, \mathcal{H}B\rangle_{\mathbb{R}^N}$. This *a priori* estimate (2.6) implies the global existence of a bounded solution A to (1.13). Hence we see that Condition B implies the weak null condition.

Theorem 2.4 ([15, 16])**.** *Let $F = F(\partial u)$ and let $(d,p) = (3,2)$ or $(2,3)$. Then Condition B implies the small data global existence for* (1.1)*.*

A typical example satisfying Condition B with $(d,p) = (2,3)$ is (1.11), because we have ${}^\star F(\omega, Y) = Y^3$ and $Y{}^\star F(\omega, Y) = Y^4 \geq 0$. Another example satisfying Condition B with $(d,p) = (3,2)$ or $(2,3)$ is

$$\begin{cases} \Box u_1 = (\partial_1 u_1)^{p-1}(\partial_t u_2), \\ \Box u_1 = -2(\partial_1 u_1)^{p-1}(\partial_t u_1), \end{cases} \tag{2.7}$$

because we have ${}^\star F(\omega, Y) = (-\omega_1^{p-1}Y_1^{p-1}Y_2, 2\omega_1^{p-1}Y_1^p)$ and $\langle Y, \mathcal{H}(\omega){}^\star F(\omega, Y)\rangle_{\mathbb{R}^2} = 0$ with $\mathcal{H}(\omega) = \mathrm{diag}(2,1)$. However the null condition is not satisfied for this system.

Recently, in a joint work with S. Masaki, the author succeeded to weaken Condition B in the following way:

Definition 2.5. Let $F = F(\partial u)$, and let $(d,p) = (3,2)$ or $(2,3)$. We say that Condition C is satisfied, if the reduced system, written in the form

$$\begin{cases} \partial_t B(t,\omega) = -\dfrac{1}{2t}{}^\star F\big(\omega, B(t,\omega)\big), & (t,\omega) \in [1,\infty) \times \mathbb{S}^{d-1}, \\ B(1,\omega) = \varepsilon B^0(\omega), & \omega \in \mathbb{S}^{d-1}, \end{cases}$$

admits a global solution B for $(t,\omega) \in [1,\infty) \times \mathbb{R}^d$ satisfying

$$|B(t,\omega)| \leq C_0\varepsilon|B^0(\omega)|, \quad (t,\omega) \in [1,\infty) \times \mathbb{S}^{d-1}$$

with a positive constant C_0 determined only by ${}^\star F$, provided that $\varepsilon > 0$ is sufficiently small. Here the parameter σ is omitted since the reduced system is autonomous with respect to σ.

It is apparent that the null condition implies Condition C, and Condition C implies the weak null condition. In view of (2.6), we see that Condition B implies Condition C.

Theorem 2.6 ([14])**.** *Let $F = F(\partial u)$, and let $(d,p) = (3,2)$ or $(2,3)$. Then Condition C implies the small data global existence for* (1.1)*.*

In Conditions A and B, some explicit structures of the nonlinearity F are assumed. In contrast, no structure of F is assumed explicitly in Condition C, but some qualitative property of the reduced system is assumed, in a similar spirit to the weak null condition. A key idea in the proof of the above theorem is that we obtain *a priori* estimates by using a new ODE lemma, which is also used for the proof of the asymptotic behavior of global solutions mentioned in the next section.

The following system for $(d,p) = (2,3)$ satisfies Condition C, but not Condition B:

$$\begin{cases} \Box u_1 = (\partial_t u_2)^3, \\ \Box u_2 = -(\partial_t u_1)^3. \end{cases} \tag{2.8}$$

Indeed, writing $B = \partial_\sigma A$, the reduced system is

$$\begin{cases} \partial_t B_1 = \dfrac{1}{2t} B_2^3, \\ \partial_t B_2 = -\dfrac{1}{2t} B_1^3. \end{cases} \tag{2.9}$$

If we put $B(1,\sigma,\omega) = \varepsilon B^0(\sigma,\omega)$, then we have $\partial_t(B_1^4 + B_2^4) = 0$, which shows

$$|B|^2 \le 2(B_1^4 + B_2^4) = 2\varepsilon^2\big((B_1^0)^4 + (B_2^0)^4\big) \le 2\varepsilon^2 |B^0|^2,$$

and Condition C is satisfied. Suppose that Condition B is satisfied with

$$\mathcal{H}(\omega) = \begin{pmatrix} a(\omega) & b(\omega) \\ b(\omega) & c(\omega) \end{pmatrix},$$

which is continuous and positive-definite. Then we have

$$\langle Y, \mathcal{H}^\star F(Y) \rangle_{\mathbb{R}^2} = aY_1Y_2^3 - bY_1^4 + bY_2^4 - cY_1^3Y_2 \ge 0, \quad Y = (Y_1, Y_2) \in \mathbb{R}^2, \omega \in \mathbb{S}^1.$$

By putting $Y = (1,0)$ and $Y = (0,1)$, we find that $b \equiv 0$. Hence we get

$$Y_1Y_2(aY_2^2 - cY_1^2) \ge 0.$$

Note that we have $a, c > 0$ because $\mathcal{H}$ is positive-definite. Let $Y_1, Y_2 \ne 0$. If we put $-Y_2$ in place of Y_2, we obtain $aY_2^2 - cY_1^2 = 0$. Now taking the limit as $Y_2 \to 0$, we get $-cY_1^2 = 0$ for $Y_1 \ne 0$, which leads to $c \equiv 0$. This is a contradiction, and Condition B is not satisfied for this system.

(2.8) shows that Condition C is strictly weaker than Condition B when $(d,p) = (2,3)$. Unfortunately, we do not know such an example for $(d,p) = (3,2)$, and there is a possibility that Conditions B and C coincides with each other when $(d,p) = (3,2)$.

3. Asymptotic Behavior

Next we turn our attention to the asymptotic behavior. We write the solution $A = A(t,\sigma,\omega)$ to (1.13)–(1.14), satisfying the boundary condition $A(\sigma,\omega) = 0$ for

$\sigma \geq R$ with some $R > 0$, as $A[\varepsilon A^0](t, \sigma, \omega)$. For $\omega = (\omega_1, \dots, \omega_d) \in \mathbb{S}^{d-1}$, and a multi-index $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_d)$, we write

$$\partial^\alpha = \partial_0^{\alpha_0} \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}, \quad \overline{\omega}^\alpha = (-1)^{\alpha_0} \omega_1^{\alpha_1} \cdots \omega_d^{\alpha_d}.$$

As mentioned in the introduction, $A[\varepsilon A^0](t, r-t, \omega)$ is expected to give the asymptotics for the solution u to (1.1), provided that A^0 is appropriately chosen.

Theorem 3.1. *Let $(d,p) = (3,2)$ or $(2,3)$, and let F satisfy (1.2) and (1.3). We assume Condition A, and choose arbitrarily small $\mu > 0$. For any $f, g \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^N)$ and sufficiently small $\varepsilon > 0$, there is $A^0 = A^0(\sigma, \omega; \varepsilon)$ such that we have the following:*

$$r^{\frac{d-1}{2}} \partial^\alpha u(t,x) = \overline{\omega}^\alpha (\partial_\sigma^{|\alpha|} A[\varepsilon A^0])(t, r-t, \omega) + O\big(\varepsilon \langle t+r \rangle^{\mu-1}\big), \quad |\alpha| = 1, 2,$$

$$r^{\frac{d-1}{2}} u^{\mathrm{g}}(t,x) = A^{\mathrm{g}}[\varepsilon A^0](t, r-t, \omega) + O\big(\varepsilon \langle t+r \rangle^{\mu-\frac{1}{p}}\big)$$

for $(t,x) \in [0,\infty) \times \mathbb{R}^d$, where u is the solution to (1.1), $r = |x|$, and $\omega = x/|x|$.

This was proved for the semilinear case in [10], and for the quasi-linear case in [13] (see also [12], which was originally intended to be the announcement of [13], for the partial result).

A similar formula holds for ∂u under Condition B, but some additional assumptions were assumed when $(d,p) = (2,3)$ (see [11] for instance). In [14], we suceeded to remove such additional assumptions, and we can show the following under Condition C.

Theorem 3.2. *Let $F = F(\partial u)$, and let $(d,p) = (3,2)$ or $(2,3)$. We assume Condition C, and choose arbitrarily small $\mu > 0$. For any $f, g \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^N)$ and sufficiently small $\varepsilon > 0$, there is $A^0 = A^0(\sigma, \omega; \varepsilon)$ such that*

$$r^{\frac{d-1}{2}} \partial_a u(t,x) = \omega_a (\partial_\sigma A[\varepsilon A^0])(t, r-t, \omega) + O\big(\varepsilon \langle t+r \rangle^{\mu-1}\big)$$

for $(t,x) \in [0,\infty) \times \mathbb{R}^d$, where u is the solution to (1.1), $r = |x|$, $\omega = x/|x|$, and $\omega_0 = -1$.

References

[1] S. Alinhac, *The null condition for quasilinear wave equations in two space dimensions I*, Invent. Math. **145** (2001), 597–618.

[2] S. Alinhac, *Semilinear hyperbolic systems with blowup at infinity*, Indiana Univ. Math. J. **55** (2006), 1209–1232.

[3] D. Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Comm. Pure Appl. Math. **39** (1986), 267–282.

[4] P. Godin, *Lifespan of solutions of semilinear wave equations in two space dimensions*, Comm. Partial Differential Equations **18** (1993), 895–916.

[5] K. Hidano and K. Yokoyama, *Global existence for a system of quasi-linear wave equations in 3D satisfying the weak null condition*, IMRN **2020** (2020), 39–70.

[6] A. Hoshiga, *The initial value problems for quasi-linear wave equations in two space dimensions with small data*, Adv. Math. Sci. Appl. **5** (1995), 67–89.

[7] A. Hoshiga, *The existence of the global solutions to semilinear wave equations with a class of cubic nonlinearities in 2-dimensional space,* Hokkaido Math. J. **37** (2008), 669–688.

[8] S. Katayama, *Global existence for systems of nonlinear wave equations in two space dimensions*, Publ. RIMS, Kyoto Univ. **29** (1993), 1021–1041.

[9] S. Katayama, *Global existence for systems of nonlinear wave equations in two space dimensions, II*, Publ. RIMS, Kyoto Univ. **31** (1995), 645–665.

[10] S. Katayama, *Asymptotic pointwise behavior for systems of semilinear wave equations in three space dimensions*, J. Hyperbolic Differ. Equ. **9** (2012), 263–323.

[11] S. Katayama, Global Solutions and the Asymptotic Behavior for Nonlinear Wave Equations with Small Initial Data, MSJ Memoirs **36**, Math. Soc. Japan, Tokyo, 2017.
[12] S. Katayama, *Global existence and the asymptotic behavior for systems of nonlinear wave equations violating the null condition*, in: The Role of Metrics in the Theory of Partial Differential Equations, pp. 215–225, ASPM **85**, Math. Soc. Japan, Tokyo, 2020.
[13] S. Katayama, *Asymptotic behavior for global solutions to systems of quasilinear wave equations under a kind of weal null condition*, preprint.
[14] S. Katayama and S. Masaki, in preparation.
[15] S. Katayama, T. Matoba and H. Sunagawa, *Semilinear hyperbolic systems violating the null condition*, Math. Ann. **31** (2015), 275–312.
[16] S. Katayama, A. Matsumura and H. Sunagawa, *Energy decay for systems of semilinear wave equations with dissipative structure in two space dimensions*, NoDEA Nonlinear Differential Equations Appl. **22** (2015), 601–628.
[17] S. Klainerman, *The null condition and global existence to nonlinear wave equations*, in: Nonlinear Systems of Partial Differential Equations in Applied Mathematics, Part 1, pp. 293–326, Lectures in Appl. Math. **23**, AMS, Providence, 1986.
[18] H. Kubo, *Asymptotic behavior of solutions to semilinear wave equations with dissipative structure*, Discrete Contin. Dynam. Systems, Supplement Volume 2007 (2007), 602–613.
[19] H. Lindblad and I. Rodnianski, *The weak null condition for Einstein's equations*, C. R. Math. Acad. Sci. Paris **336** (2003), 901–906.
[20] H. Lindblad and I. Rodnianski, *Global existence for the Einstein vacuum equations in wave coordinates*, Commun. Math. Phys. **256** (2005), 43–110.

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SMOOTHING EFFECT OF THE NONLINEAR SCATTERING OPERATOR ASSOCIATED WITH NLS

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We present some results obtained in collaboration with N. Burq (Paris-Saclay), H. Koch (Bonn), N. Tzvetkov (ENS Lyon).

In order to simplify the presentation we focus on the following Cauchy problem, however the result we shall present is rather general and can be extended for other nonlinearities as well as in higher dimensions (see section 2):

$$\begin{cases} i\partial_t u + \partial_x^2 u = u|u|^4, \quad (t,x) \in \mathbb{R} \times \mathbb{R} \\ u(0,x) = u_0 \in H^s(\mathbb{R}). \end{cases} \tag{0.1}$$

Notice that this Cauchy problem is L^2-critical, which makes very hard its analysis as we shall specify below. However our result will not exploit the fact that we are L^2 critical and we believe that in fact can be extended to generic intercritical regimes, namely NLS which are mass supercritical and energy subcritical. For instance we present in section 2 a statement concerning 3d cubic NLS.

In general the Cauchy problem associated with NLS, in any dimension and for general pure power nonlinearities, has attracted a lot the attention of a huge mathematical community. The main questions concern the existence and uniqueness of global solutions, as well as the description of the long-time behavior of the solutions. In particular the nonlinear scattering theory has been considered in several papers. The basic idea of scattering is the intuition that for large times the dynamics associated to (0.1) is drived by its nonlinear part (see below for a precise definition). In between several papers where this question has been studied we quote [1], [4], [5], [6], [7], [8], [9], [10], [11]. We also quote the very complete book by T. Cazenave [3].

The above Cauchy problem (0.1) admits one unique global solution $u(t,x)$ for any $u_0 \in H^s(\mathbb{R})$ and for any $s \geq 0$, moreover the solution scatters at $t \to \pm\infty$, namely

$$\exists u_\pm \in H^s(\mathbb{R}) \text{ s.t. } \|e^{-it\partial_x^2} u(t,x) - u_\pm\|_{H^s(\mathbb{R})} \overset{t\to\pm\infty}{\longrightarrow} 0. \tag{0.2}$$

The proof of this fact is rather delicate in the specific case of (0.1), since the problem considered is L^2-critical, the proof being provided along the work of Dodson [7]. In fact one can establish the following bound:

(0.3) $\forall u_0 \in H^s(\mathbb{R})$ for the nonlinear solution to (0.1) we have the bound

$$\|u\|_{L^p(\mathbb{R};W^{s,q}(\mathbb{R}))} < \infty,$$

$$\forall (p,q) \in [4,\infty] \times [2,\infty] \text{ s.t. } \frac{2}{p} + \frac{1}{q} = \frac{1}{2}.$$

It is worth mentioning scattering property similar to (0.2) is rather classical and much easier to prove in the intercritical regime, namely for NLS which are L^2 supercritical and H^1 subcritical.

We also point out that the scattering property of solutions to (0.1) can be easily achieved under the stronger assumption $u_0 \in \Sigma$,where

$$\Sigma = \{u_0 \in H^1 | \int |x|^2 |u_0|^2 dx < \infty\}.$$

Indeed by using the lens transform (see for instance the appendix in [2])) it is not difficult to show the following fact:

$$\forall u_0 \in \Sigma \quad \exists! u(t,x) \in \mathcal{C}(\mathbb{R};\Sigma) \text{ global solution to (0.1)},$$
$$\text{and moreover } \exists u_\pm \in \Sigma \text{ s.t. } \|e^{-it\partial_x^2} u(t,x) - u_\pm\|_\Sigma \overset{t\to\pm\infty}{\longrightarrow} 0.$$

However working in the unweighted Sobolev spaces, as in (0.2), is much harder and the corresponding result for the 1d quintic NLS has been achieved in the aforementioned paper [7].

The main novelty of our result is to establish qualitative properties of the scattering operator

$$u_0 \to u_\pm$$

when $u_0 \in H^s$.
Roughly speaking we show that

$$u_\pm = u_0 + \text{smoother term}$$

and moreover the convergence in (0.2) can be updated in a stronger topology than H^s. This smoothing propery will depend of the regularity H^s and is stronger in the case $s \geq \frac{1}{4}$ and weaker when $s \to 0$.

Next we state our first main result about the solutions to (0.1).

Theorem 0.1. *Let $s \geq \frac{1}{4}$ and u be the unique global solution to* (0.1)*, which satisfies* (0.3)*. Then there exists*

$$v_\pm \in \bigcap_{s'<s} H^{s'+1}(\mathbb{R})$$

such that

$$\|e^{-it\partial_x^2} u(t,x) - u_0 - v_\pm\|_{H^{s'+1}(\mathbb{R})} \overset{t\to+\infty}{\longrightarrow} 0, \quad \forall s' \in [0,s).$$

Concerning the low regularity case we get the following result.

Theorem 0.2. *Let $s \in [0,\frac{1}{4})$ and u be the unique global solution u to* (0.1)*, which satisfies* (0.3)*. Then there exists*

$$v_\pm \in H^{5s}(\mathbb{R})$$

such that

$$\|e^{-it\partial_x^2} u(t,x) - u_0 - v_\pm\|_{H^{5s}(\mathbb{R})} \overset{t\to\pm\infty}{\longrightarrow} 0.$$

1. Tools of the proof in the 1d case

Our proof, in the 1d case, is based on rather classical estimates (see [13]).

First of all we use the local smoothing estimate:

$$\|D^{\frac{1}{2}}e^{it\partial_x^2}f\|_{L_x^\infty L_t^2} \leq C\|f\|_{L_x^2};$$

next we use the maximal function estimate:

$$\|D^{-\frac{1}{4}}e^{it\partial_x^2}f\|_{L_x^4 L_t^\infty} \leq C\|f\|_{L_x^2};$$

finally we need the classical Strichartz estimate:

$$\|e^{it\partial_x^2}f\|_{L_{t,x}^6} \leq C\|f\|_{L_x^2}.$$

as well as on the following vector-valued Leibnitz rule (see [12]):

Proposition 1.1. *Let* $\alpha \in (0,1)$, $\alpha_1, \alpha_2 \in [0,\alpha]$ *such that*

$$\alpha_1 + \alpha_2 = \alpha,$$

and $p, q, p_1, q_1, p_2, q_2 \in (1,\infty)$ *such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2},$$

then we get the bound

$$\|D^\alpha(fg) - gD^\alpha f - fD^\alpha g\|_{L_x^p L_t^q} \leq C\|D^{\alpha_1}f\|_{L_x^{p_1}L_t^{q_1}}\|D^{\alpha_2}g\|_{L_x^{p_2}L_t^{q_2}}$$

2. The 3d cubic NLS

We consider now the following Cauchy problem in dimension three:

$$\begin{cases} i\partial_t u + \Delta u = u|u|^2, & (t,x) \in \mathbb{R} \times \mathbb{R}^3 \\ u(0,x) = u_0 \in H^s. \end{cases} \tag{2.1}$$

We assume for solutions to (2.1) the following bound:

$$\|u\|_{L_t^p W_x^{s,q}} \leq C(u_0) < \infty, \quad \forall (p,q) \in [2,\infty] \times [2,6] \text{ s.t. } \frac{2}{p} + \frac{3}{q} = \frac{3}{2}. \tag{2.2}$$

This bound has been proved by J. Ginibre and G. Velo (see [8]) in the energy space $s = 1$ and for $s > 4/5$ by J. Colliander, M. Keel, G. Staffilani, T. Tao, H. Takaoka in [4]. It is worth mentioning that the Cauchy problem (2.1) is easier from the scattering point of view than (0.1) since it is intercritical. Indeed its global well-posedness and the corresponding nonlinear scattering theory has been established in [8], with a easier proof compared to [5], [6], [7] where the L^2 critical case is considered.

Our main contribution is to establish also in the framework of (2.1) smoothing properties of the scattering operator similar to the ones established in theorem 0.1 and theorem 0.2 concerning (0.1).

Here is our smoothing result for the scattering operator associated with (2.1).

Theorem 2.1. *Let $s \in (\frac{1}{2}, 1]$ be given. Assume that there exists one unique global solution u to* (2.1)*, which satisfies* (2.2)*. Then there exists*

$$v_+ \in \bigcap_{s'<s} H^{3s'-1}(\mathbb{R}^3)$$

such that

$$\|e^{-it\Delta}u(t,x) - u_0 - v_+\|_{H^{3s'-1}(\mathbb{R}^3)} \overset{t\to+\infty}{\longrightarrow} 0, \quad \forall s' < s.$$

3. A probabilistic result

Our result on the smoothing of the scattering operator, has been inspired by a previous probabilistic result established with the same coauthors.

We denote below

$$\|f\|_{\mathcal{H}^s} = \|(-\Delta + |x|^2)^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^n)}$$

and also for every $\gamma \in \mathcal{H}^s$ we denote by μ_γ a family of gaussian measures on $\mathcal{H}^s$ defined as the law of the random vector

$$(\Omega, p, \mathcal{A}) \ni \omega \to \sum_k c_k g_k(\omega)\varphi_k(x)$$

where we are assuming:

- $\{g_k(\omega)\}$ is a family of i.i.d. centered and normalized complex valued gaussian measures on the probability space $(\Omega, p, \mathcal{A})$;
- $\varphi_k(x)$ are eigenfunctions of the harmonic oscillator $-\Delta + |x|^2$ with associated eigenvalue λ_k;
- we assume moreover that the vector $\gamma \in \mathcal{H}^s$ has the expansion

$$\gamma(x) = \sum c_k \varphi_k$$

where

$$|c_k|^2 \lesssim \frac{1}{\# I(j)} \sum_{m \in I(j)} |c_m|^2, \quad \forall k \in I(j), \quad j \geq 1$$

and

$$I(j) = \{k \in \mathbb{N} \text{ s.t. } 2j \leq \lambda_k < 2(j+1)\}.$$

One can show the following properties about the measures constructed above:

- assume $\gamma \in \mathcal{H}^s(\mathbb{R}^n) \setminus \mathcal{H}^{s+\varepsilon}(\mathbb{R}^n)$ is as above, then

$$\mu_\gamma(\mathcal{H}^{s+\varepsilon}(\mathbb{R}^n)) = 0; \tag{3.1}$$

- assume that the corresponding Fourier coefficients c_k of g_k are different from zero, then

$$\mu_\gamma(B) > 0, \quad \forall B \text{ open set in } \mathcal{H}^s(\mathbb{R}^n). \tag{3.2}$$

Next we consider the cubic NLS

$$\begin{cases} i\partial_t u + \Delta u - |u|^2 u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = u_0 \end{cases}$$

and we state are main probabilistic result, which was a source of inspiration of our previous deterministic results on the smoothing of the scattering operator.

Theorem 3.1. *Let $(n,s) \in \{2,3,4\} \times \mathbb{R}$ satisfy*

$$\begin{cases} s > 0 & \text{if } n = 2 \\ s \geq -\frac{1}{4} & \text{if } n = 3 \\ s \geq -\frac{1}{2} & \text{if } n = 4 \end{cases}$$

there exists a set $\Sigma_n^s \subset \mathcal{H}^s(\mathbb{R}^n)$ with full measure w.r.t. μ_γ such that for every $u_0 \in \Sigma_n^s$ the corresponding Cauchy problem associated with cubic NLS admits a unique global solution with the following structure

$$u(t,x) = e^{it\Delta}(u_0 + R_{u_0}(t,x)), \quad R_{u_0}(t,x) \in \mathcal{C}(\mathbb{R}; \mathcal{H}^1(\mathbb{R}^n)).$$

Moreover there exist

$$\lim_{t \to \pm\infty} R_{u_0}(t) = r_0^\pm \in \mathcal{H}^1(\mathbb{R}^n)$$

such that

$$\|e^{-it\Delta}u(t,x) - u_0 - r_0^\pm\|_{\mathcal{H}^1(\mathbb{R}^n)} \overset{t\to\pm\infty}{\longrightarrow} 0.$$

References

[1] Bourgain J., *Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case*, Journal of the American Mathematical Society, 12, 1999, 1, 145–171

[2] Burq, N. and Georgiev, V. and Tzvetkov, N. and Visciglia, N., *H^1 scattering for mass-subcritical NLS with short-range nonlinearity and initial data in Σ*, Ann. Henri Poincaré, 24, 2023, 4, pp. 1355-1376.

[3] Cazenave, T., *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, 10, 2003

[4] Colliander, J. and Keel, M. and Staffilani, G. and Takaoka, H. and Tao, T., *Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on $\mathbb{R}^3$*, Comm. Pure Appl. Math., 57, 2004, 8, 987–1014

[5] Dodson, B., *Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schrödinger equation when* $d = 2$, Duke Mathematical Journal, 165, 2016, 18, 3435–3516

[6] Dodson, B., *Global well-posedness and scattering for the defocusing, L^2-critical nonlinear Schrödinger equation when* $d \geq 3$, Journal of the American Mathematical Society, 25, 2012, 2, 429–463

[7] Dodson, B., , *Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schrödinger equation when* $d = 1$, American Journal of Mathematics, 138, 2016, 2, 531-569.

[8] Ginibre, J. and Velo, G., *Scattering theory in the energy space for a class of nonlinear Schrödinger equations*, J. Math. Pures Appl., 64, 1985, 4, 363–401

[9] Nakanishi K., *Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions* 1 *and* 2, Journal of Functional Analysis, 169, 1999, 1, 201–225

[10] Planchon F. and Vega L., *Bilinear virial identities and applications*, Annales Scientifiques de l'École Normale Supérieure. Quatrième Série, 42, 2009, 2, 261–290

[11] Visciglia N., *On the decay of solutions to a class of defocusing NLS*, Mathematical Research Letters, 16, 2009, 5, 919-926.

[12] Kenig C. and Ponce G. and Vega L., *Well-posedness and scattering results for the generalized korteweg-de vries equation via the contraction principle*, Communications on Pure and Applied Mathematics, 46, 193, 4,

[13] Kenig C. and Ponce G. and Vega L., *Oscillatory Integrals and Regularity of Dispersive Equations*, Indiana University Mathematics Journal, 40, 1991, 1, 33-69, 527-620

Recent developments on non-autonomous nonlinear wave equations *

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Abstract

In this talk, I will overview recent progresses on lifespan estimates of classical solutions of initial value problems for nonlinear wave equations in one space dimension. Here I especially focus on extensions to non-autonomous nonlinear terms which includes an application to nonlinear damped wave equations with the time-dependent critical case.

1 Introduction

In order to illustrate our purpose, let us turn back to the general theory for nonlinear wave equations in one space dimension. Here we consider the initial value problem of the form;

$$\begin{cases} u_{tt} - u_{xx} = H(u, u_t, u_x, u_{xx}, u_{xt}) & \text{in } \mathbf{R} \times (0, T), \\ u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x) & \text{for } x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where $T > 0$, $f, g \in C_0^\infty(\mathbf{R})$ and $\varepsilon > 0$ is a sufficiantly small parameter. Let $\widetilde{\lambda} = (\lambda;\ (\lambda_i), i = 0, 1;\ (\lambda_{ij}), i, j, = 0, 1, i + j \geq 1)$. Assume that $H = H(\widetilde{\lambda})$ is a sufficiently smooth function with

$$H(\widetilde{\lambda}) = O(|\widetilde{\lambda}|^{1+\alpha})$$

in a neighborhood of $\widetilde{\lambda} = 0$, where $\alpha \in \mathbf{N}$. Let us define the lifespan $\widetilde{T}(\varepsilon)$ as the maximal existence time of the classical solution of (1.1) with arbitrary fixed data. We are interested in the long-time stability of the trivial solution due to the fact that we cannot expect any time-decay of the solution of the free wave equation in one space dimension. Indeed, the general theory is to express the lower bound of $\widetilde{T}(\varepsilon)$ by means of the smallness of the

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initial data, i.e. ε, for which, Li, Yu and Zhou [17, 18] and Takamatsu [23] obtained

$$\widetilde{T}(\varepsilon) \geq \begin{cases} C\varepsilon^{-\alpha/2} & \text{in general case,} \\ C\varepsilon^{-\alpha(\alpha+1)/(\alpha+2)} & \text{if } \displaystyle\int_{\mathbf{R}} g(x)dx = 0, \\ C\varepsilon^{-\min\{\beta_0/2,\alpha\}} & \text{if } \partial_u^\beta H(0) = 0 \text{ for } 1+\alpha \leq \forall\beta \leq \beta_0, \\ C\varepsilon^{-\beta_0(\alpha+1)/(\beta_0+2)} & \text{if } \displaystyle\int_{\mathbf{R}} g(x)dx = 0 \text{ and} \\ & \quad \partial_u^\beta H(0,0,0) = 0 \text{ for } \alpha+1 \leq \forall\beta \leq \beta_0 < 2\alpha, \end{cases} \tag{1.2}$$

where C is a positive constant independent of ε.

Beyond the general theory, our interest has been going to its optimality, or to extending the general theory, by studying the model problems;

$$\begin{cases} u_{tt} - u_{xx} = A(x,t)|u_t|^p|u|^q + B(x,t)|u|^r & \text{in} \quad \mathbf{R} \times (0,T), \\ u(x,0) = \varepsilon f(x), \ u_t(x,0) = \varepsilon g(x), & x \in \mathbf{R}, \end{cases} \tag{1.3}$$

where $p, q, r > 1$ (q can be zero) and A, B are non-negative functions of space-time variables. Among such activities, we find that the principle in extending the nonlinear term H in (1.1) for the general theory to the non-autonomous one $H = H(x,t,u,u_t,u_x,u_{xx},u_{xt})$ must be initiated by variable coefficient case as below. From now on, we denote the lifespan of classical solutions of (1.3) by $T(\varepsilon)$.

2 Constant coefficients case

First I shall outline the constant coefficient case here. Let us assume that

$$A(x,t) \equiv A_0 \quad \text{and} \quad B(x,t) \equiv B_0.$$

where A_0 and B_0 are non-negative constants.

When $A_0 = 0$ and $B_0 > 0$, Zhou [30] obtained the estimates of $T(\varepsilon)$;

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-(r-1)/2} & \text{if } \displaystyle\int_{\mathbf{R}} g(x)dx \neq 0, \\ C\varepsilon^{-r(r-1)/(r+1)} & \text{if } \displaystyle\int_{\mathbf{R}} g(x)dx = 0. \end{cases} \tag{2.1}$$

Here we denote the fact that there are positive constants, C_1 and C_2, independent of ε satisfying $E(\varepsilon, C_1) \leq T(\varepsilon) \leq E(\varepsilon, C_2)$ by $T(\varepsilon) \sim E(\varepsilon, C)$. The classification by total integral of the initial speed g is caused by strong Huygens' principle. On the other hand, when $A_0 > 0$ and $B_0 = 0$, we have

$$T(\varepsilon) \sim C\varepsilon^{-(p+q-1)} \tag{2.2}$$

For $q = 0$, the upper bound in this estimate was obtained by Zhou [31], and the lower bound is due to Kitamura, Morisawa and Takamura [13]. For $q > 1$, (2.2) was verified by Zhou [32] for the upper bound with integer p, q satisfying $p \geq 1, q \geq 0, p+q \geq 2$, and by Li,Yu and Zhou [17, 18] for the lower bound with integer p, q satisfying $p+q \geq 2$

including more general but smooth terms. Note that [32] is a preprint version of Zhou [31] in which only the case of $q = 0$ is considered. But it is easy to apply its argument to the case of $q > 1$. The lower bound in this case is due to Kido, Sasaki, Takamatsu and Takamura [10].

Therefore the natural expectation is that

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-\min\{(p+q-1),(r-1)/2\}} & \text{if } \displaystyle\int_{\mathbf{R}} g(x)dx \neq 0, \\ C\varepsilon^{-\min\{(p+q-1),r(r-1)/(r+1)\}} & \text{if } \displaystyle\int_{\mathbf{R}} g(x)dx = 0 \end{cases} \tag{2.3}$$

in the case where $A_0 > 0$ and $B_0 > 0$. But, surprisingly, we have the following fact.

Theorem 2.1 (Morisawa, Sasaki and Takamura [20, 21], Kido, Sasaki, Takamatsu and Takamura [10]). *The conjecture (2.3) is true except for the case where*

$$\int_{\mathbf{R}} g(x)dx = 0 \quad \text{and} \quad \frac{r+1}{2} < p + q < r. \tag{2.4}$$

In this case, we have that

$$T(\varepsilon) \sim C\varepsilon^{-(p+q)(r-1)/(r+1)}, \tag{2.5}$$

which is strictly shorter than the second case in (2.3).

We shall call this special phenomenon by "generalized combined effect" of two nonlinearities. The original "combined effect", which means the case of $q = 0$, was first observed by Han and Zhou [4] which targeted to show the optimality of the result by Katayama [8] on the lower bound of the lifespan of classical solutions of nonlinear wave equations with a nonlinear term $u_t^3 + u^4$ in two space dimensions including more general nonlinear terms. It is known that $T(\varepsilon) \sim \exp(C\varepsilon^{-2})$ for the nonlinear term u_t^3 and $T(\varepsilon) = \infty$ for the nonlinear term u^4, but Katayama [8] obtained only a much worse estimate than their minimum as $T(\varepsilon) \geq c\varepsilon^{-18}$. Surprisingly, more than ten years later, Han and Zhou [4] showed that this result is optimal as $T(\varepsilon) \leq C\varepsilon^{-18}$. They also considered (1.3) with $q = 0$ for all space dimensions n bigger than 1 and obtain the upper bound of the lifespan. Its counter part, the lower bound of the lifespan, was obtained by Hidano, Wang and Yokoyama [5] for $n = 2, 3$. See the introduction of [5] for the precise results and references. We note that the estimate (2.5) with $q = 0$ coincides with the lifespan estimate for the combined effect in [4, 5] if one sets $n = 1$ formally. Indeed, [4] and [5] showed that

$$T(\varepsilon) \sim C\varepsilon^{-2p(r-1)/\{2(r+1)-(n-1)p(r-1)\}} \tag{2.6}$$

holds for $n = 2, 3$ provided

$$(r-1)\{(n-1)p-2\} < 4,\ 2 \leq p \leq r \leq 2p-1,\ r > \frac{2}{n-1}. \tag{2.7}$$

Later, Dai, Fang and Wang [3] improved the lower bound of lifespan for the critical case in [5]. They also show that $T(\varepsilon) < \infty$ for all $p, r > 1$ in case of $n = 1$, i.e. (1.3) with $q = 0$. For the non-Euclidean setting of the results above, see Liu and Wang [19] for example, in which the application to semilinear damped wave equations is included.

Finally we point out that all the results stated here ensure the optimality of the general theorey. See Introduction in [10], or an overviewing paper by Takamura [24] for details. Also, most of all the cases, their proofs are extensions of the iteration argument of point-wise estimates of the solution in weighted L^∞ space by John [7].

3 Variable coefficients case

Here we assume for our model (1.3) that $A(x,t)$, or $B(x,t)$, is of

$$\frac{1}{\langle t+\langle x\rangle\rangle^{1+a}\langle t-\langle x\rangle\rangle^{1+b}\langle x\rangle^{1+c}}, \text{ or } \equiv 0,$$

where $a,b,c\in\mathbf{R}$ and $\langle x\rangle:=\sqrt{1+x^2}$. The reason to take many $\langle\ \rangle$s is to ensure the differentiability of A,B to construct a classical solution. We will see that this kind of models will be a key in extending the general theory for (1.1) to the non-autonomous terms as stated at the end of Introduction. The weights of this kind were firstly introduced by Belchev, Kepka and Zhou [2], later Liu and Zhou [16] in higher space dimensions to show a blow-up result. But they set a special weight to make use of some geometric transform to absorb it, and reduced the equation to ordinary differential inequality of the functional without any argument on the local existence of the solution as well as the lifespan estimates.

3.1 Motivation of the problem

Our motivation to consider such A and B above comes from an initial value problem for the semilinear damped wave equations;

$$\begin{cases} v_{tt}-v_{xx}+\dfrac{2}{1+t}v_t=|v|^p & \text{in } \mathbf{R}\times(0,\infty),\\ v(x,0)=\varepsilon f(x), \quad v_t(x,0)=\varepsilon g(x) & \text{for } x\in\mathbf{R} \end{cases} \tag{3.1}$$

with the same setting on the initial data. This problem is a very important model as it has a critical decay in damping term, namely scaling invariant damping. The time-decay $(1+t)^{-1}$ is a threshold between heat-like with weaker decay and wave-like with stronger decay in the sense that the each critical exponent is the same as Fujita one and Strauss one respectively. In the critical case, the size of the constant in front of the damping term is important. There is also a threshold on the constant and this case "2" is in the domain of heat-like in one dimension. See the introduction of Kato, Takamura and Wakasa [9] for precise results and references. In fact, D'Abbicco [1] showed for $p>3$ that the energy solution of (3.1) exists globally-in-time, while Wakasugi [27] obtained its counter part of finite-time blow-up of the energy solution for $1<p\le 3$. This critical exponent 3 is Fujita one in one space dimension, so one may expect that the solution behaves like one of semilinear heat equations for which v_{tt} is neglected from (3.1). But, this is not true.

In fact, Liouville transform $u(x,t)=(1+t)v(x,t)$ shows that (3.1) is equivalent to

$$\begin{cases} u_{tt}-u_{xx}=\dfrac{|u|^p}{(1+t)^{p-1}} & \text{in } \mathbf{R}\times(0,\infty),\\ u(x,0)=\varepsilon f(x),\ u_t(x,0)=\varepsilon\{f(x)+g(x)\} & \text{for } x\in\mathbf{R}, \end{cases} \tag{3.2}$$

so that all the technique for semilinear wave equations are applicable to this problem, and we have the following results on the lifepsan estimates for (3.1). Wakasa [25] obtained that

$$T(\varepsilon)\sim\begin{cases} C\varepsilon^{-(p-1)/(3-p)} & \text{for } 1<p<3,\\ \exp(C\varepsilon^{-(p-1)}) & \text{for } p=3 \end{cases} \quad \text{if } \int_{\mathbf{R}}\{f(x)+g(x)\}dx\neq 0 \tag{3.3}$$

This is the heat-like estimate in the sense that it coincides with the corresponding semilinear heat equations for which v_{tt} is neglected in (3.1). In the original paper [25], the condition on the initial data is missing, but it was expected to be natural as the critical exponent is Fujita one. Later, Kato, Takamura and Wakasa [9] proved that

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-p(p-1)/(1+2p-p^2)} & \text{for} \quad 1 < p < 2, \\ Cb(\varepsilon) & \text{for} \quad p = 2, \\ C\varepsilon^{-p(p-1)/(3-p)} & \text{for} \quad 2 < p < 3, \\ \exp(C\varepsilon^{-p(p-1)}) & \text{for} \quad p = 3 \end{cases} \qquad \text{if} \int_{\mathbf{R}} \{f(x) + g(x)\}dx = 0, \tag{3.4}$$

where $b = b(\varepsilon)$ is a positive number satisfying $\varepsilon^2 b \log(1 + b) = 1$. This is the wave-like estimate. In deed, the cases $1 < p < 2$ and $p = 3$ are the same forms as those of 3-dimensional semillinear wave equations.

In this way, it is very important to study the weighted nonlinear terms, and it is natural to extend the results to more general weighted terms than (3.2). The first simple question may go to the case of x-decay.

3.2 Spatially weighted nonlinear terms

First we consider the following case of spatial weights;

$$A(x, t) \equiv 0 \quad \text{and} \quad B(x, t) = \frac{1}{\langle x \rangle^{1+c}}, \tag{3.5}$$

where $c \in \mathbf{R}$, in (1.3). This setting was first introduced by Suzuki [22] under the supervision by Prof. M. Ohta (Tokyo Univ. of Sci. Japan), but with the assumption that the initial data is of non-compact support. Later, Kubo, Osaka and Yazici [15] improved the result and Wakasa [26] finalized it. For the compactly supported case, we have the following result.

Theorem 3.1 (Kitamura, Morisawa and Takamura [12])**.** *Assume (3.5). Then, the lifespan of a classical solution of (1.3) satisfies the following estimates.*

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-(r-1)/(1-c)} & \textit{for } c < 0, \\ \phi^{-1}(C\varepsilon^{-(r-1)}) & \textit{for } c = 0, \\ C\varepsilon^{-(r-1)} & \textit{for } c > 0 \end{cases} \qquad \textit{if} \int_{\mathbf{R}} g(x)dx \neq 0 \tag{3.6}$$

and

$$T(\varepsilon) \sim \begin{cases} C\varepsilon^{-r(r-1)/(1-rc)} & \textit{for } c < 0, \\ \;^{-1}(C\varepsilon^{-r(r-1)}) & \textit{for } c = 0, \\ C\varepsilon^{-r(r-1)} & \textit{for } c > 0 \end{cases} \qquad \textit{if} \int_{\mathbf{R}} g(x)dx = 0, \tag{3.7}$$

where ϕ^{-1} andψ^{-1} are inverse functions defined by $\phi(s) = s\log(2 + s)$ and$\psi(s) = s\log^r(2 + s)$, respectively.

Remark 3.1. *Wakasa [26] established the estimate in (3.6) with the assumption that $f \in L^\infty(\mathbf{R})$, $g \in L^1(\mathbf{R})$ and $c \geq -1$. The last condition ensures the existence of local-in-time solutions with non-compactly supported data. We also remark that $|u|^r$ in the* *nonlinear term can be replaced with $|u|^{r-1}u$ in [26] due to the fact that the positiveness of the solution can be obtained easier than the compactly supported case. We note that the same problem in higher dimensions are recently considered by Wang and Lü [28] for 3 dimensional case, and by Wang and Sun [29] for 2 dimensional case.*

The strategy of the proof of Theorem 3.1 for the existence part is similar to the one of Theorem 2.1 based on weighted L^∞ estimates of the solution. For the blow-up part, the iteration argument of the point-wise estimate is employed. The functional method like the proof of Theorem 2.1 cannot be applied to this case due to the effect of the weight.

After the work [12], one may study the counter case of

$$A(x,t) = \frac{1}{\langle x\rangle^{1+c}} \quad \text{and} \quad B(x,t) \equiv 0, \tag{3.8}$$

where $c \in \mathbf{R}$, in (1.3) with $q = 0$. For this equation, one may expect that the result is not so interesting as $|x| \sim t$ in dealing with the nonlinear term $|u_t|^p$. But the result is

Theorem 3.2 (Zhou [31], Kitamura, Morisawa and Takamura [13]). *Assume (3.8) with $q = 0$. Then, the lifespan of a classical solution of (1.3) satisfies the following estimates.*

$$\begin{aligned} &T(\varepsilon) \sim \begin{cases} C\varepsilon^{-(p-1)/(-c)} & \text{for } c < 0, \\ \exp(C\varepsilon^{-(p-1)}) & \text{for } c = 0, \end{cases} \\ &T(\varepsilon) = \infty \quad \text{for } c > 0. \end{aligned} \tag{3.9}$$

We note that Zhou [31] established the blow-up part with $c = -1$, namely non-weighted case. The proof of Theorem 3.2 is similar to the one of Theorem 3.1. The main difference between (3.5) and (3.8) is a possibility to obtain the global-in-time existence for (3.8) while there is no such a situation due to the effect of the origin in space for (3.5).

According to the result on (3.1) and Theorems 3.1 and 3.2, one may expect that it is sufficient to study the weights in the nonlinear terms by powers of $(1 + t)$ or $\langle x\rangle$ for the purpose to extend the general theory, but we have to take into account of "characteristic weights" in handling nonlinear wave equations. Let us see it in the next subsection.

3.3 Weighted nonlinear terms in the characteristic directions.

It is well-known that the wave propagates along with the characteristic directions, so that it is not sufficient to study the weights of powers by $(1 + t)$ or $\langle x\rangle$ only. Therefore, as a breakthrough to this direction, we set

$$A(x,t) \equiv 0 \quad \text{and} \quad B(x,t) = \frac{1}{\langle t + \langle x\rangle\rangle^{1+a}\langle t - \langle x\rangle\rangle^{1+b}}, \tag{3.10}$$

where $a, b \in \mathbf{R}$, in (1.3). Then, we obtained the following result.

Theorem 3.3 (Kitamura, Wakasa and Takamura [14]). *Assume (3.10). Then, the lifespan $T(\varepsilon)$ of the classical solution of (1.3) satisfies the following estimates;*

$$T(\varepsilon) = \infty \quad \text{for } a + b > 0 \text{ and } a > 0, \tag{3.11}$$

and

$$T(\varepsilon) \sim \begin{cases} \exp(C\varepsilon^{-(r-1)}) & \text{for } \begin{array}{l} a+b=0 \text{ and } a>0, \\ \text{or } a=0 \text{ and } b>0, \end{array} \\ \exp(C\varepsilon^{-(r-1)/2}) & \text{for } a=b=0, \\ C\varepsilon^{-(r-1)/(-a)} & \text{for } a<0 \text{ and } b>0, \\ \phi_1^{-1}(C\varepsilon^{-(r-1)}) & \text{for } a<0 \text{ and } b=0, \\ C\varepsilon^{-(r-1)/(-a-b)} & \text{for } a+b<0 \text{ and } b<0 \end{cases} \tag{3.12}$$

if

$$\int_{\mathbf{R}} g(x)dx \neq 0,$$

where ϕ_1^{-1} *is an inverse function defined by*

$$\phi_1(s) = s^{-a} \log(2+s). \tag{3.13}$$

On the other hand, it holds that

$$T(\varepsilon) \sim \begin{cases} \exp(C\varepsilon^{-(r-1)}) & \text{for } a=0 \text{ and } b>0, \\ \exp(C\varepsilon^{-r(r-1)}) & \text{for } a+b=0 \text{ and } a>0, \\ \exp(C\varepsilon^{-r(r-1)/(r+1)}) & \text{for } a=b=0, \\ C\varepsilon^{-(r-1)/(-a)} & \text{for } a<0 \text{ and } b>0, \\ \psi_1^{-1}(C\varepsilon^{-r(r-1)}) & \text{for } a<0 \text{ and } b=0, \\ C\varepsilon^{-r(r-1)/(-ra-b)} & \text{for } a<0 \text{ and } b<0, \\ \psi_2^{-1}(C\varepsilon^{-r(r-1)}) & \text{for } a=0 \text{ and } b<0, \\ C\varepsilon^{-r(r-1)/(-a-b)} & \text{for } a+b<0 \text{ and } a>0 \end{cases} \tag{3.14}$$

if

$$\int_{\mathbf{R}} g(x)dx = 0,$$

where ψ_1^{-1} *and* ψ_2^{-1} *are inverse functions defined by*

$$\psi_1(s) = s^{-ra} \log(2+s) \text{ and } \psi_2(s) = s^{-b} \log^{r-1}(2+s). \tag{3.15}$$

Remark 3.2. *The estimates (3.14) with* $a = r-2$ *and* $b = -1$ *coincide with (3.3) and (3.4) because* $(1+t)$ *is equivalent to* $\langle t + \langle x \rangle \rangle$ *by finite propagation speed of the wave.*

The strategy of the proof of Theorem 3.3 is also almost the same as the one of Theorem 3.1, but the weights cause many technical difficulties. The main concern of Theorem 3.3 is that we have interactions among two characteristic directions as the critical line $a+b=0$ with $b \leq 0$ and $a=0$ with $b \geq 0$ which divides ab-plane into two domains of the global-in-time existence and the blow-up in finite time.

In contrast, if

$$A(x,t) = \frac{1}{\langle t+\langle x \rangle \rangle^{1+a} \langle t-\langle x \rangle \rangle^{1+b}} \quad \text{and} \quad B(x,t) \equiv 0, \tag{3.16}$$

where $a, b \in \mathbf{R}$, in (1.3) with $q=0$, then we obtained the different critical line on ab-plane from Theorem 3.3 as follows.

Theorem 3.4 (Kitamura [11]). *Assume (3.8) with $q=0$. Then, the lifespan of a classical solution of (1.3) satisfies the following estimates.*

$$
\begin{array}{l}
T(\varepsilon) \sim \left\{
\begin{array}{l}
C\varepsilon^{-(p-1)/(-a)} \\
\qquad \text{for } a<0 \text{ and } b \geq -p, \\
C\varepsilon^{-p(p-1)/\{-p(1+a)-b\}} \\
\qquad \text{for } p(1+a)+b<0 \text{ and } b<-p, \\
\exp(C\varepsilon^{-(p-1)}) \\
\qquad \text{for } a=0 \text{ and } b \geq -p, \\
\exp(C\varepsilon^{p(p-1)}) \\
\qquad \text{for } a>0 \text{ and } p(1+a)+b=0,
\end{array}
\right. \\
T(\varepsilon)=\infty \quad \text{for } a>0 \text{ and } p(1+a)+b>0.
\end{array}
\tag{3.17}
$$

We note again that all the estimates above are established whatever the value of $\displaystyle\int_{\mathbf{R}} g(x)dx$ is, due to the nonlinear term $|u_t|^p$. The strategy of the proof of Theorem 3.4 is also almost the same as the one of Theorem 3.1.

References

[1] M. D'Abbicco, *The threshold of effective damping for semilinear wave equations*, Mathematical Methods in Applied Sciences, **38** (2015), 1032-1045.

[2] E. Belchev, M. Kepka and Z. Zhou, *Finite-time blow-up of solutions to semilinear wave equations*, Special issue dedicated to the memory of I. E. Segal. J. Funct. Anal. **190** (2002), no. 1, 233-254.

[3] W. Dai, D. Fang and C. Wang, *Global existence and lifespan for semilinear wave equations with mixed nonlinear terms*, J. Differential Equations, **267** (2019), no. 5, 3228-3354.

[4] W. Han and Y. Zhou, *Blow up for some semilinear wave equations in multi-space dimensions*, Comm. Partial Differential Equations, **39** (2014), no. 4, 651-665.

[5] K. Hidano, C. Wang and K. Yokoyama, *Combined effects of two nonlinearities in lifespan of small solutions to semi-linear wave equations*, Math. Ann., **366** (2016), no. 1-2, 667-694.

[6] T. Imai, M. Kato, H. Takamura and K. Wakasa, *The sharp lower bound of the lifespan of solutions to semilinear wave equations with low powers in two space dimensions*, Asymptotic analysis for nonlinear dispersive and wave equations, 31-53, Adv. Stud. Pure Math., **81**, Math. Soc. Japan, Tokyo, 2019.

[7] F. John, *Blow-up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math., **28** (1979), no. 1-3, 235-268.

[8] S. Katayama, *Lifespan of solutions for two space dimensional wave equations with cubic nonlinearity*, Comm. Partial Differential Equations, **26** (2001), no. 1-2, 205-232.

[9] M. Kato, H. Takamura and K. Wakasa, *The lifespan of solutions of semilinear wave equations with the scale-invariant damping in one space dimension*, Differential Integral Equations, **32** (2019), no. 11-12, 659-678.

[10] R. Kido, T. Sasaki, S. Takamatsu and H. Takamura, *The generalized combined effect for one dimensional wave equations with semilinear terms including product type*, J. Differential Equations, **403** (2024), 576-618.

[11] S. Kitamura, *Semilinear wave equations of derivative type with characteristic weights in one space dimension*, arXiv:2211.12295v2 [math.AP], to appear in Differential and Integral Equations.

[12] S. Kitamura, K. Morisawa and H. Takamura, *The lifespan of classical solutions of semilinear wave equations with spatial weights and compactly supported data in one space dimension*, J. Differential Equations, **307** (2022), 486-516.

[13] S. Kitamura, K. Morisawa and H. Takamura, *Semilinear wave equations of derivative type with spatial weights in one space dimension*, Nonlinear Analysis, RWA, **72** (2023), Paper No.103764.

[14] S. Kitamura, H. Takamura and K. Wakasa, *The lifespan estimates of classical solutions of one dimensional semilinear wave equations with characteristic weights*, J. Math. Anal. Appl. **528** (2023), Paper No. 127516.

[15] H. Kubo, A. Osaka and M. Yazici, *Global existence and blow-up for wave equations with weighted nonlinear terms in one space dimension*, Interdisciplinary Information Sciences, **19** (2013), 143-148.

[16] X. Liu and Y. Zhou, *Global nonexistence of solutions to a semilinear wave equation in the Minkowski space*, Appl. Math. Lett. **21** (2008), no. 8, 849-854.

[17] T.-T. Li (D.-Q. Li), X. Yu and Y. Zhou, *Durée de vie des solutions régulières pour les équations des ondes non linéaires unidimensionnelles* (French), C. R. Acad. Sci. Paris Sér. I Math., **312** (1991), no. 1, 103-105.

[18] T.-T. Li, X. Yu and Y. Zhou, *Life-span of classical solutions to one-dimensional nonlinear wave equations*, Chinese Ann. Math., Ser. B, **13** (1992), no. 3, 266-279.

[19] M. Liu and C. Wang, *Blow up for small-amplitude semilinear wave equations with mixed nonlinearities on asymptotically Euclidean manifolds*, J. Differential Equations, **269** (2020), no. 10, 8573-8596.

[20] K. Morisawa, T. Sasaki and H. Takamura, *The combined effect in one space dimension beyond the general theory for nonlinear wave equations*, Comm. Pure Appl. Anal., **22** (2023), no. 5, 1629-1658.

[21] K. Morisawa, T. Sasaki and H. Takamura, *Erratum to "The combined effect in one space dimension beyond the general theory for nonlinear wave equations"*, Comm. Pure Appl. Anal., **22** (2023), no. 10, 3200-3202.

[22] A. Suzuki, "Global Existence and Blow-Up of solutions to Nonlinear Wave Equations in One Space Dimension" (in Japanese), Master Thesis, Saitama University, 2010.

[23] S. Takamatsu, *Improvement of the general theory for one dimensional nonlinear wave equations related to the combined effect*, arXiv: 2308.02174v2 [math.AP].

[24] H. Takamura, *Recent developments on the lifespan estimate for classical solutions of nonlinear wave equations in one space dimension*, arXiv:2309.08843v2 [math.AP], to appear in Advanced Studies in Pure Mathematics.

[25] K. Wakasa, *The lifespan of solutions to semilinear damped wave equations in one space dimension*, Communications on Pure and Applied Analysis, **15** (2016), 1265-1283.

[26] K. Wakasa, *The lifespan of solutions to wave equations with weighted nonlinear terms in one space dimension*, Hokkaido Math. J., **46** (2017), 257-276.

[27] Y. Wakasugi, *Critical exponent for the semilinear wave equation with scale invariant damping*, Fourier analysis, 375-390, Trends Math., Birkhäuser/Springer, Cham, 2014.

[28] H. Wang and F. Lü, *The life span of classical solutions to nonlinear wave equations with weighted terms in three space dimensions*, Acta Math. Sin. (Engl. Ser.) **40** (2024), no. 12, 2984-3002.

[29] H. Wang and H. Sun, *The life span of classical solutions to nonlinear wave equations with weighted function in two space dimensions* (Chinese), Math. Appl. (Wuhan) 33 (2020), no. 3, 620-633.

[30] Y. Zhou, *Life span of classical solutions to* $u_{tt} - u_{xx} = |u|^{1+\alpha}$, Chinese Ann. Math. Ser.B, **13** (1992), no. 2, 230-243.

[31] Y. Zhou, *Blow up of solutions to the Cauchy problem for nonlinear wave equations*, Chinese Ann. Math. Ser. B, **22** (2001), no. 3, 275-280.

[32] Y. Zhou, *Blow up of solutions to the Cauchy problem for nonlinear wave equations*, preprint, Fudan University, 1992.

Schrödinger equation from Galileian point of view

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This is the abstract of my talk with the same title, which is based on my recent joint-work with Hiromichi Nakazato [10, 11].

1 Galilei space-time and Galilei group

The product vector space $\mathbb{R} \times \mathbb{R}^n$ is called the Galilei space-time, where $\mathbb{R}^n$ is regarded as the n-dimensional Euclidean space. We denote by $x = (t, \boldsymbol{x}) = (x^0, x^1, \ldots, x^n)$ a point in $\mathbb{R} \times \mathbb{R}^n$, which is also represented as

$$x = (t, \boldsymbol{x}) = \sum_{j=0}^{n} x^j \boldsymbol{e}_j = t\boldsymbol{e}_0 + \sum_{j=1}^{n} x^j \boldsymbol{e}_j,$$

where $(\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ is the standard basis of $\mathbb{R} \times \mathbb{R}^n$. We introduce a scalar product for $\mathbb{R}^n$ by

$$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{j=1}^{n} x^j y^j$$

for $\boldsymbol{x} = (x^1, \ldots, x^n),\ \boldsymbol{y} = (y^1, \ldots, y^n) \in \mathbb{R}^n$. We also use the column vector representation for a point in $\mathbb{R} \times \mathbb{R}^n$ and in $\mathbb{R}^n$ to which $(1+n) \times (1+n)$- matrices and $n \times n$-matrices apply, respectively. A mapping $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ is called a Galilei transformation if it satisfies the following three conditions:

(G1) (Affinity) $f_* := f - f(0,0) \in GL(\mathbb{R} \times \mathbb{R}^n)$.

(G2) (Space isometry) For any $t \in \mathbb{R},\ p_2 \circ f_*(t, \bullet) \in \mathrm{Iso}(\mathbb{R}^n)$,
where $p_2 : \mathbb{R} \times \mathbb{R}^n \ni (t, \boldsymbol{x}) \mapsto \boldsymbol{x} \in \mathbb{R}^n$ is the projection.

(G3) (Invariance of time intervals) For any $x = (t, \boldsymbol{x}),\ y = (s, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}^n$,

$$|p_1(f(t, \boldsymbol{x}) - f(s, \boldsymbol{y}))| = |t - s|,$$

where $p_1 : \mathbb{R} \times \mathbb{R}^n \ni (t, \boldsymbol{x}) \mapsto t \in \mathbb{R}$ is the projection.

Let $\mathscr{G}$ be the set of all Galilei transformations. Then $\mathscr{G}$ forms a group under composition of two mappings. Moreover, for any $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n,\ f \in \mathscr{G}$ if and only if there exists a unique $(R, s, \boldsymbol{y}, \boldsymbol{v}) \in O(n) \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ such that for any $x = (t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^n$

$$f(t, \boldsymbol{x}) = (t - s, R\boldsymbol{x} - t\boldsymbol{v} - \boldsymbol{y}).$$

This means that any Galilei transformation is decomposed as $f = T_y \circ G_{\boldsymbol{v}} \circ (1 \otimes R)$, where $1 \otimes R$ is the space-rotation by $R \in O(n)$

$$(1 \otimes R)(t, \boldsymbol{x}) = (t, R\boldsymbol{x}),$$

$G_{\boldsymbol{v}}$ is the pure Galilei transformation by $\boldsymbol{v} \in \mathbb{R}^n$

$$G_{\boldsymbol{v}}(t, \boldsymbol{x}) = (t, \boldsymbol{x} - t\boldsymbol{v}),$$

and T_y is the space-time translation by $y = (s, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}^n$

$$T_y(t, \boldsymbol{x}) = (t - s,\, \boldsymbol{x} - \boldsymbol{y}).$$

The group $\mathscr{G}$ is called the Galilei group. It follows from the charactenzation above that $\mathscr{G}$ is generated by $(1 \otimes R; R \in O(n)), (T_y; y \in \mathbb{R} \times \mathbb{R}^n)$, and $(G_{\boldsymbol{v}}; \boldsymbol{v} \in \mathbb{R}^n)$. The transformation group generated by $(1 \otimes R; R \in \mathrm{O}(n))$ and $(G_{\boldsymbol{v}}; \boldsymbol{v} \in \mathbb{R}^n)$ is called the homogeneous Galilei group in $\mathbb{R} \times \mathbb{R}^n$.

2 The main results

The action of Galilei group on functions $u \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$ is introduced as the corresponding pull-backs:

$$(T_y^* u)(x) = u(T_y x) = u(x - y) = u(t - s, \boldsymbol{x} - \boldsymbol{y}), \tag{1}$$
$$\big((1 \otimes R)^* u\big)(x) = u((1 \otimes R)x) = u(t, R\boldsymbol{x}), \tag{2}$$
$$(G_{\boldsymbol{v}}^* u)(x) = u(G_{\boldsymbol{v}} x) = u(t, \boldsymbol{x} - t\boldsymbol{v}). \tag{3}$$

We introduce the notion of local gauge as a multiplication of functions of modulus 1 of the form $\exp(i\theta)$ with $\theta \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R})$.

We consider linear partial differential operators in $\mathbb{R} \times \mathbb{R}^n$ of order m of the form

$$L = \sum_{j+|\boldsymbol{\alpha}| \le m} a_{j\boldsymbol{\alpha}} \partial_t^j \boldsymbol{\partial}^{\boldsymbol{\alpha}}, \tag{4}$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\ge 0}$ is a multi-index with length $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n$, $\boldsymbol{\partial}^{\boldsymbol{\alpha}} = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $\partial_j = \partial/\partial x^j$, $\partial_t = \partial/\partial t$, $a_{j\boldsymbol{\alpha}} \in \mathrm{C}(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$, and $\sum\limits_{j+|\boldsymbol{\alpha}|=m} |a_{j\boldsymbol{\alpha}}| \ne 0$.

We say that L is space-time translation invariant if and only if

$$T_y^* L u = L T_y^* u \tag{5}$$

for any $y \in \mathbb{R} \times \mathbb{R}^n$ and any $u \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$. We say that L is space-rotation invariant if and only if

$$(1 \otimes R)^* L u = L (1 \otimes R)^* u \tag{6}$$

for any $R \in \mathrm{O}(n)$ and any $u \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n;\, \mathbb{C})$. We say that L is pure Galilei invariant if and only if for any $\boldsymbol{v} \in \mathbb{R}^n$, there exists $\theta_{\boldsymbol{v}} \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R})$ such that

$$e^{i\theta_{\boldsymbol{v}}} G_{\boldsymbol{v}}^* L u = L e^{i\theta_{\boldsymbol{v}}} G_{\boldsymbol{v}}^* u \tag{7}$$

for any $u \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$. We say that L is Galilei invariant if and only if L is space-time translation invariant, space-rotation invariant, and pure Galilei invariant. We say that L is homogeneous Galilei invariant if and only if L is space-rotation invariant and pure Galilei invariant. Here we remark that an extra degree of freedom $\theta_{\boldsymbol{v}}$ is necessary because the commutativity between L and pure pullback of translations by constant velocity breakes down.

We now state our main results.

Theorem 1 *For any partial differential operator in space-time $\mathbb{R} \times \mathbb{R}^n$ of the second order of the form*

$$L = \sum_{j+|\boldsymbol{\alpha}| \le 2} a_{j\boldsymbol{\alpha}} \partial_t^j \boldsymbol{\partial}^{\boldsymbol{\alpha}}, \tag{8}$$

with $a_{j\boldsymbol{\alpha}} \in \mathrm{C}(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$ and $\sum\limits_{j+|\boldsymbol{\alpha}|=2} |a_{j\boldsymbol{\alpha}}| \ne 0$, the following three statements are equivalent.

(1) L is Galilei invariant. Namely,

(i) For any $y \in \mathbb{R} \times \mathbb{R}^n$ and any $u \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$, $T_y^* Lu = LT_y^* u$.

(ii) For any $R \in \mathrm{O}(n)$ and any $u \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$, $(1 \otimes R)^* u = L(1 \otimes R)^* u$.

(iii) For any $\boldsymbol{v} \in \mathbb{R}^n$, there exists $\theta_{\boldsymbol{v}} \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R})$ such that $e^{i\theta_{\boldsymbol{v}}} G_{\boldsymbol{v}}^* Lu = Le^{i\theta_{\boldsymbol{v}}} G_{\boldsymbol{v}}^* u$ for any $u \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$.

(2) All coefficients of L are constants and L is homogeneous Galilei invariant. Namely,

(i) For any $(j, \boldsymbol{\alpha}) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^n$ with $j + |\boldsymbol{\alpha}| \leq 2$ and any $x \in \mathbb{R} \times \mathbb{R}^n$, $a_{j\boldsymbol{\alpha}}(x) = a_{j\boldsymbol{\alpha}}(0)$.

(ii) For any $R \in \mathrm{O}(n)$ and any $u \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$, $(1 \otimes R)^* Lu = L(1 \otimes R)^* u$.

(iii) For any $\boldsymbol{v} \in \mathbb{R}^n$, there exists $\theta_{\boldsymbol{v}} \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R})$ such that $e^{i\theta_{\boldsymbol{v}}} G_{\boldsymbol{v}}^* Lu = Le^{i\theta_{\boldsymbol{v}}} G_{\boldsymbol{v}}^* u$ for any $u \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$.

(3) For any $(j, \boldsymbol{\alpha})$ with $j + |\boldsymbol{\alpha}| \leq 2$, $a_{j\boldsymbol{\alpha}}$ is a complex constant and zero except $a_{0\mathbf{0}}, a_{1\mathbf{0}}, a_{0,2\boldsymbol{e}_1}, \ldots, a_{0,2\boldsymbol{e}_n}$, where $a_{0,2\boldsymbol{e}_1} = \cdots = a_{0,2\boldsymbol{e}_n} \neq 0$ hold. Let $-\alpha \in \mathbb{C}\backslash\{0\}$ be the common value: $-\alpha = a_{0,2\boldsymbol{e}_1} = \cdots = a_{0,2\boldsymbol{e}_n} \neq 0$. Then $ia_{1\mathbf{0}}/\alpha \in \mathbb{R}$, so that if $(\lambda, \beta) \in \mathbb{R} \times \mathbb{C}$ are given by $\lambda = -\frac{ia_{1\mathbf{0}}}{2\alpha}$ and $\beta = a_{0\mathbf{0}}$, L is represented as

$$L = \alpha(2i\lambda\partial_t + \triangle) + \beta, \tag{9}$$

where $\triangle = \nabla^2 = \partial_1^2 + \cdots \partial_n^2$ is the Laplacian in $\mathbb{R}^n$. The phase function $\theta_{\boldsymbol{v}}$ in (1)(iii) and (2)(iii)) is given by

$$\theta_{\boldsymbol{v}}(t, \boldsymbol{x}) = \theta_{\boldsymbol{v}}(0, \mathbf{0}) + \lambda \boldsymbol{v} \cdot \boldsymbol{x} - \frac{\lambda}{2} t |\boldsymbol{v}|^2 \quad \text{if} \quad \lambda \neq 0, \tag{10}$$

$$\theta_{\boldsymbol{v}}(t, \boldsymbol{x}) = \theta_{\boldsymbol{v}}(t, \mathbf{0}) \quad \text{if} \quad \lambda = 0. \tag{11}$$

Remark 1 *In the case $\lambda = 0$ in (3), we have $L = \alpha\triangle + \beta$ and $\theta_{\boldsymbol{v}}(t, \boldsymbol{x}) = \theta_{\boldsymbol{v}}(t, \mathbf{0})$ so that $e^{i\theta_{\boldsymbol{v}}}$, $G_{\boldsymbol{v}}^*$, and L are commutative on $\mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$.*

Remark 2 *In the case $\lambda \neq 0$ in (3), the constant β is essentially removable in the sense that*

$$\exp\Big(\frac{\beta}{2i\alpha\lambda} t\Big) L \exp\Big(\frac{-\beta}{2i\alpha\lambda} t\Big) u = \alpha(2i\lambda\partial_t + \triangle) u \tag{12}$$

for all $u \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$. If $\beta \in \mathbb{R}$, the mappings $u \mapsto \exp(\frac{\pm\beta}{2i\alpha\lambda}) u$ are global gauge transformations and L and $\alpha(2i\lambda\partial_t + \triangle)$ are gauge equivalent.

Remark 3 *The case $\lambda = 1$ is regarded as the standard choice since the essential factor $2i\partial_t + \triangle$ is the time-dependent free Schrödinger operator in the scale where the Planck constant $\hbar$ and the mass of the free particle are normalized. Moreover, this choice $\lambda = 1$ with another normalization $\theta_{\boldsymbol{v}}(0, \mathbf{0}) = 0$ yields*

$$\Big(e^{i\theta_{\boldsymbol{v}}} G_{\boldsymbol{v}}^* u\Big)(t, \boldsymbol{x}) = \exp\big(i(\boldsymbol{v} \cdot \boldsymbol{x} - \frac{t}{2} |\boldsymbol{v}|^2)\big) u(t, \boldsymbol{x} - t\boldsymbol{v}). \tag{13}$$

The transformation $u \mapsto e^{i\theta_{\boldsymbol{v}}} G_{\boldsymbol{v}}^* u$ in (13) is actually called "the Galilei transformation" in [4, 7].

If we assume that $\theta_{\boldsymbol{v}}$ has a specific form as in (10), we reformulate Theorem 1 to cover higher order partial differential operators as follows.

Theorem 2 *Let $m \in \mathbb{Z}_{>0}$ and $\lambda \in \mathbb{R}\backslash\{0\}$. For any $\boldsymbol{v} \in \mathbb{R}$, let $\theta_{\boldsymbol{v}} \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R})$ be defined as*

$$\theta_{\boldsymbol{v}}(t, \boldsymbol{x}) = c + \lambda \boldsymbol{v} \cdot \boldsymbol{x} - \frac{\lambda}{2} t |\boldsymbol{v}|^2, \tag{14}$$

where $c = \theta_{\boldsymbol{v}}(0, \boldsymbol{0}) \in \mathbb{C}$. Then for any linear partial differential operator in space-time $\mathbb{R} \times \mathbb{R}^n$ of order m of the form

$$L = \sum_{j+|\boldsymbol{\alpha}| \leq m} a_{j\boldsymbol{\alpha}} \partial_t^j \boldsymbol{\partial}^{\boldsymbol{\alpha}} \tag{15}$$

with $a_{j\boldsymbol{\alpha}} \in \mathrm{C}(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$ and $\sum_{j+|\boldsymbol{\alpha}|=m} |a_{j\boldsymbol{\alpha}}| \neq 0$, the following three statements are equivalent.

(1) L is Galilei invariant in the following sense.

(i) For any $y \in \mathbb{R} \times \mathbb{R}^n$ and any $u \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$, $T_y^* L u = L T_y^* u$.

(ii) For any $R \in \mathrm{O}(n)$ and any $u \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$, $(1 \otimes R)^* u = L (1 \otimes R)^* u$.

(iii) For any $\boldsymbol{v} \in \mathbb{R}^n$ and any $u \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$, $e^{i\theta_{\boldsymbol{v}}} G_{\boldsymbol{v}}^* L u = L e^{i\theta_{\boldsymbol{v}}} G_{\boldsymbol{v}}^* u$, where $\theta_{\boldsymbol{v}}$ is defined in (14).

(2) All coefficients of L are constants and L is homogeneous Galilei invariant in the following sense.

(i) For any $(j, \boldsymbol{\alpha}) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^n$ with $j + |\boldsymbol{\alpha}| \leq m$ and any $x \in \mathbb{R} \times \mathbb{R}^n$, $a_{j\boldsymbol{\alpha}}(x) = a_{j\boldsymbol{\alpha}}(0)$.

(ii) For any $R \in \mathrm{O}(n)$ and any $u \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$, $(1 \otimes R)^* L u = L (1 \otimes R)^* u$.

(iii) For any $\boldsymbol{v} \in \mathbb{R}^n$ and any $u \in \mathrm{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$, $e^{i\theta_{\boldsymbol{v}}} G_{\boldsymbol{v}}^* L u = L e^{i\theta_{\boldsymbol{v}}} G_{\boldsymbol{v}}^* u$, where $\theta_{\boldsymbol{v}}$ is defined in (14).

(3) m is an even integer and L is given by a polynomial in $2i\lambda\partial_t + \triangle$ of order $m/2$. Namely, there exists $(a_j; j \in \{0, \ldots, m/2\}) \subset \mathbb{C}$ such that

$$L = \sum_{j=0}^{m/2} a_j (2i\lambda\partial_t + \triangle)^j \quad \text{and} \quad a_{m/2} \neq 0.$$

Theorem 1 shows that the Galilei invariance with local gauge determines the structure of the space of linear partial differential operators of the second order that is exclusively provided with the free Schrödinger operator and the formulation of local gauge as well. Theorem 2 shows that a natural generalization of Theorem 1 is possible for higher-order operators if we fix the local gauge structure. The main results of this article show that the time-dependent free Schrödinger equation is derived from a simple observation and explicit calculations with the action of the Galilei group upon scalar fields up to local gauge, which is independent of the physical convention of substitution of the energy and momentum by the time and space derivatives, respectively. Our characterization of the time-dependent free Schrödinger equation in terms of the Galilei group with local gauge supports the foundation of Quantum Mechanics on the basis of the invariance naturally arising in Classical Mechanics.

This approach is different from the previous studies on the derivation of the free Schrödinger equation from the unitary irreducible representation of the Galilei group [2, 3, 5, 6, 8, 9, 12].

References

[1] A. Arai, “Symmetry in Physics,” Nippon Hyoron Publ., 2008 (in Japanese).

[2] V. Bargmann, On unitary ray representation of continuous groups, Ann. of Math. **59**(1954), 1–46.

[3] A. Barut, Conformal group $\to$ Schrödinger group $\to$ dynamical group, Helv. Phys. Acta. **46**(1973), 493–503.

[4] J. Ginibre, “Introduction aux Équations de Schröinger non Linéaires,” Paris Onze Édition L 161, 1998.

[5] M. Hamermesh, Galilei invariance and Schrödinger equation, Ann. Phys. **9**(1960), 518–521.

[6] J.-M. Levy-Leblond, Galilei group and nonrelativistic quantum mechanics, J. Math, Phys. **4**(1963), 776–788.

[7] F. Linares and G. Ponce, “Introduction to Nonlinear Dispersive Equations, 2nd ed.,” Universitext, Springer, 2014.

[8] Z. E. Musielak and J. L. Fry, Physical theories in Galilean space-time and the origin of Schrödinger like equations, Aun. Phys. **324** (2009), 236–308

[9] Z. E. Musielak and J. L. Fry, General dynamical equations for free particles and their Galilean invariance, Int. J. Theor. Phys. **48**(2009), 1194–1202.

[10] H. Nakazato and T. Ozawa, Characterisation of D'Alembertian by the Poincaré invariance, Proceedings of the conference “Critical Phenomena in Nonlinear Partial Differential Equations, Harmonic Analysis, and Functional Inequalities,” Springer (in press).

[11] H. Nakazato and T. Ozawa, Characterization of the time-dependent free Schrödinger operator by the Galilei invariance, ICIAM Proceedings, Springer (in press)

[12] M. Roncadelli, A simplified approach to the Schrödinger quation, Nuovo Cimento, **50A**(1979), 509–514.

[13] T. Nomura, “Spherical Harmonics and Group Representations,” Nippon Hyoron Pull, 2018 (in Japanese).

[14] N. Shimakura, “Elliplic Partial Differential Operators,” Kinokuniya Publ., 1978 (in Japanese).

Lipschitz regularity for manifold-constrained ROF elliptic systems

Salvador Moll *

Joint work with Esther Cabezas-Rivas and Vicent Pallardó-Julià

1 The ROF model

The Rudin-Osher-Fatemi (ROF) model for denoising [1] is one of the most widely used models in image processing. Roughly summarized, and using the characterization given in [2], it consists in the following inverse problem:

Given an open bounded Lipschitz set $\Omega \subset \mathbb{R}^m$ and an observed noisy grayscale image $f : \Omega \to \mathbb{R}$, assume that $f = u + n \in L^2(\Omega)$ with $u : \Omega \to \mathbb{R}$ being the true image and n a Gaussian noise, one wants to recover u as a minimizer of the following energy functional:

$$u \in L^2(\Omega) \longmapsto \int_\Omega |\nabla u|\, dx + \frac{\lambda}{2} \int_\Omega |u - f|^2\, dx.$$

The first term in the functional acts as a regularizer on the image and its role is to reduce the given noise, On the other hand, the second one is a fidelity term, whose effect is that the minimizer remains close to f in the L^2 norm. λ is a parameter that balances the effect of both terms: a small choice of λ produces that minimizers are not far from a constant (the average of f) while a large value of it produces that minimizers retain more details of the given f, even its noise.

Note that the natural effective domain of the above functional is the intersection of the space of Sobolev functions $W^{1,1}(\Omega)$ with $L^2(\Omega)$. However, $W^{1,1}(\Omega)$ is not the appropriate space to look for minimizers, as bounded subsets of $W^{1,1}(\Omega)$

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need not be relatively compact with respect to the weak topology. Therefore, one needs to replace the functional $u \mapsto \int_\Omega |\nabla u|\, dx$ with its relaxation with respect to to the L^1-convergence; i.e. the Total Variation functional:

$$TV(u) = |Du|(\Omega) := \inf\left\{ \liminf_{k\to+\infty} \int_\Omega |\nabla u_k|\, dx : u_k \in W^{1,1}(\Omega),\ u_k \stackrel{L^1(\Omega)}{\longrightarrow} u \right\}.$$

The effective domain of the $TV-$functional is the Banach space of Bounded Variation (BV) functions, denoted by $BV(\Omega)$. We assume the standard notation and results concerning BV functions as those in [3].

The Total Variation functional satisfies the following properties:

- It is a convex functional in $L^2(\Omega)$.
- It is lower semicontinuous with respect to the strong convergence in $L^1(\Omega)$.
- Bounded subsets are precompact with respect to the weak* topology in $BV(\Omega)$.
- Functions in $BV(\Omega)$ can have discontinuities along a countable union of oriented hypersurfaces (the jump set) with well defined traces on both sides. This fact is particularly interesting in image processing to avoid blurring of the minimizers. This phenomenon is known to happen when looking for minimizers of similar energy functionals in the Sobolev spaces $W^{1,p}(\Omega)$, for $1 < p$.

Finally, the ROF model reads as follows:

$$\min_{u\in L^2(\Omega)} \ \mathrm{ROF}(u) : \begin{cases} |Du|(\Omega) + \dfrac{\lambda}{2}\displaystyle\int_\Omega |u-f|^2\, dx & \text{if } u \in BV(\Omega)\cap L^2(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{1}$$

Using the properties of the $TV-$ functional and the strict convexity of the fidelity term and following the standard method in the Calculus of Variations, it is easy to obtain existence and uniqueness of minimizers. Moreover, the minimizer can be characterized in terms of the Euler-Lagrange PDE associated to the ROF functional:

$$u - f \in -\partial TV(u). \tag{2}$$

Here ∂TV is the subdifferential in $L^2(\Omega)$ of the convex functional TV, which was completely characterized in [4] in a general setting. In case that $u \in W^{1,\infty}(\Omega)$, then (2) can be rephrased as

'there exists

$$z \in \frac{\nabla u}{|\nabla u|} := \begin{cases} \dfrac{\nabla u}{|\nabla u|} & \text{if } \nabla u \neq 0 \\ B(0,1) \subset \mathbb{R}^m & \text{if } \nabla u = 0, \end{cases} \tag{3}$$

with $\operatorname{div} z \in L^2(\Omega)$ such that

$$\begin{cases} u - f = \operatorname{div} z & \text{in } \Omega \\ [z, \nu] = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

with ν being the outer unit normal to Ω and $[z,\nu]$ the weak trace of the normal component of z at $\partial\Omega$'.

It turns out that Lipschitz regularity if the best possible regularity for minimizers as the following example shows:

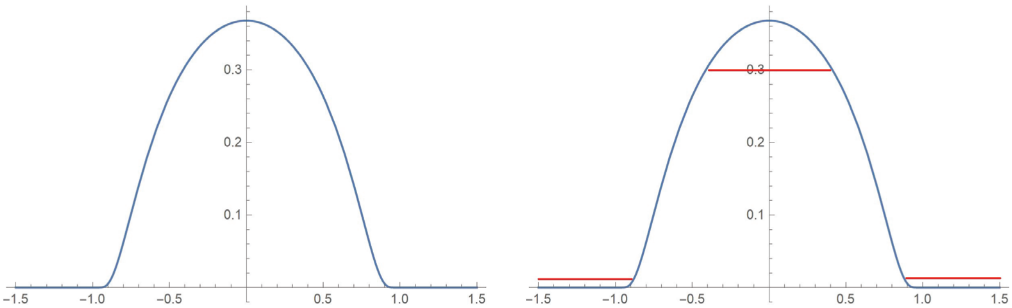

Figure 1: Left: $f \in C^\infty(]-1.5, 1.5[)$. Right: minimizer $u \in W^{1,\infty}(]-1.5, 1.5[)$.

In fact, the following result which generalizes previous results in [5] holds:

Theorem 1 ([6]). *If $f \in W^{1,\infty}(\Omega)$, then $u \in W^{1,\infty}(\Omega)$. Moreover, there exists $C > 0$ such that*

$$\|u\|_{L^\infty(B(x_0,r)} \leq \left(\|\nabla f\|_{L^\infty(B(x_0,2R))} + \frac{C}{R^2} \right),$$

if f is Lipchitz in $B(x_0, 2R) \subset \Omega$.

2 Need for manifold-constrained models.

Although the ROF model for denoising is nowadays fully understood, it has been considered in different situations other than it was designed for. In particular, in image processing, or even further, in data processing, it has become quite popular to model the objects being studied as manifold valued functions; i.e $f,u:\Omega\mapsto(\mathscr{N},h)$ with (N,h) being a Riemannian manifold (see e.g [7] or [8]). In particular, the following target manifolds have been considered in the applied literature:

- The unit sphere in dimension 3, $\mathbb{S}^2$, in the case of RGB color images or optical flow.
- $\mathbb{R}\times\mathbb{S}^1$ for LCh-color images.
- The space of movements in $\mathbb{R}^3$, $SE(3)$, for object tracking.
- The space of positive symmetric $3x3$ matrices $\mathbb{P}(3)^+$, for Diffusion Tensor Images (DTI).
- Hyperbolic spaces for Deep Neural Networks.

3 Our contribution.

In this talk, I will present some recent advances obtained in [9] about the following far-reaching generalization of the ROF model:

1. On the target space: $(\mathscr{N},h)$ an $n-$dimensional complete Riemannian manifold.
2. On the domain: (Ω,g) is a smooth compact surface such that $\partial\Omega\neq\emptyset$.

In this setting, we consider the following energy functional in $L^2(\Omega;\mathscr{N})$:

$$\mathscr{R}OF(u):=\begin{cases}\mathscr{T}\mathscr{V}(u)+\dfrac{\lambda}{2}\displaystyle\int_\Omega d_h^2(u,f)\,d\mu_g & \text{if } u\in BV(\Omega;R^N)\cap L^2(\Omega;\mathscr{N})\\ +\infty & \text{otherwise.}\end{cases}$$

where, by Nash theorem, we have considered that $\mathscr{N}$ isometrically embedded into some Euclidean space $\mathbb{R}^N$, $d\mu_g$ is the volume element corresponding to g, d_h is the intrinsic distance in $\mathscr{N}$ and

$$\mathscr{T}\mathscr{V}(u)=\inf\left\{\liminf_{k\to+\infty}\int_\Omega|\nabla u_k|\,d\mu_g : u_k\in W^{1,1}(\Omega;\mathscr{N}),\ u_k\overset{L^1(\Omega;\mathscr{N})}{\longrightarrow}u\right\}.$$

It turns out that $\mathscr{T}\mathscr{V}$ is not a geodesically convex functional in general. Moreover, for the distance term to be convex, one needs to impose both curvature and topological restrictions on the target manifold.

With some extra hypotheses on the range of f in $\mathscr{N}$, we can obtain existence of minimizers and their characterization in PDE terms similar to that in (3) and (4) for the system of Euler-Lagrange equations. In the case that $\mathscr{N}$ is non-positively curved, we also obtain uniqueness of minimizers as well as a regularity result analogous to that of Theorem 1.

Our results can also be viewed as an extension of the regularity theory for p-harmonic maps to the challenging case $p = 1$ building upon foundational works by Eells–Sampson [10] and Schoen–Uhlenbeck [11].

A key aspect of our approach is the intricate interplay between geometric and analytical techniques; we exploit both the intrinsic and the extrinsic viewpoints. In this sense, an intrinsic Cacciopoli estimate as in [12] is crucial.

Finally, we also establish several regularity results of independent interest: Hölder and Lipschitz continuity in one-dimensional settings (relevant to signal denoising) and Lipschitz regularity for a perturbed model motivated by fluid mechanics.

References

[1] L.I. Rudin, S. Osher, E. Fatemi, Nonlinear total variation based noise removal algorithms, *Phys. D* **60** (1992), no.1-4, 259–268.

[2] A. Chambolle, P. Lions, Image recovery via total variation minimization and related problems, *Numer. Math.* **76** (1997), 167–188.

[3] L. Ambrosio, N. Fusco, D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, New York, 2000.

[4] F. Andreu-Vaillo, C. Ballester, V. Caselles, J. Mazón, Minimizing total variation flow, *Differential Integral Equation* **14** (2001), no. 3, 321–360.

[5] V. Caselles, A. Chambolle, M. Novaga, Regularity for solutions of the total variation denoising problem. *Rev. Mat. Iberoam.* **27** (2011), no. 1, 233–252.

[6] A. Porretta, On the regularity of the total variation minimizers. *Commun. Contemp. Math.* **23** (2021), no.1, paper no. 1950082, 17 pp.

[7] R. Bergmann, F. Laus, J. Persch, G. Steidl, Recent advances in denoising of manifold-valued images. *Handb. Numer. Anal.*, **20**. Elsevier/North-Holland, Amsterdam, 2019, 553–578.

[8] A. Weinmann, L. Demaret, M. Storath, Total variation regularization for manifold-valued data. *SIAM J. Imaging Sci.* **7** (2014), no. 4, 2226–2257.

[9] E. Cabezas-Rivas, S. Moll, V. Pallardó-Julià, Lipschitz regularity for manifold-constrained ROF elliptic systems. https://arxiv.org/abs/2411.19166

[10] J. Eells, J.H. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.* **86** (1964), 109–160.

[11] R. Schoen, K. Uhlenbeck, A regularity theory for harmonic maps. *J. Differential Geometry* **17** (1982), no. 2, 307–335.

[12] E. Cabezas-Rivas, S. Moll, V. Pallardó-Julià, Partial regularity for manifold constrained quasilinear elliptic systems. *Nonlinear Analysis* **249** (2024), 113643

Optimal Control of Grain Boundary Dynamics with State-Dependent Mobility *

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Abstract. In this talk, we address an optimal control problem governed by a state-system of parabolic partial differential equations. The state-system is based on a mathematical model of grain boundary motion involving heat exchange. One of the key features of the state-system is that the mobility depends on the physical state of crystalline orientation order. Such state-dependent settings have a major obstacle to establishing uniqueness of solutions. Actually, in most previous studies, the mobility was assumed to be effectively constant in order to ensure uniqueness. However, for state-systems without heat exchange, recent research has shown that it is possible to guarantee complete well-posedness, including uniqueness, even under state-dependent setting of mobility. In view of these background, we will discuss about the well-posedness of the state-system and the existence and necessary conditions for optimal control, under a realistic setting where temperature is indirectly controlled via heat sources.

Introduction

Let $\Omega_T := (0,T) \times \Omega$ be a cylindrical domain, where $(0,T)$ is a bounded time interval with $0 < T < \infty$, and Ω is a spatial domain in at most three dimensions. We also denote by $\Gamma_T := (0,T) \times \Gamma$ the lateral surface of the cylinder Ω_T, i.e., the product of the time interval $(0,T)$ and the sufficiently smooth boundary $\Gamma := \partial\Omega$, which is equipped with the unit outward normal n_Γ.

In this talk, we consider a class of optimal control problems, denoted by $(\mathrm{OP})_\varepsilon$ for $\varepsilon \in [0,1]$, formulated on the following Hilbert spaces $H := L^2(\Omega)$ and $\mathscr{H} := L^2(0,T;H)$, and stated as follows.

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Keywords: Optimal control, state-dependent mobility, grain boundary motion, heat exchange

(OP)$_\varepsilon$. To find a function $f^* \in \mathscr{H}$, called *optimal control,* which minimizes the following *cost functional:*

$$\mathcal{J}_\varepsilon : f \in \mathscr{H} \mapsto \mathcal{J}_\varepsilon(f) := \frac{1}{2}\int_0^T |f(t)|_H^2 \, dt + \frac{1}{2}\int_0^T |(u - u_{\mathrm{ad}})(t)|_{L^2(\Omega)}^2 \, dt$$
$$+ \frac{1}{2}\int_0^T |(\eta - \eta_{\mathrm{ad}})(t)|_H^2 \, dt + \int_0^T |(\theta - \theta_{\mathrm{ad}})(t)|_H^2 \, dt \in [0, \infty),$$

where $[u, \eta, \theta] \in [\mathscr{H}]^3$ is a unique solution to the following parabolic system $(\mathrm{S})_\varepsilon := \{(0.1),(0.2),(0.3)\}$:

$$\begin{cases} \partial_t(u - L\eta) - \Delta u = f(t,x), \ (t,x) \in \Omega_T, \\ \nabla u \cdot n_\Gamma = 0 \text{ on } \Gamma_T, \\ u(0,x) = u_0(x), \ x \in \Omega. \end{cases} \tag{0.1}$$

$$\begin{cases} \partial_t \eta - \Delta \eta + g(\eta) + \alpha'(\eta)\sqrt{\varepsilon^2 + |\nabla\theta|^2} + u = 0 \text{ in } \Omega_T, \\ \nabla \eta \cdot n_\Gamma = 0 \text{ on } \Gamma_T, \\ \eta(0,x) = \eta_0(x), \ x \in \Omega. \end{cases} \tag{0.2}$$

$$\begin{cases} \alpha_0(\eta)\partial_t\theta - \mathrm{div}\left(\alpha(\eta)\dfrac{\nabla\theta}{\sqrt{\varepsilon^2 + |\nabla\theta|^2}} + \kappa\nabla\theta\right) = 0 \text{ in } \Omega_T, \\ \left(\alpha(\eta)\frac{\nabla\theta}{\sqrt{\varepsilon^2+|\nabla\theta|^2}} + \kappa\nabla\theta\right) \cdot n_\Gamma = 0 \text{ on } \Gamma_T, \\ \theta(0,x) = \theta_0(x), \ x \in \Omega. \end{cases} \tag{0.3}$$

The state-system $(\mathrm{S})_\varepsilon$ is based on the *WKLC system,* proposed by Warren–Kobayashi–Lobkovsky–Carter [17], which is formulated as a coupled system consisting of the initial-boundary value problem for the heat equation (0.1), and that for the phase-field model of grain boundary motion, known as the *KWC model* (cf. [11, 12]). The systems $(\mathrm{OP})_\varepsilon$ and $(\mathrm{S})_\varepsilon$ are broadly classified into a smooth setting for $\varepsilon > 0$ and a nonsmooth setting for $\varepsilon = 0$. The nonsmooth setting with $\varepsilon = 0$ is regarded as physically faithful, where the singular diffusion flux $\alpha(\eta)\frac{\nabla\theta}{|\nabla\theta|}$ appearing in (0.3) is considered a suitable mathematical expression for capturing facet structures within polycrystalline materials.

In this context, the unknown $u = u(t,x)$ represents the relative temperature, while the unknowns $\eta = \eta(t,x)$ and $\theta = \theta(t,x)$ represent the crystalline orientation order and the orientation angle, respectively, in a polycrystalline body. The function $f = f(t,x)$ denotes the heat source, and in the problem $(\mathrm{OP})_\varepsilon$, it serves as a form of heat control for grain boundary formation. The functions $u_0 = u_0(x)$, $\eta_0 = \eta_0(x)$, and $\theta_0 = \theta_0(x)$ are the initial data, while $u_{\mathrm{ad}} = u_{\mathrm{ad}}(t,x)$, $\eta_{\mathrm{ad}} = \eta_{\mathrm{ad}}(t,x)$, and $\theta_{\mathrm{ad}} = \theta_{\mathrm{ad}}(t,x)$ are fixed functions representing the admissible target profiles for u, η, and θ, respectively. The functions $\alpha_0 = \alpha_0(\eta)$ and $\alpha = \alpha(\eta)$ represent the mobilities that act as driving forces of grain boundary motion. The function $g = g(\eta)$ is a Lipschitz perturbation for η. $L > 0$ is a constant of latent heat, and $\kappa > 0$ is a fixed diffusion constant for θ.

The novelty of this study lies in the following aspects:

♯1) the coupling structure of the state-system $(\mathrm{S})_\varepsilon$ that incorporates the heat exchange term given in (0.1);

♯2) the setting in which the mobility α_0 in (0.3) depends on the unknown variable η, i.e., the physical state of the crystalline orientation order.

In particular, the state-dependent feature described in ♯2) has been a major obstacle to the uniqueness of the state-system, and even in the study of the KWC model, the uniqueness issue has remained one of the long-standing open problems, except for one-dimensional setting of Ω (cf. [8, Theorem 2.2]). Accordingly, in the optimization problems associated with the KWC model, most previous results (cf. [3, 4, 18]) have been obtained under the assumption that α_0 is effectively constant, in order to ensure the uniqueness of the state-system.

However, in recent years, the pseudo-parabolic version of the KWC system has been actively studied as one of the possible grain boundary motion models with guaranteed uniqueness (cf. [1, 5, 6]). Indeed, several previous works on the pseudo-parabolic KWC system have reported results on the well-posedness (cf. [6]), including the uniqueness, and the associated optimal control problems (cf. [5]) under the setting of state-dependent mobility as in ♯2). Furthermore, this line of research has specified sufficient conditions for the uniqueness of solutions. These uniqueness criteria provide a prospect for resolving the long-standing open problem in the original parabolic KWC system, and for establishing a corresponding optimal control theory under the same setting ♯2) (cf. [13]).

In view of these background, we now set our goal to establish a mathematical control theory based on the well-posedness of the WKLC system, under more practical settings as described in ♯1) and ♯2). The main results of this talk are discussed in accordance with the following content.

§1. Assumptions and notations.

§2. Results for the well-posedness of the state-systems.

Theorem 1: Existence for $(\mathrm{S})_\varepsilon$.

Theorem 2: Regularity result for $(\mathrm{S})_\varepsilon$.

Theorem 3: Uniqueness for $(\mathrm{S})_\varepsilon$.

Theorem 4: Continuous-dependence for $(\mathrm{S})_\varepsilon$.

§3. Results for the optimal control problems.

Theorem 5: Qualitative property of $(\mathrm{OP})_\varepsilon$.

Theorem 6: Necessary condition of optimality in the smooth case.

Theorem 7: Limiting condition of optimality to the nonsmooth case.

1 Assumptions and notations

Throughout this talk, let $T > 0$ be a fixed finite time, and let $N \in \{2, 3\}$ be a fixed dimension. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, which has a smooth boundary $\Gamma := \partial\Omega$ with the unit outer normal $n_\Gamma : \Gamma \longrightarrow \mathbb{S}^{N-1}$. Also, we let $\Omega_T := (0, T) \times \Omega$ and $\Gamma_T := (0, T) \times \gamma$, and set the base spaces as follows.

$$H := L^2(\Omega),\ V := H^1(\Omega),\ W_0 := \left\{\ w \in H^2(\Omega) \,\middle|\, \nabla w \cdot n_\Gamma = 0 \text{ in } H^{\frac{1}{2}}(\Gamma)\ \right\},$$
$$\mathscr{H} := L^2(0, T; H), \text{ and } \mathscr{V} := L^2(0, T; V).$$

Especially, we use the space W_0 as the domain of the Laplacian $\mathit{\Delta}_0 : w \in W_0 \longrightarrow \mathit{\Delta} w \in L^2(\Omega)$ subject to the zero-Neumann boundary condition.

Based on these notations, the main results of this talk are discussed under the following assumptions.

(A0) Let $L > 0$ and $\kappa > 0$ be fixed finite constants, and let $\varepsilon \in [0, 1]$ be a constant parameter that classify the optimal control problem $(\mathrm{OP})_\varepsilon$ with the state-system $(\mathrm{S})_\varepsilon$, in the smooth case for $\varepsilon > 0$ and the nonsmooth one for $\varepsilon = 0$. Additionally, let $[u_{\mathrm{ad}}, \eta_{\mathrm{ad}}, \theta_{\mathrm{ad}}] \in [\mathscr{H}]^3$ be the triplet of admissible target profiles in the optimal control problem $(\mathrm{OP})_\varepsilon$.

(A1) Let $\alpha_0 \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ and $\alpha \in C^2(\mathbb{R}) \cap C^{1,1}(\mathbb{R})$ be fixed functions such that α is convex, and $\delta_* := \inf \alpha_0(\mathbb{R}) \cup \alpha(\mathbb{R}) > 0$.

(A2) Let $g \in C^2(\mathbb{R}) \cap C^{0,1}(\mathbb{R})$ be a fixed function that has a nonnegative potential $G \in C^3(\mathbb{R}) \cap C^{1,1}(\mathbb{R})$, i.e. $G' = g$ on $\mathbb{R}$.

(A3) Let $[u_0, \eta_0, \theta_0] \in [V]^3$ be the fixed initial data.

In addition, for simplify of description, we denote by γ_ε a Lipschitz continuous function on $\mathbb{R}^N$, defined as:

$$\gamma_\varepsilon : \varpi \in \mathbb{R}^N \mapsto \gamma_\varepsilon(\varpi) := \sqrt{\varepsilon^2 + |\varpi|^2}, \text{ for } \varepsilon \in [0, 1].$$

Remark 1. In the smooth case for $\varepsilon > 0$, the subdifferential $\partial\gamma_\varepsilon$ can be identified with the standard gradient $\nabla\gamma_\varepsilon$, i.e. it coincides with the following single-valued function:

$$\partial\gamma_\varepsilon(\varpi) = \{\nabla\gamma_\varepsilon(\varpi)\} = \left\{ \frac{\varpi}{\sqrt{\varepsilon^2 + |\varpi|^2}} \right\} \text{ (singleton) in } \mathbb{R},$$
$$\text{for all } \varpi \in \mathbb{R}^N \text{ and } \varepsilon > 0.$$

In contrast, the subdifferential $\partial\gamma_0$ corresponding to the nonsmooth case for $\varepsilon = 0$ is characterized as the following set-valued function:

$$\partial\gamma_\varepsilon(\varpi) = \left\{\ \varpi^* \in \mathbb{R}^N \,\middle|\, \begin{array}{l} |\varpi^*| \le 1, \text{ and} \\ \varpi^* \cdot \varpi = \gamma_0(\varpi) = |\varpi| \end{array} \right\}$$

Finally, we introduce a state-dependent convex function defined as follows.

Definition 1. For every $\varepsilon \in [0,1]$ and $\eta \in V$, we define a proper l.s.c. and convex function $\Phi_\varepsilon(\eta;\,\cdot\,) : H \longrightarrow [0,\infty]$, depending on ε and η (state), as follows.

$$\Phi_\varepsilon(\eta;\,\cdot\,) : \theta \in H \mapsto \Phi_\varepsilon(\eta;\theta) := \begin{cases} \displaystyle\int_\Omega \alpha(\eta)\gamma_\varepsilon(\nabla\theta)\,dx + \frac{\kappa}{2}\int_\Omega |\nabla\theta|^2\,dx, \\ \qquad \text{if } \theta \in H^1(\Omega), \\ \infty \quad \text{otherwise.} \end{cases}$$

Remark 2. Let us take arbitrary $\varepsilon \in [0,1]$ and $\eta \in V$. Then, the effective domain $D(\Phi_\varepsilon(\eta;\,\cdot\,))$ of the convex function $\Phi_\varepsilon(\eta;\,\cdot\,)$ coincides with the space V.

Also, in the smooth case for $\varepsilon > 0$, it is checked that the effective domain $D(\partial\Phi_\varepsilon(\eta;\theta))$ of the subdifferential $\partial\Phi_\varepsilon(\eta;\,\cdot\,)$ coincides with the space W_0, and the subdifferential $\partial\Phi_\varepsilon(\eta;\,\cdot\,)$ can be identified with the nonlinear diffusion $-\mathrm{div}\big(\alpha(\eta)\gamma_\varepsilon(\nabla\theta) + \kappa\nabla\theta\big) \in H$, via the identification $\partial\gamma_\varepsilon = \nabla\gamma_\varepsilon$.

Meanwhile, in the nonsmooth case for $\varepsilon = 0$, the corresponding subdifferential $\partial\Phi_0(\eta;\,\cdot\,)$ is characterized as follows (cf. [1,8]).

$$D(\partial\Phi_0(\eta;\,\cdot\,)) = \left\{ \theta \in W_0 \;\middle|\; \begin{array}{l} \exists \boldsymbol{\nu}_\theta^* \in L^2_{\mathrm{div}}(\Omega;\mathbb{R}^N) \text{ s.t.} \\ \boldsymbol{\nu}_\theta^* \in \partial\gamma_0(\nabla\theta) \text{ a.e. in } \Omega \end{array} \right\} (\subset W_0), \tag{1.1}$$

and

$$\partial\Phi_0(\eta;\theta) = \left\{ \theta^* \in H \;\middle|\; \begin{array}{l} \theta^* = -\mathrm{div}\big(\alpha(\eta)\boldsymbol{\nu}_\theta^*\big) - \kappa\Delta\theta \text{ in } H, \\ \text{for some } \boldsymbol{\nu}_\theta^* \in L^2_{\mathrm{div}}(\Omega;\mathbb{R}^N) \text{ as in (1.1)} \end{array} \right\},$$
$$\text{for all } \theta \in D(\partial\Phi_0(\eta;\,\cdot\,)).$$

Now, under the above notations and assumptions, the main results of this talk will be stated in the following sections.

2 Results for the well-posedness of the state-systems

We begin with the definition of the solution to the state-system $(\mathrm{S})_\varepsilon$.

Definition 2 (Definition of solution to the state-system)**.** A triplet of functions $[u,\eta,\theta] \in [\mathscr{H}]^3$ is called a solution to the system $(\mathrm{S})_\varepsilon$ iff. $[u,\eta,\theta]$ fulfills the following conditions.

$(\mathrm{S1})_\varepsilon$ $[u,\eta,\theta] \in [W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;W_0)]^3$, subject to $[u(0),\eta(0),\theta(0)] = [u_0,\eta_0,\theta_0]$ in $[H]^3$.

$(\mathrm{S2})_\varepsilon$ $\partial_t(u - L\eta)(t) - \Delta_0 u(t) = f(t)$ in H, a.e. $t \in (0,T)$.

$(\mathrm{S3})_\varepsilon$ $\partial_t\eta(t) - \Delta_0\eta(t) + g(\eta(t)) + \alpha'(\eta(t))\gamma_\varepsilon(\nabla\theta(t)) = 0$ in H, a.e. $t \in (0,T)$.

$(\mathrm{S4})_\varepsilon$ $\exists \theta^* \in \mathscr{H}$, s.t. $\begin{cases} \theta^*(t) \in \partial\Phi_\varepsilon(\eta(t);\theta(t)), \\ \alpha_0(\eta(t))\partial_t\theta(t) + \theta^*(t) = 0, \end{cases}$ in H, a.e. $t \in (0,T)$.

Remark 3. The condition $(\mathrm{S4})_\varepsilon$ can be characterized by the following equivalent variational form:

$$\big(\alpha_0(\eta(t))\partial_t\theta(t), \theta(t) - \varpi\big)_H + \Phi_\varepsilon(\eta(t);\theta(t)) \le \Phi_\varepsilon(\eta(t);\varpi),$$
$$\text{for any } \varpi \in V, \text{ a.e. } t \in (0,T).$$

Now, the main results of this section is stated in forms of the following theorems.

Theorem 1 (Existence for $(\mathrm{S})_\varepsilon$)**.** Let us assume (A0)–(A3). Then, for any heat source $f \in \mathscr{H}$, the state-system $(\mathrm{S})_\varepsilon$ admits at least one solution $[u, \eta, \theta] \in [\mathscr{H}]^3$, such that:

$$\frac{1}{2}\int_s^t \left(\frac{1}{L}\big|\nabla u(\sigma)\big|_{[H]^N}^2 + \big|\partial_t \eta(\sigma)\big|_H^2 + \big|\sqrt{\alpha_0(\eta(\sigma))}\partial_t\theta(\sigma)\big|_H^2 \right) d\sigma$$
$$+ \mathcal{F}_\varepsilon(u(t), \eta(t), \theta(t)) \leq \mathcal{F}_\varepsilon(u(s), \eta(s), \theta(s)) + \frac{1}{L}\int_s^t \big(f(\sigma), u(\sigma)\big)\, d\sigma, \qquad (2.1)$$

for any $t \in (0, T]$ and a.e. $0 \leq s < t$, including the case of $s = 0$,

where $\mathcal{F}_\varepsilon : [H]^3 \longrightarrow [0, \infty]$ is the *free-energy* of the state-system $(\mathrm{S})_\varepsilon$, defined as follows:

$$\mathcal{F}_\varepsilon : [u, \eta, \theta] \in [H]^3 \mapsto \mathcal{F}_\varepsilon(u, \eta, \theta)$$
$$:= \begin{cases} \dfrac{1}{2L}\displaystyle\int_\Omega |u|^2\, dx + \dfrac{1}{2}\displaystyle\int_\Omega |\nabla\eta|^2\, dx + \displaystyle\int_\Omega G(\eta)\, dx + \Phi_\varepsilon(\eta; \theta), \\ \qquad \text{if } [u, \eta, \theta] \in [V]^3, \\ \infty, \quad \text{otherwise.} \end{cases}$$

Next, for the uniqueness and continuous-continuous dependence for $(\mathrm{S})_\varepsilon$, we will need the following regularity result.

Theorem 2 (Regularity result for $(\mathrm{S})_\varepsilon$)**.** In addition to (A0)–(A3), let us impose the following assumption (A4) for the initial data $[\eta_0, \theta_0]$.

(A4) $\eta_0 \in W_0 \cap H^3(\Omega)$ and $\theta_0 \in D(\partial\Phi_0(\eta_0; \cdot\,)) \subset W_0$.

Then, the component $[\eta, \theta] \in [\mathscr{H}]^2$ of the solution $[u, \eta, \theta]$ to $(\mathrm{S})_\varepsilon$ further satisfies that:

$$\begin{cases} \eta \in W^{2,2}(0, T; H) \cap W^{1,\infty}(0, T; V) \cap L^\infty(0, T; W_0), \\ \theta \in W^{1,\infty}(0, T; H) \cap W^{1,2}(0, T; V) \cap L^2(0, T; W_0). \end{cases}$$

Based on the above regularity result, the discussion of well-posedness will be completed as follows.

Theorem 3 (Uniqueness for $(\mathrm{S})_\varepsilon$)**.** Under the assumptions (A0)–(A4), let $[u^k, \eta^k, \theta^k] \in [\mathscr{H}]^3$, $k = 1, 2$, be two solutions for two heat sources $f^k \in \mathscr{H}$, $k = 1, 2$. Then, there exists a constant $C_* > 0$, such that:

$$\int_0^t \big|(u^1 - u^2)(t)\big|_H^2\, d\sigma + \left|\int_0^t \nabla(u^1 - u^2)(\sigma)\, d\sigma\right|_{[H]^N} + \big|(\eta^1 - \eta^2)(t)\big|_V^2 \qquad (2.2)$$
$$+ \big|\sqrt{\alpha_0(\eta^1(t))}(\theta^1 - \theta^2)(t)\big|_H^2 + \kappa\big|\nabla(\theta^1 - \theta^2)(t)\big|_{[H]^N}^2$$
$$\leq \exp\left[C_*\big(1 + \big|\partial_t\eta^1\big|_{\mathscr{V}}^2 + \big|\partial_t\theta^2\big|_{\mathscr{V}}^2\big)\right] \big|f^1 - f^2\big|_{\mathscr{H}}^2, \text{ for all } t \in [0, T].$$

Remark 4. The uniqueness result in Theorem 3 allows us to extend the validity of the energy inequality (2.1) to all $0 \leq s \leq t \leq T$.

Theorem 4 (Continuous-dependence for $(\mathrm{S})_\varepsilon$)**.** Let us assume (A0)–(A4). Let $\{\varepsilon_n\}_{n=1}^\infty \subset [0,1]$, $\{f_n\}_{n=1}^\infty \subset \mathscr{H}$, $\{[u_{0,n}, \eta_{0,n}]\}_{n=1}^\infty \subset V \times (W_0 \cap H^3(\Omega))$, and $\{[\theta_{0,n}, \theta_{0,n}^*] \in \partial\Phi_{\varepsilon_n}(\eta_{0,n}; \cdot)\}_{n=1}^\infty \subset H \times H$ be sequences such that:

$$\left\{\begin{array}{l} \bullet\ \varepsilon_n \to \varepsilon \text{ to some } \varepsilon \in [0,1], \\ \bullet\ f_n \to f \text{ weakly in } \mathscr{H}, \text{ for some } f \in \mathscr{H}, \\ \bullet\ [u_{0,n}, \eta_{0,n}] \to [\eta_0, \theta_0] \text{ weakly in } V \times H^3(\Omega), \\ \quad \text{for some } [u_0, \eta_0] \in V \times (W_0 \cap H^3(\Omega)), \\ \bullet\ [\theta_{0,n}, \theta_{0,n}^*] \to [\theta_0, \theta_0^*] \text{ weakly in } W_0 \times H, \text{ for} \\ \quad \text{some } [\theta_0, \theta_0^*] \in \partial\Phi_\varepsilon(\eta_0; \cdot\cdot), \end{array}\right. \qquad \text{as } n \to \infty.$$

Then, the sequence $\{[u_n, \eta_n, \theta_n]\}_{n=1}^\infty \subset [\mathscr{H}]^3$ of solutions $[u_n, \eta_n, \theta_n]$ to the state-system $(\mathrm{S})_{\varepsilon_n}$ for the initial data $[u_{0,n}, \eta_{0,n}, \theta_{0,n}]$ and the heat source f_n, for $n \in \mathbb{N}$, converges to the solution $[u, \eta, \theta] \in [\mathscr{H}]^3$ to $(\mathrm{S})_\varepsilon$ for the initial data $[u_0, \eta_0, \theta_0]$ and the heat source $f \in \mathscr{H}$, in the following sense:

$$\begin{gathered} [u_n, \eta_n, \theta_n] \to [u, \eta, \theta] \text{ in } C([0,T]; H) \times C([0,T]; V) \times C([0,T]; V), \\ \text{and weakly in } W^{1,2}(0,T; H) \times W^{1,2}(0,T; V) \times W^{1,2}(0,T; V), \text{ as } n \to \infty. \end{gathered}$$

Outline of the proofs

The existence result in Theorem 1 is proved by combining the mathematical framework for the Fix–Caginalp system of solid-liquid phase transitions, developed in [10], with the analytical techniques for the KWC model of grain boundary motion, developed in [8, 9, 13–15]. Here, the energy inequality (2.1) is deduced by multiplying the both sides of (0.1), (0.2), and (0.3) by u, $\partial_t \eta$, and $\partial_t \theta$, respectively, and then summing the resulting variational formulas.

The regularity result in Theorem 2 is based on the linearization method applied to the part of KWC model $\{(0.2),(0.3)\}$ in the smooth case for $\varepsilon > 0$ (cf. [13]). In this theorem, the linearized KWC model corresponding to a system obtained by taking the time-differentials of the PDEs in (0.2) and (0.3). This linearization approach also serves as the foundation for the proof of Theorem 6, which concerns the necessary optimality condition for $(\mathrm{OP})_\varepsilon$ in the smooth case.

In the uniqueness result Theorem 3, the uniqueness estimate (2.2) follows from Gronwall's lemma, based on the mathematical techniques developed in the previous works [10, 13]. Furthermore, the regularity result $[\partial_t \eta, \partial_t \theta] \in [\mathscr{V}]^2$, established in Theorem 2, plays a crucial role in ensuring that the estimate (2.2) is well-defined.

Finally, the continuous dependence result in Theorem 4 is established as a consequence of a uniform estimate of solution derived from the energy inequality (2.1), the compactness theory of the Aubin–Lions (or Ascoli–Arzerà) type (cf. [16]), the weak (or weak-$*$) compactness via Alaoglu's theorem (cf. [2]), and the Γ-convergence (cf. [7]) of the sequence of free-energies $\{\mathcal{F}_\varepsilon\}_{\varepsilon \in [0,1]}$, based on the previous results in [8, 9, 13–15].

3 Results for the optimal control problems

On the basis of the well-posedness results for $(\mathrm{S})_\varepsilon$, we can obtain the following results for the optimal control problem $(\mathrm{OP})_\varepsilon$.

Theorem 5 (Qualitative property of $(\mathrm{OP})_\varepsilon$)**.** Under the assumptions (A0)–(A4), the following items hold.

(I) (Existence of optimal control) For any $\varepsilon \in [0,1]$, the optimal control problem $(\mathrm{OP})_\varepsilon$ admits at least one optimal control $f^* \in \mathscr{H}$.

(II) (ε-dependence of optimality) Let $\{[u_{0,\varepsilon}, \eta_{0,\varepsilon}]\}_{\varepsilon\in[0,1]} \subset V \times (W_0 \cap H^3(\Omega))$ and $\{[\theta_{0,\varepsilon}, \theta^*_{0,\varepsilon}]\}_{\varepsilon\in[0,1]} \subset W_0 \times H$ be sequences such that:

$$\begin{cases} \bullet\ \{[u_{0,\varepsilon}, \eta_{0,\varepsilon}]\}_{\varepsilon\in[0,1]} \text{ is bounded in } V \times H^3(\Omega), \\ \bullet\ \{[\theta_{0,\varepsilon}, \theta^*_{0,\varepsilon}]\}_{\varepsilon\in[0,1]} \text{ is bounded in } W_0 \times H, \\ \bullet\ [\theta_{0,\varepsilon}, \theta^*_{0,\varepsilon}] \in \partial\Phi_\varepsilon(\eta_{0,\varepsilon};\, \cdot\,) \text{ in } H \times H, \text{ for } \varepsilon \in [0,1]. \end{cases}$$

Let $\{f^*_\varepsilon\}_{\varepsilon\in[0,1]} \subset \mathscr{H}$ be a sequence of optimal controls f^*_ε of $(\mathrm{OP})_\varepsilon$, for $\varepsilon \in [0,1]$, which are governed by the state-system $(\mathrm{S})_\varepsilon$ with the initial data $[u_{0,\varepsilon}, \eta_{0,\varepsilon,\theta_{0,\varepsilon}}]$ and the heat sources f^*_ε, for $\varepsilon \in [0,1]$. Then, for any $\varepsilon_0 \in [0,1]$, there exists a sequence $\{\varepsilon_n\}_{n=1}^\infty \subset [0,1]$ and an optimal control $f^*_0 \in \mathscr{H}$ of $(\mathrm{OP})_{\varepsilon_0}$, with the initial data $[u_0, \eta_0, \theta_0] \in V \times (W_0 \times H^3(\Omega)) \times D(\Phi_{\varepsilon_0}(\eta_0;\, \cdot\,))$ of the governing state-system $(\mathrm{S})_{\varepsilon_0}$, such that

$$\varepsilon_n \to \varepsilon_0, \text{ and } f^*_{\varepsilon_n} \to f^*_0 \text{ weakly in } \mathscr{H}, \text{ as } n \to \infty.$$

Remark 5. Since Theorem 5 (I) does not include the uniqueness of the optimal control, Theorem 5 (II) does not assert the coincidence of the optimal controls f^*_0 and $f^*_{\varepsilon_0}$ in $\mathscr{H}$.

Theorem 6 (Necessary condition of optimality in the smooth case)**.** In addition to (A0)–(A4), let us assume $\varepsilon > 0$. Let $f^*_\varepsilon \in \mathscr{H}$ be the optimal control of $(\mathrm{OP})_\varepsilon$, and let $[u^*_\varepsilon, \eta^*_\varepsilon, \theta^*_\varepsilon] \in [\mathscr{H}]^3$ be the solution to $(\mathrm{S})_\varepsilon$ for the heat source f^*_ε. Then, it holds that:

$$f^*_\varepsilon = -q^*_\varepsilon \text{ in } \mathscr{H}, \tag{3.1}$$

where q^*_ε is the first component of the unique solution $[q^*_\varepsilon, p^*_\varepsilon, z^*_\varepsilon] \in [W^{1,2}(0,T;V^*) \cap \mathscr{V}]^3$ to the following backward parabolic linear system:

$$\langle -\partial_t q^*_\varepsilon, \varphi \rangle_{\mathscr{V}} + \big(\nabla q^*_\varepsilon, \nabla\varphi\big)_{[\mathscr{H}]^N} + \big(p^*_\varepsilon, \varphi\big)_{\mathscr{H}} = \big(u^*_\varepsilon - u_{\mathrm{ad}}, \varphi\big)_{\mathscr{H}}, \text{ for any } \varphi \in \mathscr{V}, \tag{3.2}$$

$$\begin{aligned} &\langle -\partial_t (p^*_\varepsilon - Lq^*_\varepsilon) + \alpha''(\eta^*_\varepsilon)\gamma_\varepsilon(\nabla\theta^*_\varepsilon) p^*_\varepsilon + \alpha_0'(\eta^*_\varepsilon) z^*_\varepsilon \partial_t \theta^*_\varepsilon, \psi \rangle_{\mathscr{V}} + \big(\nabla p^*_\varepsilon, \nabla\psi\big)_{[\mathscr{H}]^N} \\ &+\big(g'(\eta^*_\varepsilon) p^*_\varepsilon + \alpha'(\eta^*_\varepsilon)\nabla\gamma_\varepsilon(\nabla\theta^*_\varepsilon)\cdot\nabla z^*_\varepsilon, \psi\big)_{\mathscr{H}} = \big(\eta^*_\varepsilon - \eta_{\mathrm{ad}}, \psi\big)_{\mathscr{H}}, \text{ for any } \psi \in \mathscr{V}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\langle -\partial_t\big(\alpha_0(\eta^*_\varepsilon) z^*_\varepsilon\big), \varpi \rangle_{\mathscr{V}} + \big(\alpha(\eta^*_\varepsilon)[\nabla^2\gamma_\varepsilon](\nabla\theta^*_\varepsilon)\nabla z^*_\varepsilon + \kappa\nabla z^*_\varepsilon, \nabla\varpi\big)_{[\mathscr{H}]^N} \\ &+\big(\alpha'(\eta^*_\varepsilon) p^*_\varepsilon [\nabla\gamma_\varepsilon](\nabla\theta^*_\varepsilon), \nabla\varpi\big)_{[\mathscr{H}]^N} = \big(\theta^*_\varepsilon - \theta_{\mathrm{ad}}, \varpi\big)_{\mathscr{H}}, \text{ for any } \varpi \in \mathscr{V}, \end{aligned} \tag{3.4}$$

subject to the homogeneous terminal condition:

$$[q^*_\varepsilon(T), p^*_\varepsilon(T), z^*_\varepsilon(T)] = [0,0,0] \text{ in } [H]^3. \tag{3.5}$$

Theorem 7 (Limiting condition of optimality to the nonsmooth case)**.** Let us assume (A0)–(A4), and let us set:

$$\mathfrak{X}_0 := \left\{ \ w \in W^{1,2}(0,T;V) \ \middle| \ w(0) = 0 \text{ in } H \ \right\}.$$

Then, there exists an optimal control $f^\circ \in \mathscr{H}$ of $(\mathrm{OP})_0$, in the nonsmooth case for $\varepsilon = 0$, with $\boldsymbol{\nu}^\circ \in L^\infty(\Omega_T)$, $\sigma^\circ \in \mathscr{H}$, $\xi^\circ \in \mathscr{V}^*$, $\Xi^\circ \in \mathfrak{X}_0^*$, and the solution $[u^\circ, \eta^\circ, \theta^\circ] \in [\mathscr{H}]^3$ to $(\mathrm{S})_\varepsilon$ for the heat source f°, such that:

$$f^\circ = -q^\circ \text{ in } \mathscr{H},$$

where q° is the first component of a triplet $[q^\circ, p^\circ, z^\circ] \in [W^{1,2}(0,T;V^*) \cap \mathscr{V}]^2 \times \mathscr{V}$ satisfying:

$$\langle -\partial_t q^\circ, \varphi \rangle_{\mathscr{V}} + \left(\nabla q^\circ, \nabla \varphi \right)_{[\mathscr{H}]^N} + \left(p^\circ, \varphi \right)_{\mathscr{H}} = \left(u^\circ - u_{\mathrm{ad}}, \varphi \right)_{\mathscr{H}}, \text{ for any } \varphi \in \mathscr{V},$$

$$\begin{aligned} &\langle -\partial_t (p^\circ - L q^\circ) + \alpha''(\eta^\circ) |\nabla \theta^\circ| p^\circ + \alpha_0'(\eta^\circ) \xi^\circ, \psi \rangle_{\mathscr{V}} + \left(\nabla p^\circ, \nabla \psi \right)_{[\mathscr{H}]^N} \\ &+ \left(g'(\eta^\circ) p^\circ + \alpha'(\eta^\circ) \sigma^\circ, \psi \right)_{\mathscr{H}} = \left(\eta^\circ - \eta_{\mathrm{ad}}, \psi \right)_{\mathscr{H}}, \text{ for any } \psi \in \mathscr{V}, \end{aligned}$$

$$\begin{aligned} &\left((\alpha_0(\eta^\circ) z^\circ), \partial_t w \right)_{\mathscr{H}} + \langle \Xi^\circ, w \rangle_{\mathfrak{X}_0} + \kappa \left(\nabla z^\circ, \nabla w \right)_{[\mathscr{H}]^N} \\ &+ \left(\alpha'(\eta^\circ) p^\circ \boldsymbol{\nu}^\circ, \nabla w \right)_{[\mathscr{H}]^N} = \left(\theta^\circ - \theta_{\mathrm{ad}}, w \right)_{\mathscr{H}}, \text{ for any } w \in \mathfrak{X}_0, \end{aligned}$$

and

$$|\boldsymbol{\nu}^\circ| \leq 1 \text{ and } \boldsymbol{\nu}^\circ \cdot \nabla \theta^\circ = |\nabla \theta^\circ|, \text{ a.e. in } \Omega_T. \tag{3.6}$$

Outline of the proofs

The previous Theorems 1–4 yield:

($\star$1) the lower semi-continuity of the cost $\mathscr{J}_\varepsilon$, for any $\varepsilon \in [0,1]$,

($\star$2) the continuous dependence of $\{\mathscr{J}_\varepsilon\}_{\varepsilon \in [0,1]}$, with respect to $\varepsilon \in [0,1]$, in the sense of Γ-convergence on $\mathscr{H}$.

Therefore, the qualitative properties stated in Theorem 5 are established by applying a standard minimization argument applied to the cost functionals $\mathscr{J}_\varepsilon$, for $\varepsilon \in [0,1]$.

Meanwhile, in Theorem 6, the necessary condition of optimality for $\varepsilon > 0$ will be based on the Gâteaux differential of the cost $\mathscr{J}_\varepsilon$. Then, for any direction $h \in \mathscr{H}$, the computation of directional derivative $D_h \mathscr{J}_\varepsilon(f)$ at $f \in \mathscr{H}$ will be associated with the following linearized system of the state-system $(\mathrm{S})_\varepsilon$:

$$\langle \partial_t (\varrho - L\chi), \varphi \rangle_{\mathscr{V}} + \left(\nabla \varrho, \nabla \varphi \right)_{[\mathscr{H}]^N} = \left(h, \varphi \right)_{\mathscr{H}}, \text{ for any } \varphi \in \mathscr{V}, \tag{3.7}$$

$$\begin{aligned} &\langle \partial_t \chi + \alpha''(\eta) \gamma_\varepsilon(\nabla \theta) \chi, \psi \rangle_{\mathscr{V}} + \left(\nabla \chi, \nabla \psi \right)_{[\mathscr{H}]^N} \\ &+ \left(g'(\eta^*) \chi + \alpha'(\eta) \nabla \gamma_\varepsilon(\nabla \theta) \cdot \nabla \zeta + \varrho, \psi \right)_{\mathscr{H}} = 0, \text{ for any } \psi \in \mathscr{V}, \end{aligned} \tag{3.8}$$

$$\left\langle \alpha_0(\eta)\partial_t\zeta + \alpha_0'(\eta)\partial_t\theta\chi, \varpi \right\rangle_{\mathscr{V}} + \left(\alpha(\eta)[\nabla^2\gamma_\varepsilon](\nabla\theta)\nabla\zeta + \kappa\nabla\zeta, \nabla\varpi\right)_{[\mathscr{H}]^N} \tag{3.9}$$
$$+\left(\alpha'(\eta)\chi[\nabla\gamma_\varepsilon](\nabla\theta), \nabla\varpi\right)_{[\mathscr{H}]^N} = 0, \text{ for any } \varpi \in \mathscr{V},$$

subject to the homogeneous initial condition:

$$[\varrho(0), \chi(0), \zeta(0)] = [0, 0, 0] \text{ in } [H]^3. \tag{3.10}$$

Indeed, the backward parabolic linear system $\{(3.2),(3.3),(3.4)\}$ in Theorem 6 is derived on the basis of the adjoint system of the linearized system $\{(3.7),(3.8),(3.9)\}$. In this context, $[u, \eta, \theta]$ is the solution to the state-system $(\mathrm{S})_\varepsilon$ for the heat source $f \in \mathscr{H}$, and the solution $[\varrho, \chi, \zeta] \in [W^{1,2}(0,T;H) \cap \mathscr{V}]^3$ to the linearized system $\{(3.7),(3.8),(3.9)\}$ corresponds to the linearized variable corresponding to $[u, \eta, \theta]$. In particular, the homogeneous initial condition (3.10) comes from the fixed setting of the initial data $[u_0, \eta_0, \theta_0]$ of $(\mathrm{S})_\varepsilon$, and it is reflexed to the homogeneous terminal condition (3.5).

Finally, the limiting optimality condition in Theorem 7 is derived as the limit of the conditions (3.1)–(3.5) as $\varepsilon \downarrow 0$. In this context, the vector field $\boldsymbol{\nu}^\circ \in [L^\infty(\Omega_T)]^N$ is a weak-$*$ limit of the diffusion flux $\{\nabla\gamma_\varepsilon(\nabla\theta_\varepsilon^*)\}_{\varepsilon\in(0,1)}$ in $[L^\infty(\Omega_T)]^N$, and the identity (3.6) implies that ν° corresponding to the mathematical representation of the singular diffusion flux $\frac{\nabla\theta^\circ}{|\nabla\theta^\circ|}$ $(\in \partial\gamma_0(\nabla\theta^\circ))$ in the nonsmooth state system $(\mathrm{S})_0$, for $\varepsilon = 0$. The functions $\sigma^\circ \in \mathscr{H}$ and $\xi^\circ \in \mathscr{V}^*$ are obtained as the weak limits of the sequences $\{[\nabla\gamma_\varepsilon](\nabla\theta_\varepsilon^*) \cdot \nabla z_\varepsilon^*\}_{\varepsilon\in(0,1]}$ and $\{z_\varepsilon^*\partial_t\theta_\varepsilon^*\}_{\varepsilon\in[0,1]}$ as $\varepsilon \downarrow 0$, respectively. However, since none of the sequences $\{\partial_t\theta_\varepsilon\}_{\varepsilon\in(0,1)}$, $\{[\nabla\gamma_\varepsilon](\nabla\theta_\varepsilon^*)\}_{\varepsilon\in(0,1)}$, $\{z_\varepsilon^*\}_{\varepsilon\in(0,1)}$, and $\{\nabla z_\varepsilon^*\}_{\varepsilon\in(0,1)}$ possesses compactness in any strong topology, the justification of the identities $\sigma^\circ = \boldsymbol{\nu}^\circ \cdot \nabla z^\circ$ and $\xi^\circ = z^\circ\partial_t\theta^\circ$ has not been achieved, yet.

Moreover, the element $\Xi^\circ \in \mathfrak{X}_0^*$ arises as a distributional limit corresponding to the linearization of the singular diffusion term $-\mathrm{div}\left(\alpha(\eta^\circ)\frac{\nabla\theta^\circ}{|\nabla\theta^\circ|}\right)$. Due to the singularity of the flux $\frac{\nabla\theta^\circ}{|\nabla\theta^\circ|}$, the characterization of Ξ° remains a challenging open problem in this study; in fact, no explicit conjecture has been formulated at present.

References

[1] Toyohiko Aiki, Daiki Mizuno, and Ken Shirakawa, *A class of initial-boundary value problems governed by pseudo-parabolic weighted total variation flows*, Advances in Mathematical Sciences and Applications **32** (2023), 311–341.

[2] Leon Alaoglu, *Weak topologies of normed linear spaces*, Ann. of Math. (2) **41** (1940), 252–267. MR 1455

[3] Harbir Antil, Shodai Kubota, Ken Shirakawa, and Noriaki Yamazaki, *Optimal control problems governed by 1-D Kobayashi-Warren-Carter type systems*, Math. Control Relat. Fields **11** (2021), no. 2, 253–289. MR 4218112

[4] ———, *Constrained optimization problems governed by PDE models of grain boundary motions*, Adv. Nonlinear Anal. **11** (2022), no. 1, 1249–1286. MR 4395725

[5] Harbir Antil, Daiki Mizuno, and Ken Shirakawa, *Optimal control of pseudo-parabolic KWC systems for grain boundary motion*, arXiv preprint arXiv:2506.09407.

[6] ———, *Well-posedness of a pseudo-parabolic KWC system in materials science*, SIAM J. Math. Anal. **56** (2024), no. 5, 6422–6445. MR 4797677

[7] Gianni Dal Maso, *An introduction to Γ-convergence*, Progress in Nonlinear Differential Equations and their Applications, vol. 8, Birkhäuser Boston, Inc., Boston, MA, 1993. MR 1201152

[8] Akio Ito, Nobuyuki Kenmochi, and Noriaki Yamazaki, *A phase-field model of grain boundary motion*, Appl. Math. **53** (2008), no. 5, 433–454. MR 2469586

[9] ———, *Weak solutions of grain boundary motion model with singularity*, Rend. Mat. Appl. (7) **29** (2009), no. 1, 51–63. MR 2548486

[10] Nobuyuki Kenmochi and Marek Niezgódka, *Evolution systems of nonlinear variational inequalities arising from phase change problems*, Nonlinear Anal. **22** (1994), no. 9, 1163–1180. MR 1279139

[11] Ryo Kobayashi, James A. Warren, and W. Craig Carter, *A continuum model of grain boundaries*, Phys. D **140** (2000), no. 1-2, 141–150. MR 1752970

[12] ———, *Grain boundary model and singular diffusivity*, Free boundary problems: theory and applications, II (Chiba, 1999), GAKUTO Internat. Ser. Math. Sci. Appl., vol. 14, Gakkōtosho, Tokyo, 2000, pp. 283–294. MR 1794359

[13] Daiki Mizuno and Ken Shirakawa, *Well-posedness result of parabolic KWC systems with variable-dependent mobilities*, In preparation.

[14] Salvador Moll and Ken Shirakawa, *Existence of solutions to the Kobayashi–Warren–Carter system*, Calc. Var. Partial Differential Equations **51** (2014), no. 3-4, 621–656. MR 3268865

[15] Salvador Moll, Ken Shirakawa, and Hiroshi Watanabe, *Energy dissipative solutions to the Kobayashi–Warren–Carter system*, Nonlinearity **30** (2017), no. 7, 2752–2784. MR 3670006

[16] Jacques Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. (4) **146** (1987), 65–96. MR 0916688

[17] James A. Warren, Ryo Kobayashi, A. Lobkovski, and W. Craig Carter, *Extending phase field models of solidification to polycrystalline materials*, Acta Materialia **51** (2001), 6035–6058.

[18] Noriaki Yamazaki, *Global attractors for non-autonomous phase-field systems of grain boundary motion with constraint*, Adv. Math. Sci. Appl. **23** (2013), no. 1, 267–296. MR 3155454

Gradient flows with a small parameter

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1 Introduction

When we have a system with a small parameter ε it is tempting to pass to the limit when ε goes to zero. Here, we will consider a couple of examples of gradient flows depending on a small parameter ε. Before studying them, we will explain the notion of a gradient flow of a functional E. The easiest starting point is when E is a smooth real valued function over $\mathbb{R}^N$. Then,

$$\dot{x} = -\nabla E(x), \qquad x(0) = x_0, \tag{1.1}$$

is the gradient flow of E. In the case of a Hilbert space H, when E is defined on $D(E) \subset H$ eq. (1.1) generalizes to

$$\dot{u} = -\nabla_H E(u), \qquad u(0) = u_0, \tag{1.2}$$

where $\nabla_H E$ denotes the gradient of E with respect to the topology of H. The resulting equation depends significantly on the choice of the topology of H. Here is an example, let us suppose that

$$E(u) = \int_\Omega (\frac{1}{2}|\nabla u|^2 + W(u))\, dx,$$

where $\Omega \subset \mathbb{R}^N$, $N = 1, 2, 3$, is a bounded region with smooth boundary and $W(u) = \frac{1}{4}(u^2 - 1)^2$ (W could be any smooth function with an appropriate growth). Of course, the domain of E is $H^1(\Omega)$.

Let us take $u_0, h \in D(E)$. We may compute

$$\frac{d}{d\varepsilon} E(u_0 + \varepsilon h)\big|_{\varepsilon=0} = DE(u_0)h = \int_\Omega (\nabla u_0 \cdot \nabla h + W_u(u_0)h)\, dx.$$

Now, we may find the gradient of E corresponding to different Hilbert spaces. Let us take $H = L^2(\Omega)$, then

$$DE(u_0)h = \langle \nabla_{L^2} E(u_0), h\rangle_{L^2} = \int_\Omega \xi \cdot h\, dx \qquad \forall h \in H^1(\Omega)$$

leads to the following conclusion,

$$\xi = -\Delta u_0 + W_u(u_0).$$

Thus, (1.2) becomes the well-known Allen-Cahn equation,

$$u_t = \Delta u - W_u(u).$$

On the other hand, if we require $H = H^{-1}(\Omega)$, where $H^{-1}(\Omega)$ denotes the dual of $H^1(\Omega)$, then the result is different. Namely, we can check that

$$\nabla_{H^{-1}} E(u) = \Delta(\Delta u - W_u(u)).$$

As a result, eq. (1.2) becomes the well-know Cahn-Hilliard equation,

$$u_t = \Delta(-\Delta u + W_u(u)),$$

whose behavior differs significantly from the Allen-Cahn equation.

We are going to present a couple of generalizations of (1.2) based on different premises. One is based on the analysis of the structure of the global attractor, the other one is simpler and it is suitable for convex functionals E. Then, we present one family of functionals from each category depending on a small parameter. Our goal is to investigate the dependence of the flows on this small parameter. We will ask different questions about these flows.

2 A topological definition of the gradient flow and stabilization of solutions

In [9] Savina and her coworkers derived a model of quantum dots formation during the process of molecular beam epitaxy. The original model was two-dimensional, with periodic boundary condition, however, we consider only its one-dimensional version

$$\begin{array}{ll} h_t = \frac{\delta}{2}|\nabla h|^2 + \Delta^3 h - \Delta \mathrm{div}\, DW_F(\nabla h), & x \in \mathbb{T} = \mathbb{R}/L\mathbb{Z},\ t > 0 \\ h(x,0) = h_0(x), & x \in \mathbb{T}. \end{array}$$

Here, $\mathbb{T}$ denotes the flat torus, W is a double-well potential

$$W(F) = \frac{1}{4}(F^2 - 1)^4$$

and $\delta > 0$ describes the intensity of the destabilizing flux of adatoms.

The answers to the basic questions about the global in time existence of strong solutions to this equation and their uniqueness were established in [5] and [6]. In the latter paper these results were re-written in the language of the semigroup theory.

We are interested in the asymptotic behavior of the above system. We notice that

$$\frac{d}{dt}\int_{\mathbb{T}} h(x,t)\, dx = \int_{\mathbb{T}} h_t(x,t)\, dx = \frac{\delta}{2}\int_{\mathbb{T}} |\nabla h|^2(x,t)\, dx \geq 0.$$

Thus, there is no hope to establish stabilization of solutions. However, the story changes significantly if we look at the slopes of h. Thus, we set $u = h_x$ and after taking the gradient of (2) we obtain

$$\begin{array}{ll} u_t = \delta u u_x + \Delta^3 u - \Delta^2 W_u(u), & x \in \mathbb{T},\ t > 0, \\ u(x, 0 = u_0(x), & x \in \mathbb{T}. \end{array} \tag{2.1}$$

We notice that for $\delta = 0$ equation (2.1) becomes a gradient flow

$$u_t = -\nabla_{H^{-2}} \mathcal{F}(u), \qquad u(0) = u_0 \tag{2.2}$$

of $\mathcal{F}$, where $\mathcal{F}(u) = \int_{\mathbb{T}}(\frac{1}{2}|\nabla u|^2 + W(u))\, dx$. In other words, (2.1) is a perturbation of (2.2). Since stabilization is 'obvious' for solutions to (2.2) we hope that the same thing for (2.1) is true.

In order to make this wish become reality we are going to present the tools, which are based on a generalization of the notion of the gradient flow. We have in mind an approach, where the structure of the global attractor matters most. Our starting observation is that if u is a solution to (1.2), then the function $t \mapsto E(u(t))$ is decreasing. As a result, the ω-limit set, $\omega(u_0)$, may contain only critical points of E.

This implies that the global attractor of (1.2) cannot contain any homoclinic orbit. We hope that a small perturbation will preserve this property. Another issue is how to prove stabilization of solutions.

Our premise is that solutions to equation (2.1) can be written as $u(t) = T(t)u_0$, where $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup on a Banach space Z. The notion of an attractor will play a decisive role in our analysis, we recall it below.

Definition 2.1. A set $\mathcal{A} \subset Z$ is the global attractor for a semigroup $T(\cdot) : Z \to Z$ if
(i) $\mathcal{A}$ is compact;
(ii) $\mathcal{A}$ is invariant;
(iii) $\mathcal{A}$ attracts each bounded subset of Z.

Now, we may introduce a generalization of the gradient flow.

Definition 2.2. A semigroup $T(t) : Z \to Z$, $t \geq 0$ with a global attractor $\mathcal{A}$ is a gradient flow with respect to a family $\mathcal{S} = \{\mathcal{E}^0, \dots, \mathcal{E}^k\}$ of invariant sets provided that:
1) For any global (eternal) solution $\xi : \mathbb{R} \to Z$ taking values in $\mathcal{A}$, there exist $i, j \in \{0, \dots, k\}$ such that

$$\lim_{t\to-\infty} \operatorname{dist}(\xi(t), \mathcal{E}^i) = 0 \quad \text{and} \quad \lim_{t\to\infty} \operatorname{dist}(\xi(t), \mathcal{E}^j) = 0.$$

2) The collection $\mathcal{S}$ contains no homoclinic structures.

It turns out that this definition is good, i.e. if we have a perturbed strongly continuous semigroup of operators, $\{T_\delta(t)\}_{t\geq 0}$, then T_δ is also a gradient flow. In other words, we want to show that (2.1), which is a perturbation of (2.2) is a gradient flow according to Definition 2.2. Fortunately, we have:

Theorem 2.1. (see [1, Theorem 5.26]) Let $T_0(\cdot)$ be a gradient flow on Z with respect to the finite collection $\mathcal{S}^0$ of isolated invariant sets $\{\mathcal{E}^{0,1}, \dots, \mathcal{E}^{0,k}\}$ and a global attractor $\mathcal{A}_0$. If:
(a) for each $\delta \in (0, 1]$, $T_\delta(\cdot)$ is a semigroup on Z with global attractor $\mathcal{A}_\delta$;
(b) $\{T_\delta(\cdot)\}_{\delta\in[0,1]}$ is collectively asymptotically compact. This means that if $\{t_n\}_{n=1}^\infty$ is a sequence such that $t_n \to \infty$ and $\{x_n\} \subset Z$ is any sequence such that $T_{\delta_n}(t_n)x_n$ is bounded, then $T_{\delta_n}(t_n)x_n$ contains a convergent subsequence. In addition we assume that $\overline{\bigcup_{\delta\in[0,1]} \mathcal{A}_\delta}$ is bounded;
(c) $T_\delta(\cdot)$ converges to $T_0(\cdot)$, i.e. $d(T_\delta(t)u, T_0(t)u)$ converges uniformly to zero as $\delta \to 0$ for u in compact subsets of Z; and
(d) for $\delta \in (0, 1]$, $\mathcal{A}_\delta$ contains a finite collection of isolated invariant sets $\mathcal{S}^\delta = \{\mathcal{E}^{\delta,1}, \dots, \mathcal{E}^{\delta,k}\}$ such that $\lim_{\delta\to 0} \operatorname{dist}_H(\mathcal{E}^{\delta,j}, \mathcal{E}^{0,j}) = 0$ and there exist $\eta > 0,\ \delta_1 \in (0, 1)$ such that for for all $\delta \in (0, \delta_1)$, if $\xi_\delta : \mathbb{R} \to \mathcal{A}_\delta$ is a solution, then

$$\sup_{t\in\mathbb{R}} \operatorname{dist}(\xi_\delta(t), \mathcal{E}^{\delta,j}) \leq \eta \quad \Rightarrow \quad \xi_\delta(t) \in \mathcal{E}^{\delta,j} \text{ for all } t \in \mathbb{R}.$$

Then, there exists $\delta_0 > 0$ such that, for all $\delta \in (0, \delta_0)$, $T_\delta(\cdot)$ is a gradient semigroup with respect to $\mathcal{S}^\delta$. In particular,

$$\mathcal{A}_\delta = \bigcup_{i=1}^{k} W^u(\mathcal{E}^{\delta,i})\,.$$

Once we deduce that the above theorem is indeed applicable to (2.1), rendering this equation a gradient flow in the sense of Definition 2.2, we will apply the following theorem:

Theorem 2.2 (see [4]). Let us the consider the problem

$$\dot{z} = C_1 z + H_1(z), \qquad z(0) = z_1, \tag{2.3}$$

where:

(H.1) $-C_1 : D(C_1) \subset Z \to Z$ is sectorial and there is an $\alpha > 0$ such that $H_1 : Z^\alpha \to Z$ is of class C^1 and $D(C_1) = D(C_1 + DH(z_1))$. For any $z_1 \in Z^\alpha$ its positive orbit of (2.3) is precompact and the ω-limit set $\omega(z_1)$ consists of equilibria of (2.3). For any $z_0 \in \omega(z_1)$, the semigroup $S(t) = S_{z_0}(t)$ generated by $C = C_1 + DH_1(z_0)$ satisfies the hypotheses (H2), (H3) and (H4) below.

(H.2) There is a decomposition $Z = Y_1 \oplus X \oplus Y_2$ with associated continuous projection operators P_1, P_0, P_2, which commute with $S(t)$;

(H.3) The ranges X and Y_2 of P_0 and P_2 are finite dimensional, dim $X = m_0$, dim $Y_2 = m_2$;

(H.4) The spectrum $\sigma(S(1))$ can be written as $\sigma(S(1)) = \sigma_- \cup \sigma_0 \cup \sigma_+$, where $\sigma_- = \sigma(S(1)P_1)$, $\sigma_0 = \sigma(S(1)P_0)$, $\sigma_+ = \sigma(S(1)P_2)$ lies, respectively, inside, on, outside the unit circle centered at $0 \in \mathbb{C}$. Moreover, the distance of σ_- to this unit circle is positive.

In addition we assume that σ_0 is either empty or it contains the only point 1, which is a simple eigenvalue of $S_{z_0}(1)$. Then, there is a unique point $\varphi = \varphi_{z_0}$ such that $\omega(z_1) = \{\varphi\}$. □

Of course, we have to impose some restrictions on (2.1). The important one is that we may only consider a one-dimensional version of the equation. A mild limitation is that we may consider any period $L > 0$ in (2.1) except for N, a set of Lebesgue measure zero. In my talk I will show how to deduce with the help of Theorem 2.1 and Theorem 2.2 following stabilization result for solutions of (2.1).

Theorem 2.3. Let us suppose that $L \in \mathbb{R}_+ \setminus N$, where the exceptional set N mentioned above. Then, for any $u_0 \in H^2(\mathbb{T}^1)$ with zero mean the corresponding solution to (2.1) stabilizes. In other words, there is a $\varphi \in \mathcal{S}^\delta$ such that

$$\lim_{t\to\infty} u(t) = \varphi.$$

Moreover, for any $k \in \mathbb{N}$ the convergence above is in the $\dot{H}^k$ topology. Here $\dot{H}^k$ denotes elements of H^k with the zero mean. □

Since eq.(2.1) regularizes the data, we may improve this result by assuming $u_0 \in \dot{L}^2(\mathbb{T})$.

3 The integrated evolutionary variational inequality and convergence of gradient flows

Here, we will exploit a different degeneralization of (1.1), which is based on the convex analysis. In general, if E is merely convex, then its derivative need not exist. Let us suppose that H is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$. We recall that for any convex, lower semicontinuous functional $E\colon H \to [0,\infty]$ we may define its subdifferential at u, $\partial E(u)$ as follows

$$\partial E(u) = \{\xi \in H : \forall w \in H,\ E(w) - E(u) \geq \langle \xi, w - u\rangle\}.$$

In general, if $\partial E(u)$ is not empty, then it need not be a singleton. Hence, we have to replace a differential equation (1.1) by a differential inclusion,

$$-u_t \in \partial_H E(u), \qquad u(0) = u_0. \tag{3.1}$$

We write $\partial_H E$ to stress that the form of the subdifferential depends on the topology of H.

It is well-known by the Kōmura-Brezis theory that for a convex and lower semicontinuous functional E the inclusion (3.1) generates a strongly continuous semigroup. We call the resulting map $S\colon \overline{D(E)} \to C([0,T],X)$ the gradient flow of E.

However, we would like to consider an apparently weaker notion of solution to (3.1). For this purpose, we rewrite (3.1) for any $w \in H$ as follows

$$E(w) - E(u) \geq -\langle u_t, w - u\rangle\,. \tag{3.2}$$

Let us integrate (3.2) over (s,t), where $0 < s < t \leq T$, we obtain

$$(E(w) - E(u(t))(t-s) \geq \int_s^t (E(w) - E(u(\tau))\,d\tau \geq \frac{1}{2}(\|w-u(t)\|_H^2 - \|w-u(s)\|_H^2), \tag{3.3}$$

where the first inequality follows from $E(u(t)) \leq E(u(s))$ for $t \geq s$. We will call (3.3) the integrated evolutionary variational inequality, IEVI for short. We stress that for general data $u_0 \in \overline{D(E)}$ we cannot take $s = 0$, because $\|u_t\|_H$ need not be integrable in a neighborhood of $t = 0$. This is the case when $u_0 \in D(\partial E)$. The advantage of the IEVI is that it does not involve any time derivatives.

An important observation is that (3.3) is equivalent to (3.2) i.e. to differential inclusion (3.2).

We would like to apply the IEVI to study a simplified geometric problem. Namely, our functional, Q is the Dirichlet energy

$$Q[X] = \frac{1}{2}\int_0^{2\pi} |X'(u)|^2\, du\,.$$

defined on $H_\varepsilon := H^1_\varepsilon(\mathbb{S}, \mathbb{R}^2)$ and we consider the following problem

$$X_t = -\nabla_{H_\varepsilon} Q, \qquad X(u,0) = X_0(u), \quad u \in \mathbb{S}, \tag{3.4}$$

where $\mathbb{S}$ is the unit one-dimensional sphere.

As a set H_ε equals to $H^1(\mathbb{S}, \mathbb{R}^2)$, however, the inner product depends on ε. Namely, for $X, Y \in H^1(\mathbb{S}; \mathbb{R}^2)$ we set

$$\langle X, Y\rangle_{H_\varepsilon} = \langle X, Y\rangle_{L^2} + \varepsilon^2 \langle X', Y'\rangle_{L^2} = \int_0^{2\pi} X(u)\cdot Y(u) + \varepsilon^2 X'(u)\cdot Y'(u)\, du.$$

The peculiarity of problem (3.4) is that it is a linear ODE in a Hilbert space H_ε. Its solvability is not an issue. The problem is the changing base space H_ε. Since it is relatively simple, the full power of theory of Mosco-convergence along a connecting operator developed in [2] and [3] is not needed. A simple IEVI suffices, however, we have to pay a price. Namely, we have to prove separately, that X^ε the family of solutions to (3.4) converges to a limit as $\varepsilon \to 0$. In my talk I present the details of the convergence proof and I will show that the limit

$$X = \lim_{\varepsilon\to} X^\varepsilon.$$

solves the limiting problem as well as we will identify it.

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References

[1] A.N.Carvalho, J.A. Langa, J.C.Robinson, *Attractors for infinite-dimensional non-autonomous dynamical systems*, Springer, New York, 2013.

[2] Y.Giga, M.Łasica, P.Rybka, The heat equation with the dynamic boundary condition as a singular limit of problems degenerating at the boundary, *Asymptot. Anal.* **35** (2023), no. 3-4, 463–508.

[3] Y.Giga, M.Łasica, P.Rybka, A variational framework for singular limits of gradient flows with applications, *forthcoming*.

[4] J.K.Hale, G.Raugel, Convergence in gradient-like systems with applications to PDE. *Z. Angew. Math. Phys.*, **43** (1992), no. 1, 63–124.

[5] M. Korzec, P. Nayar, P. Rybka, Global weak solutions to a sixth order Cahn-Hilliard type equation, *SIAM J. Math. Analysis,* **44**, (2012), 3369-3387.

[6] M. D. Korzec, P. Nayar and P. Rybka, Global attractors of sixth order PDEs describing the faceting of growing surfaces, *J. Dynam. Differential Equations*, **28**, (2016), 49-67.
M.D. Korzec, P.Nayar, P.Rybka, Correction to: Global Attractors of Sixth Order PDEs Describing the Faceting of Growing Surfaces *J. Dynam. Differential Equations* (2019).

[7] P.Rybka, G.Wheeler, Convergence of solutions to a convective Cahn-Hilliard type equation of the sixth order in case of small deposition rates, *SIAM Journal on Mathematical Analysis*, **55** vol. 5, (2023), 4193-5861.

[8] P.Rybka, Ph.Schrader, G.Wheeler, Convergence of Sobolev curvature flow to the curve shortening flow, *in preparation*.

[9] T.V.Savina, A.A.Golovin, S.H.Davis, A.A.Nepomnyashchy and P.W.Voorhees, Faceting of a growing crystal surface by surface diffusion, *Phys. Rev. E*, **67**, (2003), 021606.

On non-uniqueness of solutions to the two-dimensional Navier-Stokes equations in the half space

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1 Introduction

In this talk we consider the two-dimensional forced Navier-Stokes equations in the half space under the noslip boundary condition:

$$\begin{aligned} &\partial_t u - \Delta u + u \cdot \nabla u + \nabla p = F\,, \qquad t > 0,\ x \in \mathbb{R}^2_+\,, \\ &\nabla \cdot u = 0, \qquad t \geq 0,\ x \in \mathbb{R}^2_+\,, \\ &u|_{x_2=0} = 0, \qquad u|_{t=0} = 0. \end{aligned} \tag{1}$$

Here $x = (x_1, x_2)^\top \in \mathbb{R}^2_+ = \{x \in \mathbb{R}^2 \mid x_2 > 0\}$, $u = (u_1(t,x), u_2(t,x))^\top$ is the unknown velocity field of the fluid, $p = p(t,x)$ is the unknown pressure field, and $F = (F_1(t,x), F_2(t,x))^\top$ is a given force specified later. We use the standard notations of derivatives; $\partial_j = \frac{\partial}{\partial x_j}$, $\nabla = (\partial_1, \partial_2)^\top$, $\Delta = \sum_{j=1}^2 \partial_j^2$, $u \cdot \nabla = \sum_{j=1}^2 u_j \partial_j$, $\nabla \cdot u = \sum_{j=1}^2 \partial_j u_j$.

It is well known by now that if the external force F is smooth and decays fast enough at infinity, then there exists a global in time smooth solution to (1) with finite energy. The uniqueness of such solutions in a time interval $(0,T)$ can be also obtained, under the smallness condition on some scale critical norm, for example,

$$\sup_{0<t<T} t^{\frac14} \|u(t,\cdot)\|_{L^4(\mathbb{R}^2_+)}\,. \tag{2}$$

The concept of scale criticality is originated from the fact that if (u,p) is a solution to (1), then

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)\,, \quad p_\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x) \tag{3}$$

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is a solution to (1) with F replaced by

$$F_\lambda(t,x) = \lambda^3 F(\lambda^2 t, \lambda x)\,. \tag{4}$$

Conventionally, the space-time norms which are invariant under the scalings (3), (4) are called the scale critical norms for u, p, and F, respectively. As a typical example, the norm

$$\sup_{t>0} t^{\frac{3}{4}} \|F(t,\cdot)\|_{L^{\frac{4}{3}}(\mathbb{R}^2_+)} \tag{5}$$

is a scale critical norm for the external force F, and it is not difficult to show that, if the norm of (5) for given F is small enough, then there exists a global in time unique mild solution of (1) (i.e., the solution to the integral equation) with finite energy satisfying the smallness condition on the norm (2); see, e.g., Kozono and Shimizu [7] for more general results in this direction on the n-dimensional Navier-Stokes equations in the whole space.

Recently, Albritton, Brué, and Colombo proved in [1] that the three-dimensional forced Navier-Stokes equations in $\mathbb{R}^3 \times (0,T)$ admits the non-uniqueness of solutions in the Lerey-Hopf class, where the non-uniqueness is achieved around a self-similar vortex ring. The argument for the non-uniqueness in [1] makes use of the linear instability of the self-similar profile, as in the program introduced by Jia and Šverák [6]. In the two-dimensional case, Albritton and Colombo [4] construtcted the non-unique Leray-Hopf solutions for the forced two-dimensional Navier-Stokes equations with hypodissipation, i.e., $-\Delta$ in (1) is replaced by $(-\Delta)^{\frac{\beta}{2}}$ with $0 < \beta < 2$, by using the approach of [1].

We note that the external force F taken in [1] belongs to $BC((0,T); L^1(\mathbb{R}^3)^3)$ and the non-unique solutions u in [1] belong to $BC((0,T); L^3(\mathbb{R}^3)^3)$ (here, BC means *bounded and continuous*), and the norms of u and F are large in these scale critical spaces. It should be emphasized here that the spaces $BC((0,T); L^1(\mathbb{R}^3)^3)$ and $BC((0,T); L^3(\mathbb{R}^3)^3)$ rather than $BC([0,T); L^1(\mathbb{R}^3)^3)$ and $BC([0,T); L^3(\mathbb{R}^3)^3)$ (i.e., the continuity up to $t = 0$) are essential for the non-uniqueness. Indeed, it is known by Okabe and Tsutsui [11] that for given external force in $BC([0,T); L^1(\mathbb{R}^3)^3)$ there exists a local in time mild solution to the three-dimensional Navier-Stokes equations, and the uniqueness of solutions is verified in the space $BC([0,T); L^{3,\infty}(\mathbb{R}^3)^3)$ (here, $L^{3,\infty}(\mathbb{R}^3)$ is the weak L^3 space in $\mathbb{R}^3$). Recently, Zhan [15] proved that the uniqueness holds even in $BC((0,T); L^{3,\infty}(\mathbb{R}^3)^3)$ but under the additional condition that there exists a mild solution in $BC([0,T); L^{3,\infty}(\mathbb{R}^3)^3)$ with the same initial data and force. The result of [15] is regarded as a kind of weak-strong uniqueness and sharply extends the result of Lions and Masmoudi [9] (see also the book of Lemarié-Rieusset [8]) on the uniqueness in $BC([0,T); L^3(\mathbb{R}^3)^3)$ for the three-dimensional *unforced* Navier-Stokes equations so that it can be applied also to the forced Navier-Stokes equations. We also note that the non-unique solutions obtained in [1] must be large in $BC((0,T); L^3(\mathbb{R}^3)^3)$, for the bilinear estimate established by Yamazaki [14] implies that the mild solutions of the three-dimensional Navier-Stokes equations must be unique if they are small in $L^\infty(0,\infty;L^{3,\infty}(\mathbb{R}^3)^3)$.

As for the initial boundary value problem, the result of the non-uniqueness in [1] is extended by the same authors in [2] to the three-dimensional forced Navier-Stokes equations in a smooth bounded domain under the noslip boundary condition. The approach of [2] is based on the gluing method, where the non-unique solutions in [1] are pasted strictly inside the domain using a cut-off, and then glued to the smooth outer solutions. In particular, the large self-similar flow, which is the source of the non-uniqueness and has a compact support in the ball with radius of $O(\sqrt{t})$ size, is located away from the boundary with $O(1)$ distance around the initial time, and the presence of the boundary plays a negligible role in the analysis. Motivated by the work of [1, 2], in this talk we are interested in the case when the large self-similar flow concentrates around the boundary at the initial time in the sense that the distance between the support of the self-similar flow and the boundary tends to zero as $t\to 0$. In this situation it is natural to expect that the effect of the boundary is more relevant, and indeed, we need to take into account a kind of the boundary layer created by the large self-similar flow. To capture the essence, we focus on the case of the flat boundary for the two-dimensional problem as described in (1).

To state the main result, let us introduce the vorticity field $\omega = \operatorname{rot} u = \partial_1 u_2 - \partial_2 u_1$, and then the velocity is formally recovered from the vorticity field by the Biot-Savart formula

$$K[\omega] = \nabla^\perp(-\Delta_D)^{-1}\omega\,, \tag{6}$$

where $\nabla^\perp = (\partial_2, -\partial_1)^\top$ and $(-\Delta_D)^{-1}g$ denotes the unique solution to the Poisson equation $-\Delta\varphi = g$ in $\mathbb{R}^2_+$ and $\varphi|_{x_2=0} = 0$, $\lim_{|x|\to\infty}\nabla\varphi = 0$. Let $\rho(x) = e^{\frac{|x|^2}{8}}$ and let $L^2_\rho(\mathbb{R}^2_+)$ be the weighted L^2 space defined as

$$\begin{aligned} L^2_\rho(\mathbb{R}^2_+) &= \{g\in L^2(\mathbb{R}^2_+)\mid \rho g\in L^2(\mathbb{R}^2_+)\}\,,\\ \langle g,\tilde g\rangle_{L^2_\rho(\mathbb{R}^2_+)} &= \langle \rho g,\rho\tilde g\rangle_{L^2(\mathbb{R}^2_+)}\,, \quad \|g\|_{L^2_\rho} = \|\rho g\|_{L^2(\mathbb{R}^2_+)}\,. \end{aligned} \tag{7}$$

Assumption 1.1. The velocity $U_E = (U_{E,1}(x), U_{E,2}(x))$ is smooth and compactly supported in $\mathbb{R}^2_+$ (in particular, $\operatorname{dist}(\operatorname{supp} U_E, \partial\mathbb{R}^2_+) > 0$) and is a solution to the stationary Euler equations in $\mathbb{R}^2_+$: $-U_E\cdot\nabla U_E + \nabla P_E = 0$, $\nabla\cdot U_E = 0$ in $\mathbb{R}^2_+$. Moreover, U_E is linearly unstable in the sense that the operator Λ_E in $L^2_\rho(\mathbb{R}^2_+)$ defined as

$$\begin{aligned} D(\Lambda_E) &= \{\omega\in L^2_\rho(\mathbb{R}^2_+)\mid \nabla\cdot(U_E\omega)\in L^2_\rho(\mathbb{R}^2_+)\}\\ \Lambda_E\omega &= -\nabla\cdot(U_E\omega) - K[\omega]\cdot\nabla\Omega_E \end{aligned} \tag{8}$$

has an isolated eigenvalue λ_E with $\Re(\lambda_E) > 0$ (here, $\Re(\lambda)$ is the real part of the complex number λ). Here $\Omega_E = \partial_1 U_{E,2} - \partial_2 U_{E,1}$ is the vorticity of the Euler flow U_E.

Remark 1.1. (1) The transport operator $T\omega = -\nabla \cdot (U_E\omega)$ is realized as a skew-adjoint operator in $L^2(\mathbb{R}^2_+)$ in virtue of $\nabla \cdot U_E = 0$, and therefore, it generates a bounded and strongly continuous semigroup in $L^2(\mathbb{R}^2_+)$. From this fact we also have that, since the support of U_E is compact, the growth bound of the semigroup generated by T in the weighted space $L^2_\rho(\mathbb{R}^2_+)$ is also 0. Since the operator $S\omega = -K[\omega] \cdot \nabla\Omega_E$ is a compact operator in $L^2_\rho(\mathbb{R}^2_+)$, the set $\sigma(\Lambda_E) \cap \{\lambda \in \mathbb{C} \mid \Re(\lambda) > 0\}$ consists of isolated eigenvalues with finite algebraic multiplicities, where $\sigma(A)$ denotes the set of the spectrum of the operator A.
(2) Let O_{cpt} be a smooth bounded domain such that $\operatorname{supp} U_E \subset O_{cpt} \subset \overline{O_{cpt}} \subset \mathbb{R}^2_+$. Then the space

$$L^2_\rho(\mathbb{R}^2_+)|_{O_{cpt}} = \{\omega \in L^2_\rho(\mathbb{R}^2_+) \mid \operatorname{supp}\omega \subset \overline{O_{cpt}}\} \tag{9}$$

is invariant under the action of Λ_E, in the sense that if $\omega \in D(\Lambda_E) \cap L^2_\rho(\mathbb{R}^2_+)|_{O_{cpt}}$, then $\Lambda_E\omega \in L^2_\rho(\mathbb{R}^2_+)|_{O_{cpt}}$. It is clear that any eigenfunction of Λ_E belongs to $D(\Lambda_E) \cap L^2_\rho(\mathbb{R}^2_+)|_{O_{cpt}}$. Therefore, λ is an eigenvalue of Λ_E if and only if so is for $\Lambda_E|_{O_{cpt}}$, where $\Lambda_E|_{O_{cpt}}$ is the restriction of Λ_E on $L^2_\rho(\mathbb{R}^2_+)|_{O_{cpt}}$. Since the spectrum of $\Lambda_E|_{O_{cpt}}$ located in the right-half plane also consists of isolated eigenvalues, we have

$$\sigma(\Lambda_E) \cap \{\lambda \in \mathbb{C} \mid \Re(\lambda) > 0\} = \sigma(\Lambda_E|_{O_{cpt}}) \cap \{\lambda \in \mathbb{C} \mid \Re(\lambda) > 0\}\,. \tag{10}$$

We have an example of U_E satisfying Assumption 1.1.

Theorem 1.1. *There exists U_E satisfying Assumption 1.1.*

As fot the proof of Theorem 1.1, let $\tilde{U}^{st}$ be the unstable radial flow with compact support in $\mathbb{R}^2$ constructed in [1, Proposition 2.2], which is obtained from Vishik's unstable radial vortices [12, 13]; see also Albritton, Brué, Colombo, Lellis, Giri, Janisch, and Kwon [3]. Then the unstable flow U_E in the half space, stated in Theorem 1.1, is obtained as $U_E = U_{E,R}(x) := \tilde{U}^{st}(x_1, x_2 - R)$ by taking $R > 1$ large enough.

As usual, we set $L^2_\sigma(\mathbb{R}^2_+) = \overline{C^\infty_{0,\sigma}(\mathbb{R}^2_+)}^{\|\cdot\|_{L^2(\mathbb{R}^2_+)}}$, where $C^\infty_{0,\sigma}(\mathbb{R}^2_+) = \{f \in C^\infty_0(\mathbb{R}^2_+)^2 \mid \nabla \cdot f = 0\}$. Here $C^\infty_0(\mathbb{R}^2_+)$ is the set of smooth and compactly supported functions in $\mathbb{R}^2_+$. We denote by $\mathbb{P}_\sigma$ the Helmholtz projection, i.e., the orthogonal projection from $L^2(\mathbb{R}^2_+)^2$ to $L^2_\sigma(\mathbb{R}^2_+)$. Let $\mathbb{A} = \mathbb{P}_\sigma\Delta$ be the Stokes operator in $L^2_\sigma(\mathbb{R}^2_+)$, i.e.,

$$\begin{aligned} D(\mathbb{A}) &= \{f \in L^2_\sigma(\mathbb{R}^2_+) \mid f \in W^{1,2}_0(\mathbb{R}^2_+)^2 \cap W^{2,2}(\mathbb{R}^2_+)^2\}\,, \\ \mathbb{A}f &= \mathbb{P}_\sigma\Delta f, \qquad f \in D(\mathbb{A})\,. \end{aligned} \tag{11}$$

Here $W^{k,p}(\mathbb{R}^2_+)$ and $W^{k,p}_0(\mathbb{R}^2_+)$, $k \in \mathbb{N}$, $1 \le p \le \infty$, are the standard Sobolev spaces. Then $-\mathbb{A}$ is nonnegative self-adjoint in $L^2_\sigma(\mathbb{R}^2_+)$, and $\mathbb{A}$ generates a strongly continuous and analytic semigroup, called the Stokes semigroup and denoted by $\{e^{t\mathbb{A}}\}_{t\ge0}$, in $L^2_\sigma(\mathbb{R}^2_+)$.

Definition 1.1. *Let $F \in L^1_{loc}(0,\infty; L^2(\mathbb{R}^2_+)^2)$. We say that u is a mild solution of* (1) *if u satisfies the following conditions:*
(i) $u \in L^\infty(0,T; L^2_\sigma(\mathbb{R}^2_+)) \cap C((0,T); L^2_\sigma(\mathbb{R}^2_+))$ *and* $u \in L^2(\delta, T; W^{1,2}_0(\mathbb{R}^2_+)^2)$ *for any* $0 < \delta < T < \infty$.
(ii) $\lim_{t\to 0} u(t) = 0$ *weakly in* $L^2_\sigma(\mathbb{R}^2_+)$.
(iii) *For any* $0 < s < t < \infty$, *the equality*

$$u(t) = e^{(t-s)\mathbb{A}} u(s) - \int_s^t e^{(t-\tau)\mathbb{A}} \mathbb{P}_\sigma \nabla \cdot (u \otimes u)(\tau) d\tau + \int_s^t e^{(t-\tau)\Delta} \mathbb{P}_\sigma F(\tau) d\tau \tag{12}$$

holds.

The main result of this talk is the non-uniqueness of mild solutions stated as follows.

Theorem 1.2. *Let $F = F_\alpha = \alpha(\partial_t u_E - \Delta u_E)$, where α is a positive constant and $u_E = \frac{1}{\sqrt{t}} U_E(\frac{x}{\sqrt{t}})$, and U_E satisfies Assumption 1.1. Then, there exists $\alpha_E \geq 1$ such that if $\alpha \geq \alpha_E$ then there exist at least two mild solutions of* (1).

It is clear that the self-similar flow αu_E itself is one mild solution of (1). In the proof of Theorem 1.2, the other solution is constructed by making use of the linear instability of αu_E for large α. We note that the support of $\alpha u_E(t,\cdot)$ in $\mathbb{R}^2_+$ converges to the origin as $t \to 0$. In Theorem 1.2, taking α large is essential in proving the non-uniqueness. Indeed, if α is small enough, then we have the smallness such as

$$\sup_{t>0} t^{\frac{3}{4}} \|F(t,\cdot)\|_{L^{\frac{4}{3}}(\mathbb{R}^2_+)} \leq C\alpha \ll 1\,.$$

Then the semigroup estimates such as $\|e^{t\mathbb{A}}\mathbb{P}_\sigma \nabla\cdot V\|_{L^4(\mathbb{R}^2_+)} \leq Ct^{-\frac{3}{4}}\|V\|_{L^2(\mathbb{R}^2_+)}$ for $V \in L^2(\mathbb{R}^2_+)^{2\times 2}$ and $\|e^{t\mathbb{A}}\mathbb{P}_\sigma F\|_{L^4(\mathbb{R}^2_+)} \leq Ct^{-\frac{1}{2}}\|F\|_{L^{\frac{4}{3}}(\mathbb{R}^2_+)}$ for $F \in L^{\frac{4}{3}}(\mathbb{R}^2_+)^2$ enable us to construct the unique mild solution satisfying the smallness condition for (2). That is, αu_E is the only solution of (1) satisfying the smallness condition on the scale critical norm (2) if α is small enough.

To prove Theorem 1.2 we basically follow the approach of [1], and hence, it is useful to introduce the self-similar variables

$$v(\tau,\xi) = e^{\frac{\tau}{2}} u(e^\tau, e^{\frac{\tau}{2}}\xi)\,, \quad \tau \in \mathbb{R}\,,\ \xi = (\xi_1,\xi_2) \in \mathbb{R}^2_+\,. \tag{13}$$

Then v satisfies the equations

$$\partial_\tau v - (\Delta + \frac{\xi}{2}\cdot\nabla + \frac{1}{2})v + v\cdot\nabla v + \nabla q = G\,, \quad \tau \in \mathbb{R}\,,\ \xi \in \mathbb{R}^2_+$$

with the divergence free condition $\nabla\cdot v = 0$ and the no-slip boundary condition $v|_{\xi_2=0} = 0$. Here $G(\tau,\xi) = e^{\frac{3}{2}\tau} F(e^\tau, e^{\frac{\tau}{2}}\xi)$, and Δ and ∇ are now the differential operators about the ξ variables. When F is chosen as in Theorem 1.2, the stationary flow $v(\tau,\xi) = \alpha U_E(\xi)$ clearly satisfies the equations. The other solution is then constructed around αU_E, and therefore, we are interested in the linearized operator in the self-similar variables ξ around the profile αU_E. This motivates us to study the operator $\mathbb{L}_\alpha$ defined as follows.

$$\begin{aligned} D(\mathbb{L}_\alpha) &= \{f \in L^2_\sigma(\mathbb{R}^2_+) \mid f \in W^{1,2}_0(\mathbb{R}^2_+)^2 \cap W^{2,2}(\mathbb{R}^2_+)^2, \quad \xi\cdot\nabla f \in L^2(\mathbb{R}^2_+)^2\}\,, \\ \mathbb{L}_\alpha f &= \frac{1}{\alpha}\big(\mathbb{A} + \frac{\xi}{2}\cdot\nabla + \frac{1}{2}\big)f - \mathbb{P}_\sigma\big(U_E\cdot\nabla f + f\cdot\nabla U_E\big)\,, \quad f \in D(\mathbb{L}_\alpha)\,. \end{aligned} \tag{14}$$

Note that the parameter α originally describes the size of the external force, while it plays a role of the viscosity coefficient in (14).

Since λ_E is an isolated unstable eigenvalue of Λ_E, there exists $r_E \in (0, \Re(\lambda_E))$ such that $\overline{B_{r_E}(\lambda_E)} \setminus \{\lambda_E\} \subset \rho_{re}(\Lambda_E)$. Here $B_r(\zeta) = \{\lambda \in \mathbb{C} \mid |\lambda - \zeta| < r\}$ and $\rho_{re}(A)$ denotes the resolvent set of the operator A. The following theorem about the existence of unstable eigenvalues of the operator $\mathbb{L}_\alpha$ is the key to show the non-uniqueness of the nonlinear problem.

Theorem 1.3. *For any $\epsilon \in (0, \frac{r_E}{2})$ there exists $\alpha_\epsilon \geq 1$ such that if $\alpha \geq \alpha_\epsilon$ then there exists an isolated eigenvalue λ_α of $\mathbb{L}_\alpha$ satisfying $|\lambda_\alpha - \lambda_E| < \epsilon$. In particular, $\mathbb{L}_\alpha$ has an unstable eigenvalue λ_α.*

Once Theorem 1.3 is proved, Theorem 1.2 follows from the similar argument as in [1, Section 4]. Indeed, the other solution v in the self-similar variables is constructed in the form

$$v(\tau,\xi) = \alpha U_E + \Re(e^{\alpha\lambda_{\alpha,max}\tau} V_{linear}(\xi)) + w(\tau,\xi)\,, \tag{15}$$

where $\lambda_{\alpha,max}$ is the maximal unstable eigenvalue of $\mathbb{L}_\alpha$, V_{linear} is its eigenfunction, and w is the remainder (for $\tau \to -\infty$) which is constructed by solving the perturbed Navier-Stokes equation in the self-similar variables.

As for the proof of Theorem 1.3, by considering the eigenprojection around the eigenvalue λ_E of Λ_E, the analysis is essentially reduced to the construction of the solution to the resolvent problem

$$\lambda v - \mathbb{L}_\alpha v = f \tag{16}$$

for the complex number λ belonging to $\rho_{re}(\Lambda_E)$ and for given f such that $g = \operatorname{rot} f \in L^2(\mathbb{R}^2_+)$ and $\operatorname{supp} g$ is compact in $\mathbb{R}^2_+$. From the condition $\lambda \in \rho_{re}(\Lambda_E)$ and $\alpha \gg 1$ it is natural to construct the solution around $K[(\lambda I - \Lambda_E)^{-1}g]$, which is the solution to the linearized Euler equations. This can be done as in the whole space case if the boundary condition is slip-type, however, it is not so straightforward if the boundary condition is noslip, due to the discrepancy between the boundary conditions in the linearized Euler equations and in the linearized Navier-Stokes equations. To handle this issue we follow the approach used in the stability analysis for the Prandtl boundary layer (e.g., see Gérard-Varet, Maekawa, and Masmoudi [5] and references therein), where we first solve the problem under the artificial boundary condition such as the perfect slip boundary condition (i.e., the vorticity is zero on $\partial\mathbb{R}^2_+$) by analyzing the vorticity equations, and the noslip boundary condition is then recovered by constructing a suitable boundary layer. In constructing the boundary layer, we make use of the advantage that the support of the profile $U_E(\xi)$ is strictly away from the boundary in the self-similar variables. In particular, the boundary layer can be taken independently of the equation itself. The instability in the high frequency for the boundary layer does not appear here in virtue of the fact that the support of the profile $U_E(\xi)$ is strictly away from the boundary, where the observation of this kind is motivated from the work of Maekawa [10] that verifies the Prandtl boundary layer expansion in the inviscid limit when the initial vorticity is located away from the boundary.

References

[1] D. Albritton, E. Brué, M. Colombo. Non-uniqueness of Leray solutions of the forced Navier-Stokes equations. Ann. Math. 196(1), 415-455 (2022).

[2] D. Albritton, E. Brué, M. Colombo. Gluing Non-unique Navier-Stokes Solutions. Annals of PDE (2023) 9:17.

[3] D. Albritton, E. Brué, M. Colombo, C. D. Lellis, V. Giri, M. Janisch, H. Kwon. Instability and nonuniqueness for the 2d Euler equations in vorticity form, after M. Vishik. arXiv preprint. arXiv:2112.04943 (2021).

[4] D. Albritton, M. Colombo. Non-uniqueness of Leray Solutions to the Hypodissipative Navier-Stokes Equations in Two Dimensions. Comm. Math. Phys. 402 (2023), no. 1, 429-446.

[5] D. Gérard-Varet, Y. Maekawa, N. Masmoudi. Optimal Prandtl expansion around a concave boundary layer. Anal. PDE 17 (2024), no. 9, 3125-3187.

[6] H. Jia, V. Šverák. Are the incompressible 3d Navier-Stokes equations locally ill-posed in the natural energy space ? J. Funct. Anal. 268(12), 3734-3766 (2015).

[7] H. Kozono, S. Shimizu. Navier-Stokes equations with external forces in Lorentz spaces and its application to the self-similar solutions. J. Math. Anal. Appl. 458 (2018) 1693-1708.

[8] P. G. Lemarié-Rieusset. Recent Developments in the Navier-Stokes Problem. Chapman & Hall/CRC Research Notes in Math. 431, Chapman & Hall/CRC, Boca Raton, FL, 2002.

[9] P.-L. Lions, N. Masmoudi. Uniqueness of mild solutions of the Navier-Stokes system in L^N. Commun. Partial Differ. Equ. 26, 2211-2226 (2001).

[10] Y. Maekawa. On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane. Comm. Pure Appl. Math. 67 (2014), no. 7, 1045-1128.

[11] T. Okabe, Y. Tsutsui. Remark on the strong solvability of the Navier-Stokes equations in the weak L^n space. Mathematische Annalen, 1-38, 2021.

[12] M. Vishik. Instability and non-uniqueness in the Cauchy problem for the Euler equations of an ideal incompressible fluid. Part I. arXiv preprint arXiv:1805.09426 (2018).

[13] M. Vishik. Instability and non-uniqueness in the Cauchy problem for the Euler equations of an ideal incompressible fluid. Part II. arXiv preprint arXiv:1805.09440 (2018).

[14] M. Yamazaki. The Navier-Stokes equations in the weak-L^n space with time-dependent external force. Mathematische Annalen, 317(4):635-675, 2000.

[15] Z. Zhan. Uniqueness of mild solutions to the Navier-Stokes equations in weak-type L^d space. Journal of Mathematical Fluid Mechanics. Vol. 27, article number 43, (2025).

ON THE SINGULAR LIMIT OF THE WEIGHTED ALLEN-CAHN EQUATION

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1. Introduction

This is a joint work with Hiroki Harashima (TAIJU LIFE INSURANCE COMPANY LIMITED). Let $\varepsilon \in (0,1)$, $\Omega := \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ and $W(s) := \dfrac{(1-s^2)^2}{2}$. We consider the following weighted Allen–Cahn equation:

$$\begin{cases} \varepsilon \partial_t \varphi^\varepsilon = \varepsilon \Delta \varphi^\varepsilon - K^\varepsilon \dfrac{W'(\varphi^\varepsilon)}{\varepsilon}, & (x,t) \in \Omega \times (0,\infty), \\ \varphi^\varepsilon(x,0) = \varphi_0^\varepsilon(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $K^\varepsilon \in C^2(\Omega)$ satisfies $\inf_{\varepsilon\in(0,1), x\in\Omega} K^\varepsilon(x) \geq C_1$ for some $C_1 > 0$ and φ_0^ε is a smooth given function with $\max_{x\in\Omega} |\varphi_0^\varepsilon(x)| \leq 1$ for any $\varepsilon \in (0,1)$. Roughly speaking, if $K^\varepsilon(x) \equiv 1$, the zero level set of the solution $M_t^\varepsilon = \{x \in \Omega \mid \varphi^\varepsilon(x,t) = 0\}$ converges to the mean curvature flow, under suitable assumptions (see [3, 4, 5, 6, 10] for example). Here, the family of hypersurfaces $\{M_t\}_{t\in[0,T)}$ is called the mean curvature flow if

$$v = h \quad \text{on } M_t, \quad \text{for all } t \in (0,T),$$

where v and h are the normal velocity vector and the mean curvature vector, respectively.

When K^ε is independent of ε (in this case, we denote K^ε by K) and K is smooth, it was shown by [11] and [1] that $\{M_t^\varepsilon\}_{t\in[0,T)}$ converges to the mean curvature flow with transport term $\{M_t\}_{t\in[0,T)}$, that is, the flow satisfies

$$v = h - \frac{\nabla^\perp K}{2K} \qquad \text{on } M_t, \tag{1.2}$$

where $\nabla^\perp K = (\nabla K \cdot n)n$ and n is the inner unit normal vector of M_t. For reasons discussed below, (1.2) can be said to be the weighted mean curvature flow, and flows similar to (1.2) can be seen in [8, 15], for example. Note that the flow (1.2) can also be obtained by the Allen-Cahn equation with transport term $\dfrac{\nabla^\perp K}{2K}$, which is different from (1.1) (see [14]).

Remark 1.1. By using the level set method, the weighted mean curvature flow (1.2) is expressed by the (weak) solution to the following equation:

$$\frac{\partial_t u}{|\nabla u|} = \frac{1}{\sqrt{K}} \mathrm{div} \left(\sqrt{K} \frac{\nabla u}{|\nabla u|} \right) \qquad \text{in } \mathbb{R}^d \times (0,\infty).$$

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Set the Radon measure μ_t^ε on Ω by

$$\mu_t^\varepsilon(\phi) := \int_\Omega \phi \left(\frac{\varepsilon |\nabla \varphi^\varepsilon(x,t)|^2}{2} + K^\varepsilon(x) \frac{W(\varphi^\varepsilon(x,t))}{\varepsilon} \right) dx, \qquad \phi \in C_c(\Omega),$$

where φ^ε is the solution to (1.1). Qi and Zheng [12] proved that the family $\{\mu_t^\varepsilon\}_{t\in[0,\infty)}$ converges to the weak solution to the flow (1.2) in the sense of Brakke, when K^ε is independent of ε and $K \in C^2(\Omega)$ (see also [7]). The aim of this study is a generalization of [12] with respect to the weight.

Remark 1.2. It is well-known that the mean curvature flow is the gradient flow of the surface energy $\mathcal{H}^{d-1}(M_t)$, where $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. When $K(x) \equiv 1$, under suitable conditions, the measure μ_t^ε converges to $\mathcal{H}^{d-1}\lfloor_{M_t}$, that is,

$$\int_\Omega \phi \, d\mu_t^\varepsilon \to \sigma \int_{M_t} \phi \, d\mathcal{H}^{d-1} \qquad \text{for any } \phi \in C_c(\Omega) \text{ as } \varepsilon \downarrow 0,$$

where $\sigma = \int_{-1}^1 \sqrt{2W(s)}\, ds$. Since $\mu_t^\varepsilon(\Omega)$ is the energy of the equation (1.1), it is natural that μ_t^ε converges to the (weak) solution to the mean curvature flow. In the same way, the reason why (1.2) is derived can be explained as follows: under suitable conditions we have

$$\int_\Omega \phi \, d\mu_t^\varepsilon \to \sigma \int_{M_t} \phi \sqrt{K} \, d\mathcal{H}^{d-1} \qquad \text{for any } \phi \in C_c(\Omega) \text{ as } \varepsilon \downarrow 0$$

and (1.2) is the gradient flow for the energy $\int_{M_t} \sqrt{K}\, d\mathcal{H}^{d-1}$ (see Remark 1.3 below).

Next we recall the definition of the Brakke flow. Assume that the family of smooth hypersufaces $\{M_t\}_{t\in[0,T)}$ has its smooth velocity vector v. Then for any smooth function $\phi \in C_c^1(\Omega \times [0,T))$ we have

$$\frac{d}{dt} \int_{M_t} \phi \, d\mathcal{H}^{d-1} = \int_{M_t} \left(\nabla^\perp \phi - \phi h \right) \cdot v + \partial_t \phi \, d\mathcal{H}^{d-1},$$

where h is the mean curvature. Hence, if $\{M_t\}_{t\in[0,T)}$ is the classical solution to (1.2), it holds that

$$\begin{aligned} \frac{d}{dt} \int_{M_t} \phi \sqrt{K} \, d\mathcal{H}^{d-1} &= \int_{M_t} (\nabla^\perp(\phi\sqrt{K}) - (\phi\sqrt{K})h) \cdot v + \partial_t \phi \sqrt{K} \, d\mathcal{H}^{d-1} \\ &= \int_{M_t} \left[\left(\nabla^\perp \phi + \phi \frac{\nabla^\perp K}{2K} - \phi h \right) \cdot v + \partial_t \phi \right] \sqrt{K} \, d\mathcal{H}^{d-1} \\ &= \int_{M_t} \left[\nabla^\perp \phi \cdot v - \phi |v|^2 + \partial_t \phi \right] \sqrt{K} \, d\mathcal{H}^{d-1}. \end{aligned} \tag{1.3}$$

By this, for any $\phi \in C_c^1(\Omega \times [0,T); [0,\infty))$, we have

$$\frac{d}{dt} \int_{M_t} \phi \sqrt{K} \, d\mathcal{H}^{d-1} \le \int_{M_t} \left[\nabla^\perp \phi \cdot v - \phi |v|^2 + \partial_t \phi \right] \sqrt{K} \, d\mathcal{H}^{d-1}. \tag{1.4}$$

This can be regarded as the generalization of Brakke's inequality (see [2]). One can check that $\{M_t\}_{t\in[0,T)}$ is the classical solution to (1.2) if and only if (1.4) holds for any $t \in (0,T)$ and for any $\phi \in C_c^1(\Omega \times [0,T); [0,\infty))$ (see [16]). For the singular limit of μ_t^ε, we show Brakke's inequality corresponding to (1.4) in the main theorem below (see (2.3)).

Remark 1.3. Substituting $\phi \equiv 1$ into (1.3), we obtain

$$\frac{d}{dt}\int_{M_t} \sqrt{K}\, d\mathcal{H}^{d-1} = -\int_{M_t} |v|^2 \sqrt{K}\, d\mathcal{H}^{d-1}.$$

Therefore (1.2) is the gradient flow of the energy $\int_{M_t} \sqrt{K}\, d\mathcal{H}^{d-1}$.

2. Main results

In this section we show the convergence of (1.1) to the Brakke flow of (1.2) for $K \in W^{2,p}(\Omega)$ with $p > \frac{d}{2}$.

We recall the properties of the varifolds and refer to [13, 16] for more details. For $d \times d$ matrix $A = (a_{ij})$ and $B = (b_{ij})$, we denote $A \cdot B := \sum_{i,j} a_{ij} b_{ij}$. For $d, k \in \mathbb{N}$ with $k < d$, we define the space of k-dimensional subspace of $\mathbb{R}^d$ by $\mathbb{G}(d,k)$. Note that we often identify $S \in \mathbb{G}(d,k)$ with the orthogonal projection matrix of $\mathbb{R}^d$ onto S. Set $G_k(U) := U \times \mathbb{G}(d,k)$ for an open set $U \subset \mathbb{R}^d$. We say the measure V a general k-varifold on U if V is a Radon measure on $G_k(U)$. We denote the set of all general k-varifolds on U by $\mathbb{V}_k(U)$. For $V \in \mathbb{V}_k(U)$, we define the weight measure $\|V\|$ by

$$\|V\|(\phi) := \int_{G_k(U)} \phi(x)\, dV(x,S) \qquad \text{for all } \phi \in C_c(U).$$

We say the varifold $V \in \mathbb{V}_k(U)$ rectifiable if there exist a $\mathcal{H}^k$-measurable k-countably rectifiable set $M \subset U$ and positive function $\theta \in L^1_{loc}(\mathcal{H}^k\lfloor_M)$ such that

$$V(\phi) = \int_M \phi(x, T_x M)\theta(x)\, d\mathcal{H}^k \qquad \text{for all } \phi \in C_c(G_k(U)),$$

where $T_x M$ is the approximate tangent space of M at x, with respect to θ (see [13]). Moreover, we say V integral if $\theta \in \mathbb{N}$ $\mathcal{H}^k$-a.e. on M. For $V \in \mathbb{V}_k(U)$, the first variation δV is given by

$$\delta V(X) := \int_{G_k(U)} \nabla X(x) \cdot S\, dV(x,S) \qquad \text{for all } X \in C^1_c(U;\mathbb{R}^d).$$

Let δV satisfy

$$\sup\{|\delta V(X)| \mid X \in C^1_c(U;\mathbb{R}^d),\ |X| \le 1,\ \operatorname{spt} X \subset K\} < \infty$$

for any compact set $K \subset U$. Then the domain of δV can be extended to $C_c(U;\mathbb{R}^d)$ uniquely, and the Riesz representation theorem tells us that there exist a Radon measure $\|\delta V\|$ and a $\|\delta V\|$-measurable function $\sigma : U \to \mathbb{R}^d$ such that

$$\delta V(X) = \int_U X \cdot \sigma\, d\|\delta V\|, \qquad \text{for all } X \in C_c(U).$$

Moreover, if $\|\delta V\| \ll \|V\|$, then the Radon-Nikodym theorem implies that there exists a measurable vector field $\tilde{h} = -\frac{d\|\delta V\|}{d\|V\|}\sigma$ such that

$$\delta V(X) = -\int_U X(x) \cdot \tilde{h}(x)\, d\|V\|(x) \qquad \text{for all } X \in C_c(U;\mathbb{R}^d).$$

We say $\tilde{h}$ the generalized mean curvature vector of V.

Remark 2.1. In general, the classical mean curvature vector h and the generalized mean curvature vector $\tilde{h}$ do not necessarily coincide. Let $M \subset \mathbb{R}^d$ be a closed smooth hypersurface and h be the classical mean curvature vector of M. Then for smooth function g with $\inf_{x\in\mathbb{R}^d} g(x) > 0$, one can define the rectifiable varifold $V \in \mathbb{V}_{d-1}(\mathbb{R}^d)$ by

$$V(\phi) = \int_M \phi(x, T_x M) g(x)\, d\mathcal{H}^{d-1} \qquad \text{for all } \phi \in C_c(G_k(U)).$$

Then

$$\begin{aligned}\delta V(X) &= \int_M (\mathrm{div}_M X) g\, d\mathcal{H}^{d-1} = \int_M \mathrm{div}_M (gX) - \nabla^\top g \cdot X\, d\mathcal{H}^{d-1}\\ &= -\int_M \left(h + \frac{\nabla^\top g}{g}\right)\cdot X\, g\, d\mathcal{H}^{d-1} = -\int_{\mathbb{R}^d} \left(h + \frac{\nabla^\top g}{g}\right)\cdot X\, d\|V\|\end{aligned}$$

holds for any $X \in C_c^1(\mathbb{R}^d;\mathbb{R}^d)$, where $\nabla^\top g = \nabla g - \nabla^\perp g$. Therefore the generalized mean curvature vector $\tilde{h}$ for V is given by

$$\tilde{h} = h + \frac{\nabla^\top g}{g}.$$

Substituting $g = \sqrt{K}$ into the above formula, we obtain

$$\tilde{h} = h + \frac{\nabla^\top K}{2K}. \tag{2.1}$$

By using (2.1), (1.2) is rewritten to

$$v = \tilde{h} - \frac{\nabla K}{2K} \qquad \text{on } M_t. \tag{2.2}$$

Next we define the weak solution to (1.2).

Definition 2.2 (Brakke flow of (1.2))**.** Let $d \geq 2$, $p > \frac{d}{2}$ and $p \geq \frac{4}{3}$ in addition if $d = 2$. Suppose that $K \in W^{2,p}(\Omega)$ and there exists $C_1 > 0$ such that $K(x) \geq C_1$ holds for any $x \in \Omega$. Assume that a family of Radon measures $\{\mu_t\}_{t\in[0,\infty)}$ on Ω satisfies:

(1) for a.e. $t \in [0,\infty)$, there exists a rectifiable varifold V_t such that $\|V_t\| = \mu_t$.

(2) for any $T \in (0,\infty)$,

$$\sup_{t\in(0,T), r\in(0,1), x\in\Omega} \frac{\mu_t(B_r(x))}{r^{d-1}} < \infty,$$

(3) for a.e. $t \in (0,\infty)$, δV_t is locally bounded and $\|\delta V_t\| \ll \|V_t\|$,

(4) the generalized mean curvature $\tilde{h}$ of V_t satisfies

$$\int_0^T \int_\Omega |\tilde{h}|^2\, d\|V_t\| dt < \infty$$

for any $T > 0$.

The family $\{\mu_t\}_{t\in[0,\infty)}$ is called the Brakke flow of (1.2) if

$$\int_\Omega \phi\, d\mu_t \Big|_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} \int_\Omega \left[\nabla^\perp \phi - \phi\left(\tilde{h} - \frac{\nabla K}{2K}\right)\right]\cdot\left(\tilde{h} - \frac{\nabla K}{2K}\right) + \partial_t \phi\, d\mu_t dt \tag{2.3}$$

holds for any $\phi \in C_c^2(\Omega \times [0,\infty);[0,\infty))$ and for any $t_1, t_2 \in [0,\infty)$ with $t_1 < t_2$.

Remark 2.3. (1) The inequality (2.3) is the generalization of (1.4) by using (2.2).

(2) By the upper bound of the density for μ_t and the assumptions for K, we may define $\frac{\nabla K}{2K}$ as an $L^2(\mu_t;\mathbb{R}^d)$ function, since there exists $C = C(d,p) > 0$ such that

$$\int |\phi|^2 \, d\mu \leq \left(\int |\phi|^{\frac{p(d-1)}{d-p}} \, d\mu \right)^{\frac{2(d-p)}{p(d-1)}} (\mu(\operatorname{spt}\phi))^{\frac{pd+p-2d}{p(d-1)}}$$
$$\leq (CD)^{\frac{2(d-p)}{p(d-1)}} \left(\int |\nabla \phi|^p \, dx \right)^{\frac{2}{p}} (\mu(\operatorname{spt}\phi))^{\frac{pd+p-2d}{p(d-1)}}$$

for any $\phi \in C^1_c(\mathbb{R}^d)$ and for any Radon measure μ on $\mathbb{R}^d$ with $\sup_{x\in\mathbb{R}^d, r>0} \frac{\mu(B_r(x))}{\omega_{d-1} r^{d-1}} \leq D$ by the inequality studied by [9] (see [14] for more details). The assumption for p is used for $2 \leq \frac{p(d-1)}{d-p}$ (see also Remark 2.5).

Let J_δ be the standard mollifier and $K^i := J_{1/i}K$ for $K \in W^{2,p}(\Omega)$. Assume that φ^{ε_i} is the solution to (1.1) when $\varepsilon = \varepsilon_i$ and $K^\varepsilon = K^i$ are substituted into (1.1). The following is the main theorem of this study.

Theorem 2.4. Assume all of the following:

(1) $d \geq 2$, $p > \frac{d}{2}$ and $p \geq \frac{4}{3}$ in addition if $d = 2$.
(2) $K \in W^{2,p}(\Omega)$ and there exists $C_1 > 0$ such that $K(x) \geq C_1$ holds for any $x \in \Omega$.
(3) A positive sequence $\{\varepsilon_i\}_{i=1}^\infty$ satisfies $\varepsilon_i \to 0$ as $i \to \infty$ and for some $\lambda \in (0, \frac{1}{16})$ it holds that
$$\|\nabla^m K^i\|_{L^\infty} \leq \frac{1}{\varepsilon_i^{m\lambda}}$$
for any $m = 1, 2$ and for any $i \in \mathbb{N}$.
(4) The smooth initial data $\varphi_0^{\varepsilon_i}$ satisfies the following:
  (a) $\|\varphi_0^{\varepsilon_i}\|_{L^\infty} \leq 1$ for any $i \in \mathbb{N}$.
  (b) It holds that
$$\sup_{i\in\mathbb{N}, r\in(0,1), x\in\Omega} \frac{\mu_0^{\varepsilon_i}(B_r(x))}{r^{d-1}} < \infty.$$
  (c) There exists $\beta \in (\frac{3}{8}, \frac{1}{2})$ such that
$$\sup_{x\in\Omega} \left(\frac{\varepsilon_i |\nabla \varphi_0^{\varepsilon_i}(x)|^2}{2} - K^i(x) \frac{W(\varphi_0^{\varepsilon_i}(x))}{\varepsilon_i} \right) \leq \frac{1}{\varepsilon_i^\beta} \qquad \text{for any } i \in \mathbb{N}.$$

Then there exist a family of Radon measures $\{\mu_t\}_{t\in[0,\infty)}$ on Ω and a subsequence $\{\varepsilon_i\}_{i=1}^\infty$ (denoted by same index) such that the following hold:

(1) For any $t \geq 0$, $\mu_t^{\varepsilon_i} \overset{*}{\rightharpoonup} \mu_t$ as $i \to \infty$, that is,
$$\int_\Omega \phi \, d\mu_t^{\varepsilon_i} \to \int_\Omega \phi \, d\mu_t \qquad \text{for any } \phi \in C_c(\Omega).$$
(2) $\{\mu_t\}_{t\in[0,\infty)}$ is the Brakke flow of (1.2).

Remark 2.5. (1) Let $u \in W^{1,p}(\mathbb{R}^d)$ with $\|\nabla u\|_{L^p(\mathbb{R}^d)} \neq 0$, $\tilde{x} := x/r$, and $\tilde{u}(\tilde{x}) := ru(x)$ for $r > 0$. Then $\|\nabla_{\tilde{x}} \tilde{u}\|_{L^p(\mathbb{R}^d)} = r^{2-d/p} \|\nabla u\|_{L^p(\mathbb{R}^d)}$ and thus $\lim_{r\downarrow 0} \|\nabla_{\tilde{x}} \tilde{u}\|_{L^p(\mathbb{R}^d)} = 0$ if and only if $p > \frac{d}{2}$. We use the smallness of the norm at this rescale to study the problem. Related to this, Tonegawa and Tsukamoto [17] proved that the singular limit of the stationary solution to the Allen-Cahn equation with given transport term $g \in W^{1,p}(\Omega;\mathbb{R}^d)$ is the weak solution to the prescrived mean curvature equation $\tilde{h} = g^\perp$ when $p > \frac{d}{2}$.

(2) Theorem 2.4 is a partial generalization of the results of [12] (the integrality of $\frac{1}{\sigma\sqrt{K}} V_t$ is shown in [12] when $K \in C^2(\Omega)$, but Theorem 2.4 does not show it). The proof of Theorem 2.4 is provided by a revisiting of the energy estimates with respect to the monotonicity formula obtained by [12].

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References

[1] Alfaro, M., Garcke, H., Hilhorst, D., Matano, H., and Schätzle, R., *Motion by anisotropic mean curvature as sharp interface limit of an inhomogeneous and anisotropic Allen-Cahn equation*, Proc. Roy. Soc. Edinburgh Sect. A, **140** (2010), 673–706.

[2] Brakke, K. A., *The motion of a surface by its mean curvature*, Princeton University Press, Princeton, N.J., (1978).

[3] Bronsard, L. and Kohn, R. V., *Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics*, J. Differential Equations, **90** (1991), 211–237.

[4] Chen, X., *Generation and propagation of interfaces for reaction-diffusion equations*, J. Differential Equations, **96** (1992), 116–141.

[5] Evans, L. C., Soner, H. M., and Souganidis, P. E., *Phase transitions and generalized motion by mean curvature*, Comm. Pure Appl. Math., **45** (1992), 1097–1123.

[6] Ilmanen, T., *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, J. Differential Geom., **38** (1993), 417–461.

[7] Jiang, G.-C., Wang, C.-J., and Zheng, G.-F., *Convergence of solutions of some Allen-Cahn equations to Brakke's mean curvature flow*, Acta Appl. Math., **167** (2020), 149–169.

[8] Laux, T. and Lelmi, J., *Large data limit of the MBO scheme for data clustering: convergence of the dynamics*, J. Mach. Learn. Res., **24** (2023), Paper No. [344], 49.

[9] Meyers, N. G. and Ziemmer, W. P., *Integral inequalities of Poincaré and Wirtinger type for BV functions*, Amer. J. Math. **99** (1977), no. 6, 1345–1360.

[10] Mugnai, L. and Röger, M., *Convergence of perturbed Allen-Cahn equations to forced mean curvature flow*, Indiana Univ. Math. J., **60** (2011), 41–75.

[11] Nakamura, K.-I. and Matano, H., Hilhorst, D., and Schätzle, R., *Singular limit of a reaction-diffusion equation with a spatially inhomogeneous reaction term*, J. Statist. Phys., **95** (1999), 1165–1185.

[12] Qi, Y. and Zheng, G.-F., *Convergence of solutions of the weighted Allen-Cahn equations to Brakke type flow*, Calc. Var. Partial Differential Equations, **57** (2018), Paper No. 133, 41.

[13] Simon, L., *Lectures on geometric measure theory*, Proc. Centre Math. Anal. Austral. Nat. Univ. **3** (1983).

[14] Takasao, K. and Tonegawa, Y., *Existence and regularity of mean curvature flow with transport term in higher dimensions*, Math. Ann. **364** (2016), 857–935.

[15] Timonov, A., *Numerical solution of a regularized weighted mean curvature flow problem for electrical conductivity imaging*, SIAM J. Sci. Comput., **41** (2019), B1137–B1154.

[16] Tonegawa, Y., *Brakke's mean curvature flow*, SpringerBriefs in Mathematics, Springer, Singapore, (2019).

[17] Tonegawa, Y. and Tsukamoto, Y., *A diffused interface with the advection term in a Sobolev space*, Calc. Var. Partial Differential Equations, **59** (2020), Paper No. 184, 23.

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A minimizing movements approach with a level set formulation for evolving spirals by crystalline eikonal curvature flow

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1. Introduction

In this talk, we propose a numerical method for the evolution of spirals in a bounded domain $\Omega \subset \mathbb{R}^2$ by a crystalline curvature flow. We denote the evolving spirals at time $t \geq 0$ by $\Sigma(t)$, and use $\mathbf{n} \in S^1$ for a continuous unit normal vector field of $\Sigma(t)$ defining the orientation of the evolution. In this evolution, the normal velocity of $\Sigma(t)$ is formally defined by

$$V_\gamma = -\kappa_\gamma + f, \tag{1}$$

where V_γ and κ_γ respectively denote the normal velocity and the crystalline curvature of $\Sigma(t)$, and f is a given driving force. Each spiral in $\Sigma(t)$ is attached to a stationary center denoted by $a_1, \ldots, a_N \in \Omega$, and in the evolution of $\Sigma(t)$, portions of different spirals may merge. To describe the merger of spirals during the evolution, we formulate $\Sigma(t)$ by the level set method developed in [13, 12].

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary. We remove small discs $B_r(a_j) = \{x \in \mathbb{R}^2;\ |x - a_j| < r\}$ from Ω, and set $W = \Omega \setminus \overline{B_r(a_j)}$. We choose the radius $r > 0$ small enough so that $\overline{B_r(a_i)} \cap \overline{B_r(a_j)} = \emptyset$ if $i \neq j$, and $\overline{B_r(a_j)} \subset \Omega$ for $j = 1, \ldots, N$ and then ∂W is still smooth. Let m_j denote a signed number of spirals associated with a_j meaning that

- $|m_j|$ curves are attached to a_j as their endpoint,
- if $m_j > 0$ (resp. $m_j < 0$), then curves rotate around a_j with counterclockwise (resp. clockwise) rotation when $V_\gamma > 0$.

Set θ be a function defined by the formula

$$\theta(x) = \sum_{j=1}^{N} m_j \arg(x - a_j).$$

Then, $\Sigma(t)$ and $\mathbf{n}$ are given by

$$\Sigma(t) = \{x \in \overline{W};\ u(t,x) - \theta(x) \equiv 0 \mod 2\pi\mathbb{Z}\}, \quad \mathbf{n} = -\frac{\nabla(u - \theta)}{|\nabla(u - \theta)|} \tag{2}$$

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with an auxiliary function $u\colon [0,T)\times\overline{W}\to\mathbb{R}$ and $T>0$.

We refer to the survey paper [7] for the crystalline curvature flow, and the book [6] for the level set formulation of the geometric evolution equation. According to [6], when the surface energy of interface Σ is given by the functional

$$\int_\Sigma \gamma_0(\mathbf{n})d\sigma$$

with a convex function $\gamma_0\colon S^1\to(0,\infty)$, the anisotropic curvature κ_γ of (2) is formally of the form

$$\kappa_\gamma = -\mathrm{div}\Big(\xi(\nabla(u-\theta))\Big),$$

where $\xi=\nabla\gamma$ and $\gamma(p)=|p|\gamma_0(p/|p|)$ for $p\neq 0$. Moreover, we consider an anisotropic normal velocity for $\Sigma(t)$, which is corresponding to the Finsler metric given by γ, that is,

$$V_\gamma = \frac{u_t}{\gamma(\nabla(u-\theta))}.$$

Then, (1) is represented as

$$u_t-\gamma(\nabla(u-\theta))\Big\{\mathrm{div}\Big(\xi(\nabla(u-\theta))\Big)+f\Big\}=0 \quad \text{in } (0,T)\times W. \tag{3}$$

The crystalline curvature is a singular and nonlocal anisotropic curvature so that the Wulff diagram

$$\mathcal{W}_\gamma=\{p\in\mathbb{R}^2 \mid \gamma^\circ(p)\le 1\},$$

is a convex polygon, where γ° is the support function of $\{p\in\mathbb{R}^2;\ \gamma(p)\le 1\}$, that is,

$$\gamma^\circ(p)=\sup\{p\cdot q;\ \gamma(q)\le 1\}.$$

In summary, we here assume that γ satisfies the following:

(A1) $\gamma\colon\mathbb{R}^2\to[0,\infty)$ is convex,

(A2) γ is positively homogeneous of degree 1,

(A3) There exists $\Lambda_\gamma>0$ such that $\Lambda_\gamma^{-1}\le\gamma\le\Lambda_\gamma$ on S^1,

(A4) $\mathcal{W}_\gamma=\{p;\ \gamma^\circ(p)\le 1\}$ is a convex polygon.

To establish (A4), it is natural to give γ° by a piecewise linear function of the form

$$\gamma^\circ(p)=\max_{0\le j\le N_\gamma-1}\tilde{n}\cdot p, \quad \text{with} \quad \tilde{n}_j=\tilde{r}_j(\cos\tilde{\vartheta}_j,\sin\tilde{\vartheta}_j).$$

However, it implies that γ is also a piecewise linear function, that is,

$$\gamma(p)=\max_{0\le j\le N_\gamma-1} n_j\cdot p \quad \text{with} \quad n_j=r_j(\cos\vartheta_j,\sin\vartheta_j), \tag{4}$$

since $\gamma^{\circ\circ}=\gamma$ when γ is convex. We point out that it is not immediately clear how to make sense of (3) classically, as it involves formally taking the first and second derivatives of γ. Therefore, we propose a new numerical algorithm to compute the solution of (3), particularly with γ of the form (4), in the sense of a weak formulation. Here, we assume the following for n_j in (4) to guarantee (A1)–(A4):

(W1) $\vartheta_j \in [\vartheta_0, \vartheta_0 + 2\pi]$ for $j = 0, 1, 2, \ldots, N_\gamma - 1$,

(W2) $\vartheta_j < \vartheta_{j+1} < \vartheta_j + \pi$ for $j = 0, 1, \ldots, N_\gamma - 1$, where $\vartheta_{N_\gamma} = \vartheta_0 + 2\pi$,

(W3) The set $\{n_j;\ j = 0, 1, 2, \ldots, N_\gamma - 1\}$ is minimal, i.e.,

$$\gamma^\circ(p) \neq \max\{\tilde{n}_j \cdot p;\ j \neq k\}$$

for any $k \in \{0, \ldots, N_\gamma - 1\}$.

The typical example of γ and γ° is

$$\gamma(p) = \|p\|_{\ell^1} = |p_1| + |p_2|, \quad \gamma^\circ(p) = \|p\|_{\ell^\infty} = \max\{|p_1|, |p_2|\},$$

and thus $\mathcal{W}_\gamma = [-1, 1]^2$. They can be given by

$$n_j = \sqrt{2}\left(\cos\frac{\pi(2j+1)}{4}, \sin\frac{\pi(2j+1)}{4}\right), \quad \tilde{n}_j = \left(\cos\frac{\pi j}{2}, \sin\frac{\pi j}{2}\right).$$

The variational approach by [2, 5] is a powerful option for formulating our problem. Almgren, Taylor and Wang [2] launched an algorithm for the isotropic mean curvature flow by regarding it as a minimizing problem of the functional that consists of its perimeter and deformation. This idea is also extended to the problem with driving force by [9], or the crystalline case by [1]. Chambolle [5] proposed an algorithm combining the above idea and a level set formulation using a signed distance function from an interface. Oberman, Osher, Takei and Tsai [10] proposed a numerical algorithm with split Bregman method [8] for interface motion by Chambolle's scheme, and extend it to the crystalline case. Thus, we extend Chambolle's algorithm to the evolution of spirals, and apply the split Bregman method. We also present some mathematical analysis and numerical results on our proposed approach.

2. The proposed formulation

Chambolle's algorithm includes the minimization of the total variation of u, interpreted as the perimeter of each of u's level sets, and L^2 norm, which measures the deformation from one discrete step to the next. To describe the functional measuring the deformation, this algorithm uses the signed distance function from the interface. When we apply this idea to our problem, the crucial difficulty is that a spiral curve is not interfacial curve, that is, it does not divide the domain into two regions. Thus, the signed distance function does not work well for our problem. To overcome this difficulty, we reconstruct Chambolle's algorithm with a general level set formulation due to [13, 12].

2.1. Proposed minimizing movements approach

Let the spiral $\Sigma \subset \overline{W}$ be given by

$$\Sigma = \{x \in \overline{W} \mid u(x) - \theta(x) \equiv 0 \mod 2\pi\mathbb{Z}\}$$

with $u \in C(\overline{W})$. Corresponding to the level set equation defined in (3), consider the minimizing problem of the functional

$$w \mapsto \int_W \gamma(\nabla(w-\theta))dx - \int_W fwdx + \frac{1}{2h}\left\| \frac{w-u}{\sqrt{\gamma(\nabla(u-\theta))}} \right\|_{L^2}^2. \tag{5}$$

Formally, the minimizer w^* of the above satisfies

$$-\mathrm{div}\Big(\xi(\nabla(w^*-\theta))\Big) - f + \frac{w^*-u}{h\gamma(\nabla(u-\theta))} = 0,$$

which implies

$$w^* = u + h\gamma(\nabla(u-\theta))\left\{\mathrm{div}\Big(\xi(\nabla(w^*-\theta))\Big) + f\right\}. \tag{6}$$

On the other hand, the implicit Eular scheme of (3) is of the form

$$u(t+h) = u(t) + \gamma(\nabla(u(t+h)-\theta))\left\{\mathrm{div}\Big(\xi(\nabla(u(t+h)-\theta))\Big) + f\right\}. \tag{7}$$

Comparing (6) and (7), we can regard w^* as an approximation of the solution $u(t+h)$ of (3) for the initial data $u(t)$. Thus, we define

$$S_h(\Sigma) = \{x \in \overline{W} \mid w^*(x) - \theta(x) \equiv 0 \mod 2\pi\mathbb{Z}\}$$

as the result of the evolution of Σ for a short time step $h > 0$. We can now iterate the stepping as follows.

(i) For given Σ_n $(n \geq 0)$ and $u_n \in C(\overline{W})$ satisfying

$$\Sigma_n = \{x \in \overline{W};\ u_n(x) - \theta(x) \equiv 0 \mod 2\pi\mathbb{Z}\},$$

compute the minimizer w^* of (5) with $u = u_n$.

(ii) Set $u_{n+1} = w^*$ and

$$\Sigma_{n+1} = \{x \in \overline{W};\ u_{n+1}(x) - \theta(x) \equiv 0 \mod 2\pi\mathbb{Z}\}.$$

This algorithm can be applied even if γ is not smooth, and we can then define $\Sigma(t) = \Sigma_n$ when $nh \leq t < (n+1)h$.

2.2. Existence of minimizing movements

To establish our proposed approach, we shall address two essential questions:

(i) the existence of the minimizer w^* of (5),

(ii) whether the minimizer w^* of (5) has a function ∇w^* again.

For the first question, we first give a rigorous definition of the functional $w \mapsto \int_W \gamma(\nabla(w-\theta))dx$. Let us define $J_\gamma \colon L^1(W) \to [0,\infty]$ by

$$J_\gamma = \sup\left\{ -\int_W w \mathrm{div}\varphi dx - \int_W \nabla\theta \cdot \varphi dx \;\middle|\; \varphi \in C_c^1(W;\mathbb{R}^2), \gamma^\circ(\varphi) \le 1 \right\}.$$

See also [3] for the above definition. One can find that J_γ has standard properties for the minimization problem, which is as the following.

Lemma 1 *Assume that* (A1)–(A3) *hold. Then, the following hold:*

(i) J_γ *is convex, and* $J_\gamma \ge 0$.

(ii) If $w \in BV(W)$, *then* $J_\gamma < \infty$.

(iii) $\liminf_{u\to v} J_\gamma(u) \ge J_\gamma(v)$ *in* $L^1(W)$.

(iv) If $w \in W^{1,1}(W)$, *then* $J_\gamma(w) = \int_W \gamma(\nabla(w-\theta))dx$.

According to the above result, we now define the functional (5). For given $f, g \in L^2(W)$ and $\psi \colon W \to \mathbb{R}$ such that $\varphi/\sqrt{\psi} \in L^2(W)$ for every $\varphi \in L^2(W)$, define

$$E(w;g) = \begin{cases} J_\gamma(w) - \displaystyle\int_W f w dx + \frac{1}{2h}\left\| \frac{w-g}{\sqrt{\psi}} \right\|_{L^2}^2 & \text{if } w \in L^2(W)\cap BV(W), \\ \infty & \text{otherwise.} \end{cases} \tag{8}$$

Then, the minimization problem of (5) can be regarded as the existence of the minimizer of $E(w;g)$, and we obtain the following result.

Theorem 2 *Let* $f, g \in L^2(W)$, *and* $\psi \colon W \to \mathbb{R}$ *be such that* $\varphi/\sqrt{\psi} \in L^2(W)$ *for every* $\varphi \in L^2(W)$. *Assume that* (A1)–(A3) *hold, and there exist positive constants* α, A *satisfying*

$$0 < \alpha < \psi < A \quad \text{on } W. \tag{9}$$

Then, there exists a unique minimizer $w^* \in L^2(W)\cap BV(W)$ *of* $E(w;g)$.

This result is established with a standard argument. In fact, let w_n be a minimizing sequence of $E(\cdot;g)$. Then, one can find that w_n is bounded in $L^2(W)\cap BV(W)$ so that w_n includes a subsequence, which is still denoted by w_n, and $w^* \in L^2(W)\cap BV(W)$ such that $w_n \rightharpoonup w^*$ in $L^2(W)$ and $w_n \to w^*$ in $L^1(W)$. Hence, w^* is the minimizer of E by the semicontinuity of E in $L^1(W)$.

We next discuss on the second problem, that is, the minimizer $w^* \in L^2(W)\cap BV(W)$ has a function ∇w^* on W. According to the theory of functions of bounded variation, there exists a Radon measure ν and a ν-measurable function $\sigma \colon W \to \mathbb{R}^2$ satisfying

$$\int_W w^* \mathrm{div}\varphi dx = -\int_W \varphi\cdot\sigma d\nu \quad \text{for every } \varphi \in C_c^1(W;\mathbb{R}^2).$$

We now pick up the absolutely continuous part of the signed measures

$$\nu^i(U) = \int_U \sigma^i d\nu \quad (i = 1, 2)$$

by Lebesgue's decomposition theorem, which are denoted by $\nu^i_{\rm ac}$, respectively. Then, there exists $v^i \in L^1(W)$ for $i = 1, 2$ such that

$$\nu^i_{\rm ac}(U) = \int_U v^i dx$$

for every Borel set $U \subset W$. Moreover, one can find that $v = (v^1, v^2)$ plays the role of ∇w^* in a weak sense, i.e.,

$$\int_W w^* {\rm div}\varphi dx = -\int_W v \cdot \varphi dx$$

when $w^* \in W^{1,1}(W)$. Hence, we here define the weak derivative ∇w^* of $w^* \in BV(W)$ by

$$\nabla w^* = (v^1, v^2). \tag{10}$$

To establish our algorithm, it remains to guarantee the bound (9) for $\psi = \gamma(\nabla(w^* - \theta))$. We here use the cut-off of it, and consequently we obtain the following result.

Theorem 3 *Let $f \in L^2(W)$ and $u_0 \in L^2(W) \cap BV(W)$. Let $h > 0$ and $0 < \alpha < A < \infty$ be constants. Then, there exists a unique sequence $\{u_n\} \subset L^2(W) \cap BV(W)$ such that*

$$u_{n+1} = \arg \min_{L^2(W) \cap BV(W)} E(\cdot; u_n) \quad \textit{with} \quad \psi = \min\{\max\{\gamma(\nabla u_n - \nabla\theta), \alpha\}, A\}.$$

3. Split Bregman method for the proposed algorithm

This section describes how we use the split Bregman method to efficiently construct the minimizer of $E(w; g)$ defined in (8).

The key idea to find w^* is to interpret the problem as a constraint minimization problem by dividing the dependent variable: find a minimizer (w^*, d^*) of

$$F(w, d) := \int_W \gamma(d - \nabla\theta)dx - \int_W fwdx + \frac{1}{2h}\left\|\frac{w - g}{\sqrt{\psi}}\right\|^2_{L^2} \quad \text{subject to } d = \nabla w.$$

To solve this problem, we apply the Bregman iteration due to [4, 14]. Let us introduce the Bregman distance of the form

$$D_F^{p^*, q^*}(w, d; \hat{w}, \hat{d}) = F(w, d) - F(\hat{w}, \hat{d}) - \langle p^*, w - \hat{w}\rangle - \langle q^*, d - \hat{d}\rangle$$

for $p^* \in \partial_p F(\hat{w}, \hat{d})$ and $q^* \in \partial_d F(\hat{w}, \hat{d})$, where

$$\partial_w F(\hat{w}, \hat{d}) = \{p \in H^1(W)^*;\ F(w, \hat{d}) \geq F(\hat{w}, \hat{d}) - \langle p, w - \hat{w}\rangle \quad \text{for } w \in H^1(W)\},$$
$$\partial_d F(\hat{w}, \hat{d}) = \{q \in L^2(W; \mathbb{R}^2)^*;\ F(\hat{w}, d) \geq F(\hat{w}, \hat{d}) - \langle q, d - \hat{d}\rangle \quad \text{for } d \in L^2(W; \mathbb{R}^2)\}.$$

Then, we set

$$(w^{k+1}, d^{k+1}) = \arg \min_{(w,d)\in H^1(W)\times L^2(W;\mathbb{R}^2)} \left\{ D_F^{p^k,q^k}(w, d; w^k, d^k) + \frac{\mu}{2}\|d - \nabla w\|_{L^2}^2 \right\},$$
$$(w^0, d^0) = (g, 0), \qquad p^k \in \partial_w F(w^k, d^k), \quad q^k \in \partial_d F(w^k, d^k)$$

with a constant $\mu > 0$. It should hold that $(w^*, d^*) = \lim_{k\to\infty}(w^k, d^k)$.

Since $(w, d) \mapsto d - \nabla w$ is linear, then finding (w^{k+1}, d^{k+1}) of the above is rephrased as the following:

$$\begin{cases} (w^{k+1}, d^{k+1}) = \arg \min_{(w,d)\in H^1(W)\times L^2(W;\mathbb{R}^2)} \left\{ F(w, d) + \frac{\mu}{2}\|d - \nabla w - b^k\|_{L^2}^2 \right\}, \\ \qquad b^{k+1} = b^k + \nabla w^{k+1} - d^{k+1}, \\ (w^0, d^0, b^0) = (g, 0, 0). \end{cases}$$

We solve the above by the following alternate iteration:

$$w_{\ell+1}^k = \arg \min_{w\in H^1(W)} \left\{ F(w, d_\ell^k) + \frac{\mu}{2}\|d_\ell^k - \nabla w - b^k\|_{L^2}^2 \right\}, \tag{11}$$
$$d_{\ell+1}^k = \arg \min_{d\in L^2(W;\mathbb{R}^2)} \left\{ F(w_{\ell+1}^k, d) + \frac{\mu}{2}\|d - \nabla w_{\ell+1}^k - b^k\|_{L^2}^2 \right\}, \tag{12}$$
$$(w_0^k, d_0^k) = (w^k, d^k).$$

The problem (11) can be solved by solving the following Eular–Lagrange equation:

$$w - h\mu\psi\Delta w = g + h\psi(f - \mu \mathrm{div}(d_\ell^k - b^k)), \qquad \text{in } W,$$
$$(d^k - \nabla w - b^k)\cdot \vec{\nu} = 0 \qquad \text{on } \partial W,$$

where $\vec{\nu}$ is the outer unit normal vector field of ∂W. On the other hand, the problem defined in (12) can be solved by finding the minimizer of

$$d \mapsto \int_W \gamma(d - \nabla\theta)dx + \frac{\mu}{2}\|d - \nabla w_{\ell+1}^k - b^k\|_{L^2}^2.$$

It is solved by considering the minimizer of integrand, i.e.,

$$d(x) = \arg \min_{d\in\mathbb{R}^2} \left\{ \gamma(d - \nabla\theta) + \frac{\mu}{2}|d - \nabla w_{\ell+1}^k(x) - b^k(x)|^2 \right\}$$

for every $x \in \overline{W}$. In the case when γ is given of the form (4) with (W1)–(W3), the above minimizer can be found by direct calculation; see [10, 11] for details.

Let us set

$$\begin{aligned} F_{\mu,\alpha}^k(w, d; g) &:= F(w, d) + \frac{\mu}{2}\|d - \nabla w - b^k\|_{L^2}^2 \\ &= \int_W (\gamma(d - \nabla\theta))dx - \int_W fw dx + \frac{1}{2h}\left\| \frac{w - g}{\sqrt{\max\{\gamma(\nabla g - \nabla\theta), \alpha\}}} \right\|_{L^2}^2 \\ &\quad + \frac{\mu}{2}\|d - \nabla w - b^k\|_{L^2}^2. \end{aligned}$$

(Note that, at the practical stage of numerical simulations, we just give the lower bound of ψ by a cut-off.) Our algorithm is consequently summarized as follows.

(i) Let an initial curve $\Sigma_0 \subset \overline{W}$ and $u_0 \in C(\overline{W})$ for (2) be given.

(ii) Fix parameters $\mu > 0$, $\alpha > 0$, $\varepsilon_{\rm in}$, and $\varepsilon_{\rm out}$.

(iii) For each time step $n = 0, 1, \dots, [T/h] - 1$, do the following outer loop.

   (a) Set $g = u_n$, and initialize $w^0 = u_n$, $d^0 = b^0 = 0$.

   (b) For each outer loop number $k = 0, 1, \dots$, do the following inner loop.

      i. Initialize $w^k_0 = w^k$, $d^k_0 = d^k$.

      ii. For each inner loop number $\ell = 0, 1, \dots$ and given functions (w^k_ℓ, d^k_ℓ), find the minimizers $w^k_{\ell+1}$ and $d^k_{\ell+1}$ of (11) and (12), respectively.

      iii. If $|F^k_{\mu,\alpha}(w^k_{\ell+1}, d^k_{\ell+1}; u_n) - F^k_{\mu,\alpha}(w^k_\ell, d^k_\ell; u_n)| < \varepsilon_{\rm in}$, then break the inner loop. Otherwise, continue ii updating the inner loop number to $\ell + 1$.

   (c) Set $w^{k+1} = w^k_{\ell+1}$, $d^{k+1} = d^k_{\ell+1}$, and $b^{k+1} = b^k + \nabla w^{k+1} - d^{k+1}$.

   (d) $|F^k_{\mu,\alpha}(w^{k+1}, d^{k+1}; u_n) - F^k_{\mu,\alpha}(w^k, d^k; u_n)| < \varepsilon_{\rm out}$, then break the outer loop. Otherwise, return to (b) updating the outer loop number to $k + 1$.

(iv) Set $u_{n+1} = w^{k+1}$.

We conclude this section by presenting an example of numerical simulation by our approach with the following settings:

$$\begin{cases} V_\gamma = 3(1 - 0.02\kappa_\gamma), \\ \gamma(p) = |p_1| + |p_2|, \quad \gamma^\circ(p) = \max\{|p_1|, |p_2|\}, \\ a_1 = (-0.8, -0.8), \quad a_2 = (0, 0.8), \quad a_3 = (0.8, 0), \\ m_1 = -2, \quad m_2 = 1, \quad m_3 = 3. \end{cases}$$

Figure 1 shows the profiles of Σ_n and the graphs of u_n drawing Σ_n at time $t = 0, 0.1, 0.2$ and 0.4.

References

[1] Fred Almgren and Jean E. Taylor. Flat flow is motion by crystalline curvature for curves with crystalline energies. *J. Differential Geom.*, 42(1):1–22, 1995.

[2] Fred Almgren, Jean E. Taylor, and Lihe Wang. Curvature-driven flows: a variational approach. *SIAM J. Control Optim.*, 31(2):387–438, 1993.

[3] M. Amar and G. Bellettini. A notion of total variation depending on a metric with discontinuous coefficients. *Annales de l'I.H.P. Analyse non linéaire*, 11(1):91–133, 1994.

[4] L. M. Brègman. A relaxation method of finding a common point of convex sets and its application to the solution of problems in convex programming. *Ž. Vyčisl. Mat i Mat. Fiz.*, 7:620–631, 1967.

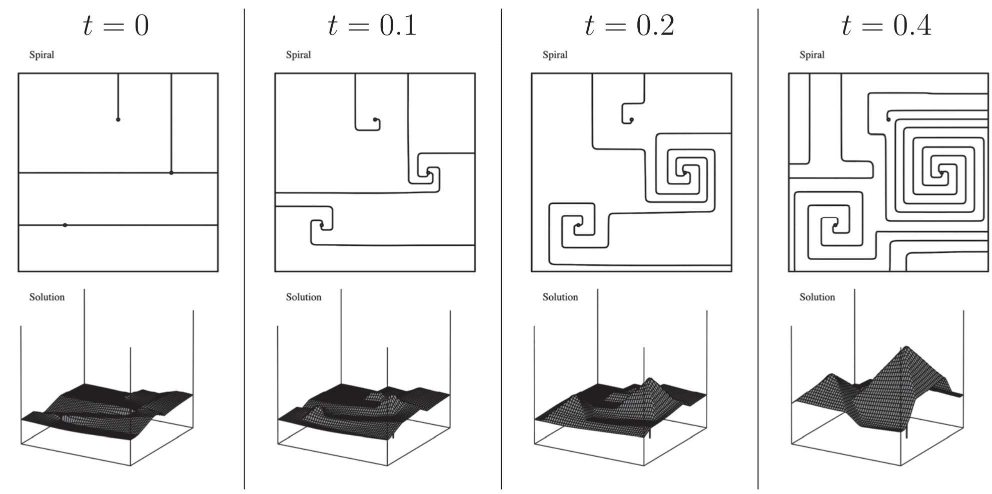

Figure 1: Numerical simulations of our approach with a square spiral setting.

[5] Antonin Chambolle. An algorithm for mean curvature motion. *Interfaces Free Bound.*, 6(2):195–218, 2004.

[6] Yoshikazu Giga. *Surface evolution equations: A level set approach.* Monographs in Mathematics. Birkhäuser Verlag, Basel, 2006.

[7] Yoshikazu Giga and Norbert Požár. Motion by crystalline-like mean curvature: A survey. *Bulletin of Mathematical Sciences*, 12(02):2230004, 2022.

[8] Tom Goldstein and Stanley Osher. The split bregman method for l1-regularized problems. *SIAM Journal on Imaging Sciences*, 2(2):323–343, 2009.

[9] Stephan Luckhaus and Thomas Sturzenhecker. Implicit time discretization for the mean curvature flow equation. *Calc. Var. Partial Differ. Equ.*, 3(2):253–271, 1995.

[10] Adam Oberman, Stanley Osher, Ryo Takei, and Richard Tsai. Numerical methods for anisotropic mean curvature flow based on a discrete time variational formulation. *Commun. Math. Sci.*, 9(3):637–662, 2011.

[11] T. Ohtsuka and Y.-H. R. Tsai. A minimizing movements approach for crystalline eikonal-curvature flows of spirals. *Interfaces Free Bound.*, 2025, published online first.

[12] T. Ohtsuka, Y.-H. R. Tsai, and Y. Giga. A level set approach reflecting sheet structure with single auxiliary function for evolving spirals on crystal surfaces. *J. Sci. Comput.*, 62(3):831–874, 2015.

[13] Takeshi Ohtsuka. A level set method for spiral crystal growth. *Adv. Math. Sci. Appl.*, 13(1):225–248, 2003.

[14] Stanley Osher, Martin Burger, Donald Goldfarb, Jinjun Xu, and Wotao Yin. An iterative regularization method for total variation-based image restoration. *Multiscale Model. Simul.*, 4(2):460–489, 2005.

On Kobayashi-Warren-Carter type total variation energy with fidelity

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1 Introduction

This is based on my joint work [GKKOS] with A. Kubo (Hokkaido University), H. Kuroda (Hokkaido University), J. Okamoto (Kyoto University) and K. Sakakibara (Kanazawa University).

We consider a total variation type energy which measures jumps different from the conventional total variation energy. The conventional total variation energy for a function u of bounded total variation in a domain $\Omega \subset \mathbb{R}^n$ (i.e., $u \in BV(\Omega)$) can be represented as

$$TV(u) = \int_{\Omega \setminus J_u} |Du| + \int_{J_u} |u^+ - u^-| \, d\mathcal{H}^{n-1},$$

where J_u denotes the (approximate) jump discontinuity of u and $u^\pm$ is a trace of u from each side of J_u; here, $\mathcal{H}^{n-1}$ denotes the $n-1$ dimensional Hausdorff measure. For a precise meaning of this formula see [AFP]. If Ω is an interval and u is monotone non-decreasing,

$$TV(u) = \int_{\Omega \setminus J_u} |Du| + \sum_{x_i \in J_u} |u(x_i + 0) - u(x_i - 0)| \, .$$

We consider more general total variation type energy

$$TV_K(u) = \int_{\Omega \setminus J_u} |Du| + \int_{J_u} K\left(|u^+ - u^-|\right) \, d\mathcal{H}^{n-1},$$

where $K(\rho)$ is a given strictly increasing continuous function for $\rho \geq 0$ with $K(0) = 0$.

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We are interested in minimizing

$$TV_{Kg}(u) = TV_K(u) + \mathcal{F}(u), \quad \mathcal{F}(u) = \frac{\lambda}{2} \int_\Omega |u - g|^2 \, dx$$

when $\lambda > 0$ is a constant and $g \in L^2(\Omega)$ is a given function. The term $\mathcal{F}$ is called a fidelity term. The functional

$$TV_g(u) = TV(u) + \mathcal{F}(u)$$

is sometimes called Rudin-Osher-Fatemi functional since the total variation flow is proposed by [ROF] to denoise the original image whose grey-level values equals g. For TV_g, there always exists a unique minimizer since the problem a strictly convex and lower semicontinuous in $L^2(\Omega)$. For general K, we assume

(K1w) $K : (0, \infty) \to [0, \infty)$ is non-decreasing and lower semicontinuous;

(K2w) K is subadditive, i.e., $K(\rho_1 + \rho_2) \leq K(\rho_1) + K(\rho_2)$;

(K3) $\lim_{\rho \downarrow 0} K(\rho)/\rho = 1$

Theorem 1 ([GKKOS]). *Let Ω be a bounded domain in $\mathbb{R}^n$. Assume (K1w), (K2w) and (K3). If $g \in L^\infty(\Omega)$, then there exists a minimizer of TV_{Kg} on $BV(\Omega)$.*

This can be proved by a direct method. It is based on lower semicontinuity result [AFP, Theorem 5.4], which is originally due to [BB] and a compact embedding $BV(\Omega) \subset L^1(\Omega)$. For compactness, we invoke the assumption $g \in L^\infty(\Omega)$. The uniqueness of a minimizer is in general not expected since the problem may not be strictly convex.

Our main concern is the profile of the minimizer u_*. For TV_g, it is known [CCN]

$$J_{u_*} \subset J_g, \quad u_*^+(x) - u_*^-(x) \leq g^+(x) - g^-(x) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in J_{u_*}$$

provided that $g \in BV(\Omega) \cap L^2(\Omega)$. In particular, if g has no jump, so does u_*. This type of result $J_{u_*} \subset J_g$ is first proved in one-dimensional setting [BF] and later extended by many authors, [CL]; see also [GKL] for a review.

However, if $K(\rho) = \rho/(1+\rho)$, then we expect that the minimizer prefer a big few jumps to many small jumps. In fact, we are able to prove in one-dimensional setting that all minimizer must be piecewise constant if g is continuous.

Let us give a precise statement. We assume

(K1) K is continuous and strictly increasing with $K(0) = 0$;

(K2) For $M > 0$, there exists a positive constant C_M such that

$$K(\rho_1) + K(\rho_2) \geq K(\rho_1 + \rho_2) + C_M \rho_1 \rho_2$$

for all $\rho_1, \rho_2 \geq 0$ with $\rho_1 + \rho_2 \leq M$.

Note that (K1) is stronger than (K1w) and (K2) is stronger than (K2w).

Theorem 2 ([GKKOS]). *Assume that K satisfies (K1), (K2) and (K3). For $g \in C[a,b]$, let $U \in BV(a,b)$ be a minimizer of TV_{Kg}. Then U must be a piecewise constant function satisfying $\inf g \leq U \leq \sup g$ on $[a,b]$. Let m be the number of jumps of U. Then*

$$m \leq [(b-a)\lambda/A_M] + 1, \quad A_M = \min(C_M, c_M/M)$$

where $\max g - \min g \leq M$ and $[r]$ denotes the integer part. Here c_M is a constant satisfying

$$\inf_{0 \leq \rho \leq M} K(\rho)/\rho \geq c_M.$$

Theorem 3. *Assume further that g is non-decreasing. Then U is non-decreasing and*

$$m \leq [(b-a)\lambda/C_M] + 1.$$

These theorems are interpreted as the upper bound for numbers of segmentation of image. For general discontinuous g, by approximation we are able to prove the existence of piecewise constant minimizer. Since there is no uniqueness result, there might exist a non-piecewise constant minimizer.

2 Source of the problem

Examples of TV_K are provided as a singular limit of the Kobayashi-Warren-Carter energy of the form

$$E^\varepsilon_{\mathrm{KWC}}(u,v) := \int_\Omega v^2 |Du| + E^\varepsilon_{\mathrm{sMM}}(v),$$
$$E^\varepsilon_{\mathrm{SMM}}(v) := \frac{\varepsilon}{2} \int_\Omega |\nabla v|^2 \, dx + \frac{1}{2^\varepsilon} \int_\Omega K(v) \, dx;$$

here $F(v)$ is a single-well potential typically of the form $F(v) = (v-1)^2$. In [KWC], Kobayashi, Warren and Carter introduce a gradient flow of $E^\varepsilon_{\mathrm{KWC}}(u,v)$ to model evolution of structures of multi-grain in the materials since. The solvability of their gradient system is non-trivial because of total variation term in $E^\varepsilon_{\mathrm{KWC}}$. There are a large literature on well-posedness; see e.g. [MSW1]. In more recent papers, the well-posedness of the system with value constraint of u in sphere or $SO(3)$ is discussed; see e.g. [MSW2]. This is a natural problem since u represents an average direction of crystalline structure of the grain.

In one-dimensional setting, the Gamma limit of $E^\varepsilon_{\mathrm{sMM}}$ (as $\varepsilon \to 0$) in the graph topology is obtained in [GOU]; see also [GOSU] for higher dimensional setting. It formally equals

$$E^0_{\mathrm{SMM}}(\Xi) = \sum_{i=1}^{\infty} 2\left(G(\xi_i^-) + G(\xi_i^+)\right), \quad G(s) = \left| \int_0^s \sqrt{F(\xi)} \, d\xi \right|,$$

if the limit of v in $\Omega = (a,b)$ equals a set-valued function Ξ of the form

$$\Xi(x) = \begin{cases} 1, & x \notin \Sigma \\ [\xi_i^-, \xi_i^+] \ (\ni 1) & \text{for} \quad x_i \in \Sigma, \end{cases}$$

where Σ is some (at most) countable set. The Gamma limit of $E_{\mathrm{KWC}}^{\varepsilon}$ equals

$$\begin{aligned} E_{\mathrm{KWC}}^0(u, \Xi) = \sum_{i=1}^{\infty} & \left((\xi_i^-)_+ \right)^2 \left| u^+(x_i) - u^-(x_i) \right| \\ & + 2 \left(G(\xi_i^-) + G(\xi_i^+) \right) + \int_{\Omega \setminus J_u} |Du|, \end{aligned}$$

where $\xi_+ = \max(\xi, 0)$ and $x_i \in \Sigma$. Here for u, L^1 type limit is considered. If we minimize $E_{\mathrm{KWC}}^0(u, \Xi)$ with fixed u, ξ^+ must be one since $[\xi^-.\xi^+] \ni 1$ and $G(1) = 0$. Thus

$$\begin{aligned} \inf_{\Xi} E_{\mathrm{KWC}}^0(u, \Xi) &= \sum_{i=1}^{\infty} \min_{\xi} \left(\xi_+^2 \left| u^+(x_i) - u^-(x_i) \right| + 2G(\xi) \right) + \int_{\Omega \setminus J_u} |Du| \\ &= TV_K(u), \end{aligned}$$

where

$$K(\rho) = \min_{\xi} \left(\xi_+^2 \rho + 2G(\xi) \right).$$

If $F(v) = (v-1)^2$, it is easy to see that

$$K(\rho) = \frac{\rho}{1+\rho}.$$

By a direct calculation, this K satisfies (K2) as well as (K1) and (K3). Indeed, for $\rho = \rho_1 + \rho_2, \rho_1, \rho_2 > 0$, we see

$$\begin{aligned} K(\rho_1) + K(\rho_2) &= \frac{\rho_1}{1+\rho_1} + \frac{\rho_2}{1+\rho_2} = \frac{\rho + 2\rho_1\rho_2}{1 + \rho + \rho_1\rho_2} \\ &= \frac{\rho}{1+\rho} + \frac{(2+\rho)\rho_1\rho_2}{(1+\rho)(1+\rho+\rho_1\rho_2)} \geq K(\rho) + C_M \rho_1 \rho_2 \end{aligned}$$

with $C_M = 2 \big/ (1+M)^3$ which yields (K2). In fact, we are able to prove [GKKOS] that (K2) is fulfilled if

$$\overline{\lim_{v \uparrow 1}} F'(v) \big/ (v-1) < \infty.$$

This kind of singular limit problem resembles to a singular limit problem of the Ambrosio-Tortorelli functional [AT] which yields the Mumford-Shah functional [MS] as a limit.

For a singular limit of original gradient flow, the reader is referred to [GKKOSU].

3 Idea of the proof

Let us give a rough idea of the proof for Theorem 2. It is important to introduce coincidence set

$$C = \left\{ x \in [a, b] \;\middle|\; U(x) = g(x) \right\}.$$

For a minimizer U of TK_{Kg}, it is not difficult to show that U is continuous on C and outside C, U is piecewise constant. Moreover,

$$\sup U \leq \sup g, \quad \inf U \geq \inf g \tag{1}$$

Furthermore, we are able to prove that U has at most one jump on (α, β) if $(\alpha, \beta) \cap C = \emptyset$ and $\alpha, \beta \in C$. Also, if $\alpha, \beta \in C$ is too close, then U must be monotone in (α, β); here (α, β) may include the point in C. To show these properties, we only need to assume subadditivity of K, i.e., (K2w) instead of (K2)

The key step in the proof of Theorem 2 is to prove that U must be constant on (α, β) if $\alpha \in C$ and $\beta \in C$ with $\alpha < \beta$ are too close.

Lemma 4. *Assume that K satisfies (K1), (K2) and (K3). Assume that $g \in C[a, b]$. Let $U \in BV(a, b)$ be a minimizer of TV_{Kg}. Assume that U is non-decreasing. Let $\alpha, \beta, \gamma \in C$ satisfy $a \leq \alpha < \gamma < \beta \leq b$ and $U(\beta) - U(\gamma) \geq U(\gamma) - U(\alpha) > 0$. Assume that $(\gamma, \beta) \cap C = \emptyset$. Then*

$$\beta - \alpha > C_M / \lambda$$

for $M \geq \max_{[\alpha,\beta]} g - \min_{[\alpha,\beta]} g$, where C_M is the constant in (K2).

We give a sketch of the proof of this lemma. Since $(\gamma, \beta) \cap C = \emptyset$, there is exactly one jump point x_1 in (γ, β). We compute $TV_{Kg}(U)$ and $TV_{Kg}(v)$ in (α, β), where v equals $U(\alpha)$ on (α, x_1) and equals $U(\beta)$ on (x_1, β); see Figure 1. It is not difficult to prove

$$TV_K(U) - TV_K(v) \geq K(\rho_1) + K(\rho_2) - K(\rho_1 + \rho_2) \geq C_M \rho_1 \rho_2 \tag{2}$$

for $\rho_1 := U(\gamma) - U(\alpha) \leq \rho_2 := U(\beta) - U(\gamma)$, since $\rho_1 + \rho_2 \leq M$ by (1). The proof for

$$\frac{2}{\lambda} \left(\mathcal{F}(U) - \mathcal{F}(v) \right) \geq -2\rho_1 \rho_2 (x_1 - \alpha) \tag{3}$$

is more involved. It is not difficult to show

$$\int_\gamma^x \left\{ (U - g)^2 - (v - g)^2 \right\} dx \geq -\rho_1 (\rho_1 + \rho_2)(x_1 - \gamma). \tag{4}$$

However, the proof for

$$\int_\alpha^\gamma \left\{ (U - g)^2 - (v - g)^2 \right\} dx \geq -\rho_1^2 (\gamma - \alpha) \tag{5}$$

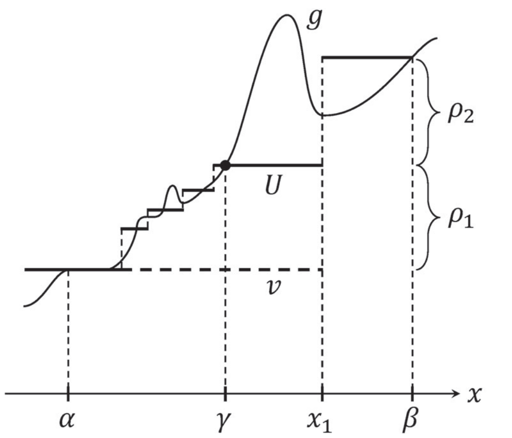

Figure 1: the graph of U, g and v FGUV

is more involved. If g is non-decreasing, we are able to prove

$$g(x) - U(x) \leq U(x_2 + 0) - U(x_2 - 0) \quad \text{for} \quad x \in F,$$

where $F = [x_0, x_2] \subset [a, b]$ is a maximal closed interval such that F is a constant in the interior of F. For general F, we only get such an inequality just in the average sense, i.e.,

$$\int_{x_0}^{x_2} (g(x) - U(x))\, dx \leq (U(x_2 + 0) - U(x_2 - 0))\,(x_2 - x_0) \tag{6}$$

The estimate (5) follows from (6). Combing (4) and (5), we obtain (3) since $\rho_2 \geq \rho_1$.

The estimates (2) and (3) yield

$$TV_{Kg}(U) - TV_{Kg}(v) \geq \rho_1\rho_2\,(C_M - (x_1 - \alpha)\lambda)\,.$$

If $\beta - \alpha \leq C_M/\lambda$, U cannot be a minimizer. This proves Lemma 4.

Similarly, we are able to prove that if U is continuous on (α, β) with $\alpha, \beta \in C$, then U is a constant on (α, β) even if α and β are not close. Lemma 4 and this observation yield Theorem 2.

References

[AFP] L. Ambrosio, N. Fusco and D. Pallara, Functions of bounded variation and free discontinuity problems. Oxford Mathematical Monographs, Clarendon Press, *Oxford University Press, New York,* 2000.

[AT] L. Ambrosio and V. M. Tortorelli, Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence. *Comm. Pure Appl. Math.* **43** (1990), no. 8, 999–1036.

[BF] M. Bonforte and A. Figalli, Total variation flow and sign fast diffusion in one dimension. *J. Differential Equations* **252** (2012), 4455–4480.

[BB] G. Bouchitté and G. Buttazzo, New lower semicontinuity results for non convex functionals defined on measures. *Nonlinear Anal.* **15** (1990), 679–692.

[CCN] V. Caselles, A. Chambolle and M. Novaga, The discontinuity set of solutions of the TV denoising problem and some extensions. *Multiscale Model. Simul.* **6** (2007), 879–894.

[CL] A. Chambolle and M. Łasica, Inclusion and estimates for the jumps of minimizers in variational denoising. *SIAM Journal on Imaging Sciences* **17** (2024), 1844–1878.

[GKKOS] Y. Giga, A. Kubo, H. Kuroda, J. Okamoto and K. Sakakibara, Piecewise constant profiles minimizing total variation energies of Kobayashi-Warren-Carter type with fidelity. *Hokkaido University Preprint Series in Mathematics* **1159** (2024).

[GKKOSU] Y. Giga, A. Kubo, H. Kuroda, J. Okamoto, K. Sakakibara and M. Uesaka, Fractional time differential equations as a singular limit of the Kobayashi-Warren-Carter system. *Proc. Roy. Soc. Edinburgh Sect. A*, published online (2024), 1–37.

[GKL] Y. Giga, H. Kuroda and M. Łasica, A few topics on total variation flows, *Hokkaido University Preprint Series in Mathematics* **1156** (2024).

[GOSU] Y. Giga, J. Okamoto, K. Sakakibara and M. Uesaka, On a singular limit of the Kobayashi-Warren-Carter energy. *Indiana Univ. Math. J.* **73** (2024), 1453–1491.

[GOU] Y. Giga, J. Okamoto and M. Uesaka, A finer singular limit of a single-well Modica-Mortola functional and its applications to the Kobayashi-Warren-Carter energy. *Adv. Calc. Var.* **16** (2023), 163–182.

[KWC] R. Kobayashi, J. A. Warren and W. C. Carter, A continuum model of grain boundaries. *Phys. D* **140** (2000), 141–150.

[MSW1] S. Moll, K. Shirakawa and H. Watanabe, Energy dissipative solutions to the Kobayashi-Warren-Carter system. *Nonlinearity* **30** (2017), no. 7, 2752–2784.

[MSW2] S. Moll, K. Shirakawa and H. Watanabe, Existence and uniqueness results for phase-field system of grain boundary motion with 3D-crystalline orientation. preprint.

[MS] D. Mumford and J. Shah, Optimal approximations by piecewise smooth functions and associated variational problems. *Comm. Pure Appl. Math.* **42** (1989), 577–685.

[ROF] L. Rudin, S. Osher and E. Fatemi, Nonlinear total variation based noise removal algorithms. Experimental mathematics: computational issues in nonlinear science (Los Alamos, NM, 1991), *Phys. D* **60** (1992), 259–268.

Semilinear parabolic equations on a simple metric graph

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§1. Introduction

A metric graph is a system consisting of line segments (of finite or infinite length) and nodes (vertices). Each node is an endpoint of a line segment. Line segments are connected at their endpoints and so becomes a connected topological space. The graph is imbedded in a certain Euclidean space and so it naturally becomes a metric space by the induced distance. Each line segment of the graph is equipped with the coordinate which is according to length parameter and so we can consider a (well-posed) system of differential equations by prescribing adequate conditions at each vertices. There are a lot of studies of PDEs (and ODEs) in metric graphs in these several decades. Some of them are related with spectral theory of physical operators, heat conductivity, wave phenomena, quantum physics and reaction diffusion in material and nerve sciences, etc (cf. [7], [8], [20], [29], [11]). In this talk I consider a single semilinear parabolic equation with bistable nonlinearity (Allen-Cahn equation, Nagumo equation[1])

$$\frac{\partial u}{\partial t} = \Delta u + f(u) \quad (t > 0, x \in \Omega) \tag{1.1}$$

(where Δ is the Laplacian in the graph Ω) in a simple metric graph Ω (star graph, quasi star graph) and discuss the solutions with behaviors like travelling front in long branch of the graph.

Here we specify the bistable nonlinearity $f \in C^1(\mathbb{R};\mathbb{R})$ which satisfies the following conditions:

$$\begin{cases} \exists a \in (0,1) s.t. f(0) = f(a) = f(1) = 0 \\ f(s) > 0 \quad (s < 0, a < s < 1), \quad f(s) < 0 \quad (0 < s < a, s > 1) \end{cases} \tag{1.2}$$

§2. Front type solution in one-dimensional space

First we consider equation (1.1) in the space $\mathbb{R}$. It is written

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u) \tag{2.1}$$

in an adequate time-space region $(0,T) \times \mathbb{R}$. One of the well-known solutions of (2.1) is the traveling solution which takes the form $u(t,x) = \phi(x - ct)$. Substitite it into (2.1) and get

$$\frac{d^2 u}{dz^2} + c\frac{d\phi}{dz} + f(\phi) = 0. \tag{2.2}$$

Here c is an unknown constant and ϕ is an unknown function. Since we want to consider front-wave type solution, we give a condition at infinity point $\pm\infty$ as $\phi(\infty) = 0, \phi(-\infty) = 1$. Under this condition, it is known that there exist a unique $c \in \mathbb{R}$ and a solution ϕ (unique up to space shift) to (2.2) (cf. [12], [2]).

Remark. It can be easily verified that the sign of c agrees to that of $\int_0^1 f(r)dr$.

[1] They appear as model equations in Material science [1] and Nerve system science[24], respectively.

Now I mention a famous result obtained in Fife-McLeod [12], [13] (which is closely related to this talk). Consider the initial value problem of (2.1) with the condition

$$u(0,x) = \varphi(x) \quad (x \in \mathbb{R}) \tag{2.3}$$

where φ is a continuous function in $\mathbb{R}$.

Theorem 2.1 ([12], [13]). If φ is bounded and continuous in $\mathbb{R}$, there exists a solution $u = u(t,x)$ to (2.1) (for $T = \infty$) and (2.3). Moreover, we assume the condition

$$\lim_{x\to-\infty} \varphi(x) > a, \quad \lim_{x\to\infty} \varphi(x) < a. \tag{2.4}$$

Then there exists $\theta \in \mathbb{R}$ such that the solution $u(t,x)$ satisfies the following property

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(t,x) - \phi(x - ct + \theta)| = 0. \tag{2.5}$$

Remark. When we replace initial condition φ by $\varphi + \xi$ where ξ is a continuous function with compact support, the condition (2.4) still holds. It should be emphasized that under such change of initial condition, c and ϕ are unchanged (while θ may be changed) in (2.5) and so the traveling solution $\phi(x - ct)$ is stable in some strong sense and tough.

§3. Front wave type solution in the stargraph

The stargraph (in this talk) Ω consists of finite number of several half lines connected one another at their endpoints and it is one of the simplest graphs. One portion of the graph (one branch) looks like the line $\mathbb{R}$ and one natural question is whether a sort of traveling solution in $\mathbb{R}$ can exist in the stargraph, or not. We specify the stargraph. Let $m \in \mathbb{N}$ and $m \geqq 3$. Ω is expressed as

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_m \cup \{O\} \tag{3.1}$$

For j, each Ω_j has a coordinate which is a variable $0 < x_j < \infty$. The point O agrees to the endpoint of Ω_j which is given by $x_j = 0$, for each $1 \leqq j \leqq m$. The equation (1.1) in Ω is written in each Ω_j ($1 \leqq j \leqq m$) with the connection condition (Kirchhoff condition) on the nodes. Actually, for the unknown function u, we put

$$u_j = u_{|\Omega_j} \qquad (1 \leqq j \leqq m)$$

and we have

$$\begin{cases} \dfrac{\partial u_j}{\partial t} = \dfrac{\partial^2 u_j}{\partial x_j^2} + f(u_j) \quad (t_0 < t < t_1, x_j > 0, 1 \leqq j \leqq m), \\ u_1(t,0) = \cdots = u_m(t,0), \quad \displaystyle\sum_{j=1}^{m} \frac{\partial u_j}{\partial x_j}(t,0) = 0 \quad (t_0 < t < t_1) \quad \text{(Kirchhoff condition).} \end{cases} \tag{3.2}$$

For the Cauchy problem for the above equation, we have the following result.

Existence: For any bounded continuous function $\varphi(x)$, there exists a solution $u = u(t,x)$ to (3.2) ($-\infty < t_0 < t_1 < \infty$, $x \in \Omega$) with the initial condition $u(t_0, x) = \varphi(x)$.

This result is obtained through the integral equation

$$u(t,x) = \int_\Omega G(t - t_0, x, y)\varphi(y)dy + \int_{t_0}^{t} \int_\Omega G(t - s, x, y) f(u(s,y))dyds \quad (t > t_0)$$

with the aid of the heat kernel (cf. T.Okada [25]). Comparison principle, super-sub solution method and well-posedness can be proved by the aid of the argument of D. Sattinger [28]. There are several other approaches for existence theorem (cf. [3], [4]).

The main subject of this talk is to make sure that traveling front wave solution can exist in the stargraph. Before that, we look at the existence of stationary solutions which is a coupled system of ODEs.

Remark. If the initial condition φ satisfies

$$\varphi_j(\infty) := \lim_{x_j \to \infty} \varphi_j(x_j) = \alpha_j \in \{0, a, 1\} \quad (1 \leqq j \leqq m),$$

then it holds that

$$u_j(t, \infty) = \alpha_j \quad \text{for} \quad t > 0 \, (1 \leqq j \leqq m). \tag{3.3}$$

This property is verified through the above integral equation.

§4. Stationary problem in the stargraph

We introduce the function $F(u) = \int_0^u f(r)\, dr$ for later use in this note. Let us mention the result about the condition about the existence of stationary problem of (3.1) under the infinity point condition on each branch Ω_j.

Theorem 4.1 ([16], [17]). Assume $F(1) > 0$. Let $\alpha_1 = 1$, $\alpha_2 = \cdots = \alpha_m = 0$. It holds that
(I) if $F(1) + ((m-1)^2 - 1)F(a) > 0$, then there exists no time independent solution to (3.2), (3.3).

(II) if $F(1) + ((m-1)^2 - 1)F(a) < 0$, then there exist two time independent solutions $w^{(1)}(x), w^{(2)}(x)$ to (3.2), (3.3) such that $0 < w^{(1)}(x) < w^{(2)}(x) < 1$ $(x \in \Omega)$ and $w^{(1)}$ is stable and $w^{(2)}$ is unstable.
(III) if $F(1) + ((m-1)^2 - 1)F(a) = 0$, then there exists one time independent solution $w^{(3)}$ to (3.2), (3.3) such that $0 < w^{(3)} < 1$ and $w^{(3)}$ is unstable.

§5. Backward (ancient) solution in the stargraph

In this section we consider the existence of a (time-) backward solution. This solution gives the important point of our motivation since it forms a front wave comming along a branch to the center of graph according to the time evolution from ancient time $t = -\infty$. In general parabolic equation are not solvable in minus time direction and so it is not so easy to discuss backward solution with a prescribed behavior at $t = -\infty$.

Theorem 5.1 ([16], [17]). Assume $F(1) > 0$. Then there exists a solution $u = u(t, x)$ to (3.2) with $t_0 = -\infty, t_1 = 0$ with (3.3) with $\alpha_1 = 1$, $\alpha_2 = \cdots = \alpha_m = 0$ and

$$\lim_{t \to -\infty} \sup_{x_1 \geqq 0} |u_1(t, x_1) - \phi(-x_1 - ct)| = 0,$$

$$\lim_{t \to -\infty} \sup_{x_j \geqq 0} |u_j(t, x_j)| = 0 \quad (2 \leqq j \leqq m).$$

(Sketch of the proof) (Step 1) We make use of ϕ to construct a subsolution $\underline{U}(t, x)$ and a supersolution $\overline{U}(t, x)$ which are defined in the region $(t \leqq t_2, x \in \Omega)$.

Recall the function ϕ is the solution of (2.2) which corresponds to traveling profile in $\mathbb{R}$. Set

$$\underline{U}(t,x) = \begin{cases} (\phi(-x_1 - ct - \xi(t)) - p(t))_+ & (0 < x_1 = x \in \Omega_1) \\ 0 \quad (0 \leqq x_j = x \in \Omega_j,\ 2 \leqq j \leqq m) \end{cases} \tag{5.1}$$

Here $z_+ = \max(0, z)$. $p(t) = p_0 e^{\mu(t-t_2)}$, $\xi(t) = (1/\beta)(1/\mu)(K - \mu)e^{\mu(t-t_2)}$ and positive constants p_0, μ, K, β and negative constant t_2 are chosen adequately so that $\underline{U}$ becomes a subsolution in the region $(-\infty, t_2] \times \Omega$.

$$\overline{U}(t,x) = \begin{cases} \phi(-x_1 - ct + \xi(t)) + q(t) & (0 < x_1 = x \in \Omega_1) \\ \phi(x_j - ct - \xi(t)) + q(t) & (0 \leqq x_j = x \in \Omega_j,\ 2 \leqq j \leqq m) \end{cases} \tag{5.2}$$

is similarly constructed so that $\overline{U}$ becomes a supersolution in $(-\infty, t_2) \times \Omega$ and $\underline{U}(t,x) \leqq \overline{U}(t,x)$.
(Step 2) Let $N \in \mathbb{N}$ and consider the initial value problem (3.2) with $t_0 = -N$ and $t_1 = t_2$ with $v(-N, x) = \underline{u}(-N, x)$. By the existence and comparison theory, we have a solution $v^{(N)}$

$$\underline{u}(t,x) \leqq v^{(N)}(t,x) \leqq \overline{u}(t,x) \quad (-N \leqq t \leqq t_2)$$

By the limitting argument, there exists a subsequence of $\{v^N\}_{N=1}^{\infty}$ which converges to a solution $v = v(t,x)$ to (3.2), (3.3) ($t_0 = -\infty, t_1 = t_2$), which is defined in $(-\infty, t_2] \times \Omega$ with the required property. □

In the case of the problem on $\mathbb{R}$, Yagisita [30] constructed an interesting backward solution which expresses an annihilation phenomenon of two fronts waves. That solution is applicable to consider supersolution in the above problem. Anyway the method of subsolution and supersolution pair is often used.

Of course this solution can be extended in time forward direction infinitely and we can get a time entire solution v. However the limit behavior for $t \to \infty$, depends on situation.

We have the following theorem which classifies the situation according to stationary problem of the section 4.

Theorem 5.2 ([16], [17]). Assume $F(1) > 0$. The solution $v = v(t,x)$ obtained in the above theorem has the following asymptotic property for $t \to \infty$ according to (I), (II), (III), as
(J) if $F(1) + ((m-1)^2 - 1)F(a) > 0$, then $\lim_{t\to\infty} v(t,x) = 1$,
(JJ) if $F(1) + ((m-1)^2 - 1)F(a) < 0$, $\lim_{t\to\infty} v(t,x) = w^{(1)}$,
(JJJ) if $F(1) + ((m-1)^2 - 1)F(a) = 0$, then $\lim_{t\to\infty} v(t,x) = w^{(3)}(x)$.

In the situation of (JJ) or (JJJ), the front solution is regarded as blocked by stationary solution.

Remark. (The work of Pauwelussen). In [26], [27], the dynamics of the semilinear parabolic equation with discontinuous diffusion coefficient with a bistable nonlinearity f (same as that of us) such as

$$\begin{cases} \dfrac{\partial u}{\partial t} = r(x)\dfrac{\partial^2 u}{\partial x^2} + f(u) \quad (t > 0, x \in \mathbb{R} \setminus \{0\}) \\ u(t, 0-0) = u(t, 0+0), \quad (\partial u/\partial x)(t, 0-0) = (\partial u/\partial x)(t, 0+0) \quad (t > 0) \end{cases}$$

where $r(x) = 1$ $(x < 0)$, $r(x) = \epsilon > 0$ $(x > 0)$ was studied. By the change of the space variable, it is related with the problem of the stargraph. The author gives a certain class of initial functions and investigated the behavior of solution and saw what occurs (as t grows up) depending on the parameter ϵ and got a quite clear classification. The notion of blocking was introduced. Therefore it is a pioneering work. It has not been recognized much long time, but eventually it is an important work since it is related to a lot of later works of front wave propagation and blocking.

§6. Solutions in Quasi Stargraph

Adding some branches to Ω, we get a little more complicated graph (tree). I can consider (front wave) solution phenomena which are similar to the case of stargraph but, by changing geometry of the graph we can observe interesting dependency of the solutions.

There are several recent development [15], [18], [19], [21], [22], [23] and more.

References

[1] S. Allen and J.W. Cahn, A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening, Acta. Metallurgica **27** (1979), 1085-1095.

[2] D. G. Aronson and H. F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve propagation, in Partial Differential Equations and Related Topics, ed. J. A. Goldstein, LNM **446**(1975), 5-49, Springer.

[3] J. von Below, Classical solvability of linear parabolic equations on networks, J. Differential Equations, **72** (1988), 316-337.

[4] J. von Below, An existence result for semi linear parabolic network equations with dynamical node conditions, Progress in partial differential equations: elliptic and parabolic problems (Pontá-Mousson, 1991), **266**, Pitman Res. Notes Math. Ser., 274-283. Longman Sci. Tech., Harlow, 1992.

[5] H. Berestyciki, J. Bouhours and G. Chapuisat, Front blocking and propagation in cylinders with varying cross section, Calc. Var. PDE, **55**, 44 (2016).

[6] H. Berestycki, F. Hamel and H. Matano, Bistable travelling waves around an obstacle, Comm. Pure Appl. Math., **62** (2009), 729-788.

[7] G. Berkolaiko and P. Kuchment, *Introduction to Quantum Graphs*, AMS Monograph, 2012.

[8] G. Berkolaiko, R. Carlson, S.A. Fulling P. Kuchment, ed. , Quantum Graphs and their Applications, Contemporary Mathematics 415 AMS 2006

[9] F. Camilli and L. Corrias, Parabolic models for chemotaxis on weighted networks, J. Math. Pures Appl., **108** (2017), 459-480.

[10] C. Cattaneo, The Spread of the potential on a homogeneous tree, Annali di Matematica Pura ed Applicata, **175** (1998), 29-57.

[11] P. Exner, J.P. Keating, P. Kuchment, T. Sunada, A. Teplyaev, ed., Analysis on Graphs and its Application, Proc. Symp. Pure Mathematics **77** AMS 2008

[12] P. C. Fife and J. B. McLeod, The approach of solutions of nonlinear diffusion equations to travelling front solutions, Arch. Rational Mech. Anal., **65** (1977), 335-361.

[13] P. C. Fife and J. B. McLeod, A phase plane discussion of convergence to travelling fronts for nonlinear diffusion, Arch. Rational Mech. Anal., **75** (1980/81), 281-314.

[14] F. Hamel, F. Lutscher and M. Zhang, Propagation and blocking in a two-patch reaction-diffusion model, J. Math. Pures Appl., **168** (2022), 213-267.

[15] S. Iwasaki, S. Jimbo and Y. Morita, Standing waves of reaction-diffusion equations on an unbounded graph with two vertices, SIAM J. Appl. Math., **82** (2022), 1733-1763.

[16] S. Jimbo and Y. Morita, Entire solutions to reaction-diffusion equations in multiple half-lines with a junction, J. Differential Equations, **267** (2019), 1247-1276.

[17] S. Jimbo and Y. Morita, Asymptotic behavior of entire solutions to reaction-diffusion equations in an infinite star graph, Discrete and Cont. Dyn. Sys., **41** (2021), 4013-4039.

[18] S. Jimbo and Y. Morita, Front propagation in the bistable reaction-diffusion equation on tree-like graphs, J. Diff. Eqs, **384** (2024), 93-119.
[19] T. Kan, Y. Morita, K.-I. Nakamura and C.-H. Wu, Asymmetric front propagation for the bistable reaction-diffusion equation on a metric graph, preprint.
[20] P. Kuchment, Graph models for waves in thin structures, Waves in Random Media **12** (2002), R1-R24.
[21] B. Lou and Y. Morita, Asymptotic behaviour for solutions to reaction-diffusion equations on a root-like metric graph, Nonlinearity, **37** (2024) 075011.
[22] H. Matano and S. Jimbo, Front propagation on a general metric graph, preprint.
[23] Y. Morita, Front profile in time backward for the bistable reaction-diffusion equation on metric graphs, J. Dyn. Diff. Equat. **37** (2025), 55-69.
[24] J. Nagumo, S. Yoshizawa and S. Arimoto, An active pulse transmission line stimulating axon, Proc. Inst. Radio Eng. **50** (1962), 2061-2070.
[25] T. Okada, Asymptotic behavior of skew conditional heat kernels on graph networks, Canadian J. Math. **45** (1993), 863-878.
[26] J. Pauwelussen, Nerve impulse propagation in a branching nerve system: A simple model, Physica D, **4** (1981), 67-88.
[27] J. Pauwelussen, One way traffic of pulses in a neuron, J. Math. Biol., **15** (1982), 151-171.
[28] D.H. Sattinger, Monotone methods in nonlinear elliptic and parabolic boundary value problems, Indiana Univ. Math. J. **21** (1972), 979-1000.
[29] G. Q. Xu and N. Mastorakis, Differential Equations On Metric Graph, WSEAS Press, 2010.
[30] H. Yagisita, Backward global solutions characterizing annihilation dynamics of travelling fronts, Publ. RIMS Kyoto Univ. **39** (2002), 117-164.

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Singular boundary condition problems for a class of fully nonlinear parabolic equations*

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1 Introduction

This paper is concerned with singular Dirichlet or Neumann boundary problem for fully nonlinear differential equation. Let $b > 0$ and $f, g \in C(\mathbb{R})$. We are mainly interested in the following equation in one space dimension:

$$\begin{cases} u_t - f(g(u_x)u_{xx}) = 0 & \text{in } (-b,b)\times(0,\infty), & (1.1)\\ \lim\limits_{x\to\pm b} u(x,t) = \infty \ \text{ or } \ \lim\limits_{x\to\pm b} u_x(x,t) = \pm\infty & \text{for any } t>0, & (1.2)\\ u(\cdot,0) = u_0 & \text{in } (-b,b), & (1.3)\end{cases}$$

where u_0 is a given continuous function in $(-b, b)$ satisfying several assumptions to be elaborated later, and f and g are given continuous function satisfying

(A1) $f \in C(\mathbb{R})$ is strictly increasing with $f(0) = 0$ and $f(s) \to \pm\infty$ as $s \to \pm\infty$.

(A2) $g \in C(\mathbb{R})$ is positive and

$$|s|^\alpha g(s) \to C_{g\pm} \quad \text{as } s \to \pm\infty \tag{1.4}$$

for some exponent $\alpha \in \mathbb{R}$ and constants $C_{g+}, C_{g-} > 0$.

One of the examples of the interior equation (1.1) is a p-Laplace type heat equation

$$u_t = |\Delta_p u + \varepsilon\Delta u|^{\beta_1-1}(\Delta_p u + \varepsilon\Delta u) \quad \text{in } (-b,b)\times(0,\infty), \tag{1.5}$$

where $p \geq 2, \beta_1 > 0$ and $\varepsilon > 0$ are constants. Since $\Delta_p u + \varepsilon\Delta u$ can be re-written as

$$\Delta_p u + \varepsilon\Delta u = \{(p-1)|u_x|^{p-2} + \varepsilon\}u_{xx},$$

it is a special case of (1.1) with

$$f(s) = |s|^{\beta_1-1}s, \quad g(s) = (p-1)|s|^{p-2} + \varepsilon$$

for $s \in \mathbb{R}$. The function g satisfies (A2) with $\alpha = 2 - p$.

Another typical example of (1.1) is the following equation:

$$u_t = (1+|u_x|^2)^{\frac{1-3\beta_2}{2}}|u_{xx}|^{\beta_2-1}u_{xx} \quad \text{in } (-b,b)\times(0,\infty), \tag{1.6}$$

where $\beta_2 > 0$ is a constant. It is a special case of (1.1) with

$$f(s) = |s|^{\beta_2-1}s, \quad g(s) = (1+s^2)^{\frac{1-3\beta_2}{2\beta_2}}$$

for $s \in \mathbb{R}$. The function g satisfies (A2) with $\alpha = 3 - \frac{1}{\beta_2}$. Such an equation has a geometric interpretation: it describes the motion of a graph-like planar curve whose normal velocity equals

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to the singed β_2-th power of its curvature. The singular Dirichlet boundary condition

$$\lim_{x\to\pm b} u(x,t) = \infty \quad \text{for } t > 0 \tag{1.7}$$

depicts that the graph $y = u(x,t)$ is a complete curve whose two ends are asymptotic to two parallel lines $x = \pm b$ at each time $t > 0$. The singular Neumann boundary condition

$$\lim_{x\to\pm b} u_x(x,t) = \pm\infty \quad \text{for } t > 0 \tag{1.8}$$

depicts tangential sliding behavior of the curve along the boundary $x = \pm b$. The singed β_2-th power type curvature flow of complete graphs was studied by [1] and the results are briefly summarized as follows:

(i) If u_0 is smooth and convex, and satisfies $\lim_{x\to\pm b} u_0(x) = \infty$, then the problem (1.6), (1.7) and (1.3) has a unique global-in-time solution u for any $\beta_2 > 0$.

(ii) If $\beta_2 > 1$, then there exist traveling wave solutions to (1.6) and (1.8) formed by $w(x,t) = W(x) + ct$ with $W \in C([-b,b]) \cap C^2((-b,b))$ (which is bounded on $[-b,b]$ due to $W \in C([-b,b])$) and $c \in \mathbb{R}$. If $\frac{1}{2} < \beta_2 \leq 1$, then there exist traveling wave solutions to (1.6) and (1.8) formed by $w(x,t) = W(x) + ct$ with $W \in C^2((-b,b))$ and $c \in \mathbb{R}$. Furthermore, in this case, w satisfies the singular Dirichlet boundary condition (1.7) (i.e. w is a traveling wave solution to (1.6) and (1.7)). For each case, the traveling wave solution is unique up to the vertical translation.

(iii) If $\beta_2 > \frac{1}{2}$, then the solution u obtained in (i) locally smoothly converges to a traveling wave solution in $(-b,b)$ as $t \to \infty$.

Our purpose in this presentation is to extend their theory for more general interior equation and without assuming the boundedness, convexity of the initial function. We first summarize properties of traveling wave solutions to (1.1) and (1.2) as follows (see [2, 3] for the proofs):

(a) If $\alpha > 2$, then there exist traveling wave solutions to (1.1) and (1.8) formed by $w(x,t) = W(x) + ct$ with $W \in C([-b,b]) \cap C^2((-b,b))$ and $c \in \mathbb{R}$. Furthermore, the traveling wave is unique up to the vertical translation.

(b) If $1 < \alpha \leq 2$, then there exist traveling wave solutions to (1.1) and (1.8) formed by $w(x,t) = W(x) + ct$ with $W \in C^2((-b,b))$ and $c \in \mathbb{R}$. Furthermore, the traveling wave satisfies (1.7) and is unique up to the vertical translation.

(c) If $\alpha \leq 1$, there is no traveling wave solution to (1.1) and (1.2) formed by $w(x,t) = W(x)+ct$ with $W \in C^2((-b,b))$ and $c \in \mathbb{R}$.

We note that the cases (a) and (b) respectively corresponds to $\beta_2 > 1$ and $\frac{1}{2} < \beta_2 \leq 1$ due to $\alpha = 3 - \frac{1}{\beta_2}$. The above traveling wave solution is expected to be stable from the result in [1]. In this presentation, before studying the stability analysis of the traveling wave solution, the existence of the solution to (1.1)–(1.3) is investigated. In particular, a natural question is how the structure of the existence of solution satisfying either boundary condition in (1.2) depends on α. The main results show that the existence of solution depends not only on α, but also on f and the boundedness/unboundedness of the initial function u_0. Therefore, we assume either of the following conditions for the initial function u_0:

(B1) u_0 is of class $C([-b,b])$

(B2) u_0 is of class $C((-b,b))$ and satisfies

$$\lim_{x\to\pm b} u_0(x) = \infty, \quad \limsup_{x\to b} u_0(x)(b-x)^\gamma < \infty, \quad \limsup_{x\to -b} u_0(x)(b+x)^\gamma < \infty \tag{1.9}$$

for some $\gamma > 0$

(B3) u_0 is of class $C((-b,b))$ and satisfies

$$\lim_{x\to b}\big(u_0(x)-D_+\psi_{\gamma_+}(b-x)\big)=\hat{C}_+ \ \text{ and } \ \lim_{x\to -b}\big(u_0(x)-D_-\psi_{\gamma_-}(b+x)\big)=\hat{C}_- \tag{1.10}$$

for some constants $\gamma_\pm \geq 0, D_\pm > 0$ and $\hat{C}_\pm \in \mathbb{R}$, where

$$\psi_\gamma(s) := \begin{cases} s^{-\gamma} & \text{if } \ \gamma > 0, \\ -\log s & \text{if } \ \gamma = 0. \end{cases} \tag{1.11}$$

We note that (B1) yields the boundedness of u_0. (B3) is a stronger assumption than (B2), which will be needed to prove the uniqueness of the solution for the singular Dirichlet boundary problem.

We finally remark that the solvability of the elliptic equation

$$\begin{cases} -\Delta u + |\nabla u|^p + \lambda u = f & \text{in } \ \Omega \subset \mathbb{R}^n, \\ u(x) = \infty \ \text{ of } \ \frac{\partial u}{\partial \nu}(x) = \infty & \text{on } \ \partial\Omega \end{cases},$$

where $p > 1, \lambda > 0$ and ν is the outward pointing unit normal vector of $\partial\Omega$, was discussed in [4]. The solvability for either boundary problems requires, as for our problem, a case study of p and the divergence rate $f(x) \to \infty$ as $x \to x_0 \in \partial\Omega$.

2 Definition of viscosity solution

Since (1.1) is a general fully nonlinear and degenerate parabolic, we need to adopt the viscosity solution theory to discuss the existence of solution. Therefore, we also need to introduce a boundary condition (1.2) in the viscosity sense.

For convenience of notation, hereafter we denote

$$Q := (-b,b)\times(0,\infty), \quad Q_0 := (-b,b)\times[0,\infty)$$

and let

$$F(p,z) := -f(g(p)z) \tag{2.1}$$

for $p, z \in \mathbb{R}$.

We fist define solutions to the singular Neumann boundary problem as follows:

Definition 2.1 (Solutions of singular Neumann problem). *An upper semicontinuous function $u \in USC(\overline{Q})$ is called a subsolution of* (1.1) *and* (1.8) *if whenever there exist $(x_0,t_0) \in Q$ and $\phi \in C^2(\overline{Q})$ such that $u-\phi$ attains a local maximum at (x_0,t_0), we have*

$$\phi_t(x_0,t_0) + F\left(\phi_x(x_0,t_0), \phi_{xx}(x_0,t_0)\right) \leq 0.$$

A lower semicontinuous function $u \in LSC(\overline{Q})$ is called a supersolution of (1.1) *and* (1.8) *if the following conditions hold:*

(i) Whenever there exist $(x_0,t_0) \in Q$ and $\phi \in C^2(\overline{Q})$ such that $u-\phi$ attains a local minimum at (x_0,t_0), we have

$$\phi_t(x_0,t_0) + F\left(\phi_x(x_0,t_0), \phi_{xx}(x_0,t_0)\right) \geq 0.$$

(ii) For any $t > 0$ and any function $\phi \in C^2(\overline{Q})$, $u-\phi$ never attains a local minimum at the point (b,t) or $(-b,t)$.

A continuous function $u \in C(\overline{Q})$ is said to be a solution of (1.1), (1.8) *and* (1.3) *if u is both a subsolution and a supersolution of* (1.1) *and* (1.8) *and satisfies* (1.3) *on $[-b,b]$.*

Remark 2.2. *For any function* $u \in C(\overline{Q}) \cap C^1(\overline{Q})$ *satisfying* (1.8) *and any function* $\phi \in C^2(\overline{Q})$, $u - \phi$ *never attains a local minimum at the point* (b, t) *or* $(-b, t)$ *for* $t > 0$. *Indeed, letting* $v(x, t) := u(x, t) - \phi(x, t)$, *we can see that*

$$\lim_{x \to \pm b} v_x(x, t) = \pm\infty \quad \textit{for } t > 0.$$

Therefore, $v(\cdot, t)$ *is strictly increasing and strictly decreasing respectively near* $x = b$ *and* $x = -b$, *and thus* v *does not attain a local minimum at the points* $(\pm b, t)$. *It yields that the condition (ii) in the definition of supersolution in Definition is a weak boundary condition of* (**??**).

On the other hand, if we assume $u \in C(\overline{Q})$ *satisfying the condition (ii) in the definition of supersolution in Definition is of class* $C^1(Q)$, *then we can see that*

$$\limsup_{x \uparrow b} u_x(x, t) = \infty, \quad \liminf_{x \downarrow -b} u_x(x, t) = -\infty \quad \textit{for } t > 0.$$

We next define solutions to the singular Dirichlet boundary problem. To define it, we denote by, respectively, u^* and u_* the upper and lower semicontinuous envelope of u defined on Q_0, that is,

$$u^*(x, t) := \lim_{r \to +0} \sup\{u(y, s) : (y, s) \in Q_0 \cap B_r((x, t))\} \quad \text{for } (x, t) \in Q_0, \tag{2.2}$$

$$u_*(x, t) := \lim_{r \to +0} \inf\{u(y, s) : (y, s) \in Q_0 \cap B_r((x, t))\} \quad \text{for } (x, t) \in Q_0, \tag{2.3}$$

where $B_r((x, t))$ is the ball with center (x, t) and radius r.

Definition 2.3. *An upper semicontinuous function* $u \in USC(Q_0)$ *is called a subsolution of* (1.1) *and* (1.7) *if whenever there exist* $(x_0, t_0) \in Q$ *and* $\phi \in C^2(Q_0)$ *such that* $u - \phi$ *attains a local maximum at* (x_0, t_0), *we have*

$$\phi_t(x_0, t_0) + F(\phi_x(x_0, t_0), \phi_{xx}(x_0, t_0)) \le 0.$$

A lower semicontinuous function $u \in LSC(Q_0)$ *is called a supersolution of* (1.1) *and* (1.7) *if the following conditions hold:*

(i) Whenever there exist $(x_0, t_0) \in Q$ *and* $\phi \in C^2(Q_0)$ *such that* $u - \phi$ *attains a local minimum at* (x_0, t_0), *we have*

$$\phi_t(x_0, t_0) + F(\phi_x(x_0, t_0), \phi_{xx}(x_0, t_0)) \ge 0.$$

(ii) For any $t > 0$, *it holds that* $\lim_{x \to \pm b} u(x, t) = \infty$.

A function u *defined in* Q_0 *is said to be a solution of* (1.1), (1.7) *and* (1.3) *if* u^* *is a subsolution,* u_* *is a supersolution of* (1.1) *and* (1.7), *and it holds that* $u^*(\cdot, 0) = u_*(\cdot, 0) = u_0$ *in* $(-b, b)$.

Remark 2.4. *The condition* $u^*(\cdot, 0) = u_*(\cdot, 0) = u_0$ *in* $(-b, b)$ *yields*

$$\lim_{(y,t) \to (x,0)} u(y, t) = u_0(x) \quad \textit{for } x \in (-b, b).$$

3 Main results

We first present the existence/nonexistence theory for the singular Neumann boundary problem under the initial function assumption (B1). The existence/nonexistence is related to the threshold of boundedness of the traveling wave solutions, resulting in cases $\alpha > 2$ and $\alpha \le 2$ separately.

The following theorem is the existence result when $\alpha > 2$. The asymptotic result is also included.

Theorem 3.1 ([[3, Theorem 1.1 and Theorem 1.4]). *Let $b > 0$. Assume that functions f, g satisfy (A1), (A2) with $\alpha > 2$ respectively. Let u_0 satisfies (B1). Then there exists a unique viscosity solution $u \in C([-b, b] \times [0, \infty))$ of* (1.1), (1.8) *and* (1.3).

Furthermore, if f^{-1} is Lipschitz away from $s = 0$, g is Lipschitz in $\mathbb{R}$ and u_0 is convex, then there exists a constant $m \in \mathbb{R}$ such that

$$\sup_{x\in[-b,b]} |u(x,t) - (W(x) + m + ct)| \to 0 \quad as \quad t \to \infty,$$

where $w(x, t) = W(x) + ct$ is a traveling wave solution for (1.1) *and* (1.8).

If $\alpha \leq 2$, we have the nonexistence result as follows:

Theorem 3.2 ([3, Theorem 1.2]). *Let $b > 0$. Assume that functions f and g satisfy (A1) and (A2) with $\alpha \leq 2$. Let u_0 satisfies (B1). Then there exist no solutions of* (1.1), (1.8) *and* (1.3) *in $C([-b, b] \times [0, \infty))$.*

In the proof of Theorem 3.2, we use the comparison principle and construct a sequence of subsolutions to prove that if a solution $u \in C([-b, b]\times[0, \infty))$ exists, then u satisfies $u(\pm b, t) = \infty$ for any $t > 0$ (which contradicts that u is of class $C([-b, b] \times [0, \infty))$). Since u_0 is bounded in $[-b, b]$, the proof shows that a solution (in a different sense than the class of solutions defined in Section 2) u to (1.1), (1.8) and (1.3) may "instantaneously" blows up at $x = \pm b$. Therefore, when $\alpha \leq 2$, we may expect that the singular Dirichlet boundary problem can be solved even if u_0 is bounded in $(-b, b)$.

We next present the existence/nonexistence theory for the singular Dirichlet boundary problem. Unlike the singular Neumann boundary problem, the threshold for the existence/nonexistence of the solution is different from the threshold for the existence/nonexistence of the traveling wave solutions for (1.1) and (1.2). More precisely, even in the case of $\alpha \leq 1$, which is the case when any traveling wave solutions for (1.1) and (1.2) do not exist, a solution to the singular Dirichlet boundary problem exists, depending on a condition of f.

When $\alpha > 2$, we need divergence condition of u_0 as in (B2) to prove the existence of solution as follows:

Theorem 3.3 ([2, Theorem1.1]). *Let $b > 0$. Assume that f, g and u_0 respectively satisfy (A1), (A2) with $\alpha > 2$. If u_0 satisfies (B2), then there exists a viscosity solution $u : (-b, b)\times[0, \infty) \to \mathbb{R}$ to* (1.1)–(1.3). *If u_0 satisfies (B3) with $\gamma_\pm \geq 0$, then the viscosity solution is continuous and unique.*

When $1 < \alpha \leq 2$, the existence of the solution can be proved if u_0 satisfies either (B1) or (B2).

Theorem 3.4 ([2, Theorem1.2]). *Let $b > 0$. Assume that f, g and u_0 respectively satisfy (A1), (A2) with $1 < \alpha \leq 2$. If u_0 satisfies (B1) or (B2), then there exists a viscosity solution $u : (-b, b) \times [0, \infty) \to \mathbb{R}$ to* (1.1)–(1.3). *If u_0 satisfies (B3) with $\gamma_\pm \geq \frac{2-\alpha}{\alpha-1}$, then the viscosity solution is continuous and unique.*

In the case $\alpha \leq 1$, the existence and nonexistence of the solution to the singular Dirichlet boundary problem depend on the divergence rate of $f(s)$ as $s \to \pm\infty$. Therefore, we assume the following additional condition for f:

(A3) f satisfies

$$|s|^{-\beta} f(s) \to \pm C_{f\pm} \quad \text{as} \quad s \to \pm\infty$$

for some $\beta > 0$ and constants $C_{f+}, C_{f-} > 0$.

A typical example of f satisfying (A1) and (A3) is $f(s) = |s|^{\beta-1}s$. Then, we have the following result.

Theorem 3.5 ([2, Theorem1.3])**.** *Let* $b > 0$. *Assume that* f *satisfies (A1) and (A3), and assume also* g *satisfies (A2) with* $\alpha \le 1$.

(a) If $\alpha < 1$, $\beta \ge \frac{1}{1-\alpha}$ *and* $u_0 \in C((-b, b))$ *satisfies* $\inf_{x\in(-b,b)} u_0(x) > -\infty$, *then there is no viscosity solution* $u : (-b, b) \times [0, \infty) \to \mathbb{R}$ *to* (1.1)–(1.3).

(b) If $\alpha = 1$ *or* $\alpha < 1$ *and* $\beta < \frac{1}{1-\alpha}$, *assume* u_0 *satisfies (B1) or (B2). Then there exists a viscosity solution* $u : (-b, b) \times [0, \infty) \to \mathbb{R}$ *to* (1.1)–(1.3).

In the proof of (a) in Theorem 3.5, we prove that $u(x, t) = \infty$ at any points $x \in (-b, b)$ and any time $t > 0$ if a viscosity solution exists, which yields formally that "instantaneous interior blow-up" occurs. The proof suggests that the divergence of the solution to infinity at the boundary points propagates rapidly to the interior due to the diffusion effects of the interior equation.

As applications of these theorems, we summarize the existence and nonexistence of solutions to the singular Dirichlet/Neumann boundary problems for the typical examples (1.5) and (1.6). First corollary is for (1.5).

Corollary 3.6. *Assume* $b > 0$. *Assume also* $p \ge 2, \beta_1 > 0$ *and* $\varepsilon > 0$.

- *If* $\beta_1 \ge \frac{1}{p-1}$ *and* $u_0 \in C((-b, b))$ *satisfies* $\inf_{x\in(-b,b)} u_0 > -\infty$, *then there is no viscosity solution to* (1.5), (1.2) *and* (1.3).
- *If* $0 < \beta_1 < \frac{1}{p-1}$ *and* u_0 *satisfies (B1) or (B2), then a viscosity solution* $u : (-b, b) \times [0, \infty) \to \mathbb{R}$ *to* (1.5), (1.7) *and* (1.3) *exists.*

Since (1.5) coincides with the heat equation $u_t = (1 + \varepsilon)u_{xx}$ if $\beta_1 = 1$ and $p = 2$, the corollary shows that the heat equation with singular Dirichlet boundary condition does not have any viscosity solutions.

For the signed power type curvature flow (1.6) with the singular Dirichlet boundary condition (1.2), the existence theory in [1] requires that u_0 is smooth and strictly convex in $(-b, b)$ and satisfies

$$\lim_{x\to\pm b} u_0(x) = \infty.$$

Our existence theory does not need the smoothness, convexity and, when $\beta_2 \le 1$, the unboundedness of u_0 although an additional assumption is necessary to obtain the uniqueness.

Corollary 3.7. *Assume* $b > 0$ *and* $\beta_2 > 0$.

- *If* $\beta_2 > 1$ *and* u_0 *satisfies (B1), then a viscosity solution* $u \in C([-b, b] \times [0, \infty))$ *to* (1.6), (1.8) *and* (1.3) *exists uniquely.*
- *If* $\beta_2 > 1$ *and* u_0 *satisfies (B2), then a viscosity solution* $u : (-b, b) \times [0, \infty) \to \mathbb{R}$ *to* (1.6), (1.7) *and* (1.3) *exists. Furthermore, if* u_0 *satisfies (B3) with* $\gamma_\pm \ge 0$, *then the solution is continuous and unique.*
- *If* $0 < \beta_2 \le 1$ *and* u_0 *satisfies (B1) or (B2), then a viscosity solution* $u : (-b, b) \times [0, \infty) \to \mathbb{R}$ *to* (1.6), (1.7) *and* (1.3) *exists. Furthermore, if* $\frac{1}{2} < \beta_2 \le 1$ *and* u_0 *satisfies (B3) with* $\gamma_\pm \ge \frac{1-\beta_1}{2\beta_2-1}$, *then the solution is continuous and unique.*

4 Outline of the proofs

We here summarize briefly the proofs of the main results. For the existence theory, Perron's method is used for both the singular Dirichlet and Neumann boundary problems. Therefore, we first perturb the initial function so that it is smooth. Let $u_{0,\varepsilon}$ be the perturbed initial function for $\varepsilon > 0$. We next construct a supersolution v_ε and w_ε starting from $u_{0,\varepsilon}$. The constructability of, in particular, this supersolution v_ε depends on α and f so that v_ε satisfies the singular Dirichlet or Neumann boundary condition in the viscosity sense. We then can construct a solution u_ε starting from $u_{\varepsilon,0}$ by

$$u_\varepsilon(x,t) := \sup\{\phi(x,t) : \psi \text{ is a subsolution to (1.1) satisfying } w_\varepsilon \leq \psi \leq v_\varepsilon\}.$$

By using the standard stability result for the viscosity solutions, we can obtain a solution u satisfying (1.1) and (1.3) in the viscosity sense as a limit of u_ε. The remained proof is to prove that u satisfies the singular Dirichlet or Neumann boundary condition. The proof is completed by choosing a "good" test function $\phi \in C^2(\overline{Q})$ for the singular Neumann boundary problem and constructing a "good" subsolution for the singular Dirichlet boundary problem (see [2, 3] for the details).

The nonexistence theory can be proved by using a "good" sequence of subsolutions via the comparison principle. We here discuss the theory for the singular Dirichlet boundary problem. We first state the comparison principle which is needed to show the nonexistence theory (a) in Theorem 3.5. One of the difficult to show the comparison principle is that $\sup_{x\in(-b,b)} v(x,t) - u(x,t)$ is not always a bounded value and possible to diverge to ∞ for supersolution v and subsolution u which diverge to ∞ at the boundary points $x = \pm b$.

Proposition 4.1. *Let $b > 0$. Assume that $f, g \in C(\mathbb{R})$ satisfy (A1) and (A2). Let u and v be respectively sub- and super-solution of* (1.1) *and* (1.7). *Assume $u(\cdot,t)$ is convex for any $t \geq 0$ and u is a continuous function satisfying*

$$u_t \leq f\left(\frac{1}{1+\delta} g(u_x)u_{xx}\right) \quad \text{for } (x,t) \in (-b,b)\times(0,\infty)$$

for some $\delta > 0$. If $u(\cdot,0) \leq v(\cdot,0)$ in $(-b,b)$, then $u \leq v$ in Q_0.

Proof. Letting

$$u_\lambda(x,t) := (1+\lambda)u\left(\frac{x}{1+\lambda}, \frac{t}{1+\lambda}\right) \quad \text{for } -b < x < b,$$

we can see that (1) u_λ is a sub-solution to (1.1) for $0 < \lambda \leq \delta$; (2) $u_\lambda(\cdot,t)$ is bounded on $(-b,b)$ for any $t \geq 0$ and $\lambda > 0$; and (3) there is a modulus of continuity ω such that $u_\lambda(\cdot,0) \leq u(\cdot,0) + \omega(\lambda)$ whether $u(\cdot,0)$ is bounded on $(-b,b)$ or $\lim_{x\to\pm b} u(x,0) = \infty$. Therefore, the standard comparison argument for the viscosity solutions yields

$$u_\lambda(x,t) - \omega(\lambda) \leq v(x,t) \quad \text{for } (x,t) \in (-b,b)\times[0,\infty).$$

Letting $\lambda \to 0$, since u_λ converges to u pointwise due to the continuity of u, we have the conclusion. □

We next discuss the idea of the construction of the "good" sequence of subsolutions to prove the nonexistence theory. Let f satisfies (A1) and (A3) with $\beta \geq \frac{1}{1-\alpha}$ and g satisfies (A2) with $\alpha < 1$. Our purpose is to construct a sequence of sub-solutions v_L to (1.1) which is uniformly bounded at $t = 0$ and diverges to ∞ as $L \to \infty$ on $(-b,b)\times(0,\infty)$. The idea is from the approximated inequality, which follows from the assumptions (A2) and (A3),

$$\frac{f(g(V_y(y))V_{yy}(y))}{V_y} \approx L^{\beta+\beta(1-\alpha)-1} y^{L(1-\beta(1-\alpha))-\beta} \geq L^{\beta+\beta(1-\alpha)-1} =: c_L$$

if $0 < y \leq 1$ and $L \gg 1$ for $V(y) = y^{-L}$. The boundedness from blow holds due to $1-\beta(1-\alpha) \leq 0$. Since $c_L \to \infty$ and $V(y) \to \infty$ as $L \to \infty$ for $0 < y < 1$, the sequence of sub-solutions can be constructed as a shifted and scaled version of traveling functions $V(c_L t - x)$. We first define the sequence of sub-solutions as follows:

Definition 4.2. *For constants $L > 0$ and $c_L > 0$, we define $v_L \in C((-b,b) \times [0,\infty))$ as*

$$v_L(x,t) := \begin{cases} (\frac{3b-x}{2b} - c_L t)^{-L} & \text{for} \quad b - 2bc_L t \leq x < b,\ 0 < t < \frac{1}{c_L}, \\ 1 & \text{for} \quad -b < x < b - 2bc_L t,\ 0 \leq t < \frac{1}{c_L}, \\ (\frac{b-x}{2b})^{-L} & \text{for} \quad -b < x < b,\ t \geq \frac{1}{c_L}. \end{cases} \tag{4.1}$$

Then, we can prove that v_L is a sub-solution to (1.1) for suitable c_L and $L > 0$ large as follows:

Proposition 4.3. *Let $b > 0, \alpha < 1$ and $\beta \geq \frac{1}{1-\alpha}$. Assume that function f satisfies (A1) and (A3) and function g satisfies (A2). Then, there exist $L_0 > 0$ such that for any $L \geq L_0$ there exists c_L such that*

$$\lim_{L\to\infty} c_L = \infty \tag{4.2}$$

and the function v_L defined by (4.1) satisfies

$$(v_L)_t \leq f\left(\frac{1}{2} g((v_L)_x)(v_L)_{xx}\right) \quad \text{on} \ (-b,b) \times (0,\infty) \tag{4.3}$$

in the viscosity sense.

We here omit the proof of Proposition 4.3 since the idea of proof is as above. Let us now prove Theorem 3.5 (a).

Proof of Theorem 3.5 (a). Let $M := \inf_{x\in(-b,b)} u_0(x)$. Assume that a solution u to (1.1)–(1.3) exists. Let v_L be the sequence of sub-solutions to (1.1) obtained in Proposition 4.3. Then, from $v_L(\cdot,0) - (M+1) \leq u_0$ and the convexity of $v_L(\cdot,t)$, we have by the comparison result in Proposition 4.1

$$v_L(x,t) - (M+1) \leq u(x,t) \quad \text{for } (x,t) \in (-b,b) \times (0,1/c].$$

Letting $L \to \infty$, we have $u(x,t) = \infty$ for $(x,t) \in (-b,b) \times (0,\infty)$, which contradicts that u is a function defined on $(-b,b) \times [0,\infty)$. □

References

[1] Beomjun Choi, Kyeongsu Choi, and Panagiota Daskalopoulos. Convergence of curve shortening flow to translating soliton. *Amer. J. Math.*, Vol. 143, No. 4, pp. 1043–1077, 2021.

[2] Takashi Kagaya, Singular Dirichlet boundary problems for a class of fully nonlinear parabolic equations in one dimension. In preparation, arXiv:2506.07415.

[3] Takashi Kagaya and Qing Liu. Singular Neumann boundary problems for a class of fully nonlinear parabolic equations in one dimension. *SIAM J. Math. Anal.*, Vol. 53, No. 4, pp. 4350–4385, 2021.

[4] J.-M. Lasry and P.-L. Lions. Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem. *Math. Ann.*, Vol. 283, No. 4, pp. 583–630, 1989.

STABILITY OF AN EQUILIBRIUM OF THE MHD EQUATIONS IN 3D DOMAINS WITH ARBITRARY GEOMETRY

HIDEO KOZONO, SENJO SHIMIZU, AND TAKU YANAGISAWA

1. Introduction

This article is a survey paper by authors [4].

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. We consider the initial-boundary value problem of the magnetohydrodynamic equations(MHD equations) in Ω;

$$
\text{(MHD)}\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u\cdot\nabla u - B\cdot\nabla B + \nabla(\frac{1}{2}|B|^2 + p) = 0 \quad \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial B}{\partial t} + \text{rot }(\text{rot } B) + \text{rot }(B\times u) = 0 \quad \text{in } \Omega\times(0,\infty), \\ \text{div } u = 0, \quad \text{div } B = 0 \quad \text{in } \Omega\times(0,\infty), \\ u = 0, \ \ B\cdot\nu = 0, \ \ \text{rot } B\times\nu = 0 \quad \text{on } \partial\Omega\times(0,\infty), \\ u|_{t=0} = u_0, \quad B|_{t=0} = B_0 \quad \text{in } \Omega, \end{cases}
$$

where ν denotes the unit outer normal vector to $\partial\Omega$.

We first introduce the linearized operator $\mathcal{L}_r$, $1<r<\infty$ with the domain $D(\mathcal{L}_r)$ defined by

$$
(1.1)\qquad \begin{cases} D(\mathcal{L}_r) = \{B\in H^{2,r}(\Omega);\ \text{rot } B\times\nu|_{\partial\Omega} = 0,\ B\cdot\nu|_{\partial\Omega} = 0\}, \\ \mathcal{L}_r B = -\Delta B = \text{rot rot } B - \nabla\text{div } B \quad \text{for } B\in D(\mathcal{L}_r). \end{cases}
$$

It is known that $\mathcal{L}_r$ has a non-trivial kernel unless Ω is simply connected. Indeed, we have that

$$
\text{Ker}(\mathcal{L}_r) = X_{\text{har}}(\Omega) \equiv \{h\in C^\infty(\bar{\Omega}); \text{rot } h = 0,\ \text{div } h = 0,\ h\cdot\nu|_{\partial\Omega} = 0\}
$$

for all $1<r<\infty$, and hence if the domain Ω has the second Betti number N, then zero is an eigenvalue of $\mathcal{L}_r$ with the multiplicity N. To get around such difficulty, we make use of the Helmholtz–Weyl decomposition given by [3, Theorem 2.1]

$$
\begin{aligned}
(1.2)\qquad L^r(\Omega) &= L^r_\sigma(\Omega)\oplus\nabla H^{1,r}(\Omega) \\
(1.3)\qquad &= X_{\text{har}}(\Omega)\oplus\text{rot } V^r_\sigma(\Omega)\oplus\nabla H^{1,r}(\Omega), \quad 1<r<\infty \quad \text{(direct sum)},
\end{aligned}
$$

where

$$
\begin{aligned}
L^r_\sigma(\Omega) &\equiv \{v\in L^r(\Omega); \text{div } v = 0,\ v\cdot\nu|_{\partial\Omega} = 0\}, \\
V^r_\sigma(\Omega) &\equiv \{w\in H^{1,r}(\Omega);\ \text{div } w = 0,\ w\times\nu|_{\partial\Omega} = 0\}.
\end{aligned}
$$

To solve (MHD) we consider an abstract setting of evolution equation on $L^r_\sigma(\Omega)\times\mathscr{X}^r_\sigma(\Omega)$, where

$$
(1.4)\qquad \mathscr{X}^r_\sigma(\Omega)\equiv\text{rot } V^r_\sigma(\Omega).
$$

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Let $R_r : L^r(\Omega) \to \mathscr{X}^r_\sigma(\Omega)$ be the projection operator along the decomposition (1.3) and (1.4)(see, e.g., [3, Corollary 2.2]). We define the magnetic Laplace operator L_r by

$$(1.5) \qquad \begin{cases} D(L_r) = D(\mathcal{L}_r) \cap \mathscr{X}^r_\sigma(\Omega), \\ L_r \bar{B} = \mathcal{L}_r \bar{B} = \text{rot rot } \bar{B} \quad \text{for } \bar{B} \in D(L_r), \end{cases}$$

where $\mathcal{L}_r$ is the same one as is defined by (1.1). L_r may be regarded as a closed operator in $\mathscr{X}^r_\sigma(\Omega)$ and the projection R_r commutes with $-\Delta$ on $D(\mathcal{L}_r)$.

2. Preliminaries

In what follows, for $1 < r < \infty$, we denote by $\|\cdot\|_r$ the usual L^r-norm on Ω. We denote by C constants which may change from the line to line. In particular, $C = C(\cdot, \cdots, \cdot)$ denotes the constant depending only on the quantities appearing in the parenthesis.

2.1. **Magnetic Laplace operator.** We first investigate the operator L_r given by (1.5) on the space $\mathscr{X}^r_\sigma(\Omega)$ defined by (1.4). By (1.3), $\mathscr{X}^r_\sigma(\Omega)$ is a closed subspace in $L^r(\Omega)$, and we may regard $\mathscr{X}^r_\sigma(\Omega)$ itself as a Banach space with the L^r-norm $\|\cdot\|_r$.

Lemma 2.1. *Let $1 < r < \infty$. Let $\mathcal{L}_r$ and L_r be operators defined by (1.1) and (1.5), respectively. Suppose that $R_r : L^r(\Omega) \to \mathscr{X}^r_\sigma(\Omega)$ is a projection associated with (1.3) and (1.4).*

(i) *For the kernel* $\mathrm{Ker}(\mathcal{L}_r)$ *of $\mathcal{L}_r$, it holds that*

$$\mathrm{Ker}(\mathcal{L}_r) = X_{har}(\Omega)$$

for all $1 < r < \infty$.

(ii) *We have $R_r D(\mathcal{L}_r) = D(L_r)$, and it holds that*

$$\mathcal{L}_r R_r = R_r \mathcal{L}_r = L_r R_r \quad \text{on } D(\mathcal{L}_r).$$

In particular, we have $Ran(L_r) \subset \mathscr{X}^r_\sigma(\Omega)$, where $Ran(L_r)$ denotes the range of L_r. Furthermore, L_r may be regarded as a densely defined closed operator in $\mathscr{X}^r_\sigma(\Omega)$.

(iii) *It holds that*

$$\|\nabla^2 w\|_r + \|w\|_r \le C\|L_r w\|_r, \quad w \in D(L_r)$$

with $C = C(\Omega, r)$.

We next investigate the resolvent set $\rho(-L_r)$ of L_r.

Proposition 2.2. *Let $1 < r < \infty$. We have that*

$$\rho(-\mathcal{L}_r) \subset \rho(-L_r),$$

and it holds that

$$(\mathcal{L}_r + \lambda)^{-1} R_r = R_r (\mathcal{L}_r + \lambda)^{-1} = (L_r + \lambda)^{-1} R_r \quad \text{on } L^r(\Omega)$$

for all $\lambda \in \rho(-\mathcal{L}_r)$.

Proposition 2.3. *Let $1 < r < \infty$.* (i) *There exist $\delta > 0$ and $\frac{\pi}{2} < \omega < \pi$ such that*

$$\Sigma_\omega \equiv \{\lambda \in \mathbb{C}; |arg\ \lambda| < \omega,\ \lambda \neq 0\} \cup B_\delta(0) \subset \rho(-L_r)$$

($B_\delta(0) \equiv \{\lambda \in \mathbb{C}; |\lambda| < \delta\}$) with the estimate

$$\|(L_r + \lambda)^{-1}\|_{\mathbb{B}(\mathscr{X}^r_\sigma(\Omega))} \le \frac{M}{1 + |\lambda|}$$

for all $\lambda \in \Sigma_\omega \cup B_\delta(0)$, where $M > 1$ is some constant depending on r.

(ii) $-L_r$ *generates a bounded analytic semi-group* $\{e^{-tL_r}\}_{t>0}$ *of class* C^0 *in* $\mathscr{X}^r_\sigma(\Omega)$ *with the estimate*

$$\|e^{-tL_r}a\|_r \le Ce^{-\gamma t}\|a\|_r \tag{2.1}$$

for all $a \in \mathscr{X}^r_\sigma(\Omega)$ *and all* $t > 0$ *with some* $\gamma > 0$*, where* $C = C(\Omega, r)$.

(iii) *The fractional power* L_r^α *of* L_r *is well defined for* $0 \le \alpha \le 1$*, and it holds that*

$$\|L_r^\alpha e^{-tL_r}a\|_r \le Ct^{-\alpha}e^{-\gamma t}\|a\|_r$$

for all $a \in \mathscr{X}^r_\sigma(\Omega)$ *and all* $t > 0$*, where* γ *and* C *are the same as in (2.1).*

We next characterize the domain $D(L_r^\alpha)$ of fractional powers L_r^α for $0 \le \alpha \le 1$.

Proposition 2.4. *Let* $1 < r < \infty$*. It holds that*

$$D(L_r^\alpha) \subset [L^r(\Omega), D(\mathcal{L}_r)]_\alpha, \quad 0 \le \alpha \le 1$$

with the estimate

$$\|u\|_{H^{2\alpha,r}(\Omega)} \le C\|L_r^\alpha u\|_r$$

for all $u \in D(L_r^\alpha)$ *with* $C = C(\Omega, r, \alpha)$*, where* $[X, Y]_\alpha$ *denotes the complex interpolation space of* X *and* Y.

The following L^r-L^q-estimates of the semi-group $\{e^{-tL}\}_{t>0}$ play an important role to construct global solutions of (MHD).

Lemma 2.5. *For every* $1 < q \le r < \infty$ *there is a constant* $C = C(\Omega, q, r)$ *such that*

$$\|e^{-tL}a\|_r \le Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})}e^{-\gamma t}\|a\|_q,$$
$$\|\nabla e^{-tL}a\|_r \le Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}e^{-\gamma t}\|a\|_q$$

for all $a \in \mathscr{X}^q_\sigma(\Omega)$ *and all* $t > 0$.

2.2. **Abstract evolution equation.** Our purpose is to show an asymptotic stability of the nontrivial equilibrium state of the form $U_* = (0, B_*)$ with $B_* \in X_{\mathrm{har}}(\Omega)$. It is shown that $u(x,t) \equiv 0$ and $B(x,t) = B_*(x)$ give a stationary solution of (MHD). In fact, since rot $B_* = 0$, we have an identity $\nabla(|B_*|^2) = 2B_* \cdot \nabla B_*$, which yields that U_* is an exact stationary solution of (MHD). If $\bar{B}_0 \equiv B_0 - B_*$ is an initial disturbance of B_*, then the perturbed magnetic field $\bar{B}(x,t) \equiv B(x,t) - B_*(x)$ is subject to the following abstract evolution equations;

$$\text{(E)} \qquad \begin{cases} \dfrac{du}{dt} + A_r u + P_r(u \cdot \nabla u) \\ \quad - P_r(B_* \cdot \nabla \bar{B} + \bar{B} \cdot \nabla B_* + \bar{B} \cdot \nabla \bar{B}) = 0 \quad \text{in } L^r_\sigma(\Omega) \times (0,\infty), \\ \dfrac{d\bar{B}}{dt} + L_r\bar{B} + \text{rot } (B_* \times u) + \text{rot } (\bar{B} \times u) = 0 \quad \text{in } \mathscr{X}^r_\sigma(\Omega) \times (0,\infty), \\ u|_{t=0} = u_0, \quad \bar{B}|_{t=0} = \bar{B}_0. \end{cases}$$

Here $P_r : L^r(\Omega) \to L^r_\sigma(\Omega)$ denotes the Helmholtz decomposition along (1.2) and A_r is the Stokes operator defined by

$$\begin{cases} D(A_r) = \{u \in H^{2,r}(\Omega);\ u|_{\partial\Omega} = 0\} \cap L^r_\sigma(\Omega), \\ A_r u = -P_r \Delta u \quad \text{for } u \in D(A_r). \end{cases}$$

For more details, we refer to Giga [2].

Let us rewrite (E) in a more abstract way. Let

$$S_r \equiv \begin{pmatrix} A_r & K_{B_*} \\ J_{B_*} & L_r \end{pmatrix} \quad \text{with the domain } D(S_r) = D(A_r) \times D(L_r),$$

where K_{B_*} and J_{B_*} are linear operators defined by

$$K_{B_*}\bar{B} \equiv -P(B_* \cdot \nabla \bar{B} + \bar{B} \cdot \nabla B_*), \quad J_{B_*}u \equiv \text{rot } (B_* \times u).$$

We may regard S_r as the densely defined closed operator in $L^r_\sigma(\Omega) \times \mathscr{X}^r_\sigma(\Omega)$ for $1 < r < \infty$. Then we have the following proposition.

Proposition 2.6. *Let $1 < r < \infty$. There is a positive constant $\delta_* = \delta_*(r)$ such that if*

$$\|B_*\|_1 \leq \delta_*, \tag{2.2}$$

then the resolvent set $\rho(-S_r)$ of $-S_r$ contains the origin $\{0\}$ and the sectorial region $\Sigma_\omega \equiv \{\lambda \in \mathbb{C}; |\arg \lambda| < \omega, \lambda \neq 0\}$ for some $\pi/2 < \omega < \pi$ with the estimate

$$\|(S_r + \lambda)^{-1}\|_{\mathbb{B}(L^r_\sigma(\Omega)\times\mathscr{X}^r_\sigma(\Omega))} \leq \frac{M_r}{1 + |\lambda|}$$

for all $\lambda \in \Sigma_\omega \cup \{0\}$ with some constant $M_r > 1$. ($\mathbb{B}(Y)$ is the class of bounded linear operators in Y.) Hence, under the assumption (2.2), $-S_r$ generates a bounded analytic semigroup $\{e^{-tS_r}\}_{t>0}$ in $L^r_\sigma(\Omega) \times \mathscr{X}^r_\sigma(\Omega)$ with the estimate

$$\|e^{-tS_r}U_0\|_{L^r_\sigma(\Omega)\times\mathscr{X}^r_\sigma(\Omega)} \leq Ce^{-\gamma t}\|U_0\|_{L^r_\sigma(\Omega)\times\mathscr{X}^r_\sigma(\Omega)}$$

for all $U_0 \in L^r_\sigma(\Omega) \times \mathscr{X}^r_\sigma(\Omega)$ and all $t > 0$ with some $\gamma > 0$, where $C = C(\Omega, r)$.

We next show the L^p-L^q-type estimates of semigroup $\{e^{-tS}\}_{t>0}$ in $L^r_\sigma(\Omega) \times \mathscr{X}^r_\sigma(\Omega)$.

Lemma 2.7. *For every $1 < q \leq r < \infty$ there is a constant $\mu(q, r)$ such that if*

$$\|B_*\|_1 \leq \mu,$$

then it holds that

$$\|e^{-tS}F\|_{L^r_\sigma(\Omega)\times\mathscr{X}^r_\sigma(\Omega)} \leq Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})}e^{-\gamma t}\|F\|_{L^q_\sigma(\Omega)\times\mathscr{X}^q_\sigma(\Omega)},$$

$$\|\nabla e^{-tS}F\|_{L^r(\Omega)\times L^r(\Omega)} \leq Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}e^{-\gamma t}\|F\|_{L^q_\sigma(\Omega)\times\mathscr{X}^q_\sigma(\Omega)}$$

for all $F = (f, g) \in L^q_\sigma(\Omega) \times \mathscr{X}^q_\sigma(\Omega)$ and all $t > 0$ with $\gamma > 0$ and $C = C(\Omega, q, r)$

2.3. **Nonlinear structure of** rot $(B \times u)$**.** Finally in this section, we show that rot $(B \times u)$ remains in $\mathscr{X}^r_\sigma(\Omega)$, which enables us to solve (E) for (u, B) in the product space $L^r_\sigma(\Omega) \times \mathscr{X}^r_\sigma(\Omega)$.

Lemma 2.8. *Let $u \in H^{1,r}(\Omega)$ and $B \in H^{1,r}(\Omega)$ for $r > 3$. If either $u = 0$ on $\partial\Omega$, or $u \cdot \nu = B \cdot \nu = 0$ on $\partial\Omega$, then we have that* rot $(B \times u) \in \mathscr{X}^r_\sigma(\Omega)$*. In case $r = 3$, it holds that* rot $(B \times u) \in \mathscr{X}^q_\sigma(\Omega)$ *for all $1 < q < 3$.*

3. Main Theorem

For $U = (u, \bar{B}) \in D(A_r) \times D(L_r) \subset L^r_\sigma(\Omega) \times \mathscr{X}^r_\sigma(\Omega)$, we define the nonlinear mapping N associated with (E) by

$$NU \equiv (P(u \cdot \nabla u) - P(\bar{B} \cdot \nabla \bar{B}), \text{ rot } (\bar{B} \times u)).$$

Then we may rewrite (E) to the following evolution equation on $L^r_\sigma(\Omega) \times \mathscr{X}^r_\sigma(\Omega)$.

$$\text{(E')} \qquad \begin{cases} \dfrac{dU}{dt} + SU + NU = 0 \quad \text{in } L^r_\sigma(\Omega) \times \mathscr{X}^r_\sigma(\Omega),\ 0 < t < \infty, \\ U(0) = U_0 \equiv (u_0, \bar{B}_0). \end{cases}$$

By the Duhamel principle, we may solve (E') by means of the following system of integral equations:

$$\text{(IE)} \qquad U(t) = e^{-tS}U_0 - \int_0^t e^{-(t-\tau)S} NU(\tau)d\tau \quad \text{in } L^r_\sigma(\Omega)\times \mathscr{X}^r_\sigma(\Omega),\ 0<t<\infty.$$

Our result now reads;

Theorem 3.1. *For every $3 < r < \infty$ there are positive constants $\delta_* = \delta_*(r)$ and $\delta = \delta(r)$ such that if $B_* \in X_{har}(\Omega)$, $u_0 \in L^3_\sigma(\Omega)$ and $\bar{B}_0 \in \mathscr{X}^3_\sigma(\Omega)$ satisfy*

$$\|B_*\|_1 \le \delta_*, \quad \|u_0\|_3 + \|\bar{B}_0\|_3 \le \delta,$$

then there exists a unique solution $U = (u, \bar{B})$ of (IE) on $(0,\infty)$ such that

$$\begin{aligned}
& u \in BC([0,\infty); L^3_\sigma(\Omega)), \quad t^{\frac{3}{2}(\frac{1}{3}-\frac{1}{r})}u \in BC([0,\infty); L^r_\sigma(\Omega)),\\
& t^{\frac{3}{2}(\frac{1}{3}-\frac{1}{r})+\frac{1}{2}}\nabla u \in BC([0,\infty); L^r(\Omega)),\\
& \bar{B} \in BC([0,\infty); \mathscr{X}^3_\sigma(\Omega)), \quad t^{\frac{3}{2}(\frac{1}{3}-\frac{1}{r})}\bar{B} \in BC([0,\infty); \mathscr{X}^r_\sigma(\Omega)),\\
& t^{\frac{3}{2}(\frac{1}{3}-\frac{1}{r})+\frac{1}{2}}\nabla \bar{B} \in BC([0,\infty); L^r(\Omega)),
\end{aligned}$$

where BC denotes the class of bounded and continuous functions. Moreover, u and $\bar{B}$ have the following decay properties

$$\|u(t)\|_r + \|\nabla u(t)\|_r = O(e^{-\alpha t}), \quad \|\bar{B}(t)\|_r + \|\nabla \bar{B}(t)\|_r = O(e^{-\alpha t})$$

with some $\alpha > 0$ as $t \to \infty$.

Remark. Our result with the formulation such as (E) is similar to that of Yoshida–Giga [7, Theorem 6.1]. They proved the corresponding theorem on existence and uniqueness of global solutions to the case $X_{\rm har}(\Omega) = \{0\}$, i.e., Ω is a simply connected domain. See also Akiyama [1, Theorem 3]. In a multi-connected domain, Mosconia–Solonnikov [6, Theorem 6.5] proved a similar result to ours in the L^2-setting. Monniaux [5, Theorem 3.3] treated u and B as differential one and two forms, respectively, and also proved a similar result. Her method is related to the de Rham–Hodge–Kodaira decomposition of differential forms on Ω. On the other hand, our decomposition (1.3) rests on that of vector fields on Ω so that boundary conditions of u and B in (MHD) and in [5] are necessarily different from each other.

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References

[1] Akiyama,T., *On the existence of L^p solutions of the magnetohydrodynamic equations in a bounded domain.* Nonlinear Analysis **54** (2003) 1165–1174.

[2] Giga, Y., *Analyticity of the semigroup generated by the Stokes operator in L_r spaces.* Math. Z. **178** (1981),297–329.

[3] Kozono, H., Yanagisawa, T., *Variational inequality for vector fields and the Helmholtz–Weyl decomposition in bounded domains.* Inidiana Univ. Math, J. **58** (2009), 1853–1920.

[4] Kozono, H., Shimizu, S., Yanagisawa, T., *Stability of harmonic vector fields as an equilibrium of the 3D MHD equations in bounded domains with arbitrary geometry.* to appear in Math. Ann..

[5] Monniaux, S., *Existence in critical spaces for the magnetohydrodynamical system in 3D bounded Lipschitz domains.* J. Elliptic Parabol. Equ. **7**(2021), 311–322.

[6] Mosconia, S.J.N., Solonnikov, V.N., *On a problem of magnetohydrodynamics in a multi-connected domain.* Nonlinear Analysis **74** (2011), 462–478.

[7] Yoshida, Z., Giga, Y., *On the Ohm-Navier-Stokes system in magnetohydrodynamics.* J. Math. Phys. **24** (1983), 2860–2864.

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